DIPLOMARBEIT

Titel der Diplomarbeit

Warped Products

in

General Relativity

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag.rer.nat.)

Verfasserin:	Nathalie Tassotti
Matrikelnummer:	0502223
Studienrichtung:	A 405 Mathematik
Betreuer:	Ao. Univ.-Prof. Michael Kunzinger

Wien, am 23.6.2010

Abstract

In general relativity one tries to find solutions to Einstein's equations

$$8\pi T = G$$

which typically is not possible in full generality because of the complexity of this system of partial differential equations. However, under certain assumptions on the spacetime and on the matter in it, solutions can be found.
Our aim is to investigate solutions which have a certain geometric structure, namely that of warped products. We will first develop general results for geometric properties of warped products and will afterwards use them in order to investigate spacetimes which are of this form. In this work we restrict our attention to Robertson-Walker-, Schwarzschild- and Reissner-Nordström spacetimes.

Zusammenfassung

Die allgemeine Relativitätstheorie beschäftigt sich mit der Suche nach Lösungen der Einstein-Gleichungen

$$8\pi T = G$$

Wegen der Komplexität dieses Systems von partiellen Differentialgleichungen ist es zumeist nicht möglich, explizite Lösungen zu bestimmen. Unter bestimmten Bedingungen an die Raumzeit und die darin enthaltene Materie wird es jedoch möglich das System zu lösen.
Das Ziel dieser Arbeit ist es, Lösungen zu untersuchen, die eine gewisse geometrische Struktur aufweisen, sogenannte warped products (verzerrte Produkte). Wir werden zunächst allgemeine Resultate für die Geometrie von warped products herleiten. Im Anschluss daran verwenden wir diese um Raumzeiten zu betrachten, deren Geometrien warped products sind. Wir beschränken uns auf Robertson-Walker, Schwarzschild- und Reissner-Nordström-Raumzeiten.

Acknowledgement

I want to thank my supervisor Michael Kunzinger for his patience, guidance and care during the development of this work.
Also I would like to thank the members of the DIANA research group for great courses and discussions during all the years.
Above all I am very grateful for the support of my family and friends during my studies. Last but not least I want to thank the people from Hofmobiliendepot and Move On Pilates Studio for becoming kind of a 'second family' for me.

Contents

Chapter 1

Introduction

In 1915, Albert Einstein stated the equations of general relativity,

$$8\pi T = G$$

where T is the stress-energy tensor of the matter in a spacetime M and G the Einstein tensor. Solving this system of 10 coupled nonlinear partial differential equations is in general not explicitly possible. However, under certain symmetry assumptions or assumptions on the matter distribution special solutions for these equations were constructed. Karl Schwarzschild for instance found in 1916 a solution for spherically symmetric vacuum spacetimes. A.G. Walker and H.P. Robertson found an exact solution of Einstein's field equations for an isotropic and homogeneous universe.

The aim of this work is to identify some of the well-known solutions of the field equations, namely the Robertson-Walker-, Schwarzschild- and Reissner-Nordström-solution, as so-called warped products. A warped product, denoted by $M = B \times_f F$, is a special kind of a product manifold with a different metric tensor than the usual one, i.e.

$$g_M = \mathrm{pr}_1{}^*(g_B) + (f \circ \mathrm{pr}_1)^2 \mathrm{pr}_2{}^*(g_F)$$

where g_B and g_F are the metric tensors on B resp. F.

In chapter 2 we will provide the technical tools for the investigation of geometric properties of warped products. In the first section we introduce the notion of lifts which will turn out to be of great importance in what follows. We then will compute the Levi-Civita connection on M and afterwards also the Riemannian tensor and Ricci-curvature.
In the following section we develop the geodesic equations,

$$\alpha'' = \langle \beta', \beta' \rangle_F f \circ \alpha \ \mathrm{grad(f)}$$

resp.

$$\beta'' = -\frac{2}{f \circ \alpha} \frac{d(f \circ \alpha)}{ds} \beta'$$

We also discuss the causal properties of M and it turns out that the causal structure on M is mainly determined by the one on B and also the converse

holds under certain assumptions on F. We will also prove that in case of Riemannian manifolds, completeness of M is equivalent to completeness of both factors B and F.
The third section of chapter 2 provides a generalization by using more fiber manifolds F_i, each one warped with a function f_i. Analogously to the singly warped product case we will derive general geometric properties and causality relations.

Chapter 3 starts with a short description of homogeneous and isotropic manifolds. Afterwards we investigate the Robertson-Walker solution of Einstein's equations for a homogeneous and isotropic universe. It turns out that the metric on such a spacetime $M(k, f) = I \times_f S$ where $I \subseteq \mathbb{R}$ and S an isotropic Riemannian manifold is of the form

$$dt^2 + f^2(t)d\sigma^2$$

hence it is a warped product. We will therefore apply the results developed in chapter 2.
The characterization of geodesics will lead to the result that for null geodesics $f(t)\frac{dt}{ds}$ is constant. This will allow us to compute the redshift parameter in terms of the warping function and so provide an explanation for the cosmological redshift. This result is the most direct observational evidence for the expansion of the universe. As a corollary we can compute the present distance between two galaxies.
After investigating causality and completeness of the spacetime, we will interpret it as a solution of Einstein's equations. It turns out that the stress-energy tensor is that of a perfect fluid. Since we assume that the energy density dominates pressure we will then restrict our investigation to the case in which no pressure is present, and therefore obtain Friedmann cosmological models.
The main result in the last section of this chapter is that a Robertson-Walker spacetime has an initial singularity and depending on f either expands indefinitely or ends in a final singularity.

The topic of chapter 4 is the Schwarzschild solution of Einstein's equation for a spherically symmetric and static universe containing a single star as a source of gravitation.
We will first derive the metric and obtain as the line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Hence we can see that again it turns out to be a warped product and we use chapter 2 for determining geometric properties.
After having developed the geodesic equations we will interpret the obtained constants on the Schwarzschild exterior, i.e. the region with $r > 2m$. Afterwards we will, in order to study the movement of material and lightlike particles, derive the orbit equation. It turns out that this equation differs from the classical Newtonian orbit equation only by a summand proportional to $\frac{1}{r^2}$. However, this difference is used for explaining the perihelion advance of Mercury's orbit. This is one of the most important experimental proofs of Einstein's theory. Other famous results in favor of it mentioned in this work are the time delay of

radar echoes and the deflection of light near a mass.
The last section in this chapter, which deals with Schwarzschild spacetime, offers a short historical overview of how physicists tried to get rid of the coordinate singularity at $r = 2m$. Different coordinate systems are introduced, but for a more detailed description we refer to [MTW].
In the final section we give another example for a warped product spacetime, namely the Reissner-Nordström spacetime. It models the spacetime surrounding a non-rotating spherically symmetric charged black hole. The line element is given by

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Therefore Schwarzschild spacetime is just the special case with $\epsilon = 0$.

Chapter 2

Warped products

In this section we want to prepare the mathematical tools which are necessary for the further study of different kinds of spacetimes.

2.1 Lifts

To relate the calculus of a product manifold $M \times N$ to that of its factors, the crucial notion is that of lifting:
If $f \in \mathcal{C}^\infty(M)$ the lift of f to $M \times N$ is $\widetilde{f} = f \circ \mathrm{pr}_1 \in \mathcal{C}^\infty(M \times N)$ where $\mathrm{pr}_1 : M \times N \to M$, $(p, q) \mapsto p$.
Let $x \in T_pM$ and $q \in N$ then the lift $\widetilde{x} \in T_{(p,q)}(M \times N)$ of x to (p, q) is the unique vector in $T_{(p,q)}(M \times N)$ such that

$$T_{(p,q)}\mathrm{pr}_1(\tilde{x}) = x$$

and

$$T_{(p,q)}\mathrm{pr}_2(\tilde{x}) = 0$$

For $X \in \mathfrak{X}(M)$ the lift of X to $M \times N$ is the vector field $\widetilde{X}$ whose value at each (p, q) is the lift of X_p to (p, q). $\widetilde{X}$ is smooth thus the lift of $X \in \mathfrak{X}(M)$ is the unique element of $\mathfrak{X}(M \times N)$ that is pr_1-related to X, i.e. $X \circ \mathrm{pr}_1 = T\mathrm{pr}_1 \circ \widetilde{X}$

$$\begin{tikzcd}
T(M \times N) \arrow[r, "T\mathrm{pr}_1"] \arrow[d, shift left, "\pi_{M\times N}"] & TM \arrow[d, shift left, "\pi_M"] \\
M \times N \arrow[r, "\mathrm{pr}_1"'] \arrow[u, shift left, "\tilde{X}"] & M \arrow[u, shift left, "X"]
\end{tikzcd}$$

and pr_2-related to the 0-vector field on N, i.e. $0 \circ \mathrm{pr}_2 = T\mathrm{pr}_2 \circ \widetilde{X}$ and so $T\mathrm{pr}_2 \circ \widetilde{X} = 0$, hence

$$\widetilde{X}(p, q) = (X_p, 0_q)$$

The set of such horizontal lifts $\widetilde{X}$ is denoted by $\mathcal{L}(M)$.
Analogously functions, tangent vectors and vector fields can be lifted to $M \times N$ via using the projection $\mathrm{pr}_2 : M \times N \to N$, $(p, q) \mapsto q$. The space of these

vertical lifts is denoted by $\mathcal{L}(N)$.

Note: $\mathcal{L}(M)$ and $\mathcal{L}(N)$ are vector subspaces of $\mathfrak{X}(M \times N)$ but neither one is invariant under multiplication with arbitrary functions $f \in \mathcal{C}^\infty(M \times N)$.

2.2 Warped products

Warped products were introduced the first time in [B.o'N.].
Let B and F be n-resp. l- dimensional semi Riemannian manifolds. On the semi Riemannian product manifold $B \times F$ the metric tensor g is given by

$$g = \mathrm{pr_1}^*(g_B) + \mathrm{pr_2}^*(g_F)$$

where g_F and g_B are the metric tensors on F resp. B.
A rich class of metrics on $B \times F$ can be obtained by homothetically warping the product metric on each fiber $p \times F$. Note that the manifold stays the same but the geometries differ.

Notation 2.2.1. On B we use the notation $g_B = \langle\ ,\ \rangle_B$, the Levi-Civita connection on B is denoted by ∇^B. Analogously, on F, $g_F = \langle\ ,\ \rangle_F$, the Levi-Civita connection is ∇^F. On $M = B \times_f F$ we write $g_M = \langle\ ,\ \rangle_M$, resp. ∇^M.

Definition 2.2.2. For semi Riemannian manifolds B, F of dimension n resp. l and $f > 0$ a smooth function on B we define the *warped product* $M = B \times_f F$ as the product manifold $B \times F$ furnished with the metric tensor

$$g_M = \mathrm{pr_1}^*(g_B) + (f \circ \mathrm{pr_1})^2 \mathrm{pr_2}^*(g_F)$$

Explicitly, for a vector $x \in T_{(p,q)}(M \times N)$ we get

$$\langle x, x\rangle_M = \langle T_{(p,q)}\mathrm{pr_1}(x), T_{(p,q)}\mathrm{pr_1}(x)\rangle_B + f^2(p)\langle T_{(p,q)}\mathrm{pr_2}(x), T_{(p,q)}\mathrm{pr_2}(x)\rangle_F$$

g_M is a metric tensor since symmetry is clear, nondegeneracy follows from nondegeneracy of g_F and g_B and because $f > 0$.
For $f = 1$, $B \times_f F$ is just the usual semi Riemannian product manifold.

Definition 2.2.3. B is called the *base* of $M = B \times_f F$ and F is called *fiber*.

Remark 2.2.4. The difference between the usual product manifold $B \times F$ and $M = B \times_f F$ is caused by the different metrics hence the geometries of the two manifolds are not the same. Tangent spaces remain unchanged since

$$\begin{aligned} T_{(p,q)}(B \times_f F) &= T_{(p,q)}(B \times F) \\ &= T_pB \times T_qF \\ &\cong (T_pB \times \{0\}) \oplus (\{0\} \times T_qF) \end{aligned}$$

Now we want to express the geometry of M in terms of the warping function f and the geometries of B and F.
The relation to the base B is almost as simple as for semi Riemannian products, but the warping function changes the relation to the fiber F.
The fibers $p \times F = \mathrm{pr_1}^{-1}(p)$ and leaves $B \times q = \mathrm{pr_2}^{-1}(q)$ are semi Riemannian submanifolds of M. We obtain the following properties:

Lemma 2.2.5.
(1) $\forall\ q \in F$, *the map* $pr_1|_{B\times q}$ *is an isometry onto* B
(2) $\forall\ p \in B$, *the map* $pr_2|_{p\times F}$ *is a positive homothety onto* F *with scale-factor* $\frac{1}{f(p)}$.
(3) $\forall\ (p,q) \in M$ *the leaf* $B \times q$ *and the fiber* $p \times F$ *are orthogonal at (p,q).*
(4) $\forall\ (p,q) \in M$ *we have that* $(T_{(p,q)}(p \times F))^{\perp} = T_{(p,q)}(B \times q)$.

Proof.
(1) $\mathrm{pr_1}^*(g_B) = g|_{B\times q}$ hence $\mathrm{pr}_1|_{B\times q}$ is an isometry.
(2) $\mathrm{pr}_2 : p \times F \to F$ is a diffeomorphism and

$$\mathrm{pr_2}^*(g_F) = \frac{1}{f(p)^2} g|_{p\times F}$$

so by using [O'N.] ,3.63 we get the result.
(3) Let $v = (v_1, 0) \in T_{(p,q)}(B \times q)$, $w = (0, w_2) \in T_{(p,q)}(p \times F)$. Then we get

$$\langle (v_1, 0), (0, w_2)\rangle_M = \langle v_1, 0\rangle_B + f^2(p)\langle 0, w_2\rangle_F = 0 + 0 = 0$$

(4)

$$\begin{aligned}
(T_{(p,q)}(p \times F))^{\perp} &= \{(z_1', z_2') \in T_{(p,q)}(B \times_f F) |\ \forall\ (0, z_2) \in T_{(p,q)}(p \times F) : \\
&\quad\ \langle 0, z_1'\rangle_B + f^2(p)\langle z_2, z_2'\rangle_F = 0\} \\
&= \{(z_1', z_2') \in T_{(p,q)}(B \times_f F) |\ \forall\ z_2 \in T_q F :\ \langle z_2, z_2'\rangle = 0\} \\
&= \{(z_1', z_2') \in T_{(p,q)}(B \times_f F) |\ z_2' = 0\} \\
&= T_p B \times 0 \\
&= T_{(p,q)}(B \times q)
\end{aligned}$$

□

Definition 2.2.6. Vectors tangent to leaves are called *horizontal*, those tangent to fibers *vertical*.
The last item of 2.2.5 will be used in the following proofs to show that a vector is horizontal or vertical.

Lemma 2.2.7. *For* $h \in \mathcal{C}^\infty(B)$ *the gradient of the lift* $h \circ pr_1$ *of* h *onto* $M = B \times_f F$ *is the lift of the gradient of* h *on* B *onto* M *(i.e. lift and gradient commute).*

Proof. Our task is to show that:

- $\mathrm{grad}(h \circ \mathrm{pr}_1)$ is pr_1-related to $\mathrm{grad}(h)$ on B
- $\mathrm{grad}(h \circ \mathrm{pr}_1)$ is horizontal. As remarked above we show this via the relation

$$(T_{(p,q)}(p \times F))^{\perp} = T_{(p,q)}(B \times q)$$

We start by showing *horizontality*:
Let $v \in T_{(p,q)}(p \times F)$, $v = (0, v_2)$ be a vertical tangent vector, then

$$\langle \mathrm{grad}(h \circ \mathrm{pr}_1), v\rangle_M = v(h \circ \mathrm{pr}_1) = T_{(p,q)}\mathrm{pr}_1(v)h = 0$$

since $T_{(p,q)}\mathrm{pr}_1(v) = 0$ because v is vertical.
So we conclude that $\mathrm{grad}(h \circ \mathrm{pr}_1)$ is horizontal.

To show *pr_1-relatedness* it is necessary to show that

$$T\mathrm{pr}_1 \circ \mathrm{grad}(h \circ \mathrm{pr}_1) = \mathrm{grad}(h) \circ \mathrm{pr}_1$$

So for x being horizontal, $x \in T_{(p,q)}(B \times q)$, $x = (x_1, 0)$, we get

$$\begin{aligned}
\langle T_{(p,q)}\mathrm{pr}_1(\mathrm{grad}(h \circ \mathrm{pr}_1)), T_{(p,q)}\mathrm{pr}_1(x)\rangle_B &\overset{\text{horizontal}}{=} \langle \mathrm{grad}(h \circ \mathrm{pr}_1), x\rangle_M \\
&= x(h \circ \mathrm{pr}_1) \\
&= T_{(p,q)}\mathrm{pr}_1(x)h \\
&= \langle \mathrm{grad}(h)|_p, T_{(p,q)}\mathrm{pr}_1(x)\rangle_B
\end{aligned}$$

so $T_{(p,q)}\mathrm{pr}_1(\mathrm{grad}(h \circ \mathrm{pr}_1)) = \mathrm{grad}(h) \circ \mathrm{pr}_1$ at each point.

□

This result allows us to simply write h for $h \circ \mathrm{pr}_1$ and $\mathrm{grad}(h)$ instead of $\mathrm{grad}(h \circ pr_1)$.

Before investigating the Levi-Civita connection ∇^M of M and how it can be related to those of B and F we state the basic properties of connections and prove some results for product manifolds which will frequently be used in what follows.

Definition 2.2.8. A *connection* on a smooth manifold M is a function $\nabla : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ such that
(D1) $\nabla_V W$ is $\mathcal{C}^\infty$-linear in V.
(D2)$\nabla_V W$ is $\mathbb{R}$-linear in W.
(D3) $\nabla_V(fW) = (Vf)W + f\nabla_V W$ for $f \in \mathcal{C}^\infty$.

Theorem 2.2.9. *On a semi Riemannian manifold M there is a unique connection ∇, the so called Levi-Civita connection of M, such that*
(D4) $[V, W] = \nabla_V W - \nabla_W V$
(D5) $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$
for all $V,\ W,\ X \in \mathfrak{X}(M)$.
∇ *is characterized by the Koszul formula*

$$\begin{aligned}
2\langle \nabla_V W, X\rangle &= V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle \\
&\quad -\langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle
\end{aligned}$$

Proof. See [O'N.], 3.11. □

Lemma 2.2.10. *Let* $X, Y \in \mathfrak{X}(B)$, $V, W \in \mathfrak{X}(F)$. *Then*
(1) $\langle \tilde{X}, \tilde{Y}\rangle_M = \langle X, Y\rangle_B$
(2) $\langle \tilde{V}, \tilde{W}\rangle_M(p, q) = f^2(p)\langle V, W\rangle_F(q)$
(3) $\langle \tilde{X}, \tilde{V}\rangle_M(p, q) = 0$
(4) $[\tilde{X}, \tilde{V}] = 0$
(5) $\tilde{X}(h \circ \mathrm{pr}_1) = X(h) \circ \mathrm{pr}_1$
(6) $\tilde{V}\langle \tilde{X}, \tilde{Y}\rangle_M = 0$

Proof.
(1)

$$\begin{aligned}
\langle \tilde{X}, \tilde{Y} \rangle_M(p,q) &= \langle T_{(p,q)}\mathrm{pr}_1(\tilde{X}), T_{(p,q)}\mathrm{pr}_1(\tilde{Y}) \rangle_B \\
&\quad + f^2(p)(\underbrace{\langle T\mathrm{pr}_2 \circ X, T\mathrm{pr}_2 \circ Y \rangle_F}_{=0}) \\
&= \langle X, Y \rangle_B(p)
\end{aligned}$$

(2) Analogously to (1)
(3)

$$\begin{aligned}
\langle \tilde{X}, \tilde{V} \rangle_M(p,q) &= \langle T_{(p,q)}\mathrm{pr}_1(\tilde{X}), T_{(p,q)}\mathrm{pr}_1(\tilde{V}) \rangle_B(p) \\
&\quad + f^2(p)\langle T_{(p,q)}\mathrm{pr}_2(\tilde{X}), T_{(p,q)}\mathrm{pr}_2(\tilde{Y}) \rangle_F(q) \\
&= 0
\end{aligned}$$

because $T_{(p,q)}\mathrm{pr}_1(\tilde{V}) = 0$ and $T_{(p,q)}\mathrm{pr}_2(\tilde{X}) = 0$.
(4) Follows since by [O'N.], 1.22., $[\tilde{X}, \tilde{V}]$ is pr_1-related to $[\tilde{X}, 0] = 0$ and pr_2-related to $[0, \tilde{V}] = 0$
(5) In general

$$T\mathrm{pr}_1 \circ \tilde{X} = X \circ \mathrm{pr}_1,$$

hence

$$\begin{aligned}
\tilde{X}(h \circ \mathrm{pr}_1) &= T(h \circ \mathrm{pr}_1)(\tilde{X}) \\
&= Th \circ T\mathrm{pr}_1 \circ \tilde{X} \\
&= Th \circ X \circ \mathrm{pr}_1 \\
&= X(h) \circ \mathrm{pr}_1
\end{aligned}$$

(6) Follows by noticing that

$$\tilde{V}\langle \tilde{X}, \tilde{Y} \rangle_M = \tilde{V}(\langle X, Y \rangle_B \circ \mathrm{pr}_1) = V\langle X, Y \rangle_B \circ \mathrm{pr}_1 = 0$$

since $V \in \mathfrak{X}(F)$ and $\langle X, Y \rangle_B$ is constant on F.

□

Proposition 2.2.11. *On $M = B \times_f F$ we have for $\tilde{X}, \tilde{Y} \in \mathcal{L}(B)$, $\tilde{V}, \tilde{W} \in \mathcal{L}(F)$ the following properties:*
(1) $\nabla^M_{\tilde{X}}\tilde{Y} \in \mathcal{L}(B)$ is the lift of $\nabla^B_X Y$ to M.
(2) $\nabla^M_{\tilde{X}}\tilde{V} = \nabla^M_{\tilde{V}}\tilde{X} = \frac{\tilde{X}f}{f}\tilde{V}$
(3) nor $\nabla^M_{\tilde{V}}\tilde{W} = II(\tilde{V}, \tilde{W}) = -\left(\frac{\langle \tilde{V}, \tilde{W} \rangle_M}{f}\right)$ grad(f)
(4) tan $\nabla^M_{\tilde{V}}\tilde{W} \in \mathcal{L}(F)$ *is the lift of $\nabla^F_V W$ to M.*

Proof. In what follows we denote by $\tilde{X}$, $\tilde{Y} \in \mathcal{L}(B)$, $\tilde{V}$,$\tilde{W} \in \mathcal{L}(F)$ the lifts of the vector fields $X, Y \in \mathfrak{X}(B)$, $V, W \in \mathfrak{X}(F)$ to M.

(1): For the statement $\nabla^M_{\tilde{X}}\tilde{Y} = \nabla^{\tilde{B}}_X Y$ we have to show:

- $T_{(p,q)}pr_1(\nabla^M_{\tilde{X}}\tilde{Y})= \nabla^B_X Y \circ \mathrm{pr}_1$
- $T_{(p,q)}pr_2(\nabla^M_{\tilde{X}}\tilde{Y}) = 0$

Now

$$\begin{aligned} T_{(p,q)}\mathrm{pr}_1(\nabla^M_{\tilde X}\tilde Y) &= \nabla^B_X Y|_p \\ &\Leftrightarrow \langle T_{(p,q)}\mathrm{pr}_1(\nabla^M_{\tilde X}\tilde Y) - \nabla^B_X Y|_p, A\rangle_B = 0 \ \ \forall\ A \in \mathfrak{X}(B) \\ &\Leftrightarrow \langle T_{(p,q)}\mathrm{pr}_1(\nabla^M_{\tilde X}\tilde Y), A\rangle_B = \langle \nabla^B_X Y|_p, A\rangle_B \ \ \forall\ A \in \mathfrak{X}(B) \end{aligned}$$

We have

$$\begin{aligned} 2\langle T_{(p,q)}\mathrm{pr}_1(\nabla^M_{\tilde X}\tilde Y), A\rangle_B &= 2\langle T_{(p,q)}\mathrm{pr}_1(\nabla^M_{\tilde X}\tilde Y), T_{(p,q)}\mathrm{pr}_1(\tilde A)\rangle_B \\ &= 2(\langle \nabla^M_{\tilde X}\tilde Y, \tilde A\rangle_M - f^2(p)\langle T_{(p,q)}\mathrm{pr}_2(\nabla^M_{\tilde X}\tilde Y), T_{(p,q)}\mathrm{pr}_2(\tilde A)\rangle_F) \end{aligned}$$

where the last term vanishes because $T_{(p,q)}\mathrm{pr}_2(\tilde A) = 0$. Using the Koszul-formula leads to

$$\begin{aligned} 2\langle \nabla^M_{\tilde X}\tilde Y, \tilde A\rangle_M &= \tilde X\langle \tilde Y, \tilde A\rangle_M + \tilde Y\langle \tilde A, \tilde X\rangle_M - \tilde A\langle \tilde X, \tilde Y\rangle_M - \\ &\quad -\langle \tilde X, [\tilde Y, \tilde A]\rangle_M + \langle \tilde Y, [\tilde A, \tilde X]\rangle_M + \langle \tilde A, [\tilde X, \tilde Y]\rangle_M \\ &= \tilde X(\langle Y, A\rangle_B \circ \mathrm{pr}_1) + \tilde Y(\langle A, X\rangle_B \circ \mathrm{pr}_1) - \tilde A(\langle X, Y\rangle_B \circ \mathrm{pr}_1) - \\ &\quad -\langle X, [Y, A]\rangle_B \circ \mathrm{pr}_1 + \langle Y, [A, X]\rangle_B \circ \mathrm{pr}_1 + \langle A, [X, Y]\rangle_B \circ \mathrm{pr}_1 \\ &= [X(\langle Y, A\rangle_B + Y\langle A, X\rangle_B - A\langle X, Y\rangle_B - \langle X, [Y, A]\rangle_B + \\ &\quad +\langle Y, [A, X]\rangle_B + \langle A, [X, Y]\rangle_B] \circ \mathrm{pr}_1 \\ &= 2\langle \nabla^B_X Y, A\rangle_B \end{aligned}$$

For the second part we calculate:

$$\begin{aligned} T_{(p,q)}\mathrm{pr}_2(\nabla^M_{\tilde X}\tilde Y) = 0 &\Leftrightarrow \langle T_{(p,q)}\mathrm{pr}_2(\nabla^M_{\tilde X}\tilde Y), A\rangle = 0 \ \ \forall\ A \in \mathfrak{X}(F) \\ &\Leftrightarrow \langle T_{(p,q)}\mathrm{pr}_2(\nabla^M_{\tilde X}\tilde Y), T_{(p,q)}\mathrm{pr}_2(\tilde A)\rangle_M = 0 \end{aligned}$$

Again using the Koszul formula we arrive at:

$$2\langle \nabla^M_{\tilde X}\tilde Y, \tilde A\rangle_M = -\tilde A\langle \tilde X, \tilde Y\rangle_M + \langle \tilde A, [\tilde X, \tilde Y]\rangle_M = 0$$

since all the other terms vanish by 2.2.10.

(2): Since

$$0 = [\tilde X, \tilde V] = \nabla^M_{\tilde X}\tilde V - \nabla^M_{\tilde V}\tilde X$$

it follows that $\nabla^M_{\tilde X}\tilde V = \nabla^M_{\tilde V}\tilde X$.
Using property (D5) of the Levi-Civita connection we get for $A \in \mathfrak{X}(B)$:

$$\langle \nabla^M_{\tilde X}\tilde V, \tilde A\rangle_M = -\langle \tilde V, \nabla^M_{\tilde X}\tilde V\rangle_M + \tilde X\langle \tilde V, \tilde A\rangle_M = 0$$

by 2.2.10 (3) because $\tilde V \in \mathcal{L}(F)$, but $\nabla^M_{\tilde X}\tilde V \in \mathcal{L}(B)$. Hence we just have to consider vector fields $W \in \mathfrak{X}(F)$ since $T_{(p,q)}M = T_pB \oplus T_qF$:

$$\begin{aligned} 2\langle \nabla^M_{\tilde X}\tilde V, \tilde W\rangle_M &\overset{\text{Koszul}}{=} -\tilde X\langle \tilde V, \tilde W\rangle_M + \tilde V\langle \tilde W, \tilde X\rangle_M - \tilde W\langle \tilde X, \tilde V\rangle_M \\ &\quad -\langle \tilde X, [\tilde V, \tilde W]\rangle_M + \langle \tilde V, [\tilde W, \tilde X]\rangle_M + \langle \tilde W, [\tilde X, \tilde V]\rangle_M \end{aligned}$$

All terms except the first vanish by 2.2.10 (3) resp. (4). The definition of the warped metric tensor leads to

(∗)

$$\langle \tilde{V}, \tilde{W} \rangle_M(p,q) = f^2(p)\langle V_q, W_q \rangle_F$$

Writing f for $f \circ \mathrm{pr}_1$ we get

$$\langle \tilde{V}, \tilde{W} \rangle_M = f^2(\langle V, W \rangle_F \circ \mathrm{pr}_2)$$

$\langle V, W \rangle_F \circ \mathrm{pr}_2$ is constant on fibers to which $\tilde{X}$ is tangent since

$$T_{(p,q)}\mathrm{pr}_1(\langle V, W \rangle_F \circ \mathrm{pr}_2) = 0,$$

so $\tilde{X}(\langle V, W \rangle_F \circ \mathrm{pr}_2) = 0$ and we get:

$$\begin{aligned} \tilde{X}\langle \tilde{V}, \tilde{W} \rangle_M &= \tilde{X}(f^2(\langle V, W \rangle_F \circ \mathrm{pr}_2) \\ &= 2fXf(\langle V, W \rangle_F \circ \mathrm{pr}_2) \\ &\overset{(*)}{=} 2\frac{Xf}{f}\langle \tilde{V}, W \rangle \end{aligned}$$

hence $\nabla^M_{\tilde{X}}\tilde{V} = \frac{Xf}{f}\tilde{V}$.

(3): Using again (D5) and (2) we get

$$\begin{aligned} \langle \nabla^M_{\tilde{V}}\tilde{W}, \tilde{X} \rangle_M &\overset{(D5)}{=} -\langle \tilde{W}, \nabla^M_{\tilde{V}}\tilde{X} \rangle_M + \underbrace{\tilde{V}\langle \tilde{W}, \tilde{X} \rangle}_{=0} \\ &\overset{(2)}{=} -\langle \tilde{W}, \frac{Xf}{f}\tilde{V} \rangle_M \\ &= -\frac{Xf}{f}\langle \tilde{V}, \tilde{W} \rangle_M \end{aligned}$$

Since by 2.2.7

$$\tilde{X}f = \langle \widetilde{\mathrm{grad(f)}}, \tilde{X} \rangle_M = \langle \mathrm{grad(f)}, X \rangle_B$$

it follows that

$$\langle \nabla^M_{\tilde{V}}\tilde{W}, \tilde{X} \rangle_M = -\langle \frac{\langle \tilde{V}, \tilde{W} \rangle_M}{f}\widetilde{\mathrm{grad(f)}}, \tilde{X} \rangle_M = -\frac{\langle \tilde{V}, \tilde{W} \rangle_M}{f}\langle \mathrm{grad(f)}, X \rangle_B$$

Because $(T_{(p,q)}(p \times F))^{\perp} = T_{(p,q)}(B \times q)$ the result follows.

(4): V and W are tangent to all fibers $p \times F$, so we have $\tan\nabla^M_{\tilde{V}}\tilde{W}$ is the fiber-covariant derivative on a fiber applied to the restrictions of V and W to that fiber, i.e. $\tan\nabla^M_{\tilde{V}}\tilde{W} = \widetilde{\nabla^F_V W}$. pr_2-relatedness then follows since homotheties preserve the Levi-Civita connection ([O'N.], 3.64).

□

Corollary 2.2.12. *Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ be arbitrary vector fields on M with X_1, $Y_1 \in B$, X_2, $Y_2 \in F$ then*

$$\nabla^M_X Y = \nabla^B_{X_1}Y_1 + \nabla^F_{X_2}Y_2 + \frac{1}{f}(X_1(f)Y_2 + Y_1(f)X_2 - \langle (0, X_2), (0, Y_2) \rangle_M \mathrm{grad}(f))$$

Proof. Applying 2.2.11 and linearity of ∇^M leads to

$$\begin{aligned}\nabla^M_X Y &= \nabla^M_{(X_1,X_2)}(Y_1,Y_2)\\ &= \nabla^M_{(X_1,0)+(0,X_2)}((Y_1,0)+(0,Y_2))\\ &= \nabla^M_{(X_1,0)}(Y_1,0)+\nabla^M_{(X_1,0)}(0,Y_2)+\nabla^M_{(0,X_2)}(Y_1,0)\nabla^M_{(0,X_2)}(0,Y_2)\\ &= (*)\end{aligned}$$

Since $(X_1,0),\ (Y_1,0)\in\mathcal{L}(B)$ and $(0,X_2),\ (0,Y_2)\in\mathcal{L}(F)$ we obtain

$$\begin{aligned}(*) &= \nabla^B_{X_1}Y_1+\frac{Y_1(f)}{f}X_2+\frac{X_1(f)}{f}Y_2-\frac{\langle(0,X_2),(0,Y_2)\rangle_M}{f}\ \mathrm{grad}(f)+\nabla^F_{X_2}Y_2\\ &= \nabla^B_{X_1}Y_1+\nabla^F_{X_2}Y_2+\frac{1}{f}\left(X_1(f)Y_2+Y_1(f)X_2-\langle(0,X_2),(0,Y_2)\rangle_M\ \mathrm{grad}(f)\right)\end{aligned}$$

□

Notation 2.2.13. The second fundamental form on leaves $B\times q$ is denoted by $II^{B\times q}$, on fibers $p\times F$ we write $II^{p\times F}$.

Corollary 2.2.14. *The leaves $B\times q$ of a warped product are totally geodesic (i.e. $II^{B\times q}=0$), the fibers $p\times F$ are totally umbillic (i.e. there exists a smooth normal vector field Z on $p\times F$ s.t. $II^{p\times F}(V,W)=\langle V,W\rangle_M Z\ \ \forall\ V,W\in\mathfrak{X}(p\times F)$).*

Proof. In general we have $II(V,W)=\text{nor}\ \nabla^M_V W$

The shape tensor of a leaf is just the projection of $\nabla^M_{\tilde{X}}\tilde{Y}$ onto F hence by using 2.2.11(1) ($\nabla^M_{\tilde{X}}\tilde{Y}=\widetilde{\nabla^B_X Y}$) we get the desired result.

By 2.2.11 (3) $II^{p\times F}(V,W)=-\left(\frac{\langle V,W\rangle_M}{f}\right)\mathrm{grad(f)}$ so

$$Z=-\frac{1}{f}\mathrm{grad(f)}$$

is the desired vector field satisfying

$$II^{p\times F}(V,W)=\langle V,W\rangle_M Z$$

□

Example 2.2.15.

- $\mathbb{R}^3\setminus\{0\}$ with spherical coordinates:

$$ds^2=dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

  For $r=1$ we get the line element of the unit sphere S^2. So $\mathbb{R}^3\setminus\{0\}$ is diffeomorphic to $\mathbb{R}^+\times S^2$ under the map $(t,p)\leftrightarrow tp$.
  Thus the formula for ds^2 shows that $\mathbb{R}^3\setminus\{0\}$ can be identified with the warped product $\mathbb{R}^+\times_r S^2$.
  Leaves are the rays from the origin and fibers are the spheres $S^2(r)$, $r>0$.
  In general $\mathbb{R}^n\setminus\{0\}$ is naturally isometric to $\mathbb{R}^+\times_r S^{n-1}$

- A surface of revolution: Leaves are different positions of the rotated curve, fibers are the circles of revolution.
  If M is built up by revolving a plane curve C about an axis in $\mathbb{R}^3$, and $f:C\to\mathbb{R}^+$ gives the distance to the axis, then $M=C\times_f S^1$

2.2.1 Geodesics

In $B \times_f F$ a curve γ can be written as

$$\gamma(s) = (\alpha(s), \beta(s))$$

with α, β being the projections of γ onto B resp. F.
We now want to characterize geodesics via the following proposition:

Proposition 2.2.16. *A curve $\gamma = (\alpha, \beta)$ in $M = B \times_f F$ is a geodesic if and only if*

1. $\alpha'' = \langle \beta', \beta' \rangle_F f \circ \alpha \operatorname{grad(f)}$ *in* B
2. $\beta'' = -\frac{2}{f\circ\alpha}\frac{d(f\circ\alpha)}{ds}\beta'$ *in* F

Proof. Since the result is local, it suffices to work in an arbitrary small interval around $s = 0$.

Case 1: $\gamma'(0)$ is neither horizontal nor vertical:
$\gamma'(0) = (\alpha'(0), \beta'(0))$, so α, β are regular.
Therefore we can suppose that α is an integral curve of X on B, β one of V on F. By denoting $\tilde{X}$, $\tilde{V}$ the lifts to M, γ is an integral curve of $\tilde{X} + \tilde{V}$. Thus

$$\gamma'' = \nabla^M_{\tilde{X}+\tilde{V}}(\tilde{X} + \tilde{V}) = \nabla^M_{\tilde{X}}\tilde{X} + \nabla^M_{\tilde{X}}\tilde{V} + \nabla^M_{\tilde{V}}\tilde{X} + \nabla^M_{\tilde{V}}\tilde{V}$$

($\Rightarrow$) $\gamma'' = 0$ therefore tan $\gamma'' = 0$ and nor $\gamma'' = 0$ with respect to the fibers

$$0 = \operatorname{nor} \gamma'' = \nabla^M_{\tilde{X}}\tilde{X} + \operatorname{nor} \nabla^M_{\tilde{V}}\tilde{V} = \underbrace{\nabla^B_X X}_{=\alpha''} - \frac{\langle \tilde{V}, \tilde{V}\rangle_M}{f}\operatorname{grad}(f)$$

and we conclude that $\alpha'' = \langle \beta', \beta' \rangle_F f \circ \alpha \operatorname{grad}(f)$.
Since

$$\begin{aligned} 0 = \tan \gamma'' &= \nabla^M_{\tilde{X}}\tilde{V} + \nabla^M_{\tilde{V}}\tilde{X} + \tan \nabla^M_{\tilde{V}}\tilde{V} \\ &= 2\frac{Xf}{f}\tilde{V}\tilde{X} + \tan \nabla^M_{\tilde{V}}\tilde{V} \\ &= 2\frac{Xf}{f}\tilde{V} + \underbrace{\nabla^F_V V}_{=\beta''} \end{aligned}$$

So

$$\beta'' = -\frac{2}{f\circ\alpha}\frac{d(f\circ\alpha)}{ds}\beta'$$

where we have used the relations $\langle \tilde{V}, \tilde{V}\rangle_M = f^2\langle V, V\rangle_F$ and $\alpha' = X$, $\beta' = V$

($\Leftarrow$) By assumption $\alpha'' - \langle \beta', \beta' \rangle_F f \circ \alpha \operatorname{grad}(f) = 0$. So

$$0 = \nabla^M_{\tilde{X}}\tilde{X} - \frac{\langle \tilde{V}, \tilde{V}\rangle_M}{f} \operatorname{grad}(f) = 0 = \operatorname{nor} \gamma''$$

By $\beta'' + \frac{2}{f\circ\alpha}\frac{d(f\circ\alpha)}{ds}\beta' = 0$ it follows that tan $\gamma'' = 0$.
Summing up leads to $\gamma'' = \operatorname{nor} \gamma'' + \tan \gamma'' = 0$

Case 2: $\gamma'(0)$ is horizontal, i.e. $\gamma'(0) = (\alpha'(0), 0)$:
($\Rightarrow$) Let γ be a geodesic in M, then, since leaves are totally geodesic,

$$(*) \qquad \gamma \text{ remains in } B \times \beta(0)$$

since $B \times \beta(0)$ is totally geodesic if and only if every geodesic in $B \times \beta(0)$ is a geodesic of M hence γ has to be a geodesic in $B \times \beta(0)$.
So β is constant (i.e. $\beta' = 0$) and therefore it remains to show that $\alpha'' = 0$ and $\beta'' = 0$. The second statement is clear since already $\beta' = 0$. To show the first recall that γ is a geodesic, hence $\gamma'' = 0 = (\alpha'', 0)$. Indeed, $I\!I = I\!I^{B\times\beta(0)} + \nabla^M_{\alpha'}\alpha' = \nabla^M_{\alpha'}\alpha'$ since $I\!I^{B\times\beta(0)} = 0$.

($\Leftarrow$) If (2) holds, then, since $\beta'(0) = 0$ by horizontality of γ, it follows with an ODE-argument that β is constant:

If we define $Z := \beta'$ then we obtain the system

$$Z'^k = \frac{dZ^k}{dt} + \Gamma^k_{ij}(\beta(t)) \underbrace{\beta'^i}_{Z^i} \underbrace{\beta'^j}_{Z^j} \overset{(2)}{=} -\frac{2}{f\circ\alpha}\frac{d(f\circ\alpha)}{ds} Z^k$$

hence

$$\frac{dZ^k}{dt} = \Gamma^k_{ij}(\beta(t)) Z^i Z^j - \frac{2}{f\circ\alpha}\frac{d(f\circ\alpha)}{ds} Z^k$$

$$Z(0) = 0$$

so $Z = 0$ is a solution and by uniqueness of solutions for ordinary differential equations, $Z = 0$ is the only solution and so $\beta' = 0$ and therefore β is constant. Then (1) implies that $\alpha'' = 0$ hence α is a geodesic in B, so by $(*)$ also $\gamma = (\alpha, \beta(0))$ is a geodesic.

Case 3: $\gamma'(0)$ is vertical and nonnull i.e. $\gamma'(0) = (0, \beta'(0))$ and $\langle\gamma', \gamma'\rangle_M \neq 0$:
We can suppose $\mathrm{grad}(f) \neq 0$ at $p = \alpha(0)$, otherwise the fiber $p \times F$ is totally geodesic since $I\!I^{p\times F}(V, W) = -\left(\frac{\langle V,W\rangle}{f}\right)\mathrm{grad}(f) = 0$. As in the case above α is constant and the result follows since $\gamma'' = (0, \beta'')$ and $I\!I(V, W) = I\!I^{p\times F} + \nabla^M_{\alpha'}\alpha' = 0$.
($\Rightarrow$) If γ is a geodesic then γ does not remain for any interval around zero in the totally umbillic fiber $p \times F = \alpha(0) \times F$ hence $\alpha'(s) \neq 0$ $\forall s \neq 0$. Otherwise

$$0 = \mathrm{nor}\nabla^M_{\gamma'}\gamma' = I\!I(\gamma', \gamma') = -\langle\gamma', \gamma'\rangle \underbrace{\frac{\mathrm{grad(f)}}{f}}_{\neq 0}$$

Therefore $\langle\gamma', \gamma'\rangle_M = 0$, a contradiction.
Hence there is a sequence $s_i \to 0$ such that for all i, $\gamma'(s_i)$ is neither horizontal nor vertical:
$\gamma'(s_i) = (\alpha'(s_i), 0)$ is not possible since $\beta'(0) \neq 0$, $\alpha'(0) = 0$ and by continuity $\gamma'(s_i) \to \gamma'(0) = (0, \beta'(0))$, therefore it cannot be horizontal.
$\gamma'(s_i) = (\alpha'(s_i), \beta'(s_i))$ where $\alpha'(s_i) \neq 0$ $\forall s \neq 0$ therefore a contradiction to

verticality.

Hence (1) and (2) follow by continuity from case 1.

($\Leftarrow$) (1) shows that

$$\alpha''(0) = \underbrace{\langle \beta'(0), \beta'(0) \rangle_F}_{\neq 0} f \circ \mathrm{grad(f)} \neq 0$$

hence there is a sequence s_i as above with $\gamma'(s_i)$ neither horizontal nor vertical. Since in the proof of case 1 we argued pointwise, any point s could be chosen instead of 0 and therefore the result follows and γ is a geodesic since if $\gamma''(s_i) = 0$ it follows that $\gamma''(0) = 0$.

Case 4: $\gamma'(0)$ is vertical and null, i.e. $\langle \gamma', \gamma' \rangle_M = 0$:
From $\gamma'(0)$ being null it follows by using

$$0 = \langle \gamma'(0), \gamma'(0) \rangle_M = f^2(p) \langle \beta'(0), \beta'(0) \rangle_F$$

that $\langle \beta'(0), \beta'(0) \rangle_F = 0$.
We now have to investigate two cases:

1. $\gamma'(0)$ is null and there is a sequence $t_i \to 0$ such that $\gamma(t_i)$ is nonnull.
2. $\gamma'(t)$ is null in an interval around 0

($\Leftarrow$)

1. Without loss of generality $\gamma'(t_i)$ is eventually vertical since otherwise the assertion follows from the cases 1 or 2. Using case 3 and continuity leads to the desired result.
2. Assuming as above that $\gamma'(t)$ is vertical for all t then $\alpha'(t) = 0$ and $\langle \beta'(t), \beta'(t) \rangle_F = 0$.
   So equations (1) and (2) decouple and because of the uniqueness of solutions we get $\alpha(t) \equiv \alpha_0$.
   Since $\beta'(0) \neq 0$ there is at least locally a vector field $V \in \mathfrak{X}(F)$ with $V = \beta'$. α is constant hence it is an integral curve of $X \equiv 0$. So

   $$\gamma'' = \underbrace{\nabla^M_{\tilde{X}} \tilde{X}}_{=0} + \underbrace{\nabla^M_{\tilde{X}} \tilde{V}}_{=0} + \underbrace{\nabla^M_{\tilde{V}} \tilde{X}}_{=0} + \nabla^M_{\tilde{V}} \tilde{V}$$

   Therefore

   $$\begin{aligned} \mathrm{nor}\, \gamma'' &= \mathrm{nor}\, \nabla^M_{\tilde{V}} \tilde{V} \\ &= -\frac{\langle \tilde{V}, \tilde{V} \rangle_M}{f} \mathrm{grad(f)} \\ &\overset{\mathrm{V}=\beta'}{=} -\langle \beta', \beta' \rangle_F f \circ \alpha \ \mathrm{grad(f)} \\ &\overset{(1)}{=} \alpha'' = 0 \end{aligned}$$

   $$\tan \gamma'' = \tan \nabla^M_{\tilde{V}} \tilde{V} = \nabla^F_V V = \beta'' = -\frac{2}{f \circ \alpha} \frac{d(f \circ \alpha)}{ds} \beta' = 0$$

   since α is constant. Hence $\gamma'' = 0$

($\Rightarrow$): We show that the equations for α'' and β'' are valid for the second possibility above (i.e $\gamma'(t)$ is null in an interval around 0) since the first one follows via continuity from the other cases which have already been investigated. By assumption,

$$0 = \langle \gamma', \gamma' \rangle_M = \langle \alpha', \alpha' \rangle_B + f^2 \langle \beta', \beta' \rangle_F \qquad (*)$$

and $\langle \alpha'(t), \alpha'(t) \rangle_B = 0$ resp. $\langle \beta'(t), \beta'(t) \rangle_F = 0 \ \forall \ t$.
Since $\gamma'(0)$ is vertical we know

$$\alpha'(t) = 0 \ \forall \ t \qquad (**)$$

hence $\alpha'' = 0$.
Using $(*)$ and $(**)$ leads to formula (1).
For (2) we observe that as above $0 = \gamma'' = \nabla^M_{\tilde{V}} \tilde{V}$, hence

$$0 = \tan \gamma'' = \tan \nabla^M_{\tilde{V}} \tilde{V} = \nabla^F_V V = \beta''$$

and, since α is constant, $\frac{d(f \circ \alpha)}{ds} = 0$ the result follows.

□

Remark 2.2.17. In the special case of a semi Riemannian product the warping function is constant so the geodesic equations reduce to $\alpha'' = 0$ resp. $\beta'' = 0$

Lemma 2.2.18. *If B and F are complete Riemannian manifolds then $M = B \times_f F$ is complete for every warping function f.*

Proof. We use the theorem by Hopf-Rinow and show that every Cauchy-sequence (p_i, q_i) converges:
Let $v \in T_{(p,q)}M$ be tangent to M, then since $f > 0$ and since F is a Riemannian manifold

$$\begin{aligned} \langle v, v \rangle_M &= \langle T_{(p,q)}\mathrm{pr}_1(v), T_{(p,q)}\mathrm{pr}_1(v) \rangle_B + f^2(p) \langle T_{(p,q)}\mathrm{pr}_2(v), T_{(p,q)}\mathrm{pr}_2(v) \rangle_F \\ &\geq \langle T_{(p,q)}\mathrm{pr}_1(v), T_{(p,q)}\mathrm{pr}_1(v) \rangle_B \end{aligned}$$

Hence for any curve segment α we have

$$L(\alpha) = \int_a^b \|\alpha'(s)\|^2 \, ds \geq \int_a^b \|(\mathrm{pr}_1 \circ \alpha)'(s)\|^2 \, ds = L(\mathrm{pr}_1 \circ \alpha)$$

So $\forall \ m, m' \in M$:

$$d(m, m') \geq d(\mathrm{pr}_1(m), \mathrm{pr}_1(m')) \quad (*)$$

which says that a curve in M is longer than its projection onto B.
Let (p_i, q_i) be a Cauchy-sequence in M then by $(*)$ p_i is a Cauchy sequence in B. B is complete hence there is a limit $p \in B$.
We can assume that the sequence lies in some compact set $K \subseteq B$ hence $f \geq c > 0$ on K. Analogously to the above argument we get

$$d(m, m') \geq c \ d(\mathrm{pr}_2(m), \mathrm{pr}_2(\text{m'})) \ \ \forall \ m, m' \in K \times F$$

So q_i is a Cauchy sequence in F and hence converges, so the original sequence (p_i, q_i) converges and thus M is complete. □

Remark 2.2.19. Also the converse of this result holds true, i.e. if M is a complete Riemannian manifold, then also B and F are complete Riemannian manifolds. A proof of this result will be given in the more general setting of multiply warped products below (see 2.3.10).

2.2.2 Curvature

Now our task is to express the curvature of a warped product $M = B \times_f F$ in terms of the warping function f and the curvatures of B resp. F. We will use the Riemannian curvature tensor, hence we first have to discuss the lifting of tensors:
For a covariant tensor A on B its lift $\tilde{A}$ to M is just its pullback ${\mathrm{pr}_1}^*(A)$ under the projection $\mathrm{pr}_1 : M \to B$
In case of a (1,s) tensor $A : \mathfrak{X}(B) \times \cdots \times \mathfrak{X}(B) \to \mathfrak{X}(B)$ we define for $v_1, \dots, v_s \in T_{(p,q)}M$ $\tilde{A}(v_1, \dots, v_s)$ to be the horizontal vector at (p, q) that projects to $A(T_{(p,q)}\mathrm{pr}_1(v_1), \dots, T_{(p,q)}\mathrm{pr}_1(v_s))$ in $T_p(B)$, i.e.

$$T_{(p,q)}\mathrm{pr}_1(\tilde{A}(v_1, \dots v_s)) = A(T_{(p,q)}\mathrm{pr}_1(v_1) \dots T_{(p,q)}\mathrm{pr}_1(v_s))$$

Thus $\tilde{A}$ is zero on vectors any one of which is vertical.
These definitions involve no geometry hence no warping function, and so they are valid for lifts from F too.

Remark 2.2.20.

- Let $\widetilde{{}^BR}$ and $\widetilde{{}^FR}$ be the lifts of the Riemannian curvature tensors ${}^BR$ and ${}^FR$ on B resp. F. Since the projection pr_1 is an isometry on each leaf, ${}^BR$ is just the Riemannian curvature of the leaf ($\widetilde{{}^BR} = {}^BR$), and analogously, since the projection pr_2 is a homothety and hence preserves Levi-Civita connection ([O'N.], 3.64), ${}^FR = \widetilde{{}^FR}$.

- Leaves are totally geodesic hence ${}^BR$ agrees with the curvature tensor ${}^MR$ of M on horizontal vectors which can be seen by using Gauss' equation

$$\begin{aligned} \langle {}^BR_{VW}X, Y\rangle_B &= \langle {}^MR_{VW}X, Y\rangle_M + \underbrace{\langle I\!I(V,X), I\!I(W,Y)\rangle}_{=0} \\ &\quad - \underbrace{\langle I\!I(V,Y), I\!I(W,Z)\rangle}_{=0} \end{aligned}$$

  The corresponding assertion fails in case of ${}^FR$ because fibers are just umbilic.

- For $h \in \mathcal{C}^\infty(B)$ the lift of the Hessian $H^h = \nabla^B(\nabla^B(h))$ of h to M is denoted by $\tilde{H}^h$ and it agrees with the Hessian of $h \circ \mathrm{pr}_1$ generally only on horizontal vectors. Indeed, for horizontal X

$$\nabla^M_{\tilde{X}}(\nabla^M_{\tilde{X}}(h \circ \mathrm{pr}_1)) = \nabla^M_{\tilde{X}}(\widetilde{\nabla^B(h)}) = \nabla^B\widetilde{(\nabla^B(h))}$$

  For vertical V we calculate:

$$\nabla^M_{\tilde{V}}(\nabla^M_{\tilde{V}}(h \circ \mathrm{pr}_1)) = \nabla^M_{\tilde{V}}\left(\frac{\tilde{V}f}{f}(h \circ \mathrm{pr}_1)\right) = \frac{\tilde{V}f}{f}\left(\frac{\tilde{V}f}{f}(h \circ \mathrm{pr}_1)\right)$$

In the following proposition some properties of R are discussed:

Proposition 2.2.21. *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor ${}^M R$. For $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{L}(B)$, $\tilde{U}, \tilde{V}, \tilde{W} \in \mathcal{L}(F)$ we have:*
(1) ${}^M R_{\tilde{X}\tilde{Y}} \tilde{Z} \in \mathcal{L}(B)$ is the lift of ${}^B R_{XY} Z$ on B.
(2) ${}^M R_{\tilde{V}\tilde{X}} \tilde{Y} = \left(\frac{H^f(\tilde{X},\tilde{Y})}{f} \right) \tilde{V}$ where H^f is the Hessian of f.
(3) ${}^M R_{\tilde{X}\tilde{Y}} \tilde{V} = {}^M R_{\tilde{V}\tilde{W}} \tilde{X} = 0$
(4) ${}^M R_{\tilde{X}\tilde{V}} \tilde{W} = \left(\frac{\langle V,W \rangle_M}{f} \right) \nabla_{\tilde{X}} (\mathrm{grad}(f))$
(5) ${}^M R_{\tilde{V}\tilde{W}} \tilde{U} = \widetilde{{}^F R_{VW} U} - \left(\frac{\langle \mathrm{grad}(f), \mathrm{grad}(f) \rangle_M}{f^2} \right) (\langle \tilde{V}, \tilde{U} \rangle_M \tilde{W} - \langle \tilde{W}, \tilde{U} \rangle_M \tilde{V})$

Remark 2.2.22. These are tensor identities hence are valid also for individual tangent vectors.

Proof.
(1) See 2.2.20
(2) $[\tilde{V}, \tilde{X}] = 0$ since $\tilde{V} \in \mathcal{L}(F)$, $\tilde{X} \in \mathcal{L}(B)$, so we obtain

$$ {}^M R_{\tilde{V}\tilde{X}} \tilde{Y} = \nabla^M_{[\tilde{V},\tilde{X}]} \tilde{Y} - [\nabla^M_{\tilde{V}}, \nabla^M_{\tilde{X}}] \tilde{Y} = -\nabla^M_{\tilde{V}} \nabla^M_{\tilde{X}} \tilde{Y} + \nabla^M_{\tilde{X}} \nabla^M_{\tilde{V}} \tilde{Y} $$

Since by 2.2.11 (1) $\nabla^M_{\tilde{X}} \tilde{Y} \in \mathcal{L}(B)$, we can apply 2.2.11 (2) to obtain

$$ \nabla^M_{\tilde{V}} (\nabla^M_{\tilde{X}} \tilde{Y}) = \left(\frac{\nabla^M_{\tilde{X}} \tilde{Y}(f)}{f} \right) \tilde{V} $$

Thus

$$ \begin{aligned} {}^M R_{\tilde{V}\tilde{X}} \tilde{Y} &= -\left(\frac{\nabla^M_{\tilde{X}} \tilde{Y}(f)}{f} \right) \tilde{V} + \frac{\tilde{X}(\tilde{Y}(f))}{f} \tilde{V} \\ &= \left(-\frac{\nabla^M_{\tilde{X}} \tilde{Y}(f)}{f} + \frac{\tilde{X}(\tilde{Y}(f))}{f} \right) \tilde{V} \\ &= \frac{\tilde{H}^f(\tilde{X},\tilde{Y})}{f} \tilde{V} \end{aligned} $$

(3) Since this is a tensor identity we can assume $[\tilde{V}, \tilde{W}] = 0$
Thus

$$ {}^M R_{\tilde{V}\tilde{W}} \tilde{X} = -\nabla^M_{\tilde{V}} \nabla^M_{\tilde{W}} \tilde{X} + \nabla^M_{\tilde{W}} \nabla^M_{\tilde{V}} \tilde{X} $$

and

$$ \nabla^M_{\tilde{V}} \nabla^M_{\tilde{W}} \tilde{X} = \nabla^M_{\tilde{V}} \left(\frac{Xf}{f} W \right) = \tilde{V} \left(\frac{Xf}{f} \right) \tilde{W} + \frac{Xf}{f} \nabla^M_{\tilde{V}} \tilde{W} $$

$\frac{Xf}{f}$ is constant on fibers, so $\tilde{V} \left(\frac{Xf}{f} \right) = 0$ and we get

$$ {}^M R_{\tilde{V}\tilde{W}} \tilde{X} = \left(\frac{Xf}{f} \right) (-\nabla^M_{\tilde{V}} \tilde{W} + \nabla^M_{\tilde{W}} \tilde{V}) = \left(\frac{Xf}{f} \right) [\tilde{V}, \tilde{W}] = 0 $$

by our assumption.
By taking care of symmetry properties of R it follows that

$$ \langle {}^M R_{\tilde{X}\tilde{Y}} \tilde{V}, \tilde{Z} \rangle_M = \langle {}^M R_{\tilde{X}\tilde{Y}} \tilde{Z}, \tilde{V} \rangle_M = 0 $$

$\forall\ \tilde{V} \in \mathcal{L}(F)$, $\tilde{Z} \in \mathcal{L}(B)$ since $\tilde{V}$ is vertical and ${}^M R_{\tilde{X}\tilde{Y}} \tilde{Z}$ hence ${}^M R_{\tilde{X}\tilde{Y}} \tilde{V} = 0$

${}^M R_{\tilde X\tilde Y}\tilde W$ is horizontal since $\langle {}^M R_{\tilde X\tilde V}\tilde W,\tilde U\rangle_M = \langle {}^M R_{\tilde W\tilde U}\tilde X,\tilde V\rangle_M$ which is 0 by (3), ${}^M R_{\tilde V\tilde W}\tilde X = 0$ and by using the Jacobi-identity:

$${}^M R_{\tilde V\tilde W}\tilde X +{}^M R_{\tilde W\tilde X}\tilde V +{}^M R_{\tilde X\tilde V}\tilde W = 0 -{}^M R_{\tilde X\tilde W}\tilde V +{}^M R_{\tilde X\tilde V}\tilde W = 0$$

Now using (2) leads to

$$\begin{aligned}\langle {}^M R_{\tilde X\tilde V}\tilde W,\tilde Y\rangle_M &= -\langle {}^M R_{\tilde V\tilde X}\tilde W,\tilde Y\rangle_M\\ &= \langle {}^M R_{\tilde V\tilde X}\tilde Y,\tilde W\rangle_M\\ &\overset{(2)}{=} \frac{\tilde H^f(\tilde X,\tilde Y)}{f}\langle\tilde V,\tilde W\rangle_M\\ &= \frac{\langle\tilde V,\tilde W\rangle_M}{f}\langle\nabla^M_{\tilde X}(\mathrm{grad}(f)),\tilde Y\rangle_M \quad \forall\,\tilde Y\end{aligned}$$

${}^M R_{\tilde X\tilde V}\tilde W$ is horizontal and the equation holds for all $\tilde Y$ so the result follows.
(4) ${}^M R_{\tilde V\tilde W}\tilde U$ is vertical since

$$\langle {}^M R_{\tilde V\tilde W}\tilde U,\tilde X\rangle_M = -\langle {}^M R_{\tilde V\tilde W}\tilde X,\tilde U\rangle_M = 0$$

pr_2 is a homothety on fibers and so, as remarked above, ${}^F R_{VW}U \in \mathcal{L}(F)$ is the application of the curvature tensor of each fiber to V, W and U. Thus ${}^F R_{VW}U$ and ${}^M R_{\tilde V\tilde W}\tilde U$ are related via the Gauss-equation:

$$\begin{aligned}\langle {}^F R_{VW}U, Y\rangle_F &= \langle {}^M R_{\tilde V\tilde W}\tilde U,\tilde Y\rangle_M + \langle I\!I(\tilde V,\tilde U), I\!I(\tilde W,\tilde Y)\rangle_M\\ &\quad -\langle I\!I(\tilde V,\tilde Y), I\!I(\tilde W,\tilde U)\rangle_M \qquad (*)\end{aligned}$$

Since the shape tensor of fibers is given by $I\!I^{p\times F}(V,W) = -\left(\frac{\langle V,W\rangle}{f}\right)\mathrm{grad}(f)$ we get

$$\begin{aligned}(*) &= \langle {}^M R_{\tilde V\tilde W}\tilde U,\tilde Y\rangle_M + \frac{1}{f^2}\langle\tilde V,\tilde U\rangle_M\langle\tilde W,\tilde Y\rangle_M\langle\mathrm{grad}f,\mathrm{grad}f\rangle_F\\ &\quad -\frac{1}{f^2}\langle\tilde V,\tilde Y\rangle_M\langle\tilde W,\tilde U\rangle_M\langle\mathrm{grad}f,\mathrm{grad}f\rangle_F\end{aligned}$$

Y is an arbitrary vertical vector field and so the result follows.

□

Finally we now turn to Ricci curvature Ric of warped products, writing ${}^B\widetilde{\mathrm{Ric}}$ for the lift (i.e. the pullback by pr_1) of the Ricci curvature of B and similarly ${}^F\widetilde{\mathrm{Ric}}$ for the Ricci curvature of F.

Corollary 2.2.23. *On $M = B\times_f F$ with $l = \dim F > 1$, let $\tilde X,\tilde Y$ be horizontal, $\tilde V,\tilde W$ be vertical. Then*
(1) ${}^M Ric(\tilde X,\tilde Y) = {}^B\widetilde{Ric}(X,Y) - \frac{l}{f}H^f(\tilde X,\tilde Y)$
(2) ${}^M Ric(\tilde X,\tilde V) = 0$
(3) ${}^M Ric(\tilde V,\tilde W) = {}^F\widetilde{Ric}(V,W) - \langle\tilde V,\tilde W\rangle_M f^*$, where

$$f^* = \frac{\Delta f}{f} + (l-1)\frac{\langle\mathrm{grad}f,\mathrm{grad}f\rangle_M}{f^2}$$

with Δf the Laplacian on B (here we need $l > 1$)

Proof. Choose a local frame

$$E_1 \dots E_n, E_{n+1} \dots E_{n+l}$$

with $E_1 \dots E_n \in \mathcal{L}(B)$ and $E_{n+1} \dots E_{n+l} \in \mathcal{L}(F)$ Then by using the results from 2.2.23 and [O'N.], 3.52. we get

(1):

$$\begin{aligned}
{}^{M}\mathrm{Ric}(\tilde{X},\tilde{Y}) &= \sum_{m\le n} \epsilon_m \langle {}^{B}R_{\tilde{X}E_m}\tilde{Y}, E_m\rangle_M + \sum_{n<m} \epsilon_m \langle {}^{M}R_{\tilde{X}E_m}\tilde{Y}, E_m\rangle_M \\
&= {}^{B}\widetilde{\mathrm{Ric}}(X,Y) - \frac{1}{f}\sum_{n<m} \epsilon_m \langle H^f(\tilde{X},\tilde{Y})E_m, E_m\rangle_M \\
&= {}^{B}\widetilde{\mathrm{Ric}}(X,Y) - \frac{l}{f} H^f(\tilde{X},\tilde{Y})
\end{aligned}$$

(2):

$${}^{M}\mathrm{Ric}(\tilde{X},\tilde{V}) = \sum_{m\le n} \epsilon_m \langle {}^{M}R_{\tilde{X}E_m}\tilde{V}, E_m\rangle_M + \sum_{n<m} \epsilon_m \langle {}^{M}R_{\tilde{X}E_m}\tilde{V}, E_m\rangle_M = 0$$

because the first sum vanishes by observing that $\langle {}^{M}R_{\tilde{X}E_m}\tilde{V}, E_m\rangle_M = 0$ and the second sum is also equal to zero. Indeed by using the symmetry properties of R we obtain $\langle {}^{M}R_{\tilde{X}E_m}\tilde{V}, E_m\rangle_M = \langle {}^{M}R_{\tilde{V}E_m}\tilde{X}, E_m\rangle_M = 0$.

(3):

$$\begin{aligned}
{}^{M}\mathrm{Ric}(\tilde{V},\tilde{W}) &= \sum_{n<m} \epsilon_m \langle {}^{F}\tilde{R}_{\tilde{V}E_m}\tilde{W}, E_m\rangle_M - \frac{\langle \mathrm{grad(f)}, \mathrm{grad(f)}\rangle}{f^2} \\
&\cdot \left(\sum_{n<m} \epsilon_m \langle \tilde{V},\tilde{W}\rangle_M \langle E_m, E_m\rangle_M - \sum_{n<m} \epsilon_m \langle E_m, \tilde{W}\rangle_M \langle \tilde{V}, E_m\rangle_M \right) \\
&\quad - \sum_{m\le n} \epsilon_m \frac{\langle \tilde{V},\tilde{W}\rangle_M}{f} \langle \nabla^M_{E_m}(\widetilde{\mathrm{grad} f}), E_m\rangle_M \\
&= {}^{F}\widetilde{\mathrm{Ric}}(V,W) + (l-1)\frac{\langle \mathrm{grad} f, \mathrm{grad} f\rangle_M}{f^2} \langle \tilde{V},\tilde{W}\rangle_M \\
&\quad + \frac{\langle \tilde{V},\tilde{W}\rangle_M}{f} \langle \nabla^M_{E_m}(\widetilde{\mathrm{grad} f}), E_m\rangle_M
\end{aligned}$$

The result follows since

$$\frac{\langle \tilde{V},\tilde{W}\rangle_M}{f} \langle \nabla^M_{E_m}(\widetilde{\mathrm{grad} f}), E_m\rangle_M = \frac{\langle \tilde{V},\tilde{W}\rangle_M}{f} \mathrm{div}(\mathrm{grad(f)}) = \frac{\langle \tilde{V},\tilde{W}\rangle_M}{f} \Delta f$$

□

2.2.3 Causal properties

In the following section we denote by (B, g_B) an n-dimensional manifold with signature $(-,+,\dots,+)$ and by (F, g_F) a Riemannian manifold of dimension l. Before investigating causal properties of warped products we provide general facts and definitions of causal relations. We will mainly follow [B.E.], ch.1., 3..

General causal properties

In what follows let (M, g) be a Lorentzian manifold.

Definition 2.2.24.

- A nonzero tangent vector $v \in T_pM$ is called *timelike* if $g(v, v) < 0$, v is called *spacelike* if $g(v, v) > 0$. We say that v is *causal* or *nonspacelike* if $g(v, v) \le 0$ and v is *null* if $g(v, v) = 0$.
- (M, g) is said to be *time-oriented* if M admits a continuous nowhere vanishing timelike vector field X, i.e. X_p is timelike for any $p \in M$. This vector field is used to separate the causal vectors at each point into two classes, the *future directed* resp. *past directed* vectors.

Causality relations

Definition 2.2.25. For $p, q \in M$

- $p \ll q$ means that there is a future directed timelike curve in M connecting p and q.
- $p \le q$ means that either $p = q$ or that there is a future directed causal curve in M from p to q.

Clearly if $p \ll q$ then $p \le q$ since every timelike curve is causal.

Definition 2.2.26. The *chronological past* and *chronological future* of p are given respectively by $I^-(p) = \{q \in M : q \ll p\}$ and $I^+(p) = \{q \in M : p \ll q\}$. For $A \subseteq M$ we have

$$I^{\pm}(A) = \{q \in M : \exists\ p \in A \text{ with } p \ll q \text{ resp. } q \ll p\}$$

The *causal past* and *causal future* of p resp. $A \subseteq M$ are defined as $J^-(p) = \{q \in M : q \le p\}$, $J^+ = \{q \in M : p \le q\}$ resp.

$$J^{\pm}(A) = \{q \in M : \exists\ p \in A \text{ with } p \le q \text{ resp. } q \le p\}$$

The *causal structure* of (M, g) may be defined as the collection of past and future sets at all points of M together with their properties.

We can see that $A \cup I^+(A) \subseteq J^+(A)$, $I^+(A) = \bigcup\{I^+(p) : p \in A\}$.
Since causality relations are transitive we obtain

Corollary 2.2.27. *If $x \ll y$ and $y \le z$ or if $x \le y$ and $y \ll z$ then $x \ll z$.*

Proof. It suffices to show $x \ll y$ and $y \le z$ implies $x \ll z$:
Let c_1 be a future pointing timelike curve from x to y, c_2 a causal curve from y to z and $c = c_1 \cup c_2$. Then c is a causal curve from x to z, but not a null geodesic. Hence there is, by [O'N.], 10.46., a timelike curve from x to z. More exactly, sufficiently close to c there is a timelike curve $\tilde{c}$ so without loss of generality $\tilde{c}'(t)$ is in the same timecone as $c_1'(t)$ for $t \in [t_0, t_1]$ where $c_1(t_0) = x, c_1(t_1) = y$ hence futurepointing since $\langle \tilde{c}'(t), c_1'(t)\rangle$ is close to $\langle c_1'(t), c_1'(t)\rangle < 0\ \forall\ t \in [t_0, t_1]$. Let X be a timelike vector field making the time orientation then $\langle \tilde{c}'(t), X(t)\rangle < 0$ in the beginning hence at every time. □

Corollary 2.2.28. *For $A \subseteq M$ the following relations hold:*

$$\begin{aligned} I^+(A) &= I^+(I^+(A)) = I^+(J^+(A)) \\ &= J^+(I^+(A)) \subseteq J^+(J^+(A)) = J^+(A) \end{aligned}$$

Proof. By [O'N.], 14.1. □

Causality conditions

Definition 2.2.29. A spacetime (M, g) is called *chronological* if $p \notin I^+(p)$ $\forall p \in M$, i.e. (M, g) does not contain any closed timelike curve. Physically this is a natural requirement since otherwise it can come to paradoxa such as an observer taking a trip from which he returns before his departure. It is the weakest causality relation which will be introduced.
If (M, g) does not contain any closed causal curve it is said to be *causal.* Equivalently (M, g) does not contain a pair of distinct points $p, q \in M$ with $p \leq q \leq p$.
We call a spacetime *distinguishing* if for all $p, q \in M$ either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$, i.e. distinct points have distinct chronological futures and chronological pasts.
Distinguishing spacetimes are called *causally continuous* if the set valued functions I^+, I^- are outer continuous i.e. if for each $p \in M$ and each compact set $K \subseteq M \setminus \overline{I^+(p)}$ there exists a neighborhood $U(p)$ of p such that $K \subseteq M \setminus I^+(q)$ $\forall\ q \in U(p)$
(M, g) is *strongly causal at* $p \in M$ if p has arbitrarily small causally convex neighborhoods , i.e. no causal curve intersects them into disconnected sets. Equivalently we can say that given any neighborhood U of p there is a neighborhood $V \subseteq U$ of p such that every causal curve segment with endpoints in V lies entirely in U. (M, g) is *strongly causal* if it is strongly causal at every $p \in M$.
(M, g) is *stably causal* if there is a fine $\mathcal{C}^0$-neighborhood $U(q)$ of g in the set of all Lorentzian metrics, Lor(M), such that each $g_1 \in U(g)$ is causal. ($|g - g_1|_0 < \delta$ if $\forall p \in M$ all of the corresponding coefficients and derivatives up to order 0 of g and g_1 are $\delta(p)$-close at p).
One of the most important causality conditions is *global hyperbolicity*, meaning that any pair of causally related points may be joined by a causal geodesic segment of maximal length. If (M, g) is strongly causal the equivalent condition is that $\forall p, q \in M$ $J^+(p) \cap J^-(q)$ is compact.
A distinguishing spacetime is *causally simple* if $J^+(p)$ and $J^-(p)$ are closed subsets of M for all $p \in M$.

One can show the following relations:

globally hyperbolic

$\Downarrow$

causally simple

$\Downarrow$

causally continuous

$\Downarrow$

$$\begin{array}{c} \text{stably causal} \\ \Downarrow \\ \text{strongly causal} \\ \Downarrow \\ \text{distinguishing} \\ \Downarrow \\ \text{causal} \\ \Downarrow \\ \text{chronological} \end{array}$$

Further results

Proposition 2.2.30. *Any compact spacetime (M, g) contains a closed timelike curve thus fails to be chronological.*

Proof. $I^+(p)$ is open so $\{I^+(p) : p \in M\}$ is an open covering of M. By compactness it follows that $M \subseteq \bigcup_{j=1}^{k} I^+(p_j)$. Now $p_1 \in I^+(p_{i(1)})$ for some $1 \leq i(1) \leq k$. Similarly $p_{i(1)} \in I^+(p_{i(2)})$ for $1 \leq i(2) \leq k$. Inductively we obtain an infinite sequence $\ldots p_{i(3)} \ll p_{i(2)} \ll p_{i(1)} \ll p_1$. k is finite so there is only a finite number of distinct $p_{i(j)}$ hence there are repetitions on the list, by transitivity of $\ll$ it follows that $p_{i(n)} \in I^+(p_{i(n)})$ for some index $i(n)$ thus M contains a closed timelike curve through $p_{i(n)}$. □

It can also be shown that a spacetime is stably causal iff it admits a global time function, i.e. a function strictly increasing along each future directed causal curve.

Lemma 2.2.31. *Let $K \subset\subset M$ be strongly causal. If α is a future inextendible curve, $\alpha : [0, b) \to M$ starting in K then there is some $s > 0$ such that $\alpha(t) \notin K$ $\forall\ t \geq s$. Hence α leaves K and does not return anymore.*

Proof. Suppose $\exists\ s_i \nearrow b$ with $\alpha(s_i) \in K\ \forall\ i$. Without loss of generality we can suppose $\alpha(s_i) \to p \in K$. Since α is future inextendible there has to be a further sequence $t_i \nearrow b$ such that $\alpha(t_i)n \to p$. Without loss of generality there is a neighborhood U of p with $\alpha(t_i) \notin U\ \forall i$. Using subsequences we can suppose $s_1 < t_1 < s_2 < t_2 \ldots$.
The causal curves $\alpha|_{[s_k, s_{k+1}]}$ start and end sufficiently close near p for k sufficiently large, but all leave U, a contradiction to strong causality. □

Lemma 2.2.32. *Let $K \subset\subset M$ be strongly causal. Let (α_n) be a sequence of futurepointing causal curves, $\alpha_n : [0, 1] \to M$, $\alpha_n([0, 1]) \subseteq K$, $\alpha_n(0) \to p$, $\alpha_n(1) = q$, $p \neq q$. Then there exists a future pointing causal broken geodesic λ from p to q and a subsequence $(\alpha_{n_k})_k$ of (α_n) with*

$$\lim_{k \to \infty} L(\alpha_{n_k}) \leq L(\lambda)$$

Proof. See [O'N.], 14.14. □

In order to study causality breakdowns and geodesic incompleteness we define:

Definition 2.2.33. Let $\gamma : [a, b] \to M$ be a curve in M. The point $p \in M$ is called *endpoint of γ corresponding to $t = b$* if

$$\lim_{t \to b^-} \gamma(t) = p$$

If $\gamma : [a, b] \to M$ is a future (resp. past) directed causal curve with endpoint p corresponding to $t = b$, then p is called a *future-* (resp. *past-*) *endpoint of* γ.
A causal curve is *future-(resp. past) inextendible* if it has no future (resp. past) endpoint and the curve is *inextendible* if it is both future and past inextendible.

Cauchy surfaces

Globally hyperbolic spacetimes may be characterized by using Cauchy surfaces, i.e. subsets of M which every inextendible causal curve intersects exactly once. It may be shown that a spacetime is globally hyperbolic iff it admits a Cauchy surface ([H.E.]). Furthermore the following theorem was established in [Ge] in 1970:

Theorem 2.2.34. *If (M, g) is a globally hyperbolic spacetime of dimension n then M is homeomorphic to $\mathbb{R} \times S$ where S is an $(n-1)$-dimensional topological submanifold of M, and for each $t \in M$, $\{t\} \times S$ is a Cauchy surface.*

Proof. [B.E.], 3.17 □

In a complete Riemannian manifold any two points may be joined by a geodesic of minimal length. The Lorentzian analogue developed by Avez and Seifert states (see [B.E.], 3.18.)

Theorem 2.2.35. *Let (M, g) be globally hyperbolic and $p \le q$. Then one can find a causal geodesic from p to q with length greater than or equal to that of any other future directed causal curve from p to q.*

Take care that the geodesic in the theorem is not necessarily unique.

Application to warped products

Lemma 2.2.36. *The warped product $M = B \times_f F$ may be time oriented iff*

- *either (B, g_B) is time-oriented (for $n \ge 2$)*
- *or $n = 1$ and g_B is a negative definite metric.*

Proof. ($\Rightarrow$):

- $n = 1$: (B, g_B) has a negative definite metric by definition.
- $n \ge 2$: M is time-orientable hence there exists a continuous timelike vector field X on M. By assumption $f > 0$ and g_F is positive definite (since F is a Riemannian manifold), therefore we obtain for $\bar{p} = (p, q) \in M$:

$$\begin{aligned}\langle T_{(p,q)}\mathrm{pr}_1(X), T_{(p,q)}\mathrm{pr}_1(X)\rangle_B &\le \langle T_{(p,q)}\mathrm{pr}_1(X), T_{(p,q)}\mathrm{pr}_1(X)\rangle_B \\ &\quad + f^2(p)\langle T_{(p,q)}\mathrm{pr}_2(X), T_{(p,q)}\mathrm{pr}_2(X)\rangle_F \\ &= \langle X|_{\bar{p}}, X|_{\bar{p}}\rangle_M < 0\end{aligned}$$

Thus for fixed but arbitrary $q_0 \in F$ the vector field $\bar{X}(p) := T_{(p,q_0)}\mathrm{pr}_1(X(p,q_0))$ provides a time orientation for (B, g_B).

$$\begin{tikzcd} T(B \times F) \arrow[r, "T\mathrm{pr}_1"] & TB \\ B \times F \arrow[u, "X"] \arrow[r, "\mathrm{pr}_1"'] \arrow[ur] & B \arrow[u, "\bar{X}"'] \end{tikzcd}$$

($\Leftarrow$)

- $n \geq 2$: Let (B, g_B) be time oriented by the vector field V. We define $\bar{V}$ at $\bar{p} = (p, q)$ via setting $\bar{V}(\bar{p}) = (V(p), 0_q)$ where we use the isomorphism $T_{\bar{p}}(M) \cong T_pB \times T_qF$.
  From the definition of the metric it follows that

$$\begin{aligned} \langle \bar{V}|_{\bar{p}}, \bar{V}|_{\bar{p}} \rangle_M &= \langle T_{(p,q)}\mathrm{pr}_1(\bar{V}), T_{(p,q)}\mathrm{pr}_1(\bar{V}) \rangle_B \\ &\quad + f^2(p) \underbrace{\langle T_{(p,q)}\mathrm{pr}_2(\bar{V}), T_{(p,q)}\mathrm{pr}_2(\bar{V}) \rangle_F}_{=0} \\ &= \langle V|_p, V|_p \rangle_B < 0 \end{aligned}$$

  hence $\bar{V}$ time orients $B \times_f F$

- $n = 1$: In the appendix of [Mi] it is shown that B is diffeomorphic to $\mathbb{S}^1$ or $\mathbb{R}$. In either case let T be a smooth vector field on B with $g_B(T, T) = -1$. Defining $\bar{T}(\bar{p}) = (T(p), 0_q)$ as above we have $T_{(p,q)}\mathrm{pr}_2(\bar{T}) = 0$ so that $\bar{T}$ time orients M.

$\Box$

Note that in case of $B = \mathbb{S}^1$ the integral curves of $\bar{T}$ in $M = B \times_f F$ are closed timelike curves hence M is not chronological.

In what follows we restrict our attention to the study of warped products $(M = B \times_f F, g_M)$ with $n \geq 2$.

Lemma 2.2.37. *Let $\bar{p} = (p, q)$ and $\bar{p}' = (p', q')$ be two points in M with $\bar{p} \ll \bar{p}'$ (resp. $\bar{p} \leq \bar{p}'$) in (M, g_M). Then $p \ll p'$ (resp. $p \leq p'$) in (B, g_B).*

Proof. Let γ be a future directed timelike curve in M from $\bar{p}$ to $\bar{p}'$, i.e. $\langle \gamma', \gamma' \rangle_M < 0$. Then $\mathrm{pr}_1 \circ \gamma$ is a future directed timelike curve in B from p to p'. Indeed,

$$0 > \langle \gamma'(t), \gamma'(t) \rangle_M = \langle T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)) \rangle_B +$$

$$+ \underbrace{f^2(p) \langle T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)) \rangle_F}_{\geq 0}$$

hence $\langle T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)) \rangle_B < 0$ and so $\mathrm{pr}_1 \circ \gamma$ is a timelike curve. The result is proven analogously for causal γ. $\Box$

Remark 2.2.38. $\mathrm{pr}_1 : B \times_f F \to B$ maps causal curves to causal curves, but it does not preserve null curves. Indeed let γ be any smooth null curve with $T_{(p,q)}\mathrm{pr}_2(\gamma'(t)) \neq 0 \ \forall\, t$. Then

$$\begin{aligned} 0 &= \langle \gamma'(t), \gamma'(t)\rangle_M = \langle T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t))\rangle_B \\ &+ f^2(\mathrm{pr}_1(\gamma(t)))\langle T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t))\rangle_F \quad \forall\, t \end{aligned}$$

Since $f^2(\mathrm{pr}_1(\gamma(t)))\langle T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t))\rangle_F > 0$ it follows that $\langle T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t))\rangle_B \neq 0$ and so $\mathrm{pr}_1 \circ \gamma$ is not a null curve.

The converse of 2.2.37 is also true if we assume that $\bar{p}, \bar{p}'$ are in the same leaf ${\mathrm{pr}_2}^{-1}(q)$ $(q \in F)$ of $M = B \times_f F$.

Lemma 2.2.39. *If $\bar{p} = (p, q), \bar{p}' = (p', q)$ are in the same leaf ${pr_2}^{-1}(q)$ then $\bar{p} \ll \bar{p}'$ (resp. $\bar{p} \leq \bar{p}'$) in (M, g_M) iff $p \ll p'$ (resp. $p \leq p'$) in (B, g_B).*

Proof. ($\Rightarrow$) Just apply 2.2.37
($\Leftarrow$) Let $\gamma_1 : [0,1] \to B$ be a future directed timelike curve in B from p to p', i.e. $\gamma_1(0) = p$, $\gamma_1(1) = p'$. Then $\gamma(t) := (\gamma_1(t), q)$, $0 \leq t \leq 1$ is a future directed timelike curve with $T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)) = 0$ in M from $\bar{p}$ to $\bar{p}'$ since

$$\begin{aligned} \langle \gamma'(t), \gamma'(t)\rangle_M &= \langle T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_1(\gamma'(t))\rangle_B \\ &\quad + f^2(\mathrm{pr}_1(\gamma(t)))\langle T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t)), T_{\gamma(t)}\mathrm{pr}_2(\gamma'(t))\rangle_F \\ &= \langle \gamma_1(t), \gamma_1(t)\rangle_B < 0 \end{aligned}$$

Therefore γ is timelike.
Again the result is proven analogously for causal curves. $\square$

2.2.39 shows that each leaf ${\mathrm{pr}_2}^{-1}(q) = B \times \{q\}$ has the same chronology and causality as M. In particular (M, g_M) contains a closed timelike curve iff (B, g_B) contains a closed timelike curve.
Hence we get

Proposition 2.2.40. *Let (B, g_B) be a spacetime and let (F, g_F) be a Riemannian manifold. Then $(M = B \times_f F, g_M)$ is chronological (resp. causal) iff (B, g_B) is chronological (resp. causal).*

The analogous statement also holds for strong causality:

Proposition 2.2.41. *$(M = B \times_f F, g_M)$ is strongly causal iff (B, g_B) is strongly causal.*

Proof. ($\Rightarrow$) Let $\bar{p} = (p, q) \in M$. By contradiction we show that if (B, g_B) is not strongly causal at p then (M, g_M) is not strongly causal at $\bar{p}$.
Since (B, g_B) is not strongly causal at p there exists an open neighborhood U of p in B and a sequence $\{\gamma_k : [0,1] \to B\}$ of future directed causal curves that intersect U in a disconnected set with $\gamma_k(0) \to p$, $\gamma_k(1) \to p$ as $k \to \infty$, but $\gamma_k(\frac{1}{2}) \notin U \ \ \forall\, k$.
Define $\sigma_k : [0,1] \to M$ via $\sigma_k(t) = (\gamma_k(t), q)$.
Let V be any open neighborhood of q in F and set $\bar{U} := U \times V \subseteq M$. Then we obtain that $\{\sigma_k\}$ is a sequence of causal future directed curves in M intersecting $\bar{U}$ in a disconnected set. Indeed, U is an open neighborhood of $\bar{p}$ and $\sigma_k(0) \to \bar{p}$,

$\sigma_k(1) \to \bar{p}$ as $k \to \infty$, but $\sigma_k\left(\frac{1}{2}\right) \notin U$ hence M is not strongly causal at $\bar{p}$.

($\Leftarrow$) Suppose that strong causality fails at $\bar{p} = (p, q) \in M$. Let $(x^1, \ldots, x^n)$ be local coordinates on B near p such that g_B has the form $\mathrm{diag}(-1, 1, \ldots, 1)$ at p and let $(x^{n+1}, \ldots, x^{n+l})$ be local coordinates on F near q such that $f^2 g_F$ has the form $\mathrm{diag}(1, 1, \ldots, 1)$ at q. Then $(x^1, \ldots, x^{n+l})$ are local coordinates for M.
Let $F_1 := x^1$, $F_2 := x^1 \circ \mathrm{pr}_1$. These are time functions, i.e. strictly increasing along any future directed causal curve.
Indeed, let $c(t) = (c_1(t), \tilde{c}(t))$, then we obtain

$$\langle \dot{c}(t), \dot{c}(t) \rangle_M = -\dot{c}_1(t)^2 + \left\|\dot{\tilde{c}}(t)\right\|^2 < 0$$

F_2 is strictly increasing, i.e. $\dot{c}_1(t) > 0 \quad \forall\, t$. The vector $X = (1, 0, \ldots,)$ time orients M. c is future pointing iff $\langle X, \dot{c} \rangle = -\dot{c}_1 < 0$. Since M is not strongly causal at $\bar{p}$ we obtain a neighborhood U of $\bar{p}$ and a sequence of future directed causal curves with

$$\gamma_k(0) \to \bar{p}$$
$$\gamma_k(1) \to \bar{p}$$

as $k \to \infty$, but for each k there exists some t_k such that $\gamma_k(t_k) \notin U$.
Without loss of generality let $U = (-\epsilon, \epsilon) \times B_{3\epsilon}(0) \subseteq \mathbb{R}^n$ where $B_{3\epsilon}(0)$ is the Euclidean ball of radius 3ϵ in $\mathbb{R}^{n-1}$. Then there is for all k some t_k such that

$$F_2(\gamma_k(t_k)) \geq \epsilon > 0 \quad \forall\, k \qquad (*)$$

Otherwise, suppose

$$F_2(\gamma_k(t_k)) < \epsilon \qquad (**)$$

Let $\bar{\Lambda}_p := \{x | x - p \quad \text{is timelike or null}\}$. Since γ_k is causal it follows by [O'N.], 5.33 that γ_k lies entirely in $\bar{\Lambda}_{\gamma_k(0)}$. By $(**)$ γ_k lies entirely in $B := \left((-\epsilon, \epsilon) \times \mathbb{R}^{n-1}\right) \cap \bar{\Lambda}_{\gamma_k(0)}$.
For $\|\mathrm{pr}_2(\gamma_k(0))\| < \epsilon$ we conclude that

$$B \subseteq \{x | \, \|\mathrm{pr}_2(\gamma_k(x))\| < 3\epsilon\}$$

By assumption $\gamma_k(t_k) \notin U$, so because of $(**)$

$$\|\mathrm{pr}_2(\gamma_k(x))\| > 3\epsilon,$$

a contradiction.
In what follows we choose $t_k = \frac{1}{2}$ by parametrizing γ_k suitably.

We now choose a neighborhood W of p in B such that W is covered by the local coordinates above and such that

$$V = \sup\{F_1(r) : r \in W\} \leq \frac{\epsilon}{2}$$

Then $\mathrm{pr}_1 \circ \gamma_k$ are future directed causal curves in B with $\mathrm{pr}_1 \circ \gamma_k(0) \to p$, $\mathrm{pr}_1 \circ \gamma_k(1) \to p$ and $\mathrm{pr}_1 \circ \gamma_k\left(\frac{1}{2}\right) \notin W$. Indeed

$$\begin{aligned} F_1\left(\mathrm{pr}_1 \circ \gamma_k\left(\frac{1}{2}\right)\right) &= x^1 \circ \mathrm{pr}_1 \circ \gamma_k\left(\frac{1}{2}\right) \\ &= F_2\left(\gamma_k\left(\frac{1}{2}\right)\right) \overset{(*)}{\geq} \epsilon \end{aligned}$$

but $\sup\{F_1(r)|r \in W\} \leq \frac{\epsilon}{2}$, hence $\mathrm{pr}_1 \circ \gamma_k \left(\frac{1}{2}\right) \notin W$. So using W and $\{\gamma_k\}$ shows the failure of strong causality at p in (B, g_B). $\square$

Finally we mention without a proof the analogous result for the condition of stable causality. The proof can be found in [B.E.].

Proposition 2.2.42. *Let (B, g_B) be a spacetime and (F, g_F) be a Riemannian manifold. Then $(M = B \times_f F, g_M)$ is stably causal iff (B, g_B) is stably causal.*

Proof. [B.E.], 2.51. $\square$

Looking at the hierarchy of causality conditions we obtain

Corollary 2.2.43. *Let (F, g_F) be a Riemannian manifold and let $B = (a, b)$ with $-\infty \leq a < b \leq \infty$ given the negative definite metric $-dt^2$. For any smooth function $f : B \to (0, \infty)$, $M = B \times_f F$ is chronological, causal, distinguishing and strongly causal.*

If $M = \mathbb{S}^1$ then $(\mathbb{S}^1 \times_f F, g_M)$ fails to be chronological as remarked above hence fails to be causal, distinguishing and strongly causal.

Now we want to investigate conditions on (B, g_B) and (F, g_F) which imply global hyperbolicity of (M, g_M).

Theorem 2.2.44. *Let (B, g_B) be a spacetime (dim $B \geq 2$) and let (F, g_F) be a Riemannian manifold. Then $(M = B \times_f F, g_M)$ is globally hyperbolic iff both of the following conditions are satisfied:*

- *(B, g_B) is globally hyperbolic*
- *(F, g_F) is a complete Riemannian manifold*

Proof. [B.E.], 2.53. $\square$

Remark 2.2.45. If $B = (a, b)$, $-\infty \leq a < b \leq \infty$ with negative definite metric $-dt^2$ then the first condition in the theorem is automatically satisfied. Hence it is sufficient for global hyperbolicity of M that (F, g_F) is a complete Riemannian manifold.

Theorem 2.2.46. *Let (F, g_F) be a Riemannian manifold and suppose that $M = \mathbb{R} \times_f F$ is given the metric $-dt^2 \oplus f^2 h$. The following statements are equivalent:*

1. *(F, g_F) is a geodesically complete Riemannian manifold*
2. *$(\mathbb{R} \times_f F, -dt^2 \oplus f^2 h)$ is geodesically complete*
3. *$(\mathbb{R} \times_f F, -dt^2 \oplus f^2 h)$ is globally hyperbolic*

Proof. (1) $\Leftrightarrow$ (3): Follows from 2.2.44
(1) $\Leftrightarrow$ (2): All geodesics in M are up to reparametrization of the form $(\lambda t, c(t))$, $(\lambda_0, c(t))$ or $(\lambda t, q)$ with $\lambda, \lambda_0 \in \mathbb{R}$, $q \in F$ and $c : \mathbb{R} \to F$ a unit speed geodesic. $\square$

Remark 2.2.47. For dim $B = 1$ and B homeomorphic to $\mathbb{R}$ we have just given necessary and sufficient conditions for the warped product $M = B \times_f F$ to be globally hyperbolic.
If B is homeomorphic to $\mathbb{S}^1$, $M = B \times_f F$ is not chronological, no matter which Riemannian metric is chosen on F. Thus no warped spacetime $(\mathbb{S}^1 \times_f F, g_M)$ can be globally hyperbolic.

We now will construct Cauchy-surfaces (CS) for globally hyperbolic warped products:

Theorem 2.2.48. *Let (F, g_F) be a complete Riemannian manifold, and let (B, g_B) be globally hyperbolic. If S_1 is a CS of B, then $S_1 \times F$ is a CS of $(M = B \times_f F, g_M)$.*

Proof. [B.E.], 2.56. □

Remark 2.2.49. If $B = (a, b)$, $-\infty \leq a < b \leq \infty$, with metric $-dt^2$ then $p \times F$ is a CS of (M, g_M) $\forall\, p \in B$.

2.3 Multiply warped products

This section is mainly based on [Ül], [D.Ü.]. We will investigate a generalization of singly warped products.

Definition 2.3.1. Let (B, g_B), (F_i, g_{F_i}) be pseudo Riemannian manifolds of dimension n resp. l_i ($i \in \{1, \dots m\}$) and let $f_i : B \to (0, \infty)$ be smooth functions for any $i \in \{1, \dots m\}$. A *multiply warped product* is the product manifold $M = B \times F_1 \times \dots \times F_m$ furnished with the metric tensor $g_M = g_B \oplus f_1^2 g_{F_1} \oplus \dots \oplus f_m^2 g_{F_m}$ defined by

$$g_M = \mathrm{pr_1}^*(g_B) + (f_1 \circ \mathrm{pr_1})^2 \mathrm{pr_2}^*(g_{F_1}) + \dots + (f_m \circ \mathrm{pr_1})^2 \mathrm{pr_{m+1}}^*(g_{F_m})$$

Where

$$\mathrm{pr_1} : B \times F_1 \times \dots \times F_m \to B \quad (p, q_1, \dots q_m) \mapsto p$$

$$\mathrm{pr_{i+1}} : B \times F_1 \times \dots \times F_m \to F_i \quad (p, q_1, \dots q_m) \mapsto q_i$$

are the usual projection maps.
Each function $f_i : B \to (0, \infty)$ is called a *warping function* and each manifold (F_i, g_{F_i}) is called *fiber manifold*. (B, g_B) is the *base manifold* of the multiply warped product. We write $M = B \times_{f_1} F_1 \times \dots \times_{f_m} F_m$

The manifold $M = B \times F_1 \times \dots \times F_m$ is a d-dimensional pseudo Riemannian manifold with $l = \sum_{i=1}^m l_i$ and $d = n + l$. We use natural product coordinate systems:
Let $(p, q_1, \dots q_m)$ be a point in M. There are coordinate charts (U, ϕ) and (V_i, ψ_i) on B and F_i respectively such that $p \in U$, $q_i \in V_i$. We can define a coordinate chart (W, μ) on M such that W is an open subset in M contained in $U \times V_1 \times \dots \times V_m$ and $(p, q_1 \dots q_m) \in W$. Then $\mu(u, v) = (\phi(u), \psi_1(v_1), \dots, \psi_m(v_m))$. The set of all (W, μ) defines an atlas on $B \times F_1 \times \dots \times F_m$.

Remark 2.3.2.

- If $m = 1$ then we obtain a singly warped product.

- If all $f_i \equiv 1$ then we deal with a (trivial) product manifold.
- If (B, g_B) and all (F_i, g_{F_i}) are Riemannian manifolds then (M, g_M) is also a Riemannian manifold.
- The multiply warped product (M, g_M) is a Lorentzian multiply warped product if all (F_i, g_{F_i}) are Riemannian manifolds and either (B, g_B) is a Lorentzian manifold or else (B, g_B) is a one-dimensional manifold with negative definite metric $-dt^2$.
- For B being an open interval $I = (t_1, t_2)$ where $-\infty \leq t_1 < t_2 \leq \infty$, equipped with the metric $-dt^2$ and (F_i, g_{F_i}) is a Riemannian manifold for any i, the Lorentzian multiply warped product (M, g_M) is called a *generalized Robertson-Walker spacetime* or a *multi-warped spacetime.* A generalized Robertson-Walker spacetime for $m = 2$ is a *generalized Reissner-Nordström spacetime.*

Now again as for singly warped products we want to study the connection and curvature on such a multiply warped product.

Proposition 2.3.3. *Let $M = B \times_{f_1} F_1 \times \ldots \times_{f_m} F_m$ be a pseudo Riemannian multiply warped product with metric $g_M = g_B \oplus f_1^2 g_{F_1} \oplus \ldots \oplus f_m^2 g_{F_m}$ and let $\tilde{X},\ \tilde{Y} \in \mathcal{L}(B)$, $\tilde{V} \in \mathcal{L}(F_i)$ and $\tilde{W} \in \mathcal{L}(F_j)$ where lifts are defined as for usual product manifolds (see 2.1) . Then*
(1) $\nabla^M_{\tilde{X}} \tilde{Y} \in \mathcal{L}(B)$ is the lift of $\nabla^B_X Y$.
(2) $\nabla^M_{\tilde{X}} \tilde{V} = \nabla^M_{\tilde{V}} \tilde{X} = \frac{\tilde{X} f_i}{f_i} \tilde{V}$
(3) For $i \neq j$: $\nabla^M_{\tilde{V}} \tilde{W} = 0$
(4) For $i = j$: $\nabla^M_{\tilde{V}} \tilde{W} = \widetilde{\nabla^{F_i}_V W} - \frac{\langle \tilde{V}, \tilde{W} \rangle_M}{f_i} \operatorname{grad}(f_i)$

Proof.
(1) Since none of the warping functions is concerned, the result follows as in 2.2.11(1).

(2) Since $0 = [\tilde{X}, \tilde{V}] = \nabla^M_{\tilde{X}} \tilde{V} - \nabla^M_{\tilde{V}} \tilde{X}$ we obtain $\nabla^M_{\tilde{X}} \tilde{V} = \nabla^M_{\tilde{V}} \tilde{X}$. Property $(D5)$ for ∇^M leads us to

$$\langle \nabla^M_{\tilde{X}} \tilde{V}, \tilde{Y} \rangle_M = -\langle \tilde{V}, \nabla^M_{\tilde{X}} \tilde{Y} \rangle_M + \tilde{X} \langle \tilde{V}, \tilde{Y} \rangle_M = 0$$

hence as in 2.2.11 all terms in the Koszul formula for $2\langle \nabla^M_{\tilde{X}} \tilde{V}, \tilde{W} \rangle_M$ vanish except $\tilde{X} \langle \tilde{V}, \tilde{W} \rangle_M$ where $\tilde{W} \in \mathcal{L}(F_i)$. Computing the scalar product we have

$$\langle \tilde{V}, \tilde{W} \rangle_M (p, q_1, \ldots q_m) = f_i^2(p) \langle V_{q_i}, W_{q_i} \rangle_{F_i}$$

and so

$$\langle \tilde{V}, \tilde{W} \rangle_M = f_i^2 (\langle V, W \rangle_{F_i} \circ \mathrm{pr_{i+1}}) \qquad (*)$$

Hence

$$\begin{aligned} \tilde{X} \langle \tilde{V}, \tilde{W} \rangle_M &= \tilde{X}(f_i^2 (\langle V, W \rangle_{F_i} \circ \mathrm{pr_{i+1}})) \\ &= 2 f_i \tilde{X} f_i (\langle V, W \rangle_{F_i} \circ \mathrm{pr_{i+1}}) \\ &\overset{(*)}{=} 2 \left(\frac{\tilde{X}(f_i)}{f_i} \right) \langle \tilde{V}, \tilde{W} \rangle_M \end{aligned}$$

So it follows that

$$2\langle\nabla^M_{\tilde X}\tilde V,\tilde W\rangle_M = 2\left(\frac{\tilde X(f_i)}{f_i}\right)\langle\tilde V,\tilde W\rangle_M \quad \forall\, W$$

hence $\nabla^M_{\tilde X}\tilde V = \left(\frac{\tilde X(f_i)}{f_i}\right)\tilde V$.

(3) In general $\nabla^M_{\tilde V}\tilde W =$ nor $\nabla^M_{\tilde V}\tilde W +$ tan $\nabla^M_{\tilde V}\tilde W$. We will compute both parts separately:
For nor $\nabla^M_{\tilde V}\tilde W$ we calculate

$$\begin{aligned}\langle\nabla^M_{\tilde V}\tilde W,\tilde X\rangle_M &= -\langle\tilde W,\nabla^M_{\tilde V}\tilde X\rangle_M + \tilde V\langle\tilde W,\tilde X\rangle_M \\ &\overset{(2)}{=} -\langle\tilde W,\frac{\tilde X(f_i)}{f_i}\tilde V\rangle_M + 0 \\ &= -\frac{\tilde X(f_i)}{f_i}\langle\tilde V,\tilde W\rangle_M \\ &= 0\end{aligned}$$

since $i \neq j$. The result for tan $\nabla^M_{\tilde V}\tilde W$ follows from 2.2.11.
(4) nor $\nabla^M_{\tilde V}\tilde W = -\frac{\tilde X(f_i)}{f_i}\langle\tilde V,\tilde W\rangle_M$.
In case of $i = j$ we compute for $\tilde X(f_i)$:

$$\tilde X(f_i) = \langle \mathrm{grad}_M f_i,\tilde X\rangle_M = \langle \mathrm{grad}_B f_i, X\rangle_B \circ \mathrm{pr_{i+1}}$$

Therefore for all X:

$$\langle\nabla^M_{\tilde V}\tilde W,\tilde X\rangle_M = -\frac{\langle \mathrm{grad}_B f_i,\tilde X\rangle_M}{f_i}\langle\tilde V,\tilde W\rangle_M$$

and so

$$\text{nor } \nabla^M_{\tilde V}\tilde W = -\frac{\langle\tilde V,\tilde W\rangle_M}{f_i}\mathrm{grad}_B f_i = I\!I(\tilde V,\tilde W)$$

tan $\nabla^M_{\tilde V}\tilde W$ is again computed as in 2.2.11. □

Proposition 2.3.4. *For $q = (q_1,\dots q_m) \in F_1\times\ldots\times F_m$ and $p \in B$, the leaf $B\times\{q\}$ is totally geodesic, the fiber $\{p\}\times F_1\times\ldots\times F_m$ is totally umbillic and it is totally geodesic if* $\mathrm{grad}_B(f_i)|_p = 0$.

Proof. The first statement is analogous to 2.2.14.
The second follows since

$$I\!I(\tilde V,\tilde W) = \text{nor } \nabla^M_{\tilde V}\tilde W = -\frac{\langle\tilde V,\tilde W\rangle_M}{f_i}\mathrm{grad}_B f_i$$

□

Now we want to compute the gradient and the Laplacian on M in terms of the corresponding ones in B resp. F_i.

Proposition 2.3.5. *Let (M, g_M) be a pseudo Riemannian multiply warped product and $\phi : B \to \mathbb{R}$, $\psi_i : F_i \to \mathbb{R}$ be smooth functions for any $i \in \{1 \dots m\}$. Then*
(1) $\mathrm{grad}_M(\phi \circ \mathrm{pr}_1) = (\mathrm{grad}_B \phi) \circ \mathrm{pr}_1$
(2) $\mathrm{grad}_M(\psi_i \circ \mathrm{pr}_{\mathrm{i+1}}) = \frac{\mathrm{grad}_{F_i} \psi_i}{f_i^2} \circ \mathrm{pr}_{\mathrm{i+1}}$
(3) $\Delta_M(\phi \circ \mathrm{pr}_1) = \Delta_B \phi + \sum_{i=1}^m l_i \frac{\langle \mathrm{grad}_B \phi, \mathrm{grad}_B f_i \rangle_B}{f_i}$ *where* $l_i = \mathrm{dim} F_i$
(4) $\Delta_M(\psi_i \circ \mathrm{pr}_{\mathrm{i+1}}) = \frac{\Delta_{F_i} \psi_i}{f_i^2}$

Remark 2.3.6. Before starting to prove the proposition we introduce an orthogonal frame field on M.
Let $\{E_1^B, \dots, E_n^B\}$ and $\{E_1^{F_i}, \dots, E_{l_i}^{F_i}\}$ be orthonormal frames on open sets $U \subseteq B$ and $V_i \subseteq F_i$. Then one can easily see that

$$\{E_1^B, \dots, E_n^B, \frac{1}{f_1} E_1^{F_1}, \dots, \frac{1}{f_1} E_{l_1}^{F_1}, \dots, \frac{1}{f_m} E_1^{F_m}, \dots, \frac{1}{f_m} E_{l_m}^{F_m}\}$$

is an orthogonal frame on an open set $W \subseteq B \times F_1 \times \dots \times F_m$ contained in $U \times V_1 \times \dots \times V_m \subseteq B \times F_1 \times \dots \times F_m$. In the following proofs we will denote it by

$$\{E_1, \dots E_n, E_1^1, \dots E_{l_1}^1, \dots E_1^m, \dots E_{l_m}^m\}.$$

Proof.
(1): Let v be tangent to $\{p\} \times F_1 \times \dots \times F_m$. We calculate:

$$\langle \mathrm{grad}_M(\phi \circ \mathrm{pr}_1), v \rangle_M = v(\phi \circ \mathrm{pr}_1) = \underbrace{T_{(p,q_1,\dots,q_m)} \mathrm{pr}_1(v)}_{=0} \phi = 0$$

hence the gradient of $\phi \circ \mathrm{pr}_1$ is horizontal.
For x_i tangent to $B \times \{q_1\} \times \dots \times \{q_m\}$:

$$\begin{aligned} \langle T_{(p,q_1,\dots,q_m)} \mathrm{pr}_1(\mathrm{grad}_M(\phi \circ \mathrm{pr}_1)) \quad , \quad & T_{(p,q_1,\dots,q_m)} \mathrm{pr}_1(x_i) \rangle_B = \\ &= \langle \mathrm{grad}_M(\phi \circ \mathrm{pr}_1), x_i \rangle_M \\ &= x_i(\phi \circ \mathrm{pr}_1) \\ &= T_{(p,q_1,\dots,q_m)} \mathrm{pr}_1(x_i) \phi \\ &= \langle \mathrm{grad}_B(\phi), T_{(p,q_1,\dots,q_m)} \mathrm{pr}_1(x_i) \rangle_B \circ \mathrm{pr}_1 \end{aligned}$$

so $\mathrm{grad}_M(\phi \circ \mathrm{pr}_1)$ and $\mathrm{grad}_B(\phi)$ are pr_1-related.

(2): Let x_i be tangent to $B \times F_1 \times \dots \times \{q_i\} \times \dots \times F_m$ then

$$\begin{aligned} \langle \mathrm{grad}_M(\psi_i \circ \mathrm{pr}_{\mathrm{i+1}}), x_i \rangle_M &= x_i(\psi_i \circ \mathrm{pr}_{\mathrm{i+1}}) \\ &= \underbrace{T_{(p,q_1,\dots,q_m)} \mathrm{pr}_{\mathrm{i+1}}(x_i)}_{=0} \psi_i \\ &= 0 \end{aligned}$$

hence it is vertical.
For v tangent to $\{p\} \times F_1 \times \dots \times F_m$ resp. to $B \times F_1 \times \dots \times \{q_j\} \dots \times F_m$, $i \neq j$

we obtain

$$\begin{aligned}
\langle T_{(p,q_1,\ldots,q_m)}\mathrm{pr}_{\mathrm{i+1}}(\mathrm{grad}_M(\psi_i\circ\mathrm{pr}_{\mathrm{i+1}})), T_{(p,q_1,\ldots,q_m)}\mathrm{pr}_{\mathrm{i+1}}(v)\rangle_{F_i} &= \frac{1}{f_i^2}\langle \mathrm{grad}_M(\psi_i\circ\mathrm{pr}_{\mathrm{i+1}}, v\rangle_M \\
&= \frac{1}{f_i^2}v(\psi_i\circ\mathrm{pr}_{\mathrm{i+1}}) \\
&= \frac{1}{f_i^2}T_{(p,q_1,\ldots,q_m)}\mathrm{pr}_{\mathrm{i+1}}(v)\psi_i \\
&= \frac{1}{f_i^2}\langle \mathrm{grad}_{F_i}(\psi_i), T_{(p,q_1,\ldots,q_m)}\mathrm{pr}_{\mathrm{i+1}}(v)\rangle_{F_i}\circ\mathrm{pr}_{\mathrm{i+1}}
\end{aligned}$$

(3): We choose the orthogonal frame introduced in 2.3.6. Then

$$\begin{aligned}
\Delta(\phi\circ\mathrm{pr}_1) &= \mathrm{div}(\mathrm{grad}_M(\phi\circ\mathrm{pr}_1)) \\
&\overset{2.2.11(2)}{=} \mathrm{div}(\mathrm{grad}_B(\phi) \\
&= \sum_{i=1}^{n}\epsilon_i\langle\nabla^M_{E_i}\mathrm{grad}_B\phi, E_i\rangle_M + \sum_{j_1=1}^{l_1}\epsilon_{j_1}\langle\nabla^M_{E^1_{j_1}}\mathrm{grad}_B\phi, E_{j_1^1}\rangle_M + \\
&\quad \ldots + \sum_{j_m=1}^{l_m}\epsilon_{j_m}\langle\nabla^M_{E^m_{j_m}}\mathrm{grad}_B\phi, E^m_{j_m}\rangle_M \quad (*)
\end{aligned}$$

The first sum is $\Delta_B(\phi)$ and for the other terms we can calculate by 2.3.3(2) since $E^k_j\in\mathcal{L}(F_k)$

$$\begin{aligned}
\nabla^M_{E^k_j}\mathrm{grad}_B\phi &= \frac{\mathrm{grad}_B\phi(f_k)}{f_k}E^k_j \\
&= \frac{1}{f_k}\langle\mathrm{grad}_B\phi, \mathrm{grad}_B f_k\rangle_B E^k_j
\end{aligned}$$

hence $\langle\nabla^M_{E^k_j}\mathrm{grad}_B\phi, E^k_j\rangle_M = \epsilon_k\frac{1}{f_k}\langle\mathrm{grad}_B\phi, \mathrm{grad}_B f_k\rangle_B$.
Summing up leads to

$$\Delta_M(\phi\circ\mathrm{pr}_1) = \Delta_B(\phi) + \sum_{i=1}^{m}\frac{1}{f_i}\langle\mathrm{grad}_B\phi, \mathrm{grad}_B f_i\rangle_B\cdot l_i$$

(4) Using (2) we obtain

$$\begin{aligned}
\Delta_M(\psi_i\circ\mathrm{pr}_{\mathrm{i+1}}) &= \mathrm{div}(\mathrm{grad}_M(\psi_i\circ\mathrm{pr}_{\mathrm{i+1}})) \\
&\overset{(2)}{=} \frac{1}{f_i^2}\mathrm{div}(\mathrm{grad}_{F_i}(\psi_i)
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{div}(\mathrm{grad}_{F_i}(\psi_i)) &= \sum_{i=1}^{n}\epsilon_i\langle\nabla^M_{E_i}\mathrm{grad}_{F_i}\psi_i, E_i\rangle_M + \sum_{j_1=1}^{l_1}\epsilon_{j_1}\langle\nabla^M_{E^1_{j_1}}\mathrm{grad}_{F_i}\psi_i, E^1_{j_1}\rangle_M + \\
&\quad \ldots + \sum_{j_m=1}^{l_m}\epsilon_{j_m}\langle\nabla^M_{E^m_{j_m}}\mathrm{grad}_{F_m}\psi_i, E^m_{j_m}\rangle_M
\end{aligned}$$

The first term vanishes as well as those for $k \neq i$ by (2) and 2.3.3 (2) resp. (3). Therefore

$$\begin{aligned}
\operatorname{div}(\operatorname{grad}_{F_i}(\psi_i)) &= \sum_{j_i=1}^{l_i} \epsilon_{j_i} \langle \nabla^M_{E^i_{j_i}} \operatorname{grad}_{F_i} \psi_i, E^i_{j_i} \rangle_M \\
&\overset{2.2.11(4)}{=} \sum_{j_i=1}^{l_i} \epsilon_{j_i} \langle \nabla^{F_i}_{E^i_{j_i}} \operatorname{grad}_{F_i} \psi_i - \frac{\langle E^i_{j_i}, \operatorname{grad}_{F_i} \psi_i \rangle_M}{f_i} \operatorname{grad}_B f_i, E^i_{j_i} \rangle_M \\
&= \sum_{j_i=1}^{l_i} \epsilon_{j_i} \langle \nabla^{F_i}_{E^i_{j_i}} \operatorname{grad}_{F_i} \psi_i, E^i_{j_i} \rangle_{F_i} \\
&= \Delta_{F_i} \psi_i
\end{aligned}$$

So $\Delta_M(\psi_i \circ \mathrm{pr_{i+1}}) = \frac{1}{f_i^2} \Delta_{F_i} \psi_i$

□

Next we turn to Riemannian and Ricci curvature

Proposition 2.3.7. *Let $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{L}(B)$, $\tilde{V} \in \mathcal{L}(F_i)$, $\tilde{W} \in \mathcal{L}(F_j)$ and $\tilde{U} \in \mathcal{L}(F_k)$. Then*
(1) ${}^M\!R(\tilde{X}, \tilde{Y})\tilde{Z} = \widetilde{{}^B\!R(X, Y)Z}$
(2) ${}^M\!R(\tilde{V}, \tilde{X})\tilde{Y} = \frac{H^{f_i}_B(\tilde{X}, \tilde{Y})}{f_i} \tilde{V}$ where $H^{f_i}_B$ is the Hessian on B of f_i.
(3) ${}^M\!R(\tilde{X}, \tilde{V})\tilde{W} = {}^M\!R(\tilde{V}, \tilde{W})\tilde{X} = {}^M\!R(\tilde{V}, \tilde{X})\tilde{W} = 0$ if $i \neq j$.
(4) ${}^M\!R(\tilde{X}, \tilde{Y})\tilde{V} = 0$
(5) ${}^M\!R(\tilde{V}, \tilde{W})\tilde{X} = 0$ if $i = j$.
(6) ${}^M\!R(\tilde{V}, \tilde{W})\tilde{U} = 0$ if $i = j$ and $j \neq k$.
(7) ${}^M\!R(\tilde{U}, \tilde{V})\tilde{W} = \langle \tilde{V}, \tilde{W} \rangle_M \frac{\langle grad_B f_i, grad_B f_k \rangle_B}{f_i f_k} U$ if $i = j$ and $j \neq k$.
(8) ${}^M\!R(\tilde{X}, \tilde{V})\tilde{W} = \frac{\langle \tilde{V}, \tilde{W} \rangle_M}{f_i} \nabla^B_X (grad_B f_i)$ if $i = j$.
(9) ${}^M\!R(\tilde{V}, \tilde{W})\tilde{U} = \widetilde{{}^{F_i}\!R(V, W)U} - \frac{\langle grad_B f_i, grad_B f_i \rangle_B}{f_i^2} (\langle \tilde{V}, \tilde{U} \rangle_M \tilde{W} - \langle \tilde{W}, \tilde{U} \rangle_M \tilde{V})$ if $i = j = k$.

Proof.
(1) As in 2.2.21
(2) 2.2.21
(3) By 2.2.11,

$$\begin{aligned}
{}^M\!R(\tilde{X}, \tilde{V})\tilde{W} &= -\nabla^M_{\tilde{X}}(\nabla^M_{\tilde{V}} \tilde{W}) + \nabla^M_{\tilde{V}}(\nabla^M_{\tilde{X}} \tilde{W}) \\
&= 0 + \nabla^M_{\tilde{V}} \left(\frac{X(f_i)}{f_i} \tilde{W} \right) \\
&= \tilde{V} \left(\frac{X(f_i)}{f_i} \right) \tilde{W} + \frac{X(f_i)}{f_i} \nabla^M_{\tilde{V}} \tilde{W} \\
&= 0
\end{aligned}$$

since $\tilde{V} \left(\frac{X(f_i)}{f_i} \right) = 0$ because f_i, $X(f_i)$ are constant in the direction of $\tilde{V}$ and the first term vanishes because of 2.3.3 (3). The other cases are proven analogously.
(4) 2.2.21
(5) 2.2.21

(6) 2.2.21
(7) Let X be an arbitrary vector field. By Gauss' equation

$$\begin{aligned}\langle {}^F R_{UV} W, X\rangle &= \langle {}^M R_{\tilde{U}\tilde{V}} \tilde{W}, \tilde{X}\rangle_M + \langle II(U,W), II(V,X)\rangle \\ &\quad -\langle II(U,X), II(V,W)\rangle = (*)\end{aligned}$$

We know that $II(U,W) = \text{nor}\ \nabla^M_{\tilde{U}} \tilde{W} \overset{2.3.3}{=} 0$ and ${}^F R_{UV} W = 0$.
Additionally, by 2.3.3 $II(V,W) = -\frac{\langle \tilde{V}, \tilde{W}\rangle}{f_i} \text{grad} f_i$, therefore

$$(*) = \langle \tilde{V}, \tilde{W}\rangle \frac{\langle \text{grad} f_i, \text{grad} f_k\rangle}{f_i f_k} \langle \tilde{U}, \tilde{X}\rangle$$

hence the result follows.
(8) 2.2.21.
(9) 2.2.21.

□

Proposition 2.3.8. *Let* $\tilde{X},\ \tilde{Y},\ \tilde{Z} \in \mathcal{L}(B)$, $\tilde{V} \in \mathcal{L}(F_i)$, $\tilde{W} \in \mathcal{L}(F_j)$. *Then*
(1) ${}^M Ric(\tilde{X}, \tilde{Y}) = {}^B \widetilde{Ric}(X,Y) - \sum_{i=1}^m \frac{l_i}{f_i} H_B^{f_i}(X,Y)$
(2) ${}^M Ric(\tilde{X}, \tilde{V}) = 0$
(3) ${}^M Ric(\tilde{V}, \tilde{W}) = 0$ *if* $i \neq j$
(4)

$$\begin{aligned}{}^M Ric(\tilde{V}, \tilde{W}) &= {}^{F_i}\widetilde{Ric}(V,W) - \Big(\frac{\Delta_B f_i}{f_i} + (l_i - 1)\frac{\langle \text{grad}_B f_i, \text{grad}_B f_i\rangle_B}{f_i^2} \\ &\quad + \sum_{k=1\ k\neq i}^m l_k \frac{\langle \text{grad}_B f_i, \text{grad}_B f_k\rangle_B}{f_i f_k}\Big)\langle \tilde{V}, \tilde{W}\rangle_M\end{aligned}$$

if $i = j$

Proof. We again choose the frame from 2.3.6, i.e.$\{E_1, \dots E_n, E_1^1, \dots E_{l_1}^1, \dots E_1^m, \dots E_{l_m}^m\}$
.

(1)

$$\begin{aligned}{}^M\text{Ric}(\tilde{X}, \tilde{Y}) &= \sum_{k=1}^n \epsilon_k \langle {}^B R_{X E_k} Y, E_k\rangle + \sum_{j_1=1}^{l_1} \epsilon_{j_1} \langle {}^M R_{\tilde{X} E^1_{j_1}} \tilde{Y}, E^1_{j_1}\rangle_M \\ &\quad + \dots + \sum_{j_m=1}^{l_m} \epsilon_{j_m} \langle {}^M R_{\tilde{X} E^m_{j_m}} \tilde{Y}, E^m_{j_m}\rangle_M \\ &= {}^B\widetilde{\text{Ric}}(X,Y) - \sum_{l_1=1}^{l_1} \frac{H^{f_1}}{f_1} \langle E^1_{j_1}, E^1_{j_1}\rangle - \dots - \sum_{l_m=1}^{l_m} \frac{H^{f_m}}{f_m} \langle E^m_{j_m}, E^m_{j_m}\rangle \\ &= {}^B\widetilde{\text{Ric}}(X,Y) - l_1 \frac{H^{f_1}}{f_1} - \dots - l_m \frac{H^{f_m}}{f_m} \\ &= {}^B\widetilde{\text{Ric}}(X,Y) - \sum_{i=1}^m l_i \frac{H^{f_i}}{f_i}\end{aligned}$$

(2) see 2.2.23
(3) see 2.2.23, and by using the symmetry properties of R, $R(X,V)W = R(V,W)X = R(V,X)W = 0$.
(4)

$$
\begin{aligned}
{}^M\mathrm{Ric}(\tilde V,\tilde W) &= \sum_{k=1}^{n}\epsilon_k\langle {}^MR_{\tilde V E_k}\tilde W, E_k\rangle_M + \sum_{j_1=1}^{l_1}\epsilon_{j_1}\langle {}^MR_{\tilde V E^1_{j_1}}\tilde W, E^1_{j_1}\rangle_M + \\
&\quad \ldots + \sum_{j_i=1}^{l_i}\epsilon_{j_i}\langle {}^MR_{\tilde V E^i_{j_i}}\tilde W, E^i_{j_i}\rangle_M + \ldots + \sum_{j_m=1}^{l_m}\epsilon_{j_m}\langle {}^MR_{\tilde V E^m_{j_m}}\tilde W, E^m_{j_m}\rangle_M
\end{aligned}
$$

Depending on the frame elements we have to distinguish the cases whether they are elements on B, F_j for $i \neq j$ or on F_i.
Let $E_k \in \mathcal{L}(B)$. Then ${}^MR_{\tilde V E_k}\tilde W = -{}^MR_{E_k\tilde V}\tilde W = -\frac{\langle \tilde V,\tilde W\rangle}{f_i}\nabla^B_{E_k}(\mathrm{grad}_B f_i)$. So summing up leads to

$$
\begin{aligned}
\sum_{k=1}^{n}\epsilon_k\langle {}^MR_{\tilde V E_k}\tilde W, E_k\rangle_M &= -\sum_{k=1}^{n}\epsilon_k\frac{\langle \tilde V,\tilde W\rangle}{f_i}\langle\nabla^B_{E_k}(\mathrm{grad}_B f_i), E_k\rangle_M \\
&= -\langle \tilde V,\tilde W\rangle\frac{\Delta_B f_i}{f_i}
\end{aligned}
$$

For $E^j_k \in \mathcal{L}(F_j)$ with $i \neq j$ we obtain since

$$
\begin{aligned}
{}^MR_{\tilde V E^j_k}\tilde W &= -{}^MR_{E^j_k\tilde V}\tilde W \\
&= -\langle \tilde V,\tilde W\rangle\frac{\langle \mathrm{grad}_B f_i, \mathrm{grad}_B f_j\rangle}{f_i f_j}E^j_k
\end{aligned}
$$

Again summation leads to

$$
\begin{aligned}
\sum_{k=1}^{l_j}\epsilon_k\langle {}^MR_{\tilde V E^j_k}\tilde W, E^j_k\rangle_M &= -\langle \tilde V,\tilde W\rangle\frac{\langle \mathrm{grad}_B f_i, \mathrm{grad}_B f_j\rangle}{f_i f_j}\sum_{k=1}^{l_j}\epsilon_k\langle E^j_k, E^j_k\rangle \\
&= -l_j\langle \tilde V,\tilde W\rangle\frac{\langle \mathrm{grad}_B f_i, \mathrm{grad}_B f_j\rangle}{f_i f_j}
\end{aligned}
$$

For $E^i_k \in \mathcal{L}(F_i)$ we get

$$
{}^MR(\tilde V, E^i_k)\tilde W = {}^{F_i}R(V,E^i_k)W + \frac{\langle \mathrm{grad}_B f_i, \mathrm{grad}_B f_i\rangle_B}{f_i^2}(\langle \tilde V,\tilde W\rangle_M E^i_k - \langle \tilde W, E^i_k\rangle_M\tilde V)
$$

and so

$$
\begin{aligned}
\sum_{k=1}^{l_i}\epsilon_k\langle {}^MR_{\tilde V E^i_k}\tilde W, E^i_k\rangle_M &= \sum_{k=1}^{l_j}\epsilon_k\langle {}^{F_i}R_{V E^i_k}W, E^i_k\rangle_{F_i} + \frac{\langle \mathrm{grad} f_i, \mathrm{grad} f_i\rangle}{f_i^2} \\
&\quad \cdot\left(\sum_{k=1}^{l_i}\langle V,W\rangle\langle E^i_k, E^i_k\rangle - \langle W, E^i_k\rangle\langle V, E^i_k\rangle\right) \\
&= {}^{F_i}\mathrm{Ric}(V,W) + (l_i - 1)\frac{\langle \mathrm{grad}_B f_i, \mathrm{grad}_B f_i\rangle_B}{f_k^2}\langle V,W\rangle_M
\end{aligned}
$$

Summing up the terms gives the anticipated result.

□

Next we state the geodesic equations for multiply warped products:

Proposition 2.3.9. *Let $M = B \times_{f_1} F_1 \times \ldots \times_{f_m} F_m$ be a pseudo Riemannian multiply warped product with metric $g_M = g_B \oplus f_1^2 g_{F_1} \oplus \ldots \oplus f_m^2 g_{F_m}$ and let $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$ be a curve in M defined on some interval $I \subseteq \mathbb{R}$. Then γ is a geodesic in M iff for any $t \in I$ the following equations hold:*
(1) $\alpha'' = \sum_{i=1}^m (\beta_i \circ \alpha)\langle \beta_i', \beta_i' \rangle_{F_i} \mathrm{grad}_B(f_i)$
(2) $\beta_i'' = \frac{-2}{f_i \circ \alpha} \frac{d(f_i \circ \alpha)}{dt} \beta_i' \ \forall\ i \in \{1, \ldots, m\}$

Proof.

1. If $\gamma'(0)$ is neither tangent to $\{p\} \times F_1 \times \ldots \times F_m$ nor to any $B \times F_1 \times \ldots \{q_i\} \times \ldots \times F_m$ the proof works just as in case (1) of 2.2.16.
2. If $\gamma'(0) = (\alpha'(0), 0, \ldots, 0)$,$\gamma'(0) = (0, \beta_1'(0), \ldots, \beta_m'(0))$ is nonnull resp. null, the statement can also be proven analogously to 2.2.16.
3. If $\gamma'(0) = (\alpha'(0), \beta_1'(0), \ldots, \beta_m'(0))$ with $\beta_i'(0) \neq 0$ for at least one i then we are again in the case of being neither horizontal nor vertical hence we get the result from case (1) of 2.2.16

□

Now we will investigate completeness. In case of a Riemannian manifold, metric completeness is equivalent to geodesic completeness (Hopf-Rinow theorem).

Theorem 2.3.10. *Let $M = B \times_{f_1} F_1 \times \ldots \times_{f_m} F_m$ be a Riemannian multiply warped product.*

1. *If (B, g_B) and (F_i, g_{F_i}) are complete Riemannian manifolds then also (M, g_M) is a complete Riemannian manifold.*
2. *Conversely, if (M, g_M) is a complete Riemannian manifold then (B, g_B) and (F_i, g_{F_i}) are complete manifolds.*

Proof. We use the Hopf-Rinow theorem and prove metric completeness.

1. Let X be tangent to M, $\tilde{X} \in \mathcal{L}(B)$. Then
$$\langle \tilde{X}, \tilde{X} \rangle_M = \langle \mathrm{T}_{(p,q_1,\ldots q_m)} \mathrm{pr}_1(\tilde{X}), \mathrm{T}_{(p,q_1,\ldots q_m)} \mathrm{pr}_1(\tilde{X}) \rangle_B \geq 0$$
So for any curve segment $\gamma = (\alpha, \beta_1, \ldots, \beta_m)$ we obtain $L(\gamma) \geq L(\alpha)$ hence
$$d_M((p,q),(p',q')) \geq d_B(p,p') \ \forall\ (p,q),\ (p',q') \in M$$
where by d_M resp. d_B we denote the Riemannian distance functions on M resp. B.
Let now $(p_n, q_n)_n$ be a Cauchy sequence in M. By the above inequality $(p_n)_n$ is a Cauchy sequence in B. Since B is complete, $(p_n)_n$ converges to some $p \in B$. We may assume that the sequence lies completely in some compact set $P \subseteq B$ therefore we get $f_i \geq k_i > 0$ on P for some $k_i > 0$, $i \in \{1, \ldots m\}$.
Analogously to the above argument we get
$$d_M((p,q),(p',q')) \geq \min\{k_i\} \sum_{i=1}^m d_{F_i}(q_i, q_i')$$

$\forall\ (p,q),\ (p',q') \in P \times F_1 \times \ldots \times F_m$.
Thus $(q_n)_n$ is a Cauchy sequence in F_i which converges by completeness of F_i. Summing up, the original Cauchy sequence converges in M and therefore M is complete.

2. Let $(p_n)_n$ be a Cauchy sequence in B. For fixed $q = (q_1, \ldots, q_m) \in F = F_1 \times \ldots \times F_m$ the sequence $(p_n, q)_n$ is a Cauchy-sequence in M, since

$$d_M((p_n,q),(p_m,q)) = d_B(p_n,p_m) + \sum_{i=1}^{m} f_i^2 d_{F_i}(q,q) = d_B(p_n,p_m) \quad (*)$$

Since M is complete there is a point $(p,q) \in M$ such that

$$\lim_n (p_n,q) = (p,q)$$

By

$$d_M((p_n,q),(p,q)) = d_B(p_n,p)$$

we obtain that $\lim(p_n) = p$ hence B is complete.
Let now (q_n^i) be a Cauchy sequence in F_i for arbitrary i, and let q^j be fixed in F_j for $i \neq j$. For fixed $p \in B$ we get the Cauchy sequence $(p, q^1, \ldots q_n^i \ldots q^m)_n$ in M since analogously to $(*)$,

$$d_M((p,q^1,\ldots q_n^i \ldots q^m),(p,q^1,\ldots q_k^i \ldots q^m)) = f_i(p) d_{F_i}(q_n^i, q_k^i)$$

Thus $\lim_n(p,q^1,\ldots q_n^i \ldots q^m) = (p,q^1,\ldots q^i \ldots q^m)$ exists where $q_i \in F_i$. So

$$d_M((p,q^1,\ldots q_n^i \ldots q^m),(p,q^1,\ldots q_k^i \ldots q^m)) = f_i(p) d_{F_i}(q_n^i, q_k^i)$$

Therefore

$$\lim_n q_n^i = q^i$$

and F_i is complete.

□

In case of Lorentzian multiply warped products the situation is more complicated and some more calculation is needed. In what follows we consider Lorentzian multiply warped products of the form $M = (c,d) \times_{f_1} F_1 \times \ldots \times_{f_m} F_m$ with metric $g_M = -dt^2 \oplus f_1^2 g_{F_1} \oplus \ldots \oplus f_m^2 g_{F_m}$ where $-\infty \leq c < d \leq \infty$.

Definition 2.3.11. A spacetime is said to be *null (resp. timelike) geodesically incomplete* if at least one future directed null (resp. timelike) geodesic cannot be extended to arbitrary negative and positive values.

From the definition we can see that if $c > -\infty$ or $d < \infty$ then (M, g_M) is timelike geodesically incomplete for any choice of warping functions f_i. The following theorems are just stated without proof. For further details see [Ül].

Theorem 2.3.12. *Let M be a Lorentzian multiply warped product with metric as above.*

- *If $\lim_{t \to c^+} \int_{w_0}^{t} f_i(s)ds < \infty$ for some $w_0 \in (c,d)$ then there exists some future directed nullgeodesics which are past incomplete hence (M, g_M) is future directed null geodesically past incomplete.*

- *If* $\lim_{t\to c^+}\int_{w_0}^{t}\frac{f_i(s)ds}{(1+f_i^2(s))^{1/2}}<\infty$ *for some* $w_0\in(c,d)$ *then* (M,g_M) *is future directed timelike geodesically past incomplete.*
- *If* $\lim_{t\to c^+}\int_{w_0}^{t}f_i(s)ds<\infty$ *and* f_i *is bounded on* (c,w_0) *for some* $w_0\in(c,d)$ *then* (M,g_M) *is future directed spacelike geodesically past incomplete.*
- *The analogous statements hold for future incompleteness if we interchange* $\lim_{t\to c^+}$ *with* $\lim_{t\to d^-}$.

Theorem 2.3.13. *Let* (F_i,g_{F_i}) *be complete. Then*

1. *If* (M,g_M) *is timelike complete then* (M,g_M) *is null complete.*
2. *If* $0<\inf(f_i)<\sup(f_i)\ \forall\ i$ *then* (M,g_M) *is null complete iff* (M,g_M) *is timelike complete.*

For the converse we get

Theorem 2.3.14.

1. *If* (M,g_M) *is null, timelike or spacelike complete then* (F_i,g_{F_i}) *is a complete Riemannian manifold for any* $i\in\{1,\ldots,m\}$.
2. *If* (M,g_M) *is null, timelike or spacelike complete then* (B,g_B) *is either a null, timelike or spacelike complete Lorentzian manifold.*

Chapter 3

Robertson-Walker spacetime

3.1 Homogeneity and isotropy

The study of our universe has shown that there is no large asymmetry in the distribution of galaxies. So our universe looks the same in all directions. Hence for a reasonably realistic spacetime model we assume our universe to be homogeneous and isotropic. Robertson-Walker spacetimes are based on such assumptions hence we will now give a precise mathematical definition of these properties.

Definition 3.1.1. Let (H, h) be a Riemannian manifold. H is said to be *homogeneous* if the isometry group $I(H)$ of (H, h) acts transitively on H, i.e. given any $p, q \in H$ there is an isometry $\phi \in I(H)$ with $\phi(p) = q$. Further (H, h) is said to be *two-point homogeneous* if given any $p_1, q_1, p_2, q_2 \in H$ with $d_0(p_1, q_1) = d_0(p_2, q_2)$ (where d_0 denotes the Riemannian distance function) there is an isometry $\phi \in I(H)$ with $\phi(p_1) = p_2$ and $\phi(q_1) = q_2$.
(H, h) is *isotropic at* p if $I_p(H) = \{\phi \in I(H) : \phi(p) = p\}$ acts transitively on the unit sphere $S_p(H) = \{v \in T_pH : h(v, v) = 1\}$. This means that given any $v, w \in S_p(H)$ there is an isometry $\phi \in I_p(H)$ with $T_p\phi(v) = w$. (H, h) is called *isotropic* if it is isotropic at every point $p \in H$.

Since one can choose $p_1 = p_2$ and $q_1 = q_2$ in the definition of two-point homogeneity above, a two-point homogeneous Riemannian manifold is also homogeneous.

Lemma 3.1.2. *If* (H, h) *is a homogeneous Riemannian manifold then* (H, h) *is complete.*

Proof. By the Hopf-Rinow theorem it suffices to show that (H, h) is geodesically complete. Therefore suppose to the contrary that there exists a unit speed geodesic $c : [a, 1) \to H$ which is inextendible to $t = 1$, i.e.

$$\lim_{t \to 1^-} c(t) \quad \text{does not exist}$$

Let $p \in H$ be an arbitrary point. Then, by 2.2.35, there exists some constant

$\alpha > 0$ such that every geodesic starting at p has length $\geq \alpha$. Set

$$\delta := \min\left(\frac{\alpha}{2}, \frac{1-a}{2}\right)$$

By homogeneity of H, for $q = c(1-\delta)$ we can find an isometry ϕ such that $\phi(p) = q = c(1-\delta)$.
Since isometries preserve geodesics we obtain a geodesic starting at q with length $\geq \alpha$ which contradicts the non-extendibility of c. □

The analogous result is in general false for homogeneous Lorentzian manifolds (without proof).

Proposition 3.1.3. *A Riemannian manifold* (H, h) *is isotropic iff it is two-point homogeneous.*

Proof. In what follows we denote by d_0 the Riemannian distance function.

($\Rightarrow$): (H, h) is isotropic, hence $\forall\ p \in H$, $\forall\ v,\ w \in S_pH$ we can find an isometry ϕ such that $T_p\phi(v) = w$

Step 1: (H, h) is homogeneous:
Let $p \in H$ and c be an inextendible geodesic, $c : (a, b) \to H$ with $c(0) = p$.
By isotropy there exists $\phi \in I_p(H)$ such that

$$T_p\phi(c'(0)) = -c'(0)$$

$\tilde{c} := t \mapsto c(-t)$ is also a geodesic with $\tilde{c}(0) = c(0) = p$ and $\tilde{c}'(0) = -c'(0)$.
Therefore

$$\tilde{c}(0) = (\phi \circ c)(0) = p$$

$$\tilde{c}'(0) = (\phi \circ c)'(0) = -c'(0)$$

Because geodesics are uniquely determined by their initial point and velocity we can conclude that

$$\tilde{c}(t) = \phi \circ c(t) = c(-t) \quad \forall\ t \in\ (a, b)$$

So we observe that

$$L(c|_{(a,0]}) = L(c|_{[0,b)})$$

Since p is an arbitrary point on c , $a = -\infty$, $b = \infty$. Therefore (H, h) is geodesically complete.
By the theorem of Hopf-Rinow any two points p_1 and p_2 can be connected by a geodesic segment c_0 of minimal length L. $c_0 : [0, L] \to H$, $c_0(0) = p_1$ and $c_0(L) = p_2$. Let p' be the midpoint of c_0, i.e. $d_0(p_1, p') = d_0(p', p_2)$.
Since (H, h) is isotropic we obtain an isometry ϕ which reverses c_0 so $\phi \circ c_0(t) = c_0(L - t)$ and thus

$$\phi(p_1) = \phi(c_0(0)) = c_0(L) = p_2$$

So (H, h) is homogeneous.

Step 2: (H, h) is two-point homogeneous:
Let $p_1,\ q_1,\ p_2,\ q_2 \in H$ such that $d_0(p_1, q_1) = d_0(p_2, q_2) = L$. We will show that

there is an isometry ϕ with $\phi(p_1) = p_2$, $\phi(q_1) = q_2$. We choose a geodesic of minimal length $c_1 : [0, L] \to H$, $c_1(0) = p_1$, $c_1(L) = q_1$ which connects p_1 with q_1, and analogously a geodesic of minimal length $c_2 : [0, L] \to H$, $c_2(0) = p_2$ $c_2(L) = q_2$ which connects p_2 and q_2.
The homogeneity of (H, h) gives an isometry ψ such that $\psi(p_1) = p_2$, by isotropy there is some $\eta \in I_{p_2}(H)$ with $T_{p_2}\eta((\psi \circ c_1)'(0)) = c_2'(0)$ so

$$\phi = \eta \circ \psi$$

is the isometry we searched for. Indeed, $\eta \circ \psi \circ c_1$ is a geodesic since $\eta \circ \psi$ is an isometry satisfying

$$\begin{aligned} \eta \circ \psi \circ c_1(0) &= \phi(p_1) = \eta(\psi(p_1)) \\ &= \eta(p_2) = p_2 \\ &= c_2(0) \end{aligned}$$

and

$$(\eta \circ \psi \circ c_1)'(0) = T_{p_2}\eta((\psi \circ c_1)'(0)) = c_2'(0)$$

Therefore

$$\eta \circ \psi \circ c_1 = c_2$$

and so

$$\phi(q_1) = \eta \circ \psi \circ c_1(L) = c_2(L) = q_2$$

($\Leftarrow$): Let $p \in M$ be fixed. We choose a convex neighborhood $U \subseteq M$ of p and $\alpha > 0$ such that

$$\exp_p(u) \in U \quad \forall\, u \in T_pH \quad \text{with } h(u, u) \leq \alpha \qquad (*)$$

Let $v,\ w \in T_pH$, $v, w \neq 0$ and

$$h(v, v) = h(w, w) < \frac{\alpha}{2}$$

For $q_1 := \exp_p(v)$ and $q_2 := \exp_p(w)$ we obtain by $(*)$ that q_1 and q_2 are in U and

$$d(p, q_1) = (h(v, v))^{1/2} = (h(w, w))^{1/2} = d(p, q_2)$$

By assumption (H, h) is two-point homogeneous, so we get some $\phi \in I(H)$ with $\phi(p) = p$, $\phi(q_1) = q_2$. Therefore $T_p\phi(v) = w$.
Indeed, by [O'N.], 3.61, (4) $\phi \circ \exp_p = \exp_{\phi(p)} \circ T_p\phi$ and by noticing that $\phi(p) = p$ we calculate

$$\begin{aligned} \exp_p(w) &= q_2 = \phi(q_1) \\ &= \phi(\exp_p(v)) \\ &= \exp_{\phi(p)} \circ T_p\phi(v) \\ &= \exp_p(T_p\phi(v)) \end{aligned}$$

Because $\exp_p$ is a diffeomorphism the result follows.
So we can conclude that $I_p(H)$ acts transitively on S_pH by choosing the above isometry ϕ and scaling appropriately.
Since p was arbitrary (H, h) is isotropic. $\square$

3.2 Robertson-Walker spacetimes

To introduce Robertson-Walker spacetimes we start with a smooth product-manifold $M = I \times S$ where $I \subseteq \mathbb{R}$ is a (possibly infinite) open interval and S is an isotropic Riemannian manifold. In what follows because of physical reasons we will just consider connected three-dimensional manifolds with curvature k. By pr_1 and pr_2 we denote the projections onto I resp. S. The lines $I \times p$ are the world lines of the galactic flow.
Let $\tilde{U} = \partial_t$ be the lift to $I \times S$ of the standard vector field $\frac{d}{dt}$ on $I \subseteq \mathbb{R}$ and for each $p \in S$ parametrize $I \times p$ by $\gamma_p(t) = (t, p)$. Then $\tilde{U}$ gives the velocity of each 'galaxy' γ_p hence they are its integral curves. We write h' for $\tilde{U}\tilde{h} = \frac{dh}{dt}$ where $\tilde{h}(t, p)$ is the lift of $h(t)$ with $h \in \mathcal{C}^\infty(I)$
Keeping t constant gives the hypersurface $S(t) = t \times S = \{(t, p) : p \in S\}$. Below we will show that these surfaces which foliate the manifold have constant curvatures and that for distinct times s and t, $S(s)$ and $S(t)$ are related via the homothety $\mu(s, p) = (t, p)$.

Now our task is to express $I \times S$ as a warped product $I \times_f S$ with smooth $f : I \to (0, \infty)$, making the following observations. We use the assumptions of isotropy and homogeneity where locally the isotropy-condition states that each (t, p) has a neighborhood U such that for given unit tangent vectors x and y to $S(t)$ at (t, p) there is an isometry $\Phi = \mathrm{id} \times \Phi_S$ of U with $T_{(t,p)}\Phi(x) = y$and with Φ_S an isometry on a neighborhood of p in S. Φ therefore also satisfies $\Phi(t, p) = (t, p)$.

1. Each particle γ_p is a curve parametrized by proper time t hence
$$\langle U, U\rangle_M = -1.$$
2. We can split the tangent space at a certain point into the particle's time axis and in the restspace. In case of isotropy it follows that the relative motion of the actual galaxies is negligible on large scale average thus we take each slice $S(t)$ to be a common restspace for γ_p requiring
$$U \perp S(t) \quad \forall\, t \in I$$
hence $S(t)$ becomes a Riemannian (i.e. a spacelike, since U is timelike, see [O'N.], 5.26.) hypersurface.
3. S has constant curvature k and every injection $j_t : S \to S(t)$ is a homothety of scale factor $f(t)$. In particular the constant curvature of $S(t)$ is $\frac{k}{f(t)^2}$ and for vectors x, y tangent to $S(t)$ we obtain
$$\langle x, y\rangle_M = f^2(t)\langle T_{(t,p)}\mathrm{pr}_2(x), T_{(t,p)}\mathrm{pr}_2(y)\rangle_S$$

The last item in the list above follows from 3.2.2 and 3.2.3 below. Before we start with the proof we state and prove *Schur's lemma*, which will be needed in what follows.

Lemma 3.2.1. *Suppose that a Riemannian manifold* (M, g) *of dimension* $n \geq 3$ *satisfies one of the following two conditions:*

1. *$K(\Pi) = f(p)$ for all 2-planes $\Pi \subseteq T_p(M)$, $p \in M$*

2. *$Ric(v) = (n-1)f(p)v \ \forall\ v \in T_pM$, $p \in M$*

Then in either case f must be constant. In other words, the metric has constant curvature.

Proof. See also [Pe]3,3.5, lemma 3.
Step 1: The first condition implies the second one, i.e.
$K(\Pi) = f(p)$ implies $\mathrm{Ric}(v) = (n-1)f(p)v$:
Let $v \in T_p(M)$, $\langle v, v\rangle = 1$.
Choose unit vectors $e_2, \dots e_n$ such that $\{v, e_2, \dots, e_n\}$ is an orthonormal base for $T_p(M)$ and extend it to an orthonormal frame near p. Since by [O'N.], 3.52.,

$$\mathrm{Ric}(v) = R(v,v)v + \sum_{i=2}^{n} R(v,e_i)e_i$$

The first term vanishes so we get

$$\langle \mathrm{Ric}(v), v\rangle = \sum_{i=2}^{n} \langle R(v,e_i)e_i, v\rangle = (n-1)f(p)\langle v, v\rangle.$$

in the last equality we used that $\langle R(v,e_i)e_i, v\rangle = K(\Pi(v,e_i)) = f(p)$ where $\Pi(v,e_i)$ is the 2-plane spanned by v and e_i.
To obtain the result it remains to show that $\langle \mathrm{Ric}(v), e_j\rangle = 0\ \forall\ j$.
In case of $i \neq j$ we calculate:

$$\begin{aligned}
2f(p) &= K(\Pi(e_i, v+e_j))\langle e_i, e_i\rangle\langle v+e_j, v+e_j\rangle \\
&= \langle R(e_i, v+e_j)e_i, v+e_j\rangle \\
&= \langle R(e_i,v)e_i, v+e_j\rangle + \langle R(e_i,e_j)e_i, v+e_j\rangle \\
&= \langle R(e_i,v)e_i, v\rangle + \langle R(e_i,v)e_i, e_j\rangle + \langle R(e_i,e_j)e_i, v\rangle + \langle R(e_i,e_j)e_i, e_j\rangle \\
&= K(\Pi(e_i,v)) + \langle R(e_i,v)e_i, e_j\rangle + \langle R(e_i,e_j)e_i, v\rangle + K(\Pi(e_i,e_j)) \\
&= 2f(p) + \langle R(e_i,v)e_i, e_j\rangle + \langle R(e_i,e_j)e_i, v\rangle
\end{aligned}$$

By using pair symmetry $\langle R(e_i,v)e_i, e_j\rangle = 0$.

For $i = j$ we obtain that $\langle R(v,e_i)e_i, e_i\rangle = \langle R(e_i,e_i)v, e_i\rangle = 0$ because $R(e_i,e_i) = 0$.
So $\langle \mathrm{Ric}(v), e_j\rangle = \sum_{i=2}^{n} \langle R(v,e_i)e_i, e_j\rangle = 0$ since every term vanishes.

Step 2: If condition (2) is satisfied then f is constant:
From [O'N.], 3.54 we know that

$$dS = 2\mathrm{div}(\mathrm{Ric})$$

Hence

$$\begin{aligned}
dS &= d(C(\mathrm{Ric})) \\
&= d(C((n-1)f\ \mathrm{Id})) \\
&= d((n-1)f\ C(\mathrm{Id})) \\
&= d((n-1)n\ f) \\
&= n(n-1)\ df
\end{aligned}$$

On the other hand

$$\begin{aligned}
2\ \mathrm{div}(\mathrm{Ric})(v) &= 2\sum\langle(\nabla_{e_i}\mathrm{Ric})(v), e_i\rangle \\
&= 2\sum\langle(\nabla_{e_i}((n-1)f\ \mathrm{Id}))(v), e_i\rangle \\
&= 2\sum\langle(n-1)(\nabla_{e_i}(f))\ \mathrm{Id}(v), e_i\rangle + 2\sum\langle(n-1)f\ (\nabla_{e_i}(\mathrm{Id}))(v), e_i\rangle \\
&= (*)
\end{aligned}$$

where in the last equality we have used the product rule. The second term vanishes since again by using product rule

$$\begin{aligned}
0 &= \delta_{1i} = \langle v, e_i\rangle \\
&= \nabla_{e_i}(g(\mathrm{Id}(v), e_i)) \\
&= (\nabla_{e_i}(g))(\mathrm{Id}(v), e_i) + g(\nabla_{e_i}(\mathrm{Id}(v)), e_i) + g(\mathrm{Id}(v), \nabla_{e_i}e_i) \\
&= g((\nabla_{e_i}\mathrm{Id})(v), e_i) + g(\mathrm{Id}(\nabla_{e_i}v), e_i) + g(\mathrm{Id}(v), \nabla_{e_i}e_i) \\
&= g((\nabla_{e_i}\mathrm{Id})(v), e_i) + \nabla_{e_i}(g(v, e_i))
\end{aligned}$$

where we used that $\nabla_{e_i}(g) = 0$, $\mathrm{Id}(\nabla_{e_i}v) = \nabla_{e_i}v$ and $\nabla_{e_i}(g(v, e_i)) = 0$. Hence $g((\nabla_{e_i}\mathrm{Id})(v), e_i) = 0$. Therefore we continue the above calculation and get

$$\begin{aligned}
(*) &= 2\sum\langle(n-1)(\nabla_{e_i}(f))\ \mathrm{Id}(v), e_i\rangle \\
&= 2(n-1)\sum\langle v, (\nabla_{e_i}f)e_i\rangle \\
&= 2(n-1)\langle v, \sum(\nabla_{e_i}f)e_i\rangle \\
&= 2(n-1)\langle v, \mathrm{grad}\ f\rangle \\
&= 2(n-1)\ df(v)
\end{aligned}$$

From the two equalities above we get $n\ df = 2\ df$ which can only be satisfied if $n = 2$ (which is not possible since we assumed $n \geq 3$) or $df = 0$, hence f is constant.

□

Proposition 3.2.2. *Under the assumption of isotropy each slice $S(t)$ in $I \times S$ has constant curvature $C(t)$*

Proof. Let Π be a 2-plane in an arbitrary tangent space to $S(t)$ in (t, p). Since as remarked in the beginning of the chapter we assume for physical reasons that $S(t)$ is three-dimensional, Π has the form $x^\perp$ for some unit vector $x \in T_pS(t)$.
Let Π' be another plane in $T_pS(t)$ with $\Pi' = x'^\perp$, where $x' \in T_pS(t)$.
We now want to show that there is an isometric isotropy $\Phi_S : S \to S$ such that $T_p\Phi_S(\Pi) = \Pi'$:
As already mentioned in the beginning of this chapter, it follows from isotropy of $I \times S$ that for all $(t, p) \in I \times S$ there is a neighborhood N of (t, p) in $I \times S$ such that for arbitrary unit vectors v, w tangent to $S(t)$, $v,\ w \in T_pS(t)$ there is an isometry $\Phi = \mathrm{id} \times \Phi_S$ with $T_{(t,p)}\Phi(v) = w$ $(*)$.
Set

$$\bar{x} = (0, x)\ \in T_{(t,p)}S(t)$$

$$\bar{x'} = (0, x')\ \in T_{(t,p)}S(t)$$

Isotropy gives some Φ as in $(*)$ with $T_{(t,p)}\Phi(\bar{x}) = \bar{x'}$.
Since $T_{(t,p)}\Phi(\bar{x}) = (T_t\mathrm{id} \times T_p\Phi_S)(0,x) = (0,x')$ it follows that $T_p\Phi_S(x) = x'$ and therefore also $T_p\Phi_S(\Pi) = \Pi'$.
Since isometries preserve curvature it follows that Π and Π' have the same sectional curvature in the geometry of $S(t)$. Π and Π' were arbitrarily chosen hence the first condition in 3.2.1 is satisfied and therefore $S(t)$ has constant curvature. $\square$

Proposition 3.2.3. *For any t, s the map $\mu : (S(s), g_s) \to (S(t), g_t)$, $\mu(s,p) = (t,p)$, where g_s and g_t are the metrics on $S(s)$ resp. $S(t)$, is a homothety.*

Proof. *Step 1*: μ is conformal, i.e. there exists a function h such that $\mu^*(g_t) = h(s,p)g_s$:
This statement is equivalent to saying that $\left\|T_{(s,p)}\ \mu(x)\right\|$ is the same for all unit vectors x tangent to $S(s)$. Indeed, for x being such a unit vector we can calculate

$$h(s,p)\underbrace{g_s(x,x)}_{=1} = g_t(T_{(s,p)}\mu(x), T_{(s,p)}\mu(x)) = \left\|T_{(s,p)}\mu(x)\right\|^2$$

So we will show the second statement:
Let y be another unit vector, $y \in T_{(s,p)}S(s)$. From isotropy of $I \times S$ we obtain an isometry $\Phi = \mathrm{id} \times \Phi_S$ such that $T_{(s,p)}\Phi(x) = y$. Furthermore, $\mu \circ \Phi = \Phi \circ \mu$ since

$$\mu \circ \Phi(s,p) = \mu(\Phi(s,p)) = \mu(s, \Phi_S(p)) = (t, \Phi_S(p)) = \Phi(t,p) = \Phi \circ \mu(s,p)$$

where t is chosen small enough that (t,p) is in the domain of Φ. We therefore obtain $T_{(s,p)}(\mu \circ \Phi) = T_{(s,p)}(\Phi \circ \mu)$ and so $T_{\Phi(s)}\mu \circ T_s\Phi = T_{\mu(s)} \circ T_s\mu$. Hence we calculate

$$\begin{aligned}
\left\|T_{(s,p)}\mu(y)\right\| &= \left\|T_{(s,p)}\mu(T_{(s,p)}\Phi(x))\right\| \\
&= \left\|T_{(s,p)}(\mu \circ \Phi)(x)\right\| \\
&= \left\|T_{(s,p)}(\Phi \circ \mu)(x)\right\| \\
&= \left\|T_{(t,p)}\Phi(T_{(s,p)}\mu(x))\right\| \\
&= \left\|T_{(s,p)}\mu(x)\right\|
\end{aligned}$$

where in the last equality we used that Φ is an isometry. Hence μ is conformal.

Step 2: Finding the conformal factor h:
Let $x \in T_{(s,p)}S(s)$ be an arbitrary unit vector. We define

$$h(s,p,t) := g_t(T_{(s,p)}\mu(x), T_{(s,p)}\mu(x))$$

then $h : I \times S \times I \to \mathbb{R}$ is $\mathcal{C}^\infty$ and the conformal factor we are searching for. Indeed let $z \in T_{(s,p)}S(s)$, then $\frac{z}{\|z\|}$ has length 1 in $S(s)$ and

$$h(s,p,t) = g_t(\frac{T_{(s,p)}\mu(z)}{\|z\|}, \frac{T_{(s,p)}\mu(z)}{\|z\|})$$

and so

$$h(s,p,t)g_s(z,z) = g_t(T_{(s,p)}\mu(z), T_{(s,p)}\mu(z))$$

Step 3: h is independent of p
It suffices to show that $x(h) = 0 \ \forall \ x \in T_{(t,p)}S(t), \ \forall \ t \in I, \ p \in S$ since then we obtain by choosing coordinates such that $\frac{\partial}{\partial x_i} = x$ that $D_i(h \circ \phi^{-1})|_{\phi(p)} = 0$ with respect to any chart ϕ. Hence h is constant on any chart-neighborhood and since S is connected, h is constant everywhere on S.
Let now σ be the geodesic in $S(s)$ with $\sigma(0) = (s,p)$ and $\sigma'(0) = x$. Again from isotropy we obtain an isometry $\Phi = \mathrm{id} \times \Phi_S$ with $T_{(s,p)}\Phi(x) = -x$.
$\Phi \circ \sigma$ is a geodesic by [O'N.] , p.91, (3), satisfying $\Phi \circ \sigma(0) = \Phi(s,p) = (s,p)$ and $(\Phi \circ \sigma)'(0) = T_{(s,p)}\Phi(\sigma'(0)) = T_{(s,p)}\Phi(x) = -x$.
Defining $\tilde{\sigma} := u \mapsto \sigma(-u)$ we get another geodesic with $\tilde{\sigma}(0) = (s,p)$ and $\tilde{\sigma}'(0) = -\sigma'(0) = -x$ hence

$$\sigma(-u) = \tilde{\sigma}(u) = \Phi \circ \sigma(u) \quad \forall \ u$$

Differentiating with respect to u leads us to

$$T_{(s,p)}\Phi(\sigma'(u)) = -\sigma'(-u)$$

Since Φ commutes with μ (as shown above) we have

$$\begin{aligned} h(\sigma(u),t) &= \left\| T_{(t,p)}\mu(\sigma'(u)) \right\|^2 \\ &= \left\| T_{(s,p)}\Phi(T_{(t,p)}\mu(\sigma'(u)))) \right\|^2 \\ &= \left\| T_{(t,p)}\mu(T_{(s,p)}\Phi(\sigma'(u)) \right\|^2 \\ &= \left\| T_{(t,p)}\mu(-\sigma'(-u)) \right\|^2 \\ &= h(\sigma(-u),t) \end{aligned}$$

Therefore $u \mapsto h(\sigma(u),t)$ is symmetric about 0 and so

$$(xh)(s,p,t) \overset{x=\sigma'(0)}{=} \frac{d}{du} h(\sigma(u),t)|_{u=0} = 0$$

□

So $I \times_f S$ is a warped product with base I and fiber S, the function f is used as a warping function.

Definition 3.2.4. A *Robertson-Walker spacetime* $(M(k,f),g)$ is any Lorentzian manifold which can be written as a Lorentzian warped product $I \times_f S$ with $I = (a,b)$ for $-\infty \leq a < b \leq \infty$ given the negative metric $-dt^2$, S an isotropic Riemannian manifold with curvature k and $f : I \to (0,\infty)$ a smooth warping function.

Explicitly, $M(k,f)$ is the manifold $I \times S$ with line element $-dt^2 + f^2(t)d\sigma^2$ where $d\sigma^2$ is the line element of S lifted to $I \times S$.
The manifold is time oriented by requiring that $\tilde{U} = \partial_t$ is future pointing.
If the constant sectional curvature of S is nonzero, the metric may be rescaled such that k is either identically $+1$ or -1, hence in the following we deal with S being a connected three-dimensional Riemannian manifold with constant curvature $k = 1, 0$ or -1.

3.3 Geometric properties

In this section we will apply the results developed in chapter 1 for general warped products to Robertson-Walker spacetimes since we have identified them as warped products with warping function f.

Definition 3.3.1. $I \subseteq \mathbb{R}$ is said to be *maximal* if $f : I \to \mathbb{R}$ cannot be extended to a smooth positive function on an interval strictly larger than I.
The Riemannian manifold S is called the *space* of $M(k, f)$. We denote its metric tensor by $\langle\ ,\ \rangle_S$, the connection by ∇^S, on I we write $\langle\ ,\ \rangle_I$ and ∇^I, analogously $\langle\ ,\ \rangle_M$ and ∇^M for the metric tensor and connection on $M(k, f)$

One can show ([B.E.], 4.12) that all possible choices for S are the simply connected ones, $\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3$, with curvature $0, -1, 1$ respectively. Expressed in coordinates the metric takes the form:
$k = 1$:

$$ds^2 = -dt^2 + f(t)^2(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\varphi^2))$$

$k = 0$:

$$\begin{aligned} ds^2 &= -dt^2 + f(t)^2(dx^2 + dy^2 + dz^2) \\ &= -dt^2 + f(t)^2(d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\varphi^2)) \end{aligned}$$

$k = -1$:

$$ds^2 = -dt^2 + f(t)^2(d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta d\varphi^2))$$

Remark 3.3.2. In special relativity and in prerelativity physics it is assumed that space has a flat structure given by $k = 0$.

Let now $\mathcal{L}(S)$ be the set of all lifts of vector fields on S to $M(k, f)$, $\mathcal{L}(I)$ those from I. Let $\tilde{U} = \partial_t$ be the flow vector field normal to each slice $S(t)$ with $\langle\tilde{U}, \tilde{U}\rangle_M = -1$ hence it is a future pointing unit vector field. We will now express the connection ∇^M in terms of the connections ∇^I, ∇^S on I resp. S and similar for the curvature.

Corollary 3.3.3. *For $\tilde{X}, \tilde{Y} \in \mathcal{L}(S)$, $\tilde{U} \in \mathcal{L}(I)$ on $M(k, f)$ we obtain:*
(1) $\nabla^M_{\tilde{U}}\tilde{U} = 0$
(2) $\nabla^M_{\tilde{U}}\tilde{X} = \nabla^M_{\tilde{X}}\tilde{U} = \frac{f'}{f}\tilde{X}$
(3) nor $\nabla^M_{\tilde{X}}\tilde{Y} = II^S(\tilde{X}, \tilde{Y}) = \langle\tilde{X}, \tilde{Y}\rangle_M \frac{f'}{f}\tilde{U}$
(4) tan $\nabla^M_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^S_X Y}$

Proof. We use the analogous statements proven for general warped products in 2.2.11.
(1) $\nabla^M_{\tilde{U}}\tilde{U} \overset{2.2.11(1)}{=} \widetilde{\nabla^I_U U} = 0$ since $\nabla^I_U U = 0$.
(2) $\nabla^M_{\tilde{U}}\tilde{X} = \nabla^M_{\tilde{X}}\tilde{U} \overset{2.2.11(2)}{=} \frac{\tilde{U}f}{f}\tilde{X} = \frac{f'}{f}\tilde{X}$.
(3) nor$\nabla^M_{\tilde{X}}\tilde{Y} = II^M(X, Y) \overset{2.2.11(3)}{=} -\frac{\langle\tilde{X},\tilde{Y}\rangle_M}{f}\mathrm{grad}_I f = (*)$.
We observe that $\mathrm{grad}_I f = -f'U$ since

$$\langle \mathrm{grad} f, X\rangle_M = df(X) = f'dt(X) = \langle -f'\tilde{U}, X\rangle_M \qquad (**)$$

So we get

$$(*) = -\frac{\langle\tilde{X}, \tilde{Y}\rangle_M}{f}(-f'\tilde{U}) = \langle\tilde{X}, \tilde{Y}\rangle_M \frac{f'}{f}\tilde{U}$$

(4) tan $\nabla^M_{\tilde{X}}\tilde{Y} = \widetilde{\nabla^S_X Y}$ follows as in 2.2.11(4). □

Remark 3.3.4. By 3.3.3 (1) each γ_p is a geodesic, indeed

$$0 = \nabla^M_{\tilde{U}}\tilde{U} = \nabla^M_{\gamma_p'}\gamma_p'$$

3.3.3 (3) gives the shape tensor of the totally umbillic fibers.

Corollary 3.3.5. *For vector fields $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{L}(S)$ we have:*
(1) ${}^M R_{\tilde{X}\tilde{Y}}\tilde{Z} = \left[\frac{f'^2}{f^2} + \frac{k}{f^2}\right][\langle\tilde{X},\tilde{Z}\rangle_M \tilde{Y} - \langle\tilde{Y},\tilde{Z}\rangle_M \tilde{X}]$
(2) ${}^M R_{\tilde{X}\tilde{U}}\tilde{U} = \frac{f''}{f}\tilde{X}$
(3) ${}^M R_{\tilde{X}\tilde{Y}}\tilde{U} = 0$
(4) ${}^M R_{\tilde{X}\tilde{U}}\tilde{Y} = \frac{f''}{f}\langle\tilde{X},\tilde{Y}\rangle_M \tilde{U}$

Proof. We use the results from 2.2.21.
(1) $S(t)$ has constant curvature $\frac{k}{f(t)^2}$ ([O'N.], 3.65, 3.2.3) so by [O'N.], 3.43

$${}^S R_X Y Z = \frac{k}{f(t)^2}[\langle\tilde{X},\tilde{Z}\rangle_M Y - \langle\tilde{Y},\tilde{Z}\rangle_M X]$$

where X, Y, Z are the corresponding vector fields on S.
Using $(**)$ in 3.3.3

$$\langle \mathrm{grad} f, \mathrm{grad} f\rangle_M = \langle -f'\tilde{U}, -f'\tilde{U}\rangle_M = f'^2\langle\tilde{U},\tilde{U}\rangle_M = -f'^2$$

we apply Gauss' equation, [O'N.], 4.5.,

$$\begin{aligned} {}^M R_{\tilde{X}\tilde{Y}}\tilde{Z} &= {}^S R_{XY} Z - \frac{-f'^2}{f^2}[\langle\tilde{X},\tilde{Z}\rangle_M \tilde{Y} - \langle\tilde{Y},\tilde{Z}\rangle_M \tilde{X}] \\ &= \left(\frac{k}{f^2} + \frac{f'^2}{f^2}\right)[\langle\tilde{X},\tilde{Z}\rangle_M \tilde{Y} - \langle\tilde{Y},\tilde{Z}\rangle_M \tilde{X}] \end{aligned}$$

(2) Is just the application of 2.2.21(2). Indeed, ${}^M R_{\tilde{X}\tilde{U}}\tilde{U} = \frac{H^f(\tilde{U},\tilde{U})}{f}\tilde{X}$. For H^f we calculate by using [O'N.], 3.48 and the fact that $\nabla^M_{\tilde{U}}\tilde{U} = 0$ by 3.3.3(1)

$$\mathrm{H}^f(\tilde{U},\tilde{U}) = \tilde{U}(\tilde{U}f) - \nabla^M_{\tilde{U}}\tilde{U} = f''$$

(3) The result is a direct consequence of 2.2.21 (3).
(4) Since ${}^M R_{\tilde{X}\tilde{U}}\tilde{Y} = -{}^M R_{\tilde{U}\tilde{X}}\tilde{Y}$ the results follows from 2.2.21 (4) by using 3.3.3$(**)$ and noticing that

$${}^M R_{\tilde{X}\tilde{U}}\tilde{Y} = -{}^M R_{\tilde{U}\tilde{X}}\tilde{Y} = -\frac{\langle\tilde{X},\tilde{Y}\rangle_M}{f}\nabla^M_{\tilde{U}}(\mathrm{grad}_I f) = -\frac{\langle\tilde{X},\tilde{Y}\rangle_M}{f}\nabla^M_{\tilde{U}}(-f'U).$$

Using the product rule leads to

$${}^M R_{\tilde{X}\tilde{U}}\tilde{Y} = \frac{\langle\tilde{X},\tilde{Y}\rangle_M}{f}(f''\tilde{U} - f'\underbrace{\nabla^M_{\tilde{U}}\tilde{U}}_{=0}) = \frac{f''}{f}\langle\tilde{X},\tilde{Y}\rangle_M \tilde{U}$$

□

Remark 3.3.6. It follows that every plane containing a $\tilde{U}$-vector has curvature $K_{\tilde{U}} = \frac{f''}{f}$ since for x tangent to $S(t)$, $u = \tilde{U}(s)$ for some s,

$$K_{\tilde{U}}(x,u) = \frac{\langle {}^M R_{xu} x, u\rangle_M}{\langle x,x\rangle_M \langle u,u\rangle_M} = -\frac{\langle {}^M R_{ux} u, x\rangle_M}{\langle x,x\rangle_M \langle u,u\rangle_M} = \frac{f''}{f}\frac{\langle x,x\rangle_M}{\langle x,x\rangle_M}.$$

The analogous calculation shows that every plane tangent to a spacelike slice has curvature $K_\sigma = \frac{(f'^2+k)}{f^2}$ since

$$K_\sigma(x,y) = \frac{\langle {}^M R_{xy} x, y\rangle_M}{\langle x,x\rangle_M \langle y,y\rangle_M - \langle x,y\rangle_M^2}$$

But be careful not to confuse K_σ and $K_{\tilde{U}}$ with the curvature of $S(t)$ which is $\frac{k}{f^2}$ in the geometry of $S(t)$. $K_{\tilde{U}}$ and K_σ are called *principal sectional curvatures of M(k,f)*.

The next geometric properties we want to have a look at are Ricci- and scalar curvature:

Corollary 3.3.7. *For a Robertson-Walker spacetime $M(k,f)$ with flow vector field $\tilde{U} = \partial_t$ we have:*
(1) Ricci-curvature
(a) $Ric^M(\tilde{U},\tilde{U}) = -3\ \frac{f''}{f}$
(b) $Ric^M(\tilde{U},\tilde{X}) = 0$
(c) $Ric^M(\tilde{X},\tilde{Y}) = \left(2\left(\frac{f'}{f}\right)^2 + 2\frac{k}{f^2} + \frac{f''}{f}\right)\langle \tilde{X},\tilde{Y}\rangle_M$ *if* $\tilde{X},\tilde{Y}\perp\tilde{U}$

(2) Scalar curvature
$S = 6\left(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} + \frac{f''}{f}\right)$

Proof. Let $(E_m) = \{E_1, E_2, E_3, \tilde{U}\}$ be a local frame field on $M(k,f)$, then by [O'N.], 3.52.,

$$\mathrm{Ric}^M(V,W) = \sum_m \epsilon_m \langle {}^M R_{VE_m} W, E_m\rangle_M$$

Therefore we can now insert the formulas developed in 3.3.5.

(1)

(a)

$$\begin{aligned}
\mathrm{Ric}^M(\tilde{U},\tilde{U}) &= \sum_{m=1}^{4} \epsilon_m \langle {}^M R_{\tilde{U}E_m}\tilde{U}, E_m\rangle_M \\
&= \sum_{m=1}^{3} \epsilon_m \langle {}^M R_{\tilde{U}E_m}\tilde{U}, E_m\rangle_M \\
&= -\sum_{m=1}^{3} \epsilon_m \langle \frac{f''}{f} E_m, E_m\rangle_M \\
&= -3\frac{f''}{f}
\end{aligned}$$

(b)

$$\mathrm{Ric}^M(\tilde{U},\tilde{X}) = \sum_{m=1}^{3} \epsilon_m \langle {}^M R_{\tilde{U}E_m}\tilde{X}, E_m\rangle_M - \epsilon_4 \langle {}^M R_{\tilde{U}\tilde{U}}\tilde{X},\tilde{U}\rangle_M = 0$$

since both $\langle {}^M R_{\tilde{U}E_m}\tilde{X}, E_m\rangle_M = -\frac{f''}{f}\langle E_m,\tilde{X}\rangle_M \langle \tilde{U}, E_m\rangle_M = 0$ and ${}^M R_{\tilde{U}\tilde{U}}\tilde{X} = 0$.

(c)

$$\mathrm{Ric}^M(\tilde{X},\tilde{Y}) = \sum_{m=1}^{3} \underbrace{\epsilon_m \langle {}^M R_{\tilde{X}E_m}\tilde{Y}, E_m\rangle_M}_{(*)} - \underbrace{\langle {}^M R_{\tilde{X}\tilde{U}}\tilde{Y},\tilde{U}\rangle_M}_{(**)}$$

For $(**)$ we calculate

$$(**) = \frac{f''}{f}\langle \tilde{X},\tilde{Y}\rangle_M \langle \tilde{U},\tilde{U}\rangle_M = -\frac{f''}{f}\langle \tilde{X},\tilde{Y}\rangle_M$$

And $(*)$ equals

$$\begin{aligned}
(*) &= \sum_{m=1}^{3} \epsilon_m \langle \left[\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}\right] (\langle \tilde{X},\tilde{Y}\rangle_M E_m - \langle E_m,\tilde{Y}\rangle_M \tilde{X}), E_m\rangle_M \\
&= \sum_{m=1}^{3} \epsilon_m \langle E_m, E_m\rangle_M \left[\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}\right] \langle \tilde{X},\tilde{Y}\rangle_M \\
&\quad - \left[\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}\right] \sum_{m=1}^{3} \epsilon_m \langle E_m,\tilde{Y}\rangle_M \langle \tilde{X}, E_M\rangle_M \\
&= 2\left[\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}\right] \langle \tilde{X},\tilde{Y}\rangle_M
\end{aligned}$$

(2)

$$\begin{aligned}
S &= C(\mathrm{Ric}) = \sum_{m=1}^{3} \epsilon_m \mathrm{Ric}^M(E_m,E_m) + \epsilon_4 \mathrm{Ric}^M(\tilde{U},\tilde{U}) \\
&= \sum_{m=1}^{3} \epsilon_m^2 \left(2\left(\frac{f'}{f}\right)^2 + 2\frac{k}{f^2} + \frac{f''}{f}\right) + 3\frac{f''}{f} \\
&= 6\left(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} + \frac{f''}{f}\right)
\end{aligned}$$

□

3.4 Geodesics

In this section we will characterize geodesics. As an application of the theory we will afterwards explain the physical effect of cosmological redshift, a phenomenon that is naturally part of every Robertson-Walker spacetime.

We start by noting that any curve α in $M(k,f) = I \times_f S$ can be written as

$$\alpha(s) = (t(s), \beta(s))$$

where $t(s)$ is the galactic time of α and β is the projection of α onto S. In what follows we write f' for $\tilde{U}f = \frac{df}{dt}$ and derivatives with respect to s are denoted by a prime.

Proposition 3.4.1. *A curve $\alpha = (t, \beta)$ in $M(k,f)$ is a geodesic iff*

1. $\frac{d^2t}{ds^2} + \langle \beta', \beta' \rangle_S f(t) f'(t) = 0$
2. $\beta'' + 2\frac{f'(t)}{f(t)}\frac{dt}{ds}\beta' = 0$

Proof. First note that since $I \subseteq \mathbb{R}$ is furnished with the standard flat metric, no Christoffel symbols are involved in the geodesic equations. Furthermore $t'(s) = t'(s)U$ and $\frac{\nabla t'(s)}{ds} = t''(s)\frac{\partial}{\partial t} = t''(s)U$. We apply 2.2.16. Therefore equations (1) and (2) are of the following form:

1. $\frac{d^2t(s)}{ds^2} = \langle \beta', \beta' \rangle_S f \circ t \; \widetilde{\mathrm{grad} f}$
2. $\beta'' = -\frac{2}{f \circ t}\frac{d(f \circ t)}{ds}\beta'$

As seen in the proof of 3.3.3, by $(**)$ $\widetilde{\mathrm{grad} f} = -f'(t)U$
For (1) we obtain by 2.2.16 that

$$\begin{aligned} t''(s)U &= -\langle \beta', \beta' \rangle_S (f \circ t) f'(t) U \\ (t''(s) + \langle \beta', \beta' \rangle_S f \circ t f'(t))U &= 0 \end{aligned}$$

and the last line is equivalent to $t''(s) + \langle \beta', \beta' \rangle_S (f \circ t) f'(t) = 0$.
Noticing that

$$\frac{d(f \circ t)}{ds} = f'(t)\frac{dt}{ds}$$

we get the anticipated result for the second equation. □

We now investigate null-geodesics:

Corollary 3.4.2. *If $\alpha = (t, \beta)$ is a null geodesic in $M(k,f)$ then the function $f(t)\frac{dt}{ds}$ is constant.*

Proof. Since $\alpha' = \left(\frac{dt}{ds}U, \beta'\right)$ we calculate

$$0 = \langle \alpha', \alpha' \rangle_M = -\left(\frac{dt}{ds}\right)^2 + \langle \beta', \beta' \rangle_S f^2$$

hence $\langle \beta', \beta' \rangle_S = \frac{1}{f^2}(\frac{dt}{ds})^2 \qquad (*)$
By using (1) in the characterization of geodesics and $(*)$ we obtain

$$\frac{d^2t(s)}{ds^2} + \langle \beta', \beta' \rangle_S f f_t(t) = 0$$

therefore $\frac{d^2t(s)}{ds^2} + \frac{1}{f} f_t \left(\frac{dt}{ds}\right)^2 = 0$. Thus multiplication with f leads to

$$0 = f\frac{d^2t(s)}{ds^2} + f_t\left(\frac{dt}{ds}\right)^2 = \frac{d}{ds}\left(f(t)\frac{dt}{ds}\right)$$

□

Corollary 3.4.3. *In $M(k,f) = I \times_f S$ let α be a curve with $t(\alpha(0)) = t_0$. Then α is a null geodesic iff*

$$\alpha(s) = (t(s), \bar\beta(h(s))$$

where $t(s)$ is the inverse function of $s = c\int_{t_0}^{t} f(t')dt'$, $\bar\beta$ is a unit speed geodesic in S and $h(s) = \int_{t_0}^{t(s)} \frac{dt'}{f(t')}$

Proof. ($\Rightarrow$): Let $\alpha(s) = (t(s), \beta(s))$.

Since α is a null geodesic we can apply 3.4.2 to obtain that

$$f(t)\frac{dt}{ds} = c \qquad (*)$$

Therefore $ds = \frac{1}{c}f(t)dt$ and we can conclude that the inverse function of t is $s = \int \frac{1}{c}f(t)dt$.
α is a null geodesic, hence

$$0 = \langle \alpha', \alpha' \rangle_M = -\left(\frac{dt}{ds}\right)^2 + f^2\langle \beta', \beta' \rangle_S$$

so the length of β is

$$L|_{s_p}^{s_0}(\beta) = \int_{s_p}^{s_0} \langle \beta', \beta' \rangle_S^{1/2} ds = \int_{s_p}^{s_0} \frac{1}{f(t)}\frac{dt}{ds} ds = \int_{t(s_p)}^{t(s_0)} \frac{dt}{f(t)} = h(s_0)$$

We now define $\bar\beta$ via $\beta(s) =: \bar\beta(h(s))$. It remains to show that $\bar\beta$ has unit speed and that it is a geodesic.
Since $h'(s) = \|\beta'(s)\|_S$ we calculate $\beta'(s) = \bar\beta'(h(s))h'(s) = \bar\beta' \|\beta'(s)\|_S$. This leads to

$$\|\bar\beta'(s)\|_S = \frac{\|\beta'(s)\|_S}{\|\beta'(s)\|_S} = 1,$$

therefore $\bar\beta$ is a unit speed curve. To show that $\bar\beta$ is a geodesic we have to prove that $\bar\beta'' = 0$. Since α is a geodesic β has to satisfy conditions (1) and (2) in 3.4.1. Inserting

$$\beta''(s) = \bar\beta''(h(s))h'(s)^2 + \bar\beta'(h(s))h''(s)$$

in (2) leads to

$$\begin{aligned}
\bar\beta''(h(s))h'(s)^2 + \bar\beta'(h(s))h''(s) &= -2\frac{f'(t(s))}{f(t(s))}\frac{dt}{ds}\beta' \\
&= -2f' \|\beta'(s)\|_S \bar\beta'(h(s))h'(s) \\
&= -2f' \|\beta'(s)\|_S \bar\beta'(h(s)) \|\beta'(s)\|_S
\end{aligned}$$

So it remains to show that

$$h''(s) = -2f' \|\beta'(s)\|_S^2$$

Keeping $(*)$ in mind we calculate

$$\begin{aligned}
h''(s) &= -\frac{1}{f(t(s))^2}f'(t(s))\left(\frac{dt}{ds}\right)^2 + \frac{1}{f(t(s))}\frac{d^2t}{ds^2} \\
&= \frac{2}{f(t(s))}\frac{d^2t}{ds^2}
\end{aligned}$$

On the other hand by 3.4.1(2) and $(*)$

$$-2f'(t(s))\,\|\beta'(s)\|_S^2 = \frac{2}{f(t(s))}\frac{d^2t}{ds^2}$$

hence both sides are equal.

($\Leftarrow$): We now have to show that α is a geodesic and that $\langle \alpha', \alpha' \rangle_M = 0$. First observe that since $ds = cf(t)dt$ it follows that $\frac{dt}{ds} = \frac{1}{cf(t(s))}$, so

$$f(t(s))\frac{dt}{ds} = \text{const.}$$

Since $\bar{\beta}$ is a unit speed geodesic, $\langle \bar{\beta}'(u), \bar{\beta}'(u) \rangle_S = 1$ and $\bar{\beta}''(u) = 0 \ \forall\ u$. Using the first of these assumptions we calculate

$$\begin{aligned}
\langle \alpha', \alpha' \rangle_M &= -\left(\frac{dt}{ds}\right)^2 + f(t(s))^2 \langle \bar{\beta}'(h(s))h'(s), \bar{\beta}'(h(s))h'(s) \rangle_S \\
&= -\left(\frac{dt}{ds}\right)^2 + f(t(s))^2 \frac{1}{f(t(s))^2}\left(\frac{dt}{ds}\right)^2 \langle \bar{\beta}'(h(s)), \bar{\beta}'(h(s)) \rangle_S \\
&= 0
\end{aligned}$$

Since by 3.4.1 α is a geodesic iff equations (1) and (2) in 3.4.1 are satisfied, we have to show that for $\beta(s) = \bar{\beta}(h(s))$

$$\frac{d^2t}{ds^2} + \langle \beta', \beta' \rangle_S f(t) f'(t) = 0$$

$$\beta'' + 2\frac{f'(t)}{f(t)}\frac{dt}{ds}\beta'(s) = 0$$

We calculate

$$(\bar{\beta}(h(s)))'' = \underbrace{\bar{\beta}''(h(s))h'^2(s)}_{=0} + \bar{\beta}'(h(s))h''(s)$$

so

$$\begin{aligned}
\beta'' + 2\frac{f'(t)}{f(t)}\frac{dt}{ds}\beta'(s) &= \bar{\beta}'(h(s))h''(s) + 2\frac{f'(t)}{f(t)}\frac{dt}{ds}\bar{\beta}'(h(s))h'(s) \\
&= \bar{\beta}'(h(s))h''(s) + 2\frac{f'(t)}{f(t)}\frac{dt}{ds}\bar{\beta}'(h(s))\frac{1}{f(t(s))}\frac{dt}{ds} \\
&= \bar{\beta}'(h(s))h''(s) + 2\frac{f'(t)}{f(t(s))^2}\left(\frac{dt}{ds}\right)^2 \bar{\beta}'(h(s))
\end{aligned}$$

Therefore it suffices to show that

$$h''(s) = -2\frac{f'(t)}{f(t(s))^2}\left(\frac{dt}{ds}\right)^2$$

For the left hand side we calculate

$$h''(s) = -\frac{1}{f(t(s))^2} f'(t(s))\left(\frac{dt}{ds}\right)^2 + \frac{1}{f(t(s))}\frac{d^2t}{ds^2}$$

Keeping in mind that $0 = \left(f(t)\frac{dt}{ds}\right)'$ the equation reduces to

$$h''(s) = -2\frac{f'(t(s))}{f(t(s))}\left(\frac{dt}{ds}\right)^2$$

which is what we wanted to show.
For the second part we calculate by recalling that $\bar{\beta}$ has unit speed,

$$\begin{aligned}\frac{d^2t}{ds^2} &+ \langle\bar{\beta}'(h(s))h'(s), \bar{\beta}(h(s))'h'(s)\rangle_S f(t)f'(t) = \\ &= \frac{d^2t}{ds^2} + 2\frac{1}{f(t(s))^2}\left(\frac{dt}{ds}\right)^2 \langle\bar{\beta}'(h(s)), \bar{\beta}'h(s)\rangle_S f(t)f't(t) \\ &= \frac{1}{f(t(s))}\left(f'(t(s))\left(\frac{dt}{ds}\right)^2 + f(t(s))\frac{d^2t}{ds^2}\right) \\ &= \frac{1}{f(t(s))}\left(f(t(s))\frac{dt}{ds}\right)' \\ &= 0\end{aligned}$$

□

Now we investigate another consequence of 3.4.2 and give a relativistic explanation of the physical effect of cosmological redshift which is the most direct observational evidence for the expansion of the universe.
The analysis of light coming from a distant galaxy obtains the characteristic pattern of spectral lines but all wavelengths λ are longer than for earth emitted light. This relative increase

$$z = \frac{\lambda_0 - \lambda_p}{\lambda_p}$$

is called *redshift parameter* of the source. We can use the warping function for the computation of the redshift parameter:

Corollary 3.4.4. *In $M(k, f)$ a photon emitted at $\gamma_p(t_p)$ and received at $\gamma_0(t_0)$ has redshift*

$$z = \frac{f(t_0)}{f(t_p)} - 1$$

Proof. For each s the galactic observers (i.e. the $\gamma_p s$) measure $\left(\frac{dt}{ds}\right)(s)$ as the energy $E(s)$ of α and $\beta'(s)$ as its momentum by decomposing $\alpha' = (\frac{dt}{ds})U + \beta'$ with U the 4-velocities of the observer. The physical relations $E = h\nu$, where h is Planck's constant as well as ν the frequency, and $\lambda\nu = 1$ remain valid hence

$$\frac{dt}{ds}(s) = E(s) = \frac{h}{\lambda(s)} \quad \forall s$$

Thus, by using 3.4.3

$$\text{const.} = f(t)\frac{dt}{ds} = f(t)\frac{h}{\lambda}$$

the result is obtained after substituting $\lambda = cf(t)$ in the definition of z. □

Now we can also compute the distance between the galaxies γ_p and γ_0. We can neglect the relative motion but their distance changes as a consequence of the overall expansion (or contraction) of the universe. $z > 0$ implies $f(t_0) > f(t_p)$, so since in our universe all distant sources have positive redshift, we assume an expanding universe. The redshift z determines the emission time t_p. Since $f(t_0)$ and z give $f(t_p)$, it is uniquely determined since f is a diffeomorphism because $f' > 0$.

Corollary 3.4.5. *Let $M(k, f)$ have space $S^3, \mathbb{R}^3$ or H^3. For a photon starting at $\gamma_p(t_p)$ and received at $\gamma_0(t_0)$ we can calculate the present distance between the galaxies γ_p and γ_0 via*

$$d = f(t_0) \int_{t_p}^{t_0} \frac{dt}{f(t)}$$

provided in case of S^3 that the value of the integral is $\leq \pi$.

Proof. The projection β of the photon onto S is a pregeodesic with speed calculated as in 3.4.3 $|\beta'|_s = \langle \beta', \beta' \rangle_S^{\frac{1}{2}} = \frac{1}{f(t)} \frac{dt}{ds}$ hence

$$L(\beta) = \int_{s_p}^{s_0} \langle \beta', \beta' \rangle_S^{1/2} ds = \int_{s_p}^{s_0} \frac{1}{f(t)} \frac{dt}{ds} ds = \int_{t_p}^{t_0} \frac{dt}{f(t)}$$

The factor $f(t_0)$ in front of the integral is caused by the fact that in general for galaxies γ_p and γ_q the distance between $\gamma_p(t)$ and $\gamma_q(t)$ in $S(t)$ is $f(t)d(p, q)$ and $d(p, q)$ is the length of β from p to q, as calculated above.
The extra hypothesis for S^3 can be avoided by setting $d(p, 0) = \min\{|L(\beta) - 2\pi m| \;|m \in \mathbb{Z}\}$. ☐

3.5 Completeness and causality

Considering null geodesics in $M(k, f) = I \times_f S$ with $I = (a, b)$, $-\infty \leq a < b \leq \infty$ we can conclude:

1. If $\int_{t_0}^{b} f(t)dt < \infty$ then 3.4.3 shows that no future pointing null geodesic can be defined on $[0, \infty)$ since $s = c \int_{t_0}^{t} f(t)dt < \infty$. Hence every null geodesic is future incomplete.
   Analogously every null geodesic is past incomplete if $\int_{a}^{t_0} f(t)dt < \infty$.

2. If both integrals mentioned above are infinite then $t(s)$ and $h(s)$ are defined for all $s \in \mathbb{R}$. If S is complete then $\bar{\beta}$ is also defined on $\mathbb{R}$ hence every inextendible null geodesic is complete, so $M(k, f)$ is null geodesically complete and inextendible.

3. For complete S the monotone function $t(s)$ of an inextendible null geodesic runs through the entire interval I. Thus Robertson-Walker photons (unless otherwise destroyed) survive from the initial singularity t_* or $-\infty$ to the final singularity t^* or ∞.

4. For complete S, 2.2.44 implies that Robertson-Walker spacetimes are globally hyperbolic hence also causal, chronological etc.

5. We also know from 2.2.48 that every level hypersurface $\{t\} \times S$ is a Cauchy-surface.

3.5.1 Examples

Einstein static universe

Let $I = \mathbb{R}$ with metric $-dt^2$, $S = \mathbb{S}^{n-1}$ with the standard spherical Riemannian metric. For $f \equiv 1$ the product Lorentzian manifold $I \times_f S$ is the n-dimensional Einstein static universe. If $n = 2$ then $M(k, f)$ is the cylinder $\mathbb{R} \times \mathbb{S}^1$ with flat metric $-dt^2 + d\theta^2$.
If $n \geq 3$ then the metric is not flat anymore since $\mathbb{S}^{n-1}$ has constant sectional curvature 1. Applying 3.4.3 for $t_0 = 0$ leads to $s = ct$, for $c = 1$ we get $t(s) = \frac{1}{s}$ and $h(s) = \int_0^t 1dt = t(s)$.

Einstein-de Sitter universe

Here we have $I = \mathbb{R}^+$, $S = \mathbb{R}^3$ and warping function $f(t) = t^{\frac{2}{3}}$. Using the corollaries above we calculate for a typical photon: $s = (3C/5)t^{\frac{5}{3}}$, taking $C = \frac{5}{3}$ gives $t(s) = s^{\frac{5}{3}}$. Then we further calculate $h(s) = \int_0^{t^{3/5}} t^{-2/3}dt = 3(t^{3/5})^{1/3} = 3s^{1/5}$
Hence a typical photon is $\alpha(s) = (s^{3/5}, 3s^{1/5}, 0, 0)$ for $s > 0$.

3.6 The flow of Robertson-Walker spacetimes

After having studied the geometric properties of Robertson-Walker spacetime we will interpret it as a solution of Einstein's equation

$$T = \frac{1}{8\pi}(Ric - \frac{1}{2}Sg)$$

where T is the stress-energy tensor, S the scalar curvature and g the metric on M. We will investigate the dynamics of the system.

3.6.1 Perfect fluids

As we will see below we can approximate the source of the gravitational field by a special kind of fluid, namely a perfect fluid. In general a fluid is a continuum, i.e. a collection of particles where instead of individuals the whole collection is described. Properties like for instance particles per unit mass, density of energy or momentum, pressure etc.vary from point to point.
In general relativity we deal with perfect fluids. Such a kind of fluid is described by a stress-energy tensor of the form

$$T^{\mu\nu} = (\rho + \gamma)U^\mu U^\nu + \gamma g^{\mu\nu}$$

where ρ is the energy density, γ the pressure, U a flow vector field and g the metric.
In order to motivate why the stress-energy tensor is of the above form we first start with the investigation of classical Newtonian results concerning the behavior of fluids and will then generalize them such that they hold in the relativistic case.

Newtonian physics:

This section is mainly based on [Pa], ch.6., [Hu], [Ba], ch. 2,3 and [CC], ch. 2,4..
To describe the motion of a fluid there are two different approaches. In the first, the Eulerian description, flow quantities are defined as functions of positions in space x and time t. The flow of a fluid is described by its velocity vector field $\vec{u}(x,t)$ where $\vec{u}$ is a smooth function depending on the position x and time t. It describes the average motion of the fluid, i.e. it gives the direction and rate at which the fluid flows. This kind of description provides a picture of the spatial distribution of fluid velocity at each instant during the motion.
The second viewpoint, named after Lagrange, uses the fact that some dynamical or physical quantities refer not only to positions in space but also to identifiable pieces of matter. Flow quantities are functions of time and of the choice of a material element of fluid. They therefore describe the dynamical history of the selected fluid element. One has to take into account that material elements change their shape as they move. So we identify the element such that its linear extension is not involved, for instance by specifying it via the position a of its center of mass at some instant t. The primary flow quantity is the velocity $\vec{v}(a,t)$.

Definition 3.6.1. A *motion* $\vec{X}$ of a body moving in $\mathbb{R}^n$ is a $\mathcal{C}^\infty$ function $\vec{X}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ where $\vec{X}(\cdot,t)$ is a diffeomorphism for each t.
We write $x = \vec{X}(a,t)$ for the location of the particle a at time t. x therefore is the *Eulerian coordinate*, also called *spatial coordinate*. a is the *Lagrangian* or *material coordinate* of the particle.
A *particle path* is the curve $x(t) = \vec{X}(a,t)$ where a is fixed. We can imagine it as the trajectories of particles carried by the fluid.
The *velocity* $\vec{v}$ of a material particle a is given by $\vec{v}(a,t) = \vec{X}_t(a,t)$ where the derivative with respect to t is taken for fixed a. The corresponding *spatial velocity* $\vec{u}(x,t)$ is defined via

$$\vec{u}(\vec{X}(a,t),t) = \vec{v}(a,t).$$

We can also reconstruct the motion $\vec{X}$ via solving the ODE-system

$$\vec{X}_t(a,t) = \vec{u}(\vec{X}(a,t),t)$$

with initial condition $\vec{X}(a,0) = a$. The solutions are the so-called *particle-paths*. In general this is a nonlinear and not explicitly solvable system.

Let now $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a function depending on (x,t) and $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ the corresponding function $F(a,t)$ satisfying $F(a,t) = f(\vec{X}(a,t),t)$. If we differentiate f with respect to t for fixed x we obtain the rate of change of f at a given spatial point, so the local rate of change at x. But in order to find the rate of change of f for a material element, i.e. the rate of change following a particle path it is necessary to add a convective rate of change caused by the movement of the molecules within the fluid due to transporting the element to a different position. This derivative is called *material derivative* and we denote it by $\frac{Df}{Dt}$. In order to express the material derivative in terms of the spatial derivative we use the chain-rule to obtain

$$\begin{aligned}\frac{Df}{Dt}(\vec{X}(a,t),t) &= \frac{\partial f}{\partial t}(\vec{X}(a,t),t) + \vec{X}_t(a,t)\nabla f(\vec{X}(a,t),t) \\ &= \frac{\partial f}{\partial t}(\vec{X}(a,t),t) + \vec{u}(\vec{X}(a,t),t)\nabla f(\vec{X}(a,t),t) \\ &= \frac{\partial f}{\partial t}(\vec{X}(a,t),t) + (\vec{u}\cdot\nabla)f\end{aligned}$$

Therefore,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla.$$

An important mathematical tool which will be used frequently in what follows is the divergence theorem:

Theorem 3.6.2. *If $\vec{v}(x)$ is continuously differentiable in a region $R \subseteq \mathbb{R}^3$ and if ∂R has $\vec{n}$ as outward unit normal then*

$$\int_{\partial R} \vec{v}\cdot\vec{n}\ dS = \int_R \operatorname{div}\ \vec{v}\ dV = \int_R \nabla\cdot\vec{v}\ dV$$

For a proof see any book about calculus resp. [Pa], p.127 f.
Line-, surface- and volume integrals of quantities which move with the fluid and consist always of the same particles are called *material integrals*. But we have to note that the integration region is changing with time. In order to deal with such material integrals we need the following result from multi variable calculus.

Definition 3.6.3. Let $P \subseteq \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^\infty$-boundary ∂P. Then

$$P_t := \{\vec{X}(a,t) : a \in P\}$$

is the spatial region which is occupied by all material particles contained in P. P_t is called *material volume*.

Theorem 3.6.4. *(Reynolds' transport theorem) Let $\vec{X} \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ be a motion with $\vec{X}(\cdot,t)$ a diffeomorphism of $\mathbb{R}^n$ for all $t \in \mathbb{R}$. Let $P \subseteq \mathbb{R}^n$ be a smooth bounded open set and $P_t = \vec{X}(P,t)$. For a smooth function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ we get*

$$\frac{d}{dt}\int_{P_t} f\ dV = \int_{P_t} \frac{\partial f}{\partial t}\ dV + \int_{\partial P_t} f\ \vec{u}\cdot\vec{n}\ dS$$

where $\vec{n}$ is the outward unit normal vector to ∂P_t, and $\vec{u}$ the spatial velocity of $\vec{X}$.

Proof. See [Hu], p.9 f. □

Many laws of fluid mechanics are conservation laws which state that the total amount of some quantity associated with a fluid is either invariant or changes because of forces.
As a first example we study the conservation of mass. Mass conservation states that there is a balance between the outflow of mass and the rate of change of mass contained in some material volume. The amount of mass contained in such a material volume P_t is given by $\int_{P_t} \rho\ dV$ where $\rho(x,t)$ is the possibly non-constant density. If the mass of the material volume does not change as the

fluid moves we see that $\frac{d}{dt}\int_{P_t}\rho\, dV = 0$. Using now 3.6.4 and the divergence theorem we obtain

$$\begin{aligned}\frac{d}{dt}\int_{P_t}\rho\, dV &= \int_{P_t}\frac{\partial\rho}{\partial t}\, dV + \int_{\partial P_t}\rho\vec{u}\cdot\vec{n}\, dS \\ &= \int_{P_t}\left(\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{u})\right) dV \\ &= 0\end{aligned}$$

This is valid for any any fixed time and arbitrary P_t therefore we can conclude that the integrand vanishes: so we have finally obtained the law of mass conservation,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

or, in components,

$$\frac{\partial\rho}{\partial t} + \sum_k \frac{\partial}{\partial x^k}(\rho u^k) = 0$$

In the special case of incompressible fluids ($\rho = \text{const}$) the law of mass conservation can be formalized as div $v = 0$.

Another conserved quantity is momentum. In general two types of forces act on a fluid. On the one hand a body force $\vec{F}$ per unit mass. This is an external force like for instance gravity or Coriolis force. On the other hand a surface pressure force acts. It arises from inter-molecular forces and thermal motion of the molecules. For simplicity we assume this force to act in the inward normal direction with magnitude per unit surface area equal to pressure γ. So we deal with the model of an inviscid fluid.

The total momentum in the material region P_t is given by $\int_{P_t}\rho\vec{u}\, dV$. By using Newton's second law we require that the rate of change of momentum of the material volume is equal to the force acting on it, i.e.

$$\frac{d}{dt}\int_{P_t}\rho\vec{u}\, dV = -\int_{\partial P_t}\gamma\vec{n}\, dS + \int_{P_t} F\, dV$$

By using 3.6.4 and the divergence theorem we arrive at

$$\begin{aligned}\int_{P_t}\frac{\partial(\rho\vec{u})}{\partial t}\, dV + \int_{\partial P_t}(\rho\vec{u})\vec{u}\cdot\vec{n}\, dS &= \frac{d}{dt}\int_{P_t}\rho\vec{u}\, dV \\ &= -\int_{\partial P_t}\gamma\vec{n}\, dS + \int_{P_t} F\, dV \\ &= -\int_{P_t}(\nabla\gamma - F)\, dV\end{aligned}$$

So we finally obtain by again using the divergence theorem that

$$\int_{P_t}\left((\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla\gamma - F\right)\, dV = 0$$

Setting $F = 0$ and noting as above that P_t was arbitrary we conclude that

$$(\rho\vec{u})_t + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla\gamma = 0$$

or,

$$\frac{\partial}{\partial t}(\rho u^i) + \sum_k \frac{\partial}{\partial x^k}(\rho u^i u^k) = \partial^i \gamma$$

In Newton's theory a source of some field has mass density ρ. In order to not violate high-precision experiments (cf. [Sch], ch. 4.7 for further details) the source of the field should consist of all energies. But this total mass energy is only one component of a tensor and using a single component as a source would lead to a non-invariant theory of gravitation. Therefore Einstein suggested that the source of the field should be a tensor T, the stress-energy tensor, containing all stresses, momenta and pressures acting as sources.
These quantities are collected by requiring that
$T(dx^\alpha, dx^\beta)$:=flux of the α-component of the 4-momentum across a surface of constant x^β
So investigating the components we can see that
T^{00} is the flux of the 0-momentum (i.e. the energy) across a surface of constant t, therefore it is just the energy density.
T^{0i} is the flux of the energy through the surface x^i and T^{ij} is the flux of the i-momentum through the surface x^j.
One can show ([Sch], 4.5) that this tensor is symmetric. Further details can be found in [Sch], 4.4.

Special relativity:

Now we want to generalize the conservation equations in a relativistic background. Therefore it is now our task to find an explicit form for $T^{\mu\nu}$ such that the conservation equations still hold if for the velocity u, $\frac{u^2}{c^2} \ll 1$ hence in the Newtonian limit. The following calculations are mainly based on [Re], 20.12. p.851 ff.

We start with investigating a fluid with $\gamma = 0$. then the conservation equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho u^k) = 0, \qquad \frac{\partial}{\partial t}(\rho u^i) + \frac{\partial}{\partial x^k}(\rho u^i u^k) = 0 \qquad (**)$$

Setting $U^0 = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}}$, $U^k = \frac{u^k}{\sqrt{1-\frac{u^2}{c^2}}}$ for $k = 1, 2, 3$ (see [Re], 20.2) and $\rho_0 = \frac{m_0}{V_0}$ (the rest-density) and defining

$$\tilde{T}^{\mu\nu} = \rho_0 U^\mu U^\nu$$

we find as a relativistic generalization of $(**)$ that

$$\partial_\nu \tilde{T}^{\mu\nu} = 0.$$

Indeed, if we set $\rho = \frac{\Delta m}{\Delta V} = \frac{\frac{\Delta m_0}{\sqrt{1-\frac{u^2}{c^2}}}}{\Delta V_0 \sqrt{1-\frac{u^2}{c^2}}} = \frac{\rho_0}{1-\frac{u^2}{c^2}}$ then we obtain

$$\begin{aligned}
\frac{\partial}{\partial x^0}\tilde{T}^{00} + \frac{\partial}{\partial x^k}\tilde{T}^{0k} &= \frac{1}{c}\frac{\partial}{\partial t}\frac{\rho_0 c^2}{1-\frac{u^2}{c^2}} + \frac{\partial}{\partial x^k}\frac{\rho_0 u^k u^i}{1-\frac{u^2}{c^2}} \\
&= c\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho u^k)\right) \\
&= 0
\end{aligned}$$

and analogously also $\frac{\partial}{\partial x^0}\tilde{T}^{i0} + \frac{\partial}{\partial x^k}\tilde{T}^{ik}$ vanish for $i = 1, 2, 3$ by using the second equation in $(**)$.

For fluids with isotropic pressure the conservation equations are

$$c\left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^k}(\rho u^k)\right) = 0, \qquad \frac{\partial}{\partial t}(\rho u^i) + \frac{\partial}{\partial x^k}(\rho u^i u^k) = \partial^i \gamma \qquad (***)$$

We have seen above that $\tilde{T}^{\mu\nu}$ satisfies the equations in $(***)$ if $\partial^i\gamma = 0$. Now we search for a tensor $S^{\mu\nu}$ whose divergence satisfies the right sides in the Newtonian limit. They can be written as the divergence of

$$(\tilde{S}^{\mu\nu}) = \gamma \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Indeed, $\partial_\nu \tilde{S}^{0\nu} = 0$ since $\tilde{S}^{0\nu} = 0$ and $\partial_\nu \tilde{S}^{i\nu} = \partial_\nu(\gamma g^{i\nu}) = g^{i\nu}\partial_\nu\gamma = \partial^i\gamma$ where g is the Minkowski metric.
The idea now is to write $S^{\mu\nu}$ as $S^{\mu\nu} = aU^\mu U^\nu + bg^{\mu\nu}$ (Δ) where a and b are to be determined. So

$$S^{\mu\nu} = \frac{a}{1 - \frac{u^2}{c^2}} \begin{pmatrix} c^2 & cu^1 & cu^2 & cu^3 \\ cu^1 & u^1u^1 & u^1u^2 & u^1u^3 \\ cu^2 & u^1u^2 & u^2u^2 & u^2u^3 \\ cu^3 & u^1u^3 & u^2u^3 & u^3u^3 \end{pmatrix} + b \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that for $u \ll c$, it has the anticipated form (Δ). Since the c^2-term dominates strongly we can ignore the terms cu^i, u^iu^k and u^iu^i to obtain

$$S^{\mu\nu} \approx a \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + b \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So we can now see that for $S^{\mu\nu} = \tilde{S}^{\mu\nu}$ it is necessary that $ac^2 + b = 0$, $b = \gamma$ and therefore $a = -\frac{\gamma}{c^2}$. If we finally define $T^{\mu\nu} := \left(\rho_0 + \frac{\gamma}{c^2}\right) U^\mu U^\nu - \gamma g^{\mu\nu}$ we again have

$$\partial_\nu T^{\mu\nu} = 0.$$

General relativity

We now have to use the covariant derivative and the conservation equations are of the form

$$T^{\mu\nu}{}_{;\nu} = 0$$

In locally flat regions T is just the tensor derived above. In what follows we will use geometric units hence $c = 1$.
After this preparation we are now ready to define perfect fluids and see which role they play in general relativity.
To begin with we show that the flow given by the vector field $U = \partial_t$ is that of a perfect fluid.

Definition 3.6.5. A *perfect fluid* on a spacetime (M, g) is a triple (U, ρ, γ) where

- U is a timelike future pointing unit vector field on M called the *flow vector field*.
- $\rho \in \mathcal{C}^\infty(M)$ is the *energy-density function*, $\gamma \in \mathcal{C}^\infty(M)$ is called *pressure function*.
- The stress energy tensor is given by

$$T = (\rho + \gamma) U^* \otimes U^* + \gamma g \qquad (*)$$

where U^* is the one-form metrically equivalent to U.

A calculation shows that

$$\begin{aligned} T(U,U) &= \rho \\ T(X,U) &= T(U,X) = 0 \\ T(X,Y) &= \gamma \langle X, Y \rangle_M \end{aligned}$$

for $X, Y \perp U$
A perfect fluid also satisfies the *energy equation* $U\rho = -(\rho + \gamma)\mathrm{div} U$ and the *force equation* $(\rho+\gamma)\nabla_U U = -\mathrm{grad}_\perp \gamma$ where $\mathrm{grad}_\perp \gamma$ is the component of $\mathrm{grad}\gamma$ orthogonal to U. For a proof see [O'N.], 12.5..

Since the stress energy tensor T is already determined by Einstein's equations our task is now to find functions ρ and γ such that T has the form $(*)$:

Theorem 3.6.6. *If U is the flow vector field on a Robertson-Walker spacetime $M(k, f)$ then (U, ρ, γ) is a perfect fluid with energy density ρ and pressure γ given by*

$$\begin{aligned} \frac{8\pi\rho}{3} &= \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} \\ -8\pi\gamma &= 2\frac{f''}{f} + \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} \end{aligned}$$

Proof. Einstein's equation gives

$$T = \frac{1}{8\pi}(\mathrm{Ric} - \frac{1}{2} S g)$$

We therefore get $T(U, X) = 0$ since by 3.3.7 $\mathrm{Ric}(U, X) = 0$ and

$$\begin{aligned} T(X,Y) &= \frac{1}{8\pi}\left(2\left(\frac{f'}{f}\right)^2 + 2\frac{k}{f^2} + \frac{f''}{f}\right)\langle X, Y\rangle_M - \\ &\quad -\frac{3}{8\pi}\left(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} + \frac{f''}{f}\right)\langle X, Y\rangle_M \\ &= \frac{1}{8\pi}\left(-\left(\frac{f'}{f}\right)^2 - \frac{k}{f^2} - 2\frac{f''}{f}\right)\langle X, Y\rangle_M \end{aligned}$$

where we have used the expressions for Ric and S as calculated in 3.3.7. We can see that the last term is just $\gamma\langle X, Y\rangle_M$ for γ as above. If

$$\begin{aligned} \rho &= T(U,U) \\ &= \frac{1}{8\pi}\left(-3\,\frac{f''}{f} - 3\left(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} + \frac{f''}{f}\right)\underbrace{\langle U,U\rangle_M}_{=-1}\right) \\ &= \frac{3}{8\pi}\left(\left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}\right) \end{aligned}$$

then T has the required form $T = (\rho+\gamma)U^* \otimes U^* + \gamma g$ where $T(U,U), T(X,Y)$ are now the same as those we received from Einstein's equation. □

Since we deal with a perfect fluid, div $T = 0$. Therefore we get for the time rate of change of the energy density:

Corollary 3.6.7. *For a Robertson-Walker perfect fluid*

$$\rho' = -3(\rho+\gamma)\frac{f'}{f}$$

Proof. Since $U\rho = \rho'$ it suffices to check that div $U = 3\frac{f'}{f}$ because the energy equation valid for U is $U\rho = -(\rho+\gamma)\text{div } U$.
Using a frame field with $E_4 = U$ we obtain by noting that $\nabla^M_U U = 0$

$$\text{div U} = \sum_{m=1}^{4} \epsilon_m \langle \nabla^M_{E_m} U, E_m\rangle_M = \sum_{m=1}^{3} \epsilon_m \langle \nabla^M_{E_m} U, E_m\rangle_M$$

Since $\nabla^M_{E_j} U = \frac{f'}{f} E_j$ by 3.3.3 we calculate

$$\text{div U} = \sum_{m=1}^{3} \frac{f'}{f}\langle E_j, E_j\rangle_M = 3\frac{f'}{f}$$

□

3.7 Friedmann cosmological models

Since except at the earliest and final era of the universe, energy density strongly dominates pressure it suffices to investigate Robertson-Walker models with $\gamma = 0$. Dusts, i.e. perfect fluids with $\gamma = 0$ and $\rho > 0$ will be the subject of our further calculations and we obtain Friedmann cosmological models.

Lemma 3.7.1. *Let $M(k, f)$ be a Robertson-Walker spacetime with f a nonconstant function. Then the following are equivalent:*
(1) The perfect fluid U is a dust.
(2) $\rho f^3 = M$ where $M > 0$ is a constant.
(3) $f'^2 + k = \frac{A}{f}$ where $A = \frac{8\pi M}{3} > 0$ (Friedmann equation).

Proof.

(2) $\Leftrightarrow$ (3): $\frac{8\pi\rho}{3} = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}$ is equivalent to $\frac{8\pi}{3}\rho f^3 = (f')^2 f + kf$, so $\rho f^3 = M$ iff $f'^2 + k = \frac{8\pi}{3}\frac{M}{f}$

(1) $\Rightarrow$ (2): If $\gamma = 0$ then, by 3.6.7, $\rho' = -3\rho\frac{f'}{f}$, so $\rho' f + 3\rho f' = 0$ hence

$$\rho' f^3 + 3\rho f' f^2 = (\rho f^3)' = 0$$

Therefore ρf^3 is constant and positive, since both ρ and f are positive.

(2) $\Rightarrow$ (1): By 3.6.7

$$\begin{aligned} & 0 = \rho' + 3(\rho + \gamma) f'/f \\ \Leftrightarrow \quad & \rho' f + 3\rho f' = -3\gamma f' \\ \Leftrightarrow \quad & \rho' f^3 + 3\rho f' f^2 = -3\gamma f' f^2 \\ \Leftrightarrow \quad & \underbrace{(\rho f^3)'}_{=0} = -3\gamma f' f^2 \end{aligned}$$

so

$$\gamma f' = 0 \qquad (*)$$

but f is not constant by assumption.
Assume $\gamma \neq 0$, then there is a maximal interval $J \subset I$ on which $\gamma \neq 0$. By $(*)$, $f' = 0$ on J and so f is constant on J, hence J cannot be the whole interval I. Since in

$$-8\pi\gamma = 2\frac{f''}{f} + \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}$$

the first two terms vanish and the third is constant on J, it follows that γ is a nonzero constant on J. Thus by continuity γ is nonzero on an interval strictly larger than J, a contradiction to the maximality of J. $\square$

Remark 3.7.2. For the scale function f we obtain in the three cases $k = 0, 1, -1$:

- $k = 0$: The solution for the Friedmann equation is provided by $f = Ct^{2/3}$, where $4C^3 = 9A$
- $k = 1$: Integration of the Friedmann equation leads to

$$t = \frac{1}{2}A(\theta - \sin\theta)$$

$$f = \frac{1}{2}A(1 - \cos\theta)$$

- $k = -1$: Now we get as a solution of $f'^2 = 1 + \frac{A}{f}$

$$t = \frac{1}{2}A(\sinh\eta - \eta)$$

$$f = \frac{1}{2}A(\cosh\eta - 1)$$

3.8 Expansion of the universe

According to Hubble (discovered in 1929) all distant galaxies are moving away from us now at a rate proportional to their distance, which shows that there is no preferred center of expansion.
For galaxies γ_p and γ_q the distance between $\gamma_p(t)$ and $\gamma_q(t)$ in $S(t)$ is $f(t)d(p,q)$ where we denote by d the usual Riemannian distance in S. So the distance scale changes with time. Hubble's discovering was that

$$H_0 = \frac{f'(t_0)}{f(t_0)} = \frac{1}{18 \pm 2 \cdot 10^9 yr}$$

f therefore has positive derivative which implies that the spaces $S(t)$ are expanding. As a consequence we obtain:

Proposition 3.8.1. *Let $M(k,f) = I \times_f S$. If $H_0 = \frac{f'(t_0)}{f(t_0)} > 0$ for some t_0 and $\rho + 3\gamma > 0$ then I has an initial endpoint t_* with $t_0 - H_0^{-1} < t_* < t_0$ and either*
(1) $f' > 0$ which means expansion or
(2) f has a maximum point after t_0 and I is a finite interval (t_*, t^*).

Proof. Since by [O'N.], 12.12., $3\frac{f''}{f} = -4\pi(\rho + 3\gamma)$ the condition $\rho + 3\gamma > 0$ implies $f'' < 0$ since $f > 0$.
Thus the graph of f is, except at t_0, below that of its tangent line at t_0. $(*)$.
This line is the graph of

$$F(t) = f(t_0) + f'(t_0)(t - t_0) = f(t_0) + H_0 f(t_0)(t - t_0)$$

$H_0 > 0$, so as t decreases from t_0, $f > 0$ must have a singularity at some t_* before reaching the zero $t_0 - H_0^{-1}$ of F since otherwise f would become less or equal to zero because of $(*)$.
Since $f'' < 0$ either f' is always positive on I or f has a maximum point ($f' = 0$) after which $f' < 0$. In this case, repeating the procedure from above we get another singularity at $t^* > t_0$. □

Thus under the assumption of homogeneity and isotropy the theory states that at a time less than H_0^{-1} ago the universe had a singularity and in the case that it does not expand forever, must, after contracting for a while, come to a final singularity.
However the result does not say that the universe begins small (i.e. $f \to 0$ as $t \to t_*$) or that in the expanding case it will last forever.

Definition 3.8.2. If the energy density approaches infinity as $t \to t_*$ (or t^*), $M(k,f)$ has a *physical singularity* at t_* (resp. t^*).
A *big bang* is an initial singularity of $M(k,f)$ at t_* if $f \to 0$ and $f' \to \infty$ as $t \to t_*$.
Similarly to the above a final singularity is called a *big crunch* if $f \to 0$ and $f' \to -\infty$ as $t \to t^*$.

Big bangs as well as big crunches are physical singularities. The converse holds as well under weaker conditions than those of 3.8.1 (2):

Theorem 3.8.3. *Let the spacetime $M(k,f) = I \times_f S$ have only physical singularities and I be maximal. If $H_0 > 0$ for some t_0, $\rho_0 > 0$ and*

$$-\frac{1}{3} < a \leq \frac{\gamma}{\rho} \leq A \qquad (*)$$

where a and A are constants, then
(1) the initial singularity is a big bang.
(2) if $k = 0$ or -1, the interval I is of the form $I = (t_*, \infty)$ and the warping function f and the energy density ρ satisfy $f \to \infty$, $\rho \to 0$ as $t \to \infty$.
(3) if $k = 1$, f reaches a maximum followed by a big crunch so the interval I is a finite, $I = (t_*, t^*)$.

Proof. Since by $(*)$

$$\rho + 3\gamma \geq \epsilon\rho > 0$$

for some $\epsilon > 0$, namely, $\epsilon = 3a + 1$, the proof of 3.8.1 guarantees $f'' < 0$.

(1) : Because $H_0 > 0$ we get $f' > 0$ on the interval (t_*, t_0).
Since $\gamma \leq A\rho$ and, by 3.6.7 $\rho' = -3(\rho + \gamma)\frac{f'}{f}$ we get

$$\rho' \geq -3(\rho + A\rho)\frac{f'}{f} = -c\rho\frac{f'}{f}$$

where $c = 3(A + 1) > 2$. So

$$(\rho f^c)' = \rho' f^c + c\rho f^{c-1} f' \geq 0$$

which means that ρf^c is increasing, hence $\rho f^c \leq \rho(t_0) f(t_0)^c$ on (t_*, t_0).
By hypothesis t_* is a physical singularity, so $\rho \to \infty$ as $t \to t_*$, hence

$$f \to 0 \qquad (**)$$

The inequality $\rho - \epsilon\rho \geq -3\gamma$ gives

$$\rho' \overset{3.6.7}{=} -3(\rho + \gamma)\frac{f'}{f} \leq (-3\rho + \rho - \epsilon\rho)\frac{f'}{f} = -(2 + \epsilon)\rho\frac{f'}{f}$$

hence $(\rho f^{2+\epsilon})' \leq 0$ on (t_*, t_0). Thus $\rho f^{2+\epsilon} \geq \rho(t_0) f(t_0)^{2+\epsilon}$ on this interval.
As $t \to t_*$ we have by $(**)$ that $f \to 0$ hence

$$\rho f^2 = \rho f^{2+\epsilon} f^{-\epsilon} \geq f^{-\epsilon} \cdot \text{const.} \ \to \infty$$

Then, by using

$$\frac{8\pi\rho}{3} = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} \qquad (\Delta)$$

(see 3.6.6), we get $f'^2 + k \to \infty$, hence $f' \to \infty$
(2) + (3):
Case 1: f has a maximum at t_m:
Since $f'(t_m) = 0$ we get $0 \overset{(\Delta)}{<} \rho(t_m) = \frac{3k}{8\pi f^2(t_m)}$ hence *k=1* because by definition of S, $k = 0, 1$ or -1 and $f > 0$.
Since $f'' < 0$, f' will be negative for $t > t_m$. Analogously to the above we get

$\rho' \le -c\rho\frac{f'}{f}$, therefore $(\rho f^c)' \le 0$.
So $\rho f^c \le \rho(t_m) f(t_m)^c$ and, by (1), $f \to 0$

$$(\rho f^{2+\epsilon})' \ge 0$$

$$\rho f^2 \ge f^{-\epsilon} \cdot \text{const} \to \infty$$

so $f'^2 + k \to \infty$.
Since $f' < 0$ it follows that $f' \to -\infty$.

Case 2: f has no maximum: $f' > 0$ on I so the results from the preceding case 1 are valid on I. The inequalities $\rho > 0$ and $\rho + 3\gamma > 0$ imply $\rho + \gamma > 0$: indeed, if $\rho + 3\gamma > 0$ it follows that $3\rho + 3\gamma > 0$, so $\rho + \gamma > 0$
Hence $\rho' = -3(\rho + \gamma)\frac{f'}{f} < 0$ since $\rho + \gamma$ as well as $\frac{f'}{f}$ are greater than zero. So there are no physical singularities as t increases if we keep in mind that ρ decreases hence cannot reach ∞.
Thus $I = (t_*, \infty)$.

Subcase A: $f \to \infty$ as $t \to \infty$:
Since $(\rho f^{2+\epsilon})' \le 0$ which is case 1, $\rho f^{2+\epsilon}$ is bounded for t large, $\rho f^{2+\epsilon} \le c$ for some c, hence $\rho f^2 \le c f^{-\epsilon} \to 0$ as $t \to \infty$.
Thus $f'^2 + k \to 0$, since $f'^2 > 0$ k has to be either 0 or -1.

Subcase B: f is bounded as $t \to \infty$:
We show that this is impossible:
Let $f \to b$ for $t \to \infty$. Since $f'' > 0$ we have $f' \to 0$.
Hence

$$\frac{8\pi\rho}{3} \to \frac{k}{f^2}$$

So $0 < \rho f^2 \to \frac{3k}{8\pi}$. Thus $k = 0$ *or* $k = 1$.
As $t \to \infty$, ρf^c is nondecreasing since f is nondecreasing, hence

$$\rho f^2 \not\to 0$$

and so $k \ne 0$.

If, finally, *k=1* then $\rho \to \frac{3}{8\pi}$ thus $\rho \ge \delta$ for some $\delta > 0$ $(*)$

Since $f' \to 0$ there is a sequence $t_i \to \infty$ such that $f''(t_i) \to 0$
indeed, take $x_i \to \infty$, $h > 0$, then $f''(t_i) = \frac{f'(x_i+h) - f'(x_i)}{h} \to 0$, $t_i \in (x_i, x_i + h)$
Because

$$-4\pi(\rho + 3\gamma) = 3\frac{f''(t_i)}{f(t_i)} \to 0$$

we get $(\rho + \gamma)(t_i) \to 0$ which is a contradiction to

$$\rho + 3\gamma \ge \epsilon\rho \overset{(*)}{\ge} \epsilon\delta > 0$$

since $-\frac{1}{3} < a \le \frac{\gamma}{\rho} \le A$. □

Hence the ultimate fate of the universe depends on the sign k of spatial curvature which, by the equation

$$\frac{8\pi\rho}{3} = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2}$$

depends on the present energy density ρ_0 and on the Hubble number H_0.

Corollary 3.8.4. *Let $\rho_c = \frac{3H_0^2}{8\pi}$ (the critical energy density).*
If $\rho_0 \le \rho_c$, then $k = 0, -1$ so the universe expands forever.
If $\rho_0 > \rho_c$ then $k = 1$, hence the universe eventually collapses.

Proof. By 3.6.6

$$\frac{8\pi\rho}{3} = \left(\frac{f'}{f}\right)^2 + \frac{k}{f^2} = H^2 + \frac{k}{f^2}$$

so

$$\rho = \frac{3H^2}{8\pi} + \frac{3k}{8\pi f^2}$$

If $\rho_0 \le \rho_c$ we obtain $\frac{8\pi\rho}{3} = H_0^2 + \frac{k}{f^2}$, therefore $0 > \rho_0 - \rho_c = \frac{k}{f^2}$.
If $\rho_0 > \rho_c$ then $\frac{8\pi\rho}{3} = H_0^2 + \frac{k}{f^2}$ and $0 \le \rho_0 - \rho_c = \frac{k}{f^2}$
So we conclude $k = \mathrm{sgn}(\rho_0 - \rho_c)$ □

Remark 3.8.5. If we apply the above statements on Friedmann cosmological models we obtain for $k = 0$ that the initial expansion continues forever with $f \to \infty$ and $f' \to 0$. For $k = 1$ the expansion reaches a maximum $f = A$ at $t = \frac{\pi A}{2}$. Afterwards contraction starts and f decreases to a final collapse at $t^* = \pi A$. If $k = -1$ we have $f' \ge 1$ hence the universe expands forever with $f \to \infty$ and $f' \to 1$.
Unless more mass is discovered in our universe 'open' models are probably more realistic than 'closed' ones. The earliest era and the final one, if it exists, are dominated by radiation thus Friedmann models give way to radiation models for which mass is zero and $\gamma = \frac{\rho}{3}$.

Chapter 4

Schwarzschild spacetime

In this chapter our task is to find a model for a universe containing a single star which is assumed to be the only source of gravitation. We also assume spherical symmetry and staticity. Below we will show that we can identify such a spacetime again as a warped product and therefore apply the results from chapter 1.
To derive the metric we first define the condition 'static' and state some results about static spacetimes needed in what follows. We start by introducing the notion of Killing vector fields.

4.1 Killing vector fields and static spacetimes

Definition 4.1.1. A *Killing vector field* on a semi Riemannian manifold M is a vector field X along which the Lie derivative of the metric tensor vanishes, i.e.

$$L_X g = 0$$

Thus the metric tensor does not change under the flow of X, i.e. moving along X does not change the metric. So one can view such a vector field as an infinitesimal isometry. Let therefore ψ be the flow of X, i.e. $\psi : M \times \mathbb{R} \to M$, $\psi(p,t) = \alpha_p(t)$ where α_p the maximal integral curve of X at p (see also [O'N.], 1.53.). Then the stages ψ_t are maps $\psi_t : M \to M$, $p \mapsto \psi(t,p)$. Hence ψ_t lets every point of M flow for time t. For further properties of flows see [O'N.], 1.54.

Proposition 4.1.2. *A vector field X is Killing if and only if the stages ψ_t of all its local flows are isometries.*

Proof. ($\Leftarrow$): If each ψ_t is an isometry then $\psi_t^*(g) = g$. Hence by using [O'N.], 9.21.,

$$L_X g = \lim_{t\to 0}[\psi_t^*(g) - g] = 0$$

($\Rightarrow$): Let $\{\psi_t\}$ be a local flow of X with X satisfying $L_X g = 0$. Let v be a tangent vector at a point p in the domain of the flow. ψ_s is defined for small s, therefore we can choose a tangent vector $w = T_p\psi_s(v)$. Via using [O'N.], 9.21., we get

$$L_X g(w,w) = \lim_{t\to 0}\frac{1}{t}(g(T_p\psi_t(w), T_p\psi_t(w)) - g(w,w)) = 0.$$

Since $\psi_s\psi_t = \psi_{s+t}$ it follows that

$$\lim_{t\to 0}\frac{1}{t}(g(T_p\psi_{s+t}(v), T_p\psi_{s+t}(v)) - g(T_p\psi_s(v), T_p\psi_s(v)) = 0.$$

So the real valued function $s \mapsto g(T_p\psi_s(v), T_p\psi_s(v))$ has identically vanishing derivative hence is constant, so by using [O'N.], 1.54., i.e. $\psi_0(t) = \mathrm{id}$,

$$g(T_p\psi_s(v), T_p\psi_s(v)) = g(T_p\psi_0(v), T_p\psi_0(v)) = g(v, v) \quad \forall\, v, s$$

□

Definition 4.1.3.

- An *observer field* on an arbitrary spacetime M is a timelike future pointing unit vector field U. Each integral curve of U is an observer parametrized by proper time. Thus U describes a family of U-*observers* filling M.
- Let $\tilde{S}$ be a hypersurface to M normal to an observer field U at every $p \in \tilde{S}$. So by [O'N.], 5.26., $\tilde{S}$ is spacelike since U is timelike. Then the infinitesimal restspace $U_p^{\perp}$ is just $T_p(M)\ \forall\, p \in \tilde{S}$ hence $\tilde{S}$ is called *restspace of* U.
- We call a spacetime M *stationary* if there is a timelike Killing vector field on M.
- A spacetime M is *static* relative to an observer field U if U is irrotational (i.e. curl $U = 0$) and if there is a smooth function $g > 0$ on M such that gU is a Killing vector field.
- Let S be a 3-dimensional Riemannian manifold, I be an open interval and $g > 0$ be a smooth function on S. Let t and σ be the projections of $I \times S$ onto I resp. S. The *standard static spacetime* $I_g \times S$ is the manifold $I \times S$ with line element $-g(\sigma)^2dt^2 + ds^2$ with $dt,\ ds$ being the lifts of the line elements of $I,\ S$.

So heuristically a spacetime is static if time is symmetric with respect to any arbitrary time-origin. This is a stronger assumption than stationarity (see [D'I], 14.3).

Remark 4.1.4.

1. Since gU is a Killing vector field, any local flow $\{\psi_t\}$ of gU consists of isometries by 4.1.2 and each ψ_t preserves U-observers but generally distorts the proper time parametrization.
2. At least locally the spatial universe always looks the same for an U-observer. Standard static spacetime has a given restspace S.
3. In contrast to Robertson-Walker spacetimes space remains the same but time is warped.

We will now show that $I_g \times S$ is static relative to $\frac{\partial_t}{g}$:

Lemma 4.1.5. *For* $M = I_g \times S$

1. *∂_t is a Killing vector field with global flow isometries given by $\psi_t(s,p) = (s+t,p)$*

2. *The observer field $U = \partial_t/g$ is synchronizable, i.e. there exist smooth functions $h > 0$ and t on M such that $U = -h \mathrm{grad} t$, hence U is irrotational.*

3. *The restspaces $t \times S$ of U are isometric under the flow isometries ψ_t and all are isometric under σ to S.*

A proof can be found in [O'N.], 12.37. We identify every static spacetime as locally standard.

Proposition 4.1.6. *A spacetime M is static relative to an observer field U iff for all $p \in M$ there is a U-preserving isometry of a standard static spacetime onto a neighborhood of p.*

For a proof see [O'N.], 12.38.

4.2 Deriving the metric of Schwarzschild spacetime

In the following section we will denote derivation with respect to t by a dot, derivation with respect to r by a prime. The calculations are mainly based on [D'I], 14.5. A different approach following [O'N.], 13.1. will be given below.

4.2.1 Staticity

The spacetime is to be static relative to observers. We take $\mathbb{R}^3$ as the restspace with line element q which is to be determined and $I = \mathbb{R}^1$. Since any static spacetime is by 4.1.6 standard (at least locally) the line element is of the form

$$ds^2 = A(x)dt^2 + q \qquad x \in \mathbb{R}^3$$

where q is the line element lifted from $\mathbb{R}^3$.

4.2.2 Spherical symmetry

Intuitively spherical symmetry means that there exists a point 0, the origin, such that the system is invariant under spatial rotation around 0. More precisely, for each $\Phi \in O(3)$ the map $(t,x) \mapsto (t, \Phi(x))$ is an isometry.
Hence we can use spherical coordinates on $\mathbb{R}^+ \times S^2 \cong \mathbb{R}^3 \setminus \{0\}$. It is shown in [D'I], 14.4, that the line element of a spherical symmetric metric can be written as

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with $\lambda = \lambda(t,r)$, $\nu = \nu(t,r)$.

4.2.3 Vacuum

Since the only source of gravitation is the star itself, which is not modeled, the spacetime must be a vacuum solution, i.e. $T_{ab} = 0$. Einstein's equations for vacuum are

$$G = 0$$

and so by [O'N.], 12.2., it follows that Ric $= G - \frac{1}{2}C(G)\ g = 0$ hence the spacetime is Ricci-flat. We use the field equations to determine λ and ν:
The covariant metric has the form

$$g_{ab} = \mathrm{diag}(-e^{\nu}, e^{\lambda}, r^2, r^2 \sin^2\theta)$$

and therefore

$$g^{ab} = \mathrm{diag}(-e^{-\nu}, e^{-\lambda}, r^{-2}, r^{-2}\sin^{-2}\theta)$$

(Note that in [D'I]14.4,5 a different sign-convention is used.)
If we calculate ${G_a}^b$ the only non-vanishing terms are

$$\begin{aligned}
{G_0}^0 &= -e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \\
{G_0}^1 &= e^{-\lambda} r^{-1}\dot{\lambda} = -e^{-\lambda-\nu}{G_1}^0 \\
{G_1}^1 &= e^{-\lambda}\left(\frac{\nu'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \\
{G_2}^2 &= {G_3}^3 = -\frac{1}{2}e^{-\lambda}\left(\frac{\nu'\lambda'}{2} - \frac{\lambda'}{r} + \frac{\nu'}{r} + \frac{\nu'^2}{2} + \nu''\right) - \frac{1}{2}e^{-\nu}\left(\ddot{\lambda} - \frac{\dot{\lambda}^2}{2} + \frac{\dot{\lambda}\dot{\nu}}{2}\right)
\end{aligned}$$

Using the contracted Bianchi-identity (for a proof see [Sch], 6.7)

$$\nabla_b {G_a}^b \equiv 0$$

shows that the equation for ${G_2}^2$ and ${G_3}^3$ automatically vanishes if the others vanish. Hence it remains to solve three independent equations, namely

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \qquad (*)$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \qquad (**)$$

$$\dot{\lambda} = 0$$

Addition of the first two equations leads to $\lambda' + \nu' = 0$ and by integration $\lambda + \nu = h(t)$. Since the third equation shows that $\lambda = \lambda(r)$, equation $(*)$ is after multiplication with r^2 the ordinary differential equation

$$e^{-\lambda} - re^{-\lambda}\lambda' = 1$$

or equivalently

$$(re^{-\lambda})' = 1$$

hence by integration

$$re^{-\lambda} = r - 2m$$

where $-2m$ is the integration constant. Therefore we obtain

$$e^{\lambda} = \left(1 - \frac{2m}{r}\right)^{-1}$$

The last step consists of getting rid of $h(r)$ which is done by transforming t to a new time coordinate t' given by

$$t' = \int_c^t e^{1/2h(u)} du$$

(c is an arbitrary constant). The only component of the metric that is changed is

$$g'_{00} = -\left(1 - \frac{2m}{r}\right)$$

since all other components are independent of t. Hence we have found Schwarzschild's line element

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

4.2.4 Minkowski at infinity

The influence of the source of gravitation becomes smaller the further we are away from it. We can see that the above form of the metric approaches Minkowski metric of empty spacetime, i.e. $ds^2 = -dt^2 + dr^2 + r^2 d\sigma^2$ at infinity. But also the converse is true, if we assume the metric to be Ricci-flat and Minkowski at infinity then the metric has to be in the above form. The following lemma is (in contrast to the previous calculations which can be found in [D'I]) based on [O'N.], 13.1.

Lemma 4.2.1. *$P = (\mathbb{R}^1 \times \mathbb{R}^+) \times_r S$ with metric $E(r)dt^2 + G(r)dr^2 + r^2 d\sigma^2$ is Ricci flat and Minkowski at infinity iff $E = -h$ and $G = h^{-1}$ where $h(r) = 1 - \frac{2m}{r}$ with m an arbitrary constant.*

Before we start proving the lemma we have to deal with semi Riemannian surfaces of dimension 2 with line element $ds^2 = E(r)dt^2 + G(r)dr^2$.

Lemma 4.2.2. *For a semi-Riemannian surface M,* dim M = 2*, with constant sectional curvature K*
(1) $R_{XY}Z = K(\langle Z, X\rangle Y - \langle Z, Y\rangle X)$
(2) $Ric = Kg$ and $S = 2K$.

Remark 4.2.3. The first result is also given in [O'N.], 3.43., which states that for a semi Riemannian manifold with constant curvature C we obtain $R_{XY}Z = C(\langle Z, X\rangle Y - \langle Z, Y\rangle X)$.

Proof.
(1) It suffices to show the statement on the basis vector fields ∂_u, ∂_v. Additionally we can without loss of generality assume $\langle \partial_u, \partial_v\rangle = 0$ hence $\{\partial_u, \partial_v\}$ is an orthonormal base for T_pM. We consider the different possible cases

- $X = Y = Z = \partial_u$ or ∂_v: The right hand side vanishes as well as the left hand side.
- $X = Y = \partial_u$, $Z = \partial_v$ resp. $X = Y = \partial_v$, $Z = \partial_u$: Both sides of the equation again vanish.
- $X = \partial_u$, $Y = \partial_v$, $Z = \partial_u$: Equality follows because $\langle\text{left hand side}, \partial_u\rangle = 0 = \langle\text{right hand side}, \partial_u\rangle$ and

$$\begin{aligned}\langle\text{left hand side}, \partial_v\rangle &= K\langle\partial_u, \partial_u\rangle\langle\partial_v, \partial_v\rangle = K(\langle\partial_u, \partial_u\rangle\langle\partial_v, \partial_v\rangle - 0) \\ &= \langle\text{right hand side}, \partial_u\rangle\end{aligned}$$

- $X = \partial_v$, $Y = \partial_u$, $Z = \partial_v$: analogously to the above
- $X = \partial_v$, $Y = \partial_v$, $Z = \partial_u$:

$$\langle\text{left hand side}, \partial_u\rangle = \langle R_{\partial_v\partial_u}\partial_u, \partial_u\rangle = 0$$

$$\langle\text{right hand side}, \partial_u\rangle = 0$$

$$\langle\text{left hand side}, \partial_v\rangle = \langle R_{\partial_v\partial_u}\partial_u, \partial_v\rangle = -R_{\partial_u\partial_v}\partial_u, \partial_v\rangle = -K\langle\partial_u, \partial_u\rangle\langle\partial_v, \partial_v\rangle$$

$$\langle\text{right hand side}, \partial_v\rangle = K(0 - \langle\partial_u, \partial_u\rangle\langle\partial_v, \partial_v\rangle)$$

(2) Choosing a local frame $\{E_1, E_2\}$ for M we have

$$\begin{aligned}\mathrm{Ric}(C, Y) &= \sum_{i=1}^{2} \epsilon_i \langle R_{XE_i}Y, E_i\rangle \\ &\overset{(1)}{=} K\sum_{i=1}^{2} \epsilon_i \langle\langle Y, X\rangle E_i - \langle Y, E_i\rangle X, E_i\rangle \\ &= K\sum_{i=1}^{2} \epsilon_i \langle X, Y\rangle\langle E_i, E_i\rangle - \langle Y, E_i\rangle\langle X, E_i\rangle \\ &= K\left(\sum_{i=1}^{2} \epsilon_i^2 \langle X, Y\rangle - \langle X, Y\rangle\right) = Kg(X, Y)\end{aligned}$$

For the scalar curvature we calculate

$$S = \sum_{i \neq j} \underbrace{K(E_i, E_j)}_{=K} = 2K$$

□

Lemma 4.2.4. In a semi Riemannian surface with coordinate system $\{t,\ r\}$ *and line element* $ds^2 = E(r)dt^2 + G(r)dr^2$ *we have*
(1) $H^r(\partial_t, \partial_t) = \frac{E_r}{2G}$, $H^r(\partial_t, \partial_r) = 0$, $H^r(\partial_r, \partial_r) = -\frac{G_r}{2G}$
(2) $\mathrm{grad} r = \frac{\partial_r}{G}$
(3) $\Delta r = \frac{G}{2}\left(\frac{E_r}{E} - \frac{G_r}{G}\right)$

Proof.
(1) In general for $f \in \mathcal{C}^\infty(M)$, $H^f = \nabla(\nabla f)$ hence by [O'N.], 3.49.,

$$H^f(X,Y) = XYf - (\nabla_X Y)f = \langle \nabla_X(\mathrm{grad}(f)), Y\rangle$$

Here we have metric coefficients $g_{tt} = E(r)$, $g_{rr} = G(r)$, $g_{rt} = g_{tr} = 0$. Hence for $X = Y = \partial_t$

$$\nabla_{\partial_t}\partial_t = \sum_k \Gamma^k_{tt}\partial_k = \Gamma^t_{tt}\partial_t + \Gamma^r_{tt}\partial_r$$

where

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{1}{2}\sum_m g^{tm}\{\partial_t g_{tm} + \partial_t g_{tm} - \partial_m g_{tt}\}\\
&= \frac{1}{2}g^{tt}(2\partial_t g_{tt} - \partial_t g_{tt})\\
&= \frac{1}{2}g^{11}\partial_t g_{tt} = 0\\
\Gamma^r_{tt} &= -\frac{1}{2}g^{rr}\partial_r g_{tt}\\
&= -\frac{1}{2}\frac{1}{G(r)}E_r
\end{aligned}$$

hence $H^r(\partial_t, \partial_t) = \frac{E_r}{2G(r)}\partial_r r = \frac{E_r}{2G(r)}$.
$H^r(\partial_t, \partial_r) = 0$ since $\nabla_{\partial_t}\partial_r = \Gamma^t_{tr}\partial_t + \Gamma^r_{tr}\partial_r$ and

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{1}{2}(\partial_r g_{tt})g^{tt} = \frac{E_r}{2E}\\
\Gamma^r_{tr} &= \frac{1}{2}(\partial_t g_{rr})g^{rr} = 0
\end{aligned}$$

hence $H^r(\partial_t, \partial_r) = \partial_t\partial_r r - \Gamma^t_{tr}\partial_t r = 0$.

The last statement follows by an analogous calculation.
(2) $\mathrm{grad} r = \sum_{i,j} g^{ij}\partial_i r\partial_j = \sum_i g^{ii}\partial_i r\partial_i = g^{rr}\partial_r = \frac{1}{G}\partial_r$.
(3) $\Delta r = \sum_{i,j} g^{ij}H^r(\partial_i, \partial_j) = g^{tt}H^r(\partial_t, \partial_t) + g^{rr}H^r(\partial_r, \partial_r) = \frac{1}{2G}\left(\frac{E_r}{E} - \frac{G_r}{G}\right)$. □

Now we will prove 4.2.1

Proof. (⇐) is clear.
(⇒) From warped product generalities, cf. 2.2.23 we obtain
${}^M\mathrm{Ric}(\tilde{X}, \tilde{Y}) = {}^P\,\widetilde{\mathrm{Ric}}(X,Y) - \frac{2}{r}H^r(\tilde{X}, \tilde{Y})$ for $\tilde{X}$,$\tilde{Y} \in \mathcal{L}(P)$ and
${}^M\mathrm{Ric}(\tilde{V}, \tilde{W}) = {}^{S^2}\,\widetilde{\mathrm{Ric}}(V,W) - \langle \tilde{V}, \tilde{W}\rangle_M r^*$ for $\tilde{V}, \tilde{W} \in \mathcal{L}(S^2)$, where

$$r^* = \frac{\Delta r}{r} + (3-1)\frac{\langle \mathrm{grad} r, \mathrm{grad} r\rangle_M}{r^2}$$

Using 4.2.4 we obtain

$$r^* = \frac{\frac{1}{2G}\left(\frac{E'}{E} - G'G\right)}{r} + \frac{\langle \frac{\partial_r}{G}, \frac{\partial_r}{G}\rangle_M}{r^2} = \frac{1}{2rG}\left(\frac{E'}{E} - \frac{G'}{G}\right) + \frac{1}{r^2 G}$$

Since we assume Ricci-flatness these equations become

1. ${}^{P}\mathrm{Ric}(X,Y) = 2H^r(\tilde{X},\tilde{Y})/r$ for $\tilde{X}$,$\tilde{Y} \in \mathcal{L}(P)$
2. ${}^{S^2}\mathrm{Ric}(V,W) = \langle \tilde{V},\tilde{W}\rangle_M r^*$ for $\tilde{V},\tilde{W} \in \mathcal{L}(S^2)$

Since leaves are totally geodesic and project isometrically it suffices to consider (1) on leaves only by Gauss' equation. Since dim $P = 2$ it follows from 4.2.2 that ${}^{P}\mathrm{Ric}(X,Y) = {}^{P}K\langle X,Y\rangle_P$. Choosing X,Y from ∂_t,∂_r by 4.2.4, equation (1) is equivalent to

$${}^{P}KE = \frac{E'}{rG} \quad {}^{P}KG = -\frac{G'}{rG}$$

Indeed, on the one hand,

$${}^{P}\mathrm{Ric}(\partial_t,\partial_t) = {}^{P}K\langle \partial_t,\partial_t\rangle_P = {}^{P}KE$$

On the other hand, by (1)

$${}^{P}\mathrm{Ric}(\partial_t,\partial_t) = 2H^r(\partial_t,\partial_t)/r = \frac{2E'/2G}{r} = \frac{E'}{rG}$$

Analogously we get the second statement if we insert ∂_r:

$${}^{P}\mathrm{Ric}(\partial_r,\partial_r) = {}^{P}K\langle \partial_r,\partial_r\rangle_P = {}^{P}KG$$

resp.

$${}^{P}\mathrm{Ric}(\partial_r,\partial_r) = \frac{2H^r(\partial_r,\partial_r)}{r} = \frac{-2G'/2G}{r} = -\frac{G'}{rG}$$

Therefore

$$-\frac{G'}{G} = {}^{P}KrG = \frac{E'}{E} \qquad (*)$$

so $EG' + E'G = (EG)' = 0$ hence EG is constant. Using the limit conditions $E \to 1$, $G \to -1$ we obtain $EG = -1$.

Next we take equation (2) into account: We observe that ${}^{S^2}\mathrm{Ric} = g$. Indeed, by using [O'N.], remark after 3.52., $\mathrm{Ric}(u,u)$ for u a unit vector at some point p, is given by $\mathrm{Ric}(u,u) = \langle u,u\rangle \sum_m K(u,e_m)$. K is the sectional curvature of the plane spanned by u and e_m, $\{e_1,\ldots,e_n\}$ a frame at p with $e_1 = u$. Thus $\mathrm{Ric}(u,u)$ is just the sum of the sectional curvatures of $n-1$ non-degenerate orthogonal planes through u. For any aritrary vectors v,w we can reconstruct $\mathrm{Ric}(v,w)$ by using polarization. In case of the unit-sphere we know that the sectional curvature is 1. So the result follows.

Hence for $(p,q) \in P \times S^2$, $V,W \in \mathfrak{X}(S^2)$ and denoting by $\tilde{V},\tilde{W}$ the lifts to M, i.e. $T_{(p,q)}\mathrm{pr}_1(\tilde{V}|_{(p,q)}) = 0 = T_{(p,q)}\mathrm{pr}_1(\tilde{W}|_{(p,q)})$ we calculate

$$\begin{aligned}
{}^{S^2}\mathrm{Ric}(V,W) &= \langle V_q,W_q\rangle_{S^2} \\
&= \langle T_{(p,q)}\mathrm{pr}_2(\tilde{V}|_{(p,q)}), T_{(p,q)}\mathrm{pr}_2(\tilde{W}|_{(p,q)})\rangle_M \\
&= \frac{1}{r^2}\langle \tilde{V}|_{(p,q)},\tilde{W}|_{(p,q)}\rangle_M - \langle T_{(p,q)}\mathrm{pr}_1(\tilde{V}|_{(p,q)}), T_{(p,q)}\mathrm{pr}_1(\tilde{W}|_{(p,q)})\rangle_P \\
&= \frac{1}{r^2}\langle \tilde{V}|_{(p,q)},\tilde{W}|_{(p,q)}\rangle_M
\end{aligned}$$

So ${}^{S^2}\mathrm{Ric}(V,W) = \frac{1}{r^2}\langle \tilde{V},\tilde{W}\rangle_M = \langle V,W\rangle r^*$ iff $r^* = \frac{1}{r^2}$ with r^* as calculated above. By $(*)$ we can replace $\frac{E'}{E}$ by $-\frac{G'}{G}$ and this leads to

$$-\frac{G'}{rG^2} + \frac{1}{r^2G} = \frac{1}{r^2}$$

which is equivalent to

$$-\frac{rG' + G}{G^2} = 1$$

hence $\left(\frac{r}{G}\right)' = 1$, and thus $\frac{r}{G} = r + \text{const}$ where we choose the constant to be $-2m$. This leads us to the anticipated form for G and E, namely

$$G = \frac{r}{r - 2m} = \left(1 - \frac{2m}{r}\right)^{-1}$$

$$E = -(G^{-1}) = -\left(1 - \frac{2m}{r}\right)$$

$\square$

Remark 4.2.5.

1. The line element exhibits the spacetime as the warped product $P \times_r S^2$ where $P = \mathbb{R} \times \mathbb{R}^+$ is the half plane $r > 0$ in the tr-plane furnished with the line element $E(r)dt^2 + G(r)dr^2$

2. In each restspace $t = \text{const}$ (i.e. $dt = 0$) the surface $r = \text{const}$ ($dr = 0$) has the line element $r^2 d\sigma^2$ and is thus the standard 2-sphere $S^2(r)$ with Gaussian curvature $1/r^2$ and area $4\pi r^2$

3. We identify m with the mass hence require $m > 0$.

4. $h(r) = 1 - \frac{2m}{r}$ rises from $-\infty$ at $r = 0$ toward the limit 1 as $r \to \infty$ and passes through 0 at $r = 2m$. Thus the line element $-hdt^2 + h^{-1}dr^2 + r^2 d\sigma^2$ has a singularity at $r = 2m$ which will turn out to be a coordinate singularity which can be omitted by a different choice of coordinates. But we have actually found two spacetimes where initially only the exterior half where $r > 2m$ seemed to be physically relevant. The neglected half provides the simplest model of a black hole.

Definition 4.2.6. For $m > 0$ let P_I and P_{II} be the regions $r > 2m$ resp. $0 < r < 2m$ in the tr-plane $\mathbb{R} \times \mathbb{R}^+$ furnished with the line element $-hdt^2 + h^{-1}dr^2$ with $h(r) = \left(1 - \frac{2m}{r}\right)$.
For S^2 being the unit sphere the warped product $N = P_I \times_r S^2$ is called *Schwarzschild exterior spacetime* and $B = P_{II} \times_r S^2$ the *Schwarzschild black hole*, both of mass m.
The projection $t : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is called *Schwarzschild time*, the projection $r : \mathbb{R} \times \mathbb{R}^+ \times S^2 \to \mathbb{R}^+$ is the so-called *Schwarzschild radius function*

Remark 4.2.7. As already seen in 4.2.1, if the Schwarzschild radius r is sufficiently large then the metric on N is nearly a Minkowski metric. We interpret t as time, r as the radial distance. As r passes below $2m$, h becomes negative hence ∂_t and ∂_r change their causal characters, ∂_t becomes spacelike, and ∂_r timelike.

4.3 Geometric properties

As remarked above, Schwarzschild spacetime can be interpreted as a warped product with warping function r hence we can apply the results from chapter 1.

As a direct consequence of 2.2.14 we obtain the following result.

Lemma 4.3.1.

1. *For each $(t,r) \in P_I$ the fiber ${pr_1}^{-1}(t,r)$ is the sphere $S^2(r)$ in the restspace of Schwarzschild time t. The sphere is totally umbillic in N and pr_2 maps it homothetically onto S^2.*

2. *The leaf ${pr_2}^{-1}(q) = P_I \times \{q\}$ is totally geodesic in N and isometric under the projection pr_1 to the half plane P_I.*

Since the metric tensors of N and B are formally the same, geometric properties can be treated simultaneously.
We start by computing the Levi-Civita connection ∇^P on $P = P_I \cup P_{II}$ with line element $ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2$.

Lemma 4.3.2. *On $P = P_I \cup P_{II}$ with line element $ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2$ we have*
(1) $\nabla^P_{\partial_t}\partial_t = \frac{mh}{r^2}\partial_r$, $\nabla^P_{\partial_t}\partial_r = \nabla^P_{\partial_r}\partial_t = \frac{m}{r^2 h}\partial_t$, $\nabla^P_{\partial_r}\partial_r = -\frac{m}{r^2}\partial_r$.
(2) $\mathrm{grad} t = -\frac{\partial_t}{h}$, $\mathrm{grad} r = h\partial_r$.
(3) $H^r = \frac{m}{r^2} g_P$ *where g_P is the metric on P.*
(4) $K = \frac{2m}{r^3}$

For the proof we need the following proposition

Proposition 4.3.3. *Let $\{u,v\}$ be an orthogonal coordinate system in a semi Riemannian surface (i.e. $F = \langle \partial_u, \partial_v \rangle = 0$). Then, for $E = \langle \partial_u, \partial_u \rangle$ and $G = \langle \partial_v, \partial_v \rangle$ we obtain*
(1) $\nabla_{\partial_u}\partial_u = \frac{E_u}{2E}\partial_u - \frac{E_v}{2G}\partial_v$
(2) $\nabla_{\partial_v}\partial_v = -\frac{G_u}{2E}\partial_u + \frac{G_v}{2G}\partial_v$
(3) $\nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \frac{E_u}{2E}\partial_u + \frac{G_v}{2G}\partial_v$

If we set $e = |E|^{1/2}$, $g = |G|^{1/2}$, $\epsilon_1 = \pm 1$ the sign of E and $\epsilon_2 = \pm 1$ the sign of G then

$$K = -\frac{1}{eg}\left(\epsilon_1 \left(\frac{g_u}{e}\right)_u + \epsilon_2 \left(\frac{e_v}{g}\right)_v\right)$$

Proof. [O'N.], 3.44. □

Now we are well-prepared to prove 4.3.5.

Proof. of 4.3.5: For the line element $ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2$ we calculate

$$E = g_{11} = -h(r) \quad F = g_{12} = g_{21} = 0 \quad G = g_{22} = h(r)^{-1}$$

Using 4.3.3 with $u = t,\ v = r$ we obtain

$$E_t = 0 \quad E_r = -h'(r) = -\frac{2m}{r^2}$$

$$G_t = 0 \quad G_r = -\frac{2m}{r^2}$$

where a $'$ denotes the derivative with respect to r. Hence

(1)

$$\begin{aligned}
\nabla^P_{\partial_t}\partial_t &= \frac{E_t}{2E}\partial_t - \frac{E_r}{2G}\partial_r \\
&= \frac{mh}{r^2}\partial_r \\
\nabla^P_{\partial_t}\partial_r &= \nabla^P_{\partial_r}\partial_t = \frac{E_r}{2E}\partial_t + \frac{G_t}{2G}\partial_r \\
&= \frac{m}{r^2 h}\partial_t + 0 \\
\nabla^P_{\partial_r}\partial_r &= -\frac{G_t}{2E}\partial_t + \frac{G_r}{2G}\partial_r \\
&= -\frac{m}{r^2}\partial_r
\end{aligned}$$

(2)

$$\begin{aligned}
\mathrm{grad}t &= \sum_{i,j} g^{ij}\frac{\partial t}{\partial x^i}\partial_j = g^{11}\frac{\partial t}{\partial x^1}\partial_1 + g^{22}\frac{\partial t}{\partial x^2}\partial_2 \\
&= -h(r)^{-1}\frac{\partial t}{\partial t}\partial_t + h(r)\frac{\partial t}{\partial r}\partial_r \\
&= -h(r)^{-1}\partial_t
\end{aligned}$$

An analogous calculation for $\mathrm{grad}r$ leads to $\mathrm{grad}r = h\partial_r$.

(3) $H^r(X,Y) = \langle \nabla_X(\mathrm{grad}f), Y\rangle = \langle \nabla_X(h\partial_r), Y\rangle$. Decomposing X and Y as $X = a\partial_t + b\partial_r$, $Y = a'\partial_t + b'\partial_r$ and inserting it in the expression above leads to the result $H^r(X,Y) = \frac{m}{r^2}\langle X, Y\rangle$.
(4) Again using 4.3.3 with $e = h(r)^{1/2}$, $g = h(r)^{-1/2}$, $\epsilon_1 = -1$, $\epsilon_2 = 1$

$$K = -\frac{1}{2}(h'(r))' = \frac{2m}{r^3}$$

□

Notation 4.3.4. By $\tilde{\partial_t}$, $\tilde{\partial_r}$ on $M = N \cup B$ we denote the lifts of the corresponding vector fields on $P_I \cup P_{II}$

We will now calculate the connection ∇^M on $M = N \cup B$ by using warped product generalities, cf. 2.2.11.

Proposition 4.3.5. *On $M = N \cup B$ we have for $\tilde{V}, \tilde{W} \in \mathcal{L}(S^2)$, $\tilde{\partial_t}$, $\tilde{\partial_r} \in \mathcal{L}(P)$*
(1) $\nabla^M_{\tilde{\partial_t}}\tilde{\partial_t} = \widetilde{\frac{mh}{r^2}\partial_r}$
(2) $\nabla^M_{\tilde{\partial_t}}\tilde{V} = \nabla^M_{\tilde{V}}\tilde{\partial_t} = 0$, $\nabla^M_{\tilde{\partial_r}}\tilde{V} = \nabla^M_{\tilde{V}}\tilde{\partial_r} = \frac{1}{r}\tilde{V}$
(3) nor $(\nabla^M_{\tilde{V}}\tilde{W}) = II(\tilde{V},\tilde{W}) = -\frac{h}{r}\langle\tilde{V},\tilde{W}\rangle_M\partial_r$
(4) tan $(\nabla^M_{\tilde{V}}\tilde{W}) = \widetilde{\nabla^{S^2}_V W}$

Proof.
(1) In 2.2.11(1) we have already shown that $\nabla^M_{\tilde{\partial}_t}\tilde{\partial}_t \in \mathcal{L}(B)$ is the lift of $\nabla^P_{\partial_t}\partial_t$ to M which is given by 4.3.2(1).
(2) Again applying the corresponding result, 2.2.11(2), we obtain

$$\nabla^M_{\tilde{\partial}_t}\tilde{V} = \nabla^M_{\tilde{V}}\tilde{\partial}_t = \frac{\partial_t r}{r}\tilde{V} = 0$$

since $\partial_t r = 0$, resp.

$$\nabla^M_{\tilde{\partial}_r}\tilde{V} = \nabla^M_{\tilde{V}}\tilde{\partial}_r = \frac{\partial_r r}{r}\tilde{V} = \frac{1}{r}\tilde{V}$$

(3) By 2.2.11(3), nor $(\nabla^M_{\tilde{V}}\tilde{W}) = II(\tilde{V},\tilde{W}) = -\frac{\langle \tilde{V},\tilde{W}\rangle_M}{r}\mathrm{grad}r = -\frac{\langle \tilde{V},\tilde{W}\rangle_M}{r}h\partial_r$ where we used 4.3.3(3), i.e. $\mathrm{grad}r = h\partial_r$.
(4) is just a direct application of 2.2.11(4).

□

Our next task is to calculate the Riemannian curvature

Proposition 4.3.6. *Let $\tilde{V},\tilde{W}$ be vertical vector fields on $M = N \cup B$, i.e. tangent to all spheres $S^2(r)$. Then we obtain*
(1) ${}^M R_{\tilde{\partial}_t\tilde{\partial}_r}\tilde{\partial}_t = -\frac{2mh}{r^3}\tilde{\partial}_r$, ${}^M R_{\tilde{\partial}_r\tilde{\partial}_t}\tilde{\partial}_t = \frac{2m}{r^3 h}\tilde{\partial}_t$
(2) ${}^M R_{\tilde{\partial}_t\tilde{V}}\tilde{\partial}_t = \frac{mh}{r^3}\tilde{V}$, ${}^M R_{\tilde{\partial}_r\tilde{V}}\tilde{\partial}_r = -\frac{m}{r^3 h}\tilde{V}$, ${}^M R_{\tilde{\partial}_t\tilde{V}}\tilde{\partial}_r = {}^M R_{\tilde{\partial}_r\tilde{V}}\tilde{\partial}_t = 0$
(3) ${}^M R_{\tilde{\partial}_r\tilde{\partial}_t}\tilde{V} = {}^M R_{\tilde{V}\tilde{W}}\tilde{\partial}_t = {}^M R_{\tilde{V}\tilde{W}}\tilde{\partial}_r = 0$
(4) ${}^M R_{\tilde{V}\tilde{W}}\tilde{X} = {}^M R_{\tilde{X}\tilde{W}}\tilde{V} = \frac{m}{r^3}\langle\tilde{V},\tilde{W}\rangle_M\tilde{X}$ *where $X = \partial_t$ or ∂_r.*
(5) For vertical U we obtain ${}^M R_{\tilde{V}\tilde{W}}\tilde{U} = \frac{2m}{r^3}(\langle\tilde{U},\tilde{V}\rangle_M\tilde{W} - \langle\tilde{U},\tilde{W}\rangle_M)$

Proof.
(1) Applying the result for warped products, 2.2.21 we obtain

$${}^M R_{\tilde{\partial}_t\tilde{\partial}_r}\tilde{\partial}_t = {}^P\widetilde{R_{\partial_t\partial_r}\partial_t} = -\frac{2mh}{r^3}\tilde{\partial}_r$$

The second equation follows analogously.
(2) By 2.2.21(2) ${}^M R_{\tilde{V}\tilde{X}}\tilde{Y} = \frac{H^r(\tilde{X},\tilde{Y})}{r}\tilde{V}$, and using 4.3.5 (3) we compute

$${}^M R_{\tilde{\partial}_t V}\tilde{\partial}_t = -\frac{m}{r^3}\langle\tilde{\partial}_t,\tilde{\partial}_t\rangle_M\tilde{V} = \frac{mh}{r^3}\tilde{V}$$

$${}^M R_{\tilde{\partial}_r V}\tilde{\partial}_r = -\frac{m}{r^3}\langle\tilde{\partial}_r,\tilde{\partial}_r\rangle_M\tilde{V} = -\frac{m}{r^3 h}\tilde{V}$$

$${}^M R_{\tilde{\partial}_r V}\tilde{\partial}_t = -\frac{m}{r^3}\langle\tilde{\partial}_r,\tilde{\partial}_t\rangle_M\tilde{V} = 0$$

where we used the fact that $\langle\tilde{\partial}_t,\tilde{\partial}_t\rangle_M = -h$, $\langle\tilde{\partial}_r,\tilde{\partial}_r\rangle_M = h^{-1}$ and $\langle\tilde{\partial}_r,\tilde{\partial}_t\rangle_M = 0$.
(3) Again using 2.2.21(3) leads to the result.
(4) Since by 2.2.21(4) ${}^M R_{\tilde{X}\tilde{V}}\tilde{W} = \left(\frac{\langle\tilde{V},\tilde{W}\rangle_M}{r}\right)\nabla^M_{\tilde{X}}(\mathrm{grad}(r))$ and by 4.3.5 (2) we obtain

$$\nabla^M_{\tilde{\partial}_t}(h\partial_r) = \frac{\partial h}{\tilde{\partial} t}\tilde{\partial}_r + h\nabla^M_{\tilde{\partial}_t}\tilde{\partial}_r = 0 + \frac{m}{r^2 h}\tilde{\partial}_t h$$

and

$$\nabla^M_{\tilde{\partial}_r}(h\tilde{\partial}_r) = \frac{\partial h}{\partial r}\tilde{\partial}_r + h\nabla^M_{\tilde{\partial}_r}\tilde{\partial}_r = \frac{2m}{r^2 h}\tilde{\partial}_r - \frac{m}{r^2}\tilde{\partial}_r = \frac{m}{r^2}\tilde{\partial}_r$$

Therefore

$$ {}^{M}R_{\tilde{X}\tilde{V}} = \frac{m}{r^3}\langle \tilde{V}, \tilde{W}\rangle_M \tilde{X} $$

for X being either ∂_t or ∂_r.

(5) By 2.2.21 ${}^{M}R_{\tilde{V}\tilde{W}}\tilde{U} = \widetilde{{}^{S^2}R_{VW}U} - \left(\frac{\langle \mathrm{grad}(r), \mathrm{grad}(r)\rangle}{r^2}\right)(\langle \tilde{V}, \tilde{U}\rangle_M \tilde{W} - \langle \tilde{W}, \tilde{U}\rangle_M \tilde{V})$.

${}^{S^2}R$ gives the curvature tensor of each fiber ${\mathrm{pr}_1}^{-1} = S^2(r)$ Since $S^2(r)$ has constant curvature $k = \frac{1}{r^2}$ we can compute by using [O'N.], 3.43.,

$$ {}^{S^2}R(V, W)U = \frac{1}{r^2}(\langle U, V\rangle W - \langle U, W\rangle V) $$

Using again 4.3.5(2), $\langle \mathrm{grad}(r), \mathrm{grad}(r)\rangle = h^2\langle \partial_r, \partial_r\rangle = h = \frac{r-2m}{r}$ therefore we get

$$ {}^{M}R_{\tilde{V}\tilde{W}}\tilde{U} = \frac{1}{r^2}(\langle \tilde{U}, \tilde{V}\rangle_M \tilde{W} - \langle \tilde{U}, \tilde{W}\rangle_M \tilde{V}) - \frac{r-2m}{r^3}(\langle \tilde{U}, \tilde{V}\rangle_M \tilde{W} - \langle \tilde{U}, \tilde{W}\rangle_M \tilde{V}) $$

which is the result.

□

4.4 Geodesics

In order to study geodesics in Schwarzschild spacetime we first introduce Schwarzschild spherical coordinates. We will use the same notation for vector fields and their lifts.

Definition 4.4.1. Let θ, ϕ be spherical coordinates on the unit sphere S^2 and t, r be the Schwarzschild time-resp. radius function on P. The product coordinate system (t, r, θ, ϕ) in $M = N \cup B$ is called *Schwarzschild spherical coordinate system.*

A curve in M is *initially equatorial* relative to spherical coordinates on S^2 provided it has $\theta(0) = \frac{\pi}{2}$, $\frac{d\theta}{ds}(0) = 0$.

Remark 4.4.2.

- The domain of these coordinates omits $\sigma^{-1}(C)$ where C is a semicircle in S^2. ∂_t, ∂_r and ∂_ϕ are well defined and smooth everywhere, ∂_θ is singular for $\theta = 0$ and $\theta = \pi$.
  ∂_t and ∂_r are tangent to radial planes, ∂_ϕ and ∂_θ are tangent to spheres.
- By rotating the coordinates suitably any curve may become initially equatorial, because of spherical symmetry the line element remains unchanged.

In what follows let

$$ x^0 = t \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi $$

Hence we can parametrize particles as $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$.

4.4.1 Geodesic equations

We can of course determine the geodesic equations by using the results for warped products, cf. 2.2.16. But it turns out that a large amount of calculations is needed hence we will now choose a different way to obtain the result.

Lemma 4.4.3. *For an orthogonal coordinate system the geodesic differential equations become*

$$\frac{d}{ds}\left(g_{ii}\left(\frac{dx^i}{ds}\right)\right) = \frac{1}{2}\sum_{j=0}^{3}\frac{\partial g_{jj}}{\partial x^i}\left(\frac{dx^j}{ds}\right)^2 \qquad (*)$$

Proof. By [O'N.], 3.21., the geodesic equations are

$$\frac{d^2(x^k\circ\gamma)}{dt^2} + \sum_{i,j}\Gamma^k_{ij}(\gamma)\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^j\circ\gamma)}{dt} = 0 \qquad (**)$$

where the Γ^k_{ij} are the Christoffel symbols. They can be computed by [O'N.], 3.13., as

$$\Gamma^k_{ij} = \frac{1}{2}g^{km}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)$$

Since we deal with an orthogonal coordinate system, $g_{ij} = 0$ for $i \neq j$. Keeping this fact in mind we see that the Christoffel symbols vanish for $i \neq j \neq k$. For $i = k \neq j$ we get $\Gamma^k_{kj} = \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^j}$ and analogously for $i \neq j = k$, $\Gamma^k_{ik} = \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^i}$, for $i = j \neq k$ we obtain $\Gamma^k_{ii} = -\frac{1}{2}g^{kk}\frac{\partial g_{ii}}{\partial x^k}$ and finally for $i = j = k$, $\Gamma^k_{kk} = \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^k}$. Now summing up we can see that $(**)$ is equivalent to

$$\begin{aligned}
-\frac{d^2(x^k\circ\gamma)}{dt^2} &= \sum_{i,j}\Gamma^k_{ij}(\gamma)\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^j\circ\gamma)}{dt} \\
&= \sum_{i\neq j=k}\frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^i}\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^k\circ\gamma)}{dt} + \sum_{k=i\neq j}\frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^j}\frac{d(x^j\circ\gamma)}{dt}\frac{d(x^k\circ\gamma)}{dt} \\
&\quad - \sum_{i=j\neq k}\frac{1}{2}g^{kk}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{d(x^i\circ\gamma)}{dt}\right)^2 + \frac{1}{2}g^{kk}\frac{\partial g_{kk}}{\partial x^k}\left(\frac{d(x^k\circ\gamma)}{dt}\right)^2 \\
&= \sum_{i\neq k}g^{kk}\frac{\partial g_{kk}}{\partial x^i}\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^k\circ\gamma)}{dt} - \sum_{i}\frac{1}{2}g^{kk}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{d(x^i\circ\gamma)}{dt}\right)^2 \\
&\quad + g^{kk}\frac{\partial g_{kk}}{\partial x^k}\left(\frac{d(x^k\circ\gamma)}{dt}\right)^2 \\
&= \sum_{i}g^{kk}\frac{\partial g_{kk}}{\partial x^i}\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^k\circ\gamma)}{dt} - \frac{1}{2}\sum_{i}g^{kk}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{d(x^i\circ\gamma)}{dt}\right)^2
\end{aligned}$$

Since we want to show the equivalence of $(*)$ and $(**)$, we start with calculating

$$\frac{d}{ds}\left((g_{kk}\circ\gamma)\frac{d(x^k\circ\gamma)}{ds}\right) = \frac{d}{ds}(g_{kk}\circ\gamma)\frac{d(x^k\circ\gamma)}{ds} + (g_{kk}\circ\gamma)\frac{d^2(x^k\circ\gamma)}{ds^2}$$

by using the chain rule. So $(*)$ is equivalent to

$$\begin{aligned}
-\frac{d^2(x^k\circ\gamma)}{dt^2} &= (g^{kk}\circ\gamma)\left(\frac{d}{ds}(g_{kk}\circ\gamma)\frac{d(x^k\circ\gamma)}{ds} - \frac{1}{2}\sum_{i}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{d(x^i\circ\gamma)}{dt}\right)^2\right) \\
&= (g^{kk}\circ\gamma)\left(\sum_{i}g^{kk}\frac{\partial g_{kk}}{\partial x^i}\frac{d(x^i\circ\gamma)}{dt}\frac{d(x^k\circ\gamma)}{dt} - \frac{1}{2}\sum_{i}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{d(x^i\circ\gamma)}{dt}\right)^2\right)
\end{aligned}$$

which is the same result as we obtained from $(**)$. $\square$

We now deal with such an orthogonal coordinate system, hence the geodesic differential equations reduce to the above form with

$$g_{00} = -h \qquad g_{11} = h^{-1} \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta$$

Proposition 4.4.4. *For a geodesic γ in $M = N \cup B$ we have*

(1) $h\frac{dt}{ds} = E$ where E is a constant.
(2) $r^2 \sin^2\theta \frac{d\phi}{ds} = L$, with L being a constant.
(3) $\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = r^2 \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2$.

Proof.
(1) This is just the geodesic equation 4.4.3 for $i = 0$.
(2) Set $i = 3$ in the geodesic equation 4.4.3.
(3) For $i = 2$ we obtain

$$\begin{aligned} \frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) &= \frac{1}{2}\frac{\partial g_{33}}{\partial\theta}\left(\frac{d\phi}{ds}\right)^2 \\ &= r^2\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 \end{aligned}$$

□

Corollary 4.4.5. *Let γ be a freely falling material particle in M with proper time τ (i.e. $|\alpha'(\tau)| = 1\ \forall\tau$) which is initially equatorial to Schwarzschild spherical coordinates. Then we get from 4.4.4*

(G1) $h\frac{dt}{d\tau} = E$
(G2) $r^2\frac{d\phi}{d\tau} = L$
(G3) $\theta = \frac{\pi}{2}$

Furthermore the energy equation holds

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 + \frac{L^2}{r^2}\right)h(r)$$

Remark 4.4.6. (G1)-(G3) also hold for lightlike particles. In this case the energy equation is of the form

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(\frac{L^2}{r^2}\right)h(r)$$

Proof. (G1)-(G3) follow from the corresponding formulas in 4.4.4 where now we use instead of the parameter s proper time τ. Indeed,

(G1) is 4.4.4 (1) unchanged
(G2) follows from 4.4.4 (2) for $\theta = \frac{\pi}{2}$.

As far as (G3) is concerned we observe that $\theta = \frac{\pi}{2}$ is the unique solution of 4.4.4 (3) satisfying the equatorial initial conditions

$$\theta(0) = \frac{\pi}{2} \qquad \frac{d\theta}{ds}(0) = 0$$

To obtain the energy condition for material particles we calculate

$$\begin{aligned}\gamma' &= \sum \frac{dx^i}{d\tau}\partial_i = \frac{dt}{d\tau}\partial_t + \frac{dr}{d\tau}\partial_r + \frac{d\phi}{d\tau}\partial_\phi + \frac{d\theta}{d\tau}\partial_\theta \\ &= \frac{E}{h}\partial_t + \frac{dr}{d\tau}\partial_r + \frac{L}{r^2}\partial_\phi\end{aligned}$$

Hence

$$\begin{aligned}-1 &= \langle \gamma', \gamma' \rangle \\ &= \frac{E^2}{h^2}\langle \partial_t, \partial_t \rangle + \left(\frac{dr}{d\tau}\right)^2 \langle \partial_r, \partial_r \rangle + \frac{L^2}{r^4}\langle \partial_\phi, \partial_\phi \rangle \\ &= \frac{E^2}{h^2}(-h) + \left(\frac{dr}{d\tau}\right)^2 h^{-1} + \frac{L^2}{r^4}r^2\end{aligned}$$

which is equivalent to

$$E^2 = h + \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2}h = \left(\frac{dr}{d\tau}\right)^2 + \left(1 + \frac{L^2}{r^2}\right)h(r)$$

□

The energy equation for lightlike particles follows by setting $\langle \gamma', \gamma' \rangle = 0$

4.4.2 Interpretation of the constants m, E and L on N

The Schwarzschild exterior N is time oriented by requiring that the timelike Killing vector field ∂_t is future pointing. Additionally N is static relative to the observer field $U = \frac{\partial_t}{h^{1/2}}$. The integral curves α of U are called *Schwarzschild observers*. For t being the average time we get by 4.1.5 (2) that

$$U = -h^{1/2}\text{grad } t$$

If τ is the proper time of α we can calculate

$$\begin{aligned}\frac{d(t \circ \alpha)}{d\tau} &= \langle \alpha', \text{grad } t \rangle \\ &= \langle U, \text{grad } t \rangle \\ &= \langle U, -h^{-1/2}U \rangle \\ &= -h^{-1/2}\langle U, U \rangle = h^{-1/2}\end{aligned}$$

This dilation is constant on α. Since, on N, $0 < h < 1$ we conclude that

$$\frac{d(t \circ \alpha)}{d\tau} > 1$$

and therefore Schwarzschild time is always faster than the proper time of the observers.

Lemma 4.4.7. *If α is a Schwarzschild observer then*

$$\alpha'' = \nabla^M_U U = \frac{m}{r^2}\partial_r \neq 0$$

hence α is not freely falling, i.e. it is not a geodesic.

Proof. Since $U = -h^{1/2}\text{grad}\ t$ and $\frac{\partial h}{\partial t} = 0$ it follows by using 4.3.5 that

$$\nabla^M_U U = h^{-1/2}(h^{-1/2}\nabla^M_{\partial_t}\partial_t) = h^{-1}\frac{mh}{r^2}\partial_r = \frac{m}{r^2}\partial_r$$

□

So each observer must accelerate to remain at rest therefore the interpretation of m as the mass of the star is justified.

Equation (G2), $r^2\frac{d\phi}{d\tau} = L$, is formally identical to Kepler's 2^{nd} law hence we call L the *angular momentum* per unit mass of a particle.

In special relativity there is a relation between the energy, the momentum and the speed of a material particle relative to a freely falling observer. Details can be found in [O'N.], 6.27., 6.28. For a particle γ with mass m and energy momentum $m\gamma'$ the Schwarzschild observers α measure its energy as (see [O'N.], 6.28. (2))

$$\begin{aligned}
-\langle m\gamma', U\rangle &= -\langle m\gamma', h^{-1/2}\partial_t\rangle \\
&= -mh^{-1/2}\langle\frac{dt}{d\tau}\partial_t + \frac{dr}{d\tau}\partial_r + \frac{d\phi}{d\tau}\partial_\phi, \partial_t\rangle \\
&= -mh^{-1/2}\frac{dt}{d\tau}\langle\partial_t, \partial_t\rangle \\
&= mh^{1/2}\frac{dt}{d\tau}
\end{aligned}$$

By (G1), $h\frac{dt}{d\tau} = E$. Since $h \to 1$ for $r \to \infty$ E is called the *energy* per unit mass at infinity of the particle. It is positive because γ' is future pointing ($\langle\gamma', U\rangle < 0$) and therefore $\frac{dt}{d\tau}$ is positive.

For lightlike particles we also interpret E and L as the energy at infinity resp. as the angular moment.

4.5 Orbits in N

4.5.1 Basic definitions

We consider free-falling (i.e. geodesic) material particles as well as lightlike particles.

Definition 4.5.1. The *standard Schwarzschild restspace* S is the region $r > 2m$ in $\mathbb{R}^+ \times \mathbb{S}^2$ with the line element

$$h^{-1}dr^2 + r^2 d\sigma^2$$

All restspaces of U are naturally isometric to S. We can decompose a particle γ in N as

$$\gamma(t) = (t, \vec{\gamma}(t))$$

where $\vec{\gamma}$ is the projection of γ onto a curve in S.
We suppose that $\vec{\gamma}$ lies in the orbital plane of γ i.e. in the plane given in equatorial coordinates by $\theta = \frac{\pi}{2}$. Therefore we can write

$$\vec{\gamma}(\tau) = (r(\tau), \phi(\tau))$$

Definition 4.5.2. The *orbit* of γ is the route followed by $\vec{\gamma}$ in the orbital plane. The particle is called *ingoing* if $\frac{dr}{d\tau} < 0$ and *outgoing* if $\frac{dr}{d\tau} > 0$.

Let now γ be a freely falling material particle with $L \neq 0$. Equation (G2) implies that $\frac{d\phi}{d\tau}$ is non vanishing hence the orbit can be expressed via the following *orbit equation*.

Proposition 4.5.3. *A freely falling material particle γ in N with $L \neq 0$ satisfies the orbit equation*

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{L^2} + 3mu^2$$

where $u = \frac{1}{r}$ and $r = r \circ \gamma$.

Proof. From (G2) we get $r^2 \frac{d\phi}{d\tau} = L$ so

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{L}{r^2}\frac{dr}{d\phi}.$$

Hence the energy equation has the form

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(\frac{L^2}{r^2}\right)h(r) = \frac{L^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right).$$

By setting $u := \frac{1}{r}$ we obtain

$$E^2 = L^2\left(\frac{du}{d\phi}\right)^2 + \left(1 + L^2u^2\right)(1 - 2mu)$$

Now a derivation with respect to ϕ leads to

$$0 = 2L^2\frac{d^2u}{d\phi^2} + 2L^2u - 2m - 6mL^2u^2$$

Dividing by $2L^2$ leads to the anticipated result. □

Remark 4.5.4.

- The relativistic orbit equation differs from the Newtonian analogue by the correction term $3mu^2$, see the next section below.
- We get an analogous equation for lightlike particles:

$$\frac{d^2u}{d\phi^2} + u = 3mu^2$$

- In contrast to a material particle, the orbit of a lightlike particle γ does not depend on E and L separately but on the *impact parameter* $b = \frac{|L|}{E}$. We therefore can write the energy equation as

$$\left(\frac{1}{L}\frac{dr}{d\tau}\right)^2 + \frac{h}{r^2} = \frac{1}{b^2}.$$

Hence by using (G2), i.e. $\frac{d\phi}{d\tau} = \frac{L}{r^2}$ we obtain for a lightlike particle with $b \neq 0$

$$\frac{1}{b^2} = \left(\frac{1}{L}\frac{dr}{d\tau}\right)^2 + \frac{h}{r^2} = (\frac{1}{L}\frac{dr}{d\phi}\frac{d\phi}{d\tau})^2 + \frac{h}{r^2} = \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h}{r^2}$$

where $\frac{h}{r^2}$ is interpreted as the effective potential.

4.5.2 Classical results

Before we study orbits in Schwarzschild spacetime we recall some well known results from classical Newtonian physics. This section is mainly based on [Fl], [O'N.], appendix C and [HF],4.6,7.
In what follows a dot denotes derivation with respect to t, a prime derivation with respect to r.

Energy

Let α be a particle whose motion from r to $r + dr$ is determined by an outer force F. We denote by $dW = Fdr$ the work dW of the particle which is caused by F. Along a path C from r_1 to r_2 the work is given by

$$W = \int_C dW = \int F dr.$$

In general W depends on r_1, r_2 and on C. The work per time is called *power*,

$$P = \frac{dW}{dt} = F\frac{dr}{dt} = F\dot{r}$$

Kinetic and potential energy

Multiplication of Newton's second law, $F = m\ddot{r}$, with $\dot{r}$ leads to

$$m\ddot{r}\dot{r} = F\dot{r}$$

therefore

$$\frac{d}{dt}\frac{m\dot{r}^2}{2} = F\dot{r} = P \qquad (*)$$

Adding or subtracting energy changes the particle's velocity hence we define

$$E_{\text{kin}} = \frac{m}{2}\dot{r}^2,$$

as the energy caused by motion.
If we separate the force F in a conservative and a dissipative part, $F = F_{\text{cons}} + F_{\text{diss}}$ then F_{cons} contains all parts which can be written as

$$F_{\text{cons}}\dot{r} = -\frac{dU(r)}{dt}$$

where $U(r)$ is the *potential*. Hence $(*)$ becomes

$$\begin{aligned} \frac{d}{dt}\frac{m\dot{r}^2}{2} &= F_{\text{cons}}\dot{r} + F_{\text{diss}}\dot{r} \\ \Leftrightarrow \quad \frac{d}{dt}\left(\frac{m\dot{r}^2}{2} + U(r)\right) &= F_{\text{diss}}\dot{r} \end{aligned}$$

If the force is conservative then $\frac{m\dot{r}^2}{2} + U(r) = E$ is constant.

Potential

If we differentiate $U(r) = U(x, y, z)$ with respect to time then we obtain

$$F_{\text{cons}}\dot{r} = -\frac{\partial U}{\partial x}\frac{dx}{dt} - \frac{\partial U}{\partial y}\frac{dy}{dt} - \frac{\partial U}{\partial z}\frac{dz}{dt} = -\dot{r}\,\text{grad}U.$$

Therefore $F_{\text{cons}} = -\text{grad}U(r) + \dot{r} \times B(r, t)$ where $B(r, t)$ is an arbitrary vector field. We will only investigate the case $F = -\text{grad}U(r)$. One can show ([Fl], p.21) that $F(r)$ can be written in such a form iff rot $F(r) = 0$.

Energy equation

On $\mathbb{R}^3\backslash\{0\}$ the gravitational force field $F = -\frac{m}{r^2}\partial_r$ is conservative with potential $V(r) = -\frac{m}{r}$. We obtain the *energy equation*

$$2E = \dot{r}^2 + \frac{L^2}{r^2} - \frac{2m}{r}$$

Indeed, if we use polar coordinates we can write $\alpha = re^{i\phi}$ and so the kinetic energy can be calculated as $\frac{\dot{\alpha}\cdot\dot{\alpha}}{2} = \frac{\dot{r}^2 + r^2\dot{\phi}^2}{2} = \frac{1}{2}\left(\dot{r}^2 + \frac{L^2}{r^2}\right)$ since $r\dot{\phi}^2 = L$. The product in this and the following formulas is the usual scalar product on $\mathbb{R}^3$. By assumption the potential is $V(\alpha) = -\frac{m}{r}$, hence

$$2E = \dot{\alpha}\cdot\dot{\alpha} - 2\frac{m}{r} = \dot{r}^2 + \frac{L^2}{r^2} - \frac{2m}{r}$$

We interpret this equation as follows: $r = r(t)$ is the position of a unit mass particle moving along $\mathbb{R}^+$ with kinetic energy $\frac{1}{2}\dot{r}^2$. Introducing the potential $V(r) = \frac{L^2}{r^2} - \frac{2m}{r}$ we obtain $E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}V(r)$.
Since $r^2 \geq 0$ it follows that $E \geq \frac{V(r)}{2}$. To see how E determines the range of r we plot the graph of $\frac{V(r)}{2}$ and draw horizontal lines at various values E. r is restricted to those parts where $E \geq \frac{V(r)}{2}$. At $E = \frac{V(r)}{2}$ we obtain *turning points* where r changes the direction, the motion is reversed and so $\dot{r}(t) = 0$ at such a point. A *bound orbit* is possible if $\{r : V(r) \leq E(r)\}$ has two limits, r_{min} and r_{max}. r then oscillates between those points.

4.5.3 Application to Schwarzschild spacetime

Now we have to deal with a different potential function. From 4.4.5 we obtain

$$E^2 = \left(\frac{dr}{d\tau}\right)^2 + V(r)$$

where $V(r) = \left(1 + \frac{L^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = 1 - \frac{2m}{r} + \frac{L^2}{r^2} - \frac{2mL^2}{r^3}$. The only term different from the classical solution is the $\frac{1}{r^3}$ term. This leads to an elliptic integral as a solution of the energy equation which cannot be solved by elementary functions. Asymptotically we expect Newtonian results. If the radius is getting smaller the $\frac{L^2}{r^2}$ term dominates and so V is positive for L not too small. As the radius gets smaller the attractive relativistic term becomes the dominant part.
Differentiating the energy equation with respect to τ we obtain

$$0 = 2\frac{dr}{d\tau}\frac{d^2r}{d\tau^2} + \frac{dV}{dr}\frac{dr}{d\tau}$$

Therefore $2\frac{d^2r}{d\tau^2} = -V'(r)$. So we can conclude that a critical point of V (i.e. $V'(r) = 0$) represents a circular orbit $r = r_0$. A maximum is unstable since a small change in r results in an acceleration away from the center while at the minimum the orbit is stable. At $V(r) = E^2$ we obtain a turning point. For a further discussion of the orbits we have to investigate $V(r)$. It is obvious that $\lim_{r\to 0} V(r) = -\infty$ and $\lim_{r\to\infty} V(r) = 1$. The shape of the graph of $V(r)$ depends on the number of critical points of $V(r)$. To compute them we have to solve

$$0 = V'(r) = \frac{2}{r^4}(mr^2 - L^2r + 3mL)$$

so $r_{1,2} = \frac{L^2}{2m}\left(1 \pm \left(1 - \frac{12m^2}{L^2}\right)^{\frac{1}{2}}\right)$. We can see that the graph of $V(r)$ depends on the ratio $\frac{L}{m}$ since it determines whether the root is positive, zero or negative. We will investigate the situation for different angular momenta:

We denote the zeros of $V(r)$ by r_1 and r_2. Depending on L and E the orbits can be crash, escape, bound or flyby orbits:

Lemma 4.5.5. *For orbits in Schwarzschild spacetime we obtain*

- I *Low angular momentum* $L^2 < 12m^2$*:*
  - *(a) If* $E^2 < 1$ *we obtain a crash orbit, ingoing particles crash directly into the star, outgoers move to a turning point and then back to crash.*
  - *(b) If* $E^2 \geq 1$ *then ingoers crash, outgoers escape to infinity*
- II *Moderate angular momentum* $12m^2 < L^2 < 16m^2$*:*
  - *(a) If* $E^2 < V(r_1)$ *and* $r(0) > r_1$ *we have a crash orbit*
  - *(b) For* $V(r_2) \leq E(r)^2 < V(r_1)$ *and* $r(0) > r_1$ *the orbit is bound*
  - *(c) For* $V(r_1) < E^2 < 1$ *the orbit again crashes*
  - *(d) If* $E^2 \geq 1$ *we obtain a crash/escape orbit*
- III *Large angular momentum* $L^2 > 16m^2$*:*
  - *(a) If* $E^2 < V(r_1)$ *and* $r(0) < r_1$ *the orbit crashes*
  - *(b) For* $V(r_2) \leq E^2 < 1$ *and* $r(0) > r_1$ *the orbit is bound*
  - *(c) For* $1 \leq E^2 < V(r_1)$ *and* $r(0) > r_1$ *we deal with a flyby orbit*
  - *(d) If* $E^2 > V(r_1)$ *we obtain a crash/escape orbit*

Proof.
I: If $L^2 < 12m^2$ then V has no critical points $r_{1,2} = \frac{L^2}{2m}\left(1 \pm \left(1 - \frac{12m^2}{L^2}\right)^{\frac{1}{2}}\right)$ since the root is negative. So we see that there are no circular orbits by recalling

that a circular orbit is represented by a critical point. Since we see from the definition of V that $\lim_{r\to 0} V(r) = -\infty$ and $\lim_{r\to\infty} V(r) = 1$ we can conclude that V is strictly monotonically increasing, i.e. $V'(r) > 0\ \forall r$. Since $2\frac{d^2r}{d\tau^2} = -V'(r)$ it follows that $\frac{d^2r(\tau)}{d\tau^2} < 0\ \ \forall\tau$.

(a) $E^2 < 1$: Since V takes values from $-\infty$ to 1 and is continuous there has to be some $r = r(\tau_m)$ satisfying $V(r(\tau_m)) = E^2$.
By definition of ingoers, r is strictly monotonically decreasing, and, since $V(r) \leq E^2$ it follows that the starting radius r_0 has to be smaller or equal to $r(\tau_m)$. So $r \to 0$ for $\tau \to \infty$.
Outgoers starting at $r_0 \leq r(\tau_m)$ are strictly monotonically increasing until they reach $r(\tau_m)$. There $V(r(\tau_m)) = E^2$ and therefore $\frac{dr(\tau)}{d\tau}|_{\tau_m} = 0$. Since $\frac{d^2r(\tau)}{d\tau^2} < 0$, $r(\tau_m)$ is a maximum, therefore a turning point.

(b) $E^2 \geq 1$: We can conclude that $E^2 \geq V(r)\ \ \forall r$. So for ingoers with $\frac{dr(\tau)}{d\tau} < 0$ r is strictly monotonically decreasing hence $r \to 0$.
For outgoers we have $\frac{dr(\tau)}{d\tau}$. E^2 never hits $V(r)$, so $r \to \infty$.

II: $12m^2 < L^2 < 16M^2$: Since the root in $r_{1,2} = \frac{L^2}{2m}\left(1 \pm \left(1 - \frac{12m^2}{L^2}\right)^{\frac{1}{2}}\right)$ is positive we obtain two critical points $r(\tau_1) < 6m < r(\tau_2)$ (at $r = 6m$ the root is zero and we deal with a circular orbit, see the remark below).
(a) $E^2 < V(r(\tau_1))$ and $r_0 < r(\tau_1)$: V is strictly increasing until $r(\tau_1)$, so as in case $I(a)$ there exists some τ_m such that $E^2 = V(r(\tau_m))$ and $\frac{dr(\tau)}{d\tau}|_{\tau_m} = 0$, again $r(\tau_m)$ is a maximum, afterwards $r \to 0$.

(b) $V(r(\tau_2)) < E^2 < V(r(\tau_1))$ and $r_0 > r(\tau_1)$: Between $r(\tau_1)$ and $r(\tau_2)$ V is strictly decreasing, afterwards again increasing hence there are a maximal and a minimal radius. Between these the particle's orbit is oscillating. At $r = r(\tau_2)$ we get a stable circular orbit.

(c) $V(r(\tau_1)) < E^2 < 1$: By observing that there is some τ_3 satisfying $V(r(\tau_3)) = V(r(\tau_1))$ and for $\tau > \tau_3$ V is strictly increasing we get a crash orbit again as in $I(a)$.

(d) $E^2 \geq 1$ is analogous to $I(b)$.

III: $L^2 > 16m^2$: The potential barrier with crest $V(r(\tau_1))$ rises.
(a) $E^2 < V(r(\tau_1))$ and $r_0 < r(\tau_1)$: We have a crash orbit as in $I(a)$.

(b) $V(r(\tau_2)) < E^2 < V(r(\tau_1))$ and $r_0 > r(\tau_1)$: This case works just as $II(b)$ and we therefore have a bound orbit.

(c)$1 \leq E^2 < V(r(\tau_1))$ and $r_0 < r(\tau_1)$: There is some τ_3 with $V(r(\tau_3)) = 1$, for all $\tau > \tau_3$ we have $V(r(\tau)) \leq 1$. Therefore $r_0 \geq r(\tau_3)$. For outgoers $\frac{dr(\tau)}{d\tau} > 0$ so $r \to \infty$.
If $\frac{dr(\tau)}{d\tau} < 0$ there is a minimal radius $r(\tau_{\min})$ satisfying $E^2 = V(r(\tau_{\min}))$. Af-

terwards r is again increasing. Hence we deal with a flyby orbit.

(d)$E^2 > V(r(\tau_1))$: Analogously to $I(b)$ we deal with a crash/escape orbit. □

Remark 4.5.6. For $L^2 = 12m^2$ we can see that $1 - \frac{12m^2}{L^2}$ vanishes, hence V has one critical point at $r = 6m$ satisfying $V(6m) = \frac{8}{9}$. It represents a circular orbit.
For $L^2 = 16m^2$ the crest of the potential barrier reaches $V(\infty) = 1$.

For lightlike particles orbits depend on the impact parameter b of γ and discussion analogous discussion to the one above leads to the following result.

Lemma 4.5.7. *For lightlike particles' orbits we have*

I *Small impact parameter* $b < 3\sqrt{3}m$*:*
Depending on the initial conditions γ *either crashes into the star or it escapes to infinity.*

II *Large impact parameter* $b > 3\sqrt{3}m$*:*

(a) *If* $r(0) < 3m$ *the orbit crashes into the star.*

(b) *If* $r(0) > 3m$ *we deal with a flyby orbit.*

- *For* $b = 3\sqrt{3}m$ *the orbit is an unstable circular orbit with* $r = 3m$*.*

In case of $L = 0$ we obtain

Lemma 4.5.8. *Let* γ *be a freely falling material particle with* $L = 0$ *and* $E < 1$*. The proper time* τ *and the Schwarzschild radius* r *of* γ *are related by*

$$\begin{aligned} \tau &= \frac{1}{2}R\sqrt{\frac{R}{2m}}(\eta + \sin\eta) \\ r &= \frac{1}{2}R(1 + \cos\eta) \end{aligned}$$

where $\tau = 0$ *at the maximum radius* R *of* γ*.*

Proof. A particle with $L = 0$ is moving radially since the ϕ and θ coordinates are constant. Hence its energy equation is

$$E^2 - 1 = \left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r} \qquad (*)$$

At the maximum radius, $\frac{dr}{d\tau} = 0$ so we obtain

$$E^2 = 1 - \frac{2m}{R} = h(R).$$

Thus inserting the term in the energy equation leads to

$$1 - \frac{2m}{R} - 1 = \left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r}$$

which is equivalent to

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{2m}{R} = \frac{2m}{r}.$$

Substituting these formulas in the equation leads to the required result. Indeed,

$$\frac{dr}{d\tau} = \frac{dr/d\eta}{d\eta/d\tau} = \frac{-\sin\eta}{\sqrt{\frac{R}{2m}}(1+\cos\eta)}$$

Inserting into the right side of $(*)$ leads to

$$\begin{aligned}\left(\frac{dr}{d\tau}\right)^2 - \frac{2m}{r} &= \frac{-\sin^2\eta}{\frac{R}{2m}(1+\cos\eta)^2} - \frac{4m}{R(1+\cos\eta)}\\ &= \frac{-2m}{R}\left(\frac{\sin^2\eta - 2(1+\cos\eta)}{1+2\cos\eta+\cos^2\eta}\right)\\ &= \frac{-2m}{R}\\ &= E^2 - 1\end{aligned}$$

□

4.6 Classical tests of General Relativity

After having studied the mathematical background of the theory we now want to mention some experimental proofs.

4.6.1 Perihelion advance

In the bound case each trip around the star has an approximately elliptic orbit. In Newtonian gravity the orbit is a perfect ellipse, after a fixed time it returns to the same point. In the relativistic case the ellipses change their shapes. This so called *precision* can be measured by the change of the perihelion, i.e. the point of closest approach to the star where $r = \min$. In order to calculate the perihelion shift of Mercury's orbit we first state some classical Newtonian results on planetary movement. For the relativistic calculations see also [Re], 27.2.1. and [St], 3.3.

Newtonian gravitation

Let m be a mass at the origin in $\mathbb{R}^3$, α be a particle of mass $\tilde{m} \ll m$ in $\mathbb{R}^3$. Newton's law of gravitation states that

$$F = -\frac{Gm\tilde{m}}{r^2}\alpha = -\frac{Gm\tilde{m}}{r^2}U$$

where F is the force experienced by α, G the gravitational constant and $r = \|\alpha\|$ and U the outward radial unit vector, i.e. $U = \frac{\alpha}{r}$. In what follows we will use geometric units hence omit G in the formulas.

By Newton's second law $F = \tilde{m}\ddot{\alpha}$ where a dot denotes derivation with respect to time. Since $\tilde{m} \ll m$ we can ignore the motion of the mass m. Then

$$\tilde{m}\ddot{\alpha} = -\frac{m\tilde{m}}{r^2}\frac{\alpha}{r} \quad \text{therefore} \quad \ddot{\alpha} = -\frac{m}{r^3}\alpha \qquad (*)$$

We will now show that the particle α moves in a plane:
By $(*)$, $\ddot{\alpha}$ and α are parallel, so the cross product on $\mathbb{R}^3$, $\alpha \times \ddot{\alpha}$ vanishes.
Let $\vec{L} := \alpha \times \dot{\alpha}$ be the angular momentum vector per unit mass of α. By using the properties of cross products we obtain

$$\frac{d}{dt}\vec{L} = (\alpha \times \dot{\alpha})^{\cdot} = \dot{\alpha} \times \dot{\alpha} + \alpha \times \ddot{\alpha} = 0$$

so $\vec{L}$ is constant. If we assume $\vec{L} \neq 0$ then $\alpha \cdot \vec{L} = \alpha \cdot (\alpha \times \dot{\alpha}) = 0$ therefore α and $\vec{L}$ are orthogonal and α lies in a plane through the origin orthogonal to $\vec{L}$. Since $\vec{L} \neq 0$ α does not pass the origin. Indeed, $\alpha(t) = 0$ for some t implies $\alpha(t) \times \ddot{\alpha}(t) = \vec{L}(t) = 0$ therefore $\vec{L} = 0$, a contradiction.
If $\vec{L} = 0$ then α lies in a line through the origin. In both cases we can assume that α lies in the xy-plane of $\mathbb{R}^3$.

Definition 4.6.1. The *angular momentum of α per unit mass* is the number L such that $\vec{L} = L\partial_z$

Our next task is to examine the shape of the orbit:
We start by rewriting $\vec{L}$:

$$\begin{aligned}
\vec{L} &= \alpha \times \dot{\alpha} = rU \times (rU)^{\cdot} \\
&= rU \times (\dot{r}U + r\dot{U}) \\
&= r^2(U \times \dot{U}) + r\dot{r}(U \times U) \\
&= r^2(U \times \dot{U}) + 0
\end{aligned}$$

Using $(*)$ leads to

$$\begin{aligned}
\ddot{\alpha} \times \vec{L} &= (-\frac{m}{r^3}\alpha) \times (r^2(U \times \dot{U})) \\
&= -\frac{m}{r^2}U \times (r^2(U \times \dot{U})) \\
&= -mU \times (U \times \dot{U}) \\
&= -m((U \cdot \dot{U})U - (U \cdot U)\dot{U})
\end{aligned}$$

Since U is a unit vector, the scalar product $U \cdot U = 1$ and $0 = (U \cdot U)^{\cdot} = 2U \cdot \dot{U}$, therefore we get

$$\ddot{\alpha} \times \vec{L} = -m((U \cdot \dot{U})U - \dot{U}) = m\dot{U}$$

By noting that $\vec{L}$ is constant we obtain $(\dot{\alpha} \times \vec{L})^{\cdot} = \ddot{\alpha} \times \vec{L} + \dot{\alpha} \times \dot{\vec{L}} = m\dot{U} + 0$. Integration of both sides leads to

$$\dot{\alpha} \times \vec{L} = mU + c$$

where c is a constant vector in the xy-plane (U and $\dot{\alpha} \times \vec{L}$ lie in this plane).
We now use polar coordinates. Let ϕ be the angle between c and α. Then (r, ϕ)

is the particle's position in these coordinates.

$$\begin{aligned}\alpha \cdot (\dot{\alpha} \times \vec{L}) &= \alpha \cdot (mU + c)\\ &= m(\alpha \cdot U) + \alpha \cdot c\\ &= mrU \cdot U + \|\alpha\| \, \|c\| \cos\phi\\ &= r(m + c\cos\phi)\end{aligned}$$

where we used that $\|\alpha\| = r$. So $r = \frac{\alpha \cdot (\dot{\alpha} \times \vec{L})}{M + c\cos\phi}$.
Since $\alpha \cdot (\dot{\alpha} \times \vec{L}) = (\alpha \times \dot{\alpha}) \cdot \vec{L} = \vec{L} \times \vec{L} = \|\vec{L}\|^2 = L^2$ we get

$$r = \frac{\frac{L^2}{m}}{1 - e\cos\phi}$$

where $e = \frac{cL^2}{m}$. This equation describes a conic section, for $0 < e < 1$ the orbit is an ellipse, for $e = 1$ a parabola and for $e > 1$ a hyperbola. Hence we have obtained Kepler's first law.
If we write $\alpha = re^{i\phi}$, differentiate twice and set the result equal to $-\frac{m}{r^3}\alpha$ by using $(*)$ we obtain, after separating real and imaginary part, the equations

$$\ddot{r} - \dot{r}^2 = -\frac{m}{r^2}$$

$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0 = (r^2\dot{\phi})^{\cdot}$$

The second equation again shows that $L = r^2\dot{\phi}$ is constant, Kepler's second law. For the orbit equation we assume $L \neq 0$ hence r^2 and $\dot{\phi}$ are never zero and so by setting $u(\phi) = \frac{1}{r(\phi)}$ the first equation above can be rewritten as

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{L^2}$$

Perihelion advance

Corollary 4.6.2. *Let γ be a freely falling particle with $m \neq 0$ in a bound orbit around a Schwarzschild star of mass M. If $L \gg M$ and $r\frac{d\phi}{d\tau} \ll 1$ then the orbit of γ is approximately elliptical with angular perihelion advance and therefore a change of period 2π to*

$$\frac{2\pi}{1-\epsilon} \approx 2\pi(1+\epsilon)$$

where $\epsilon = \frac{3m^2}{L^2}$

Proof. As shown in 4.5.3, the orbit equation in Schwarzschild spacetime differs from the classical Newtonian only by the correction term $3mu^2$. This term is small for big r hence we can solve the orbit equation

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{L^2} + 3mu^2 \qquad (*)$$

by using a perturbation method.
We define

$$\epsilon := \frac{3m^2}{L^2}$$

Denoting the derivation with respect to ϕ with a prime we obtain

$$u'' + u = \frac{m}{L^2} + \epsilon\left(\frac{L^2u^2}{m}\right) \qquad (\Delta)$$

We suppose that a solution is of the form

$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2) \qquad (*)$$

Differentiating and inserting this solution in (Δ) leads to

$$u_0'' + u_0 - \frac{m}{L^2} + \epsilon\left(u_1'' + u_1 - \frac{L^2u_0^2}{m}\right) + \mathcal{O}(\epsilon^2) = 0$$

To make the first order approximation we set the coefficients of the ϵ, $\epsilon^2, \ldots$-terms equal to zero hence we solve $u_0''+u_0 = \frac{m}{L^2}$ which is the classical Newtonian equation with the solution developed above, i.e. $u_0 = \frac{m}{L^2}(1 + e\cos(\phi))$.
Now we set the ϵ-coefficient equal to zero and insert u_0 to obtain

$$\begin{aligned} u_1'' + u_1 &= \frac{L^2u_0^2}{m} \\ &= \frac{L^2}{m}\frac{m^2}{L^4}(1 + e\cos\phi)^2 \\ &= \frac{m}{L^2}(1 + 2e\cos\phi + e^2\cos^2\phi) \end{aligned}$$

and so

$$u_1'' + u_1 = \frac{m}{L^2}\left(1 + \frac{1}{2}e^2\right) + \frac{2me}{L^2}\cos\phi + \frac{me^2}{2L^2}\cos 2\phi$$

For a solution we make the ansatz

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi$$

and get

$$A = \frac{m}{L^2}\left(1 + \frac{1}{2}e^2\right) \qquad B = \frac{me}{L^2} \qquad C = -\frac{me^2}{6L^2}$$

So $u_1 = \frac{m}{L^2}\left(1 + \frac{1}{2}e^2\right) + 2\frac{me}{L^2}\phi\sin\phi - \frac{me^2}{6L^2}\cos 2\phi$. Therefore the general solution of the first order is

$$u \approx u_0 + \epsilon u_1 = \frac{m}{L^2}(1 + e\cos\phi) + \frac{\epsilon m}{L^2}\left(1 + e\phi\sin\phi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right)\right)$$

The perihelion of this solution is at $\phi = 0$. Since we assume $L \gg m$ the dominant terms are the first and the one involving the $\phi\sin\phi$ term which is increasing after each revolution. Therefore approximating $u \approx \frac{m}{L^2}(1+e\cos\phi+\epsilon e\phi\sin\phi) \approx \frac{m}{L^2}(1 + e\cos(\phi(1-\epsilon)))$ we see that the orbit is no longer an ellipse but is still periodic with period $\frac{2\pi}{1-\epsilon} \approx 2\pi(1+\epsilon)$. So we have a shift of the perihelion of $2\pi\epsilon = \frac{6\pi m^2}{L^2}$.

□

The measurement of Mercury's precession was first accomplished in the 19^{th} century. Other planets also disturb Mercury's orbit hence the total observed precession is about $5600''$/century. The $43''$ is the only part which is not explainable by Newtonian gravity hence the exact prediction of this amount by Einstein's theory was the first direct experimental proof in favor of it.

4.6.2 Deflection of light

Analogous calculations and further details can be found in [St], 3.4.
For a lightlike particle we already showed in 4.5.3 that

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \qquad (*)$$

As the Schwarzschild radius approaches infinity, the relativistic term becomes small and if we neglect it the path of the particle is the solution of $\frac{d^2u}{d\phi^2} + u = 0$ which is given by $u = a \sin\phi$ where $a = \frac{1}{r_0}$. $r_0 \gg m$ is the perihelion at $\phi = 0$. Inserting this solution into the right hand side of $(*)$ gives

$$\frac{d^2u}{d\phi^2} + u = 3ma^2 \sin^2\phi = 3ma^2(1 - \cos^2\phi)$$

which has the particular solution $u_1 = \frac{3ma^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right)$. So the first order solution is $u = a\sin\phi + \frac{3ma^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right)$. For large r (resp. small u) ϕ is close to zero. So we take $\sin\phi \approx \phi$ and $\cos\phi \approx 1$. In the limit $r \to \infty$ we can see that $\phi \to \phi_\infty$ with

$$\phi_\infty = -2ma.$$

The total deflection δ is equal to $2\,|\phi_\infty|$, thus

$$\delta = 4ma = \frac{4m}{r_0}$$

For the sun with $m = 1.5$km, $r_0 = 7 \times 10^5$km we obtain a deflection angle of $\delta = 1.7''$. This result was predicted by Einstein and first tested on March 29, 1919 during the solar eclipse. Citing J.J. Thompson 'this is the most important result obtained in connection with the theory of gravitation since Newton's day'. For further details see also [St].

4.6.3 Time delay of radar echoes

I. Shapiro suggested in 1964 a new test for the theory of general relativity. The idea is to send radar signals from earth through a region near the sun to another planet or satellite and reflect it back to earth which should lead to a time delay. The following calculations are based on [St], 3.5.
So let the signal be transmitted from $p_1 = (r_1, \theta = \frac{\pi}{2}, \phi_1)$ and be sent to $p_2 = (r_2, \theta = \frac{\pi}{2}, \phi_2)$. The time t_{12} required by the signal to travel from p_1 to p_2 can be computed via using the energy equation (see 4.4.5)

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(1 + \frac{L^2}{r^2}\right)h(r)$$

We write

$$\frac{dr}{d\tau}\frac{dr}{dt}\frac{dt}{d\tau} = \frac{dr}{dt}\frac{E}{h(r)} \qquad (*)$$

After inserting this equation in the energy equation above we obtain

$$h(r)^{-3}\left(\frac{dr}{dt}\right)^2 = h(r)^{-1} - \frac{L^2}{E^2}\frac{1}{r^2}$$

Let now $r = r_0$ be the distance of closest approach to the sun, then $\frac{dr_0}{dt} = 0$ and so $\frac{L^2}{E^2} = \frac{r_0^2}{h(r_0)}$.
If we insert this in $(*)$ we get

$$h(r)^{-3}\left(\frac{dr}{dt}\right)^2 + \left(\frac{r_0}{r}\right)^2 h(r_0)^{-1} - h(r)^{-1} = 0$$

Now we can calculate the time which is needed by the signal to come from r_0 to an arbitrary r by

$$t(r_0, r) = \int_{r_0}^{r} h(r)^{-1}\left(1 - \frac{h(r)}{h(r_0)}\left(\frac{r_0}{r}\right)^2\right)^{-\frac{1}{2}} dr$$

Since $\frac{2m}{r}$ is small we obtain approximately

$$\begin{aligned} t(r_0, r) &\approx \int_{r_0}^{r} 1 + \frac{2m}{r}\left(1 - \left(\frac{2m}{r_0} - \frac{2m}{r}\right)\left(\frac{r_0}{r}\right)^2\right)^{-\frac{1}{2}} dr \\ &\approx \int_{r_0}^{r} \left(1 - \frac{r_0^2}{r^2}\right)^{-\frac{1}{2}}\left(1 + \frac{2m}{r} + \frac{mr_0}{r(r+r_0)}\right) dr \end{aligned}$$

Integration leads to

$$t(r, r_0) \approx \sqrt{r^2 - r_0^2} + 2m\ln\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) + m\left(\frac{r - r_0}{r + r_0}\right)^{\frac{1}{2}}$$

For $|\phi_1 - \phi_2| > \frac{\pi}{2}$ we have $t_{12} = t(r_1, r_0) + t(r_2, r_0)$. For a circuit from p_1 to p_2 and back we introduce the *Shapiro delay in coordinate time*

$$\begin{aligned} \Delta t &:= 2(t(r_1, r_0) + t(r_2, r_0) - \sqrt{r_1^2 - r_0^2} - \sqrt{r_2^2 - r_0^2}) \\ &\approx 4m\ln\left(\frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2}\right) \\ &\quad +2m\left(\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_2 - r_0}{r_2 + r_0}}\right) \end{aligned}$$

This delay is too small to be observed, for a trip from Earth to Mars and back we get e.g. $\Delta t \approx 4m\left(\ln\frac{4r_1r_2}{r_0^2} + 1\right) \approx 240\mu s$. To measure the time delay it is necessary to know $\sqrt{\frac{r_1-r_0}{r_1+r_0}} + \sqrt{\frac{r_2-r_0}{r_2+r_0}}$ extremely precisely. One must carry out the measurements of the circuit times over some time and fit the data to a complicated model describing the motion of the transmitter and receiver. For further details see [St].

4.7 Singularities

After we have investigated the region N where the metric is nonsingular we now want to take a closer look at the singularities of the metric. Since

$$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2d\sigma^2,$$

the metric becomes singular for $r = 0$ and $r = 2m$. Additionally there are the spherical coordinate singularities. These and the one at $r = 2m$ will turn out to be coordinate singularities. One can find a different coordinate system to get rid of them. As far as $r = 0$ is concerned, this singularity is a real spacetime singularity which is not removable by a coordinate transformation. We will now study the region $r = 2m$.

Definition 4.7.1. The region of the Schwarzschild geometry with $r = 2m$ is called *gravitational radius* (or Schwarzschild radius, -surface, -horizon).

The Schwarzschild metric satisfies the vacuum field equations for $r > 2m$ as well as for $r < 2m$. To determine whether the spacetime geometry is singular at the gravitational radius we send an observer into this region and see what happens: Let him fall freely and radially, then, by [MTW], p.820 his trajectory is given by

$$\begin{aligned}\frac{\tau}{2m} &= -\frac{2}{3}\left(\frac{r}{2m}\right)^{3/2} + \text{const.}\\ \frac{t}{2m} &= -\frac{2}{3}\left(\frac{r}{2m}\right)^{3/2} - 2\left(\frac{r}{2m}\right)^{1/2} + \ln\left|\frac{\left(\frac{r}{2m}\right)^{1/2}+1}{\left(\frac{r}{2m}\right)^{1/2}-1}\right|\end{aligned}$$

To reach $r = 2m$ requires a finite lapse of proper time but an infinite lapse of coordinate time (see [MTW] ch. 31 for more details).

Definition 4.7.2. For a vector $0 \neq v \in T_pM$ the *tidal force operator* $F_v : v^\perp \to v^\perp$ is given by $F_v(y) = R_{yv}v$.

F_v is a self-adjoint linear operator on $v^\perp$, and $\text{trace}F_v = -\text{Ric}(v, v)$. See [O'N.], 8.9., for a proof.
Calculating the Riemannian curvature tensor we can see that curvature remains finite at $r = 2m$. Hence the tidal forces during approaching the gravitational radius are finite in contrast to the singularity at $r = 0$ where we obtain infinite tidal forces.
But Schwarzschild coordinates behave strangely at $r = 2m$. The roles of t and r as timelike resp. spacelike coordinates are reversed. Inside the horizon the dr^2 term is the only positive term so r cannot stand still for a particle's or a photon's world line since it has to satisfy $ds^2 \geq 0$. But not standing still is a characteristic of time. The future direction is that of decreasing r.

4.7.1 Different coordinate systems

Historically Eddington was the first who constructed a coordinate system which is nonsingular at $r = 2m$ in 1924 (see below for details). But he did not seem to have recognized the significance of his results. In 1933 Lemaitre noted that the 'Schwarzschild singularity' is not a singularity. For more detailed calculations and discussions, see [MTW]31.3-31.5.

Novikov, 1963

To each test particle a specific value R of the radial coordinate is attached. It emerges from $r = 0$ carries it through $r = 2m$ until it reaches the maximal r and then back again through $r = 2m$ to $r = 0$. R is given by $R = \left(\frac{r_{\max}}{2m} - 1\right)^{1/2}$. As a time coordinate we use proper time τ of the particle with $\tau = 0$ on the peak. A complicated transformation (see [MTW], 31.4) yields to the line element

$$ds^2 = -d\tau^2 + \left(\frac{R^2+1}{R^2}\right)\left(\frac{\partial r}{\partial R}\right)^2 dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Here $r = r(\tau, R)$ is no longer a radial coordinate but given implicitly by

$$\frac{\tau}{2m} = \pm(R^2+1)\left(\frac{r}{2m} - \frac{\left(\frac{r}{2m}\right)^2}{R^2+1}\right)^{1/2} + (R^2+1)^{3/2}\cos^{-1}\left(\left(\frac{\frac{r}{2m}}{R^2+1}\right)^{1/2}\right)$$

So there are two distinct regions with $r = 0$ and two with $r \to \infty$. The concept of this coordinate system is quite simple, but the mathematical expressions for the metric components are rather complicated.

Eddington, 1924, rediscovered by Finkelstein, 1958

In contrast to Novikov not freely falling particles are the foundation of the coordinate system but freely falling photons instead. We introduce coordinates u and v where outgoing radial null geodesics are given by $u =$ const, $u \equiv t - r^*$, ingoing ones by $v =$ const, $v \equiv t + r^*$ with $r^* = r + 2m \ln\left|\frac{r}{2m} - 1\right|$. So we obtain two coordinate systems. *Ingoing EF-coordinates* where r and v are used instead of r and t with metric $ds^2 = -h(r)dv^2 + 2dv\, dr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ on the one hand and on the other hand *outgoing EF-coordinates* where we choose r and u with line element $ds^2 = -h(r)du^2 - 2du\, dr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Both systems behave better at $r = 2m$ than the Schwarzschild coordinates but describing ingoing particles with outgoing coordinates (resp. vice versa) leads to the same problems we had with Schwarzschild coordinates. But ingoing EF-coordinates describe infall very well hence they are frequently used in the description of gravitational collapse ([MTW], ch.32) or black holes ([MTW], ch. 33, 34).

Kruskal, Szekeres, 1960

The idea now is to use both u and v as coordinates. They are related to Schwarzschild coordinates by

$$\begin{aligned} v - u &= 2r^* \\ v + u &= 2t \end{aligned}$$

hence $ds^2 = -h(r)du\, dv + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. But we can see that this coordinate system is again pathological at $r = 2m$. So we try to relabel the coordinates to get rid of the disturbing factor $1 - \frac{2m}{r}$ which is allowed since any relabeling does not change the physical properties of the surfaces .

We will now join the Schwarzschild half plane P_I and the strip P_{II} to obtain a connected spacetime K. r becomes a function of u and v. Let $f(r) = (r - 2m)\exp(\frac{r}{2m} - 1)$ with $m > 0$ a constant. Then f is given by $f(r) = uv$. Since $f'(r) = \frac{r}{2m}\exp(\frac{r}{2m} - 1) > 0$ on $\mathbb{R}^+$, f is a diffeomorphism onto $(-\frac{2m}{e}, \infty)$.

Definition 4.7.3. Let $F(r) := \frac{8m^2}{r}\exp(1 - \frac{r}{2m})$. The region Q in the uv-plane given by $uv > -\frac{2m}{3}$ furnished with the line element $ds^2 = 2F(r)dudv$ is called *Kruskal plane of mass m*.

If we remove the coordinate axes we obtain the four quadrants $Q_1, \ldots, Q_4$.

Remark 4.7.4.

- For constant r we obtain the hyperbolas $uv = \text{const.}$, at $r = 2m$ the coordinate axes.
- $\{u, v\}$ is a null-coordinate system on Q, i.e. $\langle \partial_u, \partial_u \rangle = \langle \partial_v, \partial_v \rangle = 0$ and we can replace it by spacelike and timelike coordinates $\tilde{u}, \tilde{v}$ given by

$$\begin{aligned} \tilde{u} &= \frac{1}{2}(v-u) \\ \tilde{v} &= \frac{1}{2}(v+u) \end{aligned}$$

  and obtain that $d\tilde{u}^2 + d\tilde{v}^2 = du\ dv$.
- The mapping $\phi : (u, v) \mapsto (-u, -v)$ preserves uv, hence r and so also $F(r)$ and the line element. We can interpret ϕ as an isometry reversing quadrants.

To relate Schwarzschild coordinates to the new coordinates we define $t = 2m \ln \left| \frac{v}{u} \right|$. The level curves $t = \text{const.}$ are then rays from the origin in Q. In what follows we will show that $\psi : Q_1 \cup Q_2 \to P_I \cup P_{II}$ with $(u, v) \mapsto (t(u, v), r(u, v))$ is an isometry preserving the quadrants.

Lemma 4.7.5. *On Q,*
(1) $Ff = 8m^2h$, $Ff' = 4m$ and $\frac{f}{f'} = 2mh$
(2) $dt = 2m\left(\frac{dv}{v} - \frac{du}{u}\right)$, $dr = 2mh\left(\frac{du}{u} + \frac{dv}{v}\right)$ for $uv \neq 0$.
(3) $\mathrm{grad} r = \frac{1}{4m}(u\partial_u + v\partial_v)$

Proof.
(1)The first identity is a simple calculation. For the second one use $f'(r) = \frac{r}{2m}\exp(\frac{r}{2m} - 1)$ and the third identity is a consequence of the first two.
(2) We calculate $dt = 2m\frac{u}{v} d\left(\frac{v}{u}\right) = 2m\frac{u}{v}\left(\frac{udv}{u^2} - \frac{vdu}{u^2}\right) = 2m\left(\frac{dv}{v} - \frac{du}{u}\right)$.
Since $f'(r) = \frac{d(uv)}{dr}$ we obtain $f'(r)dr = vdu + udv$. From (1) we already know that $2mh = \frac{f(r)}{f'(r)} = \frac{uv}{f'(r)}$ and so the result for dr follows.
(3) The vector fields metrically equivalent to du and dv are $\frac{\partial_v}{F}$ and $\frac{\partial_u}{F}$, respectively. Hence by using (2) we obtain the result. Indeed, let X be an arbitrary vector field, then

$$\begin{aligned} \langle \mathrm{grad} r, X \rangle &= dr(X) \\ &= 2mh\left(\frac{du}{u} + \frac{dv}{v}\right)(X) \\ &= \langle 2mh\left(\frac{\partial_v}{Fu} + \frac{\partial_u}{Fv}\right), X \rangle \\ &= \langle \frac{2mh}{F}\left(\frac{v\partial_v + u\partial_u}{uv}\right), X \rangle \\ &= \langle \frac{2mh}{Ff}(u\partial_u + v\partial_v, X \rangle \\ &= \langle \frac{1}{4m}(u\partial_u + v\partial_v), X \rangle \end{aligned}$$

□

Now we will show that the quadrants Q_1 and therefore Q_3 are isometric to the Schwarzschild half plane P_I, Q_2 and Q_4 to the strip P_{II}:

Proposition 4.7.6. *The function* $\psi : Q_1 \cup Q_2 \to P_I \cup P_{II}$, $(u, v) \mapsto (t(u, v), r(u, v))$ *is an isometry preserving the quadrants and the functions* t *and* r.

Proof. We denote the natural coordinate functions on $P_I \cup P_{II}$ by $\tilde{t}$ and $\tilde{r}$ so it has the line element $-hd\tilde{t}^2 + h^{-1}d\tilde{r}^2$. By the definition of ψ we obtain that $\tilde{t} \circ \psi = t$ and $\tilde{r} \circ \psi = r$ hence $\psi^*(T_{(u,v)}\tilde{t}) = T_{(u,v)}(\tilde{t} \circ \psi) = T_{(u,v)}t$ and analogously $\psi^*(T_{(u,v)}\tilde{r}) = T_{(u,v)}r$. ψ preserves r hence also h, therefore after applying ψ^* to the line element we obtain $-hdt^2 + h^{-1}dr^2$. We can now apply 4.7.5 to obtain the line element of Q, i.e. $2F(r)dudv$.
ψ is a diffeomorphism on each quadrant with inverse function given by $u = \sqrt{f(r)}\exp(-t/4m)$ resp. $v = \sqrt{f(r)}\exp(t/4m)$. ☐

So we can finally join N and B. We will again identify this connected spacetime as a warped product:

Definition 4.7.7. Let Q be a Kruskal plane of mass m and let S^2 be the unit 2-sphere. The *Kruskal spacetime* of mass m is the warped product $K = Q \times_r S^2$ where r is the function on Q given by $f(r) = uv$.

Thus K is the product manifold $Q \times S^2$ furnished with metric tensor $2F(r)dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2)$. By 2.2.14 we already know that each leaf $\mathrm{pr}_2^{-1}(q)$ is totally geodesic and isometric to Q and each fiber $\mathrm{pr}_1^{-1}(u, v)$ is a 2-sphere of radius $r(u, v)$ which is totally umbillic in K.

Let now K_i, $i = 1, \ldots 4$, be the open submanifolds $\mathrm{pr}_1^{-1}(Q_i)$. Then K_1 and K_3 are isometric to N, K_2 and K_4 are isometric to B. Indeed, let ψ be the isometry from 4.7.6. It preserves r so $\psi \times \mathrm{id}$ is an isometry from $K_1 = Q_1 \times_r S^2$ onto $N = P_I \times_r S^2$. The isometry $(u, v) \mapsto (-u, -v)$ preserves r, so $\phi(u, v, p) = (-u, -v, p)$ is an isometry reversing K_1 and K_3 resp. K_2 and K_4. So to summarize we have

$$K_3 \overset{\phi}{\approx} K_1 \overset{\psi}{\approx} N$$

and

$$K_4 \overset{\phi}{\approx} K_2 \overset{\psi}{\approx} B$$

We therefore can see that two Schwarzschild patches are necessary to cover the entire Schwarzschild geometry, but a single Kruskal system suffices.

Definition 4.7.8. The *horizon* H consists of all points over the coordinate axes of Q. It is obtained by deleting $\mathrm{pr}_1^{-1}(Q_i)$, $i = 1, \ldots, 4$, from K.
$\mathrm{pr}_1^{-1}(0, 0)$ is called *central sphere*. The isometry $\phi : (u, v, p) \mapsto (-u, -v, p)$ introduced above is the so-called *central symmetry*.

Remark 4.7.9. Removing the central sphere from H leaves four hypersurfaces, each diffeomorphic to $\mathbb{R}^+ \times S^2$.

Since K is defined as a warped product, covariant derivative and curvature on K can be expressed in Kruskal coordinates by using warped product generalities introduced in the first chapter.

4.8 Reissner-Nordström spacetime

In this section we want to investigate the spacetime surrounding a non-rotating charged spherical symmetric black hole as another example for a warped product spaetime. It is based on [D'I], chapter 18 where also further details can be found.

4.9 Deriving the metric

We search for a static, asymptotically flat spherical symmetric solution of the Einstein-Maxwell equations

$$G_{ab} = 8\pi T_{ab}$$

where T_{ab} is Maxwell's stress energy tensor. If there is no source it is given by

$$T_{ab} = \frac{1}{4\pi}\left(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \qquad (*)$$

F^{ab} is the electromagnetic field strength tensor, generally given by

$$F^{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

T_{ab} is tracefree, hence we can rewrite Einstein's equations as

$$R_{ab} = 8\pi T_{ab}$$

Indeed, T is trace free, i.e. $g^{ab}T_{ab} = 0$ and

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}S = 8\pi T_{ab}$$

where S is the scalar curvature. Multiplication with g^{ab} of both sides of the equation leads to

$$\underbrace{g^{ab}R_{ab}}_{=S} - \frac{1}{2}\underbrace{g^{ab}g_{ab}}_{=4} S = 8\pi \underbrace{g^{ab}T_{ab}}_{=0}$$

therefore

$$0 = S$$

and

$$G_{ab} = R_{ab}$$

Furthermore F_{ab} has to satisfy Maxwell's equations for sourcefree space, i.e.

1. $\nabla_b F^{ab} = 0$

2. $\partial_{[a}F_{bc]} = 0$

Now we take our assumptions in account to determine the metric. The calculations are quite similar to the ones in 4.2.

Assumption 1: Spherical symmetry

When using the coordinates (t, r, θ, ϕ) the line element has the form

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\sigma^2$$

where $d\sigma^2$ is the line element on the unit sphere. λ and ν depend on t and r.

Assumption 2: Staticity

Since the solutions is to be static, λ and ν depend only on r, i.e.

$$\lambda = \lambda(r) \qquad \nu = \nu(r)$$

Assumption 3: Field

The field is built up by a charged particle at the origin, hence the line element and the Maxwell tensor become singular at this origin.
The charged particle produces an electrostatic field which is radial, i.e.

$$F_{ab} = E(r) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore we can calculate

$$g_{ab} = \mathrm{diag}(-e^\nu, e^\lambda, r^2, r^2 \sin^2\theta)$$

$$g^{ab} = \mathrm{diag}(-e^{-\nu}, e^{-\lambda}, r^{-2}, r^{-2} \sin^{-2}\theta)$$

Consequently, $F_{ab} = 0$ except for

$$a = 0 \qquad b = 1$$

$$a = 1 \qquad b = 0$$

So we can conclude that (2) of the Maxwell equations is automatically satisfied and (1) is reduced to the single equation

$$(e^{-1/2(\nu+\lambda)} r^2 E)' = 0$$

where a $'$ denotes derivation with respect to r. Integration of this equation leads to

$$E = \frac{e^{1/2(\nu+\lambda)}\epsilon}{r^2}$$

where ϵ is an integration constant.

Assumption 4: Asymptotic flatness:

Since $g_{ab} \to \eta_{ab}$ for $r \to \infty$ where η_a denotes the Minkowski metric, we conclude that $\nu, \lambda \to 0$ for $r \to \infty$. So $E \approx \frac{\epsilon}{r^2}$ asymptotically which is the classical result for an electric field of a particle with charge located in the origin.

Calculating T_{ab}

We already have determined $E(r)$ and therefore also F_{ab}. By $(*)$ we know how we can calculate T_{ab} from F_{ab}. If we insert T_{ab} into the field equation then the 00- and 11- equations lead to $\lambda' + \nu' = 0$, since by assumption $\nu, \lambda \to 0$ it follows that $\lambda = -\nu$. The only remaining independent equation is the 22-equation which leads to $(re^{\nu})' = 1 - \frac{\epsilon^2}{r^2}$. After integration we have $e^{\nu} = 1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}$ where again m is an integration constant.

We finally have obtained the Reissner-Nordström solution

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{\epsilon^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

This is also a warped product, and for $\epsilon = 0$ we get the Schwarzschild solution.

Bibliography

[Ba] G.K. Batchelor. *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967

[B.E.] J. K. Beem, P. E. Ehrlich. *Global Lorentzian Geometry*, Cambridge University Press, 1973

[B.o'N.] R.L.Bishop, B.O'Neill. *Manifolds of negative curvature, Trans. Am. Math. Soc. 145, 1-49 (1969)*

[CC] P.D. Mc Cormack, L. Cane. *Physical fluid Dynamics*, Academic Press, 1973

[D'I] R. d'Inverno. *Introducing Einstein's relativity*, Clarendon, 1992

[D.Ü.] F. Dobarro, B. Ünal. *Geometry of Multiply Warped Products*, arXiv:math/0406039v3

[Fl] T. Fließbach. *Mechanik, Lehrbuch zur Theoretischen Physik 1, 4. Auflage*, Spektrum Akademischer Verlag, 2003

[Ge] R. Geroch. *Domain of Dependence, J.Math.Phys. 11, No.2 437-449, (1970). MSC2000*

[H.E.] S.W. Hawking, G.F.R. Ellis. *The large scale structure of spacetime, Second Edition*, Marcel Dekker, Inc. 1996

[HF] L.N. Hand, J.D. Finch. *Analytical Mechanics* Cambridge University Press, 1998

[Hu] J.K. Hunter. *An Introduction to the Incompressible Euler Equations*, `http://www.math.ucdavis.edu/ hunter/notes/euler.pdf`, 2006

[Mi] J. Milnor. *Topology from the differentiable viewpoint*, Princeton University Press, 1965

[MTW] C.W. Misner, K.S. Thorne, J.A. Wheeler. *Gravitation*, W.H. Freeman and Company, San Francisco, 1973

[O'N.] B. O'Neill. *Semi Riemannian Geometry* Academic Press, 1983

[Pa] D.F. Parker. *Fields, Flows and Waves. An Introduction to Continuum Models*, Springer Undergraduate Mathematics Series, 2003

[Pe] P. Petersen. *Riemannian Geometry*, Springer Graduate Texts in Mathematics, 2006

[Re] E. Rebhan. *Theoretische Physik. Band 1*, Spektrum Heidelberg, 1999

[Ri] W. Rindler. *Relativity Special, General and Cosmological. Second Edition*, Oxford University Press, 2006

[Sch] B. Schutz. *A First Course in General Relativity, Second Edition*, Cambridge University Press, 2009

[St] N. Straumann. *General Relativity with Applications to Astrophysics*, Springer-Verlag Berlin Heidelberg, 2004

[Ül] B.Ünal. *Multiply Warped Products* Journal of Geometry and Physics, Vol.34, Issue 34, p.287-301

[Wa] R.M. Wald. *General Relativity* Chicago University Press, 1984

Curriculum Vitae

Personal Information

Name: Nathalie Tassotti

Date of Birth: 29th of September, 1986

Place of Birth: Klagenfurt, Austria

Education

since 10/2005: Diploma/Bachelor studiy at the University of Vienna
Mathematics

06/2005: Matura (school leaving examination) passed with honors

1997-2005: Bundesrealgymnasium Klagenfurt-Viktring, grammar school

1993-1997: Volksschule 9, Klagenfurt, primary school

Employment

since 10-2008: Tutor at the Faculty of Mathematics, University of Vienna