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„Investigation on the stability of the Trojans of Saturn around L4 (and L5)”

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Investigation on the stability of the Trojans of Saturn around L4 (and L5)

Contents

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1. Introduction

The Lagrange points (also called libration points) mark positions where the centrifugal force finds equilibrium with the combined gravitational forces exerted by the Sun and a planet (Saturn in this case) working on a third (massless) object (Restricted Three-Body-Problem). They are named after Joseph Louis Lagrange (1736 – 1813), who discovered them whilst studying the circular restricted three-body problem and found five stationary points for a massless third body. These three bodies are co-orbiting the common barycentre. Three of the Lagrange points (L1, L2 and L3) lying in the line of the two main bodies (the collinear Lagrange points) represent an indifferent equilibrium, whereas the points L4 and L5 constitute a more or less stable zone around these points where asteroids, also named Trojans, may co-orbit with their planet for very long periods. (See Wikipedia, keywords: Joseph Louis Lagrange, Lagrangian point, Three-body problem).

Trojans are stable for small oscillations as long as the mass ratio of the primaries,

$$\mu = M_2/(M_1 + M_2)$$

(where M_1 and M_2 are the larger and smaller of the primary masses, respectively), satisfying

$\mu \sim\leq 0.0385$ (Fleming and Hamilton 2000),
or roughly is smaller than 1/25 (Dvorak & Freistetter 2005, hereafter: D&F).

In our case for Saturn $\mu = 0.000286$ and so, from this point of view, Saturn should easily be able to hold Trojans.

The first asteroid detected in a Lagrange point was the Trojan named “588 Achilles”, discovered in 1906 by the German astronomer Max Wolf, in the L4 point of the Sun–Jupiter-system. “The name "Trojans" derives from the fact that, by convention, they are each named after mythological figures from the Trojan War”. (See Wikipedia, keywords: Jupiter Trojan).

The Lagrange points L4 and L5 (hereafter sometimes called the L4 point and L5 point or just L4 and L5) are co-orbiting with the planet and travel respectively 60° ahead of and 60° behind their planet in a 1:1 Mean Motion Resonance (MMR). Therefore they are also called triangular Lagrange points with L4 the leading and L5 the trailing point. Trojan asteroids are distributed in two elongated, curved regions around these Lagrange points. It is interesting that this region covers only a very tiny band of less than half an AU, whereas the longitudinal extension goes less than half the way from the equilateral point L4 to Saturn, but about twice as far in the other direction (see Fig. 1).

Trojans are trapped asteroids, librating slowly around the exact L4 and L5 points in a tadpole or horseshoe like orbit. Horseshoe Trojans travel in a large kidney bean-like orbit between the L4 and L5 points but on the opposite side of the planet to the Sun, therefore crossing the Lagrange L3 point. Tadpole-like or “banana-shaped” orbits keep to the region of their L4 and L5 point respectively. (See e.g. D&F; Wikipedia, keyword: Horseshoe orbit).

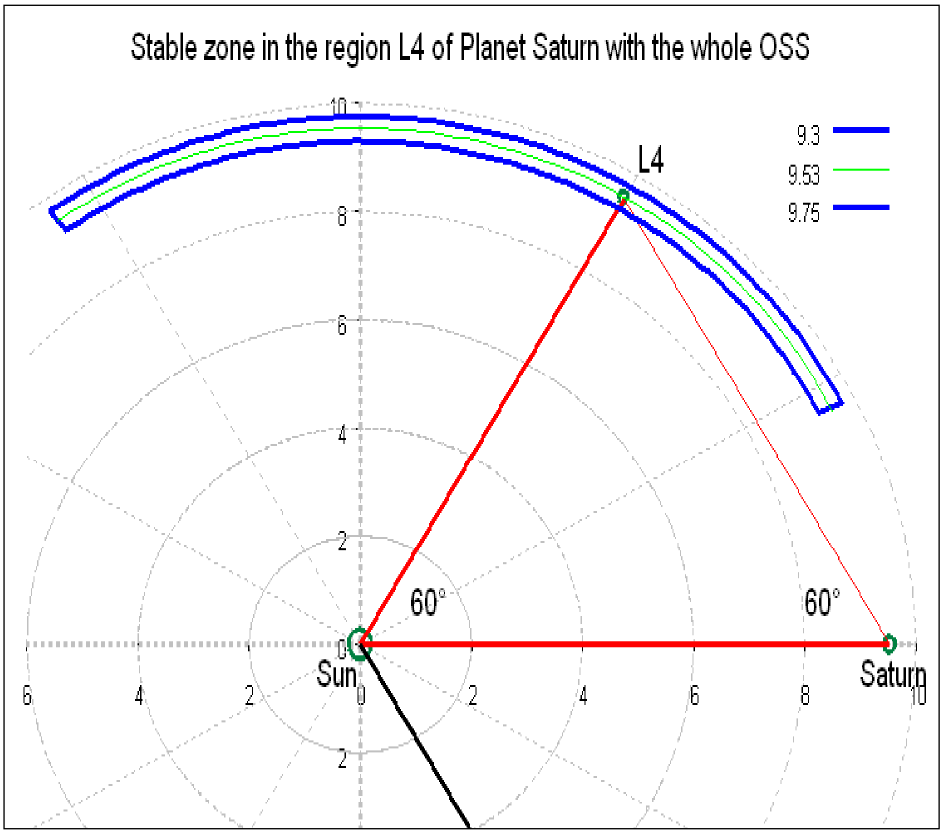

Fig. 1: Indication of the possible more or less stable region of the leading Lagrange point L4 of Saturn (blue shows the limit of the region). The revolution of the system there is anti-clockwise. The distances from the Sun are in AU. The two bold red lines indicate the flash from the Sun to the position of Saturn and to the L4 point. With the third red line they form an equilateral triangle and therefore L4 and L5 are also called the triangular Lagrange points.

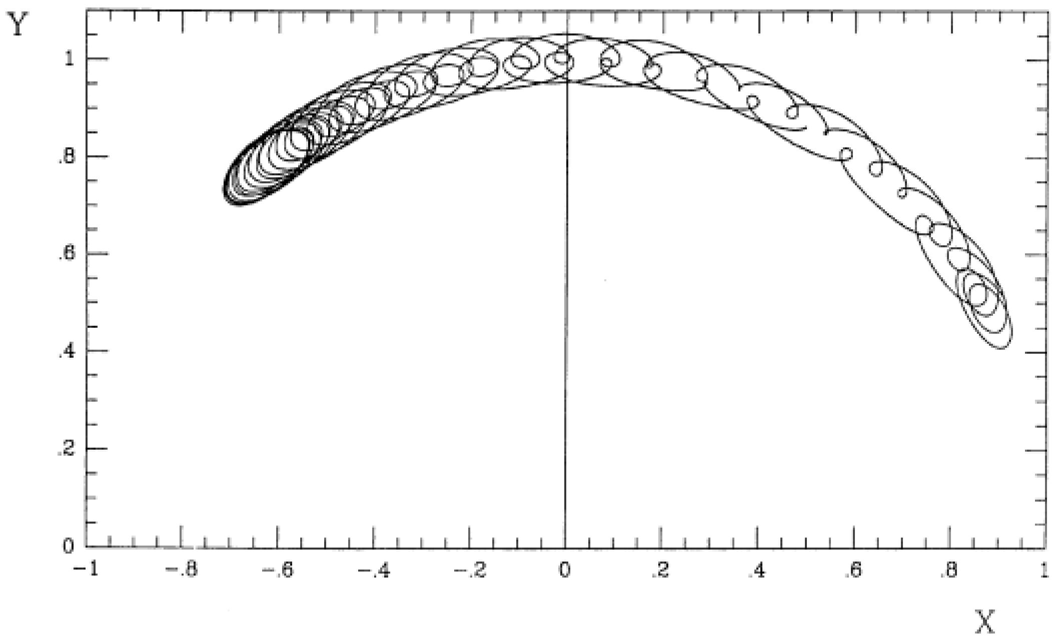

Fig. 2: The tadpole-like orbits at the Lagrange point L4 with starting point at ω_6 +60° and a_{TR} - 0.04 a_6 in 1500 yr (after Zhang and Innanen 1988)

Here, in our study, we concentrate on tadpole Trojans in the 1:1 MMR but shall not forget those becoming horseshoe orbits.

In the solar system the role of Jupiter is known as dominant, due to its great mass of 317.8 times the mass of the Earth, whereas the mass of Saturn is only 95.2 times the mass of Earth (see e.g. Unsöld and Baschek 2002; Table in Annexe 1). This means also that the mass of Jupiter is much greater than the mass of all the other planets and asteroids of the solar system in total. This difference in masses and the relative proximity with the known near MMR of Jupiter and Saturn with 5:2 must have an important influence on the orbits of Saturn and its possible Trojan asteroids. On the one hand this resonance stabilises the orbit of Saturn and on the other hand it could have an influence on the instability of its asteroids.

It is interesting, that already several thousands of Jupiter's asteroids around the Lagrange points L4 and L5 have been detected in recent years. The total number of Jupiter Trojans larger than 1 km is believed to be about 1 million (Wikipedia, keyword: Jupiter Trojan). Also some Trojans have been detected in L4 of Neptune and it is even suspected that Neptune could harbour more Trojans than Jupiter (see e.g. Zhou, Dvorak and Sun 2009, hereafter: ZDS; Nesvorny and Vokrouhlický 2009). However no such asteroids of Saturn and Uranus have been detected up to now.

In the Inner Solar System (ISS) only Mars is known to have co-orbiting Trojan asteroids, with a confirmed population of at least 4 objects of which several have orbits that are stable in solar-system timescales (Rivkin et al. 2007). Even the Earth has one known asteroid but this Trojan "2002 AA_{29}" is a horseshoe asteroid (Wajer 2009). No Trojans were found up to now for Venus, which only could hold transient (temporary) Trojans with short lifetimes (Scholl, Marzari and Tricarico 2005)

The discovery of Trojans in the Outer Solar System (OSS) as well as in the ISS means, that any planet in principle could house asteroids in the gravitational neutral regions of their Lagrange points L4 and L5. Therefore it must be interesting why Saturn, as the second massive planet of our solar system, should harbour no such asteroids or perhaps they have not yet been discovered.

Several investigations on this subject have been undertaken to understand why Saturn cannot hold Trojan asteroids and what the disturbing elements are. We divide our studies in two main chapters.

The first chapter (starting with position 2) concentrates on the fictitious L4-Trojans of Saturn. In section 2.1 of this chapter we shall introduce some prior main studies on the subject and also ideas, limits and methods from our own investigations. In section 2.2, we present the results of our different simulations with fictitious Trojans of Saturn. As we found some interesting jumps in several Trojan orbits, we investigate them in subsection 2.2.5.

The second chapter (starting with position 3) investigates Trojans with two giant planets in the orbits of Jupiter and Saturn where, in section 3.1, we placed two Jupiter in these two orbits. In section 3.2 we did the same with two Saturn. In each case we study the behaviour of L4-Trojans of both planets. Section 3.3 gives a summary of the results with the two giant planets.

Position 4 is finally devoted to a global summary and discussion.

2 Investigations on L4-Trojans of Saturn in the Outer Solar System (OSS)

2.1 Previous studies and Methods

2.1.1 Main studies on the subject

Several investigations in the past have been undertaken to understand if Saturn could hold Trojan asteroids and if not what the disturbers are.

Regarding the first studies on Saturnian Trojans, the work of Zhang and Innanen (1988) has to be mentioned. They investigated in a simple two-dimensional study several different orbits for 10^5 yr and postulated that their intuitive guess that Saturn's L4 and L5 were un-

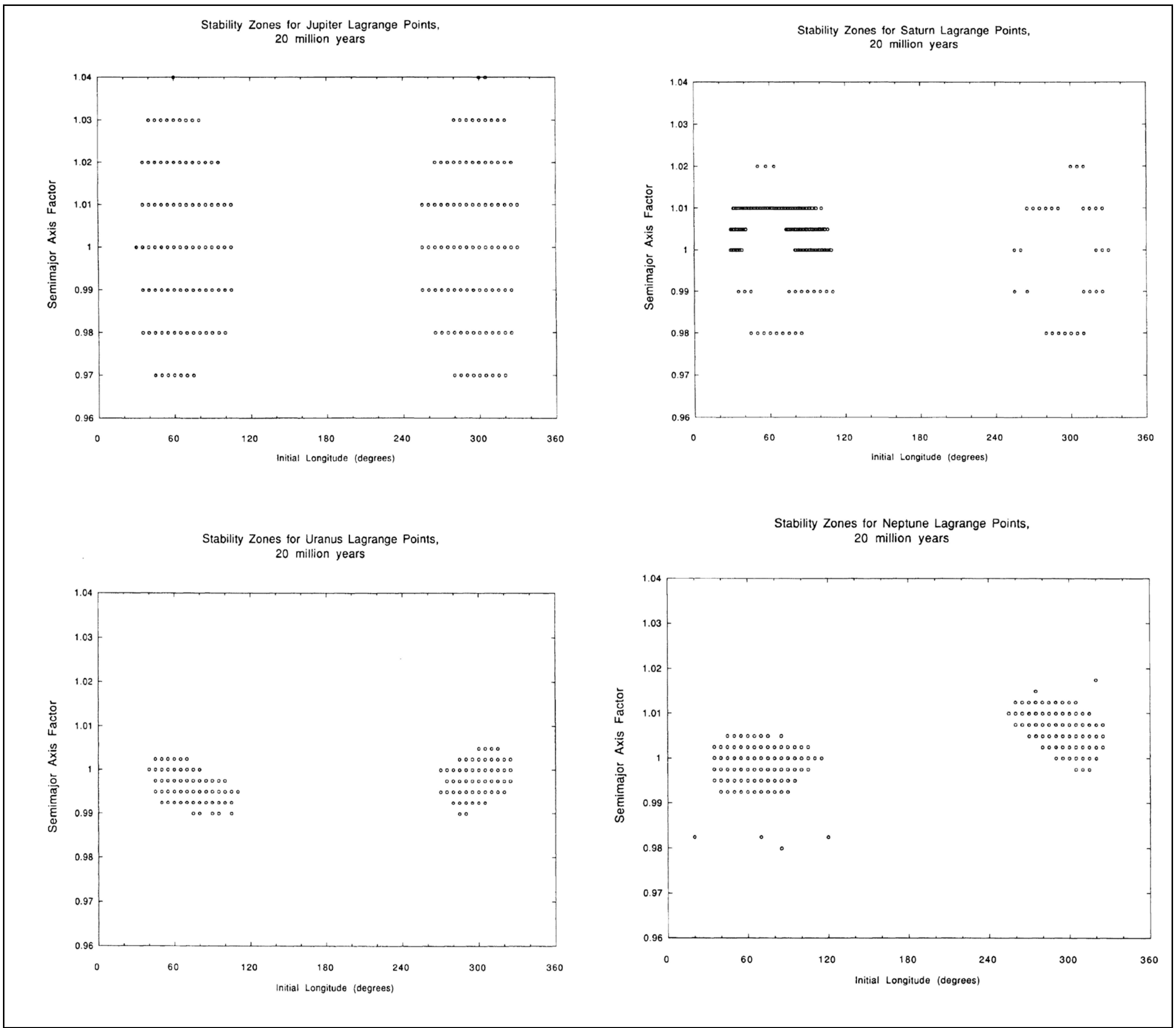

Fig. 3: A point is plotted for each test particle that survived the full 20 million yr integration. The axes show the initial displacement in longitude from the corresponding planet and factor by which that planet's semi-major axis is multiplied to initialize the semi-major axis of the test particles. A two-dimensional stable region lies near the triangular Lagrange points of each of the planets surveyed. Top left: Jupiter, top right: Saturn, bottom left: Uranus and bottom right: Neptune. (H&W)

stable due to Jupiter's perturbations were wrong and that the points of maximum stability are only slightly displaced from the "classical positions" of the Lagrange points. But just one year later Innanen and Mikkola (1989, hereafter: I&M) found already a zone of instability in the straight proximity of L4 and L5 where the Trojans approach their planet in very short time and where, on the contrary, one would anticipate the most stable orbits. I&M stated that the perturbing mechanism is the near MMR of 2:5 with Jupiter and notably postulated the "Great Inequality" of 700 to 800 yr of this near resonance with eccentricities $e > 0.15$. They studied the evolution of test particles for 10 Myr and found no real evidence that Saturn could not hold L4 and L5 Trojans beyond these zones.

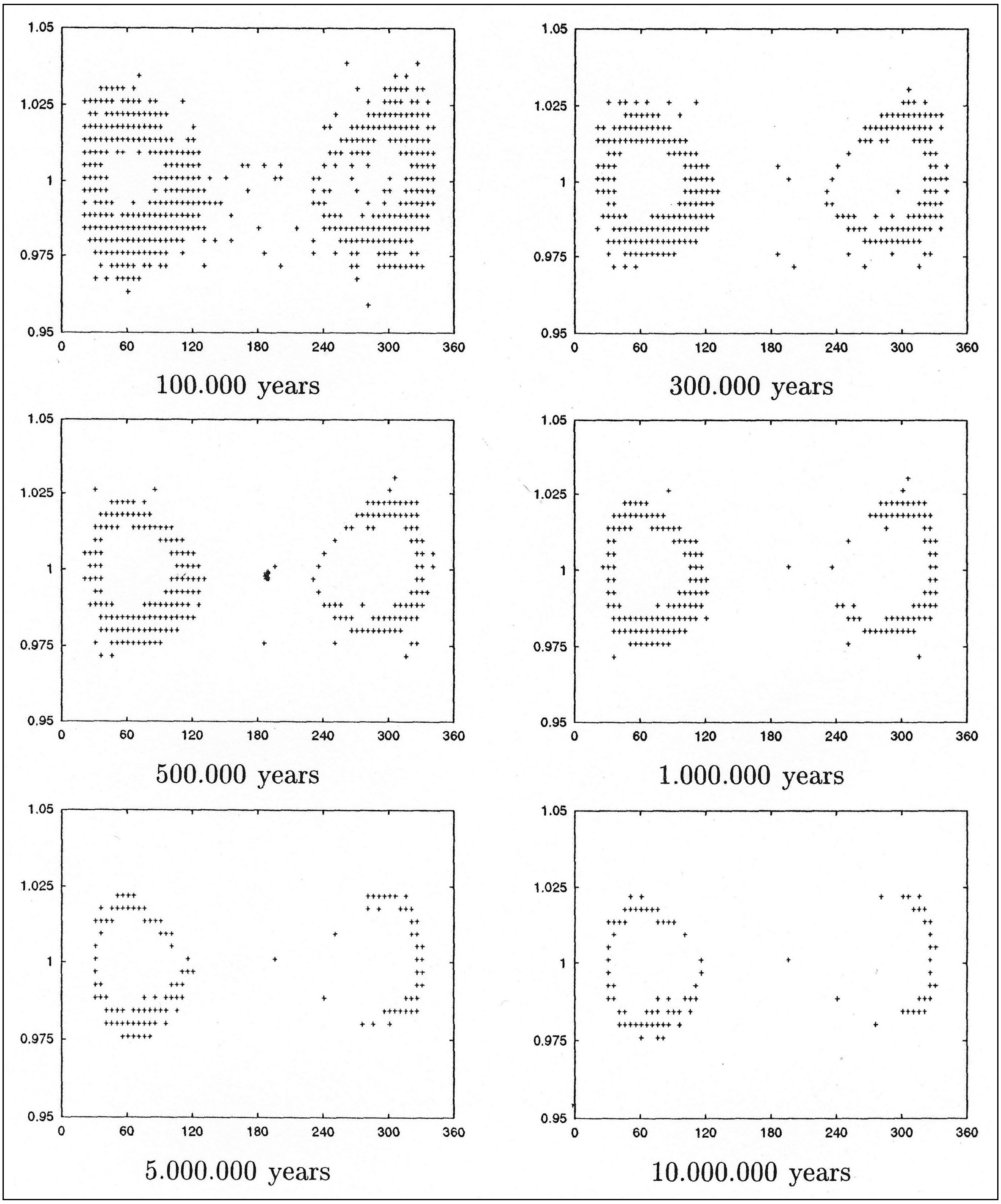

Fig. 4: The neighbourhood of the L4 and L5 triangular Lagrangean points of Saturn for different values of integration time when the mutual inclination is 0° (Téger 2000).

Holman and Wisdom (1993, hereafter: H&W) were probably the first to undertake a complete research of stable orbits in the whole OSS. They studied the evolution of about 4000 test particles for intervals up to 20 Myr near the Lagrange points of the four outer planets, Jupiter, Saturn, Uranus and Neptune, utilising for integration their symplectic mapping method, where the Sun, planets and test particles interact in the full three-dimensional n-body sense (Fig. 3). H&W outlined the phenomenon of "holes" of instability in the centre of the more or less stable regions L4 and L5 of Saturn. This is unlike the other giant planets. They realised that test-particles placed near L4 and L5 ex-

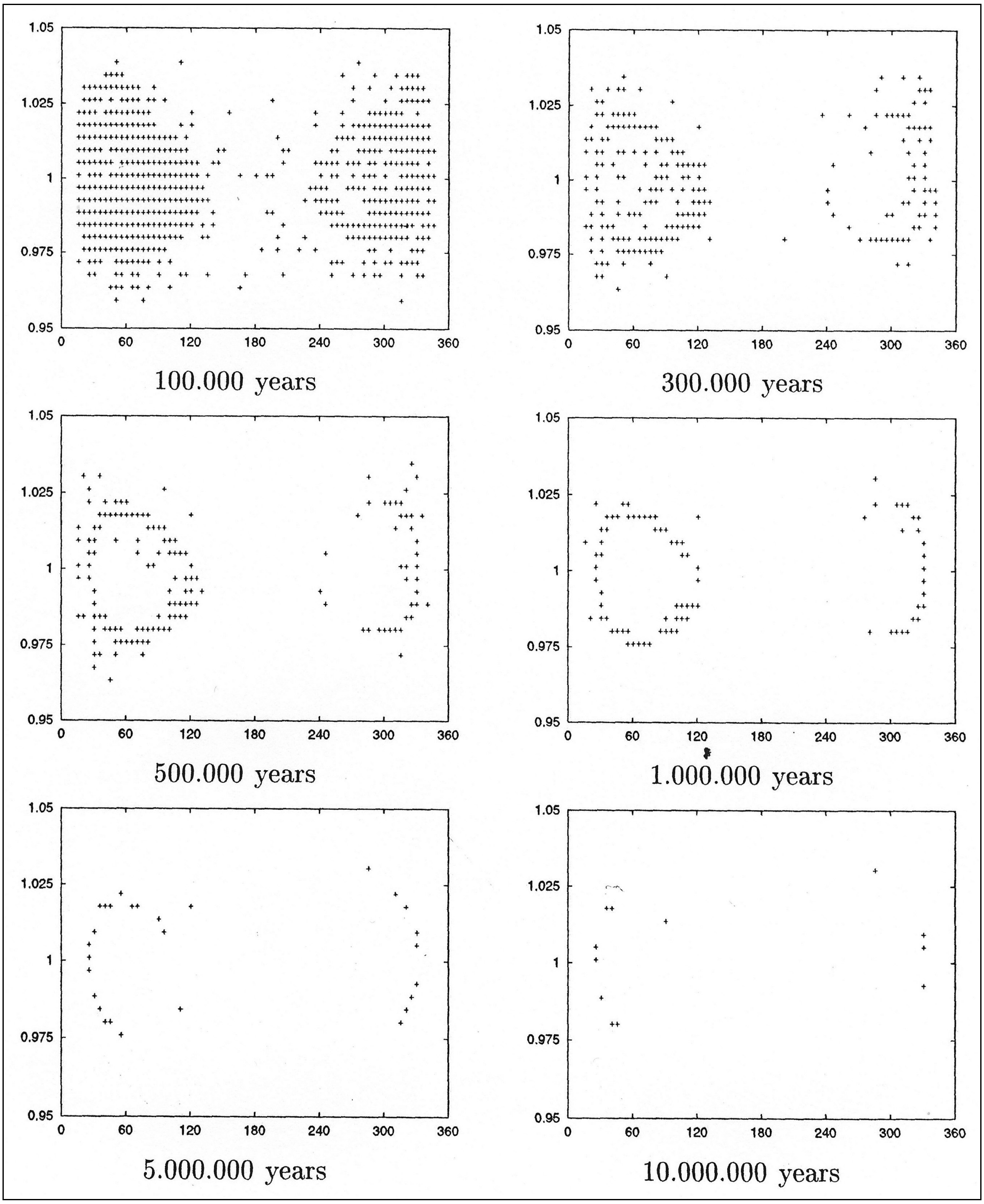

Fig. 5: The neighbourhood of the L4 and L5 triangular Lagrangean points of Saturn for different values of integration time when the mutual inclination is 15° (Téger 2000).

perience close encounters with the planet in short time-scales, but test particles further from the Lagrange points remain for the full integration time of up to 20 Myr of integration. They also mention the asymmetry of the L4 and L5 regions of Neptune, which was not confirmed by Dvorak, Lhotka and Schwarz (2010, hereafter: DLS). Finally H&W found no evidence that Saturn, Uranus and Neptune cannot retain Trojan-like asteroids.

Another study about the stability of test particles near the Lagrange points of Saturn was established by Téger (2000, hereafter: Téger). He used the same symplectic mapping method as H&W and made integrations up to 10 Myr with the whole OSS. He also underlined the existing "hole" of instability in both of the Lagrange points L4 and L5 and found that the holes are growing with integration times and with increasing inclinations. It also seems clear that the point L5 is less stable than L4 (Figs. 4 and 5).

Perhaps the most important study to date about the hypothetical Trojan population of Saturn comes from Nesvorný and Dones (2002, hereafter: N&D). They immediately found that horseshoe orbits of Saturn are strongly unstable. Concerning the tadpole orbits N&D stated that "all orbits, starting near L4 at $9.46 < a < 9.64$ AU, escape in less than a few million years". They also found with their investigations by surfaces of section, that Saturn's 2:5 MMR with Jupiter produces the large-scale chaos in Saturn's co-orbital space. "The reason why Jupiter's 2:5 MMR affects Saturn's Trojans so much while Saturn's 5:2 MMR has a negligible effect on Jupiter's Trojans … is mainly because of the mass difference between these planets, …" (N&D). After N&D the 2:5 MMR grows in size with the eccentricity as $e^{3/2}$ and is large at $e \geq 0.13$. Furthermore N&D studied the migration theory by mutually displacing Jupiter and Saturn to values presumed to be taken by Jupiter's and Saturn's semi-major axis at different epochs of their radial migration (Fig. 6). They varied the exact 2:5 MMR with Jupiter by several steps from $\Delta = 0.8$ to $\Delta = 0$ (with $\Delta \approx 0.025$ today's value of the MMR) and concluded "that the radial migration could have indeed caused a significant depletion of a pre-existing population of Saturn's primordial Trojans at small to moderate amplitudes" (N&D). Finally N&D did 4 Gyr low-resolution surveys with the symplectic integrator of H&M and suggested "that Saturn's Trojans are only marginally stable for libration amplitudes 50°-80°, $e < 0.1$ and small i" (N&D).

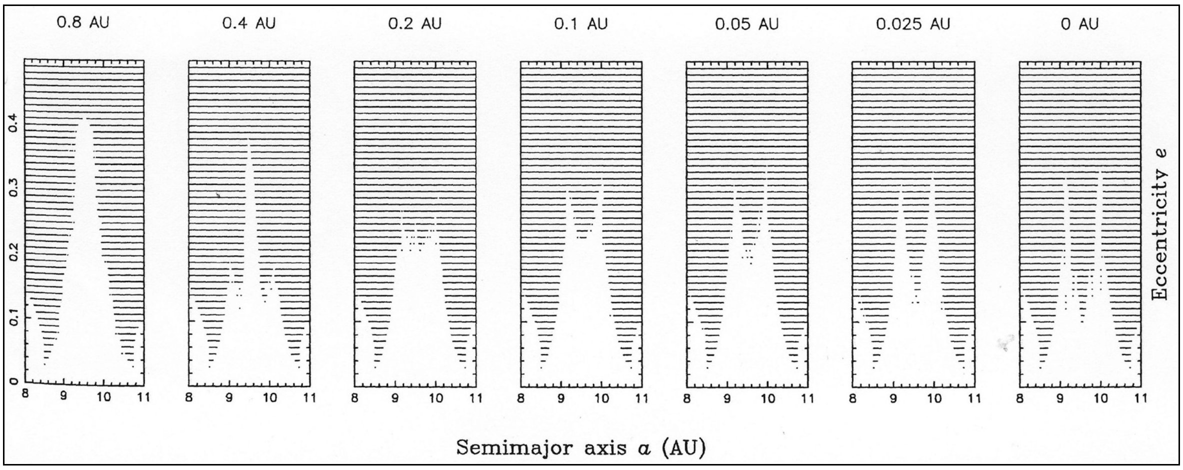

Fig. 6: "The dynamical stability of orbits of Saturn's L4 point during the planetary migration. The number above each graph indicates the distance of Saturn from the exact 2:5 MMR with Jupiter. Orbits unstable on 10^5-year time scales in the bi-circular model are shown by dots". (N&D)

An interesting investigation on the long term stability of Trojans of Saturn was undertaken by Mazari, Tricarico and Scholl (2002, hereafter: MTS) who found on frequency studies over 4.5×10^9 yr, that Trojans with inclinations lower than 15° have a half-life of about 2.5

Gyr. "Orbits with inclinations of about 20° are destabilized by a secular resonance with the forcing term $2g_6$ - g_5" (MTS), where g_6 and g_5 represent the fundamental frequencies, which are related to the revolution frequencies of the perihelion longitudes of Jupiter and Saturn, respectively. "At higher inclinations Saturn Trojan orbits are unstable on a short timescale (a few $x10^5$ yr)" (MTS).

2.1.1 Limit and method of the investigations

2.1.2.1 Borders of possible stable zones

To find reasonable borders for our investigations a first approximation can be done by calculation of the Hill's sphere of the neighbouring great planets and Saturn itself. A Trojan entering a Hill's sphere can no longer survive and will be swallowed by the planet in very short time. As it should be only a first rough estimation we can use the simple equation of the Hill's sphere neglecting eccentricity (Wikipedia, keyword: Hill sphere),

$$r \approx a \sqrt[3]{\frac{m}{3M}}$$

where r is the radius of the Hill's sphere, a the semi-major axis of the planet, m the mass of the planet and M the mass of the star (here the Sun). This gives in solar units and AU for Saturn, Jupiter and Uranus respectively

Saturn $r_{Hill} \approx 9.530 * \sqrt[3]{\frac{0.286 * 10^{-3}}{3 * 1}} = 0.435\,AU$

Jupiter $r_{Hill} \approx 5.203 * \sqrt[3]{\frac{0.955 * 10^{-3}}{3 * 1}} = 0.355\,AU$

Uranus $r_{Hill} \approx 19.235 * \sqrt[3]{\frac{0.436 * 10^{-4}}{3 * 1}} = 0.469\,AU$

This means that a stable zone for Saturn Trojans could only be found at a distance of the Sun between Jupiter and Uranus from 5.203 + 0.355 =5.558 or roughly 5.6 AU to about 19.235 – 0.469 =18.766 or roughly 19 AU and off Saturn with a radius of at least 0.435 AU. In terms of Saturn, it means that stable zones in principle could exist at a distance from the Sun between 0.68 a_6 and 1.88 a_6 but not around Saturn itself for at least 0.046 a_6 which corresponds to nearly +/- 3° in longitudinal distance of Saturn. The subscript '6' denotes Saturn and will be used hereafter for all elements of this 6^{th} planet in the Solar system. Therefore the subscripts '5', '7' and '8' represent Jupiter, Uranus and Neptune respectively. Other subscripts 'L4' and 'TR' will concern the L4 point respectively Trojans. For our purposes the L4 point is set to $a_{L4} = a_6$ and $\omega_{L4} = \omega_6$+60°, even if this is not totally correct (see Wikipedia, keyword: Lagrangian point).

To ascertain the stability zones of Trojans of Saturn, where such asteroids could be found, we also can look at the stable zone of Jupiter. As there are already more then 4000 known Jovian Trojans, we can deduce borders which should also be valid for Saturn Trojans.

In December 2008 we investigated the eccentricity and inclination of Jovian Trojans L4 and L5 (Exercises of the lecture "Architecture of Planetary Systems" (Dvorak, R. and Pilat-Lo-

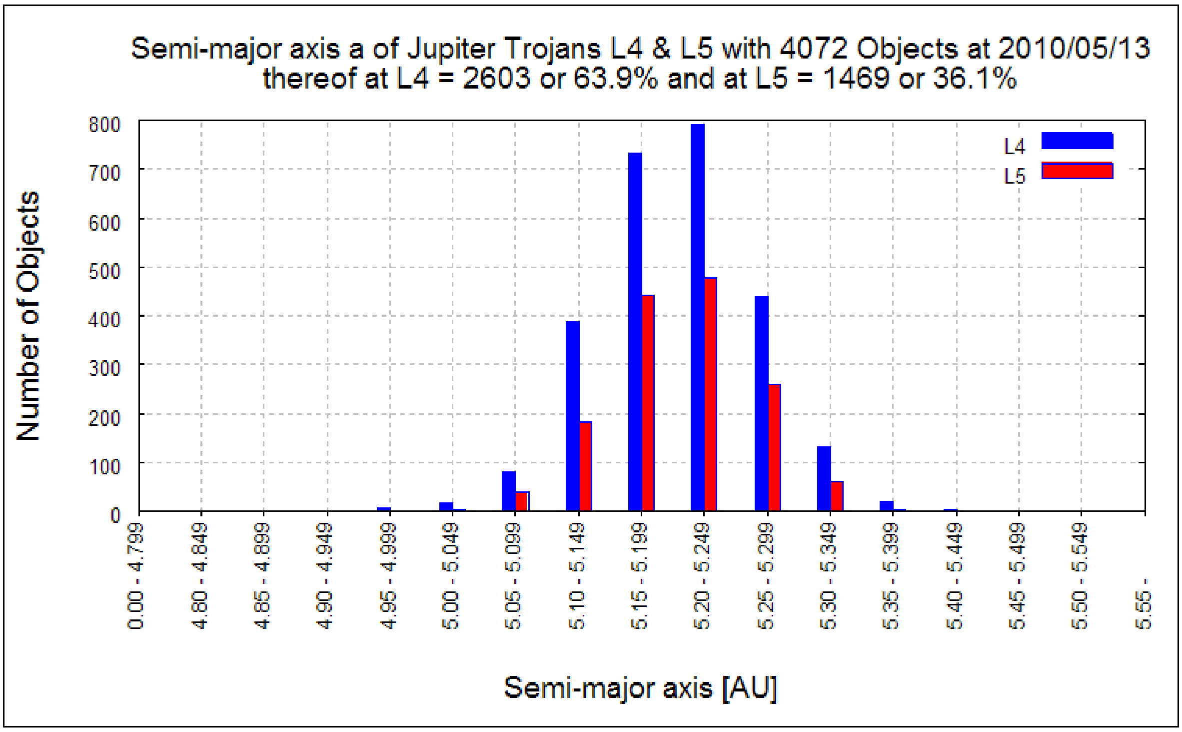

Fig. 7: Number of stable Trojans of Jupiter against their semi-major axis. There are no asteroids found below 4.95 AU and beyond 5.45 AU, corresponding to about 0.95 a_5 respectively 1.05 a_5.

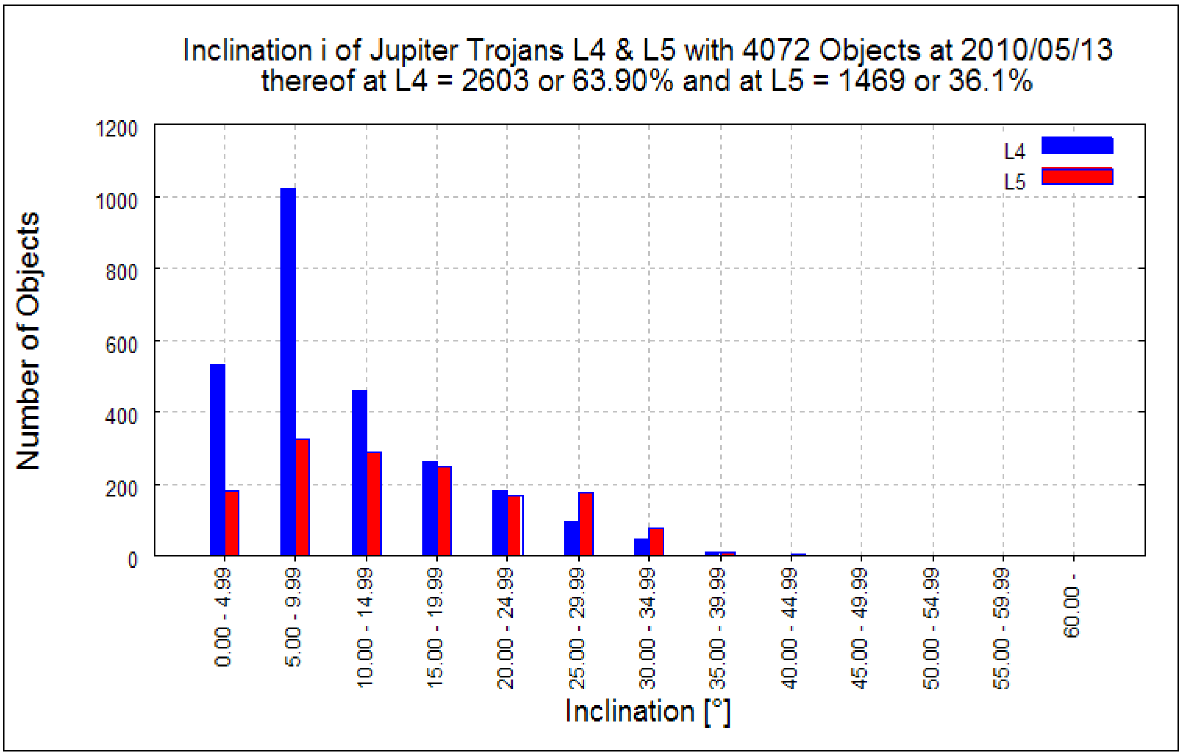

Fig. 8: Number of stable Trojans of Jupiter against their inclination. There are practically no asteroids found with an inclination of more than 40°.

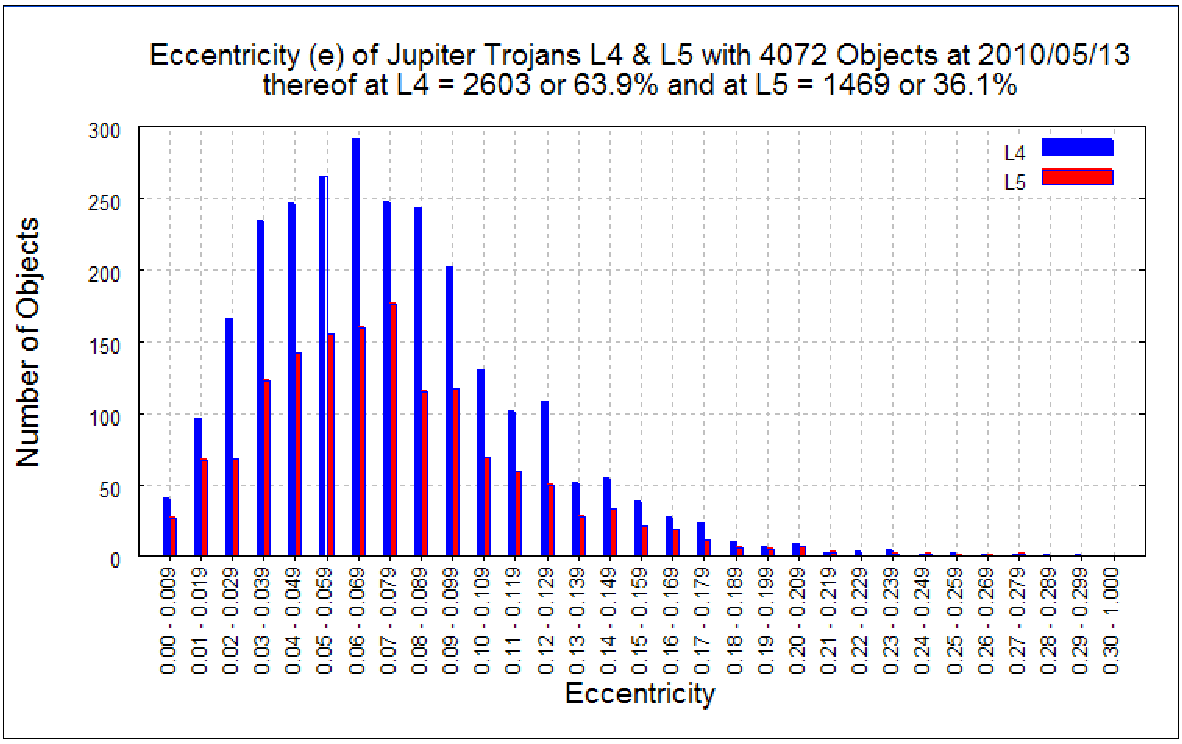

Fig. 9: Number of stable Trojans of Jupiter against their eccentricity. There are no asteroids found with more than $e_5 > 0.3$ (see also D&F).

hinger, E., winter 2008/2009). With the new at, 13th May 2010 data from the "List of Jupiter Trojans", (http://www.cfa.harvard.edu/iau/lists/JupiterTrojans.html), when already 4072 Trojans were known. We found that the asteroids are located at a semi-major axis $4.95 < a_5 < 5.45$ AU, corresponding to about 0.95 a_5 to 1.05 a_5, with an eccentricity $0.0 < e_5 < 0.3$ and an inclination $0° < i_5 < 45°$. This means that there are no stabile orbits found beyond these borders (see Figs. 7 - 9).

If we assume that for Saturn similar limits must be given, we can in principle limit our investigations for stable Trojans of Saturn to a semi-major axis a between 0.95 a_6 and 1.05 a_6, corresponding to a distance of the Sun from 9.1 to 10.0 AU on the one side and on the other side for a maximum eccentricity of $e_6 = 0.3$ as well as an inclination of no more than 45° for L4 Trojans. These borders are in good accordance with the calculated stable zones of the above mentioned studies of H&W, Téger and N&D.

The amplitude of the longitudinal extension of Jupiter Trojans is known to be very large. We base the limits of our studies, the investigated grid and the numbers of Trojans of Saturn on the results of the above mentioned studies and adapt them according to the found limits of stability.

2.1.2.2 Methods used in the investigations

All observations (see Figs. 3 - 5) and studies (e.g. H&W; Téger; ZDS) show that Trojans in L4 and L5 of all investigated planets are nearly mirror images in distances, eccentricities, inclinations, extensions, times of survival and numbers of surviving asteroids. If there is a small asymmetry it is noted in the number of surviving asteroids, where it seems that the number is always slightly higher in L4. As we want to study possible stable zones for the Lagrange points of Saturn we shall limit our studies on L4 to this planet.

For our simulations we used the three-dimensional elliptic n-body problem with the Lie-integration method with an adaptive step size (Hanselmeier and Dvorak 1984; Lichtenegger 1984; Delva 1984). This method has been extensively and successfully used in many numerical studies with L4 and L5 Trojans (e.g. ZDS; Dvorak, Bazsó and Zhou 2010, hereafter: DBZ; Schwarz, Süli and Dvorak 2002, hereafter: SSD). We limited our integrations to the Sun and the giant planets Jupiter, Saturn, Uranus, Neptune (OSS) and an adapted number of fictitious and massless Trojan asteroids of Saturn, ignoring the planets of the ISS. In general we started the fictitious Trojans in the vicinity of the L4-point with the same eccentricity ($e_{TR} = e_6$), inclination ($i_{TR} = i_6$, which means in the plane of Saturn), mean anomaly ($M_{TR} = M_6$) and longitude of the node ($\Omega_{TR} = \Omega_6$), relating to Saturn. The starting points of the planets of the OSS were taken as indicated in the Table in Annexe 1. When we write in our work about Trojans of Saturn, we always mean fictitious Trojans, as no such asteroids have been detected up to date.

To ascertain the necessary grid for our investigations, we ran a premier study for only 10^5 yr in subsection 2.2.1 which follows. In general we limit our investigations to 10^6 yr. In crosscuts, with the L4 point in the middle, we extend our investigations to 10^7 yr (in subsection 2.2.4). The Trojans there are distributed in radial directions, depending on the distance of the planet from the Sun, in steps of +/-0.002 a_{L4} ($\Delta a = 0.002\ a_{L4}$) for the Trojans of Saturn ($a_{L4} = a_6$) and in longitudinal directions, the perihelion argument ω_{TR} ($\omega_{TR} = \omega_6 + 60°$) is always varied in steps of +/-3° ($\Delta\omega = 3°$).

The escape time of a celestial body generally is defined as the time a chaotic orbit needs to leave the vicinity of an island of stability (D&F). To distinguish between stable and unstable orbits, we found, at least for the Trojans of Jupiter (Fig. 9), that Trojans do not survive at eccentricities of more than 0.3. This should also be valid for the Trojans of Saturn. We therefore define the "escape time" as the moment, when the Trojan exceeds a value of $e_{TR} >$ 0.3 and so becomes unstable. Therefore stable orbits are those where $e_{max,TR} <= 0.3$ with respect to the integration time. In the literature other escape times are also used (e.g. Tsiganis, Dvorak and Pilat-Lohinger 2000, hereafter: TDP, used the Liapunov time T_L as T_{esc}; T_L is defined as $T_L = \gamma^{-1}$, where γ is the Liapunov Characteristic Exponent (LCE). As we shall see later on, instability appears in more than one parameter.

The behaviour of the Trojans in radial and longitudinal directions is expressed in amplitudes, measured from the L4-point. Therefore we call the radial oscillations in relation to the L4 point "radial amplitudes", measured in Astronomical Units [AU], and the longitudinal oscillations in relation to the L4 point "longitudinal amplitudes", measured in Degrees [°]. In the literature, longitudinal amplitudes are normally designed as "libration amplitudes" (e.g. Marzari and Scholl 2007, hereafter: M&S; Kortenkamp, Malhotra, and Michtchenko 2004, Fleming and Hamilton 2000; TDP), or "width of libration" (D) (Tsiganis and Dvorak 2000, hereafter: T&D), with the critical argument (σ) for Trojans in the 1:1 MMR of the planet

$$\sigma = \lambda_{TR} - \lambda_{planet}\ , \text{ where}$$

$$\lambda = \omega + \Omega + M \qquad \text{and } \lambda \text{ is the mean longitude of the body.}$$

These "libration amplitudes" are, strictly speaking, only appropriate for circular orbits. We decided to calculate the positions of the bodies in their elliptic rotation and therefore understand in our work that "longitudinal amplitudes" means the width of oscillation in the elliptic motion in both longitudinal directions of the L4 point. To make it clear, we always study the relative motions of the Trojans in relation to the L4 point. Therefore our critical argument should be written as:

$$\sigma = \Gamma_{TR} - \Gamma_{L4} - 60°, \text{ where}$$

$\Gamma = \omega + \Omega + v$ and Γ is now the true longitude of the body and v is the true anomaly of the body.

From the mean anomaly (M) we obtained the true anomaly (v) as described in the Annexe 2. The longitudinal amplitude of Trojans in our work is calculated as

$$\Gamma_{TR} - \Gamma_{L4}$$

In our study on inclinations (subsection 2.2.3), we used steps of $\Delta i = 2.25°$ up to 70° above i_6 and $\Delta i = 1°$ up to 25° above i_6, the invariable plane of Saturn.

2.2 Results

2.2.1 Investigation for 10^5 years

In a first run we investigated the stable zones of the Lagrange point L4 in 100,000 yr with the whole OSS, that is with Sun, Saturn, Jupiter, Uranus and Neptune (SJUN) and 1394 starting fictitious Trojan asteroids around L4 of Saturn (Fig. 10). (In our plots we have to use 'w' instead of 'ω', and also superscripts and subscripts are not available). This grid represents a region of the semi-major axis between 0.968 a_6 and 1.036 a_6 or about 9.24 < a_{TR} < 9.87 AU and between $\omega_{TR} = \omega_6+15°$ and $\omega_{TR} = \omega_6+132°$, respectively from -45° of ω_{L4} to +72° of ω_{L4}.

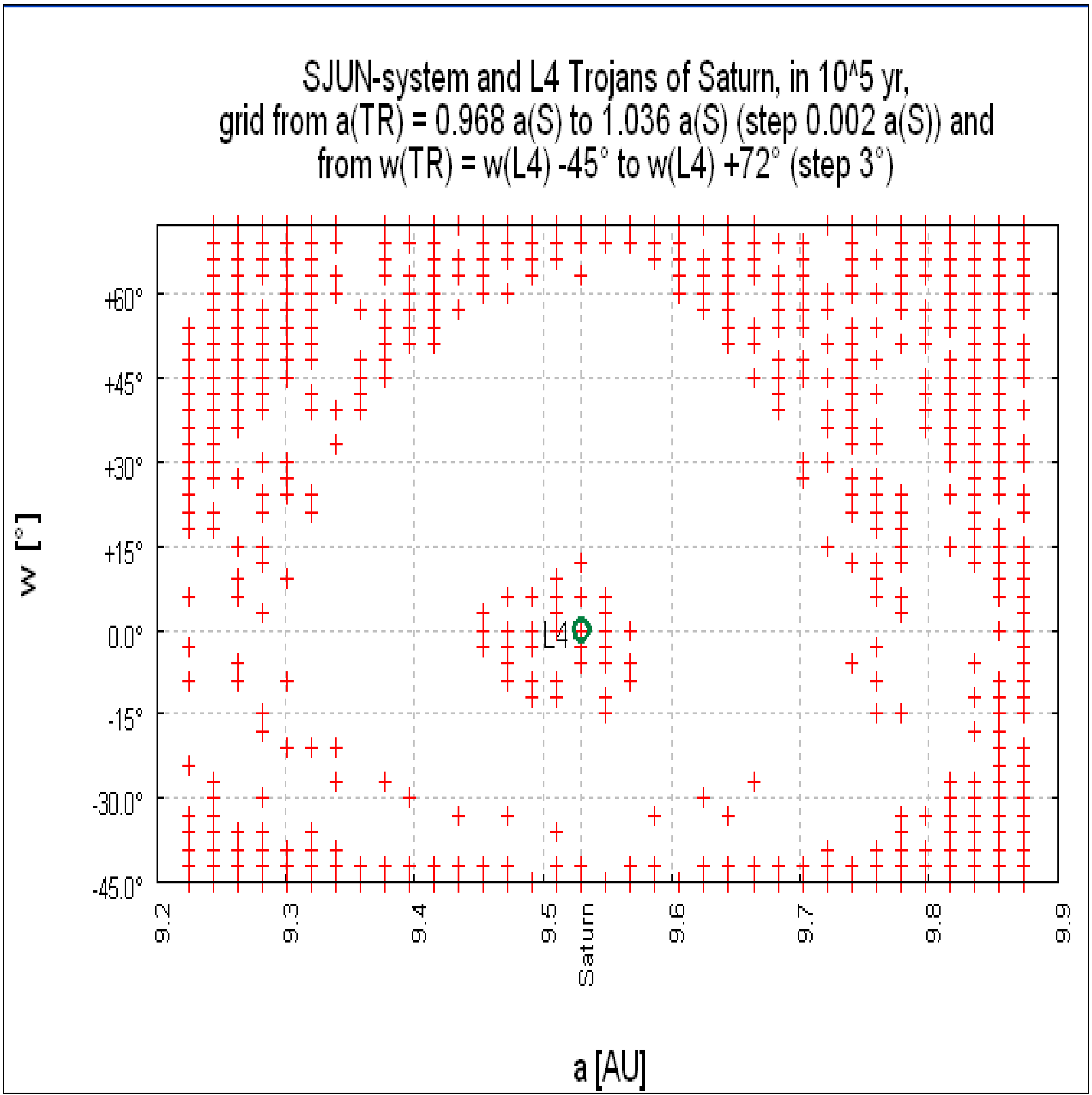

Fig. 10: The grid with 1394 fictitious Trojan asteroids around L4 of Saturn.

The integration over 10^5 yr shows stability in a wide-stretched, but narrow, oval zone, reaching from 0.968 a_6 to 1.034 a_6 respectively from about 9.23 to about 9.85 AU in radial direction and from about $\omega = \omega_6+21°$ to about $\omega = \omega_6+126°$, respectively from about -39°

of L4 to +66° of L4 in longitudinal direction (Fig. 11). The hole near L4 already starts to show with an extension between $a \sim 9.45$ AU (= 0.992 a_6) and $a \sim 9.57$ AU (=1,004 a_6) and between ω_{L4} -12° and ω_{L4} +12° respectively. Compare this with the results of Téger (Fig. 4 in subsection 2.1.1, picture upper left). The results are similar but not identical and the hole at Téger seems to be already more pronounced at that time.

This investigation can show us where we have to look for the stable regions in the following studies for more than 10^5 yr and to show the development of this region and the unstable hole.

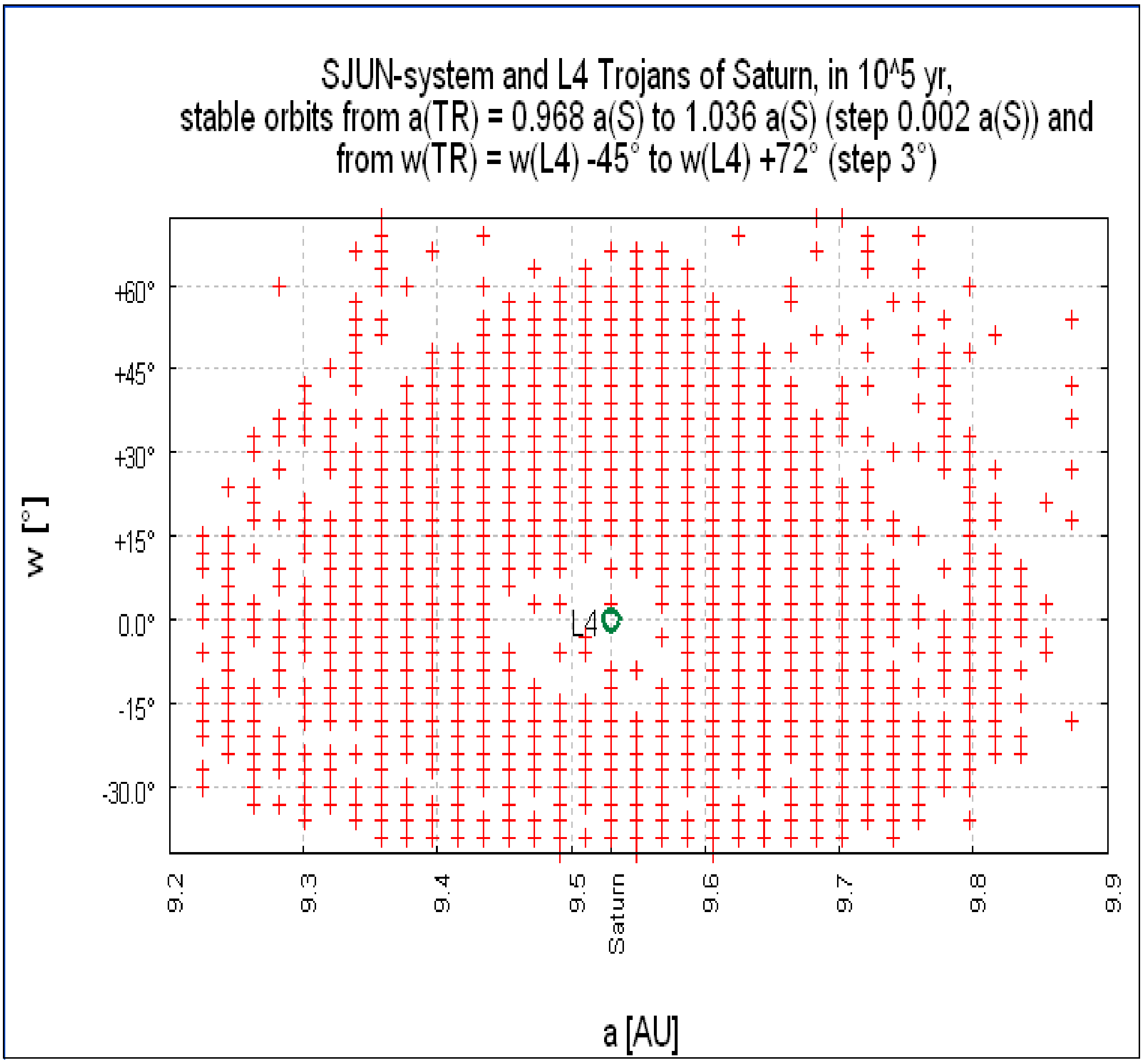

Fig. 11 represents the positive picture of Fig.10, where the stable zones over 10^5 yr are to be seen. The hole near L4 already shows with an extension between $a \sim 9.45$ and $a \sim 9.57$ AU and between ω_{L4}-12° and ω_{L4}+12°.

2.2.2 Investigations for 10^6 years with different planetary configurations

An investigation for only 10^5 yr is not very significant, so it was decided to investigate of the Lagrange point L4 for at least 10^6 yr to determine evolution of the stable and the unstable zone respectively around L4. To study the disturbing influence of the giant planets of the OSS we ran different combinations of the OSS over this period.

2.2.2.1 System with Saturn, Jupiter, Uranus and Neptune (SJUN)

SJUN means the whole OSS and 1330 starting fictitious Trojan asteroids around L4 of Saturn (Fig. 12). This represents a region of semi-major axis between 0.970 a_{L4} and 1.038 a_{L4} or about 9.24 < a < 9.89 AU (nearly the same values as for the integration over 10^5 yr) and between $\omega = \omega_6$+12° and $\omega = \omega_6$+123°, respectively from -48° of L4 to +63° of L4.

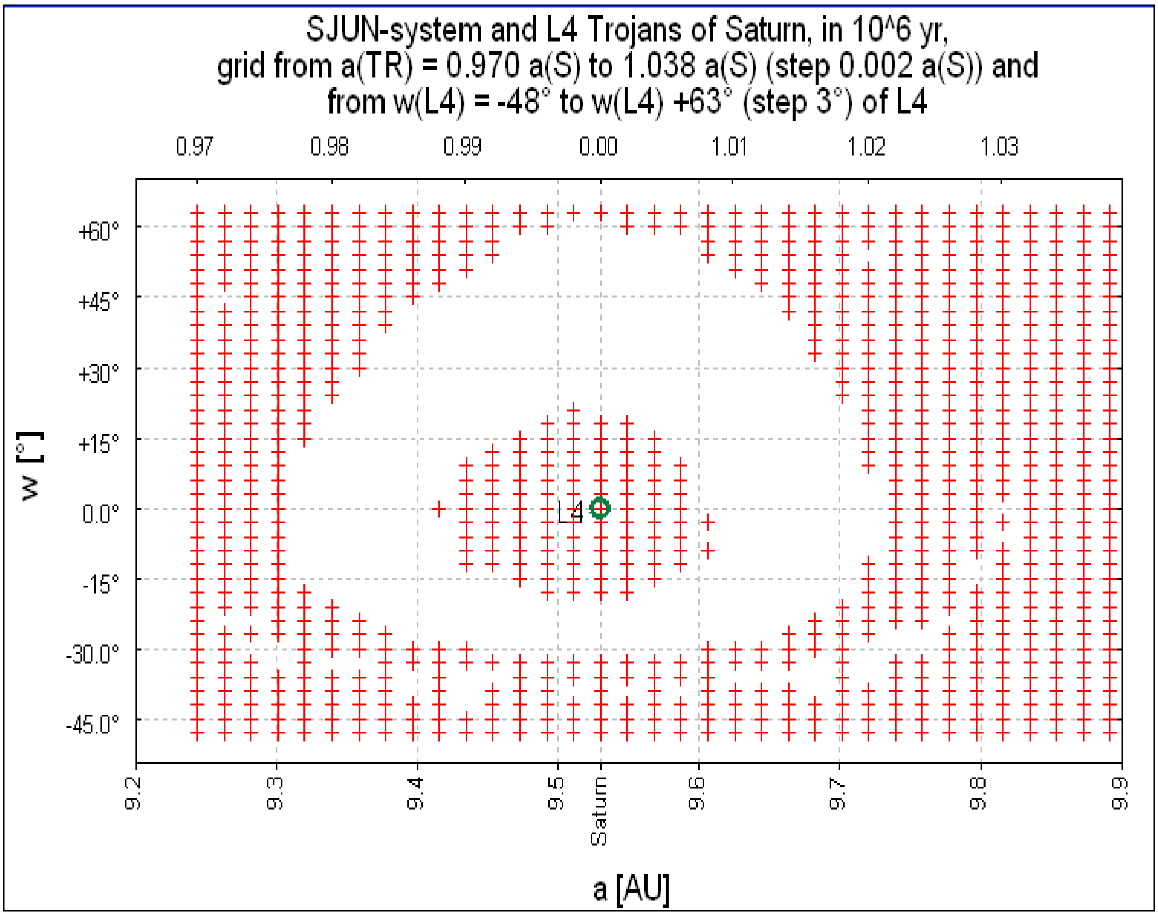

Fig. 12: The grid with 1330 fictitious Trojan asteroids around L4 of Saturn for 10^6 yr.

The integration over 10^6 yr shows a similar stable zone as the integration for 10^5 yr, but significantly smaller. The stable zone reaches now from about 0.978 a_{L4} to about 1.020 a_{L4}, respectively from about 9.32 to about 9.72 AU in the radial direction, which means a reduction of about 36% compared with the 10^5 yr integration. In the longitudinal direction the stable zone now reaches from about $\omega = \omega_6$+30° to about $\omega = \omega_6$+120°, respectively from about -30° of L4 to +60° of L4 (Fig. 13). In this direction the diminution amounts to only about 14%. The hole near L4 has grown significantly and now shows an extension between $a \sim 9.42$ and $a \sim 9.61$ AU and between ω_{L4} -18° and ω_{L4}+21° respectively. This means an increase in size of the hole, compared with the 10^5 yr investigation, of about 37% in the radial direction and 36% in the longitudinal direction.

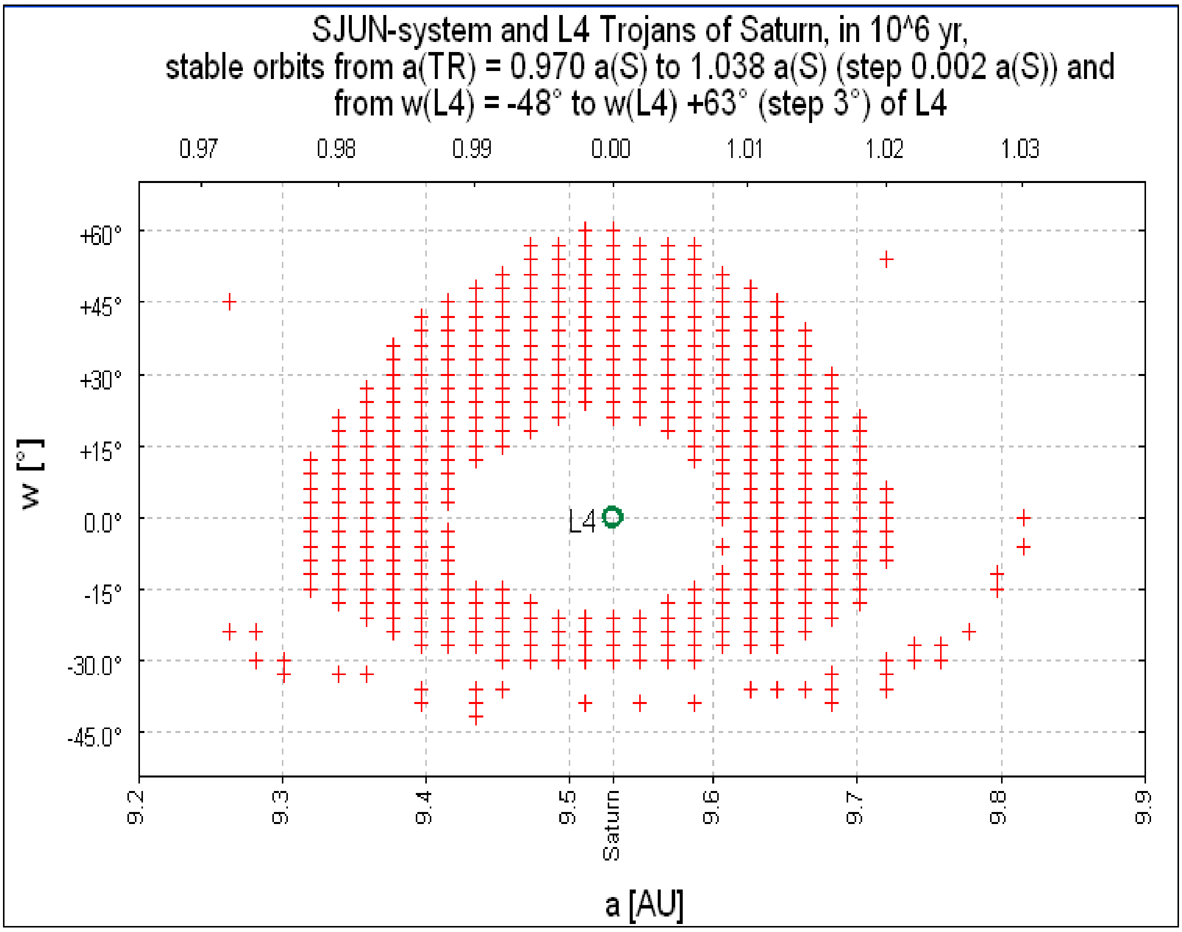

Fig. 13 represents the positive picture of Fig. 12. There the stable zones over 10^6 yr are to be seen. The hole near L4 shows now with an extension between $a \sim 9.42$ and $a \sim 9.63$ AU and between ω_{L4}-18° and ω_{L4}+18°.

2.2.2.2 System with only Saturn, Jupiter and Uranus (SJU)

The next simulation we undertook with the OSS but without Neptune, i.e. Sun, Saturn, Jupiter and Uranus (SJU) plus 1292 test particles around L4 of Saturn (Fig. 14). This represents a region of the semi-major axis between 0.970 a_{L4} and 1.036 a_{L4} or about $9.24 < a < 9.87$ AU and between $\omega = \omega_6$+15° and $\omega = \omega_6$+126°, respectively from -45° of L4 to +66° of L4.

The integration over 10^6 yr (Fig. 15) shows again a similar stable zone as SJUN. The stable zone now lies between 0.978 a_{L4} and 1.022 a_{L4} respectively reaches from about 9.32 to about 9.72 AU in the radial direction, exactly the same as in the SJUN-system (Fig. 13). In the longitudinal direction the stable zone reaches from about $\omega = \omega_6$+30° to about $\omega = \omega_6$+120°, respectively from about -30° of L4 to +60° of L4 and also covers exactly the same longitudinal extension as in the SJUN-system. The hole near L4 now has an extension between $a \sim 9.42$ and $a \sim 9.59$ AU and between ω_{L4}-18° and ω_{L4}+18° respectively. This means the extension of the hole in radial direction is nearly the same as in SJUN and only slightly smaller by about -8% in longitudinal direction compared with the SJUN-system.

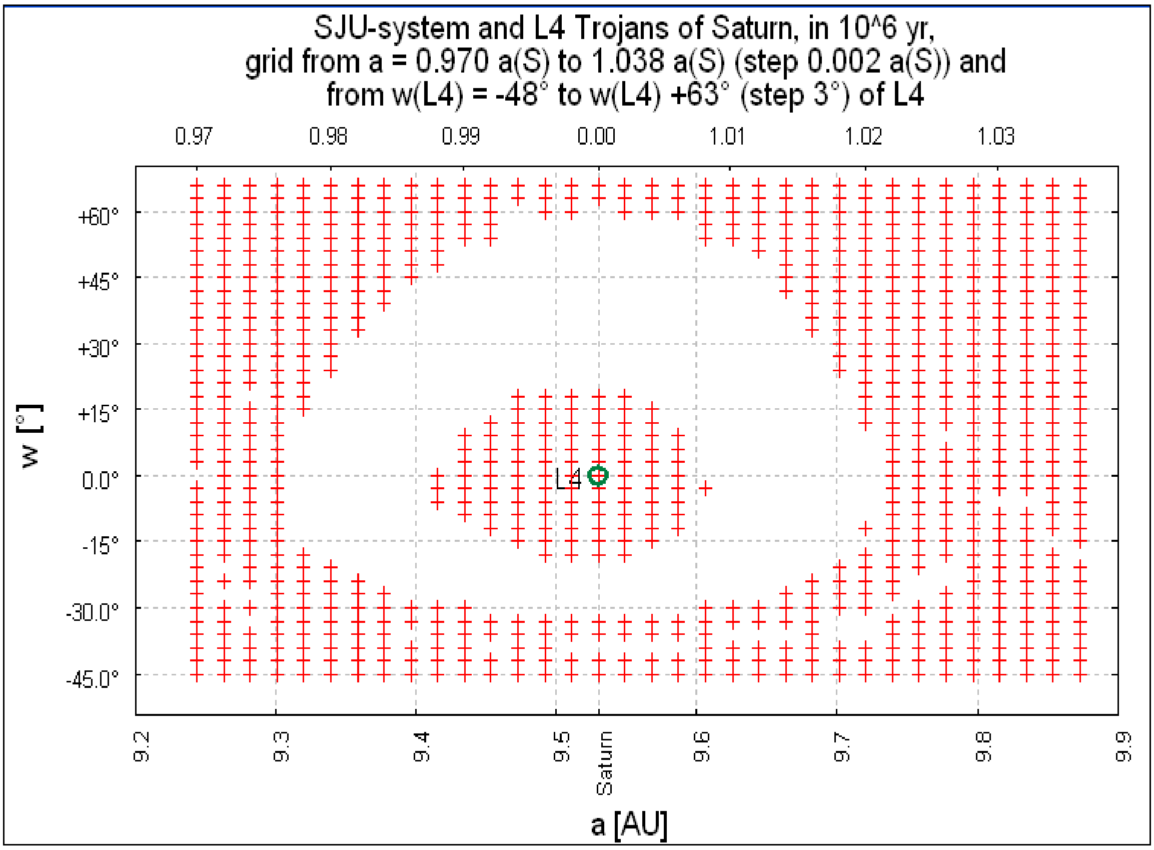

Fig. 14: The grid (SJU-system) with 1292 fictitious Trojan asteroids around L4 of Saturn.

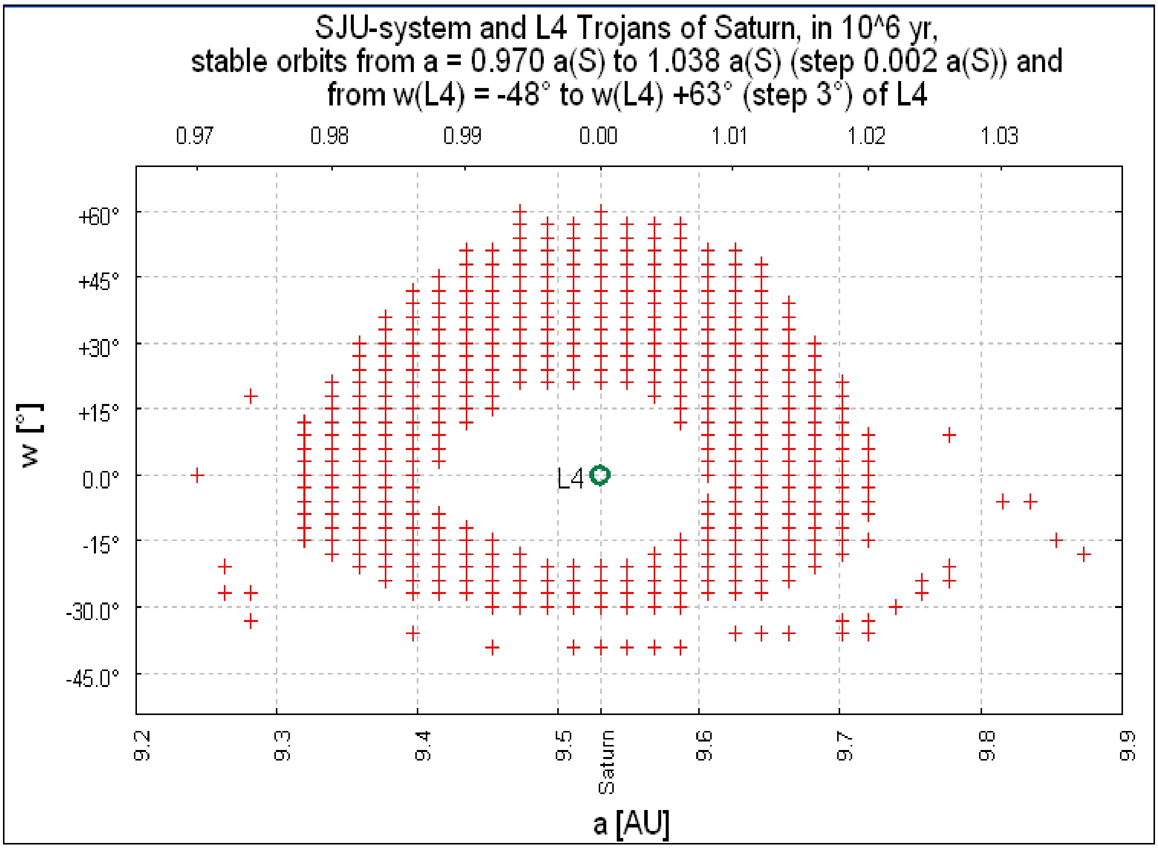

Fig. 15 represents the positive picture of Fig.14. There the stable zones over 10^6 yr are to be seen. The hole near L4 has now an extension between $a \sim 9.42$ and $a \sim 9.59$ AU and between ω_{L4}-18° and ω_{L4}+18°.

In summary one can say that there is no significant difference between the system SJUN and the system SJU. One has to keep in mind that the orbit of Neptune is more than 20 AU further out than that of Saturn. In other words, the supplementary influence of Neptune as a disturber of Trojan asteroids of Saturn is insignificant.

2.2.2.2 System with only Saturn, Jupiter and Neptune (SJN)

Next we investigated the OSS but without Uranus, i.e. Sun, Saturn, Jupiter and Neptune (SJN) plus again 1292 fictitious Trojan asteroids around L4 of Saturn. We used exactly the same grid as before (Fig. 14).

The integration over 10^6 yr of SJN gives nearly the exact picture as SJU before (see Fig. 16 and compare with Fig. 15). The hole is insignificantly larger. This means, that also the influence of Uranus as a disturber of Trojan asteroids of Saturn is insignificant, even though Uranus is much nearer to the orbit of Saturn than Neptune (about half the distance).

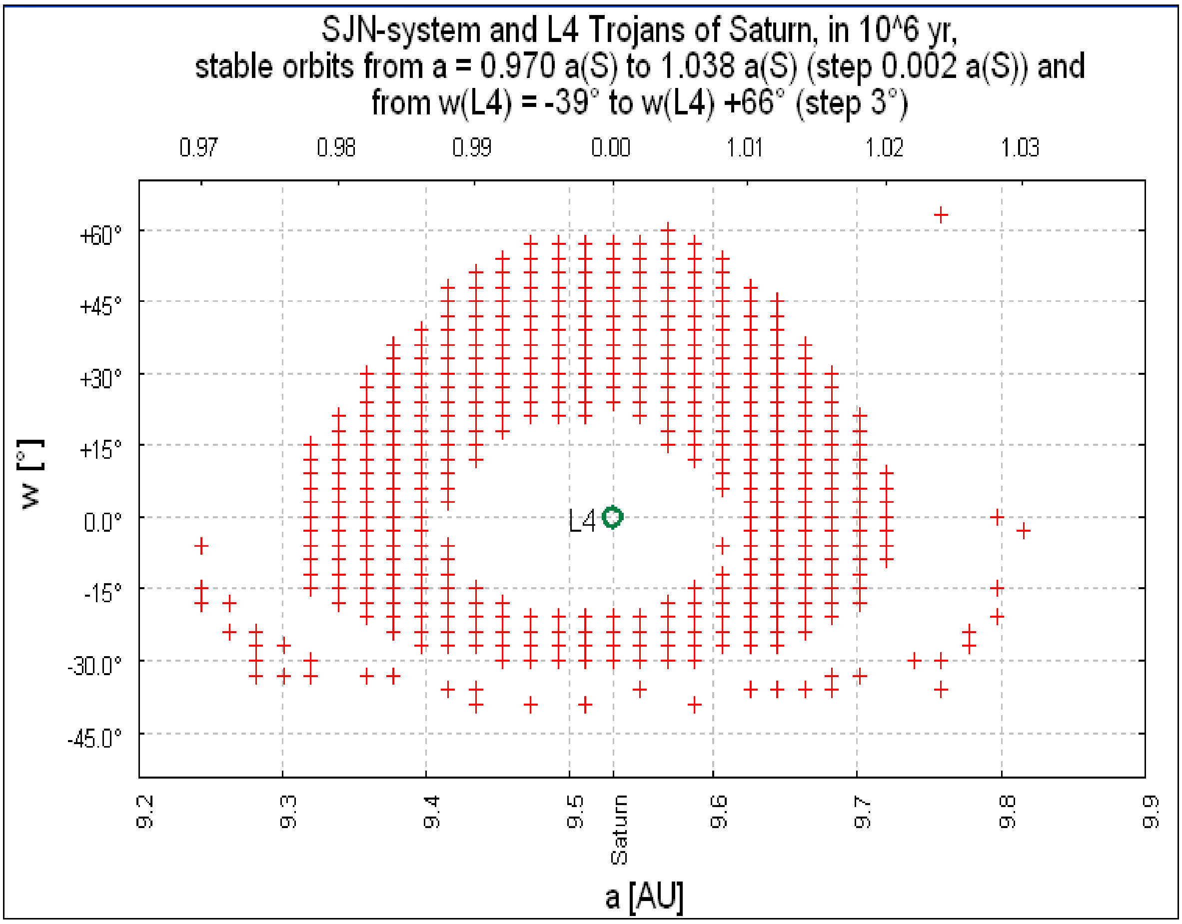

Fig. 16 represents the SJN model for the same grid as for the SJU-system, where the stable zones for 10^6 yr are to be seen. The hole near L4 has now an extension between $a \sim 9.42$ and $a \sim 9.63$ AU and between ω_{L4}-18° and ω_{L4}+18°.

2.2.2.4 System with only Saturn, Uranus and Neptune (SUN)

The next simulation was done with the OSS but without Jupiter, i.e. Sun, Saturn, Uranus and Neptune (SUN) and including 1887 fictitious Trojan asteroids around L4 of Saturn over 10^6 yr. There we had to augment the grid in a notable manner to see the whole stable zone to 1887 Trojans (Fig. 17). This represents a region of the semi-major axis between 0.968 a_{L4} and 1.036 a_{L4} or $9.23 < a < 9.87$ AU and from $\omega = \omega_6 + 15°$ to $\omega = \omega_6 + 204°$, respectively between ω_{L4} -45° and ω_{L4} +144°.

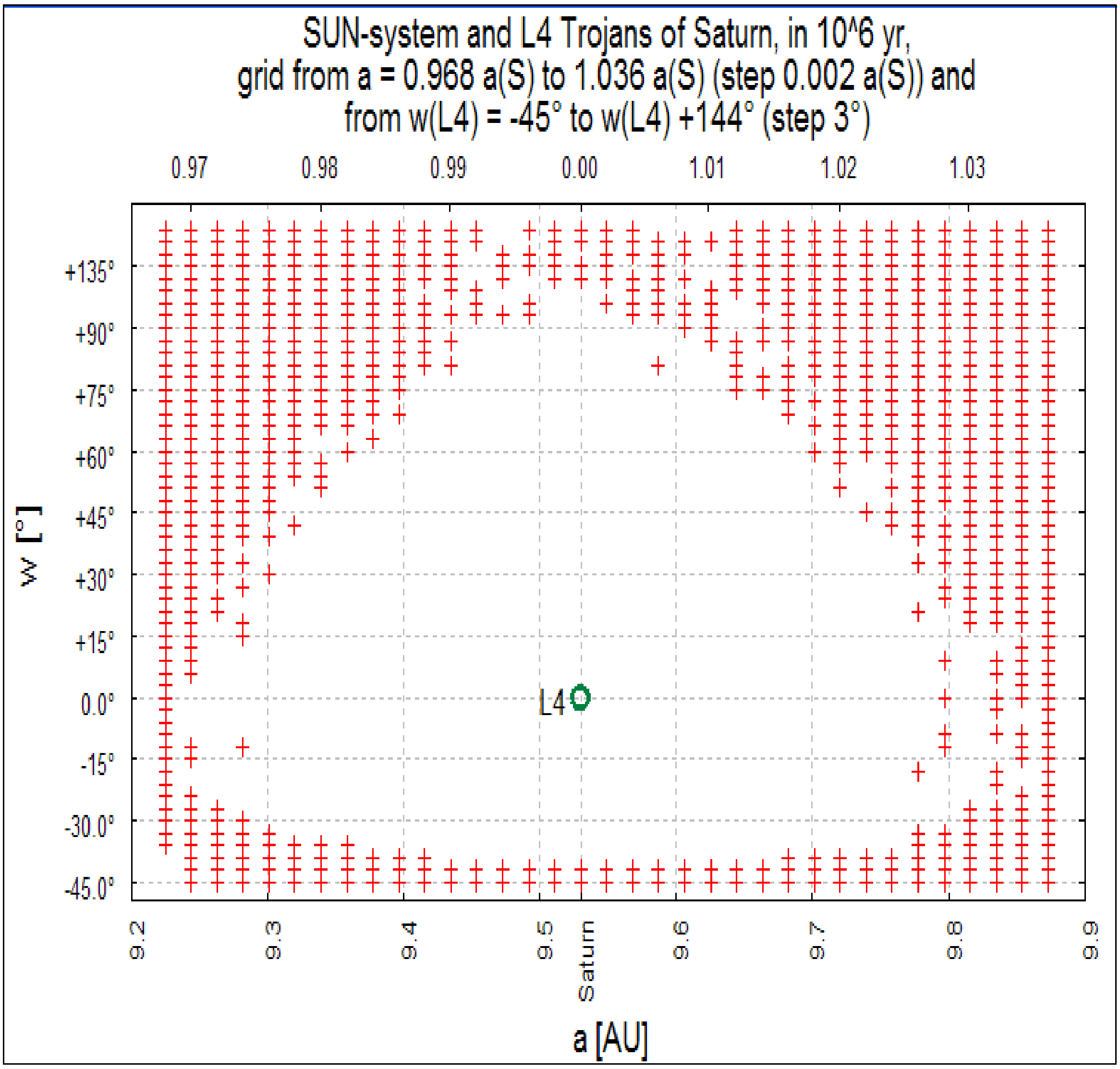

Fig. 17: The grid (SUN-system) with 1887 fictitious Trojan asteroids around L4 of Saturn.

The first surprise in the SUN-system (Fig. 18) is the complete disappearance of the hole in the centre of L4 and a second surprise is the enlarged stable region in contrast to the previous systems. In fact the stable zone is now given between 0.970 a_{L4} to 1.030 a_{L4} respectively and from about 9.24 to about 9.82 AU in the radial direction which means an enlargement of 43% to the SJUN-system. In the longitudinal direction the stable zone reaches from about $\omega = \omega_6 + 21°$ to about $\omega = \omega_6 + 159°$, respectively from about -39° to +99° of L4 that meaning an enlargement as much as 53% in respect to the SJUN-system (Fig. 13).

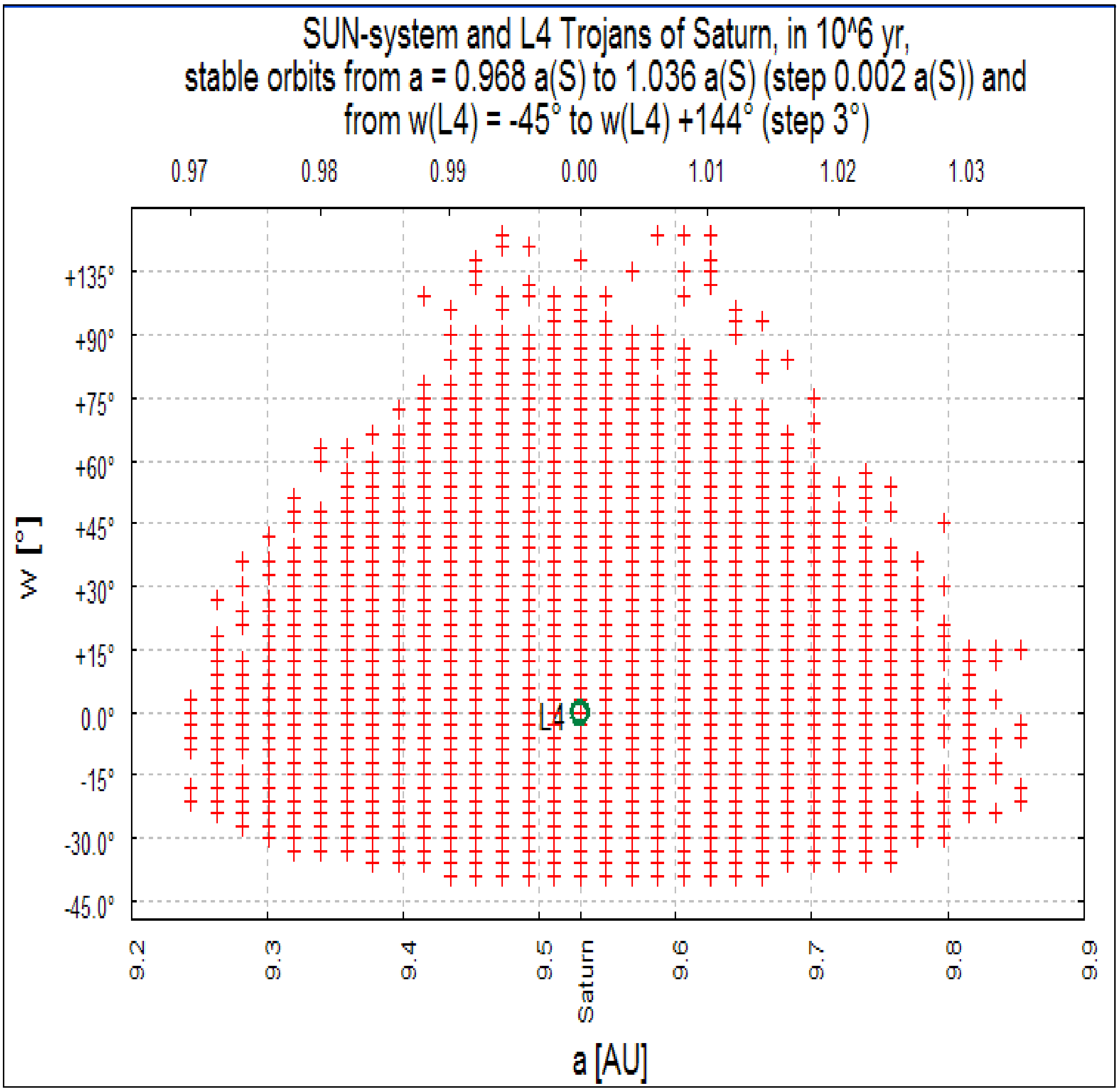

Fig. 18 represents the positive picture of Fig. 17. There the stable zones over 10^6 yr are to be seen. The central hole near L4 has completely disappeared.

It seems to be clear that the only real disturber responsible for the hole in close proximity to L4 is Jupiter. The orbit of Jupiter is also relatively close to the orbit of Saturn by a distance of only about 4.3 AU, represented by the near MMR of 5:2.

2.2.2.5 System with Saturn and Jupiter (SJ)

A final simulation was done with only Sun, Jupiter and Saturn (SJ) plus a grid of 1190 test particles around L4 of Saturn over 10^6 yr (Fig. 19). This represents a region about the semi-major axis between 0.970 a_{L4} and 1.036 a_{L4} or about 9.24 < a < 9.87 AU and from $\omega = \omega_6$+21° to $\omega = \omega_6$+123°, respectively between ω_{L4}-39° and ω_{L4}+63°. There again the hole around the centre of L4 is prominent.

The integration over 10^6 yr (Fig. 20) again shows a similar stable zone as in the SJUN-system (Fig. 13). The stable zone – except for the hole - is now given between 0.978 a_{L4} and 1.018 a_{L4} respectively from about 9.32 to about 9.72 AU in the radial direction, thus

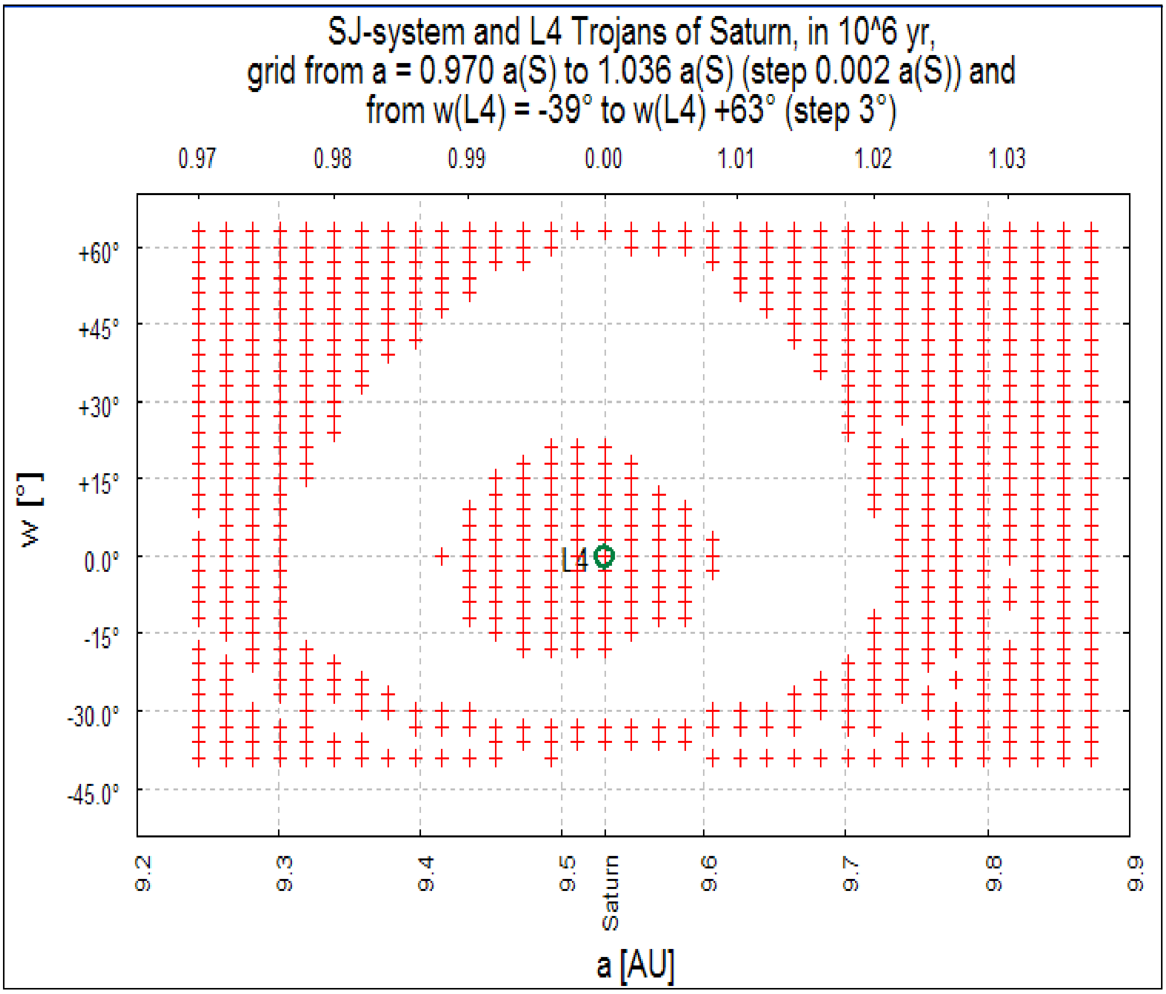

Fig. 19: The grid with 1190 fictitious Trojan asteroids around L4 of Saturn.

about the same as SJUN. In the longitudinal direction the stable zone reaches from about $\omega = \omega_6$+30° to about $\omega = \omega_6$+120°, respectively from -30° of L4 to +60° of L4 and is also exactly the same as SJUN (Fig. 13). The hole near L4 now has an extension between $a \sim$ 9.42 and $a \sim$ 9.61 AU and between ω_{L4}-18° and ω_{L4}+21° respectively. This means the extension of the hole in the radial direction is about the same as in SJUN and only slightly larger, by about 8%, in longitudinal direction compared with SJUN.

This simulation of the SJ-system underlines in an impressive manner the conclusions already drawn: **the only real disturber of fictitious Trojan asteroids of Saturn at L4 (and L5) is Jupiter and Jupiter is also solely responsible for the hole of instability near Saturn's L4 (and L5).**

The influence of Jupiter on L4 Trojans is also demonstrated by N&D and their migration theory in varying the 2:5 MMR by mutually displacing Jupiter and Saturn (see Fig. 6 and text in subsection 2.1.1).

This result was also detected by Téger, who conducted a similar simulation of the SJ-system of L4 and L5 Saturn Trojans over 10^7 yr (Fig. 21). The holes became completely dominant and only around L4 could a small ring of test particles survive. At L5 the remaining Trojans form just a sickle shape. Compared with our integration for 10^6 yr (Fig. 20), in the tenfold longer integration of Téger the fast majority of Trojans are gone.

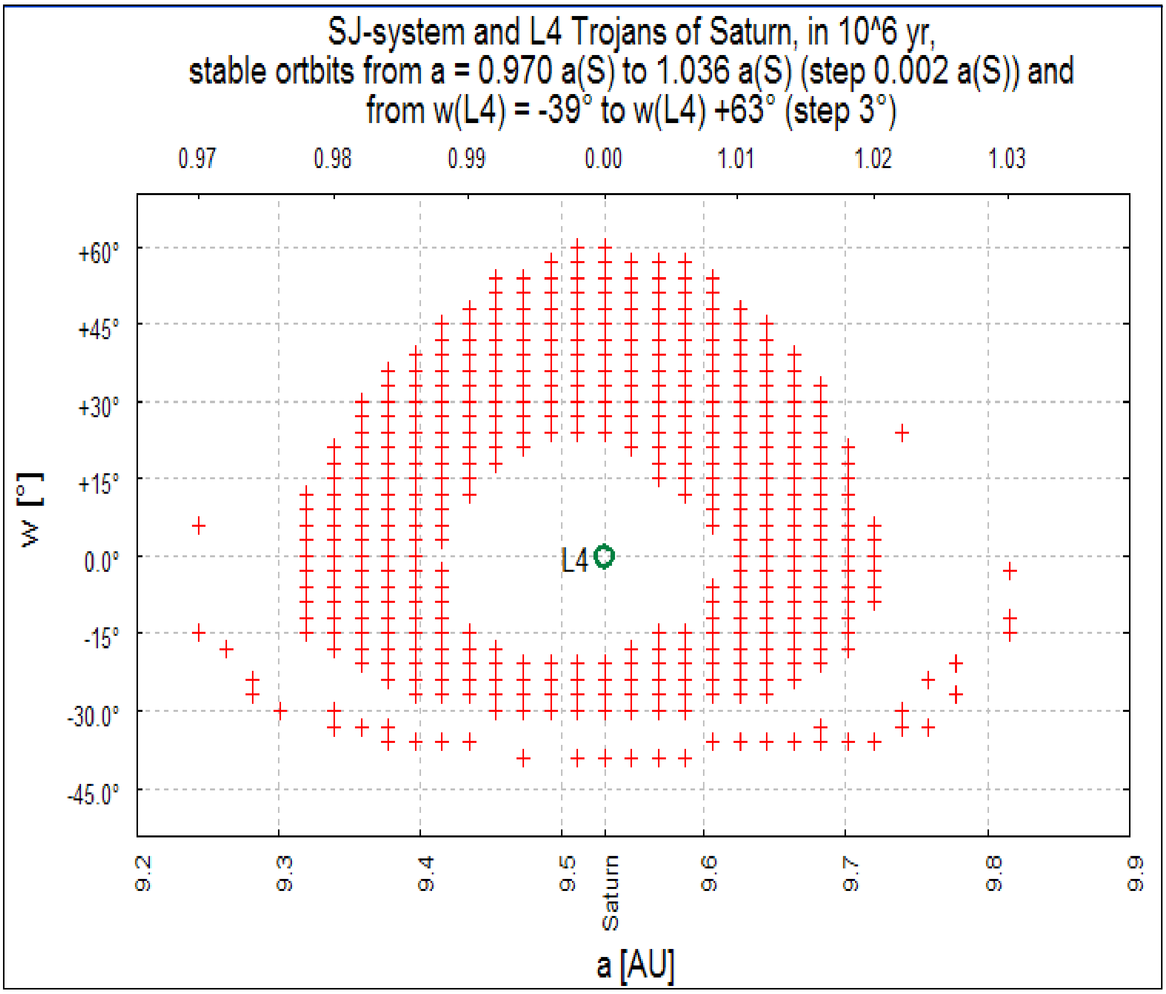

Fig. 20 represents the positive picture of Fig. 19 for the SJ-system. There the stable zones over 10^6 yr are to be seen. The hole near L4 has now an extension between $a \sim 9.42$ and $a \sim 9.63$ AU and between $\omega \sim 9.5°$ and $\omega \sim 60.5°$.

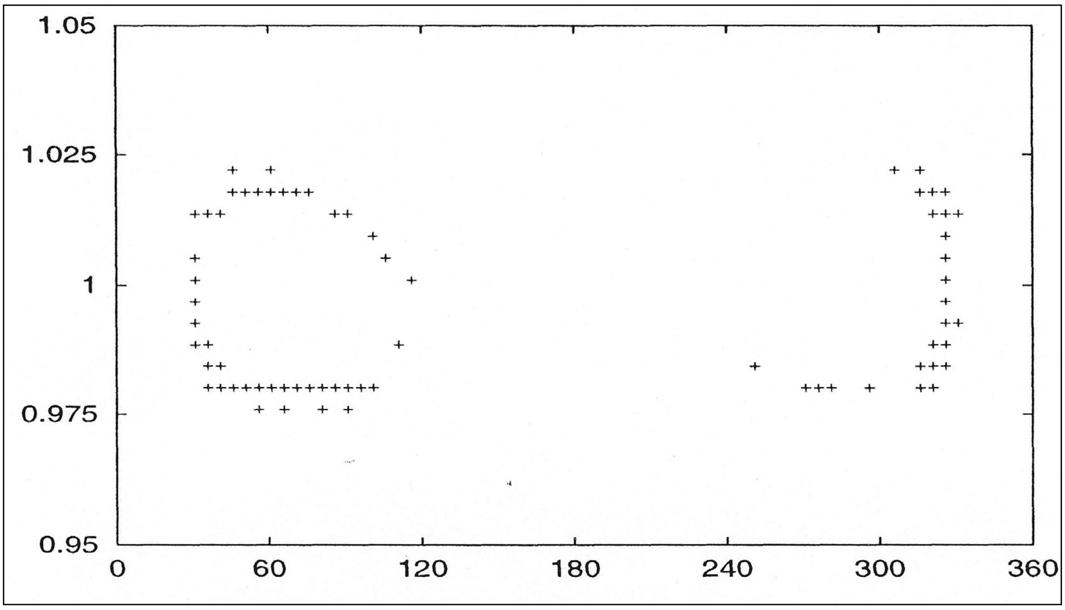

Fig. 21: Simulation of the SJ system for 10^7 yr by Téger (2000). The horizontal axis shows the initial longitude of the test particles with respect to Saturn and the vertical axis shows the ratio of the semi-major axis of Saturn and the test particles.

The question of the fast decrease of surviving test particles with time we shall investigate later in subsection 2.2.4. In the following subsection we shall study the behaviour of the Trojans around L4 of Saturn with at higher inclinations.

2.2.3 Study on inclinations up to 10^7 yr

In 2009 two special studies on inclined orbits of fictitious Trojan asteroids of planets in the OSS in the vicinity of the L4 and L5 points were done by DBZ, and ZDS.

In most cases with inclinations beyond 0.3 only zones of instability for Trojans were taken for granted as in our work. But ZDS found in their dynamical maps of fictitious Trojan asteroids, stable orbits at very high inclinations in L4 and L5 of Neptune at the libration centre σ_c. With integrations for 34 Myr three regions of the most regular orbits were found. In L4 one region A within starting inclinations i_0 of 0°-12°, a second region B within an i_0 of about 22°-36° and a third region C was discovered within an i_0 of about 51°-59°. Whereas the regions A and B are connected to each other by an area of less stable orbits, the region C is well separated from A and B in a bow-like manner by an unstable gap at i_0 ~ 44°. (See Fig. 22). The same investigation for L5 of Neptune gives a similar picture.

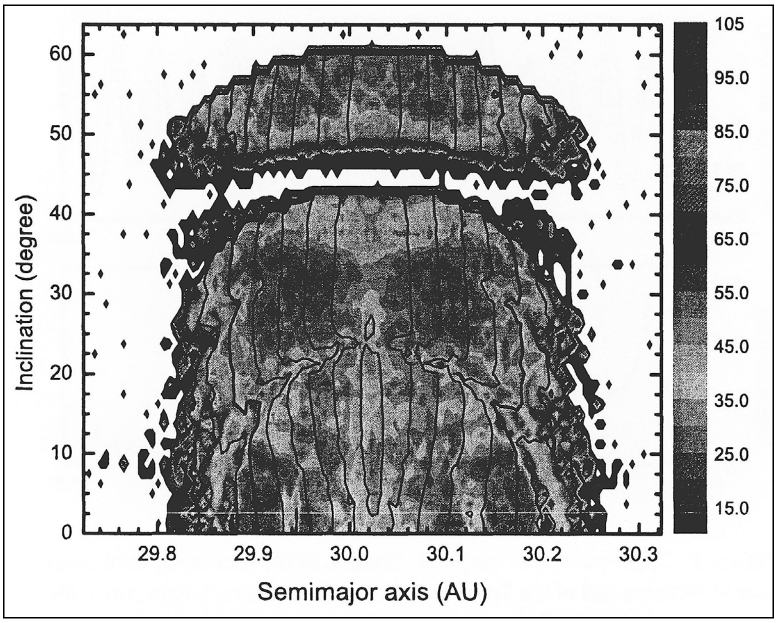

Fig. 22: The dynamical map around the L4 point of Neptune (ZDS).

For Uranus, DBZ also found in their study over 10^6 yr of fictive Trojans, different regions of stability in L4 in connection with high inclinations. They determined a region of stability within an inclination of i_0 between 0° and 14° and another more or less stable region between i_0 = 20° and i_0 = 50°. A large area of unstable test particles was found for values of i_0 between 14° and 20°. (See Fig. 23).

These discoveries caused us to investigate the possibility of stable regions of the fictive L4 Trojans of Saturn at inclinations up to 70° (for 10^6 yr) over the plane of Saturn. We also concentrated on values up to 25° (for 10^7 yr) over the plane of Saturn to eventually find similar stable orbits at higher inclinations.

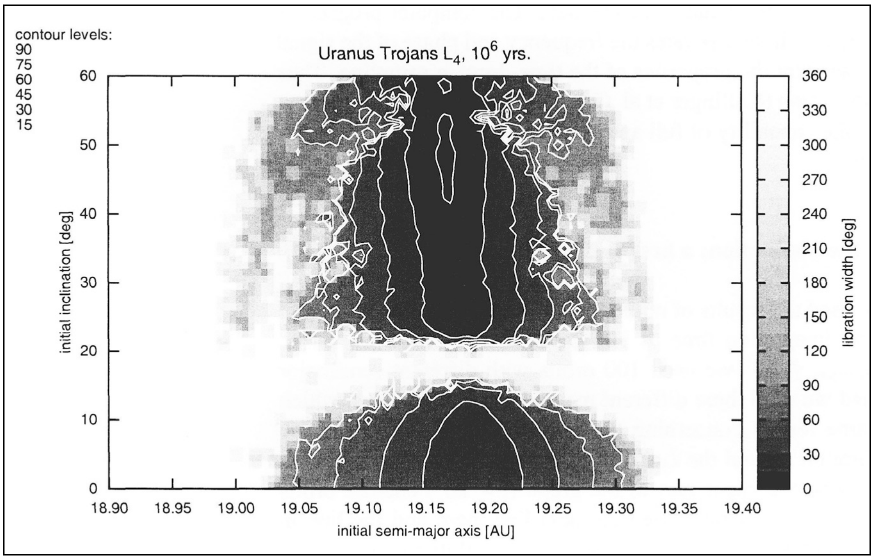

Fig. 23: The dynamical map around the L4 point of Uranus (DBZ).

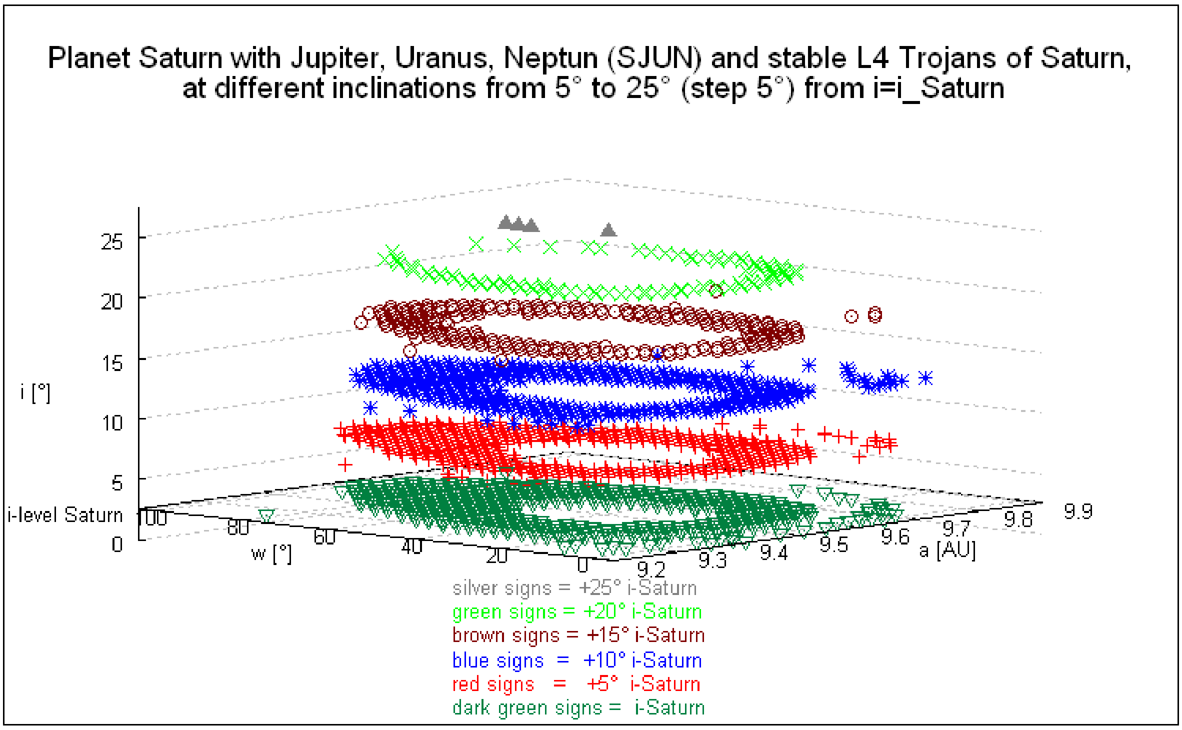

Fig. 24: Stability zones of Saturn's fictive Trojans with the whole OSS in 10^6 yr and maximal eccentricity of 0.3 with different inclinations of i from 0° to the plane of Saturn to 30° above the plane with Δi = 5°. The grid covers each level in the radial extension from 0.97 a_6 to 1.03 a_6, with Δa = 0.002 a_6 and in the longitudinal direction from –36° to +50° of the L4 point, with $\Delta\omega$ = 3° (the L4-point at about 39.5°. The graphic shows the growing hole at each level from bottom to top in inclination steps, where the hole is growing mainly from inner to outer. Over +25° there are no Trojans left.

Firstly we wanted to have a more general view of the behaviour of Trojans and the development of the hole as the inclination increased. Fig. 24 shows the stability zones of Saturn's fictive Trojans, integrated with the whole OSS for 10^6 yr. All Trojans with $e > 0.3$ are imagined as unstable. The steps of Δi = 5° up to i = 30° over the plane of Saturn show the well known hole in the near vicinity of L4. The hole is growing at each level from bottom to top mainly from the centre to the edge. In other words the stable zone is diminishing gradually, but the central hole is expanding at a greater rate. Over $i = 25°$ there are no Trojans left. This stands in excellent agreement with MTS (see subsection 2.1.1).

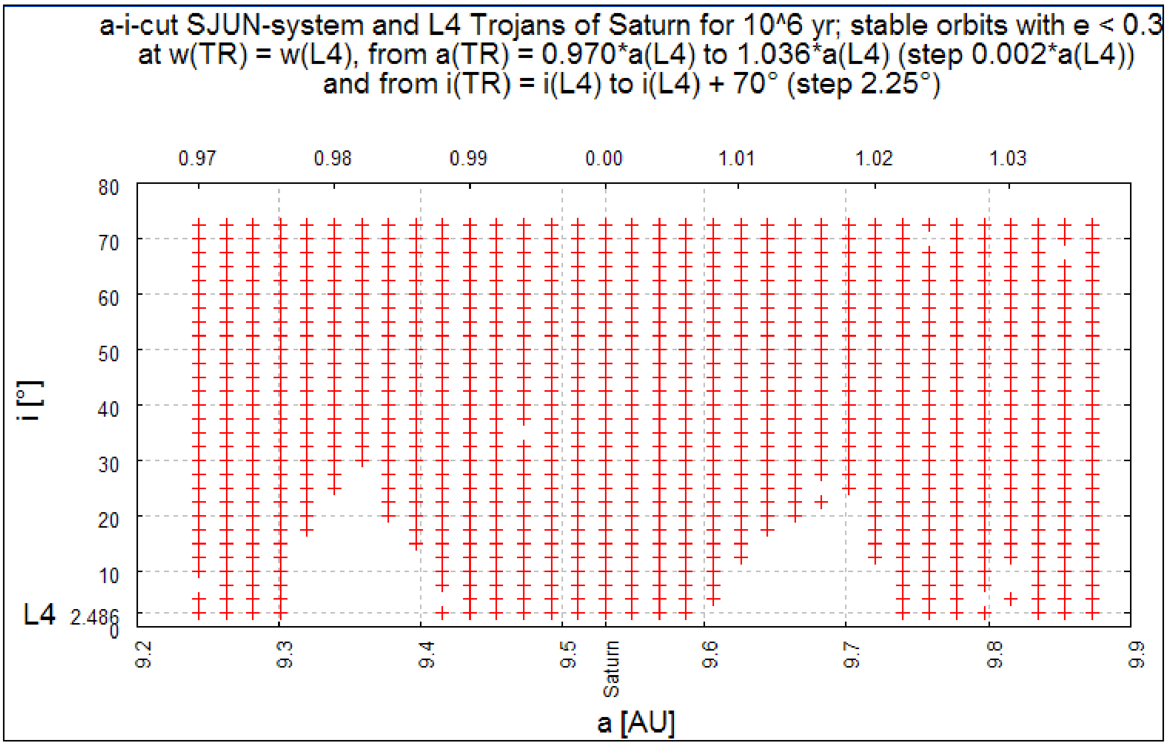

Fig. 25: The grid with 986 fictive Trojans of the *a*-*i*-cut at ω of L4 ($\omega_{L4} = +60°$ of ω_6) from 0.97 a_{L4} to 1.036 a_{L4} and $\Delta a = 0.002\ a_{L4}$ and inclinations up to 70° ($\Delta i = 2.25°$) over the plane of Saturn. Eccentricities with $e > 0.3$ are imagined as unstable.

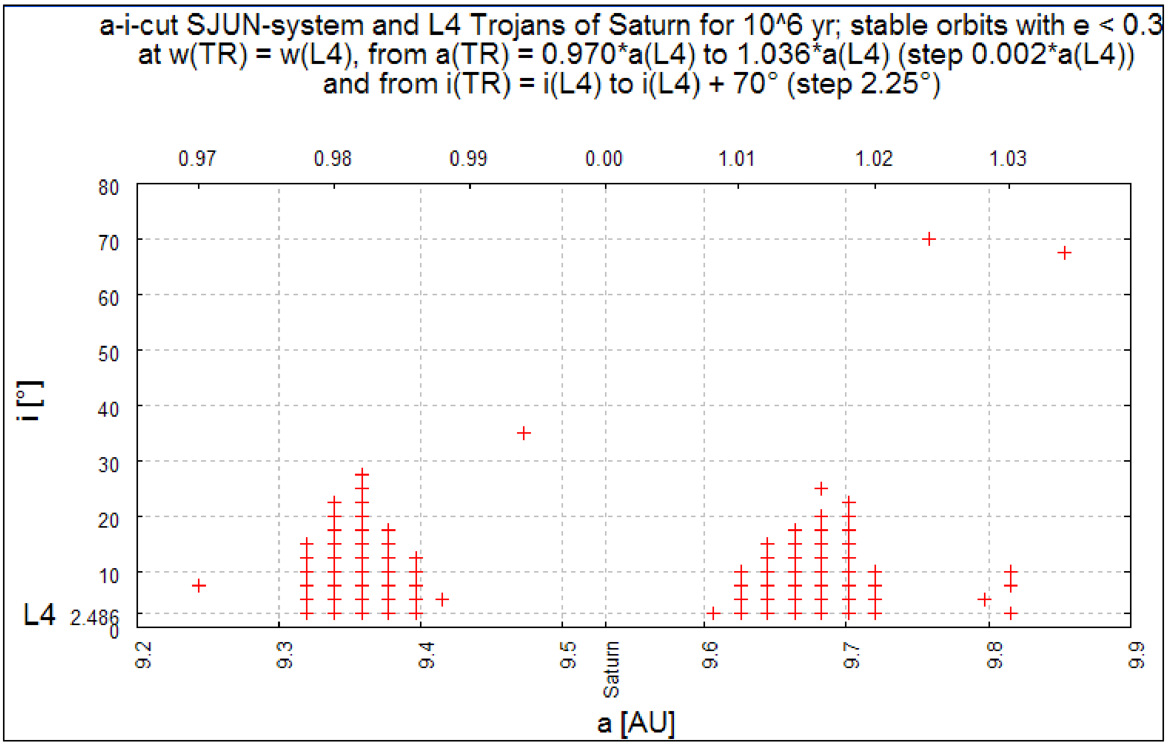

Fig. 26 shows the same map as Fig. 25 but in the positive sense, i.e. it shows the stable orbits of the *a*-*i*-cut. There are no stable regions to be seen above $i = 25°$ over the plane of Saturn.

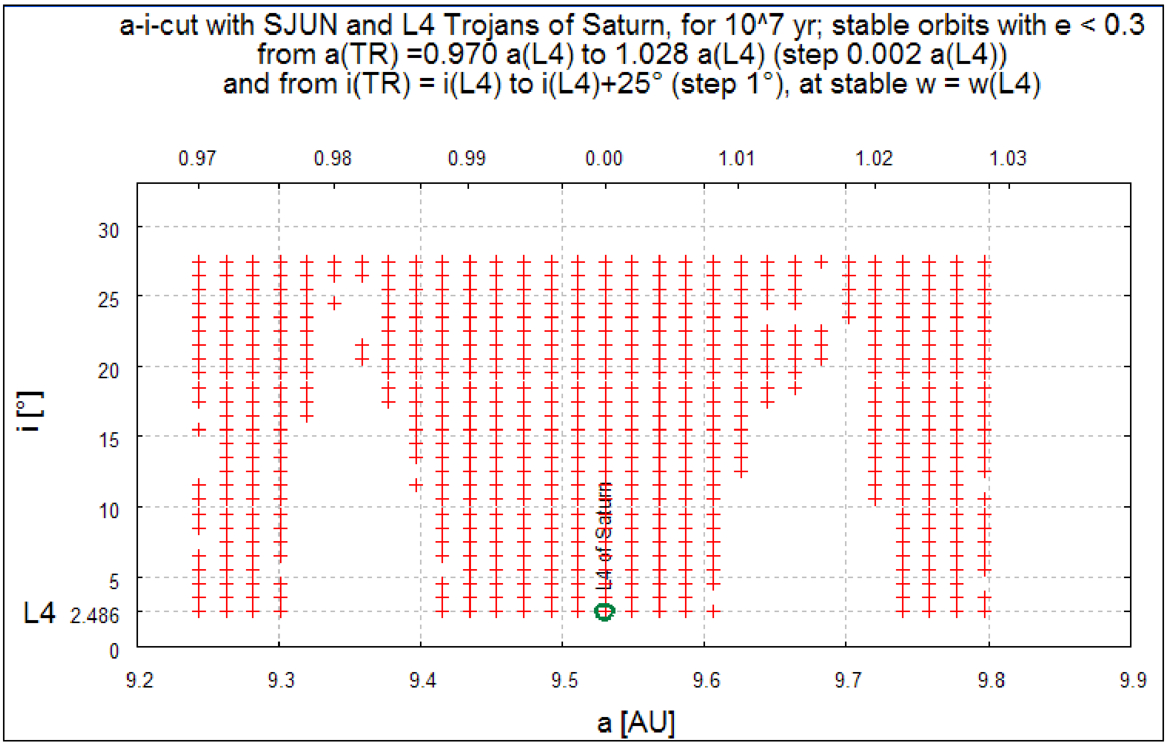

Fig. 27: The grid with 780 test particles for 10^7 yr of an *a*-*i*-cut with the whole OSS. The grid at ω of L4 (ω_{L4} = +60° of ω_6) reaches from 0.97 a_{L4} to 1.028 a_{L4} with Δa = 0.002 a_{L4} and inclinations from the plane of Saturn up to *i* of Saturn plus 25° over the plane and Δi = 1°. Eccentricities with $e > 0.3$ are imagined as unstable.

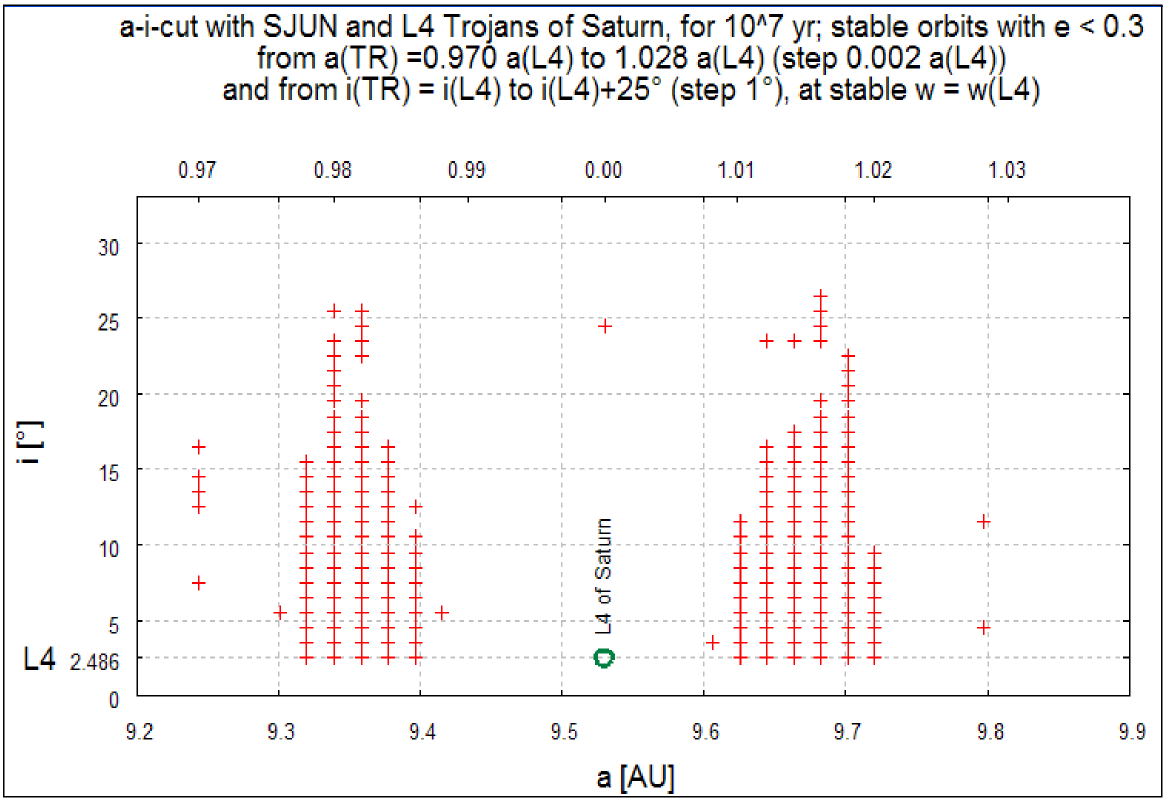

Fig. 28 shows the same map as Fig.27 but in the positive sense, i.e. it shows the possible stable orbits of the *a*-*i*-cut. There are no stable regions to be seen above about $i = 27.49°$, or 25° over the plane of Saturn.

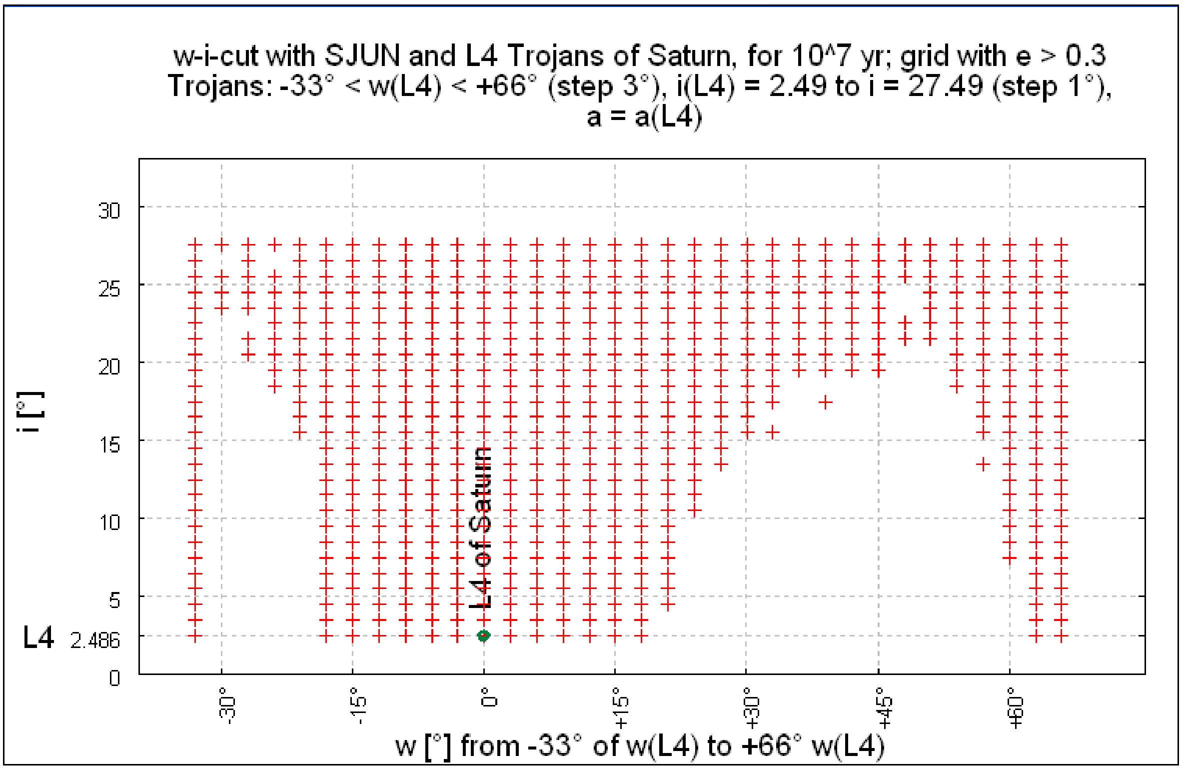

Fig. 29: The grid with 884 test particles in the ω-i-cut at a_{L4}, from ω_{L4} = -33° to ω_{L4} = +66° and from $i_{TR} = i_{L4}$ to $i_{TR} = i_{L4}$+25° and Δi = 1°.

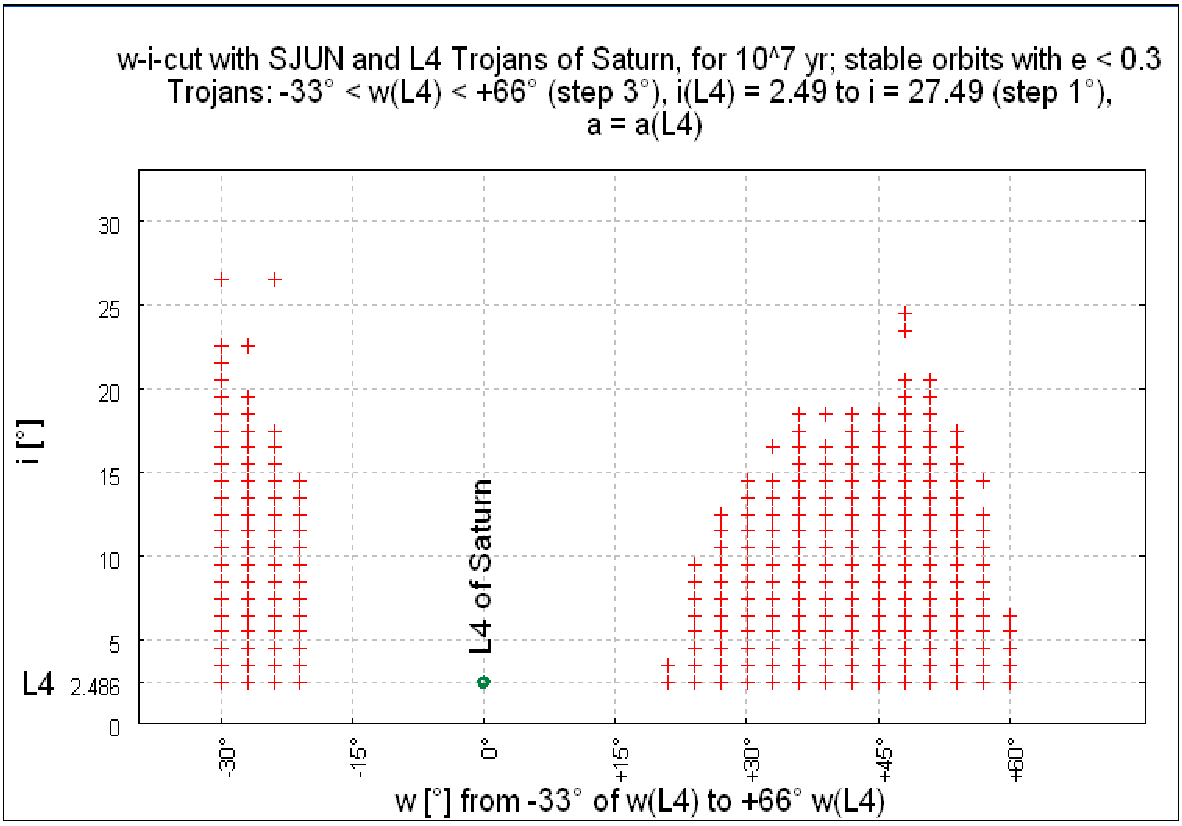

Fig. 30: The stable regions with 884 test particles in the ω-i-cut at a_{L4}, from ω_{L4} = -30° to ω_{L4} = +60° and from $i_{TR} = i_{L4}$ to $i_{TR} = i_{L4}$ + 25° and Δi = 1°. There are no stable regions to be seen above 25° over the plane of Saturn

To confirm this we investigated the stability zone with the whole OSS for 10^6 yr with inclinations above i = 30° in an a-i-plot at ω_{L4} ($\omega_{L4} = \omega_6$+60°) from 0.97 a_{L4} to 1.036 a_{L4} and Δa = 0.002 a_{L4} and with inclinations up to i = 70° above the plane of Saturn and Δi = 2.25°. Eccentricities with e > 0.3 are again imagined as unstable. Fig. 25 shows the starting grid of 986 fictive Trojans and Fig. 26 shows the positive image of Fig. 25. Fig. 26 therefore shows the possible stable orbits. There are no other stable regions to be seen like those mentioned in the papers of DBZ or ZDS for Neptune and Uranus respectively. If a similar second region should exist for Saturn it should have been found between i = 51° and i = 59° (see ZDS) or for values of i between 20° and 50° (see DBZ).

As ZDS found a less stable region (within the inclination i_0 between 12° and 22°) for Neptune's Trojans between the stable region A (i_0 between 0° and 12°), and the second region B (i_0 between 22° and 36°), we also ran an a-i-cut integration for 10^7 yr for the L4 Trojans of Saturn. The starting grid of this investigation with 780 test particles is shown in Fig. 27. Fig. 28 again shows the positive map of Fig. 27, i.e. the possible stable orbits of fictive Trojans with Δi = 1°. In fact, at the same interval for i of 12° to 22°, as in ZDS for Neptune, there appears a less stable region. Here, where i is about 12°, the hole starts to grow quickly whilst the stable region almost completely disappears between i = 12° and i = 22° above the plane of Saturn. Between i = 22° and i = 26° the hole seems to narrows slightly once more. But the bows forming the borders of the stable regions of Neptune previously described (Fig. 22, ZDS) as in the study of Uranus (Fig. 23, DBZ) are not found in our study of Saturn. Instead of bordering bows, the stable orbits there end at the upper border rather in a steep ring around the growing hole.

As before, we ran a similar integration for 10^7 yr and Δi = 1° but for an ω-i-cut in the L4 region from an ω_{L4} -30° to ω_{L4} + 66° and $\Delta\omega$ = 3°, i.e. at distance of the Sun of about 9.53 AU (=L4). Fig. 29 shows the grid of 884 test particles and Fig. 30 gives the positive view of the map with stable orbits. Also the ω-i-map shows the hole to be very large in the longitudinal direction growing from about 36° longitudinal extension in the plane to about 75° where i = 24° over the plane of Saturn. Therefore the stable zone at the higher inclination is withdrawing from the centre of L4, away from Saturn rather than towards Saturn.

In short, our investigations with higher inclinations give a quite different picture from those detected for Neptune and Uranus.

2.2.4 Investigations on "cuts" for 10^6 and 10^7 year

To get additional information on the subject of the behaviour of certain L4 Trojan orbits we ran cuts over 10^6 yr and 10^7 yr. These cuts always describe a cross in the longitudinal and one in radial direction of the L4-point in the form of a histogram. Each cut is shown against the maximum eccentricity (compare with Fig. 13 in subsection 2.2.2.1), the escape time at e_{TR} > 0.3 and the maximum amplitudes in radial and longitudinal directions.

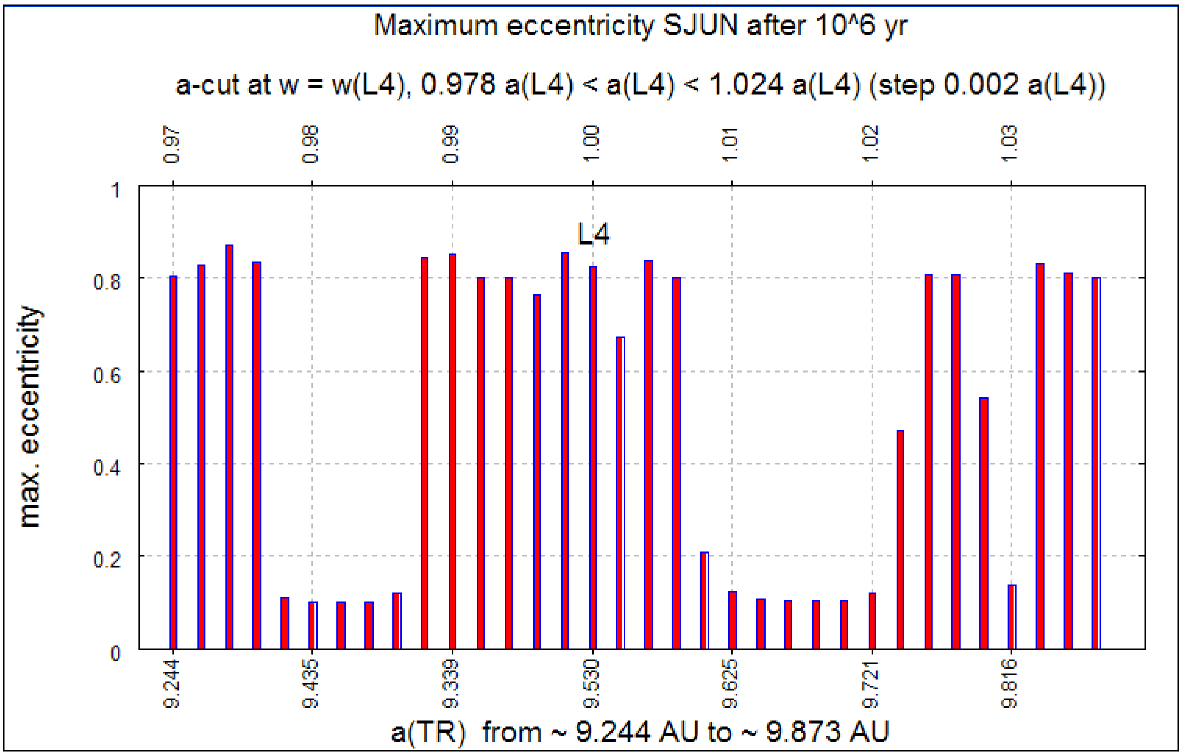

Fig. 31: The *a*-cut against maximum eccentricity of 34 Trojans in radial direction with starting orbits from values of semi-major axis a_{TR} from 0.970 a_{L4} (corresponding to about 9.244 AU) to 1.036 a_{L4} (corresponding to about 9.873 AU) and Δa = 0.002 a_{L4} at 10^6 yr. The L4-point lies at about 9.530 AU from the Sun, the same distance as Saturn ($a_{L4} = a_6$).

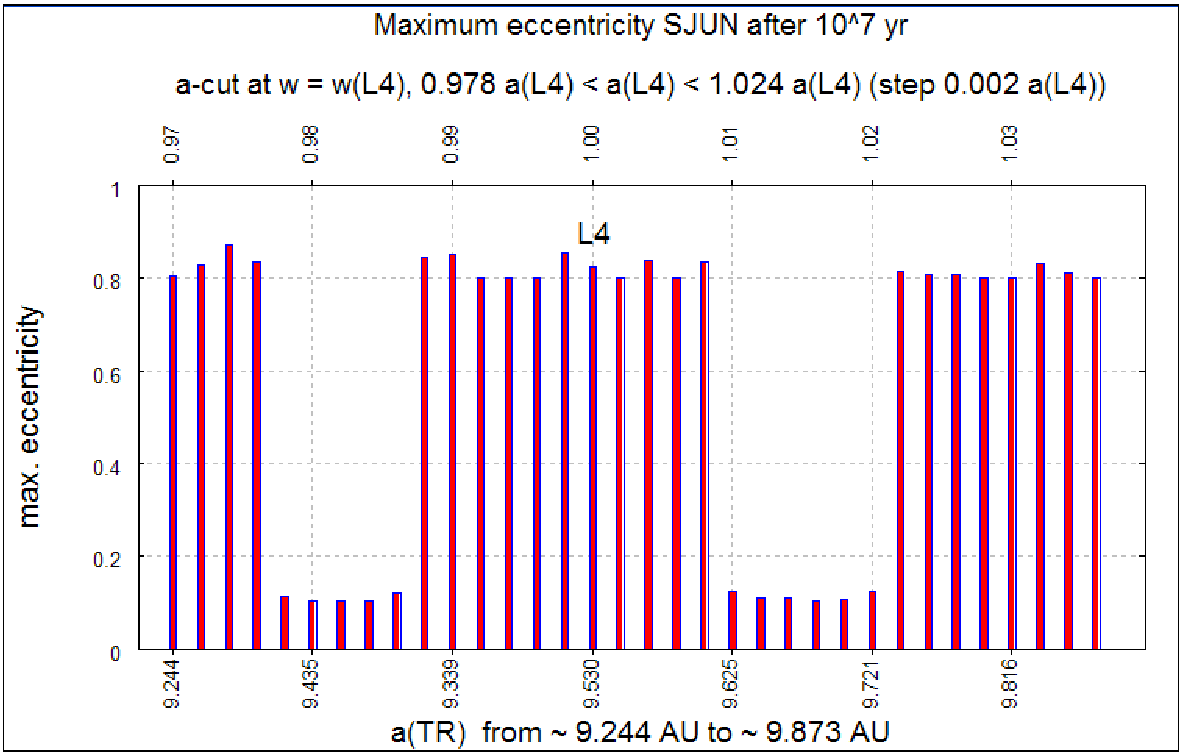

Fig. 32: The *a*-cut against maximum eccentricity as in Fig. 31 but at 10^7 yr.

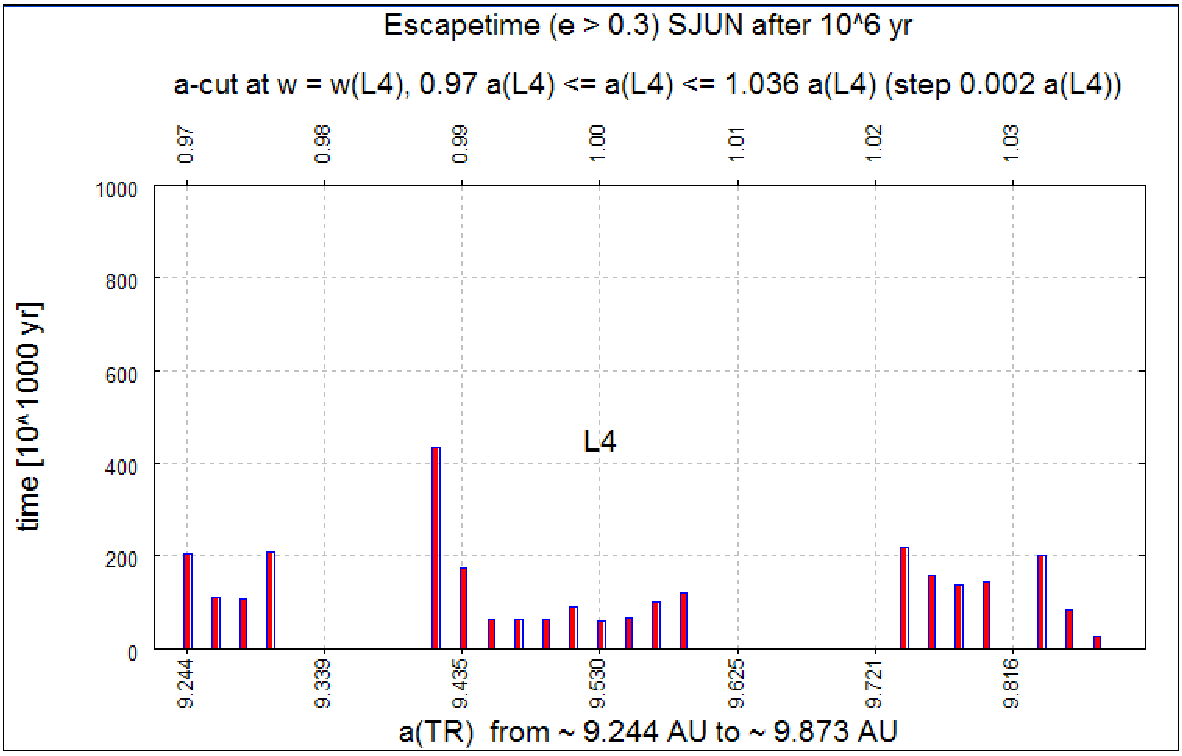

Fig. 33: The a-cut against escape time with $e_{TR} > 0.3$ for 10^6 yr and starting orbits as in the previous Figures. The escape time distinguishes in similar manner the stable from the unstable zones.

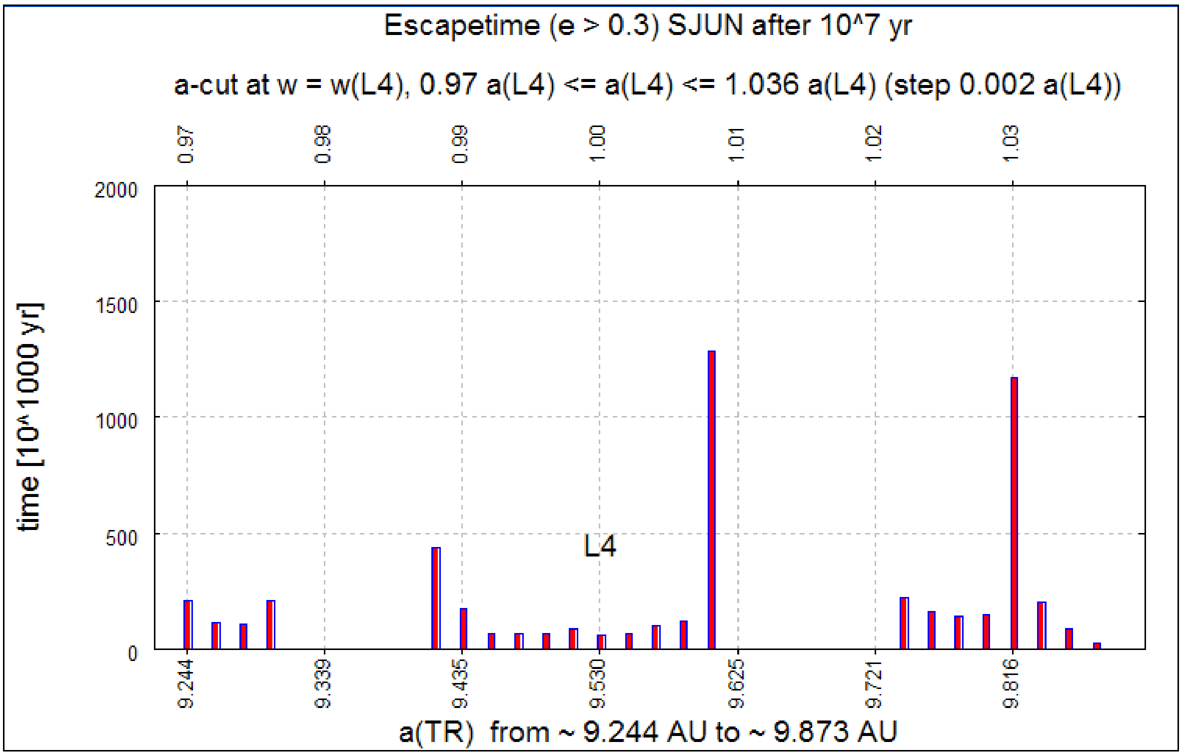

Fig. 34: The a-cut against escape time as in Fig. 33 but for 10^7 yr. The orbits at 0.008 a_{L4} and 1.030 a_{L4} now have become also unstable.

2.2.4.1 The cuts in radial direction (*a*-cut)

The first L4-cuts are made in radial directions with the same longitudinal argument ω_6 +60° from Saturn for all applicable Trojans but with a different starting semi-major axis and are therefore called "*a*-cuts". All the *a*-cuts represent 34 orbits of fictitious Trojans with starting values of semi-major axis from 0.970 a_{L4} (corresponding to about 9.244 AU) to 1.036 a_{L4} (corresponding to about 9.873 AU) and $\Delta a = 0.002\ a_{L4}$.

2.2.4.1.1 *a*-cuts against maximum eccentricity

Fig. 31 shows an *a*-cut against the maximum eccentricity for 10^6 yr. The hole around L4 naturally is also clearly remarkable in this direction between 0.988 a_{L4} and 1.006 a_{L4} with very large eccentricities. The stable zone in direction to the Sun reaches from 0.986 a_{L4} to 0.978 a_{L4}. In the other direction, off the Sun, the stable zone is a bit more expanded and reaches from about 1.008 a_{L4} to about 1.020 a_{L4}. A tiny but insignificant stable zone seems to exist near 1.03 a_{L4}, which is also to be seen in our simulation of the grid (Fig. 12 in subsection 2.2.2.1).

Fig. 32 represents the same *a*-cut but after 10^7 yr. Here the orbit at 1.008 a_{L4} has become unstable and also the former stable tiny gap at 1.03 a_{L4}. Nevertheless it is astonishing, that the stable zone is, in principle, very little changed over this period, which is ten times longer.

2.2.4.1.2 *a*-cuts against escape time.

We defined the escape time as the moment, when the eccentricity of a Trojan orbit exceeds the value of $e > 0.3$ (subsection 2.1.2.2).

The *a*-cut against escape time for 10^6 yr (Fig. 33) again shows the unstable hole near L4 with very early escapes within about 10^5 yr and about 4×10^5 yr at the border of this zone. This Figure corresponds to some extent with Fig. 31. It is typical that the escape time of orbits at the borders of the unstable zone to the stable region is always greater than in the core zone of instability.

In the same way Fig. 34 for 10^7 yr corresponds with Fig. 32. Here the orbit at 1.008 a_{L4} has become unstable and also the former stable tiny gap at 1.03 a_{L4}. Only the real stable zones within 0.978 a_{L4} and 0.986 a_{L4}, and 1.010 a_{L4} to 1.020 a_{L4} naturally have no escape times at 10^7 yr.

2.2.4.1.3 *a*-cuts against radial amplitude.

The *a*-cut against radial amplitudes signifies the maximum oscillation of the orbits in radial direction around L4.

The cut with radial amplitudes at 10^6 yr (Fig. 35) shows approximately the same shape of stable and unstable zones as maximum eccentricity (Fig. 31) and escape time (Fig. 33). In contrast to the longitudinal amplitudes the radial amplitudes in the stable region are extremely small.

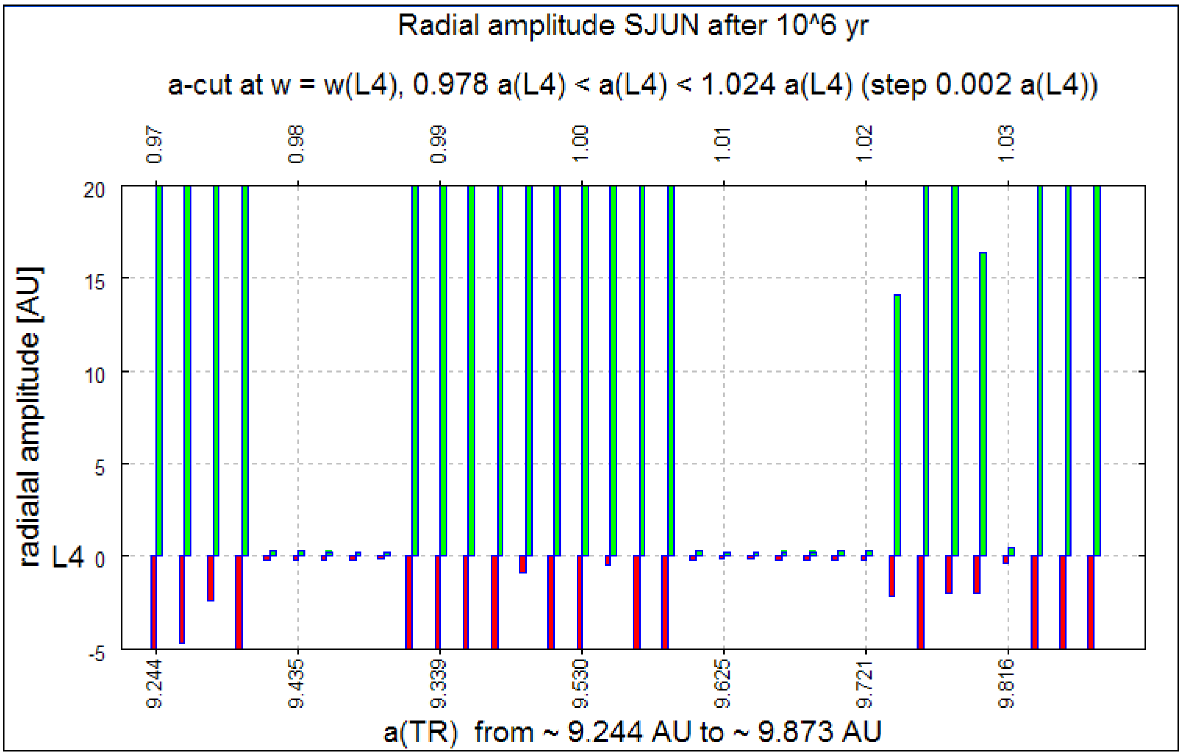

Fig. 35: The *a*-cut with the negative and positive radial amplitudes around L4 at 10^6 yr shows approximately the same shape of stable and unstable zones as maximum eccentricity (Fig. 31) and escape time (Fig. 33). In contrast to the longitudinal amplitudes the radial amplitudes are extremely small.

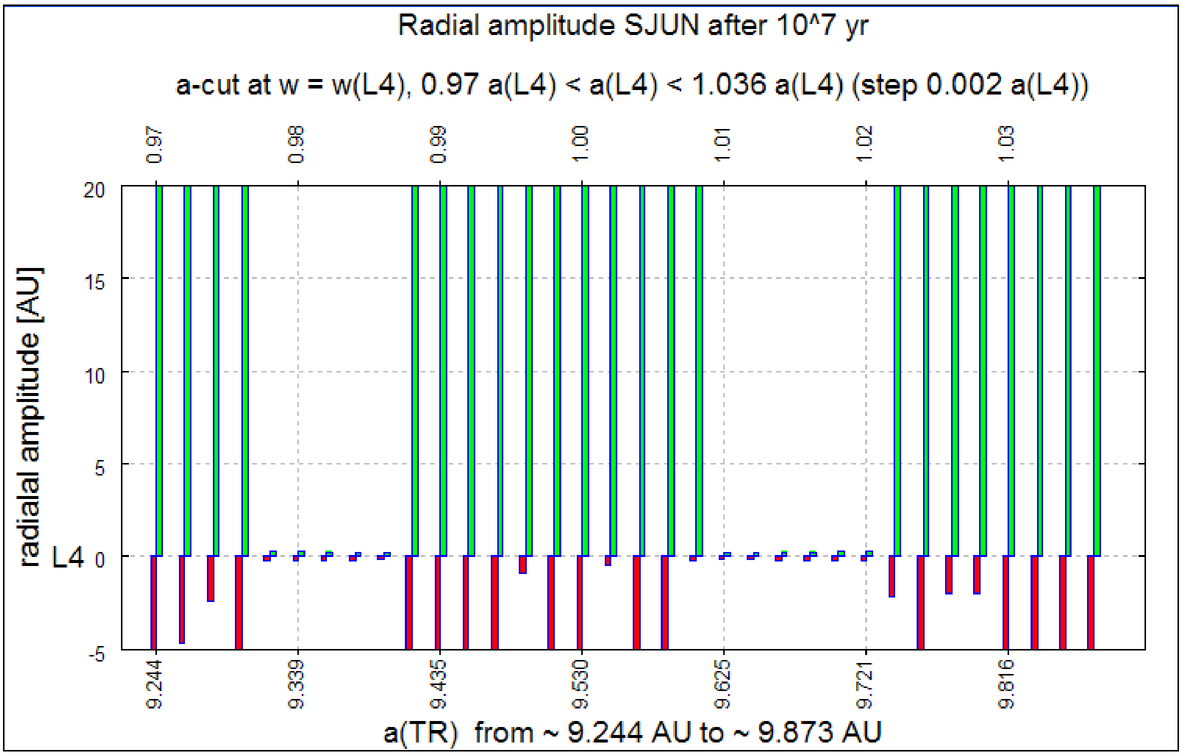

Fig. 36: The *a*-cut with the radial amplitudes at 10^7 yr again shows approximately the same shape of stable and unstable zones for maximum eccentricity (Fig. 32) and escape time (Fig. 34) at 10^7 yr. There is very little difference to Fig. 35 for a ten times greater timescale.

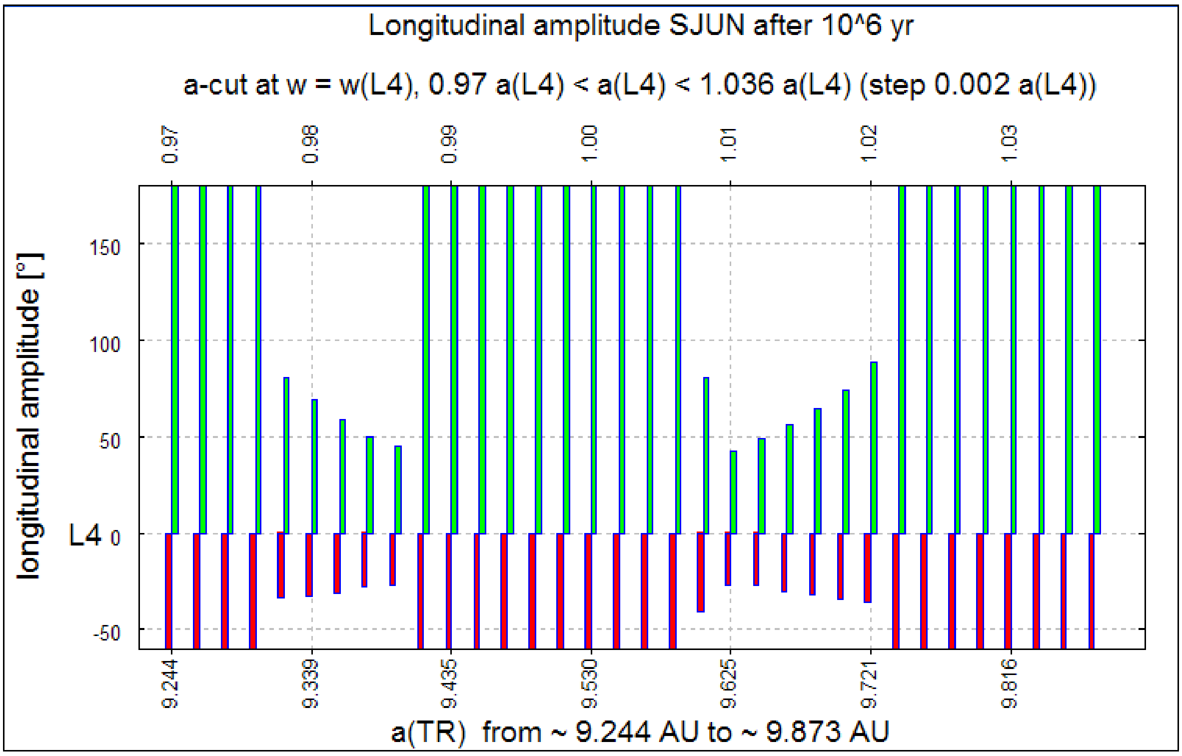

Fig. 37: The *a*-cut against maximum longitudinal amplitudes around L4 and starting orbits as in the previous Figures. At 10^6 yr approximately the same picture of stable and unstable orbits is given as for the radial amplitudes. But the longitudinal amplitudes are much larger, as 1° corresponds to about 0.17 AU.

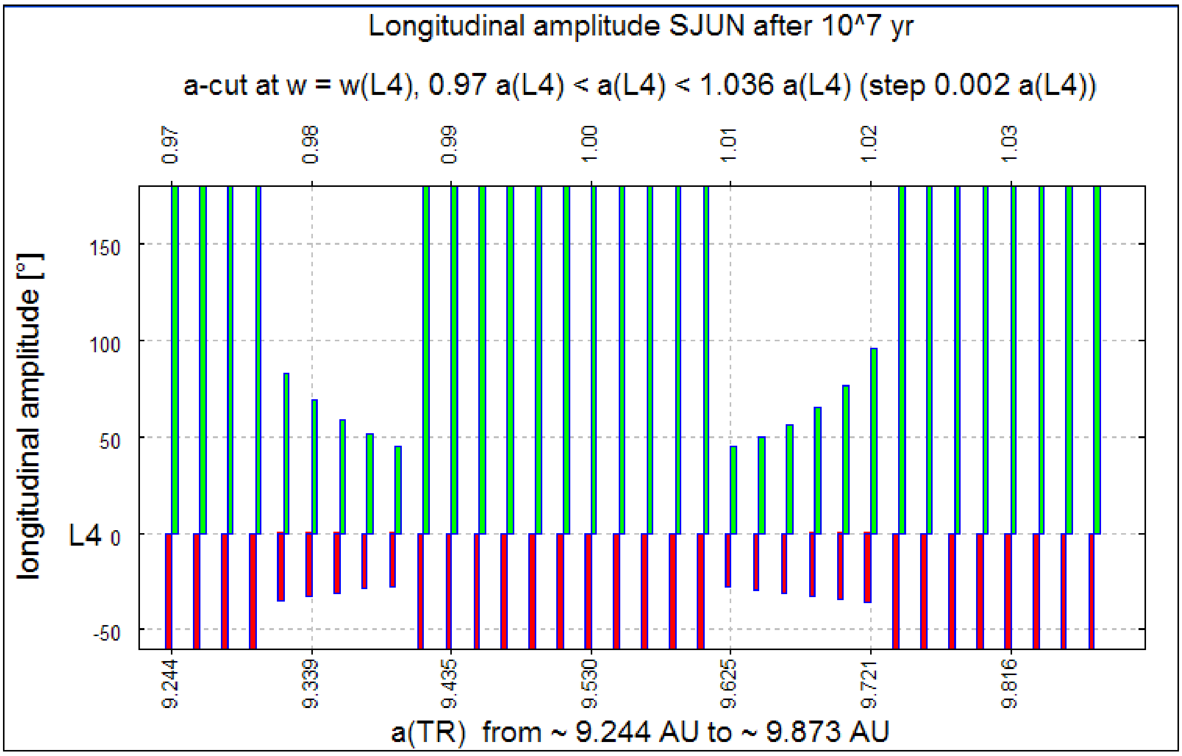

Fig. 38: The *a*-cut with the negative and positive maximal amplitude of the L4-point as Fig. 37 but at 10^7 yr. Also the longitudinal amplitudes rest more or less the same as at 10^6 yr. The stable and unstable zones are again in line with the radial amplitudes.

The a-cut with the radial amplitudes at 10^7 yr (Fig. 36) again shows the same approximate shape of stable and unstable zones at 10^7 yr as for maximum eccentricity (Fig. 32) and escape time (Fig. 34). In relation to the 10^6 yr amplitudes (Fig. 35) very little difference is diagnosed. Only the orbit at 1.008 a_{L4} has now become unstable.

2.2.4.1.4 a-cuts against longitudinal amplitude.

We defined the longitudinal amplitude as the variance of the true longitudinal lengths of the orbits with respect to that of the orbiting L4-poimt (subsection 2.1.2.2). Negative amplitude means the difference in direction towards the planet and positive amplitude in direction away from the planet. Negative values <= -60° are certainly unstable as they are then already in a close proximity to Saturn and become “planet crossers”. In the other direction amplitude values > +120° of L4 (= +/- 180° from Saturn) touches already the Lagrange L3-point. In other words, these asteroids have left the tadpole orbit and eventually form a horseshoe orbit.

The a-cut for only 10^6 yr (Fig. 37) shows nearly the same picture of stable and unstable orbits as for the radial amplitudes. But the longitudinal amplitudes are much larger, as 1° corresponds to about 0.17 AU.

The a-cut against longitudinal amplitude for 10^7 yr rest more or less the same as at 10^6 yr (Fig. 38). The stable and unstable zones are again in line with the radial amplitudes (Fig. 36), always indicating a clear separation of stability and instability.

2.2.4.2 The cuts in longitudinal direction (ω-cut)

The same cuts in longitudinal direction of L4 are made, all with the same semi-major axis with respect to Saturn, respectively L4, but with different longitudinal starting positions and therefore are called “ω-cuts”. The ω-cuts each time represent 34 orbits of Trojans with starting values of ω_{TR} from -36° from the L4-point (i.e. from +24° from Saturn) to +63° from the L4-point (i.e. +123° from Saturn) and $\Delta\omega = \pm 3°$.

2.2.4.2.1 ω-cuts against maximum eccentricity

The ω-cut against maximum eccentricity at 10^6 yr (Fig. 39) again shows the hole in the immediate proximity of L4 within -18° and +18° from ω_{L4}, the orbits gaining very large eccentricities. The stable zone in the direction of Saturn reaches from about -30° to about -21° of ω_{L4}. In the other direction the stable zone is significantly larger and reaches from about +21° to about +60° of L4.

The ω-cut against maximum eccentricity at 10^7 yr (Fig. 40) signals no difference to the same cut at 10^6 yr (Fig. 39) in relation to the stable zone. Only small or insignificant differences can be detected in consulting the basic simulation tables. This is unlike the same cuts in radial direction where differences are to be seen (Fig. 31 and Fig. 32).

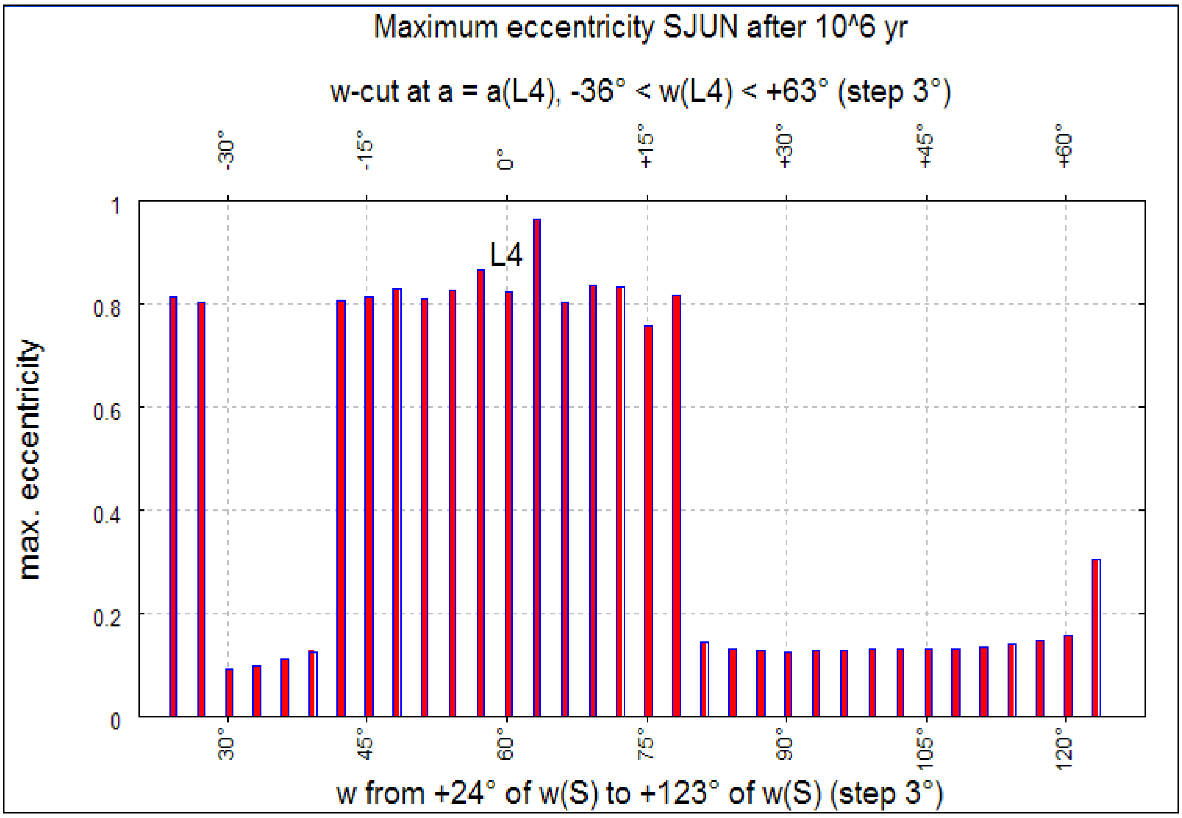

Fig. 39: The ω-cut against maximum eccentricity of 34 Trojans with starting orbits of ω_{TR} from -36° from the L4-point (i.e. ω_6 + 24°) to +63° from the L4-point (i.e. ω_6 + 123°) and $\Delta\ \omega = \pm 3°$. The ω_{L4} is always ω_6 +60°. Relative large stable zones with $e_{TR} <= 0.3$ at 10^6 yr are given.

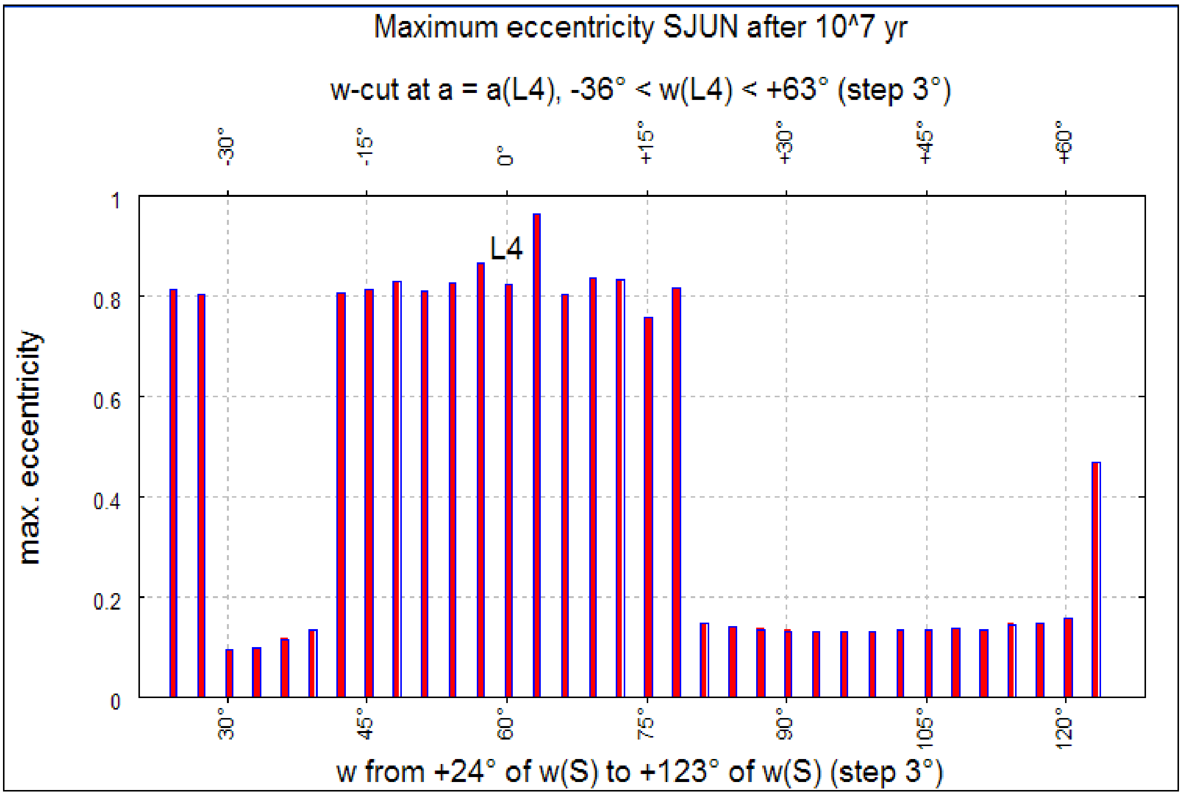

Fig. 40: The ω-cut against maximum eccentricity at 10^7 yr. There is very little difference in the stable zone during this period although it is ten times longer than the other.

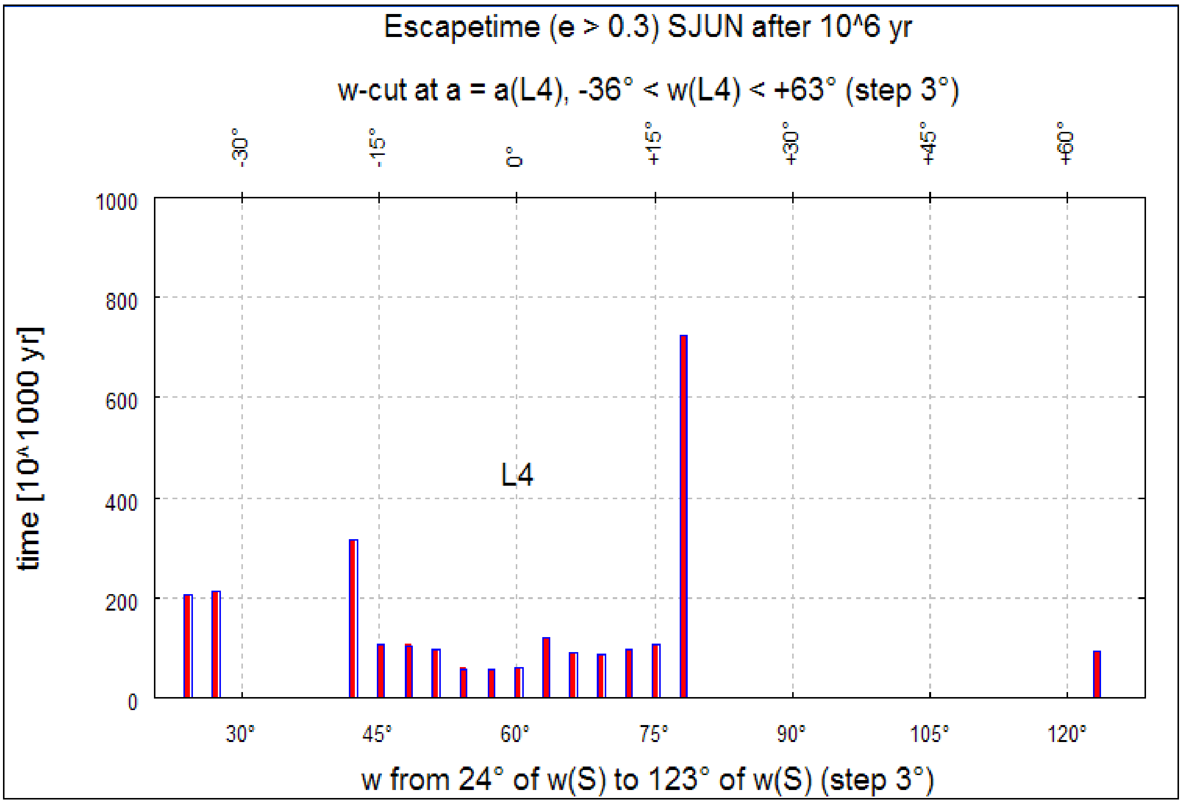

Fig. 41: The ω-cut against escape time with $e_{TR} > 0.3$ at 10^6 yr. The escape time distinguishes in a similar manner to that of the cut against maximum eccentricity between the stable and the unstable zones.

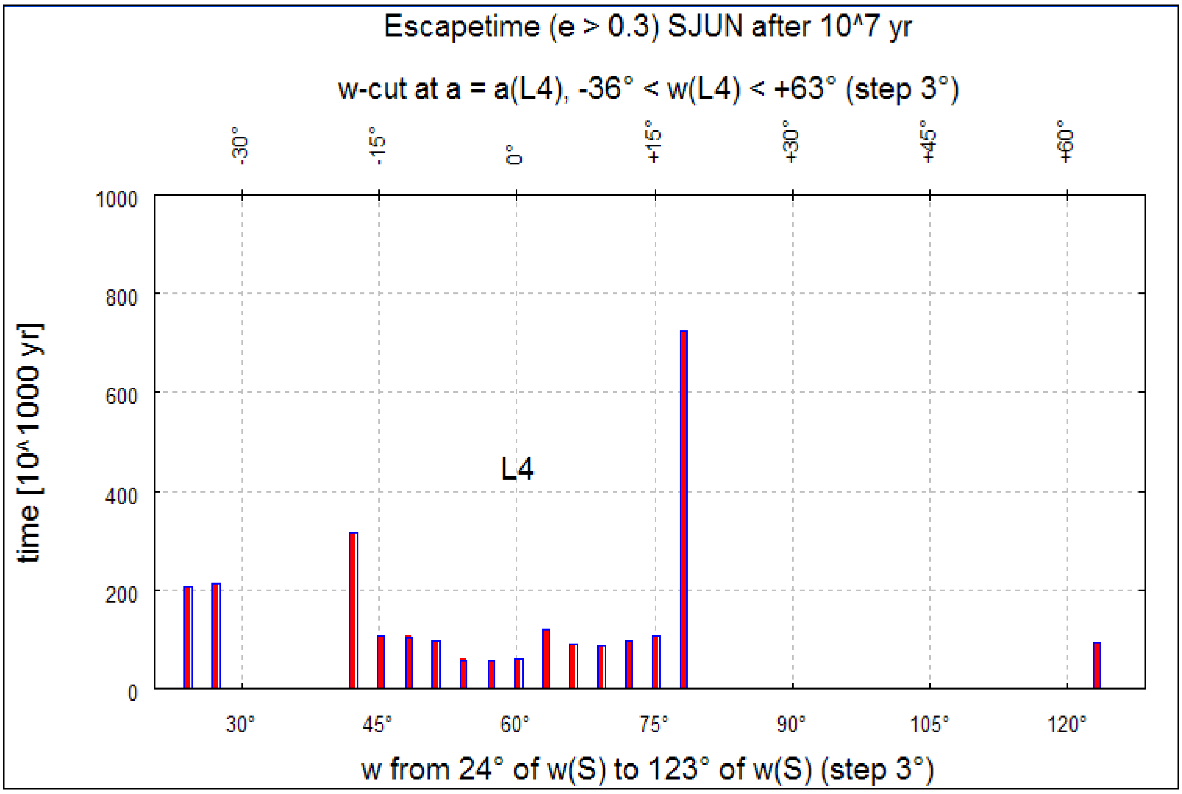

Fig. 42: The ω-cut against escape time with $e_{TR} > 0.3$ at 10^7 yr. No difference to the above 10^6 yr cut is to be seen for this 10 times longer period. All escapes have occurred within the first 10^6 yr.

2.2.4.2.2 ω-cuts against escape time

The ω-cuts against escape time at 10^6 yr (Fig. 41) and at 10^7 yr (Fig. 42) at $e_{TR} > 0.3$ signals in a certain sense the same shape as the comparable cuts against maximum eccentricity. There is absolutely no difference between the 10^6 yr and the 10^7 yr cut given, because all escapes had already occurred within the first 10^6 yr and there are no additional escapes during this tenfold longer period of 10^7 yr.

2.2.4.2.3 ω-cuts against radial amplitude.

The ω-cut against radial amplitudes signifies the maximum variability of the orbits in the radial direction around L4.

The cut for radial amplitudes at 10^6 yr (Fig. 43) shows nearly the same shape of stable and unstable zones as maximum eccentricity (Fig. 39) and escape time (Fig. 41). In contrast to the longitudinal amplitudes the radial amplitudes in the stable region are also here extremely small.

The ω-cut for the radial amplitudes at 10^7 yr (Fig. 44) again shows nearly the same shape of stable and unstable zones at 10^7 yr as for maximum eccentricity (Fig. 40) and escape time (Fig. 42). In relation to the 10^6 yr amplitudes (Fig. 43) absolutely no difference is found for this tenfold period.

2.2.4.2.4 ω-cuts against longitudinal amplitude.

The ω-cuts against longitudinal amplitude for 10^6 and 10^7 yr are to be seen in Fig. 45 and Fig. 46 respectively. As in Fig. 45 and Fig. 46 for the radial cuts, negative values <= -60° of the amplitudes are certainly unstable as well as values > +120° (= +/-180° of Saturn).

The ω-cut against longitudinal amplitudes for only 10^6 yr (Fig. 45) shows the same shape of stable and unstable orbits as the radial amplitudes (Fig. 43). But the longitudinal amplitudes are again significantly larger.

The ω-cut against longitudinal amplitudes at 10^7 yr (Fig. 46) shows for this tenfold period absolutely the same picture as at 10^6 yr (Fig. 45).

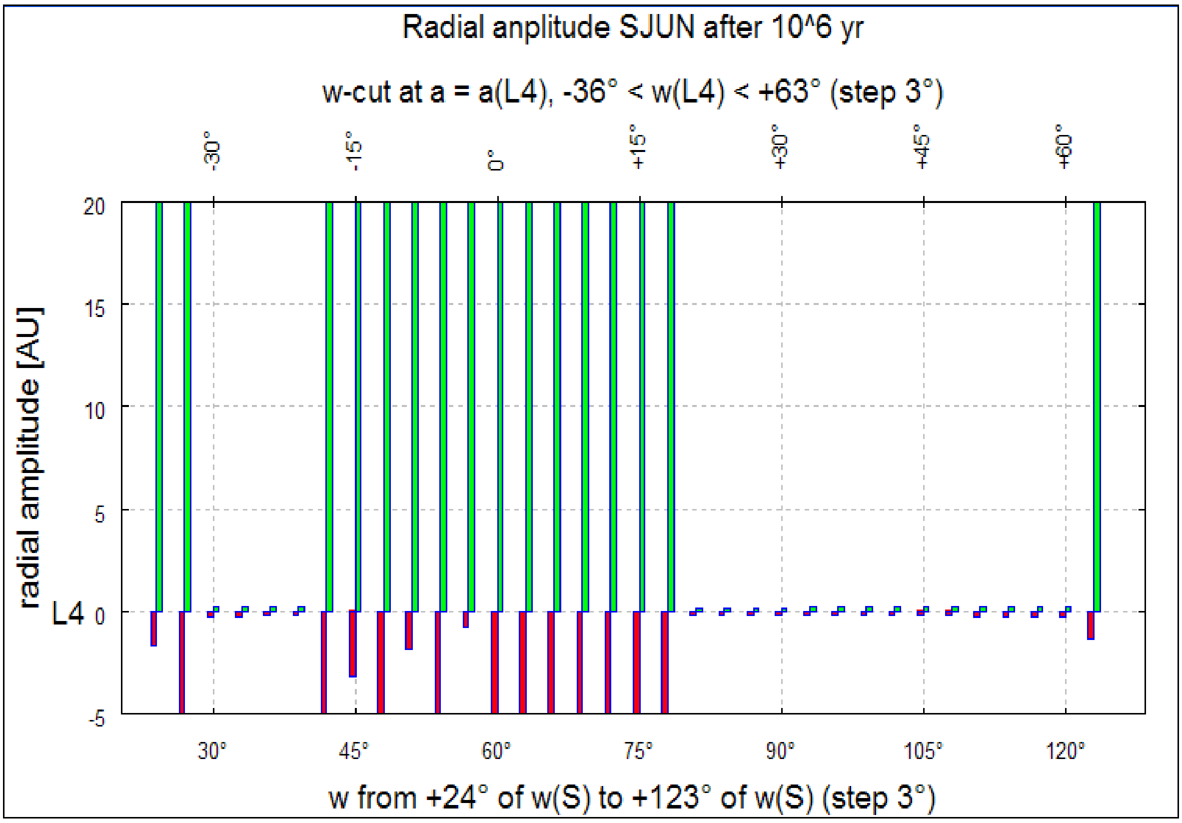

Fig. 43: The ω-cut with radial amplitudes at 10^6 yr shows large areas of very small amplitudes. Only the hole and the borders signal instabilities.

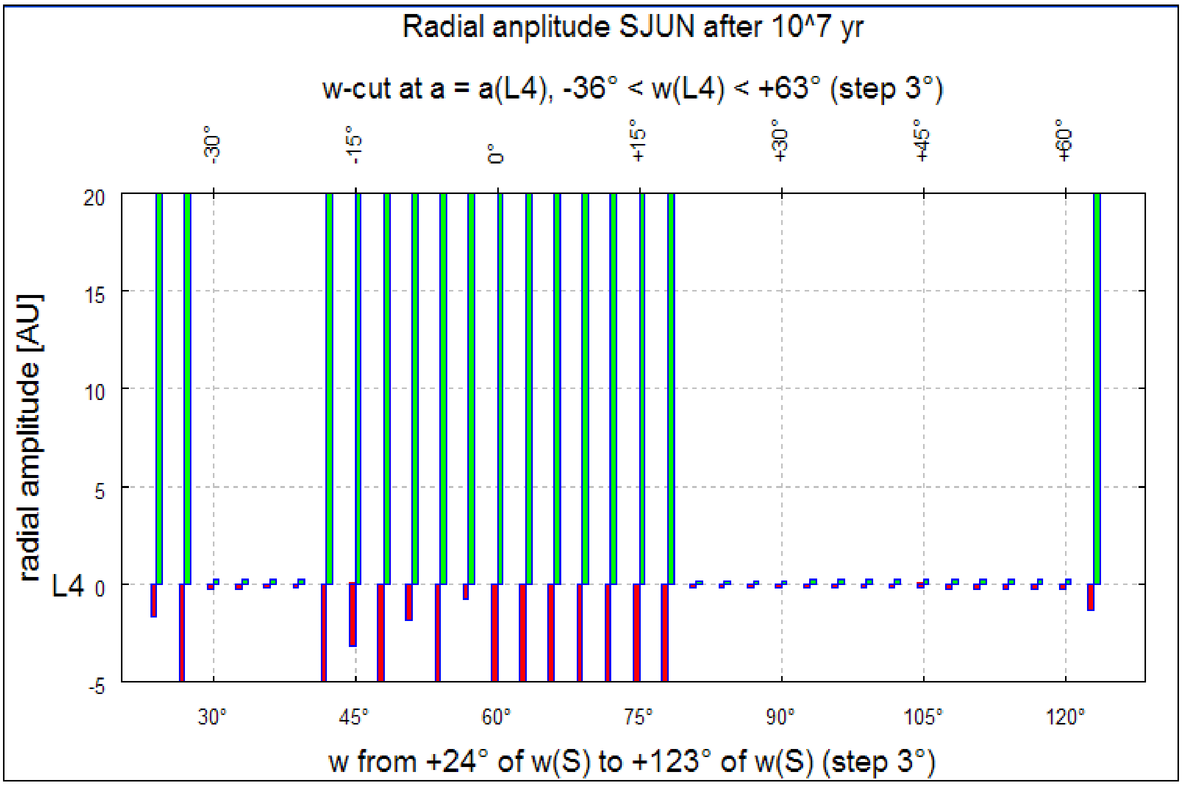

Fig. 44: The ω-cut at 10^7 yr shows absolutely the same large areas of very small radial amplitudes as the 10^6 yr cut in Fig. 43.

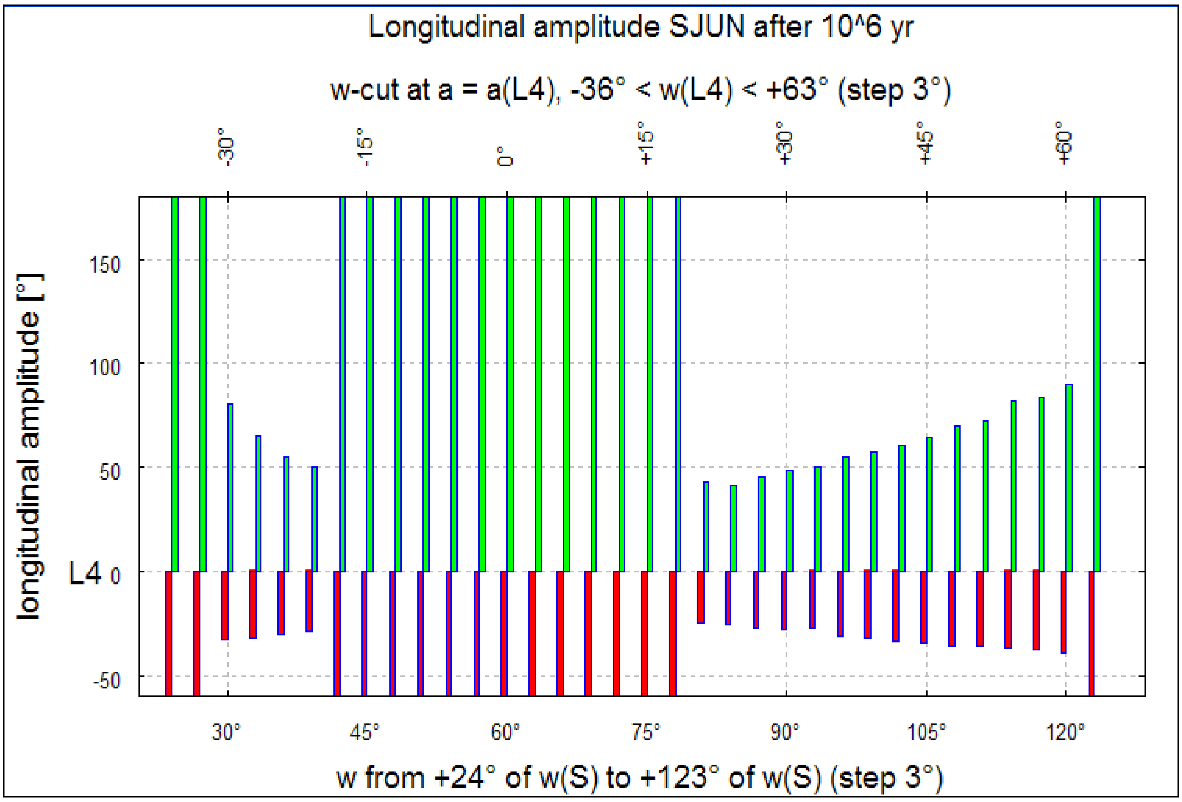

Fig. 45: The ω-cut against longitudinal amplitudes at 10^6 yr. The longitudinal amplitudes show the same shape of stable and unstable zones as the radial amplitudes but they are significantly larger.

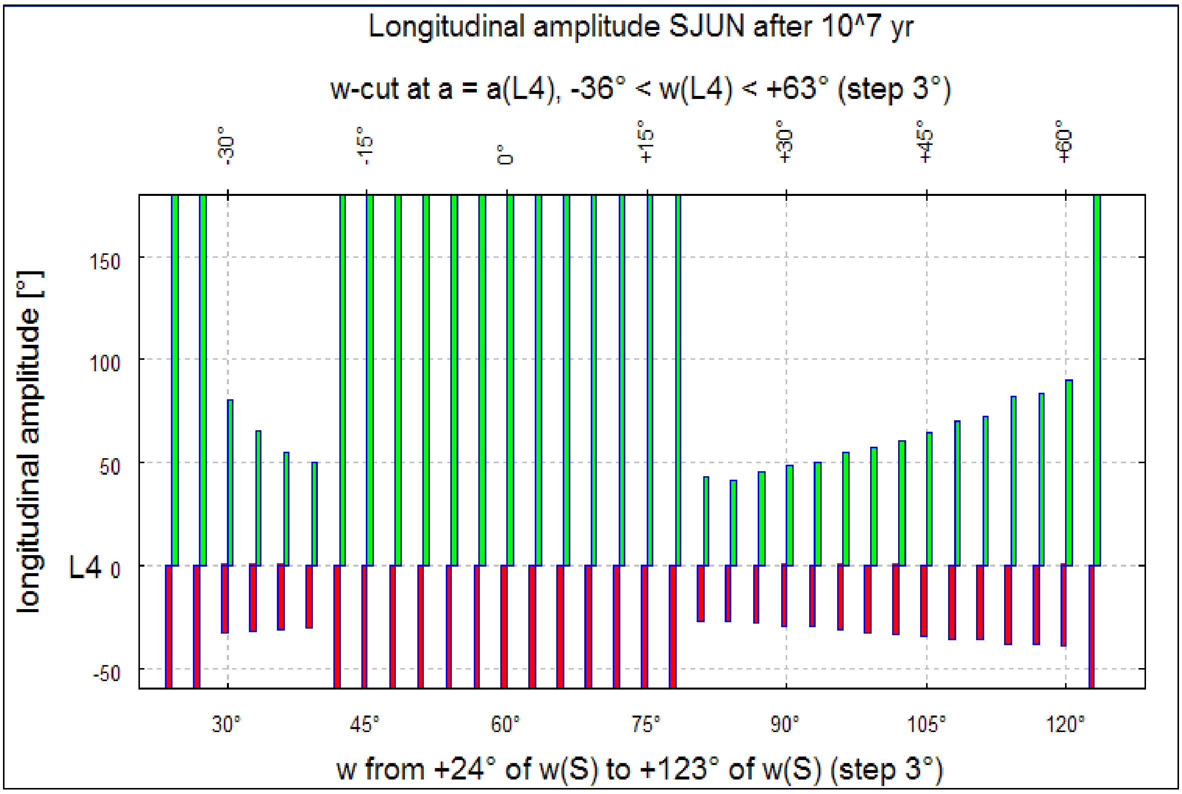

Fig. 46: The ω-cut against longitudinal amplitudes at 10^7 yr. There is no difference to be seen for this tenfold period compared with the corresponding 10^6 yr cut of Fig. 45.

2.2.4.3 Summary of the *a*-cuts and *ω*-cuts in the SJUN-model

As a summary one can note that all the parameters investigated, namely maximum eccentricity, escape time, radial and longitudinal amplitudes show more or less the same shape of stable and unstable zones for the tenfold period. This applies for the *a*-cut as well as for the *ω*-cut. The unstable hole in the near proximity of L4 expands in radial direction from about -0.988 a_{L4} to 1.008 a_{L4} and in the longitudinal direction from about -18° to about +18° of L4. The stable zone reaches only from about 0.978 a_{L4} to 0.986 a_{L4} and from 1.008 a_{L4} to 1.020 a_{L4} in radial extension and from about -30° to -24° of L4 and from about +21° to +60° of L4 in longitudinal extension.

In sum there are no major differences in the investigations for 10^6 yr and the tenfold period of 10^7 yr in the *a*-cut as well as for the *ω*-cut. In other words, the loss of stable orbits should occur mainly in the first Million of years. This stands in clear contrast to Téger, who underlined after his integrations for 10^7 yr, that with the increase of the integration time the unstable hole around L4 and L5 becomes larger and larger. This we can not confirm at all (compare with Fig. 4 in subsection 2.1.1).

To show this early loss, we computed the number of escaped Trojans against their escape time ($e_{TR} > 0.3$) for the *a*-cut and for the *ω*-cut with each of the 34 Trojans (Fig. 47). In fact we see that the survival of Trojans is fast diminishing during the first 1.5×10^6 yr (more than half of the starting Trojans) and then the number remains stable until at least 10^7 yr.

If we refer to the grid of Trojans for 10^6 yr in the SJUN-system (Fig. 12 in subsection 2.2.2), starting with the 440 asteroids surviving the first 10^3 yr, we find a hyperbolic curve of surviving Trojans with time. After 5×10^5 yr only 189 (=43%) and after 10^6 yr 172 (=39 %) of the initial 440 fictive particles survived (Fig. 48). In contrast to MTS, who found a rather steady loss till to their integration time of 4.5×10^9 yr, our curve is really asymptotic, without further losses after 1.3×10^6 yr. Another study with 50 massless particles for Trojans of Jupiter also shows such an exponential decay with an almost linear tail for t > 350 Myr (T&D).

Finally we can declare that all the cuts give a very clear picture of stable and unstable regions with the unstable hole in the middle and it will be almost impossible for Saturn to hold a great number of L4 Trojans – far away from the numbers of Jovian Trojans. If Trojans of Saturn survive for more than 1.3×10^6 yr, they only form a very small girdle around its L4-point and supposedly fewer still around its L5-point; all this only because of the relatively nearby giant planet Jupiter.

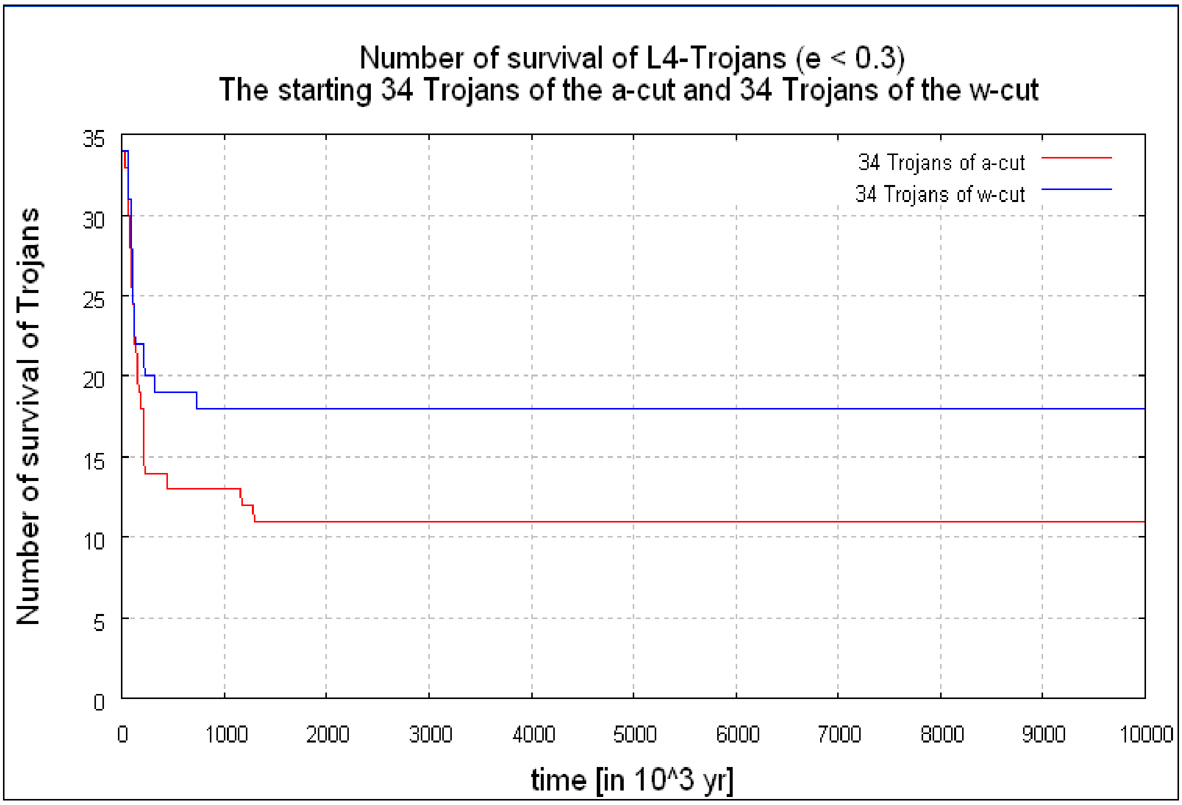

Fig. 47: Number of L4-Trojans of the *a*-cut and *ω*-cut surviving over the period of 10^7 yr.

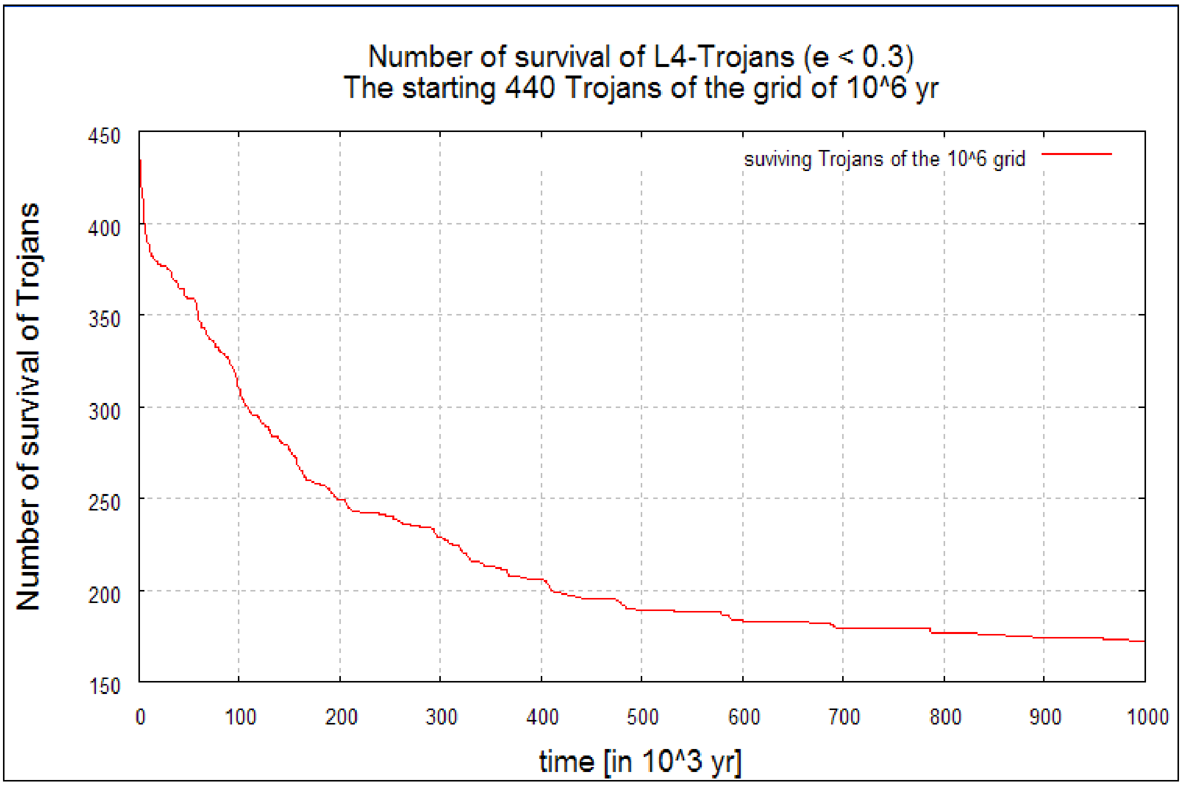

Fig. 48: Number of surviving L4-Trojans of the 10^6 yr grid of the SJUN-model with the 440 starting asteroids surviving the first 10^3 yr. It is striking to see the fast decline of surviving asteroids in the very early period.

2.2.4.4 The cuts in the SUN-system

This potentially thrilling statement has to be tempered if the near impossibility of Saturn to hold a great number of L4 and L5 Trojans is caused by the relatively nearby giant planet Jupiter as already confirmed by our investigations on the different compositions of the Solar system simulated in subsection 2.2.2. Therefore and to have a better comparison to the above cuts, we investigated cuts of the OSS but without Jupiter (the SUN-system), for 10^6 yr. In addition we expect as in the above cuts some interesting orbits which we shall investigate in the following subsection 2.2.5.

2.2.4.4.1 The *a*-cut against maximum eccentricity and escape time in the SUN-system

Fig. 49 shows the application of the *a*-cut against maximum eccentricity and escape time in the SUN-system. In comparison to the full OSS for 10^6 yr (Figs. 31 and 33 in subsection 2.2.2.1), now not only the unstable hole has completely vanished, but instead of only 12 stable orbits, a really enlarged zone with 29 stable orbits is now reaching from 0.970 a_6 to 1.026 a_6. This means, in this case, roughly 2.4 times more stable orbits. It is interesting, that also in the SUN-model the orbit at 1.030 a_6 is still stable.

2.2.4.4.2 The *a*-cut against radial and longitudinal amplitudes in the SUN-system

Here (Fig. 50) again the amplitudes are in good accordance with maximum eccentricity and escape time concerning stability (Fig. 49). It is amazing to note the characteristic devolution of the maximum amplitudes with the smallest width of values at the exact L4-point. An interesting difference between radial and longitudinal amplitudes shows in the orbits at 0.97 a_6, 0.972 a_6 and 1.03 a_6. The explication will be given in subsection 2.2.5.2.

2.2.4.4.3 The ω-cut against maximum eccentricity and escape time in the SUN-system

Fig. 51 shows the application of the ω-cut against maximum eccentricity and escape time in the SUN-system. This plot is to compare with Figs. 39 and 41 of the full OSS for 10^6 yr. Apart of the hole instead of only 18 stable orbits, a really enlarged zone with 45 stable orbits is now reaching from -39° of L4 to +93° of L4. This means, in this case, roughly 2.5 times more stable orbits, about the same as in the *a*-cut.

2.2.4.4.4 The ω-cut against radial and longitudinal amplitudes in the SUN-system

Also for the ω-cut against radial and longitudinal amplitudes (Fig. 52) the amplitudes are in good accordance with maximum eccentricity and escape time (Fig. 51). Again we find interesting differences between radial and longitudinal amplitudes, here for the orbits at -33°, -36°, +93° and +102° of ω_{L4}. For the reason we also refer to subsection 2.2.5.2. Once again, the smallest width of values is given near the exact L4-point signalling maximum stability.

These two cuts of the SUN-system compared with the SJUN-system show in a significant manner the overwhelming influence of Jupiter in not allowing Saturn to hold any Trojan asteroids in proximity to the L4-point because without Jupiter the unstable hole has vanished.

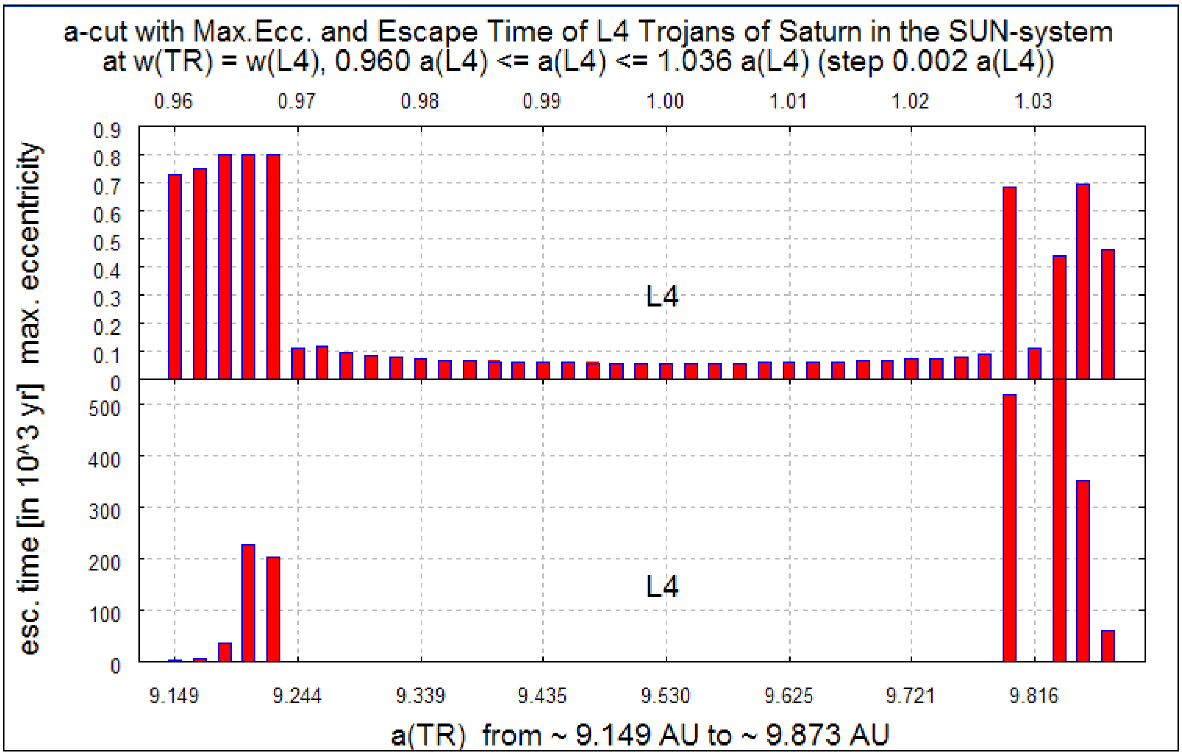

Fig 49: *a*-cut against maximum eccentricity and escape time in the SUN-system (without Jupiter). The hole does not exist at all and the stable zone is significantly broader than in the SJUN-model.

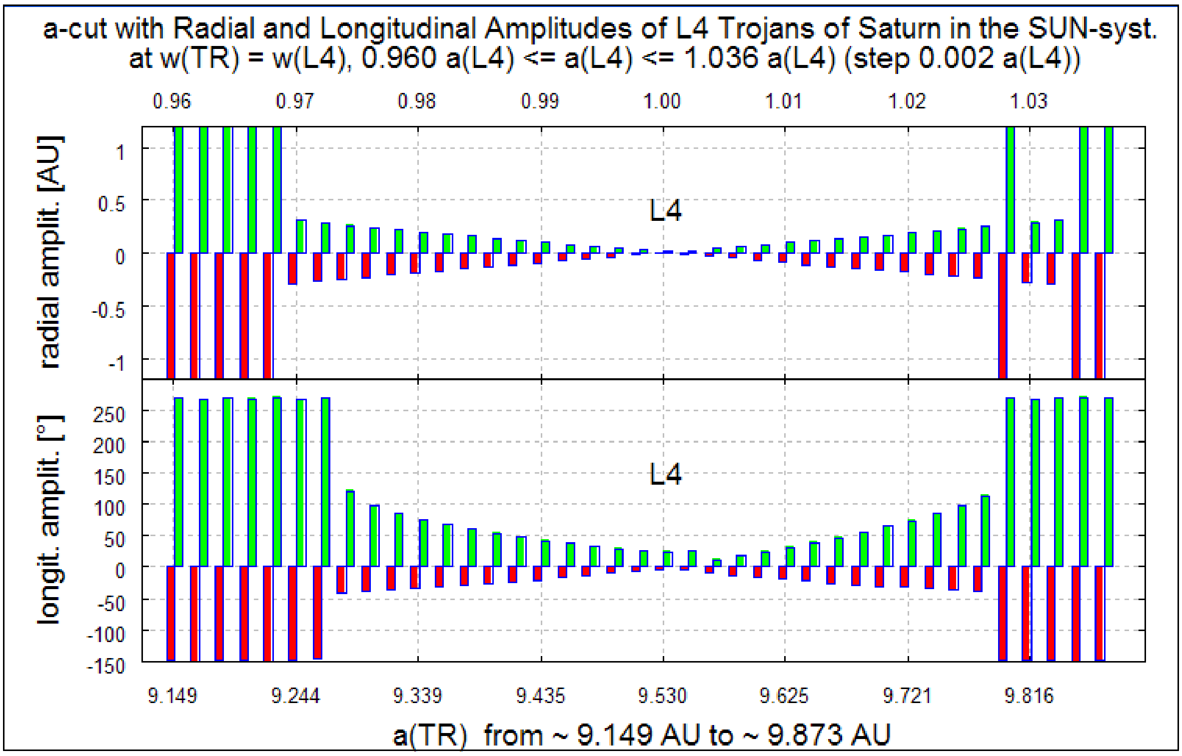

Fig 50: *a*-cut against maximum radial and longitudinal amplitudes of the SUN-system. The amplitudes are in good accordance with maximum eccentricity and escape time (Fig. 49). An interesting difference between radial and longitudinal amplitudes shows in the orbits at 0.97 a_6, 0.972 a_6 and 1.03 a_6.

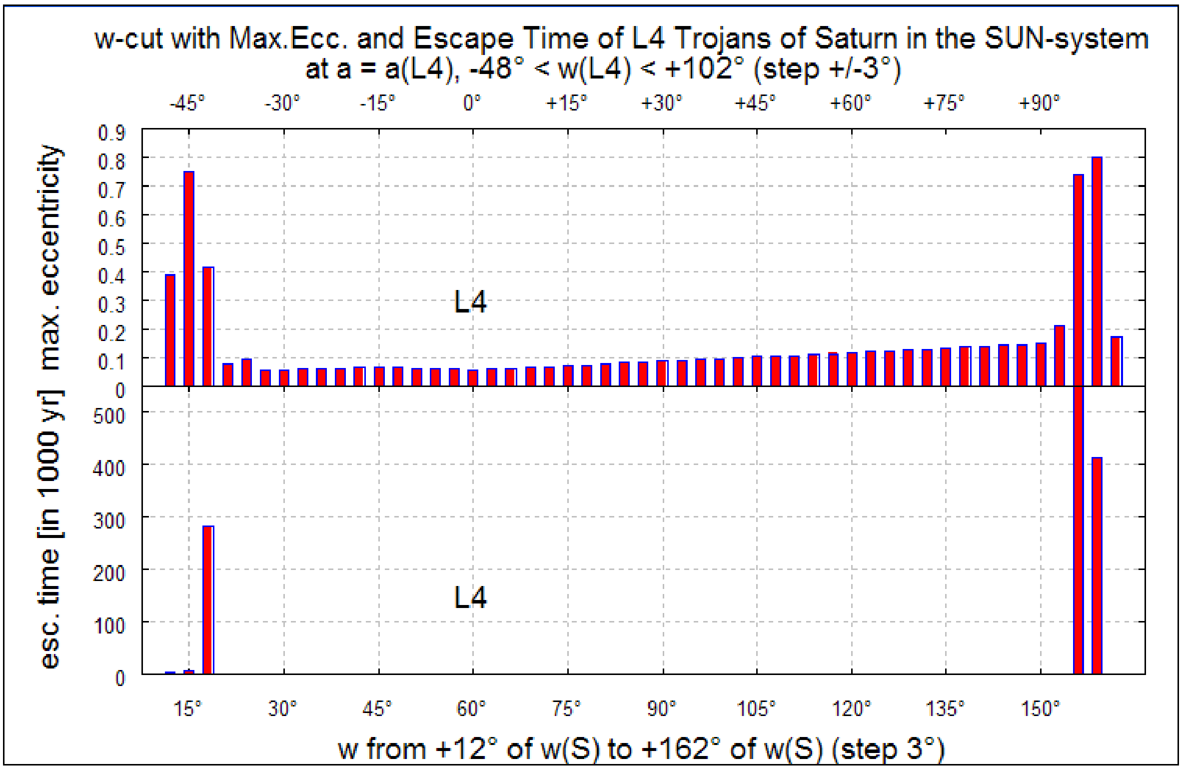

Fig 51: ω-cut against maximum eccentricity and escape time in the SUN-system (without Jupiter). The hole does not exist at all in the SUN-model. The stable zone is 2.5 times and therefore significantly broader than in the SJUN-model.

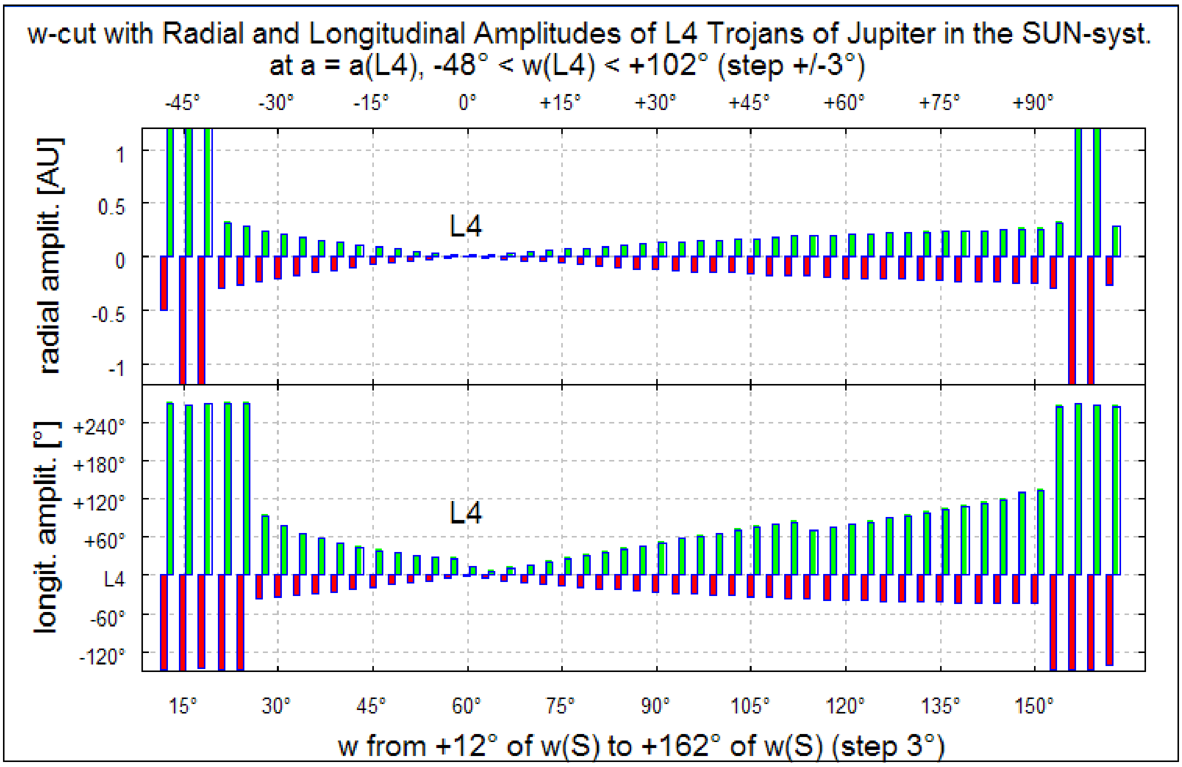

Fig 52: ω-cut against maximum radial and longitudinal amplitudes of the SUN-system. The amplitudes are in good accordance with maximum eccentricity and escape time (Fig. 51). An interesting difference between radial and longitudinal amplitudes shows in the orbits at -33°, -36°, +93° and +102° of ω_{L4}.

2.2.5 Special orbits of L4 'jumping' Trojans

As we have shown, there exists a more or less clear border between stable and unstable orbits of L4 Trojans of Saturn as well as in considering the escape time or maximum eccentricity or radial amplitudes or longitudinal amplitudes. We want to pay particular attention to the behaviour of the true longitude of some Trojans with respect to the true longitude of the L4-point to find out if the Trojans could move in horseshoe orbits. Naturally, only unstable orbits or orbits of near instability are of interest for us. To have a closer look at some special orbits, we investigated several Trojans of the *a*-cut, as well as for the *ω*–cut. We ran this integration for a period of 10^6 yr, but with much more output data (about one data-set per year) than in the previous integrations. This output step size improves coverage of the orbital period of Jupiter of 11.86 yr around the Sun and naturally also the period of Saturn with 29.46 yr (Ranzini, 2000). It also better allows us to detect orbits which eventually become horseshoe orbits. A horseshoe orbit could only be where eccentricity and radial amplitude do not exceed a certain limit (e.g. $e > 0.3$ and radial amplitude of +/-0.5 AU of L4) and remain for an appropriate time at a suitable longitudinal distance from Saturn. As our longitudinal amplitudes of the fictitious Trojans are calculated as the longitudinal distances to the L4-point, we have to be careful. A very close encounter of a Trojan with the planet (at -60°) therefore is given e.g. at <= -50° of L4 in front of the planet and at >= -70° of L4 from behind of the planet; always measured relative to the direction of the orbit of Saturn.

The time-scale in all Figures is indicated in 10^3 yr.

As we have seen that some orbits show stability in radial amplitudes, but not in longitudinal amplitudes, we shall have a closer look at the behaviour of such Trojans in the SJUN-system. Afterwards we shall do the same for such Trojans without Jupiter in the SUN-system.

2.2.5.1 Special orbits of L4 Trojans in the SJUN-system

2.2.5.1.1 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.980\ a_{L4}$

To start with, we look at the Trojan with starting values of $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.980\ a_{L4}$, which we previously knew as a stable orbit up to 10^7 yr (compare with Figs. 34, 36 and 38 in subsection 2.2.4.1). Fig. 53 shows the longitudinal oscillation of this co-rotating Trojan with the L4-point. The maximum oscillation for about +70° keeps well away from the L3-point at +120° from the L4-point (respectively at +/-180° from Saturn). The minimum with about -30° always remains in some distance from -60° of L4, where Saturn is to be found. These are the absolute limits for stable tadpole orbits relating to Saturn.

In Fig. 54 we chose a period of only 400 yr to demonstrate the stable behaviour of the orbit of this absolutely stable Trojan with respect to Saturn and the L4-point. It is nice to see the constant close proximity of the co-rotating Trojan with the L4-point.

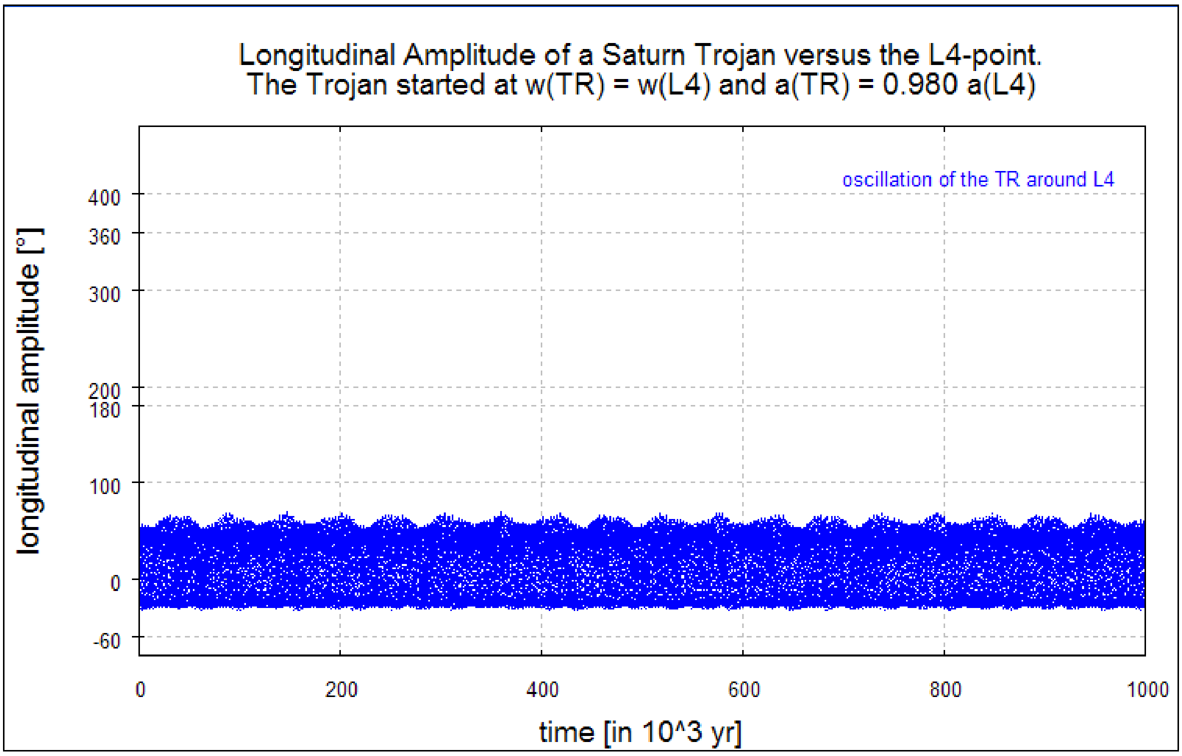

Fig. 53: Longitudinal amplitudes of a Trojan with starting position at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.980\ a_{L4}$ against the movement of the L4-point. This orbit is known to be stable up to at least 10^7 yr.

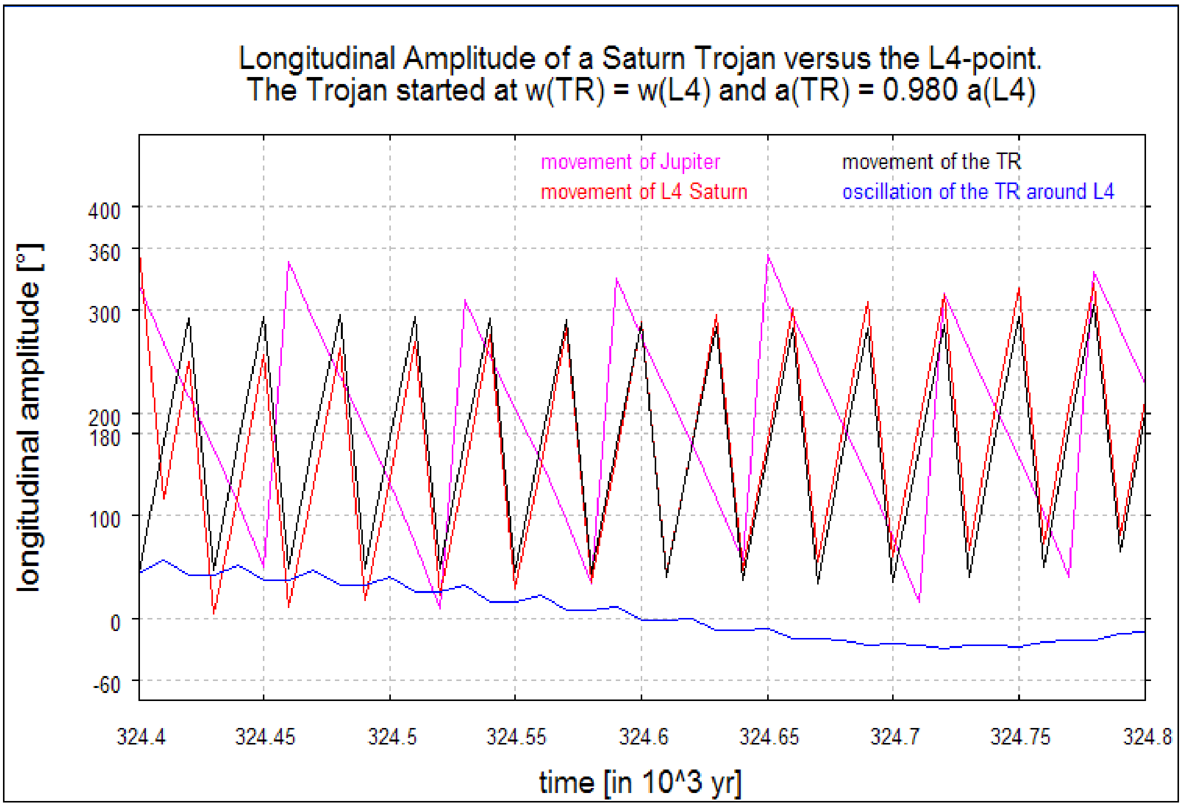

Fig. 54: Longitudinal amplitude of the same Trojan as before (Fig. 53) but with a chosen period of only 400 yr. These orbits of the Trojan keep close to the L4-point at 0°.

2.2.5.1.2 Trojan with starting $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.988\ a_{L4}$

The Trojan with starting values at $\omega_{TR} = \omega_{L4}$ but now $a_{TR} = 0.988\ a_{L4}$ was the last unstable orbit of the hole near L4 in the direction of the Sun with an escape time of about 430×10^3 yr (see Figs. 32 and 33 in subsection 2.2.4.1). In reality this Trojan already becomes unstable after 324×10^3 yr, as Fig. 55 shows, because of its longitudinal amplitudes trespassing clearly across the 120° limit as well as the -60° limit. At this time the Trojan might enter a tadpole orbit of the L5 point, because it remains for a period of about 350 yr in some distance from the L4-point; later becoming completely unstable.

In Fig. 56 we chose again the same period of only 400 yr, as in Fig. 54, for this Trojan to demonstrate the difference from a stable orbit. It seems that any time, when the true longitude of a Trojan and the true longitude of Jupiter are about the same or if the Trojan stands in opposition to Jupiter, we can detect an intermediate positive or negative maximum of elongation from L4.

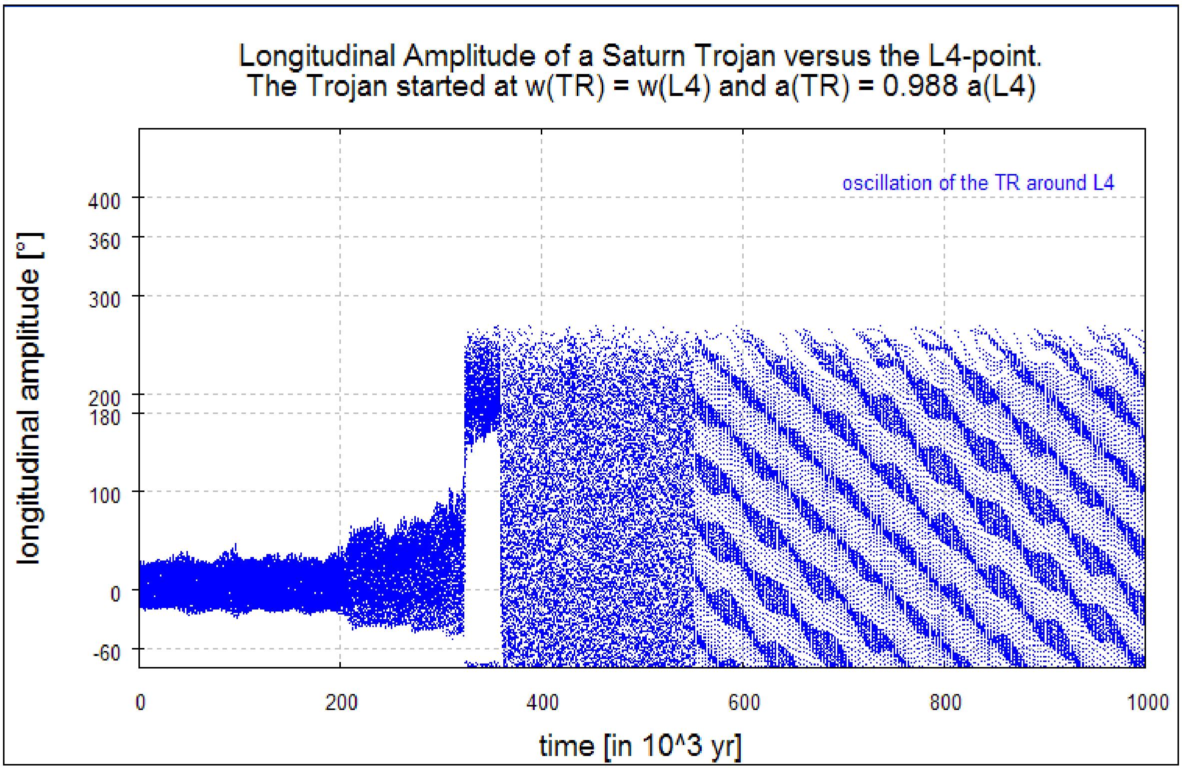

Fig. 55: Longitudinal amplitude of the same Trojan as before but over the whole integration period of 10^6 yr. At latest after 3.24×10^5 yr the Trojan is out of the control of the L4-point and jumping to the L5 point.

2.2.5.1.3 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = a_{L4}$

This Trojan is of special interest, as this test body was started exactly at the L4-point. The early escape time already noted in Fig. 33 in subsection 2.2.4.1 and Fig. 41 in subsection 2.2.4.2 is also confirmed in regard to the evolution of the longitudinal amplitude after about 58×10^3 yr (Fig. 57). The Trojan has become completely unstable as the longitudinal amplitudes reach the limits of -60° and +120° of the L4-point.

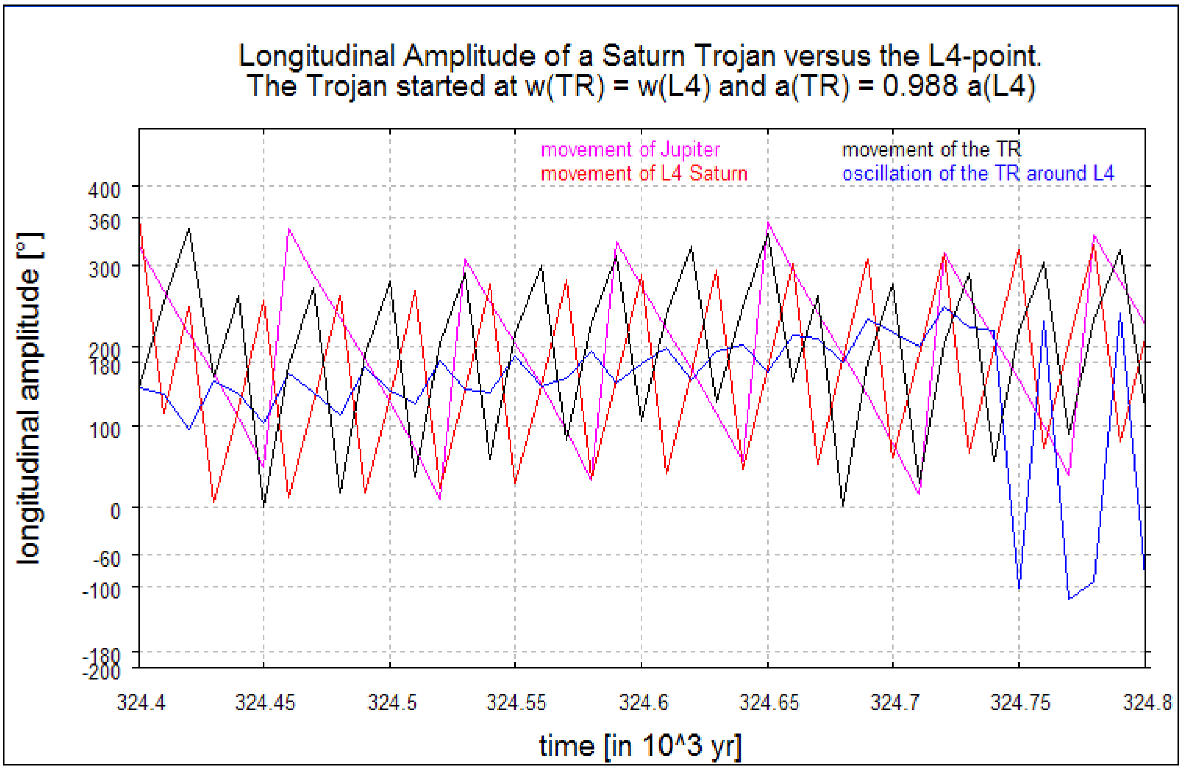

Fig. 56: Longitudinal amplitude of the same Trojan as before (Fig. 55) but with a chosen period of only 400 yr. This orbit becomes unstable after about 324 yr as the amplitudes reach the upper limit of +180° of the L4-point.

Fig. 58 represents a section of just 2000 yr of Fig. 57 and gives a closer view of the behaviour of the longitudinal amplitude at this critical moment. At $57.5x10^3$ yr the Trojan nearly touches Saturn near -60° of the L4-point and then again at about $57.8x10^3$ yr from the other side, which means that it has become for once a horseshoe orbit. Finally the Trojan becomes completely unstable at about $58.5x10^3$ yr.

Fig. 59 finally gives a complete view of the main parameters of this Trojan which started exactly at the L4-point. It seems that the Trojan indeed has developed a horseshoe orbit on reaching $57.5x10^3$ yr, as inclination, eccentricity and radial amplitude now rest within the necessary limits. After $58.5x10^3$ yr, the orbits have become completely unstable, when the radial amplitude diverges and later on also eccentricity and inclination are growing. (Similar overviews are presented e.g. by T&D).

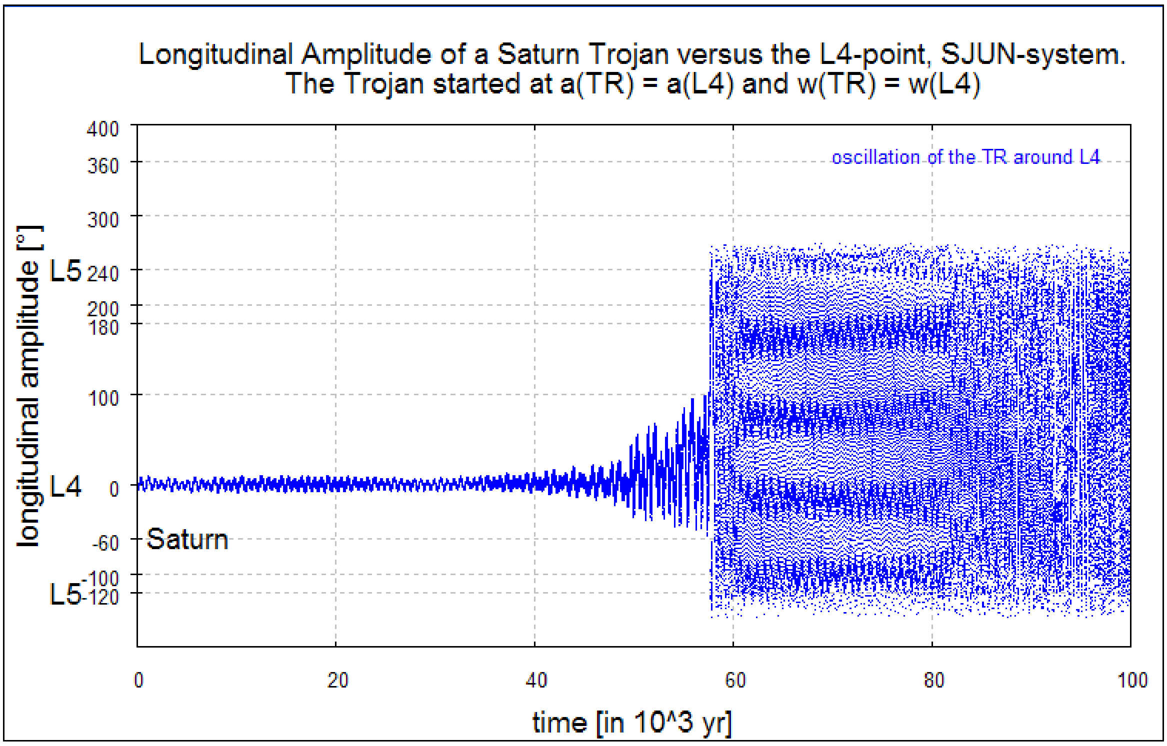

Fig. 57: Longitudinal amplitude of a Trojan, started at the exact L4-point. The Trojan becomes unstable at about $58x10^3$ yr, as the amplitudes reach the limits of -60° and +120° of the L4-point.

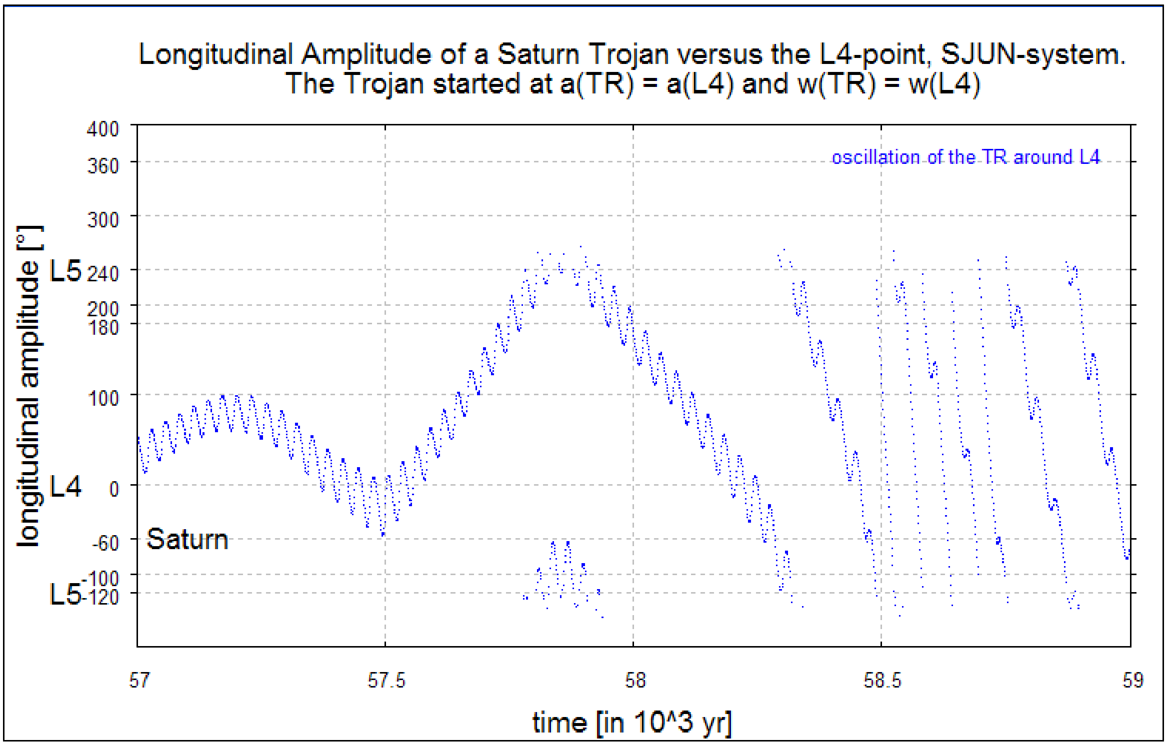

Fig. 58: Section of Fig. 57 of the longitudinal amplitudes of a Trojan, started exactly at the L4-point. At $57.5x10^3$ yr the Trojan nearly touches Saturn at -60° of the L4-point and then again from the other side at about $57.8x10^3$ yr, which means that it has become for once a horseshoe orbit. Finally the Trojan becomes completely unstable at about $58.5x10^3$ yr.

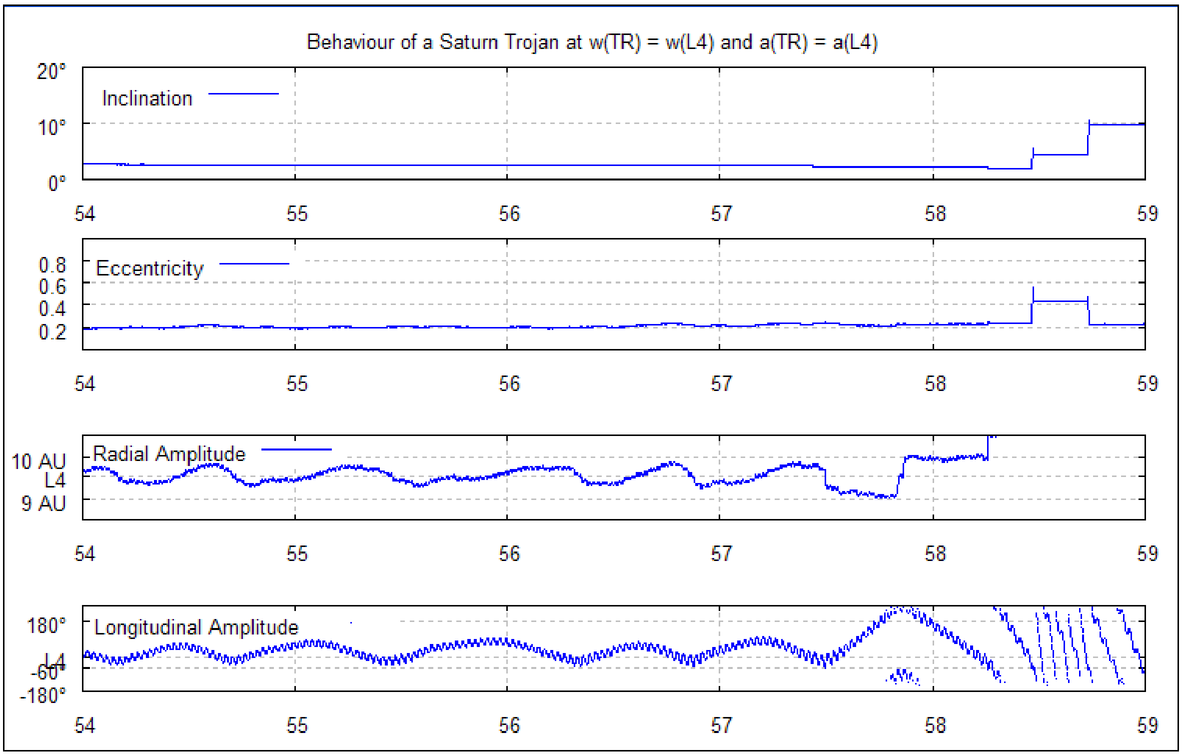

Fig. 59: Overview of some main parameters of the Trojan started at the exact L4-point. It seems that the Trojan indeed has become for once a horseshoe orbit after 57.8x10³ yr, as inclination, eccentricity and radial amplitude rest within the necessary limits. After 58.3x10³ yr, when the radial amplitude diverges and later on also eccentricity and inclination are growing, the orbits have become completely unstable.

2.2.5.1.4 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.974\ a_{L4}$

The Trojan with starting values at $\omega_{TR} = \omega_{L4}$ but now with $a_{TR} = 0.974\ a_{L4}$ was the last but one investigated relating to our cuts in direction of the Sun. This orbit had an escape time of about 10^5 yr (see Fig. 33 in subsection 2.2.4.1). The overview of this Trojan (Fig. 60) signals an interesting behaviour. It indicates an escape time somewhat earlier at 80×10^3 yr because of the diverging radial amplitude at this time and the growing inclination and eccentricity. But after 28×10^3 yr this Trojan might for once and later on have entered a temporary horseshoe orbit consistently at a great distance from the L4-point (Fig. 61).

To get a better view of this interesting behaviour we regarded three successive sections of Fig. 61, each of only 10^4 yr, in Figs. 62 – 64. Indeed it seems now to be clear, that this Trojan has entered a temporary horseshoe orbit at 28×10^3 yr and than at 36×10^3 yr with very regular jumps between the L4-point and the L5-point every 770 yr. These jumps are typical of a horseshoe orbit.

Fig. 65 finally shows a tiny section of this Trojan but for a period of only 300 yr and with the orbits of Jupiter, the L4-point and the Trojan itself. The longitudinal movement of the Trojan, with respect to those of the L4-point and the planet, is always a great. The continuous wave with the first horseshoe jump can easily be identified. The isolated points are phenomena of our computation. Therefore -100° of L4 equals +260°. The points added on top or on downwards respectively, give a continuously wave with a clear jump, when the longitudinal oscillation nears the planet from behind (seen in direction of the orbit) at about 29.3×10^3 yr and then separates to come closer to the planet from the other side (from the front).

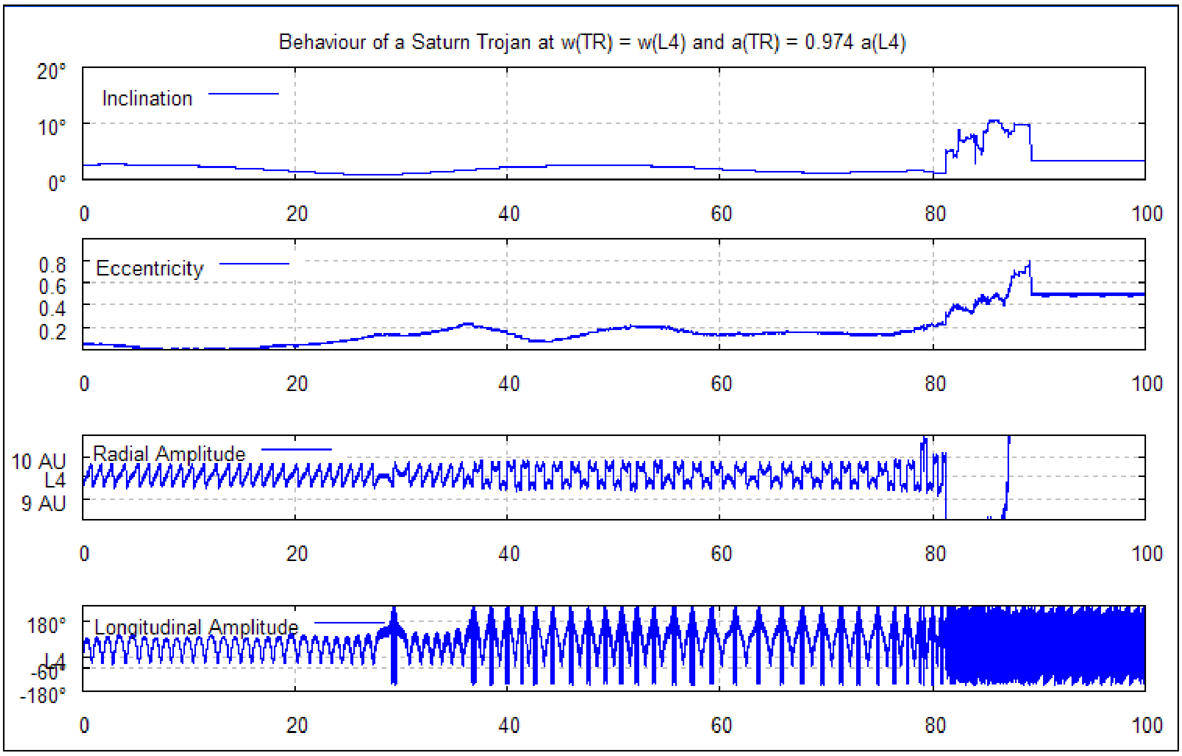

Fig. 60: Overview for the first 100,000 yr of some main parameters of the Trojan started at ω_{L4} and 0.974 a_{L4}. Until 80x10^3 yr, (from above to below) inclination, eccentricity and radial amplitude rest within the stable limits. The longitudinal amplitudes show consistently a great distance to the L4-point which indicates the possibility of a temporary horseshoe orbit.

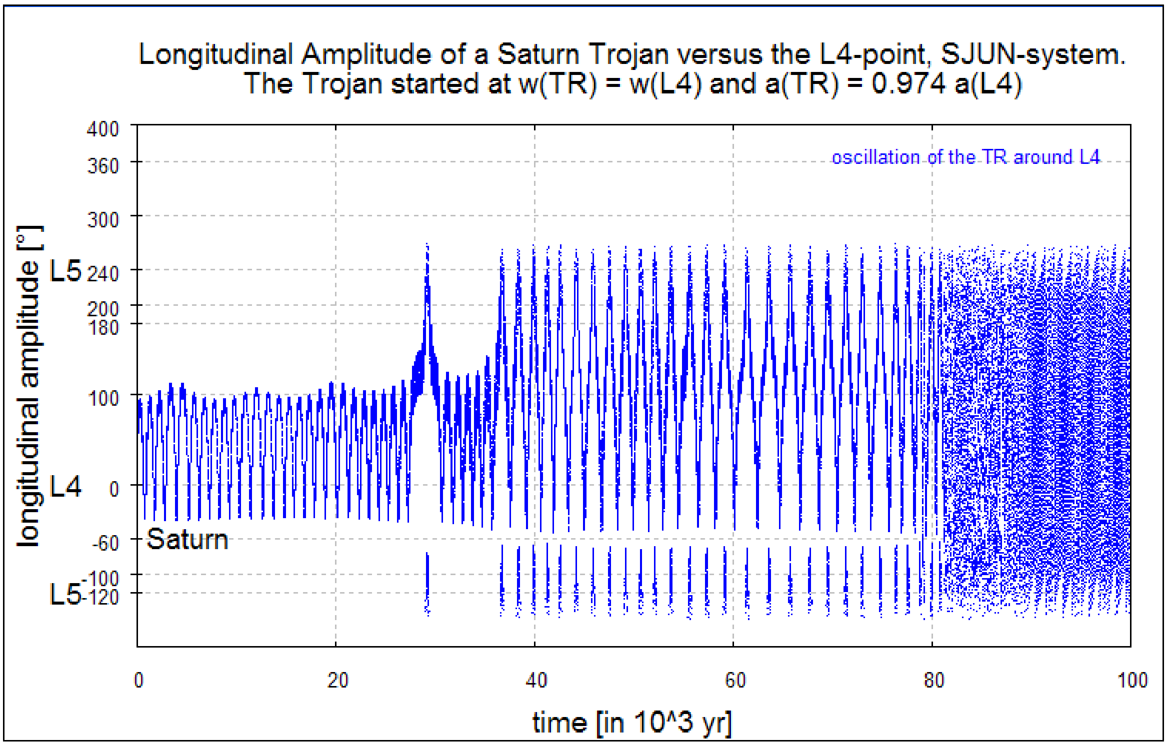

Fig. 61: Longitudinal amplitude of the Trojan started at ω_{L4} and 0.974 a_{L4} as in Fig. 60. It seems that the Trojan indeed has become a temporary horseshoe orbit after 28x10^3 yr, then returns a tadpole orbit and afterward at 36x10^3 yr in a horseshoe orbit with consistently great distances to its L4-point up to 80x10^3 yr. It is interesting to see that the Trojan never touches Saturn at -60°.

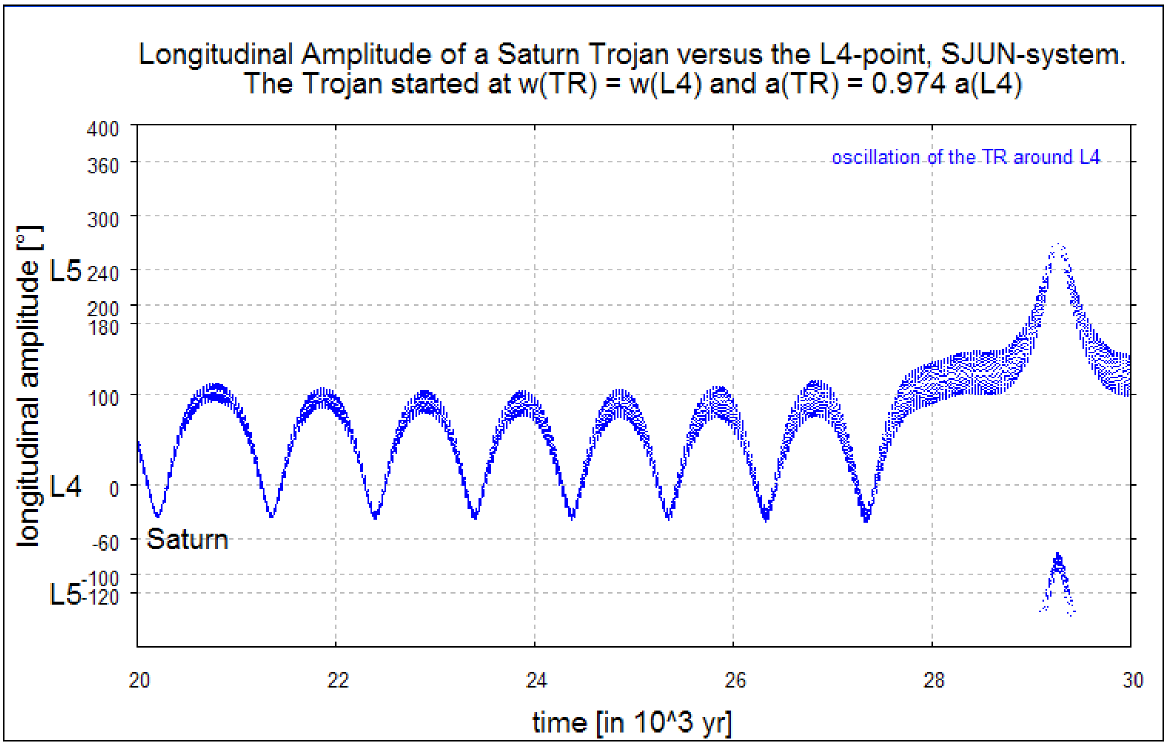

Fig. 62: Section of Fig. 61 for 10,000 yr between $20x10^3$ yr and $30x10^3$ yr with the starting horseshoe orbit at $28x10^3$ yr. It is interesting to see the regular longitudinal change every 1000 yr with a longer stay at about +100° of the tadpole period.

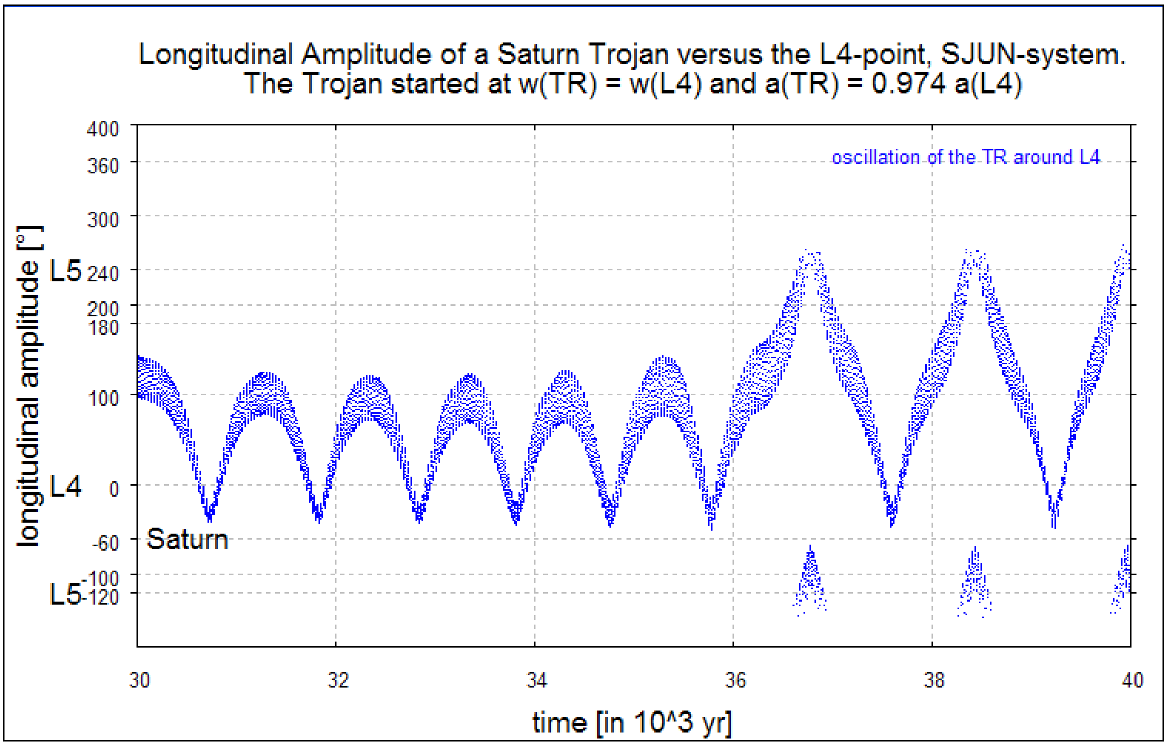

Fig. 63: Section of Fig.61 for 10,000 yr between $30x10^3$ yr and $40x10^3$ yr. The peaks at about -100° are phenomena of our computation and should or could be added on top at about +300° nearing Saturn from behind.

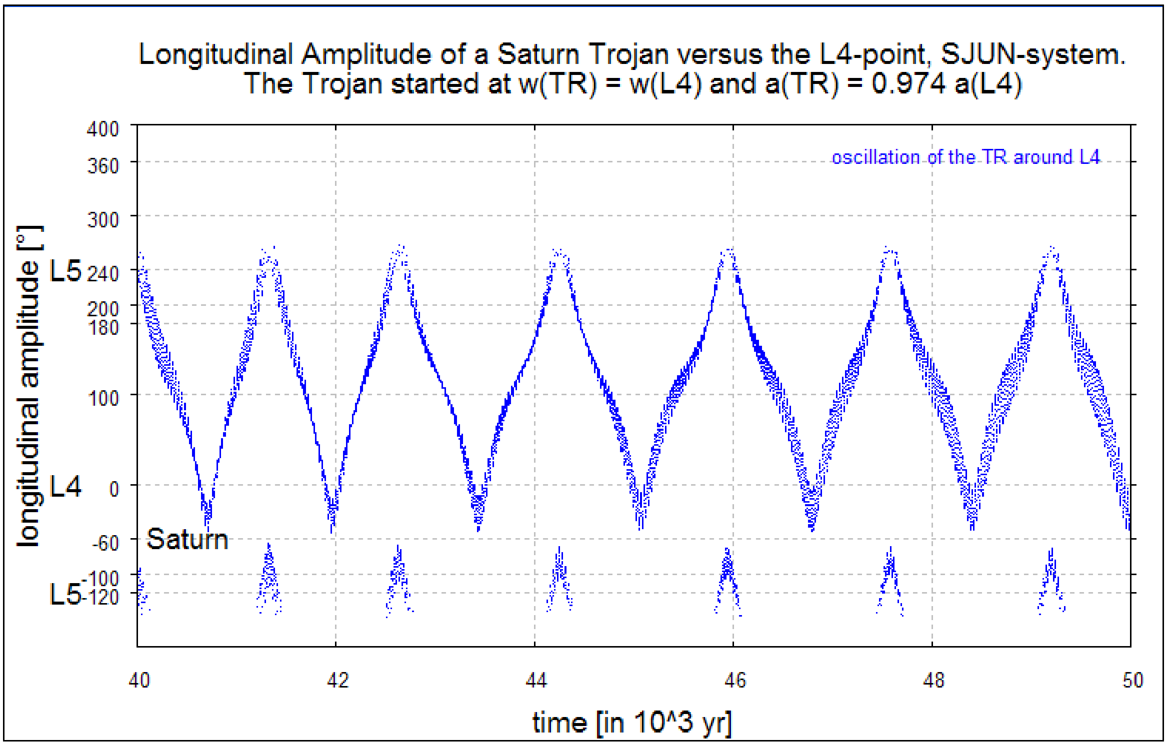

Fig. 64: Section of Fig. 61 for 10,000 yr between 40x10^3 yr and 50x10^3 yr. It is interesting to see the very regular jumps every 770 yr. The peaks at about -100° are phenomena of our computation and should be added on top at about +300°.

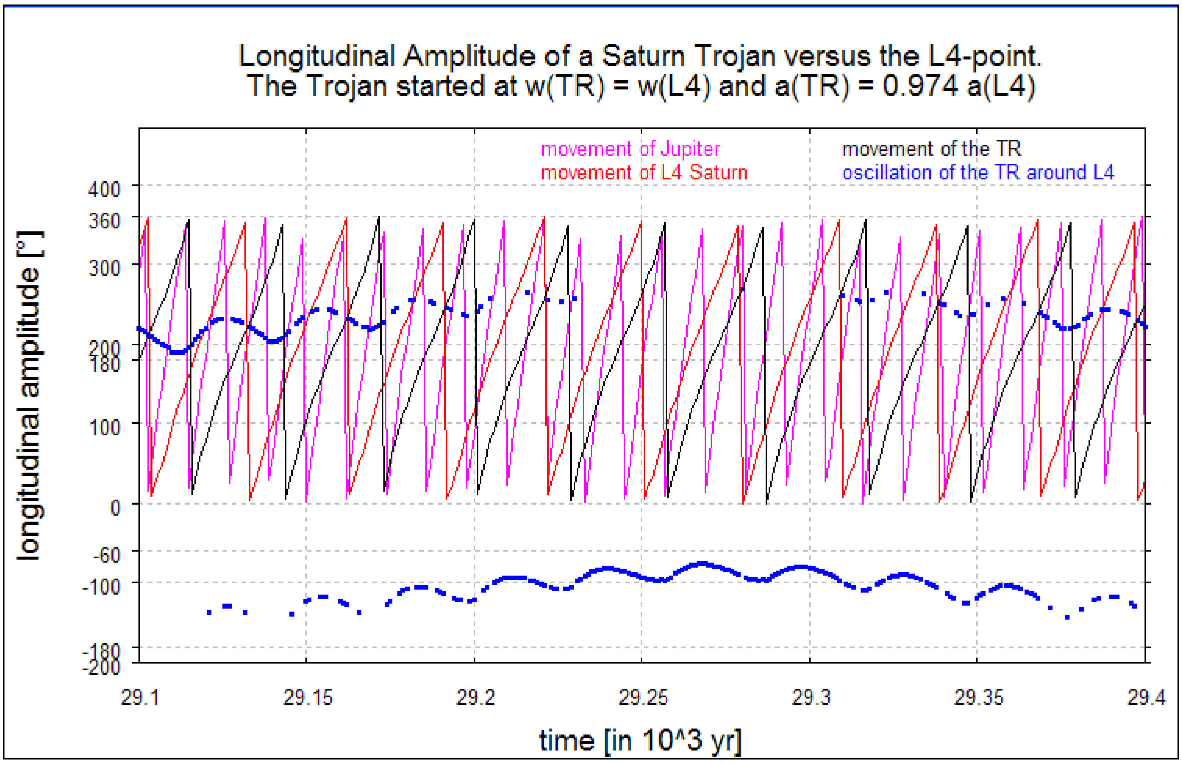

Fig. 65: Tiny section of the same Trojan as before for only 300 yr but with the orbits of Jupiter, the L4-point and the Trojan itself. There is always a large longitudinal distance of the Trojan with respect to those of the L4-point and the planet. The isolated points are phenomena of our computation. The points added on top or on downwards respectively, give a continuously wave with a clear jump.

2.2.5.1.5 Orbit starting with $a_{TR} = a_{TR}$ and $\omega_{TR} = \omega_{L4}$ -33°

This Trojan, with starting values of $a_{TR} = a_{TR}$ and $\omega_{TR} = \omega_{L4}$ -33°, is also a very interesting one. It becomes unstable after about 250x10^3 yr (Fig. 66), but there is again an interesting horizontal gap when the position of Saturn is -60°. Indeed the Trojan starts as a tadpole orbit for the first 20x10^3 yr (Fig. 67), then becomes a temporary horseshoe orbit, returns to the tadpole form and then remains for a period of about 70x10^3 yr even a tadpole orbit oscillating around L5 (at 240° of L4 which means -60° of Saturn) with several horseshoe like irregular jumps. The other parameters of this overview rest clearly within their limits.

Figs. 68 and 69 represent each a section for 20x10^3 yr of the longitudinal amplitude of Fig. 66, to have a better view of the exiting changes between horseshoe orbits and tadpole orbits of this Trojan. Afterwards it remains as a L5 tadpole orbit with no other jump for a further important period of about 55x10^3 yr.

Figs. 70 and 71 represent a still tinier section of the same Trojan, each for only 400 yr but with the orbits of Jupiter, the L4-point and the Trojan itself. In Fig. 70 the horseshoe orbit nears Saturn (in orbital sense at -60° of L4) from behind up to only 15.8°. Afterwards the amplitudes stride away and near Saturn but from the front side, up to 23.6° (Fig. 71).

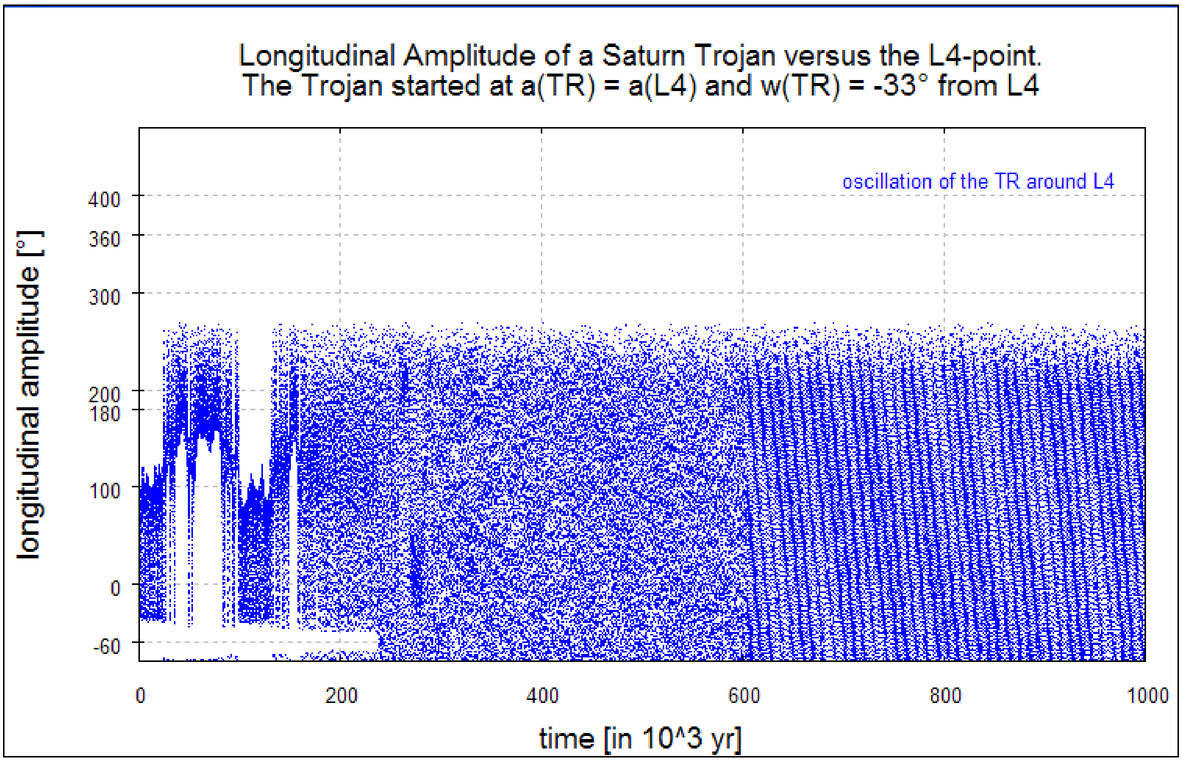

Fig. 66: The longitudinal amplitudes of the Trojan with starting $a_{TR} = a_{L4}$ and $\omega_{TR} = \omega_{L4}$-33°. The instability as a tadpole orbit soon starts. But there exists an interesting horizontal gap at -60°, the position of Saturn, which indicates a temporary horseshoe orbit.

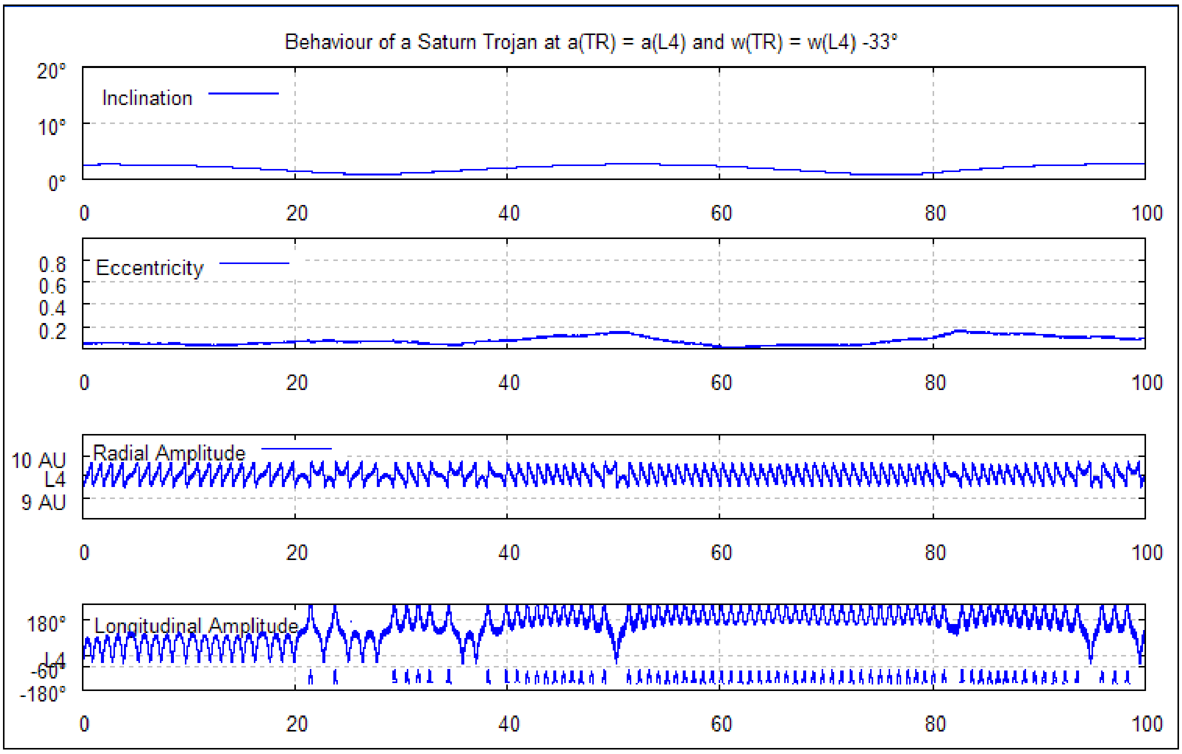

Fig. 67: The first 10^5 yr of the Trojan starting at $a_{TR} = a_{L4}$ and $\omega_{TR} = \omega_{L4}$-33°. The Trojan begins as a L4 tadpole orbit for the first $20x10^3$ yr, becomes then a temporary horseshoe orbit and remains afterwards for a period of about $70x10^3$ yr a L5 tadpole orbit with some horseshoe like irregular jumps. The other parameters in this overview rest clearly within their limits.

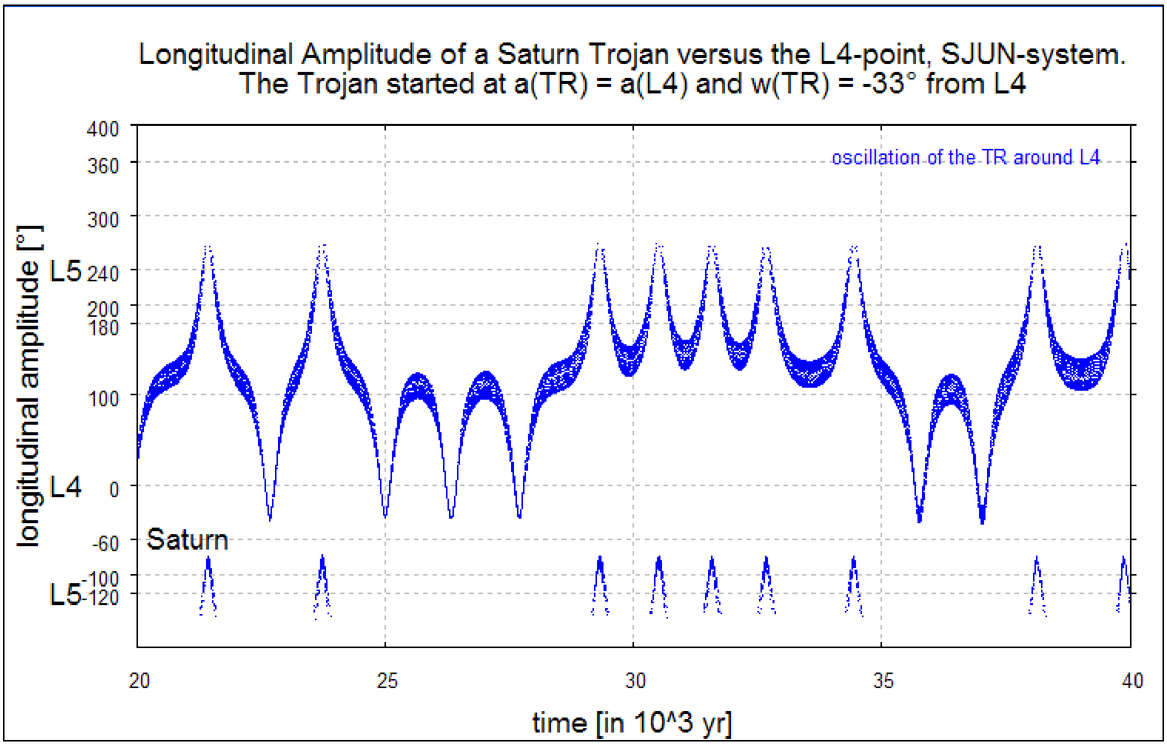

Fig. 68: A section for $20x10^3$ yr of the longitudinal amplitude of the above Fig. 66. It is interesting to note the changes between horseshoe orbits and L4 respectively L5 tadpole orbits of this Trojan.

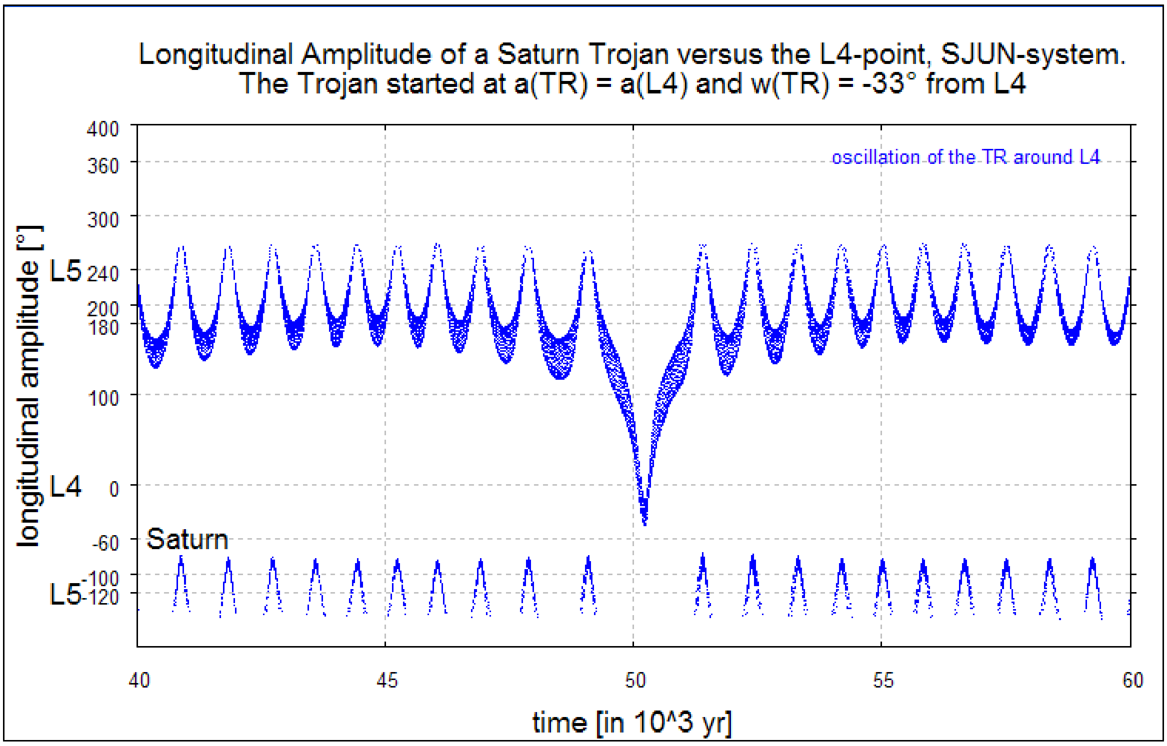

Fig. 69: The subsequent section for 20x10^3 yr of Fig. 66, showing the tadpole orbit around L5 (at 240°of L4) with the next horseshoe like jump. Afterwards remains again as a L5 tadpole orbit with no other jump for a period of about 55x10^3 yr.

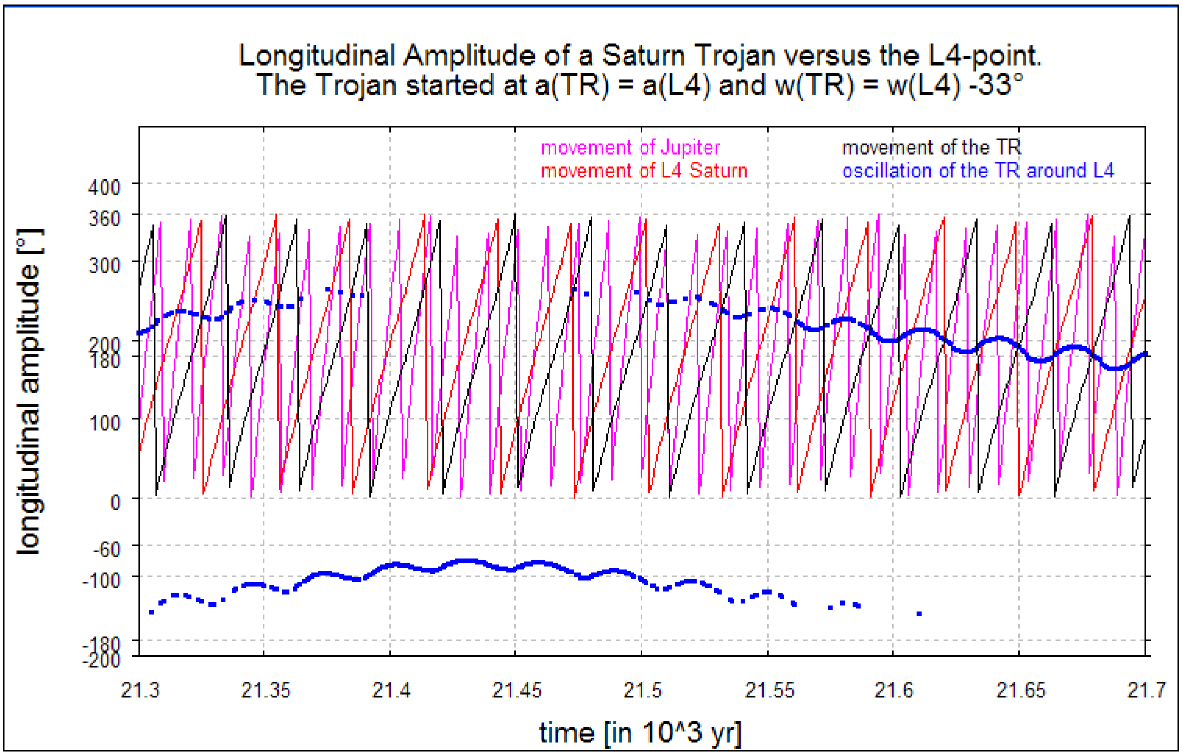

Fig. 70: Tiny section of time of the same Trojan as before (Fig. 68) for only 400 yr but with the orbits of Jupiter, the L4-point and the Trojan itself. The horseshoe orbit nears from behind Saturn (at -60° of L4) up to a degree of just 15.8°. Afterwards the orbits of the Trojan stride away.

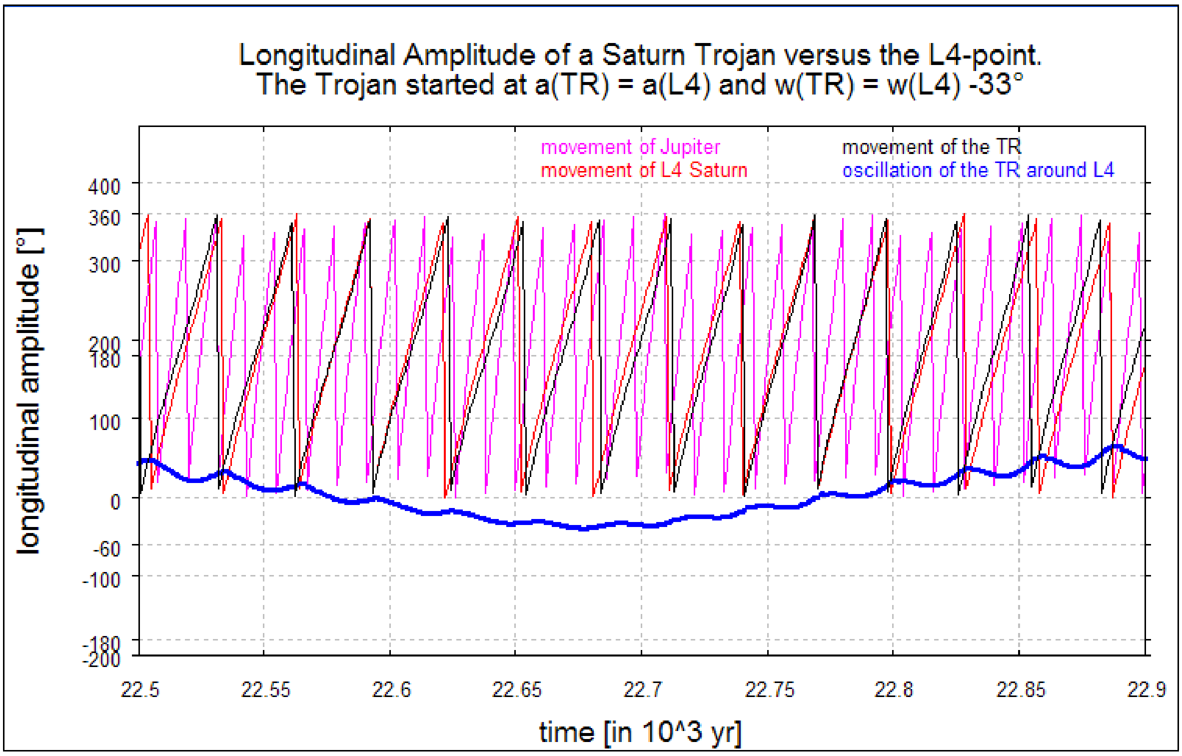

Fig. 71: Tiny section of time for only 400 yr as before (Fig. 70) when the longitudinal amplitudes come near Saturn but now from the front side, up to a degree of 23.6°.

2.2.5.1.6 Orbit starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.034\ a_{TR}$

This Trojan, with starting values of $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.034\ a_{TR}$, quickly becomes unstable after about $15\text{x}10^3$ yr in regarding eccentricity and amplitudes (Fig. 72). From the beginning there again is an interesting horizontal gap at the position of Saturn at -60° but here the Trojan starts directly with a horseshoe orbit without any changes up to $15\text{x}10^3$ yr, when it becomes completely unstable with diverging radial and longitudinal amplitudes and growing inclination and eccentricity. It is interesting to regard the shape of the radial amplitudes during the horseshoe orbit always oscillating around the planetary path. This phenomena can also be observed for the Trojan starting at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.974\ a_{TR}$ (Fig. 60).

Fig. 73 shows a tiny section from Fig. 72 of the longitudinal amplitude for 20,000 yr for the first $20\text{x}10^3$ yr. Until $15\text{x}10^3$ yr a clear horseshoe orbit is visible from the very beginning, but becoming complete unstable after $15\text{x}10^3$ yr. The horseshoe orbits never touch Saturn at -60° and the jumps occur regularly (about every 670 yr).

Figs. 74 and 75 show tiny sections of the same Trojan as before (Fig. 73) for only 400 yr but with the orbits of Jupiter, the L4-point and the Trojan itself. The horseshoe orbit approaches from behind Saturn (at -60° of L4) up to only 11.9° (Fig. 74) and later from the front up to only 15.7° (Fig. 75). We remind, that the limiting distance of the Hill's sphere of Saturn is only about 3° (see subsection 2.1.2.1).

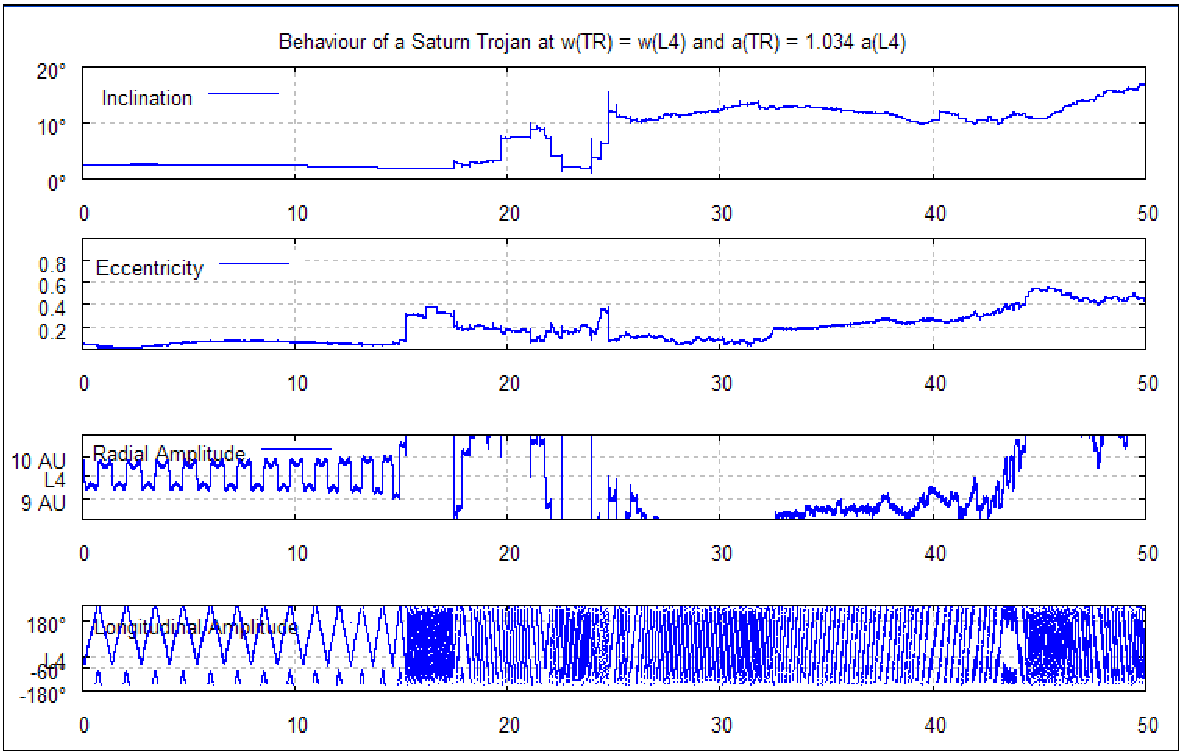

Fig. 72: Overview of some main parameters of the Trojan with starting $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.034\ a_{L4}$. The Trojan starts directly with a horseshoe orbit without any changes up to $15x10^3$ yr, when it becomes completely unstable with diverging radial and longitudinal amplitudes and growing inclination and eccentricity. It is interesting to regard the shape of the radial amplitudes during the horseshoe orbit, always oscillating around the planetary path.

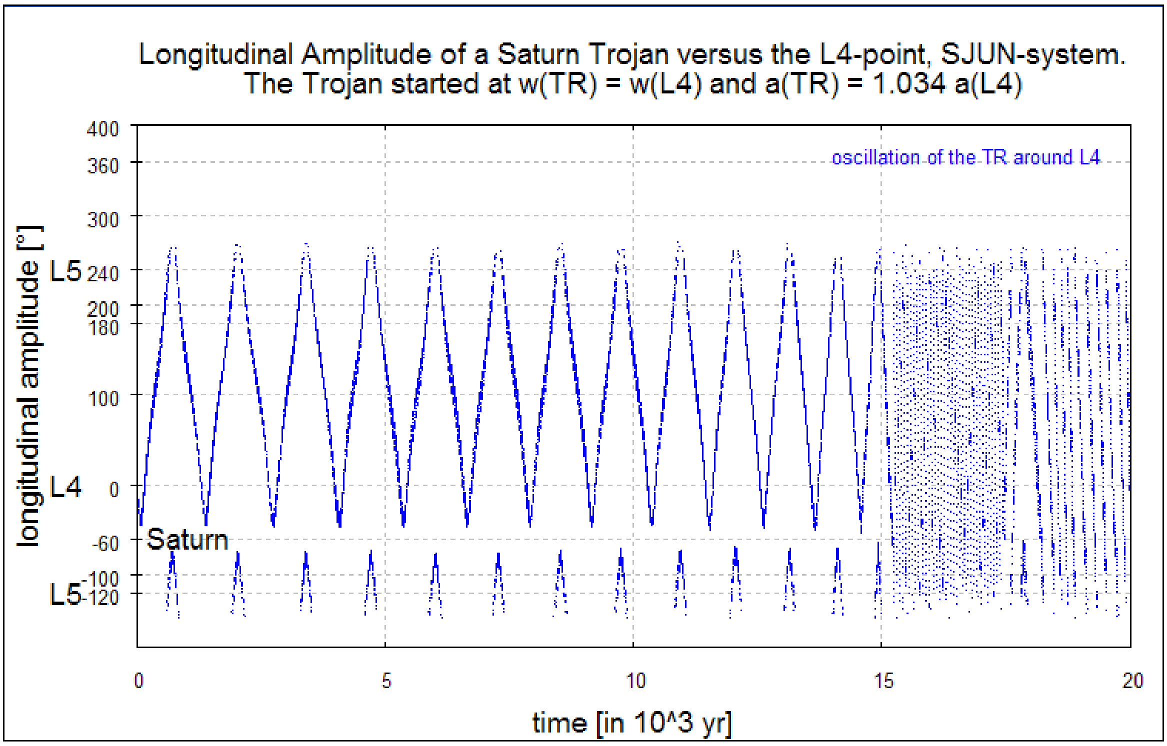

Fig. 73: Tiny section of the longitudinal amplitude in Fig. 72 for only 20,000 yr between 0 yr and $20x10^3$ yr. Until $15x10^3$ yr a clear horseshoe orbit is visible from the very beginning, becoming complete unstable after $15x10^3$ yr. The horseshoe orbits never touch Saturn at -60° and the jumps occur regularly (about every 670 yr).

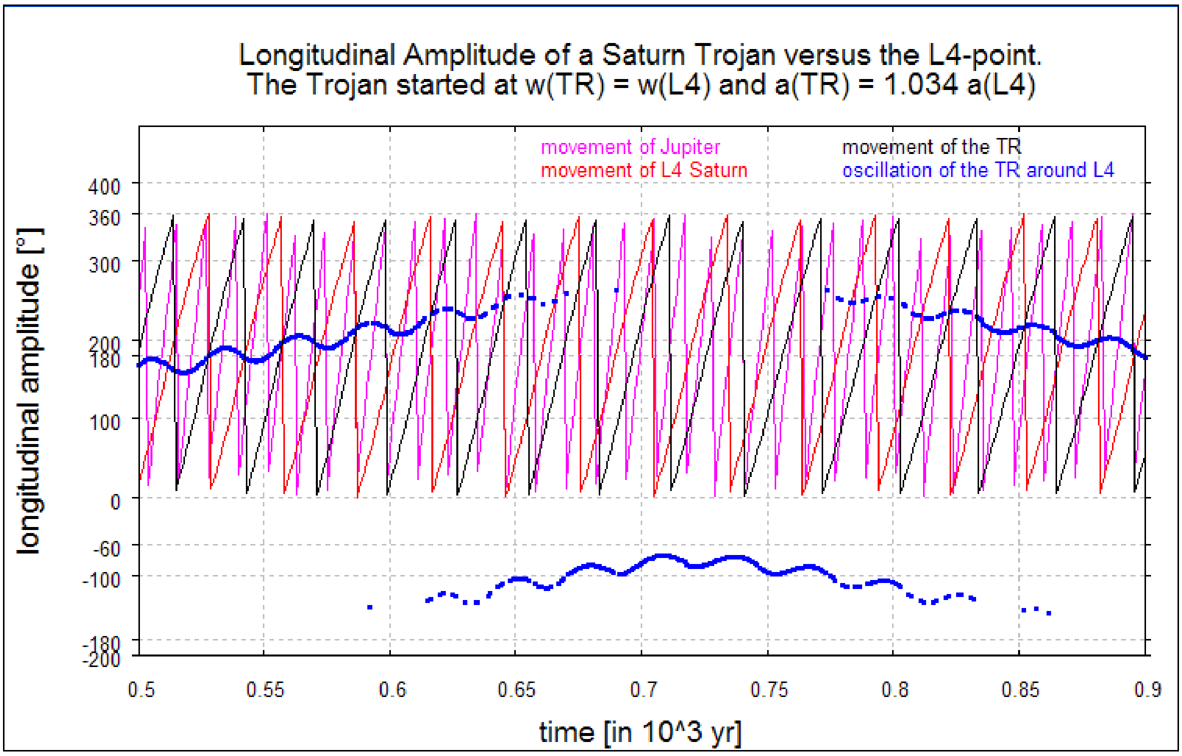

Fig. 74: Tiny section of the same Trojan as before (Figs. 72 and 73) for only 400 yr but with the orbits of Jupiter, the L4-point and the Trojan it self. The horseshoe orbit approaches from behind Saturn (at -60° of L4) up to only 11.9°.

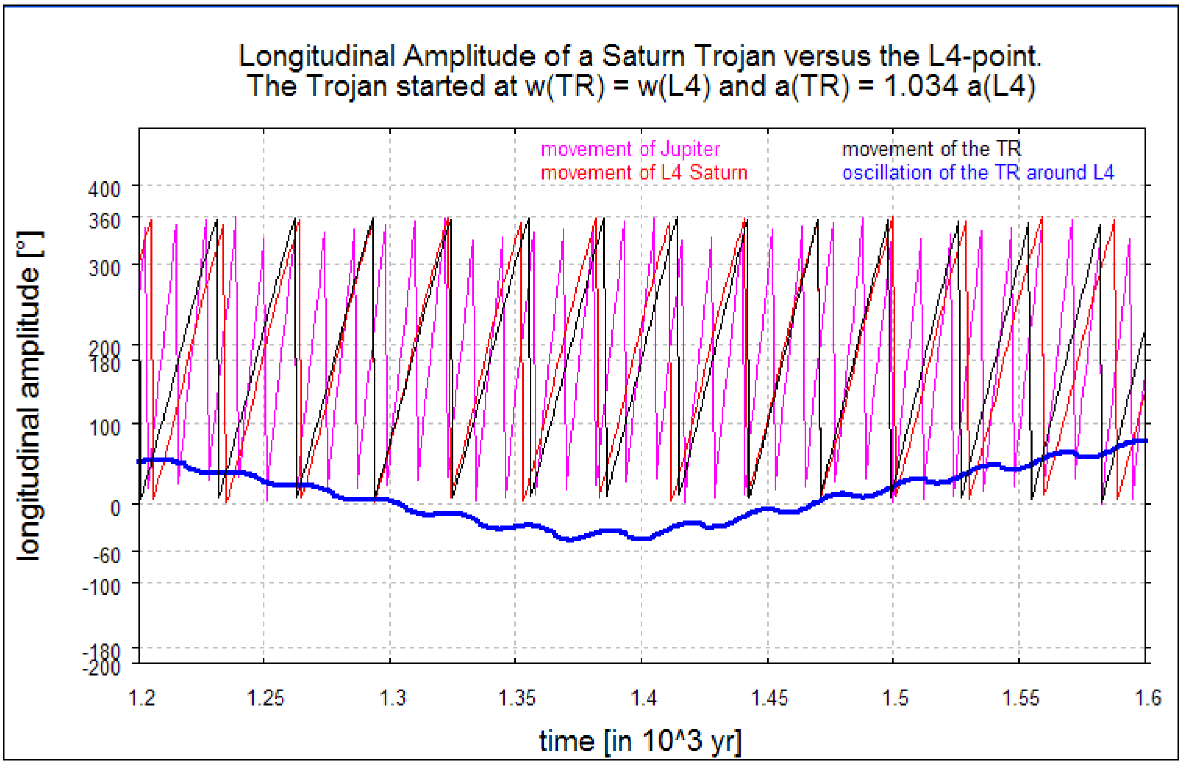

Fig. 75: Tiny section for only 400 yr as before (Fig. 74) when the longitudinal amplitudes come near Saturn but from the front, up 15.7°.

2.2.5.2 Special orbits of L4 Trojans in the SUN-system

Our investigations in the SUN-system without Jupiter in subsection 2.2.4.4 already let assume that some orbits, where a different behaviour of radial and longitudinal amplitudes is to observe, that also horseshoe orbits could there exist. Again we ran integrations for a period of 10^6 yr, with much more output data (about one data-set per year).

2.2.5.2.1 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.970\ a_{L4}$

The first Trojan we want to look at is the 6^{th} from the left in Figs. 49 and 50 of the *a*-cut (subsection 2.2.4.4), at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.970\ a_{L4}$. This Trojan has a low maximum eccentricity; with no escape time of course (Fig. 49), small maximum radial amplitudes, but large longitudinal extensions (Fig. 50).

Fig. 76 gives an overview of this Trojan with inclination, eccentricity, radial and longitudinal amplitudes for the first 10^5 yr. It is astounding to see such a regular behaviour of a Trojan in a typical horseshoe form direct from the beginning. Both, radial and longitudinal oscillations are typical for horseshoe orbits. This behaviour lasts at least for the first 10^6 yr with very regular jumps in longitudinal direction from one side of the planet to the other. Fig. 77 shows the last 10^5 yr of our integration time. To present more details of this harmony we give an overview of just the first 10^4 yr (Fig. 78) and Fig. 79 shows the detail of the longitudinal behaviour for the last 20×10^3 yr of integration.

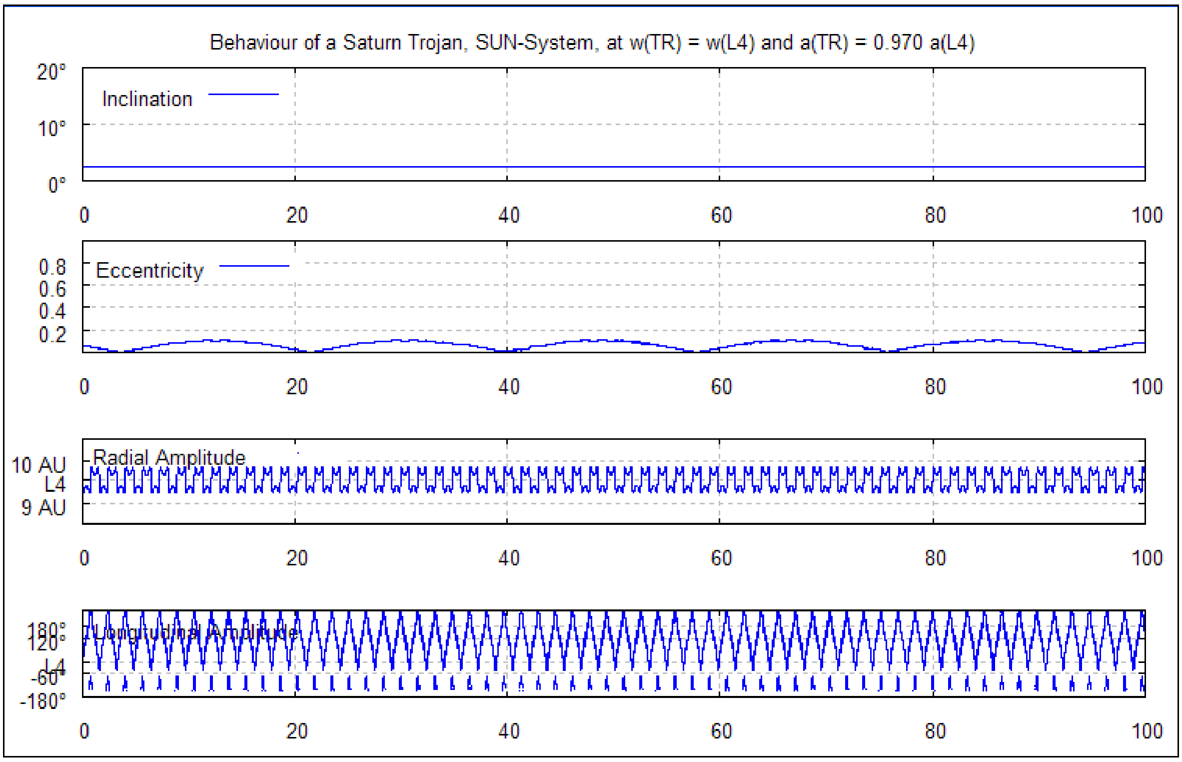

Fig. 76: Overview of a Trojan with starting position at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.970\ a_{L4}$ against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for the first 10^5 yr. It is astounding to see this regular behaviour of the Trojan in all parameters.

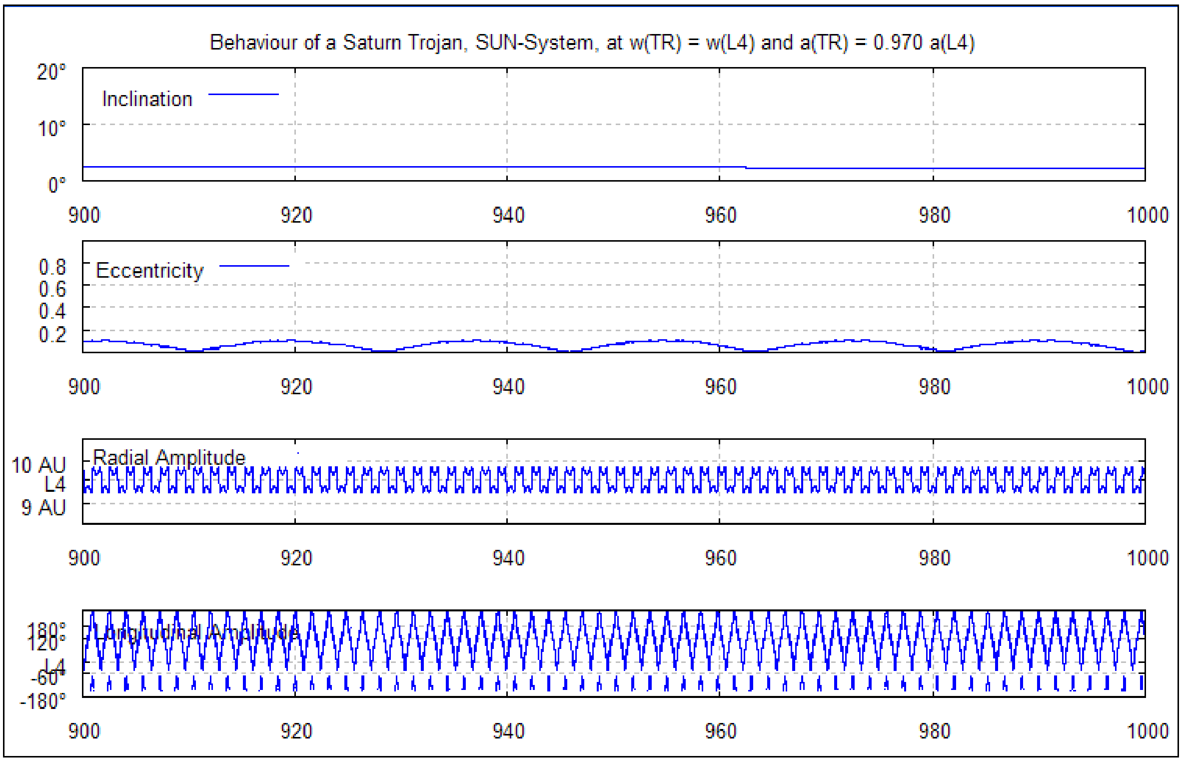

Fig. 77: Overview of the same Trojan as before but for the last 10^5 yr of our integration time. It is astounding to see this unchanged regular behaviour of the Trojan in all parameters.

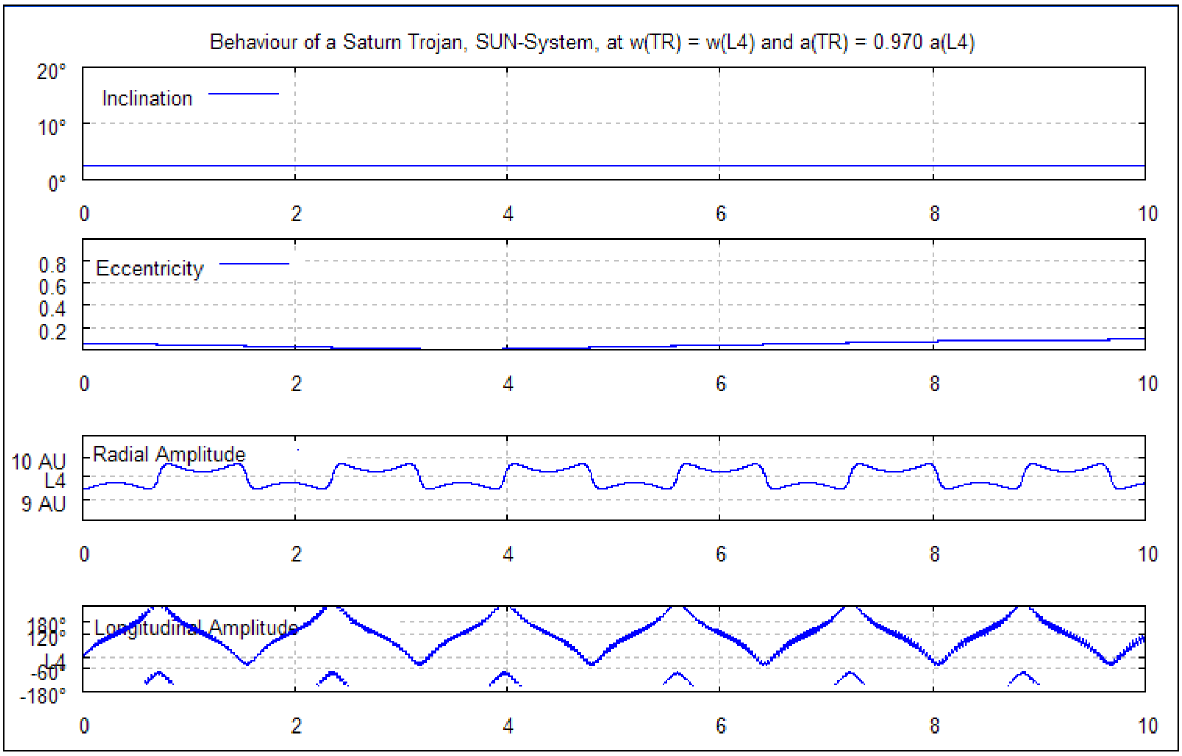

Fig. 78: The same plot as Fig. 76 with more detail for just the first 10^3 yr. Also the behaviour of the radial amplitudes with respect to the longitudinal amplitudes is typical for a horseshoe orbit.

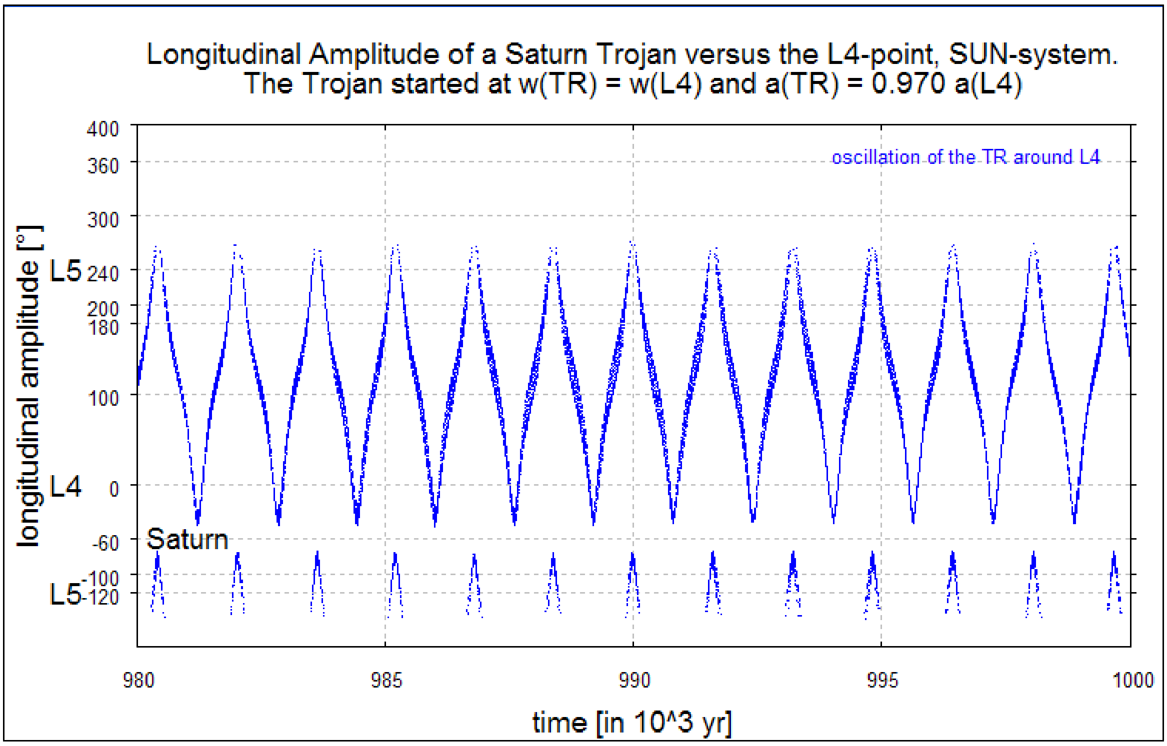

Fig. 79: Detail of the longitudinal amplitudes of Fig. 77 for just the last 20x10^3 yr of our integration time. The jumps occur regularly in longitudinal direction from one side of Saturn at -60° to the other side.

2.2.5.2.2 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.972\ a_{L4}$

Next we look at is the 7th Trojan from the left in Figs. 49 and 50 (subsection 2.2.4.4), the neighbour of the above Trojan, at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.972\ a_{L4}$ (see the overview Fig. 80). This Trojan has the same attributes as its neighbour at $a_{TR} = 0.970\ a_{L4}$, nearly being a twin. Only the longitudinal jumps occur there with slightly longer periods.

2.2.5.2.3 Trojan starting with $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.030\ a_{L4}$

Fig. 81 shows the overview of the 4th Trojan from the right side in Figs. 49 and 50 (subsection 2.2.4.4). This Trojan, at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.030\ a_{L4}$ is of special interest as its starting orbit is situated between two unstable ones. It is amazing to note that also this Trojan, really on the border of instability, shows the same harmonic features as the Trojans before.

2.2.5.2.4 Trojan starting with $a_{TR} = a_{L4}$ and $\omega_{TR} = -36°$ from ω_{L4}

Now we regard the 5th Trojan from the left in Figs. 51 and 52 (subsection 2.2.4.4), that for a Trojan of the ω-cut in the SUN-system, at $a_{TR} = a_{L4}$ and $\omega_{TR} = -36°$ from ω_{L4} (see the overview of the last 10^5 yr in Fig. 82). This Trojan shows the same main parameters as the Trojans before of the a-cut. Also here the longitudinal jumps from one side of Saturn to the other occur in regular periods (Fig. 83). All these Trojans start immediately in the horseshoe form which lasts for the full integration time.

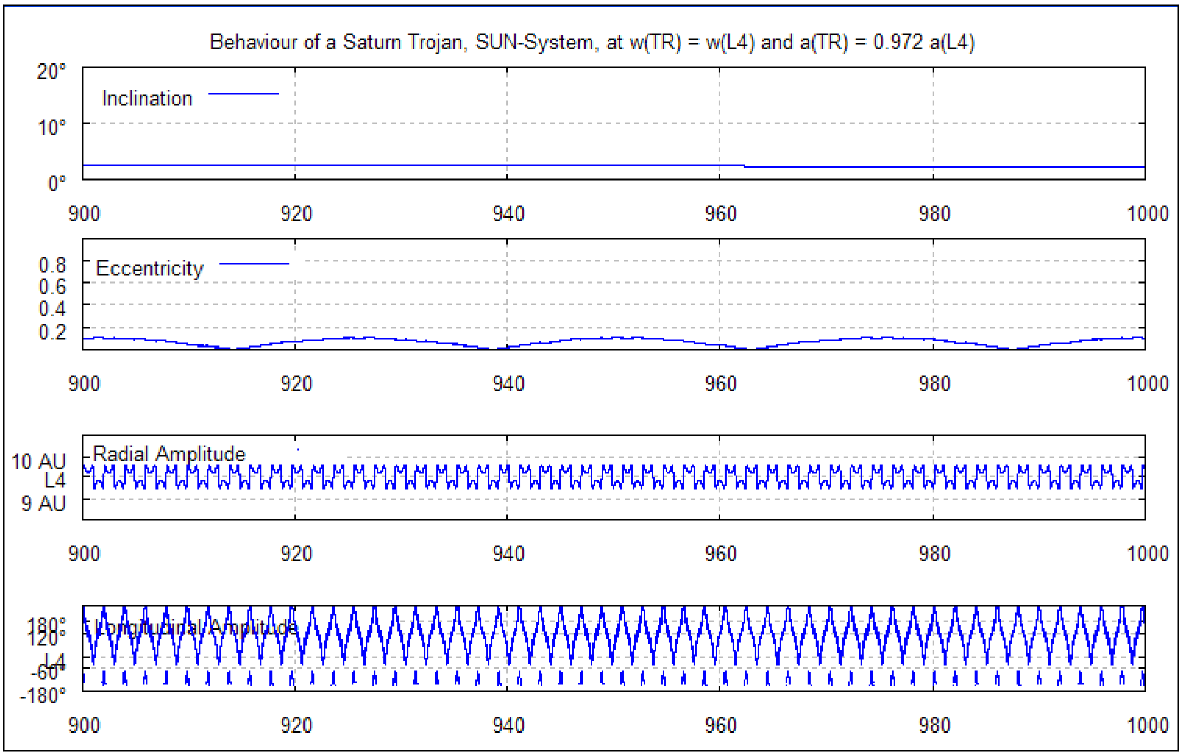

Fig. 80: Longitudinal amplitudes of a Trojan with starting position at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 0.972\ a_{L4}$ against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for the last 10^5 yr of integration. It is nice to see this regular behaviour of the Trojan.

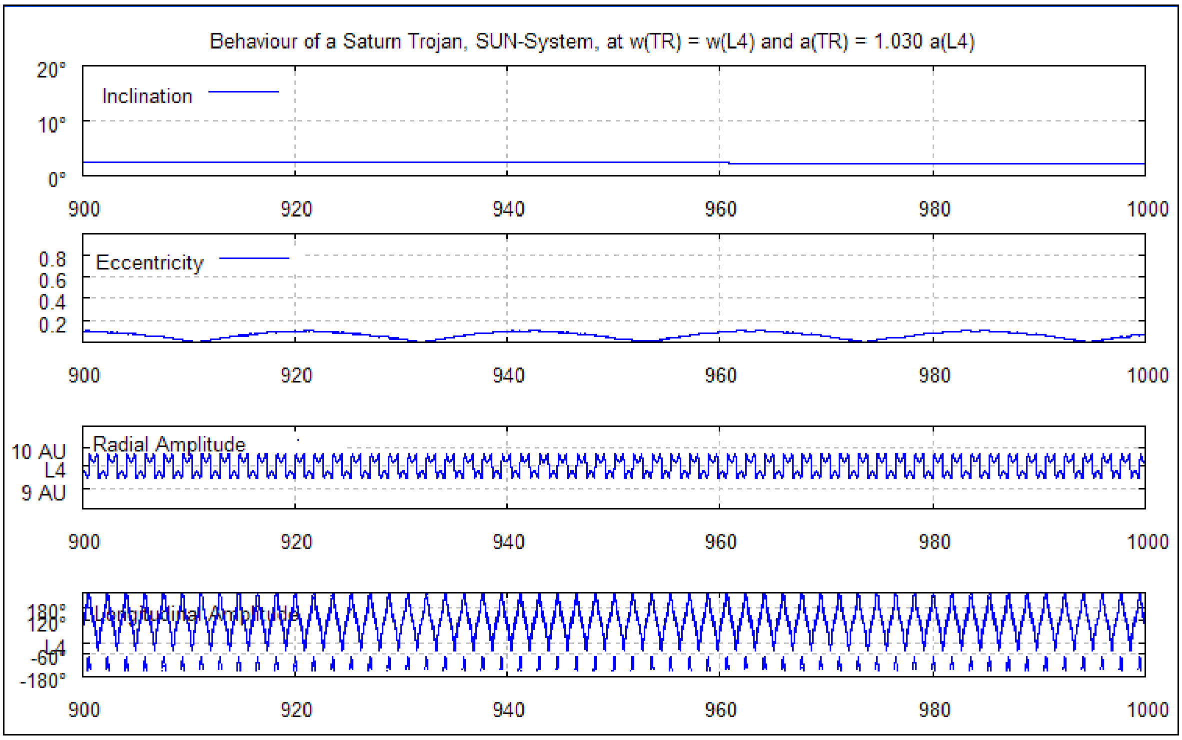

Fig. 81: Longitudinal amplitudes of a Trojan with starting position at $\omega_{TR} = \omega_{L4}$ and $a_{TR} = 1.030\ a_{L4}$ against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for the last 10^5 yr of integration. This Trojan is situated between two unstable ones, but shows a stable horseshoe orbit up to at least 10^6 yr.

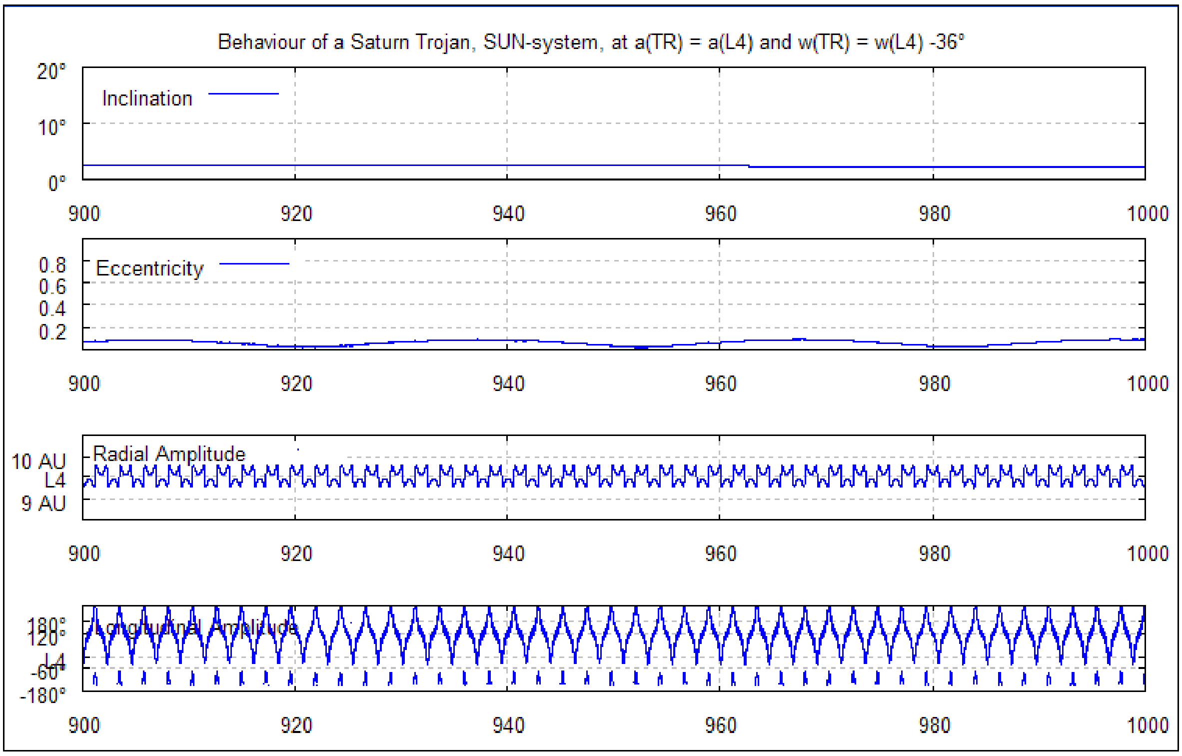

Fig. 82: Overview of a Trojan with starting position at $a_{TR} = a_{L4}$ and ω_{TR} = -36° from ω_{L4} against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for 10^5 yr. It is astounding to see again this regular horseshoe behaviour of a Trojan of the ω-cut.

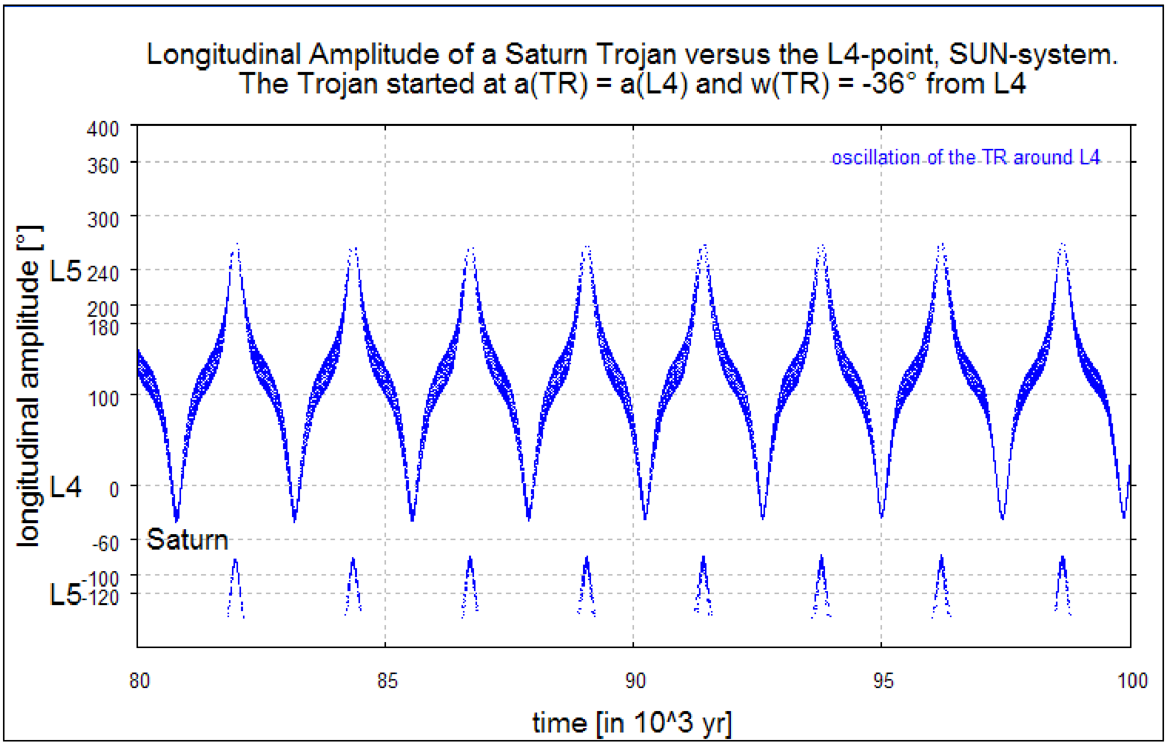

Fig. 83: Detail of the longitudinal amplitudes of Fig. 82 for just 20 x 10^3 yr of integration. The jumps occur again very regularly in longitudinal direction from one side of Saturn at -60° to the other.

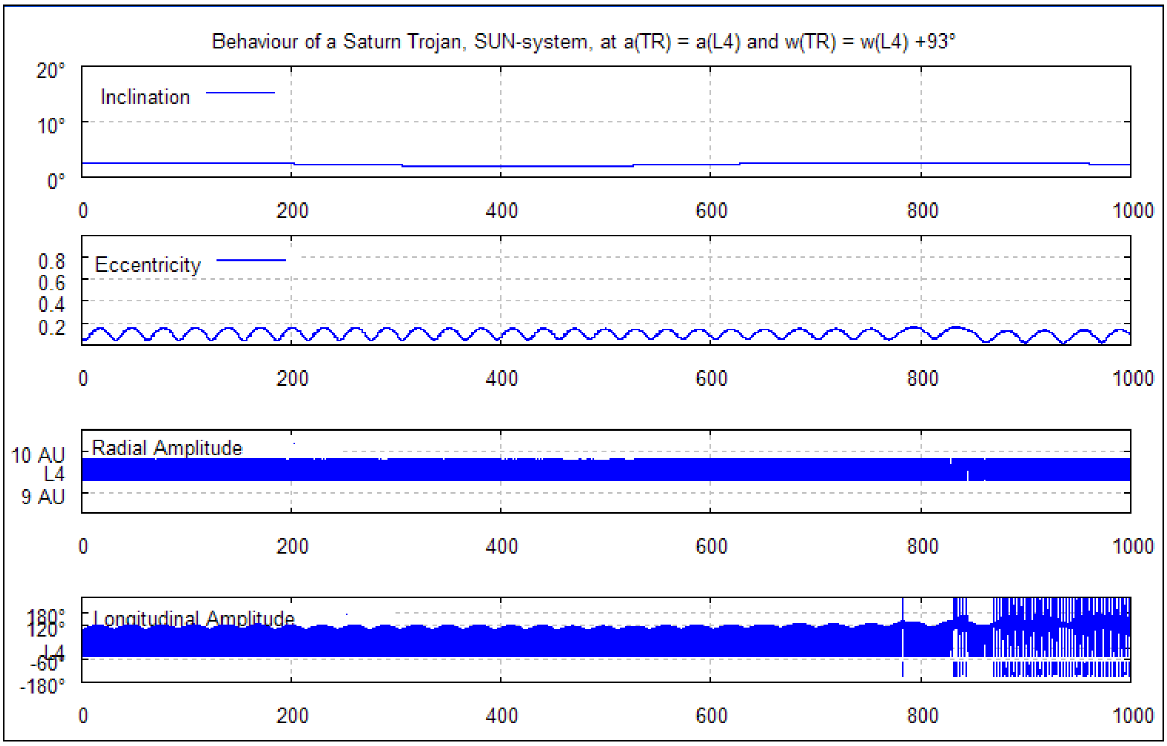

Fig. 84: Overview of a Trojan with starting position at $a_{TR} = a_{L4}$ and $\omega_{TR} = +93°$ from ω_{L4} against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for the whole integration time of 10^6 yr.

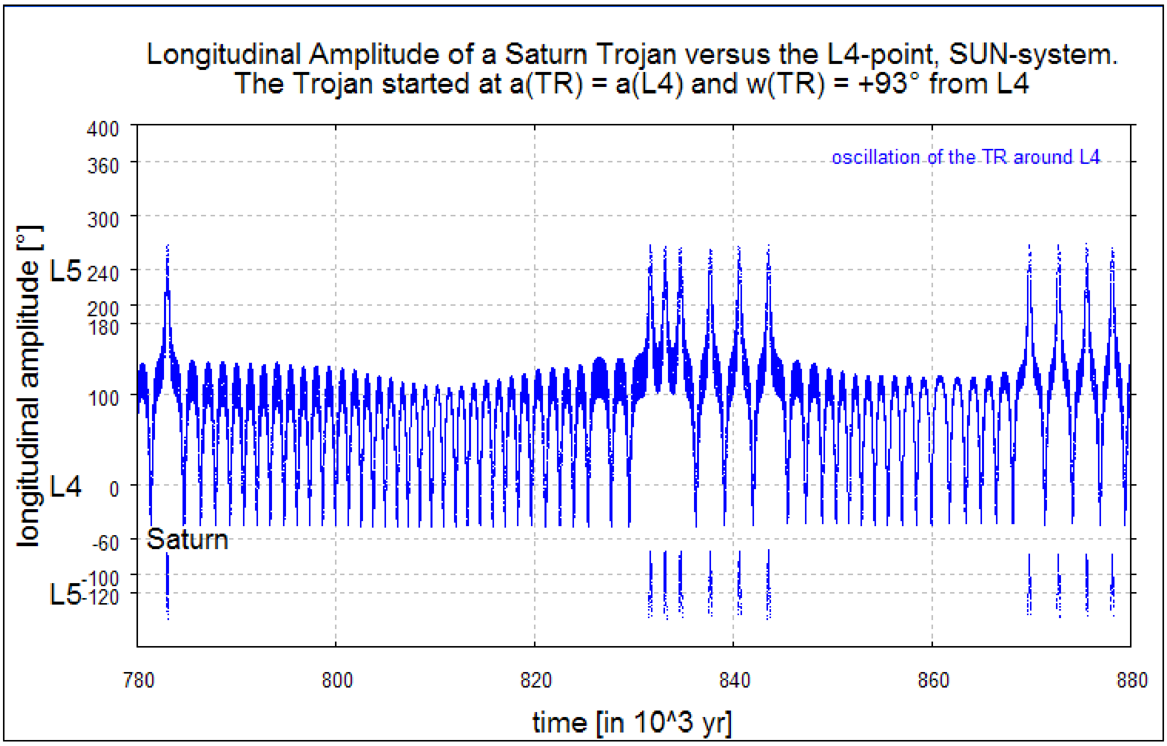

Fig. 85: Detail of the longitudinal amplitudes of Fig. 84 for the most interesting period of 10^5 yr with the jumps from horseshoe to L4 Trojan, to L5 Trojan, again to the horseshoe form, becoming anew a L4 Trojan and remaining finally a horseshoe Trojan for the rest of the integration time.

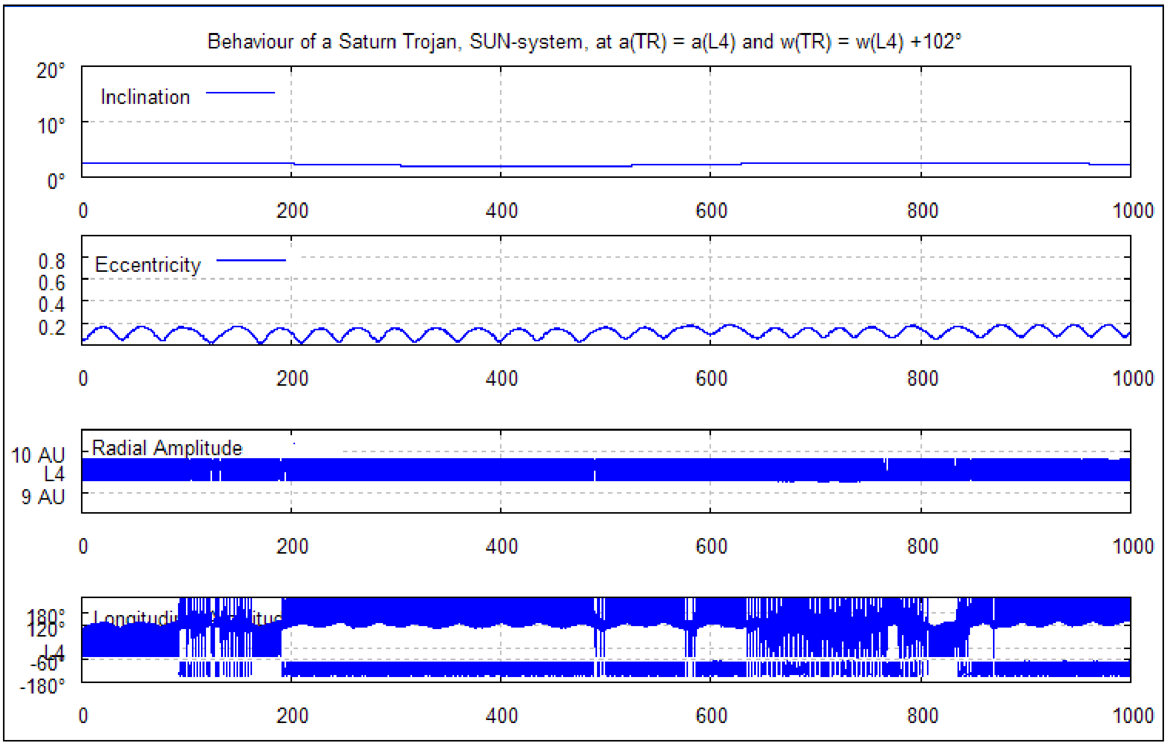

Fig. 86: Overview of a Trojan with starting position at $a_{TR} = a_{L4}$ and $\omega_{TR} = +102°$ from ω_{L4} against (from above to below) inclination, eccentricity radial and longitudinal amplitudes for the whole integration time of 10^6 yr. The longitudinal amplitudes indicate numerous jumps of all kinds.

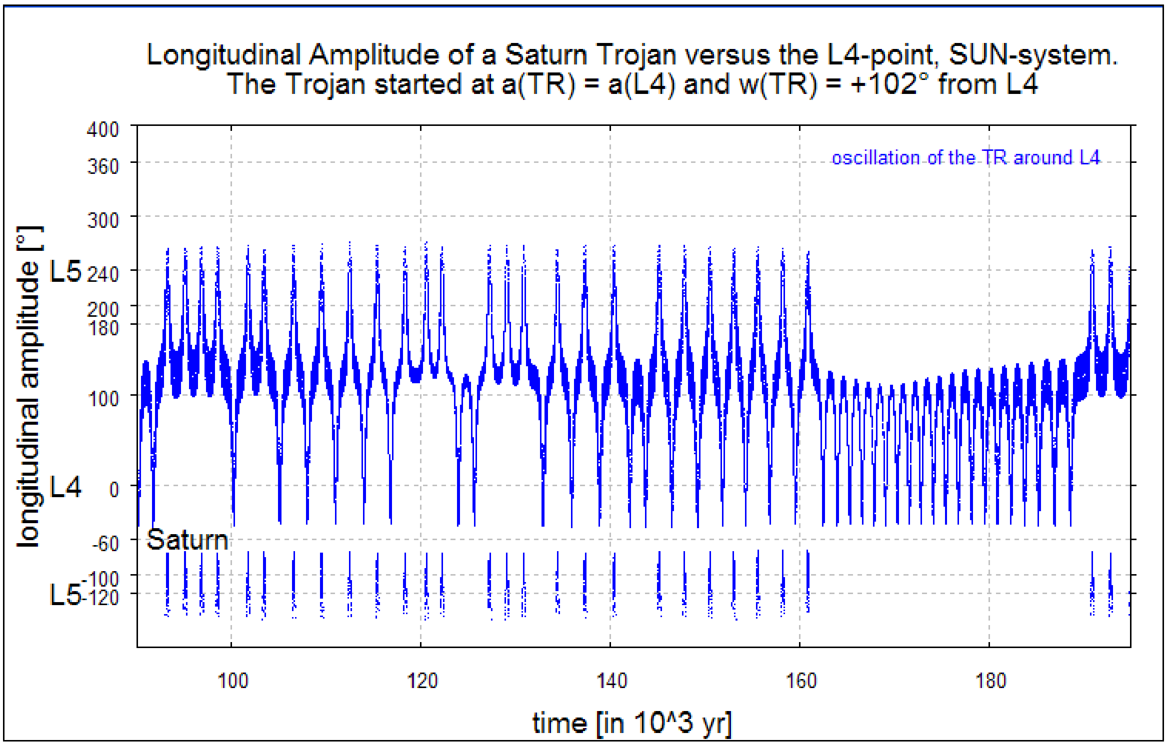

Fig. 87: Details of the same Trojan as before for the period from 90 to 195x10^3 yr. The Trojan jumps at 92x10^3 yr from a tadpole orbit at L4 into a L5 tadpole orbit and then changing several times between the horseshoe form and the L5 and L4 tadpole form in very short successions.

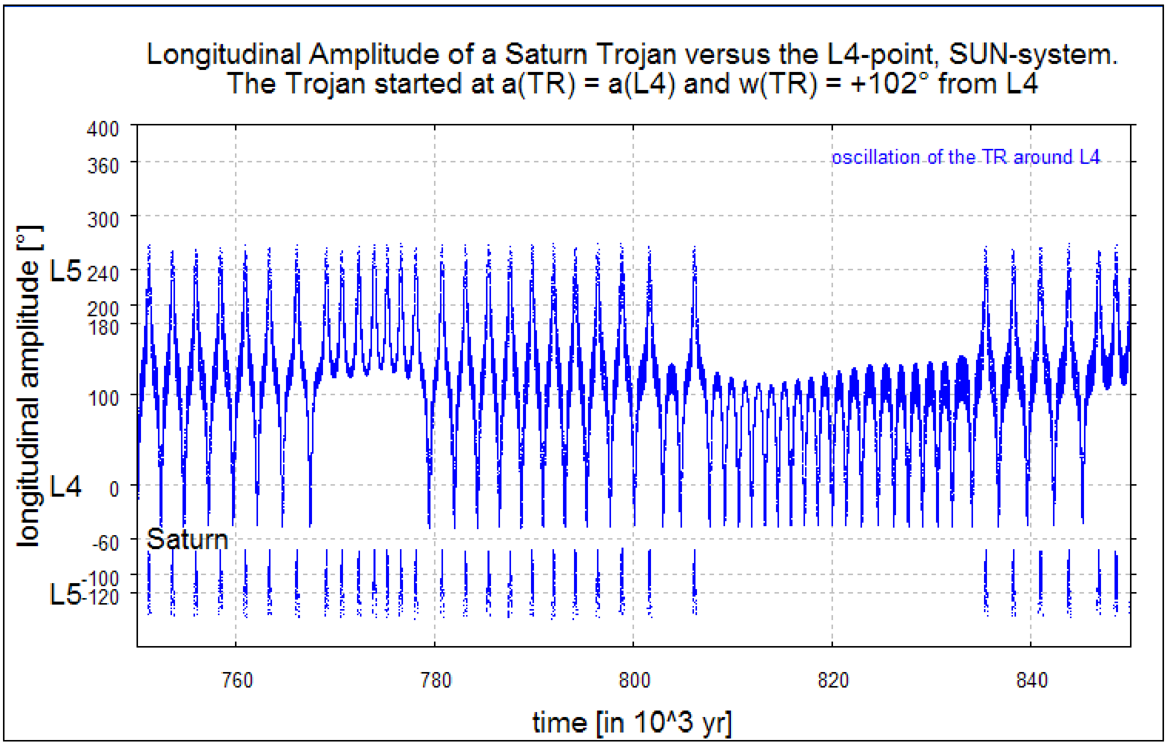

Fig. 88: Detail of the longitudinal amplitudes of the same Trojan as before but for the period of 750 to 850x10^3 yr with the most interesting jumps from horseshoe to L5 Trojan, to horseshoe, to L4 Trojan and once more to the horseshoe and finally to the L5 Trojan form.

2.2.5.2.5 Trojan starting with $a_{TR} = a_{L4}$ and ω_{TR} = -39° from ω_{L4}

The 6th Trojan from the left in Figs. 51 and 52 (subsection 2.2.4.4) at $a_{TR} = a_{L4}$ and ω_{TR} = -39° from ω_{L4} is a neighbour of the Trojan before and has absolutely the same attributes like a twin, although it is situated on the border of instability. Therefore we do not present a special overview of this Trojan.

2.2.5.2.6 Trojan starting with $a_{TR} = a_{L4}$ and ω_{TR} = +93° from ω_{L4}

This Trojan at $a_{TR} = a_{L4}$ and ω_{TR} = +93° from ω_{L4} starts as a normal L4 tadpole Trojan which lasts for 781x10^3 yr (Fig. 84). But then the Trojan is totally different and shows a very exciting behaviour of its longitudinal amplitudes, becoming once a horseshoe orbit, then again a L4 tadpole orbit, jumping at about 830x10^3 yr into a L5 Trojan, becoming after 5x10^3 yr a horseshoe orbit, changing at about 845x10^3 yr anew for a L4 Trojan and from about 870x10^3 yr on remaining a horseshoe Trojan for the rest of the integration time (see details in Fig. 85). The other parameters show no special irregularities.

2.2.5.2.7 Trojan starting with $a_{TR} = a_{L4}$ and ω_{TR} = +102° from ω_{L4}

To finish this analyses of jumping Trojans we have a look at the Trojan at $a_{TR} = a_{L4}$ and ω_{TR} = +102° from $\omega_{L4,}$ which is the outer most at right in Figs. 51 and 52 (subsection 2.2.4.4). This Trojan of the ω-cut, orbiting far away from the others and behind an unstable zone, shows also a very exciting behaviour. The overview of this Trojan (Fig. 86) indicates a great number of jumps of all kinds of the longitudinal amplitudes while the other parameters rest absolutely within their limits.

Fig. 87 shows the details of the longitudinal amplitudes between 90 and 195x10^3 yr. The Trojan starts with tadpole orbits, which last for 92x10^3 yr. Then the Trojan jumps from the L4 point to the L5 point, where it oscillates around this point again in the tadpole form for a short while and then changing several times between the horseshoe form and the L5 and L4 tadpole form in very short successions.

Afterwards there exists a long period of nearly undisturbed orbits in the horseshoe form till 768x10^3 yr. After this the Trojan changes anew into a L5 tadpole orbit, then at about 787x10^3 yr becoming again a horseshoe orbit for 20x10^3 yr and changing into a L4 tadpole form for another 22x10^3 yr (Fig. 88). At about 834x10^3 yr this Trojan changes once more into the horseshoe form and becoming at about 846x10^3 yr for the rest of the integration time a nearly undisturbed L5 Trojan.

2.2.5.2.8 Summary of the jumping L4 Trojans in the SUN-system

It is really interesting that in the configuration, without a disturbing Jupiter, also horseshoe orbits may last for a very long time in unchanged stability. This stability shows in all parameters, especially the radial amplitudes follow in their typical resonance exactly the behaviour of the longitudinal amplitudes. Only the orbit at ω_{TR} = +93° from ω_{L4} and the orbit at the far end in the ω-cut at ω_{TR} = +102° from ω_{L4}, show all forms of possible jumps. All the other investigated Trojans signal a total harmonic periodic movement from the beginning up to the end of our integration time and therefore easily could rest stable for much more than the computed timescale as a horseshoe Trojan.

3 Study on the OSS with two giant planets

Our solar system seems to be almost unique in the Milky Way. But during the last decades there several systems have been found with two or more giant planets. By end of November 2010 502 extra solar planets (so called “Exoplanets”), and thereof 59 systems with more than one planet, were detected (http://exoplanet.eu/catalog.php). It should therefore be of some interest to see the different behaviour of L4 Trojans in such a system. So we undertook a study for 10^6 yr only, replacing Jupiter and Saturn with two Jupiter in the OSS. We investigated the behaviour of L4 Trojans not only for Jupiter but also for the second Jupiter which followed the orbit of Saturn. We gave this system the short name JJUN-system. Afterwards we made the same investigation with two Saturn, where the second Saturn is now positioned instead of Jupiter. This system we shall call in short the SSUN-system. In each case we started the Trojans in crosscuts of the respective L4-points.

To distinguish the different L4-points and their Trojans, we introduce the following subscripts:
a_{5L4}, a_{5TR}, ω_{5L4} and ω_{5TR} for the semi-major axis and the perihelion argument of the L4-point of Jupiter and its Trojans respectively, both for the JJUN-system and for the SSUN-system (with a “light” Jupiter).
a_{6L4}, a_{6TR}, ω_{6L4} and ω_{6TR} for the semi-major axis and the perihelion argument of the L4-point of Saturn and its Trojans respectively, both for the JJUN-system (with the “heavy” Saturn) and for the SSUN-system.

3.1 The OSS with two Jupiter (JJUN)

In our investigation we still call the second Jupiter Saturn because of its orbit and therefore also the L4 Trojans in the orbits of Jupiter and Saturn as Jupiter Trojans and Saturn Trojans respectively, always remembering that Saturn now has the mass of Jupiter.

3.1.1 *a*-cut of L4 Trojans of Saturn in the JJUN-system

There we regard the Trojans of the *a*-cut of the L4-point of the “heavy” Saturn. Fig. 89 shows an *a*-cut against the maximum eccentricity and the escape time (at $e > 0.3$ as in our former investigations). The cut with 68 starting Trojans no longer shows an unstable hole around the L4-point and the stable region has significantly grown (compare with Fig. 31 in subsection 2.2.4.1.1). Therefore Jupiter has clearly less influence on the Trojans of the heavy Saturn. The stable zone now reaches in radial direction from 0.962 a_{6L4} to 1.040 a_{6L4}. All unstable Trojans escape within less than 7×10^6 yr, whereas stable Trojans remain within an eccentricity of only 0.1.

The amplitudes in Fig. 90 show the same picture of stable and unstable zones as Fig. 89, for both the radial amplitudes, and the longitudinal amplitudes. It is impressive to realise the very low extension of these amplitudes, growing only slowly relative to the unstable Trojans. The instability zone occurs in a very sudden manner with all parameters. In relation to the SJUN-system, apart of the central hole, the stable zone has increased by about 82%, which is enormous.

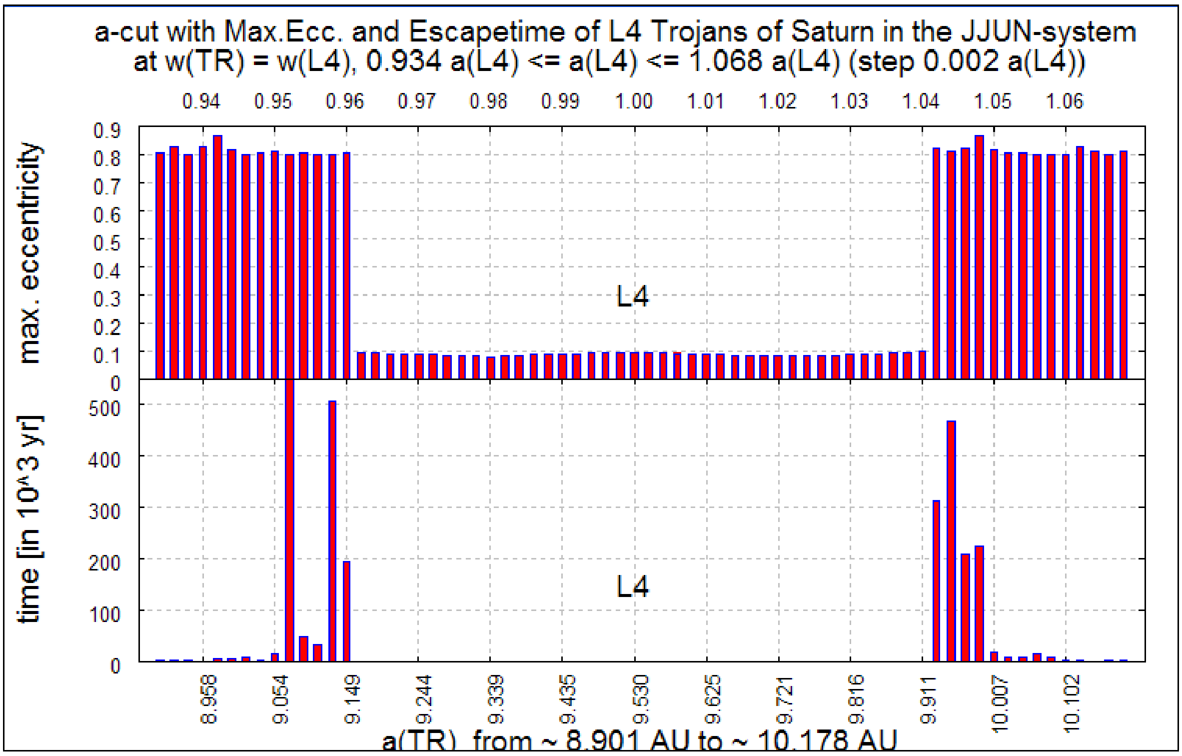

Fig. 89: The a-cut for the maximum eccentricity and the corresponding escape time (at e_{TR} > 0.3) of 68 Trojans of the heavy Saturn with starting orbits from values of the semi-major axis from a_{6TR} = 0.934 a_{6L4} (corresponding to about 8.901 AU) to 1.068 a_{6T4} (corresponding to about 10.178 AU) and Δa = 0.002 a_{6L4} for 10^6 yr. The L4-point lies at about 9.530 AU from the Sun, the same distance as Saturn (a_{6L4} = a_6).

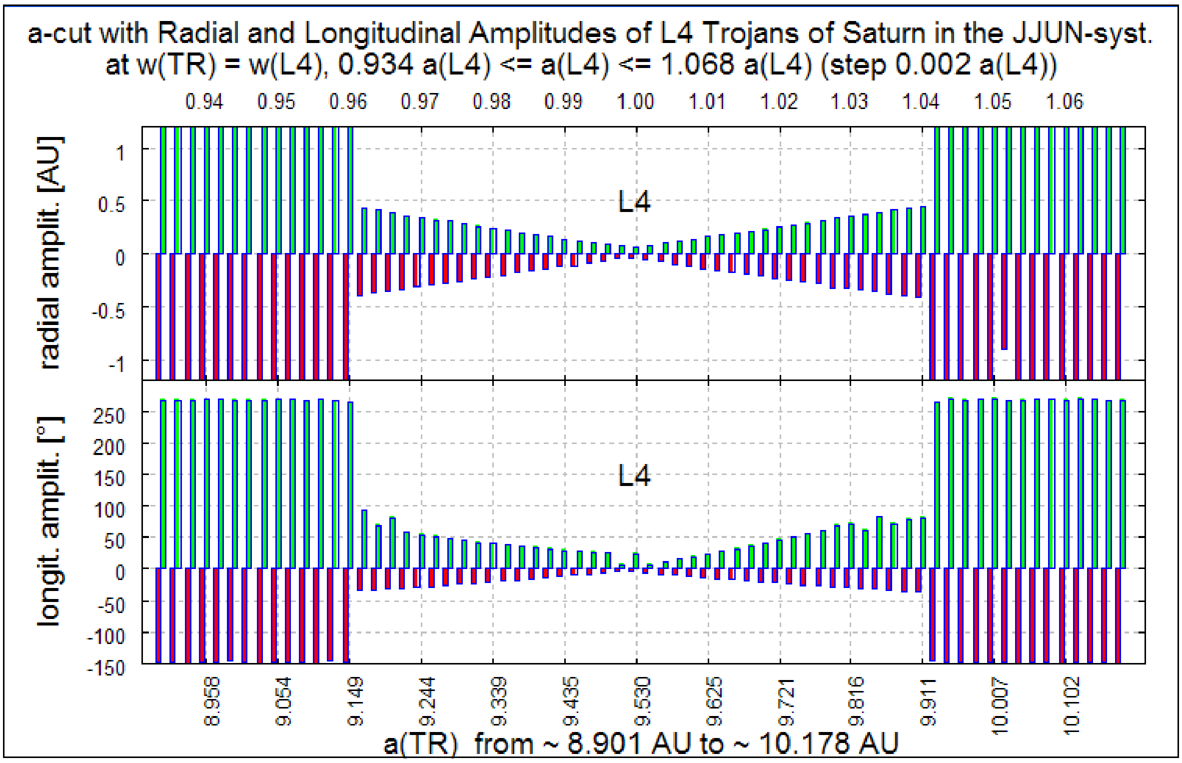

Fig. 90: The a-cut for radial and longitudinal amplitudes for the 68 Trojans of Saturn as before (Fig. 89). Both types of amplitude show the same shape of stable and unstable zones. It is remarkable to see the amplitudes growing slowly toward the unstable orbits and then the sudden jump from stable to unstable.

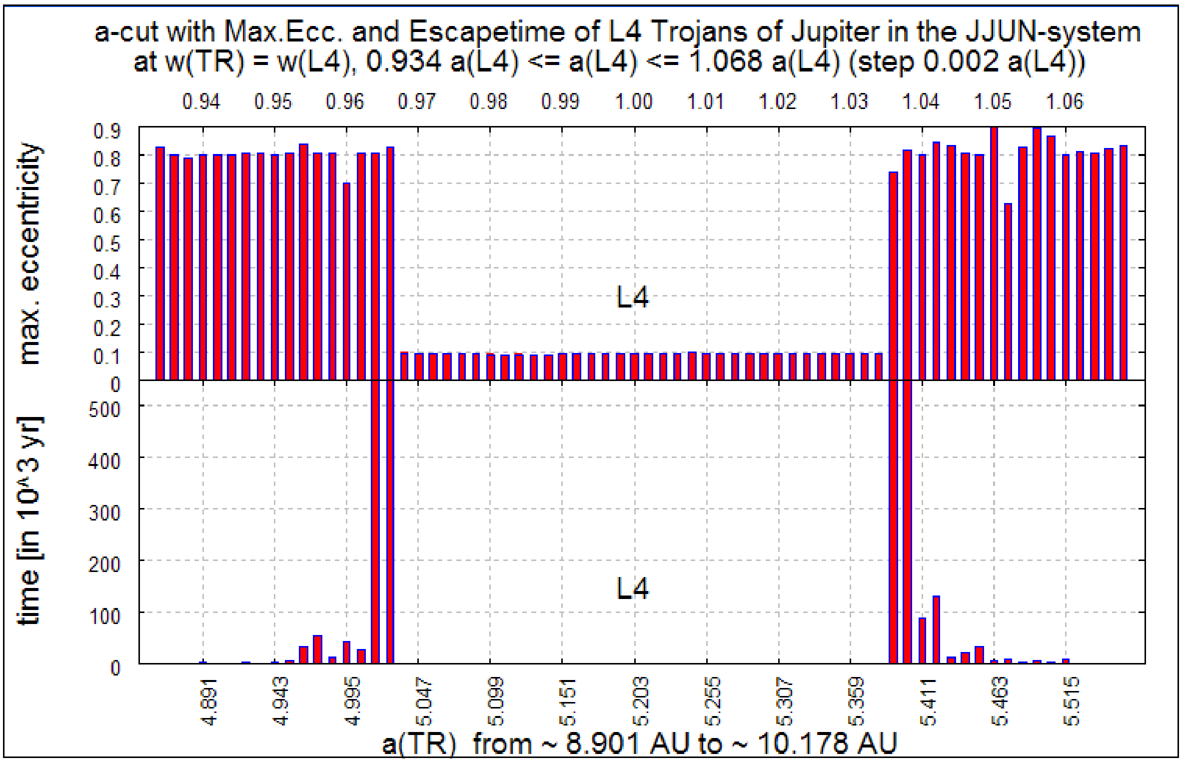

Fig. 91: The a-cut for the maximum eccentricity and the corresponding escape time (at $e_{TR} > 0.3$) of 68 Trojans of Jupiter with starting orbits from values of semi-major axis from $a_{5TR} = 0.934\ a_{5L4}$ (corresponding to about 4.859 AU) to 1.068 a_{5L4} (corresponding to about 5.557 AU) and $\Delta a = 0.002\ a_{5L4}$ for 10^6 yr. The L4-point lies at about 5.203 AU from the Sun, the same distance as Jupiter ($a_{5L4} = a_5$).

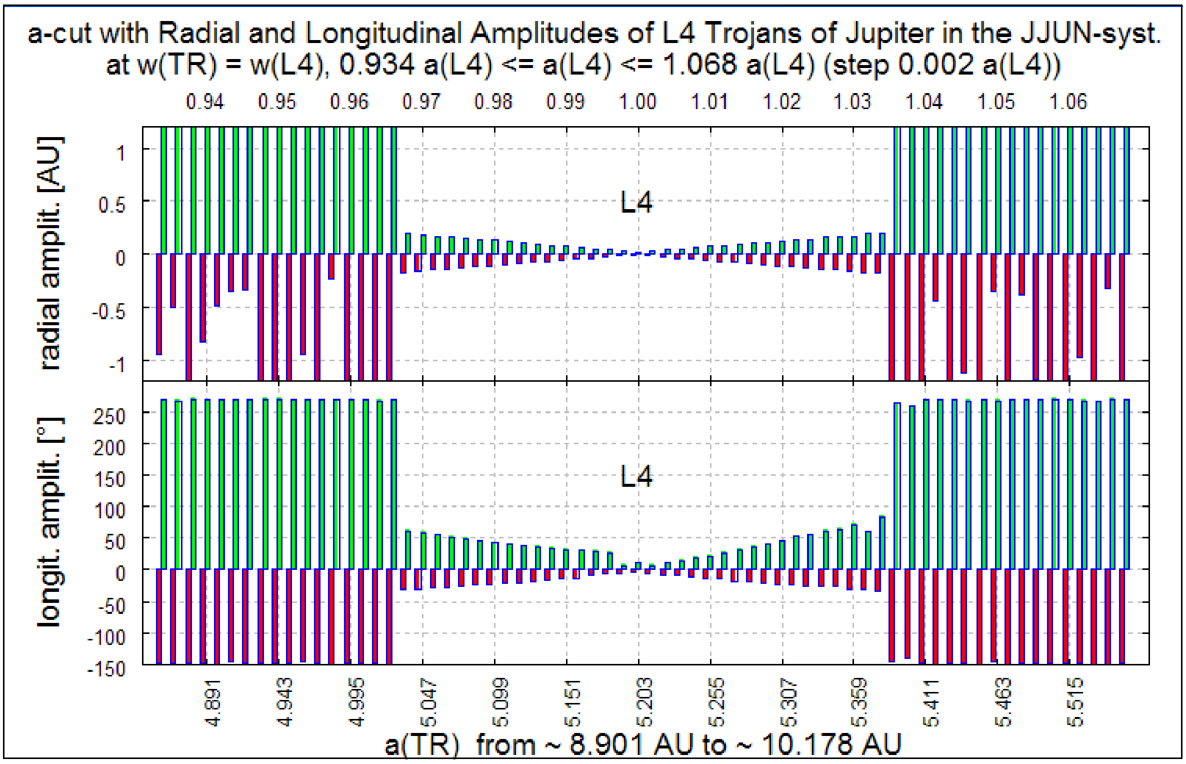

Fig. 92: The a-cut against radial and longitudinal amplitudes around the L4 of Jupiter and starting orbits as in Fig. 91 above. As before, at 10^6 yr the same shapes of stable and unstable orbits appear, marked by the separating jumps.

It is remarkable that the most stable orbit lies exactly in the L4-point of the heavy Saturn. This is in contrast to the real OSS (SJUN), where in close proximity of L4, the unstable hole appears.

3.1.2 *a*-cut of L4 Trojans of Jupiter in the JJUN-system

Now we consider the Trojans of the *a*-cut of the L4-point of Jupiter. Fig. 91 gives a very similar picture to Fig. 89 with maximum eccentricity and escape time. It seems that the heavy Saturn now augments its influence on the Trojans of Jupiter. The stable zone between 0.968 a_{5L4} (corresponding to about 5.037 AU) and 1.034 a_{5L4} (corresponding to about 5.380 AU) is a little narrower than for the L4 Trojans of the heavy Saturn. This stable zone seems to be in good agreement with our plot of the known L4 and L5 Trojans of Jupiter in Fig. 7 of subsection 2.1.2.1, where stable orbits exist between about 4.95 AU and 5.40 AU in the real OSS.

As for the Saturn Trojans, the eccentricity of the Jupiter Trojans remains within only 0.1 for the stable zone. All unstable Trojans escape within 10^6 yr.

The radial and longitudinal amplitudes in Fig. 92 confirm the stable and unstable zones of eccentricity and escape time of Fig. 91. No horseshoe orbits seem to be possible because in the unstable zone all radial amplitudes diverge.

3.1.3 *ω*-cut of L4 Trojans of Saturn in the JJUN-system

Fig. 93 represents the *ω*-cut with the maximum eccentricity and the corresponding escape time (at $e > 0.3$) of 51 Trojans of Saturn in longitudinal direction with starting orbits from values from ω_{6TR} = -48° of ω_{6L4} to +102° of ω_{6L4} and $\Delta\ \omega$ = +/-3° for 10^6 yr. The stable zone reaches from -30° to +51° from ω_{6L4}.

Fig. 94 shows the corresponding radial and the longitudinal amplitudes. In relation to the SJUN-system (see Figs. 41 and 43 in subsection 2.2.4.2) the central hole around L4 does not exist, but the orbits between +54° and +60° there are unstable. That means, that the stable zone in the JJUN-system in relation to the SJUN-system, in the radial direction, apart of the central hole, has significantly increased, but although not in the longitudinal direction. There the stable zone has shrivelled by about 10%.

3.1.4 *ω*-cut of L4 Trojans of Jupiter in the JJUN-system

Fig. 95 shows the *ω*-cut for the maximum eccentricity and the corresponding escape time ($e > 0.3$) of 51 L4-Trojans of Jupiter in longitudinal direction with starting orbits at values from ω_{5TR} = -48° of ω_{5L4} to 102° of ω_{5L4} and $\Delta\omega$ = +/-3° for 10^6 yr. in the JJUN-system. In relation to the heavy Saturn (Fig. 89) the orbits at -30, +48° and +51° from ω_{5L4}, they are already unstable. The zone of instability happens abruptly.

Fig. 96 represents the *ω*-cut against radial and longitudinal amplitudes of L4 Trojans of Jupiter corresponding to Fig. 95. In relation to Fig. 95, the amplitudes show much the same shape of stable and unstable orbits. Only the orbit at +48° seems to survive, but with an unrealistic eccentricity of $e \sim 0.8$.

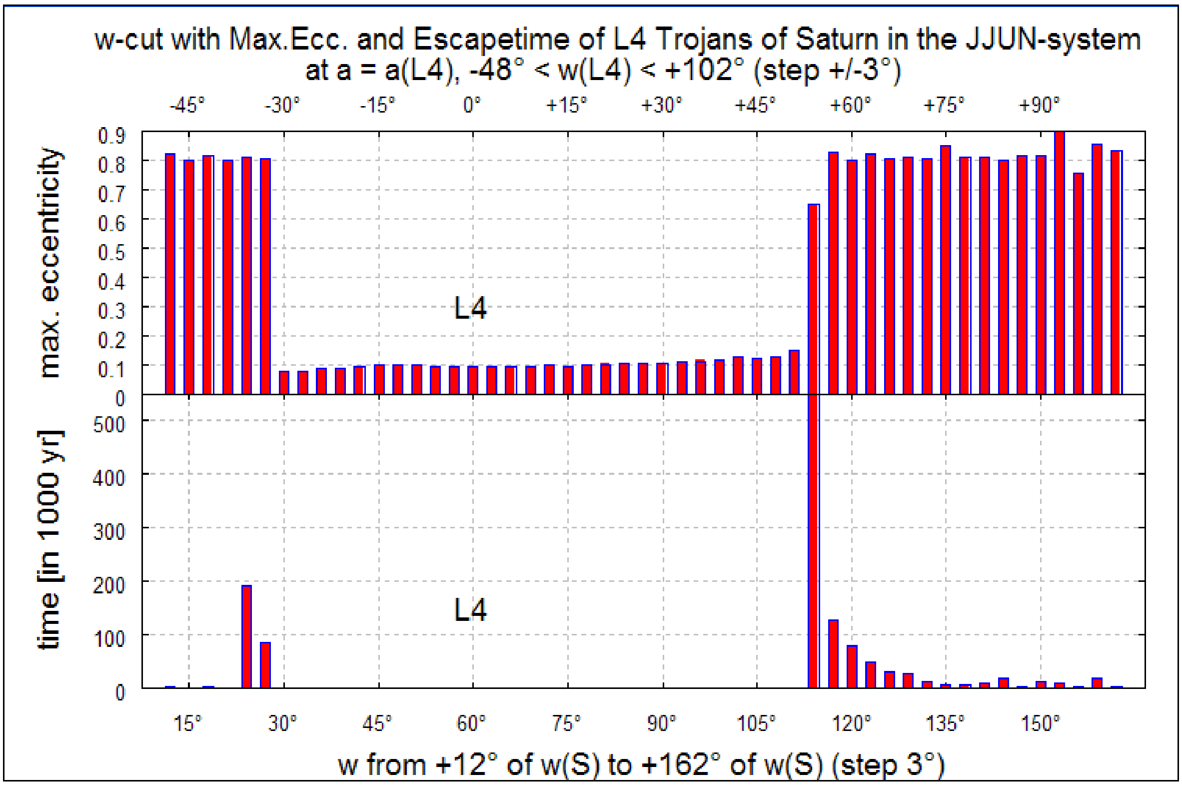

Fig. 93: The ω-cut for the maximum eccentricity and the corresponding escape time (e > 0.3) of 51 L4-Trojans of Saturn in longitudinal direction with starting orbits from values of $\omega_{6TR} = \omega_{6L4}$-48° to +102° and $\Delta\omega$ = +/-3° for 10^6 yr. in the JJUN-system.

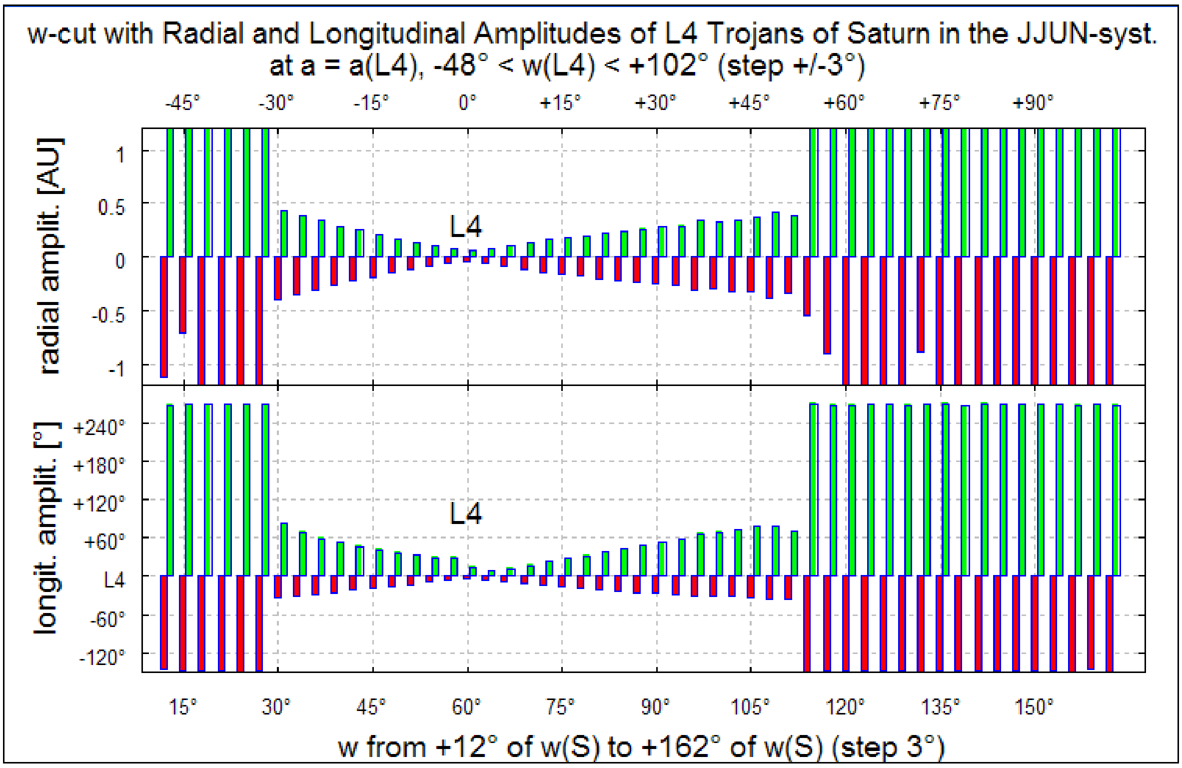

Fig. 94: The ω-cut against radial and longitudinal amplitudes of L4 Trojans of Saturn corresponding to Fig. 93. No difference is observed for the stable and unstable regions.

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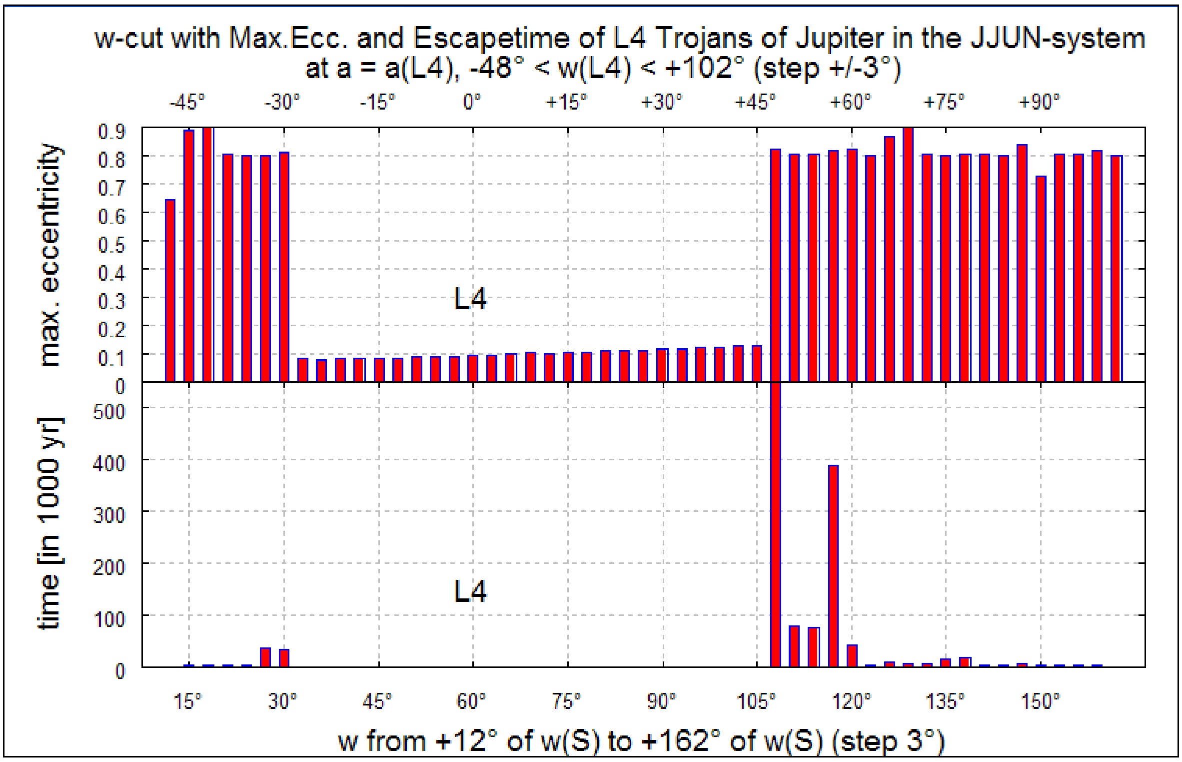

Fig. 95: The ω-cut for the maximum eccentricity and the corresponding escape time (e > 0.3) of 51 L4-Trojans of Jupiter in longitudinal direction with starting orbits from values of $\omega_{5TR} = \omega_{5L4}$-48° to +102° and $\Delta\omega$ = +/-3° for 10^6 yr. in the JJUN-system.

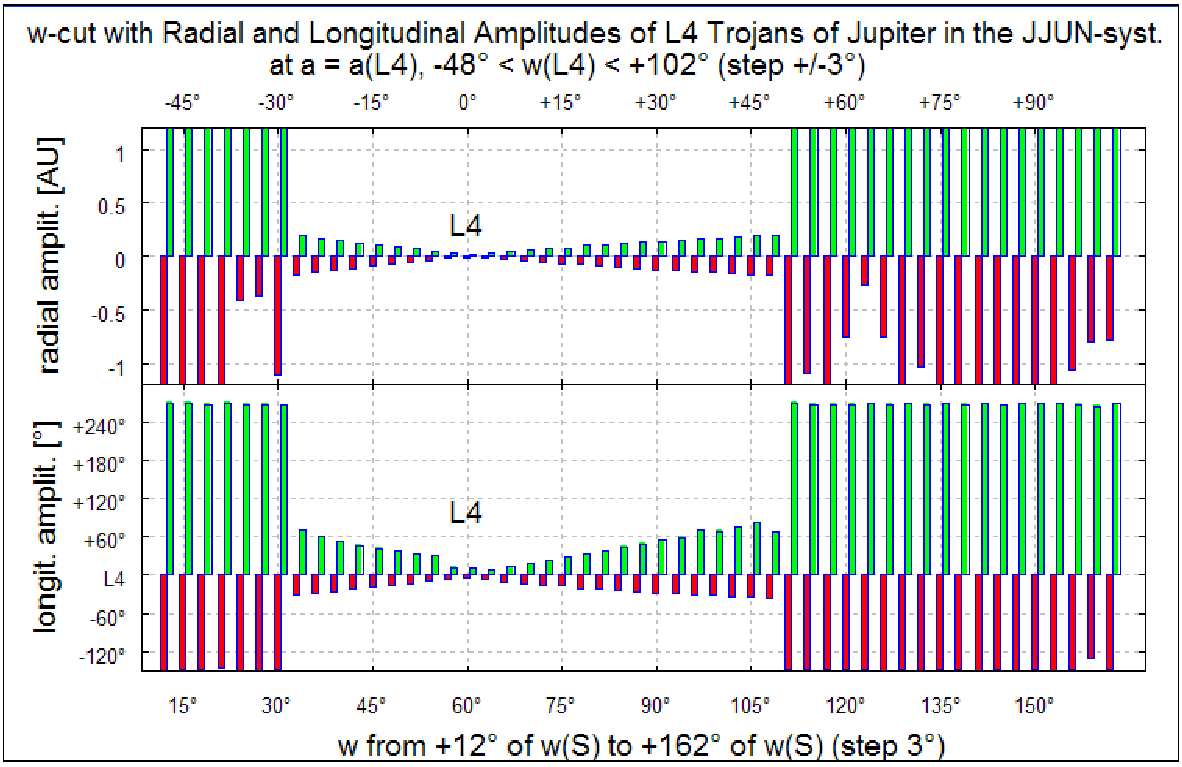

Fig. 96: The ω-cut against radial and longitudinal amplitudes of L4 Trojans of Jupiter corresponding to Fig. 95. In relation to Fig. 95, the amplitudes show much the same shape of stable and unstable orbits. Only the Trojan at +48° seems to survive, but with an unrealistic eccentricity of $e \sim 0.8$.

If we compare the two Jupiter, their L4-Trojans show much the same behaviour in relation to their stable and unstable orbits. Eventually the Trojans of the heavy Saturn show, in the *a*-cuts, as well as in the *ω*-cuts, very slightly more stability than the orbits of Jupiter. Their respective influences seem to be very alike.

3.2 The OSS with two Saturn (SSUN)

Now we shall investigate the same system but with two Saturn (SSUN-system) and still call the Saturn in the place of Jupiter, Jupiter! It is now a "light" Jupiter. Therefore we also call the L4 Trojans in the orbits of the light Jupiter and Saturn as Jupiter Trojans and Saturn Trojans respectively, always remembering that Jupiter now has only the mass of Saturn.

3.2.1 *a*-cut of L4 Trojans of Saturn in the SSUN-system

Now we look at the Trojans of the *a*-cut from the L4-point of Saturn. Fig. 97 shows an *a*-cut against the maximum eccentricity and the escape time (at e > 0.3 as in our former investigations). Also this cut, with 68 starting Trojans, no longer shows an unstable hole around the L4 point and the stable region has grown significantly (compare with Fig. 31 in subsection 2.2.4.1.1) and therefore a light Jupiter has clearly less influence on the Trojans of Saturn than a normal Jupiter. The stable zone now reaches in radial direction from 0.972 a_{6L4} to 1.026 a_{6L4}. But the stable zone in this direction has contracted by about 30% in relation to the JJUN-system (see Fig. 89). The stable Trojans rest within an eccentricity of just about 0.1. In relation to the SJUN-system (Figs. 31 and 33, in subsections 2.2.4.1) the stable zone, apart of the hole, has augmented by about 27%.

The amplitudes in Fig. 98 show approximately the same picture of stable and unstable zones as Fig. 97, and also for the radial amplitudes, as for the longitudinal amplitudes with one exception. In Fig. 97 the orbit at 0.972 a_{6L4} seems to be stable, but Fig. 98 shows that, because of the diverging amplitudes, this orbit has also become unstable. But this orbit could have changed into a horseshoe orbit or become a L5 Trojan, because the radial amplitude remains within 0.3 AU. It is also remarkable here to see the very low extension of these amplitudes, only growing slowly in the direction of the unstable Trojans. The instability zone occurs suddenly with all parameters, including eccentricity, escape time and both type of amplitudes.

3.2.2 *a*-cut of L4 Trojans of Jupiter in the SSUN-system

Now we examine the Trojans of the *a*-cut of the L4-point of Jupiter. Fig. 99 shows a very similar picture to Fig. 91 with maximum eccentricity and escape time, where the stable zone is somewhat smaller by about 12% compared with the JJUN-system.

The radial and longitudinal amplitudes in Fig. 100 confirm the stable and unstable zones of eccentricity and escape time of Fig. 99 with three exceptions: the orbits at 0.972, 1.028 and 1.030 a_{5L4}. These orbits may have changed into L5 Trojans or changed into horseshoe orbits, because their radial amplitudes rest below an eccentricity of $e < 0.2$.

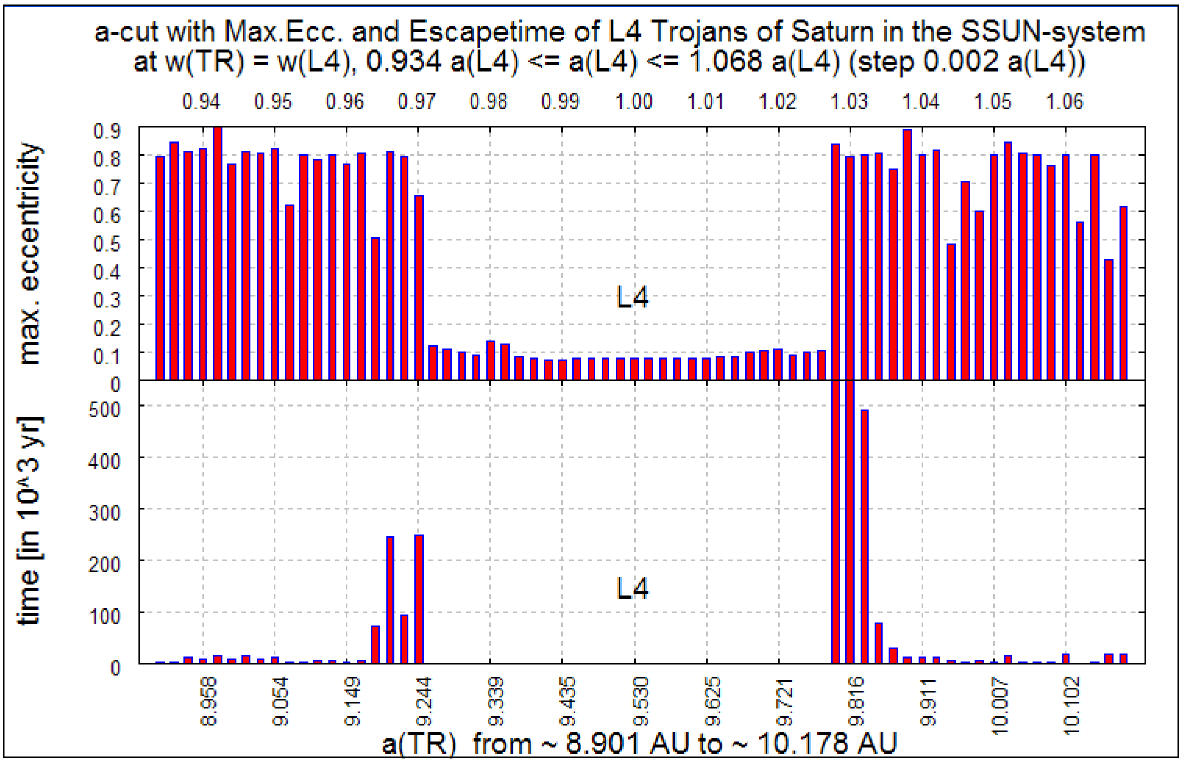

Fig. 97: The a-cut for the maximum eccentricity and the corresponding escape time (at $e_{TR} > 0.3$) of 68 Trojans of Saturn in the SSUN-system with starting orbits with values of semi-major axis from $a_{6TR} = 0.934$ a_{6L4} (corresponding to about 8.901 AU) to 1.068 a_{6T4} (corresponding to about 10.178 AU) and $\Delta a = 0.002$ a_{6L4} for 10^6 yr. The L4-point lies at about 9.530 AU from the Sun, the same distance as Saturn ($a_{6L4} = a_6$).

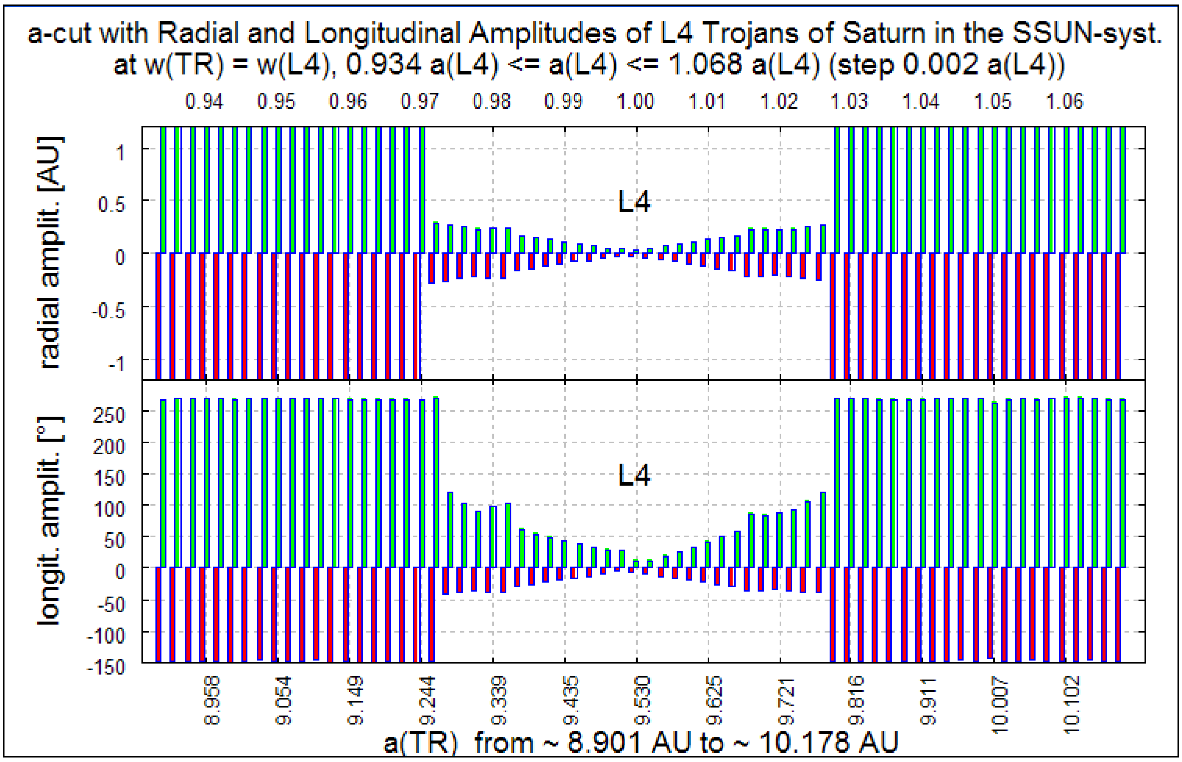

Fig. 98: The a-cut for radial and longitudinal amplitudes for the 68 Trojans of Saturn as before (Fig. 97). Both types of amplitude show the same shape of stable and unstable zones. It is remarkable to see the slowly growing amplitudes toward the unstable orbits with the sharp jump from stable to unstable.

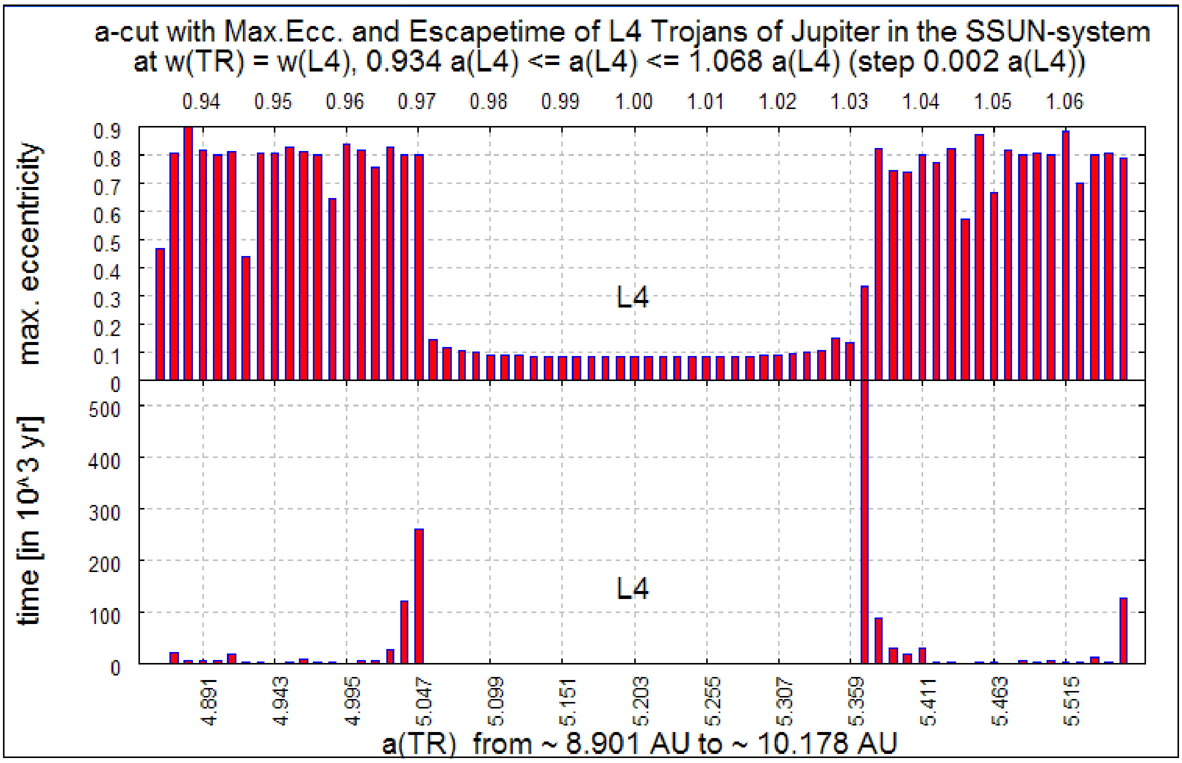

Fig. 99: The a-cut in the SSUN-system for the maximum eccentricity and the corresponding escape time (at $e_{TR} > 0.3$) of 68 Trojans of Jupiter with starting orbits from values of semi-major axis from $a_{5TR} = 0.934\ a_{5L4}$ (corresponding to about 4.859 AU) to 1.068 a_{5L4} (corresponding to about 5.557 AU) and $\Delta a = 0.002\ a_{5L4}$ for 10^6 yr.

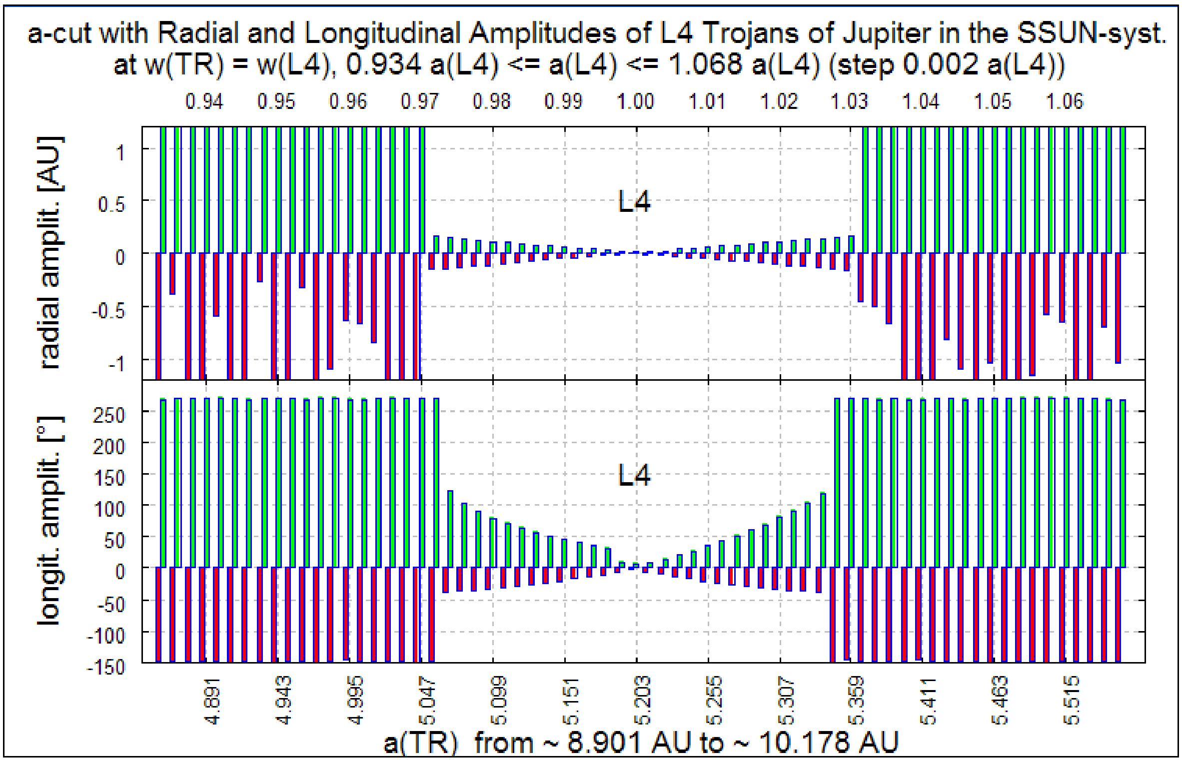

Fig. 100: The a-cut against radial and longitudinal amplitudes around the L4 of Jupiter and starting orbits as in Fig. 99 previously. At 10^6 yr the same shape of stable and unstable orbits appears, marked by the separating jumps. Three orbits at 0.972, 1.028 and 1.030 a_{5L4} may have changed into horseshoe orbits.

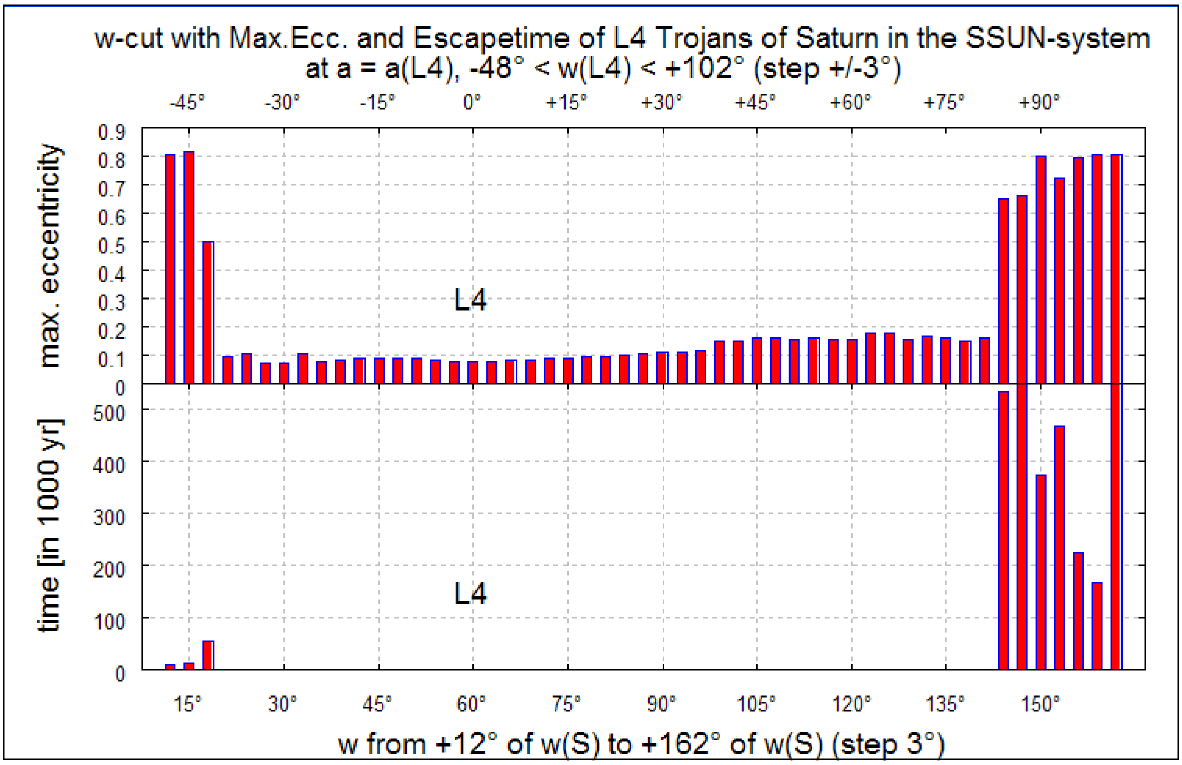

Fig. 101: The ω-cut for the maximum eccentricity and the corresponding escape time (e > 0.3) of 51 L4-Trojans of Saturn in the longitudinal direction with starting orbits from values in the range ω_{6TR} = -48° to +102° of ω_{6L4} and $\Delta\omega$ = +/-3° for 10^6 yr. in the SSUN-system.

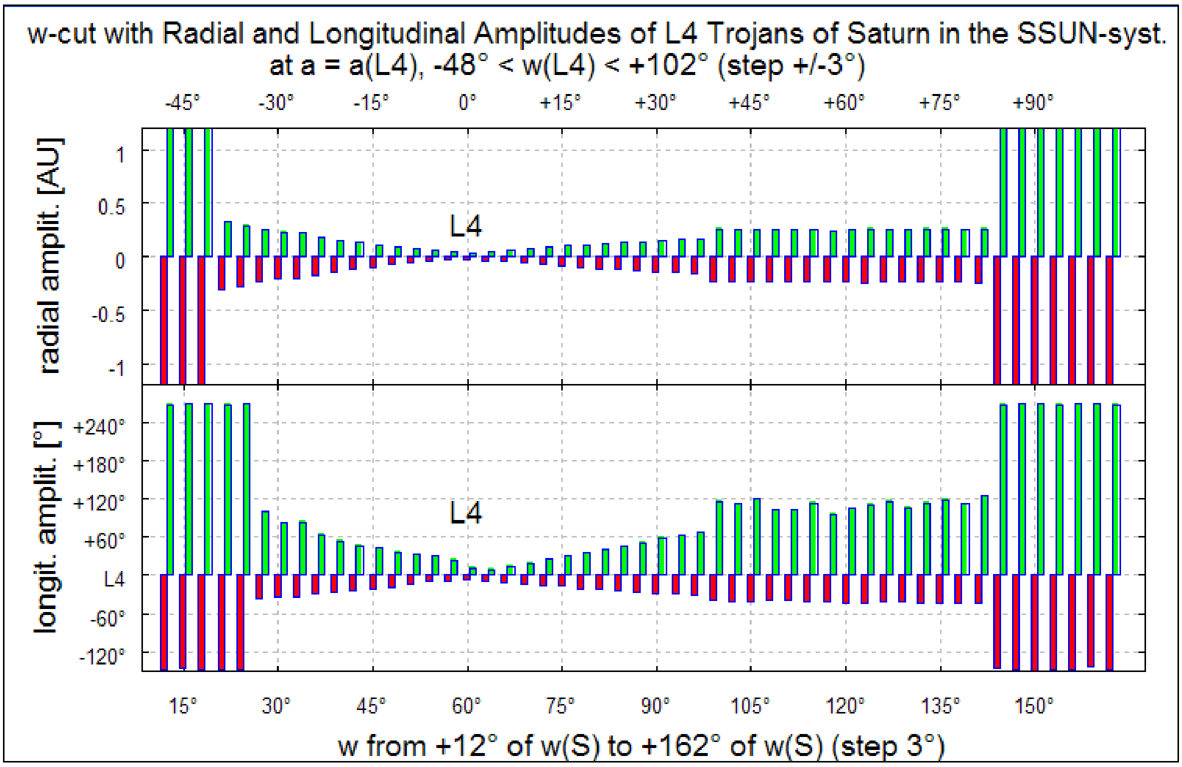

Fig. 102: The ω-cut against radial and longitudinal amplitudes of L4 Trojans of Saturn corresponding to Fig. 101. Two orbits at -36° and -39° to ω_{6L4} might have transformed into horseshoe orbits.

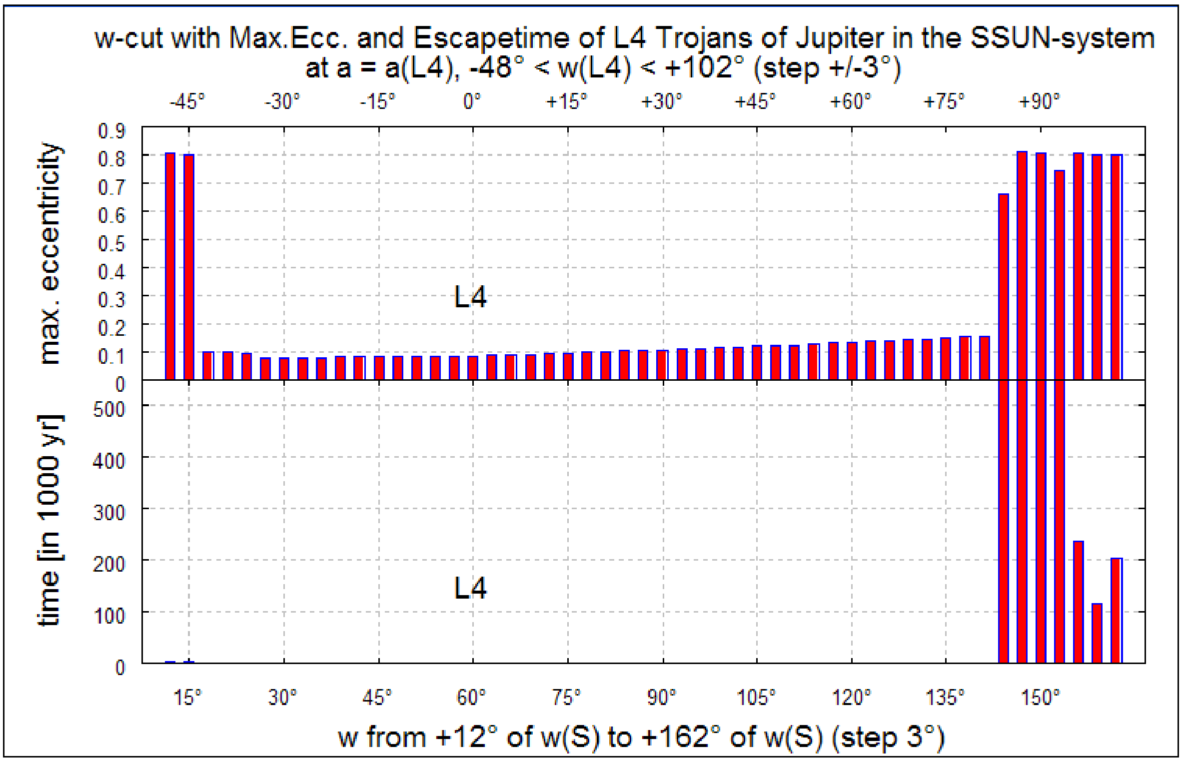

Fig. 103: The ω-cut for the maximum eccentricity and the corresponding escape time (e > 0.3) of 51 L4-Trojans of Jupiter in longitudinal direction with starting orbits having values between ω_{5TR} = -48° and 102° to ω_{5L4} and $\Delta\ \omega$ = +/-3° for 10^6 yr. in the SSUN-system.

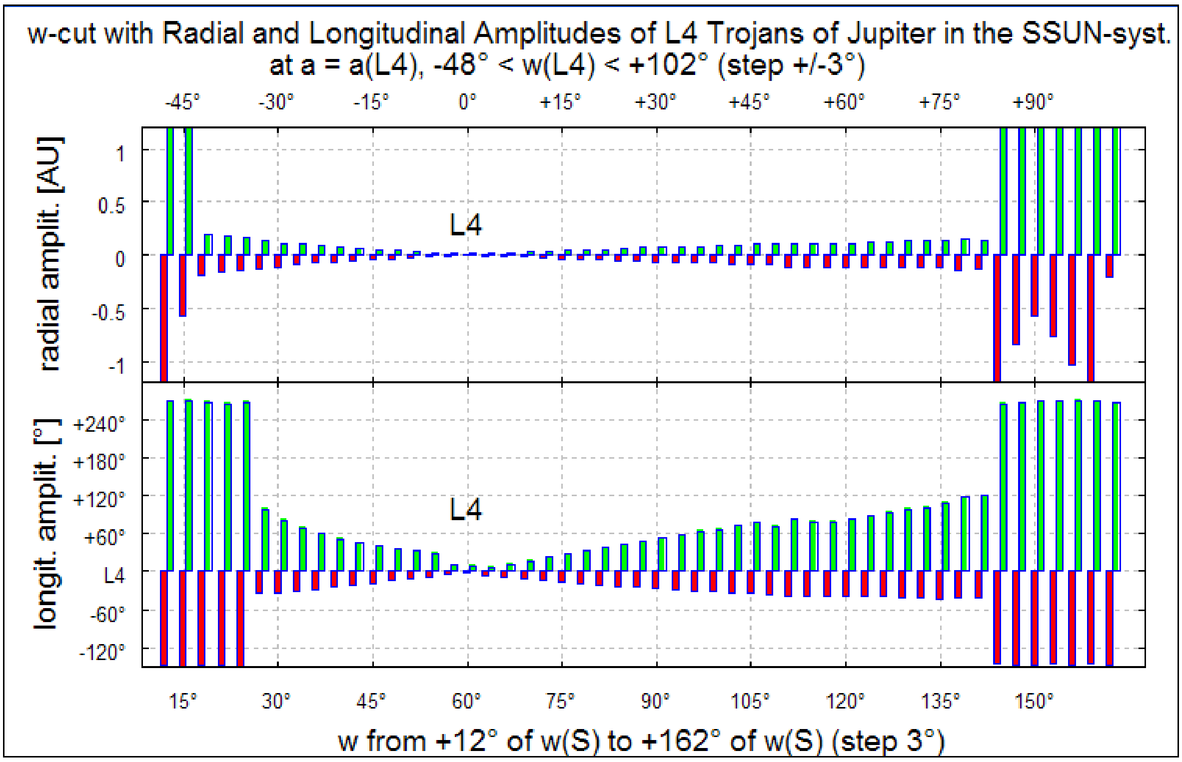

Fig. 104: The ω-cut against radial and longitudinal amplitudes of L4 Trojans of Jupiter corresponding to Fig. 103. Three orbits at -36°, -39° and -42° to ω_{5L4} might have transformed into horseshoe orbits.

3.2.3 ω-cut of L4 Trojans of Saturn in the SSUN-system

Fig. 101 represents the ω-cut for the maximum eccentricity and the corresponding escape time (where $e > 0.3$) of 51 Trojans of Saturn in the longitudinal direction with starting orbits having values between ω_{6TR} = -48° and +102° of ω_{6L4} and $\Delta\omega$ = +/-3° for 10^6 yr. All unstable orbits escape within this period.

Fig. 102 is the corresponding ω-cut to Fig. 101, but for radial and longitudinal amplitudes of L4 Trojans of Jupiter. Two orbits at -36° and -39° from ω_{6L4} might have changed into L5 Trojans or transformed into horseshoe orbits, as only their longitudinal amplitudes rise when $e < 0.1$. Apart of the hole, the longitudinal stable zone in the SSUN-system has augmented by about 32% in relation to the SJUN-system (Figs. 43 and 45 in subsections 2.2.4.2.3 and 2.2.4.2.4 respectively).

3.2.4 ω-cut of L4 Trojans of Jupiter in the JJUN-system

Fig. 103 represents the ω-cut, the maximum eccentricity and the corresponding escape time (when $e > 0.3$) of 51 Trojans of Jupiter in longitudinal direction with starting orbits having values between ω_{6TR} = -48° and +102° of ω_{6L4} and $\Delta\omega$ = +/-3° for 10^6 yr. All unstable orbits escape within this period.

Fig. 104 shows the ω-cut against radial and longitudinal amplitudes of L4 Trojans of Jupiter corresponding to Fig. 103. There are three orbits at -36°, -39° and -42° to ω_{5L4} which might have changed into L5 Trojans or transformed into horseshoe orbits, as only their longitudinal amplitudes rise when $e < 0.1$.

3.3 Summary of the orbits with two giant planets

The outstanding fact in relation to the SJUN-system is the complete disappearance of the central hole.

Apart of the hole in the SJUN-system, the stable zones in the JJUN-system in the radial direction for Saturn has been augmented enormously by about 82%, but shrivelled by about 10% in the longitudinal direction. In the SSUN-system, the stable zone for Saturn has significantly grown by about 27% in the a-cut and by about 32% in the ω-cut in relation to the SJUN-system.

In the JJUN-system the stable zone is greater by about 18% for the Trojans of the heavy Saturn in the a-cut in relation to the Trojans of Jupiter and by about 12% in the ω-cut.

In the SSUN-system the stable zone is now smaller by about 7% for the Trojans of Saturn in the a-cut in relation to the Trojans of the light Jupiter and is nearly equal in the ω-cut.

The role of the changes in the masses is striking. The Jupiter masses always ensure a greater zone of stability for Trojans. If Jupiter is placed in the orbit of Saturn, we observe the greatest stable zone. If Saturn is placed in the orbit of Jupiter then the contrary effect is demonstrated and the outer Saturn Trojans loose some of their stable zones.

4. Conclusions and Discussion

All major studies on the stability of Trojans of Saturn in the OSS found the hole of instability in close proximity to the Lagrange points L4 and L5. I&M outlined the “Great Inequality” near the 2:5 near MMR as the disturbing factor. Téger stated that the Trojans of L5 are less stable than the Trojans around L4. He also came to the conclusion that, with the increase of the integration time, the unstable hole becomes larger and larger, even as long as the integration time of 10^7 yr. We could not at all confirm this statement, as our simulations for 10^6 yr and 10^7 yr on crosscuts show a rapid decrease of stability in an asymptotic curve which, after only 1.5 x 10^5 yr appears stable without further notable decrease (see subsection 2.2.4.3).

N&D came to the conclusion that all Trojans around L4, starting where $9.46 < a < 9.64$ AU, escape within a few million years. Our investigations on a-cuts for 10^7 yr demonstrate instability at $9.32 < a < 9.72$ AU (Figs. 31 to 38 in subsection 2.2.4.1), which is not far away from the result of N&D but suggests a rather greater zone of stability. N&D also investigated the theory of radial migration by displacing the near 2:5 MMR of Saturn and Jupiter from $\Delta = 0$ to $\Delta = 0.8$ (where the unstable hole has completely disappeared). This could have caused the depletion of the Trojans of Saturn.

Concerning the influence of inclination on the stability of L4-Trojans, ZDS found three zones of stability for Neptune; the third zone being at an inclination between 51° and 59° and this zone was framed by bowlike borders. For Uranus DBZ found equal stable zones at inclinations between 20° and 50°. In our study of Trojans of Saturn, up to i_{TR} +70° above i_6 of Saturn, we could find neither stability zones at higher inclinations than 25°, nor the limiting bows (see subsection 2.2.3). Our studies show a rapidly increasing unstable hole with an increasing inclination. The outer borders of the stability zone there are less influenced by inclination, i.e. just the hole is growing.

Our main study (section 2.2) was directed to the influence of the other planets of the OSS on the L4-Trojans of Saturn. We could demonstrate with different integrations for 10^6 yr and alternative withdrawal of each of the giant planets, the overwhelming influence of Jupiter. For the models without Uranus (SJN) and without Neptune (SJU) we used the full grid of 1292 massless test particles. In a similar simulation we even withdrew Uranus and Neptune together, leaving Saturn and Jupiter alone in the OSS (SJ). We always found the unstable hole almost unchanged in addition to the stable zones. When we checked the OSS without Jupiter immediately we no longer found the unstable hole, but an enlarged stable region, with an enlargement of 40% in radial extension and an enlargement of 53% in longitudinal direction with respect to the whole SJUN-system. These investigations show in a convincing manner the immense and sole disturbing influence of Jupiter on the Trojans of Saturn, whereas the other planets, Uranus and Neptune exercise no major effect on the Trojans. Without Jupiter, only the 1:1 MMR continues to influence the stability of Trojans of Saturn.

In subsection 2.2.4 we studied crosscuts of L4 for 10^6 yr and 10^7 yr to compare the differences in the behaviour of the Trojans, concerning their maximum eccentricity, escape time, radial and longitudinal amplitudes. We could show with certainty that indeed the majority of escapes occur in the first million years, because between the cuts at 10^6 yr and the cuts at the 10^7 yr, a period ten times longer, few escapes are observed. This result is relevant for the a-cuts as well as for the ω-cuts. After 5 x 10^5 yr only about 43% of the stable Trojans at 10^3 yr could survive and after 10^6 yr 39% were still stable. After 1.5 x 10^6 yr practically no further significant escapes are visible.

A very interesting effect is the sharp boundary between stable and unstable zones. This abrupt change is seen in maximum eccentricity, escape time, radial and longitudinal amplitudes. At least for the Trojans of Saturn a maximum eccentricity of $e > 0.3$ was shown to be a good measure for instability.

To have a closer view of the behaviour of certain orbits we ran, for several Trojans, integrations to 10^6 yr, but with much more output data. This allowed us to better observe their development with time. Some authors conclude (e.g. N&D), that tadpole orbits might transform into horseshoe orbits. This was another question to consider. In subsection 2.2.5.1 we realised this whilst concentrating on orbits near the borders of instability of the SJUN-system, because there we had the best chance to find such transformations. For horseshoe orbits it is characteristic that they experience longitudinal jumps from the L4 point to the L5 point and back and so on. Another characteristic pattern is to find that their radial amplitudes rest within a certain non-divergent limit, whereas their longitudinal displacement goes to the opposite side of the planets path, touching the L3-point behind the Sun. It was interesting to note, that disturbed Trojan orbits can easily change from tadpole into horseshoe orbits becoming some times a L5 Trojan and may even return to the tadpole form at L4. As a horseshoe orbit they can even approach their planet to almost 10° of longitude from either side. Sometimes they are jumping with regular periods but not always. Mostly they eventually change into a total chaotic behaviour. “Horseshoe orbits of Saturn are strongly unstable”, as N&D have already found.

Our overviews of these specials orbits indicate that, if instability occurs, in addition to maximum eccentricity, diverging radial and longitudinal amplitudes, the inclinations also make an abrupt jump at about the same time.

We also controlled some special orbits of the SUN-system with different behaviour of radial and longitudinal amplitudes. It was a great surprise to find Trojan orbits at the edge of stability in this model without Jupiter which change right from the beginning into horseshoe orbits and stay there regularly for the full integration time of 10^6 years. Two Trojans there were of special interest because they changed temporarily several times from the tadpole (some times at L4 and some times at L5) into the horseshoe form and conversely.

Finally in chapter 3 we risked a look at the behaviour of fictive Trojans in the orbits of Saturn and Jupiter with two equal giant planets, i.e. firstly we placed a second Jupiter in the orbit of Saturn, making a heavy Saturn with the mass of Jupiter (JJUN-system) and secondly we placed a second Saturn in the orbit of Jupiter so making a light Jupiter with the mass of Saturn (SSUN-system). We investigated with crosscuts for 10^6 yr in both cases the L4-Trojans in the orbit of Saturn as well as in the orbit of Jupiter. The outstanding result in all cases was the complete disappearance of the unstable hole near the L4-point in the SJUN-system.

In the JJUN-system the heavy Saturn creates the greatest stable zone. Apart of the hole in the SJUN-system, the stable zone grew enormously by about 82% in the a-cut, but, in a surprising manner, shrivelled by about 10% in the ω-cut. This zone became greater than the stable zone of the Trojans of Jupiter by about 18% in the a-cut and by about 12% in the ω-cut.

In the SSUN-system the stable extension of the Trojans of Saturn has grown by about 27% in the *a*-cut and by about 32% in the *ω*-cut relative to the SJUN-system, apart of the hole. In relation to the light Jupiter the stable zone of the Trojans of Saturn is smaller than that of Jupiter by about 7% in the *a*-cut and is equal in the *ω*-cut.

All these investigated variants of masses, by withdrawing other planets or by replacement identical giant planets in the orbits of Saturn and Jupiter, give a feeling of the dominant relevance of masses and their influence on Trojans, especially in relation to the central unstable hole. Apart from the full SJUN-system, in all other cases without Jupiter, in place of the hole near the L4-point, the most stable orbits for Trojans are encountered.

Trojans in the Lagrangean points link either with the formation of their planet or could randomly be captured, or, perhaps, be captured during migration (chaotic capture). If they are captured, they should have high inclinations at the beginning. High inclinations tend to be indicative of capture instead of "in situ" formation (Nesvorny and Vokrouhlický 2009). This seems to be valid for steeply inclined Trojans of Jupiter, Uranus and Neptune, but not as such for Trojans of Saturn. As our investigations showed, any asteroid with $i > 25°$ becomes unstable within 10^6 yr. On the other hand Elía and Brunini (2007), from numerical experiments on Jovian Trojans, surmised that "bodies with small masses could easily be ejected, whereas massive bodies with diameters > 1 km would statistically be ejected only every 20,000 yr".

In our study of Saturn, captures are to be excluded from consideration because they become unstable with $e > 0.25$. That means that possible Trojans must have been formed in situ with the creation of Saturn. Losses due to collisions could have disrupted a swarm of Trojans, but were not considered in our study.

The only stable region of the L4-Trojans of Saturn is a large oval ring, but within this ring it is possible that Trojans might remain stable not only for 10^7 yr, as we have proven, but possibly for a much longer period, perhaps even for a solar timescale. However, up to the present day, no Trojans of Saturn have been found. Are they too small for our current instruments or has the migration of the planets caused their depletion, as N&D suggest? The question remains thrilling!

Acknowledgments by Helmut Baudisch

After leaving my Technical College, I wanted to study Chemistry or Physics. However, as a Technical High School student this was possible only by taking a circuitous route. Therefore I studied with interest Commercial Sciences in Vienna (Wirtschaftsuniversität).
When I became a Doctor of Commercial Sciences and a Magister of Social and Economic Siences, I always felt a gap in my knowledge concerning Natural Sciences. I read some popular books about Physics and Cosmology, which were very enlightening and I took an interest. When I finished my professional career, I decided to undertake some intellectual work and ultimately chose to study Astronomy at the University of Vienna.

There I met some other mature, like minded students, so called senior students, who encouraged me to study in earnest. There I want especially to thank my friend, Mr Erich Hartig, with whom I studied the first semesters in close cooperation.

In Astronomy I was always especially interested in the far Universe, in Cosmology and Galaxies. When Prof. Gerhard Hensler proposed that I write a Baccalaurean Thesis about the Dwarf Galaxies of the Milky Way, I immediately became enthusiastic. I want to thank Prof. Gerhard Hensler for his friendly and consistent mentoring. His continuous supervision and support were a great help to me.

I always thought that the crown of Astronomic Sciences is Celestial Mechanics. Through this I met Prof. Rudolf Dvorak. His clear and profound lectures, together with his friendly and helpful manner to all students, convinced me that I should ask him for a theme for my Magisterial Thesis. Prof. Dvorak first proposed and then encouraged me to work on the previously unobserved Trojans of Saturn around the Lagrangean points L4 and L5 of this, the second giant planet in our Solar System, and their possible stability, as no profound study currently existed. I was very interested and immediately started to work. Prof. Dvorak not only provided me with the software of the Lie-integration method and the starting values of the whole OSS but also helped and provided me with literature. He conducted and attended my work in a very amicable manner. I want to sincerely thank Professor Rudolf Dvorak for his skilful mentoring. Without his constant helping hand it would not have been possible for me to complete what I hope is a worthwhile Thesis.

I decided to write my Thesis in English. This decision was only possible, knowing my long time English friend, Mr Patrick Smith, a former school teacher of Mathematics, would help me by making any necessary linguistic amendments. I want to thank him very much for this exhausting “labour of love”.

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Dr. Helmut Baudisch

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Annexe 1: Table of the starting positions of integrations of the Outer Solar System (OSS)

	a [AU]	e	i [°]	ω [°]	Ω [°]	M [°]	m_{Sun}
Sun	0.0	0.0	0.0	0.0	0.0	0.0	1.0
Jupiter	5.2027870233	0.04833790226352	1.30463475	275.2010177	100.4706642588	183.897808735	0.95479066214732E-03
Saturn	9.5300498501	0.05334351875332	2.48644437	339.5198854	113.6685162395	238.293160282	0.28587764436821E-03
Uranus	19.235307728	0.04732287311718	0.77246832	99.865744927	74.0328700914	111.687917146	0.43554006968641E-04
Neptun	30.140843059	0.00624892789382	1.77180286	261.73961138	131.755959637	257.158043637	0.51775913844879E-04

Explications:

a [AU]	semimajor axis from the Sun in Astronomical Units
e	eccentricity of the orbit
i [°]	inclination (of the orbital plane of the Sun-system as an inertial system) in Degrees
ω [°]	perihelion argument in Degrees
Ω [°]	longitude of the ascending node in Degrees
M [°]	mean anomaly in Degrees
m_{Sun}	mass in solar units

Annexe 2: concerning the longitudinal amplitudes

To come from the mean anomaly (M) to the true anomaly (v) we can use the well-known Kepler equation (see D&F):

$E - e \sin E = M$, where E is the eccentric anomaly in Degrees, e the eccentricity and M the mean anomaly in Degrees.

The true anomaly finally is obtained by

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\ \tan\frac{E}{2}$$

“The Kepler equation cannot be solved in closed form” (D&F), but there exists an approximate solution, with an exponential series expansion for small e, to derive directly from M to v:

$v = M + \mathrm{C}$ (see e.g.: http://www.astro.uu.nl/~strous/AA/en/reken/kepler.html)

$M_{\mathrm{rad}} = M_{\mathrm{deg}} * \pi/180$ gives the mean anomaly in Radian.

The formula (see Stumpff, 1959) used in our computations is:

$$\begin{aligned} v_{rad} = \mathrm{M_{rad}} &+ \sin(\mathrm{M_{rad}})\left(2\mathrm{e} - \frac{1}{4}\mathrm{e}^3 + \frac{5}{96}\mathrm{e}^5 + \frac{107}{4608}\mathrm{e}^7 + \ldots\right) + \\ &+ \sin(2\mathrm{M_{rad}})\left(\frac{5}{4}\mathrm{e}^2 - \frac{11}{24}\mathrm{e}^4 + \frac{17}{96}\mathrm{e}^6 - \ldots\right) + \\ &+ \sin(3\mathrm{M_{rad}})\left(\frac{13}{12}\mathrm{e}^3 - \frac{43}{64}e^5 + \frac{95}{512}\mathrm{e}^7 - \ldots\right) + \\ &+ \sin(4\mathrm{M_{rad}})\left(\frac{103}{96}\mathrm{e}^4 - \frac{451}{480}\mathrm{e}^6 + \ldots\right) + \\ &+ \sin(5\mathrm{M_{rad}})\left(\frac{1097}{960}\mathrm{e}^5 - \frac{5957}{4608}e^7 + \ldots\right) + \\ &+ \sin(6\mathrm{M_{rad}})\left(\frac{1223}{960}\mathrm{e}^6 - \ldots\right) + \\ &+ \sin(7\mathrm{M_{rad}})\left(\frac{47237}{32256}\mathrm{e}^7 - \ldots\right) \end{aligned}$$

$v_{\mathrm{deg}} = v_{\mathrm{rad}} 180/\pi$ gives the true anomaly in Degrees.

The true longitude finally then is

$\Gamma = \omega + \Omega + v$

The longitudinal amplitude in our work is computed as

$\Gamma_{\mathrm{TR}} - \Gamma_{\mathrm{L4}}$

Annexe 3: Abstract

It was Joseph Louis Lagrange who discovered 5 points, where the centrifugal force of a massless object orbiting the Sun finds equilibrium with the combined gravitational forces exerted by the Sun and a planet (Restricted Thee-Body-Problem). Three of these points are in line with Sun and planet, representing an unstable equilibrium; whereas the Lagrangean points L4 and L5 represent a stable equilibrium, forming equilateral triangles with Sun and planet, co-orbiting with the planet the common barycenter. Powerful telescopes have already helped to detect more than 4000 asteroids, also called Trojans, around the points L4 (the leading point in the sense of the orbiting planet) and L5 (the trailing point) of Jupiter. Some Trojans were also discovered for Mars and Neptune, but not for Saturn, the second planet in size of our Sun-system. Of course, the possibility remains that they have not yet been detected. Trojans move in a long, stretched, oscillating manner around these triangular points and therefore their paths are named „Tadpole Orbits". On the other hand there exist orbits on the opposite side of the planet to the Sun. this means that they move between the points L4 and L5 and pass through the L3 point behind the Sun. Because of this their paths are named „Horseshoe Orbits".

The subject of this work is to investigate whether or not the absence of Trojans of Saturn is caused by a general instability of the libration points of Saturn respectively to ascertain the extension of possible stable regions. In general L4 points are known to offer regions of more stability than L5 points and therefore this work concentrates on the L4 point of Saturn. Trojans in this work are fictive massless bodies.

In the past some papers could prove that there exist unstable regions in close proximity to the points L4 and L5 of Saturn (the central "holes"), which were not found for Jupiter and Neptune; but around these holes there exists a fairly stable zone. This fact makes the Lagrangean points L4 and L5 of Saturn a field of special interest for investigations. The investigations made use of the restricted three-dimensional elliptic *n*-body problem with the Lie-integration method with adaptive step-size. Simulations with numerous massless test particles were made for 10^6 years of integration time and in some cases up to 10^7 years. The studies there are divided in two main chapters.

The first chapter concentrates on investigations with the four giant planets of our "Outer Solar System" (OSS), forgetting the much smaller "Inner Planets" (ISS). To discover the influences of the diverse Outer Planets on Saturn's L4 Trojans, comparable simulations were made with different configurations of the planets:

- System with Sun, Jupiter, Saturn, Uranus and Neptune (SJUN-system)
- System with Sun, Jupiter, Saturn and Uranus (SJU-system)
- System with Sun, Jupiter, Saturn and Neptune (SJN-system)
- System with Sun, Saturn, Uranus and Neptune (SUN-system)
- System with just Sun, Saturn and Jupiter (SJ-System)

It became apparent that Jupiter is by far the main disturber, whereas the other planets exercise little or no influence on the extension of the stable region. The giant planet Jupiter is the dominant factor and alone responsible for the unstable central hole. In the SUN-system, without Jupiter, the stable zone is achieving by far, the largest extension. The border between the stable and unstable regions is very clear shown for all parameters investigated. Once the eccentricity (*e*) of the Trojans exceeds 0.3, all Trojans have escaped from the stable region and the escape time signals the moment of their departure. The same occurs with the "radial amplitudes", the oscillations around the firm L4 point in radial direction, when a Trojan leaves the stable region towards Jupiter or in the other direction towards Uranus. Finally the

"longitudinal amplitudes", the movement of the Trojans in longitudinal direction, signal the same escapes when the Trojan enters the gravitational field of the planet (Hill's sphere) or, in the other direction, touches the L3 point. It is surprising that the majority of the Trojans are lost within the first $1{,}3x10^6$ years and the remainder survive for up to at least 10^7 years without further escapes.
In a subsection the question of inclinations is answered by how increasing the starting inclination with respect to the plane of Saturn leads to instabilities of the Trojans. In the case of Uranus and Neptune it has become evident that stable zones for fictive Trojans could exist even at very high inclinations. Such stable zones could not be confirmed in our studies of Saturn with simulations up to 70° above the plane of the planet.
In another subsection special orbits on the edge of stability were controlled, where the radial and longitudinal amplitudes show a very different behaviour. In fact, in the SJUN-system, orbits can be found where the Trojan changes from L4 tadpole orbits into temporarily horseshoe orbits, into L5 tadpole orbits and conversely. The same was done, without Jupiter, for such orbits in the SUN-system. It was a great surprise to find Trojan orbits on the edge of stability which change from the very beginning into horseshoe orbits and often remained there for the full integration time of 10^6 years. One Trojan was of special interest because it changed temporarily several times from the tadpole (some times at L5 and some times at L4) into the horseshoe form and conversely.

The second chapter investigates the behaviour of L4 Trojans in a system with two giant planets, which is often the case with extra solar planetary systems. There are already 48 such systems known with more than one giant planet. In one case two planets with the mass of Jupiter were placed in the orbits of Jupiter and Saturn respectively (JJUN-system) and both Trojan families near L4 were investigated. Afterwards the same simulations were done for two planets with the mass of Saturn (SSUN-system). It was somewhat surprising that in none of the four cases was a central hole manifested. In both models, apart of the unstable hole, the stable zone was significantly larger (up to +82%). With two equal heavy planets the stable zones for Trojans are much the same in extension in the orbits of Jupiter as in the orbits of Saturn. In the JJUN-model the extension of the stable zones are slightly larger as in the SSUN-system.

Annexe 4: Zusammenfassung

Joseph Louis Lagrange entdeckte fünf Punkte, bei denen sich die Zentrifugalkraft eines um die Sonne kreisenden masselosen Objektes mit der Gravitationskraft, die gemeinsam von Sonne und Planet ausgeübt wird, im Gleichgewicht befindet (Eingeschränktes Drei-Körper-Problem). Drei der fünf Punkte liegen in Linie mit Sonne und Planet und stellen ein labiles Gleichgewicht dar, während die Lagrange Punkte L4 und L5, die mit Sonne und Planet je ein gleichseitiges Dreieck bilden, ein stabiles Gleichgewicht verkörpern und mit ihrem Planeten das Baryzentrum umkreisen. Mittels leistungsstarker Teleskope konnten um die Punkte L4 (im Sinne der Planetenbewegung vorauseilend) und L5 (nacheilend) bis heute bereits mehr als 4000 Asteroiden, sogenannte Trojaner, des Jupiters entdeckt werden. Auch bei den Planeten Mars und Neptun wurden Trojaner gesichtet, nicht hingegen solche des Planeten Saturn, dem zweitgrößten Planeten in unserem Sonnensystem. Natürlich besteht die Möglichkeit, dass sie bisher nur noch nicht entdeckt wurden. Trojaner führen dabei lang gestreckte oszillierende Bewegungen um diese Dreieckspunkte durch, so dass diese Bewegung wegen ihrer Form „Kaulquappen-Umläufe“ („Tadpole Orbits“) genannt werden. Daneben gibt es auch Umläufe, die auf der dem Planeten abgewandten Seite, also hinter der Sonne, zwischen den Punkten L4 und L5 stattfinden, dabei den L3 Punkt passieren und wegen ihrer Form „Hufeisen-Umläufe“ („Horseshoe Orbits“) genannt werden.

Gegenstand dieser Arbeit ist es zu untersuchen, ob dieses „Fehlen“ von Trojanern auf eine generelle Instabilität Librationspunkte von Saturn zurückzuführen ist, bzw. herauszufinden inwieweit stabile Bereiche existieren. Im Allgemeinen sind L4 Punkte von einem stabileren Bereich umgeben als die Bereiche um L5 Punkte, weshalb sich diese Arbeit auf den Punkt L4 des Saturns konzentriert. Als Trojaner werden in dieser Arbeit fiktive masselose Objekte bezeichnet.

Einige Arbeiten konnten in der Vergangenheit nachweisen, dass im Gegensatz zu Jupiter und Neptun in der unmittelbaren Nähe der Punkte L4 und L5 des Saturn eine instabile Zone existiert (ein zentrales „Loch“), um diese herum jedoch ein weitgehend stabiler Bereich gegeben ist. Dies macht die Lagrange-Punkte L4 und L5 von Saturn zu einem besonders interessanten Untersuchungsgebiet. Die Untersuchungen erfolgten unter Anwendung des dreidimensionalen elliptischen n-Körper-Prolems und unter Verwendung der Lie-Intrationsmethode mit adaptiver Schrittweite. Es konnten numerische Simulationen mit zahlreichen masselosen Testkörpern bis zu einer Integrationszeit von 10^6 Jahren, teilweise bis zu 10^7 Jahren durchgeführt werden. Die Untersuchungen erfolgten dabei in zwei größeren Abschnitten.

Der erste Abschnitt beinhaltet Untersuchungen mit den vier großen „Äußeren Planeten“ (OSS) unseres Sonnensystems unter Vernachlässigung der viel kleineren „Inneren Planeten“ (ISS). Um den Einfluss der einzelnen Äußeren Planeten auf die Trojaner um L4 zu erkunden, wurden vergleichbare Simulationen mit unterschiedlichen Planetenkonfigurationen unternommen:

- System mit Sonne, Jupiter, Saturn, Uranus und Neptun (SJUN-System)
- System mit Sonne, Jupiter, Saturn und Uranus (SJU-System)
- System mit Sonne, Jupiter, Saturn und Neptun (SJN-System)
- System mit Sonne, Saturn, Uranus und Neptun (SUN-System)
- System nur mit Sonne, Saturn und Jupiter (SJ-System)

Es stellte sich eindeutig heraus, dass Jupiter der dominierende Störfaktor ist, während die anderen Planeten wenig bis gar keinen Einfluss auf die Größe der stabilen Region ausüben.

Der Riesenplanet Jupiter hingegen ist der dominierende Faktor und alleine für das instabile zentrale Loch verantwortlich. Im SUN-System, also ohne Jupiter, erreicht die stabile Zone ihre weitaus größte Ausdehnung. Die Grenze zwischen stabilem und instabilem Bereich hat sich als klar und eindeutig bei allen untersuchten Parametern gezeigt. Sobald die Exzentrizität (*e*) der Trojaner den Grenzwert von 0,3 übersteigt, haben sie das stabile Gebiet verlassen, und die Fluchtzeit ist der Zeitpunkt zu dem diese Exzentrizitätsgrenze überschritten wird. Dann bei den „Radialen Amplituden“ um den fixen L4 Punkt, wenn ein Trojaner den stabilen Bereich Richtung Jupiter bzw. Uranus verlassen hat. Schließlich bei den Bewegungen längs der Umlaufbahn (in longitudinaler Richtung), wenn ein Trojaner in den Gravitationsbereich (Hill's Sphäre) des Planeten bzw. in der anderen Richtung zum L3 Punkt gerät. Überraschend ist, dass die meisten Trojaner in den ersten $1{,}3 \times 10^6$ Jahren verloren gehen, die übrigen jedoch bis 10^7 Jahre ohne nennenswerte weitere Verluste überleben.
In einem Unterkapitel wird auch der Frage nachgegangen, wie weit immer größer werdende anfängliche Inklinationen im Verhältnis zur Planetenebene bei Trojanern zu Instabilitäten führen. Bei Uranus und Neptun hat sich nämlich gezeigt, dass es auch bei sehr großen Inklinationen noch stabile Felder für Trojaner geben kann, was sich aber in unserem Fall des Saturns, mit Inklinationen bis 70° über der Planetenebene, nicht gezeigt hat.
In einem weiteren Unterkapitel werden einige spezielle Orbits am Rand der stabilen Zonen untersucht, bei denen radiale und longitudinale Amplituden sehr unterschiedliches Verhalten zeigen. Tatsächlich konnten im SJUN-System Trojaner gefunden werden, die vorübergehend von „Tadpole Orbits“ um L4 in „Horseshoe Orbits“ wechseln, dann zu L5 „Tadpole Orbits“ werden und umgekehrt. Das gleiche wurde für Trojaner im SUN-System, also ohne Jupiter, durchgeführt. Zur großen Überraschung konnten Trojaner an der Stabilitätsgrenze gefunden werden, die gleich am Anfang in einen Horseshoe übergehen und meist bis zur vollen Integrationszeit von 10^6 Jahren so verharren. Zwei Trojaner waren dabei von besonderem Interesse, weil er vorübergehend mehrfach von der „Tadpole-Form“ (manchmal um L4 und manchmal um L5) in die „Horseshoe-Form“ wechselten und wieder zurück.

Der zweite Abschnitt untersucht das Verhalten von L4 Trojanern im Falle von zwei Riesenplaneten, wie dies bei extrasolaren Mehrplanetensystemen häufig der Fall ist, und man kennt schon 48 solcher Systeme mit mehr als einem Riesenplaneten. Dabei wurden einmal zwei Planeten mit Jupitermasse, also zwei Jupiter, in die Umlaufbahnen von Jupiter und Saturn gesetzt (JJUN-System) und jeweils deren L4 Trojaner untersucht. Das gleiche wurde mit zwei Planeten mit Saturnmasse (SSUN-System) simuliert. Überraschend war, dass sich in keinem dieser vier Fälle ein instabiles zentrales Loch zeigte. In beiden Modellen zeigt sich, abgesehen vom zentralen Loch, eine viel größere stabile Zone als im SJUN-System (bis zu 82% größer). Bei gleichschweren Planeten sind sowohl in der Umlaufbahn des Saturns als auch in der des Jupiters die stabilen L4 Zonen in ihrer Ausdehnung sehr ähnlich. Im JJUN-Modell erscheinen die Bereiche etwas ausgedehnter als im SSUN-Modell.

Curriculum Vitae

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Ausbildung:

Schulbildung:

1944 – 1948: 4 Klassen Volksschule
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1952 – 1957: 5 Klassen HTL Wien 10, Fachrichtung Maschinenbau; Reifeprüfung mit gutem Erfolg; 1969: Standesbezeichnung „Ingenieur“

Universitäre Bildung:

1957 – 1963: Hochschule für Welthandel in Wien , Abschluß mit Diplom; Staatsprüfungen und Diplomarbeit mit durchschnittlich gutem Erfolg, Sprachen: Englisch und Spanisch
1977 – 1979: Wirtschaftsuniversität Wien: Doktorat der Handelswissenschaften Diss.: "Das Problem der Erschließung zusätzlicher Marktpotentiale für den Schlafwagenverkehr in Westeuropa", bei Univ.-Prof. Dr. Peter FALLER und Univ.-Prof. Dr. Paul BERNECKER
1979 – 1980: Wirtschaftuniversität Wien: Magister der Sozial- und Wirtschaftswissenschaften mit Abschlussprüfung bei Univ.-Prof. Dr. Karl. KORINEK
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Sonstige Ausbildung:

1966: Französisch (Alliance Francaise Paris und im Zuge der berufl. Tätigkeit);
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1975: Hernstein-Institut: "Der Verkaufschef";
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Diverse Ferialpraktiken in Österreich, Schweden und Spanien während des Studiums
1959/60: Vertragsassistent für Maschinenbau an der HTL Wien 10
1961 - 1962: Leiter der Übungen "Kalkulationstechnik" für ausländische Hörer an der Hochschule für Welthandel in Wien
1964: Fa. Ernst Katzinger, Fachunternehmen für moderne Bürotechnik als Organisator (Programmierer)

1964/65: Vertragslehrer (Prof.) an der HTL für Chemie in Wien 17, für maschinen-technische und kaufmännische Fächer
1965/66: 9 Monate Wehrdienst beim Österr. Bundesheer (Heeres-Sport- und Nahkampfschule)
1966 - 2001: Internationale Schlafwagen- und Touristik-Gesellschaft (Wagons-Lits)
1966: Gewerberechtlicher Geschäftsführer der industriellen Waggonreparaturwerkstätte und der industriellen Wäscherei, sowie der Bahnhofswerkstätten (ca. 140 Mitarbeiter);
1974: Leiter des Eisenbahnsektors (Schlaf- und Speisewagenbetrieb, Bahnhofrestaurants, Werkstätte und Wäscherei - ca. 800 Mitarbeiter);
1977 - 1988: Marketing-Chef für alle Unternehmensbereiche (Eisenbahnsektor, Reisebürokette Wagons-Lits, Hotellerie)
Nach Neustrukturierung der Firma: Seit Anfang 1989 wieder alleiniger technischer und kaufmännischer Leiter der industriellen Waggonreparaturwerkstätte und der industriellen Wäscherei, sowie der Bahnhofswerkstätten (ca. 100 Mitarbeiter);

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1984 - 1995 Vertragslehrer am Kolleg der Fremdenverkehrsschulen der Wiener Handelskammer, 2 Wochenstunden "Verkehrswirtschaftslehre" als Maturaausgangsfach und Maturaübertrittsfach
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1990 – 2003: Generalsekretär der Österreichischen Verkehrswissenschaftlichen Gesellschaft (ÖVG). Seither Ehrenmitglied und Träger des Ehrenringes der ÖVG)