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An Off-critical Dimer Model on the Hexagonal Lattice and Massive SLE_2

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Abriss

Das Dimer Modell auf einem ebenen Graph kann als zufällige Fläche gesehen werden. Im kritischen Fall wird vermutet, dass die Schwankungen dieser zufällligen Fläche gegen ein Gaussian Free Field konvergieren. Chhita bewies in [8], dass die Schwankungen in einem off-critical Fall auf einem Quadratgitter im Grenzwert nicht gaussian sind. Wir untersuchen dieses und ein verwandtes Modell auf dem Sechseckgitter mit der verallgemeinerten Temperley Bijektion um sie mit einem massive loop-erased random walk zu verbinden, dessen scaling limit massive SLE_2 ist, wie von Chelkak und Wan in [7] bewiesen wurde. Kombiniert mit einem anderen aktuellen Resultat von Berestycki, Laslier und Ray [5] ergibt dies ein Scaling Limit für die Schwankungen der off-critical Modelle.

Abstract

The dimer model on a planar graph can be seen as a random surface, whose fluctuations are conjectured to converge to a Gaussian free field in the critical case. Chhita proved in [8] that the fluctuation for an off-critical dimer model on the square grid in the scaling limit are non-Gaussian. We study this and an analogue model on the hexagonal lattice more closely by using the generalized Temperley bijection to connect the models to the massive loop-erased random walk, whose scaling limit is massive SLE_2 as recently proven by Chelkak and Wan [7]. Combined with another recent result by Berestycki, Laslier and Ray [5] this gives a scaling limit for the height fluctuations of the off-critical models.

Contents

1 Introduction **1**
1.1 The setup and previous results . . . 3
1.2 Results and structure . . . 4

2 Background **5**
2.1 Dimers . . . 5
2.2 Temperley's Bijection . . . 8
2.3 Wilson's Algorithm . . . 10
2.4 Winding of LERW and the height function . . . 11
2.5 Schramm Loewner Evolution . . . 13

3 Brownian motion and scaling limits **14**

4 LERW with drift and massive LERW **17**
4.1 The directed triangular lattice . . . 17
4.2 Square lattice . . . 21

5 Convergence of massive LERW on the triangular lattice **24**
5.1 Convergence of domains and curves . . . 24
5.2 Absolute continuity with respect to classical SLE_2 . . . 25
5.3 Massive Martingale Observables . . . 26
5.4 Convergence of non-massive LERW . . . 28
5.5 Convergence of the Green function . . . 29
5.6 Convergence to the continuous martingales . . . 37
5.7 Proof of the main statement . . . 39

6 The main results **40**

7 Further investigation **41**

1 Introduction

A common feature in discrete statistical mechanics models is the occurrence of *critical points*: For certain values of parameters the large scale behaviour of the model changes in a discontinuous or non-smooth way. It is often the case that at these points the models are the most interesting. A very useful tool in studying such models on lattices is the *scaling limit* as the mesh-size tends to zero. Away from the critical point this limit is often trivial in some sense either converging to a deterministic point measure or some form of white noise, while at the critical point one obtains some kind of "universal object" with complex behaviour, yet independent of the particular lattice used and other microscopic details.

However, one can also consider the behaviour of the model when moving the parameter to the critical value while simultaneously letting the mesh-size approach zero. At a certain rate of convergence to the critical point one obtains an interesting limit which is neither trivial nor identical to the behaviour at the critical point. This is called a *scaling window*. Studying these limits can both give an interesting family of related examples and be useful in understanding the behaviour at the critical point itself.

The model studied in this thesis is the *dimer model*, which is defined on planar bipartite finite graphs as follows: A *dimer configuration* is a collection of edges such that each vertex is contained in exactly one of the edges. [1] If the set of dimer configuration on a graph is non-empty a probability measure on it is introduced by giving each edge in the graph a positive weight and setting the probability of each dimer configuration to be proportional to the product of the weights of its edges.

To each dimer configuration one can associate a height function h from the faces of the graph to $\mathbb{R}$ which fully determines the configuration. In the case of dimer configuration on subgraphs of the hexagonal lattice, which correspond to lozenge tilings, this height function is very visual, see Figure 1. For the definition of the height function on general graphs see section 2.1. Through this height function the dimer model can be regarded as a random surface model. It is known in a large variety of situations that the height function (rescaled linearly) converges to a deterministic limit shape given as the minimizer of a surface tension integral, see e.g. [9, 15, 16] It is also conjectured in many cases that the centered fluctuations of the height functions converge to a Gaussian Free Field but proven only in special cases, see for example [5].

[1] In graph theory this is also called a perfect matching or a 1-factor.

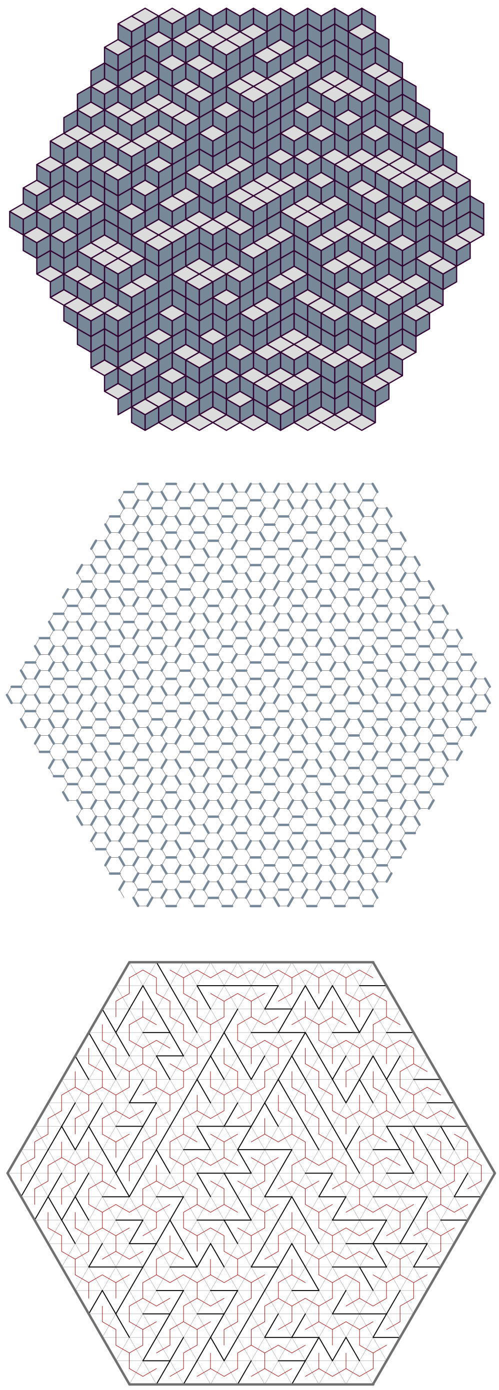

Figure 1: From top to bottom: A lozenge tiling. The corresponding dimer configuration on the hexagonal lattice. The corresponding tree and dual tree as in Tempereley's bijection.

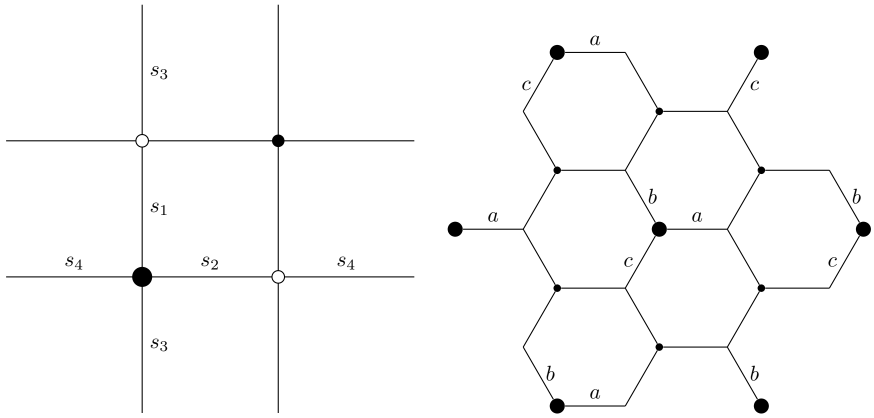

Figure 2: The drifted dimer model on the square grid, as in [8] and the for the drifted dimer model on the hexagonal lattice. All edges without weight have weight 1, the larger vertices correspond to the primal vertices (or *vertex-nodes*) in the Temperley bijection.

1.1 The setup and previous results

We study the dimer model in two settings: the square grid and the hexagonal lattice. On the square grid we consider the set of vertices obtained by translating one fixed vertex by multiples of two in both lattice directions. Around each of these vertices we set the weights s_1, s_2, s_3 and s_4 and all edges not adjacent to one of these vertices get weight 1. This is the same as the *drifted dimer model* in [8].

On the hexagonal lattice we also consider the set of vertices obtained by translating one vertex, but this time by 3 times the edge length in the directions $1, e^{2\pi/3}$ and $e^{4\pi/3}$. These are the shortest translations in those directions that leave the lattice invariant. Around each of these vertices we assign weights a, b and c and again 1 to the rest, see Figure 2.

These are called *drifted* dimer models because they are connected by Temperley's bijection and Wilson's algorithm to random walks with drift on the square grid and the triangular lattice. To emphasize this connection one can assume that $s_1+s_2+s_3+s_4 = 1$ and $a + b + c = 1$. This can be achieved by a *gauge transformations*, i.e. multiplying all weights around a vertex by the same factor, which doesn't change the dimer model measure. The critical point for these models is at $s_1 = s_2 = s_3 = s_4$ and at $a = b = c$ respectively, which correspond to the simple random walk.

The main tool for studying the dimer model is the Kasteleyn matrix and the associated theory. This thesis chooses a different approach based on the Temperley bijection itself. However, let us briefly consider the following observation: The Kasteleyn matrix for the drifted dimer model on the square grid has the following property: the term on the diagonal is $-(s_1^2+s_2^2+s_3^2+s_4^2)$, while the sum of the off-diagonal terms is $2s_1s_3+2s_2s_4$. Therefore, the sum of a row in the matrix is: $-(s_1-s_3)^2-(s_2-s_4)^2 \leq 0$. At criticality, this is 0 Whenever this is strictly less than 0 this suggests a positive mass term. Notice that this term is less then 0 if $s_1 \neq s_3$ or $s_2 \neq s_4$, which is the case exactly if there is a drift in the corresponding random walk.

Chhita's paper [8] mentions this, but by calculating Green's function for this and a related dimer model in the full plane proofs the fluctuations of the height function are non-Gaussian, therefore ruling out a massive Gaussian free field as a scaling limit. This thesis establishes a connection of the dimer models to massive SLE_2.

1.2 Results and structure

The following theorem gives the connection between the loop-erased random walk with drift and the loop-erased massive random walk:

Theorem 1.1. *The loop-erased random walk with drift on the directed triangular lattice or the square lattice started at* 0 *and run until it hits the boundary of a discrete domain* Ω^δ *has as a scaling limit which is absolutely continuous with respect to massive* SLE_2 *and its Radon-Nikodym derivative with respect to this measure is given by:*

$$\exp(2\,\langle \gamma_0, d \rangle),$$

where γ_0 *is the point of the path on the boundary of* Ω *and* d *is the drift of the Brownian motion obtained by taking the scaling limit of the walk without the loop-erasure.*

Using Schramm's finiteness theorem from [26] allows us to relate this to the random spanning tree on the weighted directed triangular lattice or the square grid:

Corollary 1.2. *The spanning tree on the weighted directed triangular lattice converges in law in the Schramm topology.*

By using [4, Theorem 2.2.] this gives the following corollary:

Corollary 1.3. *The height function of the off-critical dimer model converges in law and the limit depends only on the law of massive SLE(2). In particular it is absolutely continuous with respect to the non-critical height function.*

The structure of the thesis is as follows: In section 2, background on dimers, the Temperley bijection, Wilson's algorithm and the Schramm-Loewner evolution are recalled. Also the connection between the winding of the loop-erased random walk and the height function is explained. In section 3, we consider Brownian motion with drift and massive Brownian motion as off-critical phenomena. In section 4, a connection between the random walk with drift and the massive random walk is made: On the hexagonal lattice a random walk with drift conditioned to exit a domain at a certain boundary point is equal in law to a massive random walk conditioned to exit at the same point. For the square lattice this equality holds only asymptotically and in probability. Furthermore, the Radon-Nikodym derivative of the random walk with drift with respect to the massive random walk is shown to converge to a simple function of the exit point of the path.

In section 5 the convergence of the massive loop-erased random walk to massive SLE_2 is established following the strategy for the square lattice from [7] and adapting where necessary. Finally in section 6 the results are collected.

All simulations shown in this thesis are original images generated using Python, all diagrams were made using TikZ.

2 Background

2.1 Dimers

In this section we introduce the dimer model and the height function for planar graphs.

A graph G is a pair of vertices V and edges E. Edges will be written as pairs of vertices, even if our graphs are not directed. If $e = (x, y)$ then $-e := (y, x)$ is the reversal of e. In the undirected case (the graphs on which we consider dimer configurations are always undirected) $e \in E$ is equivalent to $-e \in E$. All graphs will be planar, meaning they can be embedded in $\mathbb{C}$ such that no two edges intersect except at their endpoints.

Every embedded planar graph splits $\mathbb{C}$ into connected components which are called faces. Each face has a boundary consisting of some edges, each edge bounds two faces (or sometimes only one, but not in our cases). To each planar graph G one can associate a dual graph $G^\dagger$ that has the faces of G as vertices and for every edge e of G there is a dual edge $e^\dagger$ of $G^\dagger$ that connects the faces that have e on their boundary and is oriented such that if $e = (x, y)$ then $e^\dagger$ has x to its left and y to its right. Note that $e^{\dagger\dagger} = -e$.

The triangular lattice can be embedded in the complex plane by considering all integer linear combinations of 1 and $e^{2\pi/3}$ as vertices and connecting pairs of vertices with distance 1 with an edge. Every vertex has degree 6 and every face is an equilateral triangle. Hence the dual of the triangular lattice has hexagonal faces and vertices of degree 3. The hexagonal lattice is bipartite and we will sometimes refer to black and white vertices, referring to a 2-coloring of the vertices, which is unique up to exchanging the two colors.

On the hexagonal lattice or finite subsets of it we will consider dimer configurations: A dimer configuration on a graph G is a subset M of edges, called *dimers*, such that every vertex of the graph is in exactly one of the edges in M and not more than that. If an edge e is in a dimer configuration one may also say it is *occupied* by a dimer. Since the hexagonal lattice is bipartite, a necessary but not sufficient condition of the existence of a dimer configuration on a finite subgraph is that there are as many black vertices as there are white ones. In the examples we will consider the existence of a dimer configuration will be given by construction.

If the set of configurations is non-empty we define a probability measure on this set: Let $w(e)$ be a positive weight associated to each edge e. Then let the probability of a dimer configuration M be

$$\frac{1}{Z_{G,w}} \sum_{e \in M} w(e), \tag{2.1}$$

where $Z_{G,w}$ is a normalizing constant. If $w(e) = 1$ for every edge e this is the uniform measure on the set of dimer configurations.

Now we define the height function. Let us first consider the entire lattice. We call an function θ on edges a flow if it is antisymmetric, i.e. $\theta(e) = -\theta(-e)$. The divergence of a flow θ at a vertex v is given by;

$$d^*\theta(v) = \sum_{(v,w) \in E} \theta(v, w).$$

This can be seen as the amount that is "flowing out" of the vertex v minus the amount that is "flowing in".

To each dimer configuration M we associate a flow ω_M of strength one from each of the white vertices to the corresponding black vertex it is matched with. This flow has divergence $+1$ at each of the white vertices and divergence -1 at each of the black vertices. We fix a reference flow ω_{ref} with that same divergence. That could be one given by a fixed dimer configuration but we will mostly choose $\omega_{\text{ref}}(e) = \frac{1}{deg(e^-)}$, which will be 1/3 on the hexagonal lattice.

As ω_{ref} and ω_M have the same divergence at every vertex the difference $\theta_M = \omega_{\text{ref}} - \omega_M$ is divergence free, i.e. has divergence 0 at every point. To each divergence free flow on a planar graph we can associate a height function on the dual graph. For each dual edge $e^\dagger$ we set $\theta_M^\dagger(e^\dagger) = \theta_M(e)$. Since θ_M is divergence free $\theta_M^\dagger$ satisfies for each cycle π in $G^\dagger$:

$$\sum_{e^\dagger \in \pi} \theta_M^\dagger(e^\dagger) = 0, \tag{2.2}$$

as this is the amount of flow out of or into the inside of the cycle π depending on the orientation of π. As θ_M is divergence free this flow is always 0.

After setting the height of an arbitrary face f^* to 0 this allows us to define the height function:

$$h(f) = \sum_{e \in \gamma_f} = \theta_M^\dagger(e^\dagger),$$

where γ_f is a path in $G^\dagger$ from f to f^*. This does not depend on the choice of γ_f because of (2.2).

This height function encodes the dimer configuration. When we pass from one face to another face in such a way that we have a white vertex on our left, the height will always increase by $\frac{2}{3}$ if there is a dimer and decrease by $\frac{1}{3}$ if there is not. For convenience we multiply the height function with 3 to obtain an integer valued height function.

Now consider a finite connected subgraph G of the hexagonal lattice, which admits at least one dimer configuration and has a connected complement. We want to define the height function on all hexagons, i.e. faces of $\mathbb{H}$ that contain any vertex of G on their boundary. To do this we set $\omega_M(e) = 0$ for all edges not in G and note that this is consistent with the dimer condition for vertices on the boundary of G: Each such vertex must be connected to exactly one other vertex in G by an edge in M and thus if we would try to extend the configuration outside of G there would never be a dimer on one of the edges connecting a vertex in G with a vertex outside of G. Using this ω_M the resulting θ_M is only divergence free on the vertices of G However, this is enough to define the height function for all faces mentioned above, if we restrict ourselves to paths γ_f which only use duals of edges and boundary edges of G.

Any cycle formed of such edges contains only contains vertices with divergence 0 inside, so this allows for a consistent definition of a height function. This is were we use that the complement is also connected, as otherwise there might be vertices inside the cycle, that are not in G and therefore might not have divergence 0. One important feature of this construction is that, if we choose as f^* one of the boundary hexes, the height of the other boundary hexes does not depend on M, as one can always use edges on the edge-boundary of G, which are always vacant. This means that the height function at the boundary is deterministic.

The effects of the boundary on the height functions are very intricate: Around certain parts of the boundary "frozen" regions might appear, where only dimers in one of the three orientations appear. Other regions might have different densities of the three types of dimer. These phenomena are mostly studied in the scaling limit: One takes a growing sequence of G_n that approximates a certain domain Ω in the sense that $\lim_{n\to\infty} \frac{1}{n} G_n = \Omega$ in some suitable sense of convergence and then $h(x) := \lim_{n\to\infty} \frac{1}{n} h_n(nx)$ is investigated, where h_n is the (random) height function on G_n, taken to be constant on each of the hexagons of the lattice. It turns out that under some additional assumptions this limit exists almost surely and is deterministic.

2.2 Temperley's Bijection

In [14] Kenyon, Propp and Wilson generalize a bijection first found by Temperley in 1972 from the square grid to more general directed and weighted graphs.

The setup is as follows: Let G be a finite connected directed graph in the plane with non-negative weights attached to its edges. A *directed spanning tree* (or *arborescence*) T of G is a collection of (directed) edges such that: They are connected, contractible (meaning there are no cycles) and every vertex except one has exactly one outgoing edge in T. This special vertex must have degree 0 in T (otherwise there would be a cycle) and is called the root of the spanning tree. The weight of a spanning tree is the product of the weights of its edges.

From G another weighted (but not directed) graph H is constructed: It has as vertices all the vertices, edges and faces of G. To avoid confusion in this section we will call its vertices *nodes* and refer to vertex-nodes, edge-nodes and face-nodes for nodes which are vertices, edges and faces in G respectively. H can be seen as the G and its planar dual $G^\dagger$ embedded in $\mathbb{C}$ simultaneously, see the last image in Figure 1. A vertex-node v is connected to an edge-node e of G if the edge e is an outgoing edge of v. This edge from v to e is given the same weight as e in G. A face-node f is connected to an edge-nodes e if e borders the face f. This is equivalent to saying that the dual edge of e contains the vertex f in $G^\dagger$. All of these edges get weight 1. Denote by $H(v, f)$ the graph obtained from deleting a vertex-node v and a face-node f such that v is on the boundary of f.

Now we can state the main theorem of [14]:

Theorem 2.1. *There is a weight-preserving bijection between spanning Trees of G rooted at v and dimer configurations of $H(v, f)$.*

Here the weight of a dimer configuration is as always the product of the weights of its edges. Note that this statement also includes the equivalence of the existence of spanning trees on G and dimer configurations on $H(v, f)$.

The bijection proceeds as follows: For each spanning tree T of G there is a dual spanning tree on $G^\dagger$, which uses exactly the duals of the edges that T does not use. From the fact that T has no cycles it follows that this gives a connected graph and from the fact that T is connected it follows that this gives a graph without cycles. Note that this is not a directed spanning tree, because the dual graph is not directed. But we can direct it towards the removed face f (which is a vertex of the dual graph) to get a directed spanning tree rooted at f.

Given a spanning tree T on G, we first construct the dual tree $T^\dagger$, orient it towards f, and then define the matching M as follows: An edge (v, e) is in M iff the edge e is in T, and an edge (f, e) is in M iff the dual edge of E in $T^\dagger$ is oriented away from f. This gives a dimer configuration, because each vertex-node except v and each face-node except f is matched to exactly one edge-node and every edge-node is matched exactly once, because either e is in T or the dual of e is in $T^\dagger$.

In this thesis one special case of this bijection is investigated: We orient each edge of the triangular lattice embedded in $\mathbb{C}$ such that each edge is oriented in the direction $1, e^{2\pi i/3}$, or $e^{4\pi i/3}$. We denote this lattice by $\mathbb{T}$. We pick a finite connected subgraph of the triangular lattice, such that its complement is also connected. Then for G we choose the graph obtained by identifying all vertices outside this finite subgraph, i.e. we use *wired* boundary conditions. This gives a graph whose dual graph is a finite connected subgraph of the hexagonal lattice. To apply the bijection we choose as v the outer vertex and as f any of the faces touching v. The resulting graph $H(v, f)$ is also a subgraph of the hexagonal lattice. This can be seen by "bending" the dual edges slightly towards tail of the original edge such that there is an angle of $2\pi/3$ at the midpoint of the dual edge, see [14, Figure 2]. The six nodes around each hexagon are always: one vertex-node from the original triangular lattice, two vertex-nodes from the dual lattice, and three edge nodes. The fact that we start with a directed graph is essential: In the undirected case the edge-nodes always have degree 4, which limits the possible graphs $H(v, f)$.

We choose three weights p, q and r for the three types of edges in the triangular lattice. This corresponds to a biased random walk on this directed graph. On the hexagonal lattice it gives weights that are biperiodic with respect to the automorphism of the hexagonal lattice which translate the hexagons in the directions of the edges, see Figure 2.

2.3 Wilson's Algorithm

In [27] David Wilson found an algorithm to generate uniform spanning trees that is not only faster than the previously known alternatives but also of theoretical importance. We will state a version for directed rooted uniform spanning trees, which are also called arborescence. In the reversible case it can be seen that the choice of root does not actually change the distribution and thus the unrooted spanning tree can be obtained by simply forgetting the root.

First, for completeness, let us define a random walk. Let G be a directed graph with positive weights $\omega(e)$ associated to its edges. A random walk $(X)_{n\in\mathbb{N}}$ on this graph is the Markov chain on the vertices of G such that the transition probability $p(x,y)$ satisfies:

$$p(x,y) = \frac{\omega(x,y)}{\sum\limits_{e^-=x} \omega(e)}, \tag{2.3}$$

where e^- denotes the outgoing vertex of the edge e. In words, the probability to move from x to y is proportional to the weight of the edge (x,y).

Given any path $P = (X_1, \dots, X_l)$ we can obtain its loop erasure $L(P)$ by deleting cycles as they appear. More precisely, let $u_0 \coloneqq X_0$. Then find the last time that X visits X_0, i.e. $i = \max j : X_j = X_0$ and set $u_1 = X_{i+1}$. Continue in this way by setting u_{k+1} always to the first vertex visited after the last visit to u_k as long as this is possible. When the last visit to u_k is X_l we terminate the process and obtain the loop erasure $(u_1, \dots, u_k)$. The result of applying this procedure to the trajectory of a random walk (that is run until some stopping time occurs) is called a loop-erased random walk or LERW for short.

Wilson's algorithm proceeds as follows: Let G be a directed graph with positive weights associated to the edges of the graph. Let r be a distinguished vertex of the graph, which will be the root of the weighted spanning trees. We will generate a growing sequence of trees T_i. To do this we first fix an ordering of the vertices of G with the root r as the first vertex (the remainder of the ordering is arbitrary). Let $T_1 \coloneqq r$. Then repeat the following for $i \geq 1$:

- If T_i is a spanning tree, we are done.

- Otherwise find the earliest vertex x in our ordering which is not contained in T_i.
- Run a random walk on G starting at x until it hits T_i, independent from the walks in previous steps.
- Create T_{i+1} from T_i by adding the vertices and edges of the loop erasure of this walk.

Theorem 2.2. *Given any directed graph G with positive weights ω, Wilson's algorithm produces a directed spanning tree T with probability proportional to $\prod_{e\in T}\omega(e)$.*

In the original paper by Wilson there is a beautiful proof of this theorem. However, it can also be proven by calculating the law of a LERW somewhat explicitly using discrete Green functions. A quite remarkable feature of the algorithm is that the arbitrary ordering of the vertices does not affect the law of the tree. This gives us the following corollary by choosing $v_2 = x$:

Corollary 2.3. *Let G be a directed weighted graph as above, x a vertex of G and T sampled according to Wilson's algorithm. The law of the unique path γ_x from x to the root in T is given by the law of the loop-erased random walk started at x and run until it hits r.*

2.4 Winding of LERW and the height function

In this section we establish the connection between the height function of the dimer configuration and the winding of the branches. This connection is established in [14]. See also [4, Section 6] for a more general version.

Let the setup be as in the Section 2.2. The graph G is a finite connected subgraph of the directed triangular lattice with connected complement and wired boundary. There is a bijection between dimer configurations on the graph $H = H(v, f)$ and directed spanning trees on G rooted at the outer vertex v. For each vertex x of the hexagonal lattice which is an inner vertex of H there is a unique path γ_x from this vertex to the boundary following edges in the tree. We want to connect the winding of this path to the height function, but to do this there are some technical difficulties to watch out for.

First the height function is defined on faces of the hexagonal lattice, while the winding is calculated for pairs of vertices. This can be fixed by adding for each face a small path from that face to the unique primal vertex on the boundary of that face. Adding this to γ_x we obtain a path from f to the boundary, let this path be called γ_f. At the boundary we use the following convention: For each such γ_f there is a boundary face

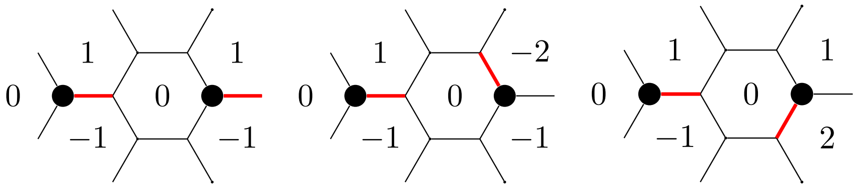

Figure 3: The three possible situations at a vertex inside γ_f from left to right: If the path is straight the height at the face opposite the dimer does not change, when the path turns left it decreases by 1 and if it turns right the height increases by 1. The larger vertices are primal vertices in Temperley's bijection, the red edges are dimers.

such that the last edge of γ_f crosses this face. We say that γ_f exits through this face. This is relevant as it is important not only at which vertex the path leaves G but also through which edge. The *intrinsic winding* of γ_x is defined as the sum of the turning angles of γ_x.

Lemma 2.4. *Let h be the height function associated to a dimer configuration on a graph H obtained from the Generalized Temperley bijection. Let T be the directed spanning tree corresponding to this dimer configuration, f be a face in H and g the face through which γ_f exits. Then:*

$$h(f) = \frac{3}{2\pi} W_{int}(\gamma_x) + h(g). \tag{2.4}$$

Proof. γ_f consists of a sequence of edges $(x_1, \ldots, x_k)$ with x_k on the boundary and an initial segment from f to x_1. Let f_i be the face bordering x_i opposite of the edge (x_i, x_{i+1}). We use a telescoping series for the height difference between f and g:

$$h(f) - h(g) = h(f) - h(f_1) + \sum_{i=1}^{k-1} (h(f_i) - h(f_{i+1})) + h(f_k) - h(g).$$

The first summand $h(f) - h(f_1)$ corresponds to the turning angle of γ_f at the vertex x_1: If γ_f is straight at x_1 then $f_1 = f$. Otherwise, if γ_f turns right, then going from f to f_1 is going clockwise around x_1 without crossing a dimer and thus decreases the height by 1. The reverse is true if γ_f turns left. By a very similar consideration it is seen that $h(f_i) - h(f_{i+1})$ is $\frac{3}{2\pi}$ times the turning angle at x_{i+1}, see Figure 3 Lastly $h(f_k) = h(g)$ because when going around the dimer, that corresponds to the last step in the path, f_k and g are the faces around that dimer that only touch the dimer at a vertex and not at an edge. Therefore to get from f_k to g there is always a step that decreases the height by 1 and one that increases the height by 1. □

Note that, as g is a boundary face $h(g)$ depends only on g and not on the dimer configuration. Thus the height at f is a function of the winding of γ_f and its exit point.

2.5 Schramm Loewner Evolution

In this section we will repeat the definition and some properties of the Schramm-Löwner evolution. For an introduction to this subject see, for example, the lecture notes [3, 18] or the monographs [17, 12]. Some standard properties of conformal maps will be used. These can for example be found in [24].

In [22] Charles Loewner defined the Loewner differential equation to study conformal mappings of the unit disk. This work was then used by Schramm in [26] to give a conjectural scaling limit of several planar processes.[2]

There are two versions of SLE: radial and chordal. One describes a random curve from a point on the boundary of a simply connected domain to a point inside the domain and other describes a random curve between two fixed boundary points. Either can be used as the limiting object for loop-erased random walks, in one case for a random walk started in the domain and condition to leave the domain at a certain point, in the other for a random walk started the boundary and conditioned on immediately entering the domain and leaving it through another point on the boundary.

In this thesis we will only consider radial SLE, as it is radial SLE, which we will need for the connection to Wilson's algorithm.

Let $D \subset \mathbb{C}$ with $0 \in D$ be a simply connected domain. By the Riemann mapping theorem there exists a unique conformal homeomorphism ψ_D from D onto the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ such that $\psi_D(0) = 0$ and $\psi'_D(0) \in \mathbb{R}^+$. Consider a continuous simple curve $\eta\colon [0,\infty] \to \overline{\mathbb{U}}$ such that $\eta(0) \in \partial\mathbb{U}$, $\eta(t) \in \mathbb{U}$ for $t > 0$ and $\eta(\infty) = 0$. For each t let $K_t := \eta[0,t], U_t := \mathbb{U} \setminus K_t$ and $g_t := \psi_{U_t}$. Because $U_t \subset \mathbb{U}$ the derivative $g'_t(0) \geq 1$. Then $\log({g_t}'(0))$ is called the capacity of K_t from 0. By the Schwarz lemma $t \mapsto g'_t(0)$ is increasing and thus η can be reparametrized such that the capacity of K_t is t. Then, we say that η_t is parametrized by capacity. By [24, Proposition 2.5] the limit

$$W(t) := \lim_{z \to \eta(t)} g_t(z) \tag{2.5}$$

[2]Oded Schramm called this process "Stochastic Loewner Evolution" but the name Schramm-Loewner evolution has since been established.

exists for each $t \in [0, \infty]$, when z approaches $\eta(t)$ from inside U_t. It is also true under these assumptions that $W\colon [0, \infty) \to \partial\mathbb{U}$ is continuous. Now, we can state Loewner's theorem:

Theorem 2.5. *Let $\eta\colon [0, \infty] \to \overline{\mathbb{U}}$ be as above and parametrized by capacity. Then the maps $g_t\colon U_t \to \mathbb{U}$ satisfy Loewner's differential equation:*

$$\partial_t g_t(z) = -g_t(z) \frac{g_t(z) + W(t)}{g_t(z) - W(t)} \tag{2.6}$$

with the initial value $g_0(z) = z$.

We call $(W(t))_{t \geq 0}$ the *driving function* of the curve η. The driving function $W(t)$ fully determines η: For every $z \in \mathbb{U}$ the Loewner equation has a solution up to some time $\tau_z \in (0, \infty]$. If $\tau_z = \infty$ the ODE has a solution at z for all times and therefore $z \in U_t$ for all $t \in [0, \infty)$. If however $\tau_z < \infty$ then $\lim_{t \uparrow \tau_z} W(t) - g_t(z) = 0$ as this is the only singularity of the differential equation. If W is obtained from a simple path as above then obtain η from W by setting $\eta(t) = g_t^{-1}(W(t))$. For an arbitrary continuous function W this might not work. One can still define $K_t = \{z \in \overline{\mathbb{U}} : \tau_z \leq t\}$, but this might not be a simple path or even no path at all.

Radial Schramm Loewner evolution with parameter κ is the process obtained from setting the driving function to $W(t) \coloneqq \exp(iB_{\kappa t})$, where $B\colon [0, \infty) \to \mathbb{R}$ is standard Brownian motion started most commonly at 0 or at a uniform point in $[0, 2\pi]$. In [25] it was shown that for $\kappa \leq 4$ almost surely K_t is a simple path, while for $\kappa > 4$ there is always a path $\eta(t)$ such that D_t is the connected component of $\mathbb{U} \setminus \eta[0, t]$ containing 0.

To obtain radial SLE_κ on a different domain D one applies ψ_D^{-1} to radial SLE_κ on $\mathbb{U}$.

3 Brownian motion and scaling limits

In this section we will understand Brownian motion with drift as an off-critical scaling limit. This will both serve as a simple example for an off-critical model and give the rates of convergence the scaling window, also for the loop-erased counterparts.

Let us consider a simple random walk on $\mathbb{Z}$. This can be written as $S_n \coloneqq \sum_{i=1}^n X_i$, where the x_i are independently and identically distributed variables taking the values -1 and 1 each with probability $\frac{1}{2}$. By the Central Limit Theorem we have:

$$\lim_{n \to \infty} \frac{S_n}{\sqrt{n}} = \mathcal{N}(0, 1), \tag{3.1}$$

where convergence is in the sense of distribution. But what if we want to look at the large scale behaviour of the path as a whole instead of just the end point?

Brownian motion is the unique Stochastic Process $(B_t)_{t\in\mathbb{R}^+}$ such that:

(i). $B_0 = 0$ almost surely.

(ii). B_t is almost surely continuous as a function of t.

(iii). B_t has stationary, independent increments.

(iv). $B_t = \mathcal{N}(0,t)$.

It is a standard result that $\left(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}}\right)_{t\in\mathbb{R}^+}$ satisfies these properties, using the central limit theorem and some additional tools for the continuity.

This is a model example for scaling limits. To bring it in line with the notation used later in this thesis we want to consider it as a scaling in time and space: Setting $\delta = \frac{1}{\sqrt{n}}$ we view $\left(\delta S_{\lfloor \delta^{-2} t\rfloor}\right)_{t\in\mathbb{R}^+}$ as a simple random walk on the scaled lattice $\delta\mathbb{Z}$ with time re-scaled by a factor of δ^2. This kind of scaling is known as *diffusive* scaling and it is the correct scaling for a variety of processes. Importantly, it also works for random walks on two and higher dimensional lattices.

For the proof of the convergence of the loop-erased random walk to SLE it can be beneficial to take a point of view from the angle of statistical mechanics: Instead of looking at the probabilities of individual steps one can consider configuration, in this case lattice paths, which are assigned weights depending often on one or more parameters and then assigning them probabilities proportional to those weights. In the example above every lattice path of length n has the same probability 2^{-n}. One of the major themes of statistical mechanics is the existence of phase transitions at critical values, i.e. values at which some behaviour in the scaling limit changes in a discontinuous or non-smooth way. Often the behaviour at the critical value is the most interesting.

One way to consider the simple random walk as an example of a model at criticality is to introduce a parameter p which is the probability that the random walk takes a step up. If $p > \frac{1}{2}$ by the law of large numbers the limit of $\left(\delta S_{\lfloor \delta^{-2} t\rfloor}\right)_{t\in\mathbb{R}^+}$ is $+\infty$ almost surely as $\delta \to 0$ for any $t > 0$. Even with adjusted scaling the limit $\left(\delta S_{\lfloor \delta^{-1} t\rfloor}\right)_{t\in\mathbb{R}^+}$ is just a straight line starting at 0 with slope $2p-1$. Of course for $p < \frac{1}{2}$ the situation is similar by symmetry and thus $p = \frac{1}{2}$ is the critical value, where neither up nor down steps dominate. For all other values the behaviour in the scaling limit is deterministic.

However, one can consider the behaviour of the model if one moves the parameter (in this case p) closer to the critical value while simultaneously scaling the lattice.

Typically, if one moves the parameter too quickly towards criticality one obtains in the limit the behaviour at criticality, while if it converges too slowly one obtains the behaviour away from criticality. But if it converges at the correct speed, one obtains a non-trivial yet distinct limiting behaviour.

Returning to our example, the correct speed can be obtained by the following calculation:

$$\mathbb{E}_p(\delta S_{\lfloor \delta^{-2} t \rfloor}) = \delta \lfloor \delta^{-2} t \rfloor (2p-1), \tag{3.2}$$

To obtain interesting behaviour we want this expectation to converge to a finite positive value and thus the correct scaling window is:

$$p = p(\delta) = 1/2 + d\delta + o(\delta), \tag{3.3}$$

where $d \in \mathbb{R}$ can be chosen arbitrarily. For each of these d one obtains a different scaling limit: Brownian motion with drift d. This is another typical feature of off-critical models: the parameter d interpolates between the critical and non-critical behaviour.

Another example of an off-critical Brownian motion that will play a role in this thesis is the scaling limit of the *massive* random walk. The massive random walk is a random walk that has a chance of p at every step to die, which means it is moved to a *cemetery* point which is added to the lattice and stays there from then on. Does this have an interesting scaling limit? Again, if we fix p and just let δ go to 0 the limit is trivial: The limiting random path has length 0 almost surely. Thus p must be moved to the critical value 0 as δ approaches 0. The probability of still being alive at time t is $(1-p)^{\lfloor \delta^{-2} t \rfloor}$. We want this to converge to a positive number less than 1 for each t. This corresponds to this rate of convergence:

$$p = p(\delta) = m^2 \delta^2 + o(\delta^2), \tag{3.4}$$

where m is a positive real parameter called the mass. Writing m^2 instead of m is a convention, but will also simplify some formulas in the following section. The probability of still being alive at time t then converges to $\exp(-m^2 t)$. Thus we can easily identify the limiting object: it is Brownian motion that dies at a random time given by an exponential variable whose expectation is m^{-2}.

The same scaling limits and scaling windows work for the massive and drifted random walk on the triangular lattice and the square grid and also in higher dimensions and on more general graphs and lattices. This thesis considers drifted and massive SLE as scaling limits of the corresponding loop-erased random walks. Because of the relation

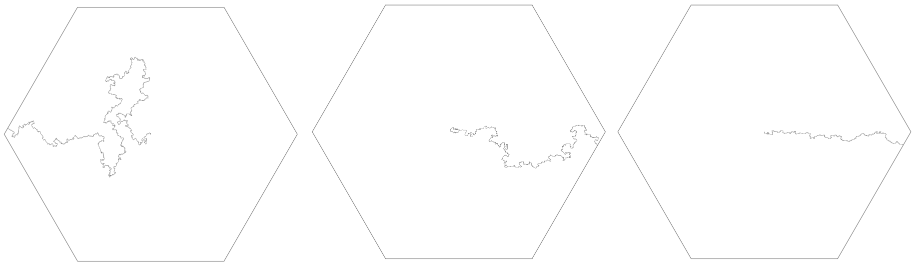

Figure 4: Three samples of loop-erased random walks on the triangular lattice in a hexagon of side-length 500. The first sample has no drift, the second one has a small drift to the right and the third one has a larger drift to the right.

between Brownian Motion and SLE_2 (see for example the remark at the beginning of [20]) it is reasonable to expect the same scaling should also be applied to these processes, and in fact this turns out to be correct.

4 LERW with drift and massive LERW

In this section, a connection between the massive random walk and the random walk with drift is established. In particular, it is shown for the triangular lattice that the laws of the processes agree after conditioning on an exit point and for the triangular and the square lattice the scaling limit of the Radon-Nikodym derivative is calculated.

4.1 The directed triangular lattice

Let Ω^δ be a sequence of domains on the scaled triangular lattice $\delta\mathbb{T}$. We consider two measures on paths starting at an interior point x^δ in Ω^δ and exiting Ω^δ at the boundary.

The *random walk with drift* is the random walk started at x^δ and taking a step in each of the directions 1,$e^{2\pi i/3}$ and $e^{4\pi i/3}$ with probabilities p, q and r respectively until it hits the boundary of Ω^δ. This assigns each path γ^δ the probability:

$$\mathbb{P}^{(d)}(\gamma) = p^{\#\text{steps to the right}} q^{\#\text{steps up-left}} r^{\#\text{steps down-left}} \tag{4.1}$$

The *massive random walk* is the walk started at x^δ and taking each of the three possible steps with equal probability, but also dying at each step with probability

$(1 - m^2\delta^2)$. The probability of a path γ^δ from x^δ to the boundary then is:

$$\mathbb{P}^{(m)}(\gamma) = (\frac{1}{3}(1 - m^2\delta^2))^{\#\text{of steps in } \gamma}.$$

To obtain a probability measure on such paths we need to condition the walk to hit the boundary before dying.

On the triangular lattice these two measures are directly related:

Lemma 4.1. *For each p, q, r and δ there exists an m such that conditioned to leave Ω^δ at y^δ the laws $\mathbb{P}^{(m)}$ and $\mathbb{P}^{(d)}$ agree.*

Proof. Consider two paths γ and γ' that start at x^δ and end at y^δ. Assume that γ takes n more steps in the 1 direction than γ'. As the only linear relation between the three steps 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$ is: $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$ this implies that γ also takes n more steps in each of the other two directions. Therefore:

$$\frac{\mathbb{P}^{(d)}(\gamma)}{\mathbb{P}^{(d)}(\gamma')} = (pqr)^n = (\sqrt[3]{pqr})^{l(\gamma) - l(\gamma')} = \frac{\mathbb{P}^{(m)}(\gamma)}{\mathbb{P}^{(m)}(\gamma')},$$

where $l(\gamma)$ is the number of steps in γ and m is such that $\frac{1}{3}(1 - m^2\delta^2) = \sqrt[3]{pqr}$. Such an m always exists because $\sqrt[3]{pqr} \le \frac{1}{3}$ by the arithmetic-geometric mean inequality or by a straightforward calculation. Because we condition on the walks exiting at the same point, which includes conditioning the massive walk not to die before doing so, this suffices to show the equility of the two probability measures. $\square$

This correspondence is also compatible with the scaling windows for the two walks discussed in section 3:

Lemma 4.2. *If p, q and r are scaled with δ such that $p = \frac{1}{3} + c_1\delta + o(\delta)$, $q = \frac{1}{3} + c_2\delta + o(\delta)$ and $r = \frac{1}{3} + c_3\delta + o(\delta)$ then the corresponding mass m from the previous proposition converges to $\sqrt{\frac{3}{2}}|d|$ where $d = c_1 + c_2 e^{2\pi i/3} + c_3 e^{4\pi i/3}$ is the drift of the scaling limit of the random walk.*

Proof. First we note that $p + q + r = 1$ implies that $c_1 + c_2 + c_3 = 0$ and also that the sum of the $o(\delta)$ terms cancel. Then

$$\begin{aligned}(1-m^2\delta^2) =&3\sqrt[3]{pqr} = \\ =&((1-3c_1\delta+o(\delta))(1-3c_2\delta+o(\delta))(1-3c_3\delta+o(\delta)))^{\frac{1}{3}} = \\ =&(1-9(c_1c_2+c_2c_3+c_3c_1)\delta^2+o(\delta^2))^{\frac{1}{3}} = \\ =&1-3(c_1c_2+c_2c_3+c_3c_1)\delta^2+o(\delta^2)\end{aligned}$$

and therefore $m^2 = -3(c_1c_2+c_2c_3+c_3c_1)+o(\delta) = \frac{3}{2}|d|^2+o(\delta)$. □

Note that, since these are the correct rates of convergence for Brownian motion with drift and massive Brownian motion, this proposition gives a relation between the drift d and the mass m for the continuous counterparts.

After conditioning the massive random walk to reach the boundary of the domain before dying, the laws of the two walks are supported on the same set and have a Radon-Nikodym derivative $\frac{d\mathbb{P}^d}{d\mathbb{P}^m}$. Because of proposition 4.1 this Radon-Nikodym derivative evaluated on a path γ^δ only depends on the point at which γ^δ leaves Ω^δ. The next proposition shows that this derivative has a simple limit.

Proposition 4.3. *Let p, q, r and m be as in the previous proposition. Let Ω^δ approximate $\Omega \subset \mathbb{C}$. Let γ^δ be a sequence of lattice paths from $x^\delta \in \Omega^\delta$ to $y^\delta \in \partial\Omega^\delta$, where x^δ and y^δ converge to x and y respectively. Then*

$$\frac{\mathbb{P}^{(d)}(\gamma^\delta)}{\mathbb{P}^{(m)}(\gamma^\delta)} \to \exp(2\langle (y-x), d\rangle),$$

where the inner product is taken by considering $y-x$ and d as vectors in $\mathbb{R}^2$.

Proof. Let a and b be such that $y = x + a + be^{2\pi i/3}$ and a^δ and b^δ such that $y^\delta = x^\delta + a^\delta\delta + b^\delta\delta e^{2\pi i/3}$. Without loss of generality assume $a^\delta, b^\delta \geq 0$, i.e. y^δ is in the upper right third of the plane relative to x^δ. Then:

$$\begin{aligned}\mathbb{P}^{(d)}(\gamma^\delta) =& (\frac{1}{3}(1-m^2\delta^2))^{l(\gamma)-(a^\delta+b^\delta)}p^{a^\delta}q^{b^\delta} \\ =&\mathbb{P}^{(m)}(\gamma^\delta)\left(\frac{p}{\frac{1}{3}(1-m^2\delta^2)}\right)^{a^\delta}\left(\frac{q}{\frac{1}{3}(1-m^2\delta^2)}\right)^{b^\delta}.\end{aligned}$$

We have that $a^\delta\delta$ converges to a and therefore $a^\delta = a\delta^{-1} + o(\delta^{-1})$. Using this we see that these factors converge:

$$\left(\frac{p}{\frac{1}{3}(1-m^2\delta^2)}\right)^{a\delta^{-1}} = \left(\frac{\frac{1}{3}+c_1\delta}{\frac{1}{3}(1-m^2\delta^2)}\right)^{a\delta^{-1}} \to \exp\left(3c_1a\right),$$

and the analogous result for the second factor. A quick calculation shows:

$$\exp(3c_1a + 3c_2b) = \exp(2\left\langle (y-x), d\right\rangle)$$

which gives the desired result. □

Note that this is stated without conditioning the massive walk to hit the boundary before dying. This conditioning would add a term the Radon-Nikodym derivative corresponding to the probability to hit the boundary before dying.

By applying the loop-erasing procedure as described in section 2.3 to both the random walk with drift and the massive random walk conditioned to hit the boundary before dying we obtain two processes we call *loop-erased random walk with drift* and *massive loop-erased random walk.*

Theorem 4.4. *Let Ω^δ approximate a bounded domain Ω and x^δ approximate x. Let $d \in \mathbb{C}$ be a drift and m be the corresponding mass. Assume that the law of the massive loop-erased random walk with mass m started at x^δ converges to massive radial SLE_2. Then the law of the loop-erased random walk with drift d started at x^δ has a limit which is absolutely continuous with respect to massive radial SLE_2 and its Radon-Nikodym derivative with respect to this law is given by:*

$$H(\gamma) = C\exp(2\left\langle (\gamma_0 - x), d\right\rangle),$$

where C is a positive constant and γ_0 is the point of γ on the boundary of Ω.

Proof. Let $\mathbb{P}^{(m,*)}$ be the law of the massive random walk conditioned to hit the boundary before dying and γ_δ be a massive loop-erased random walk from x^δ to the boundary. Because we assume convergence of the massive loop-erased random walk we can assume that these paths converge almost surely in the topology induced by the supremum norm on paths up to reparametrization by Skorokhod's embedding theorem. By the lemma 4.1 the ratio of the laws of the random walk with drift and the massive random walk depends only on the exit point of the random walk. Since the loop-erasing procedure preserves the exit point the ratio for the loop-erased walks is the same, with an additional factor because we condition the massive walk to hit the boundary before dying.

$$\frac{\mathbb{P}^{(d)}(\gamma^\delta)}{\mathbb{P}^{(m,*)}(\gamma^\delta)} = \frac{\mathbb{P}^{(d)}(\gamma^\delta)}{\mathbb{P}^{(m)}(\gamma^\delta)}\mathbb{P}^{(m)}(\tau_{\partial\Omega^\delta} < \tau_{\text{death}}).$$

The second factor converges to the probability that massive Brownian motion hits the boundary before dying, which we denote by C. The first factor converges to $\exp(2\langle y - x, d\rangle)$ by the previous proposition. Since Ω is bounded also this limit is bounded from above and away from 0, and therefore the processes are mutually absolutely continuous. □

We can interpret this result by considering Girsanov's theorem for Brownian motion with drift d. Let $\mathbb{P}^{(d)}$ and $\mathbb{P}$ denote the laws of Brownian motion with drift d and classical Brownian motion respectively. Then

$$\frac{d\mathbb{P}^{(d)}}{d\mathbb{P}|_t} = \exp\left(\langle d, B_t\rangle - \frac{|d|^2 t}{2}\right).$$

Here $\exp\left(-\frac{|d|^2 t}{2}\right)$ is the Radon-Nikodym derivative of massive Brownian motion with mass $m^2 = \frac{|d|^2 t}{2}$ with respect to classical Brownian motion. So here again the effect of the drift can be split into a mass term depending on the time t and a term depending on the position of the end point.

4.2 Square lattice

While the direct connection of the massive random walk and the random walk with drift conditioned on the exit point holds only for the triangular lattice, asymptotically a similar statement holds also for the square lattice.

On the scaled square lattice $\delta\mathbb{Z}^2$, which we identify with $\delta\mathbb{Z} + \delta\mathbb{Z}i$, we have analogues to the two random walks: The *massive random walk* is again the simple random walk on this lattice, taking steps in the four directions with equal possibility and dying at each step with probability $m^2\delta^2$.

For the *random walk with drift* we set four probabilities p_1, p_2, p_3 and p_4 to go in the directions $1, -1, i$ and $-i$ respectively. Again these probabilities converge at a certain rate: $p_i = 1/4 + c_i\delta + o(\delta)$ for $i = 1, \ldots, 4$. The resulting drift is then given by $d = (c_1 - c_2) + (c_3 - c_4)i$. For simplicity let us assume that $p_1 + p_2 = \frac{1}{2}$ and $p_3 + p_4 = \frac{1}{2}$. This implies that $c_1 + c_2 = c_3 + c_4 = 0$ but still allows for any drift $d \in \mathbb{C}$.

To proof the same result as for the triangular lattice we want to say that the walks take about as many horizontal steps as they do vertical steps in a quantifiable way:

Lemma 4.5. *Let Ω^δ be a sequence of domains on $\delta\mathbb{Z}^2$ approximating a bounded domain $\Omega \subset \mathbb{C}$. Let γ be the trajectory of a random walk with drift started at $x^\delta \in \Omega^\delta$ that is run until it hits the boundary of Ω^δ. Let $l(\gamma)$ be the number of steps of γ and $l_{\leftrightarrow}(\gamma)$ the number of horizontal steps in γ. Then for any $\alpha > 1$*

$$\mathbb{P}^{(d)}\left(|\frac{l(\gamma)}{2} - l_{\leftrightarrow}(\gamma)| > \delta^{-\alpha}\right) \to 0,$$

as $\delta \to 0$.

Proof. Let $\epsilon > 0$. Choose c such that $\mathbb{P}(\tau_{\partial\Omega} > c) < \epsilon/2$, where $\tau_{\partial\Omega}$ is the hitting time of the boundary of Ω of Brownian motion with drift d started at x. This implies that for δ small enough $\mathbb{P}^{(d)}(\tau_{\partial\Omega^\delta} < c\delta^{-2}) < \epsilon$. Let $W_n = \frac{n}{2} - l_{\leftrightarrow}(\gamma_0, \dots, \gamma_n)$. This is a simple random walk on $\frac{1}{2}\mathbb{Z}$ that goes up by $\frac{1}{2}$ if the walk takes a vertical step and goes down by $\frac{1}{2}$ if the walk takes a horizontal step, both with probability $\frac{1}{2}$. It is stopped at a random time, more specifically $\tau_{\partial\Omega^\delta}$. Thus:

$$\begin{aligned}
\mathbb{P}^{(d)}\left(|W_{\tau_{\partial\Omega^\delta}}| > \delta^{-\alpha}\right) &\leq \mathbb{P}^{(d)}\left(|W_{\tau_{\partial\Omega^\delta}}| > \delta^{-\alpha}, \tau_{\partial\Omega^\delta} < c\delta^{-2}\right) + \epsilon \\
&\leq \mathbb{P}^{(d)}\left(\sup_{t \leq c\delta^{-2}} |W_{\tau_{\partial\Omega^\delta}}| > \delta^{-\alpha}\right) + \epsilon \\
&= \mathbb{P}^{(d)}\left(\sup_{t \leq c\delta^{-2}} |\delta W_t| > \delta^{-(\alpha-1)}\right) + \epsilon.
\end{aligned}$$

Because W_n is a simple random walk on $\frac{1}{2}\mathbb{Z}$ it converges to a Brownian motion in the scaling limit. Hence $\sup_{t \leq c\delta^{-2}} |\delta W_{\tau_{\partial\Omega^\delta}}|$ converges to $\sup_{t \leq c} |\delta B_t|$, which is finite almost surely. As $\delta^{-(\alpha-1)}$ goes to infinity this implies that the right-hand side of the last display converges to ϵ as δ goes to 0. As ϵ was arbitrary this proves the lemma. □

There is no analogue to proposition 4.1 for the square lattice. However, as the previous proposition shows the ratio of horizontal to vertical steps approaches 1 quickly enough to motivate that the mass corresponding to d, and thus to $(p_1, \dots, p_4)$, should be:

$$m^2 = \frac{1 - 4\sqrt[4]{p_1 p_2 p_3 p_4}}{\delta^2} = -4(c_1 c_2 + c_3 c_4) + o(\delta^2) = 2|d|^2 + o(\delta^2) \tag{4.2}$$

Proposition 4.6. *Let p_1, p_2, p_3 and p_4 be as above and m be the corresponding mass. Let Ω^δ approximate $\Omega \subset \mathbb{C}$. Let γ^δ be the trajectory of a massive random walk started at $x^\delta \in \Omega^\delta$ and conditioned to leave the domain at $y^\delta \in \partial\Omega^\delta$, where x^δ and y^δ converge* *to x and y respectively. Then*

$$\frac{\mathbb{P}^{(d)}(\gamma^\delta)}{\mathbb{P}^{(m)}(\gamma^\delta)} \to \exp(2\langle (y-x), d\rangle),$$

in probability as δ goes to 0.

Proof. Let a, b, a^δ, and b^δ be such that $y = x + a + bi$ and $x^\delta = y^\delta + a^\delta \delta + b^\delta \delta i$, and without loss of generality let $a, b, a^\delta, b^\delta \geq 0$. Then

$$\mathbb{P}^{(d)}(\gamma^\delta) = (p_1 p_2)^{\frac{l_\leftrightarrow(\gamma^\delta)}{2} - a^\delta} (p_3 p_4)^{\frac{l_\updownarrow(\gamma^\delta)}{2} - b^\delta} p_1^{a^\delta} p_3^{b^\delta} =$$
$$= \mathbb{P}^{(m)}(\gamma^\delta)(4p_1 4p_2)^{\frac{l_\leftrightarrow(\gamma^\delta)}{2} - \frac{l(\gamma^\delta)}{4} - a^\delta} (4p_3 4p_4)^{\frac{l_\updownarrow(\gamma^\delta)}{2} - \frac{l(\gamma^\delta)}{4} - b^\delta} (4p_1)^{a^\delta} (4p_3)^{b^\delta}.$$

Here, the first two factors converge to 1 in probability: Note that $4p_1 4p_2 = (1 + 4c_1\delta + o(\delta))(1 + 4c_2\delta + o(\delta)) = (1 + 16c_1c_2\delta^2 + o(\delta^2))$, because $p_1 + p_2 = \frac{1}{2}$. Let $1 < \alpha < 2$. Then by lemma 4.5 with probability converging to 1 we have $|\frac{l_\leftrightarrow(\gamma^\delta)}{2} - \frac{l(\gamma^\delta)}{4}| < \delta^{-\alpha}$. Also, we have that $a^\delta = a\delta^{-1} + o(\delta^{-1})$ because $a^\delta \delta \to a$. Thus we get:

$$(1 + 16c_1c_2\delta^2 + o(\delta^2))^{\delta^{-\alpha} + o(\delta^{-\alpha})} \leq (4p_1 4p_2)^{\frac{l_\leftrightarrow(\gamma^\delta)}{2} - \frac{l(\gamma^\delta)}{4} - a^\delta}$$
$$\leq (1 + 16c_1c_2\delta^2 + o(\delta^2))^{-\delta^{-\alpha} + o(\delta^{-\alpha})},$$

with probability converging to 1 as δ goes to 0. As both bounds in this inequality converge to 1 this implies that the middle term converges to 1 in probability and the same works for the p_3, p_4 term. The other two factors in the product converge:

$$(4p_1)^{a^\delta} = (1 + 4c_1\delta + o(\delta))^{a\delta + o(\delta)} \to \exp(4c_1 a)$$

and because $d = (c_1 - c_2) + (c_3 - c_4)i = 2(c_1 + c_3 i)$ we get that $4c_1 a + 4c_3 b = 2\langle y - x, d\rangle$ and the statement of the theorem follows. □

Note that this convergence is only in probability as opposed to the corresponding statement for the triangular lattice which holds along any sequence of paths γ^δ.

Thus, we have the same behaviour in the limit on the square lattice as on the triangular lattice above. Because the convergence of the massive loop-erased random walk on the square grid to massive SLE_2 is proven in [7], this immediately gives a limiting object for the loop-erased random walk with drift. This also connects the drifted dimer model as defined in [8] to massive SLE.

5 Convergence of massive LERW on the triangular lattice

In [20] Lawler, Schramm and Werner proofed that the scaling limit of the loop-erased random walk in a simply connected domain converges to SLE_2. While the proof is written for the LERW on the square grid, in the last chapter it is mentioned to work in larger generality and the random walk on the directed triangular lattice is explicitly mentioned as an example of an irreversible random walk to which the proof applies. In [23] Markarov and Smirnov proposed a strategy for proving convergence of the massive LERW to massive SLE_2 building in part on ideas coming from Conformal Field Theory (see [1, 2]). This strategy was then successfully followed by Chelkak and Wan in [7], using a framework for convergence to SLE developed by Kemppainen and Smirnov in [13] and a recent addition [11] by Karilla. In this section, the arguments are repeated for the directed triangular lattice and adapted where necessary.

5.1 Convergence of domains and curves

For each discrete domain $\Omega^\delta \subset \delta\mathbb{T}$ we associate a polygonal domain $\hat{\Omega}^\delta \subset \mathbb{C}$ which is the union of open hexagons with side length δ centered at vertices of Ω^δ. Notice that vertices of $\delta\mathbb{T}$ on the boundary of $\hat{\Omega}^\delta$ are exactly vertices on the outer vertex boundary of Ω^δ.

We will assume that $\hat{\Omega}^\delta$ converges to Ω in the *Carathéodory* topology and if this is the case write, that Ω^δ approximates Ω. This means that each inner point of Ω belongs to $\hat{\Omega}^\delta$ for small enough δ and each boundary point of Ω can be approximated by boundery points of Ω^δ, see e.g. [24]. Further, we assume that $0 \in \Omega^\delta$ for each δ and we have a point $a^\delta \in \partial\Omega^\delta$ which converges to $a \in \partial\Omega$. Let $\psi_{\hat{\Omega}^\delta} : \hat{\Omega}^\delta \to \mathbb{U}$ be the unique conformal map such that $\psi_{\hat{\Omega}^\delta}(0) = 0$ and $\psi_{\hat{\Omega}^\delta}(a^\delta) = 1$. Then it can be seen (e.g. also in [24]) that Carathéodory convergence is equivalent to the uniform convergence on compacts of $\psi_{\hat{\Omega}^\delta}$ and $\psi^{-1}_{\hat{\Omega}^\delta}$ to ψ_Ω and ψ^{-1}_Ω respectively.

The main theorem of [13] states that if a family Σ of measures of random curves satisfies a certain annulus crossing condition, then the family is tight and furthermore, if $\mathbb{P}_n \in \Sigma$ is a weakly converging subsequence then its limit is a random Loewner chain. Moreover if $(W^{(n)})_{n\in\mathbb{N}}$ are the driving processes of the random curves $(\gamma^{(n)})_{n\in\mathbb{N}}$ that satisfy the annulus crossing condition which are parametrized by capacity then:

- $(W^{(n)})_{n\in\mathbb{N}}$ is tight in the space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compact subsets.

- $(\gamma^{(n)})_{n\in\mathbb{N}}$ is tight in the space of curves up to reparametrization with the supremum norm.

If the sequence converges in either of the topologies it also converges in the other one and the limit of the driving processes is the driving process of the limiting random curve.

That the annulus crossing condition is satisfied is checked for a chordal loop-erased random walk in [13, Section 4.5] with a remark that the radial case is equivalent to calculations in [20].

5.2 Absolute continuity with respect to classical SLE_2

Let $0 < \delta < m^{-1} \leq \infty$. Here, m is the mass, which we allow to be zero and δ is the scale. We consider subgraphs Ω^δ of the scaled triangular lattice $\delta\mathbb{T}$, which approximate some domain $\Omega \in \mathbb{C}$. Given such δ, m, Ω^δ as well as two vertices w^δ, z^δ we define the partition function of the massive random walk:

$$Z_{\Omega^\delta}^{(m)}(w^\delta, z^\delta) \coloneqq \sum_{\pi^\delta \in S(w^\delta, z^\delta)} \left(\frac{1}{3}(1 - m^2\delta^2)\right)^{\#\pi^\delta}, \tag{5.1}$$

where the sum is over all possible paths π^δ from w^δ to z^δ. If m is 0 this corresponds to the classical random walk and we drop the superscript (m). If w^δ is an interior vertex and z^δ is a vertex on the boundary, this is the probability that a random walk with killing rate $m^2\delta^2$ started at w^δ leaves the boundary at z without any conditioning. If both vertices are interior vertices this is the discrete massive Green function, i.e. the expected number of visits to z^δ starting from w^δ before hitting the boundary or being killed. Note that, because of the directed edges in general $Z_{\Omega^\delta}^{(m)}(w^\delta, z^\delta) \neq Z_{\Omega^\delta}^{(m)}(z^\delta, w^\delta)$. In the limit however, equality holds which follows from the results of section 5.5.

To apply the tightness results to the massive case we first need some estimates on this partition function.

Proposition 5.1. *There exists a constant $c > 0$ such that for each domain Ω^δ, interior point a^δ and boundary point b^δ, $\delta \leq \frac{1}{2}m^{-1}$, one has*

$$\frac{Z_{\Omega^\delta}^{(m)}(a^\delta, b^\delta)}{Z_{\Omega^\delta}(a^\delta, b^\delta)} \geq \exp(-cm^2R^2).$$

Proof. By Jensen's inequality:

$$\frac{Z^{(m)}_{\Omega^\delta}(a^\delta, b^\delta)}{Z_{\Omega^\delta}(a^\delta, b^\delta)} = \mathbb{E}\left((1-m^2\delta^2)^{\#\pi^\delta}\right) \geq (1-m^2\delta^2)^{\mathbb{E}(\#\pi^\delta)},$$

where the expectation is for a classical random walk π started at a^δ conditioned to leave at b^δ. There are different ways to see that $\mathbb{E}(\#\pi^\delta) \leq \text{const} \cdot \delta^{-2} R^2$. In [7] this is done by referencing [6], which also works in the directed triangular lattice, as this random walk also satisfies conditions (S) and (T) in [6]. A more fundamental reason for this inequality is the convergence of the walk to a Brownian motion under diffusive scaling, which implies that $\delta^2 \mathbb{E}(\#\pi^\delta)$ converges to the expected time until a Brownian motion leaves Ω conditioned on it leaving at b, which is less than the expected time to leave a ball of radius R. The desired estimate follows from $(1-m^2\delta^2)^{c\delta^{-2}R^2} \to \exp(-cm^2R^2)$. □

From this (just as in [7, Section 2.5]) it follows that the densities of massive LERW with respect to classical LERW are uniformly bounded from above by $\exp(cm^2R^2)$ and thus the tightness of the law of massive LERW follows. Also, (as in [7, Section 2.6]) it follows that each subsequential limit of $P^{(m)}_{\Omega^\delta}$ is absolutely continuous with respect to the SLE_2 on Ω. Thus we can use Girsanov's theorem to find the driving term of ξ_t of the Loewner evolution under $P^{(m)}_{\Omega^\delta}$.

5.3 Massive Martingale Observables

Let γ be a massive LERW from $a^\delta \in \partial\Omega^\delta$ to b^δ in Ω^δ. Then the *Massive Martingale Observables* is defined as:

$$M^{(m),\delta}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(v^\delta) \coloneqq \frac{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(v^\delta, \gamma^\delta(n))}{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(b^\delta, \gamma^\delta(n))}. \tag{5.2}$$

Note that, as said above, this is the ratio of two hitting probabilities for random walks started at v^δ and b^δ respectively.

Here we repeat the proof that this is a martingale from [20, Remark 3.6]. First we need a lemma that the law of the LERW doesn't change if we erase the loops in reverse chronological order instead of chronological order:

Lemma 5.2. *Let* $(\Gamma_0, \ldots, \Gamma_n)$ *be a Markov chain. Let* $(\gamma_1, \ldots, \gamma_l)$ *be its loop erasure and* $(\beta_1, \ldots, \beta_k)$ *be the loop erasure of the time reversal of* Γ. *Then* γ *has the same law as the time reversal of* β.

Proof. As seen in [19, Lemma 7.2.1], the idea is to cut the walk into several parts and rearrange them into a new path. This can be done in such a way that the resulting path uses the same transitions with the same multiplicities but in a different order and that the reverse-chronological loop-erasure of the new path is the loop-erasure of the original path. By the Markov property the original and the rearranged path have the same probability. Since the procedure is a bijection, the statement follows. □

Note that this is only an equality in law, it is easy to come up with examples for Γ such that γ and β are different.

Lemma 5.3. *The massive martingale observable* $M^{(m),\delta}_{\Omega^\delta\setminus\gamma^\delta[0,n]}(v^\delta)$ *as defined in 5.2 is a martingale.*

Proof. Consider two massive random walks that start at v^δ and b^δ respectively and run until they hit the boundary of Ω^δ or are killed. Perform a reverse-chronological loop erasure of these walks and call the resulting paths (α_i) and (β_i), enumerated such that α_0 and β_0 are the boundary vertices. Define

$$M_n \coloneqq \frac{\mathbb{P}_\alpha(\alpha[0,n]=\gamma[0,n])}{\mathbb{P}_\beta(\beta[0,n]=\gamma[0,n])}. \tag{5.3}$$

M_n is a random variable depending on $\gamma[0,n]$ and in the filtration of γ_n it is a martingale:

$$\begin{aligned}
&\mathbb{E}(M_{n+1}|\gamma[0,n]) && (5.4)\\
&=\sum_\omega \mathbb{P}(\gamma_{n+1}=\omega|\gamma[0,n])\frac{\mathbb{P}_\alpha(\alpha[0,n]=\gamma[0,n],\ \alpha_{i+1}=\omega)}{\mathbb{P}_\beta(\beta[0,n]=\gamma[0,n],\ \beta_{i+1}=\omega)} && (5.5)\\
&=\sum_\omega \mathbb{P}_\alpha(\alpha_{i+1}=\omega|\alpha[0,n]=\gamma[0,n])\frac{\mathbb{P}(\gamma_{n+1}=\omega|\gamma[0,n])}{\mathbb{P}_\beta(\beta_{i+1}=\omega|\beta[0,n]=\gamma[0,n])}M_n && (5.6)\\
&=\sum_\omega \mathbb{P}(\gamma_{n+1}=\omega|\gamma[0,n])M_n = M_n, && (5.7)
\end{aligned}$$

where in the second to last step we used that γ and β have the same law.

Since we are considering the reverse-chronological loop erasure, a necessary condition for $\alpha[0,n]=\gamma[0,n]$ is that the random walk used to generate α leaves $\Omega^\delta\setminus\gamma^\delta[0,n]$ through the point γ_n and the same goes for β. Because of the strong Markov property, the probability that the events $\alpha[0,n]=\gamma[0,n]$ and $\beta[0,n]=\gamma[0,n]$ occur conditioned on the respective events of leaving through γ_n are the same. Hence:

$$M_n = \frac{\mathbb{P}(\text{random walk started at } v \text{ leaves through } \gamma_n)}{\mathbb{P}(\text{random walk started at } b \text{ leaves through } \gamma_n)} \tag{5.8}$$

$$= \frac{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(v^\delta, \gamma^\delta(n))}{Z^{(m)}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(b^\delta, \gamma^\delta(n))} = M^{(m),\delta}_{\Omega^\delta \setminus \gamma^\delta[0,n]}(v^\delta) \tag{5.9}$$

and the right-hand side is also a martingale. □

5.4 Convergence of non-massive LERW

For orientation we will be sketching the proof for convergence of non-massive LERW to radial SLE_2. This was first proven in by Lawler [20], but the strategy we sketch is from [7] and is variant of the strategy in [10].

From the results in [13] it follows that the LERW measures on Ω^δ from a^δ to 0 are tight if the curves γ^δ are parametrized by capacity from 0, see Section 5.1. As the space of continuous curves is metrizable and separable, we can assume by Skorokhod's representation theorem that the γ^δ are defined on the same probability space in such a way that for each sequence $\delta_k \to 0$ along which the loop-erased random walk converges in law, it also converges almost surely. Let n_t^δ be the smallest n such that the capacity of $\gamma^\delta[0,n]$ is greater or equal to t. We set Ω_t^δ to be the connected component of $\Omega^\delta \setminus \gamma^\delta[0,n]$ including the target point and $a_t^\delta = \gamma^\delta(n_t^\delta)$. This is a discrete approximation of the capacity parametrization and by [7, Lemma 2.1] it is close to this parametrization. Further let $\Omega_t = \Omega \setminus \gamma[0,t]$.

The martingale defined in the previous section converges to a continuous limit, which will be proven in the next two sections. In the non-massive case this is just the suitably normalized Poisson kernel. These martingales are also uniformly bounded, as follows from an estimate in [6] together with convergence to the Green function, and thus their limits are also martingales.

Let ψ be the conformal map from Ω to $\mathbb{U}$ such that $\psi(0) = 0$ and $\psi(a) = 1$. Then the Poisson kernels have the following form:

$$M^\delta_{\Omega_t^\delta}(v^\delta) \to P_{\Omega_t}(v) = \Re\left(\frac{g_t(\psi(v)) + \exp(i\xi_t)}{g_t(\psi(v)) - \exp(i\xi_t)}\right). \tag{5.10}$$

Using this for two different v one can reconstruct ξ_t. This justifies the use of Itô's formula to obtain:

$$dP_{\Omega_t}(v) = d\Re\left(\frac{g_t(\psi(v)) + \exp(i\xi_t)}{g_t(\psi(v)) - \exp(i\xi_t)}\right) \tag{5.11}$$

$$= 2\Re\left(\frac{g_t(\psi(v))d\xi_t}{(g_t(\psi(v)) - \exp(i\xi_t))^2}\right) \tag{5.12}$$

$$+ 2\Re\left(\frac{g_t(\psi(v))\exp(i\xi_t)(g_t(\psi(v)) - \exp(\xi_t))(\mathrm{d}t - \frac{1}{2}\mathrm{d}\left\langle \xi, \xi\right\rangle_t)}{(g_t(\psi(v)) - \exp(i\xi_t))^3}\right). \tag{5.13}$$

By considering this for different values of v one sees that the only possibility for this to be a continuous martingale is to have ξ_t be a continuous martingale and $\langle \xi, \xi\rangle_t = 2t$. Then, by Lévy's characterisation of Brownian motion it follows that $\xi_t = B_{2t}$ with B being standard Brownian motion and therefor γ is an SLE_2 curve.

5.5 Convergence of the Green function

In this section we proof the convergence of $Z^{(m)}_{\Omega^\delta}(u, v)$ to a multiple of the massive Green function $G^{(m)}(u, v)$. To do so we use that $G^{(m)}(u, \cdot)$ is the unique function on Ω such that:

$$G^{(m)}(u, \cdot) = 0 \text{ on the boundary of } \Omega, \tag{5.14}$$

$$(-\Delta + m^2)G^{(m)}(u, \cdot) = 0 \text{ away from u, and} \tag{5.15}$$

$$G^{(m)}(u, v) = \frac{1}{2\pi}\log(|u - v|^{-1}) + O(1) \text{ as } v \to u. \tag{5.16}$$

The second condition is that $G(u, \cdot)$ is a massive harmonic function. This is the continuous analogue of the following property: Given $m > 0$ we call a function H massive discrete harmonic at $v \in \delta\mathbb{T}$ if

$$H(v) = \frac{1 - m^2\delta^2}{3}\sum_{w\in\delta\mathbb{T}:w\sim v} H(w). \tag{5.17}$$

As this is always less than the average of the values at adjacent vertices any massive discrete harmonic satisfies the maximum principle. These functions are indeed the massive counterpart to harmonic functions:

Lemma 5.4. *Let Ω^δ be a domain in $\delta\mathbb{T}$ and $(X_n)_{n\in\mathbb{N}}$ be a random walk with killing rate $m^2\delta^2$. Let H be a bounded real valued function defined on $\Omega^\delta \cup \partial\Omega^\delta$ and massive discrete harmonic at every point of Ω^δ. Denote by $\mathbb{P}^{(m)}_v$ the law of this walk started at v and by $\mathbb{E}^{(m)}_v$ the corresponding expectation. Let $\tau_{\partial\Omega^\delta}$ be the hitting time of the* *boundary. Then*

$$H(v) = \mathbb{E}_v^{(m)}(X_{\tau_{\partial\Omega^\delta}} 1_{X \text{ is not killed before } \tau_{\partial\Omega^\delta}}).$$

Proof. We add a vertex called the *cemetery* to the graph, which is connected to every vertex of Ω^δ. From every node there is a chance of $m^2\delta^2$ to visit the *cemetery* and once the walk visits the *cemetery* it never leaves it. After adding the *cemetery* to $\partial\Omega^\delta$ and extend H to be zero on the *cemetery*, H is harmonic on this adapted graph and the statement of the lemma is just the result for harmonic functions. □

To see that this is indeed the discrete equivalent to $(-\Delta + m^2)h = 0$ note that:

$$0 = \frac{1}{\delta^2}(H(v) - \frac{1 - m^2\delta^2}{\sum_{w\sim v} H(w)}) = \frac{\sum_{w\sim v}(H(w) - H(v))}{\delta^2} - m^2\frac{\sum_{w\sim v} H(w)}{3}. \quad (5.18)$$

When δ goes to 0 the first part converges to $\Delta H(v)$, as this is discrete Laplacian and the second part converges to $m^2 H(v)$.

Now we can proof the uniqueness of the Green function:

Lemma 5.5. *For each $u \in \Omega$ and $k \in \mathbb{R}^+$ there is exactly one function $G(u, \cdot) \colon \Omega \to \mathbb{R}$ that is massive harmonic away from u, 0 on the boundary, and satisfies*

$$G(u, \cdot) = k \log(|u - v|^{-1}) + o(\log|u - v||) \text{ as } v \to u.$$

Proof. Let h and g be two such functions. Then $f := h - g$ is a massive harmonic function that is massive harmonic away from u, 0 on the boundary, and is $o(\log(|u-v|))$ as v goes to u. Let $x \in \Omega$ and $B_t^{(m)}$ be massive Brownian motion with mass m started at x. Then $f(B_t^{(m)})$ is a martingale away from u, where $f(B_t^{(m)})$ is 0 if the Brownian motion has died before time t. Let $r > 0$, $B(u, r)$ be the disk of radius r, τ_r the hitting time of $B(u, r)$ and $\tau_{\partial\Omega}$ the hitting time of $\partial\Omega$. It is a well known fact about Brownian motion that the probability that $\tau_r < \tau_{\partial\Omega}$ decays logarithmically as r tends to 0. This can be seen by applying the optional stopping theorem to $\log|B_t|$, see for example [21]. By applying the optional stopping theorem to $f(B^{(m)}_{t\wedge\tau_r\wedge\tau_{\partial\Omega}})$ we obtain:

$$f(x) = |\mathbb{E}(f(B^{(m)}_{\tau_r\wedge\tau_{\partial\Omega}})) = \mathbb{E}(f(B^{(m)}_{\tau_r})|\tau_r < \tau_{\partial\Omega})\mathbb{P}(\tau_r < \tau_{\partial\Omega}).$$

Because f is $o(\log(|u - v|))$ the right-hand side converges to 0 as $r \to 0$. Thus $f(x) = h(x) - g(x) = 0$ for every x and g and h are identical. □

The existence of a function satisfying 5.14, 5.15 and 5.16 follows from the result in [7], or the convergence result below.

To prove that the partition functions $Z^{(m)}_{\Omega^\delta}(u,v)$ have a limit with those properties, we first prove that they are uniformly bounded on compact subsets of $\Omega \setminus \{u\}$ and equicontinuous to show the existence of subsequential limits. Uniform boundedness will follow from the estimates we use to proof the third property because:

$$0 \leq Z^{(m)}_{\Omega^\delta}(u,v) \leq -C\log(|u^\delta - v^\delta|) + O(1) \tag{5.19}$$

for a constant C independent of u^δ, v^δ and δ. Uniform equicontinuity follows from the following lemma 5.7.

Lemma 5.6. *Let $(X_n)_{n\in\mathbb{N}}$ be a simple random walk with killing rate $m^2\delta^2$ on $\delta\mathbb{T}$. For any annulus $A = A(v_0, r, 2r)$, let $E(A)$ be the event that X_n makes a non-trivial loop around in the annulus before leaving it, i.e. there are $0 < s < k < \tau_{\mathbb{C}\setminus A}$ such that $X_s = X_k$ and $X[s,k]$ is not contractible in A. Then there exists a positive constant independent of δ, r, v_0, and v such that:*

$$\mathbb{P}_v(E(A(v_0, r, 2r))) \geq const,$$

for all $8\delta < r \leq m^{-1}$ and all $v \in \delta\mathbb{T}$ such that $\frac{3}{2}r - \delta \leq |v_0 - v| \leq \frac{3}{2}r + \delta$.

Proof. The probability of this event can be bounded from below by considering a sequence of events of leaving disks of radius $r/8$ through a prescribed arc Γ without being killed. The probability of leaving through such an arc for the non-massive walk converges to that of Brownian motion. In particular, the probability to do so in the first $cr^2\delta^{-2}$ steps will converge to the probability that Brownian motion leaves through Γ before time cr^2 which is always positive. Thus, for the massive walk we get:

$$\begin{aligned}
&\mathbb{P}^{(m)}(X_n \text{ leaves through } \Gamma) \\
&\geq \mathbb{P}^{(m)}(X_n \text{ leaves through } \Gamma \text{ before time } r^2c\delta^{-2}) \\
&\geq \mathbb{P}(X_n \text{ leaves through } \Gamma \text{ before time } r^2c\delta^{-2})(1-m^2\delta^2)^{rc\delta^{-2}} \to \\
&\quad \mathbb{P}(B_t \text{ leaves through } \Gamma \text{ before time } r^2c)\exp(-m^2r^2c) > 0.
\end{aligned}$$

Here $\mathbb{P}$ denotes the law of the random walk and of Brownian motion without killing. Because of scale-invariance of Brownian motion the probability in the last step does not depend on r and the exponential term is bounded from below by $\exp(-c)$ because $r \leq m^{-1}$ Therefore, each of these events has a probability bounded away from 0. Because we only need a fixed number of such events to force a non-trivial loop, this also gives a lower bound for the event that the random walk forms a non-trivial loop around the annulus before exiting. □

Using lemma 5.6 we can proof a regularity estimate for discrete massive harmonic functions:

Lemma 5.7. *There are constants C and β depending on m such that for all positive massive harmonic functions H defined in $B(v_0, 2r) \cap \delta\mathbb{T}$ with $r \leq m^{-1}$ and for all $v_1, v_2 \in B(v_0, r) \cap \delta\mathbb{T}$ one has:*

$$|H(v_1) - H(v_2)| \leq C(|v_2 - v_1|/r)^{\beta} \max(H(v)).$$

Proof. See [7, Lemma 3.10], the proof only uses the corresponding lemma to Lemma 5.6. □

Finally, for the estimate we also need the following lemma about convergence of the conditioned (non-massive) random walk to a Brownian bridge:

Lemma 5.8. *Let X_n^{δ} be the simple random walk on $\delta\mathbb{T}$ started at x^{δ} converging to x. Let $y^{\delta} \in \delta\mathbb{T}$ approximate y in such a way that it is always possible to go from x^{δ} to y^{δ} in $\lfloor \delta^{-2} t \rfloor$ steps. Then the law of $(X^{\delta}_{\lfloor \delta^{-2} t \rfloor})_{t \in [0,c]}$ conditioned on $X^{\delta}_{\lfloor \delta^{-2} c \rfloor} = y^{\delta}$ converges to the law of the Brownian bridge $(b_t)_{t \in [0,c]}$ from x to y.*

Proof. We interpolate linearly between to consider $(X^{\delta}_{\lfloor \delta^{-2} t \rfloor})_{t \in [0,c]}$ as a continuous function on $[0, c]$. Let S_t^{δ} be this interpolation. Let $s < c$ and $F \colon C([0, s]) \to \mathbb{R}$ be a functional on continuous function. Then the conditioning $S_c^{\delta} = y^{\delta}$ weights every path $(S_t^{\delta})_{t \in [0,s]}$ by how likely it is to go to y^{δ} from S_s^{δ}. Thus the conditional expectation of the functional can be rewritten as:

$$\mathbb{E}(F((S_t^{\delta})_{t \in [0,s]}) | S_c^{\delta} = y^{\delta}) = \mathbb{E}(F((S_t^{\delta})_{t \in [0,s]}) \frac{\mathbb{P}(S_c^{\delta} = y^{\delta} | S_s^{\delta})}{\mathbb{P}(S_c^{\delta} = y^{\delta})}).$$

This ratio of probabilities converges to the ratio of the respective probability densities. If φ is the probability density function of the standard two dimensional Gaussian distribution, this ratio converges to $\frac{\varphi(\frac{y - B_s}{c - s})}{\varphi(\frac{y}{c})}$. Thus the expectation converges to:

$$\mathbb{E}(F((B_t)_{t \in [0,s]}) \frac{\varphi(\frac{y - B_s}{c - s})}{\varphi(\frac{y}{c})}) = \mathbb{E}(F((B_t)_{t \in [0,s]}) | B_c = y).$$

As this holds for any functional F and any $s < c$ the process $(S_t^{\delta})_{t \in [0,s]}$ conditioned to end at y^{δ} converges to the Brownian motion of the same duration conditioned to end at y. To remove the restriction of $s < c$ we consider the reversed process $(S^\delta_{c-t})_{t\in[0,s]}$. While this process takes different steps than S (in the exactly opposite direction) it also converges to Brownian motion and exactly the same consideration shows that it converges to the Brownian motion started at y and conditioned to end at x. Together this shows convergence in distribution of $(S^\delta_t)_{t\in[0,c]}$ to the Brownian bridge started at x, ending at y of duration c. □

We will use this to approximate the probability that a random walk conditioned on the point at time n leaves a domain by the corresponding probability for the Brownian motion.

Corollary 5.9. *Let Ω^δ approximate a domain $\Omega \in \mathbb{C}$ and x^δ, y^δ approximate x, y in Ω. Let $0 < c_1 < c_2$, then*

$$\mathbb{P}^\delta_{x^\delta}(\tau_{\partial\Omega^\delta} > c\delta^{-2} | X^\delta_{\lfloor c\delta^{-2}\rfloor} = y^\delta) \to \mathbb{P}(\tau_{\partial\Omega} > c | B_c = y)$$

uniformly in $c \in [c_1, c_2]$ as $\delta \to 0$.

For the estimate in the following proposition we will also need to consider the behaviour of this probability as the time c approaches 0:

Lemma 5.10. *Let $(B_t)_{t\in[0,c]}$ be standard two dimensional Brownian motion started at x. Let $P(c) = \mathbb{P}(\tau_{\partial\Omega} > c | B_c = y)$ be the probability that a Brownian bridge of length c does not leave the domain in its duration. Assume that the line between x and y is in Ω. Then:*

$$\lim_{c\to 0} P(c) = 1.$$

Proof. Let $(b_t)_{t\in[0,c]}$ be the Brownian bridge from x to y of length c. A well known representation of the Brownian bridge is $b_t = x + (y-x)\frac{t}{c} + W_t - \frac{t}{c}W_c$, where $(W_t)_{t\in[0,c]}$ is a standard two dimensional Brownian motion started at 0. By rescaling the time to the interval $[0, 1]$ we get $\hat{b}_t = b_{tc}$ for t in $[0, 1]$, which satisfies:

$$\hat{b}_t = x + (y - x)t + W_{tc} - tW_c.$$

As $c \to 0$ the second term $W_{tc} - tW_c$ converges to 0 in probability uniformly in t. Since Ω is an open set and hence also contains an open set around the line from x to y this implies that $P(c)$ converges to 1. □

Now we state the main result of this section:

Proposition 5.11. *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and $x, y \in \Omega$ be two distinct points of Ω. Assume that discrete domains $\Omega^\delta \subset \delta\mathbb{T}$ approximate Ω. Then*

$$Z_{\Omega^\delta}^{(m)}(x^\delta, y^\delta) \to \frac{\sqrt{3}}{2} G_{\Omega}^{(m)}(x, y).$$

Proof. We have that $Z_{\Omega^\delta}^{(m)}(x^\delta, \cdot)$ is equicontinuous and uniformly bounded on compact sets away from x hence subsequential limits exist, which means there is an $h(x, \cdot)$ such that:

$$Z_{\Omega^\delta}^{(m)}(x, \cdot) \to h(x, \cdot) \text{ as } \delta = \delta_k \to 0 \tag{5.20}$$

uniformly on compact subsets of $\Omega \setminus \{u\}$. Because the walk converges to Brownian motion, h is harmonic away from u. To proof that h is a multiple of the continuous Green function it suffices to check:

$$h(x, y) = -k \log(|x - y|) + O(1). \tag{5.21}$$

To do this we estimate the following sum:

$$Z_{\Omega^\delta}^{(m)}(x^\delta, y^\delta) = \sum_{n=0}^{\infty} p^n(x^\delta, y^\delta) \tag{5.22}$$

$$= \sum_{n=0}^{\infty} \#\{\text{paths of length } n \text{ from } x^\delta \text{ to } y^\delta\}(\frac{1}{3}(1 - m^2\delta^2))^n \mathbb{P}(\tau_{\partial\Omega} > n | X_n^\delta = u). \tag{5.23}$$

We split this sum into three parts: First the sum from 0 to $\lfloor |x-y|^2\delta^{-2} \rfloor$ then to $\lfloor c\delta^{-2} \rfloor$ for large c and then the rest. We will call these sums I, II and III and estimate them separately.

To estimate the first part of the sum we compare $p_n(x^\delta, y^\delta)$ with points that are closer to x: Depending on the residue of n modulo 3 a different set of vertices is reachable from x^δ but for any vertex z that is reachable and satisfies $|z - x^\delta| < \frac{1}{2}|y^\delta - z^\delta|$ it holds that $p_n(x^\delta, z) \geq p_n(x^\delta, y^\delta)$. There are approximately $\frac{1}{6\sqrt{3}}|x - y|^2\pi\delta^{-2}$ such points. Thus we have:

$$\lfloor |x-y|^2\delta^{-2} \rfloor = \sum_z \sum_{n=0}^{\lfloor |x-y|^2\delta^{-2} \rfloor} p_n(x, z) \geq \frac{1}{6\sqrt{3}}|x-y|^2\pi\delta^{-2} \sum_{n=0}^{\lfloor |x^\delta-y^\delta|^2\delta^{-2} \rfloor} p_n(x^\delta, y^\delta)$$

Which implies that $I < \frac{6\sqrt{3}}{\pi} + o(1)$ and thus I is bounded independently of $|x - y|$ as long as this distance is positive.

We choose c such that III is bounded by a constant independent of $|x-y|$ and δ. This is always possible: For every Ω there is always a k such the probability of the random walk started at y^δ conditioned to be at y^δ at time $k\delta^{-2}$ leaving Ω^δ before $k\delta^{-2}$ is bounded away from 0, since this converges to the probability that the Brownian bridge from y to y of length k leaves Ω. This implies

$$\mathbb{P}^\delta_{x^\delta}(\tau_{\partial\Omega^\delta} > nk\delta^{-2}|X_{nk\delta^{-2}} = y) \leq \mathbb{P}^\delta_{x^\delta}(\tau_{\partial\Omega^\delta} > nk\delta^{-2}|X_{mk\delta^{-2}} = y \text{ for } 0 < m \leq n),$$

as conditioning on the event that the walk visits y more often only decreases the probability that the walk leaves the domain. By the Markov property the excursions from y to y are all independent and therefore we get:

$$\mathbb{P}^\delta_{x^\delta}(\tau_{\partial\Omega^\delta} > nk\delta^{-2}|X_{mk\delta^{-2}} = y \text{ for } 0 < m \leq n) < p^{n-1}$$

for some $p < 1$ independent from δ. Thus $\mathbb{P}(\tau_{\partial\Omega} > n|X^\delta_n = u)$ decays exponentially:

$$\mathbb{P}(\tau_{\partial\Omega} > n|X^\delta_n = u) \leq Cp^{\frac{n\delta^2}{k}}$$

for a constant C independent of δ. From the estimate used for I it follows that:

$$\#\{\text{paths of length } n \text{ from } x^\delta \text{ to } y^\delta\}(\frac{1}{3})^n \leq D\delta^2$$

again for a constant D independent of δ. Putting these two estimates together we get:

$$\sum_{n=c\delta^{-2}}^{\infty} p^n(x^\delta, y^\delta) \leq CD\delta^2 \sum_{n=c\delta^{-2}}^{\infty} p^{\frac{n\delta^2}{k}} = CDp^{\frac{c}{k}} \frac{\delta^2}{1-p^{\frac{\delta^2}{k}}}.$$

This converges to 0 as $c \to 0$. It converges uniformly in δ because the fraction on the right has a limit as δ approaches 0.

For II we estimate the number of lattice paths using Sterling's formula. If $y = x+a+be^{2\pi i/3}$ with $a, b \geq 0$, then the number of such paths is 0 if $n-a-b$ is not divisible by 3. If $n-a-b$ is divisible by 3, the number of steps is given by the multinomial coefficient: $\binom{n}{\frac{n-a-b}{3},\frac{n+2a-b}{3},\frac{n-a+2b}{3}}$. This can be approximated using Sterling's formula:

$$\#\{\text{paths of length } n \text{ from } x \text{ to } x\}(\frac{1}{3})^n \tag{5.24}$$

$$= \frac{n^n\sqrt{2\pi n}}{(\frac{n-a-b}{3})^{\frac{n-a-b}{3}}(\frac{n+2a-b}{3})^{\frac{n+2a-b}{3}}(\frac{n-a+2b}{3})^{\frac{n-a+2b}{3}}(\sqrt{2\pi n/3})^3 3^n}(1+o_n(1)) \tag{5.25}$$

$$= \exp(-a-b+2a-b-a+2b)\frac{\sqrt{27}}{2\pi n}(1+o_n(1)) = \frac{\sqrt{27}}{2\pi n}(1+o_n(1)) \tag{5.26}$$

Let $P(t) = \mathbb{P}(\tau_{\partial\Omega} > t | B_t = y)$. By Corollary 5.9 we get:

$$\mathbb{P}_x(\tau_{\partial\Omega} > n | X_n^\delta = y) = P(n\delta^2)(1+o_\delta(1)),$$

for $\delta \to 0$ and all n such that $|x-y|^2\delta^{-2} \leq n \leq c\delta^{-2}$. Using this we get:

$$II = \sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} p^n(x^\delta, y^\delta) \tag{5.27}$$

$$= \frac{\sqrt{3}}{2\pi}\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} \frac{1}{n}P(n\delta^2)(1-m^2\delta^2)^n(1+o_n(1))(1+o_\delta(1)) \tag{5.28}$$

$$= \frac{\sqrt{3}}{2\pi}(1+o_\delta(1))\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} \frac{1}{n}(1-m^2\delta^2)^n P(n\delta^2), \tag{5.29}$$

where in the last step we used that the $o_n(1)$ from Sterling's formula is actually of order $\frac{1}{12n}$, so the contribution of the $o_n(1)$ term is the tail of a convergent series and thus itself $o_\delta(1)$. In the first step we get $\sqrt{3}$ instead of $\sqrt{27}$ because only every third summand is non-zero. This sum can now be seen as a Riemann sum:

$$\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} \frac{1}{n}(1-m^2\delta^2)^n P(n\delta^2) = \tag{5.30}$$

$$\frac{1}{\delta^{-2}}\sum_{n=\lfloor |x-y|^2\delta^{-2}\rfloor+1}^{\lfloor c\delta^{-2}\rfloor} ((1-m^2\delta^2)^{\delta^{-2}})^{n\delta^2}\frac{P(n\delta^2)}{n\delta^2} \to \int_{|x-y|^2}^{c} \frac{P(x)\exp(-m^2x)}{x}dx. \tag{5.31}$$

The convergence of the Riemann sum is guaranteed by the fact that the integrand is decreasing and bounded on the relevant interval. As we are interested in the behaviour when y is close to x we can assume that the straight line from x to y is in Ω and therefore lemma 5.10 applies and $P(c)$ approaches 1 as c goes to 0. Thus the behaviour of this integral as $|x-y| \to 0$ is like $-2\log(|x-y|) + O(1)$.

Putting the three estimates together we get that:

$$h(x,y) = -\frac{\sqrt{3}}{\pi}\log(|x-y|) + O(1). \tag{5.32}$$

Thus h is the unique function satisfying the desired properties. Therefore all subsequential limits are the same which proves the desired convergence of the discrete massive Green functions. □

Remark. The factor $\frac{\sqrt{3}}{2}$ can be explained as follows: Just as in the discrete case, the expected time spent by Brownian motion in a disk D is given by the integral of the Green's function. Thus, the expected amount of time spent in D of the discrete walk on the scaled lattice should satisfy:

$$\delta^2\mathbb{E}(|\{n : X_n \in D\}|) = \delta^2 \sum_{y^\delta \in A\cap\delta\mathbb{T}} Z(x^\delta, y^\delta) \to \int_D G(x,y)dy.$$

For any disk D the number of points in $D \cap \delta\mathbb{T}$ behaves like $\delta^2|A|\frac{2}{\sqrt{3}}$ as this is the density of vertices of the triangular lattice. Therefore as $Z(x,y) \to h(x,y)$ the left-hand side converges to:

$$\frac{2}{\sqrt{3}}\int_A h(x,y)dy = \int_D G(x,y)dy.$$

Therefore the constant is to be expected. It is however not harmful: the continuous objects considered below are all either identical to their counterparts on the square lattice or also just scaled by this factor.

5.6 Convergence to the continuous martingales

For the convergence of the martingale observables we need the convergence of the discrete Poisson kernel to the continuous Poisson kernel. This is proven by Yadin and Yehudahofffor all random walks on planar graphs that converge to Brownian motion in [28]:

Proposition 5.12. *Let Ω^δ approximate Ω. Let v^δ and b^δ be interior vertices of Ω^δ and a^δ be a boundary vertex. Then*

$$\frac{Z_{\Omega^\delta}(v^\delta, a^\delta)}{Z_{\Omega^\delta}(b^\delta, a^\delta)} \to \frac{P_\Omega(v,a)}{P_\Omega(b,a)},$$

as $\delta \to 0$.

We define the *massive Poisson kernel* as:

$$P^{(m)}_{\Omega_t}(z, a_t) := P_{\Omega_t}(z, a_t) - m^2 \int_{\Omega_t} G^{(m)}_{\Omega_t}(z, w) P_{\Omega_t}(w, a_t) dA(w). \tag{5.33}$$

This is motivated by the following formula for the discrete Green function:

$$(1-m^2\delta^2) Z^{(m)}_{\Omega^\delta}(w^\delta, z^\delta) = Z_{\Omega^\delta}(w^\delta, z^\delta) + m^2\delta^2 \sum_{v^\delta \in \mathrm{Int}\Omega^\delta} Z^{(m)}_{\Omega^\delta}(w^\delta, v^\delta) Z_{\Omega^\delta}(v^\delta, z^\delta), \tag{5.34}$$

which we prove by splitting each trajectory in the definition of Z_Ω into two parts, and summing over all possible ways to do so:

$$\begin{aligned}
&\sum_{v^\delta \in \mathrm{Int}\Omega^\delta} Z^{(m)}_{\Omega^\delta}(w^\delta, v^\delta) Z_{\Omega^\delta}(v^\delta, z^\delta) \\
&= \sum_{v^\delta \in \mathrm{Int}\Omega^\delta} \sum_{k \geq 0} \sum_{\substack{\pi : w \to z, \\ \pi_k = v}} (\frac{1}{3}(1 - m^2\delta^2))^k (\frac{1}{3})^{(\#\pi) - k} \\
&= \sum_{\pi : w \to z} (\frac{1}{3})^{\#\pi} \sum_{k=0}^{\#\pi} (1 - m^2\delta^2)^k \\
&= \sum_{\pi : w \to z} (\frac{1}{3})^{\#\pi} \frac{1 - (1 - m^2\delta^2)^{(\#\pi)+1}}{m^2\delta^2} \\
&= \frac{Z_{\Omega^\delta}(w^\delta, z^\delta) - (1 - m^2\delta^2) Z^{(m)}_{\Omega^\delta}(w^\delta, z^\delta)}{m^2\delta^2}.
\end{aligned}$$

Rearranging the terms gives the desired result. When z^δ is on the boundary and δ goes to 0 this formula suggests the definition above.

Lemma 5.13. *In the setup above for any* $z \in \Omega_t$ *and* $z^\delta \to z$ *as* $\delta \to 0$*, one has:*

$$\frac{Z^{(m)}_{\Omega^\delta}(z^\delta, a^\delta_t)}{Z_{\Omega^\delta}(z^\delta, a^\delta_t)} \to \frac{P^{(m)}_{\Omega_t}(z, a_t)}{P_{\Omega_t}(z, a_t)} = 1 - m^2 \int_{\Omega_t} \frac{P_{\Omega_t}(w, a_t)}{P_{\Omega_t}(z, a_t)} G^{(m)}_{\Omega_t}(z, w) dA(w).$$

The proof in [7] works also for the triangular lattice, as besides the convergence results of the last section it only requires the identity above, and estimates on the massive Green function, which follow from convergence to Brownian motion. One might at first be worried as the right-hand side of 5.34 is not linear in Z and the limit of the discrete Green function for the triangular lattice is $\frac{\sqrt{3}}{2}$ times the limit on the square grid. However, the additional factor of $\frac{\sqrt{3}}{2}$ is actually needed for the sum to converge to a Lebesgue integral, see the remark after proposition 5.11. Therefore $P^{(m)}_{\Omega_t}(z, a_t)$ is $\frac{\sqrt{3}}{2}$ times the massive Poisson kernel in [7].

Using this we can proof the convergence of the martingale observables:

Proposition 5.14. *In the setup above we have:*

$$M^{(m)}_{\Omega^\delta_t}(v^\delta) = \frac{Z^{(m)}_{\Omega^\delta_t}(v^\delta, a^\delta_t)}{Z^{(m)}_{\Omega^\delta_t}(b^\delta, a^\delta_t)} \to \frac{P^{(m)}_{\Omega_t}(v, a_t)}{P^{(m)}_{\Omega_t}(b, a_t)} =: M^{(m)}_{\Omega_t}(v)$$

Proof. We can write the martingale observable as a product:

$$\frac{Z^{(m)}_{\Omega^\delta_t}(v^\delta, a^\delta_t)}{Z^{(m)}_{\Omega^\delta_t}(b^\delta, a^\delta_t)} = \frac{Z^{(m)}_{\Omega^\delta_t}(v^\delta, a^\delta_t)}{Z_{\Omega^\delta_t}(v^\delta, a^\delta_t)} \left(\frac{Z^{(m)}_{\Omega^\delta_t}(b^\delta, a^\delta_t)}{Z_{\Omega^\delta_t}(b^\delta, a^\delta_t)} \right)^{-1} \frac{Z_{\Omega^\delta_t}(v^\delta, a^\delta_t)}{Z_{\Omega^\delta_t}(b^\delta, a^\delta_t)}$$

Applying Proposition 5.12 and Proposition 5.13 to the factors gives the desired result. □

Note that this is simpler than in [7], because in the radial setting there is no need for normalization near the endpoint. Now that this convergence has been established, all estimates and computations in section 4 of [7] apply, as these are all in the continuum, except that they need to be appropriately adapted to the radial case.

5.7 Proof of the main statement

Theorem 5.15. *Let $\Omega^\delta \subset \delta\mathbb{T}$ approximate Ω with $\in \Omega^\delta$ and a^δ a boundary point of Ω^δ which approximates $a \in \partial\Omega$. Then the massive loop-erased random walk started at 0 conditioned to hit the boundary at a^δ before dying converges in law to radial massive SLE$_2$ from a to 0.*

Proof. As seen in section 5.2 the laws of the massive loop-erased random walks are tight and all subsequential limits are absolutely continuous with respect to classical SLE$_2$. This justifies the application of Girsanov's theorem which in particular implies that the driving function ξ_t is a semi-martingale under $\mathbb{P}^{(m)}$. Moreover, the discrete martingales (5.2) have continuous limits as shown in 5.14. As the martingales are uniformly bounded, these continuous limits also have to be martingales. By brute force calculations as indicated in [23] and done in [7] this family of martingales is sufficient to determine the law ξ_t under $\mathbb{P}^{(m)}$. □

To describe the scaling limit of the massive loop-erased random walk conditioned to hit any point of the boundary instead of just one particular point one chooses a according to the hitting distribution of massive Brownian motion. This distribution, however, is more difficult to calculate, as it is no longer conformally covariant in the same way as in the classical case.

6 The main results

The results of section 3 and 5 combine to proof:

Theorem 6.1. *The scaling limit of a loop-erased random walk with drift in a domain $\Omega^\delta \subset \delta\mathbb{T}$ approximating Ω started at 0 until it hits the boundary is given by massive radial SLE$_2$ in Ω from 0 to the boundary .*

Proof. This follows from theorem 4.4 combined with the results of section 5. □

This gives the scaling limit of a single branch of the tree in Temperleys bijection. To go from this to convergence of the tree we need a version of of Schramm's finitness theorem from [26, Theorem 10.2 and Theorem 11.1(i)]:

Theorem 6.2 (Schramm's finiteness theorem)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded domain. For every $\epsilon > 0$ there exists a $\hat{\hat{\delta}}$ such that for all $0 < \delta < \hat{\hat{\delta}}$ the following holds: Let V be a set of vertices of $\delta\mathbb{T}$ (or $\delta\mathbb{Z}^2$) such that every point in Ω has distance at most $\hat{\hat{\delta}}$ from the vertices in V. Let T_δ be the subtree spanned by V, that is the union of the branches starting at the vertices in V. Then with probability $1 - \epsilon$ a branch started at any other vertex x will hit T_δ before leaving the ball of radius ϵ around x.*

Proof. The proof of this theorem given in [26] works in our setting as well, as it only uses Wilson's algorithm and a lemma which follows from the annulus crossing estimate Lemma 5.6 for the random walk with drift. Because we are only considering bounded domains Ω, the drift, while it does make the relevant probability smaller, only does so by a bounded factor, and thus the relevant probability is still bounded from below. □

Informally this theorem states that "most of the tree" is already contained in finitely many branches. This is enough to go from the convergence of the single branches to convergence of the tree in the sense of Schramm:

Theorem 6.3. *Let $\Omega \subset \mathbb{C}$ be a bounded domain approximated by Ω^δ. Then the wired directed spanning tree with weights converging at the rate given in section 4 has a scaling limit.*

Proof. This follows from Wilson's algorithm and the previous theorem. ☐

Now that the converges of the tree has been established an application of the theorem [4, Theorem 2.2] by Berestycki, Laslier and Ray gives:

Theorem 6.4. *The fluctuations of the height field of the drifted dimer model in Ω converge in law and this law depends only on the limiting law of the uniform directed spanning tree.*

This allows for a description of this limiting law by describing the winding of the branches.

In summary: The height function of the drifted dimer model to the winding of the branches of a certain random tree by Temperley's bijection. Because of Wilson's algorithm, these branches are given by a loop-erased random walk with drift. In Section 4 it was shown that the law of this loop-erased random walk is absolutely continuous with respect to the law of the massive loop-erased random walk with a simple Radon-Nikodym derivative. This can be used to find a limiting object for the random walk with drift, as the limiting object for the massive loop-erased random walk is already known for the square lattice. On the triangular lattice the convergence of the massive loop-erased random walk is shown in section 5. Furthermore, this can also be used to give an limiting object for the entire tree and therefore also for the limit of the height function.

7 Further investigation

There are a number of possible directions for further investigation. The winding of the drifted SLE could be investigated to obtain a perhaps more explicit description of limiting field of the height fluctuations. This could be done with methods similar to those in [5].

The new kind of off-critical SLE introduced here might also be of independent interest. Just as massive SLE_2 this *SLE_2 with drift* is also not conformally invariant. However, the image of Brownian motion with drift under a conformal map is Brownian motion with a certain local drift, so it is possible to interpret the conformal covariance of SLE_2 with drift in a similar way. Perhaps this could help understand the conformal covariance of massive SLE_2.

Lastly, there are definitions of massive SLE_κ for other values of κ than 2, see [23]. These are conjectured or proven to be limits of other off-critical models. Is there also a meaningful way to define SLE_κ with drift for these or other values? Does this correspond in some way to a drift in the corresponding discrete models?

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