DIPLOMARBEIT

Titel der Diplomarbeit

FUGLEDE-KADISON-DETERMINANT ENTROPY FORMULAS AND ISOMORPHISMS OF LATTICE MODELS

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Abstract

This thesis studies entropies of the following models in the thermodynamic limit: spanning trees and the associated essential spanning forest process, dimer covers, the abelian sandpile model and the harmonic model. These models arise from various fields in mathematics, but all of them can be interpreted as dynamical systems in higher dimensions and as examples from statistical mechanics. Tools of operator theory and harmonic analysis will be used to express entropy as logarithm of the Fuglede-Kadison determinant, which was originally defined on operators in a factor of type II_1. Similar methods (although in a more algebraic setting) will be used to calculate the entropy of expansive actions of discrete residually finite amenable groups by automorphisms of compact abelian groups.

Deutsche Zusammenfassung

Diese Diplomarbeit behandelt die Entropien der folgenden Modelle im thermodynamischen Limes: Spanning Trees und der damit verbundene Essential Spanning Forest Prozess, Dimer Überdeckungen, das Abelsche Sandhäufchen Modell und das Harmonische Modell. Diese Modelle aus verschiedenen Disziplinen der Mathematik können als dynamische Systeme oder als Beispiele aus der statistischen Mechanik interpretiert werden. Methoden aus der Operatoren Theorie und der harmonischen Analysis werden zur Darstellung der Entropie als Logarithmus der Fuglede-Kadison Determinante verwendet, wobei die Fuglede-Kadison Determinante ursprünglich für Operatoren in einem Faktor vom Typ II_1 eingeführt wurde. Ähnliche Methoden werden zur Berechnung der Entropie von expansiven Wirkungen von diskreten, residuell-endlichen, mittelbaren Gruppen durch Automorphismen von kompakten, abelschen Gruppen verwendet; wobei diese Wirkungen auch in einem eigenen Kapitel beschrieben werden.

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Introduction

This introduction outlines the interaction between statistical mechanics and dynamical systems, which allows one to switch between these two theories. This idea will be demonstrated by means of an example from combinatorics.

A dimer cover on the lattice $\mathbb{Z}^2$ is a subset of edges which covers every vertex exactly once. Two interpretations of this model will be discussed to analyse the complexity of dimer covers:

- Physical interpretation: If one looks at a vertex $\mathbf{n} \in \mathrm{V}(\mathbb{Z}^2)$, then there are only four possibilities a dimer can be placed on this vertex. These degrees of freedom will be represented by an alphabet $\{N, E, S, W\}$. A configuration η is a map from $\mathbb{Z}^2$ to $\{N, E, S, W\}$; with the intuitive picture that every vertex $\mathbf{n} \in \mathrm{V}(\mathbb{Z}^2)$ is in one of the four possible states $\{N, E, S, W\}$. Clearly, not every configuration is allowed because it could happen that two dimers lie on the same vertex, which is forbidden by definition. There are infinitely many points in the set $\mathcal{A}$ of allowed configurations and so the entropy per vertex will be introduced to measure the complexity of this system. Let $Q_n(\mathcal{A})$ denote the number of distinguishable quadratic boxes with side length n appearing in the points (or configurations) of $\mathcal{A}$. The entropy per vertex is defined by

$$\mathbf{h}(\mathcal{A}) = \lim_{n\to\infty} \frac{\log Q_n(\mathcal{A})}{n^2}$$

  and will be interpreted as exponential growth rate of the number of allowed configurations in finite quadratic boxes with side length n as n goes to infinity. Roughly, $Q_n(\mathcal{A})$ grows like $\mathrm{e}^{\mathbf{h}(\mathcal{A})n^2}$.

- Dynamical interpretation: The dimer model can be interpreted as a dynamical system by constructing a shift of finite type as follows:

$$\mathcal{A} = \Big\{ \eta \in \{N, E, S, W\}^{\mathbb{Z}^2} : \text{ let } \mathbf{n}, \mathbf{m} \in \mathbb{Z}^2 \text{ if } \mathbf{m} - \mathbf{n} = (1,0) \Rightarrow (\eta_{\mathbf{n}}, \eta_{\mathbf{m}}) \notin F_1; \\ \text{if } \mathbf{m} - \mathbf{n} = (0,1) \Rightarrow (\eta_{\mathbf{n}}, \eta_{\mathbf{m}}) \notin F_2 \Big\},$$

  where

$$F_1 = \{(N,W), (E,N), (E,E), (E,S)(S,W), (W,W)\}$$

  and

$$F_2 = \{(N,N), (N,E), (N,W), (S,S), (E,S), (W,S)\}.$$

  The shift action σ is defined by $(\sigma^{\mathbf{n}}\eta)_{\mathbf{m}} = \eta_{\mathbf{n}+\mathbf{m}}$, i.e. a translation of the configuration in direction $\mathbf{n}$. The entropy of this transformation will be defined informally as follows: The distance between $\eta, \eta' \in \mathcal{A}$ is given by 2^{-k}, where k is the largest integer such that $\eta_{[-k,k]^2} = \eta'_{[-k,k]^2}$. Let $B_n := \{\mathbf{m} \in \mathbb{Z}^2 : \max_{i\in\{0,1\}} m_i \leq n\}$. The orbit segments of two elements $\eta, \eta' \in \mathcal{A}$ are ε-distinguishable if there is a coordinate $\mathbf{m} \in B_n$ such that the distance between $\sigma^{\mathbf{m}}\eta$ and $\sigma^{\mathbf{m}}\eta'$ is at least ε. The value $s(B_n, \varepsilon)$ will denote the number of ε-different orbit segments with translation vectors in B_n. The quantity

$$\mathbf{h}(\sigma, \varepsilon) := \limsup_{n\to\infty} \frac{\log s(n, \varepsilon)}{n^2}$$

is the exponential growth rate of the number of ε-distinguishable orbit segments of the dynamical system as n goes to infity. The entropy is defined by

$$\mathbf{h}(\sigma) = \lim_{\varepsilon\to 0} \mathbf{h}(\sigma, \varepsilon) \, .$$

Since the shift action simply translates configurations, it is clear that this general definition of the dynamical entropy coincides with the definition of entropy given above. In the case of the dimer model $(\mathcal{A}, \sigma)$ the entropy can be expressed as the exponential growth rate of the number of periodic points. And so one gets

$$\mathbf{h}(\sigma) = \limsup_{n\to\infty} \frac{\log \mathrm{Fix}_n \sigma}{n^2} \, ,$$

where $\mathrm{Fix}_n\sigma = \{\eta \in \mathcal{A} : \sigma^{\mathbf{m}}\eta = \eta \text{ for every } \mathbf{m} \in B_n\}$.

There are many ways to define entropy, but with the above explanatory notes one should have a better idea what entropy measures. In general there is no easy way to calculate entropy. The aim of this thesis is to study entropies of several models.

In the first chapter the entropy of spanning trees on infinite graphs will be calculated with the help of von Neumann algebras and harmonic analysis. Such methods will be used in the third chapter for expansive algebraic actions, where the commutativity of the symmetry group is no longer required. In both chapters a generalised determinant, the so called Fuglede-Kadison determinant will appear and it will turn out that the logarithm of this determinant is equal to the entropy of these two models.

Entropy has another important feature: if two dynamical systems are isomorphic (in the measure theoretical or topological sense), then their entropies coincide. This property will be used in the second chapter to find the entropy of the abelian sandpile model (when restricting to finite regions the model is isomorphic to the model of spanning trees) and of the dimer model (which is isomorphic to the essential spanning forest process, which is associated to the model of spanning trees).

In the last chapter the abelian sandpile model will appear as a symbolic cover of the so called harmonic model, which is an algebraic dynamical system with a non-expansive action.

At the beginning of each chapter (or section) the main references for the chapter (or section) will be cited without any further comments.

1 Spanning Trees

In this chapter the asymptotics of the number of spanning trees on a graph will be studied. First the Matrix Tree Theorem will be proved. Then a convergence principle for sequences of graphs will be introduced to define the tree entropy, which will be an indicator for the complexity of an infinite graph. Finally, operator algebra methods will be used to find a helpful expression for the tree entropy.

1.1 Basics And The Matrix Tree Theorem

Reference(s): [16, chapter 13] and [25, section 3]

Let G be a graph with countable vertex set V and edge set E. A **loop** is an edge that connects a vertex to itself. A **cycle** is a closed path with no repeated vertices except the starting and ending vertices. Let $x, y \in \mathrm{V}$, write $x \sim y$ if x and y are connected by an edge. The **degree of a vertex** $x \in \mathrm{V}$ is the number of edges which connect x to vertices in V; this number will be denoted by $\deg_G(x)$ and it is assumed that all degrees are finite. The following matrices are indexed by the vertices of G:

Definition. The **degree matrix** D_G is the diagonal matrix, whose (x, x)-entry is equal to $\deg_G(x)$, for all $x \in \mathrm{V}$. The matrix A_G, whose (x, y)-entry is equal to the number of edges connecting x and y, will be called **adjacency matrix**. The matrix difference $D_G - A_G$ is denoted by Δ_G and is called the **graph Laplacian matrix**. The matrix product $P_G := D_G^{-1} \cdot A_G$ is the transition matrix of a Markov chain. This Markov process has state space V and is called the **simple random walk** on G. Let $p^k(x; G)$ denote the probability that the simple random walk on G started at x is back at x after k steps.

A graph H, whose vertex set is a subset of that of G and whose adjacency relation is a subset of that of G restricted to the subset of vertices of H, is called a **subgraph** of G. A subgraph H of G **spans** G if H has the same vertex set as G. A graph is a **forest** if it has no cycles and is simple (i.e. an undirected graph that has no loops and no more than one edge between any two different vertices). If a forest is connected, i.e. every pair of distinct vertices in the graph can be connected by some path, then the forest is called a **tree**. The **complexity of a graph** G is the number of spanning trees and is denoted by $\tau(G)$. The **entropy of a finite graph** G is given by $\log \tau(G)$ and will serve as a measure for the complexity.

The Matrix Tree Theorem leads to an easy calculation of the number of spanning trees of a graph G. Some definitions are necessary to formulate and prove the Matrix Tree Theorem: Let G be a graph, x and y vertices in V and e an edge in E connecting x and y. The graph $G \backslash e$ is obtained by deleting the edge e ($\mathrm{V}(G \backslash e) = \mathrm{V}(G)$, $\mathrm{E}(G \backslash e) = \mathrm{E}(G) \backslash e$). The graph G/e is constructed by identifying x and y and then deleting e. If z is a vertex which is connected to both x and y, then multiple edges will occur between z and the new identified vertex. If there are multiple edges between x and y, then they will become loops on the new identified vertex.

Let M be a symmetric matrix and $x \in \mathrm{V}$; the submatrix $M[x]$ is obtained by deleting the rows and columns indexed by x.

Theorem 1.1. *(Matrix Tree Theorem): Let G be a graph with Laplacian matrix Δ_G. If x is an arbitrary vertex of G, then $\tau(G) = \det \Delta_G[x]$.*

Proof: Induction on the number of edges of G:

- base case: $|\operatorname{E}(G)| = 0$:
  - If $|\operatorname{V}(G)| = 1$, then G has one spanning tree and $\Delta_G[x]$, $x \in \operatorname{V}$, is a 0×0-matrix the empty-matrix, which by convention has determinant 1.
  - If $|\operatorname{V}(G)| > 1$, then G has no spanning tree and $\Delta_G[x]$ is an all-zero matrix of order at least 1 which has determinant 0.
- inductive step: $|\operatorname{E}(G)| > 0$:
  If e is an edge of G, then every spanning tree either contains e or does not contain e. There is a one-to-one correspondence between spanning trees of G that contain e and spanning trees of G/e. On the other hand any spanning tree of G which does not contain e is a spanning tree of $G\backslash e$. Therefore,

  $$\tau(G) = \tau(G/e) + \tau(G\backslash e)$$

  (In this situation multiple edges are retained during contraction and loops are ignored because they cannot occur in a spanning tree).
  Let e be an edge connecting x and y and M be the $n \times n$ diagonal matrix with M_{yy} equal to 1, and all other entries equal to 0. Then

  $$\Delta_G[x] = \Delta_{G\backslash e}[x] + M$$

  from which

  $$\det \Delta_G[x] = \det \Delta_{G\backslash e}[x] + \det \Delta_{G\backslash e}[x][y]$$

  is deduced by substituting the term $(\Delta_G[x])_{yy}$ by $(\Delta_G[x])_{yy} - 1 + 1$ in the Laplace expansion of $\det \Delta_G[x]$. The first two terms of this expression lead to the first term on the right hand side and the third term leads to the second term ($\Delta_{G\backslash e}[x][y] = \Delta_G[x][y]$).
  The graph G/e will be formed by contracting x onto y, so that $\operatorname{V}(G/e) = \operatorname{V}(G)\backslash x$; this preference was chosen because x is the vertex which should be eliminated. Then $\Delta_{G/e}[y]$ has rows and columns indexed by $\operatorname{V}(G)\backslash\{x, y\}$ with the uv-entry being equal to $(\Delta_G)_{uv}$; and so one has that $\Delta_{G/e}[y] = \Delta_G[x][y]$. The formula above can be rewritten as

  $$\det \Delta_G[x] = \det \Delta_{G\backslash e}[x] + \det \Delta_{G/e}[y]\,.$$

  By induction hypothesis, $\det \Delta_{G\backslash e}[x] = \tau(G\backslash e)$ and $\det \Delta_{G/e}[y] = \tau(G/e)$; hence $\det \Delta_G[x] = \tau(G)$. □

If A and B are square $n \times n$ matrices and S a subset of $\{1, \dots, n\}$, then let A_S be the matrix obtained by replacing the rows of A indexed by elements of S with the corresponding rows of B ($A_\emptyset = A$). It can be easily seen that

$$\det(A + B) = \sum_{S \subseteq \{1,\dots,n\}} \det A_S\,,$$

by looking at the Leibniz formula of $\det(A + B)$ and expanding products and reorder sums. By applying this to $A - tI$, it is deduced that the coefficient of t^{n-k} in $\det(A - tI)$ is $(-1)^k$ times the sum of the determinants of the principal $k \times k$ submatrices of A.

Theorem 1.2. *Suppose that G is a finite connected graph with n vertices. Then*

$$\log \tau(G) = -\log(2|\operatorname{E}(G)|) + \sum_{x \in \operatorname{V}(G)} \log \deg_G(x) - \sum_{k \geq 1} \frac{1}{k} \left(\sum_{x \in \operatorname{V}(G)} p^k(x; G) - 1 \right).$$

Proof: First, the following identity will be proved:

$$\tau(G) = \frac{\prod_{x\in \mathrm{V}(G)} \deg_G(x)}{\sum_{x\in \mathrm{V}(G)} \deg_G(x)} \det{}'(I - P_G)\,, \tag{1.1}$$

where $\det' M$ denotes the product of the nonzero eigenvalues of a matrix M. Let $\phi(t) = \det\big((I - P_G) - tI\big)$ be the characteristic polynomial of $(I - P_G)$. The zeros of $\phi(t)$ are the eigenvalues of $I - P_G$. Since $\lambda_1 = 0$, the constant term of the characteristic polynomial is zero and the coefficient of t is

$$(-1)^{n-1} \prod_{i=2}^{n} \lambda_i = (-1)^{n-1} \det{}'(I - P_G)\,.$$

On the other hand, by the remark just above the theorem the coefficient of the linear term in $\phi(t)$ is

$$(-1)^{n-1} \sum_{x\in \mathrm{V}(G)} \det\big((I - P_G)[x]\big)\,.$$

The term $(I - P_G)$ is equal to $D_G^{-1}(D_G - A_G)$ and the fact that D_G^{-1} is a diagonal matrix will be used to get

$$\begin{aligned}\sum_{x\in \mathrm{V}(G)} \det\big((I - P_G)[x]\big) &= \sum_{x\in \mathrm{V}(G)} \det\big(D_G^{-1}(D_G - A_G)[x]\big)\\ &= \sum_{x\in \mathrm{V}(G)} \det\big(D_G^{-1}[x]\big)\det\big((D_G - A_G)[x]\big)\,.\end{aligned}$$

Now using the Matrix-Tree-Theorem 1.1 in the last term yields to

$$\begin{aligned}\sum_{x\in \mathrm{V}(G)} \det\big((I - P_G)[x]\big) &= \tau(G)\cdot \sum_{x\in \mathrm{V}(G)} \det\big(D_G^{-1}[x]\big) = \tau(G)\cdot \sum_{x\in \mathrm{V}(G)} \frac{1}{\prod_{y\in \mathrm{V}(G)\setminus x} \deg_G(y)}\\ &= \tau(G)\cdot \frac{\sum_{x\in \mathrm{V}(G)} \deg_G(x)}{\prod_{x\in \mathrm{V}(G)} \deg_G(x)}\,.\end{aligned}$$

The identity follows immediately from the last equation and

$$\det{}'(I - P_G) = \sum_{x\in \mathrm{V}(G)} \det\big((I - P_G)[x]\big)\,.$$

From (1.1) and the fact that the sum of the degrees of a graph equals twice the number of its edges one gets that

$$\log\tau(G) = -\log\big(2|\,\mathrm{E}(G)|\big) + \sum_{x\in \mathrm{V}(G)} \log\deg_G(x) + \log\det{}'(I - P_G)\,.$$

Let Λ be the multiset of eigenvalues of P_G other than 1 (with multiplicities). Since $\Lambda \subset [-1, 1)$,

$$\log\det{}'(I - P_G) = \sum_{\lambda\in\Lambda} \log(1 - \lambda) = -\sum_{\lambda\in\Lambda}\sum_{k\geq 1} \frac{\lambda^k}{k} = -\sum_{k\geq 1}\sum_{\lambda\in\Lambda} \frac{\lambda^k}{k} = -\sum_{k\geq 1} \frac{\operatorname{tr} P_G^k - 1}{k}\,.$$

In the last step the following facts were used: (1) The eigenvalue 1 of P has multiplicity 1 since G is connected and (2) the trace is similarity invariant. The formula now follows from $\operatorname{tr} P_G^k = \sum_{x\in \mathrm{V}(G)} p^k(x; G)$. □

1.2 Tree Entropy

Reference(s): [7, section 1] and [25, section 3]

Definition. A **graph homomorphism** $\varphi : G_1 \to G_2$ from one graph $G_1 = (\mathrm{V}_1, \mathrm{E}_1)$ to another $G_2 = (\mathrm{V}_2, \mathrm{E}_2)$ is a pair of maps $\varphi_{\mathrm{V}} : \mathrm{V}_1 \to \mathrm{V}_2$ and $\varphi_{\mathrm{E}} : \mathrm{E}_1 \to \mathrm{E}_2$ such that φ_{V} maps the endpoints of e to the endpoints of $\varphi_{\mathrm{E}}(e)$, for every edge $e \in \mathrm{E}_1$. If both maps φ_{V} and φ_{E} are bijections, then φ is called a **graph isomorphism**.

To define the limit of a sequence of graphs it is necessary to keep an eye on a base point, which will be called the root.

Definition. A **rooted graph** (G, o) is a graph G with a distinguished vertex o of G, called the root. A **rooted isomorphism** of rooted graphs is an isomorphism of the underlying graphs that maps the root of one graph to the root of the other graph. Let $\mathcal{G}_\bullet$ denote the set of rooted isomorphism classes of rooted connected locally finite graphs.

Next define the following metric on $\mathcal{G}_\bullet$. Let $(G, o), (G', o') \in \mathcal{G}_\bullet$ and denote, for all $r \in \mathbb{N}$ with $r > 0$, by $B_G(o, r)$ the closed ball of radius r about the root, i.e. the union of all minimal paths on G starting at o of length less or equal to r. Let k be the supremum of all r such that there exists a rooted isomorphism between $B_G(o, r)$ and $B_{G'}(o', r)$. A metric δ on $\mathcal{G}_\bullet$ can now be defined by setting $\delta((G, o), (G', o')) = 2^{-k}$. This metric induces a topology on $\mathcal{G}_\bullet$, which induces a Borel σ-algebra $\mathcal{S}$ on $\mathcal{G}_\bullet$.

Now let R be a positive integer, H be a finite rooted graph, and ρ a probability distribution on $(\mathcal{G}_\bullet, \mathcal{S})$. Then let $p(R, H, \rho)$ denote the probability that H is rooted isomorphic to the ball of radius R about the root of a graph chosen with distribution ρ. If (G, ρ) is a fixed graph with probability distribution ρ on its vertices, then ρ induces naturally a distribution on rooted graphs (also denoted by ρ), i.e. the probability of (G, x) is $\rho(x)$. For a finite graph G, let $U(G)$ denote the distribution of rooted graphs obtained by choosing a uniform random vertex of G as root of G.

The concept of random weak convergence is as follows:

Definition. Let (G_n) be a sequence of finite graphs with distributions $U(G_n)$ and ρ a probability measure on rooted infinite graphs (possibly induced by a probability distribution on the vertices of a fixed infinite graph). If $\lim_{n\to\infty} p\big(R, H, U(G_n)\big) = p(R, H, \rho)$, for any positive integer R and any finite rooted graph H, then ρ is called the **random weak limit** of (G_n). If ρ is induced by a distribution on the vertices of a fixed transitive graph G, then the random weak limit only depends on G and not on the root. In this case the random weak limit of (G_n) will be denoted by G.

Remark 1.3. *The name and concept of this convergence principal become clearer if one compares it with the concept of weak convergence; because the random weak limit ρ can be seen as the law of (G, o) and being the weak limit of the law of (G_n, o_n) as $n \to \infty$. But one should have in mind that random weak convergence on $(\mathcal{G}_\bullet, \mathcal{S})$ is a local concept because only neighbourhoods around the root are regarded. This definition can be generalised by requiring only convergence in probability. The convergence to ρ only depends on the component of the root, so it is wise to demand that ρ concentrates on connected graphs.*

Example. Let Γ be a group with a given generating set S and let $l(\Gamma)$ denote the length of the smallest reduced word in the generating elements that represents the identity. Suppose that Γ_n are finite groups, each generated by s elements, such that $\lim_{n\to\infty} l(\Gamma_n) = \infty$. Then the Cayley graphs G_n of Γ_n have a random weak limit equal to the usual Cayley graph G of the free group Γ on s letters, i.e. the regular tree of degree $2s$.

Definition. The **expected degree** of a probability measure ρ on rooted graphs is given by

$$\overline{\deg}(\rho) := \int \deg_G(x)\, d\rho(G,x)\,.$$

When the following integral converges, define the **tree entropy** of ρ by

$$\mathbf{h}(\rho) := \int \left(\log \deg_G(x) - \sum_{k\geq 1} \frac{p^k(x;G)}{k} \right) d\rho(G,x)\,. \tag{1.2}$$

If ρ is induced by a fixed transitive graph G of degree d, then the tree entropy can be written as

$$\mathbf{h}(G) := \mathbf{h}(\rho) = \log d - \sum_{k\geq 1} \frac{p^k(o;G)}{k}\,,$$

where o is any vertex of G.

The main theorem of this section, which will be proved with the help of several lemmas, suggests thinking of $\mathbf{h}(\rho)$ as a sequence of entropies per vertex of finite graphs.

Theorem 1.4. *If G_n are finite connected graphs with bounded average degree whose random weak limit is a probability measure ρ on infinite rooted graphs, then*

$$\lim_{n\to\infty} \frac{1}{|\operatorname{V}(G_n)|} \log \tau(G_n) = \mathbf{h}(\rho)\,.$$

Lemma 1.5. *Let P be a transition matrix of a Markov chain. For $\alpha \in [0,1)$, define the transition matrix $Q := \alpha I + (1-\alpha)P$. For a state x, let $p^k(x)$ and $q^k(x)$ denote the return probabilities to x after k steps when the Markov chain starts at x, where the transition matrices are P and Q, respectively. Then*

$$\sum_k \frac{q^k(x)}{k} = -\log(1-\alpha) + \sum_k \frac{p^k(x)}{k}\,.$$

Proof: Let $(\cdot,\cdot)$ denote the standard inner product on $\ell^2(\operatorname{V})$. For $x \in \operatorname{V}$ and $z \in (0,1)$, one gets

$$\begin{aligned}
\sum_k \frac{q^k(x)z^k}{k} &= -\big(\big[\log(I - zQ)\big]\mathbf{1}_{\{x\}}, \mathbf{1}_{\{x\}}\big) = -\big(\big[\log\big((1-z\alpha)I - z(1-\alpha)P\big)\big]\mathbf{1}_{\{x\}}, \mathbf{1}_{\{x\}}\big) \\
&= -\left(\left[\log\left(I - \frac{z(1-\alpha)}{1-z\alpha}P\right)\right]\mathbf{1}_{\{x\}}, \mathbf{1}_{\{x\}}\right) - \log(1-z\alpha)(\mathbf{1}_{\{x\}}, \mathbf{1}_{\{x\}}) \\
&= -\log(1-z\alpha) + \sum_k \frac{p^k(x)}{k}\left(\frac{z(1-\alpha)}{1-z\alpha}\right)^k.
\end{aligned}$$

The first equation follows by expressing the logarithm as integral of a geometric series. The lemma follows now by letting $z \uparrow 1$. $\square$

Definition. A Markov chain is said to be **reversible** if there is a measure π such that $\pi(x)p(x,y) = \pi(y)p(y,x)$, for all states x and y.

Lemma 1.6. [25, Lemma 3.4] *Suppose that Q is a transition matrix of a Markov chain that is reversible with respect to a positive finite measure π (π will always be normalised to be a probability measure).*

Setting $a := \inf_x Q(x,x) > 0$ and $c := \inf\{\pi(x)Q(x,y); x \neq y \text{ and } Q(x,y) > 0\} > 0$, then, for all states x and all $k \geq 0$, the following estimate holds

$$\left|\frac{Q^k(x,x)}{\pi(x)} - 1\right| \leq \min\left\{\frac{1}{ac\sqrt{k+1}}, \frac{1}{2a^2c^2(k+1)}\right\}.$$

Lemma 1.7. *Suppose that Y_n are real-valued random variables that converge in distribution to Y and that $\sup_n \mathbf{E}[|Y_n|] < \infty$. Then for all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} |f(x)|/|x| = 0$ one gets $\lim_{n\to\infty} \mathbf{E}[f(Y_n)] = \mathbf{E}[f(Y)]$.*

Proof: The given assumptions imply that $f(Y_n)$ form a uniformly integrable set of random variables. The continuity of f ensures that $f(Y_n)$ converge in distribution to $f(Y)$. The conclusion follows by combining the last two statements. □

Proof of Theorem 1.4: First, new graphs G'_n will be constructed by adding $\deg_G(x)$ loops at the vertex x, for all $x \in G_n$. If P_n is the transition matrix of G_n, then $Q_n := (I + P_n)/2$ is the transition matrix of G'_n. A consequence of this construction is that $Q_n(x,x) > 0$, for every $x \in G_n$ and so $\inf_{x\in \mathrm{V}} Q_n(x,x) > 0$; the latter inequality is a necessary condition for Lemma 1.6. The random weak limit of G'_n is ρ', where ρ' is obtained from ρ by doubling the degree of each vertex by adding loops. Lemma 1.5 implies that $\mathbf{h}(\rho) = \mathbf{h}(\rho')$ by just looking at the definition of $\mathbf{h}$. Since $\tau(G_n) = \tau(G'_n)$, it suffices to show that

$$\lim_{n\to\infty} \frac{1}{|\mathrm{V}(G'_n)|} \log \tau(G'_n) = \mathbf{h}(\rho').$$

Now let d be an upper bound for the average degree of G'_n, i.e. for all n,

$$2|\mathrm{E}(G'_n)| \leq d|\mathrm{V}(G'_n)|, \tag{1.3}$$

so that $|\mathrm{V}(G'_n)|^{-1}\log(2|\mathrm{E}(G'_n)|) \to 0$ as $n \to \infty$. Since the degree of a random vertex in G'_n converges in distribution to the degree of the root under ρ', it follows by Lemma 1.7 (with $f := \log^+$ and Y_n to be the degree of a uniform vertex in G'_n) that

$$\frac{1}{|\mathrm{V}(G'_n)|} \sum_{x\in\mathrm{V}(G'_n)} \log\deg_{G'_n}(x) \to \int \log\deg_G(x)\, d\rho'(G,x).$$

Now looking at the conclusion of Theorem 1.2 the last step is to show that

$$\lim_{n\to\infty} \sum_{k\geq 1} \frac{1}{|\mathrm{V}(G'_n)|} \frac{1}{k} \left(\sum_{x\in\mathrm{V}(G'_n)} p^k(x;G'_n) - 1\right) = \int \sum_{k\geq 1} \frac{1}{k} p^k(x;G'_n)\, d\rho'(G,x).$$

By definition and the requirement of the theorem, for every k, the following equation holds:

$$\lim_{n\to\infty} \frac{1}{|\mathrm{V}(G'_n)|} \left(\sum_{x\in\mathrm{V}(G'_n)} p^k(x;G'_n) - 1\right) = \int p^k(x;G'_n)\, d\rho'(G,x).$$

Hence, by Lemma 1.6 applied to G'_n with stationary probability measure $x \mapsto \deg_{G'_n}(x)/(2|\mathrm{E}(G'_n)|)$ and constants $a \geq 1/2$, $c \geq 1/(4|\mathrm{E}(G'_n)|)$; and by (1.3) one gets

$$\frac{1}{|\mathrm{V}(G'_n)|}\left|\sum_{x\in\mathrm{V}(G'_n)} p^k(x;G'_n) - 1\right| \leq \frac{4d}{\sqrt{k+1}}.$$

Weierstrass's M-test justifies the interchange of limit and summation and so the statement of the theorem is proved. □

1.3 Fuglede-Kadison-Determinant Entropy Formula I

Reference(s): [1, section 5], [2, section 5.2], [11], [25, section 4] and [26]

For a weighted graph some definitions must be extended: Therefor, let $w : \mathrm{E}(G) \to [0, \infty)$ be a weight function, then the new graph Laplacian is defined as follows:

$$\Delta_G(x,y) := \begin{cases} -\sum_{e \in E_1(x,y)} w(e) & \text{if } x \neq y \text{ with } E_1(x,y) := \{e : e \text{ is an edge between } x \text{ and } y\} \\ \sum_{e \in E_2^x} w(e) & \text{if } x = y \text{ with } E_2^x := \{e : e \text{ is incident to } x \text{ and not a loop}\}. \end{cases}$$

Assuming $\Delta_G(x,x) < \infty$ for all x; the associated random walk has the transition probability from x to y of $-\Delta_G(x,y)/\Delta_G(x,x)$. Finally, the tree entropy becomes

$$\mathbf{h}(\rho) := \int \left(\log \Delta_G(o,o) - \sum_{k \geq 1} \frac{p^k(o;G)}{k} \right) d\rho(G,o) \, .$$

Let $\mathcal{G}_*$ denote the set of rooted isomorphism classes of rooted weighted connected locally finite graphs. Let $\tilde{G} = (G, o; v) \in \tilde{\mathcal{G}}_* = \mathcal{G}_* \times \mathrm{V}$ be a **rooted graph with a distinguished directed edge**, where v is a neighbour of the root o of G. Next define the following involution $\iota : \tilde{\mathcal{G}}_* \to \tilde{\mathcal{G}}_*$, which maps $\tilde{G}$ to $\iota(\tilde{G})$; this is the same graph but with reversed directed edge. And so v can be interpreted as new root and o as new distinguished neighbour. Let ρ be any measure on $\mathcal{G}_*$. Now define a new measure $\tilde{\mu}$ on $\tilde{\mathcal{G}}_*$ by (1) taking the marginal measure of $\tilde{\mu}$ on $\mathcal{G}_*$ to be μ and by (2) taking the conditional measure $\tilde{\mu}_G$ on the neighbours of the root of G to be the counting measure.

Definition. A probability measure μ on $\mathcal{G}_*$ is called **unimodular** if the induced measure $\tilde{\mu}$ is involution invariant, i.e.

$$\tilde{\mu}(A) = \tilde{\mu}\big(\iota^{-1}(A)\big) \qquad \text{for all Borel subsets } A \subseteq \tilde{\mathcal{G}}_* \, .$$

If a graph G was randomly chosen, then let $p_G(o)$ be the probability that o is the root of G. The marginal measure $\tilde{\mu}_G$ puts mass $p_G(o)$ on the edge ov and mass $p_G(v)$ on the edge vo. Involution invariance implies that $p_G(o) = p_G(v)$. Clearly, if the graph G is connected, then $p_G(\cdot)$ is constant on G and so the concept of unimodularity means that every vertex is equally likely to be chosen as root.

Theorem 1.8. [1, Proposition 2.2] *Let $\mathcal{G}_{**}$ the space of isomorphism classes of weighted locally finite connected graphs with an ordered pair of distinguished vertices. Let ρ be a probability measure on $\mathcal{G}_*$. The measure ρ is unimodular if and only if*

$$\int \sum_{x \in \mathrm{V}(G)} f(G,o,x) \, d\rho(G,o) = \int \sum_{x \in \mathrm{V}(G)} f(G,x,o) \, d\rho(G,o) \, ,$$

for all measurable functions $f : \mathcal{G}_{**} \to [0,\infty)$ that are invariant under isomorphisms, i.e. $f(\phi(G), \phi(o), \phi(x)) = f(G,o,x)$ for all graph-isomorphisms.

In [2, Chapter 5] or [7, Section 3.2] it was shown that the weak limit of unimodular measures is again a unimodular measure. In the definition of random weak limits only sequences of finite graphs with uniformly distributed roots occur and so random weak limits are always unimodular.

Definition. Let H be a Hilbert space. A set of vectors $(v_i)_{i\in I}$ in H is a **quasi-orthonormal basis** if the non-zero elements of the set form an orthonormal basis of H.

Let $(X, \mathcal{S}, \mu)$ be a measure space. A **Hilbert bundle** over X is a family of Hilbert spaces $(H_x)_{x\in X}$ and a family of maps $v_i : X \to \bigsqcup_{x\in X} H_x$ with $v_i(x) \in H_x$, such that for each $x \in X$ the family $(v_i(x))$ is a quasi-orthonormal basis of H_x, and for each $i \in I$ the set of all $x \in X$ with $v_i(x) = 0$ is measurable.

Definition. A **section** is a map $s : X \to \bigsqcup_{x\in X} H_x$ with $s(x) \in H_x$, for every $x \in X$. A section is a **measurable section** if for every $j \in I$ the function $x \mapsto \langle s(x), v_j(x)\rangle$ is measurable on X, and there exists a countable set $I_s \subset I$, such that the function $x \mapsto \langle s(x), v_i(x)\rangle$ vanishes identically for every $i \notin I_s$. A measurable section s is called a **nullsection** if it vanishes outside a set of measure zero.

The **direct integral** is the vector space of all measurable sections s, which satisfy

$$\|s\|^2 := \int_X \|s(x)\|^2 \, d\mu(x) < \infty \,,$$

modulo the space of nullsections and will be denoted by

$$H = \int_X^{\oplus} H_x \, d\mu(x) \,.$$

One can show that this space is a Hilbert space with inner product

$$\langle s, t\rangle = \int_X \langle s(x), t(x)\rangle \, d\mu(x) \,.$$

Let ρ be a unimodular probability measure on rooted weighted graphs.
Choose the canonical representative for each weighted graph, i.e. a graph with $\mathrm{V}(G) = \mathbb{N}$. Let $H := \int^{\oplus} \ell^2\big(\mathrm{V}(G)\big) \, d\rho(G, o)$ be a direct integral, this is the set of ρ-equivalence classes of ρ measurable functions f defined on canonical rooted graphs (G, o) that satisfy $f_{(G,o)} \in \ell^2\big(\mathrm{V}(G)\big)$ and $\int \|f_{(G,o)}\| \, d\rho(G, o) < \infty$, which will be denoted by $f = \int^{\oplus} f_{(G,o)} \, d\rho(G, o)$. Let $T : (G, o) \mapsto T_{G,o}$ be a measurable mapping of bounded linear operators on $\ell^2\big(\mathrm{V}(G)\big) = \ell^2(\mathbb{N})$ with finite supremum of the norms $\|T_{G,o}\|$. Then T induces a bounded linear operator $T := T^{\rho} := \int^{\oplus} T_{(G,o)} \, d\rho(G, o)$ on H via

$$T^{\rho} : \int^{\oplus} f_{(G,o)} \, d\rho(G, o) \mapsto \int^{\oplus} T_{G,o} f_{(G,o)} \, d\rho(G, o) \,.$$

The norm $\|T^{\rho}\|$ of T^{ρ} is the ρ-essential supremum of $\|T_{G,o}\|$.

A graph isomorphism ϕ induces an operator $\Phi : H \to H$. Since ρ is unimodular one gets

$$\langle \Phi f, \Phi g\rangle = \int \langle \Phi_{(G,o)} f_{(G,o)}, \Phi_{(G,o)} g_{(G,o)}\rangle \, d\rho(G, o) = \int \langle f_{(G,o)}, g_{(G,o)}\rangle \, d\rho(G, o) = \langle f, g\rangle \,.$$

And so Φ is an isometry (and therefore bounded) and clearly the range of Φ is dense in H; whence Φ is unitary. Let **Alg** be the commutant of the set

$$\mathcal{U} = \{\Phi \in B(H) \;\;:\;\; \Phi \text{ is an operator induced by a graph isomorphism } \phi\} \,,$$

i.e.

$$\mathbf{Alg} := \mathcal{U}' = \{T \in B(H) \;\;:\;\; T\Phi = \Phi T \text{ for all } \Phi \in \mathcal{U}\} \,.$$

The set of operators **Alg** coincides with its bi-commutant. Therefore, **Alg** is the von Neumann algebra (i.e. a subalgebra of $B(H)$ which is closed under * and which is equal to its double commutant) of such (ρ-equivalence classes of) operators T that are equivariant in the sense that for all isomorphisms $\varphi : G_1 \to G_2$ and all $o_1, x, y \in \mathrm{V}(G_1)$ and all $o_2 \in \mathrm{V}(G_2)$: $(T_{G_1,o_1}\mathbf{1}_{\{x\}}, \mathbf{1}_{\{y\}}) = (T_{G_2,o_2}\mathbf{1}_{\{\varphi x\}}, \mathbf{1}_{\{\varphi y\}})$. This property induces that $T_{G,o}$ does not depend on the root o.

Theorem 1.9. *Let ρ be a unimodular measure and $T \in$ **Alg**, then*

$$\mathrm{Tr}_\rho(T) := \mathbf{E}\left[(T_G\mathbf{1}_{\{o\}}, \mathbf{1}_{\{o\}})\right] := \int (T_G\mathbf{1}_{\{o\}}, \mathbf{1}_{\{o\}})\, d\rho(G,o)$$

is a trace on **Alg**.

Proof: If $T \in$ **Alg** and $T \geq 0$, then $\mathrm{Tr}(T) \geq 0$ because the integrand is non-negative for $T \geq 0$. The linearity of Tr(.) follows from the linearity of the inner product and the integral. At last $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ must be proved for $S, T \in$ **Alg**:

$$\begin{aligned}\mathrm{Tr}_\rho(TS) &= \mathbf{E}\left[(S\mathbf{1}_{\{o\}}, T^*\mathbf{1}_{\{o\}})\right] = \mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} (S\mathbf{1}_{\{o\}}, \mathbf{1}_{\{x\}})(\mathbf{1}_{\{x\}}, T^*\mathbf{1}_{\{o\}})\right] \\ &= \mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} (S\mathbf{1}_{\{o\}}, \mathbf{1}_{\{x\}})(T\mathbf{1}_{\{x\}}, \mathbf{1}_{\{o\}})\right] = \mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} (S\mathbf{1}_{\{x\}}, \mathbf{1}_{\{o\}})(T\mathbf{1}_{\{o\}}, \mathbf{1}_{\{x\}})\right] \\ &= \mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} (T\mathbf{1}_{\{o\}}, \mathbf{1}_{\{x\}})(\mathbf{1}_{\{x\}}, S^*\mathbf{1}_{\{o\}})\right] = \mathrm{Tr}_\rho(ST)\,.\end{aligned}$$

The unimodularity property of the measure can be used because of absolute integrability:

$$\begin{aligned}\mathbf{E}\left[(TS\mathbf{1}_{\{o\}}, \mathbf{1}_{\{o\}})\right] &\leq \left(\mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} |(S\mathbf{1}_{\{o\}}, \mathbf{1}_{\{x\}})|^2\right] \mathbf{E}\left[\sum_{x\in\mathrm{V}(G)} |(\mathbf{1}_{\{x\}}, T^*\mathbf{1}_{\{o\}})|^2\right]\right)^{1/2} \\ &= \left(\mathbf{E}\left[\|S\mathbf{1}_{\{o\}}\|^2\right] \mathbf{E}[\|T^*\mathbf{1}_{\{o\}}\|^2]\right)^{1/2} \leq \|S\|\|T\| < \infty\end{aligned}$$

Here the identities above and the Cauchy-Schwarz-inequality for expectation values were used. □

Definition. Let VNA be a von Neumann algebra and

$$\mathrm{VNA}_+ := \{T \in \mathrm{VNA} \;\;:\;\; T \text{ is self-adjoint and } T \geq 0\}\,.$$

Let tr be a trace on VNA_+. The trace tr is called **faithful** if $\mathrm{tr}(T) = 0$ implies $T = 0$. The trace is said to be **semifinite** if, for any $0 \neq S \in \mathrm{VNA}_+$, there is a $0 \neq T \in \mathrm{VNA}_+$ with $T \leq S$ such that $\mathrm{tr}(T) < \infty$. If the trace tr fulfils

$$\mathrm{tr}(\sup_i T_i) = \sup_i \mathrm{tr}(T_i)$$

for any bounded increasing net (T_i) of VNA_+, then tr is called a **normal** trace.

A von Neumann algebra with a faithful, finite and normal trace will be called **finite**. Clearly, the von Neumann algebra **Alg** with trace Tr is finite. Next a larger class of operators will be defined and the definition of the trace will be extended to this larger class of operators. Therefor, the following definitions for unbounded operators will be recalled:

Let X, Y be two Hilbert spaces and let $T : X \to Y$ be an unbounded linear operator. The operator T is said to be **densely-defined** if its domain $\mathrm{Dom}(T)$ is a dense subset of X. If the graph of T, this is the set $\mathrm{Graph}(T) := \{(x, Tx) \in X \times Y \ : \ x \in \mathrm{Dom}(T)\}$, is a closed subset of $X \times Y$, then T is called **closed** in X. An operator T' **extends** T, which will be written as $T \subseteq T'$, when $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T')$ and $T'x = Tx$ for all $x \in \mathrm{Dom}(T)$. The operator T is said to be **closable** if the closure of the graph of T is the graph of another operator $\overline{T}$. The operator $\overline{T}$ will be called **closure** of T. A densely defined operator T is **symmetric** if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathrm{Dom}(T)$. The **adjoint** T^* of a densely-defined operator T is defined as follows. Its domain consists of those elements $y \in Y$ such that for some element $z \in X$, $\langle Tx, y\rangle = \langle x, z\rangle$ for all $x \in \mathrm{Dom}(T)$. For these elements $y \in Y$, $T^*y = z$. The operator T is said to be **self-adjoint** if $X = Y$ and $T = T^*$.

Let $\overline{\mathbf{Alg}}$ be the set of closed densely-defined operators that are affiliated with $\mathbf{Alg}$, i.e. those closed densely-defined operators that commute with all unitary operators that commute with $\mathbf{Alg}$.

The Laplacian Δ_G induces an operator

$$f \mapsto \left(x \mapsto \sum_{y \in \mathrm{V}} \Delta_G(x, y) f(y)\right)$$

for functions f on V with finite support. This operator extends by continuity to a bounded linear operator on all of $\ell^2(\mathrm{V})$ when $\sup_x \Delta_G(x, x) < \infty$. If

$$\rho - \operatorname*{ess\,sup}_{G,o} \sup_{x \in \mathrm{V}(G)} \Delta_G(x, x) < \infty\,,$$

then $(G, o) \mapsto \Delta_G$ defines an operator in $\mathbf{Alg}$. If such a uniform bound does not exist, then proceed as follows. Let

$$\mathcal{D}_0 := \{f \in H \ : \ |\mathrm{supp} f_{G,o}| < \infty \ \text{, for all } \ (G, o)\}.$$

The operator Δ is defined on the dense subspace $\mathcal{D}_0$, where it is symmetric. Let D be the diagonal weighted degree operator on $\mathcal{D}_0$, i.e. $D_G(x, x) := \Delta_G(x, x)$ and $D_G(x, y) := 0$ for $x \neq y$. Its closure $\overline{D}$ is symmetric and affiliated with $\mathbf{Alg}$ and so $\overline{D}$ is self-adjoint by the following theorem:

Theorem 1.10. [20, Exercise 6.9.53] and [19, Solution 6.9.53]
Let S be a symmetric operator affiliated with a finite von Neumann algebra. Then S is self-adjoint.

Define $\delta := \overline{D}(I - P)$; since $\overline{D} \in \overline{\mathbf{Alg}}$ and $I - P \in \mathbf{Alg}$, it follows that $\delta \in \overline{\mathbf{Alg}}$. It will be shown that Δ is closable and $\delta = \overline{\Delta}$. An easy calculation shows that δ and Δ agree on $\mathcal{D}_0$ and so δ extends Δ. The operator δ is closed, whence Δ is closable. Since $\overline{\Delta}$ is symmetric and affiliated with $\mathbf{Alg}$, it is self-adjoint by Theorem 1.10. The next theorem guarantees that $\overline{\Delta}$ and δ are equal.

Theorem 1.11. [20, Exercise 6.9.54] and [19, Solution 6.9.54]
Let A and B be operators affiliated with a finite von Neumann algebra. Suppose $A \subseteq B$, then $A = B$.

Since $\mathbf{Alg}$ is a finite von Neumann algebra and $\overline{\Delta} \subseteq \delta$, it follows that $\overline{\Delta} = \delta$. For the remainder of this section D and Δ will also denote the closures of D and Δ.

Definition. Let H be a Hilbert space $B(H)$ the algebra of bounded linear operators on H and $(X, \mathcal{S})$ a measurable space. A **spectral measure** on X is a function $E : \mathcal{S} \to B(H)$ such that

1. $E(S)$ is a projection in $B(H)$, for every $S \in \mathcal{S}$ and $E(\emptyset) = 0$.
2. $E(X) = I$, where I denotes the identity operator in $B(H)$.
3. If $S_1 \cap S_2 = \emptyset$ with $S_1, S_2 \in \mathcal{S}$, then $E(S_1)$ and $E(S_2)$ are orthogonal.
4. $E\big(\bigcup_{n=1}^{\infty} S_n\big) = \sum_{n=1}^{\infty} E(S_n)$ for every sequence (S_n) of disjoint sets in $\mathcal{S}$.

Let $T \in \overline{\mathbf{Alg}}$ be a self-adjoint operator. The operator T has a polar decomposition

$$T = U \int_0^{\infty} \lambda \, dE_{|T|}(\lambda)$$

by [20, Theorem 6.1.11], where $U \in \mathbf{Alg}$ is unitary and $E_{|T|}$ is a spectral measure. Define the measure $\mu_{\rho,|T|}$ by

$$\mu_{\rho,|T|}(B) := \mathrm{Tr}_{\rho}\left(E_{|T|}(B)\right)$$

for Borel subsets $S \subseteq \mathbb{R}$. The trace Tr will be extended by

$$\mathrm{Tr}_{\rho}(T) := \int_0^{\infty} \lambda \, d\mu_{\rho,|T|}(\lambda)$$

for positive operators $T \in \overline{\mathbf{Alg}}$ and then by linearity to all of $\overline{\mathbf{Alg}}$ when it makes sense.

Write $\widetilde{\mathbf{Alg}}$ for the set of $T \in \overline{\mathbf{Alg}}$ for which

$$\mathrm{Tr}_{\rho}(\log^{+}|T|) = \int_0^{\infty} \log^{+} \lambda \, d\mu_{\rho,|T|}(\lambda) < \infty \, .$$

For $T \in \widetilde{\mathbf{Alg}}$, define its **Fuglede-Kadison determinant** by

$$\mathrm{Det}(T) := \mathrm{Det}_{\rho}(T) := \exp \int_0^{\infty} \log \lambda \, d\mu_{\rho,|T|}(\lambda) \in [0, \infty) \, .$$

Theorem 1.12. *If ρ is a unimodular probability measure on rooted connected infinite weighted graphs with*

$$\int \log D_G(o,o) \, d\rho(G,o) \in [-\infty, \infty) \, , \tag{1.4}$$

then

$$\mathbf{h}(\rho) = \log \mathrm{Det}\Delta \in [-\infty, \infty) \, . \tag{1.5}$$

Proof: The hypothesis is equivalent to $D \in \widetilde{\mathbf{Alg}}$. Since $I - P \in \mathbf{Alg} \subseteq \widetilde{\mathbf{Alg}}$, it follows that $\Delta = D(I-P) \in \widetilde{\mathbf{Alg}}$ with

$$\mathrm{Det}\Delta = \mathrm{Det}D \cdot \mathrm{Det}(I-P) \tag{1.6}$$

by [17, Proposition 2.5]. By the assumption (1.4) of the Theorem it follows that

$$\mathrm{Det}D = \exp \int \log D_G(o,o) \, d\rho(G,o) \, . \tag{1.7}$$

Next define the measure topology of a von Neumann algebra $\overline{\mathrm{VNA}}$ to be the topology whose fundamental system of neighbourhoods around 0 is given by

$$V(\varepsilon,\delta) = \{T \in \overline{\mathrm{VNA}} \, : \, \text{there exists a projection } E \in \mathbf{Alg} \text{ s.t. } \|TE\| \le \varepsilon \text{ and } \mathrm{Tr}(1-E) \le \delta\} \, ,$$

where ε, δ run over all strictly positive numbers. Since $\|P\| \le 1$, one gets for $0 < c < 1$ that $\log|I - cP| \le (\log 2)I$ and so $|I - cP|$ converges to $|I - P|$ in the strong operator topology as

$c \uparrow 1$. Therefore $\log |I - cP|$ will converge to $\log |I - P|$ in the measure topology. Thus,

$$\mathrm{Det}(I-P) = \lim_{c\uparrow 1} \mathrm{Det}(I-cP)$$

by the generalised Monotone Convergence Theorem [13, Theorem 3.5]. For $0 < c < 1$, one has

$$\log \mathrm{Det}(I-cP) = \Re\, \mathrm{Tr} \log(I-cP)$$

by [15, Theorem 1] and

$$\log(I-cP) = -\sum_{k\geq 1} c^k P^k/k$$

(in the norm topology). Hence,

$$\log \mathrm{Det}(I-cP) = -\sum_{k\geq 1} \Re\, \mathrm{Tr}_\rho\, c^k P^k/k = -\sum_{k\geq 1} \mathrm{Tr}_\rho\, c^k P^k/k\,,$$

whose limit as $c \uparrow 1$ is

$$-\sum_{k\geq 1} \mathrm{Tr}_\rho\, P^k/k = \int -\sum_{k\geq 1} \frac{1}{k} p_k(o;G)\, d\rho(G,o) \tag{1.8}$$

by the Monotone Convergence Theorem. The entropy formula (1.5) follows now by comparing (1.2) with (1.6), (1.7) and (1.8) $\square$

The concept of an infinite periodic graph is the following: Let $K = \{0, 1, \ldots, k\}$ be a finite set and G be a graph with vertex set $\mathbb{Z}^d \times K$ and with edge set that is invariant under the natural action of $\mathbb{Z}^d$. That is, $(\mathbf{x}, i) \sim (\mathbf{y}, j)$ if and only if $(\mathbf{0}, i) \sim (\mathbf{y} - \mathbf{x}, j)$. This construction leads to the following simplification of the graph-Laplacian, i.e. for each $\mathbf{x} \in \mathbb{Z}^d$ there is a $|K| \times |K|$ matrix $L^{\mathbf{x}}$ such that

$$\Delta_G\big((\mathbf{x},u),(\mathbf{y},v)\big) = L^{\mathbf{y}-\mathbf{x}}(u,v)\,,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ and $u, v \in K$. Let ρ be a measure that puts equal mass on each rooted graph $(G, (o, u))$, with $u \in K$. Consider Δ_G as operating on $\ell^2(\mathbb{Z}^d \times K)$ as an operator T on $\ell^2\big(\mathbb{Z}^d; \ell^2(K)\big)$. The space $\ell^2\big(\mathbb{Z}^d; \ell^2(K)\big)$ is isometrically isomorphic to $L^2\big([0,1]^d; \ell^2(K)\big)$ via the Fourier transformation. The Fourier transform of the operator T becomes the matrix-valued function

$$M : (s_1, s_2, \ldots, s_d) \mapsto \sum_{\mathbf{x}\in\mathbb{Z}^d} L^{\mathbf{x}} \exp^{2\pi \mathbf{x}\cdot\mathbf{s}} \qquad \big(\mathbf{s} = (s_1, s_2, \ldots, s_d) \in [0,1]^d\big)$$

and the vector $\mathbf{1}_{\{(o,u)\}}$ becomes the function $\mathbf{1}_{\{u\}} \in \ell^2(K)$. Finally, the entropy can be rewritten as

$$\begin{aligned}\mathbf{h}(\rho) &= \frac{1}{|K|} \sum_{u\in K} \big((\log \Delta_G)\mathbf{1}_{\{(o,u)\}}, \mathbf{1}_{\{(o,u)\}}\big) = \frac{1}{|K|} \sum_{u\in K} \int_{[0,1]^d} \Big(\big(\log M(\mathbf{s})\big)\mathbf{1}_{\{u\}}, \mathbf{1}_{\{u\}}\Big)\, d\mathbf{s} \\ &= \frac{1}{|K|} \int_{[0,1]^d} \mathrm{tr}\,\big(\log M(\mathbf{s})\big)\, d\mathbf{s} = \frac{1}{|K|} \int_{[0,1]^d} \log \det M(\mathbf{s})\, d\mathbf{s}\,.\end{aligned}$$

The final part of this chapter is the calculation of the classical tree entropy of the nearest-neighbour graph and the triangle lattice graph($d = 2$) with the method above.

Example. Let G be the nearest neighbour graph on $\mathbb{Z}^2$ and $K = \{0\}$, then M becomes a 1×1

operator. When perceiving $L^{\mathbf{x}}$ as $L^{\mathbf{x}-\mathbf{0}}$ one easily gets:

$$M(\mathbf{s}) = 4 - \exp^{2\pi s_1} - \exp^{-2\pi s_1} - \exp^{2\pi s_2} - \exp^{-2\pi s_2}$$

and

$$\mathbf{h}(\rho) = \mathbf{h}(\mathbb{Z}^2) = \int_{[0,1]^2} \log\big(4 - 2\cos(2\pi s_1) - 2\cos(2\pi s_2)\big)\, ds_1\, ds_2\,.$$

Example. Let G be the triangle lattice and $K = \{0\}$. This graph does not fulfil the conditions of an infinite periodic graph but the graph is isomorphic to the nearest neighbour graph with additional edges connecting a vertex with the nearest vertices which lie in northeast and southwest direction. And so one gets,

$$M(\mathbf{s}) = 6 - 2\cos(2\pi s_1) - 2\cos(2\pi s_2) - 2\cos\big(2\pi(s_1 + s_2)\big)$$

and

$$\mathbf{h}(\rho) = \int_{[0,1]^2} \log\Big(6 - 2\cos(2\pi s_1) - 2\cos(2\pi s_2) - 2\cos\big(2\pi(s_1 + s_2)\big)\Big)\, ds_1\, ds_2\,.$$

2 Dynamical Systems That Are Related To Spanning Trees

In this chapter the essential spanning forest process, the dimer model and the abelian sandpile model will be introduced. These models are related to spanning trees or are connected by isomomorphisms. Therefore, the detailed treatment of spanning trees in the first chapter will ease the elaboration of these dynamical systems.

2.1 The Essential Spanning Forest Process

Reference(s): [25, section 5] and [27]

2.1.1 Uniform Spanning Forest Measures

In this section two probability measures will be introduced which arise as weak limit of measures of finite graphs; and some of their properties will be stated.

Given any graph $G = (\mathrm{V}, \mathrm{E})$, let $\{0,1\}^{\mathrm{E}}$ denote the measurable space of all subsets of E with the Borel σ-field that is generated by sets of the form $\{F \subseteq \mathrm{E} : e \in F\}$, with $e \in \mathrm{E}$. An elementary cylinder is an event $A \subseteq \{0,1\}^{\mathrm{E}}$ of the form

$$A = \left\{ \mathfrak{F} \in \{0,1\}^{\mathrm{E}} \ : \ B_1 \subseteq \mathfrak{F},\, B_2 \cap \mathfrak{F} = \emptyset \right\},$$

where $B_1, B_2 \subset \mathrm{E}$ are finite disjoint sets. A cylinder event is a finite union of elementary cylinders. A sequence of measures μ_n on the Borel σ-algebra converges weakly to μ if $\mu_n(C) \to \mu(C)$, for every cylinder set C. Let $G = (\mathrm{V}, \mathrm{E})$ be an infinite connected graph. Let $\mathrm{V}_1, \mathrm{V}_2, \dots$ be finite connected subsets of V with $\bigcup_{n=1}^{\infty} \mathrm{V}_n = \mathrm{V}$. Let $G_n = (\mathrm{V}_n, \mathrm{E}_n)$ be the subgraph spanned by V_n, i.e. an edge of G appears in $\mathrm{E}_n \in \mathrm{E}$ if its endpoints are in V_n. A sequence of such graphs $\langle G_n \rangle$ is called an **exhaustion** of G.

Let μ_n be the **uniform spanning tree probability measure** on G_n; that is the measure which puts equal mass to every spanning tree in G_n. Since G_n is finite there are only finitely many spanning trees. Given a finite set B of edges with $B \subseteq \mathrm{E}_n$, for n large enough. The limit $\lim_{n\to\infty} \mu_n(B \subseteq T) = \mu(B \subseteq T)$ exists because $\mu_n(B \subseteq T)$ is a decreasing sequence in n by Rayleigh‘s monotonicity principle [27, Chapter 2 and Chapter 10]. It follows from the inclusion-exclusion principle

$$\begin{aligned} \mu(B_1 \subseteq \mathfrak{F}, B_2 \cap \mathfrak{F} = \emptyset) &:= \sum_{S \subset B_2} \mu(B_1 \cup S \subseteq \mathfrak{F})(-1)^{|S|} = \sum_{S \subset B_2} \lim_{n\to\infty} \mu_n(B_1 \cup S \subseteq \mathfrak{F})(-1)^{|S|} \\ &= \lim_{n\to\infty} \mu_n(B_1 \subseteq T, B_2 \cap T = \emptyset) \end{aligned}$$

that μ is defined on all elementary cylinders and therefore for cylinder events and hence uniquely defines a probability measure μ on $\{0,1\}^{\mathrm{E}}$. This measure will be denoted by FSF and will be called **free uniform spanning forest measure** (the name will be clear later).

An alternative approach will be made because boundary conditions could have an effect on the connections within G_n. This problem will be avoided by forcing all connections outside of G_n.

Therefor, let G_n^* be the graph obtained from G by contracting the vertices outside G_n to a single vertex z_n. The vertex z_n will be called root, too, but this time with the above interpretation. Let μ_n^* be the random spanning tree measure on G_n^*. The limiting probability measure WSF does not depend on the exhaustion and it will be called the **wired spanning forest measure** (because the boundary is "wired" together).

Both FSF and WSF are invariant under any automorphisms that G may have and they are concentrated on the set of **essential** spanning forests of G; that are those spanning forests whose components are infinite trees.

An **end** of an infinite graph G is an equivalence class of infinite simple paths in G, where two paths are equivalent if, for every finite subgraph $K \subset G$, there is a connected component of $G \backslash K$ that intersects both paths.

Let G be a graph. For a subgraph H, let its (internal) vertex boundary $\partial_{\mathrm{V}} H$ be the set of vertices of H that are adjacent to some vertex not in H. The graph G is **amenable** if there is an exhaustion $G_1 \subset G_2 \subset \ldots$ with $\bigcup_n G_n = G$ and

$$\lim_{n\to\infty} \frac{|\partial_{\mathrm{V}} G_n|}{|\,\mathrm{V}(G_n)|} = 0\,,$$

where $|.|$ is the counting measure. Such an exhaustion (or the sequence of its vertex sets) is called a **Følner sequence**.

The group of all automorphisms of G will be denoted by $\mathrm{Aut}(G)$. The action of a group Γ on a graph G by automorphisms is said to be **transitive** if there is only one Γ-orbit in $\mathrm{V}(G)$ and to be **quasi-transitive** if there are only finitely many orbits in $\mathrm{V}(G)$. A graph G is transitive or quasi-transitive according as whether the corresponding action of $\mathrm{Aut}(G)$ is. A locally compact group is called **unimodular** if its left Haar measure is also right invariant. A graph G is unimodular if $\mathrm{Aut}(G)$ is unimodular, where the weak topology of $\mathrm{Aut}(G)$ is generated by its action on G.

Suppose that the finite group $\Gamma \in \mathrm{Aut}(G)$ acts on a countable or a finite vertex set of a graph and preserves the measure μ. Then the entropy of the pair (μ, Γ) is

$$\mathbf{H}(\mu,\Gamma) := |\Gamma|^{-1}\,\mathbf{H}(\mu)\,.$$

A countable locally compact group Γ with Haar measure $|.|$ is called amenable if there is a sequence of finite sets $\langle \Gamma_n \rangle$ with

$$\lim_{n\to\infty} \frac{|\gamma\Gamma_n \Delta \Gamma_n|}{|\Gamma_n|} = 0\,,$$

for all $\gamma \in \Gamma$; the sequence $\langle \Gamma_n \rangle$ will be called Følner sequence. Suppose Γ is a countable amenable finitely generated subgroup of $\mathrm{Aut}(G)$. Let μ be a probability measure on $\{0,1\}^{\mathrm{E}}$ that is preserved by Γ. Let $\langle \Gamma_n \rangle$ be a Følner sequence in Γ and H be a finite subgraph of G such that $\Gamma H = G$, provided such an H exists. Then the **metric-entropy** of the pair (μ, Γ), also called the **Γ-entropy** of μ, is

$$\mathbf{H}(\mu,\Gamma) := \lim_{n\to\infty} |\Gamma_n|^{-1}\,\mathbf{H}(\mu \downharpoonleft \Gamma_n H)\,,$$

where $\mathbf{H}(\mu \downharpoonleft (\Gamma_n H))$ is the restriction of μ to the σ-field generated by the restriction $\mathrm{E}(\Gamma_n H)$. Let X be a closed subgroup of $\mathrm{Aut}(G)$. The **stabiliser** of an element $g \in G$ is the subgroup given by $S(g) := \{x \in X : xg = g\}$. The action of a group is called **free** if the stabiliser of each element of the set is just the identity element of the group.

Lemma 2.1. [6, 18, 25] *For every quasi-transitive amenable graph G,* $\mathrm{FSF}_G = \mathrm{WSF}_G$.

Lemma 2.2. [25] *Let G be an infinite quasi-transitive unimodular connected graph. If G has 2 ends, then $\mathfrak{F}$ is a tree with exactly 2 ends* WSF*-a.s., while otherwise, for* WSF*-a.e. $\mathfrak{F}$, each component tree of $\mathfrak{F}$ has exactly one end.*

Remark 2.3. [6, 29] *Since the most examples of the following chapters live on* $\mathbb{Z}^d$ *some concrete results of graphs on* $\mathbb{Z}^d$ *will be stated: The uniform spanning forest (*USF*) has no cycles* WSF*-a.s. If* $d \leq 4$ *the* USF *is a single tree a.s.. For* $2 \leq d \leq 4$ *the* USF *has one end a.s. If* $d > 4$ *then a.s. the* USF *has infinitely many components, each component is infinite and has a single end.*

2.1.2 The Entropy Of The Essential Spanning Forest Process

Let $\mathbb{F} \subset \{0,1\}^{\mathrm{E}(G)}$ be the set of all essential spanning forests of G and $\Gamma \subseteq \mathrm{Aut}(G)$ a countable group acting freely on $\mathrm{V}(G)$ with a finite number of orbits I. The set $\mathbb{F}$ is closed, compact and Γ-invariant. The dynamical system $(\mathbb{F}, \Gamma)$ is called the **essential spanning forest process** and has entropy $\mathbf{H}(\mathrm{WSF}_G, \Gamma)$. In this section a coincidence between this Γ-entropy and the tree entropy will be shown.

Suppose that G is an infinite quasi-transitive amenable connected graph. Let $x \in \mathrm{V}$ and $\Gamma \subseteq \mathrm{Aut}(G)$, then the orbit of x is given by $\Gamma x = \{\gamma x : \gamma \in \mathrm{Aut}(G)\}$. Choose a complete set $\mathfrak{O} = \{o_1, o_2, \ldots, o_L\}$ of representatives in V of the orbits of G. Let $\mu_i := |S(o_i)|$ and normalize the Haar measure $|\cdot|$, i.e. the cardinality of a set, such that $\rho(\mathfrak{O}) = \sum_i \mu_i^{-1} = 1$, where $\rho(o_i) = \mu_i^{-1}$. By [5, Proposition 3.6] for any Følner sequence $\langle H_n \rangle$ and all i:

$$\lim_{n\to\infty} \frac{|\Gamma o_i \cap H_n|}{|H_n|} = \mu_i^{-1}\,.$$

So the relative frequency of vertices in H_n that are in the same orbit o_i converges to $\rho(o_i)$. The measure ρ is called the **natural frequency distribution** of G.

Theorem 2.4. *Let G be an infinite quasi-transitive amenable connected graph with natural frequency distribution ρ. Let G_n be a Følner sequence of finite connected subgraphs of G. Then*

$$\lim_{n\to\infty} \frac{1}{|\mathrm{V}(G_n)|} \log \tau(G_n) = \sum_{x \in \mathrm{V}(G)} \rho(x) \log \deg_G(x) - \sum_{k \geq 1} \frac{1}{k} \sum_{x \in \mathrm{V}(G)} \rho(x) p^k(x; G) = \mathbf{h}(G, \rho)\,. \tag{2.1}$$

If $\Gamma \subseteq \mathrm{Aut}(G)$ *is a countable group acting freely on* $\mathrm{V}(G)$ *with a finite number I of orbits, then*

$$\mathbf{H}(\mathrm{WSF}_G, \Gamma) = I\, \mathbf{h}(G, \rho)\,. \tag{2.2}$$

Given a finite subgraph H of a graph G and a configuration ω of $\mathrm{E}(G)$, let $H(\omega)$ denote the cylinder event consisting of those configurations of $\mathrm{E}(G)$ that agree with ω on $\mathrm{E}(H)$. Now let ω be a configuration of $\mathrm{E}(G) \setminus \mathrm{E}(H)$, then two finite graphs from certain vertex identifications on H are defined as follows: (1) $H \circ \omega$ for the graph obtained by identifying all vertices of H that are connected to each other in the graph $(\mathrm{V}(G), \omega)$; (2) and let $H * \omega$ be the graph obtained by identifying all vertices of H that are connected to each other in the graph $(\mathrm{V}(G), \omega)$ and by identifying all vertices of H that belong to any infinite connected component in $(\mathrm{V}(G), \omega)$.

Lemma 2.5. *Let G be an infinite quasi-transitive unimodular connected graph and let H be a finite connected subgraph of G. If G has 2 ends, then*

$$\mathrm{WSF}\left(H(\mathfrak{F}) \,\middle|\, \mathfrak{F} \downharpoonleft (\mathrm{E} \setminus \mathrm{E}(H))\right) = \tau\Big(H \circ (\mathfrak{F} \downharpoonleft (\mathrm{E} \setminus \mathrm{E}(H)))\Big)^{-1} \qquad \mathrm{WSF}\text{-}a.s,$$

while otherwise,

$$\mathrm{WSF}\left(H(\mathfrak{F}) \,\middle|\, \mathfrak{F} \downharpoonleft (\mathrm{E} \setminus \mathrm{E}(H))\right) = \tau\Big(H * (\mathfrak{F} \downharpoonleft (\mathrm{E} \setminus \mathrm{E}(H)))\Big)^{-1} \qquad \mathrm{WSF}\text{-}a.s.$$

Proof: The proof will only be given for the case when G has other than 2-ends. Let Z be the event that each tree of $\mathfrak{F}$ has exactly one end, Lemma 2.2 says that $\mathrm{WSF}(Z) = 1$. Let B_R be a ball of radius R about some fixed vertex of G. Choose R_H so that $H \subset B_{R_H}$. Let A_R be the following event: for all $x, y \in \partial_{\mathrm{V}} H$ and $z, w \in \partial_{\mathrm{V}} B_R$ if in $\mathfrak{F} \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big)$ x is connected to z, y is connected to w and x is not connected to w, then x and y are not connected in $\mathfrak{F} \downharpoonleft \mathrm{E}(H)$. Thus, $A_R \subseteq A_{R+1}$ for all $R \geq R_H$ and

$$Z \subseteq \bigcup_R A_R$$

whence $\lim_{R\to\infty} \mathrm{WSF}(A_R) = 1$. Define

$$C_R := \left\{ \mathfrak{F} \;\middle|\; \mathrm{WSF}(A_R \mid \mathfrak{F} \downharpoonleft (\mathrm{E}(B_R) \setminus \mathrm{E}(H))) \geq \frac{1 - \sqrt{1 - \mathrm{WSF}(A_R)}}{(\mathrm{WSF}(\mathfrak{F} \downharpoonleft (\mathrm{E}(B_R) \setminus \mathrm{E}(H))))^2} \right\},$$

and since

$$\mathrm{WSF}(A_R) = \int \mathrm{WSF}\left(A_R \;\middle|\; \mathfrak{F} \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big) \right) d\,\mathrm{WSF}\,,$$

it follows (proof by contradiction and using Chebyshev's inequality) that, for all large R,

$$\mathrm{WSF}(C_R) \geq 1 - \sqrt{1 - \mathrm{WSF}(A_R)}\,.$$

In particular, $\mathrm{WSF}(\limsup_{R\to\infty} C_R) = 1$. By definition,

$$\mathrm{WSF}\left(H(\mathfrak{F}) \;\middle|\; \mathfrak{F} \downharpoonleft \big(\mathrm{E} \setminus \mathrm{E}(H)\big) \right) = \lim_{R\to\infty} \mathrm{WSF}\left(H(\mathfrak{F}) \;\middle|\; \mathfrak{F} \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big) \right) \quad \mathrm{WSF}\text{-a.s.}$$

Fix a forest $\omega \in Z \cap \limsup_{R\to\infty} C_R$ for which the limit above holds. Choose $\varepsilon > 0$ arbitrarily small. Choose $R \geq R_H$ so large that

$$\left| \mathrm{WSF}\left(H(\omega) \;\middle|\; \omega \downharpoonleft \big(\mathrm{E} \setminus \mathrm{E}(H)\big) \right) - \mathrm{WSF}\left(H(\omega) \;\middle|\; \omega \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big) \right) \right| < \varepsilon\,, \tag{2.3}$$

that $\omega \in C_R$, that $\sqrt{1 - \mathrm{WSF}(A_R)} < \varepsilon$, and that each vertex in $\partial_{\mathrm{V}} H$ that is connected in $\omega \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big)$ to $\partial_{\mathrm{V}} B_R$ belongs to an infinite component in $\omega \downharpoonleft \big(\mathrm{E} \setminus \mathrm{E}(H)\big)$. This last requirement, in combination with $\omega \in Z$, implies that $\omega \in A_R$. Consider the cylinder set

$$D := \big(B_R \setminus \mathrm{E}(H)\big)(\omega) = \left\{ \mathfrak{F} \;\middle|\; \mathfrak{F} \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big) = \omega \downharpoonleft \big(\mathrm{E}(B_R) \setminus \mathrm{E}(H)\big) \right\}.$$

Let μ_N be the uniform spanning tree measure on B_N^*. By definition,

$$\mathrm{WSF}\big(H(\omega) \mid D\big) = \lim_{N\to\infty} \mu_N\big(H(\omega) \mid D\big)$$

and

$$\mathrm{WSF}(A_R \mid D) = \lim_{N\to\infty} \mu_N(A_R \mid D)\,.$$

Since $\omega \in C_R$ and $\sqrt{1 - \mathrm{WSF}(A_R)} < \varepsilon$, it is clear that $\mathrm{WSF}(A_R \mid D) > \frac{1-\varepsilon}{\mathrm{WSF}(D)^2} \geq 1 - \varepsilon$. Let $N > R$ be so large that

$$\big| \mathrm{WSF}\big(H(\omega) \mid D\big) - \lim_{N\to\infty} \mu_N\big(H(\omega) \mid D\big) \big| < \varepsilon$$

and $\mu_N(A_R \mid D) > 1-\varepsilon$. Since

$$\mu_N\big(H(\omega) \mid D\big) = \mu_N(A_R \mid D)\mu_N\big(H(\omega)|A_R \cap D\big) + \mu_N(A_R^C \mid D)\mu_N\big(H(\omega)|A_R^C \cap D\big)\,,$$

and so one has

$$\big|\mu_N\big(H(\omega) \mid D\big) - \mu_N\big(H(\omega) \mid A_R \cap D\big)\big| < 2\varepsilon\,.$$

Given $A_R \cap D$, the configurations inside H and outside B_R are μ_N-independent. Since $\omega \in A_R$, it follows that

$$\mu_N\big(H(\omega) \mid A_R \cap D\big) = \tau(H * \omega)^{-1}\,,$$

and so

$$\big|\,\mathrm{WSF}_N\,\big(H(\omega) \mid D\big) - \tau(H*\omega)^{-1}\big| < 3\varepsilon\,.$$

Therefore,

$$\left|\mathrm{WSF}\left(H\left(\omega\right)\,\middle|\,\omega \downharpoonright \big(\mathrm{E} \setminus \mathrm{E}\left(H\right)\big)\right) - \tau\left(H*\omega\right)^{-1}\right| < 4\varepsilon$$

by (2.3). Since ε is arbitrary and ω is an arbitrary element of a set of measure 1, the result follows. $\square$

Definition. Let A and B be random variables with state spaces $\mathcal{A}$ and $\mathcal{B}$ and joint distribution $\mu(a,b)$. The **joint entropy** of A and B is defined by

$$\mathbf{H}(A,B) := -\sum_{a\in\mathcal{A}}\sum_{b\in\mathcal{B}} \mu(a,b)\log\mu(a,b)\,.$$

Lemma 2.6. *Let Y be a finite set and m be a positive integer. Write $\alpha := m/\left|Y\right|$. Suppose that μ is a probability measure on $\{0,1\}^Y \times \{0,1\}^Y$ that is supported on the set of pairs (ω_1,ω_2) with $|\omega_1 \Delta \omega_2| \le m$. Let μ_1 and μ_2 be the coordinate marginals of μ. Then*

$$\big|\,\mathbf{H}(\mu_1) - \mathbf{H}(\mu_2)\big| \le \log\sum_{k=0}^{m}\binom{|Y|}{k} \le |Y|\big(-\alpha\log\alpha - (1-\alpha)\log(1-\alpha)\big)\,.$$

The proof mainly follows [10, Lemma 6.2].

Proof: Let A and B be binary random vectors with distribution μ_1 respectively μ_2. Define the random vector C with $C_i := \mathbf{1}_{\{A_i \ne B_i\}}$ and distribution μ_3 which puts equal mass to every allowed state of C. Then $\mathbf{H}(A) \le \mathbf{H}(A,C) = \mathbf{H}(B,C) \le \mathbf{H}(B) + \mathbf{H}(C)$. By symmetry $\big|\mathbf{H}(A) - \mathbf{H}(B)\big| \le \mathbf{H}(C)$. By definition of μ_3, $\mathbf{H}(C) = \log\sum_{k=0}^{m}\binom{|Y|}{k}$. For the last inequality see [8, Bollobas p.11]. $\square$

Lemma 2.7. *Let H be a finite connected graph and W be a subset of vertices of H. Let H' be any graph obtained from H by making certain identifications of the vertices in W with each other. When $\alpha := (|W|-1)/|\,\mathrm{E}(H)|$, then*

$$|\log\tau(H) - \log\tau(H')| \le |\,\mathrm{E}(H)|\big(-\alpha\log\alpha - (1-\alpha)\log(1-\alpha)\big)\,.$$

Proof: Let μ and μ' be the uniform spanning tree measure on H respectively H'. It follows from Feder and Mihail [14] that μ stochastically dominates μ'. By Strassen's theorem [36] this means that there is a probability measure on pairs (T,T') such that the law of T is μ, the law of T' is μ' and $T \supseteq T'$ a.s. In [27, Exercise 4.5., Proposition 4.5., Lemma 10.3., Exercise 10.7. and Theorem 10.4.] the results of the cites above are proved in the setting of graph theory. Now

$|\mathrm{E}(T)| = |\mathrm{V}(H)| - 1$ and $|\mathrm{E}(T')| = |\mathrm{V}(H')| - 1$. It follows that a.s.

$$|\,\mathrm{E}(T)\Delta\,\mathrm{E}(T')| = |\,\mathrm{V}(H)| - |\,\mathrm{V}(H')| \le |W| - 1\,.$$

Lemma 2.6 deduces that

$$|\,\mathbf{H}(\mu) - \mathbf{H}(\mu')| \le |\,\mathrm{E}(H)|\big(-\alpha\log\alpha - (1-\alpha)\log(1-\alpha)\big)\,.$$

The inequality now follows from $\mathbf{H}(\mu) = \log\tau(H)$ and $\mathbf{H}(\mu') = \log\tau(H')$. □

Lemma 2.8. *Let G be an infinite quasi-transitive unimodular connected graph and H be a finite connected subgraph of G. Write $\alpha := (|\partial_{\mathrm{V}} H| - 1)\,/\,|\mathrm{E}\,(H)|$. For* WSF*-a.e. $\mathfrak{F}$ one gets*

$$\big|\log\mathrm{WSF}\,\big(H(\mathfrak{F})\big) - \log\tau(H)^{-1}\big| \le |\,\mathrm{E}(H)|\big(-\alpha\log\alpha - (1-\alpha)\log(1-\alpha)\big)\,.$$

This result follows from Lemma 2.5 and Lemma 2.7. For details consult [25, Lemma 5.5].

Proof of 2.4: By definition of ρ, the graphs G_n have a random weak limit $(G;\rho)$. And (2.1) is a consequence of 1.4. To prove (2.2) choose a ball $B_R(o)$ of vertices and edges such that $\Gamma B_R(o) = G$. Let

$$\Gamma_n := \big\{\gamma\in\Gamma \;\big|\; \gamma B_R(o)\cap G_n \neq \emptyset\big\}$$

and put

$$G'_n := \Gamma_n B_R(o)\,.$$

Since $\langle G_n\rangle$ is a Følner sequence in G, it follows that $\langle\Gamma_n\rangle$ is a Følner sequence in Γ. Therefore,

$$\mathbf{H}(\mathrm{WSF},\Gamma) = -\lim_{n\to\infty}|\Gamma_n|^{-1}\log\mathrm{WSF}\,\big(G'_n(\mathfrak{F})\big)$$

in $L^1(\mathrm{WSF})$ by the generalised Shannon-McMillan Theorem of Kieffer [22]. Since Γ acts freely on V one gets

$$\lim_{n\to\infty}|\,\mathrm{V}(G'_n)|/|\Gamma_n| = I\,.$$

Hence,

$$\mathbf{H}(\mathrm{WSF},\Gamma) = -\lim_{n\to\infty} I|\,\mathrm{V}(G'_n)|^{-1}\log\mathrm{WSF}\,\big(G'_n(\mathfrak{F})\big) \tag{2.4}$$

in $L^1(\mathrm{WSF})$. Every quasi-transitive amenable graph is unimodular (see [31, 35] or [5]). The result now follows from Lemma 2.8 together with (2.4). □

Remark 2.9. *Sheffield [34] showed that* WSF *is the unique measure of maximal entropy.*

2.2 Dimer Model

Reference(s): [29] and [10, section 7]

A **dimer cover** or **domino tiling** or **one factor** of a graph G is a partition of the vertices into sets of size 2, where each set contains two adjacent vertices. From another point of view, a dimer covering is a subset of edges, which covers every vertex exactly once, i.e. every vertex is endpoint of exactly one edge.

2.2.1 Connection Between ESFP And Dimer Covers

Dual graphs are necessary to develop a correspondence between ESFP of a planar graph and domino tilings of a related graph: Let G be a locally finite planar graph such that every face is bounded by finitely many edges (including the exterior face). A graph with these properties is said to be **nice**. The dual graph G^* is obtained from the nice graph G by putting a vertex at each face of G and joining two such vertices by an edge if their corresponding faces in G share an edge.

Next define the superposition $\tilde{G}$ of G and G^* by the following properties: The vertex set of $\tilde{G}$ is the union of $\mathrm{V}(G)$, $\mathrm{V}(G^*)$ and $\mathrm{E}(G)$ (the "vertex set" $\mathrm{E}(G)$ is the set of intersections of $\mathrm{E}(G)$ and $\mathrm{E}(G^*)$). There is an edge of $\tilde{G}$ joining $v \in \mathrm{V}(G)$ and $e \in \mathrm{E}(G)$ if and only if e is incident to v. Likewise $v \in \mathrm{V}(G^*)$ and $e \in \mathrm{E}(G)$ are joined by an edge if v is a vertex of the edge in G^* identified with e. May someone should think of the edge set of $\tilde{G}$ as a subset of "broken" edges of the original edge sets.

If T is a subgraph of G, let T^* be the subgraph of G^* so that $e^* \in T^*$ if and only if $e \notin T$. This construction is crucial for the connection below.

A **directed essential spanning forest** is a spanning forest together with a choice of root for each component, where only the ends of the components are allowed as roots. Edges of a directed spanning forest are oriented toward this root. If $\mathfrak{F}$ and $\mathfrak{F}^*$ are dual essential spanning forests of a nice infinite planar graph, then a pair $(\mathfrak{F}, \mathfrak{F}^*)$ is called directed if a root has been chosen of each component of $\mathfrak{F}$ and of $\mathfrak{F}^*$.

Let G be a nice infinite planar graph G; a bijection between domino tilings of G and directed pairs of essential spanning forests of G and G^* is described as follows: If $(\mathfrak{F}, \mathfrak{F}^*)$ is a directed pair, then let $\Psi(\mathfrak{F}, \mathfrak{F}^*)$ be the dimer cover $A \subseteq \mathrm{E}(\tilde{G})$ such that:

1. The edge from $v \in \mathrm{V}(G)$ to $e \in \mathrm{E}(G)$ is in A if and only if $e \in \mathfrak{F}$ and ve is oriented away from v, and

2. the edge from $v^* \in \mathrm{V}(G^*)$ to $e \in \mathrm{E}(G^*) = \mathrm{E}(G)$ is in A if and only if $e \in \mathfrak{F}^*$ and is oriented away from v^*.

It is easy to see that A is a dimer cover: Each vertex $v \in \mathrm{V}(G)$ is in precisely one edge of A, corresponding to the unique edge in $\mathfrak{F}$ out of v; similarly each $v^* \in \mathrm{V}(G^*)$ is in a unique edge of A; and each $e \in \mathrm{E}(G)$ is in a unique edge of A since e is in precisely one of $\mathfrak{F}, \mathfrak{F}^*$.

Conversely, if $A \subset \mathrm{E}(G)$ is a dimer cover, then each edge $f \in A$ connects some $e \in \mathrm{E}(G)$ either to some $v \in \mathrm{V}(G)$ or some $v^* \in G^*$. Let $\Phi(f)$ be the edge e in either G or G^* accordingly and orient it away from v or v^*. The collection of all $\{\Phi(f)\colon f \in A\}$ is the union of a subgraph G' of G and the corresponding dual subgraph G'^* of G^*. If G' would have a cycle, then inside the cycle is a component of G'^*. Starting anywhere in this component of G'^* and following the orientation creates a cycle since the component is finite. This cycle encloses a cycle of G' inside the original cycle. But this cannot continue forever because of the finite-conditions on G which are required at the beginning of the chapter, whence G' (and G'^*) has no cycle. Thus G' and G'^* are essential spanning forests with each component directed toward a root.

Let Π be the map that takes a directed pair of ESFs $(\mathfrak{F}, \mathfrak{F}^*)$ and produces the undirected ESF $\mathfrak{F}$, by forgetting about $\mathfrak{F}^*$ and about the arrows. So the following correspondence was established:

$$\text{Dimer Cover} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \text{Directed ESF's} \xrightarrow{\Pi} \text{ESF's}\,.$$

2.2.2 Uniform Measure Of Maximal Entropy

Fix a $\mathbb{Z}^2$-periodic planar graph G, so the vertex set is $\mathbb{Z}^2$. There is a well defined map $\Psi \circ \Pi^{-1}$ from one-ended spanning trees of G to domino tilings of $\tilde{G}$ (which is also $\mathbb{Z}^2$ periodic).

The uniform spanning forest measure WSF on G is supported on the set of one-ended trees (see Pemantle [29]) so the preceding correspondence gives a transported measure μ on domino tilings of G. In the remainder of this chapter $\mathbf{H}(.)$ will always denote the Kolmogorov-Sinai entropy per vertex.

Theorem 2.10. *The measure μ is the unique measure of maximal entropy among all shift invariant probability measures on domino tilings and its entropy per fundamental domain is* $\mathbf{H}(\mathrm{WSF})$.

Proof: Since μ is well defined and $\mathbb{Z}^2$-invariant, it remains to prove the assertions about its entropy. Suppose that $\tilde{\nu}$ is a translation invariant probability measure on domino tilings of $\tilde{G}$. This will be transported to a measure ν on essential spanning forests of G by $\nu(B) = \tilde{\nu}(\Psi(\Pi^{-1}(B)))$.

First it will be shown that the Kolmogorov-Sinai-entropy per fundamental domain is preserved. Therefor, note that ν is translation invariant so with probability 1 the components of the essential spanning forest have one or two ends (see [9]). There is only one way that a one-ended tree may be covered with dominoes and there are two ways a two-ended tree may be covered. Thus the ambiguity in determining the domino tiling is one bit for a two-ended component in the forest in G plus one bit for each two-ended component in the dual spanning forest of G^*. Since there are $O(n)$ such components in every box $B(n)$ of side length n which has on the order of n^2 vertices, the entropy of ν and $\tilde{\nu}$ are the same since

$$\frac{1}{|B_n|}\mathbf{H}(\tilde{\nu} \downharpoonleft B_n) \leq \frac{1}{|B_n|}\mathbf{H}(\nu \downharpoonleft B_n) + \mathcal{O}\left(\frac{\log n}{n^2}\right).$$

At this point the paper of Sheffield [34] must be cited again. Now $\mathbf{H}(\tilde{\nu})$ per fundamental domain $= \mathbf{H}(\nu) \leq \mathbf{H}(\mathrm{WSF}) = \mathbf{H}(\mu)$ per fundamental domain with equality only when $\nu = \mathrm{WSF}$. But WSF is concentrated on one-ended spanning trees [see [29]] and hence μ is the only measure which transports to WSF, which establishes that μ is the unique measure of maximal entropy on domino tilings. □

The entropy per fundamental domain of the domino tiling on $\tilde{G}$ is the same as the entropy per vertex of the essential spanning forest process on G. From the construction of $\tilde{G}$ one gets $1+N_E+N_F$ for the number of vertices per fundamental domain, where and N_E is the number of edges and N_F the number of faces per fundamental domain. To convert the entropy per vertex formula of the spanning forest process to the entropy per vertex of the domino process on G it must be multiplied by $1/(1+N_E+N_F)$. If G is the nearest neighbour graph on $\mathbb{Z}^2$, then $N_E = 2$ and $N_F = 1$ and so the entropy per vertex of the dimer cover is $\mathbf{H}(\mu) = \mathbf{h}(\mathbb{Z}^2)/4$.

2.3 Abelian Sandpile Model

Reference(s): [30] and [33]

After a short introduction to the abelian sandpile model (ASM) a one-to-one correspondence on finite regions between recurrent configurations and rooted spanning trees will be given.

2.3.1 Basics

Let $d \geq 2$ and $\mathbb{Z}^d$ be the d-dimensional integer lattice and $\gamma \geq 2d$. Let $\mathfrak{C} = \mathbb{N}^{\mathbb{Z}^d}$ be the **height configuration space** and think of $\eta \in \mathfrak{C}$ as a map $\eta : \mathbb{Z}^d \to \mathbb{N}$ with the intuitive picture that every vertex is occupied with a number of grains of sand or building bricks which form a sandpile or tower.

Next a dynamical system on these height configurations will be introduced. Therefor, let $K \subseteq \mathbb{Z}^d$ be a nonempty set and denote by $N_K(\mathbf{n})$ the number of nearest neighbours (abbreviation:

n.n.) of $\mathbf{n} \in K$. The set $\Lambda_\gamma = \{0, 1, \ldots, \gamma-1\}^{\mathbb{Z}^d}$ will be called the **set of stable configurations**. Given a stable configuration, then add grains at uniformly randomly chosen vertices $\mathbf{n}$. By adding grains it could happen that $\eta(\mathbf{n}) \geq \gamma$ and the system becomes unstable. If this happens, then all unstable sites will topple. Let $\mathbf{n}$ be an unstable site of η the toppling rule $T_\mathbf{n}$ is a map $\mathfrak{C} \to \mathfrak{C}$ and acts as follows:

$$T_\mathbf{n}(\eta)_\mathbf{k} = \begin{cases} \eta_\mathbf{k} - \gamma & \text{if } \mathbf{k} = \mathbf{n} \\ \eta_\mathbf{k} + 1 & \text{if } \mathbf{k}, \mathbf{n} \text{ are n.n. in } K \\ \eta_\mathbf{k} & \text{otherwise.} \end{cases}$$

Note that at the boundary of K grains can be lost and for two different unstable sites $\mathbf{m}, \mathbf{n} \in K$, $T_\mathbf{m}$ and $T_\mathbf{n}$ will commute. If unstable sites are created by toppling these will be toppled as well. This procedure will, in general, lead to a stable configuration if K is finite or $\gamma > 2d$. In both cases the system is dissipative. The addition of a grain at $\mathbf{n}$ and the relaxation of the system afterwards will be expressed by an operator $a_\mathbf{n}$; again these operators commute. Since $\mathbf{n}$ is random one gets a Markov-process with state space Λ_γ. The set of recurrent configurations will be denoted by $\mathfrak{R}$. The presented model is the so called **abelian sandpile model** (ASM).

There are many dynamical and algebraic aspects of the abelian sandpile model which could be explored, but the discussion here will concentrate on the interface between the ASM and spanning trees and the ASM and the Harmonic model, which will be introduced in Chapter 4.

2.3.2 Connection And Entropy

Let $\gamma = 2d$. The **burning test**: Given $\eta \in \Lambda$, now "burn" all vertices $\mathbf{n}$ with $\eta_\mathbf{n} \geq N_K(\mathbf{n})$. The result is a stable configuration with vertex set $K^{(1)} \subseteq K$. This procedure will be repeated until $K^{(i)} = K^{(i+1)}$. If $K^{(i)} = K^{(i+1)} \neq \emptyset$, then η fails the burning test and is not recurrent (see [28]).

Let G be a periodic graph. The graph G will be extended by adding an extra vertex z, which will be called the root. The extended graph $G^* = (\mathrm{V}^*, \mathrm{E}^*)$ is then defined by adding extra edges from the boundary sites to the root, for $x \in \partial V$, $2d - N_\mathrm{V}(x)$ edges go from x to the root.

Given a recurrent configuration, burning times will be assigned to every vertex in the following way: Start with extending the graph by all vertices which are n.n. to a vertex in the boundary of G and give them burning time 0. The burning time 1 is given to the boundary vertices which can be burnt in the first step of the algorithm, burning time 2 is given to the vertices which can be burnt after those, etc. The edges in the spanning tree are between sites with burning time t and $t+1$, with the interpretation that the site with burning time $t+1$ receives his fire from the neighbouring site with burning time t. In G^* every vertex in V has exactly $2d$ outgoing edges. In the case of ambiguity, i.e. $t+1$ has more than one neighbour with burning time t, the edge will be chosen according to a preference rule, depending on the height. This means that the edges will be ordered, this order defines a preference (e.g. $d = 1$:left $<$ right or $d = 2$ west$<$south$<$east$<$north,...). Once a preference has been fixed, the new vertices will be identified with the root z without deletion of edges. Given the spanning tree and the preference rule, then one can reconstruct the height configuration. And so there is a bijection between $\mathfrak{R}$ and the set of wired spanning trees. Since the cardinality of $\mathfrak{R}$ is equal to $\tau(G^*)$, the number of recurrent configurations is obtained by using the Matrix-Tree-Theorem and so one gets the following theorem:

Theorem 2.11 (Dhar's formula)**.**

$$|\mathfrak{R}| = \det \Delta_{G^*}[z] = \det(2dI_{|V|\times|V|} - A_G)\,.$$

The entropy per vertex of the ASM can be obtained easily by the results of this section and Chapter 1.

2.3.3 An Infinite Volume Measure For The ASM

Recall the results of [6, 29] stated in Remark 2.3.

Theorem 2.12. *Let ν_M denote the measure on recurrent configurations $\mathfrak{R}_M$, which puts equal mass to every recurrent configuration in $M \subseteq \mathbb{Z}^d$. Then for all $d \in \mathbb{N}$ as $M \to \mathbb{Z}^d$, $\nu_M \to \nu$; in the sense that, for all local functions f,*

$$\lim_{M \uparrow \mathbb{Z}^d} \nu_M(f) \to \nu(f)\,.$$

Proof: Let M be a finite set in $\mathbb{Z}^d$, $\mathfrak{F}_M$ a rooted spanning tree on M and $A = \{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_k\}$ a finite set of vertices. The set $\bar{A}$ of vertices is obtained from A by adding all the n.n. of A. Suppose M is big enough such that $\bar{A} \subseteq M$. Let $\eta_{\mathbf{n}}(M)(\mathfrak{F}_M)$ be the height of site $\mathbf{n}$ corresponding to the rooted spanning tree $\mathfrak{F}_M$.

If an infinite tree has just one end, then the height $\eta_{\mathbf{n}}(\mathfrak{F})$ can be reconstructed. All paths starting from A or $\bar{A}$ and going to infinity (the root) coincide from some point $\mathrm{anc}(A)$ on, this means that $\mathrm{anc}(A)$ is a common ancestor of the set $\bar{A}$.

Consider the subtree $\mathfrak{F}_M(A)$ of all descendants of $\mathrm{anc}(A)$. This is a finite tree, and the height $\eta_{\mathbf{n}}$ is reconstructed from the lengths of the paths, from $\mathrm{anc}(A)$ to $\mathbf{n}$ and his n.n., and a preference rule.

Let $\mathfrak{F}_M(A, \eta)$ be the edge configuration of the tree $\mathfrak{F}_M(A)$ corresponding to the height configuration η_A on A and V a fixed finite subset of $\mathbb{Z}^d$, then one has

$$\begin{aligned}\nu_M(\eta_A) &= \mu_M\big(\mathfrak{F}_M(A) = \mathfrak{F}_M(A,\eta)\big) = \mu_M\big(\{\mathfrak{F}_M(A) = \mathfrak{F}_M(A,\eta)\} \cap \{\mathfrak{F}_M(A) \subseteq V\}\big)\\ &= \mu_M\big(\{\mathfrak{F}_M(A) = \mathfrak{F}_M(A,\eta)\} \cap \{\mathfrak{F}_M(A) \nsubseteq V\}\big)\,.\end{aligned}$$

The indicator $\mathbf{1}_{\{\mathfrak{F}_{\mathbb{Z}^d}(A) \subseteq V\}}$ is local and so

$$\limsup_{M\uparrow\infty} \nu_M(\eta_A) \le \mathrm{WSF}\left(\{\mathfrak{F}_{\mathbb{Z}^d}(A) = \mathfrak{F}_M(A,\eta)\} \cap \{\mathfrak{F}_{\mathbb{Z}^d}(A) \subseteq V\}\right) + \mathrm{WSF}\left(\{\mathfrak{F}_{\mathbb{Z}^d}(A) \nsubseteq V\}\right) \tag{2.5}$$

and

$$\liminf_{M\uparrow\infty} \nu_M(\eta_A) \ge \mathrm{WSF}\left(\{\mathfrak{F}_{\mathbb{Z}^d}(A) = \mathfrak{F}_M(A,\eta)\} \cap \{\mathfrak{F}_{\mathbb{Z}^d}(A) \subseteq V\}\right) - \mathrm{WSF}\left(\{\mathfrak{F}_{\mathbb{Z}^d}(A) \nsubseteq V\}\right)\,. \tag{2.6}$$

Therefore, since $\mathfrak{F}_{\mathbb{Z}^d}(A)$ is finite WSF-a.s., one obtains from combining (2.5) and (2.6) and letting $V \uparrow \mathbb{Z}^d$:

$$\lim_{M\uparrow\infty} \nu_M(\eta_A) = \mathrm{WSF}\left(\{\mathfrak{F}_{\mathbb{Z}^d}(A) = \mathfrak{F}_M(A,\eta)\}\right)\,.$$

So one finally gets a unique probability measure ν. Translation invariance for any $\mathbf{v} \in \mathbb{Z}^d$:

$$\lim_{M\uparrow\mathbb{Z}^d} \nu_M(\sigma_{\mathbf{v}} E) = \lim_{M\uparrow\mathbb{Z}^d} \nu_{\sigma_{-\mathbf{v}}M}(E) = \lim_{M\uparrow\mathbb{Z}^d} \nu_{\sigma_M}(E) = \nu(E).$$

In dimension $d > 4$ the correspondence above fails because the components of the USF are not connected. For a proof see [4] and the erratum [3]. □

3 Algebraic Actions

Reference(s): [12]

In this chapter generalised shifts will be studied. These shifts with alphabet $\mathbb{T}$ have expansive shift actions and symmetry groups Γ, which can be non-commutative. Algebraic properties of the dual space of $\mathbb{T}^\Gamma$ will be used to analyse the dynamics of these systems.

3.1 Introduction

Definition. Let Γ be a group and M a ring. The **group ring** $M[\Gamma]$ of Γ over M is the set of mappings $f : \Gamma \to M$ of finite support with sum $\gamma \mapsto (f+g)(\gamma) := f(\gamma) + g(\gamma)$ and convolution $\gamma \mapsto \sum_{\gamma'\gamma''=\gamma} f(\gamma')g(\gamma'')$ with $f, g \in M[\Gamma]$. The elements of $M[\Gamma]$ can be written as formal linear combinations of elements of Γ, with coefficients in M. And so an element $f \in M[\Gamma]$ can be expressed as $\sum_\gamma f(\gamma)\gamma$.

Now let Γ be a countable group with identity element 1 and integral group ring $\mathbb{Z}[\Gamma]$. Let

$$\ell^\infty(\Gamma) := \left\{ w : \Gamma \to \mathbb{R} : \|w\|_\infty = \sup_{\gamma\in\Gamma} w(\gamma) < \infty \right\},$$

and denote by w_γ the value $w(\gamma) \in \mathbb{R}$, with $w \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$. For $p \in [1, \infty)$ set

$$\ell^p := \left\{ w \in \ell^\infty(\Gamma) : \|w\|_p := \left(\sum_{\gamma\in\Gamma} |w_\gamma|^p \right)^{1/p} < \infty \right\}.$$

An element $h \in \ell^1(\Gamma)$ can be uniquely written as a convergent series

$$h = \sum_{\gamma\in\Gamma} h_\gamma e(\gamma)\,;$$

where $e(\gamma) \in \ell^1(\Gamma)$, for every $\gamma \in \Gamma$, is defined by

$$e(\gamma)_{\gamma'} := \begin{cases} 1 & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise.} \end{cases}$$

The multiplication or convolution in $\ell^1(\Gamma)$ takes the form

$$h \cdot h' = \sum_{\gamma,\gamma'\in\Gamma} h_\gamma h'_{\gamma'} e(\gamma\gamma') = \sum_{\gamma\in\Gamma} \left(\sum_{\gamma'\in\Gamma} h_{\gamma'} h'_{\gamma'^{-1}\gamma} \right) e(\gamma)\,.$$

The involution $h \mapsto h^*$ in $\ell^1(\Gamma)$ is defined by

$$h^* = \sum_{\gamma\in\Gamma} h_{\gamma^{-1}} e(\gamma) = \sum_{\gamma\in\Gamma} h_\gamma e(\gamma^{-1})\,.$$

The integral group ring $\mathbb{Z}[\Gamma]$ can be viewed as subring of $\ell^1(\Gamma)$ by identifying $\sum_{\gamma\in\Gamma} a_\gamma \gamma \in \mathbb{Z}[\Gamma]$ with $\sum_\gamma a_\gamma e(\gamma)$.

The **left shift action** $(\gamma, w) \mapsto \mathrm{L}^\gamma(w)$ and **right shift action** $(\gamma, w) \mapsto \mathrm{R}^\gamma(w)$ of Γ on $\ell^\infty(\Gamma)$ are given by

$$(\mathrm{L}^\gamma w)_{\gamma'} = w_{\gamma^{-1}\gamma'} \tag{3.1}$$
$$(\mathrm{R}^\gamma w)_{\gamma'} = w_{\gamma'\gamma}\,, \tag{3.2}$$

for every $w \in \ell^\infty(\Gamma)$ and $\gamma, \gamma' \in \Gamma$.

These actions can be extended to $(h, w) \mapsto \mathrm{L}_h w$ and $(h, w) \mapsto \mathrm{R}_h w$ of $\ell^1(\Gamma)$ on $\ell^\infty(\Gamma)$ by setting

$$\mathrm{L}_h w = \sum_{\gamma\in\Gamma} h_\gamma \mathrm{L}^\gamma w = h \cdot w \tag{3.3}$$
$$\mathrm{R}_h w = \sum_{\gamma\in\Gamma} h_\gamma \mathrm{R}^\gamma w = w \cdot h^*\,, \tag{3.4}$$

for every $h \in \ell^1(\Gamma)$, $w \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$.

For $v \in \ell^p(\Gamma)$ and $w \in \ell^q(\Gamma)$ with $\frac{1}{p} + \frac{1}{q} = 1, 1 \le p, q \le \infty$, set

$$\langle v, w\rangle = \sum_{\gamma\in\Gamma} v_\gamma w_\gamma\,.$$

A sequence v_n in $\ell^p(\Gamma)$ converges in the weak*-topology to v if and only if $\lim_{n\to\infty}\langle v_n, w\rangle = \langle v, w\rangle$ for all $w \in \ell^q(\Gamma)$.

Before defining the dynamical system the following identities will be noted:

$$\langle v, \mathrm{R}_h w\rangle = \langle v, w \cdot h^*\rangle = \langle v \cdot h, w\rangle\,, \tag{3.5}$$
$$\langle v, \mathrm{L}_h w\rangle = \langle v, h \cdot w\rangle = \langle h^* \cdot v, w\rangle\,. \tag{3.6}$$

Definition. Let Γ be a countable discrete group. An **algebraic Γ-action** is a homomorphism $\alpha : \gamma \mapsto \alpha^\gamma$ from Γ into the group $\mathrm{Aut}(X)$ of continuous automorphisms of a compact abelian group X.

Let X be the compact abelian group $\mathbb{T}^\Gamma$ under point-wise addition (that are maps from Γ to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$). Under the pairing

$$\langle f, x\rangle = e^{2\pi i \sum_{\gamma\in\Gamma} f_\gamma x_\gamma}\,,$$

where $f = \sum_{\gamma\in\Gamma} f_\gamma \gamma \in \mathbb{Z}[\Gamma]$ and $x = (x_\gamma) \in X$, the Pontryagin dual $\hat{X}$ of X can be identified with the group ring $\mathbb{Z}[\Gamma]$. The left and right shift actions L and R of Γ on $\mathbb{T}^\Gamma$ are defined analogously to (3.1) and (3.2); and just as in (3.3) and (3.4) these actions will be extended to homomorphisms L_f and R_f to actions of $\mathbb{Z}[\Gamma]$ on X with $f \in \mathbb{Z}[\Gamma]$.

For fixed $f \in \mathbb{Z}[\Gamma]$ set

$$X_f = \ker(\mathrm{R}_f) = \{x \in X : \mathrm{R}_f x = 0\} = \widehat{\mathbb{Z}[\Gamma]/\mathbb{Z}[\Gamma]f}$$

and denote by

$$\alpha_f = \mathrm{L}|_{X_f}$$

the restriction of the Γ-action L on X to X_f.

3.2 Entropy

An algebraic Γ-action on a compact abelian group X is **expansive** if there exists an open neighbourhood U of the identity $0_X = 0$ in X with $\bigcap_{\gamma\in\Gamma} \alpha^\gamma(U) = 0$.

Next a characterization of the expansive actions α_f will be given, therefor a few definitions will be needed. Set

$$\ell^\infty(\Gamma,\mathbb{Z}) = \{w \in \ell^\infty(\Gamma) : w_\gamma \in \mathbb{Z} \ \text{ for every } \ \gamma \in \Gamma\}\,.$$

The map $\rho : \ell^\infty(\Gamma) \to X$ given by

$$\rho(w)_\gamma = w_\gamma \pmod 1\,,$$

for every $w \in \ell^\infty(\Gamma)$ and $\gamma \in \Gamma$, is a continuous surjective group homomorphism with

$$\rho \circ \mathrm{L}^\gamma = \mathrm{L}^\gamma \circ \rho, \qquad\qquad \rho \circ \mathrm{R}^\gamma = \mathrm{R}^\gamma \circ \rho\,,$$

for every $\gamma \in \Gamma$. Define the linearization of X_f by

$$W_f = \rho^{-1}(X_f) = \mathrm{R}_f^{-1}\big(\ell^\infty(\Gamma,\mathbb{Z})\big) = \{w \in \ell^\infty(\Gamma) : \mathrm{R}_f w \in \ell^\infty(\Gamma,\mathbb{Z})\}\,;$$

which is weak*-closed and a L invariant subgroup with $\ker(\rho) = \ell^\infty(\Gamma,\mathbb{Z}) \subset W_f$.

For every $h \in \ell^1(\Gamma)$ and $1 \le p \le \infty$, set

$$K_p(h) = \{g \in \ell^p(\Gamma) : \mathrm{R}_h g = 0\}, \qquad\qquad V_p(h) = \mathrm{R}_h(\ell^p(\Gamma))\,.$$

Theorem 3.1. *Let Γ be a countable group and $f \in \mathbb{Z}[\Gamma]$. The following conditions are equivalent*

1. *The action α_f is expansive;*
2. *$K_\infty(f) = \{0\}$;*
3. *f is invertible in $\ell^1(\Gamma)$.*

Proof: Let δ be a metric on $\mathbb{T}$ defined by

$$\delta(s_1,s_2) = \min\{|\tilde{s_1} - \tilde{s_2}| : \tilde{s_i} \in \mathbb{R} \ \ s_i = \tilde{s_i} (\mathrm{mod}\ 1) \text{ for } \ i = 1,2\}\,.$$

$(2 \Rightarrow 1)$ Assume that there exists a nonzero element $v \in K_\infty(f)$. For every $c \in \mathbb{R}$ one gets $\rho(cv) \in X_f$. So $|c|$ can be chosen sufficiently small such that for every $\varepsilon > 0$ there exists a nonzero element $x^{(\varepsilon)} \in X_f$ with $\delta(0, x_\gamma^{(\varepsilon)}) < \varepsilon$ for every $\gamma \in \Gamma$. And so α_f is nonexpansive.
$(1 \Rightarrow 2)$ If α_f is nonexpansive, then there exists a nonzero element $x \in X_f$ with $\delta(0, x_\gamma) < (3\|f\|_1)^{-1}$ for every $\gamma \in \Gamma$. Next choose an element $\tilde{x} \in \rho^{-1}(\{x\}) \subset W_f = \rho^{-1}(X_f)$ with $\|\tilde{x}_\gamma\| < (3\|f\|_1)^{-1}$, for every $\gamma \in \Gamma$. By definition of X_f and $\mathrm{R}_f\tilde{x} \in \ell^\infty(\Gamma,\mathbb{Z})$, and the smallness of the coordinates of $\tilde{x}$ one gets $\tilde{x} \in K_\infty(f)$.
$(2 \Rightarrow 3)$ If $K_\infty(f) = \{0\}$ then $V_1(f^*)$ is dense in $\ell^1(\Gamma)$ by (3.5) and the Hahn-Banach theorem. The group of units in $\ell^1(\Gamma)$ is open [11, Lemma 2.1.5], and since $V_1(f^*)$ is dense in $\ell^1(\Gamma)$ it must intersect with the open subset of units and therefore contains a unit. Hence there exists a $g \in \ell^1(\Gamma)$ with $g \cdot f = 1$ and so f is invertible in $\ell^1(\Gamma)$ by [21, p. 122]. Sketch of the proof of the last argument: Let H be the closure of $\mathbb{C}[\Gamma]$ and $\mathcal{A}$ be the closure of $\{\mathrm{L}_f\}_{f\in\mathbb{C}[\Gamma]}$ w.r.t. the operator norm in $B(H)$. Then a trace tr on the group-C^*-algebra $\mathcal{A}$ will be defined. It can be shown that if $h \in \mathbb{C}[\Gamma]$ is an idempotent, then $\mathrm{tr}(h) = 0 \Rightarrow h = 0$ and $\mathrm{tr}(h) = 1 \Rightarrow h = 1$. The result follows by setting $h = f \cdot g$ which is an idempotent because $g \cdot f = 1$ and the defining

property $\mathrm{tr}(g \cdot f) = \mathrm{tr}(f \cdot g)$ of a trace.
$(3 \Rightarrow 2)$ If f is invertible in $\ell^1(\Gamma)$, then R_f is invertible with inverse $\mathrm{R}_{f^{-1}}$. □

Definition. Let α be an algebraic action of a countable discrete group Γ on a compact abelian group X with identity element 0. A point $x \in X$ is a α-**homoclinic** if $\lim_{\gamma\to\infty} \alpha^\gamma x = 0$, i.e. for every neighbourhood U of 0 in X there is a finite subset F of Γ with $\alpha^\gamma x \in U$ for all $\gamma \in \Gamma \backslash F$.

The set $\mathbb{H}_\alpha(X)$ of all α-homoclinic points in X is a subgroup of X, called the **homoclinic group** of α. A homoclinic point $x \in X$ is called **fundamental** if the homoclinic group $\mathbb{H}_\alpha(X)$ is generated by the orbit $\{\alpha^\gamma x : \gamma \in \Gamma\}$ of x.

Theorem 3.2. *Let Γ be a countable group and $f \in \mathbb{Z}[\Gamma]$ an element which is invertible in $\ell^1(\Gamma)$. If $w_f^{\mathbb{H}} = f^{-1} \in \ell^1(\Gamma)$ and $\xi = \rho \circ \mathrm{R}_{w_f^{\mathbb{H}}} : \ell^\infty(\Gamma, \mathbb{Z}) \to X_f$, then ξ is a surjective group homomorphism with the following properties.*

1. $\ker(\xi) = \mathrm{R}_f(\ell^\infty(\Gamma, \mathbb{Z}))$;
2. $\xi \circ \mathrm{L}^\gamma = \alpha_f^\gamma \circ \xi$ *for every* $\gamma \in \Gamma$
3. ξ *is continuous in the weak*-topology on closed, bounded subsets of* $\ell^\infty(\Gamma, \mathbb{Z})$.

Proof: Set $\tilde{w} = (f^*)^{-1} = (f^{-1})^*$. By defintion, $\mathrm{R}_f \tilde{w} = \tilde{w} \cdot f^* = e(1)$ and hence $\tilde{w} \in W_f$ and $x_f^{\mathbb{H}} = \rho(\tilde{w}) \in X_f$. Since $\tilde{w} \in \ell^1(\Gamma)$, $x_f^{\mathbb{H}} \in \mathbb{H}_{\alpha_f}(X_f)$.

The L-invariance of W_f implies that $\mathrm{R}_{w_f^{\mathbb{H}}} h = \mathrm{L}_h \tilde{w} \in W_f$, for every $h \in \mathbb{Z}[\Gamma]$. Since W_f is weak*-closed and $\mathrm{R}_{w_f^{\mathbb{H}}}$ is weak*-continuous on bounded subsets of $\ell^\infty(\Gamma, \mathbb{Z})$ it follows that

$$\mathrm{R}_{w_f^{\mathbb{H}}}(\ell^\infty(\Gamma, \mathbb{Z})) \subseteq W_f \,.$$

For proving $\mathrm{R}_{w_f^{\mathbb{H}}}(\ell^\infty(\Gamma, \mathbb{Z})) = W_f$ fix $w \in W_f$, set $v = \mathrm{R}_f w \in \ell^\infty(\Gamma, \mathbb{Z})$ and obtain that $w = \mathrm{R}_{w_f^{\mathbb{H}}} v$. The group homomorphism

$$\xi = \rho \circ \mathrm{R}_{w_f^{\mathbb{H}}} : \ell^\infty(\Gamma, \mathbb{Z}) \to X_f$$

is thus surjective, and the equivariance of ξ is obvious and (1.) follows by the definition of ρ. If $B \subset \ell^\infty(\Gamma, \mathbb{Z})$ is a closed, bounded subset, then the weak*-topology coincides with the topology of coordinate-wise convergence, and ξ is obviously continuous in that topology. □

Let Γ be a countable discrete group and $K \subset \Gamma$ a finite set. A finite set $Q \subset \Gamma$ is **left** (K, ε)**-invariant** if

$$\sum_{\gamma \in K} |\gamma Q \Delta Q| / |Q| < \varepsilon \,,$$

and **right** (K, ε)**-invariant** if

$$\sum_{\gamma \in K} |Q\gamma \Delta Q| / |Q| < \varepsilon \,.$$

If Q satisfies both these conditions it is (K, ε)**-invariant**.

A sequence $(Q_n, n \geq 1)$ of finite subsets of Γ is a **left Følner sequence** if there exists, for every finite subset $K \subset \Gamma$ and every $\varepsilon > 0$, an $N \geq 1$ such that Q_n is left (K, ε)-invariant for every $n \geq N$. The definitions of right and two-sided Følner sequences are analogous. The group Γ is **amenable** if it has a left Følner sequence.

Definition. Let X be a compact topological group with a left X-invariant metric δ. Let Γ be an amenable group that operates form the left on X^Γ by L^γ. Let Y be a closed Γ-invariant subset of X. For $F \subset \Gamma$, a subset $E \subset Y$ is called **(F, ε)-separated** if for all $x, y \in E$ with $x \neq y$ there exists a $\gamma \in F$ with $\delta(x_\gamma, y_\gamma) \geq \varepsilon$. Denote by $s_F(\varepsilon)$ the maximum of the cardinalities of all (F, ε)-separated subsets. The **topological entropy** of L is defined by

$$\mathbf{h} := \mathbf{h}_{\mathrm{sep}} := \mathbf{h}(\mathrm{L}|_Y) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{|F_n|} \log s_{F_n}(\varepsilon)\,.$$

Definition. Let $\Gamma' \subset \Gamma$ and denote by $\mathrm{Fix}_{\Gamma'}(X_f) = \{x \in X_f : \alpha_f^\gamma x = x \text{ for every } \gamma \in \Gamma'\}$ the subgroup of **Γ'-invariant points** in X_f.

The subgroup $\mathrm{Fix}_{\Gamma'}(X_f)$ is Γ'-invariant, and $\mathrm{Fix}_{\Gamma'}(X_f)$ is Γ-invariant if and only if Γ' is a normal subgroup of Γ.

Set

$$\begin{aligned}
&\ell^\infty(\Gamma)^{\Gamma'} = \{w \in \ell^\infty(\Gamma) : \mathrm{L}^\gamma w = w \text{ for every } \gamma \in \Gamma'\}\,,\\
&W_f^{\Gamma'} = W_f \cap \ell^\infty(\Gamma)^{\Gamma'} \quad \text{and}\\
&\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'} = \ell^\infty(\Gamma, \mathbb{Z}) \cap \ell^\infty(\Gamma)^{\Gamma'}.
\end{aligned}$$

Definition. A countable group Γ is said to be **residually finite** if there is a sequence Γ_n of normal subgroups of finite index with

$$\bigcap_{n \neq 1} \Gamma_n = \{1\}\,.$$

Let Γ be a countable residually finite discrete group. If $(\Gamma_n, n \geq 1)$ is a sequence of finite index normal subgroups in Γ, then

$$\lim_{n \to \infty} \Gamma_n = \{1\}$$

will denote the fact that for every finite set $K \subset \Gamma$ there exists an $N \geq 1$ with $\Gamma_n \cap (K^{-1}K) = \{1\}$, for every $n \geq N$.

Theorem 3.3. *Let Γ be a countable residually finite group, $f \in \mathbb{Z}[\Gamma]$. For every subgroup $\Gamma' \subset \Gamma$ of finite index*

$$\mathrm{Fix}_{\Gamma'}(X_f) = \xi(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}) \cong \ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'} / \mathrm{R}_f(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'})\,.$$

Proof: From the equivariance of ξ it is clear that $\xi(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}) \subseteq \mathrm{Fix}_{\Gamma'}(X_f)$. Conversely, if $x \in \mathrm{Fix}_{\Gamma'}(X_f)$, then there exists a $w \in W_f^{\Gamma'} \subseteq \ell^\infty(\Gamma)^{\Gamma'}$ with $\rho(w) = x$, and the point $v = \mathrm{R}_f w \in \ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}$ satisfies that $\xi(v) = x$. This proves that $\xi(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}) = \mathrm{Fix}_{\Gamma'}(X_f)$ and Theorem 3.2 guarantees that $\ker(\xi) \cap \ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'} = \mathrm{R}_f(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'})$. The last equation is obvious. □

Corollary 3.4. *If $|\Gamma' \backslash \Gamma| < \infty$, where $\Gamma' \backslash \Gamma$ denotes the right coset space, then $|\mathrm{Fix}_{\Gamma'}(X_f)| = |\det(\mathrm{R}_f|_{\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}})|$.*

Proof: If the right coset space $\Gamma' \backslash \Gamma$ is finite then $\ell^\infty(\Gamma)^{\Gamma'} \cong \ell(\Gamma' \backslash \Gamma, \mathbb{R})$. Let $\mathrm{R}_f|_{\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}}$ be the restriction to $\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}$. Then $\mathrm{R}_f(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}) \subseteq \ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}$ and the absolute value of the determinant $|\det(\mathrm{R}_f|_{\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'}})|$ is equal to $|\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'} / \mathrm{R}_f(\ell^\infty(\Gamma, \mathbb{Z})^{\Gamma'})|$. □

Definition. Let $\Gamma' \subset \Gamma$ be a subgroup with finite index. A finite subset $Q \subset \Gamma$ is said to be a **fundamental domain** of the right coset space $\Gamma' \backslash \Gamma$ if $\{\gamma Q \colon \gamma \in \Gamma'\}$ is a partition of Γ.

Remark 3.5. *Since α_f is expansive,* $\ker(\mathrm{R}_f) = 0$ *by Theorem 3.1, and so there is a* $\gamma \in \Gamma$ *with* $\delta(x_\gamma, x'_{\gamma}) \geq (3\|f\|_1)^{-1}$ *for every pair of distinct points* $x, x' \in X_f$*. For* $\Gamma' \subset \Gamma$ *and* Q *a fundamental domain one gets that* $\mathrm{Fix}_{\Gamma'}(X_f)$ *is* $(Q, (3\|f\|_1)^{-1})$*-separated.*

Theorem 3.6. [12, Proposition 5.5] and[37] *Let* Γ *be a countable residually finite amenable group and let* $(\Gamma_n, n \geq 1)$ *be a sequence of finite index normal subgroups with* $\lim_{n\to\infty} \Gamma_n = \{1\}$*. Then there exists, for every finite subset* $K \subset \Gamma$ *and every* $\bar{\varepsilon} > 0$*, an integer* $M = M(K, \bar{\varepsilon}) \geq 1$ *such that every* Γ_n *with* $n \geq M$ *has a* $(K, \bar{\varepsilon})$*-invariant fundamental domain* Q_n *of the coset space* $\Gamma^{(n)} = \Gamma_n \backslash \Gamma$*.*

Corollary 3.7. [12, Corollary 5.6] *Let* Γ *be a countable residually finite amenable group and let* $(\Gamma_n, n \geq 1)$ *be a sequence of finite index normal subgroups with* $\lim_{n\to\infty} \Gamma_n = \{1\}$*. Then there exists a Følner sequence* $(Q_n, n \geq 1)$ *such that* Q_n *is a fundamental domain of* $\Gamma^{(n)}$ *for every* $n \geq 1$*.*

Lemma 3.8. *For* $x \in X_f$ *there exists an element* $v \in \ell^\infty(\Gamma, \mathbb{Z})$ *with* $\xi(v) = x$ *and* $\|v\|_\infty \leq \|f\|_1/2$*.*

Proof: Choose $w \in W_f \subset \ell^\infty(\Gamma)$ with $\rho(w) = x$ and $-1/2 \leq w_\gamma \leq 1/2$ for every $\gamma \in \Gamma$. Then $v = \mathrm{R}_f w \in \ell^\infty(\Gamma, \mathbb{Z})$, $\|v\|_\infty \leq \|f\|_1/2$ and $\xi(v) = x$ by Theorem 3.2. □

Theorem 3.9. *Let* Γ *be a countable residually finite amenable group and let* $(\Gamma_n, n \geq 1)$ *be a sequence of finite index normal subgroups with* $\lim_{n\to\infty} \Gamma_n = \{1\}$*. If* $f \in \mathbb{Z}[\Gamma]$*, and if the algebraic* Γ*-action* α_f *on* X_f *is expansive, then*

$$\begin{aligned}\mathbf{h}(\alpha_f) &= \lim_{n\to\infty} \frac{1}{|\Gamma^{(n)}|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| = \lim_{n\to\infty} \frac{1}{|\Gamma^{(n)}|} \log \left|\ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n}/\mathrm{R}_f(\ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n})\right| \\ &= \lim_{n\to\infty} \frac{1}{|\Gamma^{(n)}|} \log |\det(\mathrm{R}_f|_{\ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n}})| \,.\end{aligned}$$

Proof: Choose a Følner sequence $(Q_n, n \geq 1)$ in Γ such that Q_n is a fundamental domain of $\Gamma^{(n)}$, for every $n \geq 1$, this can be done because of Corollary 3.7. Theorem 3.3 and Corollary 3.4 show that there exists, for every $n \geq 1$, a $(Q_n, (3\|f\|_1)^{-1})$-separated set of cardinality

$$|\mathrm{Fix}_{\Gamma_n}(X_f)| = |\ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n}/\mathrm{R}_f(\ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n}| = |\det(\mathrm{R}_f|_{\ell^\infty(\Gamma)^{\Gamma_n}})| \,.$$

Since $(Q_n, n \geq 1)$ is a Følner sequence and $|Q_n| = |\Gamma^{(n)}|$ this implies that

$$\mathbf{h}(\alpha_f) \geq \limsup_{n\to\infty} \frac{1}{|Q_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| \,, \tag{3.7}$$

by the definition of $\mathbf{h}$.

Conversely, let $\delta > 0$, $\varepsilon < \delta/3$, and let F_ε be a finite symmetric set with $\sum_{\gamma \in \Gamma \backslash F_\varepsilon} |w^{\mathbb{H}}_\gamma| < \varepsilon/\|f\|_1$. The sets $P_n = Q_n \cap \bigcap_{\gamma \in F_\varepsilon} Q_n\gamma, n \geq 1$, form a Følner sequence with $\lim_{n\to\infty} \frac{|P_n|}{|Q_n|} = 1$. Fix $n \geq 1$ and choose a maximal set $S_{n,\delta} \subset X_f$ which is (P_n, δ)-separated. For every $x \in S_{n,\delta}$ there is a $w(x) \in W_f \subset \ell^\infty(\Gamma)$ with $\|w(x)\|_\infty \leq \|f\|_1/2$ and $\rho(w(x)) = x$ by Lemma 3.8 and write $v(x) \in \ell^\infty(\Gamma,\mathbb{Z})^{\Gamma_n}$ for the unique point with $v(x)_\gamma = (\mathrm{R}_f w(x))_\gamma$ for every $\gamma \in Q_n$. This choice of F_ε implies that the points $\{\xi(v(x)) : x \in S_{n,\delta}\} \subset \mathrm{Fix}_{\Gamma_n}(X_f)$ are $(P_n, \delta/3)$-separated and therefore distinct, Theorem 3.3 shows that $|S_{n,\delta}| \leq |\mathrm{Fix}_{\Gamma_n}(X_f)|$. Since $(P_n, n \geq 1)$ is Følner sequence and $\lim_{n\to\infty} \frac{|P_n|}{|Q_n|} = 1$ this implies that

$$\mathbf{h}(\alpha_f) = \lim_{n\to\infty} \frac{1}{|P_n|} \log |S_{n,\delta}| \leq \liminf_{n\to\infty} \frac{1}{|Q_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| \,. \tag{3.8}$$

The theorem follows by combining (3.7) and (3.8). □

3.3 Fuglede-Kadison-Determinant Entropy Formula II

Let $\ell^p(\Gamma,\mathbb{C})$ denote the complex ℓ^p-space of Γ for $1 \le p \le \infty$ with its conjugate linear involution $w \mapsto w^*$ given by $(w^*)_\gamma = \overline{w}_{\gamma^{-1}}, \gamma \in \Gamma$. The **group von Neumann algebra** $\mathcal{N}\Gamma$ of a discrete group Γ can be defined as the algebra of left Γ-equivariant bounded operators of $\ell^2(\Gamma,\mathbb{C})$ to itself.

The homomorphism of $\mathbb{C}$-algebras with involution:

$$\mathrm{R} : \ell^1(\Gamma,\mathbb{C}) \to \mathcal{N}\Gamma$$

mapping f to the operator R_f with $\mathrm{R}_f(v) = v \cdot f^*$ is injective because $\mathrm{R}_f(e(1)) = f^*$. The von Neumann trace on $\mathcal{N}\Gamma$ is the linear form

$$\mathrm{tr}_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \to \mathbb{C}$$

mapping A to $\mathrm{tr}_{\mathcal{N}\Gamma}(A) = (A(e(1)), e(1))$.

The trace is faithful in the sense that $\mathrm{tr}_{\mathcal{N}\Gamma} A = 0$ for a positive operator A in $\mathcal{N}\Gamma$ implies that $A = 0$. Moreover $\mathrm{tr}_{\mathcal{N}\Gamma}$ vanishes on commutators and satisfies the estimate $|\,\mathrm{tr}_{\mathcal{N}\Gamma} A| \le \|A\|$. On $\ell^1(\Gamma,\mathbb{C})$ it is given by $\mathrm{tr}_{\mathcal{N}\Gamma}(w) = w(1)$.

The Fuglede-Kadison determinant $\mathrm{Det}_{\mathcal{N}\Gamma}$ of $A \in (\mathcal{N}\Gamma)^\times$ is defined as in Chapter 1. If the group Γ is finite one has $\mathcal{N}\Gamma = \mathbb{C}[\Gamma]$ and

$$\mathrm{Det}_{\mathcal{N}\Gamma} A = |\det A|^{1/|\Gamma|} \tag{3.9}$$

by [24, Examples 1.3., 2.5., 3.12.].

Theorem 3.10. *Let Γ be a countable discrete amenable and residually finite group and $f \in \mathbb{Z}[\Gamma]$ which is invertible in $L^1(\Gamma)$. Then*

$$\mathbf{h}(f) = \log \mathrm{Det}_{\mathcal{N}\Gamma} f\,.$$

Let Γ be a countable residually finite discrete group and let $(\Gamma_n, n \ge 1)$ be a sequence of finite index normal subgroups with $\lim_{n\to\infty} \Gamma_n = \{1\}$.

For f in $\ell^1(\Gamma,\mathbb{C})$ the bounded operator $\mathrm{R}_f : \ell^2(\Gamma,\mathbb{C}) \to \ell^2(\Gamma,\mathbb{C})$ given by right convolution with f^* satisfies the norm estimate

$$\|\mathrm{R}_f\| \le \|f\|_1\,. \tag{3.10}$$

The group Γ acts via L on $\ell^\infty(\Gamma,\mathbb{C})$ and so there is an isomorphism of finite dimensional $\mathbb{C}$-vector spaces

$$\ell^\infty(\Gamma,\mathbb{C})^{\Gamma_n} \cong \ell^\infty(\Gamma^{(n)},\mathbb{C})$$

given by viewing left Γ_n-invariant functions on Γ as functions on $\Gamma^{(n)}$. Since R_f is left Γ_n-equivariant it induces an endomorphism of $\ell^\infty(\Gamma,\mathbb{C})^{\Gamma_n}$ and hence an endomorphism of $\ell^\infty(\Gamma^{(n)},\mathbb{C}) = \mathbb{C}[\Gamma^{(n)}] = \ell^2(\Gamma^{(n)},\mathbb{C})$ which will be denoted by:

$$\mathrm{R}_f^{(n)} : \ell^2(\Gamma^{(n)},\mathbb{C}) \to \ell^2(\Gamma^{(n)},\mathbb{C})\,.$$

Consider the map:

$$\ell^1(\Gamma,\mathbb{C}) \to \ell^1(\Gamma^{(n)},\mathbb{C}) \tag{3.11}$$

given by sending $f:\Gamma\to\mathbb{C}$ to the function $f^{(n)}:\Gamma^{(n)}\to\mathbb{C}$ defined by

$$f^{(n)}(\delta)=\sum_{\gamma\in\delta}f(\gamma)\,,$$

for all congruence classes in δ in $\Gamma^{(n)}$. Clearly, (3.11) is a homomorphism of $\mathbb{C}$-algebras with involution such that $\|f^{(n)}\|_1\le\|f\|_1$. And so

$$\mathrm{R}_f^{(n)}=\mathrm{R}_{f^{(n)}}:\ell^2(\Gamma^{(n)},\mathbb{C})\to\ell^2(\Gamma^{(n)},\mathbb{C})\,.$$

By the estimate (3.10) applied to $f^{(n)}$ and $\Gamma^{(n)}$ one gets $\|\mathrm{R}_{f^{(n)}}\|\le\|f^{(n)}\|_1$. Using the last estimate one gets for all $n\ge1$ that

$$\|\mathrm{R}_f^{(n)}\|\le\|f\|_1\,.$$

From the definition of $\mathrm{R}_f^{(n)}$ the relation $\mathrm{R}_{fg}^{(n)}=\mathrm{R}_f^{(n)}\mathrm{R}_g^{(n)}$ follows. Hence $\mathrm{R}_f^{(n)}$ is invertible if f is invertible in $\ell^1(\Gamma,\mathbb{C})$ and one gets $(\mathrm{R}_f^{(n)})^{-1}=\mathrm{R}_{f^{-1}}^{(n)}$ and

$$\|(\mathrm{R}_f^{(n)})^{-1}\|\le\|f^{-1}\|_1\quad\text{for } f \text{ in } \ell^1(\Gamma,\mathbb{C})^\times \text{ and all } n\ge1\,.$$

By equation (3.9)

$$\log\mathrm{Det}_{\mathcal{N}\Gamma^{(n)}}f^{(n)}=\frac{1}{|\Gamma^{(n)}|}\log|\det\mathrm{R}_{f^{(n)}}|=\frac{1}{|\Gamma^{(n)}|}\log|\det(\mathrm{R}_f|_{\ell^\infty(\Gamma,\mathbb{C})^{\Gamma_n}})|\,.$$

Theorem 3.11. *Let Γ be a countable discrete residually finite group and $(\Gamma_n,n\ge1)$ a sequence of finite index normal subgroups with $\lim_{n\to\infty}\Gamma_n=\{1\}$. If f in $\ell^1(\Gamma,\mathbb{C})^\times$, then*

$$\mathrm{Det}_{\mathcal{N}\Gamma}f=\lim_{n\to\infty}\mathrm{Det}_{\mathcal{N}\Gamma^{(n)}}f^{(n)}\,.$$

Proof: Because of the relation $(ff^*)^{(n)}=f^{(n)}f^{(n)*}$ the assertion means:

$$\mathrm{tr}_{\mathcal{N}\Gamma}\log\mathrm{R}_g=\lim_{n\to\infty}\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}\log\mathrm{R}_{g^{(n)}}$$

for $g=ff^*$ in $\ell^1(\Gamma,\mathbb{C})^\times$. But $\mathrm{R}_g=\mathrm{R}_f\mathrm{R}_f^*$ and $\mathrm{R}_{g^{(n)}}=\mathrm{R}_{f^{(n)}}\mathrm{R}_{f^{(n)}}^*$ are positive operators on $\ell^2(\Gamma,\mathbb{C})$. By the remarks just above the theorem applied to g and g^{-1} instead of f it follows that the spectra $\sigma(\mathrm{R}_g)$ and $\sigma(\mathrm{R}_{g^{(n)}})$ lie in the closed interval $I=[\|g\|_1^{-1},\|g\|_1]$. Fix $\varepsilon>0$. By the Weierstrass approximation theorem there exists a real polynomial Q such that

$$\sup_{t\in I}|\log t-Q(t)|\le\varepsilon\,.$$

Since the spectra of R_g and $\mathrm{R}_{g^{(n)}}$ lie in I it follows that

$$\|\log\mathrm{R}_g-Q(\mathrm{R}_g)\|\le\varepsilon\quad\text{and}\quad\|\log\mathrm{R}_{g^{(n)}}-Q(\mathrm{R}_{g^{(n)}})\|\le\varepsilon\,.$$

Using the estimate $|\mathrm{tr}_{\mathcal{N}\Gamma}A|\le\|A\|$ one has:

$$\begin{aligned}
&|\mathrm{tr}_{\mathcal{N}\Gamma}\log\mathrm{R}_g-\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}\log\mathrm{R}_{g^{(n)}}|\\
&\le|\mathrm{tr}_{\mathcal{N}\Gamma}(\log\mathrm{R}_g-Q(\mathrm{R}_g))|+|\mathrm{tr}_{\mathcal{N}\Gamma}Q(\mathrm{R}_g)-\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}Q(\mathrm{R}_{g^{(n)}})|+|\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}(\log\mathrm{R}_{g^{(n)}}-Q(\mathrm{R}_{g^{(n)}}))|\\
&\le\|\log\mathrm{R}_g-Q(\mathrm{R}_g)\|+|\mathrm{tr}_{\mathcal{N}\Gamma}Q(\mathrm{R}_g)-\mathrm{tr}_{\mathcal{N}\Gamma/\Gamma_n}Q(\mathrm{R}_{g^{(n)}})|+\|\log\mathrm{R}_{g^{(n)}}-Q(\mathrm{R}_{g^{(n)}})\|\\
&\le2\varepsilon+|\mathrm{tr}_{\mathcal{N}\Gamma}Q(g)-\mathrm{tr}_{\mathcal{N}\Gamma^{(n)}}Q(g^{(n)})|\,.
\end{aligned}$$

The formula is a consequence of the following Lemma 3.12. □

Lemma 3.12. *For any f in $\ell^1(\Gamma, \mathbb{C})$ and any complex polynomial $Q(t)$ the following limit holds*

$$\operatorname{tr}_{\mathcal{N}\Gamma} Q(f) = \lim_{n\to\infty} \operatorname{tr}_{\mathcal{N}\Gamma^{(n)}} Q(f^{(n)}) .$$

Proof: Since $Q(f)$ is in $\ell^1(\Gamma, \mathbb{C})$ and $Q(f)^{(n)} = Q(f^{(n)})$ it suffices to prove the assertion for $Q(t) = t$, i.e.

$$\operatorname{tr}_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \operatorname{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)} ,$$

for all f in $\ell^1(\Gamma, \mathbb{C})$. Writing $f = \sum_\gamma f_\gamma e(\gamma)$, one gets $f^{(n)} = \sum_\gamma f_\gamma e(\overline{\gamma})$ where $\overline{\gamma} = \Gamma_n \gamma$. Hence,

$$\operatorname{tr}_{\mathcal{N}\Gamma} f = f_1 \quad \text{and} \quad \operatorname{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)} = \sum_{\gamma\in\Gamma_n} f_\gamma .$$

Fix some $\varepsilon > 0$. If, f is in $\ell^1(\Gamma, \mathbb{C})$, then $\sum_{\gamma\in\Gamma} |f_\gamma| < \infty$. Hence there is a finite subset K of Γ with $1 \in K$ and so one gets $\sum_{\gamma\in\Gamma\setminus K} |f_\gamma| < \varepsilon$. Since $\lim_{n\to\infty} \Gamma_n = \{1\}$ there is an index $N \geq 1$ such that $\Gamma_n \cap K^{-1}K = \{1\}$ for all $n \geq N$. Since $1 \in K$ it follows that $\Gamma_n \cap K = \{1\}$ for all $n \geq N$ as well. For $n \geq N$ the following estimate holds:

$$|\operatorname{tr}_{\mathcal{N}\Gamma} f - \operatorname{tr}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}| = |f_1 - \sum_{\gamma\in\Gamma_n} f_\gamma| \leq \sum_{\gamma\in\Gamma_n\setminus\{1\}} |f_\gamma| \leq \sum_{\gamma\in\Gamma\setminus K} |f_\gamma| < \varepsilon .$$

□

Proof of Theorem 3.10: The theorem follows by combining the results of Theorem 3.1, Theorem 3.9 and Theorem 3.11. □

At the end of this chapter the case $\Gamma = \mathbb{Z}^d$ will be regarded. An element $\mathbf{n} \in \mathbb{Z}^d$ acts isometrically on $L^2(\mathbb{T}^d)$ by pointwise multiplication with the function $\mathbb{T}^d \to \mathbb{C}$ which maps $(z_1, z_2, \ldots, z_d)$ to $(z_1^{k_1}, z_2^{k_2}, \ldots, z_d^{k_d})$. The Fourier transform provides an isometric $\mathbb{Z}^d$-equivariant isomorphism of Hilbert spaces $\mathcal{F} : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d)$.

Definition. For every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{T}^d$ let $\langle \mathbf{n}, \mathbf{t}\rangle = \sum_{i=1}^d n_i t_i$. The Fourier transform of $f \in \mathbb{C}[\mathbb{Z}^d]$ is given by

$$\mathcal{F}(f)(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} e^{2\pi i \langle \mathbf{n}, \mathbf{t}\rangle} .$$

Let $f \in L^\infty(\mathbb{T}^d)$ and define an $\mathbb{Z}^n$-equivariant operator $M_f : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ which sends $g \in L^2(\mathbb{T}^d)$ to $g \cdot f$. And so one obtains an isomorphism

$$L^\infty(\mathbb{T}^d) \xrightarrow{\cong} \mathcal{N}\mathbb{Z}^d .$$

The trace $\operatorname{tr}_{\mathcal{N}\mathbb{Z}^d}$ can be rewritten as

$$\operatorname{tr}_{\mathcal{N}\mathbb{Z}^d}(A) = (A(e(1)), e(1)) = \int_{\mathbb{T}^d} \mathcal{F}(A(e(1)))\, d\mu .$$

If $f \in \mathbb{C}[\mathbb{Z}^d]$, then $\mathcal{F}(\mathrm{R}_f(0)) = \mathcal{F}(f^*) = \overline{f}$ and so one gets the following entropy for $f \in L^1(\mathbb{Z}^d)^\times$:

$$\mathbf{h}(f) = \int_{\mathbb{T}^d} \log |f(z)|\, d\mu(z) .$$

In [23] it was shown that this entropy formula holds for general $\mathbb{Z}^d$-actions. For a detailed study of $\mathbb{Z}^d$-actions consult [32].

4 The Harmonic Model

Reference(s): [33]

First, the model will be introduced and the ASM will be used to find a symbolic cover of this dynamical system. Then the entropy of the harmonic model will be calculated.

4.1 Introduction

Let $R_d = \mathbb{Z}\left[u_1^{\pm 1}, u_2^{\pm 1}, \dots, u_d^{\pm 1}\right] \subset \ell^1(\mathbb{Z}^d)$ be the ring of Laurent polynomials. An element of this space will be written as $h = (h_{\mathbf{n}}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}}$.

Let $d > 1$. Define the shift-action α of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ by

$$(\alpha^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{m}+\mathbf{n}}\,,$$

for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d}$. To every $h \in R_d$ a group homomorphism will be associated by

$$h(\alpha) = \sum_{\mathbf{m}\in\mathbb{Z}^d} h_{\mathbf{m}} \alpha^{\mathbf{m}} : \mathbb{T}^{\mathbb{Z}^d} \longrightarrow \mathbb{T}^{\mathbb{Z}^d}\,.$$

Let $f^{(d)} \in R_d$ and $X_{f^{(d)}} \subset \mathbb{T}^{\mathbb{Z}^d}$ be the subgroup

$$X_{f^{(d)}} = \ker f^{(d)}(\alpha) = \left\{ x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : \left(f^{(d)}(\alpha)\cdot x\right)_{\mathbf{n}} = 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^d \right\}.$$

The restriction of α to $X_{f^{(d)}}$ will be denoted by $\alpha_{f^{(d)}}$. The $\mathbb{Z}^d$-action of $\alpha_{f^{(d)}}$ preserves the normalised Haar measure $\lambda_{X_{f^{(d)}}}$ of $X_{f^{(d)}}$, which is the measure of maximal entropy [32, Theorem 13.3].

Let G be an unweighted graph with vertex set $\mathbb{Z}^d$ and bounded edge degree. Now a Laurent-polynomial is associated to this graph G. Let $\mathbf{y} \in \mathbb{Z}^d$ and write $\mathbf{u}^{\mathbf{y}} = u_1^{y_1} u_2^{y_2} \cdots u_d^{y_d}$. The Laplace-polynomial of a graph G is defined by

$$\Delta_G(\mathbf{u}) = \sum_{\mathbf{y}\sim\mathbf{0}} (\mathbf{u}^{\mathbf{0}} - \mathbf{u}^{\mathbf{y}})\,.$$

The Laplace polynomial of the nearest neighbour graph on $\mathbb{Z}^2$ is given by:

$$\Delta_{\mathbb{Z}^2}(\mathbf{u}) = 4 - u_1^1 - u_1^{-1} - u_2^1 - u_2^{-1}\,.$$

For the remainder of this chapter fix $f^{(d)} = \Delta_{\mathbb{Z}^d}(\mathbf{u}) = 2d - \sum_{i=1}^d (u_i + u_i^{-1})$. Elements $x \in X_{f^{(d)}}$ can be viewed as **harmonic** (mod 1) w.r.t G because, for every $\mathbf{n} \in \mathbb{Z}^d$, $\deg_G(\mathbf{n}) \cdot x_{\mathbf{n}}$ is the sum of its $\deg_G(\mathbf{n})$ neighbouring coordinates (mod 1). The dynamical system $(X_{f^{(d)}}, \mathbb{Z}^d)$ or $(X_{f^{(d)}}, \alpha_{f^{(d)}})$ will be called the **harmonic model**.

In the next section a symbolic cover of $X_{f^{(d)}}$ will be constructed by a shift-equivariant group homomorphism from $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ to $X_{f^{(d)}}$. For this purpose a few preparations must be made. Many definitons and concepts of the last chapter will be used, but often with other descriptions or stronger restrictions. These restrictions are necessary because the action $\alpha_{f^{(d)}}$ is non-expansive.

The cartesian product $W_d = \mathbb{R}^{\mathbb{Z}^d}$ will be identified with the set of formal real power series in the variables $u_1^{\pm 1}, u_2^{\pm 1}, \ldots, u_d^{\pm 1}$ by viewing each $w = (w_{\mathbf{n}})$ as the power series

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} w_{\mathbf{n}} u^{\mathbf{n}}$$

with $w_{\mathbf{n}} \in \mathbb{R}$ and $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$, for every $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. The involution $w \mapsto w^*$ on W_d is defined by

$$w^*_{\mathbf{n}} = w_{-\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d \ .$$

It is clear that the Laurent polynomials R_d lie in W_d. For $E \subseteq \mathbb{Z}^d$ denote the projection onto the coordinates in E by $\pi_E : W_d \to \mathbb{R}^E$.

Next automorphic representations of $\mathbb{Z}^d$ will be introduced: The map $(\mathbf{m}, u) \mapsto u^{\mathbf{m}} \cdot w$ with $(u^{\mathbf{m}} \cdot w)_{\mathbf{n}} = w_{\mathbf{n}-\mathbf{m}}$ is a $\mathbb{Z}^d$-action by automorphisms of the additive group W_d which extends linearly to an R_d-action on W_d given by

$$h \cdot w = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \cdot w \,,$$

for every $h \in R_d$ and $w \in W_d$.

Next a **fundamental solution** $w^{(d)}$ of the equation

$$f^{(d)} \cdot w = e(1) \tag{4.1}$$

will be defined as follows:

1. For $d = 2$,
$$w^{(2)}_{\mathbf{n}} := \int_{\mathbf{T}^d} \frac{e^{-2\pi i\langle \mathbf{n},\mathbf{t}\rangle} - 1}{\mathcal{F}(f^{(2)})(\mathbf{t})} \text{for every } \mathbf{n} \in \mathbb{Z}^2.$$

2. For $d > 2$,
$$w^{(d)}_{\mathbf{n}} := \int_{\mathbf{T}^d} \frac{e^{-2\pi i\langle \mathbf{n},\mathbf{t}\rangle}}{\mathcal{F}(f^{(d)})(\mathbf{t})} \text{for every } \mathbf{n} \in \mathbb{Z}^d \,.$$

Let

$$I_d = \left\{g \in R_d : \ \ g \cdot w^{(d)} \in \ell^1(\mathbb{Z}^d)\right\} \supseteq (f^{(d)})\,,$$

where $(f^{(d)}) = f^{(d)} \cdot R_d$ is the principal ideal generated by $f^{(d)}$. Consult [33, Theorem 2.2] and the references cited there to see that $w^{(d)}$ is a solution of (4.1) and behaves asymptotically well (for $\|\mathbf{n}\| \to \infty$). In [33, Theorem 2.4] it was shown that the ideal I_d is of the form

$$I_d = (f^{(d)}) + \mathfrak{I}^3_d\,, \tag{4.2}$$

where

$$\mathfrak{I}^n_d := \left\{h \in R_d : \frac{\partial^{|\alpha|} h}{\partial^{\alpha_1} u_1 \ldots \partial^{\alpha_d} u_d}(\mathbf{1}) = 0, \ \text{for all } \ \alpha \in \mathbb{N}^d \ \text{ with } \ |\alpha| = \sum_{i=1}^{d} \alpha_i = n-1\right\}.$$

Next a linearization of $X_{f^{(d)}}$ will be constructed. Therefor, define the surjective map $\rho : W_d = \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{T}^{\mathbb{Z}^d}$ by

$$\rho(w)_{\mathbf{n}} = w_{\mathbf{n}} \pmod 1\,,$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $w \in W_d$ and let

$$(\sigma^{\mathbf{m}} w)_{\mathbf{n}} = (u^{-\mathbf{m}} w)_{\mathbf{n}} .$$

Set $W_d(\mathbb{Z}) = \mathbb{Z}^{\mathbb{Z}^d} \subset W_d$ and define the linearization of $X_{f^{(d)}}$ by

$$\begin{aligned} W_{f^{(d)}} &:= \rho^{-1}(X_{f^{(d)}}) = \{w \in W_d \,:\, \rho(w) \in X_{f^{(d)}}\} \\ &= \{w \in W_d \,:\, (f^{(d)}(\alpha)(\rho(w)))_{\mathbf{n}} = 0 \ \text{ for all } \mathbf{n} \in \mathbb{Z}^d\} \\ &= \{w \in W_d \,:\, (f^{(d)} \cdot w)_{\mathbf{n}} \in \mathbb{Z} \ \text{ for all } \mathbf{n} \in \mathbb{Z}^d\} = \{w \in W_d \,:\, f^{(d)} \cdot w \in W_d(\mathbb{Z})\} . \end{aligned} \tag{4.3}$$

Let

$$|t|_{\mathrm{mod}1} = \min\{|t-n| : n \in \mathbb{Z}\}, t \in \mathbb{R}$$

and

$$\mathbb{H}^{(1)}_{\alpha}(X_{f^{(d)}}) = \left\{ x \in X_{f^{(d)}} \ :\ \lim_{\|\mathbf{n}\| \to \infty} \alpha^{\mathbf{n}}_{f^{(d)}} x = 0 \ \text{ and } \ \sum_{\mathbf{n} \in \mathbb{Z}} |x_{\mathbf{n}}|_{\mathrm{mod}1} < \infty \right\}$$

be the subset of homoclinic points of $\alpha_{f^{(d)}}$ with $\alpha^{\mathbf{n}}_{f^{(d)}} x \to 0$ sufficient fast as $\|\mathbf{n}\| \to \infty$. Next recall the definition of $W_{f^{(d)}}$ and set $x^{\mathbb{H}} = \rho(w^{(d)}) \in X_{f^{(d)}}$.

Theorem 4.1. *Every homoclinic point $z \in X_{f^{(d)}}$ of $\alpha_{f^{(d)}}$ is of the form $z = \rho(h \cdot w^{(d)})$ for some $h \in R_d$ and*

$$\mathbb{H}^{(1)}_{\alpha}(X_{f^{(d)}}) = \rho(\{h \cdot w^{(d)} \,:\, h \in I_d\}) .$$

Proof: Let $z \in X_{f^{(d)}}$ be a homoclinic point of $\alpha_{f^{(d)}}$. Choose $w \in \ell^{\infty}(\mathbb{Z}^d)$ with $\rho(w) = z$ and $\lim_{\|\mathbf{n}\| \to \infty} w_{\mathbf{n}} = 0$. The smallness of most of the coordinates of w together with the fact that $f^{(d)} \cdot w \in W_d(\mathbb{Z})$ (see (4.3)) guarantee that $f^{(d)} \cdot w \in R_d$. The first part of the Theorem follows from

$$f^{(d)} \cdot w \cdot w^{(d)} = w \cdot f^{(d)} \cdot w^{(d)} = w \cdot 1 = w ,$$

where the commutativity of the convolution was used. If $z \in \mathbb{H}^{(1)}_{\alpha}(X_{f^{(d)}})$, then $w \in \ell^1(\mathbb{Z}^d)$ and hence $h \in I_d$. Conversely, if $h \in I_d$, then $h \cdot w^{(d)} \in \ell^1(\mathbb{Z})$ and so $z = \rho(h \cdot w^{(d)}) \in \mathbb{H}^{(1)}_{\alpha}(X_{f^{(d)}})$. $\square$

4.2 Symbolic Cover

The next step is to construct for every homoclinic point $z \in \mathbb{H}^{(1)}_{\alpha}(X_{f^{(d)}})$ a shift equivariant group homomorphism from $\ell^{\infty}(\mathbb{Z}^d, \mathbb{Z})$ to $X_{f^{(d)}}$, which will be used to find symbolic covers of $(X_{f^{(d)}}, \alpha_{f^{(d)}})$. Therefor fix g with $\rho(g \cdot w^{(d)}) = z$ and define the group homomorphisms $\bar{\xi}_g : \ell^{\infty}(\mathbb{Z}^d) \to \ell^{\infty}(\mathbb{Z}^d)$ and $\xi_g : \ell^{\infty}(\mathbb{Z}^d) \to \mathbb{T}^{\mathbb{Z}^d}$ by

$$\bar{\xi}_g(w) = (g \cdot w^{(d)})(\sigma)(w) = (g^* \cdot w^{(d)})(w) \ \text{ and } \ \xi_g(w) = (\rho \circ \bar{\xi}_g)(w) . \tag{4.4}$$

It is clear that

$$\bar{\xi}_g(w)_{\mathbf{n}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} w_{\mathbf{n}-\mathbf{k}} \cdot (g^* \cdot w^{(d)})_{\mathbf{k}}$$

converges for every $\mathbf{n}$ by the definitions and constructions of this chapter, hence the homomorphisms above are well defined and fulfil the following equivariance conditions:

$$\begin{aligned} \bar{\xi}_g \circ \sigma^{\mathbf{n}} &= \sigma^{\mathbf{n}} \circ \bar{\xi}_g , & \xi_g \circ \sigma^{\mathbf{n}} &= \sigma^{\mathbf{n}} \circ \xi_g , \\ \bar{\xi}_g \circ h(\sigma) &= h(\sigma) \circ \bar{\xi}_g , & \xi_g \circ h(\sigma) &= h(\sigma) \circ \bar{\xi}_g . \end{aligned} \tag{4.5}$$

Lemma 4.2. *For every $w \in \ell^\infty(\mathbb{Z}^d)$ and $g \in I_d$*

$$(f^{(d)}(\sigma) \circ \overline{\xi}_g)(w) = f^{(d)} \cdot (g^* \cdot w^{(d)}) \cdot w = g^* \cdot (f^{(d)} \cdot w^{(d)}) \cdot w = g^* \cdot w = g(\sigma) \cdot w \tag{4.6}$$

and $\xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) \subseteq X_{f^{(d)}}$.

Proof: By the remarks about $w^{(d)}$ at the beginning of the chapter one gets

$$f^{(d)} \cdot (h^* \cdot w^{(d)}) \cdot v = h^* \cdot (f^{(d)} \cdot w^{(d)}) \cdot v = h^* \cdot v\,, \tag{4.7}$$

for every $h, v \in R_d$. Fix $g \in I_d$ and let $K \geq 1$ and $V_K = \{-K+1, \dots, K-1\}^{\mathbb{Z}^d}$. Then V_K is shift-invariant and compact in the topology of coordinatewise convergence, and the set $V_K' \subset V_K$ of points with only finitely many nonzero coordinates is dense in V_K. For $v \in V_K' \subset R_d$ one has

$$\overline{\xi}_g(v) = (g^* \cdot w^{(d)}) \cdot v$$

and

$$(f^{(d)}(\sigma) \circ \overline{\xi}_g)(v) = f^{(d)} \cdot (g^* \cdot w^{(d)}) \cdot v = g^* \cdot (f^{(d)} \cdot w^{(d)}) \cdot v = g^* \cdot w = g(\sigma) \cdot v \tag{4.8}$$

by (4.4) and (4.7). Since both $\overline{\xi}_g$ and multiplication by g^* are continuous on V_K, (4.8) holds for every $v \in V_K$. By letting $K \to \infty$ one obtains (4.8), for every $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, hence for every $v \in \frac{1}{M}\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with $M \geq 1$, and finally, again by coordinatewise convergence, for every $w \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$.

For the second statement use the fact that, for every $v \in V_K'$,

$$\xi_g(v) = \rho((g^* \cdot w^{(d)}) \cdot v) = (g \cdot v^*)(\alpha)(x^{\mathbb{H}}) \in X_{f^{(d)}}\,.$$

The continuity argument above yields that $\xi_g(v) \in X_{f^{(d)}}$, for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. $\square$

Theorem 4.3. *If $g \in \widetilde{I}_d = I_d \backslash (f^{(d)})$ then*

$$\xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) = \xi_g(\Lambda_{2d}) = X_{f^{(d)}}\,.$$

Proof: Let $x \in X_{f^{(d)}}$ and define $w \in W_{f^{(d)}}$ by $\rho(w) = x$ and $0 \leq w_{\mathbf{n}} < 1$, for every $\mathbf{n} \in \mathbb{Z}^d$. If $v = f^{(d)}(\sigma)(w)$, then $-2d+1 \leq v_{\mathbf{n}} \leq 2d-1$, for every $\mathbf{n} \in \mathbb{Z}^d$. Since $\overline{\xi}_g$ commutes with $f^{(d)}(\sigma)$ by (4.5), (4.8) shows that

$$\xi_g(v) = (\rho \circ \overline{\xi}_g)(v) = g(\alpha)(x)\,. \tag{4.9}$$

Hence

$$g(\alpha)(X_{f^{(d)}}) \subseteq \xi_g(V_{2d}) \subseteq \xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) \subseteq X_{f^{(d)}}\,. \tag{4.10}$$

It will be shown that $Z = g(\alpha)(X_{f^{(d)}}) = X_{f^{(d)}}$. Consider the exact sequence

$$\{0\} \to Y = \ker g(\alpha) \cap X_{f^{(d)}} \to X_{f^{(d)}} \overset{g(\alpha)}{\to} X_{f^{(d)}} \to \{0\}\,. \tag{4.11}$$

Let α_Y and α_Z be the restrictions of α to Y and Z, and α' the action induced by α on $X_{f^{(d)}}/Z$. Yuzvinskii's addition formula [32, Theorem 14.1] implies that

$$\mathbf{h}(\alpha_{f^{(d)}}) = \mathbf{h}(\alpha_Y) + \mathbf{h}(\alpha_Z) = \mathbf{h}(\alpha') + \mathbf{h}(\alpha_Z)\,,$$

where the fact is used that topological entropies of these actions coincide with their metric entropies with respect to the Haar measure. Since $g \in \widetilde{I}_d$, g and $f^{(d)}$ have no common factors, $\mathbf{h}(\alpha_Y) = 0$ by [32, Corollary 18.5] hence $\mathbf{h}(\alpha_{f^{(d)}}) = \mathbf{h}(\alpha_Z)$ and $0 < \mathbf{h}(\alpha_{f^{(d)}}) < \infty$. The Haar measure $\lambda_{X_{f^{(d)}}}$ of $X_{f^{(d)}}$ is the unique measure of maximal entropy for $\alpha_{f^{(d)}}$ as mentioned at the beginning of this chapter and so $\lambda_{X_{f^{(d)}}}(g(\alpha)(X_{f^{(d)}})) = 1$ and $g(\alpha)(X_{f^{(d)}}) = (X_{f^{(d)}})$ as claimed. So far one has

$$g(\alpha)(X_{f^{(d)}}) = \xi_g(V_{2d}) = \xi_g(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})) = X_{f^{(d)}} \, .$$

Let $v' \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with $v'_{\mathbf{n}} = 2d - 1$, for every $\mathbf{n} \in \mathbb{Z}^d$, then $v' + V_{2d} = \Lambda_{4d-1}$ and so $\xi(\Lambda_{4d-1}) = \xi_g(V_{2d}) + \xi_g(v') = X_{f^{(d)}} + \xi_g(v') = X_{f^{(d)}}$. For $M \geq 1$ set

$$Q_M = \{-M, \dots, M\}^d \subset \mathbb{Z}^d \, .$$

For every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}_+)$ and $\mathbf{n} \in \mathbb{Z}^d$ define

$$h^{(v,\mathbf{n})} = \begin{cases} u^{\mathbf{n}} \cdot f^{(d)} & \text{if } v_{\mathbf{n}} \geq 2d \\ 0 & \text{otherwise,} \end{cases}$$

$$H^{(v,\mathbf{n})} = \sum_{\mathbf{n} \in Q_M} h^{(v,\mathbf{n})}, \quad T(v) = v - H^{(v,\mathbf{n})} \, .$$

If

$$D_M(v) = \sum_{\mathbf{n} \in Q_M} v_{\mathbf{n}} \|\mathbf{n}\|^2_{\max} \, , \tag{4.12}$$

where $\|\cdot\|_{\max}$ is the maximum norm on $\mathbb{R}^d$, then $T(v) = v$ if and only if $\mathbf{n} < 2d$ for every $\mathbf{n} \in Q_M$ and

$$D_M(T(v)) \geq D_M(v) + 2 \, .$$

Define inductively $T^n(v) = T(T^{n-1}(v))$, for $n \geq 2$, then there exists for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}_+)$ an integer $K_M(v) \geq 0$ with

$$\widetilde{v}^{(M)} = T^k(v) \, , \qquad \text{for every } k \geq K_M(v) \, .$$

For $v \in \Lambda_{4d-1}$ and any $M \geq 1$, $\widetilde{v}^{(M)}$ satisfies

$$\begin{aligned} & 0 \leq \widetilde{v}^{(M)}_{\mathbf{n}} \leq 2d - 1 && \text{if } \mathbf{n} \in Q_M \, , \\ & \widetilde{v}^{(M)}_{\mathbf{n}} \geq v_{\mathbf{n}} && \text{if } \|\mathbf{n}\| = M + 1 \, , \\ & \sum_{\{\mathbf{n} : \|\mathbf{n}\| = M+1\}} \widetilde{v}^{(M)}_{\mathbf{n}} - v_{\mathbf{n}} \leq (2d - 1) \cdot (2M + 1)^d \, , && \\ & \widetilde{v}^{(M)}_{\mathbf{n}} = v_{\mathbf{n}} && \text{if } \|\mathbf{n}\| > M + 1. \end{aligned}$$

It is clear that $v - \widetilde{v}^{(M)} \in (f^{(d)})$ by construction of $H^{(v,\mathbf{n})}$. And so $\xi_g(v - \widetilde{v}^{(M)}) = 0$ because $v - \widetilde{v}^{(M)} = f^{(d)} \cdot h$ for an $h \in R_d$ and therefore $(g^* \cdot w^{(d)}) \cdot f^{(d)} \cdot h \in \mathbb{Z}$ for every $g \in \widetilde{I}_d$ and $v \in \Lambda_{4d-1}$; hence $\xi_g(v) = \xi_g(\widetilde{v}^{(M)})$. Since $g \in \widetilde{I}_d$, (4.2) implies that there exists a constant $C > 0$ with

$$|(g^* \cdot w^{(d)})_{\mathbf{n}}| \leq C \|\mathbf{n}\|^{d-1}_{\max} \, , \qquad \text{for every nonzero } \mathbf{n} \in \mathbb{Z}^d.$$

Therefore,

$$|\overline{\xi}_g(\widetilde{v}^{(M)})_{\mathbf{0}} - \overline{\xi}_g(\overline{v}^{(M)})_{\mathbf{0}}| \leq 4d \cdot (2M+1)^d \cdot C \cdot (M+1)^{d-1} \to 0$$

as $M \to \infty$ and where

$$\overline{v}^{(M)} = \begin{cases} \widetilde{v}_{\mathbf{n}}^{(M)} & \text{if } \mathbf{n} \in Q_M \\ v_{\mathbf{n}} & \text{otherwise.} \end{cases}$$

And so

$$\lim_{M \to \infty} \overline{\xi}_g(v - \overline{v}^{(M)}) = 0$$

in the topology of coordinatewise convergence. Since

$$\overline{v}^{(M)} \in \{v \in \Lambda_{4d-1} : 0 \leq v_{\mathbf{n}} \leq 2d-1 \text{ for every } \mathbf{n} \in Q_M\},$$

for every $v \in \Lambda_{4d-1}$ and $M \geq 1$ one can conclude that $\xi_g(\Lambda_{2d})$ is dense in $X_{f^{(d)}}$. Since $\xi_g(\Lambda_{2d})$ is closed $\xi_g(\Lambda_{2d}) = X_{f^{(d)}}$. □

In the last proof a toppling argument has occurred already - in the next step the Abelian Sandpile Model will be embedded in the algebraic setting of this chapter. First recall the definition of the ASM and put, for a finite set $K \subset \mathbb{Z}^d$,

$$P_K = \left\{v \in \{0, \dots, 2d-1\}^K \ : \ v_{\mathbf{n}} \geq N_K(\mathbf{n}) \text{ for at least one } \mathbf{n} \in K\right\},$$

then recurrent configurations on K can be written as

$$\mathcal{R}_K = \bigcap_{\substack{\emptyset \neq F \subset K, \\ 0 < |F| < \infty}} P_F \,.$$

Lemma 4.4. *Let $d \geq 2$. The following conditions are equivalent for every $v \in \Lambda_{2d}$.*

1. *$v \in \mathfrak{R}_\infty$.*
2. *For every nonzero $h \in R_d$ with $h_{\mathbf{n}} \in \{0,1\}$ for every $\mathbf{n} \in \mathbb{Z}^d$, $(f^{(d)} \cdot h)_{\mathbf{n}} + v_{\mathbf{n}} \geq 2d$ for at least one $\mathbf{n} \in \operatorname{supp}(h) = \{\mathbf{m} \in \mathbb{Z}^d : h_{\mathbf{m}} \neq 0\}$.*
3. *For every nonzero $h \in R_d$ with $h_{\mathbf{n}} > 0$ for every $\mathbf{n} \in \mathbb{Z}^d$, $(f^{(d)} \cdot h)_{\mathbf{n}} + v_{\mathbf{n}} \geq 2d$ for at least one $\mathbf{n} \in \{\mathbf{m} \in \mathbb{Z}^d : h_{\mathbf{m}} > 0\}$.*

Furthermore, if $v, v' \in \mathfrak{R}_\infty$ and $0 \neq v - v' \in R_d$, then $v - v' \notin (f^{(d)})$.

Proof: ($2 \Rightarrow 1$) Fix $v \in \Lambda_{2d}$. If $h \in R_d$ with $h_{\mathbf{n}} \in \{0,1\}$, for every $\mathbf{n} \in \mathbb{Z}^d$ and $E = \operatorname{supp}(h)$, then $(f^{(d)} \cdot h)_{\mathbf{n}} + v_{\mathbf{n}} \in \{0, 1, \dots, 2d-1\}$, for every $\mathbf{n} \in E$ if and only if $v_{\mathbf{n}} \leq N_E(\mathbf{n}) - 1$. for every $\mathbf{n} \in E$, in this case $\pi_E(v) \notin P_E$ and $v \notin \mathfrak{R}_\infty$.

Let $h \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and $M_h = \max_{\mathbf{m} \in \mathbb{Z}^d} h_{\mathbf{m}} > 0$ and that $(f^{(d)} \cdot h) + v \in \Lambda_{2d}$. Set

$$\mathcal{S}_{\max}(h) = \{\mathbf{n} \in \mathbb{Z}^d : h_{\mathbf{n}} = M_h\}$$

and observe that

$$v_{\mathbf{n}} + (f^{(d)} \cdot h)_{\mathbf{n}} \geq v_{\mathbf{n}} + M_h(2d - N_{\mathcal{S}_{\max}(h)}) < 2d\,,$$

for every $\mathbf{n} \in S_{\max}(h)$ so that

$$v_{\mathbf{n}} \leq N_{\mathcal{S}_{\max}(h)} - 1\,, \qquad \text{for every } \mathbf{n} \in \mathcal{S}_{\max}(h)\,. \tag{4.13}$$

If $h \in R_d$, then $\mathcal{S}_{\max}(h)$ is finite and so (4.13) yields to contradiction to the definition of $\mathfrak{R}_\infty$. This proves the implication $(1 \Rightarrow 3)$ and the reverse implication $(3 \Rightarrow 2)$ is obvious. The last asseration is a consequence of (3). $\square$

Theorem 4.5. *For every $g \in \tilde{I}_d$, $\xi_g(\mathcal{R}_\infty) = X_{f^{(d)}}$. Furthermore, the shift-action $\sigma_{\mathcal{R}_\infty}$ of $\mathbb{Z}^d$ on $\mathcal{R}_\infty$ has topological entropy*

$$\mathbf{h}(\sigma_{\mathcal{R}_\infty}) = \lim_{N\to\infty} \frac{1}{|Q_N|} \log \left|\pi_{Q_N}(\mathcal{R}_\infty)\right| = \mathbf{h}(\alpha_{f^{(d)}})\,. \tag{4.14}$$

For every $Q \subset \mathbb{Z}^d$ $v \in W_d$ set

$$S^{(Q)}(v) = \{v' \in W_d : \pi_{\mathbb{Z}^d\setminus Q}(v') = \pi_{\mathbb{Z}^d\setminus Q}(v)\}\,.$$

If $V \subset W_d$ is a subset $S^{(Q)}_V(v) = S^{(Q)}(v) \cap V$. Fix $g \in \widetilde{I}_d$. Let $0 < \varepsilon < 1/4d$. Since $g^* \cdot w^{(d)} \in \ell^1(\mathbb{Z}^d)$, there is a $K \geq 1$ with

$$|\overline{\xi}_g(v)_{\mathbf{0}} - \overline{\xi}_g(v')_{\mathbf{0}}| < \varepsilon\,, \qquad \text{for every } v, v' \in \Lambda_{2d} \text{ with } \pi_{Q_K}(v') = \pi_{Q_K}(v)\,. \tag{4.15}$$

Lemma 4.6. *Let $v \in \Lambda_{2d}$, $Q \subset \mathbb{Z}^d$ a finite set and $v' \in S^{(Q)}_{\Lambda_{2d}}(v)$.*

1. *$\xi_g(v') = \xi_g(v)$ if and only if $v' - v \in (f^{(d)})$.*
2. *If $\xi_g(v') \neq \xi_g(v)$, then*
$$|\xi_g(v')_{\mathbf{n}} - \xi_g(v)_{\mathbf{n}}|_{\mathrm{mod}1} \geq 1/4d$$
for some $\mathbf{n} \in Q + Q_K$.

Proof: Assume that

$$|\xi_g(v')_{\mathbf{n}} - \xi_g(v)_{\mathbf{n}}|_{\mathrm{mod}1} < 1/4d\,, \tag{4.16}$$

for every $\mathbf{n} \in Q + Q_K$. Since (4.16) holds automatically for $\mathbf{n} \in \mathbb{Z}^d \setminus (Q + Q_K)$ by (4.15) it holds for every $\mathbf{n} \in \mathbb{Z}^d$. Choose $z \in W_{f^{(d)}}$ with $\rho(z) = \xi_g(v') - \xi_g(v)$ and $\|z_{\mathbf{n}}\|_{\max} < 1/4d$. Then $f^{(d)} \cdot z \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ and the smallness of the coordinates of z implies that $f^{(d)} \cdot z = 0$. Since $\rho(z) = \rho(\overline{\xi}_g(v') - \overline{\xi}_g(v))$, $z - (\overline{\xi}_g(v') - \overline{\xi}_g(v)) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$. As the coordinates of z are small and $\lim_{\|\mathbf{n}\|\to\infty} |\overline{\xi}_g(v') - \overline{\xi}_g(v)| = 0$ due to the continuity of $\overline{\xi}_g$, conclude that $h = z - (\overline{\xi}_g(v') - \overline{\xi}_g(v)) \in R_d$. According to (4.6)

$$f^{(d)} \cdot (z - (\overline{\xi}_g(v') - \overline{\xi}_g(v))) = f^{(d)} \cdot h = g^* \cdot (v' - v)\,.$$

As R_d has a unique factorization and g^* is not divisible by $f^{(d)}$, $v' - v$ must lie in the ideal $(f^{(d)}) \subset R_d$. The definitions of $w^{(d)}$ and $\overline{\xi}$ imply that $\xi_g(v') = \xi_g(v)$. $\square$

For $Q \subset \mathbb{Z}^d$ set

$$R(Q) = \{h \in R_d : \mathrm{supp}(h) \subset Q\}\,,$$
$$R^+(Q) = \{h \in R(Q) \ : \ h_{\mathbf{n}} \geq 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}\,.$$

For $L \geq 1$, $v \in \Lambda_{2d}$ and $q \geq 0$ set

$$Y_v(q) = \big\{w \in S^{(Q_{L+K+1})}(v) \ : \text{ for every } \mathbf{n} \in \mathbb{Z}^d,\ 0 \leq w_{\mathbf{n}} < 2d \text{ if } \|\mathbf{n}\|_{\max} \neq L+K+1 \text{ and } -q \leq w_{\mathbf{n}} < 2d \text{ if } \|\mathbf{n}\|_{\max} = L+K+1\big\}$$

and

$$Y'_v(q) = \{w \in Y_v(q) \ : \ \pi_{Q_{L+K}}(w) \in \pi_{Q_{L+K}}(\mathcal{R}_\infty)\}\,.$$

The proofs of the next two lemmas are very technical and will not be given, but they can be found in [33, Lemma 5.5 und Lemma 5.6].

Lemma 4.7. [33, Lemma 5.5.] *Let $L \geq 1$, $q \geq 0$ and $v \in \Lambda_{2d}$. Then*

$$Y'_v(q) = Y_v(q) \setminus \bigcup_{0 \neq h \in S^+(Q_{L+K})} (Y_v(q+1) - h \cdot f^{(d)})\,.$$

Lemma 4.8. [33, Lemma 5.6.] *For every $v \in \Lambda_{2d}$ and $L \geq 1$ there exists an $h \in R^+(Q_L)$ with $v' = v + h \cdot f^{(d)} \in Y'_v((2d-1) \cdot (2L+1)^d)$.*

Proof of Theorem 4.5. The same arguments as in the proof of Theorem 4.3 show that $\xi_g(\mathcal{R}_\infty) = X_{f^{(d)}}$. For this purpose only a few changes are required. Fix $\varepsilon > 0$ and choose K according to (4.15). From Lemma 4.8 one gets $X_{f^{(d)}} = \xi_g(\Lambda_{2d}) = \xi_g(\Lambda_{2d}(L+K+1, (2d-1)\cdot(2L+2K+1)^d))$, where

$$\begin{aligned}\Lambda_{2d}(M,q) = \Big\{ v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \ &: \ v_{\mathbf{m}} < 2d\,, \ \text{for every } \mathbf{n} \in \mathbb{Z}^d, \\ &v_{\mathbf{n}} \geq 0 \ \ \text{, for every } \mathbf{n} \in \mathbb{Z}^d \ \text{ with } \|\mathbf{n}\|_{\max} > M+1, \\ &\sum\nolimits_{\{\mathbf{n} \in \mathbb{Z}^d : \|\mathbf{n}\|_{\max} = M+1\}} v_{\mathbf{n}} \geq -q \ \text{ and } \ \pi_{Q_M}(v) \in \pi_{Q_M}(\mathcal{R}_\infty) \Big\}\,.\end{aligned}$$

Since $\xi_g(\mathcal{R}_\infty) = X_{f^{(d)}}$ one has

$$\mathbf{h}(\sigma_{\mathcal{R}_\infty}) \geq \mathbf{h}(\alpha_{f^{(d)}})\,.$$

Conversly, the map ξ_g is injective on $S^{(Q_L)}_{\mathcal{R}_\infty}(v)$, for every $v \in \mathcal{R}_\infty$ and $L \geq 1$. The set $\xi_g(S^{(Q_L)}_{\mathcal{R}_\infty}(v))$ is a $(Q_{L+K}, 1/4d)$-separated subset of $X_{f^{(d)}}$, by Lemma 4.4 and Lemma 4.6. If $\bar{v} \in \mathcal{R}_\infty$ is given by

$$\bar{v}_{\mathbf{n}} = 2d - 1 \ \text{ for every } \ \mathbf{n} \in \mathbb{Z}^d\,,$$

then $\left|\pi_{Q_L}(S^{(Q_L)}_{\mathcal{R}_\infty}(\bar{v}))\right| = \left|\pi_{Q_L}(\mathcal{R}_\infty)\right|$, for every $L \geq 1$.

For every $L \geq 0$ denote by $n(L+K)$ the maximal size of a $(Q_{L+K}, 1/4d)$-separated set in $X_{f^{(d)}}$. From the definition of topological entropy one gets that

$$\begin{aligned}\mathbf{h}(\sigma_{\mathcal{R}_\infty}) &= \lim_{L\to\infty} \frac{1}{|Q_L|} \log \left|\pi_{Q_L}(\mathcal{R}_\infty)\right| = \lim_{L\to\infty} \frac{1}{|Q_L|} \log \left|S^{(Q_L)}_{\mathcal{R}_\infty}(\bar{v})\right| \\ &= \lim_{L\to\infty} \frac{1}{|Q_L|} \log \left|\xi_g(S^{(Q_L)}_{\mathcal{R}_\infty}(\bar{v}))\right| \leq \lim_{L\to\infty} \frac{1}{|Q_L|} \log n(L+K) \\ &= \lim_{L\to\infty} \frac{1}{|Q_{L+K}|} \log n(L+K) = \mathbf{h}(\alpha_f^{(d)})\,,\end{aligned}$$

which completes the proof of the theorem. □

It is still an unresolved problem whether the mappings from the recurrent configurations of the ASM to the harmonic model are almost one-to-one or not.

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