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Foundations of Harmonic Analysis on the Heisenberg Group

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Introduction

This text focuses on the representation theory of the Heisenberg group $\mathbf{H}^n$ and some corresponding basic concepts of harmonic analysis on $\mathbf{H}^n$. The Heisenberg group is of particular interest since it plays an important role in several branches of mathematics and theoretical physics. It is a nilpotent Lie group and it is the "simplest" non-compact, non-commutative example. Consequently, the most important representations in the context of harmonic analysis, namely the irreducible unitary representations, are no longer scalar-valued (as in the commutative case) neither finite-dimensional, that is to say matrix-valued (as in the compact case). In particular, the resulting notion of Fourier transform on $\mathbf{H}^n$ is therefore an operator-valued function acting on the separable Hilbert space $L^2(\mathbb{R}^n)$.

The first chapter gives a brief introduction to some basic concepts of representation theory. As a first step we introduce the notion of strongly continuous one-parameter groups, which turn out to be a simple example of a Banach space-valued Lie group representation. We will see that each of these one-parameter groups possesses an infinitesimal generator, which in turn induces a notion of smoothness and analyticity of vectors in the representation space. In that context we meet an instance of Banach space-valued integration (discussed in Appendix A), which turns out to be one of the technical main tools throughout this text. We furthermore focus on skew-adjoint operators in Hilbert space in order to study the special relation between unitary one-parameter groups and their skew-adjoint infinitesimal generators.

The second part of Chapter 1 is dedicated to general (strongly continuous) Lie group representations on Banach spaces. After recalling the differences between finite-dimensional and infinite-dimensional representations, we construct the Haar measure, a left-invariant Borel measure on the Lie group G, since (vector-valued) integration continues to be an essential tool for many results. We shall use it, e.g., to show that the family of left translations form a Lie group representation of G on $L^p(G), p \in [1, \infty)$, called the left regular representation of G. It also permits the definition of the integrated representation, where an integral over G is used to assign an operator to certain functions on G. This procedure furthermore leads to the notion of Banach algebra representations, which we discuss in Appendix B. Finally, we return to the concept of smooth vectors and show their denseness in the representation space.

The second chapter is the core of this text. We start by constructing the Heisenberg Lie algebra $\mathfrak{h}^n$, motivated by the commutation relation from quantum mechanics. Using the exponential map on $\mathfrak{h}^n$, we discuss two approaches to the Heisenberg group, which display slightly different group laws. We then determine the automorphisms of $\mathbf{H}^n$ and, in view of future applications, its center $\mathcal{Z}$.

In the second and major part of this chapter we construct the most important representation of $\mathbf{H}^n$, the so-called Schrödinger representation on $L^2(\mathbb{R}^n)$. We then make use of the results from Chapter 1 in order to prove that the Schrödinger representation is in fact a strongly continuous unitary representation of $\mathbf{H}^n$, generated by an essentially skew-adjoint Lie algebra representation of $\mathfrak{h}^n$. Subsequently, we learn that the Schrödinger representation can be parameterized with $h \in \mathbb{R}^*$, giving a family of inequivalent unitary representations of $\mathbf{H}^n$. Next we turn to the Stone-von Neumann theorem and the classification of the irreducible unitary representations of $\mathbf{H}^n$. To this end, we study twisted convolution, a non-commutative convolution product of functions defined on the quotient $\mathbf{H}^n/\mathcal{Z} \cong \mathbb{R}^{2n}$, which preserves the structure of $\mathbf{H}^n$. Using its compatibility with the integrated Schrödinger representation (restricted to $\mathbb{R}^{2n}$) we derive the technical tools required for proving the Stone-von Neumann theorem. Together with Schur's lemma it leads to a classification of the irreducible unitary represenations of $\mathbf{H}^n$, asserting that each one is unitarily equivalent either to some Schrödinger representation of parameter $h \in \mathbb{R}^*$ or some one-dimensional representation with values in S^1.

All the results collected that far eventually culminate in the final part of this text, where we define a Fourier transform for $\mathbf{H}^n$ in analogy to the ordinary Fourier transform on $\mathbb{R}^n$. As a consequence of the Stone-von Neumann theorem the Fourier transform is either one-dimensional (agreeing with the ordinary Fourier transform) or given by an operator-valued function acting on $L^2(\mathbb{R}^n)$. The two main statements of this section are the Plancherel theorem for the Fourier transform on $L^2(\mathbf{H}^n)$, which states the existence of a measure μ that turns the Fourier transform into a unitary isomorphism. The second statement provides an explicit inversion formula of the Fourier transform for all Schwartz functions.

Some material on essential concepts used throughout the text is collected in three appendices.

Appendix A provides a compilation of facts on Bochner integration - one possible approach to Banach space-valued integration. All definitions and statements are given in such a way that they are directly applicable to the statements in Chapters 1 and 2 without any further modification. After characterizing the basic notions of Bochner measurability and Bochner integrability we prove Bochner versions of three fundamental theorems of integration theory: Fubini's theorem, the dominated convergence theorem and the fundamental theorem of calculus. We furthermore show that Bochner integration interchanges with the application of bounded linear mappings and, under some assumptions, closed linear mappings as well. Finally, we introduce the natural Bochner generalization of the Lebesgue spaces L^p to functions with values in some Banach space.

Appendix B gives a brief account on the notion of Banach algebras and their representations. Its main purpose is the illustration of the way integrated representations act on their natural domains, namely convolution algebras of integrable functions.

Last but not least, Appendix C provides some important facts on the spectral theory of self-adjoint operators in Hilbert space. We present the main aspects of functional calculus, prove Stone's theorem, which gives the relation between unitary one-parameter groups and their infinitesimal generators, and Schur's lemma.

1. Some Prerequisites from Lie Group Representation Theory

In Chapter 1 we introduce some frequently used concepts of infinite-dimensional Lie group representation theory and show some basic results that will later be used in Chapter 2 to treat the Heisenberg group. To this end, we first concentrate on strongly continuous one-parameter groups on $\mathbb{R}$, a quite particular case of representations that yet possesses a vast field of applications and interesting examples. On our way, we will encounter a cardinal feature of the theory: Banach space-valued integration, first on $\mathbb{R}$, and eventually on Lie groups, as we intent to take the step from one-parameter groups to generic Lie group representations. In general, our focus will noticeably lie on analytic rather than geometric aspects, mainly motivated by the author's personal taste. Appendices A and C provide additional material on integration theory, and on special one-parameter groups, respectively, which, we hope, will support the interested reader in deepening his or her understanding.

1.1. One-Parameter Groups of Operators

As mentioned above the treatise of one-parameter groups is a useful first step to get familiar with some concepts from representation theory due to simplicity of their underlying Lie group, $\mathbb{R}$, on the one hand. On the other hand, we try to catch a glimpse of the vastness of applications, in particular ODE- and PDE-theory, by giving some illustrating examples.

1.1.1. Definitions and Estimates

Throughout this chapter B will always denote a Banach space, and occasionally we will use B_1, B_2, etc. By $L(B)$ we denote the space of all continuous linear maps on B, which we will always equip with the operator norm defined by

$$\|A\| := \sup\{\|Au\|_B \mid u \in B,\ \|u\|_B \leq 1\}.$$

Definition 1.1. *A strongly continuous one-parameter group of operators on Banach space B is a map $V : \mathbb{R} \to L(B)$ satisfying the following conditions*

(i) $V(s+t) = V(s)V(t) \quad \forall s,t \in \mathbb{R}$,

(ii) $V(0) = I$,

(iii) $t_j \xrightarrow{\mathbb{R}} t \quad \Rightarrow \quad V(t_j)u \xrightarrow{B} V(t)u \ \ \forall u \in B$.

Property (i) is usually referred to as group homomorphism property, (iii) as strong continuity which motivates the group's name. Note that it actually suffices to ask for (iii) for $t_j \to 0$ since we can always apply $V(-t)$ to both sides.

Sometimes strong continuity is replaced by an even stronger requirement, namely convergence in the operator norm. The respective groups are called norm continuous groups. Note that one-parameter groups are often defined as semi-groups on $[0,\infty)$ to better suit the applications. Some initial value problems such as, e.g., the heat equation cannot be solved for negative time in general, thus it is important to provide a more general concept. However, our main goal will be the extension to general Lie groups, hence it is more convenient and comprehensible to introduce the concept as we have done.

Example 1.2. Our main example throughout this section (and even later in more general form) will be the family of translations on $L^p(\mathbb{R})$, $1 \le p < \infty$. We will see that this group is essentially connected to the notions of strong and weak differentiability. For $1 \le p \le \infty$, we define the translations by

$$\begin{aligned}\tau_p(t) : L^p(\mathbb{R}) &\to L^p(\mathbb{R}),\\ f(.) &\mapsto f(\,.-t).\end{aligned}$$

Properties (i) and (ii) of Definition 1.1 are immediate. Note that $\tau_p(t)$ is even an isometry since the Lebesgue integral over $\mathbb{R}$ is invariant under translations. In order to prove (iii) in case $1 \le p < \infty$, we are going to use a standard density argument. To this end, recall that the space of compactly supported smooth functions $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ (cf. Werner [22], Lemma V. 1. 10). Hence, for arbitrary $\varepsilon > 0$ and $f \in L^p(\mathbb{R})$ there exists some $\varphi \in C_c^\infty(\mathbb{R})$ such that $\|f-\varphi\|_{L^p} < \varepsilon/3$. The invariance of the Lebesgue integral under translations gives $\|\tau_p(s)f - \tau_p(s)\varphi\|_{L^p} = \|f-\varphi\|_{L^p}$ for all $s \in \mathbb{R}$. So, we are done since uniform continuity of φ yields the following estimate for small $|t_j - t|$:

$$\begin{aligned}\|\tau_p(t_j)f - \tau_p(t)f\|_{L^p} &\le \|\tau_p(t_j)f - \tau_p(t_j)\varphi\|_{L^p} + \|\tau_p(t_j)\varphi - \tau_p(t)\varphi\|_{L^p}\\ &\quad + \|\tau_p(t)\varphi - \tau_p(t)f\|_{L^p} < 3\,\frac{\varepsilon}{3} \;=\; \varepsilon. \qquad (1.1)\end{aligned}$$

Some more attention will be needed in the case of a Lie group replacing $\mathbb{R}$ to derive convergence of the middle term from uniform continuity of φ.

The case $p = \infty$ is indeed an exception: Considering the characteristic functions $\chi_{[0,h]}$ and $\chi_{[0,h']}$ for $h, h' > 0$, we see that $\left\|\chi_{[0,h]} - \chi_{[0,h']}\right\|_{L^\infty(\mathbb{R})} = 1$ whenever $h \ne h'$. From this it follows that τ_∞ is not strongly continuous on $\mathbb{R}$.

Example 1.3. Another very instructive example is the exponential of an operator $A \in L(B)$ on a Banach space B, which is defined by

$$V(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n.$$

Since this series is absolutely convergent in $L(B)$ we obtain $V(t) \in L(B)$ for all $t \in \mathbb{R}$.

The homomorphism property is shown as in the scalar case, while (ii) is immediate. Finally, we even have norm-convergence in (iii), since

$$\|V(t) - I\| = \left\| \sum_{n=1}^{\infty} \frac{t^n A^n}{n!} \right\| \leq \sum_{n=1}^{\infty} \frac{\|A\|^n t^n}{n!} = e^{t\|A\|} - 1 \to 0 \ \ (t \to 0).$$

Remark 1.4. A generalization of this example to the case of unbounded operators - which of course requires some more theoretical background - will be given in App. C.

The following proposition provides some useful estimates for establishing a series of technical results on strongly continuous one-parameter groups.

Proposition 1.5. *Let V be a strongly continuous one-parameter group. Then there exist some $M, K \in \mathbb{R}^+$, e.g., $K = \log M$, such that*

(i) $\|V(t)\| \leq M$ *for all* $t \in [-1, 1]$,

(ii) $\|V(t)\| \leq Me^{K|t|}$ *for all* $t \in \mathbb{R}$.

Proof. We will prove the local uniform bound by means of the Banach-Steinhaus Theorem, which states that if a family $\mathcal{F} \subseteq L(E)$ of bounded operators on a Banach space E satisfies $\sup_{T\in\mathcal{F}} \|Tu\| < \infty$ for all $u \in E$, then even $\sup_{T\in\mathcal{F}} \|T\| < \infty$ holds true.

The subfamily $\mathcal{F} := \{V(t) \mid t \in [-1, 1]\} \subseteq \{V(t) \mid t \in \mathbb{R}\}$ has indeed the above property: note that for every fixed $u \in B$ the map $t \mapsto \|V(t)u\|$ is continuous on $[-1, 1]$ due to strong continuity of V. Compactness of the interval yields the existence of some bound $M_u \in \mathbb{R}^+$ with $\|V(t)u\| \leq M_u$ for all $t \in [-1, 1]$. Hence by the Banach-Steinhaus Theorem, there exists a uniform bound for $\mathcal{F}$.

The rest of the proof concerns the second estimate given above. For $t \in (n, n+1]$ and $n \in \mathbb{N}$ we obtain by (i)

$$\|V(t)\| \leq \|V(t-n)\| \cdot \|V(n)\| \leq M^{n+1} \leq M^{|t|+1} \leq Me^{K|t|},$$

which proves the statement for all $t \in (-1, n+1]$. Since $n \in \mathbb{N}$ was arbitrary the estimate holds for all $t \in [-1, \infty)$. An analogous calculation yields the above also for $t \in (-\infty, -1]$, hence for all $t \in \mathbb{R}$. □

Definition 1.6. *Let V be a strongly continuous one-parameter group. Motivated by Proposition 1.5, we define the growth bound of V to be the constant*

$$K_0 := \inf\{K \in \mathbb{R}^+ \mid \exists M = M(K) \in \mathbb{R}^+ \ s.t.\ \|V(t)\| \leq Me^{K|t|} \ \forall t \in \mathbb{R}\}.$$

1.1.2. The Infinitesimal Generator

To every strongly continuous one-parameter group V on a Banach space B we can associate a "generating element", the so-called infinitesimal generator. It turns out to be a linear usually unbounded operator with closed graph defined on a dense subset of B.

Definition 1.7. *Let V be a strongly continuous one-parameter group on a Banach space B. We define its infinitesimal generator by*

$$Au := \lim_{h\to 0} \frac{1}{h}\,(V(h)u - u) \quad \forall u \in \mathcal{D}(A),$$

on its natural domain

$$\mathcal{D}(A) := \{u \in B \mid \exists \lim_{h\to 0} \frac{1}{h}\,(V(h)u - u)\}.$$

Let us illustrate this definition by discussing some examples.

Example 1.8. In analogy to the one-parameter group of translations on $L^p(\mathbb{R})$ from Example 1.2, we now define translations on the space of all continuous functions that vanish at infinity, denoted by $C_0(\mathbb{R})$. Note that this space is a Banach space with respect to uniform convergence. Denoting the translations by $\tau(t)$, $t \in \mathbb{R}$, we omit the index p since we do not refer to any $L^p(\mathbb{R})$-space in this case. Now, an informal calculation will give us an idea of how the infinitesimal generator acts on the space $C_0(\mathbb{R})$:

$$\lim_{h\to 0} \frac{1}{h}\,(\tau(t)f(t) - f(t)) = \lim_{h\to 0} \frac{f(t-h) - f(t)}{h} = -\frac{df}{dt}(t) \tag{1.2}$$

Thus, we are inclined to identify the infinitesimal generator A as the differential operator $-\frac{d}{dt} : f \mapsto -f'$. In order to check this, we localize the generator's domain.

Proposition 1.9. *Let τ be the family of translations acting on the Banach space $(C_0(\mathbb{R}), \|\,.\,\|_\infty)$. Then its infinitesimal generator is given by*

$$\begin{aligned} A &= -\frac{d}{dt}, \\ \mathcal{D}(A) &= \{f \in C_0(\mathbb{R}) \mid \exists f' \in C_0(\mathbb{R})\}. \end{aligned}$$

Proof. Let $f \in C_0(\mathbb{R})$ be differentiable with $f' \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$ arbitrary. Then due to continuity of f'

$$\begin{aligned} \left|\frac{f(t-h) - f(t)}{h} + f'(t)\right| &= \left|\frac{1}{h}\int_t^{t-h} f'(s)\,ds + f'(t)\right| \\ &\le \frac{1}{h}\int_t^{t-h} |f'(s) - f'(t)|\,ds \quad \to \quad 0, \end{aligned}$$

as $h \to 0$. Hence, $f \in \mathcal{D}(A)$, and $Af = -\frac{d}{dt}f$.

Conversely, let $f \in \mathcal{D}(A)$. Then by (1.2) f' exists, and by definition it is in $C_0(\mathbb{R})$. □

Example 1.10. Returning to the one-parameter group $V := t \mapsto e^{tA}$ with $A \in L(B)$ (cf. Example 1.3), it seems it seems quite sensible to suppose that its generator is A. In fact, a short calculation justifies our anticipation

$$\left\| \frac{1}{h}(V(h) - V(0)) - A \right\| = \left\| \frac{e^{hA} - I}{h} - A \right\| \leq \sum_{n=2}^{\infty} \frac{h^{n-1} \|A^n\|}{n!} \leq h \|A\|^2 e^{h\|A\|} \to 0,$$

as $h \to 0$. As in Example 1.3 we even have convergence in the operator norm, and completeness of $L(B)$ yields $A \in L(B)$. Hence, the infinitesimal generator is A and $\mathcal{D}(A) = B$.

Of course, boundedness of the infinitesimal generator lies at the heart of the above example: Indeed it implies uniform continuity rather than mere strong continuity.

Definition 1.11. *Let B_1 and B_2 be Banach spaces and $A : B_1 \to B_2$ be linear with domain $\mathcal{D}(A)$. We say A is closed if and only if its graph $\mathcal{G}(A) := \{(u, Au) \mid u \in \mathcal{D}(A)\}$ is closed in $B_1 \times B_2$ with respect to the norm $(u, v) \mapsto \|u\|_{B_1} + \|v\|_{B_2}$.*

Equivalently, A is closed if and only if the existence of $u_n \in \mathcal{D}(A)$, $n \in \mathbb{N}$, with $u_n \to u \in B_1$ and $Au_n \to v \in B_2$, implies $u \in \mathcal{D}(A)$ and $v = Au$.

A very nice introduction of this notion is given in Werner, [22], IV.4, where it is compared in detail to the familiar concept of continuity.

The following theorem lists the main properties of the infinitesimal generator. It involves closed operators as well as their spectra (cf. Definition 1.12), and a few results about Banach space-valued integration. As a matter of fact, there are various ways to define such an integral, and we choose the most convenient one in view of its applications, namely the Bochner integral. It is, in fact, a straightforward generalization of Lebesgue integration modifying the functions' range space, yet it does not share all of its properties. Thus, we should be careful not to just view it as the Lebesgue integral with absolute value signs replaced by norm signs. We have collected all the relevant facts to be used below in App. A and we will frequently refer to it. Nevertheless, we mention here three important arguments: First the Fundamental Theorem of Calculus says in analogy to the real-valued case that $\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} u(s)\, ds = u(t)$ if $u \in C(\mathbb{R}, B)$ (cf. Theorem A.15). Another important property is the fact that the integral interchanges with bounded operators, i.e., $T\big(\int_a^b u(s)\, ds\big) = \int_a^b Tu(s)\, ds$ for all $T \in L(B_1, B_2)$ and all Bochner integrable B_1-valued functions u (cf. Proposition A.10). Last but not least, we will use an adopted version of Lebesgue's Dominated Convergence Theorem (cf. Proposition A.14).

Our final preparation concerns the spectrum of an operator.

Definition 1.12. *Let $A : B \to B$ be a densely defined operator on a Banach space B. The set*

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \, \textit{is bijective}, \, (\lambda - A)^{-1} \in L(B)\}$$

is called the resolvent set of A. On the latter we define the resolvent of A to be the *operator-valued function*

$$\begin{aligned} R_A : \rho(A) &\to L(B) \\ z &\mapsto (z-A)^{-1}, \end{aligned}$$

and we call $\sigma(A) := \mathbb{C} \setminus \rho(A)$ *the spectrum of* A.

Remark 1.13. (i) The set of all eigenvalues of A is evidently a subset of $\sigma(A)$.

(ii) Every densely defined operator A with $\rho(A) \neq \emptyset$ turns out to be closed (cf. Werner [22], Exer. VII 5. 32). Hence, if A is not closed, then $\sigma(A) = \mathbb{C}$.

(iii) If A is a densely defined, closed operator and $\lambda - A$ is bijective for some $\lambda \in \mathbb{C}$, then by the closed graph theorem $(\lambda - A)^{-1}$ is automatically continuous.

Theorem 1.14. *If* V *is a strongly continuous one-parameter group with infinitesimal generator* A*, then the following hold*

(i) A *is closed.*

(ii) $\mathcal{D}(A)$ *is a dense subset of* B.

(iii) $V(t)\mathcal{D}(A) \subseteq \mathcal{D}(A)$ *for all* $t \in \mathbb{R}$.

(iv) $V(t)Au = AV(t)u = \frac{d}{dt}V(t)u$ *for all* $u \in \mathcal{D}(A)$.

(v) $\int_0^t V(s)u\,ds \in \mathcal{D}(A)$ *for all* $u \in B$ *and all* $t > 0$*, and* $A\left(\int_0^t V(s)u\,ds\right) = V(t)u - u$.

(vi) $\int_0^t V(s)Au\,ds = V(t)u - u$ *for all* $u \in \mathcal{D}(A)$.

(vii) $\{\lambda \mid \mathrm{Re}\,(\lambda) > K_0\} \subseteq \rho(A)$, K_0 *being the growth bound of* V *(cf. Definition 1.6).*

(viii) $(\lambda - A)^{-1}u = \int_0^\infty e^{-\lambda t}V(t)u\,dt$ *for all* $u \in B$, $\lambda \in \{\lambda \mid \mathrm{Re}\,(\lambda) > K_0\}$.

Proof. $(iii), (iv)$ Let $u \in \mathcal{D}(A)$, $t \in \mathbb{R}$. Then we have

$$\frac{1}{h}(V(h)V(t)u - V(t)u) = V(t)\frac{1}{h}(V(h)u - u).$$

We obtain $V(t)u \in \mathcal{D}(A)$ by continuity of $V(t)$ and the fact that $1/h(V(t)u - u) \to Au \in B \quad (h \to 0)$ by Definition 1.7. Moreover, we have

$$\frac{1}{h}(V(t+h)u - V(t)u) = V(t)\frac{1}{h}(V(h)u - u) \to V(t)Au,$$

as $h \to 0$, hence $\frac{d}{dt}V(t)u = V(t)Au$. Similarly, by (iii) and Definition 1.7

$$\frac{1}{h}(V(t+h)u - V(t)u) = \frac{1}{h}(V(h)V(t)u - V(t)u) \to AV(t)u,$$

so we obtain the second formula in (iv).

In order to prove (v), note to begin with that due to strong continuity the map $t \mapsto V(t)u \in C(\mathbb{R}, B)$ for any fixed $u \in B$. Furthermore, we use the first two of the above mentioned facts on Banach space-valued integration to obtain (assuming w.l.o.g. $h \leq t$)

$$\begin{aligned} \frac{1}{h}\left(V(h)\int_0^t V(s)u\,ds - \int_0^t V(s)u\,ds\right) &= \frac{1}{h}\left(\int_0^t \underbrace{V(h)V(s)}_{V(h+s)}u\,ds - \int_0^t V(s)u\,ds\right) \\ &= \frac{1}{h}\left(\int_h^{h+t} V(s)u\,ds - \int_0^t V(s)u\,ds\right) \\ &= \frac{1}{h}\left(\int_t^{t+h} V(s)u\,ds - \int_0^h V(s)u\,ds\right) \\ &\to \quad V(t)u - V(0)u \quad = \quad V(t)u - u, \end{aligned}$$

as $h \to 0$. Thus, $\int_0^t V(s)u\,ds \in \mathcal{D}(A)$, and

$$A\left(\int_0^t V(s)u\,ds\right) = V(t)u - u$$

holds true for all $t > 0$.

(vi) By (v) we obtain for $u \in \mathcal{D}(A)$

$$\begin{aligned} V(t)u - u &= A\int_0^t V(s)u\,ds \\ &= \lim_{h\to 0}\frac{1}{h}\left(V(h)\int_0^t V(s)u\,ds - \int_0^t V(s)u\,ds\right) \\ &= \lim_{h\to 0}\int_0^t V(s)\left(\frac{V(h)u-u}{h}\right)ds. \end{aligned}$$

The functions $s \mapsto V(s)\left(\frac{V(h)u-u}{h}\right)$ are continuous, thus Bochner-measurable, for fixed h on the interval $[0,t]$ (cf. Proposition A.5). The mapping $h \mapsto 1/h(V(h)u-u)$ is bounded in B if $h \in [0, h_0]$ for some $h_0 > 0$, since $\frac{V(h)u-u}{h} \to Au$ for $u \in \mathcal{D}(A)$ as $h \to 0$. Moreover, $\|V(s)\| \leq Me^{K_0 t}$ for $s \leq t$, hence, $s \mapsto V(s)\left(\frac{V(h)u-u}{h}\right)$ is uniformly bounded in $h \in [0, h_0]$, thus locally in $L^1(\mathbb{R}, B)$, with pointwise limit $s \mapsto V(s)Au$, as $h \to 0$. By Theorem A.14, we may interchange limit and integral to obtain

$$V(t)u - u = \int_0^t V(s)Au\,ds.$$

(ii) Note that $u_t := \frac{1}{t}\int_0^t V(s)u\,ds \in \mathcal{D}(A)$ by (v), and by Theorem A.15 we have $u_t \to u$, as $t \to 0$. Thus, $\overline{\mathcal{D}(A)} = B$.

In order to prove (i), consider a sequence $(u_n)_n$ in $\mathcal{D}(A)$ with $u_n \to u \in B$ and

$Au_n \to v \in B$. Then using (v) we have

$$\frac{V(h)u-u}{h} \;=\; \lim_{n\to\infty}\frac{V(h)u_n-u_n}{h} \;=\; \lim_{n\to\infty}\frac{1}{h}\int_0^h V(s)Au_n\,ds.$$

As in the proof of (iv), $s \mapsto V(s)Au_n$ is uniformly bounded for $s \in [0,h]$, thus locally in $L^1(\mathbb{R},B)$, with pointwise limit $s \mapsto V(s)v$. Hence, we may apply Theorem A.14 to obtain

$$\frac{V(h)u-u}{h} \;=\; \frac{1}{h}\int_0^h V(s)v\,ds \;\to\; v,$$

as $h \to 0$. Convergence in h is now due to Theorem A.15. Thus $u \in \mathcal{D}(A)$ and $Au = v$.

(vii), $(viii)$ Finally, consider the one-parameter group $e^{-\lambda t}V(t)$ with infinitesimal generator $A-\lambda$ and $\mathcal{D}(A-\lambda) = \mathcal{D}(A)$. Then by (v) and (vi) we obtain

$$e^{-\lambda t}V(t)u - u \;=\; \begin{cases} (A-\lambda)\displaystyle\int_0^t e^{-\lambda s}V(s)u\,ds & \forall u \in B\\ \displaystyle\int_0^t e^{-\lambda s}V(s)(A-\lambda)u\,ds & \forall u \in \mathcal{D}(A).\end{cases}$$

Let $\operatorname{Re}(\lambda) > K_0 \geq 0$ (the growth bound of V, cf. Definition 1.6), and let $t \to \infty$. Then from the above we conclude

$$u \;=\; \begin{cases} (\lambda-A)\displaystyle\int_0^\infty e^{-\lambda s}V(s)u\,ds & \forall u \in B\\ \displaystyle\int_0^\infty e^{-\lambda s}V(s)(\lambda-A)u\,ds & \forall u \in \mathcal{D}(A),\end{cases}$$

since by Theorem 1.5, $\left|e^{-\lambda t}V(t)\right| \leq e^{\operatorname{Re}(-\lambda)t}\,e^{K_0 t}\,M \to 0$, as $t \to \infty$. Hence, the operator $\lambda - A : \mathcal{D}(A) \to B$ is bijective, and $\lambda \in \rho(A)$. This proves (vii) and $(viii)$. $\square$

As it is our aim to discuss differential equations on Lie groups (especially the Heisenberg group), we will start on $\mathbb{R}$ with the solution of an abstract Cauchy problem for a Banach space-valued continuously differentiable function u and a linear operator A on B:

$$u' = Au, \quad u(0) = u_0. \tag{1.3}$$

Theorem 1.14 provides everything we need to solve (1.3) as we prove next.

Theorem 1.15. *Let V be a strongly continuous one-parameter group on a Banach space B with infinitesimal generator A. Furthermore, let $u_0 \in \mathcal{D}(A)$, the domain of A. Then the function $u : \mathbb{R} \to B$, $u(t) := V(t)u_0$ is C^1, $\mathcal{D}(A)$-valued and a solution of the Cauchy problem (1.3). Moreover, u is the unique solution and $u(t)$ depends continuously on u_0.*

Proof. By Theorem 1.14 (iii) $V(t)u_0 \in \mathcal{D}(A)$, hence $Au(t)$ is defined. Moreover, we already proved differentiability and explicitly showed that $u' = AV(t)u_0 = V(t)Au_0$.

Thus, $u(t)$ is a solution of (1.3). Furthermore, this proves continuity of u'. Let v be another solution of (1.3). Then, the product rule of differentiation for Banach space-valued functions (cf. Kriegl [14], Folgerung 6. 1. 13) yields for $t \in (-\infty, s]$ using Theorem 1.14 (iv)

$$\begin{aligned} \frac{d}{dt} V(s-t)v(t) &= (-1)AV(s-t)v(t) + V(s-t)v'(t) \\ &= -V(s-t)Av(t) + V(s-t)Av(t) \;=\; 0. \end{aligned}$$

To prove that $F : (-\infty, s] \to B$, $F(t) := V(s-t)v(t)$ is constant, let $w \in B^*$, then

$$\frac{d}{dt}\langle w, F(t)\rangle = \left\langle w, \frac{d}{dt}F(t)\right\rangle = 0,$$

(cf. Kriegl [14], Folgerung 6. 1. 10). Thus $\langle w, F(0)\rangle = \langle w, F(t)\rangle$ for all $t \in (-\infty, s]$, in particular for $t = s$. Hence, by the Hahn-Banach theorem $F(0) = F(s)$, i.e., $V(s)u_0 = v(s)$. Since s was arbitrary, uniqueness is proved.

The continuous dependence of $u(t)$ on u_0 is due to continuity of the operators $V(t)$. □

There exists a very interesting application of Theorem 1.15 proving that every strongly continuous one-parameter group is uniquely determined by its infinitesimal generator.

Proposition 1.16. *Let V and W be strongly continuous one-parameter groups on a Banach space B with the same infinitesimal generator A, then $V(t) = W(t)$ for all $t \in \mathbb{R}$.*

Proof. Let $u \in \mathcal{D}(A)$. Both $t \mapsto V(t)u$ and $t \mapsto W(t)u$ are solutions of the i.v.p.

$$u' = Au(t), \quad u(0) = u \in \mathcal{D}(A).$$

Since its solution is unique by Theorem 1.15, we obtain $V(t)|_{\mathcal{D}(A)} = W(t)|_{\mathcal{D}(A)}$ for all $t \in \mathbb{R}$. Now, continuity of each $V(t)$ and each $W(t)$ and denseness of $\mathcal{D}(A)$ in B (cf. Theorem 1.14 (ii)) give $V(t) = W(t)$ for all $t \in \mathbb{R}$. □

We conclude this subsection with another example. More precisely, we will determine the infinitesimal generator A_p of the strongly continuous one-parameter group of translations on $L^p(\mathbb{R}), 1 \leq p < \infty$, denoted by τ_p. Recall that we already found the latter in case we restricted the translations to the space $C_0(\mathbb{R})$. It turned out to be the ordinary differential operator $-\frac{d}{dx}$ (for more details see Example 1.8). In the following we will see that the $L^p(\mathbb{R})$-case requires some more effort, and a few basics about generalized functions. To begin with, note that for $f \in L^p(\mathbb{R})$ we cannot simply consider the difference quotient $h^{-1}(f(t-h) - f(t))$ for fixed $t \in \mathbb{R}$. Hence, applying the conditions of Definition 1.7 to the present case, we find $u \in L^p(\mathbb{R})$ is an element of $\mathcal{D}(A_p)$ if and only if $\lim_{h\to 0} h^{-1}(u(.-h) - f(.))$ exists in the L^p-norm. As we will see the operator A_p is nothing but the distributional derivation $-\frac{d}{dx}$ with derivatives in $L^p(\mathbb{R})$, usually referred to as the Sobolev space $W^{1,p}(\mathbb{R})$. Note that there is an equivalent description of $W^{1,p}(\mathbb{R})$ involving weak derivatives of $L^p(\mathbb{R})$-functions, and that equality of these spaces can easily be shown for the case $p = 2$ using Fourier techniques (cf. Weidmann [21], Theorem 10.8, Theorem 10.9).

Proposition 1.17. *Denote by τ_p the one-parameter group of translations on $L^p(\mathbb{R})$, $1 \leq p < \infty$, and by A_p its infinitesimal generator. Then*

$$A_p = -\frac{d}{dx},$$
$$\mathcal{D}(A_p) = W^{1,p}(\mathbb{R}) := \{u \in L^p(\mathbb{R}) \mid \frac{du}{dx} \in L^p(\mathbb{R})\}$$

where $\frac{d}{dx}$ denotes the distributional derivative.

Proof. ($\subseteq$) The condition $u \in \mathcal{D}(A_p)$ is equivalent to the existence of some $v \in L^p(\mathbb{R})$ s.t.

$$\lim_{h\to 0} \left\| \frac{u(.-h)-u}{h} - v \right\|_{L^p} = 0.$$

Let $\varphi \in C_c^\infty(\mathbb{R})$ and denote by $\langle .\,,.\rangle$ the distributional action. Then

$$\begin{aligned}\left|\left\langle -\frac{d}{dx}u, \varphi\right\rangle - \langle v,\varphi\rangle\right| &= \left|\left\langle u, \frac{d}{dx}\varphi\right\rangle - \langle v,\varphi\rangle\right| = \lim_{h\to 0}\left|\left\langle u, \frac{\varphi(.+h)-\varphi}{h}\right\rangle - \langle v,\varphi\rangle\right| \\ &= \lim_{h\to 0}\left|\left\langle \frac{u(.-h)-u}{h} - v, \varphi\right\rangle\right| \leq \lim_{h\to 0}\left\|\frac{u(.-h)-u}{h} - v\right\|_{L^p} \|\varphi\|_{L^q} \\ &= 0.\end{aligned} \tag{1.4}$$

The second equality in (1.4) is due to fact that $h^{-1}(\varphi(.+h)-\varphi) \to \varphi'$ in $C_c^\infty(\mathbb{R})$, which follows from an elementary calculation using the mean value theorem. Since φ was arbitrary, we have $\frac{du}{dx} = v \in L^p(\mathbb{R})$.

($\supseteq$) The converse inclusion is due to the following lemma, where we use the fact that $C_c^\infty(\mathbb{R})$ is weak*-dense in $L^q(\mathbb{R}), (1 < q \leq \infty)$, the dual space of $L^p(\mathbb{R}), (1 \leq p < \infty)$, and invariance of $C_c^\infty(\mathbb{R})$ under the family of translations τ_q. In fact, weak*-denseness follows from (norm) denseness of $C_c^\infty(\mathbb{R})$ in $L^q(\mathbb{R}), (1 < q < \infty)$, while in case $q = \infty$ one can argue, e.g., as follows: For given $f \in L^\infty(\mathbb{R})$ take a standard mollifier ρ_n and consider $f_n := (f * \rho_n)\chi_{B_n(0)}$, where $\chi_{B_n(0)}$ is a smooth cut-off around the ball $B_n(0)$. Indeed $f_n \in C_c^\infty(\mathbb{R})$, $\|f_n\|_{L^\infty} \leq C < \infty$, and $\int_{\mathbb{R}} g(t)(f_n - f)(t)\,dt \to 0$ for all $g \in C_c^\infty(\mathbb{R})$, which establishes the result due to Yosida [23], Theorem V. 10. $\square$

Lemma 1.18. *Let V be a one-parameter group on some Banach space B and A its infinitesimal generator. Furthermore, let $\mathcal{L}$ be a weak*-dense linear subspace of B^*, which is invariant under the adjoint maps $V(t)^*$ for all $t \in \mathbb{R}$. Now, given $u, v \in B$ such that*

$$\lim_{h\to 0}\frac{1}{h}\langle w, V(h)u - u\rangle = \langle w, v\rangle \tag{1.5}$$

for all $w \in \mathcal{L}$*, then* $u \in \mathcal{D}(A)$*, and* $Au = v$*.*

Proof. Equation (1.5) implies differentiability of $t \mapsto \langle w, V(t)u \rangle : \mathbb{R} \to \mathbb{C}$ at $t = 0$. Now fix any $t \neq 0$, then we have

$$\frac{1}{h}(\langle w, V(t+h)u \rangle - \langle w, V(t)u \rangle) = \frac{1}{h} \langle V(t)^* w, V(h)u - u \rangle \to \langle V(t)^* w, v \rangle = \langle w, V(t)v \rangle \,,$$

as $h \to 0$, thus differentiability for $t \in \mathbb{R}$. An application of the fundamental theorem of calculus to the real and complex parts of $t \mapsto \langle w, V(t)u \rangle$ together with the fact that the Bochner integral interchanges with linear bounded maps (cf. Proposition A.10) then gives

$$\begin{aligned} \langle w, V(t)u \rangle &= \int_0^t \langle w, V(s)v \rangle \; ds + \langle w, V(0)u \rangle \quad \Leftrightarrow \\ \langle w, V(t)u - u \rangle &= \left\langle w, \int_0^t V(s)v \, ds \right\rangle . \end{aligned}$$

Weak*-denseness of $\mathcal{L}$ yields $V(t)u - u = \int_0^t V(s)v \, ds$ in B. Hence, by Theorem A.15 we conclude $\lim_{h\to 0} \frac{1}{h} (V(h)u - u) = \lim_{h\to 0} \frac{1}{h} \int_0^h V(s)v \, ds = v$. □

1.1.3. Unitary One-Parameter Groups

Since we aim at an indepth discussion of representations of the Heisenberg group in Chapter 2 we now collect some results on unitary one-parameter groups on Hilbert spaces and their generators. To begin with, we recall the notions of self- and skew-adjointness. By H we will always denote a complex Hilbert space.

Definition 1.19. (*i*) *Let* $U \in L(H)$ *for some Hilbert space* H*. Then* U *is said to be unitary if* $U^* = U^{-1}$*.*

(*ii*) *A strongly continuous one-parameter group* U *is called a unitary group if* $U(t)$ *is unitary for all* $t \in \mathbb{R}$*.*

Definition 1.20. *For a densely defined linear operator* A *on a Hilbert space* H*, we define the domain of its adjoint* A^* *by*

$$\mathcal{D}(A^*) \;=\; \{u \in H \mid \exists C > 0 \;\; s.t. \;\; v \mapsto |\langle u, Av \rangle| \leq C \, \forall v \in \mathcal{D}(A)\},$$

and the operator's action by

$$\langle A^* u, v \rangle \;:=\; \langle u, Av \rangle \, .$$

Remark 1.21. The definition is sensible since by the Hahn-Banach theorem the bounded conjugate linear functional $v \mapsto |\langle u, Av \rangle|$ defined on the dense subspace $\mathcal{D}(A)$ of H extends to a bounded conjugate linear functional f on H. Then by the theorem of Riez-Fréchet there exists a unique $w \in H$ such that $f(v) = \langle w, v \rangle$ for all $v \in H$. Hence, for $u \in \mathcal{D}(A^*)$ we obtain $\langle u, Av\rangle = \langle w, v\rangle$ for all $v \in \mathcal{D}(A)$, and we may unambiguously set $A^*u := w$.

Next we may define the major classes of operators on Hilbert spaces.

Definition 1.22. *Let A be a densely defined linear operator on some Hilbert space H. We say A is symmetric if*

$$\mathcal{D}(A) \subseteq \mathcal{D}(A^*) \quad \text{and} \quad A^*u = Au \quad \forall u \in \mathcal{D}(A),$$

respectively skew-symmetric if

$$\mathcal{D}(A) \subseteq \mathcal{D}(A^*) \quad \text{and} \quad A^*u = -Au \quad \forall u \in \mathcal{D}(A).$$

Furthermore, an operator A is called self-adjoint if it is symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$. A is called skew-adjoint if it is skew-symmetric and $\mathcal{D}(A) = \mathcal{D}(A^*)$.

Remark 1.23. Whenever there exists an extension T of some linear operator $A : H \supseteq \mathcal{D}(A) \to H$ we will use the notation $A \subseteq T$. Thus, will write, e.g., $A \subseteq A^*$ if A symmetric.

Remark 1.24. In case that A is a bounded operator we have $\mathcal{D}(A) = H = \mathcal{D}(A^*)$, and A^* agrees with usual notion for self-adjointness for bounded operators. Furthermore, it is easily seen that a linear operator A is self-adjoint if and only if iA is skew-adjoint.

The next theorem will be a characterization of self-adjoint operators, but first we need some technical preparations.

Lemma 1.25. *Let A be a densely defined operator on a Hilbert space H. Then following hold:*

(i) $\ker(A^* \mp i) = \operatorname{ran}(A \pm i)^\perp$

(ii) If A is symmetric, then $A \pm i$ are injective.

(iii) If A is symmetric and closed, then $A \pm i$ are also closed with closed range.

Proof. (i) $(\supseteq)$ Let $v \in \operatorname{ran}(A+i)^\perp$ and $u \in \mathcal{D}(A)$. It follows that $\langle (A+i)u, v\rangle = 0$, hence $v \in \mathcal{D}(A^*)$ by Def.1.20, and $\langle u, (A^*-i)v\rangle = 0$ for all $u \in \mathcal{D}(A)$. Hence, $v \in \ker(A^*-i)$ $\mathcal{D}(A)$ is dense.

$(\subseteq)$ The converse inclusion follows by reading the same argument the other way round: For $v \in \ker(A^*-i)$ and $u \in \mathcal{D}(A)$ we have $0 = \langle u, (A^*-i)v\rangle = \langle (A+i)u, v\rangle$, hence $v \in \operatorname{ran}(A-i)^\perp$.

The same arguments hold for A^*+i and $A-i$.

(ii) For $u \in \mathcal{D}(A)$ we have

$$\begin{aligned}\|Ax\|^2 \pm 2\operatorname{Re}\left(\langle Ax, ix\rangle\right) + \|ix\|^2 &= \langle (A \pm i)x, (A \pm i)x\rangle = \langle (A \mp i)x, (A \mp i)x\rangle \\ &= \|Ax\|^2 \mp 2\operatorname{Re}\left(\langle Ax, ix\rangle\right) + \|-ix\|^2.\end{aligned}$$

Hence by symmetry of A

$$\|(A \pm i)x\|^2 = \|(A \mp i)x\|^2 = \|Ax\|^2 + \|x\|^2 \geq \|x\|^2, \tag{1.6}$$

thus $(A \pm i)$ is injective.

(iii) By *(ii)*, $(A+i)^{-1} : \mathrm{ran}(A+i) \underline{\to \mathcal{D}(A)}$ exists and is continuous. Let now $u_n \in \mathcal{D}(A)$ such that $(A+i)u_n \to v \in \overline{\mathrm{ran}(A+i)}$. Hence, it is a Cauchy sequence in $\mathrm{ran}(A+i)$, and due to (1.6) u_n is a Cauchy sequence in $\mathcal{D}(A)$, thus converges to some $u \in H$. Furthermore, we have $Au_n \to v - iu$, but since A is closed, it follows $Au = v - iu$. Hence $Au + iu = v \in \mathrm{ran}(A+i)$, thus $A+i$ is closed and its range, too. Analogously, one proves the case $A - i$. □

The following corollary is a special case of Lemma 1.25 (i).

Corollary 1.26. *For any densely defined operator A on H we have*

$$\ker(A^* \mp i) = \{0\} \quad \Leftrightarrow \quad \mathrm{ran}(A \pm i) \textit{ dense in } H\,.$$

Theorem 1.27 (J. von Neumann)**.** *Let A be a symmetric operator on a Hilbert space H. Then the following are equivalent:*

(i) *A is self-adjoint*

(ii) *A is closed and* $\ker(A^* \pm i) = \{0\}$

(iii) $\mathrm{ran}(A \pm i) = H$

(iv) $\pm i \in \rho(A)$

In this case, the operators

$$U_1 := (A+i)(A-i)^{-1}, \quad U_2 := (A-i)(A+i)^{-1}$$

are unitary.

Proof. $(i) \Rightarrow (ii)$ First we show that A is closed. To this end, let $v_n \in \mathcal{D}(A)$ s.t. $v_n \to v \in H$ and $Av_n \to w \in H$. Then we have

$$\langle Au, v\rangle = \lim_{n\to\infty} \langle Au, v_n\rangle = \lim_{n\to\infty} \langle u, Av_n\rangle = \langle u, w\rangle$$

for all $u \in \mathcal{D}(A)$. Hence $v \in \mathcal{D}(A^*) = \mathcal{D}(A)$ and $Av = w$, thus A is closed.

Now suppose $u \in \ker(A^* \pm i)$. By (1.6) we have $\|Au\|^2 + \|u\|^2 = \|(A \pm i)u\|^2 = 0$, hence $u = 0$.

$(ii) \Rightarrow (iii)$ follows by Lemma 1.25.

$(iii) \Rightarrow (i)$ Recalling that $A \subseteq A^*$ we only have to show $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$. Hence, let $v \in \mathcal{D}(A^*)$. Since $\mathrm{ran}(A-i) = H$, there exists some $u \in \mathcal{D}(A)$ such that $(A^*-i)v = (A-i)u$, but $A \subseteq A^*$ gives $(A^* - i)v = (A^* - i)u$. Hence $v - u \in \ker(A^* \pm i) = \mathrm{ran}(A \mp i)^\perp = H^\perp = \{0\}$ by Lemma 1.25 (i), thus $v = u \in \mathcal{D}(A)$.

$(i), (ii), (iii) \Leftrightarrow (iv)$ Since $A \pm i$ is injective and $\operatorname{ran}(A \pm i) = H$ we have $\pm i \in \rho(A)$. Conversely (iv) clearly implies (iii).

Finally, we show unitarity of U_1, skipping the analogous proof for U_2. Thus, let $u, v \in H$ be arbitrary. Then we have

$$\begin{aligned} \langle U_1 u, U_1 v \rangle &= \big\langle (A-i)(A+i)(A-i)^{-1}u, (A-i)^{-1}v \big\rangle \\ &= \big\langle (A+i)(A-i)(A-i)^{-1}u, (A-i)^{-1}v \big\rangle \\ &= \big\langle (A+i)u, (A-i)^{-1}v \big\rangle = \big\langle u, (A-i)(A-i)^{-1}v \big\rangle = \langle u, v \rangle \,, \end{aligned}$$

hence U_1 is unitary which concludes the proof. □

Next we are going to employ the above theorem to characterize the relation between skew-adjoint operators and unitary one-parameter groups.

Theorem 1.28. *If U is a strongly continuous unitary one-parameter group on a Hilbert space H, then its infinitesimal generator A is skew-adjoint.*

Proof. $U(t)$ being unitary is equivalent to $U(t)^* = U(t)^{-1} = U(-t)$. So, let $u, v \in \mathcal{D}(A)$, then

$$\frac{1}{h} \langle U(t)u - u, v \rangle \;=\; \frac{1}{h} \langle u, (U(h) - I)^* v \rangle \;=\; \frac{1}{h} \langle u, U(-h)v - v \rangle \;\to\; \langle u, -Av \rangle \,,$$

as $h \to 0$. Hence, A is skew-symmetric, and it is also closed being the infinitesimal generator of a one-parameter group by Theorem 1.14 (i). Since every unitary operator on H is an isometry, i.e., $\|U(t)\| = 1$ for all $t \in \mathbb{R}$, we can use estimate (ii) from Proposition 1.5, with $M = 1$ and $K = 0$. Then growth bound K_0 vanishes, and Theorem 1.14 gives $\pm 1 \in \rho(A)$. Hence, iA is symmetric (cf. Remark 1.24) with $\pm i \in \rho(iA)$, hence self-adjoint by Theorem 1.27. If follows that A is skew-adjoint, and we are done. □

Remark 1.29. The converse of Theorem 1.28 is also true but requires some more preparation. In the light of Example 1.3 it seems reasonable that the group U can be reconstructed from its infinitesimal generator via

$$U(t) = e^{tA}.$$

To make this statement precise, however, spectral theory of unbounded operators has to be used to define the exponential e^{tA}. In fact, we are going to prove this statement known as Stone's Theorem in App. C. We will furthermore see that for any unitary skew-adjoint operator A, $t \mapsto e^{tA}$ defines a strongly continuous unitary one-parameter group with infinitesimal generator A. Here we will just deal with the simpler analogue for bounded operators.

Proposition 1.30. *Let A be a linear bounded and skew-adjoint operator on a Hilbert* *space H. Then the family of operators $U(t)$ defined by*

$$U(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n \tag{1.7}$$

is a strongly continuous unitary one-parameter group with infinitesimal generator A.

Proof. As we know that U is a strongly continuous one-parameter group (cf. Ex.1.3) with infinitesimal generator A (cf. Example 1.10), we now show unitarity. For $u, v \in H$ we have

$$\begin{aligned} \langle u, U(t)v \rangle &= \left\langle u, \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n v \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \langle u, A^n v \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n t^n \langle A^n u, v \rangle \\ &= \left\langle e^{-tA} u, v \right\rangle = \langle U(-t)u, v \rangle = \left\langle U(t)^{-1} u, v \right\rangle , \end{aligned}$$

which holds true for all $t \in \mathbb{R}$, hence the proof is complete. □

1.1.4. Smooth and Analytic Vectors

So far we have derived several important properties of strongly continuous one-parameter groups. Inspired by continuous differentiability of the B-valued functions $F_u : \mathbb{R} \to B$, $F_u(t) := V(t)u$ for $u \in \mathcal{D}(A)$ (cf. Theorem 1.15), we now focus on iterated differentiation and smoothness. Note that a priori it is not clear for which $u \in B$, F_u might be C^k since it is not even C^1 for $u \notin \mathcal{D}(A)$. On the other hand, Example 1.10 illustrates that boundedness of the generator A implies smoothness for all F_u and also smoothness in the operator norm of $t \mapsto F_u(t)$. Therefore, it seems reasonable to suppose that k-fold differentiability is intimately related to the infinitesimal generator.

Definition 1.31. *Let V be a strongly continuous one-parameter group on the Banach space B and $u \in B$. We say that u is a smooth vector for V if F_u is a smooth function.*

The following result gives a characterization of k-fold differentiability and smooth vectors.

Proposition 1.32. *Let V be a strongly continuous one-parameter group. Then $F_u : t \mapsto F_u(t) : \mathbb{R} \to B$ is C^k (C^∞) if and only if $u \in \mathcal{D}(A^k)$ (for all $k \in \mathbb{N}$), and we have*

$$\frac{d^k F_u}{dt^k} = V(t) A^k u. \tag{1.8}$$

Proof. If F_u is k times continuously differentiable, then differentiability at $t = 0$ gives existence of $\lim_{h \to 0}(F_u(h) - F(0))$ in B, and the limit is equal to $\lim_{h \to 0}(V(h)u - u) = Au$. Hence, $u \in \mathcal{D}(A)$, and by Theorem 1.14 *(iv)* we have $\frac{d}{dt} V(t)u = V(t)Au = AV(t)u$. Moreover, we observe that $\frac{d^2}{dt^2}|_0 V(t)u = \frac{d}{dt}|_0 V(t)Au$. Thus, $\lim_{h \to 0}(V(h)Au - Au)$ exists in B, which implies $Au \in \mathcal{D}(A)$, and therefore $u \in \mathcal{D}(A^2)$. As in the proof of Theorem

1.14 (iv) it follows that $\frac{d^2}{dt^2}V(t)u = V(t)A^2u$. Proceeding inductively, we conclude that $u \in \mathcal{D}(A^k)$ and $\frac{d^k}{dt^k}V(t)u = V(t)A^k u$.

Conversely, if $u \in \mathcal{D}(A^k)$, then of course $u \in \mathcal{D}(A^j)$ for all $j \leq k$. In particular, we have $u \in \mathcal{D}(A)$, hence by Theorem 1.14 (iv) there exists $\frac{d}{dt}V(t)u$ and we have $\frac{d}{dt}V(t)u = V(t)Au$. Since u is also in $\mathcal{D}(A^2)$, Au is obviously in $\mathcal{D}(A)$. Now, the latter equation gives existence of $\frac{d}{dt}V(t)Au$ and implies that it is equal to $\frac{d^2}{dt^2}V(t)u$. Again by induction, we conclude k-fold differentiability. Continuity of the derivatives follows by strong continuity of V and (1.8). □

In analogy to smoothness of vectors, we define the notion of analyticity of a vector $u \in B$ once we have defined analyticity for Banach space-valued functions.

Definition 1.33. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, B be a Banach space over $\mathbb{K}$, and let Ω be an open subset of $\mathbb{K}$. We say that a function $f : \Omega \to B$ is analytic if for every $a \in \Omega$ there exists an $r > 0$ with $B(a, r) \subseteq \Omega$, and vectors $c_j \in B$, $j \in \mathbb{N}_0$, such that*

$$f(z) = \sum_{j=0}^{\infty} c_j \, (z-a)^j, \quad \sum_{j=0}^{\infty} \|c_j\| \; |z-a|^j < \infty \tag{1.9}$$

for all $z \in B(a, r)$. If $\mathbb{K} = \mathbb{R}$ (resp. if $\mathbb{K} = \mathbb{C}$) we say f is a real (resp. complex) analytic function.

Definition 1.34. *Let V be a strongly continuous one-parameter group on the Banach space B. A vector $u \in B$ is called analytic for V if F_u is a real analytic function of $t \in \mathbb{R}$.*

Proposition 1.35. *Let $u \in B$ be an analytic vector for the strongly continuous one-parameter group V. Then F_u extends to a complex analytic function on a strip $S := \{z \in \mathbb{C} \mid |\mathrm{Im}\,(z)| < r\}$ for some $r > 0$.*

Proof. We will prove the result in two steps: First we will show analyticity in an open neighborhood of 0. In the second step we will use the group homomorphism property of V to translate the power series.

In analogy to the case of scalar-valued real analytic functions, the corresponding power series of F_u, $p : (-r, r) \to B$, $t \mapsto \sum_{j \geq 0} c_j \, t^j$, extends to a complex power series $p' : \mathbb{C} \supseteq B(0, r) \to B$, $z \mapsto \sum_{j \geq 0} c_j \, z^j$ since it converges absolutely as long as $|z| < r$. In other words, there exist an $r > 0$ and coefficients $c_j \in B$, $j \in \mathbb{N}_0$, such that

$$F_u(z) = \sum_{j=0}^{\infty} c_j \, z^j, \quad \sum_{j=0}^{\infty} \|c_j\| \; |z|^j < \infty \quad \forall z \in B(0, r).$$

In order to obtain a power series of the form (1.9), note that we can translate p' to any point within the disc $B(0, r)$ maintaining its values and absolute convergence in a sufficiently small open neighborhood of that point. That is, for every $a \in B(0, r)$ there exist coefficients $d_j \in B, j \in \mathbb{N}_0$, such that

$$F_u(z) = \sum_{j=0}^{\infty} c_j\, z^j = \sum_{j=0}^{\infty} d_j\,(z-a)^j, \quad \sum_{j=0}^{\infty} \|d_j\|\; |z-a|^j < \infty \quad \forall z \in B(a, r-|a|).$$

The coefficients are given by (using Cauchy's Double Series Theorem)

$$d_j = \sum_{k=j}^{\infty} \binom{k}{j} c_j\, a^{k-j}. \tag{1.10}$$

Now for an arbitrary $a \in S := \{t + is \mid |s| < r\}$ there exists some $t_0 \in \mathbb{R}$ such that $a - t_0 \in B(0,r)$. Then the ball $U := B(a - t_0, r - |a - t_0|)$ is included in $B(0,r)$, and we shall consider the open set $t_0 + U = B(a, r - |a - t_0|) \subseteq S$ to show analyticity in a.

Thus, let $z \in t_0 + U$. Then $z - t_0 \in B(0,r)$, and by the above argument there exist $d_j \in B$, $j \in \mathbb{N}_0$, with

$$F_u(z - t_0) = \sum_{j=0}^{\infty} d_j\,((z-t_0) - (a-t_0))^j = \sum_{j=0}^{\infty} d_j\,(z-a)^j, \quad \sum_{j=0}^{\infty} \|d_j\|\; |z-a|^j < \infty.$$

For all $z \in S$ chosen as above, we define $F(z) := V(t_0)F(z - t_0)$. Since t_0 was only one possible choice for a to be translated into $B(0,r)$, we have to show that $F_u(z)$ is well-defined. To this end let $t_1 \in \mathbb{R}$ be such that $z - t_1 \in B(0,r)$. We have to show that $V(t_0)F_u(z-t_0) = V(t_1)F_u(z-t_1)$ which is equivalent to

$$F_u(z - t_0) = V(t_1 - t_0)F_u(z - t_1). \tag{1.11}$$

Now set $x_0 := \mathrm{Re}\,(z) - t_0$, $x_1 := \mathrm{Re}\,(z) - t_1$ and observe that

$$\sum_{j=0}^{\infty} c_j x_0^j = F_u(x_0) = V(t_1 - t_0)F_u(x_1) = V(t_1 - t_0)\sum_{j=0}^{\infty} c_j x_1^j = \sum_{j=0}^{\infty} V(t_1 - t_0)c_j x_1^j,$$

hence by the fact that $z - t_0$, $z - t_1 \in B(0,r)$, we obtain (having that $x_0 + \mathrm{Im}\,(z) = z - t_0$, $x_1 + \mathrm{Im}\,(z) = z - t_1$)

$$F_u(z - t_0) = \sum_{j=0}^{\infty} c_j(z - t_0)^j = V(t_1 - t_0)\sum_{j=0}^{\infty} c_j(z - t_1)^j = V(t_1 - t_0)F_u(z - t_1),$$

which gives (1.11).

The estimate

$$\sum_{j=0}^{\infty} \left\|V(t_0)d_j'\right\|\; |z-a|^j \le \|V(t_0)\| \sum_{j=0}^{\infty} \|d_j\|\; |z-a|^j < \infty,$$

then gives analyticity on $B(a, r - |a - t_0|) \subseteq S$. Since a was arbitrary, F_u is a complex analytic function on S. $\square$

Since the converse statement of Proposition 1.35 is obviously true, we have the following characterization:

Corollary 1.36. *A vector u is analytic for some strongly continuous one-parameter group V if and only if F_u extends to a complex analytic function on a strip $S := \{z \in \mathbb{C} \mid |\mathrm{Im}\,(z)| < r\}$ for some $r > 0$.*

The significance of the concepts of smooth and analytic vectors is underlined by the fact that their spaces are dense in B.

Proposition 1.37. *Let V be a strongly continuous one-parameter group on the Banach space B. Then the smooth vectors for V are dense in B.*

Proof. The proof is given in two steps: First we show the existence of smooth vectors of a special form, which in turn are used to prove denseness.

Existence: Fix any $u \in B$ and any $\rho \in C_c^\infty(\mathbb{R}^n)$, and set

$$u_\rho := \int_{\mathbb{R}} \rho(t) V(t) u \, dt. \tag{1.12}$$

Since $t \mapsto \rho(t)V(t)u$ is continuous due to strong continuity, it is Bochner measurable by Theorem A.4. Note that the function $t \mapsto \|\rho(t)V(t)u\|$ is also continuous, and hence bounded on the compact support of ρ. It follows that it is Lebesgue integrable, which is equivalent to $t \mapsto \rho(t)V(t)u$ being Bochner integrable by Theorem A.8. Using Proposition A.10 we obtain

$$V(s)u_\rho = \int_{\mathbb{R}} \rho(t)V(s+t)u \, dt = \int_{\mathbb{R}} \rho(t-s)V(t)u \, dt,$$

Hence, smoothness of $s \mapsto V(s)u_\rho$ can be established by observing convergence of the iterated difference quotients using dominated convergence for the Bochner integral (cf. Theorem A.14). This, by definition, says that u_ρ is a smooth vector.

Denseness: W.l.o.g., let ρ be a symmetric non-negative mollifier, and set $\rho_j(t) := j \cdot \rho(jt)$. In the following we will use the fact that $(\rho_j * f)(t) \to f(t)$, as $j \to \infty$, for all $t \in \mathbb{R}$, for continuous $f : \mathbb{R} \to \mathbb{C}$. Finally, we define g to be the continuous function $t \mapsto \|V(t)u - u\|$ and observe that

$$\begin{aligned}
\left\| \int_{\mathbb{R}} \rho_j(t)V(t)u \, dt - u \right\| &= \left\| \int_{\mathbb{R}} \rho_j(t)V(t)u \, dt - \int_{\mathbb{R}} \rho_j(t) \, dt \, u \right\| \\
&\leq \int_{\mathbb{R}} \rho_j(t) \left\| V(t)u - u \right\| \, dt \;=\; \int_{\mathbb{R}} \rho_j(-t)g(t) \, dt \\
&= (\rho_j * g)(0) \to g(0) \;=\; 0,
\end{aligned}$$

as $j \to \infty$. Hence, for all $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that $\left\| u_{\rho_j} - u \right\| < \varepsilon$ for all $j \geq j_0$. $\square$

Corollary 1.38. *Let V be a strongly continuous one-parameter group with infinitesimal generator A. Then each of the spaces $\mathcal{D}(A^k)$, $k \in \mathbb{N}$, is a dense subspace of B.*

Proof. This is immediate since by Proposition 1.32 the space of smooth vectors is a subset of each $\mathcal{D}(A^k)$, $k \in \mathbb{N}$. □

Proposition 1.39. *Let V be a strongly continuous one-parameter group on the Banach space B. Then the analytic vectors for V are dense in B.*

Proof. We will follow the same strategy as in the proof of Proposition 1.37 showing first the existence of particular analytic vectors which in the following turn out to be dense in B.

To this end set $\rho_\varepsilon(t) := (4\pi\varepsilon)^{-1/2}\, e^{-t^2/4\varepsilon}$, and note that $t \mapsto \rho_\varepsilon(t)V(t)u : \mathbb{R} \to B$ is continuous, hence Bochner measurable by Theorem A.4 for each fixed $u \in B$. By Proposition 1.5 (ii), this function is bounded in the norm by the $L^1(\mathbb{R})$-function $(4\pi\varepsilon)^{-1/2} K_0 \, \|u\| \; e^{\frac{-t^2}{4\varepsilon}+M|t|}$, thus we have Bochner integrability by Theorem A.8. It follows that

$$u_\varepsilon := \int_{\mathbb{R}} \rho_\varepsilon(t)V(t)u\, dt$$

defines a vector in B, and again by Proposition A.10 we have

$$F_{u_\varepsilon}(s) = V(s)u_\varepsilon = (4\pi\varepsilon)^{-1/2} \int_{\mathbb{R}} e^{\frac{-t^2}{4\varepsilon}} V(t+s)u\, dt = (4\pi\varepsilon)^{-1/2} \int_{\mathbb{R}} e^{\frac{-(t-s)^2}{4\varepsilon}} V(t)u\, dt. \tag{1.13}$$

As above by dominated convergence (cf. Theorem A.14), the right-hand-side of (1.13) is smooth in s, and it even extends to a holomorphic function, if we replace s by $z \in \mathbb{C}$. Thus, defining $F_{u_\varepsilon}(z)$ by (1.13), we obtain a B-valued holomorphic function on $\mathbb{C}$. (For more details see Kadison and Ringrose [11], § 3. 3.)

Denseness follows as in the proof of Proposition 1.37 since the convolution argument actually holds for all $\rho \in L^1(\mathbb{R})$ with $\rho \geq 0$, $\int_{\mathbb{R}} \rho(t)\, dt = 1$, thus in particular for ρ_ε. □

Remark 1.40. In analogy to unitary one-parameter groups, some authors use the notation $V(t) =: e^{tA}$ to denote any strongly continuous one-parameter group V on a Banach space B. This in turn motivates the definition

$$\hat{\rho}(iA)u := \int_{\mathbb{R}} \rho(t)e^{At}u\, dt = \int_{\mathbb{R}} \rho(t)V(t)u\, dt \tag{1.14}$$

for $u \in B$ and $\rho \in C_c^\infty(\mathbb{R})$ in analogy to standard Fourier transform. The vector $\hat{\rho}(iA)u$ obviously coincides with u_ρ from Theorem 1.37, and $u \mapsto \hat{\rho}(iA)u$ actually defines a bounded operator from B into B by Proposition 1.5 (ii) and the fact that we can pull norm signs into the integral (cf. Theorem A.8). Note that $\hat{\rho}_\varepsilon(iA)u$ can be defined for $\rho_\varepsilon = (4\pi\varepsilon)^{-1/2}\, e^{-t^2/4\varepsilon}$, and also $\hat{\rho}(iA)u$ for $\rho \in L^1(\mathbb{R})$ if we require uniform boundedness of V.

Remark 1.41 ($\hat{\rho}(iA)$ vs. FT)**.** The above notation is sensible since it coincides with the standard Fourier transform of $\rho \in L^1(\mathbb{R})$. Thus, we intent to interpret the strongly continuous one-parameter group $V_\xi : \mathbb{R} \to \mathbb{C}$, $x \mapsto V_\xi(x) = e^{-ix\cdot\xi}$, $\xi \in \mathbb{R}$, in terms of (1.14). In order to produce an analogue, we must actually identify ξ with the multiplication operator $M : L^2(\mathbb{R}) \supseteq \mathcal{D}(M) \to L^2(\mathbb{R})$, $u \mapsto (\xi \mapsto \xi \cdot u(\xi))$. Then $V := x \mapsto e^{-ixM}$ turns out to be a strongly continuous unitary one-parameter group on $L^2(\mathbb{R})$ with infinitesimal generator iM. Explicitly, V is given by $x \mapsto (u \mapsto (\xi \mapsto e^{-ix\cdot\xi}u(\xi)))$, and (1.14) reads

$$(\hat{\rho}(M)u)\,(\xi) := \left(\int_{\mathbb{R}} \rho(x)\, e^{-ixM} u\, dx\right)(\xi) = \int_{\mathbb{R}} \rho(x)\, e^{-ix\xi}\, u(\xi)\, dx = \hat{\rho}(\xi)\cdot u(\xi),$$

where $\hat{\rho}(\xi)$ denotes the standard Fourier transform of ρ at $\xi \in \mathbb{R}$. Hence, $\hat{\rho}(M)$ is simply the multiplication by the Fourier transform of ρ.

The properties of Fourier transform can be generalized to a large class of functions on Lie groups, and the corresponding analogues of strongly continuous one-parameter groups will be their representations. We will study these matters in case of the Heisenberg group in Chapter 2.

1.2. Representations of Lie Groups

Representation theory is a powerful approach towards studying abstract algebraic structures by representing their elements as homomorphisms of linear spaces. Especially, groups are often represented by linear operators on normed vector spaces of finite and infinite dimension. Since these spaces are well-understood it is one big asset of representation theory that purely algebraic problems can be described by means of linear algebra and functional analysis, respectively.

Applications of representation theory is a vast area comprising diverse branches of mathematics as well as theoretical physics. In particular representations are a major tool in harmonic analysis, where they are used to extend classical Fourier analysis on $\mathbb{R}^n$ to general Lie groups. In this section we try to catch a glimpse of a few essential notions of representation theory and harmonic analysis on Lie groups in order to be prepared for the special case of the Heisenberg group to be treated in Chapter 2.

Last but not least, let us mention that for sake of a convenient approach all the Lie groups throughout this text shall be second countable Hausdorff manifolds.

1.2.1. General Groups and Measure

Definition 1.42. *Let G be a Lie group with identity element e, and let B be a Banach space. A strongly continuous representation π of G on B is a map $\pi : G \to L(B)$ satisfying the following properties*

(i) $\pi(g_1 g_2) = \pi(g_1)\pi(g_2) \;\; \forall g_1, g_2 \in G$,

(ii) $\pi(e) = I$,

(iii) $g_j \xrightarrow{G} g \quad \Rightarrow \quad \pi(g_j)u \xrightarrow{B} \pi(g)u \;\; \forall u \in B$.

The space B is called the representation space of π. We shall occasionally denote it by B_π.

Definition 1.42 is obviously a generalization of the definition of strongly continuous one-parameter groups (cf. Definition 1.1), and as in case $G = \mathbb{R}$ we will refer to (i) and (iii) as group homomorphism property and strong continuity, respectively. Since we will exclusively deal with representations π of this type, we will omit the term strongly continuous and simply call them representations from now on.

Remark 1.43. Note that in case $\dim(B) = n < \infty$, the space $\mathrm{GL}\,(B)$ forms an open subgroup of all real-valued $n \times n$-matrices and by smoothness of matrix multiplication and matrix inversion, it is even a Lie group. Conditions (i) and (ii) from Definition 1.42 now translate into the statement that π is a group homomorphism from G into $\mathrm{GL}\,(B)$. (In particular, note that (i) implies (ii).) Identifying B with the corresponding isomorphic $\mathbb{R}^n$, property (iii) applied to the $\mathbb{R}^n$-standard basis $E := \{e_1, \ldots, e_n\}$ gives continuity of all component functions of the matrix representation $[\pi]_{E,E}$, and consequently continuity of π as a map from G to $\mathrm{GL}\,(B)$. Now, it is a fact that every continuous group homomorphism between two Lie groups is already smooth (cf. Kolar, Michor and Slovak [13], Theorem 4.21). Hence, every finite-dimensional Lie group representation π of G on B is a smooth group homomorphism between the Lie groups G and $\mathrm{GL}\,(B)$.

Definition 1.44. *Let $\pi : G \to B$ be a representation of some Lie group G on the Banach space B. We say π is*

(i) *non-degenerate if for every $u \in B$, $\pi(g)u = 0$ for all $g \in G$ implies $u = 0$,*

(ii) *invariant on a subspace $B_1 \subseteq B$ if $\pi(G)(B_1) \subseteq B_1$,*

(iii) *faithful if it is injective on G,*

(iv) *trivial if $\pi(g) = I$ for all $g \in G$,*

(v) *irreducible if is non-trivial and the only invariant subspaces are B and $\{0\}$,*

(vi) *equivalent to a representation ρ of G on the Banach space $\tilde{B}$ if there exists a vector-space isomorphism $V : \tilde{B} \to B$ (called equivalence) such that $\pi(g) = V\rho(g)V^{-1}$ for all $g \in G$,*

(vii) *unitary if $B = H$ is a complex Hilbert space and $\pi(g)$ is a unitary operator on H for all $g \in G$.*

(viii) *Two unitary representations $\pi : G \to H_\pi$ and $\rho : G \to H_\rho$ are said to be unitarily equivalent if they are equivalent with unitary equivalence $U : H_\rho \to H_\pi$, i.e., $\pi(g) = U\rho(g)U^*$ for all $g \in G$.*

Remark 1.45. Note that for unitary π we have the relation $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$.

As we have seen in § 1.1, an essential tool for the special case $G = \mathbb{R}$ is integration for B-valued functions. Hence, our first step in the more general case is to provide such a notion. To set the stage for the construction to come recall from the special case that one of the basic properties of the Lebesgue measure is its translation invariance, which results in translation invariance of the Lebesgue integral and the corresponding Bochner integral. The generalization of this notion is a left invariant measure on G, which we construct now.

Proposition 1.46. *Let G be a Lie group. Then there exists a smooth left invariant measure dg on G.*

Proof. Let $n := \dim(G)$ and $\alpha \in \bigwedge^n T_e^*(G)$, $\alpha \neq 0$, which is determined up to a scalar factor since $\dim(\bigwedge^n T_e^*(G)) = 1$. Now, if $\lambda_g : G \to G$, $h \mapsto gh$ denotes the the left multiplication and $X_1, \ldots, X_n \in \mathfrak{X}(G)$ are smooth vector fields, we define $\omega \in \Omega^n(G)$ by

$$\omega(g)(X_1(g), \ldots, X_n(g)) := \alpha(T\lambda_{g^{-1}} \cdot X_1(g), \ldots, T\lambda_{g^{-1}} \cdot X_n(g)).$$

This resulting n-form is invariant under left translations:

$$\begin{aligned}
(\lambda_g)^*\omega(h)(X_1(h), \ldots, X_n(h)) :&= \omega(gh)\big(T\lambda_g \cdot X_1(h), \ldots, T\lambda_g \cdot X_n(h)\big) \\
&= \alpha\big(T\lambda_{h^{-1}g^{-1}} T\lambda_g \cdot X_1(h), \ldots, T\lambda_{h^{-1}g^{-1}} T\lambda_g \cdot X_n(h)\big) \\
&= \alpha\big(T\lambda_{h^{-1}} \cdot X_1(h), \ldots, T\lambda_{h^{-1}} \cdot X_n(h)\big) \\
&= \omega(h).
\end{aligned}$$

Moreover, ω is nowhere vanishing and due to its definition determined up to a scalar factor $c \in \mathbb{R}$. Hence, it defines an orientation on G such that via integration over G we obtain a left invariant measure that is denoted by dg. Finally, if $f : G \to \mathbb{C}$ is integrable with respect to dg, we have

$$\begin{aligned}
\int_G f(hg)\, dg \;&=\; \int_G f(\lambda_h g)\, dg \;=\; \int_G f(\lambda_h g)\, (\lambda_h)^* dg \;=\; \int_G (\lambda_h)^*(f\, dg) \;=\; \int_{\lambda_h(G)} f(g)\, dg \\
&=\; \int_G f(g)\, dg.
\end{aligned}$$

∎

Remark 1.47. In case $G = \mathbb{R}^n$ the measure dg agrees with the n-dimensional Lebesgue measure up to a factor $c \in \mathbb{R}$.

Definition 1.48. *For $1 \leq p \leq \infty$, we denote by $L^p(G)$ the space of all (equivalence classes of) dg-measurable functions $f : G \to \mathbb{C}$ such that*

$$\|f\|_{L^p(G)} := \begin{cases} \left(\displaystyle\int_G |f(g)|^p\, dg\right)^{1/p} < \infty & \textit{if } 1 \leq p < \infty \\[2ex] \inf\{c \in \mathbb{R} \mid dg(\{|f| > c\}) = 0\} < \infty & \textit{if } p = \infty. \end{cases}$$

Remark 1.49. The spaces $L^p(G)$ are Banach spaces since this can be shown in general for all measure spaces $(\mathfrak{A}, \Omega, \mu)$ (cf. Taylor [19], Theorem 4. 4).

Remark 1.50. As a matter of fact Haar has proved the far more general result that on every locally compact topological group there exists a locally finite left invariant Radon measure. For detail see, e.g., Folland [7], § 2. 2. The procedure we were following above obviously benefits greatly from the additional structure available on Lie Groups and was in fact known long before Haar's work. It actually goes back to Sophus Lie. Nevertheless dg of Proposition 1.46 is called a Haar measure. A construction completely analogous to the one of dg gives a right invariant Haar measure $d_r g$ on G, which may coincide with dg depending on the nature of G. In fact, there is a way to measure right-invariance of dg. Setting

$$\Delta(h) := \frac{dg(Eh)}{dg(E)} \tag{1.15}$$

for any Borel set $E \subseteq G$ of finite positive measure, $\Delta : G \to \mathbb{R}^+$ defines a function, which is independent of the choice of E. It is also independent of the choice of dg on G and is called the modular function of G. Furthermore, it turns out to be a continuous group homomorphism from G into $(\mathbb{R}^*, +)$, which is either trivial, i.e., $\Delta = 1$, or unbounded. In the trivial case the group G is called unimodular. Obviously abelian groups are unimodular as well as discrete groups, but there are many more, in particular any compact group is unimodular. It can be shown that from (1.15) it follows that

$$\int_G f(gh^{-1})\, dg = \Delta(h) \int_G f(g)\, dg$$

and

$$\int_G f(g)\, dg = \int_G f(g^{-1})\Delta(g^{-1})\, dg \tag{1.16}$$

for all dg-measurable functions f (cf. Folland [7], § 2. 4).

1.2.2. Translations Revisited

Now, we come back to our main example in a new disguise (cf. Example 1.2).

Definition 1.51. *Let G be a Lie group and $f \in L^p(G)$, $1 \leq p < \infty$. Then the map*

$$\begin{aligned} \tau_p &: G \to L^p(G), \\ (\tau_p(h)f)(g) &:= f(h^{-1}g), \end{aligned}$$

is called the left translation on $L^p(G)$.

Theorem 1.52. *Let G be a Lie group and $p \in [1, \infty)$. Then the left translation τ_p is a strongly continuous representation of G on $L^p(G)$.*

Proof. We show that the properties of Definition 1.42 are satisfied. Properties (i) and (ii) are immediate. We split up the proof of (iii) into two parts: To begin with, we show the property for the dense subset $C_c^\infty(G) \subseteq L^p(G)$. Eventually, we use a density argument analogous to Example 1.2 to conclude the proof.

Thus, let $\varphi \in C_c^\infty(G)$. We have to show that for $h_j \to h$ in G

$$\begin{aligned} \|\tau_p(h_j)\varphi - \tau_p(h)\varphi\|_{L^p} \to 0, \\ \text{equivalently} \int_G |\varphi(h_j^{-1}g) - \varphi(h^{-1}g)|^p \, dg \to 0. \end{aligned} \tag{1.17}$$

Hence, let $\varepsilon > 0$. We recall that G, being a Hausdorff manifold, is locally compact (cf. Kunzinger [15], Proposition 2.3.6). Let V be a compact neighborhood of h and recall that $U := \operatorname{supp}(f)$ is also compact. Hence $K := V^{-1}U := \{h^{-1}g \mid h \in V, g \in U\}$ is also compact and $\tau(l)\varphi$ is supported in K if $l \in V$. Hence, for j large enough (in particular such that $h_j \in V$) we have by uniform continuity of φ on U that

$$\int_G |\varphi(h_j^{-1}g) - \varphi(h^{-1}g)|^p \, dg = \int_K |\varphi(h_j^{-1}g) - f(h^{-1}g)|^p \, dg \leq \varepsilon^p \int_K dg.$$

Recall that $\int_{K'} 1 \, dg < \infty$ for all compact sets $K' \subseteq G$ due to regularity of dg, hence (iii) holds for all $\varphi \in C_c^\infty(G)$.

Finally, using the fact that $C_c^\infty(G) \subseteq C_c(G) \subseteq L^p(G)$ is dense (cf. Hewitt and Ross [10], Theorem 12.10, all the constructions being similar to the $\mathbb{R}^n$-case and relying on regularity of dg), we may literally use the argument of (1.1) in Example 1.2 to conclude the proof. $\square$

Remark 1.53. The map τ_p is often called the left regular representation of G. Analogously, we find that the right translation $\rho_p : G \to L^p(G, d_r g), (\rho_p(h)f)(g) := f(gh^{-1})$, $1 \leq p < \infty$, ($d_r g$ denotes a right invariant Haar measure, cf. Remark 1.50) is a representation of G on $L^p(g, d_r g)$, called the right regular representation of G. In case $p = 2$, both τ_2 and ρ_2 are unitary representations.

1.2.3. The Integrated Representation

Any representation $\pi : G \to B$ of a Lie group G on a Banach space B induces an action on certain classes of functions on G. We define the latter by integrating the representation over the Lie group G. More precisely, let $f \in L_c^1(G)$, the space of integrable functions with compact support on G, and $u \in B$. Then we set

$$\pi(f)u := \int_G f(g)\pi(g)u \, dg. \tag{1.18}$$

Note that this is a Bochner integral. Its existence can be derived from the facts we have collected in Appendix A: In fact, the map $g \mapsto f(g)\pi(g)u$ is a continuous B-valued function with compact support on G for each $u \in B$. Therefore, it is Bochner measurable by Corollary A.5 (ii) and, since it is bounded in the norm, it is also Bochner integrable on G. See Theorem A.8.

Proposition 1.54. *Let G be a Lie group with Haar measure dg and let π be a representation of G on a Banach space B. If f is integrable with compact support, then $\pi(f)$ defined by (1.18) is a bounded operator on B.*

Proof. To estimate the integral (1.18), it suffices to look for a bound for π on the compact support of f. Recall that we found such a bound for strongly continuous one-parameter groups using the uniform boundedness principle by Banach and Steinhaus (cf. Proposition 1.5). In fact, the same procedure can be applied in the present situation: the restriction $\pi : K \to B$, $g \mapsto \pi(g)u$, with $K := \operatorname{supp}(f) \subset\subset G$, is continuous for every fixed $u \in B$ and therefore bounded on the compact set K. Hence, $\sup_{g\in K} \|\pi(g)u\|_B < \infty$ for all $u \in B$. The Banach-Steinhaus Theorem yields the existence of a uniform bound $M \in \mathbb{R}^+$ in the operator norm for π on K. Hence, applying inequality (A.1) from Theorem A.8, we estimate

$$\|\pi(f)u\|_B \leq \int_G \|f(g)\pi(g)u\|_B \, dg = \int_K |f(g)| \, \|\pi(g)u\|_B \, dg \leq M \, \|f\|_{L^1} \, \|u\|_B \, .$$

□

Strengthening the conditions on π (e.g., uniform boundedness), we need less strict presumptions on f to obtain boundedness of $\pi(f)$. For example, $\pi(f)$ is bounded for $f \in L^1(G)$ if π is unitary.

Now, having extended the representation's action to integrable functions, we will prove the existence of an algebraic structure on $L^1(G)$, which permits a corresponding representation. More precisely, defining the convolution $* : L^1(G) \times L^1(G) \to L^1(G)$ by

$$(f_1 * f_2)(h) := \int_G f_1(g) f_2(g^{-1}h) \, dg, \tag{1.19}$$

one sees that $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$. We will discuss this issue more thoroughly in case $G = \mathbf{H}^n$ in § 2.2.2. Some prerequisites for that matter are collected in App. B on Bochner integration.

1.2.4. Smooth Vectors on G

Just as in the case of strongly continuous one-parameter groups we can give sense to the notion of smooth vectors for a representation π, and we will see that their set is dense in B.

Definition 1.55. *Let π be a representation of G on B. A vector $u \in B$ is called smooth vector (or C^∞-vector) if $F_u(g) := \pi(g)u$ is a smooth B-valued function on G. We denote the space of smooth vectors in B for the representation π by $C^\infty(\pi)$.*

Theorem 1.56. *The space $C^\infty(\pi)$ is a dense subset of B.*

Proof. This proof proceeds along the lines of the one of Proposition 1.37. First we will construct special C^∞-vectors, then we construct an approximating sequence of C^∞-vectors for any $u \in B$.

Existence: For any $f \in C_c^\infty(G)$ we will prove that $\pi(f)u$ is a smooth vector for all $u \in B$. That is, for an arbitrary but fixed $u \in$ we have to show that $h \mapsto \pi(h)\pi(f)u$ is a smooth function from G into B. Using Proposition A.10 and applying a change of variables, we rewrite the vector by means of the Bochner integral as

$$\pi(h)\pi(f)u = \pi(h)\int_G f(g)\pi(g)u\,dg = \int_G f(g)\pi(hg)u\,dg = \int_G f(h^{-1}\tilde{g})\pi(\tilde{g})u\,d\tilde{g}. \quad (1.20)$$

Since smoothness is a local property, an explicit proof would involve local coordinates in order to compute iterated partial derivatives. Now, the final identity of (1.20) shows us that we would have to concentrate on pulling the differential quotients into the integral, applying them to the smooth function f. This, in turn, could be achieved in a straightforward manner using dominated convergence (cf. Theorem A.14).

Denseness: Following the lines of the proof of Proposition 1.37, we construct an approximating sequence by choosing a sequence of mollifiers $f_j \in C_c^\infty(G)$ with $f_j > 0$, $\int_G f_j(g)\,dg = 1$ for all $j \in \mathbb{N}$, supported in a sequence of (compact) neighborhoods of e shrinking to $\{e\}$, as $j \to \infty$. In order to establish denseness, we use the fact that $(f_j * f)(h) \to f(h)$ for all $f \in C(G)$ and for all $h \in G$, as $j \to \infty$: Let $\varepsilon > 0$, $h \in G$ and U be a sufficiently small compact neighborhood of e such that $\left|f(g^{-1}h) - f(h)\right| < \varepsilon$ for all $g \in U$. For sufficiently large j we have $\operatorname{supp}(f_j) \subseteq U$ and therefore the following estimate:

$$\begin{aligned}
|(f_j * f)(h) - f(h)| &= \left|\int_G f_j(g) f(g^{-1}h)\,dg - \int_G f_j(g)\,dg\, f(h)\right| \\
&\le \int_G \left|f(g^{-1}h) - f(h)\right| f_j(g)\,dg \\
&= \int_U \underbrace{\left|f(g^{-1}h) - f(h)\right|}_{<\varepsilon} f_j(g)\,dg + \int_{G\setminus U} \left|f(g^{-1}h) - f(h)\right| \underbrace{f_j(g)}_{=0}\,dg \\
&< \varepsilon \underbrace{\int_U f_j(g)\,dg}_{=1} \; + \; 0 \; = \; \varepsilon.
\end{aligned}$$

Now, we employ the above result to the continuous function $f := g \mapsto \left\|\pi(g^{-1})u - u\right\|_B$. Using (A.1), we obtain

$$\begin{aligned}
\|\pi(f_j)u - u\|_B &= \left\|\int_G f_j(g)\pi(g)u\,dg - \int_G f_j(g)\,dg\,u\right\|_B \le \int_G f_j(g)\,\|\pi(g)u - u\|_B\,dg \\
&= \int_G f_j(g) f_0(g^{-1})\,dg = (f_j * f_0)(e) \to f_0(e) = 0,
\end{aligned}$$

as $j \to \infty$. This completes the proof. $\square$

Remark 1.57. The set $\mathcal{G}(\pi) := \{\pi(f)u \mid u \in B, f \in C_c^\infty(G)\}$ is called the Gårding space of π, and we have shown that $\mathcal{G}(\pi) \subseteq C^\infty(\pi)$ is dense in B.

Moreover, note that if $\dim(B) < \infty$, then $\mathcal{G}(\pi)$, being a dense subspace of B, coincides with B. We conclude that all the vectors in B are smooth.

2. Foundations of Harmonic Analysis on the Heisenberg Group

2.1. The Heisenberg Group H^n

2.1.1. Motivation

In both classical and quantum mechanics, position and momentum are regarded as the two fundamental properties of a particle. In the first case, by Newton's second law, the motion of a particle is determined completely once we know the forces involved and have measured its position and momentum at one instant of time. That is, position and momentum give a complete description of the state of a particle. Moreover, all physical observables, represented by real-valued smooth functions on phase space, i.e., the space of all possible states, are completely specified by the state.

In quantum mechanics, however, there arise two complications: First, observables only take values in probability distributions rather than in the reals and, second, nature prevents us from measuring position and momentum of a particle at the same time. Hence, it does not come as a surprise that the mathematical formalisms used to describe the two theories are quite distinct. In quantum mechanics observables are no longer functions on $\mathbb{R}^{2n}$ but rather represented by self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^{2n})$. In spite of these grave formalistic differences, in both cases we have to distinguish strictly between position and momentum coordinates respectively operators. While, e.g., all the position "coordinates" behave alike, they are very different from the momentum "coordinates". In other words, these quantities must not be interchanged. More precisely, the degree of interchangeability plays an important role in physics, where the relation between position and momentum is called complementarity (cf. Messiah [16]). It is very often expressed in terms of so-called commutation relations, realized by a Lie bracket, a concept that requires some algebraic structure on the involved underlying spaces (all the details will be presented in the next subsection). To conclude this motivation, note that the commutation relations of position and momentum in quantum mechanics are derived in analogy to those of classical physics and are formally identical. Note that there also appear other complementary pairs in quantum mechanics as, for example, energy and time.

2.1.2. Lie Algebras and Commutation Relations

The commutation relations mentioned above are essentially connected to a mathematical concept called Lie algebra. Although historically derived as so-called "infinitesimal groups" from their corresponding Lie groups (then called "transformation groups"), one simple and illustrating example of a Lie algebra arises naturally in form of the vector space $\mathrm{Lin}(E)$ of all linear operators on any other vector space E equipped with the standard commutator of operators $[A, B] := AB - BA$. In general, this operation need not be commutative (unless it is trivial) nor associative, but rather antisymmetric in its two slots. Furthermore, it satisfies the so-called Jacobi identity $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$. As a matter of fact, these two properties suffice to give a complete abstract definition of Lie algebras including all the early examples that first appeared in Sophus Lie's work.

Definition 2.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-vector space E with a bilinear form $[\ ,\ ] : E \times E \to E$ is a Lie algebra if*

(i) $[\ ,\ ]$ is antisymmetric, i.e. $[X,Y] = -[Y,X]$, and

(ii) satisfies the Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$.

The bilinear form $[\ ,\ ]$ is called its corresponding Lie bracket.

Recalling that the Lie bracket on any Lie group G (or generally on any manifold, cf. Kolar, Michor and Slovak [13], § 3. 4) satisfies (i) and (ii), a corresponding bracket on its tangent space $T_eG := \mathfrak{g}$ can be obtained by simply evaluating the Lie group brackets at e, turning $\mathfrak{g}$ into a Lie algebra. Not surprisingly, this procedure goes back to Sophus Lie who sought deeper insights into transformation groups by studying their corresponding infinitesimal groups (see above).

Now, let us give some additional definitions before returning to commutation relations and physics-related concepts.

Definition 2.2. *If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, then a homomorphism of Lie algebras (or Lie algebra homomorphism) $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear mapping, which is compatible with the brackets, i.e., one that suffices $[\varphi(X), \varphi(Y)] = \varphi([X,Y])$ for all $X, Y \in \mathfrak{g}_1$.*

An isomorphism of Lie algebras is a bijective Lie algebra homomorphism.

Example 2.3. We will now place the commutation relations in the present context. To this purpose, consider $\mathrm{Lin}(L^2(\mathbb{R}^{2n}))$ equipped with the usual commutator of operators. Then, the j-th position operator is defined to be the linear mapping

$$\begin{aligned} &Q_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \\ &Q_j f(x) := X_j f(x) := x_j f(x) \end{aligned} \tag{2.1}$$

for $x \in \mathbb{R}^n$ and $j \in \{1, \ldots, n\}$. Its "dual", the k-th momentum operator, is defined by

$$\begin{aligned} &P_k : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \\ &P_k f(x) := hD_k f(x) := -ih\frac{\partial f}{\partial x_k} \end{aligned} \tag{2.2}$$

for $x \in \mathbb{R}^n$ and $k \in \{1, \ldots, n\}$, where $\partial/\partial x_j$ denotes the distributional derivative and h Planck's constant (which we will usually set equal to 1). Now, a short calculation gives

$$[P_j, P_k] \;=\; [Q_j, Q_k] \;=\; 0, \;\; [P_j, Q_k] \;=\; -ih\delta_{jk} I, \tag{2.3}$$

the so-called Heisenberg commutation relations.

2.1.3. The Heisenberg Algebra

We are now prepared to turn $\mathbb{R}^{2n+1}$ into a Lie algebra by means of a Lie bracket and to implement the commutation relations of classical and quantum mechanics. This will eventually lead us to the structures of the Heisenberg algebra and group.

Def. & Prop. 2.4. *Let the bilinear form* $[\ ,\ ] : \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ *be defined by*

$$[(t, q, p), (t', q', p')] \;:=\; (pq' - qp', 0, 0), \tag{2.4}$$

where pq' *abbreviates* $\langle p, q' \rangle$*, the standard inner product on* $\mathbb{R}^n$*. Then* $[\ ,\ ]$ *is a Lie bracket on* $\mathbb{R}^{2n+1}$*. We will call* $(\mathbb{R}^{2n+1}, [\ ,\ ])$ *the Heisenberg algebra and denote it by* $\mathfrak{h}^n$*.*

Proof. We only have to prove that $[\ ,\ ]$ is a Lie bracket: The first property is obvious. With respect to the second one, note that any expression of the form $[A, B] = AB - BA$ satisfies the Jacobi identity. So, we are done since the Lie bracket of $\mathfrak{h}^n$ reduces to this form. □

If $T, Q_1, \ldots, Q_n, P_1, \ldots, P_n$ denotes the standard basis for $\mathbb{R}^{2n+1}$, the Lie algebra structure is given by

$$[P_j, P_k] = [Q_j, Q_k] = [P_j, T] = [Q_j, T] = 0, \qquad [P_j, Q_k] = \delta_{jk} T.$$

Since these formulas are precisely the commutation relations in classical and quantum mechanics, we see that mapping the fundamental observables on $T, Q_1, \ldots, Q_n$, $P_1, \ldots, P_n$ is a Lie algebra isomorphism.

With a view to identifying the Lie group corresponding to $\mathfrak{h}^n$, it turns out to be convenient to rewrite the elements of $\mathbb{R}^{2n+1}$ in the following form: To each (t, q, p) we associate the real-valued $(2n+1) \times (2n+1)$ matrix

$$m(t, q, p) \;:=\; \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ q_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_n & 0 & \cdots & 0 & 0 \\ t & p_1 & \cdots & p_n & 0 \end{pmatrix}. \tag{2.5}$$

Furthermore, we define a second type of matrices, namely

$$M(t,q,p) := I + m(t,q,p) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ q_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ q_n & 0 & \cdots & 1 & 0 \\ t & p_1 & \cdots & p_n & 1 \end{pmatrix}. \tag{2.6}$$

Simple matrix multiplication gives the equalities

$$m(t,q,p)m(t',q',p') = m(pq',0,0), \tag{2.7}$$
$$M(t,q,p)M(t',q',p') = M(t+t'+pq',q+q',p+p'). \tag{2.8}$$

Applying the commutator to (2.7) and (2.8), it follows that

$$[m(t,q,p),m(t',q',p')] = m(pq'-qp',0,0), \tag{2.9}$$
$$[M(t,q,p),M(t',q',p')] = M(pq'-qp',0,0). \tag{2.10}$$

Hence, the linear operators $X \mapsto m(X)$ and $X \mapsto M(X)$ are a Lie algebra isomorphisms from $\mathfrak{h}^n$ to $\{m(X) \mid X \in \mathbb{R}^{2n+1}\}$ and $\{M(X) \mid X \in \mathbb{R}^{2n+1}\}$, respectively. From (2.7) we obtain by simple matrix multiplication

$$m(t,q,p)^2 = m(pq,0,0), \quad \text{and} \quad m(t,q,p)^k = 0, \quad \forall k \geq 3. \tag{2.11}$$

2.1.4. Construction of the Heisenberg Group

Finally, we employ the exponential map to identify the Lie group of the Heisenberg Lie algebra. This will lead us to the definition of the Heisenberg group. We recall from differential geometry (cf. Kolar, Michor and Slovak [13], Definition 4. 18) that the exponential map from a Lie algebra $\mathfrak{g}$ to its Lie group G is defined via the flow of the left invariant vector field $L_X := g \mapsto T_e\lambda_g X$, $X \in \mathfrak{g}$. More precisely, $\exp(X) := \mathrm{Fl}_1^{L_X}(e)$. The exponential map is a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to some neighborhood of e in G. In the particular cases of $G = GL(n,\mathbb{R})$ and $G = (\mathbb{R}^*, \cdot)$ the exponential mapping coincides with the matrix exponential and the standard exponential function, respectively.

Now, in case of the Heisenberg Lie algebra $\mathfrak{h}^n$ we have identified the initial Lie algebra $(\mathbb{R}^{2n+1}, [\ ,\ ])$ with $\{m(X) \mid X \in \mathbb{R}^{2n+1}\}$ for which the exponential map coincides with the matrix exponential. Moreover, the exponential reduces to a sum of three matrices due to (2.11):

$$\exp(m(t,q,p)) = I + m(t,q,p) + \frac{1}{2}m(pq,0,0) = M(t+\frac{1}{2}pq,q,p). \tag{2.12}$$

Thus, the exponential is a global diffeomorphism from $\mathfrak{h}^n$ onto $\{M(X) \mid X \in \mathbb{R}^{2n+1}\}$, and it is easy to show that the latter set together with the multiplication (2.8) is a group.

One possible definition of the Heisenberg group hence is given via the identification of $\{M(X) \mid X \in \mathbb{R}^{2n+1}\}$ with the vector space $\mathbb{R}^{2n+1}$, defining the group law in analogy to (2.8) by

$$(t, q, p)(t', q', p') \;=\; (t + t' + pq', q + q', p + p'). \tag{2.13}$$

The more frequent definition (we will show their equivalence right below) emanates from another group law on $\{M(X) \mid X \in \mathbb{R}^{2n+1}\}$ that is induced by the exponential function and the formula

$$\exp(m(t, q, p)) \exp(m(t', q', p')) \;=\; \exp\left(m(t + t' + \frac{1}{2}(pq' - qp'), q + q', p + p')\right), \tag{2.14}$$

which can be derived in a straightforward manner. Finally, we are ready to give the two distinct but "isomorphic" definitions of the Heisenberg group.

Def. & Prop. 2.5. *We define $\mathbf{H}^n$ to be the vector space $\mathbb{R}^{2n+1}$ with the group multiplication*

$$(t, q, p)(t', q', p') \;:=\; \left(t + t' + \frac{1}{2}(pq' - qp'), q + q', p + p'\right). \tag{2.15}$$

Then $\mathbf{H}^n$ is a Lie group and we call it the Heisenberg group. Its unit element is $(0, 0, 0)$ and the inverse to a given element (t, q, p) is $(-t, -q, -p)$. Its Lie algebra $\mathfrak{h}^n$ coincides with the Heisenberg algebra (cf. Definition 2.4), and the exponential map from $\mathfrak{h}^n$ to $\mathbf{H}^n$ is merely the identity.

Def. & Prop. 2.6. *Consider the vector space $\mathbb{R}^{2n+1}$ with the group law*

$$(t, q, p)(t', q', p') \;=\; (t + t' + pq', q + q', p + p').$$

This is also a Lie group, with unit element $(0, 0, 0)$, and the inverse to a given element (t, q, p) is of the form $(-t + pq, -q, -p)$. We call it the polarized Heisenberg group and denote it by $\mathbf{H}^n_{pol}$.

Furthermore, there exists an isomorphism $\varphi : \mathbf{H}^n \to \mathbf{H}^n_{pol}$, which is formally identical with the exponential map from $\mathfrak{h}^n$ to $\mathbf{H}^n_{pol}$ (since the Lie algebras coincide). The map φ is explicitly given by

$$(t, q, p) \mapsto (t + \frac{1}{2}pq, q, p). \tag{2.16}$$

Proof. The only thing that is neither obvious nor already proved above is the isomorphism between $\mathbf{H}^n$ and $\mathbf{H}^n_{pol}$; this, however, follows from the relation (2.12) via a straightforward calculation. □

The construction of $\mathbf{H}^n$ and $\mathbf{H}^n_{pol}$ gains another aspect if seen in the light of finite-dimensional representations of Lie groups (cf. Rem.1.43) and Lie algebras. To this end, we define

Definition 2.7. *Let $\mathfrak{g}$ be a Lie algebra and let E be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We say, $\pi : \mathfrak{g} \to \mathrm{Lin}(E)$ is a Lie algebra representation of $\mathfrak{g}$ on E if it is a Lie algebra homomorphism from $\mathfrak{g}$ into the Lie algrebra $(\mathrm{Lin}(E), [\,,\,])$.*

Taking this into account, equation (2.9) states that the map

$$m : \mathfrak{h}^n \to \{m(X) \mid X \in \mathbb{R}^{2n+1}\} \subseteq \mathrm{M}(n+2, \mathbb{R}) \cong \mathrm{Lin}(\mathbb{R}^{n+2})$$

from (2.5) is a Lie algebra representation of $\mathfrak{h}^n$ on $\mathbb{R}^{n+2}$, whereas $\exp \circ m : \mathbf{H}^n \to \mathrm{GL}\left(\mathbb{R}^{n+2}\right)$ is a Lie group representation of $\mathbf{H}^n$ on $\mathbb{R}^{n+2}$ by (2.14). Furthermore, we observe that due to (2.10) and (2.8), the map $M : \mathfrak{h}^n \to \mathrm{GL}\left(\mathbb{R}^{n+2}\right) \subseteq \mathrm{M}(n+2, \mathbb{R})$ $\mathbf{H}^n_{pol}$ from (2.6) is a Lie algebra representation of $\mathfrak{h}^n$ as well as a Lie group representation of $\mathbf{H}^n_{pol}$ on $\mathbb{R}^{n+2}$.

Definition 2.8. *Let G be a group. Then, we define the commutator of $g, h \in G$ by $[g, h] := g^{-1}h^{-1}gh$ and the commutator subgroup of G by*

$$G^c := \{[g, h] \in G \mid g, h \in G\}.$$

Furthermore, the center of G is defined to be the set of the elements that commute with all elements in the group, i.e.,

$$Z(G) := \{g \in G \mid gh = hg \quad \forall h \in G\}.$$

Remark 2.9. The group laws (2.13) and (2.15) yield that

$$\mathcal{Z} := \{(t, 0, 0) \mid t \in \mathbb{R}\} \tag{2.17}$$

is the center as well as the commutator subgroup of both $\mathbf{H}^n$ and $\mathbf{H}^n_{pol}$. In order to prove that, let (t, q, p) and (t', q', p') be elements of $\mathbf{H}^n$ and observe that by (2.15) we have

$$\begin{aligned}(t', q', p')^{-1}(t, q, p)^{-1} &= (-t', -q', -p')(-t, -q, -p)\\ &= \left(-t - t' + \frac{1}{2}(pq' - qp'), -q - q', -p - p'\right).\end{aligned}$$

Hence, the commutator of (t, q, p) and (t', q', p') is equal to $(pq' - qp', 0, 0) \in \mathcal{Z}$. Moreover, $pq' - qp' = t$ is solvable for each $t \in \mathbb{R}$ if at least one of the vectors $p, q \in \mathbb{R}^n$ does not vanish. In total we have $\mathbf{H}^{n\,c} = \mathcal{Z}$.

In order to prove $Z(\mathbf{H}^n) = \mathcal{Z}$, note that $Z(G) = \{g \in G \mid [g, h] = e \quad \forall h \in G\}$ for any group G, which in case $G = \mathbf{H}^n$ implies that any $(t, q, p) \in Z(\mathbf{H}^n)$ satisfies $pq' = qp'$ for all $(t', q', p') \in \mathbf{H}^n$. In case $p' = 0$, this yields $p \perp q'$ for all $q' \in \mathbb{R}^n$, hence $p = 0$. Thus by symmetry, both p and q must vanish, whereas t can be chosen arbitrarily. We conclude $Z(\mathbf{H}^n) \subseteq \mathcal{Z}$. The converse inclusion is obvious.

The same argument yields $Z(\mathbf{H}^n_{pol}) = \mathcal{Z} = {\mathbf{H}^n_{pol}}^c$ since by a short calculation the commutator of (t, q, p) and (t', q', p') in $\mathbf{H}^n_{pol}$ is equal to $(pq' + qp', 0, 0)$. (Now, simply set $\widetilde{q} := -q$.)

Remark 2.10. Moreover, note that the group multiplications (2.13) and (2.15) can be also be expressed in terms of vector addition on $\mathbb{R}^{2n}$. Hence, left-invariance of the Haar measure on $\mathbf{H}^n$ and $\mathbf{H}^n_{pol}$, respectively, is equivalent to left-invariance under the translations $\tau_h(f)(x) := f(-h+x) = f(x-h)$, for $x, h \in \mathbb{R}^{2n+1}$. By uniqueness of the Haar measure up to a multiplicative positive constant c (cf. § 1.2.1), this implies that the Haar measure agrees with the $(2n+1)$-dimensional Lebesgue measure up to c. For the sake of convenience, we set $c = 1$ and integrate with respect to the Lebesgue measure. Obviously, our measure is also right-invariant, and we therefore have $\Delta_{\mathbf{H}^n} = 1$ (cf. § 1.2.1).

2.1.5. The Automorphisms of the Heisenberg Group

Definition 2.11. *Let G_1 and G_2 be two Lie groups. A Lie group homomorphism from G_1 to G_2 is a smooth map $\varphi : G_1 \to G_2$ which is a group homomorphism. A Lie group isomorphism is a bijective Lie group homomorphism, and a Lie group automorphism of a Lie group G_1 is a Lie group isomorphism $\varphi : G_1 \to G_1$. The set of all automorphisms of a Lie group G_1 is denoted by $\mathrm{Aut}(G_1)$.*

Definition 2.12. *An automorphism of a Lie algebra $\mathfrak{g}$ is a Lie algebra isomorphism (cf. Definition 2.2) from $\mathfrak{g}$ onto itself. We denote the set of all Lie algebra automorphisms of $\mathfrak{g}$ by $\mathrm{Aut}(\mathfrak{g})$.*

Remark 2.13. Note that, given a Lie group G_1 and a Lie algebra $\mathfrak{g}_2$, the sets $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(\mathfrak{g}_2)$ can be turned into groups by defining the respective group multiplications to be the composition of automorphisms.

Let $\mathrm{Aut}(\mathbf{H}^n)$ and $\mathrm{Aut}(\mathfrak{h}^n)$ be the automorphism groups of $\mathbf{H}^n$ and $\mathfrak{h}^n$, respectively. Since the underlying set of both $\mathbf{H}^n$ and $\mathfrak{h}^n$ is $\mathbb{R}^{2n+1}$, $\mathrm{Aut}(\mathbf{H}^n)$ and $\mathrm{Aut}(\mathfrak{h}^n)$ are both sets of mappings from $\mathbb{R}^{2n+1}$ to itself. As a matter of fact, they are even equal. This is actually a consequence of a general result on simply connected Lie groups and their Lie algebras. We will, however, provide a direct proof.

Proposition 2.14. $\mathrm{Aut}(\mathfrak{h}^n) = \mathrm{Aut}(\mathbf{H}^n)$.

Proof. Let $\varphi \in \mathrm{Aut}(\mathfrak{h}^n)$. Note that by (2.15) we have the following relations between the respective multiplications on $\mathbf{H}^n$ and $\mathfrak{h}^n$

$$XY = X + Y + \frac{1}{2}[X, Y] \quad \text{and} \quad YX = X + Y - \frac{1}{2}[X, Y] \tag{2.18}$$

for all $X, Y \in \mathbb{R}^{2n+1}$. Now, it is a direct consequence that φ respects the group multiplication (2.15)

$$\varphi(XY) = \varphi(X) + \varphi(Y) + \frac{1}{2}[\varphi(X), \varphi(Y)] = \varphi(X)\varphi(Y).$$

So, we have $\varphi \in \operatorname{Aut}(\mathbf{H}^n)$.

In order to prove the reverse inclusion, let $\varphi \in \operatorname{Aut}(\mathbf{H}^n)$ and let $Y \in \operatorname{lin}\{X\}$ for some $X \in \mathbb{R}^{2n+1}$. Then, $[X, Y] = 0$, and (2.18) gives $XY = X + Y = YX$. Moreover

$$\begin{aligned}\varphi(X) + \varphi(Y) + \frac{1}{2}[\varphi(X), \varphi(Y)] &= \varphi(X)\varphi(Y) = \varphi(XY) = \varphi(X+Y)\\ &= \varphi(YX) = \varphi(Y)\varphi(X) = \varphi(X) + \varphi(Y) - \frac{1}{2}[\varphi(X), \varphi(Y)].\end{aligned}$$

Hence, $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ for all $Y \in \operatorname{lin}\{X\}$. In particular, we have $\varphi(nY) = n\varphi(Y)$ for all $n \in \mathbb{N}$, and by setting $Y' := nY$ we also have $\frac{1}{n}\varphi(Y') = \varphi(\frac{1}{n}Y')$. It follows that φ is linear on the dense subspace $\{qX \mid q \in \mathbb{Q}\} \subseteq \operatorname{lin}\{X\}$ and by continuity on all of $\operatorname{lin}\{X\}$. Let now X and Y be arbitrary vectors in $\mathbb{R}^{2n+1}$. Using (2.18), we observe for $t > 0$ that

$$\begin{aligned}t\varphi\left(X + Y + \frac{t}{2}[X, Y]\right) &= \varphi\left(tX + tY + \frac{t^2}{2}[X, Y]\right) = \varphi((tX)(tY))\\ &= t\varphi(X)t\varphi(Y) = t\varphi(X) + t\varphi(Y) + \frac{t^2}{2}[t\varphi(X), t\varphi(Y)].\end{aligned}$$

Diving by t and letting $t \to 0$, continuity of φ yields $\varphi(X+Y) = \varphi(X) + \varphi(Y)$ for all $X, Y \in \mathbb{R}^{2n+1}$. Thus, by the above reasoning, φ is linear on $\mathbb{R}^{2n+1}$, and we conclude the proof with the following calculation:

$$\varphi([X, Y]) = 2\varphi(XY - X - Y) = 2(\varphi(X)\varphi(Y) - \varphi(X) - \varphi(Y)) = [\varphi(X), \varphi(Y)].$$

□

The theorem to come is a classification of $\operatorname{Aut}(\mathbf{H}^n)$. As a matter of fact, each automorphism of $\mathbf{H}^n$ can be written as the composition of elements of the following four subgroups $H_i \subseteq \operatorname{Aut}(\mathbf{H}^n)$:

H_1 : The first group involves the symplectic maps $\operatorname{Sp}(n, \mathbb{R})$. For $p, q, p', q' \in \mathbb{R}^{2n}$, we define the standard symplectic form on $\mathbb{R}^{2n}$ by

$$\omega((q, p), (q', p') := pq' - qp',$$

and the symplectic maps (or symplectomorphisms) on $\mathbb{R}^{2n}$ are precisely the linear isomorphisms on $\mathbb{R}^{2n}$ which preserve ω. That is, $S \in \operatorname{GL}(2n, \mathbb{R})$ is symplectic by definition if

$$\omega(S(q, p), S(q', p') = pq' - qp'.$$

We now define H_1 to be the subgroup of $\operatorname{GL}(2n+1, \mathbb{R})$ whose elements are of the form $\begin{pmatrix} t & 0 \\ 0 & S \end{pmatrix}$. Since such a linear map respects the Lie bracket of $\mathfrak{h}^n$, it is clearly an element of $\operatorname{Aut}(\mathbf{H}^n)$ by Theorem 2.14.

H_2 : The conjugate maps. For each $(c,b,a) \in \mathbf{H}^n$ we define the corresponding conjugate map by

$$\operatorname{conj}_{(c,b,a)}((t,q,p)) := (c,b,a)(t,q,p)(c,b,a)^{-1} = (t+aq-bp,q,p) \in \mathbf{H}^n.$$

Note that H_2 is obviously a subgroup of $\operatorname{Aut}(\mathbf{H}^n)$, which is also referred to as the group of inner automorphisms of $\mathbf{H}^n$.

H_3 : The dilations on $\mathbf{H}^n$ are defined to be of the form

$$\delta_r((t,q,p)) := (r^2 t, rq, rp), \quad r > 0.$$

It is easy to check that also H_3 forms a subgroup of $\operatorname{Aut}(\mathbf{H}^n)$.

H_4 is simply the set $\{id_{\mathbf{H}^n}, \nu_{\mathbf{H}^n}\}$, which is of course another subgroup of $\operatorname{Aut}(\mathbf{H}^n)$.

Using this notation, we will prove the following:

Theorem 2.15. *Each $\varphi \in \operatorname{Aut}(\mathbf{H}^n)$ has a unique representation of the form $\varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3 \circ \varphi_4$, where φ_i is in H_i, $i = 1, \ldots, 4$.*

Proof. Recall that each $\varphi \in \operatorname{Aut}(\mathbf{H}^n)$ is also an automorphism of $\mathfrak{h}^n$, hence a linear isomorphism on $\mathbb{R}^{2n+1}$. Since the automorphisms of any group G map $Z(G)$ onto itself, φ must be of the form

$$(t,q,p) \mapsto (st + aq + bp, T(q,p))$$

for some $s \in \mathbb{R}$, $a, b \in \mathbb{R}^n$, and $T \in \mathrm{GL}(2n,\mathbb{R})$ (cf. Remark 2.17). Applying a suitable conjugation, and an inversion if necessary, we obtain $aq + bp = 0$ and $s > 0$. Hence, the composition of that map with $\delta_{s^{-1/2}}$ gives the linear map $\varphi' : (t,q,p) \mapsto (t, S(q,p))$, where $S \in \mathrm{GL}(2n,\mathbb{R})$. Note that φ', as well as φ, respects the Lie bracket on $\mathfrak{h}^n$, so we conclude the proof with the observation that S must therefore be a symplectomorphism. $\square$

2.2. Representations of $\mathbf{H}^n$

2.2.1. The Schrödinger Representation

Recall that the position operator $Q_j := X_j$ and the momentum operator $P_j := hD_j$, which we have defined in (2.1) and (2.2), respectively, satisfy the Heisenberg commutation relations (2.3) on $\mathcal{S}(\mathbb{R}^n)$. This fact has led us to the definition of the Lie bracket in $\mathfrak{h}^n$. We now see that the map A_h from the Heisenberg algebra $\mathfrak{h}^n$ into the set of skew-symmetric operators on $\mathcal{S}(\mathbb{R}^n)$ defined by

$$A_h(t,q,p) = i(tI + qX + hpD), \tag{2.19}$$

with $qX := \sum_{j=1}^n q_j X_j$ and $pD := \sum_{j=1}^n p_j D_j$, is a Lie algebra homomorphism. It is therefore even a Lie algebra representation of $\mathfrak{h}^n$ on $\mathcal{S}(\mathbb{R}^n)$ (cf. Definition 2.7).

Intuitively spoken, exponentiating this map should therefore give a Lie group representation of $\mathbf{H}^n$ on (probably) $L^2(\mathbb{R}^n)$. Due to skew-symmetry of the operators $A_h(t,q,p)$, we expect it furthermore to be unitary in analogy to unitary one-parameter groups (cf. Remark 1.29). Thus, taking $h = 1$ for the moment, we compute the operators $e^{tI+qX+pD}$: To this end let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $g : \mathbb{R}^n \times \mathbb{R} \to \mathcal{S}(\mathbb{R}^n)$ be defined by

$$g(x, s) := e^{is(tI+qX+pD)} f(x).$$

Then g is the solution to the Cauchy problem

$$\frac{\partial g}{\partial s} = i(tI + qX + pD)g, \quad \text{with initial value} \quad g(x, 0) = f(x). \tag{2.20}$$

Equivalently, g is the solution of

$$\frac{\partial g}{\partial s} - \sum_{j=1}^{n} p_j \frac{\partial g}{\partial x_j} = i(t + qx)g, \qquad g(x, 0) = f(x).$$

Hence, the expression on the left is just the directional derivative of g along the vector $(-p, 1)$. Thus setting

$$x(t) := x - sp, \qquad G(s) := g(x(s), s),$$

we obtain the ordinary differential equation

$$G'(s) = it + iq(x - sp)G(s), \qquad G(0) = f(x),$$

which is easily solved by

$$g(x - sp, s) = G(s) = e^{ist+isqx+is^2pq/2} f(x).$$

Setting $s = 1$ and replacing x by $x + p$, we obtain the desired result:

$$e^{i(tI+qX+pD)} f(x) = e^{it+iqx+ipq/2} f(x + p). \tag{2.21}$$

Since the translations τ_p and the multiplication by a complex phase, denoted by m_q, are continuous mappings (indeed isometries) on each $L^r(\mathbb{R}^n)$, $1 \le r < \infty$, the map $e^{i(tI+qX+pD)}$ extends to a continuous operator (also denoted by $e^{i(tI+qX+pD)}$) on each $L^r(\mathbb{R}^n)$. In particular, it is unitary on $L^2(\mathbb{R}^n)$, and it also extends to a continuous operator on $\mathcal{S}'(\mathbb{R}^n)$. (The maps τ_p and m_q are defined via the dual action and coincide with the classical definitions if applied to test functions. See Friedlander/Joshi [8], Corollary 8.3.1.)

Setting repeatedly two of the parameters t, q and p equal to zero in equation (2.21), we obtain the formulas

$$e^{itI} f(x) = e^{it} f(x), \tag{2.22}$$

$$e^{iqX} f(x) = e^{iqx} f(x) = m_q f(x), \tag{2.23}$$

$$e^{ipD} f(x) = f(x + p) = \tau_p f(x). \tag{2.24}$$

In analogy to the case of strongly continuous one-parameter groups, we could say that the iX and iD are the "infinitesimal generators" of the representations τ_p (cf. Theorem 1.52) and m_q (immediate, cf. Definition 1.42). However, note that they are not representations of $\mathbf{H}^n$ but of $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$. In particular, the map $t \mapsto e^{itI}$ is a representation of $\mathbb{R}$ on $L^2(\mathbb{R}^n)$, thus a strongly continuous one-parameter group with generator itI. To achieve our goal, i.e., to obtain a representation of $\mathbf{H}^n$, we prove some more operator relations using (2.21) - (2.24).

To begin with note that

$$e^{i(tI+qX+pD)}f(x) \;=\; e^{it}e^{iqx+ipq/2}f(x+p) \;=\; e^{itI}e^{i(qX+pD)}f(x). \tag{2.25}$$

Now, let $v, w \in \mathbb{R}^n$. Computing

$$e^{i[(q+w)X+(p+v)D]}f(x) \;=\; e^{i(q+w)x+i(p+v)(q+w)/2}f(x+p+v),$$

and

$$\begin{aligned} e^{i(qX+pD)}e^{i(wX+vD)}f(x) \;&=\; e^{i(qX+pD)}e^{iwx+ivw/2}f(x+v) \\ &=\; e^{iqx+ipq/2+iw(x+p)+ivw/2}f(x+p+v), \end{aligned}$$

we observe that

$$e^{i(qX+pD)}e^{i(wX+vD)} \;=\; e^{i/2(pw-qv)}e^{i[(q+w)X+(p+v)D]}, \tag{2.26}$$

which, using (2.25), yields the identity

$$e^{i(tI+qX+pD)}e^{i(sI+wX+vD)} \;=\; e^{i[(t+s+1/2(pw-qv))I+(q+w)X+(p+v)D]}.$$

Note that the exponents behave like elements of $\mathbf{H}^n$. At this point we put Planck's constant back in, by defining

$$\pi_h(t,q,p) \;:=\; e^{i(htI+qX+hpD)} \;=\; e^{ihtI}e^{i(qX+hpD)}, \tag{2.27}$$

or equivalently

$$\pi_h(t,q,p)f(x) \;:=\; e^{iht+iqx+ihpq/2}f(x+hp). \tag{2.28}$$

We will often write π instead of π_1.

Summarizing the last few pages, we have almost completely proved the following.

Theorem 2.16. *Let $h \in \mathbb{R}$. Then, the map π_h from the Heisenberg group $\mathbf{H}^n$ into the space of bounded linear operators on $L^2(\mathbb{R}^n)$, defined by*

$$\pi_h(t,q,p) \;:=\; e^{i(htI+qX+hpD)},$$

is a unitary representation of $\mathbf{H}^n$ on the Hilbert space $L^2(\mathbb{R}^n)$, called the Schrödinger representation with parameter h. Moreover, no two representations π_h and $\pi_{h'}$ are unitarily *equivalent if $h \neq h'$.*

Proof. The only remaining item of Definition 1.42 to be shown is (iii), i.e., strong continuity. In order to do so, recall that both τ_p and m_q are strongly continuous on $L^2(\mathbb{R}^n)$. So is π_h by (2.28).

Inequivalence of π_h and $\pi_{h'}$, for $h \neq h'$, follows from the distinctness of e^{iht} and $e^{ih't}$. We proceed by contradiction: Suppose there exists a unitary equivalence $U \in L(L^2(\mathbb{R}^n))$ such that $U\pi_h(t,q,p)U^* = \pi_{h'}(t,q,p)$ for all $(t,q,p) \in \mathbf{H}^n$. The choice of $(t,q,p) = (t,0,0)$ then yields

$$e^{ih'tI} = \pi_{h'}(t,0,0) = U\pi_h(t,0,0)U^* = Ue^{ihtI}U^* = e^{ihtI},$$

a contradiction. □

Remark 2.17. The Heisenberg commutation relations (2.3) are only one half of important commutation relations in quantum mechanics since they do not respect the exponentiated operators e^{iqX}, e^{ihpD}. The other half are the so-called canonical or Weyl commutation relations and state the following:

$$e^{ihpD}e^{iqX} = e^{ihqp}e^{iqX}e^{ipD}. \tag{2.29}$$

In fact, they are easily deduced from (2.26).

Once again, we turn back to the operators $A_h(t,q,p)$, which led to the definition of the Schrödinger representation. As we have seen above, unitary one-parameter groups are essentially connected to skew-adjoint operators. In fact, every unitary one-parameter group possesses a uniquely determined skew-adjoint infinitesimal generator (cf. Theorem 1.28). Parameterizing the exponent $A_h(t,q,p)$ of the Schrödinger representation, we obtain a unitary one-parameter group with skew-adjoint generator A, which we might expect to agree with $A_h(t,q,p)$ itself. However, this is not the case. As a matter of fact, the operators $A_h(t,q,p)$ are not closed on $\mathcal{S}(\mathbb{R}^n)$, but A is on its natural domain $\mathcal{D}(A)$ due to Theorem 1.14 (i). Nevertheless, there is a way to find closed extensions for certain classes of operators.

Definition 2.18. *Let $T : B \supseteq \mathcal{D}(T) \to B$ be a densely defined operator on a Banach space B. We say T is closable if the closure $\overline{\mathcal{G}(T)}$ of its graph $\mathcal{G}(T)$ is the graph of an operator $\overline{T}$. In this case, $\overline{T}$ is called the closure of T.*

Symmetric operators on Hilbert space are indeed important members of this class. They generally may possess various closed extensions; their smallest is clearly $\overline{A}$, their largest is A^*. In case $\overline{A} = A^*$, the closure is self-adjoint, and we say A is essentially self-adjoint. Via the map $A \mapsto iA$, which bijectively identifies symmetric with skew-symmetric operators, we define the notion of essential skew-adjointness for a skew-symmetric operator A. For more information see Weidmann [21], Chapter 5.

Keeping this in mind, we finally want to prove the following statement:

Theorem 2.19. *The operators $A_h(t,q,p)$ from (2.19) are essentially skew-adjoint on $L^2(\mathbb{R}^n)$.*

Proof. Recollecting the facts about $A_h(t,q,p)$ and A from above, we see that they coincide on $\mathcal{S}(\mathbb{R}^n) \subseteq \mathcal{D}(A)$. Hence, showing that A is the closure of its restriction $A|_{\mathcal{S}(\mathbb{R}^n)} = A_h(t,q,p)$, proves the assertion. In fact, this holds true due to the following two propositions and the fact that $\mathcal{S}(\mathbb{R}^n)$ is invariant under the unitary one-parameter group $s \mapsto e^{sA_h(t,q,p)}$. $\square$

Def. & Prop. 2.20. *Let $A : B \supseteq \mathcal{D}(A) \to B$ be a densely defined closed operator on a Banach space B. We say that a linear subspace $D \subseteq \mathcal{D}(A)$ is a core of A if D is dense in $\mathcal{D}(A)$ with respect to the graph norm $\|x\|_A := \|x\| + \|Ax\|$. Then we have the following:*

(i) D is a core if and only if A is the closure of $A|_D$.

(ii) If furthermore $\lambda \in \rho(A)$, then D is a core if and only if $(\lambda - A)(D)$ is dense in B.

Proof. (i) Recall that A is defined to be closed if its graph $\mathcal{G}(A)$ is closed in the product topology of $B \times B$. Since the product norm $(x,y) \mapsto \|x\| + \|y\|$ coincides with the graph norm on the graph of A and $(\mathcal{D}(A), \|\,.\|_A)$ is a Banach space due to closedness of the operator, it is evident that $\mathcal{G}(A|_D)$ is dense in $\mathcal{G}(A)$ if and only if D is dense in $\mathcal{D}(A)$ with respect to $\|\,.\|_A$.

(ii) If $\lambda \in \rho(A)$, then $(\lambda - A)$ is closed and bijective from $\mathcal{D}(A) = \mathcal{D}(\lambda - A)$ to B. Since $\|(\lambda - A)x\| \leq \|x\| + \|(\lambda - A)x\| = \|x\|_{\lambda - A}$, it is even continuous and bijective from $(\mathcal{D}(A), \|\,.\|_A)$ to B, hence a homeomorphism by the open mapping theorem. It follows that D is dense in $\mathcal{D}(A)$ if and only if $(\lambda - A)(D)$ is dense in B. $\square$

Proposition 2.21. *Let V be a strongly continuous one-parameter group on a Banach space B with infinitesimal generator A. If $D \subseteq \mathcal{D}(A)$ is a linear subspace of B, which is invariant under all $V(t)$, then $\overline{A|_D} = A$.*

Proof. By Proposition 2.20, it suffices to show that $(\lambda - A)(D)$ is dense in B for some λ with $\mathrm{Re}\,(\lambda) > K_0$ (cf. Theorem 1.14 (vii)), which we assume to be positive w.o.l.g. We proceed by contradiction: Suppose that there exists $w \in B^* \setminus \{0\}$ that annihilates $(\lambda - A)(D)$. Since $w \neq 0$ there exists a non-vanishing $u \in B$ such that $\langle w, u \rangle \neq 0$. Now, by denseness of $D \subseteq B$ and continuity of w, we conclude the existence of some $v \in D$ such that $\langle w, v \rangle \neq 0$, and w.o.l.g. we may even suppose $\langle w, v \rangle > 0$ (otherwise multiply w by a phase factor). The fact that $V(t)D \subseteq D$ leads to the computation $\frac{d}{dt} \langle w, V(t)v \rangle = \langle w, AV(t)v \rangle = \langle w, \lambda v \rangle$, hence $\langle w, V(t)v \rangle = e^{\lambda t} \langle w, v \rangle$ for all $t \in \mathbb{R}$. But this contradicts Proposition 1.5 (ii), which in the present case states $\|V(t)\| \leq Me^{K_0 t}$. So, we obtain $w = 0$, hence denseness of $(\lambda - A)(D)$. $\square$

2.2.2. Integrated Representation and Twisted Convolution

In view of classifying all unitary representations of $\mathbf{H}^n$ in § 2.2.4, it is quite useful to extend our repertoire of representations by passing from Lie group representations of $\mathbf{H}^n$ to special algebra representations of the convolution algebra $L^1(\mathbf{H}^n)$. These new representations are obtained by integrating the Lie group representations over $\mathbf{H}^n$. This is an instance of the integrated representation of Lie groups already discussed in § 1.2.3.

More precisely, we will not integrate over the whole group since equation (2.27) has taught us that the effect of the t's on π is just given by the periodic multiplicative scalar factor e^{iht}. Therefore, it is practical to simply consider an appropriate quotient; formally we define:

Definition 2.22. *The quotient group*

$$\mathbf{H}^n_{red} := \mathbf{H}^n / \{(k, 0, 0) \mid k \in 2\pi\mathbb{Z}\},$$

is called the reduced Heisenberg group.

We will still denote its elements by (t, q, p) with the understanding that t is an element of the interval $[0, 2\pi]$. Doing so, we can consider the Schrödinger representation π_h as a representation of $\mathbf{H}^n_{red}$, which is even faithful.

Another simplification, which turns out to be useful is to omit the variable t and hence the factor e^{ihtI} altogether. So, we shall use from time to time the notation

$$\pi_h(q, p) = \pi_h(0, q, p) = e^{i(qX + hpD)}. \tag{2.30}$$

Finally, we favor another simplification of the present situation by setting Planck's constant $h = 1$. All formulas appearing in the following can easily be recovered in their general versions by inserting h in its usual positions. Subsequently appearing h's shall denote an arbitrary element of $\mathbf{H}^n$.

To begin with, we introduce the integrated Schrödinger representation on the reduced Heisenberg group.

Corollary 2.23. *For $F \in L^1(\mathbf{H}^n_{red})$ the integrated representation of π on $\mathbf{H}^n_{red}$*

$$\begin{aligned} \pi(F) : L^2(\mathbb{R}^n) &\to L^2(\mathbb{R}^n), \\ f &\mapsto \int_{\mathbf{H}^n_{red}} F(g)\pi(g)f \, dg = \int_{\mathbb{R}^{2n}\times\mathbb{T}} F(t, q, p)\pi(t, q, p)f \, dt\, dq\, dp \end{aligned} \tag{2.31}$$

defines a bounded operator on $L^2(\mathbb{R}^n)$ with $\|\pi(F)\| \le \|F\|_{L^1(\mathbf{H}^n_{red})}$. Recall from (1.18) that the integral is in fact a Bochner integral (cf. Appendix A).

Proof. Boundedness is a direct consequence of Proposition 1.54, and the estimate is a direct consequence of the fact that π is unitary (cf. Theorem 2.16). □

Here we meet an instance of the general result that the vector space $L^1(\mathbf{H}^n_{red})$ endowed with convolution (cf. (1.19)) is a Banach *-algebra, which furthermore possesses a *-representation on $L^2(\mathbb{R}^n)$ that is compatible with convolution, namely the integrated representation of π (cf. Proposition B.12). Now, considering $F \in L^1(\mathbf{H}^n_{red})$ as a function on $\mathbb{T} \times \mathbb{R}^{2n}$, we can expand it in a Fourier series in its variable t:

$$F(t, q, p) = \sum_{k=-\infty}^{\infty} \widehat{F}_k(q, p)\, e^{ikt}. \tag{2.32}$$

(In fact, the above series is convergent in the $L^1(\mathbb{T})$-norm for almost all $q, p \in \mathbb{R}^n$ with limit $F(t, q, p)$ provided usual summation is replaced by summation of its Cesàro means. See Palmer [17], Theorem 1. 8. 15.) Hence, using (2.32) and dominated convergence for the Bochner integral (cf. Theorem A.14), $\pi(F)$ can be rewritten as follows:

$$\begin{aligned}
\pi(F) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{[0,2\pi]} \sum_{k=-\infty}^{\infty} \widehat{F}_k(q,p)\, e^{ikt} e^{it} \pi(q,p)\, dt\, dq\, dp \\
&= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{F}_k(q,p)\pi(q,p)\, dq\, dp \int_{[0,2\pi]} e^{i(k+1)t}\, dt \\
&= 2\pi \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{F}_{-1}(q,p)\pi(q,p)\, dq\, dp = 2\pi \cdot \pi(\widehat{F}_{-1}).
\end{aligned}$$

The equation shows that only one Fourier coefficient contributes to the effect of π on F. It is therefoe sensible to consider π also as a representation of $L^1(\mathbb{R}^{2n})$; the definition of a compatible group multiplication will be given below (cf.).

Definition 2.24. *Let $F \in L^1(\mathbb{R}^{2n})$ and let $\pi : \mathbb{R}^{2n} \to L(L^2(\mathbb{R}^n) : (q,p) \mapsto e^{i(qX+pD)}$. Then, we define the integrated representation of F to be the operator*

$$\begin{aligned}
&\pi(F) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\
&\qquad f \mapsto \int_{\mathbb{R}^{2n}} F(q,p)\pi(q,p) f\, dq\, dp \;=\; \int_{\mathbb{R}^{2n}} F(q,p) e^{i(qX+pD)} f\, dq\, dp. \qquad (2.33)
\end{aligned}$$

The defining integral is clearly another Bochner integral. Furthermore, we observe that for the above-defined $\pi(F)$ we have the same sort of statement as in case of $\pi(F)$ for $F \in L^1(\mathbf{H}^n_{red})$ (cf. Corollary 2.23). Its proof is given along the same lines.

Corollary 2.25. *Let $F \in L^1(\mathbb{R}^{2n})$. Then its integrated representation $\pi(F)$ defined by (2.33) is a bounded operator on $L^2(\mathbb{R}^n)$ with $\|\pi(F)\| \le \|F\|_{L^1(\mathbb{R}^{2n})}$.*

As a matter of fact, there is a lot to say about the properties of the operator $\pi(F)$ and the map $F \mapsto \pi(F)$. We start out giving a description of $\pi(F)$ as an integral operator by doing an informal calculation: For an arbitrary function $f \in L^2(\mathbb{R}^n)$ we compute by (2.25)

$$\begin{aligned}
\pi(F)f(x) &= \int_{\mathbb{R}^{2n}} F(q,p)\, e^{iqx+ipq/2} f(x+p)\, dq\, dp = \int_{\mathbb{R}^{2n}} F(q, y-x) e^{iq(y+x)/2} f(y)\, dq\, dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} F(q, y-x) e^{iq(y+x)/2}\, dq\, f(y)\, dy := \int_{\mathbb{R}} K_F(x,y) f(y) dy.
\end{aligned}$$

We may now also rewrite the integral kernel in the following way:

$$\begin{aligned}
K_F(x,y) &= \int_{\mathbb{R}} F(q, y-x) e^{iq(y+x)/2} f(y)\, dq \\
&= (2\pi)^n \mathcal{F}_1^{-1} F\left(\frac{y+x}{2}, y-x\right). \qquad (2.34)
\end{aligned}$$

Here, $\mathcal{F}_j$ denotes the partial Fourier transform in the j-th component. The integral operator representation of $\pi(F)$ leads the way to generalizing the map $\pi(F)$ from $L^2(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. As we will see below, equation (2.34) is in fact the key to the according theorem, but let us first recall some preparatory facts on the Schwartz kernel theorem and on Hilbert-Schmidt operators.

To begin with, we give the following definition.

Definition 2.26. *Let $u \in \mathcal{S}(\mathbb{R}^n)$ and let $v \in \mathcal{S}(\mathbb{R}^m)$. The tensor product of u and v is defined to be the smooth map*

$$\begin{aligned} u \otimes v : \mathbb{R}^n \times \mathbb{R}^m &\to \mathbb{C}, \\ (x, y) &\mapsto u(x)v(y). \end{aligned}$$

The linear space spanned by all such tensor products is denoted by $\mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^m)$.

The Schwartz kernel theorem in its $\mathcal{S}'$-version now states the following:

Theorem 2.27 (Kernel Theorem, L. Schwartz)**.** *Let $(u, v) \mapsto B(u, v) : \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^m) \to \mathbb{C}$ be a separately continuous bilinear functional of the form $B(u, v) = \langle u, Tv\rangle$ for some continuous linear map $T : \mathcal{S}(\mathbb{R}^m) \to \mathcal{S}(\mathbb{R}^n)$. Then there exists a unique tempered distribution $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^m)$ such that $B(u, v) = \langle K, u \otimes v\rangle$. The distribution K is called the kernel of the map T.*

Its proof can be found in a very general setting in Trèves [20], cf. Theorem 51. 6 and the subsequent corollary.

The second notion to be mentioned is a class of bounded operators on Hilbert spaces, which play an important role in physics, particularly in quantum mechanics.

Definition 2.28. *Let H_1, H_2 be Hilbert spaces and $A \in L(H_1, H_2)$. We say A is a Hilbert-Schmidt operator if there exists an orthonormal base $(e_\alpha)_{\alpha \in A}$ of H_1 such that*

$$\sum_{\alpha \in A} \|Ae_\alpha\|^2_{H_2} < \infty. \tag{2.35}$$

We denote the space of all Hilbert-Schmidt operators from H_1 to H_2 by $HS(H_1, H_2)$ and write $HS(H_1)$ for $HS(H_1, H_1)$.

Hilbert-Schmidt operators turn out to be even compact operators with several equivalent descriptions involving its singular values or s-numbers, arbitrary orthonormal bases of H_1, infinite matrices, etc. As a matter of fact, the value of (2.35) is independent of the choice of the orthonormal basis of H_1 and furthermore agrees with the ℓ^2-norm square of the corresponding s-numbers. That is to say, we have

$$\sum_{\alpha \in A} \|Ae_\alpha\|^2_{H_2} = \sum_{k=0}^{\infty} s_k(A)^2 < \infty$$

for all orthonormal bases $(e_\alpha)_{\alpha\in A}$ of H_1. This fact is useful to define a norm on $HS(H_1, H_2)$, the so-called Hilbert-Schmidt norm, by

$$\|A\|_{HS}^2 := \sum_{k=0}^{\infty} s_k(A)^2, \tag{2.36}$$

which turns $HS(H_1, H_2)$ into a Banach space, and it is easily seen that $\|A\| \leq \|A\|_{HS}$ for all $A \in HS(H_1, H_2)$. Furthermore A is a Hilbert-Schmidt operator if and only if A^* is and we have $\|A\|_{HS} = \|A^*\|_{HS}$, and the product of the two Hilbert-Schmidt operators is trace class. In case $H_1 = H_2 = H$ one uses these facts and shows that $\|\ \|_{HS}$ is induced by the scalar product

$$\langle A, B\rangle_{HS} := \operatorname{tr}(B^* A)$$

which turns $HS(H)$ into a Hilbert space. Moreover, it can be shown that for every bounded operator $T \in L(H)$ the compositions AT and TA are again Hilbert-Schmidt if A is, satisfying the estimates

$$\begin{aligned}\|TA\|_{HS} &\leq \|T\|\,\|A\|_{HS} \leq \|T\|_{HS}\,\|A\|_{HS}\,,\\ \|AT\|_{HS} &\leq \|T\|\,\|A\|_{HS} \leq \|T\|_{HS}\,\|A\|_{HS}\,.\end{aligned}$$

Hence, considering the vector space $L(H)$ as a normed algebra with multiplication $\circ$, the set $HS(H)$ forms a two-sided ideal within $L(H)$, which is closed under conjugation. Furthermore, it is a Banach algebra (cf. Definition B.1), and due to the identity $\|A\|_{HS} = \|A^*\|_{HS}$ it is even a Banach *-algebra with involution $A \mapsto A^*$ (cf. Definition B.4 (v)).

We require one more result on Hilbert-Schmidt operators, i.e., the characterization of Hilbert-Schmidt operators on L^2-spaces: Let $(S, \mathfrak{A}_S, \mu)$ and $(T, \mathfrak{A}_T, \nu)$ be arbitrary measure spaces. A linear map $A : L^2(S) \to L^2(T)$ is Hilbert-Schmidt if and only if there exists a function $K \in L^2(T \times S)$ such that

$$Af(t) = \int_S K(t, s) f(s)\, d\mu(s)$$

for all $f \in L^2(S)$. In that case, we furthermore have

$$\|A\|_{HS} = \|K\|_{L^2(S\times T)}\,. \tag{2.37}$$

The function K is, of course, the (distributional) kernel that corresponds to A by the Schwartz kernel theorem.

All the details to these assertions can be found in Weidmann [21], § 6. 2.

Remark 2.29. If $U : H_1 \to H_2$ is a unitary operator between the Hilbert spaces H_1 and H_2, we may define an isometric isomorphism $\tilde{U}$ from $HS(H_1)$ onto $HS(H_2)$ by $\tilde{U} := T \mapsto UTU^*$: Since $HS(H_1)$ and $HS(H_2)$ are Hilbert spaces it suffices to prove that $\tilde{U}$ is unitary. To this end, note that if $(e_\alpha)_{\alpha\in A}$ is an orthonormal basis of H_1, then $(Ue_\alpha)_{\alpha\in A}$ forms an orthonormal basis of H_2. The following computation now proves the claim:

$$\|UAU^*\|_{HS}^2 = \sum_{\alpha\in A} \|UAU^* e_\alpha\|_{H_2}^2 = \sum_{\alpha\in A} \|AU^* e_\alpha\|_{H_1}^2 = \|A\|_{HS}^2.$$

Finally, we are ready to prove the following statement.

Theorem 2.30. *The map π from $L^1(\mathbb{R}^{2n})$ to $L(L^2(\mathbb{R}^n))$, defined by*

$$\pi(F) := \int_{\mathbb{R}^{2n}} F(q,p)\pi(q,p)\,dq\,dp = \int_{\mathbb{R}^{2n}} F(q,p)e^{i(qX+pD)}\,dq\,dp,$$

extends uniquely to a vector space isomorphism of $\mathcal{S}'(\mathbb{R}^{2n})$ onto $L(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$. Moreover, π is a vector space isomorphism (even unitary up to the factor $(2\pi)^n$) of $L^2(\mathbb{R}^{2n})$ onto $HS(L^2(\mathbb{R}^n))$, the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$, satisfying $\|\pi(F)\|_{HS} = \|K_F\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n \|F\|_{L^2}$ Finally, we have the slightly weaker assertion: The operator $\pi(F)$ is compact on $L^2(\mathbb{R}^n)$ provided $F \in L^1(\mathbb{R}^{2n})$.

Proof. From equation (2.34) we can explicitly read off that K_F is obtained from F by two operations that can be extended to the distributional case (via the dual action). That is, partial Fourier transformation followed by the linear coordinate transformation

$$(x,y) \mapsto \left(\frac{y+x}{2}, y-x\right) = \begin{pmatrix} \frac{1}{2}I_n & \frac{1}{2}I_n \\ -I_n & I_n \end{pmatrix},$$

with determinant 1. This map is continuous (even an isomorphism) on $\mathcal{S}(\mathbb{R}^{2n})$ hence extends to a continuous (even isomorphic) map on $\mathcal{S}'(\mathbb{R}^{2n})$. Saturating K_F's second variable (by applying it to a Schwartz function), we obtain a continuous linear map (in the x variable) from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{C}$, thus a tempered distribution. That is to say, if $f, g \in \mathcal{S}(\mathbb{R}^n)$, then

$$\left\langle \int_{\mathbb{R}^n} K_F(\,.\,,y)f(y)\,dy, g\right\rangle = \langle\langle K_F(\cdot,y), f(y)\rangle, g\rangle = \langle K_F, g\otimes \overline{f}\rangle = \langle F, h\rangle,$$

where

$$h(q,p) := \int_{\mathbb{R}^{2n}} e^{-iqx-ipq/2}\overline{f(x+p)}g(x)\,dx.$$

Now, recall that Schwartz kernel theorem states that every continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ has exactly one representation of this form. Hence, we conclude that $F \mapsto K_F$ is bijective on $\mathcal{S}'(\mathbb{R}^{2n})$.

It is, furthermore, unitary up to the factor $(2\pi)^n$ (which emanates from the partial Fourier transform) on $L^2(\mathbb{R}^{2n})$. Hence, K_F is an L^2-kernel and so $f \mapsto \int_{\mathbb{R}^n} K_F(\cdot,y)f(y)\,dy$ defines a Hilbert-Schmidt operator, say A, with

$$\|A\|_{HS} = \|K_F\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n \|F\|_{L^2(\mathbb{R}^{2n})}.$$

For $F \in L^1(\mathbb{R}^{2n})$ choose an approximating sequence of functions F_n in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, say $F_n \in C_c^\infty(\mathbb{R}^n)$. Then each operator $\pi(F_n)$ is Hilbert-Schmidt, thus compact. Recalling the estimate $\|\pi(\Phi)\| \leq \|\Phi\|_{L^1}$ and the fact that the set of compact operators is closed in the operator norm, we obtain

$$\|\pi(F) - \pi(F_n)\| = \left\| \int_{\mathbb{R}^{2n}} (F(q,p) - F_n(q,p))\, \pi(q,p)\, dq\, dp \right\| \leq \|F - F_n\|_{L^1} \to 0,$$

as $(n \to \infty)$. This completes the proof. $\square$

Corollary 2.31. *The space of Hilbert-Schmidt oprators $HS(L^2(\mathbb{R}^n))$ is a separable Hilbert space.*

Proof. Since $L^2(\mathbb{R}^m)$ is separable for all $m \in \mathbb{N}$, separability of $HS(L^2(\mathbb{R}^n))$ is induced by the isomorphism $\pi : L^2(\mathbb{R}^{2n}) \to HS(L^2(\mathbb{R}^n))$. $\square$

Let us now turn to $L^1(\mathbb{R}^{2n})$ and define a group multiplication, which is compatible with the Schrödinger representation.

Definition 2.32. *For $F, G \in L^1(\mathbb{R}^{2n})$ we define the twisted convolution of F and G at $(q,p) \in \mathbb{R}^{2n}$ by*

$$\begin{aligned} F \sharp G(q,p) \; :&= \int_{\mathbb{R}^{2n}} F(q',p')G(q-q',p-p')e^{i/2(p'q-q'p)}\, dq'\, dp' \\ &= \int_{\mathbb{R}^{2n}} F(q-q',p-p')G(q',p')e^{i/2(pq'-qp')}\, dq'\, dp'. \end{aligned}$$

In what follows we reveal the idea behind Definition 2.32. For this purpose we define the map

$$\begin{aligned} F \mapsto F^0 : L^1(\mathbb{R}^{2n}) &\to L^1(\mathbf{H}^n_{red}) \\ F^0(t,q,p) &:= (2\pi)^{-1} e^{-it} F(q,p). \end{aligned} \tag{2.38}$$

The following proposition now provides facts on twisted convolution as well as the correspondences between usual convolution and twisted convolution, between the integrated Schrödinger representations on $\mathbb{R}^{2n}$ and $\mathbf{H}^n_{red}$, etc.

Proposition 2.33. *Let $F, G \in L^1(\mathbb{R}^{2n})$ and let $\pi(q,p) := \pi(0,q,p)$ be the Schrödinger representation restricted to $\mathbb{R}^{2n}$. Defining F^0 and G^0 as in (2.38), the following hold:*

(i) $F \sharp G \in L^1(\mathbb{R}^{2n})$ *with* $\|F \sharp G\|_{L^1(\mathbb{R}^{2n})} \leq \|F\|_{L^1(\mathbb{R}^{2n})} \|G\|_{L^1(\mathbb{R}^{2n})}$,

(ii) $F^0 * G^0 = (F \sharp G)^0$,

(iii) $\pi(F) = \pi(F^0)$,

(iv) $\pi(F \sharp G) = \pi(F)\pi(G)$,

Moreover, for $F, G \in L^2(\mathbb{R}^{2n})$ we obtain *(v)* $F \sharp G \in L^2(\mathbb{R}^{2n})$ *with* $\|F \sharp G\|_{L^2} \leq (2\pi)^n \|F\|_{L^2} \|G\|_{L^2}$.

Proof. Assertion (i) can be proved along the lines of estimate (B.2), which shows that $\|F * G\|_{L^1(G)} \leq \|F\|_{L^1(G)} \|G\|_{L^1(G)}$ for an arbitrary Lie group G.

The identityies (ii) and (iii) are due to the two following straight-forward computations:

$$
\begin{aligned}
&F^0 * G^0(t,q,p) \\
&\quad = (2\pi)^{-2} \int_{[0,2\pi]} \int_{\mathbb{R}^{2n}} F(q',p') e^{-it'} G(q-q',p-p') e^{-i(t-t')+i/2(pq'-qp')}\, dt'\, dq'\, dp' \\
&\quad = \frac{e^{-it}}{2\pi} \int_{\mathbb{R}^{2n}} F(q',p') G(q-q',p-p') e^{i/2(p'q-q'p)}\, dq'\, dp' = (\, F \sharp G)^0(t,q,p).
\end{aligned}
$$

$$
\begin{aligned}
\pi(F) &= \int_{\mathbb{R}^{2n}} F(q,p) e^{i(qX+pD)}\, dq\, dp \int_{[0,2\pi]} \frac{e^{-it}}{2\pi} e^{it}\, dt \\
&= \int_{\mathbb{R}^{2n}} \int_{[0,2\pi]} F^0(t,q,p) e^{i(qX+pD+tI)}\, dt\, dq\, dp = \pi(F^0).
\end{aligned}
$$

Keeping in mind that π, being a *-representation of $L^1(\mathbf{H}^n_{red})$ (cf. Proposition B.12), satisfies $\pi(F * G) = \pi(F)\,\pi(G)$ for all $F, G \in L^1(\mathbf{H}^n_{red})$, equation (iv) is now easily obtained by repeatedly using (ii) and (iii):

$$
\pi(F \sharp G) = \pi((F \sharp G)^0) = \pi(F^0 * G^0) = \pi(F^0)\,\pi(G^0) = \pi(F)\,\pi(G).
$$

We complete this proof by showing the estimate from (v). For this purpose, recall the identity $\|\pi(F)\|_{HS} = \|K_F\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n \|F\|_{L^2(\mathbb{R}^{2n})}$ from Theorem 2.30. Using (iv), an application of the Cauchy-Schwarz inequality finally gives

$$
\begin{aligned}
(2\pi)^n \|F \sharp G\|_{L^2} &= \|\pi(F \sharp G)\|_{HS} = \|\pi(F)\,\pi(G)\|_{HS} \\
&= \left\| \int_{\mathbb{R}^n} K_F(x,y) K_G(y,z)\, dy \right\|_{L^2(x,z)} \leq \|K_F\|_{L^2(\mathbb{R}^{2n})} \|K_G\|_{L^2(\mathbb{R}^{2n})} \\
&= (2\pi)^{2n} \|F\|_{L^2} \|G\|_{L^2}. \qquad (2.39)
\end{aligned}
$$

Hence, we are done. □

Remark 2.34. Following the lines of the proof of Proposition B.12, we see that $\big(L^1(\mathbb{R}^{2n}), \sharp, \|\,.\,\|_{L^1}\big)$ is a Banach *-algebra with *-represenation π (cf. Definitions B.4 and B.10). The image of π is included in the subalgebra $K(L^2(\mathbb{R}^n)) \subseteq L(L^2(\mathbb{R}^n))$.

Moreover, we can also turn $L^2(\mathbb{R}^{2n})$ into a Banach *-algebra: Since it is easily checked that $F \mapsto \overline{F}(-\,.\,,-\,.\,)$ defines an involution on $L^2(\mathbb{R}^{2n})$ that satisfies (i) to (v) from Definition B.4, we simply alter the algebra norm by setting $\|\,.\,\|'_{L^2} := (2\pi)^n \|\,.\,\|_{L^2}$ in order to obtain $\|F \sharp G\|'_{L^2} \leq \|F\|'_{L^2} \|G\|'_{L^2}$ from equation (2.39).

Now, combining Theorem 2.30 and Proposition 2.33 (ii), we may state that the integrated Schrödinger representation π is in fact an isometric *-isomorphism (cf. Definition B.9) of $L^2(\mathbb{R}^{2n})$ onto $HS(L^2(\mathbb{R}^n))$ since compatibility of π with the involution is easily checked. In fact, π is even a *-representation provided it is understood to be composed with the embedding $HS(L^2(\mathbb{R}^n)) \hookrightarrow L(L^2(\mathbb{R}^{2n}))$. However, is not isometric any longer.

Closing this subsection, we prove a short proposition on how the Schrödinger representation combines with the integrated Schrödinger representation. This simple technical statement will be of good use in the following subsections.

Proposition 2.35. *Let $F \in \mathcal{S}'(\mathbb{R}^{2n})$ and $a, b \in \mathbb{R}^n$. For G and H, defined by*

$$\begin{aligned} G(q,p) &= e^{i(bq-ap)/2} F(q-a, p-b), \quad \textit{and} \\ H(q,p) &= e^{i(ap-bq)/2} F(q-a, p-b), \quad \textit{respectively,} \end{aligned}$$

we have

$$\pi(a,b)\pi(F) \;=\; \pi(G) \quad \textit{and} \quad \pi(F)\pi(a,b) \;=\; \pi(H).$$

Proof. We will first prove the formulas for $F \in \mathcal{S}(\mathbb{R}^{2n})$, which by the same argument as is in the proof of Theorem 2.30 then extend to $F \in \mathcal{S}'(\mathbb{R}^{2n})$. To begin with, note that by Proposition A.10 the Bochner integral interchanges with bounded operators, such as $\pi(a,b)$ for any $a, b \in \mathbb{R}^n$. So, we compute

$$\begin{aligned} \pi(a,b)\pi(F) \;&=\; \int_{\mathbb{R}^{2n}} \pi(a,b)\pi(q,p)F(q,p)\, dq\, dp \\ &=\; \int_{\mathbb{R}^{2n}} \pi(a+q, b+p) e^{i(bq-ap)/2} F(q,p)\, dq\, dp, \end{aligned}$$

and the change of coordinates $q' := q + a$, $p' := p + b$ gives

$$\pi(a,b)\pi(F) \;=\; \int_{\mathbb{R}^{2n}} \pi(q',p') e^{i(bq'-ap')/2} F(q',p')\, dq'\, dp'.$$

The second formula is proved along the same lines using $\pi(q,p)\pi(a,b) = e^{i(ap-bq)/2}\pi(a+q, b+p)$ and the same change of coordinates. □

2.2.3. Matrix Coefficients and the Fourier-Wigner Transform

In this subsection we introduce the concept of matrix coefficients of a Lie group representation, which has proved to be of utmost importance in various applications of representation theory. The theorem of Peter and Weyl, just to give one example, states that for any compact Lie group G, the space of matrix coefficients of all finite-dimensional representations of G is dense in $C(G)$.

In our case it is the behavior of the matrix coefficients of the Schrödinger representation and, subsequently, of any unitary representation of $\mathbf{H}^n$ on a Hilbert space H we are particularly interested in.

Definition 2.36. *Let $f, g \in L^2(\mathbb{R}^n)$ and let π be the Schrödinger representation of $\mathbf{H}^n$ on $L^2(\mathbb{R}^n)$. Then, we define its matrix coefficient at (f, g) to be the function*

$$\begin{aligned} M : \mathbf{H}^n &\to \mathbb{C}, \\ (t, q, p) &\mapsto \langle \pi(t, q, p) f, g \rangle . \end{aligned}$$

Once again, we may omit the variable t since the expression for $M(t, q, p)$ reduces to $e^{it} M(0, q, p)$. Accordingly, for $f, g \in L^2(\mathbb{R}^n)$, we set

$$\begin{aligned} V(f, g)(q, p) : = \langle \pi(q, p) f, g \rangle &= \int_{\mathbb{R}^n} e^{iqx + ipq/2} f(x + p) \overline{g(x)} \, dx \\ &= \int_{\mathbb{R}^n} e^{iqy} f(y + p/2) \overline{g(y - p/2)} \, dy, \end{aligned} \tag{2.40}$$

where $y := x + \frac{1}{2}p$. The associated conjugate-linear map $(f, g) \mapsto V(f, g)$ has no standard name, but Folland [6] refers to it as the Fourier-Wigner transform, explicitly:

Definition 2.37. *Let $f, g \in L^2(\mathbb{R}^n)$ and let π be the Schrödinger representation on $\mathbb{R}^{2n}$. We define the Fourier-Wigner transform of f and g to be the map*

$$\begin{aligned} V(f, g) : \mathbb{R}^{2n} &\to \mathbb{C}, \\ (q, p) &\mapsto \langle \pi(q, p) f, g \rangle . \end{aligned}$$

For $f, g \in L^2(\mathbb{R}^n)$, $V(f, g)$ is obviously continuous. Moreover, $V(f, g)$ is bounded by the Cauchy-Schwarz inequality, explicitly $\|V(f, g)\|_\infty \leq \|f\|_{L^2} \|g\|_{L^2}$.

Our next task is to extend the Fourier-Wigner transform to distributions. We aim at following a similar road as in case of the integrated representation and rewrite $V(f, g)$ as

$$V(f, g)(q, p) = (2\pi)^n \mathcal{F}_1^{-1}(f \otimes \overline{g})(y + p/2, y - p/2).$$

Also we will be interested in viewing the Fourier-Wigner transform not only as a conjugate-linear map on $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ but to extend it to a linear map on the tensor product $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$, which - of course - is naturally isomorphic to $L^2(\mathbb{R}^{2n})$. More precisely, for $F \in L^2(\mathbb{R}^{2n})$ we define

$$\tilde{V}(F)(q, p) := \int_{\mathbb{R}^n} e^{iqy} F(y + p/2, y - p/2) \, dy \tag{2.41}$$

and also call it Fourier-Wigner transform of F. We then have for $f, g \in L^2(\mathbb{R}^n)$

$$V(f, g) = \tilde{V}(f \otimes \overline{g}).$$

Now, we see that also $\tilde{V}$ is a composition of the measure-preserving coordinate change $(y, p) \mapsto (y + p/2, y - p/2)$ and the inverse partial Fourier transform in the first component multiplied by $(2\pi)^n$ and so maps $\mathcal{S}(\mathbb{R}^{2n})$ isomorphically onto itself.

Hence, we obtain along the lines of the proof of Proposition 2.30

Proposition 2.38. *The Fourier-Wigner transform V maps $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and extends to a continuous map from $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.*

Similarly $\tilde{V}$ extends to a continuous bijection of $\mathcal{S}'(\mathbb{R}^{2n})$ onto itself, which (up to a factor $(2\pi)^n$) is unitary on $L^2(\mathbb{R}^{2n})$.

The fact that $\tilde{V}$ (up to a factor $(2\pi)^n$) is unitary can be translated back to V to yield

Corollary 2.39. *The Fourier-Wigner transform V has the property*

$$\langle V(f_1, g_1), V(f_2, g_2)\rangle \;=\; (2\pi)^{2n} \langle f_1, f_2\rangle \, \overline{\langle g_1, g_2\rangle} \tag{2.42}$$

for all $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$.

Identity (2.42) is the key to the following important result.

Proposition 2.40. *The Schrödinger representations π_h are irreducible for all $h \in \mathbb{R}^*$.*

Proof. Suppose there exists a non-trivial closed invariant subspace $\mathcal{H} \subseteq L^2(\mathbb{R}^n)$. If $f \neq 0 \in \mathcal{H}$ and $g \in \mathcal{H}^\perp$, then we have $g \perp \pi(t, q, p)f$ for all $(t, q, p) \in \mathbf{H}^n$ and consequently $V(f, g) = 0$. Now, Corollary 2.39 gives $\|f\|_{L^2} \|g\|_{L^2} = 0$, whence we conclude $g = 0$ and, therefore, $\mathcal{H} = L^2(\mathbb{R}^n)$. □

The rest of this subsection is dedicated to establish some technical results, which we will mainly use to prove the Stone-von Neumann theorem. The integrated representations of Gaussian functions are of particular interest and importance in this context.

Proposition 2.41. *For $a, b, c, d \in \mathbb{R}^n$ we have*

$$V(\pi(a, b)f, \pi(c, d)g)(q, p) = e^{i(pa+pc+bc-qb-dq-ad)/2} V(f, g)(q + a - c, p + b - d).$$

Proof. The claim follows from the identity

$$V\big(\pi(a, b)f, \pi(c, d)g\big)(q, p) \;=\; \langle \pi(-c, -d)\pi(q, p)\pi(a, b)f, g\rangle$$

and the fact that

$$\begin{aligned}(0, -c, -d)(0, q, p)(0, a, b) \;&=\; (0, -c, -d)(ap - qb, q + a, p + b)\\ &=\; (ap - qb - (q + a)d + (p + b)c, q + a - c, p + b - d)\\ &=\; (pa + pc + bc - qb - qd - ad, q + a - c, p + b - d).\end{aligned}$$

□

Corollary 2.42. *The following three identities are special cases of Proposition 2.41.*

(i) $V\big(\pi(a, b)f, g\big)(q, p) \;=\; e^{i(pa-qb)/2} V(f, g)(q + a, p + b)$,

(ii) $V(f, \pi(c, d)g)(q, p) = e^{i(pc-dq)/2} V(f, g)(q - c, p - d)$, *(iii)* $V(\pi(a,b)f, \pi(a,b)g)(q,p) = e^{i(pa-qb)}V(f,g)(q,p)$.

As several times before we have come to point where we would like to replace the variables (q,p) by an $L^1(\mathbb{R}^{2n})$-integrable function F, i.e., to go from the Schrödinger representation to the integrated representation. Keeping Proposition A.10 in mind, we see that the matrix elements of the integrated representation can also be written in terms of the Fourier-Wigner transform

$$\begin{aligned}\langle \pi(F)f, g\rangle &= \left\langle \int_{\mathbb{R}^{2n}} F(q,p)\pi(q,p)f\, dq\, dp, g\right\rangle = \int_{\mathbb{R}^{2n}} F(q,p)\, \langle \pi(q,p)f, g\rangle\, dq\, dp \\ &= \int_{\mathbb{R}^{2n}} F(q,p)V(f,g)(q,p)\, dq\, dp = \langle V(f,g), \overline{F}\rangle \end{aligned} \tag{2.43}$$

for all $f, g \in L^2(\mathbb{R}^n)$.

Remark 2.43. This formula not only shows that π is a faithful representation of $L^1(\mathbb{R}^{2n})$, but it is also the key to an interesting formula for the operators $\pi(\Phi)$, where $\overline{\Phi}$ is a Fourier-Wigner transform. In fact, it states that the operators $\pi(\Phi)$ are exactly the operators of rank one on $L^2(\mathbb{R}^n)$.

Proposition 2.44. *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$ and $\Phi := \overline{V(\varphi,\psi)}$. Then, for all $f \in L^2(\mathbb{R}^n)$ we have*

$$\pi(\Phi)f \;=\; (2\pi)^{2n}\, \langle f, \varphi\rangle\, \psi.$$

Proof. Using (2.43) and (2.42), we obtain

$$\langle \pi(\Phi)f, g\rangle \;=\; \langle V(f,g), V(\varphi,\psi)\rangle \;=\; (2\pi)^{2n}\, \langle f,\varphi\rangle\, \overline{\langle g,\psi\rangle} \;=\; (2\pi)^{2n}\, \langle f,\varphi\rangle\, \langle \psi, g\rangle\,.$$

□

With the help of Proposition 2.44 we may also describe the behavior of the Fourier-Wigner transform under twisted convolution.

Proposition 2.45. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^n)$. Then*

$$\overline{V(\varphi_1,\psi_1)} \,\sharp\, \overline{V(\varphi_2,\psi_2)} \;=\; (2\pi)^{2n}\, \langle \psi_2, \varphi_1\rangle\, \overline{V(\varphi_2,\psi_1)}.$$

Proof. Let $\Phi_j := \overline{V(\varphi_j,\psi_j)} \in L^2(\mathbb{R}^{2n})$ for $j = 1,2$ and $\Psi := \overline{V(\varphi_2,\psi_1)} \in L^2(\mathbb{R}^{2n})$. Then

$$\begin{aligned}\pi(\Phi_1 \,\sharp\, \Phi_2)f \;&=\; \pi(\Phi_1)\pi(\Phi_2)f \;=\; (2\pi)^{2n}\pi(\Phi_1)\, \langle f,\varphi_2\rangle\, \psi_2 \;=\; (2\pi)^{4n}\, \langle f,\varphi_2\rangle\, \langle \psi_2,\varphi_1\rangle\, \psi_1 \\ &=\; (2\pi)^{2n}\, \langle \psi_2,\varphi_1\rangle\, \pi(\Psi)f.\end{aligned}$$

Recalling that π is unitary up to a constant, hence faithful, on $L^2(\mathbb{R}^{2n})$, we have $\Phi_1 \,\sharp\, \Phi_2 = (2\pi)^{2n}\, \langle \psi_2, \varphi_1\rangle\, \Psi$. □

The last proposition in this subsection concerns the Fourier-Wigner transform of the Gaussian function.

Proposition 2.46. *Let $\varphi(x) := \pi^{-n/4}(2\pi)^{-n}\, e^{-x^2/2}$, a scalar multiple of the Gaussian probability distribution, let $\Phi := V(\varphi,\varphi)$ and $\Phi^{ab} := V(\varphi, \pi(a,b)\varphi)$. Then, we have*

(i) $\Phi(q,p) = (2\pi)^{-2n}\, e^{-(q^2+p^2)/4}$,

(ii) $\Phi^{ab}(q,p) = (2\pi)^{-2n}\, e^{i(pa-bq)/2}\, e^{-[(q-a)^2+(p-b)^2]/4}$,

(iii) $\pi(\Phi)\pi(a,b)\pi(\Phi) = e^{-(a^2+b^2)/4}\pi(\Phi)$,

(iv) $\Phi \,\sharp\, \Phi^{ab} = e^{-(a^2+b^2)/4}\,\Phi$.

Proof. To begin with, note that for $\gamma(x) := e^{-x^2/2}$ we have

$$\begin{aligned}(2\pi)^{-n/2}\mathcal{F}\gamma(x) \;&=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\gamma(y)e^{-ixy}\,dy \;=\; \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\gamma(y)e^{ixy}\,dy\\ &=\; (2\pi)^{n/2}\mathcal{F}^{-1}\gamma(x) \;=\; e^{-x^2/2}.\end{aligned} \tag{2.44}$$

(i) It follows that the inverse Fourier transformation (2.44) together with the change of variables $x := \sqrt{2}y$ gives

$$\begin{aligned}\Phi(q,p) &= \pi^{-n/2}(2\pi)^{-2n}\int_{\mathbb{R}^n} e^{iqy}\varphi(y+p/2)\overline{\varphi(y-p/2)}\,dy\\ &= (2\pi)^{-2n}\pi^{-n/2}\int_{\mathbb{R}^n} e^{iqy}e^{-y^2-p^2/4}\,dy\\ &= (2\pi)^{-2n}\pi^{-n/2}\,2^{-n/2}\,e^{-p^2/4}\int_{\mathbb{R}^n} e^{ixq/\sqrt{2}}e^{-x^2/2}\,dx\\ &= (2\pi)^{-2n}\pi^{-n/2}\,2^{-n/2}\,e^{-p^2/4}\,(2\pi)^n\mathcal{F}^{-1}\gamma(q/\sqrt{2})\\ &= (2\pi)^{-2n}\pi^{-n/2}\,2^{-n/2}\,e^{-p^2/4}\,(2\pi)^{n/2}\,e^{-q^2/2}\\ &= (2\pi)^{-2n}e^{-(q^2+p^2)/4}.\end{aligned}$$

(ii) Once we have (i), an application of Corollary 2.42 (ii) gives (ii):

$$\begin{aligned}\Phi^{ab}(q,p) &= V(\varphi,\pi(a,b)\varphi) = e^{i(pa-qb)/2}V(\varphi,\varphi)(q-a,p-b)\\ &= \; e^{i(pa-qb)/2}\Phi(q-a,p-b) = (2\pi)^{-2n}\,e^{i(pa-qb)/2}\,e^{-[(q-a)^2+(p-b)^2]/4}.\end{aligned}$$

Identity (iii) is due to (i) and repeated use of Proposition 2.44:

$$\begin{aligned}\pi(\Phi)\pi(a,b)\pi(\Phi)f &= (2\pi)^{2n}\pi(\Phi)\,\langle f,\varphi\rangle\,\pi(a,b)\varphi = (2\pi)^{4n}\,\langle f,\varphi\rangle\,\langle \pi(a,b)\varphi,\varphi\rangle\,\varphi\\ &= (2\pi)^{2n}V(\varphi,\varphi)(a,b)\,(2\pi)^{2n}\,\langle f,\varphi\rangle\,\varphi = e^{-(q^2+p^2)/4}\,\pi(\Phi)f.\end{aligned}$$

In order to prove (iv), we observe that $\pi(a,b)\pi(\Phi) = \pi\big(\Phi^{ab}\big)$ holds due to Proposition 2.35 and the fact that $e^{i(bq-ap)/2}\;\Phi(q-a,p-b) = \Phi^{-a,-b}(q,p)$. Hence, the identity

$$\pi\big(\Phi\,\sharp\,\Phi^{-a,-b}\big) \;=\; \pi(\Phi)\pi(a,b)\pi\big(\Phi^{-a,-b}\big) \;=\; e^{-(a^2+b^2)/4}\,\pi(\Phi)$$

and faithfulness of π prove the claim. □

2.2.4. The Stone-von Neumann Theorem

The Stone-von Neumann Theorem is a fundamental result also for theoretical physics since it shows the equivalence of two a priori seemingly different descriptions of quantum mechanics: Erwin Schrödinger's wave mechanical formulation and Werner Heisenberg's matrix mechanics (which is actually due to Max Born, Pascual Jordan, and Heisenberg). In essence, both approaches make use of representing the canonical commutator relations (2.29), that is to say the Heisenberg group itself, on particular infinite-dimensional Hilbert spaces. Schrödinger's model was strongly favored by Albert Einstein, while matrix mechanics, using a (to physicists then) new mathematical language, had been developed under the direction of Niels Bohr. The two opposing directions, however, have only been brought together when John von Neumann proved the equivalence of their approaches. More precisely, the special version of his famous theorem states that any irreducible unitary representation of $\mathbf{H}^n$ that is not trivial on the center, such as Heisenberg's, is unique in the sense of being unitarily equivalent to the some Schrödinger representation of parameter $h \in \mathbb{R}^*$. It was indeed Marshall Harvey Stone, who first asserted uniqueness, but von Neumann, who gave the complete proof (cf. Rosenberg [5]). We will prove the slightly more general version, which does not require irreducibility a priori.

One interesting aspect of the proof we will give is the important role the Gaussian functions play in such a statement, which may seem far away from classical calculus.

Theorem 2.47 (The Stone-von Neumann Theorem)**.** *Let ρ be a unitary representation of the Heisenberg group $\mathbf{H}^n$ on a Hilbert space H, such that $\rho(t,0,0) = e^{iht}I$ for some $h \in \mathbb{R}^*$. Then $H = \bigoplus_{\alpha\in A} H_\alpha$, where the H_α's are mutually orthogonal subspaces of H, each invariant under ρ, such that $\rho|_{H_\alpha}$ is unitarily equivalent to π_h for each α. In particular, if ρ is irreducible then it is equivalent to π_h.*

Proof. Once again we concentrate on the case $h = 1$ since the argument in the general case is identical. The key tools in this proof are the Fourier-Wigner transform and Gaussian functions as well as their analogues for general unitary representations of $\mathbf{H}^n$ on any Hilbert space H. Treating these objects, we adopt the notation from above:

$$\begin{aligned}
&\varphi(x) := \pi^{-n/4}(2\pi)^{-n}\, e^{-x^2/2},\\
&\varphi^{ab}(x) := \pi(a,b)\varphi(x) = \pi^{-n/4}(2\pi)^{-n}\, e^{iax+iab/2}\, e^{-(x+b)^2/2},\\
&\Phi := V(\varphi,\varphi),\\
&\Phi^{ab} := V(\varphi,\pi(a,b)\varphi) = V(\varphi,\varphi^{ab}) = (2\pi)^{-2n}\, e^{i(pa-bq)/2}\, e^{-[(q-a)^2+(p-b)^2]/4}. \qquad (2.45)
\end{aligned}$$

Let ρ be an arbitrary unitary representation of $\mathbf{H}^n$ on a Hilbert space H. In analogy to (2.30) and Definition 2.24, for $q,p \in \mathbb{R}^n$ and $F \in L^1(\mathbb{R}^{2n})$ we define

$$\rho(q,p) := \rho(0,q,p)$$

and

$$\rho(F) := \int_{\mathbb{R}^{2n}} F(q,p)\rho(q,p)\,dq\,dp.$$

Analogously it holds true that

$$\rho(q,p)\rho(s,r) = \rho\big((sp-rq)/2, q+s, p+r\big) = e^{i(sp-rq)/2}\,\rho(q+s,p+r),$$

and also the proofs of Proposition 2.33 (iv) and Proposition 2.35 are literally applicable to any unitary ρ and $F, F_1, F_2 \in L^1(\mathbb{R}^{2n})$:

$$\rho(F_1)\rho(F_2) = \rho(F_1 \sharp F_2), \tag{2.46}$$

$$\rho(a,b)\rho(F) = \rho(G), \quad \text{where} \quad H(q,p) = e^{i(bq-ap)/2}\,F(q-a,p-b), \tag{2.47}$$

$$\rho(F)\rho(a,b) = \rho(H), \quad \text{where} \quad H(q,p) = e^{i(ap-bq)/2}\,F(q-a,p-b). \tag{2.48}$$

Once we know this, we can show that the integrated representation ρ of $L^1(\mathbb{R}^{2n})$ is faithful. For this purpose, let $\rho(F) = 0$. For all $u, v \in H$ and $a, b \in \mathbb{R}^n$ we then have

$$\begin{aligned} 0 = \langle \rho(a,b)\rho(F)\rho(-a,-b)u, v\rangle_H &= \int_{\mathbb{R}^{2n}} F(q,p)\,\rho(a,b)\rho(q,p)\rho(-a,-b)\,\langle u, v\rangle_H\,dq\,dp \\ &= \int_{\mathbb{R}^{2n}} e^{i(bq-ap)}\,F(q,p)\,\langle \rho(q,p)u, v\rangle_H\,dq\,dp, \end{aligned}$$

since $(0,a,b)(0,q,p)(0,-a,-b) = (bq-ap,q,p)$. The Fourier inversion theorem now gives $F(q,p)\,\langle \rho(q,p)u, v\rangle_H = 0$ for almost all $q, p \in \mathbb{R}^n$, hence $F = 0$ a.e. since $u, v \in H$ were arbitrary.

Now, the same line of arguments as in the proof of Proposition 2.46 (iii), can be applied to the present case to obtain

$$\rho\big(\Phi \sharp \Phi^{-a,-b}\big) \;=\; e^{-(a^2+b^2)/4}\,\rho(\Phi) \;=\; \rho(\Phi)\rho(a,b)\rho\big(\Phi^{-a,-b}\big). \tag{2.49}$$

Now, setting $a = b = 0$, it follows that $\rho(\Phi)^2 = \rho(\Phi)$, and since Φ is even and real, we conclude the operator $\rho(\Phi)$ is also self-adjoint:

$$\begin{aligned} \langle \rho(\Phi)u, v\rangle_H = \left\langle \int_{\mathbb{R}^{2n}} \Phi(q,p)\rho(q,p)u\,dq\,dp, v \right\rangle_H &= \int_{\mathbb{R}^{2n}} \Phi(q,p)\,\langle u, \rho(-q,-p)v\rangle_H\,dq\,dp \\ &= \int_{\mathbb{R}^{2n}} \Phi(-q',-p')\big\langle u, \rho(q',p')v\big\rangle_H\,dq'\,dp' \\ &= \left\langle u, \int_{\mathbb{R}^{2n}} \Phi(q',p')\rho(q',p')v\,dq'\,dp' \right\rangle_H = \langle u, \rho(\Phi)v\rangle_H. \end{aligned}$$

Hence $\rho(\Phi)$ is an orthogonal projection, which is nonzero provided $\Phi \neq 0$.

Next we show that H can be written as the direct sum of invariant spaces. To this end, let $u, v \in \operatorname{ran}(\rho(\Phi))$, i.e., $\rho(\Phi)u = u$ and $\rho(\Phi)v = v$, respectively. This leads to the computation

$$\begin{aligned}\langle \rho(q,p)u, \rho(s,r)v\rangle_H &= \langle \rho(q,p)\rho(\Phi)u, \rho(s,r)\rho(\Phi)v\rangle_H = \langle \rho(\Phi)\rho(-s,-r)\rho(q,p)\rho(\Phi)u, v\rangle_H \\ &= e^{i(ps-qr)/2} \langle \rho(\Phi)\rho(q-s,p-r)\rho(\Phi)u, v\rangle_H \\ &= e^{i(ps-qr)/2} e^{-[(q-s)^2+(p-r)^2]/4} \langle u, v\rangle_H, \end{aligned} \tag{2.50}$$

where we use (2.49) in the last identity. Hence, defining H_α to be the closed linear span of $\{\rho(q,p)v_\alpha \mid q,p \in \mathbb{R}^n\}$ for any orthonormal basis $(v_\alpha)_\alpha$ of $\operatorname{ran}(\rho(\Phi))$, we immediately see that $H_\alpha \perp H_\beta$ for $\alpha \neq \beta$, and, by definition, each of these spaces is invariant under ρ. So, the space $\mathcal{N} := (\bigoplus_\alpha H_\alpha)^\perp$ is also invariant under ρ, thus $\rho|_\mathcal{N}$ defines a unitary representation of $\mathbf{H}^n$ on $\mathcal{N}$. But since $\rho(\Phi)|_\mathcal{N} = 0$ and $\rho|_\mathcal{N}$ is also faithful on $L^1(\mathbb{R}^{2n})$, the fact that $\Phi \neq 0$ implies $\mathcal{N} = 0$. It follows that $H = (\bigoplus_\alpha H_\alpha)$.

Finally we have to prove unitary equivalence of π and ρ on each H_α. To do so, we define $v^{qp} := \rho(q,p)v_\alpha$ for any fixed index $\alpha \in A$. Using (2.45) and (2.50), we conclude that

$$\langle v^{qp}, v^{sr}\rangle_H = (2\pi)^{2n} \langle \varphi^{qp}, \varphi^{sr}\rangle$$

for all $p,q,r,s \in \mathbb{R}^n$. Now, let $u \in H_\alpha$, then $u = \sum_{j,k} \alpha_{jk} v^{q_j p_k}$ for some coefficients α_{jk}. If we map u to $f := \sum_{j,k} \alpha_{jk}\varphi^{q_j p_k}$, then we obtain

$$\|u\|_H = (2\pi)^{2n} \|f\|_{L^2(\mathbb{R}^n)}$$

and we may extend this map to an isometry (up to the factor $(2\pi)^{2n}$)

$$U : H_\alpha \to L^2(\mathbb{R}^n),$$

which by construction intertwines $\rho|_{H_\alpha}$ and π, i.e., $\pi = U\rho|_{H_\alpha}U^*$ (cf. Definition 1.44 *(viii)*). □

Remark 2.48. The Stone-von Neumann theorem has been notably generalized by G. Mackey (cf. Rosenberg [5], § 3.), and indeed in many presentations the Stone-von Neumann theorem is deduced from Mackey's imprimitivity theorem. Nevertheless, it is still presented in its original version in various books due to the elegant methods involved in its proof and the more explicit character.

The last theorem in this section is a classification of all irreducible unitary representations of $\mathbf{H}^n$.

Theorem 2.49. *Every irreducible unitary representation ρ of $\mathbf{H}^n$ on a Hilbert space H is unitarily equivalent to one and only one of the following representations:*

(a) $\sigma_{(a,b)} : (t,q,p) \mapsto e^{i(aq+bp)}$, $a,b \in \mathbb{R}^n$, *acting on* $\mathbb{C}$,

(b) π_h, $h \in \mathbb{R}^*$, *acting on* $L^2(\mathbb{R}^n)$.

Proof. To begin with, recall that $Z(\mathbf{H}^n)$ denotes the center of $\mathbf{H}^n$, and by Remark 2.9 we have $Z(\mathbf{H}^n) = \mathcal{Z} = \{(t,0,0) \mid t \in \mathbb{R}\}$. Now, if $g_1 \in \mathcal{Z}$, then $\rho(g_1)$ commutes with all $\rho(g_2)$, $g_2 \in \mathbf{H}^n$, and by Schur's lemma, more precisely by Corollary C.17, we have $\rho(\mathcal{Z}) \subseteq \{e^{is}I \mid s \in [0,2\pi]\}$. Hence, $\rho(t,0,0) = e^{iht}I$ for some $h \in \mathbb{R}$. Now, if $h \neq 0$, the Stone-von Neumann theorem asserts that ρ is equivalent to some π_h. Otherwise if $h = 0$, $\mathcal{Z} \subseteq \ker(\rho)$ and ρ factors through $\mathbf{H}^n/\mathcal{Z}$, which is isomorphic to $\mathbb{R}^{2n}$. Again by Corollary C.17 any irreducible representation ρ of $\mathbb{R}^{2n}$ satisfies $\rho(\mathbb{R}^{2n}) \subseteq \mathbb{T} = \{e^{it} \mid t \in [0,2\pi]\}$. Hence ρ must be of the form $(q,p) \mapsto e^{i(aq+bp)}$ for some $a, b \in \mathbb{R}^n$. This completes the proof. $\square$

Unitary equivalence clearly defines an equivalence relation on the set of irreducible unitary representations of $\mathbf{H}^n$, which motivates the next definition.

Definition 2.50. *Let $\widehat{\mathbf{H}}^n$ be the set of equivalence classes of irreducible unitary representations of $\mathbf{H}^n$. This set is called the unitary dual (or dual set) of $\mathbf{H}^n$.*

Remark 2.51. The set $\widehat{\mathbf{H}}^n$ can be equipped with a topology (cf. Kirillov [12], § 3. 4. 5, Definition 3) such that $\mathbb{R}^*$ and $\mathbb{R}^{2n}$ with the usual topologies are homeomorphic to the topological subspaces $\widehat{\mathbf{H}}^n_\pi := \{[\pi_h] \in \widehat{\mathbf{H}}^n \mid h \in \mathbb{R}^*\}$ and $\widehat{\mathbf{H}}^n_\sigma := \{[\sigma_{(a,b)}] \in \widehat{\mathbf{H}}^n \mid a, b \in \mathbb{R}^n\}$, respectively. The topological space $\widehat{\mathbf{H}}^n$ is in fact locally compact and second countable but fails to be Hausdorff. Identifying $\widehat{\mathbf{H}}^n$ with $\mathbb{R}^* \cup \mathbb{R}^{2n}$, we may transfer an existing Borel measure μ on $\mathbb{R}^* \cup \mathbb{R}^{2n}$ to $\widehat{\mathbf{H}}^n$. In Subsection 2.2.5 we will make use of such a measure, which in particular assigns measure zero to $\widehat{\mathbf{H}}^n_\sigma$. Hence, in the following it will most of the time be sufficient to only deal with the case $[\rho] \in \widehat{\mathbf{H}}^n_\pi$.

2.2.5. The Group Fourier Transform

Recalling that the Fourier transform of $f \in L^1(\mathbb{R}^n)$ at $\xi \in \mathbb{R}^n$ is defined by the integral

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \tag{2.51}$$

we observe that it involves the unitary representation

$$\begin{aligned}\pi_\xi : \mathbb{R}^n &\to L(\mathbb{C}),\\ x &\mapsto (\lambda \mapsto e^{ix\cdot\xi} \cdot \lambda).\end{aligned}$$

(Note that by Corollary C.17 the representations $\{\pi_\xi\}_{\xi\in\mathbb{R}^n}$ are in fact the only irreducible unitary representations of $\mathbb{R}^n$.) Using this notation, we may rewrite (2.51) as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)(\pi_\xi(x))^*\, dx = \int_{\mathbb{R}^n} f(x)\pi_\xi(-x)\, dx. \tag{2.52}$$

Equation (2.52) now motivates the definition of an analogous transform for the Heisenberg group $\mathbf{H}^n$.

Definition 2.52. *Let ρ be an irreducible unitary representation of $\mathbf{H}^n$ on some Hilbert space H. For $f \in L^1(\mathbf{H}^n)$, we define its group Fourier transform at ρ to be the map*

$$\begin{aligned} \hat{f}(\rho) : H &\to H, \\ u &\mapsto \int_{\mathbf{H}^n} f(g)(\rho(g))^* u \, dg = \int_{\mathbf{H}^n} f(g)\rho(g^{-1})u \, dg. \end{aligned} \tag{2.53}$$

Note that due to linearity of the defining integral, a Bochner integral (cf. Appendix A), $\hat{f}(\rho)$ defines a linear operator on H. Moreover, recall that by Theorem 2.49 ρ is unitarily equivalent to some uniquely determined Schrödinger representation π_h, $h \in \mathbb{R}^*$. The following proposition now states that unitary equivalence is in fact inherited by the the group Fourier transform.

Proposition 2.53. *Let ρ be an irreducible unitary representation of $\mathbf{H}^n$ on some Hilbert space H and let π_h be the corresponding unitarily equivalent representation with equivalence $U : H \to L^2(\mathbb{R}^n)$. Furthermore, let $f \in L^1(\mathbf{H}^n)$. Then the corresponding group Fourier transforms defined by (2.53) are unitarily equivalent in the sense that*

$$U\hat{f}(\rho)U^* = \hat{f}(\pi_h). \tag{2.54}$$

Proof. Let $\varphi, \psi \in \tilde{H}$. Repeated use of Proposition A.10 yields

$$\begin{aligned} \left\langle U\hat{f}(\rho)U^*\varphi, \psi \right\rangle &= \left\langle U \int_{\mathbf{H}^n} f(g)\rho(g)U^*\varphi \, dg, \psi \right\rangle = \left\langle \int_{\mathbf{H}^n} f(g)U\rho(g)U^*\varphi \, dg, \psi \right\rangle \\ &= \int_{\mathbf{H}^n} f(g) \left\langle U\rho(g)U^*\varphi, \psi \right\rangle \, dg = \int_{\mathbf{H}^n} f(g) \left\langle \pi_h(g)\varphi, \psi \right\rangle \, dg \\ &= \left\langle \int_{\mathbf{H}^n} f(g)\pi_h(g)\varphi \, dg, \psi \right\rangle = \left\langle \hat{f}(\pi_h)\varphi, \psi \right\rangle. \end{aligned}$$

Since φ and ψ were arbitrary, we are done. □

In the following we identity the $\hat{f}(\rho)$'s with their corresponding $\hat{f}(\pi_h)$'s since all the results on the group Fourier transform we shall prove in this subsection carry over to the "Schrödinger"-case using (2.54). For the sake of convenience we shall write $\hat{f}(h)$ for $\hat{f}(\pi_h)$.

Consequently, we identify $\widehat{\mathbf{H}}^n$ with $\mathbb{R}^*$ and proceed with the understanding that the group Fourier transform of $f \in L^1(\mathbf{H}^n)$ at $h \in \mathbb{R}^*$ is given by the operator

$$\begin{aligned} \hat{f}(h) : L^2(\mathbb{R}^n) &\to L^2(\mathbb{R}^n), \\ \varphi &\mapsto \int_{\mathbf{H}^n} f(t,q,p)\pi_h(-t,-q,-p)\varphi \, dt \, dq \, dp. \end{aligned} \tag{2.55}$$

The group Fourier transform can therefore be viewed as a map from $\mathbb{R}^*$ into $\mathrm{Lin}(L^2(\mathbb{R}^n))$.

Remark 2.54. Recall from the proof of Proposition 2.53 that for $\varphi, \psi \in L^2(\mathbb{R}^n)$ we obtain

$$\left\langle \hat{f}(h)\varphi, \psi \right\rangle = \int_{\mathbf{H}^n} f(g)\ \langle \pi_h(g)\varphi, \psi\rangle\ dg, \tag{2.56}$$

which some texts use as a weak definition of $\hat{f}(h)$ in order to avoid Bochner integration. Note that identity (2.56) is particularly interesting since unitarity of each $\pi_h(g)$ gives $|\langle \pi_h\varphi, \psi\rangle| \le \|\varphi\|_{L^2(\mathbb{R}^n)}\, \|\psi\|_{L^2(\mathbb{R}^n)}$ and therefore

$$\left|\left\langle \hat{f}(h)\varphi, \psi \right\rangle\right| \le \|f\|_{L^1(\mathbf{H}^n)}\, \|u\|_{L^2(\mathbb{R}^n)}\, \|\psi\|_{L^2(\mathbb{R}^n)}\,. \tag{2.57}$$

Hence, we obtain boundedness of $\hat{f}(h)$ on $L^2(\mathbb{R}^n)$, with $\|\hat{f}(h)\| \le \|f\|_{L^1(\mathbf{H}^n)}$.

In fact there is much more to say about $\hat{f}(h)$ in case $f \in L^1(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$.

Proposition 2.55. *Let $f \in L^1(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$ and let $h \in \mathbb{R}^*$. Then, $\hat{f}(h)$ is a Hilbert-Schmidt operator, thus, in particular, compact on $L^2(\mathbb{R}^n)$ with*

$$\|\hat{f}(h)\|_{HS}^2 = (2\pi)^{2n}\, |h|^{-n} \int_{\mathbb{R}^{2n}} |\hat{f}(h, q, p)|^2\, dq\, dp.$$

Recall the following facts about Hilbert-Schmidt operators we have collected in § 2.2.2 following Definition 2.28: the set of all Hilbert-Schmidt operators on some Hilbert space H, denoted by $HS(H)$, is a Hilbert space itself with scalar product $\langle A, B\rangle_{HS} := \operatorname{tr}(B^*A)$ for $A, B \in HS(H)$. Furthermore, an operator on $L^2(\mathbb{R}^n)$ is Hilbert-Schmidt if and only if it is an integral operator with kernel $K \in L^2(\mathbb{R}^{2n})$. We shall use this characterization to prove Proposition 2.55.

Proof. Let $h \in \mathbb{R}^*$. In order to prove that $\hat{f}(h)$ is Hilbert-Schmidt, we must find some $K_f^h \in L^2(\mathbb{R}^{2n})$ such that for all $\varphi \in L^2(\mathbb{R}^n)$ we have

$$\hat{f}(h)\varphi(x) = \int_{\mathbb{R}^{2n}} K_f^h(x, y)\varphi(y)\, dy.$$

Thus, let φ be an arbitrary vector in $L^2(\mathbb{R}^n)$. We compute

$$\begin{aligned} \hat{f}(h)\varphi(x) &= \int_{\mathbf{H}^n} f(t, q, p)\pi_h(-t, -q, -p)\varphi(x)\, dt\, dq\, dp \\ &= \int_{\mathbf{H}^n} e^{-iht - iqx - ihpq/2} f(t, q, p)\varphi(x - hp)\, dt\, dq\, dp \\ &= |h|^{-n} \int_{\mathbf{H}^n} e^{-iht - iq(x+y)/2} f(t, q, (x - y)/h)\, dt\, dq\, \varphi(y)\, dy. \end{aligned} \tag{2.58}$$

Hence, we may explicitly read off from (2.58) that

$$\begin{aligned} K_f^h(x,y) &= |h|^{-n} \int_{\mathbb{R}^{n+1}} e^{-iht-iq(x+y)/2} f(t,q,(x-y)/h)\, dt\, dq && (2.59)\\ &= |h|^{-n}\, \mathcal{F}_{1,2}\, f(h,(x+y)/2,(x-y)/h), && (2.60)\end{aligned}$$

Now, K_f^h is clearly in $L^2(\mathbb{R}^{2n})$ since $f \in L^2(\mathbf{H}^n) \cong L^2(\mathbb{R}) \times L^2(\mathbb{R}^{2n})$ and both the change of coordinates and the Fourier transform are isomorphisms on L^2.

Finally, we compute the Hilbert-Schmidt norm of $\hat{f}(h)$. To this end, note that by (2.37) the latter is determined by $\|\hat{f}(h)\|_{HS} = \|K_f^h\|_{L^2(\mathbb{R})^{2n}}$. Thus, identity (2.60) leads to

$$\begin{aligned} \|\hat{f}(h)\|_{HS}^2 &= |h|^{-2n} \int_{\mathbb{R}^{2n}} \left|\mathcal{F}_{1,2}\, f(h,(x+y)/2,(x-y)/h)\right|^2\, dx\, dy\\ &= |h|^{-n} \int_{\mathbb{R}^{2n}} \left|\mathcal{F}_{1,2}\, f(h,q,p)\right|^2\, dq\, dp\\ &= (2\pi)^{2n}\, |h|^{-n} \int_{\mathbb{R}^{2n}} \left|\mathcal{F}_{1} f(h,q,p)\right|^2\, dq\, dp, && (2.61)\end{aligned}$$

where we have applied a linear coordinate transformation with determinant $|h|^n$ followed by the standard Plancherel formula for $L^2(\mathbb{R}^{2n})$. This completes the proof. □

Collecting the facts, we have seen that the group Fourier transform of $f \in L^1(\mathbf{H}^n)$ is an operator-valued map on $\mathbb{R}^*$ with values in $L(L^2(\mathbb{R}^{2n}))$, and in particular in $HS(L^2(\mathbb{R}^{2n}))$ provided $f \in L^1(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$. The following theorem now states the existence of a measure μ on $\mathbb{R}^*$ such that the map $\hat{f} : \mathbb{R}^* \to HS(L^2(\mathbb{R}^n))$ not is only Bochner-measurable (cf. Definition A.1) but also square-integrable (cf. Definition A.16) with respect to the Hilbert-Schmidt norm $\|\,.\,\|_{HS} = \sqrt{\langle\,.\,,\,.\,\rangle_{HS}}$. It furthermore establishes an analogue to the Plancherel identity we know from the ordinary Fourier transform on $\mathbb{R}^n$.

Theorem 2.56. *Let the measure μ on $\mathbb{R}^*$ be defined by $d\mu := (2\pi)^{-(n+1)}\, |h|^n\, dh$. Then, the group Fourier transform $f \mapsto \hat{f}$ restricted to $L^1(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$ extends to a unitary isomorphism $\mathcal{F}$ of $L^2(\mathbf{H}^n)$ onto $L^2(\mathbb{R}^*, HS(L^2(\mathbb{R}^n)); \mu)$. In particular, we have*

$$\|f\|_{L^2(\mathbf{H}^n)}^2 = \int_{\mathbb{R}^*} \|\mathcal{F}(f)(h)\|_{HS}^2\, d\mu(h)$$

for all $f \in L^2(\mathbf{H}^n)$.

Proof. From Definition 2.52 it is clear that $f \mapsto \hat{f}$ is linear on $L^1(\mathbf{H}^n)$. In order to show that it is isometric for all $f \in L^1(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$, we first integrate identity (2.61) with respect to h. An application of the Plancherel theorem for $L^2(\mathbb{R})$ then gives

$$\begin{aligned}(2\pi)^{-2n}\int_{\mathbb{R}}\|\hat{f}(h)\|_{HS}^2\,|h|^n\,dh &= \int_{\mathbb{R}}\int_{\mathbb{R}^{2n}}|\mathcal{F}_1 f(h,q,p)|^2\,dq\,dp\,dh\\ &= (2\pi)\int_{\mathbf{H}^n}|f(h,q,p)|^2\,dq\,dp\,dh,\end{aligned}$$

or equivalently the desired isometry

$$\|f\|_{L^2(\mathbf{H}^n)}^2 = \|\hat{f}\|_{L^2(\mathbb{R}^*,HS(L^2(\mathbb{R}^n));\mu)}^2. \tag{2.62}$$

Note that we have only shown so far that $h\mapsto\|\hat{f}(h)\|_{HS}$ is square-integrable with respect to μ. But since $HS(L^2(\mathbb{R}^n))$ is a separable Hilbert space (cf. Corollary 2.31), $h\mapsto\hat{f}(h)$ is Bochner-measurable, and thus in $L^2(\mathbb{R}^*,HS(L^2(\mathbb{R}^n));\mu)$, due to Proposition A.21.

Summarizing the facts, we have seen that the group Fourier transform $f\mapsto\hat{f}$ is an isometric linear map from the dense subspace $L^1(\mathbf{H}^n)\cap L^2(\mathbf{H}^n)$ of $L^2(\mathbf{H}^n)$ into $L^2(\mathbb{R}^*,HS(L^2(\mathbb{R}^n));\mu)$. So, we may in fact extend the group Fourier transform to an isometric operator $\mathcal{F}$ defined on the whole space $L^2(\mathbf{H}^n)$.

The mapping $\mathcal{F}$ is furthermore onto: By reading the above manipulations backwards, given $f\in L^2(\mathbb{R}^*,HS(L^2(\mathbb{R}^n));\mu)$, we construct $F_f\in L^2(\mathbf{H}^n)$ such that $\mathcal{F}(F_f)=f$. In fact, f corresponds to a family of Hilbert-Schmidt operators with $L^2(\mathbb{R}^{2n})$-kernels $\{K_f^h\}_{h\in\mathbb{R}^*}$. Now, for $h\in\mathbb{R}^*$ we set

$$F_f(h,q,p) := |h|^n\,\mathcal{F}_{1,2}^{-1}K_f(h,(2q+hp)/2,(2q-hp)/2).$$

Then, for $\varphi\in L^2(\mathbb{R}^n)$ identity (2.60) leads to

$$\begin{aligned}\mathcal{F}(F_f)(h)\varphi(x) &= \int_{\mathbb{R}^n}K_{F_f}^h(x,y)\varphi(y)\,dy = \int_{R^n}|h|^{-n}\,\mathcal{F}_{1,2}F_f(h,\frac{x+y}{2},\frac{x-y}{h})\varphi(y)\,dy\\ &= \int_{\mathbb{R}^n}K_f(h,\frac{2\frac{x+y}{2}+h\frac{x-y}{h}}{2},\frac{2\frac{x+y}{2}-h\frac{x-y}{h}}{2})\varphi(y)\,dy\\ &= \int_{\mathbb{R}^n}K_f(h,x,y)\varphi(y)\,dy = f(h)\varphi(x).\end{aligned}$$

Thus, $\mathcal{F}(F_f)=f$, and we are done. □

We have also an inversion theorem for the group Fourier transform with an explicit formula for the inverse transform.

Theorem 2.57. *For all $f\in\mathcal{S}(\mathbf{H}^n)$, the inverse Fourier transform $\mathcal{F}^{-1}:\mathcal{F}(f)=\hat{f}\mapsto f$ is given by the formula*

$$f(t,q,p) = \int_{\mathbb{R}^*}\operatorname{tr}\left(\hat{f}(h)\pi_h(t,q,p)\right)\,d\mu(h). \tag{2.63}$$

Proof. As in the proof of Proposition 2.55 we will show that $\operatorname{tr}\Big(\hat{f}(h)\pi_h(t,q,p)\Big)$ possesses an integral operator representation, which we subsequently use to prove identity (2.63). To begin with, we compute

$$\begin{aligned}
\hat{f}(h)\pi_h(t,q,p) &= \int_{\mathbf{H}^n} f(t',q',p')\,\pi_h(-t',-q',-p')\pi_h(t,q,p)\,dt'\,dq'\,dp' \\
&= \int_{\mathbf{H}^n} f(t',q',p')\,\pi_h(t-t'+(qp'-pq')/2, q-q', p-p')\,dt'\,dq'\,dp' \\
&= \int_{\mathbf{H}^n} e^{ih(q\tilde{p}-p\tilde{q})/2} f(\tilde{t}+t,\tilde{q}+q,\tilde{p}+p)\,\pi_h(-\tilde{t},-\tilde{q},-\tilde{p})\,d\tilde{t}\,d\tilde{q}\,d\tilde{p},
\end{aligned}$$

where $\tilde{t} := t'-t$, $\tilde{q} := q'-q$, and $\tilde{p} := p'-p$. Hence, we obtain $\hat{f}(h)\pi_h(t,q,p) = \hat{g}(h)$ for the Schwartz function g defined by

$$g(\tilde{t},\tilde{q},\tilde{p}) := e^{ih(q\tilde{p}-p\tilde{q})/2}\, f(\tilde{t}+t,\tilde{q}+q,\tilde{p}+p).$$

Consequently, using the integral operator representation from (2.59), we may express the integral kernel K_g^h of $\hat{g}(h)$ in terms of the integral

$$\begin{aligned}
&K_g^h(x,y) \\
&\quad = |h|^{-n} \int_{\mathbb{R}^{n+1}} e^{-ih\tilde{t}-i\tilde{q}(x+y)/2} e^{i[q(x-y)-hp\tilde{q}]/2} f(\tilde{t}+t,\tilde{q}+q,h^{-1}(x-y)+p)\,d\tilde{t}\,d\tilde{q}.
\end{aligned}$$

Note that $\hat{g}(h)$ is trace class with $\operatorname{tr}(\hat{g}(h)) = \int_{\mathbb{R}^n} K_g^h(x,x)\,dx$ (cf. Kirillov [12], Remark 3 in Subsection IV. 2. 2). The latter formula leads to

$$\begin{aligned}
\int_{\mathbb{R}^n} K_g^h(x,x)\,dx &= |h|^{-n} \int_{\mathbf{H}^n} e^{-ih\tilde{t}-i\tilde{q}x} e^{-ihp\tilde{q}/2} f(\tilde{t}+t,\tilde{q}+q,p)\,d\tilde{t}\,d\tilde{q}\,dx \\
&= |h|^{-n} \int_{\mathbf{H}^n} e^{-ih(t'-t)-i(q'-q)x} e^{-ihp(q'-q)/2} f(t',q',p)\,dx\,dq'\,dt' \\
&= (2\pi)^n\,|h|^{-n} \int_{\mathbb{R}^{n+1}} e^{-ih(t'-t)} e^{-ihp(q'-q)/2}\delta(q'-q) f(t',q',p)\,dq'\,dt' \\
&= (2\pi)^{2n}\,|h|^{-n} \int_{\mathbb{R}} e^{-ih(t'-t)} f(t',q,p)\,dt' \\
&= (2\pi)^{2n}\,|h|^{-n}\, e^{iht}\mathcal{F}_1 f(h,q,p),
\end{aligned}$$

where we have twice used the Fourier inversion theorem (for $\mathbb{R}^n$). Another application finally gives

$$f(t,q,p) = (2\pi)^{-1} \int_{\mathbb{R}} e^{iht}\mathcal{F}_1 f(h,q,p)\,dh = \int_{\mathbb{R}} \operatorname{tr}(\hat{g}(h))\,(2\pi)^{-(2n+1)}\,|h|^n\,dh,$$

and thus the desired result. $\square$

We close this subsection with the observation that convolution under the group Fourier transform behaves as under the ordinary Fourier transform on $\mathbf{H}^n$. That is, for $f_1, f_2 \in L^1(\mathbf{H}^n)$ we have

$$\widehat{f_1 * f_2}(h) = \widehat{f}_1(h)\, \widehat{f}_2(h). \tag{2.64}$$

In order to prove this, note that Proposition B.12 implies that the integrated Schrödinger representation satisfies $\pi_h(f_1 * f_2) = \pi_h(f_1)\, \pi_h(f_2)$ for all $f_1, f_2 \in L^1(\mathbf{H}^n)$. Recalling $\Delta_{\mathbf{H}^n} = 1$ (cf. Remark 2.10) and denoting $\tilde{f} := g \mapsto f(g^{-1})$ for $f \in L^1(\mathbf{H}^n)$, identity (1.16) then yields

$$\begin{aligned}\widehat{f_1 * f_2}(h) &= \int_{\mathbf{H}^n} (f_1 * f_2)(g)\, \pi_h(g^{-1})\, dg = \int_{\mathbf{H}^n} (f_1 * f_2)(g^{-1})\, \pi_h(g)\, dg \\ &= \pi_h(\tilde{f}_1 * \tilde{f}_2) = \pi_h(\tilde{f}_1)\, \pi_h(\tilde{f}_2) = \widehat{f}_1(h)\, \widehat{f}_2(h),\end{aligned}$$

which proves the claim.

A. The Bochner Integral

A.1. Motivation

The topic Banach space-valued functions is an essential one throughout the present text. Since it turns out to be particularly important to integrate these functions, we have to use an appropriate notion of integrability. The concept of our choice is the one due to S. Bochner. His approach can be viewed, in a certain sense, as a straightforward generalization of the Lebesgue integral, which we assume the reader to be familiar with. We will see in the following that the basic definitions are closely related to the corresponding definitions and characterizations from Lebesgue's theory. As a matter of fact, the Bochner integral agrees with Lebesgue's integral in case the functions are complex-valued. However, some important results as, e.g., the Radon-Nikodym theorem, fail for generic Banach spaces. The interested reader is referred to Arendt et al. [2], § 1. 2.

A.2. Measure Space

In order to give proper definitions of measurability and integrability we have to say a few words about requirements on our measure space.

In view of integrating on (second countable) Lie groups (cf. § 1.2.1), we will always suppose that our measure space $(\Omega, \mathfrak{A}, \mu)$ is a locally compact Hausdorff space equipped with a Borel sigma algebra $\mathfrak{A}$ and a regular Borel measure μ. For details on topological spaces we refer to L. A. Steen and J. A. Seebach [18], and to Michael E. Taylor [19] for details in measure theory. One of our main tools for integration, namely Pettis's characterization of Bochner measurable functions, requires that our measure space is σ-finite, i.e., it can be written as a countable union of sets of finite measure. Now, by second countability we are even given the existence of a sequence of compact subsets $\{K_n\}_n$ of Ω such that $K_n \subseteq K_{n+1}^\circ$ for each n, and $\Omega = \bigcup_{n=0}^\infty K_n = \bigcup_{n=0}^\infty K_n^\circ$ (cf. Aliprantis and Border [1], Lemma 2. 76 and Corollary 2. 77). We will refer to this type of sequence as a fundamental sequence for Ω. By definition of regularity of μ, all compact subsets are of finite measure, which implies that Ω is σ-finite.

A.3. Measurability

In accordance with the notation of our main references J. Diestel and J. J. Uhl, [4], and W. Arendt et al. [2], X will always denote a Banach space over $\mathbb{C}$ and X^* its dual space.

Definition A.1. *A function $f : \Omega \to X$ is called* *(i) simple if it is of the form $f = \sum_{i=1}^{n} x_i \cdot \chi_{E_i}$ for some $n \in \mathbb{N}$, $x_i \in X$, and χ_{E_i} being the characteristic function of $E_i \in \mathfrak{A}$ with $\mu(E_i) < \infty$.*

(ii) Bochner measurable (with respect to μ) if there exists a sequence of simple functions $(g_n)_n$ such that $f(\omega) = \lim_{n\to\infty} g_n(\omega)$ for (μ-)almost every $\omega \in \Omega$.

Remark A.2. Note that some well-known properties from the case $X = \mathbb{C}$ continue to hold for arbitrary Banach spaces: if $f, g : \Omega \to X$ and $h : \Omega \to \mathbb{C}$ are Bochner measurable, then $f + g$ and $h \cdot f$ are also Bochner measurable, and for a continuous map k from X into another Banach space Y, $k \circ f : I \to Y$ is also Bochner measurable. In particular, $\|f\|$ is μ-measurable.

In fact, measurability is often checked by an equivalent criterion due to Pettis, which involves a few new notions.

Definition A.3. *We say $f : \Omega \to X$ is*

(i) countably valued if there exists a countable partition $\{E_n\}_n$ of Ω such that f is constant on each E_n.

(ii) almost (or μ-essentially) separably valued if there exists a null set $E_0 \subseteq \Omega$ such that $f(\Omega \setminus E_0)$ is separable or equivalently, $f(\Omega \setminus E_0)$ is contained in a separable closed subspace of X.

(iii) weakly measurable if $x^* \circ f : \Omega \to \mathbb{C}$ is μ-measurable for all $x^* \in X^*$.

Theorem A.4 (Pettis's Theorem). *A function $f : \Omega \to X$ is Bochner measurable if and only if it is weakly measurable and almost separably valued.*

Proof. By Definition A.1 there exist simple functions g_n and a null set E_0 such that $g_n \to f$ on $\Omega \setminus E_0$. Note that for each $x^* \in X^*$, the scalar functions $x^* \circ g_n$ are also simple and converge pointwise to $x^* \circ f$ on $\Omega \setminus E_0$. Hence, f is weakly measurable. The implication is now established by observing that $\bigcup_n g_n(\Omega \setminus E_0) =: D$ is a countable set and $f(\Omega \setminus E_0) \subseteq \overline{D}$, hence f is almost separably valued.

The idea behind the proof of the converse statement is that μ-essential separability of $f(\Omega)$ implies that we can approximate f by countably valued functions which in turn can be cut off to simple functions using a fundamental sequence of compact subsets. By assumption there exists a null set $E_0 \in \mathfrak{A}$ such that f is separably valued on $\Omega \setminus E_0$, and for the sake of convenience we replace X by the smallest closed subspace which contains $f(\Omega \setminus E_0)$. Now, let $\{x_n\}_n$ be a countable dense subset of X. By the Hahn-Banach theorem, there exist unit vectors $x_n^* \in X^*$ with $x_n^*(x_n) = \|x_n\|$. Hence, for each $x \in X$ and each $\varepsilon > 0$ there exists x_k such that $\|x - x_k\| < \varepsilon$. This gives

$$\begin{aligned}\sup_n |x_n^*(x)| \;&\leq\; \|x\| \;\leq\; \|x_k\| + \varepsilon \;=\; |x_k^*(x_k)| + \varepsilon \;\leq\; |x_k^*(x - x_k)| + |x_k^*(x)| + \varepsilon \\ &\leq\; \sup_n |x_n^*(x)| + 2\varepsilon.\end{aligned}$$

Thus, we have
$$\|x\| = \sup_n |x_n^*(x)|$$
for all $x \in X$. By assumption each $|x_n^*(f(\,.\,) - x)|$ is μ-measurable for all $x \in X$, and so is $\sup_n |x_n^*(f(\,.\,) - x)| = \|f(\,.\,) - x\|$.

For fixed $\varepsilon > 0$ and $n \in \mathbb{N}$ set
$$A_{n,\varepsilon} := \{\omega \in \Omega \setminus E_0 \mid \|f(\omega) - x_n\| < \varepsilon\}.$$
By the above these sets are measurable and $\Omega = \bigcup_n A_{n,\varepsilon}$. Furthermore, we observe that the sequence of disjoint sets $E_{1,\varepsilon} := A_{1,\varepsilon}$ and $E_{n,\varepsilon} := A_{n,\varepsilon} \setminus \bigcup_{k<n} A_{k,\varepsilon}$, $n \geq 2$, are in $\mathfrak{A}$, and $\bigsqcup_{n=1}^{\infty} E_{n,\varepsilon} = \Omega$. By $h_\varepsilon := \sum_{n=1}^{\infty} x_n \chi_{E_{n,\varepsilon}}$ we define measurable, countably valued functions which approximate f μ-almost everywhere: For $\omega \in \Omega \setminus E_0$ there exists $n \in \mathbb{N}$ such that $\omega \in E_{n,\varepsilon}$. Thus, we have
$$\|f(\omega) - h_\varepsilon(\omega)\| < \varepsilon \quad \forall \omega \in \Omega \setminus E_0.$$

In order to prove Bochner measurability we now find a way to cut off all but finitely many summands in the representations of the h_ε without losing pointwise convergence a.e. To this end, let $(K_j)_j$ be a fundamental sequence of compact subsets of Ω and let $n \in \mathbb{N}$. Then by regularity of μ and compactness of K_n there exists $k_n \in \mathbb{N}$ such that by setting $H_n := K_n \cap \bigcup_{i=1}^{k_n} E_{i,2^{-n}}$ we obtain $\mu(K_n \setminus H_n) < 2^{-n}$. Next set $g_n := h_{2^{-n}} \chi_{H_n}$, which is obviously simple, and observe that $\omega \in \bigcap_{n=k}^{\infty} H_n$ implies
$$\|f(\omega) - g_n(\omega)\| = \|f(\omega) - h_{2^{-n}}(\omega)\| < 2^{-n},$$
for all $n \geq k$, hence $g_n(\omega) \to f(\omega)$ if $\omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n$. Since for each $k \geq j$ we have
$$\mu\left(K_j \setminus \bigcap_{n=k}^{\infty} H_n\right) \leq \sum_{n=k}^{\infty} \mu(K_n \setminus H_n) < 2^{-k+1},$$
it follows that
$$\mu\left(K_j \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n\right) = \mu\left(\bigcap_{k=1}^{\infty} (K_j \setminus \bigcap_{n=k}^{\infty} H_n)\right) = 0$$
for each $j \in \mathbb{N}$. Thus, we obtain $\lim_{n\to\infty} h_n(\omega) = f(\omega)$ for almost every $\omega \in \Omega$. $\square$

Corollary A.5. *Let $f : \Omega \to X$. Then the following statements hold:*

(i) If X is separable, the f is measurable if and only if it is weakly measurable.

(ii) If f is continuous, then it is measurable.

(iii) If $f_n : \Omega \to X$ are measurable functions and $f_n \to f$ pointwise a.e., then f is measurable.

(vi) *The function is measurable if and only if it is the uniform limit almost everywhere of a sequence of measurable, countably valued functions.*

Proof. (i) is immediate by Theorem A.4.

(ii) If f is continuous, then $x^* \circ f$ is continuous, hence μ-measurable for all $x^* \in X^*$, i.e., weakly measurable. Recall that second countability implies separability and choose a countable dense subset E of Ω. Then by continuity of f, $f(E)$ is a countable dense subset in $f(\Omega)$. Hence, by Theorem A.4 f is Bochner measurable.

(iii) In order to prove measurability, note that the functions $x^* \circ f_n$ are μ-measurable for all $n \in \mathbb{N}$. Since their pointwise limit $x^* \circ f$ is also μ-measurable, f is weakly measurable. Furthermore, there exist null sets E_n such that $f_n(\Omega \setminus E_n)$ is separable. We define $E_0 := \bigcup_{n=1}^{\infty} E_n$. Then $\mu(E_0) = 0$ and $\Delta := \bigcup_{n=1}^{\infty} f_n(\Omega \setminus E_0)$ is separable. Hence, the smallest closed subspace containing Δ is separable and includes $f(\Omega \setminus E_0)$, thus f is almost separably valued, and equivalently measurable by Theorem A.4.

(iv) Both implications were shown in the proof Theorem A.4. □

A.4. Integrability

For a simple function $g : \Omega \to X$, $g = \sum_{i=1}^{n} x_i \chi_{E_i}$, we define its integral by

$$\int_\Omega g(\omega)\, d\mu(\omega) := \sum_{i=1}^{n} x_i \mu(E_i) \in X.$$

As in the scalar-valued case it is routine to verify that the definition is independent of the representation $g = \sum_{i=1}^{n} x_i \chi_{E_i}$ and that the integral is linear in g.

Next we define integrability for arbitrary measurable functions.

Definition A.6. *A function $f : \Omega \to X$ is called Bochner integrable if there exist simple functions g_n such that $g_n \to f$ pointwise a.e., and $\lim_{n\to\infty} \int_\Omega \|f(\omega) - g_n(\omega)\|\, d\mu(\omega) = 0$. If f is Bochner integrable, then the Bochner integral of f on Ω is*

$$\int_\Omega f\, d\mu \;:=\; \int_\Omega f(\omega)\, d\mu(\omega) \;:=\; \lim_{n\to\infty} \int_\Omega g_n(\omega)\, d\mu(\omega).$$

Remark A.7. It is easy to see that the definition is independent of the choice of $(g_n)_n$: Let $(h_n)_n$ be another sequence satisfying the above condition and denote by $\int_\Omega^g f\, d\mu$ and $\int_\Omega^h f\, d\mu$ the corresponding limits. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\int_\Omega \|f - g_n\|\, d\mu < \varepsilon$ and $\int_\Omega \|f - h_m\|\, d\mu < \varepsilon$ for all $n, m \geq N$, which gives $\int_\Omega \|g_n - h_m\|\, d\mu < 2\varepsilon$. Hence, $\left(\int_\Omega g_1\, d\mu, \int_\Omega h_1\, d\mu, \int_\Omega g_2\, d\mu, \ldots\right)$ is Cauchy in the complete space X with convergent subsequence $\left(\int_\Omega g_n\, d\mu\right)_n$, so the whole sequence converges to $\int_\Omega^g f\, d\mu$ and the two integrals of f agree.

Furthermore, we observe the following facts: The integral $\int_\Omega f\, d\mu$ lies in the closed linear span of $\{f(\omega) \mid \omega \in \Omega\}$. The set of all Bochner integrable functions from Ω to X form a linear space denoted by $\mathcal{L}(\Omega, X, d\mu)$ and the Bochner integral is a linear mapping from $\mathcal{L}(\Omega, X, d\mu)$ into X. In case $X = \mathbb{C}$, the definitions of Bochner integrability and integrals agree with those of Lebesgue integration theory.

It is one of the great virtues of the Bochner integral that the class of Bochner integrable functions is very easily characterized.

Theorem A.8 (Bochner)**.** *A function $f : \Omega \to X$ is Bochner integrable if and only if it is Bochner measurable and $\|f\|$ is μ-integrable. If f is Bochner integrable, we have*

$$\left\| \int_\Omega f(\omega)\, d\mu(\omega) \right\| \le \int_\Omega \|f(\omega)\|\, d\mu(\omega). \tag{A.1}$$

Proof. Note that for Bochner integrable f there exists an approximating sequence of simple functions g_n and so f and $\|f\|$ are measurable in the respective sense. Furthermore, we obtain integrability of $\|f\|$ by

$$\int_\Omega \|f(\omega)\|\, d\mu(\omega) \le \int_\Omega \|g_n(\omega)\|\, d\mu(\omega) + \int_\Omega \|f(\omega) - g_n(\omega)\|\, d\mu(\omega)$$

and using the estimate

$$\int_\Omega \big| \|f(\omega)\| - \|g_n(\omega)\| \big|\, d\mu(\omega) \le \int_\Omega \|f(\omega) - g_n(\omega)\|\, d\mu(\omega) \to 0,$$

as $n \to \infty$. Finally we establish estimate (A.1) via

$$\begin{aligned} \left\| \int_\Omega f(\omega)\, d\mu(\omega) \right\| &= \lim_{n\to\infty} \left\| \int_\Omega g_n(\omega)\, d\mu(\omega) \right\| \le \lim_{n\to\infty} \int_\Omega \|g_n(\omega)\|\, d\mu(\omega) \\ &= \int_\Omega \|f(\omega)\|\, d\mu(\omega). \end{aligned}$$

Conversely, since f is measurable, let $(h_n)_n$ be a sequence of simple functions approximating f pointwise on $\Omega \setminus E_0$ for some null set E_0. We define another sequence of simple functions by

$$g_n(\omega) := \begin{cases} h_n(\omega) & \text{if} \quad \|h_n(\omega)\| \le \|f(\omega)\| \, (1 + \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|g_n(\omega)\| \le \|f(\omega)\| \,(1 + \frac{1}{n})$ and $\lim_{n\to\infty} \|f(\omega) - g_n(\omega)\| = 0$ for all $\omega \in \Omega \setminus E_0$. Note that $\|f - g_n\|$ is μ-measurable, and even μ-integrable since $\|f(\omega) - g_n(\omega)\| < 3\,\|f\|$ holds for almost every ω. Finally, we apply Lebesgue's Dominated Convergence Theorem (in the scalar case) to obtain

$$\lim_{n\to\infty} \int_\Omega \|f(\omega) - g_n(\omega)\|\, d\mu(\omega) = 0,$$

thus f is Bochner integrable. □

Examples A.9. (*i*) For $X = L^\infty((0,1))$ we define a function $f : (0,1) \to X$ by $f(t) :=$

$\chi_{(0,t)}$. Then, by Theorem A.4 f is not Bochner measurable with respect to the Lebesgue measure dt hence not Bochner integrable since it is not almost separably valued: Indeed by the fact that $\|f(t) - f(s)\| = 1$ for all $t \neq s$, $f((0,1))$ is not separable.

(*ii*) Let X be the space $c_0(\mathbb{N}) := \{x = (x_n)_n \in \mathbb{C}^{\mathbb{N}} \mid \lim_{n\to\infty} x_n = 0\}$ equipped with the norm $\|x\| := \sup_n |x_n|$. Note that by isometric isomorphy we may identify X^* with $\ell^1(\mathbb{N})$ (cf. Yosida [23], Chapter 9 Example 1). Now, define $f : [0,1] \to c_0(\mathbb{N})$ by $f(t) := (f_n(t))_n$, where $f_n(t) := n\chi_{(0,\frac{1}{n}]}(t)$. For each $x^* = (a_n)_n \in \ell^1(\mathbb{N})$, $x^* \circ f$ is Lebesgue measurable on $[0,1]$ since it is given by $t \mapsto \sum_{n=1}^{\infty} a_n n \chi_{(0,\frac{1}{n}]}(t)$. Hence, separability of $c_0(\mathbb{N})$ together with Corollary A.5 (*i*) gives Bochner measurability of f. (Note that the elements of c_0 can be approximated by finite sequences of rationals, and there exist only countably many of them.) Moreover, we observe that

$$\int_0^1 |x^*(f(t))|\, dt \leq \sum_{n=1}^{\infty} |a_n| = \|x^*\|_{\ell^1} < \infty.$$

However, since $\|f(t)\| = n$ for $t \in (\frac{1}{n+1}, \frac{1}{n}]$, we have

$$\int_0^1 \|f(t)\|\, dt = \sum_{n=1}^{\infty} n \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n+1} = +\infty.$$

Hence, f is not Bochner integrable.

A.5. Main Results for the Bochner Integral

The first half of this section is devoted to the behavior of the Bochner integral under linear operators. We will show that we may interchange integration with the application of bounded linear operators, which of course includes all elements of X^*. In the following we will see that this is also possible for closed operators under certain conditions.

In the second half we will prove Bochner-versions of Fubini's Theorem, Lebesgue's Dominated Convergence Theorem and the Fundamental Theorem of Calculus.

Proposition A.10. *Let X, Y be two Banach spaces, $T \in L(X,Y)$ and let $f : \Omega \to X$ be Bochner integrable. Then $T \circ f : \Omega \to Y$ is also Bochner integrable, and $T\left(\int_\Omega f(\omega)\, d\mu(\omega)\right) = \int_\Omega T(f(\omega))\, d\mu(\omega)$.*

Proof. Let $(g_n)_n$ be an approximating sequence of simple functions for f. Then $T \circ f$ is Bochner measurable (cf. Remark A.2) and also Bochner integrable since we have

$$\int_\Omega \|T(f(\omega)) - T(g_n(\omega))\|\, d\mu(\omega) \leq \|T\| \int_\Omega \|f(\omega) - g_n(\omega)\|\, d\mu(\omega) \to 0$$

and

$$T \circ g_n = T\left(\sum_{i_n=1}^{N_n} x_{i_n} \chi_{E_{i_n}}\right) = \sum_{i_n=1}^{N_n} T(x_{i_n}) \chi_{E i_n}.$$

Finally equality of the integrals follows from

$$\begin{aligned} T\left(\int_\Omega f(\omega)\, d\mu(\omega)\right) &= T\left(\lim_{n\to\infty} \int_\Omega g_n(\omega)\, d\mu(\omega)\right) = \lim_{n\to\infty} T\left(\int_\Omega g_n(\omega)\, d\mu(\omega)\right) \\ &= \lim_{n\to\infty} T\left(\sum_{i_n=1}^{N_n} x_{i_n} \mu(E_{i_n})\right) = \lim_{n\to\infty} \sum_{i_n=1}^{N_n} T(x_{i_n}) \mu(E_{i_n}) \\ &= \lim_{n\to\infty} \int_\Omega T(g_n(\omega))\, d\mu(\omega) = \int_\Omega T(f(\omega))\, d\mu(\omega). \end{aligned}$$

□

Remark A.11. This will be of particular interest when it comes to interchange the integral and the dual action on B or, in case $B = H$ is a Hilbert space, the scalar product on H.

Proposition A.12. *Let A be a closed operator from some Banach space X into another Banach space Y, and let $f : \Omega \to X$ be Bochner integrable. Furthermore, suppose that $f(\Omega) \subseteq \mathcal{D}(A)$ and $A \circ f : \Omega \to Y$ is also Bochner integrable. Then $\int_\Omega f(\omega)\, d\mu(\omega) \in \mathcal{D}(A)$, and*

$$A\left(\int_\Omega f(\omega)\, d\mu(\omega)\right) = \int_\Omega A(f(\omega))\, d\mu(\omega). \tag{A.2}$$

Proof. Recall that the linear space $X \times Y$ equipped with the norm $\|(x,y)\|_1 := \|x\|_X + \|y\|_Y$ is complete and its topology induced by $\|(\,.\,,\,.\,)\|_1$ is equivalent to the product topology (cf. Werner [22], Theorem I. 3. 3). Furthermore, recall that the graph $G(A)$ of A is a closed subspace of $X \times Y$ (cf. Definition 1.11). The function g defined by $g : \Omega \to G(A)$, $g(\omega) := (f(\omega), Af(\omega))$ is obviously Bochner measurable and also Bochner integrable by Theorem A.8 and the estimate

$$\int_\Omega \|g(\omega)\|_1 \; d\mu(\omega) = \int_\Omega \|f(\omega)\|_X \; d\mu(\omega) + \int_\Omega \|Af(\omega)\|_Y \; d\mu(\omega) < \infty.$$

Now, we apply Proposition A.10 to the two (continuous) projection maps of $X \times Y$ onto X and Y, respectively, to obtain

$$\int_\Omega g(\omega)\, d\mu(\omega) = \left(\int_\Omega f(\omega)\, d\mu(\omega), \int_\Omega Af(\omega)\, d\mu(\omega)\right).$$

The fact $\int_\Omega g(\omega)\, d\mu(\omega) \in \overline{\operatorname{lin}\{g(\omega) \mid \omega \in \Omega\}} \subseteq G(A)$ then gives (A.2). □

Theorem A.13 (Fubini). *Let $(\Omega_1, \mathfrak{A}_1, \mu_1)$ and $(\Omega_2, \mathfrak{A}_2, \mu_2)$ be measure spaces, and let $f : \Omega_1 \times \Omega_2 \to X$ be Bochner measurable with respect to the product measure $\mu_1 \otimes \mu_2$. Furthermore, suppose that*

$$\int_{\Omega_1} \int_{\Omega_2} \|f(\omega_1, \omega_2)\| \; d\mu_1(\omega_2)\, d\mu_2(\omega_1) \; < \; \infty. \tag{A.3}$$

Then f is Bochner integrable and we have

$$\begin{aligned} \int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2(\omega_2)\, d\mu_1(\omega_1) \; &= \; \int_{\Omega_2} \int_{\Omega_1} f(\omega_1, \omega_2)\, d\mu_1(\omega_1)\, d\mu_2(\omega_2) \\ &= \; \int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2)\, d(\mu_1 \otimes \mu_2)(\omega_1, \omega 2). \end{aligned}$$

Proof. As in the proof of Theorem A.4, we will suppose w.o.l.g. that X is separable. By the the scalar version of Fubini's Theorem for σ-finite spaces (cf. Taylor [19], Theorem 6.4), (A.3) implies that $\|f\|$ is integrable on $\Omega_1 \times \Omega_2$ and that $\int_{\Omega_2} \|f(\omega_1, \omega_2)\| \; d\mu_2(\omega_2)$ exists for almost all $\omega_1 \in \Omega_1$, hence by Theorem A.8, $f : \Omega_1 \times \Omega_2 \to X$ is Bochner integrable and $\int_{\Omega_2} f(\omega_1, \omega_2)\, d\mu_2(\omega_2)$ exists for almost all $\omega_1 \in \Omega_1$. Now, we observe that for any $x^* \in X^*$, $x^* \circ f$ is μ-integrable on $\Omega_1 \times \Omega_2$ since $|x^* \circ f| \leq \|x^*\| \; \|f\|$, and again by the classical Fubini theorem we have

$$\int_{\Omega_1} \int_{\Omega_2} x^* \circ f \, d\mu_2\, d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} x^* \circ f \, d\mu_1\, d\mu_2 = \int_{\Omega_1 \times \Omega_2} x^* \circ f \, d(\mu_1 \otimes \mu_2), \tag{A.4}$$

which by Proposition A.10 furthermore agree with the integral $\int_{\Omega_1} x^* \left(\int_{\Omega_2} f \, d\mu_2 \right) d\mu_1$. Hence, by Thm.A.4, $\omega_1 \mapsto \int_{\Omega_2} f(\omega_1 \omega_2)\, d\mu_2(\omega_2)$ is Bochner measurable with respect to μ_1, and (A.1) yields

$$\int_{\Omega_1} \left\| \int_{\Omega_2} f \, d\mu_2 \right\| \, d\mu_1 \leq \int_{\Omega_1} \int_{\Omega_2} \|f\| \; d\mu_2\, d\mu_1 < \infty,$$

thus the existence of $\int_{\Omega_1} \left(\int_{\Omega_2} f \, d\mu_2 \right) d\mu_1$ by Theorem A.8. The same argument holds for $\int_{\Omega_2} \left(\int_{\Omega_1} f \, d\mu_1 \right) d\mu_2$ for we know that

$$\int_{\Omega_1} \int_{\Omega_2} \|f\| \; d\mu_2\, d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} \|f\| \; d\mu_1\, d\mu_2.$$

An application of the Hahn-Banach theorem completes the proof since by (A.4) we have

$$\begin{aligned}
x^*\left(\int_{\Omega_1}\int_{\Omega_2} f\,d\mu_2\,d\mu_1\right) &= \int_{\Omega_1}\int_{\Omega_2} x^*\circ f\,d\mu_2\,d\mu_1 = \int_{\Omega_2}\int_{\Omega_1} x^*\circ f\,d\mu_1\,d\mu_2 \\
&= x^*\left(\int_{\Omega_2}\int_{\Omega_1} f\,d\mu_1\,d\mu_2\right) = \int_{\Omega_1\times\Omega_2} x^*\circ f\,d(\mu_1\otimes\mu_2) \\
&= x^*\left(\int_{\Omega_1\times\Omega_2} f\,d(\mu_1\otimes\mu_2)\right)
\end{aligned}$$

for all $x^* \in X^*$. □

Theorem A.14 (Dominated Convergence)**.** *Let $f_n : \Omega \to X$, $n \in \mathbb{N}$, be Bochner integrable functions such that $f(\omega) := \lim_{n\to\infty} f_n(\omega)$ exists for almost all $\omega \in \Omega$. If there exists a μ-integrable function $g : \Omega \to \mathbb{R}$ such that $\|f_n(\omega)\| \le g(\omega)$ a.e. for all $n \in \mathbb{N}$, then f is Bochner integrable, and $\lim_{n\to\infty} \int_\Omega f_n(\omega)\,d\mu(\omega) = \int_\Omega f(\omega)\,d\mu(\omega)$. Moreover, we have $\lim_{n\to\infty} \int_\Omega \|f(\omega) - f_n(\omega)\|\;d\mu(\omega) = 0$.*

Proof. Being the pointwise limit of measurable functions a.e., by Corollary A.5 f is Bochner measurable, and therefore $f - f_n$, too. We observe that both $\|f\|$ and $h_n := \|f - f_n\|$ are dominated by $2g$ a.e., hence they are Bochner integrable by Theorem A.8. Since h_n vanishes, as $n \to \infty$, the scalar version Lebesgue's Dominated Convergence Theorem implies that $\int_\Omega \|f(\omega) - f_n(\omega)\|\;d\mu(\omega) \to 0$. Equation (A.1) finally yields

$$\left\|\int_\Omega f(\omega)\,d\mu(\omega) - \int_\Omega f_n(\omega)\,d\mu(\omega)\right\| \to 0.$$

□

Theorem A.15 (Fundamental Theorem of Calculus)**.** *Let $f : \mathbb{R} \supseteq [a,b] \to X$ be continuous and let $\varphi(x) := \int_a^x f(t)\,dt$, $x \in [a,b]$. Then φ is continuously differentiable on $[a,b]$ with $\varphi' = f$.*

Conversely, for continuously differentiable $f : \mathbb{R} \supseteq [a,b] \to X$, we have

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Proof. In order to show differentiability of φ at $x \in [a,b]$ we must prove the existence of some $\varphi'(x) \in L(\mathbb{R}, X)$ such that $\|\varphi(x+h) - \varphi(x) - \varphi'(x)h\| \to 0$, as $h \to 0$ (cf. Werner [22], Definition III. 5. 1 and Lemma III. 5. 2). Of course, we choose $\varphi' = f$ to prove this condition: Let $h > 0$ be sufficiently small so that $\|f(x) - f(t)\| < 1$ for all

$t \in B(x,h)$. Then by (A.1) we have

$$\begin{aligned}\|\varphi(x+h)-\varphi(x)-f(x)h\| &= \left\| \int_x^{x+h} (f(t)-f(x))\,dt \right\| \\ &\le \int_x^{x+h} \|(f(t)-f(x))\|\ dt < h \to 0.\end{aligned}$$

The proof of the second statement involves the fundamental theorem of calculus for real-valued C^1-functions. To this end, let $x^* \in X^*$. We may assume that $x^*(X) \subseteq \mathbb{R}$ since otherwise we can treat real and imaginary parts separately. Applying the fundamental theorem of calculus to $x^* \circ f$ and Proposition A.10 to the integral, we obtain

$$\begin{aligned} x^*\left(f(b)-f(a)\right) = x^*(f(b)) - x^*(f(a)) = \int_a^b (x^* \circ f)'(t) = \int_a^b x^* \circ f'(t)\,dt = \\ = x^*\left(\int_a^b f(t)\,dt\right). \end{aligned}$$

Since x^* was arbitrary, the Hahn-Banach theorem gives the desired result. □

A.6. $L^p(\Omega, X; \mu)$-spaces

Definition A.16. *For $1 \le p \le \infty$, we denote by $L^p(\Omega, X; \mu)$ the space of all (equivalence classes of) Bochner measurable functions $f : \Omega \to X$ such that*

$$\|f\|_{L^p(\Omega,X;\mu)} := \begin{cases} \left(\int_\Omega \|f(\omega)\|_X^p\ d\mu(\omega)\right)^{1/p} < \infty & \text{if } 1 \le p < \infty, \\ \inf\{c \in \mathbb{R} \mid \mu(\{\|f\|_X > c\}) = 0\} < \infty & \text{if } p = \infty. \end{cases}$$

The spaces $L^p(\Omega, X; \mu)$ are usually called Bochner-Lebesgue or Lebesgue-Bochner spaces.

Proposition A.17. *The spaces $L^p(\Omega, X; \mu)$, $1 \le p \le \infty$, are normed vector spaces.*

The proof is omitted due to similarity to the scalar-valued version.

As a matter of fact, these spaces are even Banach spaces as shown in the following.

Theorem A.18. *The normed vector space $L^p(\Omega, X; \mu)$, $1 \le p \le \infty$, is complete.*

Proof. $1 \le p < \infty$:

Let $(f_n)_n$ be a Cauchy sequence in $L^p(\Omega, X; \mu)$. Then choose indices $n_1 < n_2 < \dots$ so that $\|f_n - f_m\|_{L^p(\Omega,X;\mu)} < \frac{1}{2^k}$ for all $n, m \ge n_k$, $k = 1, 2, \dots$, and define the Bochner measurable functions $g_i := f_{n_{i+1}} - f_{n_i}$. We observe that

$$\left\| \sum_{i=1}^N \|g_i\|_X \right\|_{L^p(\Omega,\mathbb{C})} \le \sum_{i=1}^N \| \|g_i\|_X \|_{L^p(\Omega,\mathbb{C})} = \sum_{i=1}^N \|g_i\|_{L^p(\Omega,X;\mu)} < 1 \tag{A.5}$$

for all $N \in \mathbb{N}$. The scalar-valued lemma of Fatou together with (A.5) gives

$$\int_\Omega \left(\sum_{i=1}^{\infty} \|g_i\|_X \right)^p d\mu \leq \lim_{N\to\infty} \int_\Omega \left(\sum_{i=1}^{N} \|g_i\|_X \right)^p d\mu \leq 1.$$

Thus, $\sum_{i=1}^{\infty} \|g_i\|_X \in L^p(\Omega, \mathbb{C})$, which implies that $\sum_{i=1}^{\infty} \|g_i(\omega)\|_X < \infty$ for almost all $\omega \in \Omega$. Hence, $\sum_{i=1}^{\infty} g_i(\omega)$ converges a.e. in X, and $\sum_{i=1}^{\infty} g_i$ is Bochner measurable by Corollary A.5 *(iii)*. Next, we define $f_{n_k} := f_{n_1} + \sum_{i=1}^{k} g_i = f_{n_1} + \sum_{i=1}^{k} f_{n_{i+1}} - f_{n_i}$, and

$$f(\omega) := \begin{cases} \lim_{k\to\infty} f_{n_k}(\omega) & \text{if it exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, f is Bochner measurable, and we will show that it is also the limit of $(f_n)_n$. To this end let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|f_{n_k} - f_m\|_{L^p(\Omega,X;\mu)} < \varepsilon$ for all $n_k, m \geq N$. Another application of the scalar-valued lemma of Fatou gives

$$\int_\Omega \|f - f_m\|_X^p \, d\mu \leq \liminf_{k\to\infty} \int_\Omega \|f_{n_k} - f_m\|_X^p \, d\mu < \varepsilon.$$

That is, $f - f_m \in L^p(\Omega, X; \mu)$. Hence $f_n \to f \in L^p(\Omega, X; \mu)$, as $n \to \infty$.

$p = \infty$:

Let $(f_n)_n$ be a Cauchy sequence in $L^\infty(\Omega, X)$, hence there exists $M \in \mathbb{R}^+$ such that $\|f_n\|_{L^\infty(\Omega,X)} \leq M$ for all $n \in \mathbb{N}$. Moreover, we know that $\|f_m - f_n\|_{L^\infty(\Omega,X)} = \operatorname{ess\,sup} \|f_m(\omega) - f_n(\omega)\|_X \to 0$, as $m, n \to \infty$. Thus, there exists a null set $E \in \mathfrak{A}$ such that $(f_n(\omega))_n$ is a Cauchy sequence for all $\omega \in E^c$, and we may define

$$f(\omega) := \begin{cases} \lim_{n\to\infty} f_n(\omega) & \text{if } \omega \in E^c, \\ 0 & \text{otherwise.} \end{cases}$$

So, f is Bochner measurable by Corollary A.5 *(iii)*. Since it is bounded by M almost everywhere it is an element of $L^\infty(\Omega, X)$, and obviously $f_n \to f \in L^\infty(\Omega, X)$. □

Remark A.19. Let $X = H$ be a Hilbert space with scalar product $\langle \,.\, , \,.\, \rangle_H$. Then $L^2(\Omega, H; \mu)$ is also a Hilbert space with scalar product

$$\langle f, g \rangle_{L^2(\Omega,H;\mu)} := \int_\Omega \langle f(\omega), g(\omega) \rangle_H \, d\mu(\omega)$$

for $f, g \in L^2(\Omega, H; \mu)$.

Remark A.20. Furthermore, it is clear that $L^p(\Omega, X; \mu)$, $1 \leq p \leq \infty$, agrees with the usual Lebesgue space $L^p(\Omega, \mu)$ in case $X = \mathbb{C}$.

We close this chapter with a proposition on square-integrable Hilbert space-valued functions.

Proposition A.21. *Let H be a separable Hilbert space and let $f : \Omega \to H$. If $\|f\|_H$ is μ-measurable with $\int_\Omega \|f(\omega)\|_H^2 \, d\mu(\omega) < \infty$, then f is Bochner-measurable, and we have $f \in L^2(\Omega, H; \mu)$.*

Proof. Using the polarization identity for $\langle \,.\, , \,.\, \rangle_{L^2(\Omega,H;\mu)}$, we obtain $|\langle f, g\rangle_{L^2(\Omega,X;\mu)}| < \infty$ for all $g \in L^2(\Omega, X; \mu)$. Hence, $\omega \mapsto \langle f(\omega), g(\omega)\rangle_H$ is μ-integrable, thus is particular μ-measurable, for all $g \in L^2(\Omega, X; \mu)$. Now, if $(K_j)_j$ is a fundamental sequence of compact subsets of Ω, it follows that $\omega \mapsto \big\langle f(\omega), x \cdot \chi_{K_j}(\omega)\big\rangle_H = \big\langle (f \cdot \chi_{K_j})(\omega), x\big\rangle_H$ is μ-measurable for all $x \in H \cong H'$ and all $j \in \mathbb{N}$. Hence, all $f_j := f \cdot \chi_{K_j}$ are weakly measurable, and since H is separable, they are also Bochner-measurable by Corollary A.5 (i). Corollary A.5 (iii) finally yields Bochner measurability of f, being the pointwise limit a.e. of the measurable functions $(f_j)_j$. □

For more details on Bochner integration in case Ω is a generic interval $I \subseteq \mathbb{R}$ we refer to Arendt et al. [2], § 1.

B. On the representations of Banach algebras

The main object of this appendix is a brief discussion of different types of Banach algebras and their representations with the aim of comparing them with Lie group representations. The author's motivation for this illustration is given by the correspondence between unitary Lie group representations $\pi : G \to H$ and induced algebra representations of the convolution algebra $L^1(G)$ on H.

B.1. Banach Algebras

Recall that an algebra $(A, +, \cdot\,)$ is defined to be vector space A over some field $\mathbb{F}$ with a bilinear multiplicative operation $A \times A \to A : (x, y) \mapsto xy$ which satisfies

$$\begin{aligned} (xy)z &= x(yz) \quad \text{(associativity)}, \\ x(y+z) &= xy + xz \quad \text{(distributivity)}, \\ \lambda\mu(xy) &= (\lambda x)(\mu y) \end{aligned}$$

for all $x, y \in A$ and all $\lambda, \mu \in \mathbb{F}$. From its definition, it is clear that $(A, \cdot\,)$ only forms a semi-group. If there exists a unit element $e \in A$ such that $ex = xe = x$ for all $x \in A$, then A is called algebra with unit or unital algebra. For applications it is often very important that an algebra carries a norm. On the other hand, some normed spaces turn out to provide a multiplicative operation, which in case of function spaces is often given by pointwise multiplication. Since continuity of vector addition and scalar multiplication is an essential feature of normed spaces (which is satisfied by definition), it is sensible to postulate continuity for multiplication. This can be provided, e.g., by imposing a bound on the product.

Definition B.1. *A normed algebra (Banach algebra) A is a normed vector space (Banach space) over $\mathbb{K}$, which is also an algebra satisfying*

$$\|xy\| \leq \|x\| \cdot \|y\| \tag{B.1}$$

for all $x, y \in A$.

In fact, condition (B.1) implies (joint) continuity of multiplication since for $x_n \to x$ and $y_n \to y$ we have

$$\|x_n y_n - xy\| = \|x_n y_n - x y_n + x y_n - xy\| \leq \|x - x_n\| \, \|y_n\| + \|x\| \, \|y_n - y\| \to 0.$$

If A has a multiplicative unit e, applications often require $\|e\| = 1$.

Definition B.2. *A commutative unital Banach algebra A with $\|e\| = 1$ is called a normed ring.*

For more information on normed rings see Yosida [23], Chapter XI.

As a matter of fact, some very well-known Banach spaces are even Banach algebras. A few of them and other prominent examples shall be illustrated next.

Examples B.3. (i) The archetype of Banach algebra is $L(B)$, the set of bounded operators, on a (real or complex) Banach space B, equipped with the operator norm. Scalar multiplication and addition of its elements are defined pointwise, whereas multiplication is given by composition of the operators. It clearly satisfies condition (B.1). Note that $L(B)$ is non-commutative for $\dim(B) > 1$. Moreover, it possesses a multiplicative unit element, namely the identity map. Furthermore, note that the special case $L(H)$, where H is a complex Hilbert space, in fact inspired various important notions in the context of Banach algebras (cf. Definition B.4 and Definition B.5).

(ii) The space $K(B)$ of compact linear operators on B is a closed two-sided ideal in $L(B)$. Hence, it is a closed Banach sub-algebra of $L(B)$ and, in particular, a Banach algebra itself, which is of course non-commutative for $\dim(B) > 1$. Nevertheless, it does not share the multiplicative unit with $L(B)$ if $\dim(B) = \infty$. (Otherwise, the closed unit sphere S_B would be mapped compactly onto itself, which contradicts the Lemma of Riesz and its consequences, cf. Werner [22], Lemma I. 2. 6 and Theorem I. 2. 7.)

(iii) Let X be a compact Hausdorff space. Then $(C(X,\mathbb{C}), \|\,.\,\|_\infty)$ together with pointwise addition, multiplication and scalar multiplication defines a normed ring with unit $X \to \mathbb{C} : x \mapsto 1$.

(iv) For locally compact X we define $C_0(X) := \{f \in C(X) \mid \forall \varepsilon\, \exists K \subset\subset X \text{ s.t. } \left\|f|_{X\setminus K}\right\|_\infty < \varepsilon\}$. Carrying the sup-norm it is a commutative Banach algebra without unit.

(v) Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and let $A(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$ (which are continuous on $\overline{\mathbb{D}}$). If we define multiplication again pointwise, then $(A(\mathbb{D}), \|\,.\,\|_\infty)$ is a normed ring. (Completeness of $A(\mathbb{D})$ is due to the fact that $A(\mathbb{D}) \subseteq C(\overline{\mathbb{D}})$ is closed in the sup-norm.)

(vi) Let $(\Omega, \mathfrak{A}, \mu)$ be a measure space. Then $L^\infty(\Omega, \mu)$ is also a normed ring with unit 1 with multiplication defined pointwise almost everywhere.

(vii) Consider the Banach space $\ell^1(\mathbb{Z})$ of bilateral absolutely convergent complex series. Then convolution

$$x * y(n) := \sum_{k+l=n} x(k)y(l), \quad n \in \mathbb{Z},$$

defines a continuous commutative algebra multiplication on $\ell^1(\mathbb{Z})$ (for continuity see ($viii$)) and the Banach algebra $(\ell^1(\mathbb{Z}), +, *, \|\,.\,\|_{\ell^1})$ possesses a unit element given by $(\ldots, 0, 1, 0, \ldots)$, where 1 is the 0-th digit.

(*viii*) Now, we extend Example (*vii*) to arbitrary Lie groups G. To begin with, we define convolution for $f_1, f_2 \in L^1(G)$ (cf. (1.19)) by

$$f_1 * f_2(g) := \int_G f_1(h) f_2(h^{-1}g)\, dh.$$

We observe that convolution satisfies (B.1) since we may apply Tonelli's theorem for σ-finite spaces (cf. § refMeasSp and Taylor [19], Theorem 6. 3) to obtain

$$\begin{aligned}\|f_1 * f_2\|_{L^1} &= \int_G \left| \int_G f_1(h) f_2(h^{-1}g)\, dh \right| dg \le \int_G \int_G \left| f_1(g) f_2(h^{-1}g) \right| dh\, dg \\ &= \int_G |f_1(h)| \int_G \left| f_2(h^{-1}g) \right| dg\, dh = \|f_1\|_{L^1} \|f_2\|_{L^1}\end{aligned}$$

Hence, $(L^1(G), +, *, \|\cdot\|_{L^1})$ forms a Banach algebra which is commutative if G is commutative. The existence of a multiplicative unit is not given in general, not even in case $G = (\mathbb{R}, +)$.

In analogy to the map $L(H) \to L(H) : T \mapsto T^*$, we define another operation on real and complex algebras.

Definition B.4. *Let A be an algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $A \to A : x \mapsto x^*$ with the properties*

(i) $(x+y)^* = x^* + y^*$,

(ii) $(\lambda x)^* = \overline{\lambda} x^*$,

(iii) $(xy)^* = y^* x^*$,

(iv) $(x^*)^* = x$,

for all $x, y \in A$ and all $\lambda \in \mathbb{K}$ (where complex conjugation is to be understood as the identity on $\mathbb{R}$ in the real case) is called an involution (on A). A is then called involutive algebra or *-algebra with involution $x \mapsto x^*$.

If A is also a normed algebra (Banach algebra) with

(v) $\|x^*\| = \|x\|$ *for all $x \in A$,*

then A is called a normed *-algebra (Banach *-algebra).

Trivially, every real (normed) algebra A is a (normed) *-algebra with involution $id : A \to A$.

Definition B.5. *Let A be a Banach *-algebra. Then A is said to be a C^*-algebra if it also satisfies*

(vi) $\|x^* x\| = \|x\|^2$ *for all $x \in A$.*

Remark B.6. Note that every C^*-algebra is a Banach *-algebra: $\|x\|^2 = \|x^*x\| \leq \|x^*\|\,\|x\|$ implies $\|x\| \leq \|x^*\|$, and by the above have $\|x^*\| \leq \|x^{**}\| = \|x\|$.

Conditions (v) and (vi) are also inspired by the key-example $L(H)$. In fact, $L(H)$ is a non-commutative unital C^*-algebra. Note that the algebras from Example B.3 (iii), (iv) and (vi) are commutative C^*-algebras with involution $f \mapsto \overline{f}$, and $L^1(G)$ from Example B.3 $(viii)$ is a Banach *-algebra as we will see in the proof of Proposition B.12.

B.2. Algebra Representations

In the following we introduce the notion of algebra representations and compare them with Lie group representations. Moreover, we give a concrete example of a representation of the Banach *-algebra $L^1(G)$.

Definition B.7. *Let A and B be algebras over $\mathbb{F}$. A map $\pi : A \to B$ is called an algebra homomorphism if it satisfies the conditions $\pi(x+y) = \pi(x) + \pi(y)$, $\pi(xy) = \pi(x)\,\pi(y)$, and $\pi(\lambda x) = \lambda\pi(x)$ for all $x, y \in A$ and all $\lambda \in \mathbb{F}$. A bijective algebra homomorphism is called algebra isomorphism.*

Definition B.8. *Let A be an algebra over $\mathbb{F}$ and E be an $\mathbb{F}$-vector space. A representation is an algebra homomorphism π from A to the algebra $\mathrm{Lin}(E)$.*

We say π is

(i) non-degenerate if for every $u \in E$, $\pi(x)u = 0$ for all $x \in A$ implies $u = 0$.

(ii) invariant on a subspace $E_1 \subseteq E$ if $\pi(A)(E_1) \subseteq E_1$

(iii) faithful if it is injective on A,

(iv) trivial if $\pi(x) = 0 \in L(E)$ for all $x \in A$,

(v) irreducible if is non-trivial and the only invariant subspaces are E and $\{0\}$,

(vi) equivalent to a representation ρ of A on a vector space F if there exists a vector-space isomorphism $V : E \to F$ (called equivalence) such that $\pi(x) = V\rho(x)V^{-1}$ for all $x \in A$.

In case that A is also involutive, we must evidently adapt the definition. Since the adjoint of an operator $T \in \mathrm{Lin}(E)$ is a linear map on the algebraic dual of E, we choose E to be a Hilbert space. If we furthermore wish to have some notion of continuity for our representations, it is sensible to stick to Banach $*$-algebras.

Definition B.9. *Let A and B be involutive algebras. A *-homomorphism (isomorphism) from A to B is an algebra homomorphism (isomorphism) $\pi : A \to L(H)$ that satisfies $\pi(x^*) = \pi(x)^*$ for all $x \in A$.*

Definition B.10. *A *-representation of a Banach *-algebra A on a Hilbert space H is a *-homomorphism from A into the Banach *-algebra $L(H)$.*

Remark B.11. A comparison of Definition 1.42 and Definition B.10 shows that an algebra representation formally satisfies (i) and (ii) of Definition 1.42 for the commutative group $(A,+)$, and (i) for $(A,\,\cdot\,)$. If A is unital and π is non-degenerate, then $\pi(e)=I$ (cf. Arveson [3], §. 2. 5, Example 2), hence (ii) also holds for $(A,\,\cdot\,)$. For sure, the most peculiar difference between Definition 1.42 and Definition B.10 is that we do not postulate any form of continuity of π as a map from the Banach space A into $L(H)$. This is due to the remarkable fact that every *-representation $\pi : A \to L(H)$ automatically satisfies $\|\pi\| \leq 1$ (cf. Arveson [3], Theorem 2. 5. 5). From this it is clear that every *-representation is even norm continuous.

In what follows we will show that $L^1(G)$ is a Banach *-algebra and that each unitary Lie group representation π of G on H induces a non-degenerate *-representation of $L^1(G)$ on H, which is called the integrated representation of π (cf. (1.18)). In particular, this holds for the Schrödinger representation of the Heisenberg group $\mathbf{H}^n$ on $L^2(\mathbb{R}^{2n})$ (cf. (2.31)).

To begin with, we will show that $L^1(G)$ is a Banach algebra by proving condition (B.1) from Definition B.1. To this end, let $f_1, f_2 \in L^1(G)$. An application of the Fubini-Tonelli theorem and left-invariance of the Haar measure dg then yield the required inequality

$$\begin{aligned}\|f_1 * f_2\|_{L^1(G)} &= \int_G \left|\int_G f_1(h) f_2(h^{-1}g)\,dh\right| dg \leq \int_G\int_G \left|f_1(h)f_2(h^{-1}g)\right| dh\,dg \\ &= \int_G\int_G \left|f_2(h^{-1}g)\right| dg\,|f_1(h)|\,dh = \|f_1\|_{L^1(G)}\,\|f_2\|_{L^1(G)}\,. \end{aligned} \tag{B.2}$$

Next we prove the existence of an involution on $L^1(G)$. Recalling from Remark 1.50 that the modular function $\Delta_G : G \to (\mathbb{R}^*,\,\cdot\,)$ is a continuous group homomorphism with $\int_G f(g)\,dg = \int_G f(g^{-1})\Delta(g^{-1})\,dg$, we state the map

$$\begin{aligned} {}^* :\; L^1(G) &\to L^1(G), \\ f &\mapsto (g \mapsto f^*(g) := \overline{f}(g^{-1})\Delta(g^{-1})) \end{aligned}$$

satisfies the conditions (i) - (v) from Definition B.4.

(i) and (ii) are easily seen.

(iii) Note that a change of variables and left-invariance of the Haar integral gives the following identity for the convolution:

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)\,dh = \int_G f_1(gh) f_2(h^{-1})\,dh.$$

Hence, we have

$$\begin{aligned}(f_2^* * f_1^*)(g) &= \int_G f_2^*(gh) f_1^*(h^{-1})\,dh = \int_G \overline{f_2}(h^{-1}g^{-1})\overline{f_1}(h)\,\Delta(h^{-1}g^{-1})\Delta(h)\,dh \\ &= \overline{\int_G f_1(h) f_2(h^{-1}g^{-1})\,dh}\,\Delta(g^{-1}) = \overline{f_1 * f_2}(g^{-1})\,\Delta(g^{-1}) = (f_1 * f_2)^*(g).\end{aligned}$$

for all $g \in G$.

(iv) $(f^*)^*(g) = \overline{f^*}(g^{-1})\,\Delta(g^{-1}) = \overline{\overline{f}}(g)\,\Delta(g)\Delta(g^{-1}) = f(g)$ for all $g \in G$.

(v) $\|f^*\|_{L^1} = \int_G |f^*(g)|\,dg = \int_G \left|\overline{f}(g^{-1})\right|\,\Delta(g^{-1})\,dg = \int_G |f(g)|\,dg = \|f\|_{L^1}$.

Now, for $f \in L^1(G)$ and $v \in H$, we define the induced representation by

$$\begin{aligned} \pi : L^1(G) &\to L(H), \\ f &\mapsto (v \mapsto \int_G f(g)\pi(g)v\,dg). \end{aligned}$$

The map π is linear and bounded, with $\|\pi(f)\| \leq \|f\|_{L^1}$. Moreover π defines an algebra representation since it respects convolution: Let $f_1, f_2 \in L^1(G)$ and $v \in H$. By the $L^1(G)$-estimate on $f(g,h) := f_1(h)f_2(h^{-1}g)$ from Example B.3 $(viii)$ and the fact that $\|\pi(g)\| \leq 1$ for all $g \in G$ we may apply Theorem A.13 to change the order of integration in

$$\begin{aligned} \pi(f_1 * f_2)v &= \int_G \int_G f_1(h)f_2(h^{-1}g)\,dh\,\pi(g)v\,dg \\ &= \int_G f_1(h)\pi(h) \int_G f_2(h^{-1}g)\pi(h^{-1}g)v\,dg\,dh = \pi(f_1)\pi(f_2)v \end{aligned} \tag{B.3}$$

We conclude $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$.

Non-degeneracy is due to the following observation: Let $f_j \in C_c^\infty(G)$, $j \in \mathbb{N}$, be an approximating identity as in the proof of Theorem 1.56. Supposing $\pi(f)u = 0$ for all $f \in L^1(G)$, we obtain

$$0 = \lim_{j\to\infty} \pi(f_j)u = \lim_{j\to\infty} \int_G f_j(g)\pi(g)u\,dg = \pi(e)u = u,$$

which is precisely the condition we had to prove.

A final calculation shows that π also commutes with involution. To this end, let u and v be arbitrary vectors in H and $f \in L^1(G)$. Repeated applications of Proposition A.10 give

$$\begin{aligned} \langle u, \pi(f)^*v \rangle &= \left\langle \int_G f(g)\pi(g)u\,dg, v \right\rangle = \int_G \langle \pi(g)u, v \rangle\, f(g)\,dg \\ &= \int_G \left\langle \pi(g^{-1})u, v \right\rangle f(g^{-1})\,\Delta(g^{-1})\,dg = \int_G \left\langle \pi(g)^*u, \overline{f}(g^{-1})\,\Delta(g^{-1})v \right\rangle\,dg \\ &= \left\langle u, \int_G f^*(g)\pi(g)v\,dg \right\rangle = \langle u, \pi(f^*)v \rangle\,. \end{aligned}$$

Hence, we have proved the following proposition.

Proposition B.12. *Each unitary Lie group representation $\pi : G \to H$ induces a non-degenerate *-representation of the Banach *-algebra $L^1(G)$ on H.*

C. Spectral Theory and Applications

This appendix is thought to give a brief overview of the main results of spectral theory of self-adjoint operators (cf. Definition 1.22), which we eventually use to prove Stone's Theorem and Schur's Lemma. Note that throughout this chapter, H denotes a complex Hilbert space.

C.1. Spectral Theorem

Since a complete discussion of spectral theory clearly is out of reach in this work, we will not prove any result of what is to be presented in this section but give exact references to our main source Weidmann [21]. For the sake of brevity we will presume some familiarity with spectral theory of compact operators.

Recall that the spectral theorem for compact self-adjoint operators (cf. Weidmann [21] Theorem 7. 1) states that for a compact self-adjoint operator A on H there exist non-zero eigenvalues $|\lambda_0| \geq |\lambda_1| \geq \ldots$ such that

$$Ax = \sum_{k=0}^{\nu(A)} \lambda_k \langle x, e_k \rangle e_k \quad \forall x \in H, \tag{C.1}$$

where the e_k are the corresponding eigenvectors and $\nu(A)$ is the number of non-zero eigenvalues of A counted with their multiplicities. Equivalently, if we count every eigenvalue λ_k only once and define $P_k : H \to H$ to be the orthogonal projection onto the eigenspace of λ_k, we can rewrite (C.1) as

$$A = \sum_{k=0}^{\nu(A)} \lambda_k P_k. \tag{C.2}$$

Now the question is what happens in the more general case of bounded or unbounded self-adjoint operators. As so often in mathematics, the discrete sum has to be replaced by an integral, to be precise a projection-valued integral, since the spectrum of A (cf. Definition 1.12) is that simple any longer. To this purpose, we introduce a family of projections that will eventually induce our spectral measure.

Definition C.1. *A spectral family on H is a mapping $E : \mathbb{R} \to L(H)$ with the following properties:*

(a) $E(t)$ is an orthogonal projection for every $t \in \mathbb{R}$,

(b) $E(s) \leq E(t)$ for $s \leq t$,

(c) $\lim_{\varepsilon\to 0^+} E(t+\varepsilon)x = E(t)x \ \ \forall x \in H$, *i.e.,* E *is continuous from the right in the strong operator topology (we will write s-lim),*

(d) $s - \lim_{t\to-\infty} E(t) = 0$, $s - \lim_{t\to+\infty} E(t) = I$.

Example C.2. Since Definition C.1 was motivated by generalizing (C.2), it is not surprising that we find a spectral family for a given compact self-adjoint operator A, namely

$$E(t) := \sum_{\{k|\lambda_k\le t\}} P_k, \quad t \in \mathbb{R}.$$

The proof of properties (a), (b), and (d) is immediate. Thus, it remains to prove (c). For $x \in H$, $t \in \mathbb{R}$ and $\varepsilon > 0$ we compute

$$\|E(t+\varepsilon)x - E(t)x\|^2 = \sum_{\{k|t<\lambda_k\le t+\varepsilon\}} \|P_k x\|^2 . \tag{C.3}$$

Since $\sum_k \|P_k x\|^2$ is convergent and since for every $N \in \mathbb{N}$ there exists some $\varepsilon > 0$ such that $\lambda_k \notin (t, t+\varepsilon]$ for all $k \le N$, the sum in (C.3) converges to zero as $\varepsilon \to 0$.

Setting $\rho_x(t) := \langle E(t)x, x\rangle = \|E(t)x\|^2$, we obtain a non-negative, bounded, non-decreasing, right-continuous function on $\mathbb{R}$ with $\lim_{t\to-\infty}\rho_x(t) = 0$ and $\lim_{t\to+\infty}\rho_x(t) = \|x\|^2$. Hence, $\mu_x^*((a,b]) := \rho_x(b) - \rho_x(a)$ defines a pre-measure on the algebra of subintervals of $\mathbb{R}$ which extends to a Lebesgue-Stieltjes measure μ_x on $\mathcal{B}_{\mathbb{R}}$, the Borel sigma algebra of $\mathbb{R}$ (cf. Taylor [19], Theorem 5. 4 and Exercise 1, Chapter 5).

We say $u : \mathbb{R} \to \mathbb{C}$ is E-measurable if and only if it is μ_x-measurable for all $x \in H$. In particular, all continuous functions, all simple functions and their pointwise limits are E-measurable for every spectral family E. For a step function $u = \sum_{j=1}^n c_j \chi_{I_j}$, with intervals I_j and $c_j \in \mathbb{C}$, we define its integral by

$$\int u(t)\, dE(t) := \int_{\mathbb{R}} u(t)\, dE(t) := \sum_{j=1}^n c_j E(I_j),$$

where we set

$$\begin{aligned} E((a,b]) &:= E(b) - E(a), \quad E((a,b)) := E(b^-) - E(a) \\ E([a,b]) &:= E(b) - E(a^-), \quad E([a,b)) := E(b^-) - E(a^-), \end{aligned}$$

and $E(t^-) := s - \lim_{\varepsilon\to 0^+} E(t-\varepsilon)$.

This integral is obviously a bounded operator on H and satisfies

$$\left\| \int u(t)\, dE(t)x \right\|^2 = \int |u(t)|^2\, d\rho_x(t) \tag{C.4}$$

for all step functions u and all $x \in H$. If $u \in L^2(\mathbb{R}, \rho_x)$ for some $x \in H$, then there exists a sequence $(u_n)_n$ of step functions for which $u_n \to u$ in $L^2(\mathbb{R}, \rho_x)$. Moreover, for this sequence we have

$$\left\| \int u_n(t)\, dE(t)x - \int u_m(t)\, dE(t)x \right\|^2 = \left\| \int (u_n(t) - u_m(t))\, dE(t)x \right\|^2$$
$$= \int |u_n(t) - u_m(t)|^2 \, d\rho_x(t) \quad \to \quad 0 \quad (n, m \to \infty),$$

whence we conclude that $\left(\int u_n(t)\, dE(t)x\right)_n$ is a Cauchy sequence in H. This allows us to define

$$\int u(t)\, dE(t)x \;:=\; \lim_{n\to\infty} \int u_n(t)\, dE(t)x, \tag{C.5}$$

which is easily seen to be independent of the choice of $(u_n)_n$. By continuity of the norms, (C.4) results in

$$\left\| \int u(t)\, dE(t)x \right\|^2 \;=\; \int |u(t)|^2 \, d\rho_x \;=\; \|u\|^2_{L^2(\mathbb{R}, d\rho_x)} \,. \tag{C.6}$$

So, the integral defined by (C.5) is a linear isometry from $L^2(\mathbb{R}, \rho_x)$ into H. On the other hand, if we concentrate on the action of x and fix u, equation (C.5) gives rise to the operator

$$\begin{aligned} \hat{E}(u) : H \supseteq \mathcal{D}(\hat{E}(u)) &\to H, \\ x &\mapsto \int u(t)\, dE(t)x, \end{aligned} \tag{C.7}$$

where its domain is specified by

$$\mathcal{D}(\hat{E}(u)) \;:=\; \{x \in H \mid u \in L^2(\mathbb{R}, \rho_x)\}.$$

The operator $\hat{E}(u)$ is an essential tool in spectral theory and its applications. In what follows we give a list of its properties, some of which will be needed for the proof of Stone's Theorem.

Theorem C.3. *Let E be spectral family on H and let $u : \mathbb{R} \to \mathbb{C}$ be an E-measurable function. Then (C.7) defines a normal operator on $\mathcal{D}(\hat{E}(u))$. If $v : \mathbb{R} \to \mathbb{C}$ is another E-measurable function, $a, b \in \mathbb{C}$, and $\varphi_n : \mathbb{C} \to \mathbb{C}$,*

$$\varphi_n(z) \;=\; \begin{cases} z & \text{if} \;\; |z| \le n, \\ 0 & \text{otherwise}, \end{cases}$$

Then then following hold true:

(a) For all $x \in \mathcal{D}(\hat{E}(u))$ and $y \in \mathcal{D}(\hat{E}(v))$ we have

$$\left\langle \hat{E}(v)y, \hat{E}(u)x \right\rangle \;=\; \lim_{n\to\infty} \int \varphi_n(v(t))\overline{\varphi_n(u(t))}\, d\left\langle y, E(t)x \right\rangle .$$

We will write $\int v(t)\overline{u(t)}\, d\langle y, E(t)x\rangle$ *whenever the limit exists.*

(b) *For all* $x \in \mathcal{D}(\hat{E}(u))$ *we have*

$$\left\|\hat{E}(u)x\right\|^2 = \int |u(t)|^2\, d\rho_x(t) = \|u\|^2_{L^2(\mathbb{R},d\rho_x)}.$$

(c) *If* u *is bounded, then* $\hat{E}(u) \in L(H)$ *and* $\left\|\hat{E}(u)\right\| \le \sup\{|u(t)| \mid t \in \mathbb{R}\}$.

(d) *If* $u = 1$*, then* $\hat{E}(u) = I$.

(e) *For all* $x \in \mathcal{D}(\hat{E}(u))$ *and all* $y \in H$ *we have*

$$\left\langle \hat{E}(u)x, y\right\rangle = \int u(t)\, d\langle E(t)x, y\rangle.$$

(f) *If* $u \ge C$*, then* $\left\langle x, \hat{E}(u)x\right\rangle \ge C\|x\|^2$ *for all* $x \in \mathcal{D}(\hat{E}(u))$.

(g) $\hat{E}(au + bv) \supseteq a\hat{E}(u) + b\hat{E}(v)$ *and* $\mathcal{D}(\hat{E}(u) + \hat{E}(v)) = \mathcal{D}(\hat{E}(|u| + |v|))$.

(h) $\hat{E}(uv) \supseteq \hat{E}(u)\hat{E}(v)$ *and* $\mathcal{D}(\hat{E}(u)\hat{E}(v)) = \mathcal{D}(\hat{E}(v)) \cap \mathcal{D}(\hat{E}(uv))$.

(i) $\mathcal{D}(\hat{E}(u))$ *is dense in* H*. Moreover,* $\mathcal{D}(\hat{E}(u)) = \mathcal{D}(\hat{E}(\overline{u}))$ *and* $\hat{E}(\overline{u}) = \hat{E}(u)^*$.

(j) *If* χ_S *is the characteristic function for some set* $S \subseteq \mathbb{R}$ *such that* χ_S *is* E*-measurable, then* $\hat{E}(\chi_S) =: E(S)$ *is an orthogonal projection and the map*

$$\begin{aligned} E : \mathcal{P}(\mathbb{R}) &\to L(H),\\ S &\mapsto \hat{E}(\chi_S) \end{aligned}$$

defines a projection-valued measure on $\mathbb{R}$.

The proof of (a) - (i) can be found in Weidmann [21], Chapter 7, and (j) follows from (c), (h) and (i).

Theorem C.4 (Spectral Theorem, J. von Neumann)**.** *For every self-adjoint operator* A *on a Hilbert space* H *there exists exactly one spectral family* E *such that*

$$T = \hat{E}(id) = \int t\, dE(t).$$

Moreover, the spectral family is explicitly given by Stone's formula

$$\langle y, (E(b) - E(a))x\rangle = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{a+\delta}^{b+\delta}\langle y, (R_T(t - i\varepsilon) - R_T(t + i\varepsilon))x\rangle\, dt$$

for all $x, y \in H$ *and* $-\infty < a \le b < +\infty$.

We provide a second version of the spectral theorem, which is more descriptive at first sight since it states that every self-adjoint operator has a representation as an operator of multiplication on some $L^2(\mathbb{R}, d\sigma)$ or the direct sum of such spaces.

Theorem C.5 (Spectral representation theorem)**.** *Let T be a self-adjoint operator on a Hilbert space H. Then there exists a family $\{\rho_\alpha \mid \alpha \in A\}$ of right-continuous non-decreasing functions and a unitary operator $U : H \to \oplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ such that*

$$T = U^{-1} M_{id} U,$$

where M_{id}, defined by

$$M_g((f_\alpha)_\alpha) := (g f_\alpha)_\alpha$$

for all measurable functions $g : \mathbb{R} \to \mathbb{C}$, is the maximal operator of multiplication on $\oplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$.

The spectral family E of T is given by

$$E(t) = U^{-1} M_{\chi_{(-\infty,t]}} U.$$

If H is separable, then $\oplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ is unitarily isomorphic to some $L^2(M, d\sigma)$, and there exist a σ-measurable function $a : \mathbb{R} \to \mathbb{C}$ and a unitary operator $V : H \to L^2(\mathbb{R}, d\sigma)$ such that

$$T = V^{-1} M_a V,$$

where M_a is the standard operator of multiplication by a.

Theorem C.6. *Let T be a self-adjoint operator on H and let $u(t) := \sum_{j=0}^{N} c_j t^j$ for some $c_j \in \mathbb{C}$. Then we have $\hat{E}(u) = \sum_{j=0}^{N} c_j T^j$, where $T^0 := I$.*

Theorem C.7. *Let T be a self-adjoint operator on H. Then T is bounded if and only if there exist $\gamma_1, \gamma_2 \in \mathbb{R}$ such that*

$$E(t) := \begin{cases} 0 & \text{if } t \leq \gamma_1 \\ I & \text{if } t \geq \gamma_2 \end{cases}$$

In that case we can set

$$\gamma_1 := m := \inf\{\langle x, Tx \rangle \mid x \in \mathcal{D}(T),\ \|x\| = 1\},$$
$$\gamma_2 := M := \sup\{\langle x, Tx \rangle \mid x \in \mathcal{D}(T),\ \|x\| = 1\}.$$

Moreover, we have $E(t) \neq 0$ and $E(t) \neq I$ if $m < t < M$.

For the proofs see Weidmann[21], Theorem 7. 17 - 7. 19 and 7. 21, whereas last statement of Theorem C.5 is proved in Haslinger [9], Satz 12. 9.

C.2. Stone's Theorem

This section is devoted to Stone's theorem, which characterizes the relation between self-adjoint operators and strongly continuous unitary one-parameter groups. Based on the facts illustrated above, we will give a complete proof of Stone's theorem. As in §C.1, H denotes a complex Hilbert space.

Theorem C.8. *Let A be a self-adjoint operator on H with spectral family E and let*

$$U(t) = e^{itA} = \int e^{its}\, dE(s)$$

for all $t \in \mathbb{R}$. Then $t \mapsto U(t)$ is a strongly continuous unitary one-parameter group with infinitesimal generator iA.

Proof. First note by Proposition C.3 (c) that $U(t) \in L(H)$ for all $t \in \mathbb{R}$, hence $\mathcal{D}(\hat{E}(e^{it\cdot})) = H$. By property (h) of the same theorem, we have $I = \hat{E}(e^{it\cdot}\, e^{-it\cdot}) \supseteq \hat{E}(e^{it\cdot})\hat{E}(e^{-it\cdot})$ and $\mathcal{D}(\hat{E}(e^{it\cdot})\hat{E}(e^{-it\cdot})) = \mathcal{D}(\hat{E}(e^{-it\cdot})) \cap \mathcal{D}(\hat{E}(e^{it\cdot}\, e^{-it\cdot})) = H$. It follows that $U(t)U(-t) = I$, thus $U(-t) = U(t)^{-1}$. The same argument holds for the product of the operators $U(t)$ and $U(s)$, i.e., $U(t)U(s) = U(t+s)$ for all $s, t \in \mathbb{R}$. Moreover, we observe that $U(-t) = \hat{E}(\overline{e^{it\cdot}}) = \hat{E}(e^{it\cdot})^* = U(t)^*$ due to (i), hence U is a unitary one-parameter group.

In the next step we prove strong continuity. For this purpose note that

$$\left|e^{ix} - e^{iy}\right| = \left|e^{-i(x+y)/2}\right| \left|e^{ix} - e^{iy}\right| = \left|e^{i(x-y)/2} - e^{-i(x-y)/2}\right| = 2\left|\sin\left(\frac{x-y}{2}\right)\right|$$

for all $x, y \in \mathbb{R}$ and recall that

$$\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 1 \quad \text{and} \quad \sin\left(\frac{x-y}{2}\right) \to 0 \ \ (x \to y).$$

Putting these facts together, we obtain using (C.6)

$$\begin{aligned} \left\|(U(t) - U(t'))x\right\|^2 &= \left\|\int \left(e^{its} - e^{it's}\right)\, dE(s)x\right\|^2 = \int \left|e^{its} - e^{it's}\right|^2 \, d\rho_x(s) \\ &= 2\int \left|\sin\left(\frac{(t-t')s}{2}\right)\right|^2 \, d\rho_x(s), \end{aligned}$$

which yields strong continuity using dominated convergence.

Finally, we will identify the infinitesimal generator of U (as iA). To this end, we observe that

$$\frac{1}{t}(U(t)x - Ix) = \frac{1}{t}\int \left(e^{its} - 1\right)\, dE(s)x \tag{C.8}$$

holds for all $t > 0$ and all $x \in H$. Keeping in mind that

$$\left|e^{its} - 1\right| \leq |ts| \quad \Leftrightarrow \quad \left|\frac{1}{t}\left(e^{its} - 1\right)\right| \leq |s| \quad \forall s, t \in \mathbb{R},\, t \neq 0$$

and that $t^{-1}(e^{its} - 1) \to is$ for all $s \in \mathbb{R}$, as $t \to 0$, we again apply dominated convergence and (C.6) to obtain

$$\lim_{t\to 0}\left\|\frac{1}{t}(U(t)x - Ix) - iAx\right\|^2 = \lim_{t\to 0}\left\|\int t^{-1}\left(e^{its} - 1 - its\right)\, dE(s)x\right\|^2$$
$$= \lim_{t\to 0}\int\left|\frac{e^{its}-1}{t} - is\right|^2 d\rho_x(s) = 0,$$

whenever $(s \mapsto s) \in L^2(\mathbb{R}, \rho_x)$. That is,

$$\lim_{t\to 0}\frac{1}{t}(U(t)x - Ix) \;=\; iAx$$

whenever $x \in \mathcal{D}(\hat{E}(id)) = \mathcal{D}(A)$. Hence, iA is the infinitesimal generator of U, which finishes the proof. ☐

Now, we are prepared to prove Stone's theorem.

Theorem C.9 (Stone)**.** *Let U be a strongly continuous unitary one-parameter group on a Hilbert space H. Then there exists a uniquely determined self-adjoint operator A on $\mathcal{D}(A) \subseteq H$ such that*

$$U(t) \;=\; e^{itA}$$

for all $t \in \mathbb{R}$.

Proof. By Theorem 1.28, every strongly continuous unitary one-parameter group U possesses a skew-adjoint infinitesimal generator T. Setting $-iT := A$, we observe that A is self-adjoint. Now, by Theorem C.8 the map $t \mapsto e^{itA}$ defines another strongly continuous one-parameter group with the same infinitesimal generator, hence the two groups agree by Theorem 1.16. ☐

C.3. Schur's Lemma

In the final section of this text we will present another important application of spectral theory in the context of representations. More precisely, it is our aim to characterize those operators, which commute with all irreducible unitary Lie group representations. The main theorem, known as Schur's Lemma, was in turn used a key result to characterize all irreducible representations of the Heisenberg group (cf. Theorem 2.49).

Throughout this section, T denotes a self-adjoint operator on a Hilbert space H with spectral family E.

Proposition C.10. *Let $(f_n)_n$ be sequence of uniformly bounded complex-valued Borel measurable functions on $\mathbb{R}$.*

(i) If $f_n \to f$ uniformly, then $\hat{E}(f_n) \to \hat{E}(f)$ in the operator norm.

(ii) If $f_n \to f$ pointwise, then $\hat{E}(f_n)x \to \hat{E}(f)x$ for all $x \in H$, i.e., in the strong operator topology.

Proof. Note that f is bounded, and therefore $f - f_n$, too.
(i) Since for any $x \in \{y \in H \mid \|y\| = 1\}$ we have

$$\left\|\hat{E}(f)x - \hat{E}(f_n)x\right\|^2 = \int_{\mathbb{R}} |f(t) - f_n(t)|^2 \, d\rho_x(t) \leq \|x\|^2 \|f - f_n\|_\infty^2 \to 0,$$

the same holds for the supremum over all such x.
(ii) Dominated convergence yields

$$\lim_{n\to\infty} \left\|\hat{E}(f)x - \hat{E}(f_n)x\right\|^2 = \int_{\mathbb{R}} \lim_{n\to\infty} |f(t) - f_n(t)|^2 \, d\rho_x(t) = 0$$

for all $x \in H$. □

Proposition C.11. *If T is bounded, then the norm closure of the algebra $\mathcal{A}$ generated by T contains $\hat{E}(f)$ for all continuous functions f on $[-\|T\|, \|T\|]$.*

Proof. The Weierstrass approximation theorem states that each such f can be uniformly approximated by polynomials on $[-\|T\|, \|T\|]$. By Proposition C.10 (i) we conclude that $\hat{E}(f)$ lies in $\overline{\mathcal{A}}$. □

Remark C.12. Note that by Proposition C.10 (ii), $\hat{E}(f)$ is also contained in the strong closure of $\mathcal{A}$.

Proposition C.13. *If T is bounded, then for any interval $S \subseteq \mathbb{R}$, $E(S) = \hat{E}(\chi_S)$ is in the strong closure of $\mathcal{A}$.*

Proof. We clearly may approximate χ_S pointwise by continuous functions, so Proposition C.10 (ii) and Remark C.12 give the result. □

Proposition C.14. *Let T be bounded and $A \in L(H)$ such that $TA = AT$. Then $E(S)A = AE(S)$ for all intervals $S \subseteq \mathbb{R}$.*

Proof. It is easy to verify that $\mathcal{A}' := \{T' \in L(H) \mid T'Ax = AT'x \;\; \forall x \in H\}$ forms an algebra, and we observe that it is strongly closed since $T_n'x \to T'x$, $x \in H$, implies

$$T'Ax = \lim_{n\to\infty} T_n'Ax = \lim_{n\to\infty} AT_n'x = A \lim_{n\to\infty} T_n'x = AT'x.$$

Furthermore, we know that $T \in \mathcal{A}'$, so $\overline{\mathcal{A}} \subseteq \mathcal{A}'$. By Proposition C.13 we conclude that $E(S) \in \mathcal{A}'$. □

Definition C.15. *Let π_1 and π_2 be unitary representations of some Lie group G on the Hilbert spaces H_1 and H_2, respectively. A bounded operator $T : H_1 \to H_2$ is called intertwining operator between π_1 and π_2 if it satisfies $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. The set of intertwining operators between π_1 and π_2 is denoted by $\mathcal{C}(\pi_1, \pi_2)$, and we say π_1 and π_2 are unitarily equivalent if $\mathcal{C}(\pi_1, \pi_2)$ contains at least one unitary operator T.*

Recall that a Lie group representation π is said to be irreducible if $\{0\}$ and H are the only π-invariant closed subspaces of H.

Theorem C.16 (Schur's Lemma)**.** *Let π be a unitary representation of some Lie group G on some Hilbert space H. Then π is irreducible if and only if $\mathcal{C}(\pi,\pi)=\{cI \mid c\in\mathbb{C}\}$.*

Proof. Both directions of the statement are proved by contradiction.

Thus, suppose there exists some non-trivial closed invariant subspace M and denote by P_M the orthogonal projection onto M. Then we have

$$\begin{aligned}\langle \pi(g)P_M x, y\rangle &= \langle \pi(g)P_M x, P_M y\rangle \\ &= \left\langle P_M x, \pi(g^{-1})P_M y\right\rangle = \left\langle x, \pi(g^{-1})P_M y\right\rangle = \langle P_M\pi(g)x, y\rangle\end{aligned}$$

for all $x,y\in H$. Hence, $P_M\in\mathcal{C}(\pi,\pi)=\{cI\mid c\in\mathbb{C}\}$, a contradiction.

Conversely, suppose that $T\in\mathcal{C}(\pi,\pi)$, $T\neq cI$ for all $c\in\mathbb{C}$. Unitarity of π yields

$$T^*\pi(g)=T^*\pi(g^{-1})^*=\left(\pi(g^{-1})T\right)^*=\left(T\pi(g^{-1})\right)^*=\pi(g)T^*$$

for all $g\in G$, hence $T^*\in\mathcal{C}(\pi,\pi)$. If we set $\mathcal{A}_g:=\{T'\in L(H)\mid T'\pi(g)=\pi(g)T'\}$, then $\mathcal{C}(\pi,\pi)$ is given by the closed algebra $\cap_{g\in G}\mathcal{A}_g$. It follows that the operators $T_1:=\frac{1}{2}(T^*+T)$ and $T_2:=\frac{1}{2}i(T^*-T)$ are in $\mathcal{C}(\pi,\pi)$. Two straightforward calculations show that they are self-adjoint and not multiples of each other. Thus, at least one of them, say T_1, is not a multiple of I, either. If E_1 denotes the spectral family of T_1, then by Theorem C.7 there exists an interval S such that $E_1(S)$ is neither the zero operator nor the identity, and by Proposition C.13, $E_1(S)\in\mathcal{C}(\pi,\pi)$. Hence, $\operatorname{ran}(E(S))\subset H$ is a non-trivial invariant closed subspace, again a contradiction. □

The following two statements are immediate consequences of Schur's Lemma.

Corollary C.17. *Let π be an irreducible unitary representation of the Lie group G on some Hilbert space H. If G_1 is a commutative subgroup of G, then we have*

$$\pi(G_1)\subseteq\{e^{it}I\mid t\in[0,2\pi]\}.$$

Theorem C.18. *Let π_1 and π_2 be irreducible unitary representations of G on H. Then the algebra $\mathcal{C}(\pi_1,\pi_2)$ has either dimension 1 if π_1 and π_2 are unitarily equivalent or dimension 0 otherwise.*

Proof. Note that if $T\in\mathcal{C}(\pi_1,\pi_2)$, then by the same argument as in the proof of Theorem C.16, we have $T^*\in\mathcal{C}(\pi_1,\pi_2)$, and $T^*T\in\mathcal{C}(\pi_1,\pi_1)$. By Schur's Lemma the (self-adjoint) operator T^*T must be of the form λI for some real number λ. Furthermore, we observe

$$\langle Tx,Ty\rangle=\langle T^*Tx,y\rangle=\langle\lambda x,y\rangle=\left\langle\sqrt{\lambda}x,\sqrt{\lambda}y\right\rangle,$$

hence $T=\sqrt{\lambda}U$ for some unitary operator $U\in\mathcal{C}(\pi_1,\pi_2)$.

Now, in case that π_1 and π_2 are not unitarily equivalent, U - and thus T - must be trivial, hence $\mathcal{C}(\pi_1,\pi_2)=\{0\}$.

Otherwise, there exists a unitary operator $U_1\in\mathcal{C}(\pi_1,\pi_2)$, and $U_1^{-1}T\in\mathcal{C}(\pi_1,\pi_1)$. As above we have $U_1^{-1}T=cI$ for some $c\in\mathbb{C}$, or equivalently $T=cU_1$. We conclude that $\mathcal{C}(\pi_1,\pi_2)=\{cU_1\mid c\in\mathbb{C}\}$. □

List of Symbols

$\mathbb{F}$	field
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$
$E, E_1, F...$	vector space of $\mathbb{F}$
$B, B_1, B_2, ...$	real or complex Banach spaces
$H, H_1, H_2, ...$	complex Hilbert spaces
$E, E_1, F...$	vector space of $\mathbb{F}$
B^*	dual space of B
$\mathrm{Lin}(E, F)$	space of linear maps from E_1 to E_2
$\mathrm{Lin}(E)$	space of linear maps from E to E
$L(B_1, B_2)$	space of bounded linear maps from B_1 to B_2
$L(B)$	space of bounded linear maps from B to B
$HS(H_1, H_2)$	space of Hilbert-Schmidt operators from H_1 to H_2
$HS(H)$	space of Hilbert-Schmidt operators from H to H
$\lVert T\rVert$	the operator norm of an operator T
$\lVert T\rVert_{HS}$	the Hilbert-Schmidt norm of an operator T
$\mathrm{tr}\,(T)$	the trace of an operator T
$\rho(T)$	the resolvent set of an operator T
$\sigma(T)$	the spectrum of an operator T
R_T	the resolvent of an operator T
$\mathcal{S}(\mathbb{R}^n)$	Schwartz space
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions
$G, G_1, G_2, ...$	second countable Lie groups
$\mathfrak{g}, \mathfrak{g}_1, \mathfrak{g}_2, ...$	Lie algebras
$\mathbb{T}$	the circle group $\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
$\mathbf{H}^n$	the $(2n+1)$-dimensional Heisenberg group
$\mathfrak{h}^n$	the $(2n+1)$-dimensional Heisenberg Lie algebra
$\mathfrak{X}(G)$	space of smooth vector fields on G
$\bigwedge^n T_e^*(G)$	set of alternating n-forms on the tangent space $T_e^*(G)$
$\Omega^n(G)$	space of alternating differential n-forms on G
$\int f(t)\,dt, \int f(x)\,dx, ...$	the (Bochner-) Lebesgue integral on $\mathbb{R}$, $\mathbb{R}^n$,...
$\int f(g)\,dg$	the (Bochner-) Haar integral on the Lie group G
$\int f(\omega)\,d\mu(g)$	the Bochner-μ integral on the measure space $(\Omega, \mathfrak{A}, \mu)$
$(\Omega, \mathfrak{A}, \mu)$	measure space
$L^p(\Omega, \mu)$	Lebesgue space arising from $(\Omega, \mathfrak{A}, \mu)$
$L^p(\Omega, X; \mu)$	Bochner-Lebesgue space arising from $(\Omega, \mathfrak{A}, \mu)$

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Abstract

This text provides a detailed introduction to the concepts and basic tools of harmonic analysis on the Heisenberg group $\mathbf{H}^n$. The Heisenberg group is the "simplest" and most studied non-compact, non-commutative Lie group and it was the first of this type to be thoroughly studied. For a non-compact, non-commutative Lie group representation theory is no longer essentially finite-dimensional. Consequently, to study unitary representations of $\mathbf{H}^n$, we have to leave the realm of linear algebra and enter the territory of functional analysis.

We collect all prerequisites in the first chapter. Starting with infinite-dimensional representations of the (additive) real line we prove some basic facts on the latter and eventually generalize to the case of Lie groups and their representations.

The second chapter, which is the core of this text, starts with the construction of the Heisenberg group, involving from the very beginning what eventually turns out to be its main representation, the so-called Schrödinger representation. Apart from some interesting subgroups of $\mathbf{H}^n$, its Lie algebra $\mathfrak{h}^n$, etc., we learn about closely related convolution algebras and their corresponding algebra representations, which are induced by the Schrödinger representation. The main result of this chapter is a classification of the irreducible unitary representations of the Heisenberg group, known as the Stone-von Neumann theorem. Finally, we make use of this classification to define the group Fourier transform for $\mathbf{H}^n$, which is an operator-valued analogue of the ordinary Fourier transform on $\mathbb{R}^n$ and shares some of its most important properties.

Along the way we put special emphasis on the Bochner integral, which is one possible way of integrating Banach space-valued functions defined on a second-countable locally compact measure space, thus, in particular, on second-countable Lie groups. Appendix A gives a detailed presentation of the basic results, which are eventually used to extend some classical theorems to the infinite-dimensional case.

Appendix B collects some important facts on Banach algebras and their representations in view of the convolution algebras appearing in Chapter 2.

Appendix C gives a brief account on the spectral theory of self-adjoint operators in Hilbert space including detailed proofs of Stone's theorem and Schur's Lemma.

Zusammenfassung

Die vorliegende Diplomarbeit ist der Einführung grundlegender Konzepte der harmonischen Analyse auf der Heisenberggruppe $\mathbf{H}^n$ gewidmet. Da die Heisenberggruppe eine nichtkommutative, nichtkompakte Lie-Gruppe ist (sie ist gewissermaßen das "einfachste" und daher Musterbeispiel einer solchen), sind die für die harmonische Analyse benötigten irreduziblen, unitären Darstellungen von $\mathbf{H}^n$ nicht länger notwendigerweise endlichdimensional. Zum Studium dieser Darstellungen werden daher Techniken der Funktionalanalyis anstatt der sonst üblichen linearen Algebra verwendet.

Die wichtigsten Hilfsmittel und Begriffe werden im ersten Kapitel eingeführt. Als Ausgangspunkt dazu dienen unendlichdimensionale Darstellungen der Gruppe $(\mathbb{R}, +)$, für die zuerst einige essentielle Begriffe eingeführt und hilfreiche technische Resultate bewiesen werden. Im Laufe des Kapitels werden schließlich sowohl erstere als auch letztere auf den allgemeineren Fall einer beliebigen Lie-Gruppe und deren Darstellungen erweitert.

Das zweite Kapitel bildet den Hauptteil der Arbeit. Angefangen mit der von der quantenmechanischen Unschärferelation inspirierten Definition der Heisenberg-Lie-Algebra $\mathfrak{h}^n$ wird mithilfe der Matrixexponentialfunktion aus $\mathfrak{h}^n$ die Heisenberggruppe $\mathbf{H}^n$ konstruiert. Dabei stellt sich heraus, dass die wichtigste Darstellung von $\mathbf{H}^n$, die sogenannte Schrödingerdarstellung, als "Exponential" jener Liealgebrendarstellung von $\mathfrak{h}^n$ gewonnen wird, welche die Definition der Heisenberg-Lie-Algebra erst rechtfertigte. Wie sich herausstellt, kann man diese durch $h \in \mathbb{R}^*$ parametrisieren und erhält eine ganze Familie von unitären Darstellungen, die von zentraler Bedeutung ist. Im weiteren Verlauf des Kapitels werden verschiedene Arten der Konvolution von Funktionen auf $\mathbf{H}^n$ diskutiert und daraus resultiernde Konvolutionsalgebren vorgestellt, deren Darstellungen durch Integrieren der Schrödingerdarstellungen gewonnen werden. Die dazu vorgestellten Techniken dienen später dem Beweis des Satzes von Stone und von Neumann, eines grundlegend wichtigen Theorems, das eine Klassifizierung aller irreduziblen, unitären Darstellungen von $\mathbf{H}^n$ liefert. Diese bereits erwähnten Darstellungen werden im letzten Teil des zweiten Kapitels schließlich zur Einführung einer Fouriertransformierten auf $\mathbf{H}^n$ verwendet. Diese definiert eine operatorwertige Funktion, die einige der wichtigsten Eigenschaften der Fouriertransformierten auf $\mathbb{R}^n$ aufweist.

Den Abschluss bilden drei Appendizes, deren erster eine detaillierte Darstellung des Bochnerintegrals bietet. Diese Integrationsmethode, die es erlaubt, banachraumwertige Funktionen über Lie-Gruppen zu integrieren, ist von besonderer Bedeutung für einige der wichtigsten Resultate der Arbeit. Appendix A führt dementsprechend die grundlegenden Konzepte der Bochnermessbarkeit und -integrierbarkeit ein und bietet darüber hinaus auch interessante Bochnererweiterungen klassischer Sätze wie etwa dem Satz von Fubini, Lebesgues Satz der dominierten Konvergenz oder dem Hauptsatz der Analysis. Die Appendizes B und C sind Zusammenfassungen wichtiger Fakten zu Banach-Algebren und ihren Darstellungen im Hinblick auf Konvolutionsalgebren, einerseits, und der Spektraltheorie selbstadjungierter Operatoren einschließlich der detaillierten Beweise des Satzes von Stone und des Lemmas von Schur, andererseits.

Curriculum Vitae

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