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Contents

Introduction

Combinatorial torsion is an invariant of CW-complexes and homotopy equivalences between them. Like homology or cohomology, the theory is to a large extent purely algebraic, and we will first describe this algebraic part, which involves a far-reaching generalization of the determinant of a matrix.

First, instead of finite dimensional vector spaces over fields, one considers finitely generated projective modules over arbitrary (associative) rings. Define

$$K_1(R) = \left(\varinjlim \operatorname{Aut}(R^n)\right)^{\mathrm{ab}}$$

i.e. $K_1(R)$ is the abelization of the direct limit of the general linear groups of R. The image of an automorphism in $K_1(R)$ may be viewed as a generalized determinant, first, because in the case where R is a field det induces an isomorphism $K_1(R) \cong R^*$, and second, because many of the properties carry over to the general case. The group $K_1(R)$ is one of the lower algebraic K-groups studied in algebraic K-theory, the other being $K_0(R)$.

The determinant of an automorphism f of a vector space V may be invariantly defined as the top exterior power $\Lambda^n f : \Lambda^n V \to \Lambda^n V$, where $n = \dim V$. Note, however, that this definition extends from automorphisms to isomorphisms. This generalization of the determinant produces not an element in an abelian group, but a morphism in the groupoid $\mathcal{G}$ of one dimensional vector spaces. This groupoid has an important additional structure: the tensor product, a bifunctor $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, turning $\mathcal{G}$ into a "2-group", a special case of a monoidal category in which the objects are invertible, in a certain sense.

The aim of the first chapter is to show, as in [4], how the two generalizations described above can be combined, which turns out to be essentially a category-theoretic problem.

For applications to algebraic topology it is necessary to extend the construction from finitely generated projective modules and isomorphisms to chain complexes of such modules and quasi-isomorphisms between them. This is done in the second chapter using basic homological algebra.

In the third chapter we apply the algebraic machinery of the first two chapters to CW-complexes and homotopy equivalences between them. To obtain a non-trivial theory, one considers G-spaces and equivariant maps for some fixed (discrete) group G. In the classical case, G is the fundamental group acting on a universal cover. The result is refined Whitehead torsion, in the sense of Turaev [30].

The group $K_1(R)$ is often difficult to use for concrete computations. One strategy is choosing a homomorphism $R \to \mathrm{Aut}(V)$, V a vector space, which gives a functor $V \otimes .$ from modules to vector spaces. In the above context, this translates Whitehead torsion to Reidemeister torsion, more precisely the "refined" variant of Turaev.

Applications of the theory, such as the topological classification of lens spaces or connections with knot theory, are not discussed. We refer the reader to [17], [31] and [23] for overviews.

In the final chapter we study combinatorial torsion of smooth manifolds. We discuss two approaches to obtain a CW-complex: smooth triangulations and Morse functions.

Chapter 1

2-Group Completion

Given a commutative monoid M, the *Grothendieck group* or *group completion* is an abelian group G with a homomorphism $i : M \to G$ which is universal with respect to homorphisms from M to abelian groups. The aim of this chapter is to present a "categorified" version of this construction where M is a symmetric monoidal groupoid and G is a symmetric 2-group together with a functor $V : M \to G$.

Definitions related to monoidal categories, as well as coherence for such categories, are discussed in the first section.

The next section gives the definition of 2-groups and related elementary observations.

In the third section the 2-group completion of a symmetric monoidal groupoid is constructed. Applied to the groupoid of finitely generated projective modules over a fixed ring, a relation to the lower K-groups, $K_0(R)$ and $K_1(R)$, is established.

The determinant line functor, essentially the top exterior power on finite dimensional vector spaces and isomorphisms, is shown to be a special case of the 2-group completion in the fourth section.

1.1 Monoidal categories

A *monoidal category*, as introduced by Mac Lane [20], is a category $\mathcal{C}$ with an object $1 \in \mathcal{C}$, a functor $. \otimes . : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and for $X, Y, Z \in \mathcal{C}$ natural isomorphisms

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$
$$\lambda_X : 1 \otimes X \to X$$
$$\rho_X : X \otimes 1 \to X$$

such that the diagrams

$$\begin{tikzcd}
 & & (X \otimes Y) \otimes (Z \otimes W) \arrow[drr, "\alpha_{X \otimes Y, Z, W}"] & & \\
X \otimes (Y \otimes (Z \otimes W)) \arrow[urr, "\alpha_{X,Y,Z \otimes W}"] \arrow[dr, "\mathrm{id}_X \otimes \alpha_{Y,Z,W}"'] & & & & ((X \otimes Y) \otimes Z) \otimes W \\
 & X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "\alpha_{X, Y \otimes Z, W}"'] & & (X \otimes (Y \otimes Z)) \otimes W \arrow[ur, "\alpha_{X,Y,Z} \otimes \mathrm{id}_W"'] &
\end{tikzcd} \tag{1.1}$$

and

$$\begin{tikzcd}
X \otimes (1 \otimes Y) \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, "\mathrm{id}_X \otimes \lambda_Y"'] & & (X \otimes 1) \otimes Y \arrow[dl, "\rho_X \otimes \mathrm{id}_Y"] \\
 & X \otimes Y &
\end{tikzcd} \tag{1.2}$$

commute for all $X, Y, Z, W \in \mathcal{C}$.

A *symmetric monoidal category* is a monoidal category with natural isomorphisms

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

for $X, Y \in \mathcal{C}$ such that the diagrams

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\mathrm{id}_X \otimes \sigma_{Y,Z}"] \arrow[d, "\alpha_{X,Y,Z}"'] & X \otimes (Z \otimes Y) \arrow[r, "\alpha_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[d, "\sigma_{X,Z} \otimes \mathrm{id}_Y"] \\
(X \otimes Y) \otimes Z \arrow[r, "\sigma_{X \otimes Y, Z}"'] & Z \otimes (X \otimes Y) \arrow[r, "\alpha_{Z,X,Y}"'] & (Z \otimes X) \otimes Y
\end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd}
1 \otimes X \arrow[rr, "\sigma_{1,X}"] \arrow[dr, "\lambda_X"'] & & X \otimes 1 \arrow[dl, "\rho_X"] \\
 & X &
\end{tikzcd} \tag{1.4}$$

and

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\sigma_{X,Y}"] \arrow[dr, "\mathrm{id}_{X \otimes Y}"'] & Y \otimes X \arrow[d, "\sigma_{Y,X}"] \\
 & X \otimes Y
\end{tikzcd} \tag{1.5}$$

commute for all $X, Y, Z \in \mathcal{C}$.

Any category with finite products has a symmetric monoidal structure, where $X \otimes Y$ is a product of X and Y, 1 is a terminal object, and the natural isomorphisms are given by the universal property of products, which also implies commutativity of the various diagrams. Dually, the same is true for categories with finite coproducts, where 1 is an initial object.

A *(strong) monoidal functor* is a functor $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories together with natural isomorphisms

$$\Phi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$$

such that

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\Phi_{X,Y} \otimes \mathrm{id}_{F(Z)}"] \arrow[d, "\alpha_{F(X),F(Y),F(Z)}"] & F(X \otimes Y) \otimes F(Z) \arrow[r, "\Phi_{X \otimes Y,Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "F(\alpha_{X,Y,Z})"'] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\mathrm{id}_{F(X)} \otimes \Phi_{Y,Z}"'] & F(X) \otimes F(Y \otimes Z) \arrow[r, "\Phi_{X,Y \otimes Z}"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd} \tag{1.6}$$

commutes for all $X, Y, Z \in \mathcal{C}$, and an isomorphism

$$\phi : 1_D \to F(1_C)$$

such that the diagrams

$$\begin{tikzcd}
1 \otimes F(X) \arrow[r, "\lambda_{F(X)}"] \arrow[d, "\phi \otimes \mathrm{id}_{F(X)}"'] & F(X) \\
F(1) \otimes F(X) \arrow[r, "\Phi_{1,X}"'] & F(1 \otimes X) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes 1 \arrow[r, "\rho_{F(X)}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes \phi"'] & F(X) \\
F(X) \otimes F(1) \arrow[r, "\Phi_{X,1}"'] & F(X \otimes 1) \arrow[u, "F(\rho_X)"']
\end{tikzcd} \tag{1.7}$$

commute for any $X \in \mathcal{C}$. For a functor between symmetric monoidal categories we additionally require that

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\sigma_{F(X),F(Y)}"] \arrow[d, "\Phi_{X,Y}"'] & F(Y) \otimes F(X) \arrow[d, "\Phi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F(\sigma_{X,Y})"'] & F(Y \otimes X)
\end{tikzcd} \tag{1.8}$$

commutes.

Let (F, Φ, ϕ) and (G, Γ, γ) be monoidal functors. A natural transformation

$\beta : F \to G$ is *monoidal* if the diagrams

$$\begin{tikzcd} F(X)\otimes F(Y) \arrow[r, "\beta_X\otimes\beta_Y"] \arrow[d, "\Phi_{X,Y}"'] & G(X)\otimes G(Y) \arrow[d, "\Gamma_{X,Y}"] \\ F(X\otimes Y) \arrow[r, "\beta_{X\otimes Y}"'] & G(X\otimes Y) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d, "\phi"'] \arrow[dr, "\gamma"] & \\ F(1) \arrow[r, "\beta_1"'] & G(1) \end{tikzcd} \tag{1.9}$$

commute.

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence* of monoidal categories if there exists a monoidal functor $G : \mathcal{D} \to \mathcal{C}$ and monoidal natural isomorphisms $\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}$, $\eta : \mathrm{id}_{\mathcal{C}} \to GF$. For $\mathcal{C}, \mathcal{D}$ symmetric we require F and G to be symmetric as well.

Theorem 1.1. *A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if it is essentially surjective, full and faithful.*

Proof. It is well known that an equivalence (of ordinary categories) is essentially surjective, full, and faithful, see for example [21].

Conversely, let (F, Φ, ϕ) be a monoidal functor which is essentially surjective, full and faithful. For each object Y of $\mathcal{D}$ choose an object $G(Y)$ of $\mathcal{C}$ and an isomorphism $\varepsilon_Y : F(G(Y)) \to Y$. For a morphism $g : Y_1 \to Y_2$ of $\mathcal{D}$ let $G(g)$ be the unique morphism making the square

$$\begin{tikzcd} F(G(Y_1)) \arrow[r, "\varepsilon_{Y_1}"] \arrow[d, "F(G(g))"'] & Y_1 \arrow[d, "g"] \\ F(G(Y_2)) \arrow[r, "\varepsilon_{Y_2}"'] & Y_2 \end{tikzcd} \tag{1.10}$$

commutative. For two objects Y_1, Y_2 of $\mathcal{D}$ let Γ_{Y_1,Y_2} be the unique isomorphism making the diagram

$$\begin{tikzcd} F(G(Y_1))\otimes F(G(Y_2)) \arrow[r, "\varepsilon_{Y_1}\otimes\varepsilon_{Y_2}"] \arrow[d, "F(\Gamma_{Y_1,Y_2})\circ\Phi_{G(Y_1),G(Y_2)}"'] & Y_1\otimes Y_2 \arrow[d, equal] \\ F(G(Y_1\otimes Y_2)) \arrow[r, "\varepsilon_{Y_1\otimes Y_2}"'] & Y_1\otimes Y_2 \end{tikzcd} \tag{1.11}$$

commute. Let γ be the unique isomorphism such that

$$\begin{tikzcd} 1 \arrow[d, "F(\gamma)\circ\phi"'] \arrow[dr, equal] & \\ F(G(1)) \arrow[r, "\varepsilon_1"'] & 1 \end{tikzcd} \tag{1.12}$$

commutes. For an object X of $\mathcal{C}$ define $\eta_X : X \to G(F(X))$ by the equation

$$F(\eta_X) = \varepsilon^{-1}_{F(X)}. \tag{1.13}$$

Note that the fact that ε is a monoidal natural equivalence follows directly from the definitions above. It remains to be shown that (G, Γ, γ) is a monoidal functor and η is a monoidal natural equivalence. This a lengthy but routine computation which will be omitted. □

A monoidal category is *strict* if α, λ, ρ are the identity on any object. In particular, $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$, $1 \otimes X = X$ and $X \otimes 1 = X$. The following coherence theorem is due to Mac Lane, a proof can be found in [21].

Theorem 1.2. *Every monoidal category is equivalent (through monoidal functors and monoidal natural transformations) to a strict monoidal category.*

An analogous theorem for symmetric monoidal categories is not true, since the maps $\sigma_{X,X} \in \mathrm{Aut}(X \otimes X)$ are usually not the identity, and this property is preserved by equivalence. However, as shown in [21], one can canonically identify various permutations of a (formal) product such as $X \otimes Y \otimes Z$.

1.2 2-Groups

Let $\mathcal{C}$ be a monoidal category. An *inverse* of an object X in $\mathcal{C}$ is an object X^{-1} together with isomorphisms $X \otimes X^{-1} \to 1$ and $X^{-1} \otimes X \to 1$.

Definition. A *2-group* is a monoidal groupoid in which each object has an inverse. A *symmetric 2-group* is a symmetric monoidal category which is also a 2-group.

Proposition 1.3. *A monoidal groupoid $\mathcal{C}$ is a 2-group if and only if the endofunctors $L_X = X \otimes .$, $R_X = . \otimes X$ are equivalences for every object X.*

Proof. Assume first that $\mathcal{C}$ is a 2-group and let X be an object with inverse X^{-1}, then using α, the isomorphism $X \otimes X^{-1} \to 1$ and λ

$$L_X \circ L_{X^{-1}} \cong L_{X \otimes X^{-1}} \cong L_1 \cong \mathrm{id}_{\mathcal{C}} \tag{1.14}$$

and similarly $L_{X^{-1}} \circ L_X \cong \mathrm{id}_{\mathcal{C}}$, thus L_X is an equivalence. Analogously, one shows that each R_X is an equivalence.

Conversely, assume that L_X and R_X are equivalences. In particular, L_X and R_X are essentially surjective, hence there exist objects Y, Z such that $X \otimes Y \cong 1$ and $Z \otimes X \cong 1$. The usual proof shows that $Y \cong Z$, hence X has an inverse. □

Remark. The definition of a 2-group here is that of a "weak 2-group" in [1]. Proposition 1.3 shows that this is the same thing as a "Picard category" (cf. [4]) with coherent λ, ρ.

Let $\mathcal{G}$ be a 2-group. We note that given an isomorphism $f : X \to Y$ in $\mathcal{G}$, objects X^*, Y^*, and isomorphisms $X \otimes X^* \to 1$, $Y \otimes Y^* \to 1$ there is a unique isomorphism $f^* : X^* \to Y^*$ making the diagram

$$\begin{tikzcd} X \otimes X^* \arrow[d, "f \otimes f^*"'] \arrow[dr] & \\ Y \otimes Y^* \arrow[r] & 1 \end{tikzcd} \tag{1.15}$$

commute. This allows one to define a funtor $I : \mathcal{G} \to \mathcal{G}$ such that $X \otimes I(X)$ is naturally isomorphic to 1.

If $\mathcal{G}$ is a symmetric 2-group, we may choose an inverse of an object X so that the diagram

$$\begin{tikzcd} X \otimes X^{-1} \arrow[rr, "\sigma_{X,X^{-1}}"] \arrow[dr] & & X^{-1} \otimes X \arrow[dl] \\ & 1 & \end{tikzcd} \tag{1.16}$$

commutes. Such an inverse is then unique up to unique isomorphism making (1.15) with $f = \mathrm{id}_X$ commute.

For an essentially small groupoid, $\mathcal{G}$, we define $\pi_0\mathcal{G}$ to be the set of isomorphism classes of objects in $\mathcal{G}$ and for an object X of a locally small groupoid $\mathcal{G}$ define $\pi_1(\mathcal{G}, X) = \mathrm{Aut}(X)$. Assume that $\mathcal{G}$ is an essentially small monoidal category, then $\pi_0\mathcal{G}$ is a monoid. Clearly, $\mathcal{G}$ is a 2-group if and only if $\pi_0\mathcal{G}$ is a group.

For any locally small monoidal category, $\mathcal{C}$, $\pi_1(\mathcal{G}) = \pi_1(\mathcal{G}, 1)$ has two binary operations $\circ$ and $\otimes$ related by the distributivity relation

$$(a \circ b) \otimes (c \circ d) = (a \otimes c) \circ (b \otimes d) \tag{1.17}$$

which follows from functoriality of $\otimes$, and $1 = \mathrm{id}_1$ is a unit for both. The Eckmann–Hilton argument ([5])

$$\begin{aligned} a \otimes b &= (1 \circ a) \otimes (b \circ 1) = (1 \otimes b) \circ (a \otimes 1) \\ &= b \circ a \\ &= (b \otimes 1) \circ (1 \otimes a) = (b \circ 1) \otimes (1 \circ b) \\ &= b \otimes a \end{aligned}$$

shows that $\pi_1(\mathcal{G})$ is abelian.

Let $\mathcal{G}$ be a 2-group and X an object in $\mathcal{G}$, then the map

$$\mathrm{Aut}(1) \to \mathrm{Aut}(X), f \mapsto \lambda_X \circ (f \otimes \mathrm{id}_X) \circ \lambda_X^{-1} \tag{1.18}$$

is an isomorphism. This allows us to identify all the automorphism groups of $\mathcal{G}$ to a single group, denoted $\mathrm{Aut}_{\mathcal{G}}$, which is abelian by the above argument.

Remark. The map

$$\mathrm{Aut}(1) \to \mathrm{Aut}(X), f \mapsto \rho_X \circ (\mathrm{id}_X \otimes f) \circ \rho_X^{-1} \tag{1.19}$$

is in general different from (1.18), they are however identical in the symmetric case, cf. [11].

1.3 The main theorem

In the following all functors and natural transformations are assumed to be symmetric monoidal.

Theorem 1.4. *Let $\mathcal{C}$ be an essentially small symmetric monoidal groupoid, then there exists a symmetric 2-group, $\mathcal{K}(\mathcal{C})$, together with a functor $V : \mathcal{C} \to \mathcal{K}(\mathcal{C})$ such that for every symmetric 2-group $\mathcal{D}$, the functor*

$$. \circ V : \mathrm{Hom}(\mathcal{K}(\mathcal{C}), \mathcal{D}) \to \mathrm{Hom}(\mathcal{C}, \mathcal{D})$$

given by composition with V, is an equivalence of functor categories.

The proof of the theorem requires two preliminaries. First, we recall a special case of a contruction of Quillen from [8], then we define the localization of a monoidal category with respect to all morphisms.

Let $\mathcal{C}$ be a small symmetric monoidal category. We will also make the assumption that $\mathcal{C}$ is strict as a monoidal category, though this is not essential in what follows. Let $G(\mathcal{C})$ be the following symmetric monoidal category. Objects are pairs (X_1, X_2) of objects of $\mathcal{C}$. A morphism $(X_1, X_2) \to (Y_1, Y_2)$ is represented by a triple (A, f_1, f_2) where A is an object of $\mathcal{C}$ and $f_i : A \otimes X_i \to Y_i$ are morphisms. Two triples (A, f_1, f_2) and (B, g_1, g_2) represent the same morphism in $G(\mathcal{C})$ if and only if there exists an isomorphism $h : A \to B$ such that the triangle

$$\begin{tikzcd} A \otimes X_i \arrow[rr, "h \otimes \mathrm{id}_{X_i}"] \arrow[dr, "f_i"'] & & B \otimes X_i \arrow[dl, "g_i"] \\ & Y_i & \end{tikzcd} \tag{1.20}$$

commutes for $i = 1, 2$. Composition is given by

$$(A, f_1, f_2) \circ (B, g_1, g_2) = (A \otimes B, f_1 \circ (\mathrm{id}_A \otimes g_1), f_2 \circ (\mathrm{id}_A \otimes g_2)) \tag{1.21}$$

and is well defined and associative, and $\mathrm{id}_{(X_1,X_2)} = (1, \mathrm{id}_{X_1}, \mathrm{id}_{X_2})$. The monoidal product of $G(\mathcal{C})$ is defined by

$$(X_1, X_2) \otimes (Y_1, Y_2) = (X_1 \otimes Y_1, X_2 \otimes Y_2) \tag{1.22}$$

on objects, and for morphisms

$$(A, f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2), \quad (B, g_1, g_2) : (Z_1, Z_2) \to (W_1, W_2)$$

by

$$\begin{aligned} &(A, f_1, f_2) \otimes (B, g_1, g_2) = \\ &= (A \otimes B, (f_1 \otimes g_1) \circ (\mathrm{id}_A \otimes \sigma_{B,X_1} \otimes \mathrm{id}_{Z_1}), (f_2 \otimes g_2) \circ (\mathrm{id}_A \otimes \sigma_{B,X_2} \otimes \mathrm{id}_{Z_2})). \end{aligned} \tag{1.23}$$

Finally, set

$$\sigma_{(X_1,X_2),(Y_1,Y_2)} = (1, \sigma_{X_1,Y_1}, \sigma_{X_2,Y_2}) \tag{1.24}$$

making $G(\mathcal{C})$ a strict symmetric monoidal category.

There is a canonical (strict) symmetric monoidal functor $I : \mathcal{C} \to G(\mathcal{C})$ sending X to $(X, 1)$ and f to $(1, f, \mathrm{id}_1)$. To see that I is monoidal, note that $\sigma_{1,X} = \sigma_{X,1} = \mathrm{id}_X$ in any strict symmetric monoidal category.

Lemma 1.5. *Let $\mathcal{C}$, $G(\mathcal{C})$, I be as above, then for every symmetric 2-group $\mathcal{D}$, the functor*

$$. \circ I : \mathrm{Hom}(G(\mathcal{C}), \mathcal{D}) \to \mathrm{Hom}(\mathcal{C}, \mathcal{D})$$

given by composition with I, is an equivalence of functor categories.

Proof. We may assume that $\mathcal{D}$ is strict. Choose an inversion functor $A \mapsto A^{-1}$, $f \mapsto f^*$ as described in the previous section. Let $(F, \Phi, \phi) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Define a monoidal functor $(G, \Gamma, \gamma) : \mathcal{G}(\mathcal{C}) \to \mathcal{D}$ with $G \circ I \cong F$ as follows. On objects

$$G(X_1, X_2) = F(X_1) \otimes F(X_2)^{-1} \tag{1.25}$$

and for a morphism $f = (A, f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2)$ define $G(f)$ as the composition of

$$F(f_1) \otimes F(f_2)^* : F(A \otimes X_1) \otimes F(A \otimes X_2)^{-1} \to F(Y_1) \otimes F(Y_2)^{-1}$$

and the canonical map

$$\begin{aligned} F(X_1) \otimes F(X_2)^{-1} &\to F(X_1) \otimes F(A) \otimes F(A)^{-1} \otimes F(X_2)^{-1} \\ &\to F(X_1) \otimes F(A) \otimes (F(X_2) \otimes F(A))^{-1} \\ &\to F(X_1 \otimes A) \otimes F(X_2 \otimes A)^{-1} \end{aligned}$$

involving Φ. To see that G is well defined on morphisms, note that given an isomorphism $h : A \to B$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[dr] & F(A) \otimes F(A)^{-1} \arrow[d, "F(h) \otimes F(h)^*"] \\ & F(B) \otimes F(B)^{-1} \end{tikzcd}$$

commutes by definition of $F(h)^*$. Let Γ be defined by the composition of natural isomorphisms

$$\begin{aligned} G((X_1, X_2) \otimes (Y_1, Y_2)) &= F(X_1 \otimes Y_1) \otimes F(X_2 \otimes Y_2)^{-1} && (1.26) \\ &\cong F(X_1) \otimes F(Y_1) \otimes (F(X_2) \otimes F(Y_2))^{-1} && (1.27) \\ &\cong G(X_1, X_2) \otimes G(Y_1, Y_2) && (1.28) \end{aligned}$$

and γ by

$$G(1,1) = F(1) \otimes F(1)^{-1} \cong F(1) \cong 1 \tag{1.29}$$

then (G, Γ, γ) is a monoidal functor.

Suppose $F, G : G(\mathcal{C}) \to D$ are monoidal functors and $\alpha : F \circ I \to G \circ I$ is a monoidal natural transformation. The corresponding natural transformation $\beta : F \to G$ is then given by

$$\begin{aligned} F(X_1, X_2) &\cong F(X_1, 1) \otimes F(1, X_2) & (1.30)\\ &\cong F(X_1, 1) \otimes F(X_2, 1)^{-1} & (1.31)\\ &\cong G(Y_1, 1) \otimes G(Y_2, 1)^{-1} & (1.32)\\ &\cong G(Y_1, Y_2) & (1.33)\end{aligned}$$

where the second isomorphism is $\alpha_{X_1} \otimes \alpha_{X_2}^{-1}$. This completes the proof of the lemma. □

The forgetful functor from the category of small groupoids to the category of small categories has a left adjoint which assigns to a small category $\mathcal{C}$ the groupoid $\mathcal{C}^\circ := \mathrm{Ar}(\mathcal{C})^{-1}\mathcal{C}$, i.e. the localization (see Theorem A.1) with respect to the set $\mathrm{Ar}(\mathcal{C})$ of all morphisms of $\mathcal{C}$. We will show that an analogous result holds for (symmetric) monoidal categories.

Proposition 1.6. *Let $\mathcal{C}$ be a monoidal category, then there exists a monoidal groupoid $\mathcal{C}^\circ$ and a monoidal functor $Q : \mathcal{C} \to \mathcal{C}^\circ$ such that for any monoidal groupoid $\mathcal{G}$ the functor*

$$. \circ Q : \mathrm{Hom}(\mathcal{C}^\circ, \mathcal{G}) \to \mathrm{Hom}(\mathcal{C}, \mathcal{G})$$

is an equivalence of functor categories. The same statement hold for symmetric monoidal categories.

Proof. Note first that given a functor $F : \mathcal{C} \to \mathcal{D}$ of small categories there is an induced functor $F^\circ : \mathcal{C}^\circ \to \mathcal{D}^\circ$, likewise a natural transformation $\alpha : F \to G$ induces a natural isomorphism $\alpha^\circ : F^\circ \to G^\circ$, functorial with respect to the various compositions of functors and natural transformations. (In the terminology of [21], $A \mapsto A^\circ$ is a 2-functor from the 2-category of small categories to the 2-category of small groupoids.) Furthermore, by Proposition A.2, the functor $A \mapsto A^\circ$ commutes with finite products of categories. Hence, the monoidal product induces a bifunctor $\otimes^\circ : \mathcal{C}^\circ \times \mathcal{C}^\circ \to \mathcal{C}^\circ$ and the natural isomorphisms α, λ, ρ transfer to $\mathcal{C}^\circ$. Thus, $\mathcal{C}^\circ$ has the structure of a monoidal category and the canonical functor $Q : \mathcal{C} \to \mathcal{C}^\circ$ is a strict monoidal functor.

Suppose $\mathcal{G}$ is a monoidal groupoid and $(F, \Phi, \phi) : \mathcal{C} \to \mathcal{G}$ a monoidal functor. Let $G : \mathcal{C} \to \mathcal{G}$ be the unique functor with $G \circ Q = F$, let Γ be the unique natural isomorphism with $\Gamma \circ (Q \times Q) = \Phi$, and let $\gamma = \phi$, then (G, Γ, γ) is the unique monoidal functor with $G \circ Q = F$ as monoidal fuctors. Furthermore, given a monoidal natural transformation $\alpha : G \circ Q \to F \circ Q$ there is a unique monoidal natural transformation $\beta : G \to F$ with $\beta \circ Q = \alpha$. □

Proof of Theorem 1.4. We may assume that $\mathcal{C}$ is both strict and small. Let $\mathcal{K}(\mathcal{C}) = G(\mathcal{C})^\circ$ and $V = Q \circ I$. Note that for any object (X_1, X_2) of $G(\mathcal{C})$ there is a morphism

$$(X_1 \otimes X_2, \mathrm{id}_{X_1 \otimes X_2}, \sigma_{X_1,X_2}) : (1,1) \to (X_1, X_2) \otimes (X_2, X_1)$$

and hence a corresponding isomorphism in $G(\mathcal{C})^\circ$, showing that $\mathcal{K}(\mathcal{C})$ is a symmetric 2-group. The fact that

$$. \circ V : \mathrm{Hom}(\mathcal{K}(\mathcal{C}), \mathcal{D}) \to \mathrm{Hom}(\mathcal{C}, \mathcal{D})$$

is an equivalence for any 2-group $\mathcal{D}$ follows directly from Lemma 1.5 and Proposition 1.6. □

In the sequel we will use the more compact notation $\widehat{X} = V(X)$ for an object X of C and $\hat{f} = V(f)$ for a morphism of C.

Remark. Let R be a ring and let Proj_R be the symmetric monoidal groupoid of finitely generated projective R-modules with direct sum. As shown in [8], there is a homotopy equivalence

$$BG(\mathrm{Proj}_R) \sim K_0(R) \times BGL(R)^+ \tag{1.34}$$

where $B\mathcal{C}$ denotes the classifying space of a small category $\mathcal{C}$. On the other hand

$$\pi_1(B\mathcal{C}, X) = \pi_1(\mathcal{C}^\circ, X) \tag{1.35}$$

for any small category C and object X of $\mathcal{C}$, as shown in [26]. Combining these results,

$$\pi_0\mathcal{K}(\mathrm{Proj}_R) = K_0(R) \qquad \pi_1\mathcal{K}(\mathrm{Proj}_R) = K_1(R) \tag{1.36}$$

which motivates the construction in the proof of Theorem 1.4. We will give an alternative proof of (1.36) below.

Proposition 1.7. *Let $\mathcal{C}$, $\mathcal{K}(\mathcal{C})$ be as in Theorem 1.4, then*

$$\pi_0\mathcal{K}(\mathcal{C}) \cong K(\pi_0\mathcal{C}) \tag{1.37}$$

i.e. $\pi_0\mathcal{K}(\mathcal{C})$ is the group completion (Grothendieck group) of $\pi_0\mathcal{C}$, where the universal homomorphism is $\pi_0 V$.

Proof. Let A be an abelian group and $f : \pi_0\mathcal{C} \to A$ a homomorphism. Note that A corresponds to a symmetric 2-group $\mathcal{D}$ with only identity morphisms, while f defines a monoidal functor $F : \mathcal{C} \to \mathcal{D}$, where we use the fact that $\mathcal{C}$ is a groupoid. The universal property of $\mathcal{K}(\mathcal{C})$ gives a functor $G : \mathcal{K}(\mathcal{C}) \to D$ such that $G \circ V \cong F$, hence $(\pi_0 G) \circ (\pi_0 V) = f$. □

Proposition 1.8. *Let $\mathcal{C}$, $\mathcal{K}(\mathcal{C})$ be as in Theorem 1.4, then $\pi_1\mathcal{K}(\mathcal{C})$ is isomorphic to the abelian group P with generators $f \in \mathrm{Aut}(X)$, $X \in \mathrm{Ob}(\mathcal{C})$ and relations*

$$f \circ g = f + g \qquad f, g \in \mathrm{Aut}(X) \tag{1.38}$$
$$f = g \circ (\mathrm{id}_Y \otimes f) \circ g^{-1} \qquad f \in \mathrm{Aut}(X), g \in \mathrm{Hom}(Y \otimes X, Z) \tag{1.39}$$

for $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$.

Proof. Note first that for $f \in \mathrm{Aut}(X), g \in \mathrm{Aut}(Y)$, computing in P,

$$\begin{aligned} f \otimes g &= (f \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes g) && (1.40)\\ &= \sigma_{X,Y} \circ (f \otimes \mathrm{id}_Y) \circ \sigma_{Y,X} + g && (1.41)\\ &= (\mathrm{id}_Y \otimes f) + g && (1.42)\\ &= f + g && (1.43) \end{aligned}$$

hence, any element of P is represented by an automorphism of $\mathcal{C}$.

Define

$$\varphi : P \to \pi_1\mathcal{K}(\mathcal{C}) \qquad f \in \mathrm{Aut}(X) \mapsto (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (X, f, \mathrm{id}_X) \tag{1.44}$$

which is well defined since for $f, g \in \mathrm{Aut}(X)$

$$\begin{aligned} \varphi(f \circ g) &= (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (X, f \circ g, \mathrm{id}_X) && (1.45)\\ &= (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (1, f, \mathrm{id}_X) \circ (X, g, \mathrm{id}_X) && (1.46)\\ &= (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (X, f, \mathrm{id}_X) \circ (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (X, g, \mathrm{id}_X) && (1.47)\\ &= \varphi(f) \circ \varphi(g) && (1.48) \end{aligned}$$

and for $f \in \mathrm{Aut}(X), g \in \mathrm{Hom}(Y \otimes X, Z)$

$$\begin{aligned} \varphi(f) &= (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (X, f, \mathrm{id}_X) && (1.49)\\ &= (X, \mathrm{id}_X, \mathrm{id}_X)^{-1} \circ (Y, g, g)^{-1} \circ (Y, g, g) \circ (X, f, \mathrm{id}_X) && (1.50)\\ &= (Y \otimes X, g, g)^{-1} \circ (Y \otimes X, g \circ (\mathrm{id}_Y \otimes f), g) && (1.51)\\ &= (Z, \mathrm{id}_Z, \mathrm{id}_Z)^{-1} \circ (Z, g \circ (\mathrm{id}_Y \otimes f) \circ g^{-1}, \mathrm{id}_Z) && (1.52)\\ &= \varphi(g \circ (\mathrm{id}_Y \otimes f) \circ g^{-1}). && (1.53) \end{aligned}$$

Suppose $X_1, \ldots, X_n$ are objects of $\mathcal{C}$, p is a permutation of $1, \ldots, n$, $X = X_1 \otimes \ldots \otimes X_n$, $X_p = X_{p(1)} \otimes \ldots \otimes X_{p(n)}$ and $f \in \mathrm{Hom}(X, X_p)$, then f composed with either the canonical map $X_p \to X$ or its inverse $X \to X_p$ represent the same element in P. This identification is implicit in the following definition.

Let $f = f_1 \circ \ldots \circ f_n$ be an element of $\pi_1\mathcal{K}(\mathcal{C})$ where

$$f_k = (Y_k, f_{k,1}, f_{k,2})^{\epsilon_k} : (X_{k,1}, X_{k,2}) \to (X_{k+1,1}, X_{k+1,2})$$

and $\epsilon_k \in \{1, -1\}$, define

$$\psi : \pi_1\mathcal{K}(\mathcal{C}) \to P \qquad f \mapsto \bigotimes_{k=1}^{n}(f_{k,1} \otimes f_{k,2}^{-1})^{\epsilon_k} \tag{1.54}$$

where domain and codomain of $\psi(f)$ are up to permutations of the factors

$$\left(\bigotimes_{k=2}^{n} X_{k,1} \otimes X_{k,2}\right) \otimes \bigotimes_{k=1}^{n} Y_k$$

since $X_{0,1} = X_{0,2} = X_{n+1,1} = X_{n+1,2} = 1$. To see that ψ does not depend on the choice of representative in $G(\mathcal{C})$, note that for

$$(Z, f_1, f_2), (W, g_1, g_2) : (X_1, X_2) \to (Y_1, Y_2)$$

with $f_i = g_i \circ (h \otimes \mathrm{id}_{X_i})$ we have

$$f_1 \otimes g_1^{-1} \otimes f_2^{-1} \otimes g_2 = 0 \tag{1.55}$$

in P. Furthermore, it is easily seen that ψ factors through the defining relations of the localization, hence is well-defined.

We claim that $\psi = \varphi^{-1}$. The identity $\psi \circ \varphi = \mathrm{id}_P$ is trivial, it remains to show $\varphi \circ \psi = \mathrm{id}_{\pi_1\mathcal{K}(\mathcal{C})}$. Note that $G(\mathcal{C})$ has the following property: given morphisms $f : A \to B$ and $g : A \to C$ there exist an object D and morphisms $f' : B \to D$, $g' : C \to D$ such that $f' \circ f = g' \circ g$. This implies that every element in $\pi_1\mathcal{K}(\mathcal{C})$ can be represented by a morphism of the form $g^{-1} \circ f$ where $f : (1, 1) \to (X_1, X_2)$ and $g : (1, 1) \to (X_1, X_2)$ are morphisms of $G(\mathcal{C})$. It follows easily that φ is surjective, proving the claim. $\square$

We continue with an important special case. Consider the category Proj_R of finitely generated projective modules over a fixed associative ring R. The coproduct, i.e. direct sum, of modules gives Proj_R a symmetric monoidal structure. As is well known, see for example [29], Proj_R is essentially small. One defines (cf. Rosenberg [29])

$$K_0(R) = K(\pi_0\mathrm{Proj}_R)$$
$$K_1(R) = \left(\varinjlim \mathrm{Aut}(R^n)\right)^{\mathrm{ab}}$$

where $\varinjlim$ denotes the direct limit and ab the abelianization.

Theorem 1.9.

$$K_0(R) = \pi_0\mathcal{K}(\mathrm{Proj}_R) \qquad\qquad K_1(R) = \pi_1\mathcal{K}(\mathrm{Proj}_R) \tag{1.56}$$

Proof. The first identity follows directly from Proposition 1.7. For the second identity use Proposition 1.8 and the fact that every projective module is a direct summand in a free module. $\square$

1.4 Determinant lines

Let $\mathbb{K}$ be a field. The category of *weighted* $\mathbb{K}$*-lines*, $\mathscr{L}_{\mathbb{K}}$ has as objects pairs (L,n), where L is a one-dimensional $\mathbb{K}$-vector space and $n \in \mathbb{Z}$, and the morphisms from (K,m) to (L,n) are isomorphisms from K to L if $m=n$, and none if $m \neq n$. The tensor product

$$(K,m) \otimes (L,n) = (K \otimes L, m+n) \tag{1.57}$$

defines a functor $\mathscr{L}_{\mathbb{K}} \times \mathscr{L}_{\mathbb{K}} \to \mathscr{L}_{\mathbb{K}}$. The unit object is $(\mathbb{K},0)$ and the inverse of (L,n) is $(L^*,-n)$, where L^* denotes the dual line. The natural isormophisms α,λ,ρ are defined in the usual way, while for σ we use

$$\sigma_{(K,m),(L,n)} : (K,m)\otimes(L,n) \to (L,n)\otimes(K,m) : \quad v\otimes w \mapsto (-1)^{mn} w\otimes v \tag{1.58}$$

known as the *Koszul transposition.* With this structure, $\mathscr{L}_{\mathbb{K}}$ is a symmetric 2-group.

We define a functor, Det, from the category of finite dimensional $\mathbb{K}$-vector spaces and isomorphisms, $\mathrm{Vect}_{\mathbb{K}}$, to $\mathscr{L}_{\mathbb{K}}$ as follows. On objects

$$\mathrm{Det}(V) = (\Lambda^{\dim V} V, \dim V) \tag{1.59}$$

and for an isomorphisms $f : V \to W$

$$\mathrm{Det}(f) = \Lambda^{\dim V} f \tag{1.60}$$

where Λ^k is the k-th exterior power, as usual. The choice of sign in (1.58) makes

$$\begin{tikzcd}
\mathrm{Det}(V)\otimes\mathrm{Det}(W) \arrow[r, "\sigma_{\mathrm{Det}(V),\mathrm{Det}(W)}"] \arrow[d, "\Phi_{V,W}"'] & \mathrm{Det}(W)\otimes\mathrm{Det}(V) \arrow[d, "\Phi_{W,V}"] \\
\mathrm{Det}(V\oplus W) \arrow[r, "\mathrm{Det}(\sigma_{V,W})"'] & \mathrm{Det}(W\oplus V)
\end{tikzcd}$$

commutative and Det a symmetric monoidal functor.

Theorem 1.10. *The extended functor*

$$\mathrm{Det} : \mathcal{K}(\mathrm{Vect}_{\mathbb{K}}) \to \mathscr{L}_{\mathbb{K}}$$

is an equivalence of 2-groups.

Proof. It suffices to show that Det induces isomorphisms on π_0 and π_1, since this implies essential surjectivity and bijectivity on hom-sets, hence that Det is an equivalence by Theorem 1.1. In view of Theorem 1.9 this is contained in the proof that

$$K_0(\mathbb{K}) = \mathbb{Z} \qquad\qquad K_1(\mathbb{K}) = \mathbb{K}^* \tag{1.61}$$

see for example [29]. ☐

Remark. Theorem 1.10 holds more generally for commutative local rings, see [29].

Chapter 2

Algebraic Theory of Torsion

In the previous chapter we constructed a functor $\mathrm{Proj}_R \to \mathcal{K}(\mathrm{Proj}_R)$ generalizing the top exterior power in linear algebra. The aim of this chapter is to show that this functor extends to bounded complexes of modules in Proj_R and quasi-isomorphisms between them. To achieve this, we use ideas from [12] where a related extension is performed for vector bundles over schemes.

2.1 Short exact sequences

Throughout this section, R is a fixed associative ring. As before, Proj_R denotes the category of finitely generated projective (left) R-modules. A proof of the following useful characterization can be found in [29].

Proposition 2.1. *An R-module M is finitely generated and projective if and only if it is a direct summand in finitely generated free module, i.e. there exists a module N such that $M \oplus N \cong R^n$.*

Corollary 2.2. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules. Assume the sequence splits and $B \in \mathrm{Proj}_R$, then $A, C \in \mathrm{Proj}_R$.

We continue with the study of $\mathcal{K}(\mathrm{Proj}_R)$. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{2.1}$$

be a short exact sequence with $A, B, C \in \mathrm{Proj}_R$. In particular, since C is projective, the sequence splits, i.e. there is a section $s : C \to B$ with $g \circ s = \mathrm{id}_C$. The sum $f \oplus s : A \oplus C \to B$ is an isomorphism.

Lemma 2.3. *The induced isomorphism $\widehat{f \oplus s} : \widehat{A \oplus C} \to \widehat{B}$ depends only on f and g, not on the choice of section s.*

Proof. Let $\tilde{s}$ be another section of g and $t : B \to A$ the projection of $(f \oplus \tilde{s})^{-1}$ to A. Let $h = (f \oplus \tilde{s})^{-1} \circ (f \oplus s)$, which has the form

$$h = \begin{pmatrix} 1 & ts \\ 0 & 1 \end{pmatrix} \tag{2.2}$$

with respect to the sum $A \oplus C$. The identity

$$\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -s & 1 \end{pmatrix} = \begin{pmatrix} 1 & ts & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.3}$$

shows that $h \oplus \mathrm{id}_B$ is a commutator in $\mathrm{Aut}(A \oplus C \oplus B)$, hence $\widehat{h \oplus \mathrm{id}_B}$, and thus $\hat{h}$, is trivial in $\pi_1(\mathcal{K}(\mathrm{Proj}_R))$. ☐

Proposition 2.4. *Suppose*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{11} \arrow[r, "f_1"] \arrow[d, "f^1"'] & A_{12} \arrow[r, "g_1"] \arrow[d, "f^2"'] & A_{13} \arrow[r] \arrow[d, "f^3"'] & 0 \\
0 \arrow[r] & A_{21} \arrow[r, "f_2"] \arrow[d, "g^1"'] & A_{22} \arrow[r, "g_2"] \arrow[d, "g^2"'] & A_{23} \arrow[r] \arrow[d, "g^3"'] & 0 \\
0 \arrow[r] & A_{31} \arrow[r, "f_3"] \arrow[d] & A_{32} \arrow[r, "g_3"] \arrow[d] & A_{33} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.4}$$

is a commutative diagram with $A_{ij} \in \mathrm{Proj}_R$ and exact columns and rows. Then the following diagram of induced isomorphisms

$$\begin{tikzcd}
\widehat{A}_{11} \otimes \widehat{A}_{13} \otimes \widehat{A}_{31} \otimes \widehat{A}_{33} \arrow[r] \arrow[d] & \widehat{A}_{21} \otimes \widehat{A}_{23} \arrow[d] \\
\widehat{A}_{12} \otimes \widehat{A}_{32} \arrow[r] & \widehat{A}_{22}
\end{tikzcd} \tag{2.5}$$

commutes.

Proof. Choose sections s^1 of g^1, s_1 of g_1, and s of $g_3 g^2 = g^3 g_2$. Let $s^3 = g_2 s$, $s_3 = g^2 s$ and define

$$s^2 : A_{32} \to A_{22}, \quad f_3(x) + s_3(y) \mapsto f_2(s^1(x)) + s(y) \tag{2.6}$$

$$s_2 : A_{23} \to A_{22}, \quad f^3(x) + s^3(y) \mapsto f^2(s_1(x)) + s(y) \tag{2.7}$$

then $f_2 s^1 = s^2 f_3$, $f^2 s_1 = s_2 f^3$, $s^2 s_3 = s_2 s^3 = s$ by construction. Let

$$\tau : A_{11} \oplus A_{13} \oplus A_{31} \oplus A_{33} \to A_{11} \oplus A_{31} \oplus A_{13} \oplus A_{33}, \quad (x, y, z, w) \mapsto (x, z, y, w) \tag{2.8}$$

then

$$\begin{aligned}
(f^2 \oplus s^2) \circ ((f_1 \oplus s_1) \oplus (f_3 \oplus s_3)) &= f^2 f_1 \oplus f^2 s_1 \oplus s^2 f_3 \oplus s^2 s_3 && (2.9)\\
&= f_2 f^1 \oplus s_2 f^3 \oplus f_2 s^1 \oplus s_2 s^3 && (2.10)\\
&= (f_2 f^1 \oplus f_2 s^1 \oplus s_2 f^3 \oplus s_2 s^3) \circ \tau && (2.11)\\
&= (f_2 \oplus s_2) \circ ((f^1 \oplus s^1) \oplus (f^3 \oplus s^3)) \circ \tau && (2.12)
\end{aligned}$$

which shows that (2.5) commutes. □

2.2 Bounded complexes of finite projective modules

For a bounded chain complex

$$0 \longrightarrow C_n \xrightarrow{\partial} \cdots \longrightarrow C_1 \xrightarrow{\partial} C_0 \longrightarrow 0$$

with $C_k \in \mathrm{Proj}_R$ we define

$$\widehat{C_\bullet} = \bigotimes_{i=0}^{n} \widehat{C_i}^{(-1)^i}. \tag{2.13}$$

Let $C^{\mathrm{b}}(\mathrm{Proj}_R)$ denote the category of bounded complexes of finitely generated projective R-modules and chain maps.

Proposition 2.5. *Given a short exact sequence*

$$0 \longrightarrow A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \longrightarrow 0$$

in $C^{\mathrm{b}}(\mathrm{Proj}_R)$ *there is a canonical isomorphism*

$$\widehat{A_\bullet} \otimes \widehat{C_\bullet} \longrightarrow \widehat{B_\bullet}.$$

Proof. Applying Lemma 2.3 to the short exact sequences

$$0 \longrightarrow A_k \xrightarrow{f_k} B_k \xrightarrow{g_k} C_k \longrightarrow 0$$

gives an isomorphism

$$\widehat{A_k} \otimes \widehat{C_k} \longrightarrow \widehat{B_k}$$

for each k. Applying the inversion functor for odd k we obtain isomorphisms

$$\widehat{A_k}^{(-1)^k} \otimes \widehat{C_k}^{(-1)^k} \longrightarrow \widehat{B_k}^{(-1)^k}$$

and their monoidal product is an isomorphism

$$\begin{aligned}&\widehat{A_\bullet} \otimes \widehat{C_\bullet} = \\ &= \cdots \otimes \widehat{A_k}^{(-1)^k} \otimes \widehat{C_k}^{(-1)^k} \otimes \widehat{A_{k+1}}^{(-1)^{k+1}} \otimes \widehat{C_{k+1}}^{(-1)^{k+1}} \otimes \cdots \to \\ &\to \cdots \otimes \widehat{B_k}^{(-1)^k} \otimes \widehat{B_{k+1}}^{(-1)^{k+1}} \otimes \cdots = \widehat{B_\bullet}\end{aligned}$$

where the natural isomorphism σ is used several times in the identification of the first line with the second. □

Similarly, Proposition 2.4 extends to complexes. We will omit the (straightforward) details.

Let $(C_\bullet, \partial)$ be a complex and $n \in \mathbb{Z}$, then the *shifted complex*, $(C_\bullet[n], \partial[n])$ is given by $C_k[n] = C_{k+n}$ with differential $\partial_k[n] = (-1)^n \partial_{k+n}$. Additionally, setting $f_k[n] = f_{k+n}$ for a chain map f, we obtain an endofunctor on the category of chain complexes and chain maps (of some additive category). Assume now that $C_\bullet$ is a bounded complex with $C_k \in \mathrm{Proj}_R$, then in $\mathcal{K}(\mathrm{Proj}_R)$ we find

$$\widehat{C_\bullet[1]} = \widehat{C_\bullet}^{-1} \qquad \widehat{C_\bullet[k+2]} = \widehat{C_\bullet[k]}. \tag{2.14}$$

Proposition 2.6. *Let $(C_\bullet, \partial)$ be a bounded chain complex with $C_k \in \mathrm{Proj}_R$ and assume that the $H_k \in \mathrm{Proj}_R$ as well, where H_k is the k-th homology of $C_\bullet$. Then there is a canonical isomorphism*

$$\widehat{C_\bullet} \longrightarrow \widehat{H_\bullet}. \tag{2.15}$$

Proof. Set $Z_\bullet = \mathrm{Ker}(\partial)$, $B_\bullet = \mathrm{Im}(\partial)$, and $H_\bullet = Z_\bullet / B_\bullet$, regarded as complexes with differential 0, then the sequences

$$0 \to Z_\bullet \to C_\bullet \xrightarrow{\partial} B_\bullet[-1] \to 0, \qquad 0 \to B_\bullet \to Z_\bullet \to H_\bullet \to 0 \tag{2.16}$$

are exact. Clearly, $Z_\bullet$, $B_\bullet$, $H_\bullet$ are all bounded. We claim that $Z_k, B_k \in \mathrm{Proj}_R$ for all k. Indeed, assume by induction that $B_k \in \mathrm{Proj}_R$. Corollary 2.2 and the first short exact sequence in (2.16) imply that $Z_{k+1} \in \mathrm{Proj}_R$. Using the second short exact sequence, we then find that $B_{k+1} \in \mathrm{Proj}_R$.

Applying Proposition 2.5 to (2.16), there are isomorphisms

$$\widehat{C_\bullet} \to \widehat{Z_\bullet} \otimes \widehat{B_\bullet}^{-1} \to \widehat{B_\bullet} \otimes \widehat{H_\bullet} \otimes \widehat{B_\bullet}^{-1} = \widehat{H_\bullet}.$$

□

Let

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

be a short exact sequence of bounded complexes of finite projective modules and assume that the respective homologies H', H, H'' are also finite projective. Then long exact sequence in homology

$$\cdots \longrightarrow H'_k \xrightarrow{f_*} H_k \xrightarrow{g_*} H''_k \xrightarrow{\delta} H'_{k-1} \longrightarrow \cdots$$

is a bounded acyclic complex, $\mathcal{H}$, of finite projective modules. Hence, there is an isomorphism

$$\widehat{0} \to \widehat{\mathcal{H}} = \widehat{H'} \otimes \widehat{H}^{-1} \otimes \widehat{H''} \tag{2.17}$$

or equivalently

$$\widehat{H} \to \widehat{H'} \otimes \widehat{H''}. \tag{2.18}$$

Proposition 2.7. *Given a short exact sequence of chain complexes as above, the following diagram involving the above isomorphism and the isomorphisms from propositions 2.6, 2.3, commutes.*

$$\begin{tikzcd}
\widehat{C} \arrow[r] \arrow[d] & \widehat{C'} \otimes \widehat{C''} \arrow[d] \\
\widehat{H} \arrow[r] & \widehat{H'} \otimes \widehat{H''}
\end{tikzcd}$$

Proof. The proof here is based on the one in [18]. As above, let $Z := \mathrm{Ker}(d : C \to C)$ be cycles, $B := \mathrm{Im}(d : C \to C)$ boundaries, and $H := Z/B$ the homology. Analogously, define Z', B', H' and Z'', B'', H'' for C' and C'' respectively. Additionally set $K := \mathrm{Ker}(B \to B'')$, $I := \mathrm{Im}(Z \to Z'')$, $R := \mathrm{Coker}(B' \to B)$ and

$$X' := \mathrm{Ker}(f : H' \to H) = \mathrm{Im}(\delta : H''[1] \to H') \tag{2.19}$$

$$X := \mathrm{Ker}(g : H \to H'') = \mathrm{Im}(f : H' \to H) \tag{2.20}$$

$$X'' := \mathrm{Ker}(\delta : H'' \to H'[-1]) = \mathrm{Im}(g : H \to H'') \tag{2.21}$$

For notational purposes we write $\overline{C}$ for $C[-1]$ for the remainder of the proof. Consider the following commutative diagrams with exact columns and rows.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{B}' \arrow[r] \arrow[d] & \overline{B} \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B' \arrow[r, equal] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.22}$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & I \arrow[r, equal] \arrow[d] & I \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & \overline{B}'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \overline{X}' \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & \overline{B}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{2.23}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B' \arrow[r, equal] \arrow[d] & B' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B'' \arrow[r, equal] \arrow[d] & B'' \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & \overline{X}' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X'' \arrow[r] \arrow[d] & H'' \arrow[r] \arrow[d] & \overline{X}' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{2.24}
$$

Combining the resulting commutative diagrams for the graded determinant lines, plus a few tautological diagrams, we obtain a large commutative diagram. For notational purposes we omit “$\frown$” and “$\otimes$” which are implicit everywhere.

$$
\begin{tikzcd}
 & C \arrow[r] \arrow[d] & C'C'' \arrow[r] \arrow[d] & Z'\overline{B}'Z''\overline{B}'' \arrow[d] \arrow[ddr] & \\
 & Z\overline{B} \arrow[r] \arrow[d] \arrow[ddl] & Z'I\overline{B}'\overline{R} \arrow[r] & Z'I\overline{B}'\overline{X}'\overline{B}'' \arrow[d] & \\
 & Z'I\overline{K}\overline{B}'' \arrow[r] \arrow[urr] \arrow[dr] & K\overline{X}'X\overline{B}'I\overline{B}'' \arrow[ur] \arrow[d] & H'X''\overline{X}' \arrow[r] & H'H'' \\
H \arrow[rr] & & XX'' \arrow[ur] & &
\end{tikzcd}
$$

This completes the proof. $\square$

2.3 Quasi-isomorphisms

Let $f : X \to Y$ be an injective quasi-isomorphism between complexes in $C^{\mathrm{b}}(\mathrm{Proj}_R)$ such that $\mathrm{Coker}(f) \in C^{\mathrm{b}}(\mathrm{Proj}_R)$ also. The long exact sequence of

$$0 \longrightarrow X \xrightarrow{f} Y \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

shows that Coker f is acyclic. Combining the isomorphisms from Propositions 2.5 and 2.6, gives a morphism $\hat{f} : \widehat{X} \to \widehat{Y}$ induced by f.

More generally, suppose f is a not necessarily injective quasi-isomorphism. Consider the *mapping cylinder*

$$M_f = X \oplus X[-1] \oplus Y, \qquad d_{M_f} = \begin{pmatrix} \partial & -1 & 0 \\ 0 & -\partial & 0 \\ 0 & f & \partial \end{pmatrix} \tag{2.25}$$

for which the standard inclusions ι_X, ι_Y are chain maps. The map ι_Y has a left inverse

$$\rho_Y : M_f \to Y, \qquad (x, x', y) \mapsto f(x) + y. \tag{2.26}$$

with $f = \rho_Y \circ \iota_X$. Note that both $\mathrm{Coker}(\iota_X)$ and $\mathrm{Coker}(\iota_Y)$ are acyclic and we may define

$$\hat{f} : \widehat{X} \to \widehat{Y}, \qquad \hat{f} = (\hat{\iota}_Y)^{-1} \circ \hat{\iota}_X. \tag{2.27}$$

We claim that both definitions agree on injective quasi-isomorphisms $f : X \to Y$ with $\mathrm{Coker}(f) \in C^{\mathrm{b}}(\mathrm{Proj}_R)$. This follows from propositions 2.4 and 2.7 applied to the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathrm{Coker}(\iota_Y) \arrow[r, equal] \arrow[d, "\alpha"] & \mathrm{Coker}(\iota_Y) \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & M_f \arrow[r] \arrow[d, "\rho_Y"] & \mathrm{Coker}(\iota_X) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{Coker}(f) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.28}$$

where

$$\alpha = \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f & 0 & 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \tag{2.29}$$

Proposition 2.8. *Let $f : X \to Y$, $g : Y \to Z$ be quasi-isomorphisms with $X, Y, Z \in C^{\mathrm{b}}(\mathrm{Proj}_R)$, then*

$$\widehat{g \circ f} = \hat{g} \circ \hat{f}. \tag{2.30}$$

Proof. Assume first that f and g are injective and their cokernels are in $C^{\mathrm{b}}(\mathrm{Proj}_R)$. Applying Proposition 2.4 for complexes to the diagram of short exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & \mathrm{Coker}(f) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & X \arrow[r, "g\circ f"] \arrow[d] & Z \arrow[r] \arrow[d] & \mathrm{Coker}(g \circ f) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{Coker}(g) \arrow[r, equal] \arrow[d] & \mathrm{Coker}(g) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{2.31}$$

and Proposition 2.7 to the short exact sequence of cokernels gives $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.

In the general case consider the complex

$$M = X \oplus X[-1] \oplus Y \oplus Y[-1] \oplus Z, \qquad \partial_M = \begin{pmatrix} \partial & -1 & 0 & 0 & 0 \\ 0 & -\partial & 0 & 0 & 0 \\ 0 & f & \partial & -1 & 0 \\ 0 & 0 & 0 & -\partial & 0 \\ 0 & 0 & 0 & g & \partial \end{pmatrix} \tag{2.32}$$

and the inclusions

$$\iota_{M_f} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \iota_{M_g} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \iota_{M_{g\circ f}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.33}$$

making

$$\begin{tikzcd}
 & M_f \arrow[dr, "\iota_{M_f}"] & & X \arrow[ll] \arrow[dr] & \\
Y \arrow[ur] \arrow[dr] & & M & & M_{g\circ f} \arrow[ll, "\iota_{M_{g\circ f}}"'] \\
 & M_g \arrow[ur, "\iota_{M_g}"] & & Z \arrow[ll] \arrow[ur] &
\end{tikzcd} \tag{2.34}$$

a commutative diagram of injective quasi-isomorphisms. The proposition now follows from the special case considered above. □

Proposition 2.9. *Let $f, g : X \to Y$ be homotopic quasi-isomorphisms with $X, Y \in C^{\mathrm{b}}(\mathrm{Proj}_R)$, then*

$$\hat{f} = \hat{g}. \tag{2.35}$$

Proof. Let $H : X \to Y[1]$ be a homomotopy from g to f, i.e.

$$f - g = \partial H + H\partial \tag{2.36}$$

then

$$h : M_f \to M_g, \qquad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & H & 1 \end{pmatrix} \tag{2.37}$$

is an isomorphism of complexes and

$$\begin{tikzcd} X \arrow[r, "\iota_X"] \arrow[dr, "i_X"'] & M_f \arrow[d, "h"] & \\ & M_g & Y \arrow[l, "i_Y"] \arrow[ul, "\iota_Y"'] \end{tikzcd} \tag{2.38}$$

commutes. Hence, using functoriality,

$$\begin{aligned} \hat{f} &= (\hat{\iota}_Y)^{-1}\hat{\iota}_X && (2.39)\\ &= (\widehat{h\iota_Y})^{-1}\widehat{h\iota_X} && (2.40)\\ &= \hat{g}. && (2.41) \end{aligned}$$

□

Proposition 2.10. *Let*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "j"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & X' \arrow[r, "i'"] & Y' \arrow[r, "j'"] & Z' \arrow[r] & 0 \end{tikzcd} \tag{2.42}$$

be a commutative diagram of complexes in $C^{\mathrm{b}}(\mathrm{Proj}_R)$ with exact rows and f, g, h quasi-isomorphisms. Then the square

$$\begin{tikzcd} \widehat{Y} \arrow[r] \arrow[d, "\hat{g}"] & \widehat{X} \otimes \widehat{Z} \arrow[d, "\hat{f}\otimes\hat{h}"] \\ \widehat{Y'} \arrow[r] & \widehat{X'} \otimes \widehat{Z'} \end{tikzcd} \tag{2.43}$$

commutes.

Proof. Assume first that f, g, h are injective with projective cokernels. In this case the statement follows from Proposition 2.4 for complexes applied to the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Coker}(f) \arrow[r] \arrow[d] & \mathrm{Coker}(g) \arrow[r] \arrow[d] & \mathrm{Coker}(h) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

of short exact sequences and Proposition 2.7.

In the general case the mapping cylinder construction gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\iota_X"] & Y \arrow[r] \arrow[d, "\iota_Y"] & Z \arrow[r] \arrow[d, "\iota_Z"] & 0 \\
0 \arrow[r] & M_f \arrow[r] & M_g \arrow[r] & M_h \arrow[r] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[u, "\iota_{X'}"] & Y' \arrow[r] \arrow[u, "\iota_{Y'}"] & Z' \arrow[r] \arrow[u, "\iota_{Z'}"] & 0
\end{tikzcd}$$

where $M_f \to M_g$ (resp. $M_g \to M_h$) is the obvious map induced by i, i' (resp. j, j'), hence is reduced to the injective case already considered. $\square$

Lemma 2.11. *Let $f : X \to Y$ be a quasi-isomorphism with $X, Y \in C^{\mathrm{b}}(\mathrm{Proj}_R)$. Assume that $H(X) \cong H(Y)$ is projective in each degree, then the diagram*

$$\begin{tikzcd}
\widehat{X} \arrow[r, "\hat{f}"] \arrow[d] & \widehat{Y} \arrow[d] \\
\widehat{H(X)} \arrow[r, "\widehat{Hf}"] & \widehat{H(Y)}
\end{tikzcd} \tag{2.44}$$

commutes.

Proof. For f injective this follows directly from Proposition 2.7, in particular, for the injections ι_X, ι_Y into the mapping cylinder defined in the general case, hence also for general f by functoriality. $\square$

Chapter 3

Combinatorial Torsions

The aim of this chapter is to define "refined" Reidemeister and Whitehead torsion as in Turaev's paper [30]. The algebraic formalism, as developed in the first chapter, is however quite different from the one used in [30] and, we hope, more conceptual. Also we do not assume connectedness or vanishing of the Euler characteristic. Furthermore, as for example in [18], we consider CW-complexes with a group acting freely on the cells, generalizing the action of π_1 on a universal cover.

The Whitehead group is defined in the first section in terms of the functor $\mathcal{K}$ defined in the first chapter.

In the second section, Whitehead torsion is defined for homotopy equivalences between CW-complexes, where a fixed group G acts on the spaces and maps between them are equivariant.

Refined Reidemeister torsion is defined in the third section for a CW-complex together with a representation of G.

3.1 The Whitehead group

For a group G let Set_G^f be the category of sets with a free (left) action of G having a finite number of orbits, and bijective equivariant maps as morphisms. The coproduct (disjoint union) gives Set_G^f the structure of a symmetric monoidal groupoid.

Theorem 3.1.

$$\pi_0\mathcal{K}(\mathrm{Set}_G^f) = \mathbb{Z} \tag{3.1}$$

$$\pi_1\mathcal{K}(\mathrm{Set}_G^f) = \mathbb{Z}/2 \times G^{\mathrm{ab}} \tag{3.2}$$

where G^{ab} denotes the abelianization, $G/[G,G]$, of G.

Proof. We will use the description of $\pi_i\mathcal{K}(\mathrm{Set}_G^f)$ given by Proposition 1.7 and 1.8 respectively. Note first that Set_G^f is equivalent to the full subcategory $\mathcal{S}$ generated by the G-sets of the form

$$\bigsqcup_n G = \{(g,i) \mid g \in G, 1 \leq i \leq n\} \tag{3.3}$$

$n \geq 0$, with the left action induced by composition in G. Since $K(\mathbb{Z}_{\geq 0}) = \mathbb{Z}$ this shows the first part.

Use the notation $\mathbb{Z}/2 = \{1, -1\}$. Define

$$\phi : \mathbb{Z}/2 \times G^{\mathrm{ab}} \to \pi_1\mathcal{K}(\mathcal{S}) \tag{3.4}$$
$$(1, g) \mapsto (G \to G, x \mapsto gx) \tag{3.5}$$
$$(-1, e) \mapsto (\sigma_{G,G} : G \sqcup G \to G \sqcup G). \tag{3.6}$$

which is a well defined function, since $\pi_1 = \pi_1\mathcal{K}(\mathcal{S})$ is abelian. Using the relation $g \circ f = g \otimes f$ in π_1, one verifies that ϕ is a homomorphism of groups. To show that ϕ is an isomorphism, we construct an inverse, ψ, as follows. Let f be an automorphism of $\bigsqcup_n G$. Let ϵ be the sign of the bijection induced by f on the orbit space $(\bigsqcup_n G)/G$ and let g_k be such that $f(e,k) = (g_i, l)$. Define $\psi(f) = (\epsilon, g_1 \cdots g_n)$, then $\psi = \phi^{-1}$. □

Let $\mathfrak{F} : \mathrm{Set}_G^f \to \mathrm{Proj}_{\mathbb{Z}[G]}$ be the functor assigning to a G-set the free abelian group generated by that set with the $\mathbb{Z}[G]$-module structure induced by the action of G. Note that $\mathfrak{F}$ is symmetric monoidal since it preserves coproducts. Consider the induced homomorphism

$$\pi_1\mathcal{K}\mathfrak{F} : \pi_1\mathcal{K}(\mathrm{Set}_G^f) \to \pi_1\mathcal{K}(\mathrm{Proj}_{\mathbb{Z}[G]})$$

Proposition 3.2. $\mathrm{Ker}(\pi_1\mathcal{K}\mathfrak{F}) = 0$

Proof. We use the identification $\pi_1\mathcal{K}(\mathrm{Set}_G^f) \cong G^{\mathrm{ab}} \times \mathbb{Z}/2$ from Theorem 3.1. The canonical homomorphism $\rho : G \to G^{ab}$ induces a map

$$\rho_* : K_1(\mathbb{Z}[G]) \to K_1(\mathbb{Z}[G^{\mathrm{ab}}])$$

and

$$\det \circ \rho_* \circ \pi_1\mathfrak{F}|G^{\mathrm{ab}}$$

is just the inclusion $\iota : G^{\mathrm{ab}} \hookrightarrow \mathbb{Z}[G^{\mathrm{ab}}]$, which is injective, therefore $\pi_1\mathfrak{F}|G^{\mathrm{ab}}$ is injective.

Let $\epsilon : \mathbb{Z}[G^{\mathrm{ab}}] \to \mathbb{Z}$ be induced by the trivial map $G^{\mathrm{ab}} \to \{e\}$. Let $\tau \in K_1(\mathbb{Z}[G])$ be represented by the automorphism $(x,y) \mapsto (y,x)$ of $\mathbb{Z}[G]^2$, then

$$\mathrm{Im}(\epsilon \circ \iota) = \{1\}, \qquad (\epsilon \circ \det \circ \rho_*)(\tau) = -1 \tag{3.7}$$

which shows that τ is not in the image of $\pi_1\mathfrak{F}|G^{\mathrm{ab}}$, hence $\pi_1\mathfrak{F}$ is injective. □

Definition. The *Whitehead group*, $\mathrm{Wh}(G)$, is the cokernel of $\pi_1\mathfrak{F}$.

Computing $\mathrm{Wh}(G)$ in terms of G is in general difficult, cf. Milnor [17]. For the general theory, we will only need the following.

Theorem 3.3. *The Whitehead group of the trivial group vanishes, hence*

$$\mathcal{K}\mathfrak{F} : \mathcal{K}(\mathrm{Set}^f) \to \mathcal{K}(\mathrm{Proj}_{\mathbb{Z}})$$

is an equivalence.

Proof. We identify integer matrices and automorphisms of $\mathbb{Z}^n$ as usual. Let $E_{i,j}$ be the matrix with 1 in the (i,j)-th place and zeros elsewhere. For $i, j, k \in \mathbb{Z}$ all distinct, the identity

$$(I + aE_{i,k}) = (I + aE_{i,j})(I + E_{j,k})(I - aE_{i,j})(I - E_{j,k}) \tag{3.8}$$

holds, hence the matrices of the form $I + aE_{i,j}$, $i \neq j$ vanish in $K_1(\mathbb{Z})$. Applying the Gaussian elimination algorithm, any invertible matrix is equivalent in $K_1(\mathbb{Z})$ to a diagonal matrix, necessarily with ± 1 in the diagonal. Since the map $x \to -x$ is the image of any transposition under $\pi_1\mathfrak{F}$, the claim follows. □

3.2 Whitehead torsion

A *relative CW-complex* (X, A) is a topological space X and a subset $A \subseteq X$ so that X/A is a CW-complex with A/A a 0-cell. Let $p : X \to X/A$ be the canonical map and define the k-skeleton $X^{(-1)} = A$, $X^{(k)} = p^{-1}((X/A)^{(k)})$ for $k \geq 0$. Relative CW-complexes are only a mild generalization of CW-complexes and many results carry over to the relative case (see e.g. May [19]).

Remark. By "CW-complex" we mean "CW-complex with a fixed decomposition into cells", so that it makes sense to refer to the k-skeleton and the k-cells of a (relative) CW-complex.

Let G be a (discrete) group. We say (X, A) is a *relative G-complex* if

1. (X, A) is a relative CW-complex
2. G acts on X by continuous cellular maps
3. G acts freely on the set of cells of (X, A)
4. $(X/G, A/G)$ has a finite number of cells

Note that these conditions ensure that the set of cells of (X, A), which we will denote by $C(X, A)$, is an object in Set^f_G. We only consider those maps $f : (X, A) \to (Y, B)$ of relative G-complexes which are G-equivariant and satisfy $f(A) \subseteq B$.

If (X, A) is a relative CW-complex with a finite number of cells such that X admits a universal cover $p : \widetilde{X} \to X$, then $(\widetilde{X}, p^{-1}(A))$ is a relative G-complex, where $G \cong \pi_1(X)$ is the group of deck-transformations of $\widetilde{X}$. More generally, one can consider principal G-bundles over a relative CW-complex for an arbitrary discrete group G.

Let (X, A) be a relative G-complex. We have a decomposition

$$C(X, A) = \bigsqcup_{k \geq 0} C_k(X, A) \tag{3.9}$$

where $C_k(X, A)$ is the set of cells of dimension k. The image of the object

$$\widehat{C(X, A)} = \bigotimes_{k \geq 0} \widehat{C_k(X, A)}^{(-1)^k} \tag{3.10}$$

in $\pi_0(\mathcal{K}(\mathrm{Set}^f_G)) = \mathbb{Z}$ is the (relative) Euler characteristic of the pair $(X/G, A/G)$, $\chi(X, A)$.

Consider the $\mathbb{Z}$-graded $\mathbb{Z}[G]$-module $\mathbb{Z}C(X, A)$. Assuming each cell is oriented, there is an isomorphism of $\mathbb{Z}[G]$-modules

$$\mathbb{Z}C_k(X, Y) \to H_k(X^{(k)}, X^{(k-1)}) \tag{3.11}$$

sending each cell to its fundamental class. Here H_k is the k-th singular homology with coefficients in $\mathbb{Z}$, with the $\mathbb{Z}[G]$-module structure given by the induced action of G. The boundary operator of the exact sequence of the triple (X^k, X^{k-1}, X^{k-2}) is a homomorphism $\partial_k : H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2})$ with $\partial_k \partial_{k+1} = 0$, making $\mathbb{Z}C(X, Y)$ a complex. The homology of this complex is canonically isomorphic to singular homology $H_\bullet(X, A)$, see for example [19].

Let (X, A), (Y, B) be relative G-complexes and $f : (X, A) \to (Y, B)$ a cellular homotopy equivalence (equivariant), hence inducing a chain homotopy equivalence

$$f_* : \mathbb{Z}C(X, A) \to \mathbb{Z}C(Y, B). \tag{3.12}$$

Definition. An cellular homotopy equivalence $f : (X, A) \to (Y, B)$ is a *simple homotopy equivalence* if the induced isomorphism

$$\widehat{f}_* : \mathbb{Z}\widehat{C(X, A)} \to \mathbb{Z}\widehat{C(Y, B)}$$

is in the image of the functor

$$\mathcal{K}\mathfrak{F} : \mathcal{K}(\mathrm{Set}^f_G) \to \mathcal{K}(\mathrm{Proj}_{\mathbb{Z}[G]})$$

The cellular approximation theorem and Proposition 2.9 show that $\widehat{f}_*$ depends only on the homotopy class of f. As a consequence, $\widehat{f}_*$ is well defined for all equivariant homotopy equivalences, not necessarily cellular.

Choose any $g : \widehat{C(Y, B)} \to \widehat{C(X, A)}$, then the image of $(\mathcal{K}\mathfrak{F}g) \circ \widehat{f}_*$ in $\mathrm{Wh}(G)$, the *Whitehead torsion* of f (cf. Whitehead [32]), does not depend on g and vanishes if and only if f is a simple homotopy equivalence. Note however, that the *refined*

Whitehead torsion $\widehat{f_*}$ can not be recovered from the corresponding element in $\mathrm{Wh}(G)$.

By Proposition 3.2 the functor $\mathcal{K}\mathfrak{F}$ is faithful (i.e. injective on homsets), thus, if f is a simple homotopy equivalence then there is a unique isomorphism $\widehat{C(X,A)} \to \widehat{C(Y,B)}$ mapped to $\widehat{f_*}$ by $\mathcal{K}\mathfrak{F}$.

Proposition 3.4 (Multiplicativity of Whitehead torsion)**.** *Let* (X, A) *and* (Y, B) *be relative* G*-complexes with relative* G*-subcomplexes* (X', A') *and* (Y', B') *respectively. Let* $f : (X, A) \to (Y, B)$ *be a cellular homotopy equivalence inducing homotopy equivalences* $g : (X', A') \to (Y', B')$ *and* $h : (X, X' \cup A) \to (Y, Y' \cup B)$*, then*

$$\widehat{f_*} = \widehat{g_*} \otimes \widehat{h_*}. \tag{3.13}$$

In particular, if two of the maps f, g, h *are simple homotopy equivalences, then so is the third.*

Proof. Apply Proposition 2.10 to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}C(X',A') \arrow[r] \arrow[d, "g_*"] & \mathbb{Z}C(X,A) \arrow[r] \arrow[d, "f_*"] & \mathbb{Z}C(X,X'\cup A) \arrow[r] \arrow[d, "h_*"] & 0 \\
0 \arrow[r] & \mathbb{Z}C(Y',B') \arrow[r] & \mathbb{Z}C(Y,B) \arrow[r] & \mathbb{Z}C(Y,Y'\cup B) \arrow[r] & 0
\end{tikzcd}$$

□

Lemma 3.5. *Let* $f : (X, A) \to (Y, B)$ *be a homotopy equivalence of finite relative CW-complexes. For any group* G *the product map*

$$f \times \mathrm{id}_G : (X \times G, A \times G) \to (Y \times G, B \times G)$$

is a simple homotopy equivalence of G*-complexes.*

Proof. By Theorem 3.3 the functor

$$\mathcal{K}\mathfrak{F} : \mathcal{K}(\mathrm{Set}^f) \to \mathcal{K}(\mathrm{Proj}_{\mathbb{Z}})$$

is full, hence there exists a set S and a bijection

$$g : C(X,A) \sqcup S \to C(Y,B) \sqcup S$$

such that

$$\mathcal{K}\mathfrak{F}(\hat{g}) = \hat{f}_* \tag{3.14}$$

therefore

$$\mathcal{K}\mathfrak{F}(\widehat{g \times \mathrm{id}_G}) = (\widehat{f \times \mathrm{id}_G})_* \tag{3.15}$$

i.e. $f \times \mathrm{id}_G$ is a simple homotopy equivalence. □

Lemma 3.6. *Let $f : (X, A) \to (Y, B)$ be a cellular equivariant homotopy equivalence of G-complexes. If each component of $(X \setminus A)/G$ and $(Y \setminus B)/G$ is simply connected, and f induces a bijection of these components, then f is a simple homotopy equivalence.*

Proof. Since G acts freely on the cells of (X, A) by asssumption, $X \setminus A$ is a covering of $(X \setminus A)/G$, in fact trivial, since each component of the latter space is simply connected. The same argument applies to (Y, B). Hence, on the level of chain complexes, the situation is as in the previous lemma, and the claim follows. □

Theorem 3.7 (Invariance under subdivisions)**.** *Assume $f : (X, A) \to (Y, B)$ is a cellular equivariant homeomorphism of relative G-complexes. Then f is a simple homotopy equivalence.*

Proof. Geometrically, (Y, B) is a subdivision of (X, A), where f maps an (open) cell onto the cells of smaller of equal dimension it is divided into.

We prove the theorem by induction on dimension n of (X, A). In the case $n = 0$ both (X, A) and (Y, B) are essentially objects in Set_G^f and f an equivariant bijection between them. Trivially, $\mathcal{K}\mathfrak{F}(\hat{f}) = \widehat{f_*}$.

Assume now that (X, A) has dimension n and that the theorem is true for all complexes of strictly smaller dimension. The map $(X^{(n-1)}, A) \to (f(X^{(n-1)}), B)$ induced by f is a simple homotopy equivalence by induction, while the map $(X, A \cup X^{(n-1)}) \to (Y, f(A \cup X^{(n-1)}))$ is a simple homotopy equivalence by Lemma 3.6. Hence, f is a simple homotopy equivalence by Proposition 3.4. □

For a relative G-complex (X, A) consider the cylinder

$$(Y, B) := (X \times [0, 1], (A \times [0, 1]) \cup (X \times 0))$$

which is a relative G-complex with cells $e \times (0, 1)$, $e \times 1$ for each cell e of (X, A).

Lemma 3.8. *With (Y, B) as above, the inclusion $\iota : (X \times 0, X \times 0) \hookrightarrow (Y, B)$ of the trivial complex is a simple homotopy equivalence.*

Proof. Note that

$$\widehat{C(Y, B)} = \widehat{C(X, A)} \otimes \widehat{C(X, A)}^{-1} \tag{3.16}$$

hence there is a canonical map $\tau : \hat{\emptyset} \to \widehat{C(Y, B)}$. To see that $\mathcal{K}\mathfrak{F}\tau = \widehat{\iota_*}$ consider first the case when (X, A) has a single cell and then use Proposition 3.4 for the general case. □

3.3 Reidemeister torsion

Let G be a discrete group and (X, A) be a relative G-complex. In this section we adopt the convention that G acts on X on the right, thus making $\mathbb{Z}C(X, A)$ a right $\mathbb{Z}[G]$-module. As noted before,

$$\widehat{C(X, A)} = \chi(X, A) \qquad \text{in } \pi_0(\mathcal{K}(\mathrm{Set}_G^f)) \tag{3.17}$$

hence

$$\mathrm{Tur}(X,A) := \mathrm{Hom}(\widehat{G}^{\chi(X,A)}, \widehat{C(X,A)}) \tag{3.18}$$

is non-empty, and therefore a $\pi_1(\mathcal{K}(\mathrm{Set}_G^f))$-torsor. An element of $\mathrm{Tur}(X,A)$, which we refer to as a *Turaev structure*, corresponds to a choice of *Euler structure* and *homology orientation* (cf. Turaev [30], [31]).

Let V be a representation of G, i.e. V is a finite dimensional vector space over a field $\mathbb{K}$ together with a group homomorphism $\rho : G \to \mathrm{Aut}(V)$. We extend ρ to a ring homomorphism $\mathbb{Z}[G] \to \mathrm{End}(V)$, giving V he structure of a left $\mathbb{Z}[G]$-module. The tensor product $. \otimes V$ defines a functor from $\mathrm{Proj}_{\mathbb{Z}[G]}$ to $\mathrm{Vect}_{\mathbb{K}}$ preserving coproducts, and hence a functor of 2-groups

$$\mathcal{K}(\mathrm{Proj}_{\mathbb{Z}[G]}) \to \mathcal{K}(\mathrm{Vect}_{\mathbb{K}}).$$

By Theorem 1.10, we can equivalently use $\mathscr{L}_{\mathbb{K}}$, the category of weighted $\mathbb{K}$-lines, as the target category. Applying the above functors to an element $t \in \mathrm{Tur}(X,A)$ produces a non-zero element

$$\tau_{X,A;V}(t) \in \mathrm{Det}(X,A;V) := \mathrm{Det}(H(X,A;V)) \otimes \mathrm{Det}(V)^{-\chi(X,A)} \tag{3.19}$$

where

$$H(X,A;V) := H(\mathbb{Z}C(X,A) \otimes V, \partial \otimes \mathrm{id}_V). \tag{3.20}$$

The function $\tau_{X,A;V}$ is the *Reidemeister torsion* of the relative G-complex (X,A) with coefficients in V. Note that $\mathrm{Det}(X,A;V) = \mathbb{K}$ if $H(X,A;V) = 0$.

Let $f : (X,A) \to (Y,B)$ be a simple homotopy equivalence. Then f defines isomorphisms $\mathrm{Tur}(X,A) \to \mathrm{Tur}(Y,B)$ and $H(X,A;V) \to H(Y,B;V)$.

Proposition 3.9. *For f as above, $\tau_{X,A;V} = \tau_{Y,B;V}$ under the indentifications given by f, i.e. the Reidemeister torsion is invariant under simple homotopy equivalences.*

Proof. For $t \in \mathrm{Tur}(X,A)$ then the corresponding element in $\mathrm{Tur}(Y,B)$ is by definition $\hat{f} \circ t$. On the other hand, the induced isomorphism $Hf : H(X,A;V) \to H(Y,B;V)$ gives a map $\widehat{Hf} \otimes \mathrm{id} : \mathrm{Det}(X,A;V) \to \mathrm{Det}(Y,B;V)$. The proposition follows from functoriality of $\mathcal{K}\mathfrak{F}$, $\mathcal{K}(\otimes V)$, and Lemma 2.11. $\square$

Let (Y,A) be a G-invariant subcomplex of (X,A). There are canonical identifications

$$\mathrm{Tur}(X,A) = \mathrm{Tur}(X,Y) \times_{\pi_1(\mathcal{K}(\mathrm{Set}_G^f))} \mathrm{Tur}(Y,A) \tag{3.21}$$

$$\mathrm{Det}(X,A;V) = \mathrm{Det}(X,Y;V) \otimes \mathrm{Det}(Y,A;V) \tag{3.22}$$

Proposition 3.10. *With the notation and identifications as above,*

$$\tau_{X,A;V} = \tau_{X,Y;V} \otimes \tau_{Y,A;V}. \tag{3.23}$$

Proof. Let $x \in \operatorname{Tur}(X,Y)$, $y \in \operatorname{Tur}(Y,A)$, then the product $x \otimes y$ defines an element $z \in \operatorname{Tur}(X,A)$ such that

$$\begin{tikzcd} \widehat{G}^{\chi(X,Y)} \otimes \widehat{G}^{\chi(Y,A)} \arrow[r] \arrow[d, "x\otimes y"'] & \widehat{G}^{\chi(X,A)} \arrow[d, "z"] \\ \widehat{C(X,Y)} \otimes \widehat{C(Y,A)} \arrow[r] & \widehat{C(X,A)} \end{tikzcd} \tag{3.24}$$

is commutative. Applying the fact that $\mathcal{K}\mathfrak{F}$ and $\mathcal{K}(\otimes V)$ are monoidal functors and Proposition 2.7 to the above diagram gives commutativity of

$$\begin{tikzcd} \operatorname{Det}(V)^{\chi(X,Y)} \otimes \operatorname{Det}(V)^{\chi(Y,A)} \arrow[r] \arrow[d] & \operatorname{Det}(V)^{\chi(X,A)} \arrow[d] \\ \operatorname{Det}(H(X,Y;V)) \otimes \operatorname{Det}(H(Y,A;V)) \arrow[r] & \operatorname{Det}(H(X,A;V)) \end{tikzcd} \tag{3.25}$$

hence $\tau_{X,Y;V}(x) \otimes \tau_{Y,A;V}(y) = \tau_{X,A;V}(z)$. $\square$

Chapter 4

Torsion of Manifolds

A manifold can be given the structure of a CW-complex by constructing a triangulation on it. Using the fact that any two triangulations have isomorphic subdivisions, it follows that the combinatorial torsions defined in the previous chapter are independent of the choice of triangulation. This is explained in more detail in the first section.

In the second section we review the definition of the Thom–Smale complex associated with a Morse function and a Riemannian metric, and show that this complex represents the simple homotopy type of the manifold, in a certain sense.

As an application of these results we show that the inclusion of the base space into the unit disc bundle of a vector bundle is a simple homotopy equivalence when the base is a manifold without boundary.

4.1 Diffeomorphism invariance

A *bordism* is a compact smooth manifold M such that ∂M is the disjoint union of two distinguished closed subsets $\partial_+ M$ and $\partial_- M$. Morphisms $f : M \to N$ between bordisms are smooth maps with $f(\partial_\pm M) \subseteq \partial_\pm N$. More generally let G be a discrete group, then a G-bordism, M, is a smooth manifold with a strictly discontinuous smooth action of G such that M/G is a bordism, i.e. M/G is compact and the boundary of M is the disjoint union of two distinguished G-invariant submanifolds.

A *smooth triangulation* of a manifold M is a simplicial complex $K \subset \mathbb{R}^N$ and a homeomorphism $f : K \to M$ which restricts to a smooth immersion on each simplex. For more details and a proof of the following result we refer to Munkres [22].

Theorem 4.1. *Every smooth manifold, possibly non-compact and with boundary, has a smooth triangulation.*

A triangulation of a G-bordism M can be chosen G-invariant by lifting a triangulation of M/G. Such a triangulation gives $(M, \partial_- M)$ the structure of a relative G-complex.

Theorem 4.2. *Let $f : M \to N$ be a G-equivariant diffeomorphism between G-bordisms. Assume that both M and N are triangulated, then f is a simple homotopy equivalence.*

Proof. By a theorem of Whitehead and Munkres [22] there are subdivisions of the triangulations of M and N such that f is isotopic to an isomorphism of the respective simplicial complexes. Hence the claim follows from Theorem 3.7 (invariance under subdivision) and the homotopy invariance of Whitehead torsion. □

In view of the above results we can canonically identify the objects $C(\widehat{M, \partial_- M})$ for different smooth triangulations using $\widehat{(\mathrm{id}_M)}_*$, and the refined Whitehead torsion of a homotopy equivalence to or from M does not depend on the triangulation. As a consequence, $\mathrm{Tur}(M, \partial_- M)$ and the Reidemeister torsion

$$\tau_{M,\partial_- M;V} : \mathrm{Tur}(M, \partial_- M) \to \mathrm{Det}(M, \partial_- M; V)$$

of a representation V of G is well defined without reference to a particular triangulation of M (this uses Proposition 3.9).

Remark. The difficult part of the proof of Theorem 4.2 is of course the demonstration of the results on triangulations, cf. [22]. Alternatively, the result can be based on geodesically convex coverings as in [28] or on the bifurcation analysis of Morse functions, see [2].

4.2 The Thom–Smale complex

In this section we assume some familiarity with Morse theory, see for example [15], [9] or the more recent [24].

Definition. A *Morse function with boundary values $a < b \in \mathbb{R}$* on a G-bordism M is a G-invariant smooth function $f : M \to [a, b]$ such that

1. all critical points of f are non-degenerate,
2. $f = a$ on $\partial_- M$ and $f = b$ on $\partial_+ M$,
3. a and b are regular values of f.

We denote the set of critical points by $\mathrm{Cr}(f)$, the subset of critical points of index k by $\mathrm{Cr}_k(f)$. One can show existence of Morse functions using Sard's theorem and partition of unity arguments, cf. [16], [25]. In fact, a much stronger result holds: The set of Morse functions with boundary values $a < b$ is C^2-open and C^∞-dense in the set of smooth functions $f : M \to [a, b]$ with $f|\partial_- M = a$, $f|\partial_+ M = b$.

Let g be a G-invariant Riemannian metric on M. The gradient, ∇f, of f with respect to g has a partially defined flow Φ_t. For each $p \in \mathrm{Cr}(f)$ define the *stable set*

$$W^s(p) = \{q \in M \mid \Phi_t(q) \text{ exists for all } t \geq 0 \text{ and } \lim_{t\to\infty} \Phi_t(q) = p\} \tag{4.1}$$

and the *unstable set*

$$W^u(p) = \{q \in M \mid \Phi_t(q) \text{ exists for all } t \leq 0 \text{ and } \lim_{t\to-\infty} \Phi_t(q) = p\}. \tag{4.2}$$

It follows from the Hadamard-Perron theorem for hyperbolic zeros of vector fields and the fact that f is a Lyapunov function for ∇f that $W^s(p)$ is a contractible submanifold of dimension k and $W^u(p)$ is a contractible submanifold of codimension k. A detailed proof can be found in [25].

We assume that the metric g is chosen so that

$$W^s(p) \pitchfork W^u(q) \qquad \text{for } p, q \in \mathrm{Cr}(f) \tag{4.3}$$

where $\pitchfork$ denotes transverse intersection. By a theorem of Smale this holds for generic g.

Choose an orientation on each stable manifold. Since $W^s(p)$ and $W^u(p)$ have transverse intersection $\{p\}$, this also gives a coorientation (i.e. an orientation of the normal bundle) on each unstable manifold. Define $C_k(X)$ as the free abelian group generated by $\mathrm{Cr}_k(f)$. The action of G induces a $\mathbb{Z}[G]$-module structure on $C_\bullet(X)$. Let $x \in \mathrm{Cr}_k(f)$ and $y \in \mathrm{Cr}_{k-1}(f)$ and assume that $f(x) > f(y)$. Choose a regular value r of f with $f(x) > r > f(y)$. Note that $S = W^s(x) \cap \{f = r\}$ is oriented as the boundary of $W^s(x) \cap \{f \geq r\}$, $U = W^u(y) \cap \{f = r\}$ is cooriented in the level set $\{f = r\}$, and their intersection is transverse in $\{f = r\}$. Thus, the intersection number of U and S in $\{f = r\}$, $n(x, y)$, is defined. Let

$$\partial(x) = \sum_{y \in \mathrm{Cr}_{k-1}(f)} n(x, y) y \tag{4.4}$$

which extends to a homorphism of $\mathbb{Z}[G]$-modules $\partial : C_k(X) \to C_{k-1}(X)$. It is well known that $\partial^2 = 0$, see for example [16] or [25], hence $(C_\bullet(X), \partial)$ is a chain complex. The homology of this complex is canonically isomorphic to $H_\bullet(M, \partial_- M)$ as $\mathbb{Z}[G]$-modules. We will describe this isomorphism in more detail and show that it is induced by a simple homotopy equivalence, in the algebraic sense.

A Morse function with boundary values $a < b$ is *self-indexing* if $a = -\frac{1}{2}$, $b = n + \frac{1}{2}$ and $\mathrm{Cr}_k(f) \subseteq f^{-1}(k)$. By a result of Smale, there is a self indexing Morse function $\tilde{f}$ and a metric $\tilde{g}$ so that $\nabla^g f = \nabla^{\tilde{g}} \tilde{f}$, hence we may assume, without loss of generality, that f is self-indexing.

Let

$$M_k = \{f \leq k + \frac{1}{2}\} \qquad\qquad E_k = \{k - \frac{1}{2} \leq f \leq k + \frac{1}{2}\} \tag{4.5}$$

then

$$\partial_- M = M_{-1} \subset M_0 \subset \ldots \subset M_n = M \tag{4.6}$$

and E_k (resp. M_k) contains exactly those critical points with index equal (resp. less than or equal) to k. Note also that the M_k and E_k are G-bordisms. Let

$$N_k = \bigcup_{p \in \mathrm{Cr}_k(f)} W^s(p) \cup M_{k-1} \tag{4.7}$$

then by Morse theory, the inclusion $N_k \to M_k$ is a homotopy equivalence relative M_{k-1}, and (N_k, M_{k-1}) is a relative G-complex where the cells are the stable discs $D^s(p) = W^s(p) \cap E_k$ of the critical points of index k. Hence, the map

$$C_k(X) \to H_k(M_k, M_{k-1}) \tag{4.8}$$

sending $p \in \mathrm{Cr}_k(f)$ to the fundamental class of $D^s(p)$ is an isomorphism of $\mathbb{Z}[G]$-modules. Moreover, the boundary operator of the exact sequence of the triple (M_k, M_{k-1}, M_{k-1})

$$\delta : H_k(M_k, M_{k-1}) \to H_{k-1}(M_{k-1}, M_{k-2})$$

coincides with the boundary operator $\partial : C_k(X) \to C_{k-1}(X)$ defined in terms of intersection numbers (see for example [16] or [25]).

Lemma 4.3. *The inclusion*

$$\left(\bigcup_{p \in \mathrm{Cr}_k(f)} D^s(p) \cup \partial_- E_k, \partial_- E_k \right) \hookrightarrow (E_k, \partial_- E_k)$$

is a simple homotopy equivalence.

Proof. Define the unstable discs $D^u(p) = W^u(p) \cap E_k$. Let D^s (resp. D^u) be the union of the $D^s(p)$ (resp. $D^u(p)$) for $p \in \mathrm{Cr}_k(f)$. The flow of ∇f can be used to construct a diffeomorphism

$$E_k \setminus (D^s \cup D^u) \cong (\partial_- E_k \setminus \partial D^s) \times [0,1]. \tag{4.9}$$

Let $\partial_- F_k \subset \partial_- E_k$ be a submanifold (with boundary) obtained by removing an open tubular neighbourhood of ∂D^s in $\partial_- E_k$ and let F_k be the submanifold corresponding to $\partial_- F_k \times [0,1]$ under the above diffeomorphism. Choose a triangulation of E_k which restricts to the product triangulation on F_k.

The inclusion

$$(D^s \cup \partial_- E_k, \partial_- E_k) \hookrightarrow (E_k, F_k \cup \partial_- E_k)$$

is a simple homotopy equivalence by Lemma 3.6, while the inclusion

$$(\partial_- E_k, \partial_- E_k) \hookrightarrow (F_k \cup \partial_- E_k, \partial_- E_k)$$

of the trivial complex is a simple homotopy equivalence by Lemma 3.8, since the product triangulation is a subdivision of the product cell structure considered in that lemma. Hence, by Proposition 3.4 the inclusion

$$(D^s \cup \partial_- E_k, \partial_- E_k) \hookrightarrow (E_k, \partial_- E_k)$$

is also a simple homotopy equivalence. □

The next theorem states, roughly, that the Thom–Smale complex has the same simple homotopy type as (triangulations of) the G-bordism $(M, \partial_- M)$.

Theorem 4.4 (Milnor)**.** *With M, f, $X = \nabla f$ as above, there exists a G-complex (Y, A) together with a simple homotopy equivalence $\rho : (M, \partial_- M) \to (Y, A)$ such that there is a canonical identification of complexes*

$$C_*(X) \cong \mathbb{Z}C(Y, A).$$

Proof. Inductively we will construct simple homotopy equivalences

$$\rho_k : (M_k, \partial_- M) \to (Y^{(k)}, A)$$

together with the k-skeleton $(Y^{(k)}, A)$ of (Y, A). Let $(Y^{(0)}, A) = (N_0, \partial_- M)$ and ρ_0 be a homotopy inverse of the inclusion into E_0, then ρ_0 is a simple homotopy equivalence by the previous lemma.

Assume now that $(Y^{(k-1)}, A)$ and ρ_{k-1} have already been constructed. Let $\varphi : \partial D^s \to \partial_+ M_k$ be the attaching map of the k-discs. By a lemma in [15] there exists a homotopy equivalence

$$\alpha : N_k = M_{k-1} \cup_\varphi D^s \to Y^{(k-1)} \cup_{\rho_{k-1} \circ \varphi} D^s =: Y^{(k)}$$

extending ρ_{k-1}. Define ρ_k as the composition of α and a homotopy inverse of the inclusion $N_k \to M_k$. By Lemma 4.3, the induction hypothesis, and Proposition 3.4, ρ_k is a simple homotopy equivalence. □

Remark. One would like to show, as in the special case of the E_k above, that the stable manifolds are the cells of a relative CW-complex and that the inclusion of this complex into M is a simple homotopy equivalence. A proof of the former statement is sketched in [2], see also [13]. The approach here, on the other hand, is closer to the standard results in Morse theory.

Remark. Theorems 4.2 and 4.4 combined show that the Thom–Smale complexes of any two different Morse–Smale pairs have the same simple homotopy type. A more direct, Morse-theoretic proof involves bifurcation analysis of one-parameter families of Morse–Smale pairs, cf. [2], [13], [10].

4.3 Unit disc bundles

Throughout this section let M be a smooth manifold without boundary and G a discrete group acting smoothly and strictly discontinuously on M such that M/G is compact.

Let $p : V \to M$ be a real G-vector bundle over M, or equivalently, the pullback of a vector bundle over M/G. Choose a G-invariant metric g^V on V. The unit disc bundle

$$D(V) = \{v \in V \mid g^V(v, v) \leq 1\} \tag{4.10}$$

is a G-bordism with $\partial_+ D(V) = \partial D(V)$, $\partial_- D(V) = \emptyset$.

We can extend a Morse function f on M to a Morse function $\bar{f}$ on $D(V)$ as follows. Choose $b \in \mathbb{R}$ such that $f < b$ and define a rescaled metric

$$\tilde{g}^V_x = (b - f(x))g^V_x \tag{4.11}$$

and set

$$\bar{f}(v) = f(p(v)) + \tilde{g}^V(v, v). \tag{4.12}$$

Clearly, $f = b$ on $\partial D(V)$ and $\bar{f}$ has the same critical points as f. A computation shows that the Hessian of $\bar{f}$ at $x \in \mathrm{Cr}(f)$ is the sum of the Hessian of f and $\tilde{g}^V$ at x, hence $\mathrm{Cr}_k(\bar{f}) = \mathrm{Cr}_k(f)$ for all k. Moreover, the stable manifolds of $\bar{f}$ and f are the same.

Using this construction we prove

Theorem 4.5. *The inclusion $M \hookrightarrow D(V)$ is a simple homotopy equivalence.*

Proof. Choose a self-indexing Morse function f on M. We may assume that $f < n + \frac{1}{2}$, where $n = \dim M$. Construct the extension $\bar{f}$ of f to $D(V)$ as above with $b = n + \frac{1}{2}$, then $\bar{f}$ is also self-indexing.

As in the previous section define

$$M_k = \{f \leq k + \frac{1}{2}\} \tag{4.13}$$

$$N_k = \bigcup_{p \in \mathrm{Cr}_k(f)} W^s(p) \cup M_{k-1} \tag{4.14}$$

and similarly $\overline{M}_k$ and $\overline{N}_k$ for $\bar{f}$. Consider the commutative square of inclusions

$$\begin{array}{ccc} (N_k, M_{k-1}) & \longrightarrow & (M_k, M_{k-1}) \\ \downarrow & & \downarrow \\ (\overline{N}_k, \overline{M}_{k-1}) & \longrightarrow & (\overline{M}_k, \overline{M}_{k-1}) \end{array} \tag{4.15}$$

The horizontal maps are simple homotopy equivalences by Lemma 4.3, while the left vertical map is an isomorphism of complexes, since $\overline{N}_k = N_k \cup \overline{M}_{k-1}$, hence also a simple homotopy equivalence. Thus, the inclusions $(M_k, M_{k-1}) \hookrightarrow (\overline{M}_k, \overline{M}_{k-1})$

are simple homotopy equivalences for all k. This implies, by repeated application of Proposition 3.4, that the inclusion $M \hookrightarrow \overline{M} = D(E)$ is a simple homotopy equivalence. $\square$

Remark. Construct $\bar{f}$ as above, then $-\bar{f}$ is a Morse function on $D(V)$ as a bordism with the opposite convention $\partial_- M = \partial D(V)$, $\partial_+ M = \emptyset$. Let r be the rank of V. Examination of the Thom–Smale complex of $-\bar{f}$ shows that

$$H_\bullet(D(V), \partial D(V)) \cong H_\bullet(M)[r] \tag{4.16}$$

which is of course the Thom isomorphism. However, this suggests that the Thom isomorphism is a simple homotopy equivalence, at least in an algebraic sense.

Appendix A

Localization of Categories

Proposition A.1 (Gabriel–Zisman [7]). *Let $\mathcal{C}$ be a small category and S a subset of the set of morphisms of $\mathcal{C}$, then there exists a small category $S^{-1}\mathcal{C}$ and a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that:*

1. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism for every $s \in S$ then there exist a unique functor $G : S^{-1}\mathcal{C} \to \mathcal{D}$ with $F = G \circ Q$.*
2. *For functors $G_1, G_2 : S^{-1}\mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : G_1 \circ Q \to G_2 \circ Q$ there exists a unique natural transformation $\beta : G_1 \to G_2$ with $\alpha = \beta \circ Q$.*

Proof. Let $\mathcal{G}$ be the following directed graph: Vertices are objects of $\mathcal{C}$ and arrows from X to Y are morphisms $X \to Y$ in $\mathcal{C}$ and morphisms $Y \to X$ in S. Denote the arrow from X to Y corresponding to $s \in S$, $s : Y \to X$, by $\bar{s}$. Let $\mathcal{F}$ be the free category (cf. [21]) on $\mathcal{G}$, i.e. the category of paths in $\mathcal{G}$. We denote the composition of paths $f : X \to Y$ and $g : Y \to Z$ by $g \cdot f$ and $\emptyset_X$ is the empty path starting and ending at X. Define $S^{-1}\mathcal{C}$ to be the quotient of $\mathcal{F}$ (cf. [21]) by the equivalence relation generated by

$$\begin{aligned} f \circ g &\sim f \cdot g && \text{for morphisms } f, g \text{ of } \mathcal{C} \\ \mathrm{id}_X &\sim \emptyset_X && \text{for objects } X \text{ of } \mathcal{C} \\ \bar{s} \cdot s &\sim \mathrm{id}_X && \\ s \cdot \bar{s} &\sim \mathrm{id}_Y && \text{for morphisms } s : X \to Y \text{ in } S \end{aligned}$$

The inclusion of generators into $\mathcal{F}$ factors to a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphisms for $s \in S$. Define $G : S^{-1}\mathcal{C} \to \mathcal{D}$ on generators by

$$G(f) = F(f) \quad \text{for morphisms } f \text{ of } \mathcal{C}, \qquad G(\bar{s}) = F(s)^{-1} \quad \text{for } s \in S$$

then $F = G \circ Q$ and G is the only functor with this property.

Let $G_1, G_2 : S^{-1}\mathcal{C} \to \mathcal{D}$ be functors and $\alpha : G_1 \circ Q \to G_2 \circ Q$ a natural transformation. Since Q is an isomorphisms on objects there is a unique $\beta : G_1 \to G_2$ with $\alpha = \beta \circ Q$, where naturality of β follows easily from naturality of α. $\square$

Proposition A.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be small categories, S_1, S_2 subsets of the sets of morphisms of $\mathcal{C}_1, \mathcal{C}_2$ respectively. Assume that S_1, S_2 contain all identity morphisms, then the canonical functor*

$$F : (S_1 \times S_2)^{-1}\mathcal{C}_1 \times \mathcal{C}_2 \to S_1^{-1}\mathcal{C}_1 \times S_2^{-1}\mathcal{C}_2$$

is an isomorphism of categories.

Proof. The plan is to construct an inverse of F. Let

$$f = (f_1, f_2) : (X_1, X_2) \to (Y_1, Y_2)$$

be a morphism of $S_1^{-1}\mathcal{C}_1 \times S_2^{-1}\mathcal{C}_2$. For $i = 1, 2$ factor

$$f_i = f_{i,1} \circ \ldots \circ f_{i,n_i}$$

such that each $f_{i,j}$ is the image of a morphism in $\mathcal{C}_i$ or $\bar{s}$ for an $s \in S_i$. Define a morphism $g_{1,j}$ in $(S_1 \times S_2)^{-1}\mathcal{C}_1 \times \mathcal{C}_2$ as follows. If $f_{1,j}$ is a morphism in $\mathcal{C}_1$ let $g_{1,j}$ be (f_1, id_{X_2}), and if $f_{1,g}$ is $\bar{s}$ for $s \in S_1$ let $g_{1,j}$ be $(f_1, \overline{\mathrm{id}_{X_2}})$. Similarly, $g_{2,j}$ is defined, where id_{X_2} is replaced by id_{Y_1}. Set

$$G(f) = g_{2,1} \circ \ldots \circ g_{2,n_2} \circ g_{1,1} \circ \ldots \circ g_{1,n_1}$$

then G is well-defined an inverse of F. $\square$

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Zusammenfassung

Die kombinatorische Torsion ist eine Invariante von CW-Komplexen und Homotopieäquivalenzen zwischen diesen. Ähnlich wie Homologie oder Kohomologie, besitzt die Theorie der Torsion einen beachtlichen algebraischen Teil, welcher grob gesprochen als eine weitreichende Verallgemeinerung der Determinante einer Matrix aufgefasst werden kann.

Für einen assoziativen Ring R definiert man

$$K_1(R) = \left(\varinjlim \operatorname{Aut}(R^n)\right)^{\mathrm{ab}}$$

d.h. $K_1(R)$ ist die Abelisierung des direkten Limes der allgemeinen linearen Gruppen von R. Das Bild einer linearen Abbildung $f \in \operatorname{Aut}(R^n)$ in $K_1(R)$ ist eine Verallgemeinerung der Determinante in folgendem Sinn: Falls R ein Körper ist, induziert die gewöhliche Determinante einen Isomorphismus $K_1(R) \cong R^*$. Die Gruppe $K_1(R)$ gehört zu einer Reihe $K_i(R)$, $i \geq 0$ von Gruppen welche von zentraler Bedeutung in der algebraischen K-Theorie sind.

Für einen Automorphismus f eines n-dimensionalen Vektorraumes kann die Determinante auch über das äußere Produkt als $\Lambda^n f : \Lambda^n V \to \Lambda^n V$ definiert werden, was auch allgemeiner für einen beliebigen Isomorphismus $f : V \to W$ sinnvoll ist. Diese verallgemeinerte Determinante ist aber nicht mehr ein Element in einem Körper, sondern ein Morphismus im Gruppoid $\mathcal{G}$ der eindimensionalen Vektorräume und Isomorphismen. Dieser Gruppoid hat eine zusätzliche Struktur: Das Tensorprodukt, ein Bifunktor $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ der aus $\mathcal{G}$ eine "2-Gruppe" macht, eine spezielle Tensorkategorie (*monoidal category*) in welcher die Objekte in gewissem Sinn invertierbar sind.

Im ersten Kapitel zeigen wir, analog zu [4], wie die eben erwähnten Erweiterungen kombiniert werden können, wobei vom Konzept der Tensorkategorie wesentlich Gebrauch gemacht wird.

Für Anwendungen in der algebraischen Topologie ist es notwendig Kettenkomplexe von Moduln, beziehungsweise Quasiisomorphismen zwischen diesen zu betrachten. Diese Erweiterung wird im zweiten Kapitel beschrieben.

Im dritten Kapitel wird die algebraische Theorie der ersten beiden Kapitel auf CW-Komplexe und Homotopieäquivalenzen angewendet. Dabei betrachtet man G-Räume und G-äquivariante Abbildungen für eine diskrete Gruppe G. Der wichtigste Spezialfall ist die Wirkung der Fundamentalgruppe auf einer universellen Überlagerung. In diesem Fall erhält man eine Variante der Whitehead-Torsion von V. Turaev [30].

Die Gruppe $K_1(R)$ ist für konkrete Berechnungen oft ungeeignet, stattdessen wählt man eine Darstellung von R, welche eine Funktor von R-Moduln zu Vektorräumen definiert. Im obigen Zusammenhang übersetzt sich so die Whitehead-Torsion in die Reidemeister-Torsion.

Anwendungen der Torsion, zu denen die topologische Klassifikation von Linsenräumen und Zusammenhänge mit der Knotentheorie gehören, werden in dieser Arbeit nicht besprochen. Einen Überblick dazu bieten zum Beispiel [17], [31] oder [23].

Im letzten Kapitel werden noch zwei Zugänge zur kombinatorischen Torsion auf Mannigfaltigkeiten genauer diskutiert: Dreieckszerlegungen und Morse-Funktionen.

Curriculum Vitae

Name	Fabian Haiden
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Education

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