

DIPLOMARBEIT

Titel der Diplomarbeit

The Mutual Fund Theorem and Separation in financial
markets

angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag.rer.nat.)

Verfasser: Thomas Weitz

Matrikelnummer: 0550620

Studienkennzahl lt. Studienblatt: A 405

Studienrichtung lt. Studienblatt: Mathematik

Betreuer: Univ.-Prof. Mag. Dr. Walter Schachermayer

Wien, Mai 2010

“It is a good idea to be ambitious, to have goals, to want to be good at what you do, but it is a terrible mistake to let drive and ambition get in the way of treating people with kindness and decency.”

Robert Merton

Acknowledgments

Three kinds of people were invaluable in writing this thesis: the ones behind the actual writing of the thesis, the ones behind the ideas and the ones that made both possible.

First and foremost I would like to express my gratitude towards my thesis advisor, Prof. Walter Schachermayer for supporting me throughout the months I have been writing this thesis. Both his personal advice and his academic publications proved to be tremendously valuable.

Secondly, the last few years have been the most exciting of my life. To all my teachers, colleagues and friends who joined me in this exciting journey - I couldn't have done it without you!

Finally, there are my parents. They enabled me to indulge in the studies of mathematics and encouraged and supported me all my life. My thanks to them know no limits.

Thomas Weitz, Vienna, May 2010

Abstract

The most important versions of the Mutual Fund Theorem and Separation results in discrete and continuous time are presented. We further investigate whether the predictions of the Mutual Fund Theorem hold true in actual financial markets by analyzing proposed asset allocations of Fidelity and Vanguard.

Contents

1	Notation	3
2	Basics and fundamentals	5
2.1	Portfolio Selection	5
2.2	A simple version of the Mutual Fund Theorem	8
3	The Mutual Fund Theorem and Separation in discrete time	10
3.1	Separation for certain utility functions	10
3.2	The Mutual Fund Theorem in a $\mu - \sigma$ world	14
4	The Mutual Fund Theorem and Separation in continuous time	19
4.1	Separation under log-normally distributed asset returns	22
4.2	Separation for certain asset return distributions	24
4.3	Separation and martingale representation	27
5	The Mutual Fund Theorem and Separation in recent papers	31
5.1	A generalization of the Mutual Fund Theorem	31
5.2	A further generalization of the Mutual Fund Theorem	35
6	Empirical study: Can we observe the Mutual Fund Theorem?	39
7	Conclusion	44

Introduction

Given a set of n securities, the aim of this thesis is to study the necessary conditions, under which several mutual funds are sufficient to capture all investor demands. We define a mutual fund $H = (h_1, \dots, h_n)$ with $\sum_{i=1}^n h_i = 1$ as a linear combination of the available securities. The basic idea of the Mutual Fund Theorem is that an investor is indifferent between investing in the n original assets or only in a few mutual funds. Consequently it is enough to study only the mutual funds, since they encompass all relevant investment opportunities.

In chapter 1 we outline the notation we will use throughout the whole thesis. This is intended to increase legibility for the reader since many of the cited papers use different notations for the same idea and even more confusing, in some cases the same notation for different ideas.

Chapter 2 covers the first separation results and basics of the Portfolio Theory of Markowitz and Tobin ([18] and [32]). The great achievement of those authors was to be the first to describe the mean variance trade off and in the process of doing so laying the foundation for modern Portfolio Theory and finance in general.

In the following chapter we discuss the paper of Cass and Stiglitz on which utility functions lead to separation in general discrete-time financial markets [6] and look into Merton's findings concerning the Mutual Fund Theorem in a $\mu - \sigma$ world [20]. Chapter 4 is devoted to results in continuous time. Since the technical requirements are more advanced than in the previous chapters, we start this chapter with giving

a few central definitions and presenting some key ideas, such as the Itô Integral and the definition of a martingale. In the spirit of Robert Merton ([19] and [21]) we investigate separation for log-normally distributed returns. Furthermore we study the approach of Ross to separation for certain types of asset returns [25] and consider Chamberlain's results based on the martingale representation of Brownian Motion [7].

Chapter 5 will cover recent findings, such as Khanna and Kulldorff's generalization of the Mutual Fund Theorem [14] and a unifying treatment of many of the previously mentioned versions of the Mutual Fund Theorem, as presented by Schachermayer et al. [26].

Finally, in chapter 6, we investigate whether the market conditions necessary for separation are fulfilled in real financial markets. We look at suggested asset allocations of Fidelity and Vanguard. We check the prediction of the Mutual Fund Theorem, namely that the ratio of stocks to bonds (risky assets) should remain constant, even if the total share of stocks increases.

Chapter 1

Notation

In order to ensure the legibility of this thesis, we decided to unify the different notions the authors use in the cited papers. Throughout the thesis we will strive to use the following notations whenever possible:

- Value of the i^{th} security in the j^{th} state: S_{ij}
- Return of the i^{th} security in j^{th} state: s_{ij}
- Expected return of the i^{th} security: μ_i
- Expected return of the portfolio: μ_P
- Share of the portfolio invested in the i^{th} security: h_i
- Amount of money invested in the i^{th} security: H_i
- Number of shares of the i^{th} security: N_i
- Value of the portfolio in the i^{th} state: W_i
- There are m states in the probability space. The state that occurred: θ
- Covariance between the i^{th} and j^{th} security: σ_{ij}

- Correlation between the i^{th} and j^{th} security: ρ_{ij}
- Risk-free rate of return: r

Chapter 2

Basics and fundamentals

2.1 Portfolio Selection

According to Harry Markowitz, the portfolio selection process¹ consists of two steps:

- (i) Observing the available securities and forecasting their future performance
- (ii) Equipped with the knowledge of step one, choosing an appropriate portfolio

In his seminal paper [18] Markowitz discusses the second step of the portfolio selection process. Let us consider a market with n securities. The return of a security is a random variable and the m different possible values for the i^{th} security S_{ij} occur with probability p_{ij} . Hence the expected return for the i^{th} security is:

$$\mu_i = \sum_{j=1}^m p_{ij} S_{ij} \quad (2.1)$$

The investor chooses to allocate his initial wealth among the n available securities. We denote the share of his portfolio invested in the i^{th} security as h_i and don't allow

¹Process of choosing the optimal mix of securities, both risky and riskless, given the investor's initial endowment

short sales², so $h_i \geq 0$. Then the **expected return of the portfolio** is:

$$\mu_P = \sum_{i=1}^n E(s_i)h_i = \sum_{i=1}^n \mu_i h_i \quad (2.2)$$

If we denote the covariance between i^{th} and the j^{th} security by σ_{ij} , we obtain the following formula for the variance of the portfolio:

$$\sigma_P^2 = \sum_{i=1}^n \sum_{j=1}^m h_i h_j \sigma_{ij} \quad (2.3)$$

For a given set of n securities, the investor can allocate his funds to achieve different combinations of (μ_P, σ_P^2) .

Assumption 2.1. *The investor is risk-averse. This means that he tries to minimize the portfolio variance for a given expected return. For a given level of variance, he tries to maximize expected return.*

“The investor considers expected return a desirable thing and variance of return an undesirable thing.” - Harry Markowitz

If there were 2 portfolios with the same expected return but different variances, a risk averse investor would clearly prefer the less risky portfolio (the one with lower variance). Hence only for some portfolios there isn't a clearly superior investment opportunity readily available. We call those portfolios **efficient**.

Let us consider a financial market with only 3 securities. In this case the relevant equations for the portfolio take the form of:

$$\begin{aligned} \mu_P &= \sum_{i=1}^3 E(s_i)h_i = \sum_{i=1}^3 \mu_i h_i \\ \sigma_P^2 &= \sum_{i=1}^3 \sum_{j=1}^3 h_i h_j \sigma_{ij} \end{aligned} \quad (2.4)$$

²Selling securities the investor doesn't own, $h_i < 0$. This is done by borrowing them temporarily from a third party with the intention of buying them back at a later date.

with $h_i \geq 0$ and $\sum_{i=1}^3 h_i = 1$. If we make the straightforward replacement $h_3 = 1 - h_1 - h_2$, both μ_P and σ_P^2 depend only on h_1 and h_2 .

Definition. An *isomean curve* is the set of all portfolios which have the same expected return.

Definition. An *isovariance curve* is the set of all portfolios which have the same variance.

Fairly straightforward calculations in [18] show that the isomean curves are straight lines, whereas all isovariance portfolios have the shape of an ellipse. Figure 2.1 illustrates this fact.

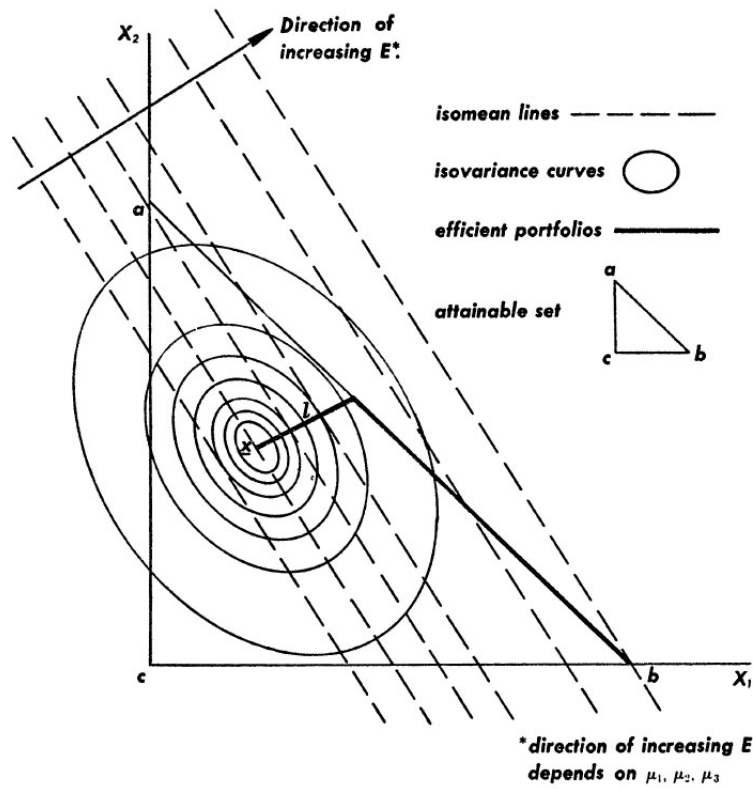


Figure 2.1: Isomean and isovariance curves

Source: [18]

The set of efficient portfolios are the points where the isomean lines are tangent to the isovariance ellipses. Since those asset allocations are superior to all other available investment strategies, the investor only chooses among them. For all other portfolios we can find one with either the same mean and lower variance or with a higher mean and the same variance - both are clearly advantageous to any risk averse investor.

The great achievement of Harry Markowitz is to give a theoretical foundation to the investment decision. His $\mu - \sigma$ principle, which states that risk averse investors should maximize expected return for a given level of acceptable variance can be used both in theoretical analysis as well as in actual portfolio selection. It therefore represents the starting point for all future inquiries into this matter, including the Mutual Fund Theorem.

2.2 A simple version of the Mutual Fund Theorem

Tobin covers in his paper on liquidity preference [32] a simple version of the Mutual Fund Theorem. Let us consider a market in discrete time with risk-free and risky assets. Tobin refers to the latter as conolos. Adding to the similarities to Markowitz's paper, Tobin considers curves of equal mean and variance, which he refers to as **constant-return locus** and **constant-risk locus**. In the 2-securities case, the isomean curve takes the shape of a straight line, whereas the isovariance curve is a quarter ellipse. We obtain the equation from 2.4:

$$h_1^2\sigma_{11} + 2h_1h_2\sigma_{12} + h_2^2\sigma_{22} = constant \quad (2.5)$$

Again, we assume that the investor is risk averse. Since for a given level of return he seeks to minimize variance (or equivalently for a given level of variance tries to maximize expected return) only the portfolios where an isomean curve is tangent

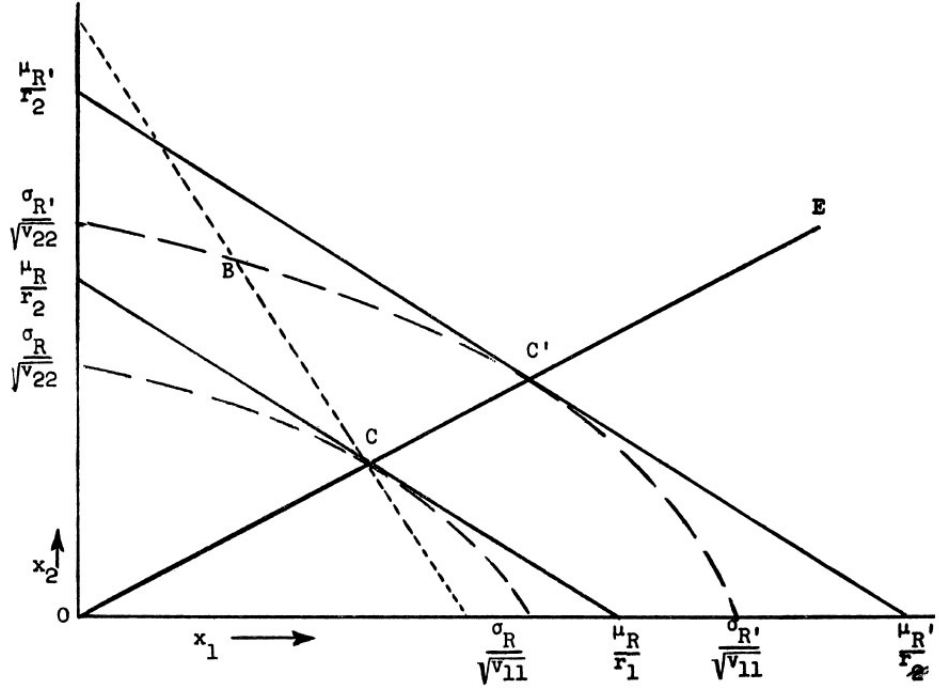


Figure 2.2: Efficient portfolios Tobin

Source: [32]

to an isovariance curve are efficient. Not surprisingly, this is the same result we found in [18]. Tobin defines a dominant combination of assets as a set of h_i which minimize σ_P^2 for a given expected return μ_P . All efficient portfolios lie on a ray from the origin. Hence the composition of the portfolio of risky assets is the same for all investors. They only choose how to allocate their funds among risky and riskless assets. This constitutes our first Mutual Fund result and is depicted in figure 2.2.

Chapter 3

The Mutual Fund Theorem and Separation in discrete time

3.1 Separation for certain utility functions

We distinguish between the following types of separation and investigate for which utility functions they are attained in our 2-period model:

Definition. *A utility function exhibits the **generalized separability property** if and only if for any arbitrary set of n original securities there are $m < n$ mutual funds.*

Definition. *If generalized separation obtains and when in addition the number of mutual funds $m = 2$ we say that the utility function exhibits the **separability property** or that there is **separation**.*

Definition. *If separation obtains and when in addition one of the mutual funds formed is money ¹, we say that **money separation** obtains.*

¹Riskfree asset yielding the same in every state

In the spirit of [6], we consider an investor with initial wealth W_0 which he can invest in n different assets. We will refer to his terminal wealth as W . The investor seeks to maximize the portfolio's utility. This utility takes the form of:

$$EU(W_\theta) = EU \left(\sum_i H_i s_{i,\theta} \right) \quad (3.1)$$

considering his budgetary constraint $\sum_{i=1}^n H_i \leq W_0$. Furthermore we assume U to be twice differentiable and strictly concave. The economic interpretation of this is decreasing marginal utility of wealth, which means that an additional unit of wealth increases total utility less if the investor has more wealth to begin with.

Let us firstly consider the special case when Arrow-Debreu securities are available. For this small subset of all possible distributions of asset returns, we will show which utility functions lead to separation and money separation. We will see that the set of utility functions which lead to separation even under those specific circumstances is rather limited, so we only need to consider those few utility functions when we investigate which ones lead to separation in general markets².

Definition. An **Arrow-Debreu security** is a claim against every possible state. For every possible state θ there is a security which returns:

$$s_\theta = \begin{cases} s_\theta > 0 & \text{if } \theta \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.1. Given Arrow-Debreu securities, a necessary and sufficient condition for separation is that marginal utility U' satisfy:

$$AU'(W)^\alpha + BU'(W)^\beta = W \quad (3.2)$$

or

$$U'(W)^\alpha (A + B \log U'(W)) = W \quad (3.3)$$

²In general markets we don't assume securities to have a particular distribution of returns, e.g. log-normal or Arrow-Debreu)

Proof. The proof is done in three steps:

- (i) We show that separability is equivalent to the demand function for the i^{th} security to be of the form

$$H_i = A_i W_0 + B_i h(W_0) \quad (3.4)$$

with $\sum_{i=1}^n A_i = \sum_{i=1}^n B_i = 1$

- (ii) We further show that for the demand function to be of the form 3.4, a necessary condition is that $U'^{-1} = G = 0$ satisfy

$$G(xy) = \bar{f}(x)\tilde{f}(y) + \bar{g}(x)\tilde{g}(y) \quad (3.5)$$

- (iii) Finally we show that the only solutions to 3.5 are

$$\bar{f}(x) = \frac{\tilde{f}(x)}{A} = x^\alpha, \bar{g}(x) = \frac{\tilde{g}(x)}{B} = x^\beta \text{ and } G(x) = Ax^\alpha + Bx^\beta \quad (3.6)$$

and

$$\bar{f}(x) = \tilde{g}(x) = x^\alpha \left(\frac{A}{2} + B \log x \right), \tilde{f}(x) = \bar{g}(x) = x^\alpha \text{ and } G(x) = x^\alpha (A + B \log x) \quad (3.7)$$

For the details of the proof, we refer the reader to [6].

□

Remark. If we plug $\alpha = 1$ into 3.2, we obtain $A + BU'(W)^\beta = W$ and hence $U'(W) = \left(-\frac{A}{B} + \frac{W}{B}\right)^{\frac{1}{\beta}}$, which we can rewrite as

$$U'(W) = (a + bW)^c \quad (3.8)$$

Well known special cases of 3.8 are:

- (i) The **constant relative risk aversion function**

$$U'(W) = dW^c \quad (a = 0, d = b^c) \quad (3.9)$$

- (ii) The **quadratic utility function**

$$U'(W) = a + bW \quad (c = 1) \quad (3.10)$$

An interesting special case of 3.3 is the **constant absolute risk aversion function**

$$U'(W) = ae^{bW} \quad (3.11)$$

The names of the above utility functions derives from the **Arrow-Pratt measure of risk-aversion** in [24]. We define the measure for Absolute Risk Aversion (ARA) and Relative Risk Aversion (RRA):

$$ARA = -\frac{U''(W)}{U'(W)} \quad \text{and} \quad RRA = -W \frac{U''(W)}{U'(W)} \quad (3.12)$$

Calculating RRA for equation 3.9 we obtain:

$$RRA = -W \frac{dcW^{c-1}}{dW^c} = -c \quad (3.13)$$

which is constant, hence the name constant relative risk aversion function.

Calculating ARA for equation 3.11 we obtain:

$$ARA = \frac{abe^{bW}}{ae^{bW}} = -b \quad (3.14)$$

which is constant, hence the name constant absolute risk aversion function.

Theorem 3.2. *Given Arrow-Debreu securities, a necessary and sufficient condition for monetary separation is that marginal utility U' takes the form of 3.8 or of a constant absolute risk aversion function (3.11).*

Proof. The proof is done for the special case of a 3-securities market. For the proof see [6]. □

These findings can be generalized if we don't assume Arrow-Debreu securities or a risk-free asset to exist.

Theorem 3.3. *A necessary and sufficient condition for separation in general markets is that marginal utility U' takes the form of the constant relative risk aversion (3.9) or that there is a quadratic utility function (3.10).*

Proof. We have already seen which utility functions lead to separation in the special case of Arrow-Debreu markets. Naturally the utility functions that lead to separation in general markets are a subset of those. We first of all check the subcases 3.8 and 3.11. Only then do we check the more general cases 3.2 and 3.3 (excluding the subcases). The proof for the 3-securities case can be found in [6]. \square

The set of utility functions which lead to separation in general markets is surprisingly limited. It is understandable though, given that separation has to attain for all possible distributions of security returns.

We have just learned that when marginal utility satisfies $AU'(W)^\alpha + BU'(W)^\beta = W$ or $U'(W)^\alpha(A + B \log U'(W)) = W$ (3.2 or 3.3) in Arrow-Debreu markets, we can construct 2 mutual funds that are sufficient to represent all relevant market opportunities of the original securities. In order for one of the mutual funds to be money in Arrow-Debreu markets, marginal utility must be of the form $U'(W) = (a + bW)^c$ or $U'(W) = ae^{bW}$ (3.8 or 3.11, the constant absolute risk aversion function).

3.2 The Mutual Fund Theorem in a $\mu - \sigma$ world

Merton derives the efficient portfolio frontier explicitly for more than three securities [20]. This is a significant new finding, given that before the frontier was only described qualitatively and in graphs. The most important implication for us is the following theorem:

Theorem 3.4. *Given n risky assets among which no one can be represented as a linear combination of the other securities (the variance-covariance matrix is non-singular), we can construct two mutual funds, such that all risk averse investors,*

whose utility function depends only on the mean and variance of the portfolio (μ_P and σ_P^2), will be indifferent between investing in the n original securities or the two mutual funds derived from them.

Proof. We will use the following result: the frontier portfolio consists of

$$h_k = \frac{\mu_P \sum_{j=1}^n v_{kj}(C\mu_j - A) + \sum_{j=1}^n v_{kj}(B - A\mu_j)}{D} \quad (3.15)$$

For certain coefficients v_{ij} , the elements of the inverse variance-covariance matrix and A, B, C, D functions of v_{kj} and μ_k and $\mu_P = \sum_{i=1}^n h_i \mu_i$. For the derivation and the exact definitions we refer the reader to [20]. Let a_i and b_i denote the proportion of the first and respectively second mutual fund's value invested in the i^{th} security. In order to simplify 3.15 we define $f_k = \frac{\sum_{j=1}^n v_{kj}(B - A\mu_j)}{D}$ and $g_k = \frac{\sum_{j=1}^n v_{kj}(C\mu_j - A)}{D}$. Plugging those into 3.15, we can simplify the equation:

$$h_k = \mu_P g_k + h_k = \lambda a_k + (1 - \lambda) b_k \quad \text{for } k = 1, \dots, n \quad (3.16)$$

This holds true since we assume the portfolio to only consist of a combination of the two mutual funds. All solutions of this equation will satisfy $\lambda = \delta \mu_P - \alpha$, for some α and δ which are constant and $\delta \neq 0$. Plugging this into 3.16, we obtain

$$h_k = \lambda a_k + (1 - \lambda) b_k = (\delta \mu_P - \alpha) a_k + (1 - \delta \mu_P + \alpha) b_k = \mu_P (\delta a_k - \delta b_k) + (b_k + \alpha b_k - \alpha a_k) \quad (3.17)$$

Assuming further that a_k and b_k are independent of μ_P , from comparing the above equation with 3.16, we get that $g_k = \delta(a_k - b_k)$ and $f_k = b_k - \alpha(a_k - b_k)$, for $k = 1, \dots, m$. For $\delta \neq 0$ we can solve this system of 2 equations:

$$\begin{aligned} a_k &= f_k + \alpha \frac{g_k}{\delta} + \frac{g_k}{\delta} = b_k + \frac{g_k}{\delta} \\ b_k &= f_k + \alpha \frac{g_k}{\delta} \end{aligned} \quad (3.18)$$

The vectors a and b are linearly independent basis for the vector space of the frontier portfolios. Using 3.18, we can derive both mutual fund compositions from their

expected return. Plugging a_k and b_k into $\mu_a = \sum_{i=1}^n a_i \mu_i$ and $\mu_b = \sum_{i=1}^n b_i \mu_i$ we obtain the following expressions for the expected return of the mutual funds (we use $\sum_k f_k \mu_k = 0$ and $\sum_k g_k \mu_k = 1$):

$$\begin{aligned}\mu_a &= \sum_k a_k \mu_k = \sum_k b_k \mu_k + \frac{1}{\delta} \sum_k g_k \mu_k = \frac{1 + \alpha}{\delta} \\ \mu_b &= \sum_k b_k \mu_k = \sum_k f_k \mu_k + \sum_k g_k \mu_k = \frac{\alpha}{\delta}\end{aligned}\tag{3.19}$$

Alternatively we also can write:

$$\begin{aligned}\alpha &= \frac{\mu_b}{\mu_a - \mu_b} \\ \delta &= \frac{1}{\mu_a - \mu_b}\end{aligned}\tag{3.20}$$

From this it is apparent that the choice of α and δ does not depend in the individual investor's preference. The investor can choose α and δ arbitrarily ($\delta \neq 0$). Then the system of equations 3.18 describes the composition of the individual funds. Consequently it is sufficient for the individual investor to know the mean, variance and covariance of the two mutual funds to determine how to split the money among the two funds, or in other words to choose λ .

□

Remark. *Merton showed that $\sigma_{ij}^2 \geq \sigma_i^2 > 0$. The interpretation of this is that all efficient portfolios are positively correlated. When we add one riskless security, we can say even more: in this case all efficient portfolios are perfectly correlated. Since showing this is beyond the scope of this thesis, the authors refer the interested reader to [20].*

In this case theorem 3.4 obviously holds, since any 2 distinct portfolios on the efficient portfolio frontier are appropriate mutual funds. We can, however, achieve a stronger result and describe the mutual funds in greater detail.

Theorem 3.5. *Given n risky assets and one riskless asset, we can construct a unique pair of mutual funds where one is risky and the other one is riskless, such that all risk averse investors, whose utility function depends only on the mean and variance of the portfolio (μ_P and σ_P^2), will be indifferent between investing in the $n + 1$ original securities or the two mutual funds derived from them if and only if the risk-free rate of return is less than the expected return on the minimum variance portfolio.*

Proof. The proof follows a very similar logic as the proof for theorem 3.4 and can be found in [20]. □

Before Merton's findings it was common to derive the mutual fund by drawing the efficient portfolio frontier and then drawing the highest possible tangent of the frontier which goes through the portfolio which consists only of riskless assets $(0, r)$. The following graph illustrates the procedure:

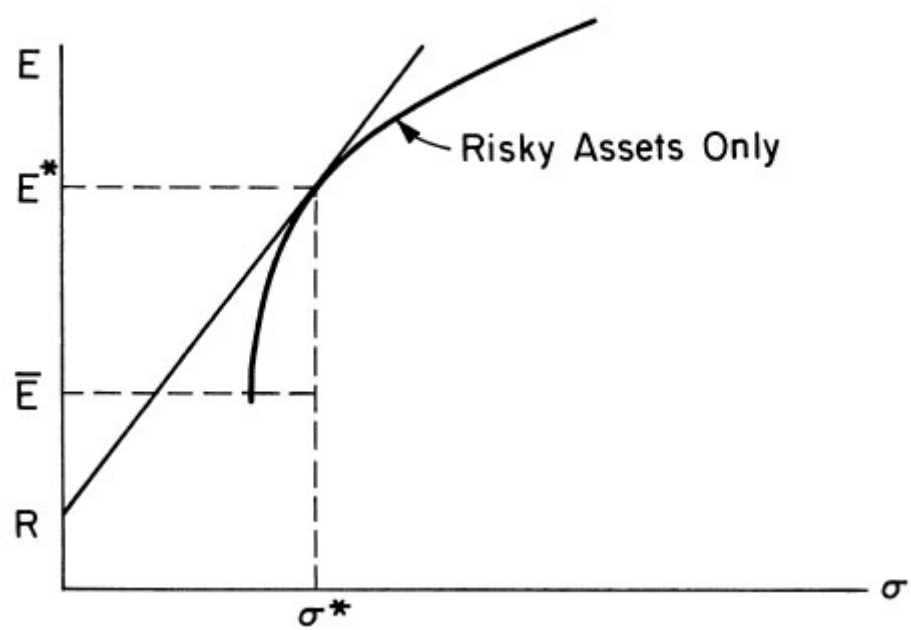


Figure 3.1: Mutual fund with risk-free asset

Source: [20]

Chapter 4

The Mutual Fund Theorem and Separation in continuous time

The remaining chapters will be more demanding from a technical point. Therefore we begin by introducing a few key concepts and definitions.

Definition. Let Ω be a nonempty set and let \mathcal{F} be a collection of sets of subsets of Ω . Then \mathcal{F} is a σ -**algebra** if the following conditions are met:

- (i) the empty set \emptyset belongs to \mathcal{F}
- (ii) if $A \in \mathcal{F}$, then also the complement $A^c \in \mathcal{F}$
- (iii) if $A_1, A_2, \dots \in \mathcal{F}$, then also their union $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$

Definition. Let Ω be a nonempty set and let \mathcal{F} be a collection of sets of subsets of Ω . Then a **probability measure** \mathbb{P} is a function that assigns every set $A \in \mathcal{F}$ a value from the interval $[0,1]$. We call that number the probability of A or $\mathbb{P}(A)$. Furthermore we require:

- (i) $\mathbb{P}(\Omega) = 1$
- (ii) if $A_1, A_2, \dots \in \mathcal{F}$, then $\mathbb{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$

We refer to the triple $(\Omega, \mathcal{F}, \mathbb{P})$ as a **probability space**.

We have used the concept of a random variable before. Now we can define it formally.

Definition. A **random variable** is a real valued function X defined on Ω with the property that for every Borel subset¹ B of \mathbb{R} the subset of Ω given by $X \in B = \{\omega \in \Omega; X(\omega) \in B\}$ is in the σ algebra \mathcal{F} .

Definition. A stochastic process (X_n) is called a **martingale** with respect to the filtration \mathcal{F} , if (X_n) is integrable and $E(X_{n+1}|\mathcal{F}_n) = X_n$, and in continuous time: $E(X_t|\mathcal{F}_s) = X_s$ for $0 \leq s \leq t$. A martingale is what we think of as a “fair game”. We call a stochastic process a submartingale if $E(X_t|\mathcal{F}_s) \geq X_s$ and a supermartingale if $E(X_t|\mathcal{F}_s) \leq X_s$, both for $0 \leq s \leq t$.

Definition. For each $\omega \in \Omega$ suppose there is a continuous function $W(t)$ with $t \geq 0$ that satisfies $W(0) = 0$ and that depends on ω . Then $W(t)$ is a **Brownian Motion**, if for all $0 = t_0 < t_1 \dots < t_n$ the increments $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent and each of these increments is normally distributed with

$$\begin{aligned} E[W(t_{i+1}) - W(t_i)] &= 0 \\ \text{Var}[W(t_{i+1}) - W(t_i)] &= t_{i+1} - t_i \end{aligned} \tag{4.1}$$

To put this in words, the increments have mean zero and their variance equals the duration of the interval. For a definition of the **Filtration for the Brownian Motion** we refer the reader to [30].

Next we will try to understand the expression $\int_0^T X(t)dW(t)$, where $X(t)$ is an adapted stochastic process ($X(t)$ is \mathcal{F}_t -measurable) and $W(t)$ a Brownian Motion.

¹Borel sets are the sets that can be constructed from open or closed sets by repeatedly taking countable unions and intersections.

Since we can't interpret this expression directly, we define the **Itô Integral** $I(T)$ of $X(T)$ for $t_n \leq T \leq t_{n+1}$:

$$I(T) = \int_0^T X(t) dW(t) = \sum_{i=0}^{n-1} X(t_i)[W(t_{i+1}) - W(t_i)] + X(t_n)[W(T) - W(t_n)] \quad (4.2)$$

A few key properties of the Itô Integral are:

- (i) **Adaptedness:** For each T , $I(T)$ is $\mathcal{F}(T)$ -measurable
- (ii) **Linearity:** If $J(T) = \int_0^T Y(t) dW(t)$, then $I(T) \pm J(T) = \int_0^T (X(t) \pm Y(t)) dW(t)$
and $cI(T) = \int_0^T cX(t) dW(t)$
- (iii) **Martingale:** $I(T)$ is a martingale
- (iv) **Continuity:** $I(T)$ is a continuous function of the upper limit of integration T

Further details and proofs for those properties can be found both in the lecture notes and the book of Shreve ([28] and [30]).

Let us consider Itô's Lemma next:

Theorem 4.1. Itô's Lemma

Let (X_t) be an Itô Process and $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is C^2 , then

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s) dX_s + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s) \sigma_s^2 ds \quad (4.3)$$

Which is equivalent to

$$d(f(t, X_t)) = \frac{\partial f}{\partial t}(t, X_t) dt + \frac{\partial f}{\partial x}(t, X_t) dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \sigma_t^2 dt \quad (4.4)$$

Since the Itô Integral plays such an important role in modern stochastics, we highlight another notation for the same ideas, as presented in [23]:

$$dF(S_t, t) = F_s dS_t + F_t dt + \frac{1}{2} F_{SS} \sigma_t^2 dt \quad (4.5)$$

A concept central to changing the probability measure is the following:

Definition. When two probability measures \mathbb{P} and $\tilde{\mathbb{P}}$ give positive probability to every member of Ω , we call the **Radon-Nikodym derivative** of $\tilde{\mathbb{P}}$ with respect to \mathbb{P}

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad (4.6)$$

4.1 Separation under log-normally distributed asset returns

We assume a perfect market in continuous time with no transaction costs and that the share prices $S_i(t)$ are generated by the Itô Processes:

$$\frac{dS_i}{S_i} = \alpha_i(S, t)dt + \sigma_i(S, t)dW_i \quad (4.7)$$

In this context the α_i are the instantaneous expected percentage change in price of the i^{th} asset per time unit and σ_i^2 is the instantaneous variance per unit time. dW_i represents the change in a one-dimensional Wiener process. The second term supplies the unpredictable, erratic movements which are scaled by the diffusion factor σ . If α and σ are constant (we refer to α as μ in this context), the equation takes the following form (we drop the subscripts for ease of reading):

$$\frac{dS}{S} = \mu dt + \sigma dW \quad (4.8)$$

We call this process a **Geometric Brownian Motion**.

[30] presents the following result: The process generating 4.8 is of the form:

$$S(t) = S(0) \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \right) \quad (4.9)$$

which is a log-normally distributed random variable with expected value $E[\mu_{S(t)}] = e^{\mu t} S(0)$. The intuitive interpretation for the expected value is if we discount $S(t)$ with the interest rate μ , we get the present value $S(0)$.

Theorem 4.2. *Given n assets whose price changes are log-normally distributed (α_i and σ_i are constant), there exists a unique pair of mutual funds independent of the individual utility functions and wealth distributions. Furthermore, the prices of both mutual funds will also be log-normally distributed.*

Proof. Merton found the optimal portfolio to be $h_i^* = a_i + m(W, t)b_i$ with $\sum_{i=1}^n h_i = 1$ and $\sum_{i=1}^n b_i = 0$ and both a_i and b_i are constants for all i . For exact definitions of the coefficients and further details see [19]. This is the representation of a line in the hyperplane defined by $\sum_1^n h_k = 1$. Consequently there are two independent vectors (our mutual funds) which are sufficient to construct all portfolios investors could possibly want to invest in. So investors are indifferent between investing in the n original assets and the mutual funds.

For any of the mutual funds we can write the return:

$$\frac{dS_{MF}}{S_{MF}} = \sum_{i=1}^n h_i \frac{dS_i}{S_i} = \sum_{i=1}^n h_i \mu_i dt + \sum_{i=1}^n h_i \sigma_i dW_i \quad (4.10)$$

Applying Itô's Lemma to this equation, we get:

$$S_{MF}(t) = S_{MF}(0) \exp \left[\left(\sum_{i=1}^n h_i \mu_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n h_i h_j \sigma_{ij} \right) t + \sum_{i=1}^n h_i \sigma_i \int_0^t dW_i \right] \quad (4.11)$$

We recognize equation 4.9 in this, so $S_{MF}(t)$ is log-normally distributed. □

Remark. *One of the many advantages of this theorem is that whenever log-normally distributed prices are assumed, it is sufficient to only consider the two mutual funds. If we drop the assumption that α_i and σ_i are constant, theorem 4.2 changes in such a way, that all investors are indifferent between the original assets and 3 mutual funds. These mutual funds do not depend on the individual utility functions. For the proof we refer the reader to [21].*

4.2 Separation for certain asset return distributions

In his paper [25], Ross investigates which distribution of asset returns will lead to separation. Central to his argument is the following concept:

Definition. A set of random returns $s = (s_1, \dots, s_n)$ is said to **stochastically dominate** an alternative \tilde{s} if for all monotone, increasing, concave utility functions U if the following holds true

$$E[U(\tilde{s})] \leq E[U(s)] \quad (4.12)$$

The economic interpretation of this is that a risk averse investor would always prefer s to \tilde{s} . Another way of expressing this notion is:

$$\tilde{s} = s + z + \epsilon \quad (4.13)$$

where $z \leq 0$ a non positive random return and ϵ a random noise term with $E[\epsilon|s + z] = 0$. This simply means that we add a non positive random variable to s and also add a random noise term. The second condition implies the first. Ross shows in his paper [25] that these two concepts are actually equivalent.

Definition. A set of returns s exhibits **strong k-fund separability** if there exist k mutual funds, such that for any given portfolio P consisting of the n original assets there exists a portfolio $H = (\tilde{h}_1, \dots, \tilde{h}_k)$ of k mutual funds which stochastically dominates P .

There also exists a weaker form of separability, which can be defined by:

Definition. A set of returns s exhibits **weak k-fund separability** if the above defined portfolio H depends also on the utility function.

Theorem 4.3. A vector s of asset returns exhibits 1 fund separation (the investor is indifferent between investing in the n original securities and the 1 mutual fund)

if and only if the following conditions are satisfied:

$\exists z, \epsilon :$

$$(i) \quad s_i = z + \epsilon_i$$

$$(ii) \quad E[\epsilon_i|z] = 0$$

$$(iii) \quad \exists \alpha : \alpha \epsilon = 0$$

Where the s_i represent the returns of the i^{th} security and α is our mutual fund.

Proof. In order to prove sufficiency, we show that the portfolio α stochastically dominates any alternative β . We define η by

$$\beta = \alpha + \eta \text{ with } \eta e = \eta_1 + \dots + \eta_n = 0$$

From (1) and (3) we obtain

$$\begin{aligned} \alpha_i s_i &= \alpha_i(z + \epsilon_i) \text{ and } \alpha s = \alpha(z + \epsilon) = \\ &= \alpha z + \alpha \epsilon = (\alpha_1 + \dots + \alpha_n)z = z \\ \text{and } E[\beta s | \alpha s] &= E[(\alpha + \eta)(z + \epsilon) | z] = \\ &= z + E[\eta \epsilon | z] = z \end{aligned}$$

This is exactly the situation of 4.13. Hence αs stochastically dominates βs . The proof for necessity can be found in [25]. \square

It is very easy to construct n securities that exhibit 1-fund separability. We choose any random return z (which is the same for all securities) and $n-1$ random variables ϵ_i with $E[\epsilon_i|z] = 0$. Then we choose ϵ_n , such that $\alpha \epsilon = 0$. Hence all the conditions of the theorem are fulfilled and 1-fund separability obtains.

Remark. If the s_i have finite variances, α simply represents the minimum variance portfolio. See [25] for a proof of this.

Theorem 4.4. *A vector s of asset returns exhibits 2-fund separation if and only if the following conditions are satisfied:*

$\exists y, z, \epsilon :$

$$(i) \quad s_i = \mu_i + y + b_i z + \epsilon_i$$

$$(ii) \quad \forall \lambda \quad E[\epsilon_i | \lambda y + (1 - \lambda)z] = 0$$

$$(iii) \quad \exists \alpha, \beta : \alpha \epsilon = \beta \epsilon = 0$$

Where $\mu_i = E[s_i] = a_0 + a_1 b_i$ and all μ_i lie on a straight line and are a function of b_i . If b is not a constant vector, then $\alpha b \neq \beta b$. α and β are our mutual funds.

Proof. The proof is done in a similar way as the proof for theorem 4.3. We show that for any possible portfolio there is a linear combination of the mutual funds $\lambda \alpha + (1 - \lambda) \beta$ that stochastically dominates the alternative portfolio. The details can found in [25]. \square

Remark. *For any λ the linear combination of the two funds $\lambda \alpha + (1 - \lambda) \beta$ minimizes variance for a given expected return. This hints at the fact that the mutual funds are in general by no means unique. More generally, any two linear combinations of them can be used as mutual fund. In fact, any basis of the space of optimal portfolios is a set of mutual funds.*

In theorems 4.3 and 4.4 we didn't distinguish between weak and strong separability. Naturally strong separability implies weak separability. Let us look at a simple case to show that the two are actually equivalent. The figure “Equivalence of weak and strong 2-fund separability” illustrates this matter well.

In the case of two fund separation with a riskless asset, all risk averse investors will choose a portfolio consisting of the riskless asset and the market portfolio M . For any possible portfolio P consisting of the n original assets, there is a portfolio on the line L (which connects the two portfolios that only consist of the risk-free asset

and only of the market portfolio) which stochastically dominates P , hence strong separation obtains. For weak but not strong separability to obtain we would need different points on the line L to stochastically dominate the portfolio P . But no single portfolio would be ideal for all utility functions of risk averse investors. This is clearly wrong, so in this special case the two kinds of separation are equivalent. For a general proof we refer the reader to [25].

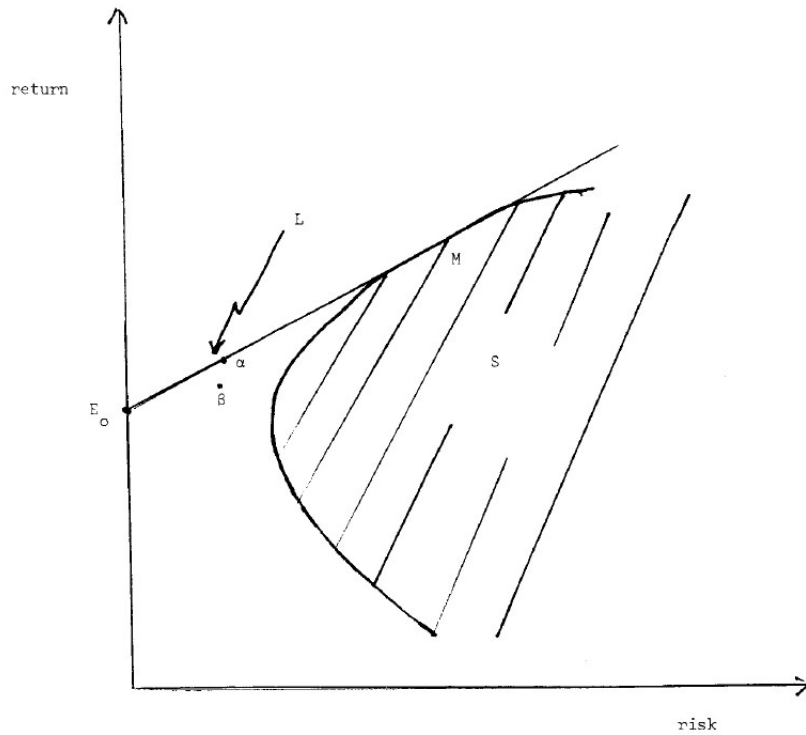


Figure 4.1: Equivalence of weak and strong 2-fund separability

Source: [25]

4.3 Separation and martingale representation

The following theorem, which we will present without giving a proof, has a central role in stochastics [30]:

Theorem 4.5. One-dimensional martingale representation

Let $W(t)$ with $0 \leq t \leq T$ be a Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t)$ be the filtration generated by the Brownian Motion. Let $M(t)$ be a martingale with respect to this filtration (for every t , $M(t)$ is $\mathcal{F}(t)$ -measurable and for $0 \leq s \leq t \leq T$: $E[M(t)|\mathcal{F}(s)] = M(s)$). Then there is an adapted process $\Gamma(u)$ with $0 \leq u \leq T$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) dW(u) \quad (4.14)$$

The martingale representation theorem asserts that when the filtration is the one generated by the Brownian Motion (the only information in \mathcal{F} is the one derived from the Brownian Motion up to time t), then every martingale with respect to this filtration is a certain starting value plus an Itô Integral with respect to the Brownian Motion.

Theorem 4.6. Multi-dimensional martingale representation

Let $W(t)$ with $0 \leq t \leq T$ be a d -dimensional Brownian Motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t)$ be the filtration generated by the Brownian Motion. Let $M(t)$ be a martingale with respect to this filtration under \mathbb{P} . Then there is an adapted process $\Gamma(u) = (\Gamma_1(u), \dots, \Gamma_d(u))$, such that

$$M(t) = M(0) + \int_0^t \Gamma(u) \cdot dW(u) \quad (4.15)$$

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Hilbert space H that consists of \mathcal{F} -measurable random variables which additionally are square integrable ($\in L^2$). We refer to the trading strategy as $\theta_t = (\theta_{0t}, \dots, \theta_{nt})$, which tells us how many shares of each security the investor holds at time t . The 0^{th} security is the risk-free asset and there are n risky securities. We say a claim $x \in H$ is marketed and write $x \in M$, if there is a trading strategy θ , such that $\theta_T Z_T = x$ with probability 1 at maturity. This means that we can replicate the payoff of the claim x by trading only in the

underlying securities.

Furthermore we assume that there exists a random variable $\rho \in H$, such that $\rho > 0$ a.s. and $\pi(x) = E(\rho x)$ for any $x \in M$, where $\pi(x)$ denotes the price at time $t = 0$ of the claim x , so $\pi(x) = \theta_0 S_0$. $S_t = (S_{0t}, \dots, S_{nt})$ is the value of the securities at time t . Then we can define a new probability measure $P^* = \int_A \rho dP$ and $E^*(x) = \int x dP^*$. The i^{th} investor has a utility function $u_i : \mathbb{R} \times H \rightarrow \mathbb{R}$ and is risk averse. Hence we try to maximize the following expression:

$$\max u_i(c, x) \text{ with } c \in \mathbb{R}, x \in M, \text{ subject to } c + \pi(x) \leq \text{initial endowment} \quad (4.16)$$

The economic interpretation of this is that the investor wants to consume part of his wealth now and invest the remaining initial endowment. In our notation, the investor consumes the amount c at $t = 0$ and invests in the claim x with uncertain future pay-offs. For the remainder of this section, let us assume that this maximization problem has the solution (c_i^*, x_i^*) . Let us further refer to the set of all claims measurable with respect to ρ as $H(\rho) = \{x \in H : x = g(\rho) \text{ a.s. for some measurable function } g : \mathbb{R} \rightarrow \mathbb{R}\}$. We also require the following notion: let Π_2 be the set of all predictable processes α with $E\left(\int_0^T \alpha_s^2 ds\right) < \infty$.

Lemma 4.1. *If $H(\rho) \subset M$, then $x_i^* \in H(\rho)$.*

Proof. We define $\hat{x}_i = E(x_i^* | \rho)$. Therefore we get $x_i^* = \hat{x}_i + e$ with $E(e | \rho) = 0$. Consequently $\hat{x}_i \in H(\rho) \subset M$ and (c_i^*, \hat{x}_i) satisfies the budget constraint $\pi(\hat{x}_i) = \pi(x_i^*) - E[\rho(x_i^* - \hat{x}_i)] = \pi(x_i^*)$. Furthermore, from $E(e | \hat{x}_i^*) = 0$ we know that (c_i, \hat{x}_i^*) is strictly preferred to (c_i^*, x_i^*) unless $e = 0$ a.s. \square

We will use the condition (R_N) in the next Lemma and Theorem: there is an N -dimensional vector W of independent Brownian Motions which are square-integrable (\mathcal{F}, P) Martingales such that $\rho \in \mathcal{F}^W$ (the σ -algebra generated by W).

Lemma 4.2. *Suppose that condition (R_N) holds. Then there is a $\gamma = (\gamma_1, \dots, \gamma_N) \in \Pi_2$ such that:*

(i) $Y_t = W_t + \int_0^t \gamma_s ds$ is an N -dimensional vector of independent Brownian motions that is at the same time a square-integrable $(\mathbb{P}^*, \mathcal{F})$ martingale.

(ii) If $x \in H(\rho)$, then $x = E^*(x) + \int_0^T \alpha_s dY_s$ for some α in Π_2

Proof. For the rather technical proof that utilizes results from Kunita-Watanabe and Girsanov, we refer to the appendix of [7]. \square

Theorem 4.7. Suppose that condition (R_N) holds and that $H(\rho) \subset M$. If we assume that the i^{th} investor choses the claim x_i^* by following the trading strategy θ_i^* , so that $\theta_{iT}^* Z_T = x_i^*$ a.s. Then there exists an $\alpha_i \in \Pi_2$, such that

$$\theta_{it}^* Z_t = \theta_{i0}^* Z_0 + \int_0^t \alpha_{is} dY_s \text{ with } (0 \leq t \leq T) \quad (4.17)$$

Proof. From Lemma 4.1 we get $x_i^* \in H(\rho)$.

From Lemma 4.2 we get $x_i^* = E^*(x_i^*) + \int_0^T \alpha_{is} dY_s$ with $(0 \leq t \leq T)$.

Since both $\theta_i^* Z$ and $\int \alpha_{is} dY_s$ are martingales, we have:

$$\theta_{it}^* Z_t = E^*(\theta_{iT}^* Z_T | \mathcal{F}_t) = E^*(x_i^* | \mathcal{F}_t) = E^*(x_i^*) + \int_0^t \alpha_{is} dY_s \quad (4.18)$$

This shows that the value of any optimal portfolio can be calculated as the stochastic integral over a martingale Y , which is the same for all optimal portfolios plus some initial price.

\square

Chapter 5

The Mutual Fund Theorem and Separation in recent papers

5.1 A generalization of the Mutual Fund Theorem

Let us assume a continuous market with a risk-free asset and n risky assets. The price dynamics of the 0^{th} and risk-free asset is given by:

$$dS_0(t) = r(t)S_0(t)dt \quad (5.1)$$

while the price of the i^{th} risky asset follows the diffusion process:

$$dS_i(t) = b_i(t)S_i(t)dt + S_i(t) \sum_{j=1}^k \sigma_{i,j}(t)dW_j(t) \quad (5.2)$$

where $W(t)$ is a k -dimensional standard Brownian Motion. Both the interest rate $r(t)$ and the volatility matrix $\sigma(t)$ are known at time t . We further assume the matrix $\sigma(t)$ to be non-singular. The latter assumption is not very restrictive, since any security that is linearly dependent on the other securities can simply be omitted in our calculations. Let $c(t)$ denote the instantaneous consumption rate. Then we can

calculate the value of the portfolio at time t :

$$X(t) = \sum_{i=0}^n N_i(t) S_i(t) \quad (5.3)$$

where $N_i(t)$ is the number of shares of the i^{th} security the investor holds at time t .

The changes in portfolio value is:

$$dX(t) = \sum_{j=1}^n H_j(t) \left(b_j(t)dt + \sum_{j=1}^n \sigma_{i,j}(t) dW_j(t) \right) + \left(X(t) - \sum_{i=1}^n H_i(t) \right) r(t)dt - c(t)dt \quad (5.4)$$

If we use vector notation we get the following expression for changes in portfolio value:

$$dX(t) = H(t) [(b(t) - r(t) \cdot 1)dt + \sigma(t)dW(t)] + [r(t)X(t) - c(t)]dt \quad (5.5)$$

In the spirit of [14], we will henceforth only consider strategies $(H(t), c(t))$, for which equation 5.5 has a unique solution. Furthermore we require the associated wealth process to be always non-negative and refer to strategies that fulfill this condition as feasible. Since we assume the variance matrix $\sigma(t)$ to be invertible, we can define $\gamma(t) = \sigma^{-1}(t)(b(t) - r(t) \cdot 1)dt$. Plugging γ into equation 5.5, we get:

$$dX(t) = H(t)\sigma(t)[\gamma(t) + dW(t)] + [r(t)X(t) - c(t)]dt \quad (5.6)$$

Let us define the discount factor $\beta(t) = \exp\left(-\int_0^t r(s)ds\right)$ and the discounted wealth process $Y(t) = \beta(t)X(t)$. We get:

$$dY(t) = \beta(t)H(t)\sigma(t)[\gamma(t)dt + dW(t)] - \beta(t)c(t)dt \quad (5.7)$$

where we can drop the summand $r(t)X(t)$ from equation 5.6 due to discounting. We can further simplify this by substituting $C(t) = c(t)\beta(t)$ and $s(t) = \beta(t)H(t)\sigma(t)$. Hence 5.7 takes the form of:

$$dY(t) = s(t)[\gamma(t)dt + dW(t)] - C(t)dt \quad (5.8)$$

Khanna and Kulldorf point out that the transformed differential equation 5.8 is a special case of 5.5 with the following properties:

- (i) zero interest rate on bond: $r(t) = 0$
- (ii) all stocks are stochastically independent: $\sigma_{i,j} = 0$, for $i \neq j$
- (iii) infinitesimal variance for each stock is 1: $\sigma_{i,i} = 1$
- (iv) infinitesimal drift for the i^{th} stock at time t is $\gamma_i(t)$

It is sufficient to show mutual fund results for equation 5.8, the results for 5.5 follow directly. We will refer to any solutions of the original problem as $(H(X, t), c(X, t))$ and to the new problem of equation 5.8 as $(s(Y, t), C(Y, t))$.

We don't pose any restrictions on the utility functions. Consequently we aim to show the following: if there exists an optimal solution, then there exists one with only 2 mutual fund, one of which is the riskless asset. If, however there isn't an optimal solution to begin with, then for any arbitrary investment strategy we can find one that is at least as good or better using only the mutual funds.

Lemma 5.1. *In a complete market, if there is an optimal investment strategy $(s(Y, t), c(Y, t))$, there is one of the form $(s^*(Y, t), c^*(Y, t))$ with*

$$s^*(Y, t) = K(t)\gamma(t) \tag{5.9}$$

for some scalar $K(t)$. If no optimal strategy exists, then for any feasible strategy there exists another one of the above form which is just as good or even advantageous to an investor.

Proof. The proof follows the following logic: Given a strategy $(s(Y, t), C(Y, t))$, we construct another investment and consumption strategy $(s^*(t), C^*(t))$ which only invests in the risk-free asset and a mutual fund consisting of the original securities. The proportion of the individual securities depends on the value of $\gamma(t) = \sigma^{-1}(t)(b(t) - r(t) \cdot 1)dt$. We proceed to show that the wealth processes

$Y(t)$ and $Y^*(t)$ have the same distribution and $E[U(Y(T))] = E[U(Y^*(T))]$. Since the discounted consumption $C^*(Y, t) \geq C(Y, t)$ we get the result that the new strategy $(s^*(Y, t), C^*(Y, t))$ is at least as good or even better than the original one. For the details of the proof, see [14].

□

Theorem 5.1. *In a complete market, if there is an optimal investment strategy $(H(t), c(t))$, then there is one of the form $(H^*(t), c^*(t))$ with*

$$H^*(X, t) = K'(X, t)\gamma(t)\sigma^{-1}(t) \quad (5.10)$$

for some scalar $K'(t)$. If no optimal strategy exists, then for any feasible strategy there exists another one of the form above which is just as good or even advantageous to an investor.

Proof. Transforming the result of Lemma 5.1, [14] achieves this result. □

Remark. *At time t only the current and past values for $r(t)$, $b(t)$ and $\sigma(t)$ are known and needed to construct the mutual fund. Hence individual investor's expectations about future values of those coefficients do not influence present investment decisions in any way.*

Remark. *If the stock returns are independent, theorem 5.1 tells us to invest in the i^{th} security in proportion to $\gamma_i(t) = \frac{(b_i(t) - r(t))}{\sigma_{i,i}}$. The economic interpretation of this is to divide the i^{th} security's excess drift (drift - interest rate) by the security's variance, in other words: the allocation of initial wealth to the i^{th} security increases with excess drift and decreases with volatility.*

5.2 A further generalization of the Mutual Fund Theorem

In this section we strive to tie all our previous findings together. We will rely heavily on [26] and consider a market on the finite time interval $[0, T]$ with one risk-free and n risky assets. We assume the process that drives stock prices S to be a locally bounded semimartingale (we can rewrite the process as the sum of a local martingale and an adapted finite-variation process). We define a portfolio as a pair (x, H) where x represents initial wealth and H the trading strategy. We can express the portfolio value:

$$W_t = x + \int_0^t H_u dS_u \text{ with } 0 \leq t \leq T \quad (5.11)$$

We call to the set of all wealth processes with nonnegative capital at all times and with initial value of x of the form of equation 5.11 $\chi(x)$. As $\hat{W}(x, U)$ we denote the optimal wealth process that maximizes the following expression:

$$\sup_{W \in \chi(x)} E[U(W_T)] \quad (5.12)$$

We will use the term mutual fund similarly as in previous chapters:

Definition. *A mutual fund for the market is any positive wealth process M with initial capital equal to one. $M \in \chi(1)$*

We say the market satisfies the Mutual Fund Theorem if there exists a mutual fund M such that for all utility functions the optimal wealth process can be written as a certain starting value x and a stochastic integral with respect to M . In other words, there exists a process $k = k(x, U)$ such that:

$$\hat{X}_t(x, U) = x + \int_0^t k_u dM_u \quad (5.13)$$

We will continue to use the notation in [26] and define:

- $G_t^W = \sigma(W_t)$

- set of all bounded European options ¹ on the numéraire N expiring at time T : $L^\infty(G_T^N)$
- set of all bounded random variables that are replicable by trading in the whole market $R(S)$
- set of all bounded random variables that are replicable by trading in the mutual fund and the risk-free asset: $R(M)$

For the following results, we need a few key assumptions.

Assumption 5.1. *The set of equivalent local martingale measures² is non-empty.*

Assumption 5.2. *The utility function is strictly increasing, strictly concave and differentiable on $(0, \infty)$ and the Inada conditions hold true:*

$$\lim_{x \rightarrow 0} U'(x) \rightarrow \infty \quad \text{and} \quad \lim_{x \rightarrow \infty} U'(x) \rightarrow 0 \quad (5.14)$$

We also require the reasonable asymptotic elasticity condition

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1 \quad (5.15)$$

We can read in [15] that many popular utility functions like $U(x) = \ln(x)$ or $U(x) = \frac{x^\alpha}{\alpha}$ for $\alpha < 1$, fulfill this condition. It is also here the the concept of asymptotic elasticity was first introduced.

Assumption 5.3. *We assume that $u(x) < \infty$ for some $x > 0$, where*

$$u(x) = \sup_{W \in \chi(x)} E[U(W_T)] \quad (5.16)$$

Assumption 5.4. $\sup_{W \in \chi(1)} E[\ln(W_T)] < \infty$

¹An option that is not path-dependent and has fixed maturity

²A probability measure $Q \sim P$ is called an equivalent local martingale measure if S is a local martingale under Q

We will refer to the optimal wealth process for initial wealth 1 and utility function \ln as N and the numéraire portfolio. In short: $N = \hat{X}(1, \ln)$.

Assumption 5.5. *The process $Z_t = \frac{1}{N_t}$ is a martingale.*

Now we are equipped to present one of the key results of this thesis:

Theorem 5.2. *Let us assume the financial market fulfills assumptions 5.1, 5.4 and 5.5. If every European option on the numéraire portfolio can be replicated by trading only in the mutual fund M and cash, then the Mutual Fund Theorem holds true with respect to all utility functions that fulfill assumptions 5.2 and 5.3. More formally:*

$$\text{If there exists a mutual fund } M \text{ such that } L^\infty(G_T^N) \subset R(M) \quad (5.17)$$

then the Mutual Fund Theorem holds true with respect to all utility functions that fulfill assumptions 5.2 and 5.3.

We refer to the replicability condition 5.17 as **(R)** and to the Mutual Fund Theorem as **(MFT)**.

Proof. We refer the reader to [26] for the proof. □

Remark. *As a direct consequence of theorem 5.2, if we can replicate every European option on the numéraire portfolio by trading only in the numéraire and cash, then the Mutual Fund Theorem naturally holds true with the numéraire as mutual fund. We call this condition **(R^N)** and write:*

$$L^\infty(G_T^N) \subset R(N) \quad (5.18)$$

We have seen that the Mutual Fund Theorem holds true if we assume either **(R)** or **(R^N)**. Now it is only natural to ask if the inverse also holds true, in other words are **(R)** and maybe even **(R^N)** consequences of the Mutual Fund Theorem? The answer in general is no. If we consider a financial market with only one risky asset,

naturally the (MFT) holds true with the one asset being the mutual fund. But the replicability condition doesn't necessarily hold true, see [26] example 4.5 for a counterexample. In order to get the implication in the other direction, we need to make further assumptions. The following is a promising candidate:

$$L^\infty(G_T^N) \subset R(S) \quad (5.19)$$

We will refer to this weak completeness condition as **(WC)**. This simply means that any European option on the numéraire can be replicated by trading in all available stocks and the risk-free asset. Clearly (R) implies (WC) : if we can replicate all European options by trading only in a mutual fund, we can do so by trading all available securities as well.

Theorem 5.3. *Let us assume the financial market fulfills assumptions 5.1, 5.4 and 5.5 as well as weak completeness 5.19. Further, we presume the Mutual Fund Theorem holds true with respect to all utility functions that fulfill assumptions 5.2 and 5.3. Then the replicability condition (R) 5.17 holds true.*

Proof. See [26] for the proof. □

Let us summarize the findings of the previous two theorems. Theorem 5.2 tells us that under certain assumptions, $(R) \rightarrow (MFT)$. Furthermore, as we already pointed out, (R) implies (WC) . Conversely, theorem 5.3 shows that (WC) and the (MFT) imply (R) . Hence there is equivalency. This is the central result of [26]:

$$(MFT) + (WC) \leftrightarrow (R) \quad (5.20)$$

Chapter 6

Empirical study: Can we observe the Mutual Fund Theorem?

We have discussed various versions of the Mutual Fund Theorem and numerous Separation results in great detail. Now it is only natural to ask whether we can observe those results in real financial markets.

In one of the most common versions, the 2-fund Separation Theorem (Capital Asset Pricing Model, see [24],[14] and [19]) predicts that all risk averse investors should hold only one risky mutual fund and cash. Since the mutual fund is the same for all investors, everyone should hold the risky assets in the same proportion. Only the ratio of risky to riskless assets varies with different degrees of risk aversion. This is one of the central ideas of financial mathematics, which makes it even more stunning how few empirical studies have been done to verify this. One of them, [5] surprises the reader with the following introduction:

“Popular financial advisors appear not to follow the mutual-fund separation theorem. When these advisors are asked to allocate portfolios among stocks, bonds, and cash, they recommend more complicated strategies than indicated by the theorem. Moreover, these strategies differ from

the theorem in a systematic way. According to these advisors, more risk averse investors should hold a higher ratio of bonds to stocks. This advice contradicts the conclusion that all investors should hold risky assets in the same proportion.”

Let us investigate this assertion. Basically what the CAPM states is that all investors should hold the risky assets (in our case we assume them to be risky bonds and stocks) in the same proportion with the only variable being how much of risk-free cash they hold. Hence the ratio of $\frac{\text{bonds}}{\text{stocks}}$ should remain constant for all investors. But does it? Fidelity and Vanguard, by every imaginable measure two of the foremost investment advisors, supply their clients with a plethora of asset allocation suggestions among stocks, bonds and cash (see tables 7.1 - 7.4 in the appendix). If their advice were consistent with the Mutual Fund Theorem and CAPM we would expect the ratio of bonds to stocks to remain constant.

We now plot the share of total investment in stocks as a proxy for risk tolerance on the X-axis against the ratio of bonds to stocks on the Y-axis. Again, according to the Mutual Fund Theorem we would expect to see a straight line parallel to the X-axis (even if the share of stocks increases, the composition of the risky assets remains the same and with them the bond to stock ratio). This is not, however, what we observe. In fact, figure 6.1 paints quite a different picture. Clearly all four trend lines are downward sloping, which means that as the portfolio gets riskier (higher share in stocks), the composition of risky assets gets riskier as well (decrease in the ratio of bonds to stocks).

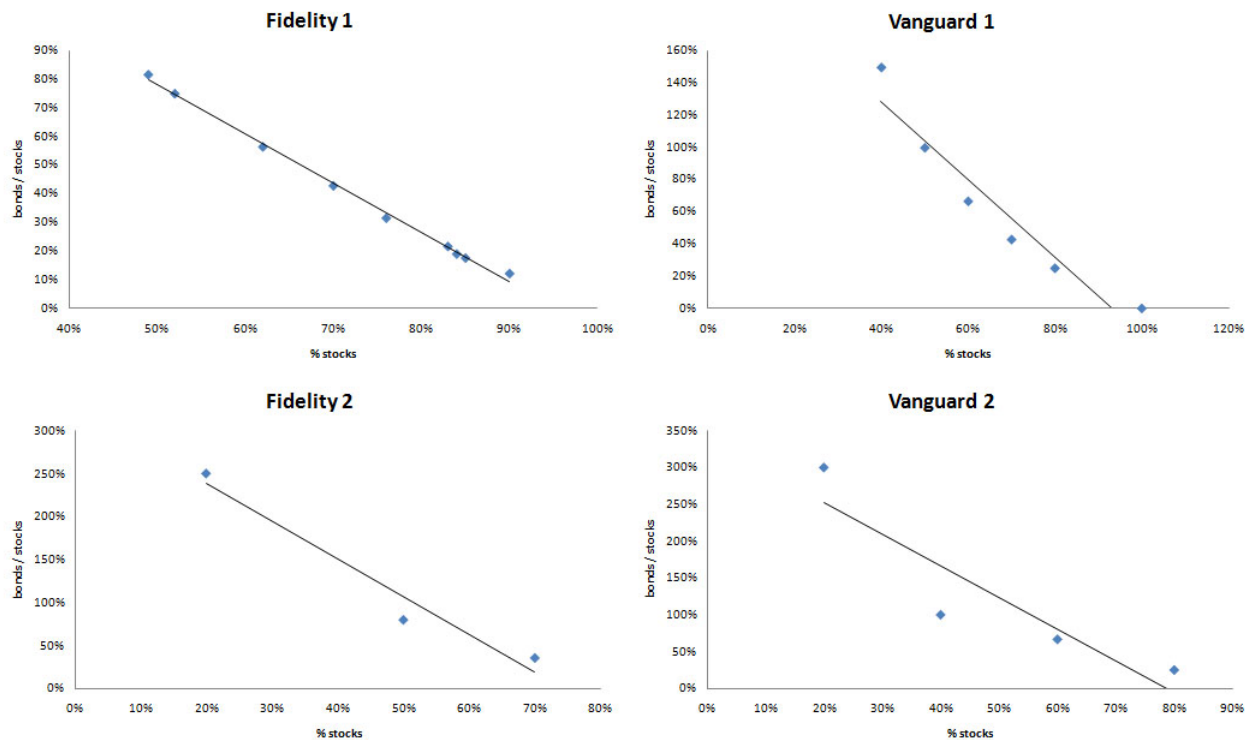


Figure 6.1: Isomean and isovariance curves

Data: see tables 7.1 - 7.4 in the appendix

What explains the discrepancy between the elegant theory and recommendations from asset managers? One obvious answer is that investors simply don't buy just one or two mutual funds. There are thousands of different funds. Still, their existence can be explained by different expectations about the distribution of future returns. Yet this does not justify our findings, since we observe a negative slope in all four graphs even though every single one of them derives from recommendations of one single asset manager and it is safe to assume that they apply the same expectations across all proposed asset combinations.

Another reason could be that people don't follow the theory, because it is too complicated. Yet in this case the conclusion of the Mutual Fund Theorem to hold the risky

assets always in the same proportion is clearly simpler than the advice of Vanguard and Fidelity. So this doesn't explain the difference either.

An alternative to assuming that people make the wrong investment choices (and asset managers propose the wrong mix of assets) is to argue that the model behind the theory is faulty. This becomes even more likely when we consider that the findings of the Mutual Fund Theorem and investment choices in the real world differ systematically. All theories rest on assumptions, in the case of CAPM it's the following five:

- (i) All assets can be freely traded
- (ii) Investors consider a one-period investment horizon
- (iii) Long and short positions in all assets are possible
- (iv) Investors consider only mean and variance of securities
- (v) A riskless asset exists

Canner et al. show in [5], that relaxing (iv) and (v) doesn't explain the disparity. Relaxing assumption (iii) is unlikely to do the trick either, since in about a third (8 out of 22) of the presented asset allocation the share of cash is > 0 . In those cases short-selling of cash clearly wasn't an option. The same is true for stocks and bonds. This effect is even more pronounced in [5], where only 2 out of 12 sample portfolios hold no cash whatsoever. Concerning (ii), if the asset returns are independently distributed over time, relaxing this assumption won't supply an explanation either (heteroskedastic stock returns and serially correlated interest rates make independence very unlikely, see [4] and [3]). If they are not, however, further research is required. There are assets that we simply can't trade, e.g. human capital. It is only natural to assume that younger people have "more human capital, so they should hold less stocks relative to bonds (since returns on human capital and on

stocks should be positively correlated). This is the opposite of what we see in figure 6.1 and table 7.1, so relaxing (i) won't help us either. We have to conclude that current research can't explain the apparent divergence between the predictions of the Mutual Fund Theorem and advice of asset managers. Clearly further research into this matter is required.

Chapter 7

Conclusion

The goal of this thesis was to give the reader an extensive overview over the development of the Mutual Fund Theorem in scientific literature. We covered almost 60 years, starting with Markowitz and Tobin ([18] and [32]) and finishing with the recent results of Schachermayer et al. [26].

The literature on the Mutual Fund Theorem is extensive, yet surprisingly few studies into whether the predictions actually hold true have been done. Further studies into why asset allocation suggestions and investment choices in general systematically seem to differ from the predictions of the Mutual Fund Theorem are needed. It is the firm belief of the authors, that such studies would help further the understanding of investment choices in particular and human decision making in general. There are exciting times ahead of us.

Appendix

Portfolio	Shares	Bonds	Cash	<u>Bonds Shares</u>
Retirement Portfolio 2050	90%	11%	0%	12%
Retirement Portfolio 2045	85%	15%	0%	18%
Retirement Portfolio 2040	84%	16%	0%	19%
Retirement Portfolio 2035	83%	18%	0%	22%
Retirement Portfolio 2030	76%	24%	0%	32%
Retirement Portfolio 2025	70%	30%	0%	43%
Retirement Portfolio 2020	62%	35%	3%	56%
Retirement Portfolio 2015	52%	39%	9%	75%
Retirement Portfolio 2010	49%	40%	10%	82%

Table 7.1: Fidelity 1

Source: Fidelity webpage

Portfolio	Shares	Bonds	Cash	<u>Bonds Shares</u>
Growth	70%	25%	5%	36%
Balanced	50%	40%	10%	80%
Conservative	20%	50%	30%	250%

Table 7.2: Fidelity 2

Source: Fidelity webpage

Portfolio	Shares	Bonds	Cash	$\frac{\text{Bonds}}{\text{Shares}}$
Balanced 1	40%	60%	0%	150%
Balanced 2	50%	50%	0%	100%
Balanced 3	60%	40%	0%	67%
Growth 1	70%	30%	0%	43%
Growth 2	80%	20%	0%	25%
Growth 3	100%	0%	0%	0%

Table 7.3: Vanguard 1

Source: Vanguard webpage

Portfolio	Shares	Bonds	Cash	$\frac{\text{Bonds}}{\text{Shares}}$
Growth	80%	20%	0%	25%
Moderate Growth	60%	40%	0%	67%
Conservative Growth	40%	40%	20%	100%
Income	20%	60%	20%	300%

Table 7.4: Vanguard 2

Source: http://www.bogleheads.org/wiki/Vanguard_LifeStrategy_Funds

List of Figures

2.1	Isomean and isovariance curves	7
2.2	Efficient portfolios Tobin	9
3.1	Mutual fund with risk-free asset	18
4.1	Equivalence of weak and strong 2-fund separability	27
6.1	Isomean and isovariance curves	41

Curriculum vitæ

CONTACT Gusshausstrasse 23/2/21 *Phone:* +43 (680) 20 55 975
1040 Vienna *Email:* tom.weitz@gmail.com

EDUCATION 10/2005 - present: **University of Vienna**, Vienna

- Mathematics (Diplom), Majors: Probability theory, Stochastics
- January 2008: Completed undergraduate studies

10/2005 - present: **WU Vienna**, Vienna

- International BA (Diplom), Majors: Finance, Entrepreneurship
- January 2008: Completed undergraduate studies

02/2009 - 06/2009: **Université Pierre et Marie Curie**, Paris

- Mathematics (Licence and Master)
- Erasmus exchange term

09/1996 - 06/2004: **Gymnasium Theresianische Akademie**, Vienna

- A-Levels with honors

WORK 11/09 - present: Teaching assistant at **WU Vienna**, Vienna

- Supporting several advanced finance classes

09/2009 - 10/2009: Internship at **JPMorgan**, London and Frankfurt

- Credit and Rates Sales, Credit and Equity Structuring

07/2009 - 08/2009: Internship at **Nomura**, London

- Credit Structuring, FX Sales

10/2008 - 12/2008 - 08/2009: Internship at **Lazard**, Frankfurt

- M&A

07/2008 - 09/2008 - 08/2009: Internship at **Oliver Wyman**, The Hague and Frankfurt

- Consulting

03/07 - 06/08: Teaching assistant at **WU Vienna**, Vienna

- Supporting a 1st year class in finance

Bibliography

- [1] A. Bain. Stochastic calculus, 2007.
- [2] F. Black, M. Jensen, and M. Scholes. The capital asset pricing model: Some empirical tests. *Studies in the theory of Capital Markets*, 1972.
- [3] S. Brown and J. Warner. Using daily stock returns. *Journal of financial economics*, 1985.
- [4] J. Y. Campbell and L. Hentschel. No news is good news: An asymmetric model of changing volatility in stock returns. *Journal of Financial Economics*, 31(3):281–318, Jun 1992.
- [5] N. Canner, N. G. Mankiw, and D. N. Weil. An asset allocation puzzle. *The American Economic Review*, 87(1):pp. 181–191, Mar 1997.
- [6] D. Cass and J. Stiglitz. The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds. *Journal of Economic Theory*, 2:122–160, 1970.
- [7] G. Chamberlain. Asset pricing in multiperiod securities markets. *Econometrica*, 56:1283–1300, Nov 1988.
- [8] F. Delbaen and W. Schachermayer. *The Mathematics of Arbitrage*. Springer, 2006.

- [9] H. Föllmer and A. Schied. *Stochastic Finance*. Walter de Gruyter, 2004.
- [10] E. D. Giorgi. Advanced mathematics and statistics. *Lecture Notes*, HSG, 2010.
- [11] N. Hakansson. Risk disposition and the separation property in portfolio selection. *The Journal of Financial and Quantitative Analysis*, 4(4):401–416, Dec 1969.
- [12] F. Hofbauer. Höhere Wahrscheinlichkeitstheorie. *Lecture notes*, Vienna University.
- [13] J. Hull. *Options, Futures and Other Derivatives*. Prentice Hall, 6th edition, 2006.
- [14] A. Khanna and M. Kulldorff. A generalization of the mutual fund theorem. *Finance and Stochastics*, 3(2), 1999.
- [15] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9(3):pp. 904–950, Aug 1999.
- [16] L. Kruschwitz. *Finanzierung und Investition*. Oldenburg, 4th edition, 2004.
- [17] J. Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *The Review of Economics and Statistics*, 47(1):pp. 13–37, Feb 1965.
- [18] H. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, Mar 1952.
- [19] R. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

- [20] R. Merton. An analytic derivation of the efficient portfolio. *The Journal of Financial and Quantitative Analysis*, 7(4):1851–1872, Sep 1972.
- [21] R. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41(5):867–887, Sep 1973.
- [22] J. Mossin. Equilibrium in a capital asset market. *Econometrica*, 34(4):pp. 768–783, Oct 1966.
- [23] S. Neftci. *An Introduction to the Mathematics of Financial Derivatives*. Academic Press, 2000.
- [24] J. Pratt. Risk aversion in the small and in the large. *Econometrica*, 32(1/2):122–136, Jan - Apr 1964.
- [25] S. Ross. Mutual fund separation and financial theory the separating distributions. *Journal of Financial Economics*, 17:254–86, 1978.
- [26] W. Schachermayer, M. Sirbu, and E. Taflin. In which financial markets do mutual fund theorems hold true? *Finance and Stochastics*, 13(1):44–77, Jan 2009.
- [27] W. F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance*, 19(3):pp. 425–442, Sep 1964.
- [28] S. Shreve. Stochastic calculus and finance. *Lecture notes*, Carnegie Mellon, 1997.
- [29] S. Shreve. *Stochastic Calculus for Finance 1*. Springer, 2004.
- [30] S. Shreve. *Stochastic Calculus for Finance 2*. Springer, 2004.
- [31] M. Steele. *Stochastic Calculus and Financial Applications*. Springer, 2001.

- [32] J. Tobin. Liquidity preference as behavior towards risk. *The Review of Economic Studies*, 25:65–86, Feb 1958.