Dissertation

Titel der Dissertation

Scattering Theory and Cauchy Problems

Verfasserin

Katrin Grunert

angestrebter akademischer Grad

Doktorin der Naturwissenschaften (Dr.rer.nat.)

Wien, im Juni 2010

Studienkennzahl lt. Studienblatt:	A 091 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuer:	ao. Univ.-Prof. Gerald Teschl

Abstract

In the first part of this thesis we investigate the kernels of the transformation operators for one–dimensional Schrödinger operators with potentials, which are asymptotically close to Bohr almost periodic infinite–gap potentials. Based on this we can develop scattering theory in the steplike case.
Furthermore we present an application of direct and inverse scattering theory for the Korteweg–de Vries equation, by solving the associated Cauchy problem with initial conditions, which are steplike Schwartz–type perturbations of finite–gap quasi–periodic potentials under the assumption that the respective spectral bands either coincide or are disjoint.
The second and last part is devoted to the Camassa-Holm equation, for which we study the stability of solutions of the Cauchy problem by deriving a Lipschitz metric.

Acknowledgements

First of all I want to thank my advisor Gerald Teschl for all the support I had when writing this thesis. Starting from choosing interesting problems for me to work on, over the time he spent discussing mathematical questions with me, to the possibility of working together with some of his colleagues. These are Ira Egorova from the Institute of Low Temperature in Kharkov, who discussed a lot with me and who gave me hints for the literature, Helge Holden at the NTNU Trondheim, who advised me during my research stay in Norway, and Xavier Raynaud from the University Oslo, who always believes that things can be made working.
I would also like to thank my colleagues within the Start Project and the PhD students at the NTNU Trondheim, who created a great working atmosphere during the last two years.
This work has been supported by the Austrian Science Fund (FWF) under Start Project No. Y330 and by the Research Council of Norway under the Yggdrasil Project No. 195792/V11, together with the Faculty of Mathematics at the University of Vienna and the Math department at the NTNU Trondheim, which provided me with excellent working conditions.
Last but not least I want to thank my family, who supported me all along the way.

Danksagung

Als erstes möchte ich mich bei meinem Betreuer Gerald Teschl, für die tolle Unterstützung während meines Doktorrats bedanken. Beginnend bei der Wahl von interessanten Themen, über die Zeit in der er mit mir über mathematische Fragen diskutiert hat, bis zur Möglichkeit mit einigen seiner Kollegen zusammen zu arbeiten. Dies sind, Ira Egorova vom Institute for Low Temperature in Kharkov, die oft mit mir diskutiert hat und mir bei der Literaturrecherche geholfen hat; Helge Holden von der NTNU Trondheim, der mich während meines Forschungsaufenthaltes in Norwegen betreut hat; und Xavier Raynaud von der Universität Oslo, der immer daran glaubt, dass es eine Lösung zu einem mathematischen Problem gibt.
Des Weiteren möchte ich mich bei meinen Kollegen im Start Project und den PhD Studenten in Trondheim bedanken, die eine tolle Arbeitsumgebung geschaffen haben.
Diese Arbeit wurde vom FWF durch das Start Projekt Y330 und vom Norwegischen Forschungsrat durch das Yggdrasil Projekt 195792/V11 unterstützt, gemeinsam mit der Fakultät für Mathematik der Universität Wien und dem Department für Mathematik der NTNU Trondheim, die mir exzellente Arbeitsbedingungen geschaffen haben.
Zum Abschluss möchte ich mich auch noch ganz herzlich bei meiner Familie bedanken, die mich auf meinem Weg unterstützt hat.

Contents

Chapter 1

Introduction

In this thesis, we will at first have a look at transformation operators, which play an important role in the investigation of direct and inverse scattering problems. They first showed up in the context of generalized shift operators in the work of Delsarte [31], and were constructed for arbitrary Sturm–Liouville equations by Povzner [86]. Afterwards transformation operators have been applied for the first time when considering inverse spectral problems, for example by Marchenko [81]. Soon after that Gel'fand and Levitan [44] found a method of recovering Sturm-Liouville equations from its spectral functions, using transformation operator techniques.

Another important step was the introduction of transformation operators, which preserve the asymptotic behavior of solutions at infinity by Levin [76]. Since that, these transformation operators are the main tool for solving different kinds of scattering problems, mainly in the case of constant backgrounds. They have been partly studied for periodic infinite-gap backgrounds by Firsova [39], [40], without estimates, which are necessary for solving related inverse scattering problems and they have been recently investigated in the finite-gap case by Boutet de Monvel, Egorova, and Teschl [10].

Since the seminal work of Gardner, Green, Kruskal, and Miura [43] in 1967, one of the main tools for solving various Cauchy problems is the inverse scattering transform and therefore, since then, a large number of articles has been devoted to direct and inverse scattering theory.

In much detail direct and inverse scattering have been studied (see e.g Marchenko [81]) in the case where the initial condition is asymptotically close to $p_\pm(x) = 0$. Taking this as a starting point, there are two natural cases, which have also been considered in the past. On the one hand the case of equal quasi-periodic, finite-gap potentials $p_-(x) = p_+(x)$ and on the other hand the case of steplike constant asymptotics $p_\pm(x) = c_\pm$ with $c_- \neq c_+$. Very recently, the combination of these two cases, namely the case that the initial condition is asymptotically close to steplike quasi-periodic finite-gap potentials $p_-(x) \neq p_+(x)$, has been investigated by Boutet de Monvel, Egorova and Teschl [10]. Of much interest is also the case of asymptotically periodic solutions, which has first been considered by Firsova [40].

As the Korteweg–de Vries (KdV) equation

$$q_t = -q_{xxx} + 6qq_x \tag{1.1}$$

is one of the most famous examples of a completely integrable nonlinear wave equation, a lot of articles have been devoted to the corresponding Cauchy problem, since the seminal work of Gardner, Green, Kruskal, and Miura [43] in 1967, where the inverse scattering transform is one of the main tools for solving the KdV equation. In particular, the case when the initial condition is asymptotically close to 0 is well understood and we just refer to the monographs by Eckhaus and Van Harten [32], Marchenko [81], Novikov, Manakov, Pitaevskii, and Zakharov [84] or Faddeev and Takhtajan [37]. The same is true for the case of steplike initial conditions which are asymptotically constant (with different constants in different directions), where we refer to Buslaev and Fomin [17], Cohen [22], Cohen and Kappeler [23] and Kappeler [65]. In fact, even the case where the asymptotics are given by some power-like behavior (including some unbounded initial conditions) were investigated by Bondareva, Kappeler, Perry, Shubin and, Topalov [7], [8], [66]. On the other hand, essentially nothing is known about the Cauchy problem for initial conditions which are asymptotically periodic. The first to consider a periodic background seem to be Kuznetsov and A.V. Mikhaĭlov, [74], who informally treated the Korteweg–de Vries equation with the Weierstraß elliptic function as background solution. The only known results, concerning to the existence of the solution seem to be by Ermakova [35], [36] and Firsova [41] (where the evolution of the scattering data for periodic background was given). However, both works are incomplete from the point of view of a rigorous application of the inverse scattering method. Surprisingly, much more is know about the asymptotical behavior (assuming existence) of such solutions, see for example [1], [4]–[6], [59], [67]–[71], [85]. Finally we mention that in the discrete case (Toda lattice) the same problem was completely solved in [34] (for corresponding long-time asymptotics see [9], [30], [61], [62], [63], [64], [73], [93]).

The second part of this thesis is devoted to the Camassa–Holm equation and the corresponding Cauchy problem. The Camassa-Holm (CH) equation, also known as the dispersive shallow water equation, is given through

$$u_t + 2ku_x - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad t < 0, \quad x \in \mathbb{R}, \tag{1.2}$$

where $u = u(x,t)$ is the fluid velocity in the x direction, and $k > 0$ is a constant related to the critical shallow water wave speeds, and the subscripts denote the partial derivatives. This equation first appeared in a list by B. Fuchssteiner and A. Fokas [42] and was first introduced as a model for shallow water waves by R. Camassa and D. Holm [18] and R. Camassa et al. [19]. More on the hydrodynamical relevance of this model can be found in the recent articles by R. Johnson [60] and A. Constantin and D. Lannes [27]. With

$$w := u - u_{xx} + \varkappa, \tag{1.3}$$

called the "momentum", equation (1.2) can be expressed as the condition of compatibility between

$$\frac{1}{w}\left(-f'' + \frac{1}{4}f\right) = \lambda f, \tag{1.4}$$

and

$$\partial_t f = -\left(\frac{1}{2\lambda} + u\right) f' + \frac{1}{2}u' f, \tag{1.5}$$

that is,

$$\partial_t \partial_{xx} f = \partial_{xx} \partial_t f$$

is the same as to say that (1.2) holds. Equation (1.4) is the spectral problem associated to (1.2). It were again R. Camassa and D. Holm [18], who noted that wave breaking can occur, by investigating the momentum in the case $k = 0$. Later existence of solutions, long-time behavior, and the blow up phenomena for $k \geq 0$ were studied using various methods, see for example [11], [13], [24], [25], and [26] and the references therein. The question of how to continue solutions beyond wave breaking has been of special interest and has been considered for example by Bressan and Constantin [13], and Holden and Raynaud [54], [56]. One way to do that is to continue the solution in such a way that the energy is conserved for almost all times. This can be described figurative in the context of peakon and antipeakon solutions by saying that peakon and antipeakon pass through each other when colliding. The questions that naturally arise in that context are the ones about stability of solutions and how to measure distances between two solutions.

The structure of this thesis, which is composed of the following four papers [48], [49], [33], and [50], is as follows:
In Chapter 2 we will investigate the kernels of transformation operators for one–dimensional Schrödinger operators with potentials, which are asymptotically close to Bohr almost periodic infinite–gap potentials.
Based on this, we develop, in Chapter 3, direct scattering theory for one–dimensional Schrödinger operators, which are asymptotically close to different Bohr almost periodic infinite–gap potentials on different half–axes.
Chapter 4 presents an application of the direct and inverse scattering theory in the case of one-dimensional Schrödinger operators with steplike potentials, which are asymptotically close to different finite–gap potentials on different half–axes. In more detail, we will solve the Cauchy problem for the KdV equation with initial conditions, which are steplike Schwartz–type perturbations of finite–gap potentials under the assumption that the mutual spectral bands either coincide or are disjoint.
After that we will consider the Camassa–Holm equation in the periodic case (Chapter 5) and study the stability of the solution of the corresponding Cauchy problem. In particular, we derive a Lipschitz metric, which states that two solutions, whose initial conditions are close, stay close.

Chapter 2

The transformation operator for Schrödinger operators on almost periodic infinite-gap backgrounds

2.1 Introduction

In this chapter we propose an investigation of the transformation operator in the case of Bohr almost periodic infinite–gap backgrounds, which belong to the so called Levitan class. The reason why this is a difficult problem is that the background Weyl solutions have countable many poles and the same will be valid for the Jost solutions. As a special case the Levitan class includes the set of smooth, periodic infinite–gap operators, in which case the background Weyl solutions have been thoroughly investigated, but not in the case of the more general Levitan class.

To set the stage, we need:

Hypothesis H. 2.1. *Let*

$$0 \le E_0 < E_1 < \cdots < E_n < \dots$$

be an increasing sequence of points on the real axis which satisfies the following conditions:

(i) for a certain $l > 1$, $\sum_{n=1}^{\infty} (E_{2n-1})^l (E_{2n} - E_{2n-1}) < \infty$ *and*

(ii) $E_{2n+1} - E_{2n-1} > Cn^{\alpha}$, *where* C *and* α *are some fixed, positive constants.*

We will call, in what follows, the intervals (E_{2j-1}, E_{2j}) for $j = 1, 2, \dots$ gaps. In each closed gap $[E_{2j-1}, E_{2j}]$, $j = 1, 2, \dots,$ we choose a point μ_j and an arbitrary sign $\sigma_j \in \{\pm 1\}$.

Consider next the system of differential equations for the functions $\mu_j(x)$, $\sigma_j(x)$, $j = 1, 2, ...$, which is an infinite analogue of the well-known Dubrovin equations, given by

$$\frac{d\mu_j(x)}{dx} = -2\sigma_j(x)\sqrt{-(\mu_j(x)-E_0)}\sqrt{\mu_j(x)-E_{2j-1}}\sqrt{\mu_j(x)-E_{2j}}\times \prod_{k=1,k\neq j}^{\infty} \frac{\sqrt{\mu_j(x)-E_{2k-1}}\sqrt{\mu_j(x)-E_{2k}}}{\mu_j(x)-\mu_k(x)} \tag{2.1}$$

with initial conditions $\mu_j(0) = \mu_j$ and $\sigma_j(0) = \sigma_j$, $j = 1, 2, \ldots$ [1]. Levitan [77], [78], and [79], proved, that this system of differential equations is uniquely solvable, that the solutions $\mu_j(x)$, $j = 1, 2, \ldots$ are continuously differentiable and satisfy $\mu_j(x) \in [E_{2j-1}, E_{2j}]$ for all $x \in \mathbb{R}$. Moreover, these functions $\mu_j(x)$, $j = 1, 2, \ldots$ are Bohr almost periodic [2]. Using the trace formula (see for example [77])

$$p(x) = E_0 + \sum_{j=1}^{\infty}(E_{2j-1} + E_{2j} - 2\mu_j(x)), \tag{2.2}$$

we see that also $p(x)$ is real Bohr almost periodic. The operator

$$\tilde{L} := -\frac{d^2}{dx^2} + p(x), \tag{2.3}$$

is then called an almost periodic infinite-gap Schrödinger operator of the Levitan class. It has as absolutely continuous spectrum the set

$$\sigma = [E_0, E_1] \cup \cdots \cup [E_{2j}, E_{2j+1}] \cup \ldots,$$

and has spectral properties analogous to the quasi-periodic finite-gap Schrödinger operator. In particular, it is completely defined by the series $\sum_{j=1}^{\infty}(\mu_j, \sigma_j)$, which we call the Dirichlet divisor. Analogously to the finite–gap case this divisor is connected to a Riemann surface of infinite genius, which is associated to the function $Y^{1/2}(z)$, where

$$Y(z) = -(z - E_0)\prod_{j=1}^{\infty}\frac{(z - E_{2j-1})}{E_{2j-1}}\frac{(z - E_{2j})}{E_{2j-1}}, \tag{2.4}$$

and where the cuts are taken along the spectrum. It is known, that the spectral equation

$$\Big(-\frac{d^2}{dx^2} + p(x)\Big)y(x) = \lambda y(x)$$

with any continuous, bounded potential $p(x)$ has two Weyl solutions $\psi_\pm(z, x)$ normalized by

$$\psi_\pm(z, 0) = 1 \quad \text{and } \psi_\pm(z, .) \in L^2(\mathbb{R}_\pm), \text{ for } z \in \mathbb{C}\backslash\sigma.$$

In our case of Bohr almost periodic potentials of the Levitan class, these solutions have complementary properties similar to properties of the Baker-Akhiezer functions in the finite-gap case. We will briefly discuss them in the next section.

[1] We will use the standard branch cut of the square root in the domain $\mathbb{C} \setminus \mathbb{R}_+$ with $\operatorname{Im}\sqrt{z} > 0$.

[2] For informations about almost periodic functions we refer to [80].

The objects of interest, for us, are the Jost solutions of the one-dimensional Schrödinger operator

$$L := -\frac{d^2}{dx^2} + q(x), \tag{2.5}$$

with the real potential $q(x) \in C(\mathbb{R})$ satisfying the following condition

$$\int_{\mathbb{R}} (1+|x|^2)|q(x)-p(x)|dx < \infty. \tag{2.6}$$

We will prove the following result

Theorem 2.2. *Assume Hypothesis* 2.1. *Let p, defined as in* (2.2), *belong to the Levitan class, and q satisfy* (2.6), *then the Jost solutions $\phi_\pm(z,x)$ can be represented in the following form*

$$\phi_\pm(z,x) = \psi_\pm(z,x) \pm \int_x^{\pm\infty} K_\pm(x,y)\psi_\pm(z,y),$$

where, the solutions of (2.68), *are real valued, continuously differentiable with respect to both parameters and for $\pm y > \pm x$ they satisfy*

$$|K_\pm(x,y)| \leq \pm C_\pm(x) \int_x^{\pm\infty} |q(x)-p(x)|dx.$$

Here $C_\pm(x)$ are continuous positive functions, which are monotonically decreasing as $x \to \infty$.

Here it should be pointed out that a subset of the operators belonging to the Levitan class consists of operators with periodic potentials. Assume that the sequence $\{E_j\}_{j=1}^{\infty}$ fulfills Hypothesis 2.1, then there is a criteria when this sequence is the set of band edges of the spectrum of some Schrödinger operator with periodic potential $p(x+a) = p(x) \geq 0$ with $p \in W_2^3[0,a]$. Namely, Marchenko and Ostrovskii proved in [82], that

$$p \in W_2^k[0,a], \qquad \text{iff} \sum_{j=1}^{\infty} j^{2k+2}(\sqrt{E_{2j}-E_0} - \sqrt{E_{2j-1}-E_0})^2 < \infty,$$

for $k = 0, 1, \ldots$. As it is well–known that in the periodic case $E_{2j-1} = j^2 + O(1)$ and $E_{2j} = j^2 + O(1)$ as $j \to \infty$, we obtain for large j that

$$\begin{aligned} E_{2j-1}(E_{2j} - E_{2j-1}) \leq{}& E_{2j-1}(\sqrt{E_{2j}-E_0} - \sqrt{E_{2j-1}-E_0})\times \\ & (\sqrt{E_{2j}-E_0} - \sqrt{E_{2j-1}-E_0}) \\ \leq{}& 2j^3(\sqrt{E_{2j}-E_0} - \sqrt{E_{2j-1}-E_0}) =: I_j. \end{aligned}$$

As Hypothesis 2.1 is satisfies, we have $\sum_{j=1}^{\infty} j^{2k-4} I_j^2 < \infty$ for $k > 2$ and hence the Cauchy inequality implies that $\sum_{j=1}^{\infty} I_j < \infty$ in this case. This means, that Hypothesis 2.1 is satisfied for any a-periodic potential $p(x) \geq 0$ with $p \in W_2^k[0,a]$ for $k > 2$.

2.2 Background Schrödinger operators

In this section we want to summarize some facts for the background Schrödinger operator of Levitan class. We present these results, obtained in [77], [88], and [89], in a form similar to the finite-gap case used in [10] and [45].

Let $\tilde{L}$ be as in (2.3). Denote by $s(z,x)$, $c(z,x)$ the sin- and cos-type solutions of the corresponding equation

$$\left(-\frac{d^2}{dx^2}+p(x)\right)y(x)=zy(x),\quad z\in\mathbb{C},\tag{2.7}$$

associated with the initial conditions

$$s(z,0)=c'(z,0)=0,\quad c(z,0)=s'(z,0)=1,$$

where prime denotes the derivative with respect to x. Then $c(z,x)$, $c'(z,x)$, $s(z,x)$, and $s'(z,x)$ are holomorphic with respect to $z\in\mathbb{C}\backslash\sigma$. They can be represented in the following form

$$c(z,x)=\cos(\sqrt{z}x)+\int_0^x\frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}}p(y)c(z,y)dy,$$

$$s(z,x)=\frac{\sin(\sqrt{z}x)}{\sqrt{z}}+\int_0^x\frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}}p(y)s(z,y)dy.$$

The background Weyl solutions are given by

$$\psi_\pm(z,x)=c(z,x)+m_\pm(z,0)s(z,x),\tag{2.8}$$

where

$$m_\pm(z,x)=\frac{\psi'_\pm(z,x)}{\psi_\pm(z,x)}=\frac{H(z,x)\pm Y^{1/2}(z)}{G(z,x)},\tag{2.9}$$

are the Weyl functions of $\tilde{L}$ (cf, [77]), where $Y(z)$ is defined by (2.4),

$$G(z,x)=\prod_{j=1}^\infty\frac{z-\mu_j(x)}{E_{2j-1}},\quad\text{and}\quad H(z,x)=\frac{1}{2}\frac{d}{dx}G(z,x).\tag{2.10}$$

Using (2.1) and (2.10), we have

$$H(z,x)=\frac{1}{2}\frac{d}{dx}G(z,x)=G(z,x)\sum_{j=1}^\infty\frac{\sigma_j(x)Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x),x)(z-\mu_j(x))}.\tag{2.11}$$

The Weyl functions $m_\pm(z,x)$ are Bohr almost periodic as the following argument shows: For each $j\in\mathbb{N}$ the functions $\mu_j(x)$ are almost periodic and hence, as a finite product of almost periodic functions is again almost periodic, also

$$G_n(z,x)=\prod_{j=1}^n\frac{z-\mu_j(x)}{E_{2j-1}}\tag{2.12}$$

is almost periodic for fixed $z\in\mathbb{C}$. Moreover, we have

$$|G_n(z,x)-G(z,x)|=\left|\prod_{j=1}^n\frac{z-\mu_j(x)}{E_{2j-1}}\Big(1-\prod_{j=n+1}^\infty\frac{z-\mu_j(x)}{E_{2j-1}}\Big)\right|,\tag{2.13}$$

and for every fixed $z \in \mathbb{C}$ there exists a $m \in \mathbb{N}$ such that $|z| \leq E_{2m-1}$. Then for $n > m$, we obtain on the one hand

$$\exp\Big(- \sum_{j=n+1}^{\infty} \frac{|z|}{E_{2j-1} - |z|}\Big) \leq \prod_{j=n+1}^{\infty} \frac{\mu_j(x) - |z|}{E_{2j-1}} \leq \prod_{j=n+1}^{\infty} |\frac{z - \mu_j(x)}{E_{2j-1}}|, \quad (2.14)$$

and on the other hand

$$\prod_{j=n+1}^{\infty} |\frac{z-\mu_j(x)}{E_{2j-1}}| \leq \prod_{j=n+1}^{\infty} \frac{\mu_j(x) + |z|}{E_{2j-1}}$$
$$\leq \exp\Big(\frac{1}{E_{2n+1}} \sum_{j=n+1}^{\infty} (E_{2j} - E_{2j-1}) + |z| \sum_{j=n+1}^{\infty} \frac{1}{E_{2j-1}}\Big), \quad (2.15)$$

where we used that $\log(1+x) \leq x$ and $\log(1-x) \geq \frac{-x}{1-x}$ for $x > 0$. Noticing that the second condition in Hypothesis 2.1 implies that $\sum_{j=1}^{\infty} \frac{1}{E_{2j-1}}$ converges, all terms are well-defined, and it follows that the product $\prod_{j=n+1}^{\infty} \frac{z-\mu_j(x)}{E_{2j-1}}$ converges to 1 as $n \to \infty$. Furthermore, $\prod_{j=1}^{\infty} \frac{z-\mu_j(x)}{E_{2j-1}}$ is uniformly bounded with respect to x for any fixed z. As all our estimates are independent of x, we have that $G_n(z,x)$ converges uniformly for fixed z against $G(z,x)$ and thus, the function $G(z,x)$ is almost periodic with respect to x. Furthermore by definition $\frac{H(z,x)}{G(z,x)} = \frac{1}{2}\frac{G'(z,x)}{G(z,x)} = \frac{1}{2}(\log(G(z,x))'$ and therefore $\frac{H(z,x)}{G(z,x)}$ is also almost periodic, where we use that $\log(G(z,x)) \neq 0$ for $z \notin [E_1, E_2] \cup \cdots \cup [E_{2j-1}, E_{2j}] \cup \dots$ together with [80, Property 3,4,5], and hence $m_\pm(z,x)$ are also almost periodic functions.

Lemma 2.3. *The background Weyl solutions, for $z \in \mathbb{C}$, can be represented in the following form*

$$\psi_\pm(z,x) = \exp\left(\int_0^x m_\pm(z,y)dy\right) = \left(\frac{G(z,x)}{G(z,0)}\right)^{1/2} \exp\left(\pm \int_0^x \frac{Y^{1/2}(z)}{G(z,y)}dy\right). \quad (2.16)$$

If for some $\varepsilon > 0$, $|z - \mu_j(x)| > \varepsilon$ for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$, then the following holds: For any $C > 0$ there exists an $R > 0$ such that

$$|\psi_\pm(z,x)| \leq \mathrm{e}^{\mp(1-C)x\,\mathrm{Im}(\sqrt{z})}\Big(1 + \frac{D}{|z|}\Big), \ \text{for any } |z| \geq R, \quad (2.17)$$

where D denotes some constant dependent on R.

Proof. First we will show that

$$f_\pm(z,x) = \left(\frac{G(z,x)}{G(z,0)}\right)^{1/2} \exp\left(\pm \int_0^x \frac{Y^{1/2}(z)}{G(z,y)}dy\right) \quad (2.18)$$

fulfills

$$\left(-\frac{d^2}{dx^2} + p(x)\right) y(x) = zy(x). \quad (2.19)$$

Using (2.9) we obtain that $f'_\pm = m_\pm f_\pm$ and $f''_\pm = (m'_\pm + m_\pm^2)f_\pm$. Hence (2.19) will be satisfies if and only if

$$m'_\pm + m_\pm^2 = p(x) - z. \tag{2.20}$$

This can be checked using the following relations, which are proved in [77],

$$G(z,x)N(z,x) + H(z,x)^2 = Y(z), \tag{2.21}$$

where

$$N(z,x) = -(z-\tau_0(x))\prod_{j=1}^{\infty}\frac{z-\tau_j(x)}{E_{2j-1}}, \tag{2.22}$$

with $\tau_0(x) \in (-\infty, E_0]$ and $\tau_j(x) \in [E_{2j-1}, E_{2j}]$ and

$$\frac{d^2}{dx^2}G(z,x) = 2((p(x)-z)G(z,x) - N(z,x)). \tag{2.23}$$

Moreover, for z outside an ε neighborhood of the gaps we have $f(z,0)=1$, and we can make the following considerations

$$\frac{G(z,x)}{G(z,0)} = \prod_{j=1}^{\infty}\frac{z-\mu_j(x)}{z-\mu_j(0)} = \exp\Big(\sum_{j=1}^{\infty}\log\Big(1+\frac{\mu_j(0)-\mu_j(x)}{z-\mu_j(0)}\Big)\Big). \tag{2.24}$$

Thus we obtain

$$\left|\frac{G(z,x)}{G(z,0)}\right| \leq \exp\Big(\sum_{j=1}^{\infty}\log\Big(1+\left|\frac{\mu_j(0)-\mu_j(x)}{z-\mu_j(0)}\right|\Big)\Big) \leq \exp\Big(\sum_{j=1}^{\infty}\left|\frac{\mu_j(0)-\mu_j(x)}{z-\mu_j(0)}\right|\Big)\Big), \tag{2.25}$$

where we used that $\log(1+x) \leq x$ for $x > 0$. Moreover

$$\left|\frac{\mu_j(0)-\mu_j(x)}{z-\mu_j(0)}\right| = \frac{1}{\left|\frac{z-\mu_j(0)}{\mu_j(0)-\mu_j(x)}\right|}, \tag{2.26}$$

which implies for $|z| \leq 2E_{2j}$ that

$$\left|\frac{(\mu_j(0)-\mu_j(x))z}{z-\mu_j(0)}\right| \leq \left|\frac{2(\mu_j(0)-\mu_j(x))E_{2j}}{\varepsilon}\right|. \tag{2.27}$$

For $|z| > 2E_{2j}$ we can estimate the terms by

$$\begin{aligned}\left|\frac{(\mu_j(0)-\mu_j(x))z}{z-\mu_j(0)}\right| &\leq |\mu_j(0)-\mu_j(x)|\left|\frac{1}{1-\frac{\mu_j(0)}{z}}\right| \leq |\mu_j(0)-\mu_j(x)|\frac{1}{1-\left|\frac{\mu_j(0)}{z}\right|} \qquad (2.28)\\ &\leq |\mu_j(0)-\mu_j(x)|\frac{1}{1-\left|\frac{E_{2j}}{z}\right|} \leq 2|\mu_j(0)-\mu_j(x)|.\end{aligned}$$

Combining the estimates from above, we obtain

$$\begin{aligned}\left|\frac{(\mu_j(0)-\mu_j(x))z}{z-\mu_j(0)}\right| &\leq 2\max(1,\frac{E_{2j}}{\varepsilon})|\mu_j(0)-\mu_j(x)| \qquad (2.29)\\ &\leq 2\max(1,\frac{E_{2j}}{\varepsilon})(E_{2j}-E_{2j-1}),\end{aligned}$$

and for any fixed $\varepsilon > 0$ there exists a k independent of x and z such that $\frac{E_{2n}}{\varepsilon} > 1$ for all $n > k$, and therefore

$$\left|\frac{G(z,x)}{G(z,0)}\right| \le \exp\Big(\frac{1}{|z|}\sum_{j=1}^{\infty}\left|\frac{(\mu_j(0)-\mu_j(x))z}{z-\mu_j(0)}\right|\Big) \le \exp\Big(C\frac{1}{|z|}\Big). \tag{2.30}$$

where C is a constant independent of x and z. Analogously one can now investigate $\frac{Y^{1/2}(z)}{G(z,x)}$. Using

$$Y^{1/2}(z) = \mathrm{i}\sqrt{z-E_0}\prod_{j=1}^{\infty}\frac{\sqrt{z-E_{2j-1}}\sqrt{z-E_{2j}}}{E_{2j-1}}, \tag{2.31}$$

where the roots are defined as follows

$$\sqrt{z-E} = \sqrt{|z-E|}\mathrm{e}^{\mathrm{i}\arg(z-E)/2}, \tag{2.32}$$

together with $\frac{Y^{1/2}(z)}{G(z,x)}$ is a Herglotz function and

$$\sqrt{z-E_0} = \sqrt{z}(1+O(\frac{1}{z})), \qquad \text{as } z \to \infty, \tag{2.33}$$

we obtain the following estimate, which is uniform with respect x,

$$\frac{Y^{1/2}(z)}{G(z,x)} = \mathrm{i}\sqrt{z}(1+O(\frac{1}{z})), \qquad \text{as } z \to \infty. \tag{2.34}$$

Using now that $\int_0^x \frac{Y^{1/2}(z)}{G(z,\tau)}d\tau = x\frac{Y^{1/2}(z)}{G(z,\zeta)}$, where $\zeta \in (0,x)$ by the mean value theorem, we finally obtain that $f_\pm(z,x)$ has the following asymptotic expansion outside a small neighborhood of the gaps as $z \to \infty$

$$f_\pm(z,x) = \mathrm{e}^{\pm\mathrm{i}\sqrt{z}x(1+O(\frac{1}{z}))}\Big(1+O(\frac{1}{z})\Big),$$

where we use the branch cut of the square root in the domain $\mathbb{C}\backslash\mathbb{R}_+$ with $\mathrm{Im}(\sqrt{z}) > 0$. Thus $f_\pm(z,.) \in L^2(\mathbb{R}_\pm)$ for $z \in \mathbb{C}$ outside a small neighborhood of the gaps and therefore away from the gaps $f_\pm(z,x)$ must coincide with $\psi_\pm(z,x)$. Taking limits for the values on the real axis, we get that $\psi_\pm(z,x)$ can be represented for any $z \in \mathbb{C}$ by (2.16). □

As the spectrum consists of infinitely many bands, let us cut the complex plane along the spectrum σ and denote the upper and lower sides of the cuts by σ^u and σ^l. The corresponding points on these cuts will be denoted by z^u and z^l, respectively. In particular, this means

$$f(z^u) := \lim_{\varepsilon\downarrow 0} f(z+\mathrm{i}\varepsilon), \qquad f(z^l) := \lim_{\varepsilon\downarrow 0} f(z-\mathrm{i}\varepsilon), \qquad z \in \sigma.$$

Define the Green function (see e.g. [2], [28], and [88])

$$g(z) = -\frac{G(z,0)}{2Y^{1/2}(z)}, \tag{2.35}$$

where the branch of the square root is chosen in such a way that

$$\frac{1}{\mathrm{i}} g(z^u) = \mathrm{Im}(g(z^u)) > 0 \quad \text{for} \quad \lambda \in \sigma, \tag{2.36}$$

then we obtain after a short calculation

$$W(\psi_-(z), \psi_+(z)) = m_+(z) - m_-(z) = -g(z)^{-1}, \tag{2.37}$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the usual Wronskian determinant.

For every Dirichlet eigenvalue $\mu_j = \mu_j(0)$, the Weyl functions $m_\pm(z)$ might have singularities. If μ_j is in the interior of its gap, precisely one Weyl function m_+ or m_- will have a simple pole. Otherwise, if μ_j sits at an edge, both will have a square root singularity. Hence we divide the set of poles accordingly:

$$\begin{aligned}
M_+ &= \{\mu_j \mid \mu_j \in (E_{2j-1}, E_{2j}) \text{ and } m_+ \text{ has a simple pole}\},\\
M_- &= \{\mu_j \mid \mu_j \in (E_{2j-1}, E_{2j}) \text{ and } m_- \text{ has a simple pole}\},\\
\hat{M} &= \{\mu_j \mid \mu_j \in \{E_{2j-1}, E_{2j}\}\}.
\end{aligned}$$

In particular, the following properties of the Weyl solutions are valid (see, e.g. [29], [77], [88], [91]):

Lemma 2.4. *The Weyl solutions have the following properties:*

(i) The functions $\psi_\pm(z,x)$ are holomorphic as a function of z in the domain $\mathbb{C} \setminus (\sigma \cup M_\pm)$, real valued on the set $\mathbb{R} \setminus \sigma$, and have simple poles at the points of the set $M_\pm$. Moreover, they are continuous up to the boundary $\sigma^u \cup \sigma^l$ except at the points from $\hat{M}$ and

$$\psi_+(\lambda^{\mathrm{u}}) = \psi_-(\lambda^{\mathrm{l}}) = \overline{\psi_+(\lambda^{\mathrm{l}})}, \quad \lambda \in \sigma. \tag{2.38}$$

For $E \in \hat{M}$ the Weyl solutions satisfy

$$\psi_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \text{as } z \to E \in \hat{M}.$$

The same is true for $\psi'_\pm(z,x)$.

(ii) The functions $\psi_\pm(z,x)$ form an orthonormal basis on the spectrum with respect to the weight

$$d\rho(z) = \frac{1}{2\pi\mathrm{i}} g(z)dz, \tag{2.39}$$

and any $f(x) \in L^2(-\infty,\infty)$ can be expresses through

$$f(x) = \oint_\sigma \left(\int_{\mathbb{R}} f(y)\psi_+(z,y)dy \right) \psi_-(z,x) d\rho(z). \tag{2.40}$$

Here we use the notation

$$\oint_\sigma f(z)d\rho(z) := \int_{\sigma^u} f(z)d\rho(z) - \int_{\sigma^l} f(z)d\rho(z). \tag{2.41}$$

Proof. (i) Having in mind (2.16), we will show as a first step that $\int_0^x \frac{Y^{1/2}(z)}{G(z,\tau)}d\tau$ is purely imaginary as $z \to E_{2j}$, with $z \in \sigma$, (the case $z \to E_{2j-1}$ can be handled in the same way). For fixed $x \in \mathbb{R}$ we can separate the interval $[0,x]$ into smaller intervals $[0,x_1] \cup [x_1,x_2] \cup \cdots \cup [x_k,x]$ such that $\mu_j(x_l) \in \{E_{2j-1}, E_{2j}\}$ and $\mu_j(x_l) \neq \mu_j(x_{l+1})$. Assuming $\mu_j(x_l) = E_{2j-1}$ and $\mu_j(x_{l+1}) = E_{2j}$, and setting

$$\tilde{Y}_j(z,x) := \sqrt{-(z-E_0)}\sqrt{z-E_{2j-1}}\prod_{j\neq l}\left(\frac{z-E_{2l-1}}{z-\mu_l(x)}\frac{z-E_{2l}}{z-\mu_l(x)}\right)^{1/2},$$

where $\tilde{Y}_j(z,x)$ is bounded for any z inside the j'th gap, we can conclude,

$$\begin{aligned}
&\int_{x_l}^{x_{l+1}} \frac{Y^{1/2}(z)}{G(z,\tau)}d\tau = \sqrt{z-E_{2j}}\int_{x_l}^{x_{l+1}}\frac{\tilde{Y}_j(z,\tau)}{z-\mu_j(\tau)}d\tau \\
&= \sqrt{z-E_{2j}}\Big(\int_{x_l}^{x_{l+1}}\frac{\tilde{Y}_j(\mu_j(\tau),\tau)}{z-\mu_j(\tau)}d\tau + \int_{x_l}^{x_{l+1}}\frac{\tilde{Y}_j(z,\tau)-\tilde{Y}_j(\mu_j(\tau),\tau)}{z-\mu_j(\tau)}d\tau\Big) \\
&= \sqrt{z-E_{2j}}\Big(\int_{x_l}^{x_{l+1}}\frac{-\frac{d\mu_j(\tau)}{d\tau}}{2\sigma_j(\tau)\sqrt{\mu_j(\tau)-E_{2j}}(z-\mu_j(\tau))}d\tau + \\
&\quad + \int_{x_l}^{x_{l+1}}\frac{d}{dz}\tilde{Y}_j(z,\tau)|_{z=\zeta_j(\tau)}d\tau\Big),
\end{aligned}$$

where $\zeta_j(\tau) \in (\mu_j(\tau), z)$. Note that the function $\frac{d}{dz}\tilde{Y}_j(z,\tau)$ is uniformly bounded for $z \in [E_{2j-1}+\varepsilon, E_{2j}+\varepsilon]$ for some $\varepsilon > 0$ and that $\frac{\tilde{Y}_j(z,\tau)-\tilde{Y}_j(\mu_j(\tau),\tau)}{z-\mu_j(\tau)}$ is uniformly bounded for $\mu_j \in [E_{2j-1}, E_{2j-1}+\varepsilon]$ and z near E_{2j}, which yields

$$\sqrt{z-E_{2j}}\int_{x_l}^{x_{l+1}}\frac{d}{dz}\tilde{Y}_j(z,\tau)|_{z=\zeta_j(\tau)}d\tau = O\left(\sqrt{z-E_{2j}}\right). \tag{2.42}$$

On each of the intervals $[x_l, x_{l+1}]$ the function $\sigma_j(x)$ is constant and therefore

$$\begin{aligned}
&\sqrt{z-E_{2j}}\left(\int_{x_l}^{x_{l+1}}\frac{-\frac{d\mu_j(\tau)}{d\tau}}{2\sigma_j\sqrt{\mu_j(\tau)-E_{2j}}(z-\mu_j(\tau))}d\tau\right) \qquad (2.43)\\
&= \sqrt{z-E_{2j}}\left(-\int_{\mu_j(x_l)}^{\mu_j(x_{l+1})}\frac{1}{2\sigma_j\sqrt{y-E_{2j}}(z-y)}dy\right) \\
&= \sqrt{E_{2j}-z}\left(\int_{\sqrt{E_{2j}-E_{2j-1}}}^{0}\frac{1}{\sigma_j(z-E_{2j}+s^2)}ds\right) \\
&= -\sigma_j \mathrm{i}\arctan\left(\frac{\sqrt{E_{2j}-E_{2j-1}}}{\sqrt{z-E_{2j}}}\right).
\end{aligned}$$

A close look shows that the same method can be applied to compute $\int_0^{x_1}\frac{Y^{1/2}(z)}{G(z,\tau)}d\tau$ and $\int_{x_k}^{x}\frac{Y^{1/2}(z)}{G(z,\tau)}d\tau$
This implies that $\int_{x_l}^{x_{l+1}}\frac{Y^{1/2}(z)}{G(z,\tau)}d\tau \to -\frac{1}{2}\sigma_j \mathrm{i}\pi$ as $z \to E_{2j}$ and thus $\int_0^x \frac{Y^{1/2}(z)}{G(z,\tau)}d\tau \in \mathrm{i}\mathbb{R}$.

In more detail one obtains that

$$\exp\Big(\int_0^x \frac{Y^{1/2}(z)}{G(z,\tau)}d\tau\Big) = \begin{cases} \pm 1, & \mu_j(0)\neq E, \mu_j(x)\neq E,\\ \pm 1, & \mu_j(0)= E, \mu_j(x)= E,\\ \pm \mathrm{i}, & \mu_j(0)= E, \mu_j(x)\neq E,\\ \pm \mathrm{i}, & \mu_j(0)\neq E, \mu_j(x)= E. \end{cases} \tag{2.44}$$

For the product term in (2.16), we have the following estimate

$$\begin{aligned} \frac{z-\mu_j(x)}{z-\mu_j(0)} \exp\Big(-\frac{2}{\beta}\sum_{j=1}^{\infty}(E_{2j}-E_{2j-1})\Big) &\leq \prod_{l=1}^{\infty}\frac{z-\mu_l(x)}{z-\mu_l(0)}\\ &\leq \frac{z-\mu_j(x)}{z-\mu_j(0)} \exp\Big(\frac{2}{\beta}\sum_{j=1}^{\infty}(E_{2j}-E_{2j-1})\Big), \end{aligned}$$

for z inside the interval $[E_{2j-1}-\frac{\beta}{2}, E_{2j}+\frac{\beta}{2}]\cap\sigma$ for some $\varepsilon>0$, where $\beta=\min_{l,j;l\neq j}\{|E_{2j}-E_{2l}|,|E_{2j-1}-E_{2l}|\}$. This finishes the proof of the first claim, as (2.38) follows directly from (2.36). For the second claim consider, using (2.16),

$$\psi'_{\pm}(z,x) = m_{\pm}(z,x)\psi_{\pm}(z,x). \tag{2.45}$$

By the investigations from before, it suffices to analyze $m_{\pm}(z,x)$, given by (2.9), which has the following representation,

$$m_{\pm}(z,x) = \sum_{n=1}^{\infty}\frac{\sigma_n(x)Y^{1/2}(\mu_n(x))}{\frac{d}{dz}G(\mu_n(x),x)(z-\mu_n(x))} \pm \frac{Y^{1/2}(z)}{G(z,x)}. \tag{2.46}$$

For $j\neq n$ we have

$$\Big|\frac{Y^{1/2}(\mu_n(x)}{\frac{d}{dz}G(\mu_n(x),x)(E_{2j}-\mu_n(x)}\Big| \leq C_1\frac{\sqrt{E_{2n}-E_0}}{\beta}(E_{2n}-E_{2n-1}), \tag{2.47}$$

where $C_1 := \exp\Big(\frac{1}{\beta}\sum_{j=1}^{\infty}(E_{2j}-E_{2j-1})\Big)$. Thus; using Hypothesis 2.1, we obtain that $\sum_{n\neq j}\frac{\sigma_j(x)Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x),x)(z-\mu_j(x))}$ converges uniformly and is uniformly bounded with respect to x.
For $j=n$, we obtain

$$\Big|\frac{Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x),x)(z-\mu_j(x))}\Big| \leq C_1\frac{\sqrt{E_{2j}-E_0}\sqrt{(\mu_j(x)-E_{2j-1})(E_{2j}-\mu_j(x))}}{z-\mu_j(x)},$$

and analogously

$$\Big|\frac{Y^{1/2}(z)}{G(z,x)}\Big| \leq C_1\frac{\sqrt{z-E_0}\sqrt{(z-E_{2j-1})(z-E_{2j})}}{z-\mu_j(x)}.$$

Multiplying now the last two terms of interest by $\psi_{\pm}(z,x)$ and let $z\to E_{2j}$ with $z\in\sigma$ as in the proof of the first claim, finishes the proof.

(ii) For a proof we refer to [29] and [77] or [92].

□

2.3 Derivation of the integral equations for the transformation operators and estimates

Consider the equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right) y(x) = zy(x), \quad z \in \mathbb{C}, \tag{2.48}$$

with a potential $q(x)$ satisfying (2.6). Suppose that this equation has two solutions $\phi_\pm(z,x)$, which we will call the Jost solutions, which are asymptotically close for fixed z as $x \to \pm\infty$ to $\psi_\pm(z,x)$, the background Weyl solutions defined in (2.8). Set

$$J(z,x,y) = \frac{\psi_+(z,y)\psi_-(z,x) - \psi_+(z,x)\psi_-(z,y)}{W(\psi_+(z),\psi_-(z))} \tag{2.49}$$

and

$$\tilde{q}(x) = q(x) - p(x). \tag{2.50}$$

Then the Jost solutions have to satisfy the following integral equations

$$\phi_\pm(z,x) = \psi_\pm(z,x) - \int_x^{\pm\infty} J(z,x,y)\tilde{q}(y)\phi_\pm(z,y)dy \tag{2.51}$$

Suppose, that solutions of this form, also have the following representation

$$\phi_\pm(z,x) = \psi_\pm(z,x) \pm \int_x^{\pm\infty} K_\pm(x,y)\psi_\pm(z,y)dy, \tag{2.52}$$

where $K_\pm(x,y)$ are real valued functions, then we have to show that these functions exist and for showing the existence of the Jost solutions we must also show that these functions are decaying fast enough for fixed x, when y tends to infinity in a certain sense. For simplicity and because both representations can be obtained using the same techniques we will only investigate the $+$ case.

Assume that there exist $K_+(x,y)$ with $K_+(x,.) \in L^2(\mathbb{R})$ and $K_+(x,y) = 0$ for $y < x$, such that $\phi_+(z,x)$ can be represented by (2.52). Then substituting (2.52) into (2.51), multiplying it with $\psi_-(z,x)g(z)$, integrating over the set $\sigma^{u,l}$, using the identity (2.40), and taking into account that $K_+(x,y) = 0$, $x > y$, we obtain

$$K_+(x,s) + \int_x^\infty dy\, \tilde{q}(y) \oint_\sigma J(\lambda,x,y)\psi_+(\lambda,y)\psi_-(\lambda,s)d\rho(\lambda) \tag{2.53}$$

$$+ \int_x^\infty dy\, \tilde{q}(y) \int_y^\infty dt\, K_+(y,t) \oint_\sigma J(\lambda,x,y)\psi_+(\lambda,t)\psi_-(\lambda,s)d\rho(\lambda) = 0.$$

Set

$$\Gamma_+(x,y,t,s) = \oint_\sigma \psi_+(\lambda,x)\psi_-(\lambda,y)\psi_+(\lambda,t)\psi_-(\lambda,s)g(\lambda)d\rho(\lambda), \tag{2.54}$$

where the integral has to be understood as a principal value.

Then substituting (2.37), (2.39), (2.49), and (2.54) into (2.53) we obtain

$$K_+(x,s) + \int_x^\infty (\Gamma_+(x,y,y,s) - \Gamma_+(y,x,y,s))\,\tilde{q}(y)\,dy \tag{2.55}$$
$$+ \int_x^\infty dy\,\tilde{q}(y) \int_y^\infty K_+(y,t)\,(\Gamma_+(x,y,t,s) - \Gamma_+(y,x,t,s))\,dt = 0.$$

A simple calculation using (2.36) and (2.38) shows that (2.54) satisfies

$$\overline{\Gamma_+(x,y,t,s)} = -\Gamma_+(y,x,s,t). \tag{2.56}$$

Combining (2.16), (2.35), and (2.39), one obtains that the only poles of the integrand are given at the band edges. Using ideas from Cauchy's theorem it turns out to be necessary to investigate the following series

$$D_+(x,y,r,s) = -\frac{1}{4} \sum_{E\in\partial\sigma} f_+(E,x,y,r,s), \tag{2.57}$$

where

$$f_+(E,x,y,r,s) = \lim_{z\to E} \frac{G(z,0)^2}{\frac{d}{dz}Y(z)} \psi_+(z,x)\psi_-(z,y)\psi_+(z,r)\psi_-(z,s). \tag{2.58}$$

Lemma 2.5. *The series $D_+(x,y,r,s)$ defined by (2.57) and (2.58) converges, is continuous, and uniformly bounded with respect to all variables.*

Proof. First note that by (2.16) we have

$$f_+(E,x,y,r,s) = \frac{(G(E,x)G(E,y)G(E,r)G(E,s))^{1/2}}{\frac{d}{dz}Y(E)} \times$$
$$\lim_{z\to E} \exp\left(\int_y^x \frac{Y^{1/2}(z)}{G(z,\tau)} d\tau \int_s^r \frac{Y^{1/2}(z)}{G(z,\tau)} d\tau \right),$$

where the limit is taken from inside the spectrum. For computing the integral terms we refer to the proof of Lemma 2.4. We will now investigate

$$M_+(E_{2l},x,y,r,s) = \frac{(G(E_{2l},x)G(E_{2l},y)G(E_{2l},r)G(E_{2l},s))^{1/2}}{\frac{d}{dz}Y(E_{2l})}$$
$$= -\frac{((E_{2l}-\mu_l(x))(E_{2l}-\mu_l(y))(E_{2l}-\mu_l(r))(E_{2l}-\mu_l(s)))^{1/2}}{(E_{2l}-E_0)(E_{2l}-E_{2l-1})} \times \tag{2.59}$$
$$\prod_{j=1,j\neq l}^\infty \frac{((E_{2l}-\mu_j(x))(E_{2l}-\mu_j(y))(E_{2l}-\mu_j(r))(E_{2l}-\mu_j(s)))^{1/2}}{(E_{2l}-E_{2j-1})(E_{2l}-E_{2j})},$$

which can be estimated as follows

$$\prod_{j=1,j\neq l}^\infty \left|\frac{(E_{2l}-\mu_j(x))}{(E_{2l}-E_{2j})}\frac{(E_{2l}-\mu_j(y))}{(E_{2l}-E_{2j-1})}\right| \le \prod_{j=1}^{l-1} \frac{(E_{2l}-\mu_j(x))}{(E_{2l}-E_{2j})} \prod_{j=+11}^\infty \frac{(E_{2l}-\mu_j(y))}{(E_{2l}-E_{2j-1})}$$
$$\le \exp\left(\sum_{j=1}^{l-1} \log\left(1 + \frac{E_{2j}-\mu_j(x)}{E_{2l}-E_{2j}}\right) + \sum_{j=l+1}^\infty \log\left(1+\frac{\mu_j(y)-E_{2j-1}}{E_{2j}-E_{2l}}\right)\right)$$
$$\le \exp\left(\frac{1}{\beta}\sum_{j=1}^\infty (E_{2j}-E_{2j-1})\right) < \infty,$$

where we used $\log(1+x) \le x$ for $x > 0$ and $\beta = \min_{l,j;l\neq j}\{|E_{2j} - E_{2l}|, |E_{2j-1} - E_{2l}|\}$. Moreover,

$$\frac{((E_{2l} - \mu_l(x))(E_{2l} - \mu_l(y))(E_{2l} - \mu_l(r))(E_{2l} - \mu_l(s)))^{1/2}}{(E_{2l} - E_0)(E_{2l} - E_{2l-1})} \le \frac{(E_{2l} - E_{2l-1})}{(E_{2l} - E_0)}. \tag{2.60}$$

This implies,

$$0 \le |f_+(E_k, x, y, r, s)| = |M_+(E_k, x, y, r, s)| \le C_1 \frac{E_{2l} - E_{2l-1}}{E_{2l} - E_0}, \tag{2.61}$$

where $k \in \{2l-1, 2l\}$ and $C_1 := \exp\left(\frac{1}{\beta}\sum_{j=1}^{\infty}(E_{2j} - E_{2j-1})\right)$, and therefore our series converges and hence $D_+(x, y, r, s)$ is well-defined and uniformly bounded with respect to all variables.

The continuity follows immediately, by using (2.16) and (2.44). □

Lemma 2.6. *The function $D_+(x, y, r, s)$, defined through* (2.57) *and* (2.58), *has first partial derivatives, which are uniformly bounded in $\mathbb{R}^4$.*

Proof. Consider the integral representation (2.16) of the background Weyl solutions, then the derivative with respect to x is given by

$$\psi_+'(z, x) = \left(\frac{H(z, x) + Y^{1/2}(z)}{G(z, x)}\right)\left(\frac{G(z, x)}{G(z, 0)}\right)^{1/2} \exp\left(\int_0^x \frac{Y^{1/2}(z)}{G(z, x)}\right). \tag{2.62}$$

Note that $G(z, 0)^{1/2}\frac{d}{dx}\psi_+(z, x)$, by Lemma 2.4 has neither poles nor square root singularities at the band edges, which allows us to pass to the limit in the following expression

$$\lim_{z\to E}(z - E)\frac{H(z, x) + Y^{1/2}(z)}{G^{1/2}(z, x)}\frac{(G(z, y)G(z, r)G(z, s))^{1/2}}{Y(z)},. \tag{2.63}$$

Therefore we will slightly abuse the notation by omitting the limit and replacing z by E. W.l.o.g. we will assume that $E = E_{2n}$ (the case $E = E_{2n-1}$ can be treated similarly). Using (2.11), we have

$$\frac{H(E_{2n}, x)}{G^{1/2}(E_{2n}, x)}\frac{(G(E_{2n}, y)G(E_{2n}, r)G(E_{2n}, s))^{1/2}}{\frac{d}{dz}Y(E_{2n})} = \sum_{j=1}^{\infty}\frac{\sigma_j(x)Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x), x)(E_{2n} - \mu_j(x))}M_+(E, x, y, r, s).$$

Due to (2.61) we will at first consider $\sum_{j\neq n}\frac{Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x),x)(E_{2n}-\mu_j(x))}$, which can be done using the same ideas as in Lemma 2.5. Namely, for $j \neq n$

$$0 \le \left|\frac{Y^{1/2}(\mu_j(x))}{\frac{d}{dz}G(\mu_j(x), x)(E_{2n} - \mu_j(x))}\right| \le \frac{1}{\beta}\sqrt{(E_{2j} - E_0)}(E_{2j} - E_{2j-1})C_1^{1/2}, \tag{2.64}$$

which implies that the corresponding sum converges.
For $j = n$, there are two cases to distinguish:

(i) If $\mu_n(x) = E_{2n}$, then

$$\begin{aligned}
0 &\le |\frac{Y^{1/2}(E_{2n})}{G(E_{2n},x)} M_+(E_{2n},x,y,r,s)| \\
&\le C_1^{3/2} \frac{((E_{2n}-\mu_n(y))(E_{2n}-\mu_n(r))(E_{2n}-\mu_n(s)))^{1/2}}{\sqrt{(E_{2n}-E_0)(E_{2n}-E_{2n-1})}} \\
&\le C_1^{3/2} \frac{E_{2n}-E_{2n-1}}{\sqrt{z-E_0}}.
\end{aligned}$$

(ii) If $\mu_n(x) \neq E_{2n}$, we have

$$\begin{aligned}
0 &\le |\frac{Y^{1/2}(\mu_n(x))}{\frac{d}{dz}G(\mu_n(x),x)(z-\mu_n(x))} M_+(E,x,y,r,s)| \\
&\le C_1^{3/2}(\mu_n(x)-E_0)^{1/2} \frac{((\mu_n(x)-E_{2n-1})(E_{2n}-\mu_n(y)))^{1/2}}{(E_{2n}-E_0)} \\
&\quad \frac{((E_{2n}-\mu_n(r))(E_{2n}-\mu_n(s)))^{1/2}}{(E_{2n}-E_{2n-1})} \\
&\le C_1^{3/2} \frac{(E_{2n}-E_{2n-1})}{\sqrt{E_{2n}-E_0}}.
\end{aligned}$$

Next we consider

$$\frac{Y^{1/2}(E_{2n})}{G^{1/2}(E_{2n},x)} \frac{(G(E_{2n},y)G(E_{2n},r)G(E_{2n},s))^{1/2}}{\frac{d}{dz}Y(E_{2n})}, \tag{2.65}$$

which can be investigated as before.

(i) If $\mu_n(x) = E_{2n}$, then

$$\begin{aligned}
0 &\le |\frac{Y^{1/2}(E_{2n})}{G(E_{2n},x)} M_+(E_{2n},x,y,r,s)| \\
&\le C_1^{3/2} \frac{((E_{2n}-\mu_n(y))(E_{2n}-\mu_n(r))(E_{2n}-\mu_n(s)))^{1/2}}{\sqrt{(E_{2n}-E_0)(E_{2n}-E_{2n-1})}} \\
&\le C_1^{3/2} \frac{(E_{2n}-E_{2n-1})}{\sqrt{(E_{2n}-E_0)}}.
\end{aligned}$$

(ii) If $\mu_n(x) \neq E_{2n}$,

$$\frac{Y^{1/2}(E_{2n})}{G(E_{2n},x)} M_+(E_{2n},x,y,r,s) = 0,$$

because

$$\lim_{z \to E_{2n}} \frac{(-(z-E_{2n-1})(z-E_{2n})(z-\mu_n(y))(z-\mu_n(r))(z-\mu_n(s)))^{1/2}}{\sqrt{(z-E_0)(z-\mu_n(x))}(z-E_{2n-1})} = 0. \tag{2.66}$$

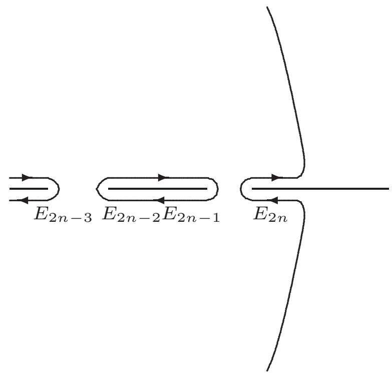

Figure 2.1: Contour C_n

This finishes the proof, as all our estimates are uniformly and

$$0 \le |f_{+,x}(E_k,x,y,r,s)| \le C\frac{(E_{2n}-E_{2n-1})}{\sqrt{(E_{2n}-E_0)}}, \tag{2.67}$$

where $k \in \{2n-1, 2n\}$ and C denotes a constant independent of n. □

Lemma 2.7. *The kernels $K_\pm(x,s)$ of the transformation operators satisfy the integral equation*

$$\begin{aligned} K_\pm(x,s) = &-2\int_{\frac{x+s}{2}}^{\pm\infty} \tilde{q}(y)D_\pm(x,y,y,s)dy \\ &\mp 2\int_x^{\pm\infty} dy \int_{s+x-y}^{s+y-x} D_\pm(x,y,r,s)K_\pm(y,r)\tilde{q}(y)\,dr, \quad \pm s > \pm x, \end{aligned} \tag{2.68}$$

where $D_\pm(x,y,r,s)$ are defined by (2.57). In particular,

$$K_\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty}(q(s)-p(s))ds. \tag{2.69}$$

Proof. Suppose $(x-y+r-s) > 0$, where x,y,r,s are considered as fixed parameters, and take a series of closed contours C_n consisting of a circular arc R_n centered at the origin with radius $(E_{2n}+E_{2n+1})/2$ together with some parts wrapping around each of the first $n+1$ bands of the spectrum σ, but not intersecting it, as indicated in figure 2.1.

On the circle R_n we have the following asymptotic behavior as $n \to \infty$ and therefore $z \to \infty$,

$$g^{1/2}(z)\psi_+(z,x) = \mathrm{e}^{\mathrm{i}\sqrt{z}x(1+O(\frac{1}{z}))}O\left(\frac{1}{z^{1/4}}\right), \tag{2.70}$$

where we used Lemma 2.3 and the fact that these asymptotics are valid as long as we are outside a small neighborhood of the gaps. This yields

$$g(z)^2\psi_+(z,x)\psi_-(z,y)\psi_+(z,r)\psi_-(z,s) = \mathrm{e}^{\mathrm{i}\sqrt{z}(x-y+r-s)(1+O(\frac{1}{z}))}O\left(\frac{1}{z}\right), \tag{2.71}$$

as $z \to \infty$.

Hence one can apply Jordan's lemma to conclude that the contribution of the circle R_n vanishes as $n \to \infty$.

Shrinking the loops around the bands of the spectrum, the integral converges to

$$\Gamma_+(x, y, t, s) = D_+(x, y, t, s), \quad \text{for} \quad (x - y + t - s) > 0. \tag{2.72}$$

Note that $f_+(E, x, y, r, s)$ is real for any $E \in \partial\sigma$, because (2.44) and $G(E, x) = 0$, if $\mu_j(x) = E$, imply that $f_+(E, x, y, r, s) = 0$. Moreover $f_+(E, x, y, r, s) = f_+(E, y, x, s, r)$. Thus $D_+(x, y, r, s)$ is also real and satisfies

$$D_+(x, y, r, s) = D_+(y, x, s, r). \tag{2.73}$$

Now let $(x-y+r-s) < 0$, that is $-(x-y+r-s) > 0$. Then (2.56), (2.72), and (2.73) imply

$$\Gamma_+(x, y, t, s) = -\overline{\Gamma_+(y, x, s, r)} = -\overline{D_+(x, y, r, s)} = -D_+(x, y, r, s). \tag{2.74}$$

Therefore,

$$\Gamma(x, y, r, s) = D(x, y, r, s,) \operatorname{sign}(x - y + r - s). \tag{2.75}$$

Combining all the informations, the domain, where the first integrand in (2.55) does not vanish is given by

$$\operatorname{sign}(x - s) = -\operatorname{sign}(2y - x - s), \quad s > x. \tag{2.76}$$

In the second integral the domain of integration is

$$\operatorname{sign}(x - y + t - s) = -\operatorname{sign}(y - x + t - s), \text{ with } \quad s > x, \quad t > y > x.$$

Solving (2.76) and (2.77), proves (2.68).

Setting now $s = x$ in (2.68), the second summand vanishes, because we set $K_+(y, r) = 0$ for $r < y$. Hence

$$K_+(x, x) = -2 \int_x^\infty \tilde{q}(y) D_+(x, y, y, x). \tag{2.77}$$

Thus we obtain

$$\begin{aligned} D_+(x, y, y, x) &= \frac{1}{4} \sum_{E \in \partial\sigma} \operatorname*{Res}_E \frac{1}{z - E_0} \prod_{j=1}^\infty \frac{(z - \mu_j(x))(z - \mu_j(y))}{(z - E_{2j-1})(z - E_{2j})} \\ &= \frac{1}{4} \sum_{l=0}^\infty \lim_{z \to E_l} \frac{z - E_l}{z - E_0} \prod_{j=1}^\infty \frac{(z - \mu_j(x))(z - \mu_j(y))}{(z - E_{2j-1})(z - E_{2j})}, \end{aligned} \tag{2.78}$$

and we already know that this function is bounded by Lemma 2.5. Considering now the following sequence

$$\begin{aligned} D_{+,n}(x, y, y, x) &= \frac{1}{4} \sum_{E \in \partial\sigma} \operatorname*{Res}_E \frac{1}{z - E_0} \prod_{j=1}^n \frac{(z - \mu_j(x))(z - \mu_j(y))}{(z - E_{2j-1})(z - E_{2j})} \\ &= \frac{1}{4} \sum_{l=0}^{2n} \lim_{z \to E_l} \frac{z - E_l}{z - E_0} \prod_{j=1}^n \frac{(z - \mu_j(x))(z - \mu_j(y))}{(z - E_{2j-1})(z - E_{2j})}, \end{aligned} \tag{2.79}$$

which corresponds to the case where we only have n gaps and crossed out all the other ones. We will now estimate

$$|D_+(x,y,y,x) - D_{n,+}(x,y,y,x)| \leq \frac{1}{4}\sum_{l=0}^{2n} \lim_{z\to E_l} \left| \frac{z-E_l}{z-E_0} \prod_{j=1}^{n} \left(\frac{z-\mu_j(x)}{z-E_{2j-1}} \frac{z-\mu_j(y)}{z-E_{2j}} \right) \right| \times \tag{2.80}$$

$$\left| \left(\prod_{j=n+1}^{\infty} \left(\frac{z-\mu_j(x)}{z-E_{2j-1}} \frac{z-\mu_j(y)}{z-E_{2j}} \right) - 1 \right) \right|$$

$$+ \frac{1}{4}\sum_{l=2n+1}^{\infty} \lim_{z\to E_l} \left| \frac{z-E_l}{z-E_0} \prod_{j=1}^{\infty} \left(\frac{z-\mu_j(x)}{z-E_{2j-1}} \frac{z-\mu_j(y)}{z-E_{2j}} \right) \right|,$$

using the same techniques as in the proof of Lemma 2.5. We fix $z = E_{2l}$, (the case $z = E_{2l-1}$ can be handled analogously). If $l < n$, we have

$$\begin{aligned}
&\lim_{z\to E_{2l}} \left| \frac{z-E_{2l}}{z-E_0} \prod_{j=1}^{n} \left(\frac{z-\mu_j(x)}{z-E_{2j-1}} \frac{z-\mu_j(y)}{z-E_{2j}} \right) \right| \\
&\quad = \left| \frac{E_{2l}-\mu_l(x)}{E_{2l}-E_0} \frac{E_{2l}-\mu_j(y)}{E_{2l}-E_{2j-1}} \prod_{j=1,j\neq l}^{n} \frac{E_{2l}-\mu_j(x)}{E_{2l}-E_{2j-1}} \frac{E_{2l}-\mu_j(y)}{E_{2l}-E_{2j}} \right| \\
&\quad \leq \frac{E_{2l}-E_{2l-1}}{E_{2l}-E_0} \prod_{j=1}^{l-1} \frac{E_{2l}-\mu_j(x)}{E_{2l}-E_{2j}} \prod_{j=l+1}^{n} \frac{E_{2l}-\mu_j(x)}{E_{2l}-E_{2j-1}} \\
&\quad \leq \frac{E_{2l}-E_{2l-1}}{E_{2l}-E_0} \exp\Big(\frac{1}{\beta} \sum_{j=1}^{n} (E_{2j}-E_{2j-1}) \Big)
\end{aligned}$$

and

$$\begin{aligned}
\exp\Big(-\frac{1}{\beta} \sum_{j=n+1}^{\infty} (E_{2j}-E_{2j-1}) \Big) &\leq \prod_{j=n+1}^{\infty} \frac{\mu_j(x)-E_{2l}}{E_{2j}-E_{2l}} \\
&\leq \prod_{j=n+1}^{\infty} \frac{\mu_j(x)-E_{2l}}{E_{2j-1}-E_{2l}} \frac{\mu_j(y)-E_{2l}}{E_{2j}-E_{2l}} \\
&\leq \prod_{j=n+1}^{\infty} \frac{\mu_j(x)-E_{2l}}{E_{2j-1}-E_{2l}} \leq \exp\Big(\frac{1}{\beta} \sum_{j=n+1}^{\infty} (E_{2j}-E_{2j-1}) \Big),
\end{aligned} \tag{2.81}$$

where the last estimate implies that the first sequence in (2.80) converges uniformly to zero as n tends to ∞ as in the investigation of (2.13). Analogously, one can estimate the second sequence, which also converges uniformly to zero, as we are working in the Levitan class and hence $D_{n,+}(x,y,y,x)$ converges uniformly against $D_+(x,y,y,x)$.

Moreover, it is known (see e.g. [10]), that

$$D_{n,+}(x,y,y,x) = -\frac{1}{4} \tag{2.82}$$

for each fixed n. and hence we finally obtain

$$D_+(x,y,y,x) = \lim_{n\to\infty} D_{n,+}(x,y,y,x) = \lim_{n\to\infty} -\frac{1}{4} = -\frac{1}{4}. \tag{2.83}$$

Therefore we can now conclude, using (2.77), that

$$K_\pm(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} (q(s)-p(s))ds. \tag{2.84}$$

□

Lemma 2.8. *Suppose* (2.6)*, then* (2.68) *has a unique solution* $K_\pm(x,y)$*, such that* $K_\pm(x,y)$ *has first order partial derivatives with respect to both variables. Moreover for* $\pm y \geq \pm x$ *the following estimates are valid*

$$|K_\pm(x,y)| \leq C_\pm(x)Q_\pm(x+y), \tag{2.85}$$

$$\left|\frac{\partial K_\pm(x,y)}{\partial x}\right| + \left|\frac{\partial K_\pm(x,y)}{\partial y}\right| \leq C_\pm(x)\left(\left|\tilde{q}\left(\frac{x+y}{2}\right)\right| + Q_\pm(x+y)\right), \tag{2.86}$$

where

$$Q_\pm(x) = \pm\int_{\frac{x}{2}}^{\pm\infty} |\tilde{q}(s)|ds, \quad \tilde{q}(x) = q(x)-p(x), \tag{2.87}$$

and $C_\pm(x)$ *are positive continuous functions for* $x\in\mathbb{R}$*, which decrease as* $x\to\pm\infty$ *and depend on the corresponding background data and on the first moment of the perturbation.*

Proof. Using the method of successive approximation one can prove existence and uniqueness of the solution $K_\pm(x,y)$ of (2.68). We restrict our considerations to the + case. After the following change of variables

$$2\alpha := s+r,\ 2\beta := r-s,\ 2u := x+y,\ 2v := y-x, \tag{2.88}$$

(2.68) becomes

$$\begin{aligned} H(u,v) = &-2\int_u^\infty \tilde{q}(s)D_1(u,v,s)ds \\ &-4\int_u^\infty d\alpha\int_0^v \tilde{q}(\alpha-\beta)D_2(u,v,\alpha,\beta)H(\alpha,\beta)d\beta, \end{aligned} \tag{2.89}$$

with

$$\begin{aligned} H(u,v) = K_+(u-v,u+v), \quad D_1(u,v,s) &= D_+(u-v,s,s,u+v), \\ D_2(u,v,\alpha,\beta) &= D_+(u-v,\alpha-\beta,\alpha+\beta,u+v). \end{aligned} \tag{2.90}$$

As the functions D_1 and D_2 are bounded uniformly with respect to all their variables by a constant C, we can apply the method of successive approximation to estimate $H(u,v)$, which yields

$$|H(u,v)| \leq C(u-v)Q_+(2u), \tag{2.91}$$

where

$$C(u-v) = 2C\exp\left(4C\int_{2u-2v}^{\infty} Q_+(x)dx\right). \tag{2.92}$$

To obtain the second estimate remember that the partial derivatives with respect to all variables exist for D_1 and D_2 and that they are also bounded with respect to all variables. Thus, using

$$\begin{aligned}
&\frac{\partial H(u,v)}{\partial u} - 2\tilde{q}(u)D_1(u,v,u) = \qquad (2.93)\\
&= +4\int_0^v \tilde{q}(u-\beta)D_2(u,v,u,\beta)H(u,\beta)d\beta - 2\int_u^{\infty}\tilde{q}(s)\frac{\partial D_1(u,v,s)}{\partial u}ds\\
&- 4\int_u^{\infty} d\alpha\int_0^v \tilde{q}(\alpha-\beta)\frac{\partial D_2(u,v,\alpha,\beta)}{\partial u}H(\alpha,\beta)d\beta,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial H(u,v)}{\partial v} = \qquad (2.94)\\
&= -2\Big(\int_u^{\infty}\tilde{q}(s)\frac{\partial D_1(u,v,s)}{\partial v}ds + 2\int_u^{\infty}\tilde{q}(\alpha-\beta)D_2(u,v,\alpha,v)H(\alpha,v)d\alpha\\
&+ 2\int_u^{\infty} d\alpha\int_0^v \tilde{q}(\alpha-\beta)\frac{\partial D_2(u,v,\alpha,\beta)}{\partial v}H(\alpha,\beta)d\beta\Big),
\end{aligned}$$

one obtains.

$$\begin{aligned}
&|\frac{\partial}{\partial u}H(u,v)| \le C_1(u-v)(|\tilde{q}(u)| + Q_+(2u)), \qquad (2.95)\\
&|\frac{\partial}{\partial v}H(u,v)| \le C_1(u-v)(|\tilde{q}(u)| + Q_+(2u)),
\end{aligned}$$

where $C_1(u-v)$ is a positive continuous function for $x = u - v \in \mathbb{R}$, which decreases as $x \to \pm\infty$ and depends on the corresponding background data. Using

$$\left|\frac{dK_+(x,y)}{dx}\right| + \left|\frac{dK_+(x,y)}{dy}\right| \le \left|\frac{dH(u,v)}{du}\right| + \left|\frac{dH(u,v)}{dv}\right|, \tag{2.96}$$

completes the proof. □

To finish the proof of Theorem 2.2, we have to show the following:

Lemma 2.9. *The functions*

$$\tilde{\phi}_{\pm}(z,x) = \psi_{\pm}(z,x) \pm \int_x^{\pm\infty} K_{\pm}(x,s)\psi_{\pm}(z,s), \tag{2.97}$$

where $K_{\pm}(x,s)$ is defined by (2.68), *satisfy*

$$\Big(-\frac{d^2}{dx^2} + q(x)\Big)\tilde{\phi}_{\pm}(z,x) = z\tilde{\phi}(z,x). \tag{2.98}$$

Proof. Again we will only consider the + case as the other one can be treated similarly and drop the + whenever possible. On the one hand we obtain, using (2.77), that

$$\begin{aligned}\Big(-\frac{d^2}{dx^2}+q(x)\Big)\tilde{\phi}(z,x) &= -\psi''(z,x)+p(x)\psi(z,x)\\ &\quad+\frac{1}{2}\tilde{q}(x)\psi(z,x)+K(x,x)\psi'(z,x)+K_x(x,x)\psi(z,x)\\ &\quad+\int_x^\infty (q(x)K(x,s)-K_{xx}(x,s))\psi(z,s)ds, \qquad (2.99)\end{aligned}$$

and on the other hand, using that $\psi_\pm(z,x)$ are the background Weyl solutions, we have

$$\begin{aligned}z\tilde{\phi}(z,x) &= z\psi(z,x)+K(x,x)\psi'(z,x)-K_y(x,x)\psi(z,x)\\ &\quad+\int_x^\infty (p(s)K(x,s)-K_{ss}(x,s))\psi(z,s)ds. \qquad (2.100)\end{aligned}$$

Applying (2.77) once more, we see that (2.98) is satisfied if and only if

$$\int_x^\infty (K_{xx}(x,s)-K_{ss}(x,s))\psi(z,s)ds=\int_x^\infty (q(x)-p(s))\psi(z,s)ds. \qquad (2.101)$$

For proving this identity we use the integral equation (2.53) instead of (2.68) for $K(x,s)$, which yields for $x<s$

$$\begin{aligned}K_{ss}(x,s) &= -\int_x^\infty dy\tilde{q}(y)\oint_\sigma J(\lambda,x,y)\psi_+(\lambda,y)\psi_-''(\lambda,s)d\rho(\lambda)\\ &\quad-\int_x^\infty dy\tilde{q}(y)\int_y^\infty dtK(y,t)\oint_\sigma J(\lambda,x,y)\psi_+(\lambda,t)\psi_-''(\lambda,s)d\rho(\lambda)\\ &= (p(s)-p(x))K(x,s)-\int_x^\infty dy\tilde{q}(y)\oint_\sigma J_{xx}(\lambda,x,y)\psi_+(\lambda,y)\psi_-(\lambda,s)d\rho(\lambda)\\ &\quad-\int_x^\infty dy\tilde{q}(y)\int_y^\infty dtK(y,t)\oint_\sigma J_{xx}(\lambda,x,y)\psi_+(\lambda,t)\psi_-(\lambda,s)d\rho(\lambda)\\ &= (p(s)-p(x))K(x,s)+K_{x,x}(x,s)-\tilde{q}(x)\oint_\sigma J_x(\lambda,x,x)\psi_+(\lambda,x)\psi_-(\lambda,s)d\rho(\lambda)\\ &\quad-\tilde{q}(x)\int_x^\infty dtK(x,t)\oint_\sigma J_x(\lambda,x,x)\psi_+(\lambda,t)\psi_-(\lambda,s)d\rho(\lambda)\\ &= (p(s)-q(x))K(x,s)+K_{xx}(x,s). \qquad (2.102)\end{aligned}$$

Here it should be noticed that $\oint_\sigma J(\lambda,x,y)\psi_+(\lambda,t)\psi_-''(\lambda,s)d\rho(\lambda)$ exists, because we can again estimate the sum of the absolute values of the residues by using that $-\psi_\pm''(z,x)+p(x)\psi_\pm(z,x)=z\psi_\pm(z,x)$ and the same techniques as in Lemma 2.5. Analogously for $\oint_\sigma J_{xx}(\lambda,x,y)\psi_+(\lambda,t)\psi_-(\lambda,s)d\rho(\lambda)$. Thus (2.101) is fulfilled and therefore also (2.98).

□

Lemma 2.10. *The functions $K_\pm(x,y)\in L^2(\mathbb{R}_\pm)$ as a function of y for fixed x.*

Proof. Using (2.6), we can conclude

$$\begin{aligned}\int_{\mathbb{R}} |K_\pm(x,y)|^2 dy &\le \pm C_\pm(x)^2 \int_x^{\pm\infty} Q_\pm(x+y)^2 dy \\ &= C_\pm(x)^2 Q_\pm(2x) \int_x^{\pm\infty} \int_{\frac{x+y}{2}}^{\pm\infty} |\tilde{q}(s)| ds dy \\ &= C_\pm(x)^2 Q_\pm(2x) \int_x^{\pm\infty} (2s-2x)|\tilde{q}(s)| ds < \infty.\end{aligned} \tag{2.103}$$

□

It should also be noticed that for any function $f_\pm(x) \in L^2(\mathbb{R}_\pm)$,

$$h_\pm(x) = f_\pm(x) \pm \int_x^{\pm\infty} K_\pm(x,y) f_\pm(y) dy \in L^2(\mathbb{R}_\pm). \tag{2.104}$$

Thus as a consequence we obtain

Corollary 2.11. *The normalized Jost solutions $\phi_\pm(z,x)$ coincide with the Weyl solutions of the Schrödinger operator (*2.5*).*

Chapter 3

Scattering theory for Schrödinger operators on steplike, almost periodic infinite-gap backgrounds

3.1 Introduction

One of the main tools for solving various Cauchy problems, since the seminal work of Gardner, Green, Kruskal, and Miura [43] in 1967, is the inverse scattering transform and therefore, since then, a large number of articles has been devoted to direct and inverse scattering theory.

In much detail the case where the initial condition is asymptotically close to $p_\pm(x) = 0$, has been studied (see e.g Marchenko [81]). Taking this as a starting point, there are two natural cases, which have also been considered in the past. On the one hand the case of equal quasi-periodic, finite-gap potentials $p_-(x) = p_+(x)$ and on the other hand the case of steplike constant asymptotics $p_\pm(x) = c_\pm$ with $c_- \neq c_+$. Very recently, the combination of these two cases, namely the case that the initial condition is asymptotically close to steplike quasi-periodic finite-gap potentials $p_-(x) \neq p_+(x)$, has been investigated by Boutet de Monvel, Egorova, and Teschl [10].

Of much interest is also the case of asymptotically periodic solutions, which has been first considered by Firsova [40]. In the present work we propose a complete investigation of the scattering theory on Bohr almost periodic infinite-gap backgrounds, which belong to the so–called Levitan class. It should be noticed, that this class, as a special case, includes the set of smooth, periodic infinite–gap operators.

To set the stage, we need:

Hypothesis H. 3.1. *Let*

$$0 \le E_0^\pm < E_1^\pm < \cdots < E_n^\pm < \dots \tag{3.1}$$

be two increasing sequences of points on the real axis which satisfy the following conditions:

(i) for a certain $l > 1$, $\sum_{n=1}^{\infty}(E_{2n-1}^{\pm})^l(E_{2n}^{\pm} - E_{2n-1}^{\pm}) < \infty$ and

(ii) $E_{2n+1}^{\pm} - E_{2n-1}^{\pm} > Cn^{\alpha_{\pm}}$, where C and $\alpha^{\pm}$ are some fixed, positive constants.

We will call, in what follows, the intervals $(E_{2j-1}^{\pm}, E_{2j}^{\pm})$ for $j = 1, 2, \dots$ gaps. In each closed gap $[E_{2j-1}^{\pm}, E_{2j}^{\pm}]$, $j = 1, 2, \dots$, we choose a point $\mu_j^{\pm}$ and an arbitrary sign $\sigma_j^{\pm} \in \{-1, 1\}$.

Next consider the system of differential equations for the functions $\mu_j^{\pm}(x)$, $\sigma_j^{\pm}(x)$, $j = 1, 2, ...$, which is an infinite analogue of the well-known Dubrovin equations, given by

$$\frac{d\mu_j^{\pm}(x)}{dx} = -2\sigma_j^{\pm}(x)\sqrt{-(\mu_j^{\pm}(x) - E_0^{\pm})}\sqrt{\mu_j^{\pm}(x) - E_{2j-1}^{\pm}}\sqrt{\mu_j^{\pm}(x) - E_{2j}^{\pm}} \times \prod_{k=1, k\neq j}^{\infty} \frac{\sqrt{\mu_j^{\pm}(x) - E_{2k-1}^{\pm}}\sqrt{\mu_j^{\pm}(x) - E_{2k}^{\pm}}}{\mu_j^{\pm}(x) - \mu_k^{\pm}(x)} \tag{3.2}$$

with initial conditions $\mu_j^{\pm}(0) = \mu_j^{\pm}$ and $\sigma_j^{\pm}(0) = \sigma_j^{\pm}$, $j = 1, 2, \dots$ [1]. Levitan [77], [78], and [79], proved, that this system of differential equations is uniquely solvable, that the solutions $\mu_j^{\pm}(x)$, $j = 1, 2, \dots$ are continuously differentiable and satisfy $\mu_j^{\pm}(x) \in [E_{2j-1}^{\pm}, E_{2j}^{\pm}]$ for all $x \in \mathbb{R}$. Moreover, these functions $\mu_j^{\pm}(x)$, $j = 1, 2, \dots$ are Bohr almost periodic [2]. Using the trace formula (see for example [77])

$$p_{\pm}(x) = E_0^{\pm} + \sum_{j=1}^{\infty}(E_{2j-1}^{\pm} + E_{2j}^{\pm} - 2\mu_j^{\pm}(x)), \tag{3.3}$$

we see that also $p_{\pm}(x)$ are real Bohr almost periodic. The operators

$$L_{\pm} := -\frac{d^2}{dx^2} + p_{\pm}(x), \tag{3.4}$$

are then called an almost periodic infinite-gap Schrödinger operator of the Levitan class. They have as absolutely continuous spectrum the set

$$\sigma_{\pm} = [E_0^{\pm}, E_1^{\pm}] \cup \dots \cup [E_{2j}^{\pm}, E_{2j+1}^{\pm}] \cup \dots, \tag{3.5}$$

and have spectral properties analogous to the quasi-periodic finite-gap Schrödinger operator. In particular, they are completely defined by the series $\sum_{j=1}^{\infty}(\mu_j^{\pm}, \sigma_j^{\pm})$, which we call the Dirichlet divisor. These divisors are associated to a Riemann surfaces of infinite genius, which are connected with the functions $Y_{\pm}^{1/2}(z)$, where

$$Y_{\pm}(z) = -(z - E_0^{\pm})\prod_{j=1}^{\infty}\frac{(z - E_{2j-1}^{\pm})}{E_{2j-1}^{\pm}}\frac{(z - E_{2j}^{\pm})}{E_{2j-1}^{\pm}}, \tag{3.6}$$

where the cuts are taken along the spectrum. It is known, that the spectral equation

$$\Big(-\frac{d^2}{dx^2} + p_{\pm}(x)\Big)y(x) = zy(x) \tag{3.7}$$

[1] We will use the standard branch cut of the square root in the domain $\mathbb{C} \setminus \mathbb{R}_+$ with $\operatorname{Im}\sqrt{z} > 0$.

[2] For informations about almost periodic functions we refer to [80].

with any continuous, bounded potential $p_\pm(x)$ has two Weyl solutions $\psi_\pm(z,x)$ and $\breve{\psi}_\pm(z,x)$, which satisfy

$$\psi_\pm(z,.) \in L^2(\mathbb{R}_\pm), \quad \text{resp.} \quad \breve{\psi}_\pm(z,.) \in L^2(\mathbb{R}_\mp), \tag{3.8}$$

for $z \in \mathbb{C}\backslash\sigma_\pm$ and which are normalized by $\psi_\pm(z,0) = \breve{\psi}_\pm(z,0) = 1$. In our case of Bohr almost periodic potentials of the Levitan class, these solutions have complementary properties similar to the properties of the Baker-Akhiezer functions in the finite-gap case. We will briefly discuss them in the next section.

The object of interest, for us, is the one-dimensional Schrödinger operator

$$L := -\frac{d^2}{dx^2} + q(x), \tag{3.9}$$

with the real potential $q(x) \in C(\mathbb{R})$ satisfying the following condition

$$\pm \int_0^{\pm\infty} (1+|x|^2)|q(x) - p_\pm(x)|dx < \infty, \tag{3.10}$$

for which we will characterize the corresponding scattering data with the help of the transformation operator, which has been investigated in Chapter 4.100.

3.2 The Weyl solutions of the background operators

In this section we want to summarize and recall some facts for the background Schrödinger operators $L_\pm$ of Levitan class and introduce the notation we will use from now on. We present these results, obtained in Chapter 4.100, [77], [88], and [89], in a form, similar to the finite-gap case used in [10] and [45].

Let $L_\pm$ be the quasi-periodic one-dimensional Schrödinger operators associated with the potentials $p_\pm(x)$. Let $s_\pm(z,x)$, $c_\pm(z,x)$ be sin- and cos-type solutions of the equation

$$\left(-\frac{d^2}{dx^2} + p_\pm(x)\right) y(x) = zy(x), \quad z \in \mathbb{C}, \tag{3.11}$$

associated with the initial conditions

$$s_\pm(z,0) = c'_\pm(z,0) = 0, \quad c_\pm(z,0) = s'_\pm(z,0) = 1, \tag{3.12}$$

where prime denotes the derivative with respect to x. Then $c_\pm(z,x)$, $c'_\pm(z,x)$, $s_\pm(z,x)$, and $s'_\pm(z,x)$ are holomorphic with respect to $z \in \mathbb{C}\backslash\sigma_\pm$. Moreover, they can be represented in the following form

$$c_\pm(z,x) = \cos(\sqrt{z}x) + \int_0^x \frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}} p_\pm(y) c_\pm(z,y)dy, \tag{3.13}$$

$$s_\pm(z,x) = \frac{\sin(\sqrt{z}x)}{\sqrt{z}} + \int_0^x \frac{\sin(\sqrt{z}(x-y))}{\sqrt{z}} p_\pm(y) s_\pm(z,y)dy. \tag{3.14}$$

The background Weyl solutions are given by

$$\begin{aligned} &\psi_\pm(z,x) = c_\pm(z,x) + m_\pm(z,0)s_\pm(z,x), \\ &\text{resp. } \breve{\psi}_\pm(z,x) = c_\pm(z,x) + \breve{m}_\pm(z,0)s_\pm(z,x), \end{aligned} \tag{3.15}$$

where

$$m_{\pm}(z,x) = \frac{H_{\pm}(z,x) \pm Y_{\pm}^{1/2}(z)}{G_{\pm}(z,x)}, \quad \breve{m}_{\pm}(z,x) = \frac{H_{\pm}(z,x) \mp Y_{\pm}^{1/2}(z)}{G_{\pm}(z,x)}, \tag{3.16}$$

are the Weyl function of $L_{\pm}$ (cf [77]), where $Y_{\pm}(z)$ are defined by (3.6),

$$G_{\pm}(z,x) = \prod_{j=1}^{\infty} \frac{z - \mu_j^{\pm}(x)}{E_{2j-1}^{\pm}}, \quad \text{and} \quad H_{\pm}(z,x) = \frac{1}{2}\frac{d}{dx} G_{\pm}(z,x). \tag{3.17}$$

Using (3.2) and (3.17), we have

$$H_{\pm}(z,x) = \frac{1}{2}\frac{d}{dx} G_{\pm}(z,x) = G_{\pm}(z,x) \sum_{j=1}^{\infty} \frac{\sigma_j^{\pm}(x) Y_{\pm}^{1/2}(\mu_j^{\pm}(x))}{\frac{d}{dz} G(\mu_j^{\pm}(x), x)(z - \mu_j^{\pm}(x))}. \tag{3.18}$$

The Weyl functions $m_{\pm}(z,x)$ and $\breve{m}_{\pm}(z,x)$ are Bohr almost periodic.

Lemma 3.2. *The background Weyl solutions, for $z \in \mathbb{C}$, can be represented in the following form*

$$\psi_{\pm}(z,x) = \exp\left(\int_0^x m_{\pm}(z,y)dy\right) = \left(\frac{G_{\pm}(z,x)}{G_{\pm}(z,0)}\right)^{1/2} \exp\left(\pm \int_0^x \frac{Y_{\pm}^{1/2}(z)}{G_{\pm}(z,y)} dy\right), \tag{3.19}$$

and

$$\breve{\psi}_{\pm}(z,x) = \exp\left(\int_0^x \breve{m}_{\pm}(z,y)dy\right) = \left(\frac{G_{\pm}(z,x)}{G_{\pm}(z,0)}\right)^{1/2} \exp\left(\mp \int_0^x \frac{Y_{\pm}^{1/2}(z)}{G_{\pm}(z,y)} dy\right). \tag{3.20}$$

If for some $\varepsilon > 0$, $|z - \mu_j^{\pm}(x)| > \varepsilon$ for all $j \in \mathbb{N}$ and $x \in \mathbb{R}$, then the following holds: For any $C > 0$ there exists an $R > 0$ such that

$$|\psi_{\pm}(z,x)| \le \mathrm{e}^{\mp(1-C)x\,\mathrm{Im}(\sqrt{z})}\left(1 + \frac{D}{|z|}\right), \ \textit{for any } |z| \ge R, \tag{3.21}$$

and

$$|\breve{\psi}_{\pm}(z,x)| \le \mathrm{e}^{\pm(1-C)x\,\mathrm{Im}(\sqrt{z})}\left(1 + \frac{D}{|z|}\right), \ \textit{for any } |z| \ge R, \tag{3.22}$$

where D denotes some constant dependent on R.

As the spectra $\sigma_{\pm}$ consist of infinitely many bands, let us cut the complex plane along the spectrum $\sigma_{\pm}$ and denote the upper and lower sides of the cuts by $\sigma_{\pm}^{\mathrm{u}}$ and $\sigma_{\pm}^{\mathrm{l}}$. The corresponding points on these cuts will be denoted by λ^{u} and λ^{l}, respectively. In particular, this means

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon \downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon \downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \qquad \lambda \in \sigma_{\pm}.$$

Defining

$$g_{\pm}(\lambda) = -\frac{G_{\pm}(\lambda,0)}{2Y_{\pm}^{1/2}(\lambda)}, \tag{3.23}$$

where the branch of the square root is chosen in such a way that

$$\frac{1}{\mathrm{i}} g_\pm(\lambda^{\mathrm{u}}) = \mathrm{Im}(g_\pm(\lambda^{\mathrm{u}})) > 0 \quad \text{for} \quad \lambda \in \sigma_\pm, \tag{3.24}$$

it follows from Lemma 3.2 that

$$W(\breve{\psi}_\pm(z), \psi_\pm(z)) = m_\pm(z) - \breve{m}_\pm(z) = \mp g_\pm(z)^{-1}, \tag{3.25}$$

where $W(f,g)(x) = f(x)g'(x) - f'(x)g(x)$ denotes the usual Wronskian determinant.

For every Dirichlet eigenvalue $\mu_j^\pm = \mu_j^\pm(0)$, the Weyl functions $m_\pm(z)$ and $\breve{m}_\pm(z)$ might have poles. If $\mu_j^\pm$ is in the interior of its gap, precisely one Weyl function $m_\pm$ or $\breve{m}_\pm$ will have a simple pole. Otherwise, if $\mu_j^\pm$ sits at an edge, both will have a square root singularity. Hence we divide the set of poles accordingly:

$$\begin{aligned} M_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in (E_{2j-1}^\pm, E_{2j}^\pm) \text{ and } m_\pm \text{ has a simple pole}\}, \\ \breve{M}_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in (E_{2j-1}^\pm, E_{2j}^\pm) \text{ and } \breve{m}_\pm \text{ has a simple pole}\}, \\ \hat{M}_\pm &= \{\mu_j^\pm \mid \mu_j^\pm \in \{E_{2j-1}^\pm, E_{2j}^\pm\}\}, \end{aligned}$$

and we set $M_{r,\pm} = M_\pm \cup \breve{M}_\pm \cup \hat{M}_\pm$.

In particular, we obtain the following properties of the Weyl solutions (see, e.g. [29], [77], [91]):

Lemma 3.3. *The Weyl solutions have the following properties:*

(i) *The function $\psi_\pm(z,x)$ (resp. $\breve{\psi}_\pm(z,x)$) is holomorphic as a function of z in the domain $\mathbb{C} \setminus (\sigma_\pm \cup M_\pm)$ (resp. $\mathbb{C} \setminus (\sigma_\pm \cup \breve{M}_\pm)$, real valued on the set $\mathbb{R} \setminus \sigma_\pm$, and have simple poles at the points of the set $M_\pm$ (resp. $\breve{M}_\pm$). Moreover, they are continuous up to the boundary $\sigma_\pm^u \cup \sigma_\pm^l$ except at the points from $\hat{M}_\pm$ and*

$$\psi_\pm(\lambda^{\mathrm{u}}) = \breve{\psi}_\pm(\lambda^{\mathrm{l}}) = \overline{\psi_\pm(\lambda^{\mathrm{l}})}, \quad \lambda \in \sigma_\pm. \tag{3.26}$$

For $E \in \hat{M}_\pm$ the Weyl solutions satisfy

$$\psi_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \breve{\psi}_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \quad \text{as } z \to E \in \hat{M}_\pm.$$

The same is true for $\psi'_\pm(z,x)$ and $\breve{\psi}'_\pm(z,x)$.

(ii) *At the edges of the spectrum these functions possess the properties*

$$\psi_\pm(z,x) - \breve{\psi}_\pm(z,x) = O(\sqrt{z-E}) \quad \text{near} \quad E \in \partial\sigma_\pm \setminus \hat{M}_\pm, \tag{3.27}$$

and

$$\psi_\pm(z,x) + \breve{\psi}_\pm(z,x) = O(1) \quad \text{near} \quad E \in \hat{M}_\pm. \tag{3.28}$$

(iii) *The functions $\psi_\pm(z,x)$ form an orthonormal basis on the spectrum with respect to the weight*

$$d\rho_\pm(z) = \frac{1}{2\pi\mathrm{i}} g_\pm(z) dz, \tag{3.29}$$

and any $f(x) \in L^2(-\infty,\infty)$ can be expresses through

$$f(x) = \oint_{\sigma_\pm} \left(\int_{\mathbb{R}} f(y)\psi_\pm(z,y)dy \right) \breve{\psi}_\pm(z,x)d\rho(z). \tag{3.30}$$

Here we use the notation

$$\oint_{\sigma_\pm} f(z)d\rho_\pm(z) := \int_{\sigma_\pm^u} f(z)d\rho_\pm(z) - \int_{\sigma_\pm^l} f(z)d\rho_\pm(z). \tag{3.31}$$

3.3 The direct scattering problem

Consider the equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right) y(x) = zy(x), \quad z \in \mathbb{C}, \tag{3.32}$$

with a potential $q(x)$ satisfying the following condition

$$\pm \int_0^{\pm\infty} (1+x^2)|q(x) - p_\pm(x)|dx < \infty. \tag{3.33}$$

Then, as in Chapter 4.100, there exist two solutions, the so called Jost solutions $\phi_\pm(z,x)$, which are asymptotically close to the background Weyl solutions $\psi_\pm(z,x)$ of equation (3.11) as $x \to \pm\infty$ and they can be represented as

$$\phi_\pm(z,x) = \psi_\pm(z,x) \pm \int_x^{\pm\infty} K_\pm(x,y)\psi_\pm(z,y)dy. \tag{3.34}$$

Here $K_\pm(x,y)$ are real-valued functions, which are continuously differentiable with respect to both parameters and satisfy the estimate

$$|K_\pm(x,y)| \le C_\pm(x)Q_\pm(x+y) = \pm C_\pm(x) \int_{\frac{x+y}{2}}^{\pm\infty} |q(t) - p_\pm(t)|dt, \tag{3.35}$$

where $C_\pm(x)$ are continuous, positive, monotonically decreasing functions, and therefore bounded as $x \to \pm\infty$. Furthermore,

$$\left|\frac{dK_\pm(x,y)}{dx}\right| + \left|\frac{dK_\pm(x,y)}{dy}\right| \le C_\pm(x)\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right| + Q_\pm(x+y)\right) \tag{3.36}$$

and

$$\pm \int_a^{\pm\infty} (1+x^2)\left|\frac{d}{dx}K_\pm(x,x)\right| dx < \infty, \quad \forall a \in \mathbb{R}. \tag{3.37}$$

Moreover, for $\lambda \in \sigma_\pm^u \cup \sigma_\pm^l$ a second pair of solutions of (3.32) is given by

$$\overline{\phi_\pm(\lambda,x)} = \breve{\psi}_\pm(\lambda,x) \pm \int_x^{\pm\infty} K_\pm(x,y)\breve{\psi}_\pm(\lambda,y)dy, \quad \lambda \in \sigma_\pm^u \cup \sigma_\pm^l. \tag{3.38}$$

Note $\breve{\psi}_\pm(\lambda,x) = \overline{\psi_\pm(\lambda,x)}$ for $\lambda \in \sigma_\pm$.

Unlike the Jost solutions $\phi_\pm(z,x)$, these solutions only exist on the upper and lower cuts of the spectrum and cannot be continued to the whole complex plane. Combining (3.25), (3.34), (3.35), and (3.38), one obtains

$$W(\phi_\pm(\lambda), \overline{\phi_\pm(\lambda)}) = \pm g(\lambda)^{-1}. \tag{3.39}$$

In the next lemma we want to point out, which properties of the background Weyl solutions are also inherited by the Jost solutions.

Lemma 3.4. *The Jost solutions $\phi_\pm(z,x)$ have the following properties:*

(i) The function $\phi_\pm(z,x)$ considered as a function of z, is holomorphic in the domain $\mathbb{C}\backslash(\sigma_\pm \cup M_\pm)$, and has simple poles at the points of the set $M_\pm$. They are continuous up to the boundary $\sigma_\pm^u \cup \sigma_\pm^l$ except at the points from $\hat{M}_\pm$. Moreover, we have

$$\phi_\pm(z,x) \in L^2(\mathbb{R}_\pm), \quad z \in \mathbb{C}\backslash\sigma_\pm \tag{3.40}$$

For $E \in \hat{M}_\pm$ they satisfy

$$\phi_\pm(z,x) = O\left(\frac{1}{\sqrt{z-E}}\right), \qquad \text{as } z \to E \in \hat{M}_\pm. \tag{3.41}$$

(ii) At the band edges we have the following behavior:
$\phi_\pm(z,x) - \overline{\phi_\pm(z,x)} = O(\sqrt{z-E})$ for $E \in \partial\sigma_\pm\backslash\hat{M}_\pm$, and
$\phi_\pm(z,x) + \overline{\phi_\pm(z,x)} = O(1)$ for $E \in \hat{M}_\pm$.

Proof. Everything follows from the fact that these properties are only dependent on z and therefore the transformation operator does not influence them. ☐

Now we want to characterize the spectrum of our operator L, which consists of an (absolutely) continuous part, $\sigma = \sigma_+ \cup \sigma_-$ and an at most countable number of discrete eigenvalues, which are situated in the gaps, $\sigma_d \subset \mathbb{R}\backslash\sigma$. For our purposes it will be convenient to write

$$\sigma = \sigma_-^{(1)} \cup \sigma_+^{(1)} \cup \sigma^{(2)}, \tag{3.42}$$

with

$$\sigma^{(2)} := \sigma_- \cap \sigma_+, \quad \sigma_\pm^{(1)} = \operatorname{clos}(\sigma_\pm\backslash\sigma^{(2)}). \tag{3.43}$$

It is well-known that a point $\lambda \in \mathbb{R}\backslash\sigma$ corresponds to the discrete spectrum if and only if the two Jost solutions are linearly dependent, which implies that we should investigate

$$W(z) := W(\phi_-(z,.), \phi_+(z,.)), \tag{3.44}$$

the Wronskian of the Jost solutions. This is a meromorphic function in the domain $\mathbb{C}\backslash\sigma$, with possible poles at the points $M_+ \cup M_- \cup (\hat{M}_+ \cap \hat{M}_-)$ and possible square root singularities at the points $\hat{M}_+ \cup \hat{M}_- \backslash(\hat{M}_+ \cap \hat{M}_-)$. Moreover it should be pointed out (cf. [72] and [87]) that every gap can only contain a finite number of discrete eigenvalues and thus they cannot cluster. For investigating the function $W(z)$ in more detail, we will multiply the possible poles and square root singularities away. Thus we define locally in a small neighborhood $U_j^{\pm}$ of the j'th gap $[E_{2j-1}^{\pm}, E_{2j}^{\pm}]$, where $j = 1, 2, \dots$

$$\tilde{\phi}_{j,\pm}(z,x) = \delta_{j,\pm}(z)\phi_{\pm}(z,x), \tag{3.45}$$

where

$$\delta_{j,\pm}(z) = \begin{cases} z - \mu_j^{\pm}, & \text{if } \mu_j^{\pm} \in M_{\pm}, \\ 1, & \text{else} \end{cases} \tag{3.46}$$

and

$$\hat{\phi}_{j,\pm}(z,x) = \hat{\delta}_{j,\pm}(z)\phi_{\pm}(z,x), \tag{3.47}$$

where

$$\hat{\delta}_{j,\pm}(z) = \begin{cases} z - \mu_j^{\pm}, & \text{if } \mu_j^{\pm} \in M_{\pm}, \\ \sqrt{z - \mu_j^{\pm}}, & \text{if } \mu_j^{\pm} \in \hat{M}_{\pm}, \\ 1, & \text{else.} \end{cases} \tag{3.48}$$

Correspondingly, we set

$$\tilde{W}(z) = W(\tilde{\phi}_-(z,.), \tilde{\phi}_+(z,.)), \quad \hat{W}(z) = W(\hat{\phi}_-(z,.), \hat{\phi}_+(z,.)). \tag{3.49}$$

Here we use the definitions

$$\tilde{\phi}_{\pm}(z,x) = \begin{cases} \tilde{\phi}_{j,\pm}(z,x), & \text{for } z \in U_j^{\pm}, j = 1, 2, \dots, \\ \phi_{\pm}(z,x), & \text{else .} \end{cases}, \tag{3.50}$$

$$\hat{\phi}_{\pm}(z,x) = \begin{cases} \hat{\phi}_{j,\pm}(z,x), & \text{for } z \in U_j^{\pm}, j = 1, 2, \dots, \\ \phi_{\pm}(z,x), & \text{else .} \end{cases} \tag{3.51}$$

and we will choose $U_j^+ = U_m^-$, if $[E_{2j-1}^+, E_{2j}^+] \cap [E_{2m-1}^-, E_{2m}^-] \neq \emptyset$. Analogously, one can define $\delta_{\pm}(z)$ and $\hat{\delta}_{\pm}(z)$.

Note that the function $\hat{W}(z)$ is holomorphic in the domain $U_j^{\pm} \cap (\mathbb{C}\backslash\sigma)$ and continuous up to the boundary. But unlike the functions $W(z)$ and $\tilde{W}(z)$ it may not take real values on the set $\mathbb{R}\backslash\sigma$ and complex conjugated values on the different sides of the spectrum $\sigma^u \cup \sigma^l$ inside the domains $U_j^{\pm}$. That is why we will characterize the spectral properties of our operator L in terms of the function $\tilde{W}(z)$ which can have poles at the band edges.

Since the discrete spectrum of our operator L is at most countable, we can write it as

$$\sigma_d = \bigcup_{n=1}^{\infty} \sigma_n \subset \mathbb{R}\backslash\sigma, \tag{3.52}$$

where

$$\sigma_n = \{\lambda_{n,1}, \dots, \lambda_{n,k}\}, \quad n \in \mathbb{N}, \tag{3.53}$$

and $k(n)$ denotes the number of eigenvalues in the n'th gap of σ.

For every eigenvalue $\lambda_{n,m}$ we can introduce the corresponding norming constants

$$(\gamma_{n,m}^{\pm})^{-2} = \int_{\mathbb{R}} \tilde{\phi}_{\pm}^2(\lambda_{n,m}, x)dx. \tag{3.54}$$

Now we begin with the study of the properties of the scattering data. Therefore we introduce the scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda,x) = \overline{\phi_\mp(\lambda,x)} + R_\mp(\lambda)\phi_\mp(\lambda,x), \quad \lambda\in\sigma_\mp^{u,l}, \tag{3.55}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) := \frac{W(\overline{\phi_\pm(\lambda)},\phi_\pm(\lambda))}{W(\phi_\mp(\lambda),\phi_\pm(\lambda))}, \quad R_\pm(\lambda) := -\frac{W(\phi_\mp(\lambda),\overline{\phi_\pm(\lambda)})}{W(\phi_\mp(\lambda),\phi_\pm(\lambda))}, \quad \lambda\in\sigma_\pm^{u,l} \tag{3.56}$$

Theorem 3.5. *For the scattering matrix the following properties are valid:*

(i) $T_\pm(\lambda^u) = \overline{T_\pm(\lambda^l)}$ *and* $R_\pm(\lambda^u) = \overline{R_\pm(\lambda^l)}$ *for* $\lambda\in\sigma_\pm$.

(ii) $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda)$ *for* $\lambda\in\sigma_\pm^{(1)}$.

(iii) $1-|R_\pm(\lambda)|^2 = \dfrac{g_\pm(\lambda)}{g_\mp(\lambda)}|T_\pm(\lambda)|^2$ *for* $\lambda\in\sigma^{(2)}$.

(iv) $\overline{R_\pm(\lambda)}T_\pm(\lambda) + R_\mp(\lambda)\overline{T_\pm(\lambda)} = 0$ *for* $\lambda\in\sigma^{(2)}$.

Proof. (i) and (iv) follow from (3.34), (3.38), (3.56), and Lemma 3.3
For showing (ii) observe that $\tilde\phi_\mp(\lambda,x)\in\mathbb{R}$ as $\lambda\in\operatorname{int}(\sigma_\pm^{(1)})$, which implies (ii).
Now assume $\lambda\in\operatorname{int}\sigma^{(2)}$, then by (3.55)

$$|T_\pm|^2 W(\phi_\mp,\overline{\phi_\mp}) = (|R_\pm|^2-1)W(\phi_\pm,\overline{\phi_\pm}). \tag{3.57}$$

Thus using (3.39) finishes the proof. □

Theorem 3.6. *The transmission and reflection coefficients have the following asymptotic behavior, as* $\lambda\to\infty$ *for* $\lambda\in\sigma^{(2)}$ *outside a small* ε *neighborhood of the band edges of* $\sigma^{(2)}$*:*

$$R_\pm(\lambda) = O(|\lambda|^{-1/2}), \tag{3.58}$$
$$T_\pm(\lambda) = 1+O(|\lambda|^{-1/2}). \tag{3.59}$$

Proof. The asymptotics can only be valid for $\lambda\in\sigma^{(2)}$ outside an ε neighborhood of the band edges, because the Jost solutions $\phi_\pm$ might have square root singularities there. At first we will investigate $W(\phi_-(\lambda,0),\phi_+(\lambda,0))$:

$$\begin{aligned}\phi_-(\lambda,0)\phi_+'(\lambda,0) = {}&\left(1+\int_{-\infty}^0 K_-(0,y)\psi_-(\lambda,y)dy\right)\times \\ &\left(m_+(\lambda)-K_+(0,0)+\int_0^\infty K_{+,x}(0,y)\psi_+(\lambda,y)dy\right).\end{aligned} \tag{3.60}$$

$$\int_{-\infty}^0 K_-(0,y)\psi_-(\lambda,y)dy = \int_{-\infty}^0 \frac{K_-(0,y)}{m_-(\lambda,y)}\psi_-'(\lambda,y) \tag{3.61}$$

where we used (cf. (3.19))

$$\psi_\pm'(\lambda,x) = m_\pm(\lambda,x)\psi_\pm(\lambda,x).$$

Hence

$$\int_{-\infty}^{0} K_-(0,y)\psi_-(\lambda,y)dy = \frac{K_-(0,0)}{m_-(\lambda)} + I_1(\lambda), \tag{3.62}$$

$$I_1(\lambda) = -\int_{-\infty}^{0} \left(K_{-,y}(0,y)\frac{\psi_-(\lambda,y)}{m_-(\lambda,y)} - K_-(0,y)\psi_-(\lambda,y)\frac{m'_-(\lambda,y)}{m_-(\lambda,y)^2} \right) dy. \tag{3.63}$$

Here it should be noticed that $m_\pm(\lambda)^{-1}$ has no pole, because (see e.g. [77])

$$G_\pm(z)N_\pm(z) + H_\pm(z)^2 = Y_\pm(z), \tag{3.64}$$

where

$$N_\pm(z) = -(z-\tau_0^\pm)\prod_{j=1}^{\infty}\frac{z-\tau_j^\pm}{E_{2j-1}^\pm}, \tag{3.65}$$

with $\tau_0^\pm \in (-\infty.E_0^\pm]$ and $\tau_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$. Thus we obtain

$$m_\pm(\lambda)^{-1} = \frac{G_\pm(\lambda)}{H_\pm(\lambda) \pm Y_\pm(\lambda)^{1/2}} = -\frac{H_\pm(\lambda) \mp Y_\pm(\lambda)^{1/2}}{N_\pm(\lambda)}, \tag{3.66}$$

and therefore $\frac{K_-(0,0)}{m_-(\lambda)} = O(\frac{1}{\sqrt{\lambda}})$.

Moreover $I_1(\lambda) = O\big(\frac{1}{\sqrt{\lambda}}\big)$ as the following estimates show:

$$\begin{aligned} |I_1(\lambda)| &\le \int_{-\infty}^{0} |K_{-,y}(0,y)\frac{\psi_-(\lambda,y)}{m_-(\lambda,y)}|dy + \int_{-\infty}^{0} |K_-(0,y)\psi_-(\lambda,y)\frac{m'_-(\lambda,y)}{m_-(\lambda,y)^2}|dy \\ &\le \frac{C}{\sqrt{\lambda}}\int_{-\infty}^{0}(|q(y)-p_-(y)| + Q_-(y))dy, \end{aligned} \tag{3.67}$$

where we used that $|\psi_\pm(\lambda,y)| = |\frac{G_\pm(\lambda,y)}{G_\pm(\lambda,0)}| = O(1)$ and $m_\pm^{-1}(\lambda,y) = O\left(\frac{1}{\sqrt{\lambda}}\right)$ for all y by the quasi-periodicity, together with (3.7) and

$$\psi''_\pm(\lambda,x) = m_\pm(\lambda,x)^2\psi_\pm(\lambda,x) + m'_\pm(\lambda,x)\psi_\pm(\lambda,x).$$

Making the same conclusions as before, one obtains

$$\int_0^\infty K_{+,x}(0,y)\psi_+(\lambda,y)dy = O(1). \tag{3.68}$$

In a similar manner one can investigate

$$\begin{aligned} \phi'_-(\lambda,0)\phi_+(\lambda,0) = &\left(m_-(\lambda) + K_-(0,0) + \int_{-\infty}^{0} K_{-,x}(0,y)\psi_-(\lambda,y)dy\right)\times \\ &\left(1 + \int_0^\infty K_+(0,y)\psi_+(\lambda,y)dy\right), \end{aligned} \tag{3.69}$$

where

$$\int_{-\infty}^{0} K_{-,x}(0,y)\psi_-(\lambda,y)dy = O(1), \tag{3.70}$$

$$\int_0^\infty K_+(0,y)\psi_+(\lambda,y)dy = -\frac{K_+(0,0)}{m_+(\lambda)} + I_2(\lambda), \tag{3.71}$$

$$I_2(\lambda) = -\int_0^\infty \left(K_{+,y}(0,y)\frac{\psi_+(\lambda,y)}{m_+(\lambda,y)} - K_+(0,y)\psi_+(\lambda,y)\frac{m'_+(\lambda,y)}{m_+(\lambda,y)^2}\right) dy, \tag{3.72}$$

with $I_2(\lambda) = O\big(\frac{1}{\sqrt{\lambda}}\big)$. Thus combining all the informations we obtained so far yields

$$\begin{aligned} W(\phi_-(\lambda),\phi_+(\lambda)) = m_+(\lambda) - m_-(\lambda) + K_-(0,0)\left(\frac{m_+(\lambda)-m_-(\lambda)}{m_-(\lambda)}\right) \\ + K_+(0,0)\left(\frac{m_-(\lambda)-m_+(\lambda)}{m_+(\lambda)}\right) + O(1). \end{aligned} \tag{3.73}$$

and therefore, using (3.39),

$$T_\pm(\lambda) = 1 + O\left(\frac{1}{\sqrt{\lambda}}\right). \tag{3.74}$$

Analogously one can investigate the behavior of $W(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)}$ to obtain $R_\pm(\lambda) = O\big(\frac{1}{\sqrt{\lambda}}\big)$. □

Theorem 3.7. *The functions $T_\pm(\lambda)$ can be extended analytically to the domain $\mathbb{C}\backslash(\sigma \cup M_\pm \cup \breve{M}_\pm)$ and satisfy*

$$\frac{-1}{T_+(z)g_+(z)} = \frac{-1}{T_-(z)g_-(z)} =: W(z), \tag{3.75}$$

where $W(z)$ possesses the following properties:

(i) The function $\tilde{W}$ is holomorphic in the domain $U_j^\pm \cap (\mathbb{C}\backslash\sigma)$, with simple zeros at the points λ_k, where

$$\left(\frac{d\tilde{W}}{dz}(\lambda_k)\right)^2 = (\gamma_{n,k}^+\gamma_{n,k}^-)^{-2}. \tag{3.76}$$

Besides it satisfies

$$\overline{\tilde{W}(\lambda^u)} = \tilde{W}(\lambda^l), \quad \lambda \in U_j^\pm \cap \sigma \quad \text{and} \quad \tilde{W}(\lambda) \in \mathbb{R}, \quad \lambda \in U_j^\pm \cap (\mathbb{R}\backslash\sigma). \tag{3.77}$$

(ii) The function $\hat{W}(z)$ is continuous on the set $U_j^\pm \cap \mathbb{C}\backslash\sigma$ up to the boundary $\sigma^l \cup \sigma^u$. It can have zeros on the set $\partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$ and does not vanish at any other points of σ. If $\hat{W}(E) = 0$ as $E \in \partial\sigma \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)})$, then $\hat{W}(z) = \sqrt{z-E}(C(E) + o(1))$, $C(E) \neq 0$.

Proof. (i) Except for (3.76) everything follows from the corresponding properties of $\phi_\pm(z,x)$. Therefore assume $\hat{W}(\lambda_0) = 0$ for some $\lambda_0 \in \mathbb{C}\backslash\sigma$, then

$$\tilde{\phi}_\pm(\lambda_0,x) = c_\pm\tilde{\phi}_\mp(\lambda_0,x), \tag{3.78}$$

for some constants $c_\pm$, which satisfy $c_- c_+ = 1$. Moreover, every zero of $\tilde{W}$ (or $\hat{W}$) outside the continuous spectrum, is a point of the discrete spectrum of L and vice versa.

Denote by $\gamma_\pm$ the corresponding norming constants defined in (3.54) for some fixed point λ_0 of the discrete spectrum. Proceeding as in [81] one obtains

$$W\big(\tilde{\phi}_\pm(\lambda_0,0), \frac{d}{d\lambda}\tilde{\phi}_\pm(\lambda_0,0)\big) = \int_0^{\pm\infty} \tilde{\phi}^2_\pm(\lambda_0,x)dx. \tag{3.79}$$

Thus using (3.78) and (3.79) yields

$$\begin{aligned}\gamma_\pm^{-2} &= \mp c_\pm^2 \int_0^{\mp\infty} \tilde{\phi}^2_\mp(\lambda_0,x)dx \pm \int_0^{\pm\infty} \tilde{\phi}^2_\pm(\lambda_0,x)dx \qquad (3.80)\\ &= \mp c_\pm^2 W\big(\tilde{\phi}_\mp(\lambda_0,0), \frac{d}{d\lambda}\tilde{\phi}_\mp(\lambda_0,0)\big) \pm W\big(\tilde{\phi}_\pm(\lambda_0,0), \frac{d}{d\lambda}\tilde{\phi}_\pm(\lambda_0,0)\big)\\ &= c_\pm \frac{d}{d\lambda} W(\tilde{\phi}_-(\lambda_0), \tilde{\phi}_+(\lambda_0)),\end{aligned}$$

applying now $c_- c_+ = 1$, we obtain (3.76).

(ii) The continuity of $\hat{W}(z)$ up to the boundary follows immediately from the corresponding properties of $\hat{\phi}_\pm(z,x)$. Now we will investigate the possible zeros.

Assume $W(\lambda_0) = 0$ for some $\lambda_0 \in \operatorname{int}(\sigma^{(2)})$. Then $\phi_+(\lambda_0,x) = c\phi_-(\lambda_0,x)$ and $\overline{\phi_+(\lambda_0,x)} = \overline{c}\overline{\phi_-(\lambda_0,x)}$. Thus $W(\phi_+,\overline{\phi_+}) = |c|^2 W(\phi_-,\overline{\phi_-})$ and therefore $\operatorname{sign} g_+(\lambda_0) = -\operatorname{sign} g_-(\lambda_0)$ by (3.39), contradicting (3.24).

Next let $\lambda_0 \in \operatorname{int}(\sigma_\pm^{(1)})$ and $\tilde{W}(\lambda_0) = 0$, then $\phi_\pm(\lambda_0,x)$ and $\overline{\phi_\pm(\lambda_0,x)}$ are linearly independent and bounded, moreover $\tilde{\phi}_\mp(\lambda_0,x) \in \mathbb{R}$. Therefore $\tilde{W}(\lambda_0) = 0$ implies that $\tilde{\phi}_\mp = c_1^\pm \phi_\pm = c_2^\pm \overline{\phi_\pm}$ and thus $W(\phi_\pm, \overline{\phi_\pm}) = 0$, which is impossible by (3.39). Note that in this case λ_0 can coincide with a pole $\mu \in M_\mp$.

Now introduce the local parameter $\tau = \sqrt{z-E}$ in a small neighborhood of each point $E \in \partial\sigma_\pm$ and define $\dot{y}(z,x) = \frac{d}{d\tau}y(z,x)$. A simple calculation shows that $\frac{dz}{d\tau}(E) = 0$, hence for every solution $y(z,x)$ of (3.32), its derivative $\dot{y}(E,x)$ is again a solution of (3.32). Therefore, the Wronskian $W(y(E),\dot{y}(E))$ is independent of x.

For each $x \in \mathbb{R}$ in a small neighborhood of a fixed point $E \in \partial\sigma_\pm$ we introduce the function

$$\hat{\psi}_{\pm,E}(z,x) = \begin{cases} \psi_\pm(z,x), & E \in \partial\sigma_\pm \backslash \hat{M}_\pm, \\ \tau\psi_\pm(z,x), & E \in M_\pm. \end{cases}$$

Proceeding as in [10] Lemma B.1 one obtains

$$W\Big(\hat{\psi}_{\pm,E}(E), \frac{d}{d\tau}\hat{\psi}_{\pm,E}(E)\Big) = \pm \lim_{z\to E} \frac{\alpha\tau^\alpha}{2g_\pm(z)}, \tag{3.81}$$

where $\alpha = -1$ if $E \in \partial\sigma_\pm \backslash \hat{M}_\pm$ and $\alpha = 1$ if $E \in \hat{M}_\pm$.

Using representation (3.19) for $\psi_\pm(z,x)$ one can show (cf [48]),

$$\psi_\pm(E,x) = \left(\frac{G_\pm(E,x)}{G_\pm(E,0)}\right)^{1/2} \exp\left(\pm \lim_{z\to E}\int_0^x \frac{Y_\pm(z)^{1/2}}{G_\pm(z,\tau)}d\tau\right), \quad E\in\partial\sigma \tag{3.82}$$

where

$$\exp\left(\pm \lim_{z\to E}\int_0^x \frac{Y_\pm(z)^{1/2}}{G_\pm(z,\tau)}d\tau\right) = \begin{cases} (\mathrm{i})^{2s+1}, & \mu_j\neq E, \mu_j(x)=E,\\ (\mathrm{i})^{2s+1}, & \mu_j = E, \mu_j(x)\neq E,\\ (\mathrm{i})^{2s}, & \mu_j = E, \mu_j(x)=E,\\ (\mathrm{i})^{2s}, & \mu_j \neq E, \mu_j(x)\neq E,\end{cases} \tag{3.83}$$

for $s\in\{0,1\}$. Defining

$$\hat\phi_{\pm,E}(\lambda,x) = \begin{cases}\phi_\pm(\lambda,x), & E\in\partial\sigma_\pm\backslash\hat M_\pm,\\ \tau\phi_\pm(\lambda,x), & E\in\hat M_\pm,\end{cases} \tag{3.84}$$

we can conclude using (3.34) that

$$\overline{\phi_\pm(E,x)} = \phi_\pm(E,x), \quad \text{for } E\in\partial\sigma_\pm\backslash\hat M_\pm. \tag{3.85}$$

Moreover, for $E\in\hat M_\pm$,

$$\begin{cases}\overline{\hat\phi_{\pm,E}(E,x)} = -\hat\phi_{\pm,E}(E,x), & \text{a left band edge from } \sigma_\pm,\\ \overline{\hat\phi_{\pm,E}(E,x)} = \hat\phi_{\pm,E}(E,x), & \text{a right band edge from } \sigma_\pm.\end{cases}$$

If $\lambda_0 = E\in\partial\sigma^{(2)}\cap\mathrm{int}(\sigma_\pm)\subset\mathrm{int}(\sigma_\pm)$, then $\hat W(E)=0$ if and only if $W(\psi_\pm,\hat\psi_{\mp,E})(E)=0$. Therefore, as $\hat\phi_{\mp,E}(E,.)$ are either pure real or pure imaginary, $W(\overline{\phi_\pm},\hat\phi_{\mp,E})(E)=0$, which implies that $\overline{\phi_\pm}(E,x)$ and $\phi_\pm(E,x)$ are linearly dependent, a contradiction.

Thus the function $\hat W(z)$ can only be zero at points E of the set $\partial\sigma\cup(\partial\sigma_+^{(1)}\cap\partial\sigma_-^{(1)})$. We will now compute the order of the zero. First of all note that the function $\hat W(\lambda)$ is continuously differentiable with respect to the local parameter τ. Since $\frac{d}{d\tau}(\delta_+\delta_-)(E)=0$, the function $W(\hat\phi_{+,E},\hat\phi_{-,E})$ has the same order of zero at E as $\hat W(\lambda)$. Moreover, if $\hat\delta_\pm(E)\neq 0$,then $\frac{d}{d\tau}\hat\delta_\pm(E)=0$ and if $\hat\delta_-(E)=\hat\delta_+(E)=0$, then $\frac{d}{d\tau}(\tau^{-2}\hat\delta_+\hat\delta_-)(E)=0$. Hence $\frac{d}{d\tau}\hat W(E)=0$ if and only if $\frac{d}{d\tau}W(\hat\phi_{+,E},\hat\phi_{-,E})=0$.

Combining now all the informations we obtained so far, we can conclude as follows: if $\hat W(E)=0$, then $\hat\phi_{\pm,E}(E,.)=c_\pm\hat\phi_{\mp,E}(E,.)$, with $c_-c_+=1$. Furthermore we can write

$$\begin{aligned}\dot W(\hat\phi_{+,E},\hat\phi_{-,E})(E) &= W(\frac{d}{d\tau}\hat\phi_{+,E},\hat\phi_{-,E})(E) - W(\frac{d}{d\tau}\hat\phi_{-,E},\hat\phi_{+,E})(E)\\ &= c_-W(\frac{d}{d\tau}\hat\phi_{+,E},\hat\phi_{+,E})(E) - c_+W(\frac{d}{d\tau}\hat\phi_{-,E},\hat\phi_{-,E})(E) \qquad (3.86)\\ &= c_-W(\frac{d}{d\tau}\hat\psi_{+,E},\hat\psi_{+,E})(E) - c_+W(\frac{d}{d\tau}\hat\psi_{-,E},\hat\psi_{-,E})(E).\end{aligned}$$

Using (3.81), (3.85), (3.86), and distinguishing several cases as in [10] finishes the proof.

□

Theorem 3.8. *(i) The reflection coefficient $R_\pm(\lambda)$ is a continuous function on the set* $\mathrm{int}(\sigma_\pm^{u,l})$.

(ii) If $E \in \partial\sigma_+ \cap \partial\sigma_-$ and $\hat{W}(E) \neq 0$, then the function $R_\pm(\lambda)$ is also continuous at E. Moreover,

$$R_\pm(E) = \begin{cases} -1 & \text{for } E \notin \hat{M}_\pm, \\ 1 & \text{for } E \in \hat{M}_\pm. \end{cases} \tag{3.87}$$

Proof. (i) At first it should be noted that by Lemma 3.5 the reflection coefficient is bounded, as $\frac{g_\pm(\lambda)}{g_\mp(\lambda)} > 0$ for $\lambda \in \mathrm{int}(\sigma^{(2)})$. Thus, using the corresponding properties of $\phi_\pm(z,x)$, finishes the first part.

(ii) By (3.56) the reflection coefficient can be represented in the following form:

$$R_\pm(\lambda) = -\frac{W(\overline{\phi_\pm(\lambda)}, \phi_\mp(\lambda))}{W(\phi_\pm(\lambda), \phi_\mp(\lambda))} = \pm\frac{W(\overline{\phi_\pm(\lambda)}, \phi_\mp(\lambda))}{W(\lambda)}, \tag{3.88}$$

and is therefore continuous on both sides of the set $\mathrm{int}(\sigma_\pm)\backslash(M_\mp \cup \hat{M}_\mp)$. Moreover,

$$|R_\pm(\lambda)| = \left| \frac{W(\overline{\hat{\phi}_\pm(\lambda), \hat{\phi}_\mp(\lambda))}}{\hat{W}(\lambda)} \right|, \tag{3.89}$$

where the denominator does not vanish, by assumption and hence $R_\pm(\lambda)$ is continuous on both sides of the spectrum in a small neighborhood of the band edges under consideration.

Next, let $E \in \{E_{2j-1}^\pm, E_{2j}^\pm\}$ with $\hat{W}(E) \neq 0$. Then, if $E \notin \hat{M}_\pm$, we can write

$$R_\pm(\lambda) = -1 \mp \frac{\hat{\delta}_{j,\pm}(\lambda) W(\phi_\pm(\lambda) - \overline{\phi_\pm(\lambda)}, \hat{\phi}_\mp(\lambda))}{\hat{W}(\lambda)}, \tag{3.90}$$

which implies $R_\pm(\lambda) \to -1$, since $\phi_\pm(\lambda) - \overline{\phi_\pm(\lambda)} \to 0$ by Lemma 3.4 as $\lambda \to E$. Thus we proved the first case.

If $E \in \hat{M}_\pm$ with $\hat{W}(E) \neq 0$, we use (3.88) in the form

$$R_\pm(\lambda) = 1 \pm \frac{\hat{\delta}_{j,\pm}(\lambda) W(\phi_\pm(\lambda) + \overline{\phi}_\pm(\lambda), \hat{\phi}_\mp(\lambda))}{\hat{W}(\lambda)}, \tag{3.91}$$

which yields $R_\pm(\lambda) \to 1$, since $\hat{\delta}_{j,\pm}(\lambda) \to 0$ and $\phi_\pm(\lambda) + \overline{\phi}_\pm(\lambda) = O(1)$ by Lemma 3.4 as $\lambda \to E$. This settles the second case.

□

3.4 The Gel'fand-Levitan-Marchenko Equation

The aim of this section is to derive the Gel'fand-Levitan-Marchenko (GLM) equation, which is also called the inverse scattering problem equation and to obtain some additional properties of the scattering data, as a consequence of the GLM equation.

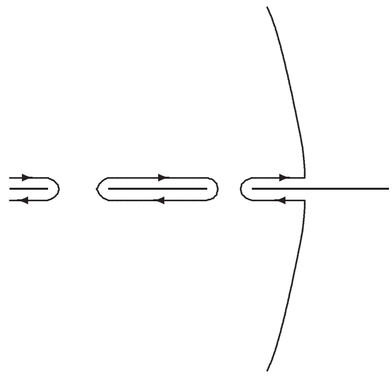

Figure 3.1: Contours $\Gamma_{\varepsilon,n}$

Therefore consider the function

$$\begin{aligned} G_\pm(z,x,y) &= T_\pm(z)\phi_\mp(z,x)\psi_\pm(z,y)g_\pm(z) - \breve{\psi}_\pm(z,x)\psi_\pm(z,y)g_\pm(z) \\ &:= G'_\pm(z,x,y) + G''_\pm(z,x,y), \quad \pm y > \pm x, \end{aligned} \tag{3.92}$$

where x and y are considered as fixed parameters. As a function of z it is meromorphic in the domain $\mathbb{C}\backslash\sigma$ with simple poles at the points λ_k of the discrete spectrum. It is continuous up to the boundary $\sigma^u \cup \sigma^l$, except for the points of the set, which consists of the band edges of the background spectra $\partial\sigma_+$ and $\partial\sigma_-$, where

$$G_\pm(z,x,y) = O((z-E)^{-1/2}) \quad \text{as} \quad E \in \partial\sigma_+ \cup \partial\sigma_-. \tag{3.93}$$

Outside a small neighborhood of the gaps of σ_+ and σ_-, the following asmptotics as $z \to \infty$ are valid:

$$\begin{aligned} \phi_\mp(z,x) &= \mathrm{e}^{\mp \mathrm{i}\sqrt{z}x(1+O(\frac{1}{z}))}\left(1+O(z^{-1/2})\right), \quad g_\pm(z) = \frac{-1}{2\mathrm{i}\sqrt{z}} + O(z^{-1}), \\ \breve{\psi}_\pm(z,x) &= \mathrm{e}^{\mp \mathrm{i}\sqrt{z}x(1+O(\frac{1}{z}))}\left(1+O(z^{-1})\right), \quad T_\pm(z) = 1 + O(z^{-1/2}), \\ \psi_\pm(z,y) &= \mathrm{e}^{\pm \mathrm{i}\sqrt{z}y(1+O(\frac{1}{z}))}\left(1+O(z^{-1})\right), \end{aligned}$$

and the leading term of $\phi_\mp(z,x)$ and $\breve{\psi}_\pm(z,x)$ are equal, thus

$$G_\pm(z,x,y) = \mathrm{e}^{\pm \mathrm{i}\sqrt{z}(y-x)(1+O(\frac{1}{z}))}O(z^{-1}), \quad \pm y > \pm x. \tag{3.94}$$

Consider the following sequence of contours $\Gamma_{\varepsilon,n,\pm}$, where $\Gamma_{\varepsilon,n,\pm}$ consists of two parts for every $n \in \mathbb{N}$ and $\varepsilon \geq 0$:

(i) $C_{\varepsilon,n,\pm}$ consists of a part of a circle which is centered at the origin and has as radii the distance from the origin to the midpoint of the largest band of $[E_{2n}^\pm, E_{2n+1}^\pm]$, which lies inside $\sigma^{(2)}$, together with a part wrapping around the corresponding band of σ at a small distance, which is at most ε, as indicated by figure 1.

(ii) Each band of the spectrum σ, which is fully contained in $C_{\varepsilon,n,\pm}$, is surrounded by a small loop at a small distance from σ not bigger than ε.

W.l.o.g. we can assume that all the contours are non-intersecting.
Using the Cauchy theorem, we obtain

$$\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{\varepsilon,n,\pm}} G_\pm(z,x,y)dz = \sum_{\lambda_k\in \operatorname{int}(\Gamma_{\varepsilon,n,\pm})} \operatorname*{Res}_{\lambda_k} G_\pm(z,x,y), \quad \varepsilon>0. \tag{3.95}$$

By (3.93) the limit value of $G_\pm(z,x,y)$ as $\varepsilon\to 0$ is integrable on σ, and the function $G''_\pm(z,x,y)$ has no poles at the points of the discrete spectrum, thus we arrive at

$$\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{0,n,\pm}} G_\pm(z,x,y)dz = \sum_{\lambda_k\in \operatorname{int}(\Gamma_{0,n,\pm})} \operatorname*{Res}_{\lambda_k} G'_\pm(z,x,y), \quad \pm y>\pm x. \tag{3.96}$$

Estimate (3.94) allows us now to apply Jordan's lemma, when letting $n\to\infty$, and we therefore arrive, up to that point only formally, at

$$\frac{1}{2\pi \mathrm{i}}\oint_{\sigma} G_\pm(\lambda,x,y)d\lambda = \sum_{\lambda_k\in \sigma_d} \operatorname*{Res}_{\lambda_k} G'_\pm(\lambda,x,y), \quad \pm y>\pm x. \tag{3.97}$$

Next, note that the function $G''_\pm(\lambda,x,y)$ does not contribute to the left part of (3.97), since $G''_\pm(\lambda^u,x,y)=G''_\pm(\lambda^l,x,y)$ for $\lambda\in\sigma_\mp^{(1)}$ and, hence $\oint_{\sigma_\mp^{(1)}} G''_\pm(\lambda,x,y)d\lambda = 0$. In addition, $\oint_{\sigma_\pm} G''_\pm(\lambda,x,y)d\lambda=0$ for $x\neq y$ by Lemma 3.3 (iv).

Therefore we arrive at the following equation,

$$\frac{1}{2\pi \mathrm{i}}\oint_{\sigma_\pm} G'_\pm(\lambda,x,y)d\lambda = \sum_{\lambda_k\in \sigma_d} \operatorname*{Res}_{\lambda_k} G'_\pm(\lambda,x,y), \quad \pm y>\pm x. \tag{3.98}$$

For making our argument rigorous, we have to apply Jordan's lemma, which implies that the contribution of the integral along the circle of $C_{0,n,\pm}$, converges against zero as $n\to\infty$ and we have to show that the series of integrals along the parts of the spectrum contained in $C_{0,n,\pm}$ converges as $n\to\infty$. This will be done next.

Using (3.30), (3.34), (3.38), (3.55), and Lemma 3.3 (iv) we obtain

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}}\oint_{\sigma_\pm} G'_\pm(\lambda,x,y)d\lambda &= \oint_{\sigma_\pm} T_\pm(\lambda)\phi_\mp(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&= \oint_{\sigma_\pm}\Big(R_\pm(\lambda)\phi_\pm(\lambda,x)+\overline{\phi_\pm(\lambda,x)}\Big)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&= \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,y)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) + \oint_{\sigma_\pm}\breve{\psi}_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)\\
&\quad\pm\int_x^{\pm\infty} dt K_\pm(x,t)\Big(\oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,t)\psi_\pm(\lambda,y)d\rho_\pm(\lambda)+\delta(t-y)\Big)\\
&= F_{r,\pm}(x,y)\pm\int_x^{\pm\infty} K_\pm(x,t)F_{r,\pm}(t,y)dt + K_\pm(x,y),
\end{aligned} \tag{3.99}$$

where

$$F_{r,\pm}(x,y) = \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,y)\psi_\pm(\lambda,y)d\rho_\pm(\lambda). \tag{3.100}$$

Now properties (ii) and (iii) from Lemma 3.5 imply that

$$|R_\pm(\lambda)| < 1 \quad \text{for} \quad \lambda \in \operatorname{int}(\sigma^{(2)}), \quad |R_\pm(\lambda)| = 1 \quad \text{for} \quad \lambda \in \sigma_\pm^{(1)}. \tag{3.101}$$

and by (3.19) we can write

$$\begin{aligned} F_{r,\pm}(x,y) &= \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) \\ &= -\oint_{\sigma_\pm} R_\pm(\lambda)\frac{(G_\pm(\lambda,x)G_\pm(\lambda,y))^{1/2}}{2Y_\pm(\lambda)^{1/2}}\exp(\eta_\pm(\lambda,x)+\eta_\pm(\lambda,y))d\lambda, \end{aligned}$$

with

$$\eta_\pm(\lambda,x) := \pm\int_0^x \frac{Y_\pm(\lambda)^{1/2}}{G_\pm(\lambda,\tau)}d\tau \in \mathrm{i}\mathbb{R}. \tag{3.102}$$

We will show

Lemma 3.9. *The series*

$$\begin{aligned} F_{r,\pm}(x,y) &= \sum_{n=0}^{\infty}\oint_{(E_{2n}^\pm,E_{2n+1}^\pm)} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) \\ &= \lim_{n\to\infty}\oint_{\sigma_\pm\cap\Gamma_{0,n,\pm}} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) = F_{r,n,\pm}(x,y), \end{aligned} \tag{3.103}$$

is convergent and uniformly bounded with respect to x and y.

Proof. For $\lambda \in \sigma_\pm$ as $\lambda \to \infty$ we have the following asymptotic behavior

(i) in a small neighborhood $V_n^\pm$ of $E = E_n^\pm$

$$|R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda)| = O\Big(\frac{\sqrt{E_{2j}^\pm - E_{2j-1}^\pm}}{\sqrt{\lambda(\lambda-E)}}\Big), \tag{3.104}$$

(ii) in a small neighborhood $W_n^\pm$ of $E = E_n^\mp$, if $E \in \sigma_\pm$

$$R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda) = \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y)(1+O(\frac{1}{\lambda}))O(\frac{1}{\sqrt{\lambda}}), \tag{3.105}$$

(iii) and for $\lambda \in \sigma_\pm \setminus \bigcup_{j\in\mathbb{N}}(V_n^\pm \cup W_n^\pm)$

$$R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda) = \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y)(1+O(\frac{1}{\lambda})))\Big(\frac{C}{\lambda}+O\Big(\frac{1}{\lambda^{3/2}}\Big)\Big). \tag{3.106}$$

These estimates are good enough to show that $F_{r,\pm}(x,y)$ exists, if we choose $V_n^\pm$ and $W_n^\pm$ in the following way: We choose $V_n^\pm \subset \sigma_\pm^{(1)} \cup \sigma^{(2)}$, if $E_n^\pm$ is a band edge of $\sigma_\pm^{(1)}$, such that $V_n^\pm$ consists of the corresponding band of $\sigma_\pm^{(1)}$ together with the following part of $\sigma_\pm^{(2)}$ with length $E_n^\pm - E_{n-1}^\pm$, if n is even and $E_{n+1}^\pm - E_n^\pm$, if n is odd. If E_n^+ is a band edge of $\sigma^{(2)}$, we choose $V_n^\pm \subset \sigma^{(2)}$, where the length of $V_n^\pm$ is equal to the length of the gap pf $\sigma_\pm$ next to it. We set $W_n^\pm \subset \sigma^{(2)}$ with length $3(E_n^\mp - E_{n-1}^\mp)$, if n is even and $3(E_{n+1}^\mp - E_n^\mp)$, if n is odd, centered at the midpoint of the corresponding gap in $\sigma_\mp$. As we are working in the Levitan class and we therefore know that $\sum_{n=1}^\infty (E_{2n-1}^\pm)^l (E_{2n}^\pm - E_{2n-1}^\pm) < \infty$ for some $l > 1$, we obtain that the sequences belonging to $V_n^\pm$ and $W_n^\pm$ converge.

For the last sequence, observe first that

$$|\exp(\pm\mathrm{i}\sqrt{\lambda}(x+y)O(\frac{1}{\lambda}))| \leq (x+y)O(\frac{1}{\sqrt{\lambda}}), \text{ if } (x+y)O(\frac{1}{\sqrt{\lambda}}) \geq \pi/2, \tag{3.107}$$

respectively

$$|\exp(\pm\mathrm{i}\sqrt{\lambda}(x+y)O(\frac{1}{\lambda}))| \leq 1+(x+y)O(\frac{1}{\sqrt{\lambda}}), \text{ if } (x+y)O(\frac{1}{\sqrt{\lambda}}) \leq \pi/2. \tag{3.108}$$

Furthermore

$$\begin{aligned}\int_a^b \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))\frac{C}{\lambda}d\lambda &= \pm\exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))\frac{C}{2\lambda^{1/2}(x+y)}\Big|_a^b \\ &\pm \int_a^b \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))\frac{C}{4\lambda^{3/2}(x+y)}d\lambda,\end{aligned} \tag{3.109}$$

and

$$\begin{aligned}\int_a^b \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))(x+y)O(\frac{1}{\lambda^{3/2}})d\lambda &= \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))O(\frac{1}{\lambda})\Big|_a^b \\ &+ \int_a^b \exp(\pm\mathrm{i}\sqrt{\lambda}(x+y))O(\frac{1}{\lambda^2})d\lambda\end{aligned} \tag{3.110}$$

For showing the convergence of the corresponding series, we can use the following argument: Integrate (3.106), where C can be computed explicitly, from $E_0^\pm$ to ∞ and subtract the parts corresponding to the gaps, and use

$$\frac{\exp(\pm\mathrm{i}\sqrt{E_{2j}^\pm}(x+y)) - \exp(\pm\mathrm{i}\sqrt{E_{2j-1}^\pm}(x+y))}{x+y} = O(\sqrt{E_{2j}^\pm} - \sqrt{E_{2j-1}^\pm}), \tag{3.111}$$

and

$$\sqrt{E_{2j}^\pm} - \sqrt{E_{2j-1}^\pm} = \frac{E_{2j}^\pm - E_{2j-1}^\pm}{\sqrt{E_{2j}^\pm} + \sqrt{E_{2j-1}^\pm}}. \tag{3.112}$$

Thus, putting all together, also the last part of the integral is finite and the bound is only dependent on x. □

For investigating the other terms, we will need the following lemma, which is taken from [40]:

Lemma 3.10. *Suppose in an integral equation of the form*

$$f_\pm(x,y) \pm \int_x^{\pm\infty} K_\pm(x,t)f_\pm(t,y)dt = g_\pm(x,y), \quad \pm y > \pm x, \tag{3.113}$$

the kernel $K_\pm(x,y)$ and the function $g_\pm(x,y)$ are continuous for $\pm y > \pm x$,

$$|K_\pm(x,y)| \leq C_\pm(x)Q_\pm(x+y), \tag{3.114}$$

and for $g_\pm(x,y)$ one of the following estimates hold

$$|g_\pm(x,y)| \le C_\pm(x) Q_\pm(x+y), \qquad or \tag{3.115}$$

$$|g_\pm(x,y)| \le C_\pm(x)(1+\max(0,\pm x)). \tag{3.116}$$

Furthermore assume that

$$\pm \int_0^{\pm\infty} (1+|x|^2)|q(x)-p_\pm(x)|dx < \infty. \tag{3.117}$$

Then (3.113) *is uniquely solvable for $f_\pm(x,y)$. The solution $f_\pm(x,y)$ is also continuous in the half-plane $\pm y > \pm x$, and for it the estimate* (3.115) *respectively* (3.116) *is reproduced.*

Moreover, if a sequence $g_{n,\pm}(x,y)$ satisfies (3.115) *or* (3.116) *uniformly with respect to n and pointwise $g_{n,\pm}(x,y) \to 0$, for $\pm y > \pm x$, then the same is true for the corresponding sequence of solutions $f_{n,\pm}(x,y)$ of* (3.113).

Proof. For a proof we refer to [38, Lemma 6.3]. □

Remark 3.11. *An immediate consequence of this lemma is the following. If $|g_\pm(x,y)| \le C_\pm(x)$, where $C_\pm(x)$ denotes a bounded function, then $|g_\pm(x,y)| \le C_\pm(x)(1+\max(0,\pm x))$ and therefore $|f_\pm(x,y)| \le C_\pm(x)(1+\max(0,\pm x))$. Rewriting this integral equation as follows*

$$f_\pm(x,y) = g_\pm(x,y) \mp \int_x^{\pm\infty} K_\pm(x,t) f_\pm(t,y)dt, \tag{3.118}$$

we obtain that the absolute value of the right hand side is smaller than a bounded function $\tilde{C}_\pm(x)$ by using (3.10) *and* (3.35), *and hence the same is true for the left hand side. In particular if $C_\pm(x)$ is a decreasing function the same will be true for $\tilde{C}_\pm(x)$.*

We will now continue the investigation of our integral equation.

Lemma 3.12. *The series*

$$\begin{aligned} F_{h,\pm}(x,y) &= \int_{\sigma_\mp^{(1),u}} |T_\mp(\lambda)|^2 \psi_\pm(\lambda,x)\psi_\pm(\lambda,y) d\rho_\mp(\lambda) \\ &= \lim_{n\to\infty} \int_{\sigma_\pm^{(1),u} \cap \Gamma_{0,n,\pm}} |T_\mp(\lambda)|^2 \psi_\pm(\lambda,x)\psi_\pm(\lambda,y) d\rho_\mp(\lambda) = F_{h,n,\pm}(x,y) \end{aligned} \tag{3.119}$$

converges uniformly and for every $n \in \mathbb{N}$ we have

$$|F_{h,n,\pm}(x,y)| \le C_\pm(x), \qquad and \qquad |F_{h,\pm}(x,y)| \le C_\pm(x) \tag{3.120}$$

where $C_\pm(x)$ are monotonically decreasing functions.

Proof. On the set $\sigma_\mp^{(1)}$ both the numerator and the denominator of the function $G'_\pm(\lambda,x,y)$ have poles (resp. square root singularities) at the points of the set $\sigma_\mp^{(1)} \cap (M_\pm \cup (\partial\sigma_+^{(1)} \cap \partial\sigma_-^{(1)}))$ (resp. $\sigma_\mp^{(1)} \cap (M_\mp \backslash (M_\mp \cap M_\pm))$, but multiplying them, if necessary away, we can avoid singularities. Hence, w.l.o.g., we can suppose $\sigma_{\mp}^{(1)} \cap (M_{r,+} \cup M_{r,-}) = \emptyset$. Thus we can write

$$\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} G'_{\pm}(\lambda, x, y) d\lambda = \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} T_{\pm}(\lambda)\phi_{\mp}(\lambda, x)\psi_{\pm}(\lambda, y) g_{\pm}(\lambda) d\lambda. \tag{3.121}$$

For investigating this integral we will consider, using (3.75),

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} & T_{\mp}(\lambda)\phi_{\mp}(\lambda, x)\phi_{\pm}(\lambda, y) g_{\mp}(\lambda, y) d\lambda \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} \phi_{\mp}(\lambda, x)\Big(\overline{\phi_{\mp}(\lambda, y)} + R_{\mp}(\lambda)\phi_{\mp}(\lambda, y)\Big) g_{\mp}(\lambda) d\lambda.\end{aligned}$$

First of all note that the integrand, because of the representation on the right hand side, can only have square root singularities at the boundary $\partial\sigma_{\mp}^{(1)}$ and we therefore have

$$\begin{aligned}\int_{\sigma_{\mp}^{(1)} \cap [E_{2n-1}^{\pm}, E_{2n}^{\pm}]} & |\phi_{\mp}(\lambda, x)\Big(\overline{\phi_{\mp}(\lambda, y)} + R_{\mp}(\lambda)\phi_{\mp}(\lambda, y)\Big) g_{\mp}(\lambda)| d\lambda \\ &\le 2 \int_{\sigma_{\mp}^{(1)} \cap [E_{2n-1}^{\pm}, E_{2n}^{\pm}]} |\phi_{\mp}(\lambda, x)\phi_{\mp}(\lambda, y) g_{\mp}(\lambda)| d\lambda \\ &\le C_{\pm}(y) C_{\pm}(x) \left(\frac{(E_{2n}^{\pm} - E_{2n-1}^{\pm})}{\sqrt{\lambda - E_0^{\mp}}} + \frac{\sqrt{E_{2n}^{\pm} - E_{2n-1}^{\pm}}}{\sqrt{\lambda - E_0^{\mp}}} \right),\end{aligned}$$

where $E_{2n-1}^{\pm}$ and $E_{2n}^{\pm}$ denote the edges of the gap of $\sigma_{\pm}$ in which the corresponding part of $\sigma_{\mp}^{(1)}$ lies and $C_{\pm}(x)$ denote monotonically decreasing functions from now on. Therefore as we are working in the Levitan class and by separating $\sigma_{\mp}^{(1)}$ into the different parts, one obtains that

$$|\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} T_{\pm}(\lambda)\phi_{\mp}(\lambda, x)\phi_{\pm}(\lambda, y) g_{\pm}(\lambda) d\lambda| \le C_{\pm}(y) C_{\pm}(x).$$

Thus we can now apply Lemma 3.10, and hence

$$|\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} T_{\pm}(\lambda)\phi_{\mp}(\lambda, x)\psi_{\pm}(\lambda, y) g_{\pm}(\lambda) d\lambda| \le C_{\pm}(y) C_{\pm}(x)(1 + \max(0, \pm y)).$$

Note that we especially have, because of (3.35),

$$|\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} T_{\pm}(\lambda)\phi_{\mp}(\lambda, x)\phi_{\pm}(\lambda, y) g_{\pm}(\lambda) d\lambda| \le C_{\pm}(x)$$

Therefore we can conclude that for fixed x and y the left hands side of (3.98) exists and satisfies

$$|\frac{1}{2\pi\mathrm{i}} \oint_{\sigma_{\mp}^{(1)}} T_{\pm}(\lambda)\phi_{\mp}(\lambda, x)\psi_{\pm}(\lambda, y) g_{\pm}(\lambda) d\lambda| \le C_{\pm}(x), \tag{3.122}$$

and hence

$$|\frac{1}{2\pi\mathrm{i}} \oint_{\sigma} G_{\pm}(z, x, y) dz| \le C_{\pm}(x). \tag{3.123}$$

Furthermore, since $\psi_\pm(\lambda,x)\in\mathbb{R}$ as $\lambda\in\sigma_\mp^{(1)}$, we have

$$\frac{1}{2\pi \mathrm{i}}\oint_{\sigma_\mp^{(1)}} G'_\pm(\lambda,x,y)d\lambda = \frac{1}{2\pi \mathrm{i}}\int_{\sigma_\mp^{(1),u}} \psi_\pm(\lambda,y)\Big(\frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}} - \frac{\phi_\mp(\lambda,x)}{W(\lambda)}\Big)d\lambda \quad (3.124)$$

Moreover, (3.55) and Lemma 3.5 (ii) imply

$$\overline{\phi_\mp(\lambda,x)} = T_\mp(\lambda)\phi_\pm(\lambda,x) - \frac{T_\mp(\lambda)}{\overline{T_\mp(\lambda)}}\phi_\mp(\lambda,x). \quad (3.125)$$

Therefore,

$$\begin{aligned}\frac{\phi_\mp(\lambda,x)}{W(\lambda)} - \frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}} &= \phi_\mp(\lambda,x)\Big(\frac{1}{W(\lambda)} + \frac{T_\mp(\lambda)}{\overline{T_\mp(\lambda)}\overline{W(\lambda)}}\Big) - \frac{T_\mp(\lambda)\phi_\pm(\lambda,x)}{\overline{W(\lambda)}} \quad (3.126)\\ &= \phi_\mp(\lambda,x)\frac{2\,\mathrm{Re}(T_\mp^{-1}(\lambda)\overline{W(\lambda)})T_\mp(\lambda)}{|W(\lambda)|^2} - \frac{T_\mp(\lambda)\phi_\pm(\lambda)}{\overline{W(\lambda)}}.\end{aligned}$$

But by (3.75)

$$T_\mp^{-1}(\lambda)\overline{W(\lambda)} = |W(\lambda)|^2 g_\mp(\lambda)\in \mathrm{i}\mathbb{R}, \qquad \text{for } \lambda\in\sigma_\mp^{(1)}, \quad (3.127)$$

and therefore the first summand of (3.126) vanishes. Using now $\overline{W} = (\overline{T_\mp} g_\mp)^{-1}$ we arrive at

$$\frac{\overline{\phi_\mp(\lambda,x)}}{\overline{W(\lambda)}} - \frac{\phi_\mp(\lambda,x)}{W(\lambda)} = |T_\mp(\lambda)|^2 g_\mp(\lambda)\phi_\pm(\lambda,x) \quad (3.128)$$

and hence

$$\frac{1}{2\pi \mathrm{i}}\oint_{\sigma_\mp^{(1)}} G'_\pm(\lambda,x,y)d\lambda = F_{h,\pm}(x,y) \pm \int_x^{\pm\infty} K_\pm(x,t)F_{h,\pm}(t.y)dt, \quad (3.129)$$

where

$$F_{h,\pm}(x,y) = \int_{\sigma_\pm^{(1),u}} |T_\mp(\lambda)|^2\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\mp(\lambda), \quad (3.130)$$

and

$$|F_{h,\pm}(x,y)| \le C_\pm(x)C_\pm(y) \quad (3.131)$$

by Lemma 3.10. The partial sums $F_{h,n,\pm}(x,y)$ can be investigated similarly $\square$

We will now investigate the r.h.s. of (3.96) and (3.98). Therefore we consider first the question of the existence of the right hand side:

To prove the boundedness of the corresponding series on the left hand side, it is left to investigate the series, which correspond to the circles. We will derive the necessary estimates only for the part of the n'th circle $K_{R_{n,\pm}}$, where $R_{n,\pm}$ denotes the radius, in the upper half plane as the part in the lower half plane can be considered similarly. We have

$$\begin{aligned}
|\int_{K_{R_n,\pm}} G_\pm(z,x,y)dz| &\leq \int_0^\pi C\mathrm{e}^{\pm\sqrt{R}(x-y)(1-\nu)\sin(\theta/2)}d\theta \\
&\leq \int_0^{\pi/2} C\mathrm{e}^{\pm\sqrt{R}(x-y)(1-\nu)\sin(\eta)}d\eta \\
&\leq \int_0^{\pi/2} C\mathrm{e}^{\pm\sqrt{R}(x-y)(1-\nu)2\frac{\eta}{\pi}}d\eta \\
&\leq C\frac{1}{\sqrt{R}(x-y)(1-\nu)}\mathrm{e}^{\pm\sqrt{R}(x-y)(1-\nu)2\frac{\eta}{\pi}}|_0^{\frac{\pi}{2}},
\end{aligned}$$

where C and ν denote some constant, which are dependent on the radius (cf. Lemma 3.2). Therefore as already mentioned the part belonging to the circles converges against zero and hence the same is true for the corresponding series.

Additionally we have to estimate

$$\begin{aligned}
\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm\cap\Gamma_{0,n,\pm}} G'_\pm(\lambda,x,y)d\lambda &= \oint_{\sigma_\pm\cap\Gamma_{0,n,\pm}} T_\pm(\lambda)\phi_\mp(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) \\
&= \oint_{\sigma_\pm\cap\Gamma_{0,n,\pm}} \Big(R_\pm(\lambda)\phi_\pm(\lambda,x)+\overline{\phi_\pm(\lambda,x)}\Big)\psi_\pm(\lambda,y)d\rho_\pm(\lambda).
\end{aligned} \tag{3.132}$$

Therefore observe that both terms can be investigated using the same techniques as in the proof of Lemma 3.9.

Thus we obtain that this sequence of partial sums is uniformly bounded and we are therefore able to proof the following result:

Lemma 3.13.

$$\begin{aligned}
F_{d,\pm}(x,y) &= \sum_{\lambda_k\in\sigma_d}(\gamma_k^\pm)^2\tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y) \\
&= \sum_{\lambda_k\in\sigma_d\cap\Gamma_{0,n,\pm}}(\gamma_k^\pm)^2\tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y) = F_{d,n,\pm}(x,y),
\end{aligned} \tag{3.133}$$

exists and satisfies for every $n\in\mathbb{N}$

$$|F_{d,n,\pm}(x,y)| \leq C_\pm(x), \quad \textit{and} \quad |F_{d,\pm}(x,y)| \leq C_\pm(x) \tag{3.134}$$

where $C_\pm(x)$ are monotonically decreasing functions.

Proof. Applying (3.34), (3.49), (3.50), (3.78), and (3.80) to the right hand side of (3.98), yields

$$\begin{aligned}
\sum_{\lambda_k\in\sigma_d}\operatorname*{Res}_{\lambda_k} G'_\pm(\lambda,x,y) &= -\sum_{\lambda_k\in\sigma_d}\operatorname*{Res}_{\lambda_k}\frac{\tilde{\phi}_\mp(\lambda,x)\tilde{\psi}_\pm(\lambda,y)}{\tilde{W}(\lambda)} \\
&= -\sum_{\lambda_k\in\sigma_d}\frac{\tilde{\phi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y)}{\tilde{W}'(\lambda_k)c_{k,\pm}} \\
&= -\sum_{\lambda_k\in\sigma_d}(\gamma_k^\pm)^2\tilde{\phi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y) \qquad (3.135) \\
&= -F_{d,\pm}(x,y)\mp\int_x^{\pm\infty}K_\pm(x,t)F_{d,\pm}(t,y)dt,
\end{aligned}$$

where

$$F_{d,\pm}(x,y) := \sum_{\lambda_k\in\sigma_d} (\gamma_k^\pm)^2 \tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y). \tag{3.136}$$

Thus we obtained the following integral equation,

$$\begin{aligned} F_{d,\pm}(x,y) = -K_\pm(x,y) - F_{c,\pm}(x,y) \mp \int_x^{\pm\infty} K_\pm(x,t)F_{c,\pm}(t,y)dt \qquad (3.137)\\ \mp \int_x^{\pm\infty} K_\pm(x,t)F_{d,\pm}(t,y)dt, \end{aligned}$$

which we can now solve for $F_{d,\pm}(x,y)$ using again Lemma 3.10 and hence $F_{d,\pm}(x,y)$ exists and satisfies the given estimates. The corresponding partial sums can be investigated analogously using the considerations from above. □

Thus we have proved the following theorem

Theorem 3.14. *The GLM equation has the form*

$$K_\pm(x,y) + F_\pm(x,y) \pm \int_x^{\pm\infty} K_\pm(x,t)F_\pm(t,y)dt = 0, \quad \pm(y-x) > 0, \tag{3.138}$$

where

$$\begin{aligned} F_\pm(x,y) = &\oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\pm(\lambda) \qquad (3.139)\\ &+ \int_{\sigma_\mp^{(1),u}} |T_\mp(\lambda)|^2 \psi_\pm(\lambda,x)\psi_\pm(\lambda,y)d\rho_\mp(\lambda)\\ &+ \sum_{k=1}^{\infty} (\gamma_k^\pm)^2 \tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda,y). \end{aligned}$$

Moreover, we have

Lemma 3.15. *The function $F_\pm(x,y)$ is continuously differentiable with respect to both variables and there exists a real-valued function $q_\pm(x)$, $x\in\mathbb{R}$ with*

$$\pm\int_a^{\pm\infty} (1+x^2)|q_\pm(x)|dx < \infty, \quad \text{for all } a\in\mathbb{R}, \tag{3.140}$$

such that

$$|F_\pm(x,y)| \le \tilde{C}_\pm(x)Q_\pm(x+y), \tag{3.141}$$

$$\left|\frac{d}{dx}F_\pm(x,y)\right| \le \tilde{C}_\pm(x)\left(\left|q_\pm\left(\frac{x+y}{2}\right)\right| + Q_\pm(x+y)\right), \tag{3.142}$$

$$\pm\int_a^{\pm\infty} \left|\frac{d}{dx}F_\pm(x,x)\right|(1+x^2)dx < \infty, \tag{3.143}$$

where

$$Q_\pm(x) = \pm\int_{\frac{x}{2}}^{\pm\infty} |q_\pm(t)|dt, \tag{3.144}$$

and $C_\pm(x) > 0$ is a continuous function, which decreases monotonically as $x\to\pm\infty$.

Proof. Applying once more Lemma 3.10, one obtains (3.141). Now, for simplicity, we will restrict our considerations to the + case and omit + whenever possible. Proceeding as in [10], we set $Q_1(u) = \int_u^\infty Q(t)dt$. Then, using (3.33), the functions $Q(x)$ and $Q_1(x)$ satisfy

$$\int_a^\infty Q_1(t)dt < \infty, \qquad \int_a^\infty Q(t)(1+|t|)dt < \infty. \tag{3.145}$$

Differentiating (3.138) with respect to x and y yields

$$|F_x(x,y)| \le |K_x(x,y)| + |K(x,x)F(x,y)| + \int_x^\infty |K_x(x,t)F(t,y)|dt, \tag{3.146}$$

$$F_y(x,y) + K_y(x,y) + \int_x^\infty K(x,t)F_y(t,y)dt = 0. \tag{3.147}$$

We already know that the functions $Q(x)$, $Q_1(x)$, $C(x)$, and $\tilde{C}(x)$ are monotonically decreasing and positive. Moreover,

$$\int_x^\infty \Big(\Big|q_+\Big(\frac{x+t}{2}\Big)\Big| + Q(x+t)\Big)Q(t+y)dt \le (Q(2x)+Q_1(2x))Q(x+y), \tag{3.148}$$

thus we can estimate $F_x(x,y)$ and $F_y(x,y)$ can be estimates using (3.36) and the method of successive approximation. It is left to prove (3.143). Therefore consider (3.138) for $x = y$ and differentiate it with respect to x:

$$\frac{dF(x,x)}{dx} + \frac{dK(x,x)}{dx} - K(x,x)F(x,x) + \int_x^\infty (K_x(x,t)F(t,x) + K(x,t)F_y(t,x))dt = 0. \tag{3.149}$$

Next (3.35) and (3.141) imply

$$|K(x,y)F(x,x)| \le \tilde{C}(a)C(a)Q^2(2x), \qquad \text{for } x > a, \tag{3.150}$$

where $\int_a^\infty (1+x^2)Q^2(2x)dx < \infty$. Moreover, by (3.36) and (3.142)

$$|K_x'(x,t)F(t,x)| + \big|K(x,t)F_y'(t,x)\big| \le 4\tilde{C}(a)\hat{C}(a)\Big\{\Big|q\Big(\frac{x+t}{2}\Big)\Big|Q(x+t) + Q^2(x+t)\Big\},$$

together with the estimates

$$\int_a^\infty dx\, x^2 \int_x^\infty Q^2(x+t)dt \le \int_a^\infty |x|Q(2x)dx \sup_{x\ge a}\int_x^\infty |x+t|Q(x+t)dt < \infty,$$

$$\int_a^\infty x^2 \int_x^\infty \Big|q\Big(\frac{x+t}{2}\Big)\Big|Q(x+t)dt \le$$

$$\le \int_a^\infty Q(2x)dx \sup_{x\ge a}\int_x^\infty \Big|q\Big(\frac{x+t}{2}\Big)\Big|(1+(x+t)^2)dt < \infty,$$

and (3.37), we arrive at (3.143).

□

In summary, we have obtained the following necessary conditions for the scattering data:

Theorem 3.16. *The scattering data*

$$\mathcal{S} = \Big\{ R_+(\lambda),\, T_+(\lambda),\, \lambda \in \sigma_+^{\mathrm{u,l}};\, R_-(\lambda),\, T_-(\lambda),\, \lambda \in \sigma_-^{\mathrm{u,l}};$$
$$\lambda_1, , \lambda_2, \cdots \in \mathbb{R} \setminus (\sigma_+ \cup \sigma_-),\, \gamma_1^{\pm}, \gamma_2^{\pm}, \cdots \in \mathbb{R}_+ \Big\} \tag{3.151}$$

possess the properties listed in Theorem 3.5, 3.6, 3.7, *and* 3.8, *and Lemma* 3.9, 3.12, *and* 3.13. *The functions* $F_\pm(x, y)$ *defined in* (3.139), *possess the properties listed in Lemma* 3.15.

Chapter 4

The Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data

4.1 Introduction

The aim of this chapter is to provide a rigorous treatment of the inverse scattering transform for the Korteweg–de Vries (KdV) equation

$$q_t = -q_{xxx} + 6qq_x \tag{4.1}$$

in the case of initial conditions which are steplike Schwartz–type perturbations of finite–gap solutions. The reason which makes the periodic case much more difficult are the poles of the Baker–Akhiezer functions which arise from the fact that the underlying hyperelliptic Riemann surface is no longer simply connected. In particular, we include a complete discussion of the problems arising from these poles. We will consider the case of Schwartz–type perturbations together with the additional assumption that the mutual spectral bands either coincide or are disjoint. While this last assumption excludes the classical case of steplike constant background, it clearly includes the case of short range perturbations of arbitrary finite-gap solutions.

More precisely, we will prove the following result

Theorem 4.1. *Let $p_\pm(x,t)$ be a real-valued finite-gap solution of the KdV equation corresponding to the initial condition $p_\pm(x) = p_\pm(x,0)$. Suppose that the mutual spectral bands of the one-dimensional Schrödinger operators associated with p_+ and p_- either coincide or are disjoint.*

Let $q(x)$ be a real-valued smooth function such that (the Schwartz class)

$$\pm\int_0^{\pm\infty} \left| \frac{d^n}{dx^n}\big(q(x) - p_\pm(x)\big)\right| (1+|x|^m)dx < \infty, \quad \forall m,n \in \mathbb{N}\cup\{0\}, \tag{4.2}$$

then there is a unique smooth solution $q(x,t)$ of the KdV equation corresponding to the initial condition $q(x,0) = q(x)$ and satisfying

$$\pm \int_0^{\pm\infty} \left| \frac{\partial^n}{\partial x^n} \big(q(x,t) - p_\pm(x,t) \big) \right| (1 + |x|^m) dx < \infty, \quad \forall m, n \in \mathbb{N} \cup \{0\}, \tag{4.3}$$

for all $t \in \mathbb{R}$.

4.2 Some general facts on the KdV flow

Let $q(x,t)$ be a classical solution of the KdV equation, that is, all partial derivatives appearing in equation (4.1) exist and are continuous. Moreover, suppose $q(x,t)$ and $q_x(x,t)$ are bounded with respect to x for all $t \in \mathbb{R}_+$.

Introduce the Lax pair [75]

$$L_q(t) = -\partial_x^2 + q(x,t), \tag{4.4}$$

$$P_q(t) = -4\partial_x^3 + 6q(x,t)\partial_x + 3q_x(x,t). \tag{4.5}$$

Note that $L_q(t)$ is self-adjoint on $\mathfrak{D}(L_q(t)) = H^2(\mathbb{R})$ and $P_q(t)$ is skew-adjoint on $\mathfrak{D}(P_q(t)) = H^3(\mathbb{R})$. Moreover, the KdV equation is equivalent to the Lax equation

$$\partial_t L_q(t) = [P_q(t), L_q(t)]$$

on $H^5(\mathbb{R})$.

The following result follows from classical theory of ordinary differential equations.

Lemma 4.2. *Let $c(\lambda,x,t)$ and $s(\lambda,x,t)$ be the solutions of the differential equation $L_q(t)u = \lambda u$ corresponding to the initial conditions $c(\lambda,0,t) = s_x(\lambda,0,t) = 1$ and $c_x(\lambda,0,t) = s(\lambda,0,t) = 0$.*

Then $c(\lambda,x,t)$ and $c_x(\lambda,x,t)$ are holomorphic with respect to $\lambda \in \mathbb{C}$ (for fixed x and t) and continuously differentiable with respect to t (provided $q(x,t)$ is). Similarly for $s(\lambda,x,t)$ and $s_x(\lambda,x,t)$.

Next, note the following property

Lemma 4.3. *Suppose $q(x,t)$ is three times differentiable with respect to x and once with respect to t. If $L_q(t)u = \lambda u$ holds, then*

$$(L_q(t) - \lambda)(u_t - P_q(t)u) = -(q_t + q_{xxx} - 6qq_x)u \tag{4.6}$$

Proof. Suppose $L_q u = \lambda u$, then we have $P_q u = (2(q + 2\lambda)\partial_x - q_x)u$ and thus

$$(L_q(t) - \lambda)P_q(t)u = (q_{xxx} - 6qq_x)u$$

respectively

$$(L_q(t) - \lambda)u_t = -q_t u$$

which proves the claim. □

Corollary 4.4 ([81], corollary to Lemma 4.1.1'). *Suppose $q(x,t)$ is three times differentiable with respect to x and once with respect to t. The function $q(x,t)$ satisfies the KdV equation (4.1) if and only if the operator*

$$\mathcal{A}_q(t) = \partial_t - 2(q(x,t) + 2\lambda)\partial_x + q_x(x,t) \tag{4.7}$$

transforms solutions of equation $(L_q(t) - \lambda)u = 0$ into solutions of the same equation.

Furthermore, we obtain

Lemma 4.5. *Let $q(x,t)$ be a classical solution of the KdV equation (4.1). The system of differential equations*

$$L_q(t)u = \lambda u, \tag{4.8}$$
$$u_t = P_q(t)u \tag{4.9}$$

has a unique solution $u(\lambda,x,t)$ for any given initial conditions $u(\lambda,0,0) = a_0(\lambda)$ and $u_x(\lambda,0,0) = b_0(\lambda)$. It will be continuous with respect to λ if a_0, b_0 are.

Proof. Write
$$u(\lambda,x,t) = a(\lambda,t)c(\lambda,x,t) + b(\lambda,t)s(\lambda,x,t),$$
then clearly $L_q(t)u = \lambda u$ holds by construction, and Lemma 4.3 implies

$$(L_q - \lambda)(u_t - P_q u) = 0.$$

Hence $u_t = P_q u$ will hold if and only if

$$a_t c + a c_t + b_t s + b s_t = a(P_q c) + b(P_q s) = 2(2\lambda + q)(a c_x + b s_x) - q_x(ac + bs)$$

holds together with its x derivative at $x = 0$, that is,

$$\begin{aligned}
a_t(\lambda,t) &= -a(\lambda,t)q_x(0,t) + b(\lambda,t)(4\lambda + 2q(0,t)),\\
b_t(\lambda,t) &= b(\lambda,t)q_x(0,t) + a(\lambda,t)\left(2(2\lambda + q(0,t))(q(0,t) - \lambda) - q_{xx}(0,t)\right),\\
a(\lambda,0) &= a_0(\lambda),\\
b(\lambda,0) &= b_0(\lambda).
\end{aligned} \tag{4.10}$$

This is a system of ordinary differential equations for the unknown functions $a(\lambda,t)$, $b(\lambda,t)$ and hence the claim follows. □

Let $c(\lambda,x,t) + m_\pm(\lambda,t)s(\lambda,x,t)$ be a pair of Weyl solutions for operator $L_q(t)$, where $m_\pm(\lambda,t)$ are the Weyl m-functions associated with L_q.

Lemma 4.6. *The functions*

$$u_\pm(\lambda,x,t) = a_\pm(\lambda,t)\big(c(\lambda,x,t) + m_\pm(\lambda,t)s(\lambda,x,t)\big), \tag{4.11}$$

where

$$a_\pm(\lambda,t) = \exp\left(\int_0^t \Big(2\big(q(0,s) + 2\lambda\big)m_\pm(\lambda,s) - q_x(0,s)\Big)ds\right), \tag{4.12}$$

solve (4.8), (4.9).

Proof. Let u denote one of the Weyl solutions $u_+(\lambda, x, t)$ or $u_-(\lambda, x, t)$ and let $\bar{u}$ be the other one. Then Lemma 4.3 implies that $u_t - P_q u$ is again a solution of $L_q u = \lambda u$. Consequently $u_t - P_q u = \beta u + \gamma \bar{u}$, where $\beta = \beta(\lambda, t)$, $\gamma = \gamma(\lambda, t)$. Since the Weyl solution decays sufficiently fast with respect to x on the corresponding half-axis when $\lambda \in \mathbb{C} \setminus \sigma$, then $u_t - P_q u$ also decays on the same half-axis. Therefore, $\gamma = 0$ and $u_t - P_q u = \beta u$ and the function $\hat{u}(\lambda, x, t) = \exp(-\int_0^t \beta(\lambda, s) ds) u(\lambda, x, t)$ satisfies the system (4.8), (4.9).

It remains to compute $\beta(\lambda, t)$. Using $u(\lambda, x, t) = c(\lambda, x, t) + m(\lambda, t) s(\lambda, x, t)$, where $m(\lambda, t)$ is the corresponding Weyl function, we obtain

$$\begin{aligned} c_t + m_t s + m s_t =& -4c_{xxx} - 4m s_{xxx} + 6q(c_x + m s_x) + 3q_x(c + ms) + \beta(c + ms) \\ =& 4(\lambda c_x - q_x c - c_x q) + 4m(\lambda s_x - q_x s - s_x q) + 6q(c_x + m s_x) \\ & + 3q_x(c + ms) + \beta(c + ms). \end{aligned}$$

For $x = 0$ this equation reads $0 = 2(q(0, t) + 2\lambda) m(\lambda, t) - q_x(0, t) + \beta(\lambda, t)$. □

Let $\mathsf{W}(f, g)(x) = f(x) g'(x) - f'(x) g(x)$ denote the Wronski determinant. The next lemma is a straightforward calculation.

Lemma 4.7. *Let u_1, u_2 be two solutions of* (4.8), (4.9)*, then the Wronskian $\mathsf{W}(u_1, u_2)$ does neither depend on x nor on t.*

4.3 Some general facts on finite-gap potentials

Since we want to study the initial value problem for the KdV equation in the class of initial conditions which asymptotically look like (different) finite-gap solutions, we need to recall some necessary background from finite-gap solutions first. For further information and for the history of finite-gap solutions we refer to, for example, [45], [46], [81], or [84].

Let $L_\pm(t) := L_{p_\pm}(t)$ be two one-dimensional Schrödinger operators associated with two arbitrary quasi-periodic finite-gap solutions $p_\pm(x, t)$ of the KdV equation. We denote by

$$\psi_\pm(\lambda, x, t) = c_\pm(\lambda, x, t) + m_\pm(\lambda, t) s_\pm(\lambda, x, t) \tag{4.13}$$

the corresponding Weyl solutions of $L_\pm(t)\psi_\pm = \lambda \psi_\pm$, normalized according to $\psi_\pm(\lambda, 0, t) = 1$ and satisfying $\psi_\pm(\lambda, ., t) \in L^2((0, \pm\infty))$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

It is well-known that the spectra $\sigma_\pm := \sigma(L_\pm(t))$ are t independent and consist of a finite number, say $r_\pm + 1$, bands:

$$\sigma_\pm = [E_0^\pm, E_1^\pm] \cup \cdots \cup [E_{2j-2}^\pm, E_{2j-1}^\pm] \cup \cdots \cup [E_{2r_\pm}^\pm, \infty). \tag{4.14}$$

Then $p_\pm$ are uniquely determined by their associated Dirichlet divisors

$$\left\{ (\mu_1^\pm(t), \sigma_1^\pm(t)), \dots, (\mu_{r_\pm}^\pm(t), \sigma_{r_\pm}^\pm(t)) \right\},$$

where $\mu_j^\pm(t) \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm(t) \in \{+1, -1\}$.

Let us cut the complex plane along the spectrum $\sigma_\pm$ and denote the upper and lower sides of the cuts by $\sigma_\pm^{\mathrm{u}}$ and $\sigma_\pm^{\mathrm{l}}$. The corresponding points on these cuts will be denoted by λ^{u} and λ^{l}, respectively. In particular, this means

$$f(\lambda^{\mathrm{u}}) := \lim_{\varepsilon \downarrow 0} f(\lambda + \mathrm{i}\varepsilon), \qquad f(\lambda^{\mathrm{l}}) := \lim_{\varepsilon \downarrow 0} f(\lambda - \mathrm{i}\varepsilon), \qquad \lambda \in \sigma_\pm.$$

Set

$$Y_\pm(\lambda) = -\prod_{j=0}^{2r_\pm}(\lambda - E_j^\pm), \tag{4.15}$$

and introduce the functions

$$g_\pm(\lambda,t) = -\frac{\prod_{j=1}^{r_\pm}(\lambda - \mu_j^\pm(t))}{2Y_\pm^{1/2}(\lambda)}, \tag{4.16}$$

where the branch of the square root is chosen such that

$$\frac{1}{\mathrm{i}} g_\pm(\lambda^{\mathrm{u}}) = \mathrm{Im}(g_\pm(\lambda^{\mathrm{u}})) > 0 \quad \text{for} \quad \lambda \in \sigma_\pm. \tag{4.17}$$

The functions $\psi_\pm$ admit two other well-known representations that will be used later on. The first one is

$$\psi_\pm(\lambda,x,t) = u_\pm(\lambda,x,t)\mathrm{e}^{\pm\mathrm{i}\theta_\pm(\lambda)x} \quad \lambda \in \mathbb{C}\setminus\sigma_\pm \tag{4.18}$$

where $\theta_\pm(\lambda)$ are the quasimoments and the functions $u_\pm(\lambda,x,t)$ are quasiperiodic with respect to x with the same basic frequencies as the potentials $p_\pm(x,t)$. The quasimoments are holomorphic for $\lambda \in \mathbb{C}\setminus\sigma_\pm$ and normalized according to

$$\frac{d\theta_\pm}{d\lambda} > 0 \quad \text{for} \quad \lambda \in \sigma_\pm^{\mathrm{u}}, \qquad \theta_\pm(E_0^\pm) = 0. \tag{4.19}$$

This normalization implies (cf. (4.17))

$$\frac{d\theta_\pm}{d\lambda} = \frac{\mathrm{i}\prod_{j=1}^{r_\pm}(\lambda - \zeta_j^\pm)}{Y_\pm^{1/2}(\lambda)}, \qquad \zeta_j^\pm \in (E_{2j-1}^\pm, E_{2j}^\pm), \tag{4.20}$$

and therefore, the quasimoments are real-valued on $\sigma_\pm$. Note, in the case where $p_\pm(x,t) \equiv 0$ we have $\theta_\pm(\lambda) = \sqrt{\lambda}$ and $u_\pm(\lambda,x,t) \equiv 1$.

Furthermore, the Weyl solutions possess more complicated properties, for example, they can have poles, as we see from the other representation. Namely, let $\mathbb{P}_\pm$ be the Riemann surfaces, associated with the functions $Y_\pm^{1/2}(\lambda)$ and let $\pi_\pm$ be parameters on these surfaces, corresponding to the spectral parameter λ, where π_+ (resp. π_-) is the parameter on the upper (resp., lower) sheet of $\mathbb{P}_+$ (resp. $\mathbb{P}_-$). Then

$$\psi_\pm(\pi_\pm,x,t) = \exp\left(\int_0^x m_\pm(\pi_\pm,y,t)dy\right), \tag{4.21}$$

where $m_\pm(\pi_\pm,x,t)$ are shifted Weyl functions (cf. [77]). Note, that the Weyl function $m_+(\lambda,t)$ is the branch, corresponding to values of $m_+(\pi_+,0,t)$ and $m_-(\lambda,t) = m_-(\pi_-,0,t)$. Denote the divisor of poles (the Dirichlet divisor) of the shifted Weyl functions by $\sum_{j=1}^{r_\pm}(\mu_j^\pm(x,t),\sigma_j^\pm(x,t))$. Then the functions $\mu_j^\pm(x,t)$ satisfy the system of Dubrovin equations ([45, Lem. 1.37])

$$\frac{\partial\mu_j^\pm(x,t)}{\partial x} = -2\sigma_j^\pm(x,t)Y_{\pm,j}(\mu_j^\pm(x,t),x,t), \tag{4.22}$$

$$\frac{\partial\mu_j^\pm(x,t)}{\partial t} = -4\sigma_j^\pm(x,t)(p_\pm(x,t) + 2\mu_j^\pm(x,t))Y_{\pm,j}(\mu_j^\pm(x,t),x,t), \tag{4.23}$$

where

$$Y_{\pm,j}(\lambda,x,t)=\frac{Y_\pm^{1/2}(\lambda)(\lambda-\mu_j^\pm(x,t))}{G_\pm(\lambda,x,t)} \tag{4.24}$$

and

$$G_\pm(\lambda,x,t)=\prod_{j=1}^{r_\pm}(\lambda-\mu_j^\pm(x,t)). \tag{4.25}$$

In (4.23) $p_\pm(x,t)$ have to be replaced by the trace formulas

$$p_\pm(x,t)=\sum_{j=0}^{2r_\pm}E_j^\pm-2\sum_{j=1}^{r_\pm}\mu_j^\pm(x,t). \tag{4.26}$$

Moreover, the following formula holds ([45, (1.165)])

$$m_\pm(\lambda,x,t)=\frac{H_\pm(\lambda,x,t)\pm Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,x,t)}, \tag{4.27}$$

where

$$H_\pm(\lambda,x,t)=\frac{1}{2}\frac{\partial}{\partial x}G_\pm(\lambda,x,t). \tag{4.28}$$

We will also use

$$\breve{m}_\pm(\lambda,x,t)=\frac{H_\pm(\lambda,x,t)\mp Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,x,t)}, \tag{4.29}$$

to denote the other branches of the Weyl functions on the Riemann surfaces $\mathbb{P}_\pm$, that is, $\breve{m}_\pm(\lambda,x,t)=m_\pm(\pi_\pm^*,x,t)$. In addition,

$$m_\pm(\lambda,t)-\breve{m}_\pm(\lambda,t)=\frac{\pm 2Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,0,t)}. \tag{4.30}$$

Lemma 4.8. *The following asymptotic expansion for large λ is valid*

$$\psi_\pm(\lambda,x,t)=\exp\left(\pm\mathrm{i}\sqrt{\lambda}x+\int_0^x\kappa_\pm(\lambda,y,t)dy\right), \tag{4.31}$$

where

$$\kappa_\pm(\lambda,x,t)=\sum_{k=1}^{\infty}\frac{\kappa_k^\pm(x,t)}{(\pm 2\mathrm{i}\sqrt{\lambda})^k}, \tag{4.32}$$

with coefficients defined recursively via

$$\kappa_1^\pm(x,t)=p_\pm(x,t),\quad \kappa_{k+1}^\pm(x,t)=-\frac{\partial}{\partial x}\kappa_k^\pm(x,t)-\sum_{m=1}^{k-1}\kappa_{k-m}^\pm(x,t)\kappa_m^\pm(x,t). \tag{4.33}$$

Proof. By (4.27) we conclude that

$$m_\pm(\lambda,x,t)=\pm\mathrm{i}\sqrt{\lambda}+\kappa_\pm(\lambda,x,t),$$

where $\kappa_\pm(\lambda,x,t)$ has an asymptotic expansion of the type (4.32). Inserting this expansion into the Riccati equation

$$\frac{\partial}{\partial x}\kappa_\pm(\lambda,x,t)\pm 2\mathrm{i}\sqrt{\lambda}\kappa_\pm(\lambda,x,t)+\kappa_\pm^2(\lambda,x,t)-p_\pm(x,t)=0 \tag{4.34}$$

and comparing coefficients shows (4.33). □

As a special case of Lemma 4.6 we obtain

Lemma 4.9. *The functions*

$$\hat{\psi}_\pm(\lambda, x, t) = \mathrm{e}^{\alpha_\pm(\lambda,t)}\psi_\pm(\lambda, x, t), \tag{4.35}$$

where

$$\alpha_\pm(\lambda, t) := \int_0^t \left(2(p_\pm(0, s) + 2\lambda)m_\pm(\lambda, s) - \frac{\partial p_\pm(0, s)}{\partial x}\right) ds, \tag{4.36}$$

satisfy the system of equations

$$L_\pm(t)\hat{\psi}_\pm = \lambda\hat{\psi}_\pm, \tag{4.37}$$

$$\frac{\partial\hat{\psi}_\pm}{\partial t} = P_\pm(t)\hat{\psi}_\pm, \tag{4.38}$$

where $P_\pm(t) := P_{p_\pm}(t)$.

We note that ([45, (1.148)])

$$\alpha_\pm(\lambda, t) = \frac{1}{2}\log\left(\frac{G_\pm(\lambda, 0, t)}{G_\pm(\lambda, 0, 0)}\right) \pm 2Y_\pm^{1/2}(\lambda)\int_0^t \frac{p_\pm(0, s) + 2\lambda}{G_\pm(\lambda, 0, s)}ds \tag{4.39}$$

and corresponding to $\breve{m}_\pm(\lambda, t)$ we also introduce

$$\begin{aligned}\breve{\alpha}_\pm(\lambda, t) &:= \int_0^t \left((2p_\pm(0, s) + 4\lambda)\breve{m}_\pm(\lambda, s) - \frac{\partial p_\pm(0, s)}{\partial x}\right) ds \\ &= \frac{1}{2}\log\left(\frac{G_\pm(\lambda, 0, t)}{G_\pm(\lambda, 0, 0)}\right) \mp 2Y_\pm^{1/2}(\lambda)\int_0^t \frac{p_\pm(0, s) + 2\lambda}{G_\pm(\lambda, 0, s)}ds.\end{aligned} \tag{4.40}$$

Note

$$\overline{\alpha_\pm(\lambda, t)} = \breve{\alpha}_\pm(\lambda, t), \qquad \lambda \in \sigma_\pm. \tag{4.41}$$

In order to remove the singularities of the functions $\psi_\pm(\lambda, x, t)$ we set

$$\begin{aligned} M_\pm(t) &= \{\mu_j^\pm(t) \mid \mu_j^\pm(t) \in (E_{2j-1}, E_{2j}) \text{ and } m_\pm(\lambda, t) \text{ has a simple pole}\}, \\ \hat{M}_\pm(t) &= \{\mu_j^\pm(t) \mid \mu_j^\pm(t) \in \{E_{2j-1}, E_{2j}\}\}, \end{aligned} \tag{4.42}$$

and introduce the functions

$$\begin{aligned}\delta_\pm(\lambda, t) &:= \prod_{\mu_j^\pm(t)\in M_\pm(t)} (\lambda - \mu_j^\pm(t)), \\ \hat{\delta}_\pm(\lambda, t) &:= \prod_{\mu_j^\pm(t)\in M_\pm(t)} (\lambda - \mu_j^\pm(t)) \prod_{\mu_j^\pm(t)\in \hat{M}_\pm(t)} \sqrt{\lambda - \mu_j^\pm(t)},\end{aligned} \tag{4.43}$$

where $\prod = 1$ if the index set is empty.

Lemma 4.10. *For each* $t \geq 0$ *and* $\lambda \in \mathbb{C} \setminus \sigma_\pm$ *the functions* $\alpha_\pm(\lambda, t)$ *possess the properties*

$$\exp\big(\alpha_\pm(\lambda, t) + \breve{\alpha}_\pm(\lambda, t)\big) = \frac{G_\pm(\lambda, 0, t)}{G_\pm(\lambda, 0, 0)}, \tag{4.44}$$

$$\exp\big(\alpha_\pm(\lambda, t)\big) = \frac{\hat{\delta}_\pm(\lambda, t)}{\hat{\delta}_\pm(\lambda, 0)}f_\pm(\lambda, t), \tag{4.45}$$

where the functions $f_\pm(\lambda,t)$ are holomorphic in $\mathbb{C}\setminus\sigma_\pm$, continuous up to the boundary and $f_\pm(\lambda,t)\neq 0$ for all $\lambda\in\mathbb{C}$.

Furthermore, let $E\in\{E^\pm_{2j-1},E^\pm_{2j}\}$, then

$$\lim_{\lambda\to E}(\alpha_\pm(\lambda,t)-\breve{\alpha}_\pm(\lambda,t))=\begin{cases}0, & \mu_j^\pm(t)\neq E, \mu_j^\pm(0)\neq E,\\ 0, & \mu_j^\pm(t)=E, \mu_j^\pm(0)=E,\\ \mathrm{i}\pi, & \mu_j^\pm(t)=E, \mu_j^\pm(0),\neq E,\\ \mathrm{i}\pi, & \mu_j^\pm(t)\neq E, \mu_j^\pm(0)=E,\end{cases} \pmod{2\pi\mathrm{i}}. \tag{4.46}$$

Proof. To shorten notations let us denote the derivative with respect to t by a dot and the derivative with respect to x by a prime. Equations (4.36) and (4.39) immediately give (4.44) and

$$\alpha_\pm(\lambda,t)-\breve{\alpha}_\pm(\lambda,t)=\pm 4Y_\pm^{1/2}(\lambda)\int_0^t\frac{p_\pm(0,s)+2\lambda}{G_\pm(\lambda,s)}ds, \tag{4.47}$$

where we have abbreviated

$$G_\pm(\lambda,t):=G_\pm(\lambda,0,t).$$

This function is well-defined on the set $\mathbb{C}\setminus\cup_{j=1}^{r_\pm}[E^\pm_{2j-1},E^\pm_{2j}]$, but may have singularities inside gaps. Note, that

$$\alpha_\pm(\lambda,t)-\breve{\alpha}_\pm(\lambda,t)\in\mathbb{R},\quad \text{for}\quad \lambda\in\mathbb{R}\setminus\sigma_\pm. \tag{4.48}$$

Consider the behavior of this function in the jth gap. By splitting the integral $\int_0^t$ in the definition of $\alpha_\pm(\lambda,t)$ (resp. $\breve{\alpha}_\pm(\lambda,t)$) into a sum of smaller integrals $\int_{t_0}^{t_1}$ it suffices to consider the cases where $\mu_j^\pm(s)\notin\{E^\pm_{2j-1},E^\pm_{2j}\}$ for $s\in[t_0,t_1)$ or $s\in(t_0,t_1]$. We will only investigate the first case (the other being completely analogous) and assume $t_0=0$ without loss of generality. In other words, it suffices to consider the case where $\mu_j^\pm(0)\in(E^\pm_{2j-1},E^\pm_{2j})$ and the time $t>0$ is so small, that $\sigma_j^\pm(s)=\sigma_j^\pm(0)$ for $s\le t$. Consequently, $\mu_j^\pm(t)\in(E^\pm_{2j-1},E^\pm_{2j})$ and there exists some $\varepsilon=\varepsilon(t)$ such that

$$\mu_j^\pm(s)\in(E^\pm_{2j-1}+2\varepsilon,E^\pm_{2j}-2\varepsilon),\quad 0\le s\le t. \tag{4.49}$$

Consider (e.g.) the case where the point $\mu_j^\pm(s)$ moves to the right, that is $\mu_j^\pm(0)<\mu_j^\pm(t)$. If $\lambda\notin(\mu_j^\pm(0)-\varepsilon,\mu_j^\pm(t)+\varepsilon)$, then the integral (4.47) is well-defined and by definition (4.15) the first case of (4.46) is fulfilled. Now let

$$\lambda\in(\mu_j^\pm(0)-\varepsilon,\mu_j^\pm(t)+\varepsilon). \tag{4.50}$$

From equation (4.23) we have

$$\dot{\mu}_j^\pm(s)=-\sigma_j^\pm(s)\tilde{Y}_{\pm,j}(\mu_j^\pm(s),s), \tag{4.51}$$

where

$$\tilde{Y}_{\pm,j}(\lambda,s)=4(p_\pm(s)+2\lambda)Y_{\pm,j}(\lambda,0,s) \tag{4.52}$$

and the functions $Y_{\pm,j}(\lambda,0,s)$ are defined by (4.24). Recall that $\sigma_j^\pm(s) = \text{const}$. Thus

$$\begin{aligned}\int_0^t \frac{\pm 4(p_\pm(s)+2\lambda)Y_\pm^{1/2}(\lambda)}{G_\pm(\lambda,s)} &= \pm\int_0^t \frac{\tilde Y_{\pm,j}(\lambda,s)}{\lambda-\mu_j^\pm(s)}ds\\ &= \pm\int_0^t \frac{\tilde Y_{\pm,j}(\mu_j^\pm(s),s)}{\lambda-\mu_j^\pm(s)}ds \pm \int_0^t \frac{\partial}{\partial\lambda}\tilde Y_{\pm,j}(\lambda,s)_{|\lambda=\xi_j^\pm(s)}\,ds,\end{aligned} \tag{4.53}$$

where $\xi_j^\pm(s) \in (E_{2j-1}^\pm+\varepsilon, E_{2j}^\pm-\varepsilon)$. Therefore $\frac{\partial}{\partial\lambda}\tilde Y_{\pm,j}(\lambda,s)$ is bounded here. But

$$\begin{aligned}\pm\int_0^t \frac{\tilde Y_{\pm,j}(\mu_j^\pm(s),s)}{\lambda-\mu_j^\pm(s)}ds &= \mp\sigma_j^\pm(0)\int_0^t \frac{\dot\mu_j^\pm(s)}{\lambda-\mu_j^\pm(s)}ds\\ &= \pm\sigma_j^\pm(0)\log\frac{\lambda-\mu_j^\pm(t)}{\lambda-\mu_j^\pm(0)}.\end{aligned}$$

Thus, in the case under consideration we have

$$\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t) = \log\frac{(\lambda-\mu_j^\pm(t))^{\pm\sigma_j^\pm(t)}}{(\lambda-\mu_j^\pm(0))^{\pm\sigma_j^\pm(0)}} + \tilde f_\pm(\lambda,\varepsilon), \tag{4.54}$$

where $\tilde f_\pm(\lambda,\varepsilon)$ is a smooth function, bounded by virtue of (4.50). Combining this formula with (4.44) we arrive at the following representation:

$$\exp\big(2\alpha_\pm(\lambda,t)\big) = \frac{(\lambda-\mu_j^\pm(t))^{\pm\sigma_j^\pm(t)+1}}{(\lambda-\mu_j^\pm(0))^{\pm\sigma_j^\pm(0)+1}} f_\pm^{(1)}(\lambda,t), \quad f_\pm^{(1)}(\lambda,t)\neq 0, \tag{4.55}$$

which is valid provided (4.49) and (4.50) hold. According to our notations $\mu_j^\pm(s)\in M_\pm(s)$ iff $\pm\sigma_j^\pm(s)=1$. Thus, if $\mu_j^\pm(t)\in M_\pm(t)$ (resp. $\mu_j^\pm(0)\in M_\pm(0)$), then the function $\exp(\alpha_\pm(\lambda,t))$ has a first order zero (resp. pole) at such a point and does not have any other poles or zeros inside the gap $(E_{2j-1}^\pm, E_{2j}^\pm)$. But if $\pm\sigma_j^\pm(t)=-1$ (resp. $\pm\sigma_j^\pm(0)=-1$), then the function $\exp(\alpha_\pm(\lambda,t))$ has no zero (resp. pole) at this point.

Now let us turn to the case $\mu_j^\pm(t)$ or $\mu_j^\pm(0)\in\{E_{2j-1}^\pm, E_{2j}^\pm\}$. Here we cannot use the decomposition (4.53) since the function $\frac{\partial}{\partial\lambda}\tilde Y_{\pm,j}(\lambda,s)$ is not bounded at the edges of the spectrum $\sigma_\pm$. Suppose, that $\mu_j^\pm(0)\in(E_{2j-1}^\pm,E_{2j}^\pm)$, the point $\mu_j^\pm(s)$ moves to the right, and the time $t>0$ is such, that $\sigma_j^\pm(s)=\sigma_j^\pm(0)$ for $s<t$ and $\mu_j^\pm(t)=E_{2j}^\pm$. Set $\varepsilon < 1/2(\mu_j^\pm(0)-E_{2j-1}^\pm)$ and let λ be such that

$$E_{2j-1}^\pm+\varepsilon<\lambda<E_{2j}^\pm+\varepsilon<E_{2j+1}^\pm.$$

Represent the function $\tilde Y_{\pm,j}(\lambda,s)$, defined by (4.52), as

$$\tilde Y_{\pm,j}(\lambda,s) = \sqrt{\lambda-E_{2j}^\pm}\;\breve Y_{\pm,j}(\lambda,s), \tag{4.56}$$

with

$$\breve Y_{\pm,j}(\lambda,s) = \breve Y_{\pm,j}(\mu_j^\pm(s),s) + (\lambda-\mu_j^\pm(s))\frac{\partial}{\partial\lambda}\breve Y_{\pm,j}(\zeta_j^\pm(s),s) \tag{4.57}$$

where $\frac{\partial}{\partial\lambda}\breve{Y}_{\pm,j}$ is evidently bounded. From (4.51) it follows that

$$\breve{Y}_{\pm,j}(\mu_j^{\pm}(s),s) = -\frac{\sigma_j^{\pm}(0)\dot{\mu}_j^{\pm}(s)}{\sqrt{\mu_j^{\pm}(s)-E_{2j}^{\pm}}}, \quad 0\le s\le t,$$

and

$$\begin{aligned}\int_0^t \frac{\tilde{Y}_{\pm,j}(\lambda,s)}{\lambda-\mu_j^{\pm}(s)}ds &= -\sigma_j^{\pm}(0)\sqrt{\lambda-E_{2j}^{\pm}}\left(\int_0^t \frac{\dot{\mu}_j^{\pm}(s)}{\sqrt{\mu_j^{\pm}(s)-E_{2j}^{\pm}}(\lambda-\mu_j^{\pm}(s))}ds + f_j^{\pm}(t,\varepsilon)\right)\\ &= -\sigma_j^{\pm}\sqrt{E_{2j}^{\pm}-\lambda}\left(\int_{\mu_j^{\pm}(0)}^{E_{2j}^{\pm}}\frac{d\tau}{(\lambda-\tau)\sqrt{E_{2j}^{\pm}-\tau}}+f_j^{\pm}(t,\varepsilon)\right) =\\ &= \sigma_j^{\pm}\sqrt{E_{2j}^{\pm}-\lambda}\int_{\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}}^0\frac{2dy}{y^2+\lambda-E_{2j}^{\pm}}+O\left(\sqrt{\lambda-E_{2j}^{\pm}}\right).\end{aligned} \tag{4.58}$$

To compute the first summand in (4.58) we will distinguish two cases. First let $\lambda\in\sigma_\pm$, that is, $\lambda > E_{2j}^{\pm}$. Then the first summand in (4.58) is equal to

$$-2\sigma_j^{\pm}(0)\mathrm{i}\arctan\frac{\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}}{\sqrt{\lambda-E_{2j}^{\pm}}}\to -\sigma_j^{\pm}(0)\mathrm{i}\pi, \quad \text{as} \quad \lambda\to E_{2j}^{\pm}, \quad \lambda\in\sigma_\pm.$$

This proves the two lower cases in (4.46). Next, consider the case when $\lambda\in(\mu_j^{\pm}(0),E_{2j}^{\pm})$. Then

$$\sigma_j^{\pm}(0)\sqrt{E_{2j}^{\pm}-\lambda}\int_{\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}}^0\frac{2dy}{y^2+\lambda-E_{2j}^{\pm}} =$$

$$= \sigma_j^{\pm}(0)\left(-\log\frac{\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}-\sqrt{E_{2j}^{\pm}-\lambda}}{\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}+\sqrt{E_{2j}^{\pm}-\lambda}}+\log(-1)\right) =$$

$$= -\sigma_j^{\pm}(0)\log\frac{\lambda-\mu_j^{\pm}(0)}{\left(\sqrt{E_{2j}^{\pm}-\mu_j^{\pm}(0)}+\sqrt{E_{2j}^{\pm}-\lambda}\right)^2}+\sigma_j^{\pm}(0)\mathrm{i}\pi. \tag{4.59}$$

If $\lambda\to E_{2j}^{\pm}$, then the first summand in (4.59) vanishes, and we arrive again at (4.46). If λ is in a small vicinity of $\mu_j^{\pm}(0)$, then

$$\pm\int_0^t\frac{\tilde{Y}_{\pm,j}(\lambda,s)}{\lambda-\mu_j^{\pm}(s)}ds = \mp\sigma_j^{\pm}(0)\log(\lambda-\mu_j^{\pm}(0))+O(1),$$

that confirm (4.45) for the case under consideration. □

4.4 Scattering theory

First we collect some facts from scattering theory for Schrödinger operators with step-like finite-gap potentials (cf. [10]). To shorten notations we omit the dependence on t throughout this section.

Let $L_\pm$ be two Schrödinger operators with real-valued finite-gap potentials $p_\pm(x)$, corresponding to the spectra (4.14) and the Dirichlet divisors $\sum_{j=1}^{r^\pm}(\mu_j^\pm, \sigma_j^\pm)$, where $\mu_j^\pm \in [E_{2j-1}^\pm, E_{2j}^\pm]$ and $\sigma_j^\pm \in \{-1, 1\}$.

Let $q(x)$ be a real-valued smooth function satisfying condition (4.2). The case $m = 2$ and $n = 0$ was rigorously studied in [10]. In this section we point out the necessary modifications for the Schwartz case. Let

$$L_q := -\frac{d^2}{dx^2} + q(x), \quad x \in \mathbb{R}, \tag{4.60}$$

be the "perturbed" operator with a potential $q(x)$, satisfying (4.2). The spectrum of L_q consists of a purely absolutely continuous part $\sigma := \sigma_+ \cup \sigma_-$ plus a finite number of eigenvalues situated in the gaps, $\sigma_d \subset \mathbb{R} \setminus \sigma$. We will use the notation $\mathrm{int}(\sigma_\pm)$ for the interior of the spectrum, that is, $\mathrm{int}(\sigma_\pm) := \sigma_\pm \setminus \partial\sigma_\pm$. The set $\sigma^{(2)} := \sigma_+ \cap \sigma_-$ is the spectrum of multiplicity two, and $\sigma_+^{(1)} \cup \sigma_-^{(1)}$ with $\sigma_\pm^{(1)} = \mathrm{clos}(\sigma_\pm \setminus \sigma_\mp)$ is the spectrum multiplicity one.

The Jost solutions of the equation

$$\left(-\frac{d^2}{dx^2} + q(x)\right) y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{4.61}$$

that are asymptotically close to the Weyl solutions of the background operators as $x \to \pm\infty$, can be represented with the help of the transformation operators as

$$\phi_\pm(\lambda, x) = \psi_\pm(\lambda, x) \pm \int_x^{\pm\infty} K_\pm(x, y)\psi_\pm(\lambda, y)dy, \tag{4.62}$$

where $K_\pm(x, y)$ are real-valued functions, that satisfy the integral equations

$$\begin{aligned} K_\pm(x, y) = &-2\int_{\frac{x+y}{2}}^{\pm\infty} (q(s) - p_\pm(s))\, D_\pm(x, s, s, y)ds \\ &\mp 2\int_x^{\pm\infty} ds \int_{y\pm x\mp s}^{y\pm s\mp x} D_\pm(x, s, r, y)K_\pm(s, r)\,(q(s) - p_\pm(s))\, dr, \quad \pm y > \pm x, \end{aligned} \tag{4.63}$$

where

$$D_\pm(x, y, r, s) = \mp\frac{1}{4} \sum_{E\in\partial\sigma_\pm} \frac{f_\pm(E, x, y)f_\pm(E, r, s)}{\frac{d}{d\lambda}Y_\pm(E)}, \tag{4.64}$$

with

$$f_\pm(E, x, y) = \lim_{\lambda\to E} \left(\prod_{j=1}^{r_\pm}(\lambda - \mu_j^\pm)\right) \psi_\pm(\lambda, x)\breve{\psi}_\pm(\lambda, y). \tag{4.65}$$

In particular,

$$K_\pm(x, x) = \pm\frac{1}{2}\int_x^{\pm\infty} (q(s) - p_\pm(s))ds. \tag{4.66}$$

Since

$$\frac{\partial^{n+l}}{\partial x^l \partial y^n} f_\pm(E, x, y) \in L^\infty(\mathbb{R} \times \mathbb{R}),$$

condition (4.2) and the method of successive approximations imply smoothness of the kernels for the transformation operators and the following estimate

$$\left| \frac{\partial^{n+l}}{\partial x^n \partial y^l} K_\pm(x, y) \right| < \frac{C_\pm(n, l, m)}{|x + y|^m}, \quad x, y \to \pm\infty, \quad m, n, l \in \mathbb{N} \cup \{0\}, \tag{4.67}$$

where $C_\pm(n, l, m)$ are positive constants (see Section 4.7 for the details).

Representation (4.62) shows, that the Jost solutions inherit all singularities of the background Weyl m-functions $m_\pm(\lambda)$. Hence we set (recall (4.43))

$$\tilde{\phi}_\pm(\lambda, x) = \delta_\pm(\lambda)\phi_\pm(\lambda, x) \tag{4.68}$$

such that the functions $\tilde{\phi}_\pm(\lambda, x)$ have no poles in the interior of the gaps of the spectrum σ. Let

$$\sigma_d = \{\lambda_1, \dots, \lambda_p\} \subset \mathbb{R} \setminus \sigma$$

be the set of eigenvalues of the operator L_q. For every eigenvalue we introduce the corresponding norming constants

$$\left(\gamma_k^\pm\right)^{-2} = \int_{\mathbb{R}} \tilde{\phi}_\pm^2(\lambda_k, x) dx. \tag{4.69}$$

Furthermore, introduce the scattering relations

$$T_\mp(\lambda)\phi_\pm(\lambda, x) = \overline{\phi_\mp(\lambda, x)} + R_\mp(\lambda)\phi_\mp(\lambda, x), \quad \lambda \in \sigma_\mp^{\mathrm{u,l}}, \tag{4.70}$$

where the transmission and reflection coefficients are defined as usual,

$$T_\pm(\lambda) := \frac{\mathsf{W}(\overline{\phi_\pm(\lambda)}, \phi_\pm(\lambda))}{\mathsf{W}(\phi_\mp(\lambda), \phi_\pm(\lambda))}, \qquad R_\pm(\lambda) := -\frac{\mathsf{W}(\phi_\mp(\lambda), \overline{\phi_\pm(\lambda)})}{\mathsf{W}(\phi_\mp(\lambda), \phi_\pm(\lambda))}, \quad \lambda \in \sigma_\pm^{\mathrm{u,l}}. \tag{4.71}$$

Lemma 4.11. *Suppose* (4.86). *Then the scattering data*

$$\mathcal{S} := \Big\{ R_+(\lambda),\ T_+(\lambda),\ \lambda \in \sigma_+^{\mathrm{u,l}};\ R_-(\lambda),\ T_-(\lambda),\ \lambda \in \sigma_-^{\mathrm{u,l}}; \\ \lambda_1, \dots, \lambda_p \in \mathbb{R} \setminus \sigma,\ \gamma_1^\pm, \dots, \gamma_p^\pm \in \mathbb{R}_+ \Big\} \tag{4.72}$$

have the following properties:

I. (a) $T_\pm(\lambda^{\mathrm{u}}) = \overline{T_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda \in \sigma_\pm$.
$R_\pm(\lambda^{\mathrm{u}}) = \overline{R_\pm(\lambda^{\mathrm{l}})}$ *for* $\lambda \in \sigma_\pm$.

(b) $\dfrac{T_\pm(\lambda)}{\overline{T_\pm(\lambda)}} = R_\pm(\lambda)$ *for* $\lambda \in \sigma_\pm^{(1)}$.

(c) $1 - |R_\pm(\lambda)|^2 = \dfrac{g_\pm(\lambda)}{g_\mp(\lambda)} |T_\pm(\lambda)|^2$ *for* $\lambda \in \sigma^{(2)}$ *with* $g_\pm(\lambda)$ *from* (4.16).

(d) $\overline{R_\pm(\lambda)} T_\pm(\lambda) + R_\mp(\lambda) \overline{T_\pm(\lambda)} = 0$ *for* $\lambda \in \sigma^{(2)}$.

(e) $T_\pm(\lambda) = 1 + O\Big(\frac{1}{\sqrt{\lambda}}\Big)$ *for* $\lambda \to \infty$.

(f) *$R_\pm(\lambda) = O\Big(\frac{1}{(\sqrt{\lambda})^{n+1}}\Big)$ for $\lambda\to\infty$ and for all $n\in\mathbb{N}$.*

II. *The functions $T_\pm(\lambda)$ can be extended as meromorphic functions into the domain $\mathbb{C}\setminus\sigma$ and satisfy*

$$\frac{1}{T_+(\lambda)g_+(\lambda)} = \frac{1}{T_-(\lambda)g_-(\lambda)} =: -W(\lambda), \tag{4.73}$$

where the function $W(\lambda)$ possesses the following properties:

(a) *The function $\tilde W(\lambda) = \delta_+(\lambda)\delta_-(\lambda)W(\lambda)$, where $\delta_\pm(\lambda)$ is defined by (4.43), is holomorphic in the domain $\mathbb{C}\setminus\sigma$, with simple zeros at the points λ_k, where*

$$\left(\frac{d\tilde W}{d\lambda}(\lambda_k)\right)^2 = (\gamma_k^+\gamma_k^-)^{-2}. \tag{4.74}$$

In addition, it satisfies

$$\overline{\tilde W(\lambda^{\mathrm{u}})} = \tilde W(\lambda^{\mathrm{l}}), \quad \lambda\in\sigma \quad \text{and} \quad \tilde W(\lambda)\in\mathbb{R} \quad \text{for} \quad \lambda\in\mathbb{R}\setminus\sigma. \tag{4.75}$$

(b) *The function $\hat W(\lambda) = \hat\delta_+(\lambda)\hat\delta_-(\lambda)W(\lambda)$, where $\hat\delta_\pm(\lambda)$ is defined by (4.43), is continuous on the set $\mathbb{C}\setminus\sigma$ up to the boundary $\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}$. Moreover, the function $\hat W(\lambda)$ is infinitely many times differentiable with respect to λ on the set $\big(\sigma^{\mathrm{u}}\cup\sigma^{\mathrm{l}}\big)\setminus\partial\sigma$ and continuously differentiable with respect to the local variable $\sqrt{\lambda-E}$ for $E\in\partial\sigma$. It can have zeros on the set $\partial\sigma$ and does not vanish at the other points of the set σ. If $\hat W(E)=0$ as $E\in\partial\sigma$, then $\hat W(\lambda)=\sqrt{\lambda-E}(C(E)+o(1))$, $C(E)\neq 0$.*

III. **(a)** *The reflection coefficients $R_\pm(\lambda)$ are continuously differentiable infinitely many time functions on the sets $\operatorname{int}(\sigma_\pm^{\mathrm{u,l}})$.*

(b) *If $E\in\partial\sigma$ and $\hat W(E)\neq 0$ then the functions $R_\pm(\lambda)$ are also continuous at E. Moreover, in this case*

$$R_\pm(E) = \begin{cases} -1 & \text{for } E\notin\hat M_\pm, \\ 1 & \text{for } E\in\hat M_\pm. \end{cases} \tag{4.76}$$

Proof. For the case $m=2$ and $n=0$ this lemma was proven in [10]. In particular, except for the differentiability properties of the scattering data and item **I.(f)** everything follows from Lemma 3.3 in [10].

Differentiability of $\hat W(\lambda)$ and $R_\pm(\lambda)$ is a direct consequence of differentiability of the Jost solutions. In fact, since $\frac{\partial^l\psi_\pm(\lambda,y)}{\partial\lambda^l} = O(|y|^l)$ for $\lambda\in\operatorname{int}\sigma_\pm$ as $y\to\pm\infty$, equations (4.62), (4.67), and (4.2) imply, that $\phi_\pm(\lambda,x)$ are continuously differentiable infinitely many times with respect to $\lambda\in\operatorname{int}\sigma_\pm$ since $\psi_\pm(\lambda,x)$ are. Moreover, note, that at the points $E_j^\pm$ these solutions are continuously differentiable with respect to the local parameter $\sqrt{\lambda-E_j^\pm}$ since this holds for $\psi_\pm(\lambda,x)$. Furthermore, since $\operatorname{Im}\theta_\pm(\lambda)>0$ for $\lambda\in\mathbb{R}\setminus\sigma_\pm$, we infer that $\psi_\pm(\lambda,y)$ are exponentially decaying together with all derivatives as $y\to\pm\infty$ if $\lambda\in\mathbb{R}\setminus\sigma_\pm$.

It remains to show **I.(f)**. To this end, represent the Jost solutions in the form

$$\phi_\pm(\lambda,x) = \psi_\pm(\lambda,x)\exp\left(-\int_x^{\pm\infty}\tilde\kappa_\pm(\lambda,y)dy\right), \tag{4.77}$$

where

$$\tilde\kappa_\pm(\lambda,x) = \sum_{k=1}^\infty \frac{\tilde\kappa_k^\pm(x)}{(\pm 2\mathrm{i}\sqrt{\lambda})^k}. \tag{4.78}$$

To derive a differential equation for $\tilde\kappa_\pm(\lambda,x)$ we substitute (4.77) into (4.61) and use (4.31) and (4.34). This yields the differential equations

$$\frac{\partial}{\partial x}\tilde\kappa_\pm(\lambda,x) + \tilde\kappa_\pm^2(\lambda,x) \pm 2(\mathrm{i}\sqrt{\lambda} + \kappa_\pm(\lambda,x))\tilde\kappa_\pm(\lambda,x) + p_\pm(x) - q(x) = 0, \tag{4.79}$$

from which we obtain the recurrence formulas

$$\tilde\kappa_1^\pm(x) = q(x) - p_\pm(x), \quad \tilde\kappa_{k+1}^\pm(x) = -\frac{\partial}{\partial x}\tilde\kappa_k^\pm(x) - \sum_{m=1}^{k-1}\tilde\kappa_{k-m}^\pm(x)(\tilde\kappa_m^\pm(x) + 2\kappa_m^\pm(x)). \tag{4.80}$$

Using (4.71) we now derive an asymptotic formula for $R_+(\lambda)$ (for R_- the considerations are analogous). By (4.77) and (4.78)

$$\mathsf{W}(\phi_-(\lambda),\phi_+(\lambda)) = \phi_-(\lambda,0)\phi_+(\lambda,0)\left(2\mathrm{i}\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right)\right) = 2\mathrm{i}\sqrt{\lambda}(1+o(1)) \tag{4.81}$$

and

$$\mathsf{W}(\phi_-(\lambda),\overline{\phi_+(\lambda)}) = \phi_-(\lambda,0)\overline{\phi_+(\lambda,0)}\left(\overline{y_+(\lambda,0)} - y_-(\lambda,0)\right), \tag{4.82}$$

where we have set $y_\pm(\lambda,x) := \tilde\kappa_\pm(\lambda,x) + \kappa_\pm(\lambda,x)$. Equations (4.34) and (4.79) imply

$$\frac{\partial}{\partial x}y_\pm(\lambda,x) \pm 2\mathrm{i}\sqrt{\lambda}y_\pm(\lambda,x) + y_\pm^2(\lambda,x) - q(x) = 0. \tag{4.83}$$

Therefore, the functions $\tilde y_+(\lambda,x) := \overline{y_+(\lambda,x)}$ and $\tilde y_-(\lambda,x) := y_-(\lambda,x)$ satisfy one and the same equation. Moreover, $\kappa_1^\pm(x) + \tilde\kappa_1^\pm(x) = q(x)$. Hence, since $q(x)$ is smooth, the functions $\tilde y_\pm$ admit asymptotic expansions

$$\tilde y_\pm(\lambda,x) = \sum_{k=1}^\infty \frac{\tilde y_k^\pm(x)}{(-2\mathrm{i}\sqrt{\lambda})^k},$$

where $\tilde y_k^+(x)$ and $\tilde y_k^-(x)$ satisfy the same recurrence equations

$$\tilde y_1^\pm(x) = q(x), \quad \tilde y_{k+1}^\pm(x) = -\frac{\partial}{\partial x}\tilde y_k^\pm(x) - \sum_{l=1}^{k-1}\tilde y_{k-l}^\pm(x)\tilde y_l^\pm(x). \tag{4.84}$$

Therefore,

$$\overline{y_+(\lambda,0)} - y_-(\lambda,0) = O(\lambda^{-n/2})$$

for $\lambda \to \infty$ and for all $n \in \mathbb{N}$ and the same is true for $R_+(\lambda)$ by (4.81) and (4.82). □

To complete the characterization of scattering data $\mathcal{S}$, consider the associated Gelfand-Levitan-Marchenko (GLM) equations.

Lemma 4.12. *The kernels $K_\pm(x,y)$ of the transformation operators satisfy the Gelfand-Levitan-Marchenko equations*

$$K_\pm(x,y) + F_\pm(x,y) \pm \int_x^{\pm\infty} K_\pm(x,s)F_\pm(s,y)ds = 0, \quad \pm y > \pm x, \tag{4.85}$$

where [1]

$$\begin{aligned} F_\pm(x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{\sigma_\pm} R_\pm(\lambda)\psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\pm(\lambda)d\lambda + \\ + \frac{1}{2\pi \mathrm{i}} \int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda)|^2 \psi_\pm(\lambda,x)\psi_\pm(\lambda,y)g_\mp(\lambda)d\lambda \\ + \sum_{k=1}^{p} (\gamma_k^\pm)^2 \tilde{\psi}_\pm(\lambda_k,x)\tilde{\psi}_\pm(\lambda_k,y). \end{aligned} \tag{4.86}$$

IV. *The functions $F_\pm(x,y)$ are differentiable infinitely many times with respect to both variables and satisfy*

$$\left| \frac{\partial^{l+n}}{\partial x^l \partial y^n} F_\pm(x,y) \right| \le \frac{C_\pm(m,n,l)}{|x+y|^m} \quad as \quad x,y \to \pm\infty, \quad m,l,n = 0,1,2,\dots \tag{4.87}$$

Proof. Formulas (4.85) and (4.86) are obtained in [10], estimate (4.87) follows directly from (4.85) and (4.67). □

Properties **I–IV** from above are characteristic for the scattering data $\mathcal{S}$, that is

Theorem 4.13 (characterization, [10]). *Properties* **I–IV** *are necessary and sufficient for a set $\mathcal{S}$ to be the set of scattering data for operator L with a potential $q(x)$ from the class* (4.2).

In addition, we will now describe a procedure of solving of the inverse scattering problem.

Let $L_\pm$ be two one-dimensional finite-gap Schrödinger operators associated with the potentials $p_\pm(x)$. Let $\mathcal{S}$ be given scattering data (4.72) satisfying **I–IV** and define corresponding kernels $F_\pm(x,y)$ via (4.86). As it shown in [10], condition ***IV*** the GLM equations (4.85) have unique smooth real-valued solutions $K_\pm(x,y)$, satisfying estimate of type (4.67), possibly with some other constants $C_\pm$, than in (4.87). In particular,

$$\pm \int_0^{\pm\infty} (1+|x|^m) \left| \frac{d^n}{dx^n} K_\pm(x,x) \right| dx < \infty, \qquad \forall m,n \in \mathbb{N}. \tag{4.88}$$

Now introduce the functions

$$q_\pm(x) = \mp 2 \frac{d}{dx} K_\pm(x,x) + p_\pm(x), \quad x \in \mathbb{R} \tag{4.89}$$

[1] Here we have used the notation $\oint_{\sigma_\pm} f(\lambda)d\lambda := \int_{\sigma_\pm^{\mathrm{u}}} f(\lambda)d\lambda - \int_{\sigma_\pm^{\mathrm{l}}} f(\lambda)d\lambda$.

and note that the estimate (4.88) reads

$$\pm\int_0^{\pm\infty} |\frac{d^n}{dx^n}(q_\pm(x)-p_\pm(x))|(1+|x|^m)dx<\infty, \quad \forall n,m\in\mathbb{N}\cup\{0\}. \tag{4.90}$$

Moreover, define functions $\phi_\pm(\lambda,x)$ by formula (4.62), where $K_\pm(x,y)$ are the solutions of (4.85). Then these functions solve the equations

$$\left(-\frac{d^2}{dx^2}+q_\pm(x)\right)\phi_\pm(\lambda,x)=\lambda\phi_\pm(\lambda,x). \tag{4.91}$$

The only remaining difficulty is to show that in fact $q_-(x)=q_+(x)$:

Theorem 4.14 ([10])**.** *Let the scattering data $\mathcal{S}$, defined as in* (4.72)*, satisfy the properties* **I–IV***. Then the functions $q_\pm(x)$, defined by* (4.89) *coincide, $q_-(x)\equiv q_+(x)=: q(x)$. Moreover, the data $\mathcal{S}$ are the scattering data for the Schrödinger operator with potential $q(x)$ from the class* (4.2).

4.5 The inverse scattering transform

As our next step we show how to use the solution of the inverse scattering problem found in the previous section to give a formal scheme for solving the initial-value problem for the KdV equation with initial data from the class (4.2).

Suppose first that our initial-value problem has a solution $q(x,t)$ satisfying (4.3) for each $t>0$. Then all considerations from the previous section apply to the operator $L_q(t)$ if we consider t as an additional parameter. In particular, there are time-dependent transformation operators with kernels $K_\pm(x,y,t)$ satisfying the estimates

$$\left|\frac{\partial^{l+n}}{\partial x^l\partial y^n}K_\pm(x,y,t)\right|\le\frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y\to\pm\infty, \quad l,n,m=0,1,2,\dots. \tag{4.92}$$

and

$$\left|\frac{\partial^{n+l+1}}{\partial x^n\partial y^l\partial t}K_\pm(x,y,t)\right|\le\frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y\to\pm\infty, \quad l,n,m=0,1,2,\dots. \tag{4.93}$$

These estimates follows from the fact that the kernels $D_\pm(x,y,s,r,t)$ of the time-dependent equations (4.63) are smooth with respect to all variables, and each partial derivative is uniformly bounded with respect to $x,y,s,r,t\in\mathbb{R}$. Consequently, the Jost solutions

$$\phi_\pm(\lambda,x,t)=\psi_\pm(\lambda,x,t)\pm\int_x^{\pm\infty}K_\pm(x,y,t)\psi_\pm(\lambda,y,t)dy, \tag{4.94}$$

are also differentiable with respect to t and satisfy

$$\frac{\partial}{\partial t}\phi_\pm(\lambda,x,t)=\frac{\partial}{\partial t}\psi_\pm(\lambda,x,t)(1+o(1)) \qquad \text{as } x\to\pm\infty, \tag{4.95}$$

$$\frac{\partial^n}{\partial x^n}\phi_\pm(\lambda,x,t)=\frac{\partial^n}{\partial x^n}\psi_\pm(\lambda,x,t)(1+o(1)) \qquad \text{as } x\to\pm\infty. \tag{4.96}$$

By Lemma 4.3 we know that the functions $P_q(t)\phi_\pm(\lambda,x,t)$ solves the equation $L_q(t)u = \lambda u$. Asymptotics (4.95) and (4.96) show, that

$$P_q(t)\phi_\pm(\lambda,x,t) = \beta_\pm(\lambda,t)\phi_\pm(\lambda,x,t),$$

where $\beta_\pm(\lambda,t)$ is the same factor as in $P_\pm(t)\psi_\pm(\lambda,x,t) = \beta_\pm(\lambda,t)\psi_\pm(\lambda,x,t)$. From Lemma 4.6 we obtain then

Lemma 4.15. *Let $\alpha_\pm(\lambda,t)$ be defined by* (4.36) *and let $q(x,t)$ be a solution of the KdV equation satisfying* (4.2). *Then the functions*

$$\hat\phi_\pm(\lambda,x,t) = \mathrm{e}^{\alpha_\pm(\lambda,t)}\phi_\pm(\lambda,x,t) \tag{4.97}$$

solve the system (4.8), (4.9).

Before we proceed further we note that equation (4.45) implies

Corollary 4.16. *The function $\hat\phi_\pm(\lambda,x,t)$, defined by formula* (4.97), *have simple poles on the set $M_\pm(0)$, square root singularities on the set $\hat M_\pm(0)$, and no other singularities.*

Next, consider the time-dependent scattering relations

$$T_\mp(\lambda,t)\phi_\pm(\lambda,x,t) = \overline{\phi_\mp(\lambda,x,t)} + R_\mp(\lambda,t)\phi_\mp(\lambda,x,t), \quad \lambda\in\sigma_\mp^{\mathrm{u,l}}. \tag{4.98}$$

Then, using the previous lemma in combination with Lemma 4.7 to evaluate (4.71) we infer

Lemma 4.17. *Let $q(x,t)$ be a solution of the KdV equation satisfying* (4.2). *Then $\lambda_k(t) = \lambda_k(0) \equiv \lambda_k$;*

$$R_\pm(\lambda,t) = R_\pm(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t)}, \quad \lambda\in\sigma_\pm, \tag{4.99}$$

$$T_\mp(\lambda,t) = T_\mp(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\mp(\lambda,t)}, \quad \lambda\in\mathbb{C}, \tag{4.100}$$

$$\left(\gamma_k^\pm(t)\right)^2 = \left(\gamma_k^\pm(0)\right)^2\frac{\delta_\pm^2(\lambda_k,0)}{\delta_\pm^2(\lambda_k,t)}\mathrm{e}^{2\alpha_\pm(\lambda_k,t)}, \tag{4.101}$$

where $\alpha_\pm(\lambda,t)$, $\breve\alpha_\pm(\lambda,t)$, $\delta_\pm(\lambda,t)$ are defined in (4.36), (4.40), (4.43), *respectively.*

Proof. First of all set $\hat W(\lambda,t) = \hat\delta_+(\lambda,t)\hat\delta_-(\lambda,t)W(\lambda,t)$ (recall (4.73)). Then, since $\mathsf{W}(\hat\phi_-(\lambda,t),\hat\phi_+(\lambda,t))$ does not depend on t by Lemma 4.7, it follows from (4.73) and (4.45) that

$$f(\lambda,t)\hat W(\lambda,t) = \hat W(\lambda,0), \quad f(\lambda,t) = f_-(\lambda,t)f_+(\lambda,t) \neq 0. \tag{4.102}$$

This implies, that the discrete spectrum of the operator $L(t)$, which is the set of zeros of the function $\hat W(\lambda,t)$ on the set $\mathbb{R}\setminus\sigma$, does not depend on t.

Similarly, if we replace the functions $\phi_\pm$ by $\hat\phi_\pm$ in all Wronskians of formulas (4.71), the result will be a constant with respect to t. Together with (4.97) it implies (4.99) and (4.100). To obtain (4.101) we set $\check\phi(\lambda,x,t) = \delta_\pm(\lambda,0)\hat\phi_\pm(\lambda,x,t)$ (which is continuous near λ_k) and compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\check\phi_\pm(\lambda_k,x,t)^2dx &= 2\int_{\mathbb{R}}\check\phi_\pm(\lambda_k,x,t)\partial_t\check\phi_\pm(\lambda_k,x,t)dx\\ &= \int_{\mathbb{R}}\check\phi_\pm(\lambda_k,x,t)P_q(t)\check\phi_\pm(\lambda_k,x,t)dx = 0,\end{aligned}$$

since P_q is skew-adjoint and $\check{\phi}_\pm(\lambda_k, x, t)$ is real-valued. Note that interchanging differentiation and integration is permissible by the dominated convergence theorem (recall that the quasimoments $\theta_\pm(\lambda)$ are independent of t). Thus, (4.68) and (4.69) imply

$$\frac{d}{dt}\frac{\delta_\pm(\lambda_k,0)\ \mathrm{e}^{\alpha_\pm(\lambda_k,t)}}{\delta_\pm(\lambda_k,t)\ \gamma_k^\pm(t)} = 0,$$

which finishes the proof. □

Hence the solution $q(x,t)$ can be computed from the time-dependent scattering data as follows. Construct one of the functions $F_+(x,y,t)$ or $F_-(x,y,t)$ via

$$\begin{aligned} F_\pm(x,y,t) =& \frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm} R_\pm(\lambda,t)\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)g_\pm(\lambda,t)d\lambda + \\ &+ \frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}} |T_\mp(\lambda,t)|^2\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)g_\mp(\lambda,t)d\lambda \\ &+ \sum_{k=1}^{p}(\gamma_k^\pm(t))^2\tilde{\psi}_\pm(\lambda_k,x,t)\tilde{\psi}_\pm(\lambda_k,y,t). \end{aligned} \tag{4.103}$$

Solve the corresponding GLM equation

$$K_\pm(x,y,t) + F_\pm(x,y,t) \pm \int_x^{\pm\infty} K_\pm(x,s,t)F_\pm(s,y,t)ds = 0, \quad \pm y > \pm x, \tag{4.104}$$

and obtain the solution by

$$q(x,t) = \mp 2\ \frac{d}{dx}K_\pm(x,x,t) + p_\pm(x,t), \quad x \in \mathbb{R}. \tag{4.105}$$

Theorem 4.14 guarantees, that both formulas give one and the same solution.

Up to now we have assumed that $q(x,t)$ is a solution the KdV equation satisfying (4.2). Now we can get rid of this assumption. We will proceed as follows. Suppose the initial condition $q(x)$ satisfies (4.2) with some finite-gap potential $p_\pm(x)$. Consider the corresponding scattering data $\mathcal{S} = \mathcal{S}(0)$ which obey conditions **I**–**IV**. Let $p_\pm(x,t)$ be the finite-gap solution of the KdV equation with initial condition $p_\pm(x)$ and let $m_\pm(\lambda,t)$, $\psi_\pm(\lambda,x,t)$, and $\alpha_\pm(\lambda,t)$ be the corresponding quantities as in Section 4.3.

Introduce the set of scattering data $\mathcal{S}(t)$, where $R_\pm(\lambda,t)$, $T_\pm(\lambda,t)$ and $\gamma_k^\pm(t)$ are defined by formulas (4.99)–(4.101). In the next section we prove, that these data satisfies conditions **I**–**III**, and the functions $F_\pm(x,y,t)$, defined via (4.103), satisfy **IV** under the assumption that the respective bands of the spectra $\sigma_\pm$ either coincide or otherwise do not intersect at all, that is

$$\sigma^{(2)} \cap \sigma_\pm^{(1)} = \emptyset \quad \text{and} \quad \sigma_+^{(1)} \cap \sigma_-^{(1)} = \emptyset. \tag{4.106}$$

The typical situation is depicted in Figure 4.1.

Then Theorem 5.3 from [10] ensures the unique solvability for each of the GLM equations (4.104)with the solutions $K_\pm(x,y,t)$ that satisfy the estimate of type (4.92). Moreover, since $F_\pm(x,y,t)$ are differentiable with respect to t with (4.87) valid for this derivative, then (4.85) implies (4.93). Consequently,

Figure 4.1: Typical mutual locations of σ_- and σ_+.

the function $q(x,t)$, defined by formula (4.105), has a continuous derivative with respect to t and satisfies (4.3) and

$$\pm\int_0^{\pm\infty}\left|\frac{\partial}{\partial t}\big(q(x,t)-p_\pm(x,t)\big)\right|(1+|x|^m)dx<\infty. \tag{4.107}$$

Moreover, the functions $\phi_\pm(\lambda,x,t)$, defined via (4.94), solve equation (4.8) with $q(x,t)$, defined by (4.105). To prove, that this $q(x,t)$ solves the KdV equation, we will apply Corollary 4.4 as follows.

Since $\phi_+(\lambda,x,t)$ and $\phi_-(\lambda,x,t)$ are independent for all $\lambda\in\mathbb{C}$ but a finite number of values, it is sufficient to check that both functions $(\mathcal{A}_q\phi_\pm)(\lambda,x,t)$ solve (4.8), where $\mathcal{A}_q$ is defined by (4.7) with $q(x,t)$ from (4.105). But due to (4.94) and the estimates (4.92), (4.93) we have (4.95) and (4.96). This implies one should show that

$$(\mathcal{A}_q\phi_\pm)(\lambda,x,t)=\beta_\pm(\lambda,t)\phi_\pm(\lambda,x,t), \tag{4.108}$$

for some $\beta_\pm(\lambda,t)$. Letting $x\to\pm\infty$ in (4.108) and comparing with

$$(\mathcal{A}_{p_\pm}\psi_\pm)(\lambda,x,t)=-\frac{\partial\alpha_\pm(\lambda,t)}{\partial t}\psi_\pm(\lambda,x,t) \tag{4.109}$$

(which is evident form Lemma 4.9), gives

$$\beta_\pm(\lambda,t)=-\frac{\partial\alpha_\pm(\lambda,t)}{\partial t}=-2(p_\pm(0,t)+2\lambda)m_\pm(\lambda,t)+\frac{\partial p_\pm(0,t)}{\partial x}. \tag{4.110}$$

Finally, as already pointed out before, (4.108) is equivalent to the KdV equation for $q(x,t)$ by Corollary 4.4. Equality (4.108) will be proved in the next section.

4.6 Justification of the inverse scattering transform

Our first task is to check, that if $\mathcal{S}(0)$ satisfies **I**–**III**, then the time-dependent scattering data $\mathcal{S}(t)$, defined by (4.99)–(4.101) satisfy the same conditions (with $g_\pm(\lambda)=g_\pm(\lambda,t)$). Properties **I**, **(a)**–**(f)** are straightforward to check. Using

$$g_\pm(\lambda,t)=g_\pm(\lambda,0)\mathrm{e}^{\alpha_\pm(\lambda,t)+\breve{\alpha}_\pm(\lambda,t)}, \tag{4.111}$$

which follows from (4.16) and (4.44), we see that $W(\lambda,t)$ defined as in (4.73) satisfies

$$W(\lambda,t)=W(\lambda,0)\mathrm{e}^{-\alpha_-(\lambda,t)-\alpha_+(\lambda,t)}. \tag{4.112}$$

Hence Lemma 4.10 implies that properties **II**, **(a)** and **(b)** hold.

Property **III**, **(a)** is evident, and property **III**, **(b)** follows from (4.46). In summary,

Lemma 4.18. *Let the set $\mathcal{S}(0)$ satisfy properties* **I**–**III** *and let the set $\mathcal{S}(t)$ be defined by* (4.99)–(4.101)*. Then the set $\mathcal{S}(t)$ satisfies* **I**–**III** *with $g_\pm(\lambda,t)$ defined by* (4.111).

Now substitute formulas (4.99)–(4.101), (4.35), (4.44), and (4.111) into (4.103), then we obtain the following representation for the kernels of GLM equations

$$\begin{aligned}F_\pm(x,y,t) =&\frac{1}{2\pi\mathrm{i}}\oint_{\sigma_\pm} R_\pm(\lambda,0)\,\hat\psi_\pm(\lambda,x,t)\hat\psi_\pm(\lambda,y,t)g_\pm(\lambda,0)d\lambda \\ &+\frac{1}{2\pi\mathrm{i}}\int_{\sigma_\mp^{(1),\mathrm{u}}}|T_\mp(\lambda,0)|^2\hat\psi_\pm(\lambda,x,t)\hat\psi_\pm(\lambda,y,t)g_\mp(\lambda,0)d\lambda \\ &+\sum_{k=1}^p(\gamma_k^\pm(0))^2\tilde\psi_\pm(\lambda_k,x,t)\tilde\psi_\pm(\lambda_k,y,t),\end{aligned} \tag{4.113}$$

where the functions

$$\tilde\psi_\pm(\lambda,x,t):=\delta_\pm(\lambda,0)\hat\psi_\pm(\lambda,x,t) \tag{4.114}$$

are well-defined (bounded, continuous) for $\lambda\in\mathbb{C}\setminus\sigma_\pm$. Recall that the functions $\hat\psi_\pm(\lambda,x,t)$ inherit all singularities from the functions $\psi_\pm(\lambda,x,0)$, that is, they have simple poles on the set $M_\pm(0)$, square-root singularities on the set $\hat M_\pm(0)$, and no other singularities. Therefore, formula (4.113) consists of three well-defined summands, the singularities of the integrands are integrable (cf. [10, Sect. 5]), and it remains to verify **IV**.

Due to our assumption (4.106) the second and third summands in (4.113) (or (4.103)) satisfies **IV** for all m and n, and hence we only need to investigate the first summand in (4.103). To this end, we use (4.18)–(4.20) to obtain the representation

$$\begin{aligned}F_{\pm,R}(x,y,t)&:=2\,\mathrm{Re}\int_{\sigma_\pm^{\mathrm{u}}}R_\pm(\lambda,t)\psi_\pm(\lambda,x,t)\psi_\pm(\lambda,y,t)\frac{g_\pm(\lambda,t)}{2\pi\mathrm{i}}d\lambda \\ &=\mathrm{Re}\int_0^\infty \mathrm{e}^{\pm\mathrm{i}(x+y)\theta_\pm}\rho_\pm(\theta_\pm,x,y,t)d\theta_\pm,\end{aligned} \tag{4.115}$$

where

$$\rho_\pm(\theta_\pm,x,y,t):=\frac{1}{2\pi}\Psi_\pm(\theta_\pm,x,y,t)\mathrm{e}^{\alpha_\pm(\lambda,t)-\breve\alpha_\pm(\lambda,t)}R_\pm(\lambda,0), \tag{4.116}$$

$$\Psi_\pm(\theta_\pm,x,y,t):=u_\pm(\lambda,x,t)u_\pm(\lambda,y,t)\prod_{j=1}^{r_\pm}\frac{\lambda-\mu_j^\pm(t)}{\lambda-\zeta_j^\pm}, \tag{4.117}$$

and $\lambda=\lambda(\theta_\pm)$. We will integrate (4.115) by parts m times for arbitrary m. Since the integrand is not continuous for $\theta_\pm\in[0,\infty)$, we regard this integral as

$$F_{\pm,R}(x,y,t)=\mathrm{Re}\sum_{k=0}^{r_\pm}\int_{\theta_\pm(E_{2k}^\pm)}^{\theta_\pm(E_{2k+1}^\pm)}\mathrm{e}^{\pm\mathrm{i}(x+y)\theta}\rho_\pm(\theta,x,y,t)d\theta, \tag{4.118}$$

where we set $E_{2r_\pm+1}^\pm=+\infty$ for notational convenience. Then the boundary terms during integration by parts will be

$$\mathrm{Re}\lim_{\lambda\to E}\frac{e^{\pm\mathrm{i}\theta_\pm(E)(x+y)}\frac{\partial^s\rho_\pm(\lambda(\theta_\pm),x,y,t)}{\partial\theta_\pm^s}}{(\mathrm{i}(x+y))^{s+1}},\quad s=0,1,\dots,\ E\in\partial\sigma_\pm, \tag{4.119}$$

and we will prove that they vanish for all s=0,1,....

Lemma 4.19. *Let $E \in \partial\sigma_\pm$. The following limits exists for all $s = 0, 1, \ldots$ and take real or pure imaginary values:*

$$\lim_{\lambda\to E,\, \lambda\in\sigma_\pm} \frac{d^s}{d\theta_\pm^s} R_\pm(\lambda(\theta_\pm), 0) \in \mathrm{i}^s\,\mathbb{R}, \tag{4.120}$$

$$e^{\pm\mathrm{i}\,\theta_\pm(E)\,(x+y)} \lim_{\lambda\to E} \frac{\partial^s}{\partial\theta_\pm^s} \Psi_\pm(\theta_\pm, x, y, t) \in \mathrm{i}^s\mathbb{R}, \tag{4.121}$$

$$\lim_{\lambda\to E} \frac{\partial^s}{\partial\theta_\pm^s} \exp\{\alpha_\pm(\lambda, t) - \breve{\alpha}_\pm(\lambda, t)\} \in \mathrm{i}^s\mathbb{R}. \tag{4.122}$$

Proof. The proof is the same for $+$ and $-$ cases, we will give it for $+$ case and omit the sign $+$ in notations, except of notation for spectrum σ_+.

Let ε be a positive value smaller than the minimal length of all bands in σ_+ and abbreviate

$$\mathcal{O}(E) = (E-\varepsilon, E+\varepsilon) \cap \sigma_+.$$

Let

$$\mathcal{F}(E) = C^\infty(\mathcal{O}(E), \mathbb{R})$$

be the class of all functions $f(\lambda)$ which are smooth and real-valued on $\mathcal{O}(E)$ and let

$$\mathcal{G}(E) = \{f_1(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_2(\lambda) \,|\, f_1, f_2 \in \mathcal{F}(E)\}.$$

From (4.20) we see that $\frac{d\lambda}{d\theta}$ is a real-valued and bounded function on the set σ_+ and $\frac{d\lambda}{d\theta}(E) = 0$. This function is smooth with respect to θ on the set $\mathcal{O}(E)$. From (4.19) we conclude, that

$$\frac{d^2\lambda}{d\theta^2} = \frac{d}{d\lambda}\left(\frac{\mathrm{i}\,Y^{1/2}(\lambda)}{\prod(\lambda-\zeta_j)}\right) \frac{\mathrm{i}\,Y^{1/2}(\lambda)}{\prod(\lambda-\zeta_j)} \in \mathcal{F}(E) \text{ and } \left(\frac{d\lambda}{d\theta}\right)^2 \in \mathcal{F}(E). \tag{4.123}$$

In particular, the last two formulas imply that $\mathcal{G}(E)$ is an algebra. Moreover, from (4.123) it follows, that

$$\frac{d^{2k}\lambda}{d\theta^{2k}}(E) \in \mathbb{R}, \quad \frac{d^{2k+1}\lambda}{d\theta^{2k+1}}(E) = 0. \tag{4.124}$$

Now let

$$g(\lambda) = f_1(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_2(\lambda) \in \mathcal{G}(E), \tag{4.125}$$

then (4.123) shows that

$$\frac{dg(\lambda)}{d\theta} = \mathrm{i}\left(\frac{df_2}{d\lambda}\left(\frac{d\lambda}{d\theta}\right)^2 + f_2\frac{d^2\lambda}{d\theta^2} - \mathrm{i}\frac{df_1}{d\lambda}\frac{d\lambda}{d\theta}\right) \in \mathrm{i}\mathcal{G}(E). \tag{4.126}$$

Hence (4.125) and (4.126) imply

$$\frac{d^s g}{d\theta^s}(E) \in \mathrm{i}^s\mathbb{R}, \quad s = 0, 1, \ldots, \tag{4.127}$$

where the values are to be understood as limits at E from within the spectrum. In particular, for any $f(\lambda) \in \mathcal{F}(E)$,

$$\frac{d^{2k} f}{d\theta^{2k}}(E) \in \mathbb{R}, \quad \frac{d^{2k+1} f}{d\theta^{2k+1}}(E) = 0, \quad k = 0, 1, \ldots . \tag{4.128}$$

The idea of the proof of (4.120) and (4.121) is to write $R(\lambda, 0)$ and

$$\hat{\Psi}(\theta, x, y, t) := \psi(\lambda, x, t)\psi(\lambda, y, t) \prod_{j=1}^{r} \frac{\lambda - \mu_j(t)}{\lambda - \zeta_j} \tag{4.129}$$

in the form (4.125). We start with $\hat{\Psi}(\theta)$ (where x, y, t play the role of parameters). From (4.28), (4.22), (4.24), and (4.25) we see, that the function $\frac{H(\lambda,0,t)}{G(\lambda,0,t)}$ is a holomorphic function in a vicinity of E even if $\mu_j(t) = E$. Thus,

$$\frac{H(\lambda, 0, t)}{G(\lambda, 0, t)} \in \mathcal{F}(E). \tag{4.130}$$

Since $\zeta_j \in (E_{2j-1}, E_{2j})$, then $\prod(\lambda - \zeta_j)^{-1} \in \mathcal{F}$. Also $s(\lambda, x, t)$, $c(\lambda, x, t) \in \mathcal{F}(E)$. Using in (4.129) the representations (4.13), (4.27), and (4.28) we conclude that the function $\hat{\Psi}(\theta, x, y, t)$ admits a representation of the type (4.125). Therefore

$$\lim_{\lambda \to E} \frac{\partial^s}{\partial \theta^s} \hat{\Psi}(\theta, x, y, t) \in \mathrm{i}^s \mathbb{R}, \quad s = 0, 1, \ldots . \tag{4.131}$$

Note that in this formula it is in fact irrelevant from what side the limit is taken.

Now consider the function $\Psi(\lambda, x, y, t)$ defined by formula (4.117). As is known (cf.[3], [45]) for each t and λ this function is a quasiperiodic bounded function with respect to x and y. Therefore, if its derivatives with respect to the quasimomentum variable exist, then they will be bounded with respect to x and y. Taking into account (4.131) we obtain

$$\lim_{\lambda \to E} \frac{\partial^s}{\partial \theta^s} \Psi(\theta, x, y, t) = U_s(E, x, y, t) \mathrm{e}^{-\mathrm{i}\theta(E)(x+y)},$$

where $U_s(E, x, y, t) \in \mathrm{i}^s \mathbb{R}$, $s = 0, 1, \ldots$, are functions which are bounded with respect to $x, y \in \mathbb{R}$ for each t. This proves (4.117). Note that $\mathrm{e}^{-\mathrm{i}\theta(E)(x+y)}$ has modulus one, but it is in general not real-valued.

To prove (4.120) we will distinguish the resonant and nonresonant cases. We start with nonresonant case $\hat{W}(E, t) \neq 0$ (cf. **II, (b)** and note that by (4.102) this is independent of t).

Suppose, that $E \in \partial\sigma_+ \cap \partial\sigma^{(2)}$ is a left edge of the spectrum σ, that is,

$$E = E_{2j}^+ = E_{2k}^-. \tag{4.132}$$

Consider the reflection coefficient $R_+(\lambda, 0)$, defined by formula (4.71) and let $\theta := \theta_+$. Suppose, that $\mu_j^+(0) \neq E$, $\mu_k^-(0) \neq E$. Then from (4.13), (4.130), (4.62), (4.67),(4.2), and (4.20) we see, that the Jost solution $\phi_+(\lambda, x)$ plus its derivative $\frac{\partial}{\partial x}\phi_+(\lambda, x)$ is in $\mathcal{G}(E)$. Moreover, by (4.20) and (4.132)

$$\frac{d\theta_+}{d\theta_-} = \frac{d\theta_+}{d\lambda}\frac{d\lambda}{d\theta_-} \in \frac{\sqrt{(\lambda - E_{2k}^-)(\lambda - E_{2k+1}^-)}}{\sqrt{(\lambda - E_{2j}^-)(\lambda - E_{2j+1}^-)}} \mathcal{F}(E) = \mathcal{F}(E)).$$

Therefore, the same is true for $\phi_-(\lambda, x)$ and hence we also have

$$\mathsf{W}(\phi_-(\lambda), \phi_+(\lambda)),\ \mathsf{W}(\phi_-(\lambda), \overline{\phi_+(\lambda)}) \in \mathcal{G}(E)$$

Since $\mathsf{W}(\phi_-,\, \phi_+)(E) \neq 0$ we conclude $R_+(\lambda, 0) \in \mathcal{G}(E)$ and (4.120) is proven in this case.

If $\mu_j^+(0) \neq E$ but $\mu_k^-(0) = E$ we replace $\phi_-(\lambda, x)$ by

$$\phi_-^{(1)}(\lambda, x) := \mathrm{i}\frac{d\lambda}{d\theta}\, \phi_-(\lambda, x)$$

which is in $\mathcal{G}(E)$ and proceed as before (observe that the extra factor cancels in the definition of $R_+(\lambda, 0)$). The cases $\mu_j^+(0) = E$, $\mu_k^-(0) \neq E$ and $\mu_j^+(0) = \mu_k^-(0) = E$ can be handled similarly.

In the nonresonant case, when $E \in \partial\sigma_+^{(1)} \cap \partial\sigma$ the consideration are even simpler, because in this case (cf. (4.68)) $\tilde{\phi}_-(\lambda, x) \in \mathcal{F}(E)$. We assume $\mu_j^+(0) \neq E$, if $\mu_j^+(0) = E$ one only needs to replace $\phi_+(\lambda)$ by $\phi_-^{(1)}(\lambda)$ as pointed out before. Thus

$$R_+(\lambda, 0) = \frac{f_1(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_2(\lambda)}{f_3(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_4(\lambda)}, \qquad \text{where } f_i(\lambda) \in \mathcal{F}(E),\ i = 1, 2, 3, 4. \tag{4.133}$$

This finishes the proof of formula (4.120) in the nonresonant case, because in this case we have $f_3(E) \neq 0$ and, therefore $R_+(\lambda, 0) \in \mathcal{G}(E)$.

In the resonance case we have $\hat{W}(E) = 0$ but $\frac{d\hat{W}}{d\theta}(E) \neq 0$ (cf. **II, (b)**). Hence we have (4.133) with $f_1(E) = f_3(E) = 0$ and $f_4(E) \neq 0$. Let us show that the derivative of the right-hand side of (4.133) satisfies

$$\frac{d}{d\theta} \frac{f_1(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_2(\lambda)}{f_3(\lambda) + \mathrm{i}\frac{d\lambda}{d\theta} f_4(\lambda)} \in \mathrm{i}\mathcal{G}(E). \tag{4.134}$$

Namely, denote by dot the derivative with respect to θ and by prime - with respect to λ. Then

$$\frac{d}{d\theta} \frac{g_1(\lambda) + \mathrm{i}\dot{\lambda} g_2(\lambda)}{g_3(\lambda) + \mathrm{i}\dot{\lambda} g_4(\lambda)} = \mathrm{i}\Big(\ddot{\lambda}(g_2 g_3 - g_4 g_1) + (\dot{\lambda})^2 (g_1' g_4 - g_3' g_2 + g_2' g_3 - g_4' g_1) +$$

$$+ \mathrm{i}\dot{\lambda}\Big(g_3' g_1 - g_1' g_3 + (\dot{\lambda})^2 (g_2' g_4 - g_4' g_2)\Big)\Big) \Big(-(\dot{\lambda})^2\, g_4^2 + g_3^2 + \mathrm{i}\dot{\lambda}(2 g_4 g_3)\Big)^{-1}.$$

Functions g_1, g_3 and $(\dot{\lambda})^2$ have zeros of the first order with respect to λ at the point E and $g_4(E)\ddot{\lambda}(E) \neq 0$. It means, that we can divide nominator and denominator in the r.h.s. of the last equality by $(\dot{\lambda})^2$ and using (4.123) we arrive at (4.134). The last one implies (4.120) for $s \geq 1$. To prove the remaining case $s = 0$ we have to check that $R_+(E, 0) \in \mathbb{R}$ in the resonance case. Since the nominator and denominator in (4.133) vanishes,

$$\lim_{\lambda \to E} R_+(\lambda, 0) = \lim_{\lambda \to E} \frac{(f_1' + \mathrm{i} f_2)\dot{\lambda} + \mathrm{i}\ddot{\lambda} f_2}{(f_3' + \mathrm{i} f_4)\dot{\lambda} + \mathrm{i}\ddot{\lambda} f_4} = \frac{f_2(E)}{f_4(E)} \in \mathbb{R}.$$

this completes the proof of (4.120).

To prove (4.122) we use the same approach. Again the prove will be done for the + case. From (4.46) it follows, that

$$\lim_{\lambda\to E}\exp\left(\alpha_+(\lambda,t)-\overline{\alpha_+}(\lambda,t)\right)\in\mathbb{R},$$

therefore it suffices to show that for

$$h(\lambda):=(\alpha_+(\lambda,t)-\overline{\alpha_+}(\lambda,t))$$

the derivative $\dot h(\lambda)=\frac{dh}{d\theta}$ satisfies

$$\dot h(\lambda)=\mathrm{i}f(\lambda),\quad f(\lambda)\in\mathcal{F}(E). \tag{4.135}$$

To simplify notations, we will omit sign + until the end of this lemma.

Suppose first, that

$$\mu_j(t)\neq E=E_{2j},\quad \mu_j(0)\neq E \tag{4.136}$$

Let $0<t_1<...<t_N<t$ be the set of points, where $\mu_j(t_k)=E$. Choose $\delta>0$ so small, that

$$\mu_j(E\pm\delta)>\max\{\mu_j(0),\mu_j(t),(E_{2j-1}+E)/2\}.$$

Denote

$$\Delta=[0,t]\setminus\cup_{k=1}^N(t_k-\delta,t_k+\delta).$$

Let $\lambda>E$ be a point in the spectrum, close to E. Then for $s\in\Delta$ $|\mu_j(s)-\lambda|>const(E)>0$ we have (see (4.47))

$$4Y^{1/2}(\lambda)\int_\Delta\frac{p_\pm(0,s)+2\lambda}{G_\pm(\lambda,s)}ds=\mathrm{i}\dot\lambda f_1(\lambda),\quad f_1\in\mathcal{F}(E). \tag{4.137}$$

On the remaining set we use the representations (4.56) and (4.57). Proceeding as in (4.58) we obtain

$$4Y^{1/2}(\lambda)\int_{t_k-\delta}^{t_k}\frac{p_+(0,s)+2\lambda}{G_+(\lambda,s)}ds=-\sigma_j\mathrm{i}\left(\arctan\frac{\sqrt{E-\mu_j(t_k-\delta)}}{\sqrt{\lambda-E}}\right)+$$

$$+\sqrt{\lambda-E}\int_{t_k-\delta}^{t_k}\frac{\partial}{\partial\lambda}\breve G_j(\xi_j(s,\lambda),s)ds,\quad \sigma_j\in\{-1,1\}, \tag{4.138}$$

where $\xi(\lambda,s)\in\mathcal{F}(E)$ such that $\mu_j(t_k-\delta)\le\xi(\lambda,s)\le\lambda$ for $t_k-\delta\le s\le t_k$. Furthermore, note that the function

$$\breve G(\xi,s)=\frac{Y^{1/2}(\xi)}{\sqrt{\xi-E}\prod_{l\neq j}(\xi-\mu_l)}$$

is smooth with respect to ξ in the domain $\mu_j(t_k-\delta)\le\xi\le\lambda$ and takes pure imaginary values there. Namely,

$$\begin{array}{llll} Y^{1/2}(\xi)\in\mathrm{i}\mathbb{R}, & \sqrt{\xi-E}\in\mathbb{R} & \text{for} & E\le\xi\le\lambda,\\ Y^{1/2}(\xi)\in\mathbb{R}, & \sqrt{\xi-E}\in\mathrm{i}\mathbb{R} & \text{for} & \mu_j(t_k-\delta)\le\xi\le E.\end{array}$$

Thus,

$$\frac{\partial^s \breve{G}(\xi, s)}{\partial \xi^s} \in \mathrm{i}\mathbb{R} \quad \text{for} \quad \mu_j(t_k - \delta) \le \xi \le \lambda, \quad s = 0, 1, \ldots. \tag{4.139}$$

The same considerations show

$$\sqrt{\lambda - E} = \dot{\lambda} f_2(\lambda) \quad \text{where} \quad f_2(\lambda) \in \mathcal{F}(E),\, f(E) \ne 0. \tag{4.140}$$

Combining this with (4.139) we obtain

$$\sqrt{\lambda - E} \int_{t_k-\delta}^{t_k} \frac{\partial}{\partial \lambda} \breve{G}_j(\xi_j(s, \lambda), s) ds = \mathrm{i}\dot{\lambda}\, f_3(\lambda), \quad f_3(\lambda) \in \mathcal{F}(E).$$

Thus

$$\frac{d}{d\theta} \left(\sqrt{\lambda - E} \int_{t_k-\delta}^{t_k} \breve{G}_j'(\xi_j(s, \lambda), s) ds \right) = \mathrm{i} f_4(\lambda), \quad f_4(\lambda) \in \mathcal{F}(E). \tag{4.141}$$

Using (4.140) one can also represent the argument of arctan in the first summand of (4.138) as $\frac{f_5(\lambda)}{\dot{\lambda}}$, where $f_5(\lambda) \in \mathcal{F}(E)$ and $f_5(E) \ne 0$. Therefore,

$$-\sigma_j \mathrm{i} \frac{d}{d\theta} \left(\arctan \frac{\sqrt{E - \mu_j(t_k - \delta)}}{\sqrt{\lambda - E}} \right) = -\sigma_j \mathrm{i} \frac{f_5'(\dot{\lambda})^2 - \ddot{\lambda}\, f_5}{(\dot{\lambda})^2 + f_5^2} \in \mathrm{i}\mathcal{F}(E). \tag{4.142}$$

The same is valid for the interval $(t_k, t_k + \delta)$. Combining (4.137), (4.141), and (4.142) we obtain (4.135). These considerations also show that the restriction (4.136) is unessential. □

Our next goal is to prove formula (4.108). Since for any solution of the equation $L_v(t)u = \lambda u$ the equality $\mathcal{A}_v u = u_t - P_v(t)u$ is valid, it suffices to prove the following

Lemma 4.20. *Let $K_\pm(x, y, t)$ be the solutions of the GLM equations* (4.104) *with the kernels* (4.103)*, corresponding to the scattering data* (4.99)–(4.101)*. Let the functions $\phi_\pm(\lambda, x, t)$ be defined by* (4.94) *and let $q(x, t)$ be defined by* (4.105)*. Then $\phi_\pm(\lambda, x, t)$ satisfy*

$$\Big(\frac{\partial}{\partial t} - P_q(t) \Big) \phi_\pm(\lambda, x, t) = \beta_\pm(\lambda, t) \phi_\pm(\lambda, x, t), \tag{4.143}$$

where $\beta_\pm(\lambda, t)$ is defined by (4.110).

Proof. As before we prove this lemma only for the $+$ case. To simplify notations, set $P = P_q(t)$, $P_0 = P_+(t)$, $\phi = \phi_+(\lambda, x, t)$, $\psi = \psi_+(\lambda, x, t)$, $p = p_+$,

$$(\mathcal{K} f)(x, t) = \int_x^{+\infty} K_+(x, y, t) f(y, t) dy$$

$$(\dot{\mathcal{K}} f)(x, t) = \int_x^{+\infty} \frac{\partial}{\partial t} K_+(x, y, t) f(y, t) dy, \tag{4.144}$$

and denote by a dot the derivative with respect to t and by a prime the derivative with respect to spatial variables. Moreover, we will omit the variable t whenever it is possible and use the notations

$$D_{x^l y^m}(x) := \left(\frac{\partial^l}{\partial x^l} + \frac{\partial^m}{\partial y^m}\right) D(x,y)|_{y=x}, D_{x^0 y^0}(x) = D(x).$$

Since $\dot{\psi} - P_0\psi = \beta\psi$, then

$$\dot{\phi} - P\phi = \beta\phi + (P_0 - P)\psi + \dot{\mathcal{K}}\psi + \mathcal{K}P_0\psi - P\mathcal{K}\psi. \tag{4.145}$$

Differentiating the last term and integrating by parts gives

$$\begin{aligned}(P\mathcal{K}\psi)(x) = &\{-2(q'(x) - p'(x)) + 4K_{xy}(x) + 8K_{x^2}(x) - 6q(x)K(x)\}\,\psi(x)\\ &- \{4(q(x) - p(x)) - 4K_x(x)\}\,\psi'(x) + 4K(x)\psi''(x) +\\ &+ \int_x^\infty (-4K_{x^3}(x,y) + 6q(x)K_x(x,y) + 3q'(x)K(x,y))\,\psi(y)dy,\end{aligned} \tag{4.146}$$

and

$$\begin{aligned}(\mathcal{K}P_0\psi)(x) = &\big(4K_{y^2}(x) - 6K(x)p(x)\big)\,\psi(x) - 4K_y(x)\psi'(x) + 4K(x)\psi''(x)\\ &+ \int_x^\infty \big(4K_{y^3}(x,y) - 6K_y(x,y)p(y) - 3K(x,y)p'(y)\big)\,\psi(y)dy.\end{aligned} \tag{4.147}$$

Besides,

$$(P - P_0)\psi(x) = 6(q(x) - p(x))\psi'(x) + 3(q'(x) - p'(x))\psi(x). \tag{4.148}$$

Combining (4.144)–(4.148) and taking into account the formula (cf. [38])

$$-K_{xx}(x,y) + q(x)K(x,y) = -K_{yy}(x,y) + p(y)K(x,y), \tag{4.149}$$

where we put $x = y$, we arrive at the representation

$$(\dot{\phi} - P\phi - \beta\phi)(x) = A(x)\psi(x) + B(x)\psi'(x) + \int_x^\infty (\tau^{xy}K(x,y))\psi(y)dy = 0, \tag{4.150}$$

where

$$\begin{aligned}A(x) &= p'(x) - q'(x) - 2K_{x^2}(x) - 4K_{xy}(x) - 2K_{y^2}(x),\\ B(x) &= 2(p(x) - q(x)) - 4(K_x(x) + K_y(x)),\end{aligned}$$

and

$$\tau^{xy} := \frac{\partial}{\partial t} + \tau_q^x + \tau_p^y, \quad \tau_q^x := 4\frac{\partial^3}{\partial x^3} - 6q(x)\frac{\partial}{\partial x} - 3q'(x). \tag{4.151}$$

But according to (4.105)

$$p(x) - q(x) = 2K_x(x) + 2K_y(x), \quad p'(x) - q'(x) = 2K_{x^2}(x) + 4K_{xy}(x) + 2K_{y^2}(x), \tag{4.152}$$

and therefore, $A(x) = B(x) = 0$. Thus, to prove (4.143) one has to check, that

$$\begin{aligned} D(x,y) :=& \tau^{xy} K(x,y) = K_t(x,y) + 4K_{y^3}(x,y) + 4K_{x^3}(x,y) - 6q(x)K_x(x,y) \\ &- 6p(y)K_y(x,y) - 3q'(x)K(x,y) - 3p'(y)K(x,y) \equiv 0. \end{aligned} \tag{4.153}$$

To this end, let us derive an equation for the function $F = F_+(x,y,t)$, defined by formula (4.103). This function can be represented (see (4.113)) as

$$F(x,y,t) = \int_{\mathbb{R}} \hat{\psi}(\lambda,x,t)\hat{\psi}(\lambda,y,t)d\rho(\lambda),$$

where the measure

$$\begin{aligned} d\rho(\lambda) =& \Big(\frac{1}{\pi \mathrm{i}} R_+(\lambda,0) g_+(\lambda,0)\chi_{\sigma_+^u}(\lambda) + \frac{1}{2\pi\mathrm{i}} |T_-(\lambda,0)|^2 g_-(\lambda,0)\chi_{\sigma_-^{(1)}}(\lambda) \\ &+ \sum_k (\gamma_k^+)^2(0)\delta(\lambda-\lambda_k)\delta_+(\lambda_k,0)^2\Big)d\lambda \end{aligned}$$

does not depend on t. Using (4.38) we conclude, that

$$\tau_0^{xy} F(x,y) = 0, \quad \tau_0^{xy} = \frac{\partial}{\partial t} + \tau_p^x + \tau_p^y. \tag{4.154}$$

Now set $V(x) = q(x) - p(x)$ and apply the operator τ^{xy} to the GLM equation (4.104). Taking into account (4.153), (4.154) and the equality

$$\tau^{xy} - \tau_0^{xy} = -6V(x)\frac{\partial}{\partial x} - 3V'(x)$$

we obtain

$$\begin{aligned} D(x,y) =& \int_x^\infty \big\{K(x,s)\tau_p^s\,[F(s,y)] - K_t(x,s)F(s,y)\big\}\,ds \\ &- \tau_q^x\left[\int_x^\infty K(x,s)F(s,y)ds\right] + 6V(x)F_x(x,y) + 3V'(x)F(x,y), \end{aligned}$$

or

$$D(x,y) + \int_x^\infty D(x,s)F(s,y)ds = r(x,y), \tag{4.155}$$

where

$$r(x,y) = \int_x^\infty \big\{\tau_p^s\,[K(x,s)]\,F(s,y) + K(x,s)\tau_p^s\,[F(s,y)]\big\}\,ds+ \tag{4.156}$$

$$+\int_x^\infty \tau_q^x\,[K(x,s)]\,F(s,y)ds - \tau_q^x\left[\int_x^\infty K(x,s)F(s,y)ds\right] +$$

$$+6V(x)F_x(x,y) + 3V'(x)F(x,y).$$

It is proved in [10], that the equation $D(x,y)+\int_x^\infty D(x,s)F(s,y)ds = 0$, where x plays the role of a parameter, has only the trivial solution in the space $L^1(x,\infty)$. Since the function $D(x,\cdot)$ evidently belongs to this space, then to prove (4.153) it is sufficient to prove that $r(x,y) = 0$.

Taking into account, that $V(x) = -2\frac{d}{dx}K(x,x)$, direct computations imply

$$\int_x^\infty \tau_q^x\left[K(x,s)\right]F(s,y)ds - \tau_q^x\left[\int_x^\infty K(x,s)F(s,y)ds\right] + \tag{4.157}$$
$$+6V(x)F_x(x,y) + 3V'(x)F(x,y) = 4K(x,x)F_{x^2}(x,y)+$$
$$+4K_x(x,x)F_x(x,y) + 8K_{x^2}(x,x)F(x,y) + 4K_{xy}(x,x)F(x,y)+$$
$$+V'(x)F(x,y) + 2V(x)F_x(x,y) - 6q(x)K(x,x)F(x,y).$$

From the other side, integration by parts gives

$$\int_x^\infty \left\{\tau_p^s\left[K(x,s)\right]F(s,y) + K(x,s)\tau_p^s\left[F(s,y)\right]\right\}ds = \tag{4.158}$$
$$= -4\left\{K_{s^2}(x,s)F(x,y) + K(x,x)F_{s^2}(s,y) - K_s(x,s)F_s(s,y)\right\}|_{s=x}+$$
$$+6p(x)K(x,x)F(x,y).$$

Substituting last to formulas to (4.156) gives

$$r(x,y) = F_x(x,y)(4K_x(x,x) + 4K_y(x,x) + 2V(x))+$$
$$+F(x,y)\left(-6V(x)K(x,x) + 8K_{x^2}(x,x) + 4K_{xy}(x,x) - 4K_{y^2}(x,x) + V'(x)\right).$$

Taking into account (4.152) we obtain

$$r(x,y) = F(x,y)\left(-6V(x)K(x,x) + 6K_{x^2}(x,x) - 6K_{y^2}(x,x)\right),$$

and (4.149) implies $r(x,y) = 0$. □

4.7 Appendix

In this section we thoroughly investigate the integral equations for the kernels $K_\pm(x,y,t)$ of the transformation operators. We will obtain the necessary estimates for them and their derivatives with respect to t, x and y. This will allow us to state the necessary and sufficient conditions on the functions $F_\pm(x,y,t)$ and to solve the scattering problem in the prescribed class of perturbations (4.2).

Throughout this section we will assume that

$$\pm\int_0^{\pm\infty}\left|\frac{\partial^n}{\partial x^n}\big(q(x,t) - p_\pm(x,t)\big)\right|(1+|x|^m)dx < \infty, \quad \forall m,n \in \mathbb{N}\cup\{0\}, \tag{4.159}$$

for all $t \in \mathbb{R}$. First we recall some facts for the operator kernel $K_\pm(x,y,t)$ which have been proved in [10]:

Lemma 4.21. *The kernels $K_\pm(x,y,t)$ of the transformation operators satisfy the integral equation*

$$K_\pm(x,y,t) = -2\int_{\frac{x+y}{2}}^{\pm\infty} q_\pm(s,t)D_\pm(x,s,s,y,t)ds$$
$$\mp 2\int_x^{\pm\infty} ds\int_{y+s-x}^{y+x-s} D_\pm(x,s,r,y,t)K_\pm(s,r,t)q_\pm(s,t)ds, \quad \pm y > \pm x, \tag{4.160}$$

where

$$D_\pm(x,y,r,s,t) = \mp\frac{1}{4}\sum_{E\in\partial\sigma_\pm}\frac{f_\pm(E,x,y,t)f_\pm(E,r,s,t)}{\frac{d}{dz}Y_\pm(E)}, \tag{4.161}$$

with

$$f_\pm(E,x,y,t) = \lim_{z\to E} G_\pm(z,0,t)\psi_\pm(z,x,t)\breve{\psi}_\pm(z,y,t), \tag{4.162}$$

$$q_\pm(x,t) = q(x,t) - p_\pm(x,t). \tag{4.163}$$

In particular,

$$K_\pm(x,x,t) = \pm\frac{1}{2}\int_x^{\pm\infty}(q(s,t)-p_\pm(s,t))ds. \tag{4.164}$$

Lemma 4.22. *Assume* (4.159)*. Then*

$$\frac{\partial^{n+l}}{\partial x^l\partial y^n}f_\pm(E,x,y,t)\in L^\infty(\mathbb{R}\times\mathbb{R}),$$

for any fixed $t\in\mathbb{R}$*.*

Proof. It is well known that the background Weyl solutions can be represented as

$$\psi_\pm(z,x,t) = c_\pm(z,x,t) + m_\pm(z,t)s_\pm(z,x,t), \tag{4.165}$$

where $c_\pm(z,x,t)$ and $s_\pm(z,x,t)$ satisfy the initial conditions $c(z,0,t) = s_x(z,0,t) = 1$ and $c_x(z,0,t) = s(z,0,t) = 0$. Furthermore they are solutions of

$$-\frac{d^2}{dx^2}y_\pm(x,t) + p_\pm(x,t)y_\pm(x,t) = zy_\pm(x,t) \tag{4.166}$$

Thus one can conclude by differentiating this expressions that

$$c_\pm^{(2)}(z,x,t) = (p_\pm(x)-z)c_\pm(z,x,t), \tag{4.167}$$

$$c_\pm^{(3)}(z,x,t) = (p_\pm(x)-z)c_\pm^{(1)}(z,x,t) + p_\pm'(x,t)c_\pm(z,x,t), \tag{4.168}$$

$$s_\pm^{(2)}(z,x,t) = (p_\pm(x)-z)s_\pm(z,x,t), \tag{4.169}$$

$$s_\pm^{(3)}(z,x,t) = (p_\pm(x)-z)s_\pm^{(1)}(z,x,t) + p_\pm'(x,t)s_\pm(z,x,t), \tag{4.170}$$

which also implies that every derivative of $c_\pm(z,x,t)$ (resp. $s_\pm(z,x,t)$) with respect to x can be written as a combination of $c_\pm(z,x,t)$, $c_\pm^{(1)}(z,x,t)$ (resp. $s_\pm(z,x,t)$, $s_\pm^{(1)}(z,x,t)$) and the derivatives of $p_\pm(x,t)$ with respect to x. Thus knowing that all derivatives with respect to x of $p_\pm(x,t)$ are uniformly bounded for $x\in\mathbb{R}$ and $t\in\mathbb{R}$ fixed which can be obtained from

$$p_\pm(x,t) = \sum_{j=0}^{2r_\pm}E_j^\pm - 2\sum_{j=1}^{r_\pm}\mu_j^\pm(x,t), \tag{4.171}$$

we only have to show that $f_\pm(E,x,y,t)$, $\frac{df_\pm(E,x,y)}{dx}$, and $\frac{df_\pm(E,x,y,t)}{dx}$ are uniformly bounded with respect to x and y for any fixed time t. Therefore we will use the following representation of the background Weyl solutions

$$\psi_\pm(z,x,t) = \exp\left(\int_0^x m_\pm(z,y,t)dy\right), \tag{4.172}$$

$$\breve{\psi}_\pm(z,x,t) = \exp\left(\int_0^x \breve{m}_\pm(z,y,t)dy\right). \tag{4.173}$$

This can be rewritten as

$$\psi_\pm(z,x,t) = \left(\frac{G_\pm(z,x,t)}{G_\pm(z,0,t)}\right)^{1/2} \exp\left(\pm\int_0^x \frac{Y_\pm^{1/2}(z)}{G_\pm(z,\tau,t)}d\tau\right), \tag{4.174}$$

$$\breve{\psi}_\pm(z,x,t) = \left(\frac{G_\pm(z,x,t)}{G_\pm(z,0,t)}\right)^{1/2} \exp\left(\mp\int_0^x \frac{Y_\pm^{1/2}(z)}{G_\pm(z,\tau,t)}d\tau\right). \tag{4.175}$$

In the next step we want to show that $\int_0^x \frac{Y_\pm^{1/2}(z)}{G\pm(z,\tau,t)}d\tau$ is purely imaginary as $z \to E_{2j}^\pm$ (the case $z \to E_{2j-1}^\pm$ can be handled in the same way). For fixed $t \in \mathbb{R}$ we can separate the interval $[0,x]$ into smaller intervals $[0,x_1]\cup[x_1,x_2]\cup\dots\cup[x_k,x]$ such that $x_l \in \{E_{2j-1}^\pm, E_{2j}^\pm\}$ and $x_l \neq x_{l+1}$. On each of these intervals the function $\sigma_j(x,t)$ is constant, thus we can conclude

$$\int_{x_l}^{x_{l+1}} \frac{Y_\pm^{1/2}(z)}{G(z,\tau,t)}d\tau = \int_{x_l}^{x_{l+1}} \frac{Y_j^\pm(z,\tau,t)}{z-\mu_j^\pm(\tau,t)}d\tau = \sqrt{z-E_{2j}^\pm}\int_{x_l}^{x_{l+1}} \frac{\tilde{Y}_j^\pm(z,\tau,t)}{z-\mu_j^\pm(\tau,t)}d\tau$$
$$= \sqrt{z-E_{2j}^\pm}\left(\int_{x_l}^{x_{l+1}} \frac{\tilde{Y}_j^\pm(\mu_j^\pm(\tau,t),\tau,t)}{z-\mu_j^\pm(\tau,t)}d\tau + \int_{x_l}^{x_{l+1}} \frac{\tilde{Y}_j^\pm(z,\tau,t)-\tilde{Y}_j^\pm(\mu_j^\pm(\tau,t),\tau,t)}{z-\mu_j^\pm(\tau,t)}d\tau\right)$$
$$= \sqrt{z-E_{2j}^\pm}(\int_{x_l}^{x_{l+1}} \frac{-\frac{d\mu_j^\pm(\tau,t)}{d\tau}}{2\sigma_j^\pm(\tau,t)\sqrt{\mu_j^\pm(\tau,t)-E_{2j}^\pm}(z-\mu_j^\pm(\tau,t))}d\tau$$
$$+\int_{x_l}^{x_{l+1}} \frac{d}{dz}\tilde{Y}_j^\pm(z,\tau,t)_{|z=\mu_j^\pm(s,t)})$$
$$= \sqrt{z-E_{2j}^\pm}(-\int_{\mu_j^\pm(x_l,t)}^{\mu_j^\pm(x_{l+1},t)} \frac{1}{2\sigma_j^\pm(\tau,t)\sqrt{y-E_{2j}^\pm}(z-y)}dy$$
$$+\int_{x_l}^{x_{l+1}} \frac{d}{dz}\tilde{Y}_j^\pm(z,\tau,t)_{|z=\mu_j^\pm(s,t)})$$
$$= \sqrt{E_{2j}^\pm - z}\left(\int_0^{E_{2j}^\pm-E_{2j-1}^\pm} \frac{1}{\sigma_j^\pm(\tau,t)(z-E_{2j}^\pm+s^2)}ds\right) + \sqrt{z-E_{2j}^\pm}M$$
$$= \sigma_j^\pm \mathrm{i}\arctan\left(\frac{\sqrt{E_{2j}^\pm - E_{2j-1}^\pm}}{z-E_{2j}^\pm}\right) + \sqrt{z-E_{2j}^\pm}M,$$

where we define $Y_j^\pm(z,x,t) = \sqrt{z-E_{2j}^\pm}\tilde{Y}_j^\pm(z,x,t)$. This implies that $\int_{x_l}^{x_{l+1}} \frac{Y_\pm^{1/2}(z)}{G(z,\tau,t)}d\tau \to \frac{1}{2}\sigma_j^\pm \mathrm{i}\pi$ as $z \to E_{2j}^\pm$ and thus $\int_0^x \frac{Y_\pm^{1/2}(z)}{G\pm(z,\tau,t)}d\tau \in \mathrm{i}\mathbb{R}$. Combining all the informations we have proved that

$$|f_\pm(E,x,y,t)| = |(G_\pm(E,x,t)G_\pm(E,y,t))^{1/2}| \tag{4.176}$$

and thus $f_{\pm}(E,x,y,t)$ is uniformly bounded with respect to x and y for fixed $t \in \mathbb{R}$. Returning to our representation we can conclude that

$$\begin{aligned}
&\frac{d}{dx}\psi_{\pm}(z,x,y) = m_{\pm}(z,x,t)\psi_{\pm}(z,x,t)\\
&\quad = \left(\frac{H_{\pm}(z,x,t) \pm Y_{\pm}^{1/2}(z)}{G_{\pm}(z,x,t)}\right)\left(\frac{G_{\pm}(z,x,t)}{G_{\pm}(z,0,t)}\right)^{1/2} \exp\left(\pm \int_0^x \frac{Y_{\pm}^{1/2}(z)}{G \pm (z,\tau,t)} d\tau\right).
\end{aligned}$$

Thus we can write

$$\frac{d}{dx} f_{\pm}(z,x,y,t) = \left(\frac{H_{\pm}(z,x,t) \pm Y_{\pm}^{1/2}(z)}{G_{\pm}(z,x,t)^{1/2}}\right) G_{\pm}(z,y,t)^{1/2} \exp\left(\pm \int_y^x \frac{Y_{\pm}^{1/2}(z)}{G \pm (z,\tau,t)} d\tau\right),$$

which also implies that $\frac{d}{dx} f_{\pm}(E,x,y,t)$ is uniformly bounded with respect to x and y for any fixed $t \in \mathbb{R}$, where we use the well known fact that $\frac{H_{\pm}(z,x,t) \pm Y_{\pm}^{1/2}(z)}{G_{\pm}(z,x,t)^{1/2}}$ is uniformly bounded. Thus the claim follows by combining all the informations we obtained so far. □

Lemma 4.23. *Assume (*4.159*). Then*

$$\frac{\partial^{n+l+1}}{\partial x^l \partial y^n \partial t} f_{\pm}(E,x,y,t) \in L^{\infty}(\mathbb{R} \times \mathbb{R} \times [0,T)),$$

for some constant $T \in [0,\infty)$.

Proof. We already know that every derivative with respect to x and y is uniformly bounded for any fixed $t \in \mathbb{R}$. To proof our assumption remember the following lemma. The functions

$$\hat{\psi}_{\pm}(z,x,t) = \mathrm{e}^{\alpha_{\pm}(z,t)}\psi_{\pm}(z,x,t), \tag{4.177}$$

where

$$\alpha_{\pm}(z,t) := \int_0^t \left(2(p_{\pm}(0,s) + 2z)m_{\pm}(z,s) - \frac{\partial p_{\pm}(0,s)}{\partial x}\right) ds, \tag{4.178}$$

satisfy the system of equations

$$L_{\pm}(t)\hat{\psi}_{\pm} = z\hat{\psi}_{\pm}, \tag{4.179}$$

$$\frac{\partial \hat{\psi}_{\pm}}{\partial t} = P_{\pm}(t)\hat{\psi}_{\pm}, \tag{4.180}$$

where

$$L_{p_{\pm}}(t) = -\partial_x^2 + p_{\pm}(x,t), \tag{4.181}$$

$$P_{p_{\pm}}(t) = -4\partial_x^3 + 6p_{\pm}(x,t)\partial_x + 3p_{\pm,x}(x,t). \tag{4.182}$$

This implies

$$\begin{aligned}
\frac{d\psi_{\pm}(z,x,t)}{dt} &= -4\psi_{\pm}^{(3)}(z,x,t) + 6p_{\pm}(x,t)\psi_{\pm}^{(1)}(z,x,t) \\
&\quad + 3p_{\pm}^{(1)}(x,t)\psi_{\pm}(z,x,t) - \dot{\alpha}(z,t)\psi_{\pm}(z,x,t),
\end{aligned} \tag{4.183}$$

with

$$\dot{\alpha}(z,t) = (2p_\pm(0,t) + 4z)m_\pm(z,t) - \frac{\partial p_\pm(0,t)}{\partial x}. \tag{4.184}$$

Similarly one obtains: The functions

$$\hat{\breve{\psi}}_\pm(z,x,t) = \mathrm{e}^{\breve{\alpha}_\pm(z,t)}\breve{\psi}_\pm(z,x,t), \tag{4.185}$$

where

$$\breve{\alpha}_\pm(z,t) := \int_0^t \left(2(p_\pm(0,s) + 2z)\breve{m}_\pm(z,s) - \frac{\partial p_\pm(0,s)}{\partial x} \right) ds, \tag{4.186}$$

satisfy the system of equations

$$L_\pm(t)\hat{\breve{\psi}}_\pm = z\hat{\breve{\psi}}_\pm, \tag{4.187}$$

$$\frac{\partial \hat{\breve{\psi}}_\pm}{\partial t} = P_\pm(t)\hat{\breve{\psi}}_\pm, \tag{4.188}$$

where

$$L_{p_\pm}(t) = -\partial_x^2 + p_\pm(x,t), \tag{4.189}$$

$$P_{p_\pm}(t) = -4\partial_x^3 + 6p_\pm(x,t)\partial_x + 3p_{\pm,x}(x,t). \tag{4.190}$$

Thus we can conclude that the only critical term of $\frac{d}{dt}\psi_\pm(E,x,t)$ is given by $2(p_\pm(0,t) + 2E)m_\pm(E,0,t)\psi_\pm(E,x,t)$ and of $\frac{d}{dt}\breve{\psi}_\pm(E,x,t)$ it is given by $2(p_\pm(0,t) + 2E)\breve{m}_\pm(E,0,t)\breve{\psi}_\pm(E,x,t)$. Using now (4.23) and (4.25), we obtain that the critical terms cancel out in $\frac{df_\pm(E,x,y,t)}{dt}$ and therefore we obtain that

$$\frac{df_\pm(E,x,y)}{dt} \in L^\infty(\mathbb{R} \times \mathbb{R} \times [0,T)). \tag{4.191}$$

Computing now $\frac{d^2\psi_\pm(z,y,t)}{dxdt}$ using (4.183) yields

$$\begin{aligned}\frac{d^2\psi_\pm(z,x,t)}{dxdt} &= -4\psi_\pm^{(4)}(z,x,t) + 6p_\pm(x,t)\psi_\pm^{(2)}(z,x,t) + 3p_\pm^{(1)}(x,t)\psi_\pm^{(1)}(z,x,t)\\ &\quad - \dot{\alpha}(z,t)\psi_\pm^{(1)}(z,x,t) + 6p_\pm^{(1)}(x,t)\psi_\pm^{(1)}(z,x,t) + 3p_\pm^{(2)}(x,t)\psi_\pm(z,x,t).\end{aligned}$$

Hence the critical term of $\frac{d^2\psi_\pm(E,x,t)}{dxdt}$ is given by $2(p_\pm(0,t)+2E)m_\pm(E,0,t)\psi_\pm^{(1)}(E,x,t)$ and of $\frac{d^2\breve{\psi}_\pm(E,x,t)}{dxdt}$ is given by $2(p_\pm(0,t)+2E)\breve{m}_\pm(E,0,t)\breve{\psi}_\pm^{(1)}(E,x,t)$. This again implies, using (4.23) and (4.25), that the critical terms cancel out in $\frac{d^2f_\pm(E,x,y,t)}{dxdt}$. Thus we obtain

$$\frac{d^2f_\pm(E,x,y)}{dxdt} \in L^\infty(\mathbb{R} \times \mathbb{R} \times [0,T)) \tag{4.192}$$

.

Again knowing that the derivatives with respect to x and t are bounded, and combining all the informations we obtained so far, proofs the claim. $\square$

Lemma 4.24. *Let*

$$Q_\pm(x,t) = \pm\int_{\frac{x}{2}}^{\pm\infty} |q_\pm(s,t)|ds \quad \textit{and} \quad Q_{\pm,t}(x,t) = \pm\int_{\frac{x}{2}}^{\pm\infty} |q_{\pm,t}(s,t)|ds. \tag{4.193}$$

Then $K_\pm(x,y,t)$ has partial derivatives of any order with respect to both variables x and y. Moreover, for large x the following estimates are valid

$$|\frac{\partial^{m+n}}{\partial x^m \partial y^n} K_\pm(x,y,t)| \leq C_{\pm,m,n,0}(x,t)(Q_\pm(x+y,t) + \sum_{l=0}^{m+n-1} |q_\pm^{(l)}(\frac{x+y}{2},t)) \tag{4.194}$$

where $C_{\pm,m,n,0}(x,t)$ are positive continuous functions for $x \in \mathbb{R}$ which depend on the corresponding background data, on the first moment and on the derivatives of $q_\pm(x,t)$ for large x. For every such function there exists an $x_0 \in \mathbb{R}$ such that $C_{\pm,m,n,0}(x,t)$ is decreasing for all $x > x_0$.
Furthermore for large x we have

$$\begin{aligned} |\frac{\partial^{m+n+1}}{\partial x^m \partial y^n \partial t} K_\pm(x,y,t)| \leq C_{\pm,m,n,1}(x,t)(Q_\pm(x+y,t) + Q_{\pm,t}(x+y,t) + \\ + \sum_{l=0}^{m+n-1} (|q_\pm^{(l)}(\frac{x+y}{2},t)| + |q_{\pm,t}^{(l)}(\frac{x+y}{2},t)|)) \end{aligned} \tag{4.195}$$

where $C_{\pm,m,n,1}(x,t)$ inherit the same properties as $C_{\pm,m,n,0}(x,t)$.

Proof. We restrict our considerations to the + case and omit for the proof of the first part the time dependence. After the following change of variables

$$2\alpha := s + r,\ 2\beta := r - s,\ 2u := x + y,\ 2v := y - x, \tag{4.196}$$

(4.160) yields

$$\begin{aligned} H(u,v) = &-2\int_u^\infty q_+(s) D_1(u,v,s) ds \\ &- 4\int_u^\infty d\alpha \int_0^v q_+(\alpha-\beta) D_2(u,v,\alpha,\beta) H(\alpha,\beta) d\beta, \end{aligned} \tag{4.197}$$

with

$$\begin{aligned} H(u,v) = K_+(u-v,u+v), \quad D_1(u,v,s) = D_+(u-v,s,s,u+v), \\ D_2(u,v,\alpha,\beta) = D_+(u-v,\alpha-\beta,\alpha+\beta,u+v). \end{aligned} \tag{4.198}$$

A simple calculation shows that

$$|\frac{\partial^{m+n}}{\partial x^m \partial y^n} K_+(x,y)| \leq \frac{1}{2^{m+n}} \sum_{j=0}^{n+m} |\binom{m+n}{j} \frac{\partial^{m+n}}{\partial u^j \partial v^{n+m-j}} H(u,v)|. \tag{4.199}$$

Moreover, by induction one can formally show,

$$
\begin{aligned}
&\frac{\partial^{n+m}}{\partial v^n \partial u^m} H(u,v) = -2\int_u^\infty q_+(s)\frac{\partial^{n+m}}{\partial v^n \partial u^m} D_1(u,v,s)ds \\
&+2\sum_{l=0}^{m-1}\frac{\partial^l}{\partial u^l}(q_+(u)(\frac{\partial^{n-1+m-l}}{\partial v^n \partial u^{m-1-l}} D_1(u,v,s))_{s=u}) \\
&-4\int_u^\infty d\alpha \int_0^v q_+(\alpha-\beta)\frac{\partial^{n+m}}{\partial v^n \partial u^m} D_2(u,v,\alpha,\beta)H(\alpha,\beta)d\beta \\
&-4\int_u^\infty d\alpha (\sum_{k=0}^{n-1}\frac{\partial^k}{\partial v^k}(q_+(\alpha-\beta)(\frac{\partial^{m+n-1-k}}{\partial u^m \partial v^{n-1-k}} D_2(u,v,\alpha,\beta))_{\beta=v}H(\alpha,\beta))) \qquad (4.200)\\
&+\int_0^v \sum_{l=0}^{m-1}\frac{\partial^l}{\partial u^l}(q_+(u-\beta)(\frac{\partial^{n+m-1-l}}{\partial v^n \partial u^{m-1-l}} D_2(u,v,\alpha,\beta))_{\alpha=u}H(u,\beta))d\beta \\
&+4\sum_{k=0}^{m-1}\sum_{l=0}^{n-1}\frac{\partial^{l+k}}{\partial v^l \partial u^k}(q_+(u-v)(\frac{\partial^{m-1-k+n-1-l}}{\partial u^{m-1-k} \partial v^{n-1-l}} D_2(u,v,\alpha,\beta))_{\beta=v,\alpha=u}H(u,v)).
\end{aligned}
$$

As the functions D_1 and D_2 are bounded uniformly with respect to all their variables, we can apply the method of successive approximation to estimate $H(u,v)$, which is given by

$$|H(u,v)| \le C(u-v)Q_+(2u). \tag{4.201}$$

To obtain the other estimates observe that the partial derivatives with respect to all variables exist for D_1 and D_2 and they are again bounded with respect to all variables. Thus one can show

$$
\begin{aligned}
&|\frac{\partial}{\partial u}H(u,v)| \le C_1(u-v)(|q_+(u)| + Q_+(2u)), \qquad (4.202)\\
&|\frac{\partial}{\partial v}H(u,v)| \le C_1(u-v)(|q_+(u)| + Q_+(2u)),
\end{aligned}
$$

where $C(u-v)$ is a positive continuous function for $x = u-v \in \mathbb{R}$, which decreases for large x and depends on the corresponding background data. This is the starting point for the induction to show the claim for arbitrary derivatives with respect to x and y. Here we use that $q_+^{(n)}(x) \to 0$ as $x \to \infty$, which implies that there exists an $x_0 \in \mathbb{R}$ such that $q_+^{(n)}(x)$ is decreasing for all $x > x_0$, with $n \in \mathbb{N}$. Thus we obtain

$$|\frac{\partial^n}{\partial u^m \partial v^{n-m}}H(u,v)| \le C_{m,n-m,0}(u-v)(Q_\pm(2u) + \sum_{l=0}^{n-1}|q_+^{(l)}(u)|), \tag{4.203}$$

where $C(u-v)$ has the same properties $C_1(u-v)$, but also depends on the derivatives of $q_+(x)$ up to order $n-1$ for large x. Thus combining these estimates we arrive at (4.199). The same method can be used to obtain

$$|\frac{\partial^{n+1}}{\partial u^m \partial v^{n-m} \partial t} H(u,v,t)| \leq C_{m,n-m,1}(u-v,t)(Q_+(2u,t)+Q_{+,t}(2u,t)+ \\ \sum_{l=0}^{n-1}(|q_+^{(l)}(u,t)|+|q_{+,t}^{(l)}(u,t)|)), \tag{4.204}$$

which proves the second part of the claim, where again the function $C_{m,n-m,1}(u-v,t)$ is decreasing. □

As an immediate consequence of the last lemma we obtain

Corollary 4.25. *The functions $K_\pm(x,y,t)$ are infinitely many times differentiable with respect x and y, and*

$$\left|\frac{\partial^{l+n}}{\partial x^l \partial y^n} K_\pm(x,y,t)\right| \leq \frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y \to \pm\infty, \quad l,n,m = 0,1,2,\ldots, \tag{4.205}$$

Moreover they are also differentiable with respect to t and satisfy

$$\left|\frac{\partial^{n+1}}{\partial x^n \partial t} K_\pm(x,y,t)\right| \leq \frac{C_\pm(m,n,l,t)}{|x+y|^m}, \quad x,y \to \pm\infty, \quad l,n,m = 0,1,2,\ldots. \tag{4.206}$$

Proof. Again we restrict our considerations to the + case. We only need to apply (4.159) as follows

$$\int_{\frac{x+y}{2}}^{\infty} |q_+(s,t)|ds \leq \frac{2^m}{|x+y|^m} \int_0^\infty (1+|s|^m)|q_+(s,t)|ds < \infty, \tag{4.207}$$

and

$$|q_+^{(n)}(y,t)| \leq \frac{1}{|y|^m} \int_y^\infty (1+|s|^m)|q_+^{(n+1)}(s,t)|ds < \infty, \tag{4.208}$$

where we used that $q_+^{(n)}(x,t) \to 0$ as $x \to \infty$. Analogously we can show

$$|q_{+,t}^{(n)}(y,t)| \leq \frac{1}{|y|^m} \int_y^\infty (1+|s|^m)|q_{+,t}^{(n+1)}(s,t)|ds < \infty, \tag{4.209}$$

if we assume that

$$\int_0^\infty (1+|x|^m)|q_{+,t}^{(n)}(x,t)|dx < \infty, \quad \forall n,m \in \mathbb{N} \text{ and } t \in \mathbb{R}. \tag{4.210}$$

This proves the claim. □

With the help of this lemma we can now derive the corresponding estimates for the GLM equation.

Lemma 4.26. *The kernel $F_\pm(x,y,t)$ of the GLM equation (4.104) has partial derivatives of any order with respect to each variable. Furthermore, for large x it satisfies the following estimates*

$$|\frac{\partial^{m+n}}{\partial x^m \partial y^n} F_\pm(x,y,t)| \leq C_{\pm,0}(x,t)(Q_\pm(x+y,t) + \sum_{l=0}^{m+n-1} |q_\pm(\frac{x+y}{2},t)|), \quad (4.211)$$

and

$$|\frac{\partial^{m+n+1}}{\partial x^m \partial y^n \partial t} F_\pm(x,y,t)| \leq C_{\pm,1}(x,t)(Q_\pm(x+y,t) + Q_{\pm,t}(x+y,t) + \quad (4.212)$$
$$+ \sum_{l=0}^{m+n-1} |q_\pm^{(l)}(\frac{x+y}{2},t)| + |q_{\pm,t}^{(l)}(\frac{x+y}{2},t)|)$$

where the functions $q_\pm(x,t)$, $Q_\pm(x,t)$ and $Q_{\pm,t}(x,t)$ are defined as in Lemma 4.24 Again $C_{\pm,0}(x,t)$ and $C_{\pm,1}(x,t)$ are positive continuous functions which decrease as $x \to \pm\infty$.

Proof. Again we restrict our consideration to the $+$ case and omit for the proof of the first part the time dependence. We consider the GLM equation

$$K_+(x,y) + F_+(x,y) + \int_x^\infty K_+(x,s)F_+(s,y)ds. \quad (4.213)$$

Furthermore

$$\frac{\partial^n}{\partial y^n} F_+(x,y) = -\frac{\partial^n}{\partial y^n} K_+(x,y) - \int_x^\infty K(x,s)\frac{\partial^n}{\partial y^n} F_+(s,y)ds, \quad (4.214)$$

which implies that we must use in this cases the method of successive approximation to obtain

$$|\frac{\partial^n}{\partial y^n} F_+(x,y)| \leq C(x)(Q_+(x+y) + \sum_{l=0}^{n-1} |q_+^{(l)}(\frac{x+y}{2},t)|), \quad \forall n \in \mathbb{N}_0. \quad (4.215)$$

For the other derivatives one can show by induction that

$$\frac{\partial^{n+m}}{\partial x^n \partial y^m} F_+(x,y) = -\frac{\partial^{n+m}}{\partial x^n \partial y^m} K_+(x,y) + \sum_{k=0}^{n-1} \frac{\partial^k}{\partial x^k}((\frac{\partial^{n-1-k}}{\partial x^{n-1-k}} K_+(x,y))_{y=x} \frac{\partial^m}{\partial y^m} F_+(x,y))$$
$$- \int_x^\infty (\frac{\partial^n}{\partial x^n} K_+(x,s))\frac{\partial^m}{\partial y^m} F_+(s,y)ds. \quad (4.216)$$

Therefore one can now show by induction with respect to n that the statement is true for every fixed $y \in \mathbb{N}_0$. The same method can be used to proof the second part of the lemma. ☐

Remark 4.27. *Note the result of this lemma is in some sense invertible, as we can obtain the properties of the kernels $K_\pm(x,y,t)$ from the properties of the functions $F_\pm(x,y,t)$ by using (4.213) and (4.216).*

Chapter 5

Lipschitz metric for the periodic Camassa–Holm equation

5.1 Introduction

The ubiquitous Camassa–Holm (CH) equation [18, 19]

$$u_t - u_{xxt} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{5.1}$$

where $\kappa \in \mathbb{R}$ is a constant, has been extensively studied due to its many intriguing properties. The aim of this paper is to construct a metric that renders the flow generated by the Camassa–Holm equation Lipschitz continuous on a function space in the conservative case. To keep the presentation reasonably short, we restrict the discussion to properties relevant for the current study.

More precisely, we consider the initial value problem for (5.1) with periodic initial data $u|_{t=0} = u_0$. Since the function $v(t,x) = u(t, x - \kappa t/2) + \kappa/2$ satisfies equation (5.1) with $\kappa = 0$, we can without loss of generality assume that κ vanishes. For convenience we assume that the period is 1, that is, $u_0(x+1) = u_0(x)$ for $x \in \mathbb{R}$. The natural norm for this problem is the usual norm in the Sobolev space $H^1_{\rm per}$ as we have that

$$\frac{d}{dt}\|u(t)\|^2_{H^1_{\rm per}} = \frac{d}{dt}\int_0^1 \big(u^2 + u_x^2\big)dx = 2\int_0^1 \big(uu_t + u_x u_{xt}\big)dx = 0 \tag{5.2}$$

(by using the equation and several integration by parts as well as periodicity) for smooth solutions u. Even for smooth initial data, the solutions may develop singularities in finite time and this breakdown of solutions is referred to as wave breaking. At wave breaking the H^1 and L^∞ norms of the solution remain finite while the spatial derivative u_x becomes unbounded pointwise. This phenomenon can best be described for a particular class of solutions, namely the multipeakons. For simplicity we describe them on the full line, but similar results can be described in the periodic case. Multipeakons are solutions of the

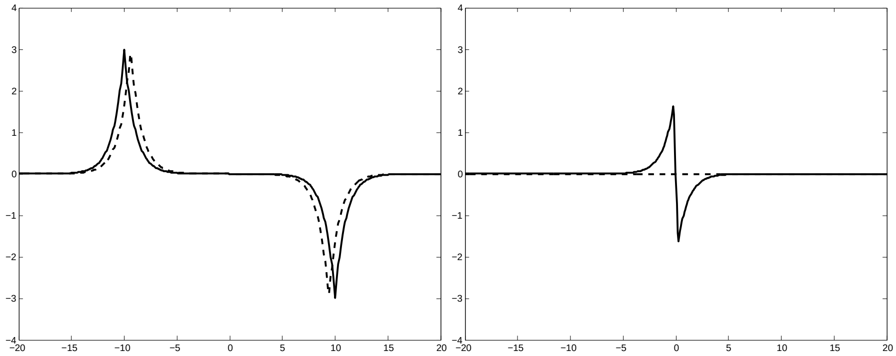

Figure 5.1: The dashed curve depicts the antisymmetric multipeakon solution $u(t,x)$, which vanishes at t^*, for $t=0$ (on the left) and $t=t^*$ (on the right). The solid curve depicts the multipeakon solution given by $u^\varepsilon(t,x)=u(t-\varepsilon,x)$.

form (see also [55])

$$u(t,x)=\sum_{i=1}^{n} p_i(t)e^{-|x-q_i(t)|}. \tag{5.3}$$

Let us consider the case with $n=2$ and one peakon $p_1(0)>0$ (moving to the right) and one antipeakon $p_2(0)<0$ (moving to the left). In the symmetric case ($p_1(0)=-p_2(0)$ and $q_1(0)=-q_2(0)<0$) the solution u will vanish pointwise at the collision time t^* when $q_1(t^*)=q_2(t^*)$, that is, $u(t^*,x)=0$ for all $x\in\mathbb{R}$. Clearly the well-posedness, in particular, Lipschitz continuity, of the solution is a delicate matter. Consider, e.g., the multipeakon u^ε defined as $u^\varepsilon(t,x)=u(t-\varepsilon,x)$, see Figure 5.1. For simplicity, we assume that $\|u(0)\|_{H^1}=1$. Then, we have

$$\lim_{\varepsilon\to 0}\|u(0)-u^\varepsilon(0)\|_{H^1}=0 \text{ and } \|u(t^*)-u^\varepsilon(t^*)\|_{H^1}=\|u^\varepsilon(t^*)\|_{H^1}=1,$$

and the flow is clearly not Lipschitz continuous with respect to the H^1 norm.

Our task is here to identify a metric, which we will denote by $d_{\mathbb{D}}$ for which conservative solutions satisfy a Lipschitz property, that is, if u and v are two solutions of the Camassa–Holm equation, then

$$d_{\mathbb{D}}(u(t),v(t))\le C_T d_{\mathbb{D}}(u_0,v_0),\quad t\in[0,T]$$

for any given, positive T. For nonlinear partial differential equations this is in general a quite nontrivial issue. Let us illustrate it in the case of hyperbolic conservation laws

$$u_t+f(u)_x=0,\quad u|_{t=0}=u_0.$$

In the scalar case with $u=u(x,t)\in\mathbb{R}$, $x\in\mathbb{R}$, it is well-known [52] that the solution is L^1-contractive in the sense that

$$\|u(t)-v(t)\|_{L^1(\mathbb{R})}\le\|u_0-v_0\|_{L^1(\mathbb{R})},\quad t\in[0,\infty).$$

In the case of systems, i.e., for $u\in\mathbb{R}^n$ with $n>1$ it is known [52] that

$$\|u(t)-v(t)\|_{L^1(\mathbb{R})}\le C\|u_0-v_0\|_{L^1(\mathbb{R})},\quad t\in[0,\infty),$$

for some constant C. More relevant for the current study, but less well-known, is the recent analysis [16] of the Hunter–Saxton (HS) equation

$$u_t + uu_x = \frac{1}{4}\Big(\int_{-\infty}^{x} u_x^2\,dx - \int_{x}^{\infty} u_x^2\,dx\Big), \quad u|_{t=0} = u_0, \tag{5.4}$$

or alternatively

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \quad u|_{t=0} = u_0, \tag{5.5}$$

which was first introduced in [58] as a model for liquid crystals. Again the equation enjoys wave breaking in finite time and the solutions are not Lipschitz in term of convex norms. The Hunter–Saxton equation can in some sense be considered as a simplified version of the Camassa–Holm equation, and the construction of the semigroup of solutions via a change of coordinates given in [16] is very similar to the one used here and in [56] for the Camassa–Holm equation. In [16] the authors constructed a Riemannian metric which renders the conservative flow generated by the Hunter–Saxton equation Lipschitz continuous on an appropriate function space.

For the Camassa–Holm equation, the problem of continuation beyond wave breaking has been considered by Bressan and Constantin [13, 14] and Holden and Raynaud [54, 56, 57] (see also Xin and Zhang [95, 96] and Coclite, Karlsen, and Holden [20, 21]). Both approaches are based on a reformulation (distinct in the two approaches) of the Camassa–Holm equation as a semilinear system of ordinary differential equations taking values in a Banach space. This formulation allows one to continue the solution beyond collision time, giving either a global conservative solution where the energy is conserved for almost all times or a dissipative solution where energy may vanish from the system. Local existence of the semilinear system is obtained by a contraction argument. Going back to the original function u, one obtains a global solution of the Camassa–Holm equation.

In [15], Bressan and Fonte introduce a new distance function $J(u, v)$ which is defined as a solution of an optimal transport problem. They consider two multipeakon solutions $u(t)$ and $v(t)$ of the Camassa–Holm equation and prove, on the intervals of times where no collisions occur, that the growth of $J(u(t), v(t))$ is linear (that is, $\frac{dJ}{dt}(u(t), v(t)) \le CJ(u(t), v(t))$ for some fixed constant C) and that $J(u(t), v(t))$ is continuous across collisions. It follows that

$$J(u(t), v(t)) \le e^{CT} J(u(0), v(0)) \tag{5.6}$$

for all times t that are not collision times and, in particular, for almost all times. By density, they construct solutions for any initial data (not just the multipeakons) and the Lipschitz continuity follows from (5.6). As in [15], the goal of this article is to construct a metric which makes the flow Lipschitz continuous. However, we base the construction of the metric directly on the reformulation of the equation which is used to construct the solutions themselves, and we use some fundamental geometrical properties of this reformulation (relabeling invariance, see below). The metric is defined on the set $\mathbb{D}$ which includes configurations where part of the energy is concentrated on sets of measure zero; a natural choice for conservative solutions. In particular, we obtain that the Lipschitz continuity holds for all times and not just for almost all times as in [15].

Let us describe in some detail the approach in this paper, which follows [56] quite closely in setting up the reformulated equation. Let $u = u(t, x)$ denote the solution, and $y(t, \xi)$ the corresponding characteristics, thus $y_t(t, \xi) = u(t, y(t, \xi))$. Our new variables are $y(t, \xi)$,

$$U(t, \xi) = u(t, y(t, \xi)), \quad H(t, \xi) = \int_{y(t,0)}^{y(t,\xi)} (u^2 + u_x^2)\, dx \tag{5.7}$$

where U corresponds to the Lagrangian velocity while H could be interpreted as the Lagrangian cumulative energy distribution. In the periodic case one defines

$$\begin{aligned} Q = {} & \frac{1}{2(e-1)} \int_0^1 \sinh(y(\xi) - y(\eta))(U^2 y_\xi + H_\xi)(\eta)\, d\eta \\ & - \frac{1}{4} \int_0^1 \operatorname{sign}(\xi - \eta) \exp\big(- \operatorname{sign}(\xi - \eta)(y(\xi) - y(\eta))\big)(U^2 y_\xi + H_\xi)(\eta)\, d\eta, \end{aligned} \tag{5.8}$$

$$\begin{aligned} P = {} & \frac{1}{2(e-1)} \int_0^1 \cosh(y(\xi) - y(\eta))(U^2 y_\xi + H_\xi)(\eta)\, d\eta \\ & + \frac{1}{4} \int_0^1 \exp\big(- \operatorname{sign}(\xi - \eta)(y(\xi) - y(\eta))\big)(U^2 y_\xi + H_\xi)(\eta)\, d\eta. \end{aligned} \tag{5.9}$$

Then one can show that

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = [U^3 - 2PU]_0^\xi, \end{cases} \tag{5.10}$$

is equivalent to the Camassa–Holm equation. Global existence of solutions of (5.10) is obtained starting from a contraction argument, see Theorem 5.5. The issue of continuation of the solution past wave breaking is resolved by considering the set $\mathbb{D}$ (see Definition 5.22) which consists of pairs (u, μ) such that $(u, \mu) \in \mathbb{D}$ if $u \in H^1_{\mathrm{per}}$ and μ is a positive Radon measure with period one, and whose absolutely continuous part satisfies $\mu_{\mathrm{ac}} = (u^2 + u_x^2)\, dx$. With three Lagrangian variables (y, U, H) versus two Eulerian variables (u, μ), it is clear that there can be no bijection between the two coordinate systems. If two Lagrangian variables correspond to one and the same solution in Eulerian variables, we say that the Lagrangian variables are relabelings of each other. To resolve the relabeling issue we define a group of transformations which acts on the Lagrangian variables and lets the system of equations (5.10) invariant. We are able to establish a bijection between the space of Eulerian variables and the space of Lagrangian variables when we identify variables that are invariant under the action of the group. This bijection allows us to transform the results obtained in the Lagrangian framework (in which the equation is well-posed) into the Eulerian framework (in which the situation is much more subtle). To obtain a Lipschitz metric in Eulerian coordinates we start by constructing one in the Lagrangian setting. To this end we start by identifying a set $\mathcal{F}$ (see Definition 5.2) that leaves the flow (5.10) invariant, that is, if $X_0 \in \mathcal{F}$ then the solution $X(t)$ of (5.10) with $X(0) = X_0$ will remain in $\mathcal{F}$, i.e., $X(t) \in \mathcal{F}$. Next, we identify a subgroup G, see Definition 5.6, of the group of homeomorphisms on the unit interval, and we interpret G as the set of relabeling functions. From this we define a natural group action of G on $\mathcal{F}$, that is, $\Phi(f,X) = X \bullet f$ for $f \in G$ and $X \in \mathcal{F}$, see Definition 5.6 and Proposition 5.8. We can then consider the quotient space $\mathcal{F}/G$. However, we still have to identify a unique element in $\mathcal{F}$ for each equivalence class in $\mathcal{F}/G$. To this end we introduce the set $\mathcal{H}$, see (5.64), of elements in $\mathcal{F}$ for which $\int_0^1 y(\xi)d\xi = 0$ and $y_\xi + H_\xi = 1 + \|H_\xi\|_{L^1}$. This establishes a bijection between $\mathcal{F}/G$ and $\mathcal{H}$, see Lemma 5.10, and therefore between $\mathcal{H}$ and $\mathbb{D}$. Finally, we define a semigroup $\bar{S}_t(X_0) = X(t)$ on $\mathcal{H}$ (Definition 5.12), and the next task is to identify a metric that makes the flow $\bar{S}_t$ Lipschitz continuous on $\mathcal{H}$. We use the bijection between $\mathcal{H}$ and $\mathbb{D}$ to transport the metric from $\mathcal{H}$ to $\mathbb{D}$ and get a Lipschitz continuous flow on $\mathbb{D}$.

In [56], the authors define the metric on $\mathcal{H}$ by simply taking the norm of the underlying Banach space (the set $\mathcal{H}$ is a nonlinear subset of a Banach space). They obtain in this way a metric which makes the flow continuous but not Lipschitz continuous. As we will see (see Remark 5.20), this metric is stronger than the one we construct here and for which the flow is Lipschitz continuous. In [16], for the Hunter–Saxton equation, the authors use ideas from Riemannian geometry and construct a semimetric which identifies points that belong to the same equivalence class. The Riemannian framework seems however too rigid for the Camassa–Holm equation, and we have not been able to carry out this approach. However, we retain the essential idea which consists of finding a semimetric which identifies equivalence classes. Instead of a Riemannian metric, we use a discrete counterpart. Note that this technique will also work for the Hunter–Saxton and will give the same metric as in [16]. A natural candidate for a semimetric which identifies equivalence classes is (cf. (5.71))

$$J(X,Y) = \inf_{f,g \in G} \|X \bullet f - Y \bullet g\|,$$

which is invariant with respect to relabeling. However, it does not satisfy the triangle inequality. Nevertheless it can be modified to satisfy all the requirements for a metric if we instead define, see Definition 5.14, the following quantity[1]

$$d(X,Y) = \inf \sum_{i=1}^{N} J(X_{n-1}, X_n) \tag{5.11}$$

where the infimum is taken over all finite sequences $\{X_n\}_{n=0}^N \in \mathcal{F}$ which satisfy $X_0 = X$ and $X_N = Y$. One can then prove that $d(X,Y)$ is a metric on $\mathcal{H}$, see Lemma 5.19. Finally, we prove that the flow is Lipschitz continuous in this metric, see Theorem 5.21. To transfer this result to the Eulerian variables we reconstruct these variables from the Lagrangian coordinates as in [56]: Given $X \in \mathcal{F}$, we define $(u,\mu) \in \mathbb{D}$ by (see Definition 5.24) $u(x) = U(\xi)$ for any ξ such that $x = y(\xi)$, and $\mu = y_\#(\nu d\xi)$. We denote the mapping from $\mathcal{F}$ to $\mathbb{D}$ by M, and the inverse restricted to $\mathcal{H}$ by L. The natural metric on $\mathbb{D}$, denoted $d_\mathbb{D}$, is then defined by $d_\mathbb{D}((u,\mu),(\tilde{u},\tilde{\mu})) = d(L(u,\mu), L(\tilde{u},\tilde{\mu}))$ for two elements $(u,\mu),(\tilde{u},\tilde{\mu})$ in $\mathbb{D}$, see Definition 5.28. The main theorem, Theorem 5.30, then states that the metric $d_\mathbb{D}$ is Lipschitz continuous on all states with finite energy. In the last section, Section 5.6, the metric is compared with the standard norms. Two results are proved: The mapping $u \mapsto (u, (u^2 + u_x^2)dx)$ is continuous from H^1_{per} into $\mathbb{D}$ (Proposition 5.31). Furthermore, if (u_n, μ_n) is a sequence in $\mathbb{D}$ that converges to (u,μ) in $\mathbb{D}$. Then $u_n \to u$ in L^∞_{per} and $\mu_n \overset{*}{\rightharpoonup} \mu$ (Proposition 5.32).

[1] This idea is due to A. Bressan (private communication).

The problem of Lipschitz continuity can nicely be illustrated in the simpler context of ordinary differential equations. Consider three differential equations:

$$\dot{x} = a(x), \qquad x(0) = x_0, \quad a \text{ Lipschitz}, \tag{5.12a}$$
$$\dot{x} = 1 + \alpha H(x), \qquad x(0) = x_0, \quad H \text{ the Heaviside function}, \alpha > 0, \tag{5.12b}$$
$$\dot{x} = |x|^{1/2}, \qquad x(0) = x_0, \qquad t \mapsto x(t) \text{ strictly increasing}. \tag{5.12c}$$

Straightforward computations give as solutions

$$x(t) = x_0 + \int_0^t a(x(s))\, ds, \tag{5.13a}$$
$$x(t) = (1 + \alpha H(t - t_0))(t - t_0), \quad t_0 = -x_0/(1 + \alpha H(x_0)), \tag{5.13b}$$
$$x(t) = \operatorname{sign}\big(\frac{t}{2} + v_0\big)\big(\frac{t}{2} + v_0\big)^2 \text{ where } v_0 = \operatorname{sign}(x_0)|x_0|^{1/2}. \tag{5.13c}$$

We find that

$$|x(t) - \bar{x}(t)| \le e^{Lt}|x_0 - \bar{x}_0|, \quad L = \|a\|_{\mathrm{Lip}}, \tag{5.14a}$$
$$|x(t) - \bar{x}(t)| \le (1 + \alpha)|x_0 - \bar{x}_0|, \tag{5.14b}$$
$$x(t) - \bar{x}(t) = t(x_0 - \bar{x}_0)^{1/2} + |x_0 - \bar{x}_0|, \quad \text{when } \bar{x}_0 = 0,\, t > 0,\, x_0 > 0. \tag{5.14c}$$

Thus we see that in the regular case (5.12a) we get a Lipschitz estimate with constant e^{Lt} uniformly bounded as t ranges on a bounded interval. In the second case (5.12b) we get a Lipschitz estimate uniformly valid for all $t \in \mathbb{R}$. In the final example (5.12c), by restricting attention to strictly increasing solutions of the ordinary differential equations, we achieve uniqueness and continuous dependence on the initial data, but without any Lipschitz estimate at all near the point $x_0 = 0$. We observe that, by introducing the Riemannian metric

$$d(x, \bar{x}) = \Big|\int_x^{\bar{x}} \frac{dz}{|z|^{1/2}}\Big|, \tag{5.15}$$

an easy computation reveals that

$$d(x(t), \bar{x}(t)) = d(x_0, \bar{x}_0). \tag{5.16}$$

Let us explain why this metric can be considered as a Riemannian metric. The Euclidean metric between the two points is then given

$$|x_0 - \bar{x}_0| = \inf_x \int_0^1 |x_s(s)|\, ds \tag{5.17}$$

where the infimum is taken over all paths $x\colon [0,1] \to \mathbb{R}$ that join the two points x_0 and $\bar{x}_0$, that is, $x(0) = x_0$ and $x(1) = \bar{x}_0$. However, as we have seen, the solutions are not Lipschitz for the Euclidean metric. Thus we want to measure the infinitesimal variation x_s in an alternative way, which makes solutions of equation (5.12c) Lipschitz continuous. We look at the evolution equation that governs x_s and, by differentiating (5.12c) with respect to s, we get

$$\dot{x}_s = \frac{\operatorname{sign}(x) x_s}{2\sqrt{|x|}},$$

and we can check that

$$\frac{d}{dt}\left(\frac{|x_s|}{\sqrt{|x|}}\right) = 0. \tag{5.18}$$

Let us consider the real line as a Riemannian manifold where, at any point $x \in \mathbb{R}$, the Riemannian norm is given by $|v|/\sqrt{|x|}$ for any tangent vector $v \in \mathbb{R}$ in the tangent space of x. From (5.18), one can see that at the infinitesimal level, this Riemannian norm is exactly preserved by the evolution equation. The distance on the real line which is naturally inherited by this Riemannian is given by

$$d(x_0, \bar{x}_0) = \inf_x \int_0^1 \frac{|x_s|}{\sqrt{|x|}}\, ds$$

where the infimum is taken over all paths $x\colon [0,1] \to \mathbb{R}$ joining x_0 and $\bar{x}_0$. It is quite reasonable to restrict ourselves to paths that satisfy $x_s \geq 0$ and then, by a change of variables, we recover the definition (5.15).

The Riemannian approach to measure a distance between any two distinct points in a given set (as defined in (5.17)) requires the existence of a smooth path between points in the set. In the case of the Hunter–Saxton (see [16]), we could embed the set we were primarily interested in into a convex set (which is therefore connected) and which also could be regularized (so that the Riemannian metric we wanted to use in that case could be defined). In the case of the Camassa–Holm equation, we have been unable to construct such a set. However, there exists the alternative approach which, instead of using a smooth path to join points, uses finite sequences of points, see (5.11). We illustrate this approach with equation (5.12c). We want to define a metric in $(0,\infty)$ which makes the semigroup of solutions Lipschitz stable. Given two points $x, \bar{x} \in (0,\infty)$, we define the function $J\colon (0,\infty) \times (0,\infty) \to [0,\infty)$ as

$$J(x,\bar{x}) = \begin{cases} \frac{x-\bar{x}}{\bar{x}^{1/2}} & \text{if } x \geq \bar{x}, \\ \frac{\bar{x}-x}{x^{1/2}} & \text{if } x < \bar{x}. \end{cases}$$

The function J is symmetric and $J(x,\bar{x}) = 0$ if and only if $x = \bar{x}$, but J does not satisfy the triangle inequality. Therefore we define (cf. (5.11))

$$d(x,\bar{x}) = \inf \sum_{n=0}^{N} J(x_n, x_{n+1}) \tag{5.19}$$

where the infimum is taken over all finite sequences $\{x_n\}_{n=0}^N$ such that $x_0 = x$ and $x_N = \bar{x}$. Then, d satisfies the triangle inequality and one can prove that it is also a metric. Given $x_n, x_{n+1} \in E$ such that $x_n \leq x_{n+1}$, we denote $x_n(t)$ and $x_{n+1}(t)$ the solution of (5.12c) with initial data x_n and x_{n+1}, respectively. After a short computation, we get

$$\frac{d}{dt} J(x_n(t), x_{n+1}(t)) = -\frac{1}{2x_n}(x_n + x_{n+1} - 2\sqrt{x_n x_{n+1}}) \leq 0.$$

Hence, $J(x_n(t), x_{n+1}(t)) \leq J(x_n, x_{n+1})$ so that

$$d(x(t), \bar{x}(t)) \leq d(x,\bar{x})$$

and the semigroup of solutions to (5.12c) is a contraction for the metric d. It follows from the definition of J that, for $x_1, x_2, x_3 \in E$ with $x_1 < x_2 < x_3$, we have

$$J(x_1, x_2) + J(x_2, x_3) < J(x_1, x_3). \tag{5.20}$$

It implies that $d(x, \bar{x})$ satisfies

$$d(x, \bar{x}) = \inf_{\delta} \sum_{n=0}^{N} J(x_n, x_{n+1})$$

where $\delta = \min_n |x_{n+1} - x_n|$, which is also the definition of the Riemann integral, so that

$$d(x, \bar{x}) = \int_x^{\bar{x}} \frac{1}{\sqrt{z}}\, dz$$

and the metric we have just defined coincides with the Riemannian metric we have introduced. Note that if we choose

$$\bar{J}(x, \bar{x}) = \begin{cases} \frac{x-\bar{x}}{x^{1/2}} & \text{if } x \geq \bar{x} \\ \frac{\bar{x}-x}{\bar{x}^{1/2}} & \text{if } x < \bar{x}, \end{cases}$$

then (5.20) does not hold; we have instead $\bar{J}(x_1, x_3) < \bar{J}(x_1, x_2) + \bar{J}(x_2, x_3)$, which is the triangle inequality. Thus, for $\bar{d}$ as defined by (5.19) with J replaced by $\bar{J}$, we get

$$\bar{d}(x, \bar{x}) = \bar{J}(x, \bar{x}) \neq \int_x^{\bar{x}} \frac{1}{\sqrt{z}}\, dz.$$

It is also possible to check that, for $\bar{J}$, we cannot get that $\bar{J}(x_n(t), x_{n+1}(t)) \leq C\bar{J}(x_n, x_{n+1})$ for any constant C for any x_n and x_{n+1} and $t \in [0, T]$ (for a given T), so that the definition of $\bar{J}$ is inappropriate to obtain results of stability for (5.12c).

5.2 Semi-group of solutions in Lagrangian coordinates

The Camassa–Holm equation for $\kappa = 0$ reads

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{5.21}$$

and can be rewritten as the following system[2]

$$u_t + uu_x + P_x = 0, \tag{5.22}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{5.23}$$

We consider periodic solutions of period one. Next, we rewrite the equation in Lagrangian coordinates. Therefore we introduce the characteristics

$$y_t(t, \xi) = u(t, y(t, \xi)). \tag{5.24}$$

[2]For κ nonzero, equation (5.22) is simply replaced by $P - P_{xx} = \kappa u + u^2 + \frac{1}{2}u_x^2$.

We introduce the space V_1 defined as

$$V_1 = \{f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}) \mid f(\xi+1) = f(\xi) + 1 \,\text{for all } \xi \in \mathbb{R}\}.$$

Functions in V_1 map the unit interval into itself in the sense that if u is periodic with period 1, then $u \circ f$ is also periodic with period 1. The Lagrangian velocity U reads

$$U(t,\xi) = u(t, y(t,\xi)). \tag{5.25}$$

We will consider $y \in V_1$ and U periodic. We define the Lagrangian energy cumulative distribution as

$$H(t,\xi) = \int_{y(t,0)}^{y(t,\xi)} (u^2 + u_x^2)(t,x)\,dx. \tag{5.26}$$

For all t, the function H belongs to the vector space V defined as follows:

$$V = \{f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}) \mid \text{there exists } \alpha \in \mathbb{R} \\ \text{such that } f(\xi+1) = f(\xi) + \alpha \text{ for all } \xi \in \mathbb{R}\}.$$

Equip V with the norm

$$\|f\|_V = \|f\|_{L^\infty([0,1])} + \|f_\xi\|_{L^1([0,1])}.$$

As an immediate consequence of the definition of the characteristics we obtain

$$U_t(t,\xi) = u_t(t,y) + y_t(t,\xi)u_x(t,y) = -P_x \circ y(t,\xi). \tag{5.27}$$

This last term can be expressed uniquely in term of U, y, and H. We have the following explicit expression for P,

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(u^2(t,z) + \frac{1}{2}u_x^2(t,z))\,dz. \tag{5.28}$$

Thus,

$$P_x \circ y(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(y(t,\xi)-z)e^{-|y(t,\xi)-z|}(u^2(t,z) + \frac{1}{2}u_x^2(t,z))\,dz,$$

and, after the change of variables $z = y(t,\eta)$,

$$P_x \circ y(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \Big[\operatorname{sign}(y(t,\xi)-y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|} \\ \times \Big(u^2(t,y(t,\eta)) + \frac{1}{2}u_x^2(t,y(t,\eta))\Big)\, y_\xi(t,\eta)\Big]\,d\eta. \tag{5.29}$$

We have

$$H_\xi = (u^2 + u_x^2)\circ y y_\xi =: \nu. \tag{5.30}$$

Note that ν is periodic with period one. Then, (5.29) can be rewritten as

$$P_x \circ y(\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(y(\xi)-y(\eta))\exp(-|y(\xi)-y(\eta)|)\big(U^2 y_\xi + \nu\big)(\eta)\,d\eta, \tag{5.31}$$

where the t variable has been dropped to simplify the notation. Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x \circ y$ is equivalent to Q where

$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi-\eta)\exp\big(-\operatorname{sign}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + \nu\big)(\eta)\,d\eta, \tag{5.32}$$

and, slightly abusing the notation, we write

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sign}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + \nu\big)(\eta)\,d\eta. \tag{5.33}$$

The derivatives of Q and P are given by

$$Q_\xi = -\frac{1}{2}\nu - \left(\frac{1}{2}U^2 - P\right) y_\xi \quad\text{and}\quad P_\xi = Q y_\xi, \tag{5.34}$$

respectively. For $\xi \in [0,1]$, using the fact that $y(\xi+1) = y(\xi)+1$ and the periodicity of ν and U, the expressions for Q and P can be rewritten as

$$\begin{aligned} Q &= \frac{1}{2(e-1)}\int_0^1 \sinh(y(\xi)-y(\eta))(U^2 y_\xi+\nu)(\eta)\,d\eta \\ &- \frac{1}{4}\int_0^1 \operatorname{sign}(\xi-\eta)\exp\big(-\operatorname{sign}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2 y_\xi+\nu)(\eta)\,d\eta, \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} P &= \frac{1}{2(e-1)}\int_0^1 \cosh(y(\xi)-y(\eta))(U^2 y_\xi+\nu)(\eta)\,d\eta \\ &\qquad + \frac{1}{4}\int_0^1 \exp\big(-\operatorname{sign}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2 y_\xi+\nu)(\eta)\,d\eta. \end{aligned} \tag{5.36}$$

Thus $P_x \circ y$ and $P \circ y$ can be replaced by equivalent expressions given by (5.32) and (5.33) which only depend on our new variables U, H, and y. We obtain a new system of equations, which is at least formally equivalent to the Camassa–Holm equation:

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = [U^3 - 2PU]_0^\xi. \end{cases} \tag{5.37}$$

After differentiating (5.37) we find

$$\begin{cases} y_{\xi t} = U_\xi, \\ U_{\xi t} = \frac{1}{2}\nu + \left(\frac{1}{2}U^2 - P\right) y_\xi, \\ H_{\xi t} = -2Q\,U y_\xi + \big(3U^2 - 2P\big)\,U_\xi. \end{cases} \tag{5.38}$$

From (5.37) and (5.38), we obtain the system

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ \nu_t = -2Q\,U y_\xi + \big(3U^2 - 2P\big)\,U_\xi. \end{cases} \tag{5.39}$$

We can write (5.39) more compactly as

$$X_t = F(X), \quad X = (y, U, \nu). \tag{5.40}$$

Let

$$W^{1,1}_{\mathrm{per}} = \{f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}) \mid f(\xi+1) = f(\xi) \text{ for all } \xi \in \mathbb{R}\}.$$

We equip $W^{1,1}_{\mathrm{per}}$ with the norm of V, that is,

$$\|f\|_{W^{1,1}_{\mathrm{per}}} = \|f\|_{L^\infty([0,1])} + \|f_\xi\|_{L^1([0,1])},$$

which is equivalent to the standard norm of $W^{1,1}_{\mathrm{per}}$ because $\|f\|_{L^1([0,1])} \le \|f\|_{L^\infty([0,1])} \le \|f\|_{L^1([0,1])} + \|f_\xi\|_{L^1([0,1])}$. Let E be the Banach space defined as

$$E = V \times W^{1,1}_{\mathrm{per}} \times L^1_{\mathrm{per}}.$$

We derive the following Lipschitz estimates for P and Q.

Lemma 5.1. *For any $X = (y, U, \nu)$ in E, we define the maps $\mathcal{Q}$ and $\mathcal{P}$ as $\mathcal{Q}(X) = Q$ and $\mathcal{P}(X) = P$ where Q and P are given by (5.32) and (5.33), respectively. Then, $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps on bounded sets from E to $W^{1,1}_{\mathrm{per}}$. More precisely, we have the following bounds. Let*

$$B_M = \{X = (y, U, \nu) \in E \mid \|U\|_{W^{1,1}_{\mathrm{per}}} + \|y_\xi\|_{L^1} + \|\nu\|_{L^1} \le M\}. \tag{5.41}$$

Then for any $X, \tilde{X} \in B_M$, we have

$$\|\mathcal{Q}(X) - \mathcal{Q}(\tilde{X})\|_{W^{1,1}_{\mathrm{per}}} \le C_M \|X - \tilde{X}\|_E \tag{5.42}$$

and

$$\|\mathcal{P}(X) - \mathcal{P}(\tilde{X})\|_{W^{1,1}_{\mathrm{per}}} \le C_M \|X - \tilde{X}\|_E \tag{5.43}$$

where the constant C_M only depends on the value of M.

Proof. Let us first prove that $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps from B_M to L^∞_{per}. Note that by using a change of variables in (5.35) and (5.36), we obtain that $\mathcal{P}$ and $\mathcal{Q}$ are periodic with period 1. Let now $X = (y, U, \nu)$ and $\tilde{X} = (\tilde{y}, \tilde{U}, \tilde{\nu})$ be two elements of B_M. Since the map $x \mapsto \cosh x$ is locally Lipschitz, it is Lipschitz on $[-M, M]$. We denote by C_M a generic constant that only depends on M. Since, for all ξ, η in $[0,1]$ we have $|y(\xi) - y(\eta)| \le \|y_\xi\|_{L^1}$, we also have

$$\begin{aligned}|\cosh(y(\xi) - y(\eta)) - \cosh(\tilde{y}(\xi) - \tilde{y}(\eta))| &\le C_M |y(\xi) - \tilde{y}(\xi) - y(\eta) + \tilde{y}(\eta)| \\ &\le C_M \|y - \tilde{y}\|_{L^\infty}.\end{aligned}$$

It follows that, for all $\xi \in [0,1]$,

$$\begin{aligned}&\|\cosh(y(\xi) - y(\,\cdot\,))(U^2 y_\xi + \nu)(\,\cdot\,) - \cosh(\tilde{y}(\xi) - \tilde{y}(\,\cdot\,))(\tilde{U}^2 \tilde{y}_\xi + \tilde{\nu})(\,\cdot\,)\|_{L^1} \\ &\qquad \le C_M \big(\|y - \tilde{y}\|_{L^\infty} + \|U - \tilde{U}\|_{L^\infty} + \|y_\xi - \tilde{y}_\xi\|_{L^1} + \|\nu - \tilde{\nu}\|_{L^1}\big)\end{aligned}$$

and the map $X = (y, U, \nu) \mapsto \frac{1}{2(e-1)} \int_0^1 \cosh(y(\xi) - y(\eta))(U^2 y_\xi + \nu)(\eta)\, d\eta$ which corresponds to the first term in (5.36) is Lipschitz from B_M to L^∞_{per} and the Lipschitz constant only depends on M. We handle the other terms in (5.36) in the same way and we prove that $\mathcal{P}$ is Lipschitz from B_M to L^∞_{per}. Similarly, one proves that $\mathcal{Q}\colon B_M \to L^\infty_{\mathrm{per}}$ is Lipschitz for a Lipschitz constant which only depends on M. Direct differentiation gives the expressions (5.34) for the derivatives P_ξ and Q_ξ of P and Q. Then, as $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz from B_M to L^∞_{per}, we have

$$\begin{aligned}
&\big\|\mathcal{Q}(X)_\xi - \mathcal{Q}(\tilde{X})_\xi\big\|_{L^1}\\
&\quad= \|y_\xi \mathcal{P}(X) - \tilde{y}_\xi \mathcal{P}(\tilde{X}) - \frac{1}{2}(U^2 y_\xi - \tilde{U}^2\tilde{y}_\xi) - \nu + \tilde{\nu}\|_{L^1}\\
&\quad\leq C_M\big(\|\mathcal{P}(X) - \mathcal{P}(\tilde{X})\|_{L^\infty} + \|U - \tilde{U}\|_{L^\infty} + \|y_\xi - \tilde{y}_\xi\|_{L^1} + \|\nu - \tilde{\nu}\|_{L^1}\big)\\
&\quad\leq C_M\|X - \tilde{X}\|_E.
\end{aligned}$$

Hence, we have proved that $\mathcal{Q}\colon B_M \to W^{1,1}_{\mathrm{per}}$ is Lipschitz for a Lipschitz constant that only depends on M. We prove the corresponding result for $\mathcal{P}$ in the same way. ∎

The short-time existence follows from Lemma 5.1 and a contraction argument. Global existence is obtained only for initial data which belong to the set $\mathcal{F}$ as defined below.

Definition 5.2. *The set $\mathcal{F}$ is composed of all $(y, U, \nu) \in E$ such that*

$$y \in V_1,\ (y, U) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}),\ \nu \in L^\infty, \tag{5.44a}$$

$$y_\xi \geq 0,\ \nu \geq 0,\ y_\xi + \nu \geq c \text{ almost everywhere, for some constant } c > 0, \tag{5.44b}$$

$$y_\xi \nu = y_\xi^2 U^2 + U_\xi^2 \text{ almost everywhere.} \tag{5.44c}$$

Lemma 5.3. *The set $\mathcal{F}$ is preserved by the equation* (5.39)*, that is, if $X(t)$ solves* (5.39) *for $t \in [0, T]$ with initial data $X_0 \in \mathcal{F}$, then $X(t) \in \mathcal{F}$ for all $t \in [0, T]$.*

Proof. The proof is basically the same as in [56], and we will repeat this proof with the necessary adaptions here for completeness. Solutions of (5.40) can be rewritten as

$$X(t) = \overline{X} + \int_0^t F(X(\tau))d\tau, \tag{5.45}$$

where $\overline{X}$ denotes the initial condition. Proceeding as in the proof of Lemma 5.1, one obtains that F is Lipschitz on bounded sets from $E \to E$, which implies the existence of short time solutions.

Next we want to show that for initial data in $[W^{1,\infty}]^2 \times L^\infty$ we have short time solutions in $[W^{1,\infty}]^2 \times L^\infty$. Therefore observe first that $y, U, P, Q \in W^{1,1}$ implies that $y, U, P, Q \in L^\infty$, and we therefore have to consider the following system of ordinary differential equations for y_ξ, U_ξ, and ν:

$$\begin{cases}
\dfrac{d}{dt}\alpha(t,\xi) = \beta(t,\xi),\\
\dfrac{d}{dt}\beta(t,\xi) = \dfrac{1}{2}\gamma(t,\xi) + \left(\dfrac{1}{2}U^2 - P\right)(t,\xi)(1 + \alpha(t,\xi)),\\
\dfrac{d}{dt}\gamma(t,\xi) = -2Q\,U(t,\xi)(1 + \alpha(t,\xi)) + \big(3U^2 - 2P\big)(t,\xi)\beta(t,\xi).
\end{cases} \tag{5.46}$$

where we substituted ζ_ξ, U_ξ, and ν by α, β, and γ. We specify the initial conditions for this system by defining $\mathcal{A}$ as the following set

$$\mathcal{A} = \{\xi \in \mathbb{R} | |\overline{U_\xi}(\xi)| \leq \|\overline{U_\xi}(\xi)\|_{L^\infty}, |\overline{\zeta_\xi}(\xi)| \leq \|\overline{\zeta_\xi}(\xi)\|_{L^\infty}, |\overline{\nu}(\xi)| \leq \|\overline{\nu}\|_{L^\infty}\}.$$

Since we assume $\overline{X} \in [W^{1,\infty}]^2 \times L^\infty$, we have that $\mathcal{A}$ has full measure, that is, $\text{meas}(\mathcal{A}^c) = 0$. For $\xi \in \mathcal{A}$, we define $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (\overline{\zeta}_\xi(\xi), \overline{U}_\xi(\xi), \overline{\nu}(\xi))$. For $\xi \in \mathcal{A}^c$ we take $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (0,0,0)$. This allows us to work in the Banach space of everywhere bounded periodic functions B^∞_{per}, whose norm is given by $\|f\|_{B^\infty_{per}} = \sup_{\xi \in [0,1]} |f(\xi)|$. We define (α, β, γ) as the solutions of (5.46) in $[B^\infty_{per}]^3$ with initial data as given above. As to any function in L^∞_{per} we can find a function in B^∞_{per}, so that the two functions coincide almost everywhere, it is no problem to work in this slightly different setting. As before we can use a contraction argument to show the short time existence of solutions in $[B^\infty_{per}]^3$ and applying Gronwall's lemma yields that this solution exists on $[0,T]$, the interval on which (ζ, U, ν) exist. For showing that (α, β, γ) coincide with (ζ_ξ, U_ξ, ν) almost everywhere for any $t \in [0,T]$, we need the following proposition, which is adapted from [94, p.134, Corollary 2].

Proposition 5.4. *Let R be a bounded linear operator on a Banach space X into a Banach space Y. Let f be in $C([0,T],X)$. Then, Rf belongs to $C([0,T],Y)$ and therefore is Riemann integrable, and $\int_{[0,T]} Rf(t)dt = R\int_{[0,T]} f(t)dt$.*

For any given ξ, the map $f \to f(\xi)$ from B^∞_{per} to $\mathbb{R}$ is linear and continuous. Hence after applying this map to each term in (5.46) written in integral from and using Proposition 5.4, we recover the original definition of α, β, and γ as solutions, for any given $\xi \in \mathbb{R}$, of the system (5.46) of ordinary differential equations in $\mathbb{R}^3$. The derivation map $\frac{d}{d\xi}$ is continuous form $W^{1,1}_{per}$ to L^1_{per}. We can apply it to (5.39), written in integral form, and by Proposition 5.4, this map commutes with the integral. Thus we end up with

$$\begin{cases} \zeta_\xi(t) = \overline{\zeta}_\xi + \displaystyle\int_0^t U_\xi(\tau)d\tau, \\ U_\xi(t) = \overline{U}_\xi + \displaystyle\int_0^t \Big(\frac{1}{2}\nu + \Big(\frac{1}{2}U^2 - P\Big)(1+\zeta_\xi)\Big)(\tau)d\tau, \\ \nu(t) = \overline{\nu} - \displaystyle\int_0^t \Big(2Q\,U(1+\zeta_\xi) + \big(3U^2 - 2P\big)\,U_\xi\Big)(\tau)d\tau. \end{cases} \tag{5.47}$$

The map from B^∞_{per} to L^1_{per} is also continuous, we can apply it to (5.46) written in integral from, and again use Proposition 5.4. Then, we subtract each equation in (5.47) from the corresponding one in (5.46), take the norm and add them. After introducing $Z(t) = \|\alpha(t,.) - \zeta_\xi(t,.)\|_{L^1} + \|\beta(t,.) - U_\xi(t,.)\|_{L^1} + \|\gamma(t,.) - \nu(t,.)\|_{L^1}$, we end up with the following equation

$$Z(t) \leq Z(0) + C\int_0^t Z(\tau)d\tau, \tag{5.48}$$

where C is a constant which again, only depends on the $C([0,T], W^{1,1})$ norms of U, P, and Q. By assumption we have $Z(0) = 0$ and therefore by Gronwall's lemma, we get $Z(t) = 0$ for all $t \in [0,T]$. Thus we showed that (5.44a) is preserved.

Denote by $\mathcal{B}$ the set where the absolute values of $\overline{\zeta}_\xi(\xi)$, $\overline{U}_\xi(\xi)$, and $\overline{\nu}(\xi)$ all are smaller than $\|\overline{X}\|_{[W^{1,\infty}]^2\times L^\infty}$ and where (5.44b) and (5.44c) are satisfied for $\overline{y}_\xi$, $\overline{U}_\xi$, and $\overline{\nu}$. By assumption we have that meas $\mathcal{B}^c = 0$, and we set $(\overline{\zeta}_\xi, \overline{U}_\xi, \overline{\nu})$ equal to zero on $\mathcal{B}^c$.

Using that $\mathcal{B} \subset \mathcal{A}$, we obtain that $X(t)$ satisfies (5.44a) for all $t \in [0,T]$. Next we will show that (5.44b) and (5.44c) hold for any $\xi \in \mathcal{B}$ and hence almost everywhere. We consider a fixed ξ in $\mathcal{B}$ and drop it in the notation when there is no ambiguity. From (5.39), we have, on the one hand,

$$(y_\xi \nu)_t = y_{\xi t}\nu + \nu_t y_\xi = U_\xi \nu + (3U^2 U_\xi - 2y_\xi QU - 2PU_\xi)y_\xi,$$

and, on the other hand,

$$\begin{aligned}(y_\xi^2 U^2 + U_\xi^2)_t &= 2y_{\xi t} y_\xi U^2 + 2y_\xi^2 U_t U + 2U_{\xi t} U_\xi \\ &= 3U_\xi U^2 y_\xi - 2PU_\xi y_\xi + \nu U_\xi - 2y_\xi^2 QU.\end{aligned}$$

Thus $(y_\xi \nu - y_\xi^2 U^2 - U_\xi^2)_t = 0$, and since $y_\xi \nu(0) = (y_\xi^2 U^2 - U_\xi^2)(0)$, we have proved (5.44c). Let us introduce $t^\star$ given by

$$t^\star = \sup\{t \in [0,T] | y_\xi(t') \geq 0 \text{ for all } t' \in [0,t]\}.$$

Here we recall that we consider a fixed $\xi \in \mathcal{B}$ and drop it in the notation. Assume $t^\star < T$. Since $y_\xi(t)$ is continuous with respect to time, we have

$$y_\xi(t^\star) = 0. \tag{5.49}$$

Hence, from (5.44c), that we just proved, we get $U_\xi(t^\star) = 0$ and by (5.39),

$$y_{\xi t}(t^\star) = U_\xi(t^\star) = 0. \tag{5.50}$$

Form (5.39), since $y_\xi(t^\star) = U_{\xi t}(t^\star) = 0$, we get

$$y_{\xi tt}(t^\star) = U_{\xi t}(t^\star) = \frac{1}{2}\nu(t^\star). \tag{5.51}$$

If $\nu(t^\star) = 0$, then $(y_\xi, U_\xi, \nu) = (0,0,0)$ and, by the uniqueness of the solution of (5.39), seen as a system of ordinary differential equations, we must have $(y_\xi, U_\xi, \nu) = 0$ for all $t \in [0,T]$. This contradicts the fact that $y_\xi(0)$ and $\nu(0)$ cannot vanish at the same time. If $\nu(t^\star) < 0$, then $y_{\xi tt}(t^\star) < 0$, and because of (5.49) and (5.50), there exists a neighborhood U of $t^\star$ such that $y_\xi(t) < 0$ for all $t \in U \backslash \{t^\star\}$. This contradicts the definition of $t^\star$. Hence, $\nu(t^\star) > 0$, and, since we now have $y_\xi(t^\star) = y_{\xi t}(t^\star) = 0$ and $y_{\xi tt}(t^\star) > 0$, there exists a neighborhood of $t^\star$, which we again denote by U such that $y_\xi(t) > 0$ for all $t \in U\backslash\{t^\star\}$. This contradicts the fact that $t^\star < T$, and we have proved the first inequality in (5.44c), namely that $y_\xi(t) \geq 0$ for all $t \in [0,T]$. Let us prove that $\nu(t) \geq 0$ for all $t \in [0,T]$. This follows from (5.44c) when $y_\xi(t) > 0$, Now if $y_\xi(t) = 0$, then $U_\xi(t) = 0$ from (5.44c) and we have seen that $\nu(t) < 0$ would imply that $y_\xi(t') > 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence $\nu(t) \geq 0$ and we have proved the second inequality in (5.44b). Assume that the third inequality in (5.44b) does not hold. then, by continuity, there exists a time $t \in [0,T]$ such that $(y_\xi + \nu)(t) = 0$. Since y_ξ and ν are positive, we must have $y_\xi(t) = \nu(t) = 0$ and , by (5.44c), $U_\xi(t) = 0$. Since zero is a solution of (5.39), this implies that $y_\xi(0) = U_\xi(0) = \nu(0) = 0$, which contradicts $(y_\xi + H_\xi)(0) > 0$.

□

Theorem 5.5. *For any $\bar{X} = (\bar{y}, \bar{U}, \bar{\nu}) \in \mathcal{F}$, the system (5.39) admits a unique global solution $X(t) = (y(t), U(t), \nu(t))$ in $C^1(\mathbb{R}_+, E)$ with initial data $\bar{X} = (\bar{y}, \bar{U}, \bar{\nu})$. We have $X(t) \in \mathcal{F}$ for all times. Let the mapping $S\colon \mathcal{F} \times \mathbb{R}_+ \to \mathcal{F}$ be defined as*

$$S_t(X) = X(t).$$

Given $M > 0$ and $T > 0$, we define B_M as before, that is,

$$B_M = \{X = (y, U, \nu) \in E \mid \|U\|_{W^{1,1}_{\mathrm{per}}} + \|y_\xi\|_{L^1} + \|\nu\|_{L^1} \leq M\}. \tag{5.52}$$

Then there exists a constant C_M which depends only on M and T such that, for any two elements X_α and X_β in B_M, we have

$$\|S_t X_\alpha - S_t X_\beta\|_E \leq C_M \|X_\alpha - X_\beta\|_E \tag{5.53}$$

for any $t \in [0, T]$.

Proof. By using Lemma 5.1, we proceed using a contraction argument and obtain the existence of short time solutions to (5.39). Let T by the maximal time of existence and assume $T < \infty$. Let (y, U, ν) be a solution of (5.39) in $C^1([0, T), E)$ with initial data (y_0, U_0, ν_0). We want to prove that

$$\sup_{t \in [0,T)} \|(y(t, \cdot\,), U(t, \cdot\,), \nu(t, \cdot\,))\|_E < \infty. \tag{5.54}$$

From (5.39), we get

$$\begin{aligned}
\int_0^1 \nu(t,\xi)\, d\xi &= \int_0^1 \nu(0,\xi)\, d\xi + \int_0^1 \int_0^t \left(-2Q\, U y_\xi + \left(3U^2 - 2P\right) U_\xi\right)(t,\xi)\, dt d\xi \\
&= \int_0^1 \nu(0,\xi)\, d\xi + \int_0^t \int_0^1 (U^3 - 2PU)_\xi(t,\xi)\, d\xi dt \\
&= \int_0^1 \nu(0,\xi)\, d\xi.
\end{aligned} \tag{5.55}$$

Hence, $\|\nu(t, \cdot\,)\|_{L^1} = \|\nu(0, \cdot\,)\|_{L^1}$. This identity corresponds to the conservation of the total energy. We now consider a fixed time $t \in [0, T)$ which we omit in the notation when there is no ambiguity. For ξ and η in $[0, 1]$, we have $|y(\xi) - y(\eta)| \leq 1$ because y is increasing and $y(1) - y(0) = 1$. From (5.44c), we infer $U^2 y_\xi \leq \nu$ and, from (5.35), we obtain

$$|Q| \leq \frac{1}{e-1} \int_0^1 \sinh(|y(\xi) - y(\eta)|)\nu(\eta)\, d\eta + \int_0^1 e^{-|y(\xi)-y(\eta)|}\nu(\eta)\, d\eta.$$

Hence,

$$\|Q(t, \cdot\,)\|_{L^\infty} \leq C\|\nu(t, \cdot\,)\|_{L^1} = C\|\nu(0, \cdot\,)\|_{L^1} \tag{5.56}$$

for some constant C. Similarly, one prove that $\|P(t, \cdot\,)\|_{L^\infty} \leq C\|\nu(0, \cdot\,)\|_{L^1}$ and therefore $\sup_{t\in[0,T)} \|Q(t, \cdot\,)\|_{L^\infty}$ and $\sup_{t\in[0,T)} \|P(t, \cdot\,)\|_{L^\infty}$ are finite. Since $U_t = -Q$, it follows that

$$\|U(t, \cdot\,)\|_{L^\infty} \leq \|U(0, \cdot\,)\|_{L^\infty} + CT\|\nu(0, \cdot\,)\|_{L^1} \tag{5.57}$$

and $\sup_{t\in[0,T)}\|U(t,\,\cdot\,)\|_{L^\infty} < \infty$. Since $y_t = U$, we have that $\sup_{t\in[0,T)}\|y(t,\,\cdot\,)\|_{L^\infty}$ is also finite. Thus, we have proved that

$$C_1 = \sup_{t\in[0,T)} \{\|U(t,\,\cdot\,)\|_{L^\infty} + \|P(t,\,\cdot\,)\|_{L^\infty} + \|Q(t,\,\cdot\,)\|_{L^\infty}\}$$

is finite and depends only on T and $\|U(0,\,\cdot\,)\|_{L^\infty} + \|\nu(0,\,\cdot\,)\|_{L^1}$. Let $Z(t) = \|y_\xi(t,\,\cdot\,)\|_{L^1} + \|U_\xi(t,\,\cdot\,)\|_{L^1} + \|\nu(t,\,\cdot\,)\|_{L^1}$. Using the semi-linearity of (5.38) with respect to (y_ξ, U_ξ, ν), we obtain

$$Z(t) \le Z(0) + C\int_0^t Z(\tau)\,d\tau$$

where C is a constant depending only on C_1. It follows from Gronwall's lemma that $\sup_{t\in[0,T)} Z(t)$ is finite, and this concludes the proof of the global existence.

Moreover we have proved that

$$\|U(t,\,\cdot\,)\|_{W^{1,1}_{\mathrm{per}}} + \|y_\xi(t,\,\cdot\,)\|_{L^1} + \|\nu(t,\,\cdot\,)\|_{L^1} \le C_M \tag{5.58}$$

for a constant C_M which depends only on T and $\|U(0,\,\cdot\,)\|_{W^{1,1}_{\mathrm{per}}} + \|y_\xi(0,\,\cdot\,)\|_{L^1} + \|\nu(0,\,\cdot\,)\|_{L^1}$. Let us prove (5.53). Given T and $X_\alpha, X_\beta \in B_M$, from Lemma 5.1 and (5.58), we get that

$$\|U_\alpha(t,\,\cdot\,) - U_\beta(t,\,\cdot\,)\|_{L^\infty} + \|Q_\alpha(t,\,\cdot\,) - Q_\beta(t,\,\cdot\,)\|_{L^\infty} \le C_M\|X_\alpha(t) - X_\beta(t)\|_E$$

where C_M is a generic constant which depends only on M and T. Using again (5.38) and Lemma 5.1, we get that for a given time $t \in [0,T]$,

$$\begin{aligned}\|U_{\alpha\xi} - U_{\beta\xi}\|_{L^1} + \|\frac{1}{2}\nu_\alpha + \left(\frac{1}{2}U_\alpha^2 - P_\alpha\right) y_{\alpha\xi} - \frac{1}{2}\nu_\beta - \left(\frac{1}{2}U_\beta^2 - P_\beta\right) y_{\beta\xi}\|_{L^1}\\ + \|-2Q_\alpha U_\alpha y_{\alpha\xi} + \left(3U_\alpha^2 - 2P_\alpha\right) U_{\alpha\xi} + 2Q_\beta U_\beta y_{\beta\xi} - \left(3U_\beta^2 - 2P_\beta\right) U_{\beta\xi}\|_{L^1}\\ \le C_M\|X_\alpha - X_\beta\|_E.\end{aligned}$$

Hence, $\|F(X_\alpha(t)) - F(X_\beta(t))\|_E \le C_M\|X_\alpha(t) - X_\beta(t)\|_E$ where F is defined as in (5.40). Then, (5.53) follows from Gronwall's lemma applied to (5.40). $\square$

5.3 Relabeling invariance

We denote by G the subgroup of the group of homeomorphisms on the unit interval defined as follows:

Definition 5.6. *Let G be the set of all functions f such that f is invertible,*

$$f \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}),\ f(\xi+1) = f(\xi) + 1 \textit{ for all } \xi \in \mathbb{R}, \textit{ and} \tag{5.59}$$

$$f - \mathbb{I} \textit{ and } f^{-1} - \mathbb{I} \textit{ both belong to } W^{1,\infty}_{\mathrm{per}}. \tag{5.60}$$

The set G can be interpreted as the set of relabeling functions. Note that $f \in G$ implies that

$$\frac{1}{1+\alpha} \le f_\xi \le 1 + \alpha$$

for some constant $\alpha > 0$. This condition is also almost sufficient as Lemma 3.2 in [56] shows. Given a triplet $(y, U, \nu) \in \mathcal{F}$, we denote by h the total energy $\|\nu\|_{L^1}$. We define the subsets $\mathcal{F}_\alpha$ of $\mathcal{F}$ as follows

$$\mathcal{F}_\alpha = \{X = (y, U, \nu) \in \mathcal{F} \mid \frac{1}{1+\alpha} \leq \frac{1}{1+h}(y_\xi + \nu) \leq 1 + \alpha\}.$$

The set $\mathcal{F}_0$ is then given by

$$\mathcal{F}_0 = \{X = (y, U, \nu) \in \mathcal{F} \mid y_\xi + \nu = 1 + h\}. \tag{5.61}$$

We have $\mathcal{F} = \cup_{\alpha \geq 0} \mathcal{F}_\alpha$. We define the action of the group G on $\mathcal{F}$.

Definition 5.7. *We define the map* $\Phi \colon G \times \mathcal{F} \to \mathcal{F}$ *as follows*

$$\begin{cases} \bar{y} = y \circ f, \\ \bar{U} = U \circ f, \\ \bar{\nu} = \nu \circ f f_\xi, \end{cases}$$

where $(\bar{y}, \bar{U}, \bar{\nu}) = \Phi(f, (y, U, \nu))$. *We denote* $(\bar{y}, \bar{U}, \bar{\nu}) = (y, U, \nu) \bullet f$.

Proposition 5.8. *The map* Φ *defines a group action of* G *on* $\mathcal{F}$.

Proof. By the definition it is clear that Φ satisfies the fundamental property of a group action, that is $X \bullet f_1 \bullet f_2 = X \bullet (f_1 \circ f_2)$ for all $X \in \mathcal{F}$ and f_1, $f_2 \in G$. It remains to prove that $X \bullet f$ indeed belongs to $\mathcal{F}$. We denote $\hat{X} = (\hat{y}, \hat{U}, \hat{\nu}) = X \bullet f$, then it is not hard to check that $\hat{y}(\xi + 1) = \hat{y}(\xi) + 1$, $\hat{U}(\xi + 1) = \hat{U}(\xi)$, and $\hat{\nu}(\xi + 1) = \hat{\nu}(\xi)$ for all $\xi \in \mathbb{R}$. By definition we have $\hat{v} = v \circ f f_\xi$, and we will now prove that

$$\hat{y}_\xi = y_\xi \circ f f_\xi, \quad \text{and} \quad \hat{U}_\xi = U_\xi \circ f f_\xi,$$

almost everywhere. Let B_1 be the set where y is differentiable and B_2 the set where $\hat{y}$ and f are differentiable. Using Rademacher's theorem, we get that $\mathrm{meas}(B_1^c) = \mathrm{meas}(B_2^c) = 0$. For $\xi \in B_3 = B_2 \cap f^{-1}(B_1)$, we consider a sequence ξ_i converging to ξ with $\xi_i \neq \xi$ for all $i \in \mathbb{N}$. We have

$$\frac{y(f(\xi_i)) - y(f(\xi))}{f(\xi_i) - f(\xi)} \frac{f(\xi_i) - f(\xi)}{\xi_i - \xi} = \frac{\hat{y}(\xi_i) - \hat{y}(\xi)}{\xi_i - \xi}. \tag{5.62}$$

Since f is continuous, $f(\xi_i)$ converges to $f(\xi)$ and, as y is differentiable at $f(\xi)$, the left-hand side of (5.62) tends to $y_\xi \circ f(\xi) f_\xi(\xi)$, the right-hand side of (5.62) tends to $\hat{y}_\xi(\xi)$, and we get

$$y_\xi(f(\xi)) f_\xi(\xi) = \hat{y}_\xi(\xi), \tag{5.63}$$

for all $\xi \in B_3$. Since f^{-1} is Lipschitz continuous, one-to-one, and $\mathrm{meas}(B_1^c) = 0$, we have $\mathrm{meas}(f^{-1}(B_1)^c) = 0$ and therefore (5.63) holds almost everywhere. One proves the other identity similarly. As $f_\xi > 0$ almost everywhere, we obtain immediately that (5.44b) and (5.44c) are fulfilled. That (5.44a) is also satisfied follows from the following considerations: $\|\hat{y}_\xi\|_{L^1} = \|y_\xi\|_{L^1}$, as y_ξ is periodic with period 1. The same argument applies when considering $\|\hat{U}_\xi\|_{L^1}$ and $\|\hat{\nu}\|_{L^1}$. As U is periodic with period 1, we can also conclude that $\|\hat{U}\|_{L^\infty} = \|U\|_{L^\infty}$. As $f \in G$, one obtains that $\|\hat{y}\|_{L^\infty}$ is bounded, but not equal to $\|y\|_{L^\infty}$. $\square$

Note that the set B_M is invariant with respect to relabeling while the E-norm is not, as the following example shows: Consider the function $y(\xi) = \xi \in V_1$, and $f \in G$, then this yields

$$\|y(f(\xi))\|_{L^\infty([0,1])} = \|f(\xi)\|_{L^\infty([0,1])}.$$

Hence, the L^∞-norm of $y(f(\xi))$ will always depend on f.

Since G is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/G$ of $\mathcal{F}$ with respect to the group action. Let us introduce the subset $\mathcal{H}$ of $\mathcal{F}_0$ defined as follows

$$\mathcal{H} = \{(y, U, \nu) \in \mathcal{F}_0 \mid \int_0^1 y(\xi)\, d\xi = 0\}. \tag{5.64}$$

It turns out that $\mathcal{H}$ contains a unique representative in $\mathcal{F}$ for each element of $\mathcal{F}/G$, that is, there exists a bijection between $\mathcal{H}$ and $\mathcal{F}/G$. In order to prove this we introduce two maps $\Pi_1 \colon \mathcal{F} \to \mathcal{F}_0$ and $\Pi_2 \colon \mathcal{F}_0 \to \mathcal{H}$ defined as follows

$$\Pi_1(X) = X \bullet f^{-1} \tag{5.65}$$

with $f = \frac{1}{1+h}(y + \int_0^\xi \nu(\eta)\, d\eta) \in G$ and $X = (y, U, \nu)$, and

$$\Pi_2(X) = X(\xi - a) \tag{5.66}$$

with $a = \int_0^1 y(\xi)\, d\xi$. First, we have to prove that f indeed belongs to G. We have

$$\begin{aligned} f(\xi+1) &= \frac{1}{1+h}\big(y(\xi+1) + \int_0^{\xi+1} \nu(\eta)\, d\eta\big) \\ &= \frac{1}{1+h}\big(y(\xi) + 1 + \int_0^{\xi} \nu(\eta)\, d\eta + h\big) = f(\xi) + 1 \end{aligned}$$

and this proves (5.59). Since $(y, U, \nu) \in \mathcal{F}$, there exists a constant $c \geq 1$ such that $\frac{1}{c} \leq f_\xi \leq c$ for almost every ξ and therefore (5.60) follows from an application of Lemma 3.2 in [56]. After noting that the group action lets the quantity $h = \|\nu\|_{L^1}$ invariant, it is not hard to check that $\Pi_1(X)$ indeed belongs to $\mathcal{F}_0$, that is, $\frac{1}{1+h}(\bar{y}_\xi + \bar{\nu}) = 1$ where we denote $(\bar{y}, \bar{U}, \bar{\nu}) = \Pi_1(X)$. Let us prove that $(\bar{y}, \bar{U}, \bar{\nu}) = \Pi_2(y, U, \nu)$ belongs to $\mathcal{H}$ for any $(y, U, \nu) \in \mathcal{F}_0$. On the one hand, we have $\frac{1}{1+h}(\bar{y}_\xi + \bar{\nu}) = 1$ because $\bar{h} = h$ and $\frac{1}{1+h}(y_\xi + \nu) = 1$ as $(y, U, \nu) \in \mathcal{F}_0$. On the other hand,

$$\int_0^1 \bar{y}(\xi)\, d\xi = \int_{-a}^{1-a} y(\xi)\, d\xi = \int_0^1 y(\xi)\, d\xi + \int_{-a}^0 y(\xi)\, d\xi + \int_1^{1-a} y(\xi)\, d\xi \tag{5.67}$$

and, since $y(\xi+1) = y(\xi) + 1$, we obtain

$$\int_0^1 \bar{y}(\xi)\, d\xi = \int_0^1 y(\xi)\, d\xi + \int_{-a}^0 y(\xi)\, d\xi + \int_0^{-a} y(\xi)\, d\xi - a = \int_0^1 y(\xi)\, dx - a = 0. \tag{5.68}$$

Thus $\Pi_2(X) \in \mathcal{H}$. Note that the definition (5.66) of Π_2 can be rewritten as

$$\Pi_2(X) = X \bullet \tau_a$$

where $\tau_a : \xi \mapsto \xi - a$ denotes the translation of length a so that $\Pi_2(X)$ is a relabeling of X.

Definition 5.9. *We denote by* Π *the projection of* $\mathcal{F}$ *into* $\mathcal{H}$ *given by* $\Pi_1 \circ \Pi_2$.

One checks directly that $\Pi \circ \Pi = \Pi$. The element $\Pi(X)$ is the unique relabeled version of X which belongs to $\mathcal{H}$ and therefore we have the following result.

Lemma 5.10. *The sets* $\mathcal{F}/G$ *and* $\mathcal{H}$ *are in bijection.*

Given any element $[X] \in \mathcal{F}/G$, we associate $\Pi(X) \in \mathcal{H}$. This mapping is well-defined and is a bijection.

Lemma 5.11. *The mapping* S_t *is equivariant, that is,*

$$S_t(X \bullet f) = S_t(X) \bullet f. \tag{5.69}$$

Proof. For any $X_0 = (y_0, U_0, \nu_0) \in \mathcal{F}$ and $f \in G$, we denote $\bar{X}_0 = (\bar{y}_0, \bar{U}_0, \bar{\nu}_0) = X_0 \bullet f$, $X(t) = S_t(X_0)$, and $\bar{X}(t) = S_t(\bar{X}_0)$. We claim that $X(t) \bullet f$ satisfies (5.39) and therefore, since $X(t) \bullet f$ and $\bar{X}(t)$ satisfy the same system of differential equations with the same initial data, they are equal. We denote $\hat{X}(t) = (\hat{y}(t), \hat{U}(t), \hat{\nu}(t)) = X(t) \bullet f$. Then we obtain

$$\hat{U}_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi-\eta)\exp\big(-\operatorname{sign}(\xi-\eta)(\hat{y}(\xi)-y(\eta))\big)\big[U^2 y_\xi + \nu\big](\eta)d\eta.$$

As $\hat{y}_\xi(\xi) = y_\xi(f(\xi))f_\xi(\xi)$ and $\hat{\nu}(\xi) = \nu(f(\xi))f_\xi(\xi)$ for almost every $\xi \in \mathbb{R}$, we obtain after the change of variables $\eta = f(\eta')$,

$$\hat{U}_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi-\eta)\exp\big(-\operatorname{sign}(\xi-\eta)(\hat{y}(\xi)-\hat{y}(\eta))\big)\big[\hat{U}^2 \hat{y}_\xi + \hat{\nu}\big](\eta)d\eta.$$

Treating similarly the other terms in (5.39), it follows that $(\hat{y}, \hat{U}, \hat{\nu})$ is a solution of (5.39). Thus, since $(\hat{y}, \hat{U}, \hat{\nu})$ and $(\bar{y}, \bar{U}, \bar{\nu})$ satisfy the same system of ordinary differential equations with the same initial conditions, they are equal and (5.69) is proved. ☐

From this lemma we get that

$$\Pi \circ S_t \circ \Pi = \Pi \circ S_t. \tag{5.70}$$

Definition 5.12. *We define the semigroup* $\bar{S}_t$ *on* $\mathcal{H}$ *as*

$$\bar{S}_t = \Pi \circ S_t.$$

The semigroup property of $\bar{S}_t$ follows from (5.70). Using the same approach as in [56], we can prove that $\bar{S}_t$ is continuous with respect to the norm of E. It follows basically of the continuity of the mapping Π but Π is not Lipschitz continuous and the goal of the next section is to improve this result and find a metric that makes $\bar{S}_t$ Lipschitz continuous.

5.4 Lipschitz metric for the semigroup $\bar{S}_t$

Definition 5.13. *Let* $X_\alpha, X_\beta \in \mathcal{F}$, *we define* $J(X_\alpha, X_\beta)$ *as*

$$J(X_\alpha, X_\beta) = \inf_{f,g \in G} \|X_\alpha \bullet f - X_\beta \bullet g\|_E. \tag{5.71}$$

Note that, for any $X_\alpha, X_\beta \in \mathcal{F}$ and $f, g \in G$, we have

$$J(X_\alpha \bullet f, X_\beta \bullet g) = J(X_\alpha, X_\beta). \tag{5.72}$$

It means that J is invariant with respect to relabeling. The mapping J does not satisfy the triangle inequality, which is the reason why we introduce the mapping d.

Definition 5.14. *Let $X_\alpha, X_\beta \in \mathcal{F}$, we define $d(X_\alpha, X_\beta)$ as*

$$d(X_\alpha, X_\beta) = \inf \sum_{i=1}^{N} J(X_{n-1}, X_n) \tag{5.73}$$

where the infimum is taken over all finite sequences $\{X_n\}_{n=0}^{N} \in \mathcal{F}$ which satisfy $X_0 = X_\alpha$ and $X_N = X_\beta$.

For any $X_\alpha, X_\beta \in \mathcal{F}$ and $f, g \in G$, we have

$$d(X_\alpha \bullet f, X_\beta \bullet g) = d(X_\alpha, X_\beta), \tag{5.74}$$

and d is also invariant with respect to relabeling.

Remark 5.15. *The definition of the metric $d(X_\alpha, X_\beta)$ is the discrete version of the one introduced in [16]. In [16], the authors introduce the metric that we denote here as $\tilde{d}$ where*

$$\tilde{d}(X_\alpha, X_\beta) = \inf \int_0^1 |||X_s(s)|||_{X(s)}\, ds$$

where the infimum is taken over all smooth path $X(s)$ such that $X(0) = X_\alpha$ and $X(1) = X_\beta$ and the triple norm of an element V is defined at a point X as

$$|||V||| = \inf_g \|V - gX_\xi\|$$

where g is a scalar function, see [16] for more details. The metric $\tilde{d}$ also enjoys the invariance relabeling property (5.74)*. The idea behind the construction of d and $\tilde{d}$ is the same: We measure the distance between two points, in a way where two relabeled versions of the same point are identified. The difference is that in the case of d we use a set of points whereas in the case of $\tilde{d}$ we use a curve to join two elements X_α and X_β. Formally, we have*

$$\lim_{\delta \to 0} \frac{1}{\delta} J(X(s), X(s+\delta)) = |||X_s|||_{X(s)}. \tag{5.75}$$

We need to introduce the subsets of bounded energy in $\mathcal{F}_0$.

Definition 5.16. *We denote by $\mathcal{F}^M$ the set*

$$\mathcal{F}^M = \{X = (y, U, \nu) \in \mathcal{F} \mid h = \|\nu\|_{L^1} \leq M\}$$

and let $\mathcal{H}^M = \mathcal{H} \cap \mathcal{F}^M$.

The important property of the set $\mathcal{F}^M$ is that it is preserved both by the flow, see (5.55), and relabeling. Let us prove that

$$B_M \cap \mathcal{H} \subset \mathcal{H}^M \subset B_{\bar{M}} \cap \mathcal{H} \tag{5.76}$$

for $\bar{M} = 6(1+M)$ so that the sets $B_M \cap \mathcal{H}$ and $\mathcal{H}^M$ are in this sense equivalent. From (5.61), we get $\|y_\xi\|_{L^\infty} \leq 1+M$ which implies $\|y_\xi\|_{L^1} \leq 1+M$. By (5.44c), we get that $U_\xi^2 \leq y_\xi \nu \leq \frac{1}{2}(y_\xi^2 + \nu^2) \leq \frac{1}{2}(y_\xi + \nu)^2 \leq \frac{1}{2}(1+h)^2$ and therefore $\|U_\xi\|_{L^1} \leq 1+M$. Since $\int_0^1 y_\xi(\eta)\, d\eta = 1$ and $y_\xi \geq 0$, the set $\{\xi \in [0,1] \mid y_\xi(\xi) \geq \frac{1}{2}\}$ has strictly positive measure. For a point ξ_0 in this set, we get, by (5.44c), that $U^2(\xi_0) \leq \frac{\nu(\xi_0)}{y_\xi(\xi_0)} \leq 2(1+M)$. Hence, $\|U\|_{L^\infty} \leq |U(\xi_0)| + \|U_\xi\|_{L^1} \leq 3(1+M)$ and, finally,

$$\|U\|_{W^{1,1}_{\mathrm{per}}} + \|y_\xi\|_{L^1} + \|\nu\|_{L^1} \leq 6(1+M),$$

which concludes the proof of (5.76).

Definition 5.17. *Let d_M be the metric on $\mathcal{H}^M$ which is defined, for any $X_\alpha, X_\beta \in \mathcal{H}^M$, as*

$$d_M(X_\alpha, X_\beta) = \inf \sum_{i=1}^{N} J(X_{n-1}, X_n) \tag{5.77}$$

where the infimum is taken over all finite sequences $\{X_n\}_{n=0}^N \in \mathcal{H}^M$ which satisfy $X_0 = X_\alpha$ and $X_N = X_\beta$.

Lemma 5.18. *For any $X_\alpha, X_\beta \in \mathcal{H}^M$, we have*

$$\|y_\alpha - y_\beta\|_{L^\infty} + \|U_\alpha - U_\beta\|_{L^\infty} + |h_\alpha - h_\beta| \leq C_M d_M(X_\alpha, X_\beta) \tag{5.78}$$

for some fixed constant C_M which depends only on M.

Proof. First, we prove that for any $X_\alpha, X_\beta \in \mathcal{H}^M$, we have

$$\|y_\alpha - y_\beta\|_{L^\infty} + \|U_\alpha - U_\beta\|_{L^\infty} + |h_\alpha - h_\beta| \leq C_M J(X_\alpha, X_\beta) \tag{5.79}$$

for some constant C_M which depends only on M. By a change of variables in the integrals, we obtain

$$\begin{aligned}|h_\alpha - h_\beta| &= |\int_0^1 \nu_\alpha \circ f f_\xi\, d\xi - \int_0^1 \nu_\beta \circ g g_\xi\, d\xi| \\ &\leq \|X_\alpha \bullet f - X_\beta \bullet g\|_E.\end{aligned}$$

We have

$$\begin{aligned}&\|y_\alpha - y_\beta\|_{L^\infty} + \|U_\alpha - U_\beta\|_{L^\infty} \\ &\qquad\leq \|X_\alpha \bullet f - X_\beta \bullet g\|_E + \|y_\beta \circ f - y_\beta \circ g\|_{L^\infty} + \|U_\beta \circ f - U_\beta \circ g\|_{L^\infty} \\ &\qquad\leq \|X_\alpha \bullet f - X_\beta \bullet g\|_E + (\|y_{\beta\xi}\|_{L^\infty} + \|U_{\beta\xi}\|_{L^\infty})\|f - g\|_{L^\infty}.\end{aligned} \tag{5.80}$$

From the definition of $\mathcal{H}^M$ we get that, for any element $X = (y, U, \nu) \in \mathcal{H}^M$, we have $\|y_\xi\|_{L^\infty} + \|\nu\|_{L^\infty} \leq 2(1+M)$. Since $U_\xi^2 \leq y_\xi \nu$, from (5.44c), it follows that $\|U_\xi\|_{L^\infty} \leq 2(1+M)$. Thus, (5.80) yields

$$\|y_\alpha - y_\beta\|_{L^\infty} + \|U_\alpha - U_\beta\|_{L^\infty} \leq \|X_\alpha \bullet f - X_\beta \bullet g\|_E + 4(1+M)\|f-g\|_{L^\infty}. \tag{5.81}$$

We denote by C_M a generic constant which depends only on M. The identity (5.79) will be proved when we prove

$$\|f-g\|_{L^\infty} \leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E. \tag{5.82}$$

By using the definition of $\mathcal{H}$, we get that

$$\begin{aligned}
\|f_\xi - g_\xi\|_{L^1} &= \|\frac{1}{1+h_\alpha}(y_{\alpha\xi}\circ f + \nu_\alpha \circ f) f_\xi - \frac{1}{1+h_\beta}(y_{\beta\xi}\circ g + \nu_\beta\circ g) g_\xi\|_{L^1} \\
&\leq \frac{|h_\alpha - h_\beta|}{1+h_\beta} + \frac{1}{1+h_\beta}\|X_\alpha \bullet f - X_\beta \bullet g\|_E \\
&\leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E.
\end{aligned} \tag{5.83}$$

Let $\delta = g(0) - f(0)$. Similar to (5.67) and (5.68), we can conclude that

$$\begin{aligned}
\int_0^1 y_\beta \circ (f+\delta) f_\xi d\xi &= \int_{f(0)+\delta}^{f(0)+1+\delta} y_\beta d\xi \\
&= \int_{f(0)+\delta}^0 y_\beta d\xi + \int_0^1 y_\beta d\xi + \int_1^{1+f(0)+\delta} y_\beta d\xi \\
&= \int_{f(0)+\delta}^0 y_\beta d\xi + \int_0^1 y_\beta d\xi + \int_0^{f(0)+\delta} y_\beta d\xi + f(0) + \delta \\
&= f(0) + \delta.
\end{aligned}$$

Thus we have $\delta = \int_0^1 y_\beta \circ (f+\delta) f_\xi \, d\xi - f(0)$ and analogously $0 = \int_0^1 y_\beta \circ (f) f_\xi d\xi - f(0)$. Hence,

$$|\delta| = |\int_0^1 y_\beta \circ (f+\delta) f_\xi \, d\xi - \int_0^1 y_\alpha \circ f f_\xi \, d\xi|. \tag{5.84}$$

By (5.83), we get that

$$\|g - f - \delta\|_{L^\infty} \leq \|f_\xi - g_\xi\|_{L^1} \leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E. \tag{5.85}$$

Then, since

$$\begin{aligned}
\|y_\beta \circ (f+\delta) - y_\beta \circ g\|_{L^\infty} &\leq \|y_{\beta\xi}\|_{L^\infty} \|f + \delta - g\|_{L^\infty} \\
&\leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\|y_\alpha \circ f - y_\beta \circ (f+\delta)\|_{L^\infty} &\leq \|y_\alpha \circ f - y_\beta \circ g\|_{L^\infty} + \|y_\beta \circ g - y_\beta \circ (f+\delta)\|_{L^\infty} \\
&\leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E.
\end{aligned} \tag{5.86}$$

Then, (5.84) yields

$$|\delta| \leq C_M \|X_\alpha \bullet f - X_\beta \bullet g\|_E. \tag{5.87}$$

From (5.85) and (5.87), (5.82) and therefore (5.79) follows. For any $\varepsilon > 0$, we consider a sequence $\{X_n\}_{n=0}^N$ in $\mathcal{H}^M$ such that $X_0 = X_\alpha$ and $X_N = X_\beta$ and

$\sum_{i=1}^{N} J(X_{n-1}, X_n) \leq d_M(X_\alpha, X_\beta) + \varepsilon$. We have

$$\begin{aligned}\|y_\alpha - y_\beta\|_{L^\infty} + &\|U_\alpha - U_\beta\|_{L^\infty} + |h_\alpha - h_\beta| \\ &\leq \sum_{n=1}^{N} \|y_{n-1} - y_n\|_{L^\infty} + \|U_{n-1} - U_n\|_{L^\infty} + |h_{n-1} - h_n| \\ &\leq C_M \sum_{n=1}^{N} J(X_{n-1}, X_n) \\ &\leq C_M(d_M(X_\alpha, X_\beta) + \varepsilon).\end{aligned}$$

Since ε is arbitrary, we get (5.78). □

From the definition of d, we obtain that

$$d(X_\alpha, X_\beta) \leq \|X_\alpha - X_\beta\|_E, \tag{5.88}$$

so that the metric d is weaker than the E-norm.

Lemma 5.19. *The mapping $d_M : \mathcal{H}^M \times \mathcal{H}^M \to \mathbb{R}_+$ is a metric on $\mathcal{H}^M$.*

Proof. The symmetry is embedded in the definition of J while the construction of d_M from J takes care of the triangle inequality. From Lemma 5.18, we get that $d_M(X_\alpha, X_\beta) = 0$ implies that $y_\alpha = y_\beta$, $U_\alpha = U_\beta$ and $h_\alpha = h_\beta$. Then, the definition (5.61) of $\mathcal{F}_0$ implies that $\nu_\alpha = \nu_\beta$. □

Remark 5.20. *In [56], a metric on $\mathcal{H}$ is obtained simply by taking the norm of E. The authors prove that the semigroup is continuous with respect to this norm, that is, given a sequence X_n and X in $\mathcal{H}$ such that $\lim_{n\to\infty}\|X_n - X\|_E$, we have $\lim_{n\to\infty}\|\bar{S}_t X_n - \bar{S}_t X\|_E = 0$. However, $\bar{S}_t$ is not Lipschitz in this norm. From* (5.88)*, we see that the distance introduced in [56] is stronger than the one introduced here. (The definition of E in [56] differs slightly from the one employed here, but the statements in this remark remain valid).*

We can now prove the Lipschitz stability theorem for $\bar{S}_t$.

Theorem 5.21. *Given $T > 0$ and $M > 0$, there exists a constant C_M which depends only on M and T such that, for any $X_\alpha, X_\beta \in \mathcal{H}^M$ and $t \in [0, T]$, we have*

$$d_M(\bar{S}_t X_\alpha, \bar{S}_t X_\beta) \leq C_M d_M(X_\alpha, X_\beta). \tag{5.89}$$

Proof. By the definition of d_M, for any $\varepsilon > 0$, there exists a sequences $\{X_n\}_{n=0}^N$ in $\mathcal{H}^M$ and functions $\{f_n\}_{n=1}^{N-1}$, $\{g_n\}_{n=1}^{N-1}$ in G such that $X_0 = X_\alpha$, $X_N = X_\beta$ and

$$\sum_{i=1}^{N} \|X_{n-1} \bullet f_{n-1} - X_n \bullet g_{n-1}\|_E \leq d_M(X_\alpha, X_\beta) + \varepsilon. \tag{5.90}$$

Since $\mathcal{H}^M \subset B_{\bar{M}}$ for $\bar{M} = 6(1 + M)$, see (5.76), and $B_{\bar{M}}$ is preserved by relabeling, we have that $X_n \bullet f_n$ and $X_n \bullet g_{n-1}$ belong to $B_{\bar{M}}$. From the Lipschitz stability result given in (5.53), we obtain that

$$\|S_t(X_{n-1} \bullet f_{n-1}) - S_t(X_n \bullet g_{n-1})\|_E \leq C_M \|X_{n-1} \bullet f_{n-1} - X_n \bullet g_{n-1}\|_E, \tag{5.91}$$

where the constant C_M depends only on M and T. Introduce

$$\bar{X}_n = X_n \bullet f_n,\ \bar{X}_n^t = S_t(\bar{X}_n),\ \text{for } n = 0, \dots, N-1,$$

and

$$\tilde{X}_n = X_n \bullet g_{n-1},\ \tilde{X}_n^t = S_t(\tilde{X}_n),\ \text{for } n = 1, \dots, N.$$

Then (5.90) rewrites as

$$\sum_{i=1}^{N} \|\bar{X}_{n-1} - \tilde{X}_n\|_E \leq d_M(X_\alpha, X_\beta) + \varepsilon \tag{5.92}$$

while (5.91) rewrites as

$$\|\bar{X}_{n-1}^t - \tilde{X}_n^t\|_E \leq C_M \|\bar{X}_{n-1} - \tilde{X}_n\|_E. \tag{5.93}$$

We have

$$\Pi(\bar{X}_0^t) = \Pi \circ S_t(X_0 \bullet f_0) = \Pi \circ (S_t(X_0) \bullet f_0) = \Pi \circ S_t(X_0) = \bar{S}_t(X_\alpha)$$

and similarly $\Pi(\tilde{X}_N^t) = \bar{S}_t(X_\beta)$. We consider the sequence in $\mathcal{H}^M$ which consists of $\{\Pi \bar{X}_n^t\}_{n=0}^{N-1}$ and $\bar{S}_t(X_\beta)$. The set $\mathcal{F}^M$ is preserved by the flow and by relabeling. Therefore, $\{\Pi \bar{X}_n^t\}_{n=0}^{N-1}$ and $\bar{S}_t(X_\beta)$ belong to $\mathcal{H}^M$. The endpoints are $\bar{S}_t(X_\alpha)$ and $\bar{S}_t(X_\beta)$. From the definition of the metric d_M, we get

$$\begin{aligned} d_M(\bar{S}_t(X_\alpha), \bar{S}_t(X_\beta)) &\leq \sum_{n=1}^{N-1} \left(J(\Pi \bar{X}_{n-1}^t, \Pi \bar{X}_n^t)\right) + J(\Pi \bar{X}_{N-1}^t, \bar{S}_t(X_\beta)) \\ &= \sum_{n=1}^{N-1} \left(J(\bar{X}_{n-1}^t, \bar{X}_n^t)\right) + J(\bar{X}_{N-1}^t, \tilde{X}_N^t)) \qquad \text{by (5.72).} \end{aligned} \tag{5.94}$$

By using the equivariance of S_t, we obtain that

$$\begin{aligned} \tilde{X}_n^t &= S_t(\tilde{X}_n) = S_t((\bar{X}_n \bullet f_n^{-1}) \bullet g_{n-1}) \\ &= S_t(\bar{X}_n) \bullet (f_n^{-1} \circ g_{n-1}) = \bar{X}_n^t \bullet (f_n^{-1} \circ g_{n-1}). \end{aligned} \tag{5.95}$$

Hence, by using (5.72), that is, the invariance of J with respect to relabeling, we get from (5.94) that

$$\begin{aligned} d_M(\bar{S}_t(X_\alpha), \bar{S}_t(X_\beta)) &\leq \sum_{n=1}^{N-1} \Big(J(\bar{X}_{n-1}^t, \tilde{X}_n^t)\Big) + J(\bar{X}_{N-1}^t, \tilde{X}_N^t) \\ &\leq \sum_{n=1}^{N} \|\bar{X}_{n-1}^t - \tilde{X}_n^t\|_E \qquad \text{by (5.88)} \\ &\leq C_M \sum_{n=1}^{N} \|\bar{X}_{n-1} - \tilde{X}_n\|_E \qquad \text{by (5.93)} \\ &\leq C_M (d_M(X_\alpha, X_\beta) + \varepsilon). \end{aligned}$$

After letting ε tend to zero, we obtain (5.89). □

5.5 From Lagrangian to Eulerian coordinates

We now introduce a second set of coordinates, the so–called Eulerian coordinates. Therefore let us first consider $X = (y, U, \nu) \in \mathcal{F}$. We can define the Eulerian coordinates as in [56] and also obtain the same mappings between Eulerian and Lagrangian coordinates. For completeness we will state the results here.

Definition 5.22. *The set $\mathbb{D}$ consists of all pairs (u, μ) such that*

(i) $u \in H^1_{\mathrm{per}}$*, and*

(ii) μ *is a positive Radon measure whose absolute continuous part,* μ_{ac}*, satisfies*

$$\mu_{\mathrm{ac}} = (u^2 + u_x^2)dx. \tag{5.96}$$

We can define a mapping, denoted by L, from $\mathbb{D}$ to $\mathcal{H} \subset \mathcal{F}$:

Definition 5.23. *For any (u, μ) in $\mathbb{D}$, let*

$$\begin{aligned} h &= \mu([0,1)), \\ y(\xi) &= \sup\{y \mid F_\mu(y) + y < (1+h)\xi\}, \\ \nu(\xi) &= (1+h) - y_\xi(\xi), \\ U(\xi) &= u \circ y(\xi), \end{aligned} \tag{5.97}$$

where

$$F_\mu(x) = \begin{cases} \mu([\,0,x)) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([\,x,0)) & \text{if } x < 0. \end{cases} \tag{5.98}$$

Then $(y, U, \nu) \in \mathcal{F}_0$. We define $L(u, \mu) = \Pi(y, U, \nu)$.

Thus from any initial data $(u_0, \mu_0) \in \mathbb{D}$, we can construct a solution of (5.39) in $\mathcal{F}$ with initial data $X_0 = L(u_0, \mu_0) \in \mathcal{F}$. It remains to go back to the original variables, which is the purpose of the mapping M, defined as follows.

Definition 5.24. *For any $X \in \mathcal{F}$, then (u, μ) given by*

$$\begin{aligned} u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \\ \mu = y_\#(\nu d\xi), \end{aligned} \tag{5.99}$$

belongs to $\mathbb{D}$. We denote by M the mapping from $\mathcal{F}$ to $\mathbb{D}$ which for any $X \in \mathcal{F}$ associates the element $(u, \mu) \in \mathbb{D}$ given by (5.99).

The mapping M satisfies

$$M = M \circ \Pi. \tag{5.100}$$

The inverse of L is the restriction of M to $\mathcal{H}$, that is,

$$L \circ M = \Pi, \quad \text{and} \quad M \circ L = \mathbb{I}. \tag{5.101}$$

Next we show that we indeed have obtained a solution of the CH equation. By a weak solution of the Camassa–Holm equation we mean the following.

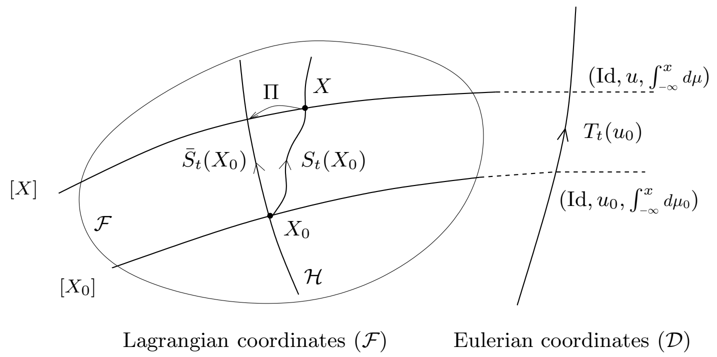

Figure 5.2: A schematic illustration of the construction of the semigroup. The set $\mathcal{F}$ where the Lagrangian variables are defined is represented by the interior of the closed domain on the left. The equivalence classes $[X]$ and $[X_0]$ (with respect to the action of the relabeling group G) of X and X_0, respectively, are represented by horizontal curves. To each equivalence class there corresponds a unique element in $\mathcal{H}$ and $\mathbb{D}$ (the set of Eulerian variables). The sets $\mathcal{H}$ and $\mathbb{D}$ are represented by the vertical curves.

Definition 5.25. *Let $u\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Assume that u satisfies*
(i) $u \in L^\infty([0,\infty), H^1_{\mathrm{per}})$,
(ii) the equations

$$\iint_{\mathbb{R}_+\times\mathbb{R}} -u(t,x)\phi_t(t,x) + (u(t,x)u_x(t,x) + P_x(t,x))\phi(t,x)dxdt \\ = \int_{\mathbb{R}} u(0,x)\phi(0,x)dx, \quad (5.102)$$

and

$$\iint_{\mathbb{R}_+\times\mathbb{R}} (P(t,x) - u^2(t,x) - \frac{1}{2}u_x^2(t,x))\phi(t,x) + P_x(t,x)\phi_x(t,x)dxdt = 0, \quad (5.103)$$

hold for all $\phi \in C_0^\infty([0,\infty),\mathbb{R})$. Then we say that u is a weak global solution of the Camassa–Holm equation.

Theorem 5.26. *Given any initial condition $(u_0,\mu_0) \in \mathbb{D}$, we denote $(u,\mu)(t) = T_t(u_0,\mu_0)$. Then $u(t,x)$ is a weak global solution of the Camassa–Holm equation.*

Proof. After making the change of variables $x = y(t,\xi)$ we get on the one hand

$$
\begin{aligned}
-\iint_{\mathbb{R}_+\times\mathbb{R}} u(t,x)\phi_t(t,x)dxdt &= -\iint_{\mathbb{R}_+\times\mathbb{R}} u(t,y(t,\xi))\phi_t(t,y(t,\xi))y_\xi(t,\xi)d\xi dt \\
&= -\iint_{\mathbb{R}_+\times\mathbb{R}} U(t,\xi)[(\phi(t,y(t,\xi))_t - \phi_x(t,y(t,\xi)))y_t(y,\xi)]y_\xi(t,\xi)d\xi dt \\
&= -\iint_{\mathbb{R}_+\times\mathbb{R}} [U(t,\xi)y_\xi(t,\xi)(\phi(t,y(t,\xi)))_t - \phi_\xi(t,y(t,\xi))U(t,\xi)^2]d\xi dt \\
&= \int_{\mathbb{R}} U(0,\xi)\phi(0,y(0,\xi))y_\xi(0,\xi)d\xi \\
&\quad + \iint_{\mathbb{R}_+\times\mathbb{R}} [U_t(t,\xi)y_\xi(t,\xi) + U(t,\xi)y_{\xi t}(t,\xi)]\phi(t,y(t,\xi))d\xi dt \\
&\quad + \iint_{\mathbb{R}_+\times\mathbb{R}} U^2(t,\xi)\phi_\xi(t,y(t,\xi))d\xi dt \\
&= \int_{\mathbb{R}} u(0,x)\phi(0,x)dx \\
&\quad - \iint_{\mathbb{R}_+\times\mathbb{R}} (Q(t,\xi)y_\xi(t,\xi) + U_\xi(t,\xi)U(t,\xi))\phi(t,y(t,\xi))d\xi dt,
\end{aligned}
\tag{5.104}
$$

while on the other hand

$$
\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} (u(t,x)u_x(t,x) + P_x(t,x))\phi(t,x)dxdt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} (U(t,\xi)U_\xi(t,\xi) + P_x(t,y(t,\xi))y_\xi(t,\xi))\phi(t,y(t,\xi))d\xi dt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} (U(t,\xi)U_\xi(t,\xi) + Q(t,\xi)y_\xi(t,\xi))\phi(t,y(t,\xi))d\xi dt,
\end{aligned}
\tag{5.105}
$$

which shows that (5.102) is fulfilled. Equation (5.103) can be shown analogously

$$
\begin{aligned}
&\iint_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi_x(t,x)dxdt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} Q(t,\xi)y_\xi(t,\xi)\phi_x(t,y(t,\xi))d\xi dt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} Q(t,\xi)\phi_\xi(t,y(t,\xi))d\xi dt \\
&\qquad = -\iint_{\mathbb{R}_+\times\mathbb{R}} Q_\xi(t,\xi)\phi(t,y(t,\xi))d\xi dt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} [\frac{1}{2}\nu(t,\xi) + (\frac{1}{2}U^2(t,\xi) - P(t,\xi))y_\xi(t,\xi)]\phi(t,y(t,\xi))d\xi dt \\
&\qquad = \iint_{\mathbb{R}_+\times\mathbb{R}} [\frac{1}{2}u_x^2(t,x) + u^2(t,x) - P(t,x)]\phi(t,x)dxdt.
\end{aligned}
\tag{5.106}
$$

In the last step we used the following

$$\int_0^1 u^2 + u_x^2 dx = \int_{y(0)}^{y(0)+1} u^2 + u_x^2 dx = \int_{y(0)}^{y(1)} u^2 + u_x^2 dx \tag{5.107}$$

$$= \int_{\{\xi\in[0,1] \mid y_\xi(t,\xi)>0\}} U^2 y_\xi + \frac{U_\xi^2}{y_\xi} d\xi = \int_0^1 \nu dx, \tag{5.108}$$

the last equality holds only for almost all t because for almost every $t \in \mathbb{R}_+$ the set $\{\xi \in [0,1] \mid y_\xi(t,\xi) > 0\}$ is of full measure and therefore

$$\int_0^1 (u^2 + u_x^2) dx = \int_0^1 \nu d\xi = h, \tag{5.109}$$

which is bounded by a constant for all times. Thus we proved that u is a weak solution of the Camassa–Holm equation. □

Next we return to the construction of the Lipschitz metric on $\mathbb{D}$.

Definition 5.27. *Let*

$$T_t := M\bar{S}_t L \colon \mathbb{D} \to \mathbb{D}. \tag{5.110}$$

Note that, by the definition of $\bar{S}_t$ and (5.100), we also have that

$$T_t = MS_tL.$$

Next we show that T_t is a Lipschitz continuous semigroup by introducing a metric on $\mathbb{D}$. Using the bijection L transport the topology from $\mathcal{H}$ to $\mathbb{D}$.

Definition 5.28. *We define the metric $d_{\mathbb{D}} \colon \mathbb{D} \times \mathbb{D} \to [0,\infty)$ by*

$$d_{\mathbb{D}}((u,\mu),(\tilde{u},\tilde{\mu})) = d(L(u,\mu), L(\tilde{u},\tilde{\mu})). \tag{5.111}$$

The Lipschitz stability of the semigroup T_t follows then naturally from Theorem 5.21. The stability holds on sets of bounded energy that we now introduce in the following definition.

Definition 5.29. *Given $M > 0$, we define the subsets $\mathbb{D}^M$ of $\mathbb{D}$, which corresponds to sets of bounded energy, as*

$$\mathbb{D}^M = \{(u,\mu) \in \mathbb{D} \mid \mu([0,1)) \leq M\}. \tag{5.112}$$

On the set $\mathbb{D}^M$, we define the metric $d_{\mathbb{D}^M}$ as

$$d_{\mathbb{D}^M}((u,\mu),(\tilde{u},\tilde{\mu})) = d_M(L(u,\mu), L(\tilde{u},\tilde{\mu})) \tag{5.113}$$

where the metric d_M is defined in (5.77).

The definition (5.113) is well-posed as we can check from the definition of L that if $(u,\mu) \in \mathbb{D}^M$ then $L(u,\mu) \in \mathcal{H}^M$. We can now state our main theorem.

Theorem 5.30. *The semigroup $(T_t, d_{\mathbb{D}})$ is a continuous semigroup on $\mathbb{D}$ with respect to the metric $d_{\mathbb{D}}$. The semigroup is Lipschitz continuous on sets of bounded energy, that is: Given $M > 0$ and a time interval $[0,T]$, there exists a constant C which only depends on M and T such that, for any (u,μ) and $(\tilde{u},\tilde{\mu})$ in $\mathbb{D}^M$, we have*

$$d_{\mathbb{D}^M}(T_t(u,\mu), T_t(\tilde{u},\tilde{\mu})) \leq C d_{\mathbb{D}^M}((u,\mu),(\tilde{u},\tilde{\mu}))$$

for all $t \in [0,T]$.

Proof. First, we prove that T_t is a semigroup. Since $\bar{S}_t$ is a mapping from $\mathcal{H}$ to $\mathcal{H}$, we have

$$T_t T_{t'} = M\bar{S}_t L M \bar{S}_{t'} L = M\bar{S}_t \bar{S}_{t'} L = M\bar{S}_{t+t'} L = T_{t+t'}$$

where we also use (5.101) and the semigroup property of $\bar{S}_t$. We now prove the Lipschitz continuity of T_t. By using Theorem 5.21, we obtain that

$$\begin{aligned} d_{\mathbb{D}^M}(T_t(u,\mu), T_t(\tilde{u},\tilde{\mu})) &= d_M(LM\bar{S}_t L(u,\mu), LM\bar{S}_t L(\tilde{u},\tilde{\mu})) \\ &= d_M(\bar{S}_t L(u,\mu), \bar{S}_t L(\tilde{u},\tilde{\mu})) \\ &\leq C d_M(L(u,\mu), L(\tilde{u},\tilde{\mu})) \\ &= C d_{\mathbb{D}^M}((u,\mu),(\tilde{u},\tilde{\mu})). \end{aligned}$$

□

5.6 The topology on $\mathbb{D}$

Proposition 5.31. *The mapping*

$$u \mapsto (u, (u^2 + u_x^2)dx) \tag{5.114}$$

is continuous from H^1_{per} into $\mathbb{D}$. In other words, given a sequence $u_n \in H^1_{\mathrm{per}}$ converging to $u \in H^1_{\mathrm{per}}$, then $(u_n, (u_n^2 + u_{nx}^2)dx)$ converges to $(u, (u^2 + u_x^2)dx)$ in $\mathbb{D}$.

Proof. Let $X_n = (y_n, U_n, \nu_n)$ be the image of $(u_n, (u_n^2 + u_{n,x}^2)dx)$ given as in (5.97) and $X = (y, U, \nu)$ the image of $(u, (u^2 + u_x^2)dx)$ given as in (5.97). We will at first prove that u_n converges to u in H^1_{per} implies that X_n converges against X in E. Denote $g_n = u_n^2 + u_{nx}^2$ and $g = u^2 + u_x^2$, then g_n and g are periodic functions. Moreover, as X_n, $X \in \mathcal{F}_0$, we have $y_{n,\xi} + \nu_n = 1 + h_n$ and $y_\xi + \nu = 1 + h$, where $h_n = \|\nu_n\|_{L^1}$ and $h = \|\nu\|_{L^1}$. By Definition 5.23, we have that $y_n(0) = 0$ and $y(0) = 0$, and hence

$$\begin{aligned} \int_0^{y_n(\xi)} g_n(x)dx + y_n(\xi) &= \int_0^{\xi} \nu_n(x)dx + y_n(\xi) = (1 + h_n)\xi, \\ \int_0^{y(\xi)} g(x)dx + y(\xi) &= \int_0^{\xi} \nu(x)dx + y(\xi) = (1 + h)\xi. \end{aligned} \tag{5.115}$$

By assumption $u_n \to u$ in H^1_{per}, which implies that $u_n \to u$ in L^∞, $g_n \to g$ in L^1, and $h_n \to h$. Therefore we also obtain that $y_n \to y$ in L^∞. We have

$$U_n - U = u_n \circ y_n - u \circ y = u_n \circ y_n - u \circ y_n + u \circ y_n - u \circ y. \tag{5.116}$$

Then, since $u_n \to u$ in L^∞, also $u_n \circ y_n \to u \circ y_n$ in L^∞ and as u is in H^1_{per}, we also obtain that $u \circ y_n \to u \circ y$ in L^∞. Hence, it follows that $U_n \to U$ in L^∞. By definition, the measures $(u^2 + u_x^2)dx$ and $(u_n^2 + u_{nx}^2)dx$ have no singular part, and we therefore have almost everywhere

$$y_\xi = \frac{1 + h}{1 + g \circ y} \quad \text{and} \quad y_{n\xi} = \frac{1 + h_n}{1 + g_n \circ y_n}. \tag{5.117}$$

Hence

$$\begin{aligned}y_\xi - y_{n\xi} &= y_\xi y_{n\xi}\Big(\frac{1+g_n\circ y_n}{1+h_n} - \frac{1+g\circ y}{1+h}\Big) \\ &= y_\xi y_{n\xi}\Big(\frac{1+g_n\circ y_n}{1+h_n} - \frac{1+g_n\circ y_n}{1+h}\Big) \\ &\quad + \frac{y_\xi y_{n\xi}}{1+h}(g_n\circ y_n - g\circ y_n + g\circ y_n - g\circ y).\end{aligned} \tag{5.118}$$

In order to show that $\zeta_{n,\xi}\to\zeta_\xi$ in L^1_{per}, it suffices to investigate

$$\int_0^1 |g\circ y_n - g\circ y| y_\xi y_{n,\xi} d\xi, \tag{5.119}$$

and

$$\int_0^1 |g_n\circ y_n - g\circ y_n| y_\xi y_{n,\xi} d\xi, \tag{5.120}$$

as we already know that $h_n\to h$ and therefore $y_{n,\xi}$ and y_ξ are bounded. Since $0\le y_\xi\le 1+h$, we have

$$\int_0^1 |g\circ y_n - g_n\circ y_n| y_\xi y_{n,\xi} d\xi \le (1+h)\|g-g_n\|_{L^1}. \tag{5.121}$$

For the second term, let $C=\sup_n(1+h_n)\ge 1$. Then for any $\varepsilon>0$ there exists a continuous function v with compact support such that $\|g-v\|_{L^1}\le\varepsilon/3C^2$ and we can make the following decomposition

$$\begin{aligned}(g\circ y - g\circ y_n)y_{n,\xi}y_\xi &= (g\circ y - v\circ y)y_{n,\xi}y_\xi \\ &\quad + (v\circ y - v\circ y_n)y_{n,\xi}y_\xi + (v\circ y_n - g\circ y_n)y_{n,\xi}y_\xi.\end{aligned} \tag{5.122}$$

This implies

$$\int_0^1 |g\circ y - v\circ y| y_{n,\xi}y_\xi d\xi \le C\int_0^1 |g\circ y - v\circ y| y_\xi d\xi \le \varepsilon/3, \tag{5.123}$$

and analogously we obtain $\int_0^1 |g\circ y_n - v\circ y_n| y_{n,\xi}y_\xi d\xi \le \varepsilon/3$. As $y_n\to y$ in L^∞ and v is continuous, we obtain, by applying the Lebesgue dominated convergence theorem, that $v\circ y_n\to v\circ y$ in L^1, and we can choose n so big that

$$\int_0^1 |v\circ y_n - v\circ y| y_{n,\xi}y_\xi d\xi \le C^2\|v\circ y - v\circ y_n\|_{L^1} \le \varepsilon/3. \tag{5.124}$$

Hence, we showed, that $\int_0^1 |g\circ y - g\circ y_n| y_{n,\xi}y_\xi d\xi \le \varepsilon$ and therefore, using (5.122),

$$\lim_{n\to\infty}\int_0^1 |g\circ y - g\circ y_n| y_{n,\xi}y_\xi d\xi = 0. \tag{5.125}$$

Combing now (5.118), (5.121), and (5.122), yields $\zeta_{n\xi}\to\zeta_\xi$ in L^1, and therefore also $\nu_n\to\nu$ in L^1. Because $\zeta_{n,\xi}$ and ν_n are bounded in L^∞, we also have that $\zeta_{n,\xi}\to\zeta_\xi$ in L^2 and $\nu_n\to\nu$ in L^2. Since $y_{n,\xi}$, ν_n and U_n tend to y_ξ, ν and U in L^2 and $\|U_n\|_{L^\infty}$ and $\|y_{n,\xi}\|_{L^\infty}$, are uniformly bounded, it follows from (5.44c) that

$$\lim_{n\to\infty}\|U_{n,\xi}\|_{L^2} = \|U_\xi\|_{L^2}. \tag{5.126}$$

Once we have proved that $U_{n,\xi}$ converges weakly to U_ξ, this will imply that $U_{n,\xi} \to U_\xi$ in L^2. For any smooth function ϕ with compact support in $[0,1]$ we have

$$\int_{\mathbb{R}} U_{n,\xi}\phi d\xi = \int_{\mathbb{R}} u_{n,x} \circ y_n y_{n,\xi}\phi d\xi = \int_{\mathbb{R}} u_{n,x}\phi \circ y_n^{-1} d\xi. \tag{5.127}$$

By assumption we have $u_{n,\xi} \to u_\xi$ in L^2. Moreover, since $y_n \to y$ in L^∞, the support of $\phi \circ y_n^{-1}$ is contained in some compact set, which can be chosen independently of n. Thus, using Lebesgue's dominated convergence theorem, we obtain that $\phi \circ y_n^{-1} \to \phi \circ y^{-1}$ in L^2 and therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}} U_{n,\xi}\phi d\xi = \int_{\mathbb{R}} u_x \phi \circ y^{-1} d\xi = \int_{\mathbb{R}} U_\xi \phi d\xi. \tag{5.128}$$

Form (5.44c) we know that $U_{n,\xi}$ is bounded and therefore by a density argument (5.128) holds for any function ϕ in L^2 and therefore $U_{n,\xi} \to U_\xi$ weakly and hence also in L^2. Using now that

$$\|U_{n,\xi} - U_\xi\|_{L^1} \le \|U_{n,\xi} - U_\xi\|_{L^2}, \tag{5.129}$$

shows that we also have convergence in L^1. Thus we obtained that $X_n \to X$ in E. As a second and last step, we will show that Π_2 is continuous, which then finishes the proof. We already know that $y_n \to y$ in L^∞ and therefore $a_n = \int_0^1 y_n(\xi)d\xi$ converges to $a = \int_0^1 y(\xi)d\xi$. Thus we obtain as an immediate consequence

$$\begin{aligned} &\|U_n(\xi - a_n) - U(\xi - a)\|_{L^\infty} \\ &\qquad \le \|U_n(\xi - a_n) - U(\xi - a_n)\|_{L^\infty} + \|U(\xi - a_n) - U(\xi - a)\|_{L^\infty}, \end{aligned} \tag{5.130}$$

and hence the same argumentation as before shows that $U_n(\xi - a_n) \to U(\xi - a)$ in L^∞. Moreover,

$$\begin{aligned} &\int_0^1 |U_{n,\xi}(\xi - a_n) - U_\xi(\xi - a)| d\xi \\ &\qquad \le \int_0^1 |U_{n,\xi}(\xi - a_n) - U_\xi(\xi - a_n)| d\xi + \int_0^1 |U_\xi(\xi - a_n) - U_\xi(\xi - a)| d\xi \\ &\qquad \le \|U_{n,\xi} - U_\xi\|_{L^1} + \|U_\xi(\xi - a_n) - U_\xi(\xi - a)\|_{L^1}, \end{aligned} \tag{5.131}$$

and again using the same ideas as in the first part of the proof, we have that $U_{n,\xi}(\xi - a_n) \to U_\xi(\xi - a)$ in L^1, which finally proves the claim, because of (5.88) ∎

Proposition 5.32. *Let (u_n, μ_n) be a sequence in $\mathbb{D}$ that converges to (u, μ) in $\mathbb{D}$. Then*

$$u_n \to u \text{ in } L^\infty_{\text{per}} \text{ and } \mu_n \overset{*}{\rightharpoonup} \mu. \tag{5.132}$$

Proof. Let $X_n = (y_n, U_n, \nu_n) = L(u_n, \mu_n)$ and $X = (y, U, \nu) = L(u, \mu)$. By the definition of the metric $d_{\mathbb{D}}$, we have $\lim_{n\to\infty} d(X_n, X) = 0$. We immediately obtain that

$$X_n \to X \text{ in } L^\infty(\mathbb{R}), \tag{5.133}$$

by Lemma 5.18. Denote by $C = \sup_n(1 + h_n)$. For any $x \in \mathbb{R}$ there exists ξ_n and ξ, which may not be unique, such that $x = y_n(\xi_n)$ and $x = y(\xi)$. We set $x_n = y_n(\xi)$. Then we have

$$u_n(x) - u(x) = u_n(x) - u_n(x_n) + U_n(\xi) - U(\xi), \tag{5.134}$$

and hence

$$\begin{aligned}
|u_n(x) - u_n(x_n)| &= |\int_{\xi}^{\xi_n} U_{n,\xi}(\eta)d\eta| \qquad (5.135)\\
&\leq \sqrt{|\xi_n - \xi|}\Big(\int_{\xi}^{\xi_n} U_{n,\xi}^2(\eta)d\eta\Big)^{1/2}\\
&\leq \sqrt{|\xi_n - \xi|}\Big(\int_{\xi}^{\xi_n} y_{n,\xi}\nu_n(\eta)d\eta\Big)^{1/2}\\
&\leq C\sqrt{|\xi - \xi_n|}\sqrt{|y_n(\xi_n) - y_n(\xi)|}\\
&= C\sqrt{|\xi - \xi_n|}\sqrt{|y(\xi) - y_n(\xi)|}\\
&\leq C\sqrt{|\xi - \xi_n|}\|y - y_n\|_{L^\infty}^{1/2}.
\end{aligned}$$

W.l.o.g., we can assume that $\|y_n - y\|_{L^\infty} < 1$, and $|\xi_n - \xi| < 1$ as y_n is increasing. Thus

$$|u_n(x) - u_n(x_n)| \leq C\|y_n - y\|_{L^\infty}^{1/2}. \tag{5.136}$$

Since $y_n \to y$ and $U_n \to U$ in L^∞, it follows that $u_n \to u$ in L^∞.

By weak-star convergence, we mean that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi d\mu_n = \int_{\mathbb{R}} \phi d\mu, \tag{5.137}$$

for all continuous functions with compact support. Using Definition 5.24, it follows that

$$\int_{\mathbb{R}} \phi d\mu_n = \int_{\mathbb{R}} \phi \circ y_n \nu_n d\xi \quad \text{and} \quad \int_{\mathbb{R}} \phi d\mu = \int_{\mathbb{R}} \phi \circ y \nu d\xi. \tag{5.138}$$

Since $y_n \to y$ in L^∞, the support of $\phi \circ y_n$ is contained in some compact set, which can be chosen independently of n and from Lebesgue's dominated convergence theorem, we obtain that $\phi \circ y_n \to \phi \circ y$ in L^2. As $\nu_n \to \nu$ in L^2, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi \circ y_n \nu_n d\xi = \int_{\mathbb{R}} \phi \circ y \nu d\xi. \tag{5.139}$$

This finishes the proof. □

Bibliography

[1] V. B. Baranetskii, V. P. Kotlyarov, *Asymptotic behavior in a back front domain of the solution of the KdV equation with a "step type" initial condition*, Teoret. Mat. Fiz., **126**, no. 2, 214–227 (2001).

[2] V. Batchenko and F. Gesztesy, *On the spectrum of Schrödinger operators with quasi-periodic algebro geometric KdV potentials*, J. Anal. Math. **95**, 333–387, (2005).

[3] E. D. Belokolos, A. I. Bobenko, V. Z. Enolskii, A. R. Its, and V. B. Matveev, *Algebro Geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[4] R.F. Bikbaev, *Time asymptotics of the solution of the nonlinear Schrödinger equation with boundary conditions of "step-like" type*, Teoret. Mat. Fiz.,**81:1**, 3–11 (1989).

[5] R.F. Bikbaev, *Structure of a shock wave in the theory of the Korteweg-de Vries equation*, Phys. Lett. A **141:5-6**, 289–293 (1989).

[6] R.F. Bikbaev and R. A. Sharipov, *The asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for the Korteweg-de Vries equation in a class of potentials with finite-gap behavior as $x \to \pm\infty$*, Teoret. Mat. Fiz. **78:3**, 345–356 (1989).

[7] I. N. Bondareva, *The Korteweg–de Vries equation in classes of increasing functions with prescribed asymptotic behavior as $|x| \to \infty$*, Mat. USSR Sb. **50:1**, 125–135 (1985).

[8] I. Bondareva and M. Shubin, *Increasing asymptotic solutions of the Korteweg–de Vries equation and its higher analogues*, Sov. Math. Dokl. **26:3**, 716–719 (1982).

[9] A. Boutet de Monvel and I. Egorova, *The Toda lattice with step-like initial data. Soliton asymptotics*, Inverse Problems **16:4**, 955–977 (2000).

[10] A. Boutet de Monvel, I. Egorova, and G. Teschl, *Inverse scattering theory for one-dimensional Schrödinger operators with steplike finite-gap potentials*, J. d'Analyse Math. **106:1**, 271–316, (2008).

[11] A. Boutet de Monvel, A. Kostenko, D. Shepelsky, and G. Teschl, *Long-time asymptotics for the Camassa–Holm equation*, (preprint).

[12] A. Bressan, A. Constantin, *Global solutions of the Hunter–Saxton equation*, SIAM J. Math. Anal. **37**, 996–1026 (2005).

[13] A. Bressan and A. Constantin, *Global conservative solutions of the Camassa–Holm equation*, Arch. Rational Mech. Anal. **183**, 215–239 (2007).

[14] A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa–Holm equation*, Analysis and Applications **5**, 1–27 (2007).

[15] A. Bressan and M. Fonte, *An optimal transportation metric for solutions of the Camassa–Holm equation*, Methods Appl. Anal. **12**, 191–220 (2005).

[16] A. Bressan, H. Holden and X. Raynaud, *Lipschitz metric for the Hunter–Saxton equation*, J. Math. Pures Appl. (to appear).

[17] V. S. Buslaev and V. N. Fomin, *An inverse scattering problem for the one-dimensional Schrödinger equation on the entire axis*, Vestnik Leningrad. Univ. **17:1**, 56–64 (1962).

[18] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Letters **71**, 1661–1664 (1993).

[19] R. Camassa, D. Holm, and J. M. Hyman, *A new integrable shallow water equation*, Adv. Appl. Mech. **31**, 1–33 (1994).

[20] G.M. Coclite, H. Holden, and K.H. Karlsen, *Well-posedness for a parabolic-elliptic system*, Discrete Contin. Dyn. Syst. **13**, 659–682 (2005).

[21] G.M. Coclite, H. Holden, and K.H. Karlsen, *Global weak solutions to a generalized hyperelastic-rod wave equation*, SIAM J. Math. Anal. **37**, 1044–1069 (2005).

[22] A. Cohen, *Solutions of the Korteweg-de Vries equation with steplike initial profile*, Comm. Partial Differential Equations **9:8**, 751–806 (1984).

[23] A. Cohen and T. Kappeler, *Solutions to the Korteweg–de Vries equation with initial profile in* $L_1^1(R) \cap L_N^1(R^+)$, SIAM J. Math. Anal. **18:4**, 991–1025 (1987).

[24] A, Constantin and J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. **181**, 229–243 (1998).

[25] A. Constantin and J. Escher, *Global existence and blow-up for a shallow water equation*, Annali Sc. Norm. Sup. Pisa **26**, 303–328 (1998).

[26] A. Constantin and J. Escher, *On the blow-up rate and the blow-up set of breaking waves for a shallow water equation*, Math. Z. **233**, 75–91 (2000).

[27] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations*, Arch. Rat. Mech. Anal., (to appear).

[28] W. Craig, *The trace formula for Schrödinger operators on the line*, Comm. Math. Phys. **126**, no.2, 379–407, (1989).

[29] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, Mc Graw-Hill, New York, 1955.

[30] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Comm. Pure Appl. Math. **49**, no. 1, 35–83 (1996).

[31] J. Delsarte, *Sur certaines transformations fonctionnelles relatives aux equations lineaires aux derivees partielles du second ordre*, C. R. Hebd. Acad. Sci. **206**, 1780–1782, (1938).

[32] W. Eckhaus and A. Van Harten, *The Inverse Scattering Transformation and Solitons: An Introduction*, Math. Studies 50, North-Holland, Amsterdam, 1984.

[33] I. Egorova, K. Grunert, and G. Teschl, *On the Cauchy Problem for the Korteweg–de Vires equation with Steplike Finite-Gap Initial Data*, Nonlinearity **22**, 1431–1457 (2009).

[34] I. Egorova, J. Michor, and G. Teschl, *Inverse scattering transform for the Toda hierarchy with quasi-periodic background*, Proc. Amer. Math. Soc. **135**, 1817–1827 (2007).

[35] V. D. Ermakova, *The inverse scattering problem on the whole axis for the Schrödinger equation with nondecreasing potential of special form*, Vestnik Khar'kov. Univ. **230**, 50–60 (1982).

[36] V. D. Ermakova, *The asymptotics of the solution of the Cauchy problem for the Korteweg-de Vries equation with nondecreasing initial data of special type*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **7**, 3–6 (1982).

[37] L. D. Faddeev, L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[38] N. E. Firsova, *An inverse scattering problem for the perturbed Hill operator*, Mat. Zametki **18:6**, 831–843 (1975).

[39] N.E. Firsova, *Resonances of a perturbed Hill operator with exponentially decreasing extrinsic potential*, Mat. Zametki **36**, no. 5, 711–724, (1984).

[40] N.E. Firsova, *The direct and inverse scattering problems for the one-dimensional perturbed Hill operator*, Math. USSR Sbornik **58**, no.2, 351–388, (1987).

[41] N. E. Firsova, *Solution of the Cauchy problem for the Korteweg-de Vries equation with initial data that are the sum of a periodic and a rapidly decreasing function*, Math. USSR-Sb. **63:1**, 257–265 (1989).

[42] B. Fuchssteiner and A. Fokas, *Symplectic structures, their Bäklund transforms, and hereditary symmetries*, Phys. D **4**, 47–66 (1981).

[43] C. S. Gardner, J. M. Green, M. D. Kruskal, R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett., **19**, 1095–1097 (1967).

[44] I.M. Gel'fand and B.M. Levitan, *On the determination of a differential equation from its spectral function*, Am. Math. Soc. Trans. **1**, 253-304, (1962).

[45] F. Gesztesy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions. Volume I:* $(1+1)$*-Dimensional Continuous Models.*, Cambridge Studies in Advanced Mathematics, Vol. 79, Cambridge University Press, Cambridge, 2003.

[46] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in "*Proceedings of the International Conference on Applications of Operator Theory*", (eds. I. Gohberg, P. Lancaster, and P. N. Shivakumar), Oper. Theory Adv. Appl., 87, Birkhäuser, Basel, 125–163 (1996).

[47] F. Gesztesy and P. Yuditskii, *Spectral properties of a class of reflectionless Schrödinger operators*, J. Funct. Anal. **241**, no.2, 486–527, (2006).

[48] K. Grunert, *The transformation operator for Schrödinger operators on almost periodic infinite-gap backgrounds*, (Preprint).

[49] K. Grunert, *Scattering theory for Schrödinger operators on steplike, almost periodic infinite-gap backgrounds*, (Preprint).

[50] K. Grunert, H. Holden, and X. Raynaud, *Lipschitz metric for the periodic Camassa–Holm equation*, arXiv:1005.3440.

[51] H. Holden, K. H. Karlsen, and N.H. Risebro, *Convergent schemes for the Hunter–Saxton equation*, Mathematics of Computations **76**, 699–744 (2007).

[52] H. Holden, N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws.*, Springer-Verlag, New York, 2007.

[53] H. Holden and X. Raynaud, *Global conservative solutions of the generalized hyperelastic-rod wave equation*, J. Differential Equations **233**, 448–484 (2007).

[54] H. Holden and X. Raynaud, *Global conservative solutions of the Camassa–Holm equation — a Lagrangian point of view*, Comm. Partial Differential Equations **32**, 1511–1549 (2007).

[55] H. Holden and X. Raynaud, *Global conservative multipeakon solutions of the Camassa–Holm equation*, J. Hyperbolic Differ. Equ. **4**, 39–64 (2007).

[56] H. Holden and X. Raynaud, *Periodic conservative solutions of the Camassa-Holm*, Ann. Inst. Fourier (Grenoble) **58**, 945–988 (2008).

[57] H. Holden and X. Raynaud, *Dissipative solutions for the Camassa–Holm equation*, Discrete Contin. Dyn. Syst. **24**, 1047–1112 (2009).

[58] J.K. Hunter, R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. **51**, 1498–1521 (1991).

[59] A. R. Its and A. F. Ustinov, *Time asymptotics of the solution of the Cauchy problem for the nonlinear Schrödinger equation with boundary conditions of finite density type*, Dokl. Akad. Nauk SSSR **291:1**, 91–95 (1986).

[60] R. S. Johnson, *Camassa–Holm, Korteweg–de Vries and related models for water waves*, J. Fluid Mech. **455**, 63–82 (2002).

[61] S. Kamvissis, *On the Toda shock problem*, Physica D **65**, 242–266 (1993).

[62] S. Kamvissis and G. Teschl, *Stability of periodic soliton equations under short range perturbations*, Phys. Lett. A, **364:6**, 480–483 (2007).

[63] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice under short range perturbations*, arXiv:0705.0346.

[64] S. Kamvissis and G. Teschl, *Stability of the periodic Toda lattice: Higher order asymptotics*, arXiv:0805.3847.

[65] T. Kappeler, *Solutions of the Korteweg de Vries equation with steplike initial data*, J. Differential Equations **63:3**, 306–331 (1986).

[66] T. Kappeler, P. Perry, M. Shubin and P. Topalov, *Solutions of mKdV in classes of functions unbounded at infinity*, J. Geom. Anal. **18**, 443–477 (2008).

[67] E. Ya. Khruslov, *Asymptotics of the Cauchy problem solution to the KdV equation with step-like initial data* Matem.sborn. **99**, 261–281 (1976).

[68] E. Ya. Khruslov and V. P. Kotlyarov, *Soliton asymptotics of nondecreasing solutions of nonlinear completely integrable evolution equations*, Advances in Soviet Mathematics **19**, 129–181 (1994).

[69] E. Ya. Khruslov and V. P. Kotlyarov, *Time asymptotics of the solution of the Cauchy problem for the modified Korteweg- de Vries equation with nondecreasing initial data*, Dokl. Akad. Nauk Ukrain. SSR Ser. A ,**10**, 61–64 (1986).

[70] E. Ya. Khruslov and V. P.Kotlyarov *Solitons of the nonlinear Schrödinger equation, which are generated by the continuous spectrum*, Teoret. Mat. Fiz., **68:2**, 172–186 (1986).

[71] E. Ya. Khruslov and H. Stephan, *Splitting of some nonlocalized solutions of the Korteweg-de Vries equation into solitons*, Mat. Fiz. Anal. Geom. **5:1-2**, 49–67 (1998).

[72] H. Krüger and G. Teschl, *Effective Prüfer angles and relative oscillation criteria*, J. Diff. Eq. **245**, 3823–3848 (2008).

[73] H. Krüger and G. Teschl, *Long-time asymptotics for the periodic Toda lattice in the soliton region*, arXiv:0807.0244.

[74] E. A. Kuznetsov and A. V. Mikhailov, *Stability of stationary waves in nonlinear weakly dispersive media*, Soviet Phys. JETP **40:5**, 855–859 (1975).

[75] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math., **7**, 159–193 (1968).

[76] B. Levin, *Fourier and Laplace types of transformations by means of differential equations of the second order*, Dokl. Akad. Nauk. SSSR **106**, no. 2, 187–190, (1956), (Russian).

[77] B.M. Levitan, *Inverse Sturm-Liouville problems*, VNU Science Press, Utrecht, 1987.

[78] B.M. Levitan, *Almost periodicity of infinite-zone potentials*, Math. USSR-Izv **18**, no. 2, 249–273, (1982).

[79] B.M. Levitan, *Approximation of infinite-zone potentials by finite-zone potentials*, Math. USSR-Izv **20**, no. 1, 55–87, (1983).

[80] B.M. Levitan and V.V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, Cambridge, 1982.

[81] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[82] V.A. Marchenko and I.V. Ostrovskii, *A characterization of the spectrum of the Hill Operator*, Math. USSR Sb. **26**, no.4, 493–554, (1975).

[83] N.I. Muskhelishvili, *Singular Integral Equations*, P. Noordhoff Ltd., Groningen, 1953.

[84] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons. The Inverse Scattering Method*, Springer, Berlin, 1984.

[85] V. Yu. Novokshenov, *Time asymptotics for soliton equations in problems with step initial conditions*, J. Math. Sci. **125:5**, 717–749 (2005).

[86] A. Povzner, *On differential equations of Sturm-Liouville type on a half-axis*, Mat. Sbornik **23**, no. 1, 3–52. (1948).

[87] F.S. Rofe–Beketov and A.M. Kohlkin, *Spectral Analysis of Differential Operators; Interplay Between Spectral and Oscillatory Properties*, World Scientific Monograph Series in Mathematics **7**, World Scientific, Singapore, 2005.

[88] M. Sodin and P. Yuditskii, *Almost periodic Sturm–Liouville operators with Cantor homogeneous spectrum*, Comment. Math. Helv. **70**, no.4, 639–658, (1995).

[89] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite–dimensional Jacobi inversion, and Hardy spaces of character–automorphic functions*, J. Geom. Anal. **7**, no.3, 387–435, (1997).

[90] E.M. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis **II**, Princeton University Press, Princeton, 2003.

[91] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, Graduate Studies in Mathematics, Volume **99**, Amer. Math. Soc., Providence, 2009.

[92] E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, Vol. 2, Clarendon Press, Oxford, 1958.

[93] S. Venakides, P. Deift, and R. Oba, *The Toda shock problem*, Comm. in Pure and Applied Math. **44**, 1171–1242 (1991).

[94] K. Yosida, *Functional Analysis*, Classics in Mathematics, Reprint of the sixth edition, Springer, Berlin, 1995.

[95] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Comm. Pure Appl. Math. **53**, 1411–1433 (2000).

[96] Z. Xin and P. Zhang, *On the uniqueness and large time behavior of the weak solutions to a shallow water equation*, Comm. Partial Differential Equations **27** 1815–1844 (2002).

Appendix A

Zusammenfassung

Im ersten Teil dieser Dissertation untersuchen wir den Kern von Transformationsoperatoren für eindimensionale Schrödingeroperatoren mit Potentialen, die asymptotisch nahe bei Bohr fast-periodischen Potentialen sind, deren Schrödingeroperatoren Spektren mit unendlich vielen Lücken besitzen. Darauf basierend werden wir direkte Streutheorie für den Fall von stufenartigen Hintergründen entwickeln.
Außerdem präsentieren wir eine Anwendung von direkter und indirekter Streutheorie auf die Korteweg–de Vries Gleichung, in der wir das zugehörige Cauchy-Problem für Anfangsbedingungen lösen, die Störungen vom Schwartz-Typ von quasi–periodischen Potentialen sind, deren Schrödingeroperatoren ein Spektrum mit endlich vielen Lücken besitzen, unter der Voraussetzung, dass die zugehörigen Teile der Spektren gleich oder disjunkt sind.
Im zweiten und letzten Teil beschäftigen wir uns mit der Camassa–Holm Gleichung und studieren die Stabilität von Lösungen des zugehörigen Cauchy-Problems in dem wir eine Lipschitz Metrik konstruieren.

Appendix B

Curriculum Vitae

Name:	Katrin Grunert
Date of birth:	June 6th, 1986
Place of birth:	Krems an der Donau, Austria
Parents:	Walter Grunert
	Herta Grunert, née Bauer

Education:

2008–2010:	Doctoral studies in mathematics at the University of Vienna
May 2008:	Mag.rer.nat. with distinction
	Diploma thesis: Long-time asymptotics for the KdV equation, advised by Gerald Teschl
2004–2008:	Diploma studies in mathematics at the University of Vienna
June 2004:	Matura (school leaving examination)
1996–2004:	BG Rechte Kremszeile, Krems an der Donau
1992–1996:	Volksschule, Kirchberg am Wagram

Employment:

January 2010 -present:	Research assistant fully supported by the FWF Start project Y330
August 2008 - December 2009:	Research stay at the NTNU Trondheim supported by the Yggdrasil project 195792/V11 of the Research Council of Norway
March 2008 - July 2009:	Research assistant fully supported by the FWF Start project Y330
October 2006 - February 2008	Tutor at the Faculty of Mathematics

Prices:

2009:	Studienpreis of the Austrian Mathematical Society for my Diploma thesis Long-time asymptotics for the KdV equation

Participation in:

17^{th} Annual Vojtech Jarnik International Mathematical Competition (2007)
14^{th} International Mathematics Competition for University Students (2007)

Publications and Preprints:

- Lipschitz metric for the periodic Camassa–Holm equation, together with H. Holden and X. Raynaud, arXiv:1005.3440.
- Scattering theory for one–dimensional Schrödinger operators on steplike, almost periodic infinite–gap backgrounds (Preprint).
- The transformation operator for Schrödinger operators on almost periodic infinite-gap backgrounds (Preprint).
- Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent, together with G. Teschl, Math. Phys. Anal. Geom. 12, 287–324 (2009).
- On the Cauchy problem for the Korteweg-de Vries equation with steplike finite-gap initial data I, Schwartz-type pertubations, together with I. Egorova and G. Teschl, Nonlinearity 22, 1431–1457 (2009).

Talks:

- DIFTA Seminar, NTNU Trondheim, November 4, 2009.
- Joint Mathematical Conference CSASC 2010, Prague, Czech Republic, January 22–27, 2010.