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"Manifolds of J-holomorphic fibre bundle sections and their dimension"

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Markus Steenbock

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Contents

Introduction

Let N be an almost complex manifold of dimension n with almost complex structure J and let (Σ, j) be a Riemannian surface. A J-holomorphic curve is a map $\Sigma \to N$ which satisfies the Cauchy-Riemann equation $Tu + J(u)Tu \circ j = 0$. Every curve can be interpreted as a section in a trivial fibre bundle. In this picture, a J-holomorphic curve is a section u in the fibre bundle $\Sigma \times N$ over Σ, which satisfies the Cauchy-Riemann equation

$$pr_2 Tu + J(u) pr_2 Tu \circ j = 0,$$

a nonlinear first order partial differential equation. There is no reason to restrict to almost complex structures constant in $z \in \Sigma$. So let J be a section in $\Sigma \times L(TN)$ over Σ, restricting to an almost complex structure in each fibre. The projection $pr_2 : T\Sigma \times TN \to \Sigma \times TN$ is the canonical connection on $\Sigma \times N$. Let ϕ be a second connection. It makes sense to write down the Cauchy-Riemann equation with respect to the derivative $D_\phi := \phi \circ T$, i.e.

$$D_\phi u + J(u) D_\phi u \circ j = 0.$$

In this diploma thesis we are interested in the space of solutions to this nonlinear partial differential equation. We show that the solution space is a finite dimensional manifold for almost all $\mathcal{C}^\infty$ connections (up to a set of first category in the space of $\mathcal{C}^\infty$ connections, a countable intersection of nowhere dense subsets). We can compute the dimension of its connected components with help of Riemann-Roch's theorem.

However, we cannot presume the space of J-holomorphic curves - the solution space at the canonical connection pr_2 - to be a finite dimensional manifold, but it has been shown that this is possible for almost all almost complex structures on N, see [19].

For the development of the theory it is necessary to equip spaces of continuously differentiable and Sobolev class sections in fibre bundles with the geometric structure of infinite dimensional Banach manifolds. We use Γ as place-holder for spaces sections of differentiability class $\mathcal{C}^l$ or $H^{l,p}$, for $lp > 2$, where a section is in Γ if local representations are in this class.

Once we have done this, differentiation of sections can be understood as a section in the vector bundle $\Gamma(T^*\Sigma \otimes TN)$ of Banach spaces $\Gamma(T^*\Sigma \otimes u^*TN)$ over the Banach manifold $\Gamma(\Sigma \times N)$. Thus a partial differential operator may be understood as a vector bundle section. The Cauchy-Riemann equation is a section in the vector bundle $\Gamma((T^{1,0}\Sigma)^* \otimes_J TN)$ of J-antilinear sections. If this section is transversal to the zero section and the implicit function theorem can be applied, the solution space would be an infinite dimensional manifold. We can however not expect so; but we can perturb the equation, aiming at producing a section in a vector bundle over $\Gamma(\Sigma \times N) \times \mathfrak{P}$, which is transversal to the zero section. The parameter space $\mathfrak{P}$ will typically be a Banach manifold or Banach space.

The equation we are studying is parametrized on the space of connections on $\Sigma \times N$, an affine space modelled on the space of sections in the bundle $T^*\Sigma \otimes TN$ over $\Sigma \times N$. For all connections ϕ on $\Sigma \times N$, there is a section A in $T^*\Sigma \otimes TN$ such that $\phi = pr_2 + A$. This yields that the Cauchy-Riemann equation defines a section

$$(u, A) \mapsto pr_2 Tu + J(u) pr_2 Tu \circ j + A(u) + J(u) A(u) \circ j$$

in the vector bundle $\Gamma((T^{1,0}\Sigma)^* \otimes_J TN) \times \Gamma(T^*\Sigma \otimes TN) \to \Gamma(\Sigma \times N) \times \Gamma(T^*\Sigma \otimes TN)$, which is transversal to the zero section.

The differential of the restriction of this section to $\Gamma(\Sigma \times N)$ at u is a first order elliptic operator, whose first order part is a Cauchy-Riemann operator, for all u. Thus the differential at u induces a Fredholm operator on Sobolev spaces, a linear operator between Banach spaces whose kernel and cokernel are finite dimensional subspaces. Then the preimage of the zero section is a Banach manifold by the implicit function theorem. An application of Smale's Sard theorem implies that the solution space of the Cauchy-Riemann equation is a finite dimensional manifold for all connections in a set with complement of first category. We say that a connection is generic, if it is is in a set with complement of first category.

The dimensions of the connected components of the solution space at u coincide with the index of the Cauchy-Riemann operator. The index of a Fredholm operator is the dimension of the kernel minus the dimension of the cokernel.
The index of a Cauchy-Riemann operator on a Riemannian surface can be computed by Riemann-Roch's theorem. Let g be the genus of the Riemannian surface. Then

$$n(1-g) + \big\langle c_1(u^*TN), [\Sigma] \big\rangle$$

is the dimension of the connected component of u in the solution space at a generic $\mathcal{C}^\infty$ connection.

There is no reason to restrict to trivial fibre bundles, so we will develop the theory for fibre bundles $P \to \Sigma$, which admit a vertical almost complex structure, a vector bundle endomorphism of the vertical bundle restricting to almost complex structures on the fibres of P. Then

$$n(1-g) + \big\langle c_1(u^*VP), [\Sigma] \big\rangle$$

is the dimension of the connected component of u in the solution space at a generic $\mathcal{C}^\infty$ connection.

The first chapter introduces the reader to the theory of Fredholm maps we need. Fredholm maps are continuously differentiable maps between Banach manifolds with local representations whose differential is a Fredholm operator for each fibre. The index of a Fredholm map is the index of its differential. Sard's theorem generalizes to Fredholm maps between Banach manifolds. This generalization is known as Smale's Sard theorem. Along with it, transversality results known from finite dimensional geometry carry over.
More general, a Fredholm section in a vector bundle of Banach spaces over a Banach manifold is a section having local representations which are Fredholm maps. A perturbed Fredholm section is a section in a vector bundle of Banach spaces over a product of Banach manifolds $\mathfrak{M} \times \mathfrak{P}$ whose restriction to $\mathfrak{M} \times \{p\}$ is a Fredholm section for all $p \in \mathfrak{P}$. For all perturbed Fredholm sections transversal to the zero section, the preimage of the zero section is an infinite dimensional Banach manifold by the implicit function theorem. Moreover the projection of the zero space onto the parameter space is a Fredholm map whose index coincides with the index of the Fredholm section.
We can apply Smale's Sard theorem to the situation we just explained. Up to a set of first category in the parameter space, the preimage of the zero section at a fixed parameter is a finite dimensional manifold. The dimension of its connected components coincides with the index of the Fredholm section. If the

projection is proper, i.e. the solution space at a generic element in $\mathfrak{P}$ is compact, the solution space defines an element in the group of non-oriented cobordism classes. Compactness results, however, are beyond the scope of this work.

In the second chapter of this work, the infinite dimensional geometric set up will be established. Function spaces of class $\mathcal{C}^l$ and Sobolev spaces $H^{l,p}$ of vector bundle sections and maps are discussed. Let $lp > m$. Sobolev spaces of maps of class $H^{l,p}$ consist of functions or sections with local representations whose weak derivatives up to order l are L^p integrable. We have to assume that $lp > m$ because the Sobolev class of local representations is not compatible with chart changes otherwise. The Sobolev embedding theorem is needed here. All these spaces may be equipped with a canonical infinite dimensional manifold structure. Spaces of $\mathcal{C}^l$ or $H^{l,p}$ maps or sections are Banach manifolds.
Spaces of $\mathcal{C}^l$ and $H^{l,p}$ vector bundle sections over closed manifolds are separable Banach spaces. For closed manifolds M, $C^l(M,N)$ and $H^{l,p}(M,N)$ are paracompact Hausdorff spaces. Moreover, their topology is generated by a countable basis.
Let (P,p) be a fibre bundle over M. The space of sections $\Gamma(P \to M)$ is a closed subspace in $C(M,P)$. It is equipped with a canonical Banach manifold structure. An atlas is given as follows: The pullback of the vertical bundle VP along a smooth section u is a vector bundle over M. There are bundle diffeomorphisms which map this bundle to a neighbourhood of the image of u in P. Composition with such diffeomorphisms is a chart for $\Gamma(P)$.
The collection of all such maps is an atlas centred at smooth sections. The chart changes are $\mathcal{C}^\infty$. The main argument is that if a smooth fibre preserving map ϕ over the identity between vector bundles E and F is given, then the composition with ϕ is smooth $\Gamma(E) \to \Gamma(F)$. Especially for Sobolev spaces this statement is non trivial as the Sobolev embedding theorem comes in.
If E is a vector bundle over P, $\Gamma(E \to M)$ is a vector bundle over $\Gamma(P)$. We show that $\Gamma(VP)$ is canonically isomorphic to the tangent bundle of $\Gamma(P)$.

In the third chapter, we treat the Cauchy-Riemann equations on fibre bundle sections. As the space of connections is affine, the derivative in the direction of a connection is very easy to compute.

In the appendix, the index of the Cauchy-Riemann operator is computed. Atiyah-Singer's index theorem for Dirac operators is explained. We sketch a proof using the heat kernel method, as it has been presented in [26]. Atiyah-Singer's index theorem implies Riemann-Roch's theorem.

The reader should have a good background in differential geometry. The necessary material is completely covered in [23] for example. Moreover, the reader should already have had contact with the theory of (elliptic) partial differential equations and Sobolev spaces.

1 Fredholm Maps and Sard's Theorem

In this chapter we introduce the concept of Fredholm maps f between Banach manifolds. Fredholm maps are important as they allow to reduce infinite dimensional to finite dimensional problems. Theorems on transversal maps such as Sard's theorem are generalized to the setting of Banach manifolds and Fredholm maps. Thus the set of critical values of a smooth Fredholm map is a set of first category; this is a countable union of nowhere dense subsets. In combination with the implicit function theorem we can conclude that the preimage $f^{-1}(y)$ is a finite dimensional manifold up to a set of first category (1.0.21). Its dimension equals the index of f. The aim of this chapter is the presentation of these results, which are due to S. Smale. In view of the application to nonlinear partial differential equations, we further investigate perturbed Fredholm sections in vector bundles of Banach spaces. If such a section is transversal to the zero section, the solution space at some fixed parameter is a finite dimensional manifold up to a set of first category in the parameter space. Moreover the dimension of this manifold coincides with the index of the Fredholm section (1.0.25).
References for this chapter are the paper [34] by S. Smale, [36] and [4, 4.3].

We assume all manifolds to be Hausdorff and second countable, i.e. manifolds have a countable basis for their topology. In this section $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathcal{C}^\infty$ Banach manifolds. We start with a short introduction of Banach manifolds and vector bundles over Banach manifolds. More can be found in [16] and [31]. In [16], the reader may also find an introduction on calculus in Banach spaces.

1.0.1 Definition (Banach manifold)*.* A Banach manifold $\mathfrak{M}$ is a Hausdorff manifold locally homeomorphic to a Banach space, that is, for all $x \in \mathfrak{M}$ there is a open neighbourhood $U \subseteq \mathfrak{M}$ of x, a open neighbourhood V in a Banach space, and a homeomorphism $\mathfrak{m} : U \to V$. The pair $(U, \mathfrak{m})$ is called chart for $\mathfrak{M}$ at x then. A collection $(U_i, \mathfrak{m}_i)$ of charts covering $\mathfrak{M}$ is a $\mathcal{C}^l$ atlas for $\mathfrak{M}$, if for all charts the chart change $\mathfrak{m}_i \circ \mathfrak{m}_j^{-1}$ is a $\mathcal{C}^l$ diffeomorphism. $\mathfrak{M}$ is a $\mathcal{C}^l$ or $\mathcal{C}^\infty$ Banach manifold if it is equipped with a $\mathcal{C}^l$ or $\mathcal{C}^\infty$ atlas. Moreover, we assume that Banach manifolds are second countable, i.e. their topology has a countable basis.

A submanifold $\mathfrak{A}$ in $\mathfrak{M}$ is a subset such that for all $x \in \mathfrak{A}$ there are charts $(U, \mathfrak{m})$ of $\mathfrak{M}$ whose restriction to $\mathfrak{A} \cap U$ maps to a splitting subspace of the local model at $x \in \mathfrak{M}$. So $\mathfrak{m}(U) = V_1 \times V_2$ and $\mathfrak{m} : \mathfrak{A} \cap U \to V_1$ is a homeomorphism. Such charts are referred to as submanifold charts for $\mathfrak{A}$.

A map $f : \mathfrak{M} \to \mathfrak{N}$ is $\mathcal{C}^l$ if for all $x \in \mathfrak{M}$, there is a chart $(U, \mathfrak{m})$ at x and a chart $(V, \mathfrak{n})$ at $f(x)$, such that $f(U) \subseteq V$ and the local representation $f_{UV} := \mathfrak{n} \circ f \circ \mathfrak{m}^{-1}$ is $\mathcal{C}^l$.

1.0.2 (Vector bundles)*.* A $\mathcal{C}^l$ vector bundle over $\mathfrak{M}$, with fibres Banach spaces E, is a $\mathcal{C}^l$ manifold $\mathcal{E}$ and a $\mathcal{C}^l$ projection $\pi : \mathcal{E} \to \mathfrak{M}$, such that for all $x \in \mathfrak{M}$ there are fibrewise linear $\mathcal{C}^l$ vector bundle charts, i.e. $\mathcal{C}^l$ diffeomorphisms $\psi_i^{\mathcal{E}} : U_i \times E_i \to \pi^{-1}(U_i) = \mathcal{E}|_{U_i}$, such that $pr_1 = \pi \circ \psi^{\mathcal{E}}$, the transition functions $\psi_{ji}^{\mathcal{E}} := (\psi_j^{\mathcal{E}})^{-1} \circ \psi_i^{\mathcal{E}} : U_i \cap U_j \to \mathcal{L}_b(E_i, E_j)$ are $\mathcal{C}^l$, and $\psi_{ji}^{\mathcal{E}}(x)$ are linear and continuous isomorphisms $E_i \to E_j$ for all $x \in U_i \cap U_j$.
The transition functions satisfy the cocycle conditions $\psi_{ij}^{\mathcal{E}}(x)\psi_{jl}^{\mathcal{E}}(x) = \psi_{il}^{\mathcal{E}}(x)$, $\psi_{ii}^{\mathcal{E}}(x) = id_E$ and $\psi_{ij}^{\mathcal{E}}(x) = (\psi_{ji}^{\mathcal{E}}(x))^{-1}$.

On the other hand, if we are given a covering $\mathfrak{U} = (U_i)$ of $\mathfrak{M}$, Banach spaces E_i, and a family of $\mathcal{C}^l$ functions $\psi_{ij}^{\mathcal{E}} : U_i \cap U_j \to \mathcal{L}_b(E_i, E_j)$, such that $\psi_{ij}^{\mathcal{E}}(x)$ are linear and continuous isomorphisms for all x, and satisfy the cocycle conditions, then we may reconstruct a $\mathcal{C}^l$ vector bundle structure. Namely, we define the fibres $E_x := \{(x, U, v) : x \in U \in \mathfrak{U}, v \in E\} \big/ \sim$, where the equivalence relation $\sim$ is defined by $(x, U_i, v_i) \sim (x, U_j, v_j) \Leftrightarrow v_i = \psi_{ij}^{\mathcal{E}}(x)(v_j)$. Then E_x carries a Banach space structure, induced by the bijection $E_x \to E, [(x, U, v)] \mapsto v$. $\mathcal{E} := \bigsqcup_{x \in \mathfrak{M}} E_x$ together with the obvious projection onto $\mathfrak{M}$ is a $\mathcal{C}^l$ vector bundle, which is unique up to isomorphisms. Vector bundle charts are given by $\psi_i^{\mathcal{E}} : U_i \times E \to \pi^{-1}(U_i), (x, v) \mapsto [(x, U_i, v)]$, which are $\mathcal{C}^l$, because $(\psi_j^{\mathcal{E}})^{-1} \psi_i^{\mathcal{E}} : (x, v) \mapsto [(x, U_i \cap U_j, v)] = [(x, U_i \cap U_j, \psi_{ji}^{\mathcal{E}}(x) v] \mapsto (x, \psi_{ji}^{\mathcal{E}}(x) v)$ is $\mathcal{C}^l$. [16, III§1 1.2], cf. [23, 8.3], [14, 25.7 6.].

1.0.3 Definition (Tangent bundle). Let $(U_i, \mathfrak{m}_i)$ be a $\mathcal{C}^l$ atlas for a Banach manifold $\mathfrak{M}$. Then the differentials $d\left(\mathfrak{m}_j \mathfrak{m}_i^{-1}\right) : \mathfrak{m}_i(U_i \cap U_j) \to \mathcal{L}_b(E)$ are $\mathcal{C}^{l-1}$ and satisfy the cocycle conditions. The tangent bundle $T\mathfrak{M}$ is the vector bundle over $\mathfrak{M}$ with respect to this transition functions. Its fibres $T_x\mathfrak{M}$ are the tangent spaces of $\mathfrak{M}$. The equivalence classes in $T_x\mathfrak{M}$ are called tangent vectors.

If $f : \mathfrak{M} \to \mathfrak{N}$ is $\mathcal{C}^l$, $T_x f : T_x\mathfrak{M} \to T_{f(x)}\mathfrak{N}$ is the map which maps $[(x, U, v)] \to [(f(x), V, df_{UV}(x)v)]$, whenever $f(U) \subseteq V$, where f_{UV} is the local representation $\mathfrak{n} \circ f \circ \mathfrak{m}^{-1}$ of f at x. The tangential $Tf : T\mathfrak{M} \to T\mathfrak{N}$ is fibrewise given by $T_x f$. Local representations of Tf from $U \times E \to V \times F$ are given by $(x, v) \mapsto (f_{UV}(x), df_{UV}(x)v)$.

We recall the inverse and implicit function theorem for mappings between Banach spaces and refer to [15] or [7] for a discussion on differentiability beyond Banach spaces and for a more general version of the implicit function theorem.

1.0.4 Theorem (Inverse function theorem)**.** *[16, thm. 5.2] Let $f : E \supseteq_{open} U \to F$ a $\mathcal{C}^1$ function between Banach spaces, such that at a point $x_0 \in U$ the differential $df(x_0)$ is a bounded and invertible linear map. Then f is a local $\mathcal{C}^1$ diffeomorphism. If f is $\mathcal{C}^l$, f is a local $\mathcal{C}^l$ diffeomorphism. If f is $\mathcal{C}^\infty$, f is a local $\mathcal{C}^\infty$ diffeomorphism.*

1.0.5 Theorem (Implicit function theorem)**.** *[16, thm. 5.9] Let $f : E \times F \to F$ be $\mathcal{C}^1$ in a neighbourhood of $(x_0, y_0) \in E \times F$ and such that $d_2 f(x_0, y_0) : F \to F$ is an isomorphism. Then there are neighbourhoods U of x_0, V of y_0 and W of $f(x_0, y_0)$ and a $\mathcal{C}^1$ mapping $y : U \times W \to V$, $(x, z) \to y$ such that $f(x, y(x, z)) = z$. If f is $\mathcal{C}^l$, y is $\mathcal{C}^l$. If f is $\mathcal{C}^\infty$, y is $\mathcal{C}^\infty$.*
If $f(x, y) = z_0$, the neighbourhood U of x_0 can be chosen so that $y : U \times \{z_0\} \to V$ is unique.

1.0.6 Remark. Let $f : E \times F \to F$, so that $f(x, y) = z$ is a regular equation. Then $f^{-1}(z)$ is a submanifold in $E \times F$.

1.0.7 Definition (Fredholm operator). A *Fredholm operator* is a linear and bounded operator between Banach spaces, such that kernel and cokernel are finite dimensional subspaces.

$\mathfrak{F}(E, F)$ denotes the subspace of Fredholm operators in the space of linear and bounded operators $\mathcal{L}_b(E, F)$ between Banach spaces E and F.

1.0.8 Remark. [33, VII, theorem 1] The range of a Fredholm operator is closed.

1.0.9 Definition (Index). The index of $f \in \mathfrak{F}(E, F)$ is the number

$$\operatorname{index}(f) := \dim(\ker(f)) - \dim(\operatorname{coker}(f)).$$

We state some important properties of Fredholm operators and the index.

1.0.10 Proposition (Properties of Fredholm operators). [17, III][33, VII] Let E, F and G be Banach spaces.

(1) $\mathfrak{F}(E,F)$ is open in $\mathcal{L}_b(E,F)$.

(2) Fredholm operators are exactly those operators which are invertible modulo compact ones: A is a Fredholm operator iff there is a Fredholm operator B and compact operators K_1, K_2 such that $AB = id - K_1$ and $BA = id - K_2$.

(3) The index is constant on connected components of $\mathfrak{F}(E,F)$. This means that index : $\mathfrak{F}(E,F) \to \mathbb{Z}$ is continuous.

(4) The composition $AB : E \to G$ of Fredholm operators $A : D \to F$ and $B : F \to G$ is again Fredholm and $\text{index}(AB) = \text{index}(A) + \text{index}(B)$.

1.0.11 Remark. Let A be a Fredholm operator. For all compact operators K, $A + K$ is a Fredholm operator of index(A).

1.0.12 Remark. [33, VII, theorem 2] We can strengthen one direction of (2). If A is a Fredholm operator there is an operator B such that $AB - id$ and $BA - id$ are projections onto finite dimensional subspaces.

1.0.13 Definition (Fredholm map). A $\mathcal{C}^1$ map $f : \mathfrak{M} \to \mathfrak{N}$ between Banach manifolds is called *Fredholm map* if its differential $T_x f : T_x\mathfrak{M} \to T_{f(x)}\mathfrak{M}$ is a Fredholm operator for all $x \in \mathfrak{M}$. The index of f at x is the number $\text{index}_x(f) = \text{index}(T_x f)$. $\text{index}_x(f)$ is locally constant on $\mathfrak{M}$, as the index is continuous $\mathfrak{F} \to \mathbb{Z}$.

Fredholm maps are important in nonlinear and infinite dimensional geometry because the implicit function theorem can often be applied. The following are immediate consequences of the inverse or implicit function theorem.

1.0.14 Lemma. [4, 4.2.19] [36, 17.4] Let $l > 0$ and $f : \mathfrak{M} \to \mathfrak{N}$ be a $\mathcal{C}^l$ Fredholm map.
For all $x \in \mathfrak{M}$, there are charts such that the local representation of f at x is of the form

$$\begin{array}{ccc} \ker(T_x f) \times \text{range}(T_x f) \supseteq U & \to & \text{coker}(T_x f) \times \text{range}(T_x f) \\ (y^1, y^2) & \mapsto & (\phi(y^1, y^2), y^2) \end{array},$$

where $\phi : \ker(T_x f) \times \text{range}(T_x f) \to \text{coker}(T_x f)$ is $\mathcal{C}^l$. Consequently

$$df_i = \begin{pmatrix} d_1\phi & d_2\phi \\ 0 & id \end{pmatrix}.$$

Proof. Let f be a local representation at x_0. As the kernel of $T_{x_0} f$ is finite dimensional we can split $E = T_{x_0}\mathfrak{M}$ into two closed subspaces, i.e. $E = \ker(T_{x_0} f) \times E_0$ and $x = (x^1, x^2)$. Similarly $F = T_{f(x_0)}\mathfrak{N} = \text{coker}(T_{x_0} f) \times \text{range}(T_{x_0} f)$. As usual we denote $f_1 : E \supseteq U \to \text{coker}(T_{x_0} f)$ and $f_2 : E \supseteq U \to \text{range}(T_{x_0} f)$.
Define

$$\varphi^{-1} : (x^1, x^2) \mapsto (x^1, f_2(x^1, x^2)).$$

As $d_2 f_2(x_0)$ is invertible, $d\varphi^{-1}(x_0)$ is invertible. φ is a local diffeomorphism by the inverse function theorem.
$f \circ \varphi$ is of the required form. Indeed let $\varphi(y^1, y^2) = (x^1, x^2)$. Then

$$(y^1, y^2) = \varphi^{-1}(x^1, x^2) = \big(x^1, f_2(x^1, x^2)\big)$$

and thus

$$y^1 = x^1; y^2 = f_2(x^1, x^2) = f_2(\varphi(y^1, y^2)).$$

Consequently

$$(f \circ \varphi)(y^1, y^2) = \big(f_1(\varphi(y^1, y^2)), f_2(\varphi(y^1, y^2))\big) = \big((f_1 \circ \varphi)(y^1, y^2), y^2\big).$$

Set $\phi = f_1 \circ \varphi$. □

1.0.15 Lemma. [34, 1.6] Let $f : \mathfrak{M} \to \mathfrak{N}$ be a $\mathcal{C}^1$ Fredholm map. Then f is locally proper, i.e. for all $x \in \mathfrak{M}$ there is a neighbourhood U of x such that the restriction of f to U is proper.

Proof. [34] Let f be a local representation. The partial $d_2 f(x_0^1, x_0^2) : E_0 \to F$ is continuous and invertible. By lemma 1.0.14 we can produce a neighbourhood $K \times W \subseteq \ker(T_{x_0} f) \times E_0$ of (x_0^1, x_0^2) such that K is compact and $f(k, .)$ is the identity for each $k \in K$. This implies that the preimage of convergent sequences have a convergent subsequence and hence imply that f restricted to $K \times W$ is proper. □

1.0.16 Definition (Transversality). [23, 2.16] Let $f_1 : \mathfrak{M}_1 \to \mathfrak{N}$ and $f_2 : \mathfrak{M}_2 \to \mathfrak{N}$ be $\mathcal{C}^1$. We say that f_1 is *transversal* to f_2 at $y \in \mathfrak{N}$, if

$$T_{x_1} f_1(T_{x_1}\mathfrak{M}_1) + T_{x_2} f_2(T_{x_2}\mathfrak{M}_2) = T_y\mathfrak{N}, \text{ whenever } y = f_1(x_1) = f_2(x_2).$$

If f_1 is transversal to f_2 at all $y \in \mathfrak{N}$, we say that f_1 is transversal to f_2.

$f : \mathfrak{M} \to \mathfrak{N}$ is called transversal to a submanifold $\mathcal{I}$ in $\mathfrak{M}$ if f is transversal to the natural embedding of $\mathcal{I}$.

1.0.17 Proposition. Let $\mathcal{I}$ be a m dimensional submanifold in $\mathfrak{N}$. Let $f : \mathfrak{M} \to \mathfrak{N}$ be a Fredholm map transversal to $\mathcal{I}$. Then $f^{-1}(\mathcal{I})$ is a submanifold of $\mathfrak{M}$. Its dimension equals $\operatorname{index}(f) + m$.

Proof. The proof is similar to the one for the classic case [14, 21.22]. We may assume $\mathfrak{M}$ connected. Let F be the local model of $\mathfrak{N}$. Let Ψ be a submanifold chart for $\mathcal{I}$ centred at $f(x) \in \mathcal{I}$.
Locally $x' \in f^{-1}(f(x))$ means that $\pi\Psi f(x') = 0$, where π is the projection onto the complement of $\mathbb{R}^m$ in F. This equation is regular by transversality. $\pi\Psi f$ is a Fredholm map with $\operatorname{index}(\pi\Psi f) = \operatorname{index}(\pi) + \operatorname{index}(f)$. The implicit function theorem now implies, that $f^{-1}(\mathcal{I})$ is an $\operatorname{index}(f) + m$ dimensional submanifold of $\mathfrak{M}$. □

We use these results to generalize important classic results on regular values and transversal maps to the Fredholm setting. The main theorem 1.0.21 is a generalization of Sard's theorem. We recall the the following facts from [14, 178ff] and [10].

1.0.18 Theorem (Sard). *If M and N are finite dimensional manifolds and $f : M \to N$ l-times differentiable such that $l > \max\{0, m-n\}$, then the critical values of f is a zero set.*
The set of critical values of a smooth map between paracompact finite dimensional manifolds is a zero set.

A consequence of this is: *If $m < n$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ is $\mathcal{C}^1$, $f(\mathbb{R}^m)$ has zero measure in $\mathbb{R}^n$.*
If $m \geq n$, the preimage of a regular value of a $\mathcal{C}^l$-map is a regular $\mathcal{C}^l$ submanifold of dimension $m-n$ by the implicit function theorem.

1.0.19 Definition. Let $f : \mathfrak{M} \to \mathfrak{N}$ be a $\mathcal{C}^1$ map.

We say that $x \in \mathfrak{M}$ is a *regular point* if $T_x f : T_x\mathfrak{M} \to T_{f(x)}\mathfrak{N}$ is surjective. Otherwise x is called *critical point*.

We say that $y \in \mathfrak{N}$ is a *regular value* if f is transversal to y. Otherwise y is called *critical value*.

1.0.20 Remark. The set of critical points of a Fredholm map is closed, because the set of surjective, linear and bounded maps is open in $\mathcal{L}_b$.

1.0.21 Theorem (Smale-Sard). *[34, 1.3]*
Let $\mathfrak{M}$ and $\mathfrak{N}$ be connected Hausdorff Banach manifolds with a countable basis for the topology. Let $f : \mathfrak{M} \to \mathfrak{N}$ be a $\mathcal{C}^l$ Fredholm map.

(1) If $\infty \geq l > \max\{0, \mathrm{index}(f)\}$, the set of critical values of f are of first category in $\mathfrak{N}$. That means it is a countable union of nowhere dense sets in $\mathfrak{N}$.

(2) $\mathrm{range}(f)$ *is of first category if* $\mathrm{index}(f) < 0$.

(3) If $\infty \geq l > \max\{0, \mathrm{index}(f)\}$, $f^{-1}(y)$ is a finite dimensional $\mathcal{C}^l$-submanifold of $\mathfrak{M}$ for all y in the complement of a set of first category in $\mathfrak{N}$. Then we say that $f^{-1}(y)$ is a finite dimensional manifold up to a set of first category in $\mathfrak{N}$. The dimension of this manifold equals $\mathrm{index}(f)$*, if $y \in$* $\mathrm{range}(f)$*. Otherwise, it is empty.*

Proof. Let $f_i : U_i \to V_i \subseteq F = T_{f(x_0)}\mathfrak{N}$ be the local representation of lemma 1.0.14 at x_0 for charts U_i for $\mathfrak{M}$ and V_i for $\mathfrak{N}$. f_i is of the form

$$(x^1, x^2) \mapsto (\phi(x^1, x^2), x^2)$$

for a smooth map $\phi : U_i \to \mathrm{coker}(T_{x_0} f) \cap V_i$. Let π denote the projection $F \to \mathrm{coker}(T_{x_0} f)$.

We show that the critical values of f_i are of first category in V_i.
Let $x \in U_i$. Then every open neighbourhood $V \subseteq V_i$ of $f_i(x)$ contains a regular value: If the assumptions on the differentiability of f are satisfied, by Sard's theorem the set of critical values for

$$\phi(., x^2) : \big(\ker(T_{x_0} f) \cap U_i\big) \times \{x^2\} \to \mathrm{coker}(T_{x_0} f) \cap V_i$$

is a zero set. The set V_ϕ^r of regular values of $\phi(., x^2)$ is open and dense in $\pi(V_i)$. If $\phi(z, x^2)$ is regular for $\phi(., x^2)$, $y = (\phi(z, x^2), x^2)$ is a regular value for f_i. In fact $d_1\phi(z, x^2)$ is surjective then. Thus $\pi^{-1}(V_\phi^r) \cap V$ is an open and nonempty set of regular values of f.

As the critical points of f are closed, the critical values of f are closed by 1.0.15. Together this implies that the set of regular values of $f_i : U_i \to V_i$ is open and dense in V_i; its complement, the set of critical values consequently is of first category.

Since the topology of $\mathfrak{N}$ is countably generated and a countable union of sets of first category is of first category, the theorem is proved. □

As a first application of theorem 1.0.21 we have

1.0.22 Theorem (Transversality Theorem)**.** *[34, 3.3] Let P be a m dimensional $\mathcal{C}^1$ manifold. Let ι be an embedding of P in $\mathfrak{N}$ and $f : \mathfrak{M} \to \mathfrak{N}$ Fredholm of class $\mathcal{C}^l$, such that $l > \max\{\mathrm{index}(f) + m, 0\}$.*
Then ι can be approximated by a $\mathcal{C}^1$ map ι', such that f is transversal to $\iota'(P)$.

Proof. Let $\mathfrak{V}$ be a countable, locally finite atlas of $\mathfrak{N}$ and $\mathfrak{U}$ an atlas for $\mathfrak{M}$ f-adapted, so that the image of a neighbourhood in $\mathfrak{U}$ is contained in a neighbourhood in $\mathfrak{V}$.
Let $\mathfrak{W}$ be a atlas for P, such that $V_i = \iota_i(W_i) \times V$ and $\iota_i(W_i) = (\iota_i(W_i), 0)$, where V is an open subset of a Banach space. Let $(\widetilde{W}_i, \mathfrak{w}_i)$ be a subordinated $\mathcal{C}^l$-partition of unity.
Let $\widetilde{V} \subseteq V$ be open. Since $l > \max\{\mathrm{index}(f) + m, 0\}$, we can use the Smale-Sard's theorem for the map $pr_2 \circ f_i : U_i \to \widetilde{V}$ and choose a regular value $z \in \widetilde{V}$.
Now define $\iota_i' := \mathfrak{w}_i(\iota_i + z)$. ι_i' is transversal to f_i.
Then $\iota' = \sum_i \iota_i'$ is a $\mathcal{C}^1$ approximation of ι such that f is transversal to ι'. □

The following idea to find a mapping degree is described in the paper of Smale in (3.5).

1.0.23 Theorem. *[34, 3.5] Let $f : \mathfrak{M} \to \mathfrak{N}$, $\mathfrak{N}$ connected, be a proper $\mathcal{C}^l$ Fredholm map with* $\mathrm{index}(f) = m$ *and y a regular value. If $l > m$, the non-oriented cobordism class of $f^{-1}(y)$ in the group of non-oriented cobordism classes of manifolds of dimension m is independent of the regular value. If it is not zero, f is surjective.*

Sketch. By the transversality theorem, we can find a path $\mathfrak{c}$ connecting two different regular values y_0 and y_1, such that f is transversal to $\mathfrak{c}$. By proposition 1.0.17 $f^{-1}(\mathfrak{c})$ is a $(m+1)$-dimensional compact manifold relating the compact manifolds $f^{-1}(y_0)$ and $f^{-1}(y_1)$. □

1.0.24 Definition. Let $\mathcal{E}$ be a vector bundle of Banach spaces over $\mathfrak{M}$. A *Fredholm section* in $\mathcal{E}$ is a section $\Phi : \mathfrak{M} \to \mathcal{E}$ such that local representations of Φ are Fredholm maps $U \to \mathcal{E}_x$.

It need not to be true, that any local representation of a Fredholm section is a Fredholm map. At zero however, the index of a section does not depend on the choice of a chart.
In fact, let $s : U \to E$ be a local representation of a Fredholm section, $x \in s^{-1}(0)$, and let $\phi \in \mathcal{G}l(E)$ be a transition function. Then $d(\phi \cdot s)(x) = d\phi(x)(s(x)) + \phi\, ds(x) = 0 + \phi(x)ds(x)$, whose index is the index of ds at x.

Elliptic partial differential operators may be interpreted as Fredholm sections in a vector bundle $\mathcal{E} \to C(M, N)$ of Banach spaces. We can disturb some parameter $y \in P$ of the partial differential equation in order to get a section in a vector bundle over the manifold $C(M, N) \times C(M, P)$ which is transversal to the zero section. The following theorem is an application of the above theory to this situation.

1.0.25 Theorem. *[4, 4.3] Let $\mathfrak{M}$ and $\mathfrak{P}$ be Banach manifolds. Let $\mathcal{E} \to \mathfrak{M} \times \mathfrak{P}$ be a vector bundle. Suppose that there is a section Φ in $\mathcal{E} \to \mathfrak{M} \times \mathfrak{P}$, whose restriction to $\mathfrak{M} \times \{p\}$ is a $\mathcal{C}^l$ Fredholm section in $\mathcal{E}|_{\mathfrak{M}\times\{p\}}$ for all $p \in \mathfrak{P}$. Moreover assume Φ transversal to the zero section.*
Then $\Phi^{-1}(0)$ is a $\mathcal{C}^l$ submanifold in $\mathfrak{M} \times \mathfrak{P}$. Its tangent space over (x, y) is the kernel of $d\Phi_{x,y} : T_x\mathfrak{M} \times T_y\mathfrak{P} \to \mathcal{E}_{x,y}$. The projection $\Pi : \Phi^{-1}(0) \to \mathfrak{P}$ is a Fredholm map. Its index equals the index of Φ_y.
Thus, if $l > \max\{0, \mathrm{index}(\Phi_y)\}$, $\Pi^{-1}(y)$ is a $\mathcal{C}^l$ manifold of dimension $\mathrm{index}(\Phi_y)$ for all y contained in the complement of a set of first category in $\mathfrak{P}$.

Proof. The preimage of the zero section is a submanifold by the implicit function theorem. Let $(x, y) \in \Phi^{-1}(0)$. To apply the implicit function theorem, $\ker(d\Phi_{x,y})$ has to be a splitting subspace. Let $A : T\mathfrak{P} \to \mathcal{E}$ be the vector bundle map such that the diagram given on the right commutes. Let $T_x\mathfrak{M} = E_0 \oplus \ker(d_1\Phi_{(x,y)})$. The fibre $\mathcal{E}_{(x,y)}$ splits into $\mathrm{range}(d_1\Phi_{(x,y)})$ and the finite dimensional subspace $\mathrm{coker}(d_1\Phi_{(x,y)})$. By surjectivity of $d\Phi_{(x,y)}$, $\mathrm{coker}(d_1\Phi_{(x,y)})$ is contained in the image of A. There is a finite dimensional subspace F in $T_y\mathfrak{P}$ such that A maps F isomorphically onto $\mathrm{coker}(d_1\Phi_{(x,y)})$.

$$\begin{tikzcd}
 & & 0 & & \\
 & & \mathcal{E}_{(x,y)} \arrow[u] & & \\
T_y\mathfrak{P} \arrow[urr, dotted, "A"] & & T_x\mathfrak{M} \times T_y\mathfrak{P} \arrow[ll, "pr_2"'] \arrow[rr, "pr_1"] \arrow[u, "d\Phi_{(x,y)}"] & & T_x\mathfrak{M} \arrow[ull, "d_1\Phi_{(x,y)}"'] \\
 & & \ker(d\Phi_{(x,y)}) \arrow[ull, "d\Pi_{(x,y)}"] \arrow[u, hook] \arrow[urr] & &
\end{tikzcd}$$

$E_0 \oplus F$ has a topological complement. Moreover, $E_0 \oplus F$ is a topological complement to the kernel of $d\Phi_{(x,y)}$. In fact $d\Phi_{(x,y)} : E_0 \oplus F \to \mathcal{E}_{(x,y)}$ is surjective.
Since the kernel of $d\Phi_{x,y}$ is in the complement of $E_0 \oplus F$, it is also injective.

We need to proof that the projection $\Pi : \Phi^{-1}(0) \to \mathfrak{P}$ is a Fredholm map. The remaining statements then follow from Smale-Sard's theorem.
The kernel of $d\Phi_{(x,y)}$ contains $\{0\} \times \ker(d_1\Phi_{(x,y)})$. Hence, the dimensions of the kernel of both, $d\Pi_{(x,y)}$ and $d_1\Phi_{(x,y)}$, equal. We have that

$$A(d\Pi_{(x,y)}(\xi, \eta)) = d_1\Phi_{(x,y)}(\xi),$$

iff $(\xi, \eta) \in \ker(d\Phi_{(x,y)})$. Thus $A(\mathrm{coker}(d\Pi_{(x,y)}))$ equals $\mathrm{coker}(d_1\Phi_{(x,y)})$. The dimension of the cokernel of $d\Pi_{(x,y)}$ could exceed the dimension of $\mathrm{coker}(d_1\Phi_{(x,y)})$. By surjectivity of $d\Phi_{(x,y)}$ the dimensions coincide. In fact, as $E_0 \oplus F$ is a complement to $\ker(d\Phi_{(x,y)})$, F is a complement to $d\Pi_{(x,y)}\big(\ker(d\Phi_{(x,y)})\big)$. $\square$

2 Spaces of Sections and Maps

In this chapter we explain a setting of infinite dimensional geometry suitable for the theory of nonlinear partial differentiable equations on manifolds in the next chapter.

Let M be a m dimensional closed manifold. Let E be a vector bundle over M and N a finite dimensional paracompact manifold. The first two sections treat spaces of sections $\Gamma(E)$ and maps $C(M, N)$, where Γ and C are used as place-holders for one of the classes $\mathcal{C}^l$ or $H^{l,p}_{loc}$. A section or map is of class C if its local representations are of this class. One has to show that the class of a local representation does not change under chart changes, which is non trivial for Sobolev spaces of maps. In fact, we have to assume $l - m/p > 0$, for composition with a nonlinear smooth map preserves the Sobolev class and induces even a continuous map then. This uses the Sobolev embedding theorems. We discuss this result in the more general setting of Sobolev vector bundle sections in section 3 of this chapter.

We equip these spaces with a topology. Sobolev spaces are studied in most detail. $\Gamma^l(E)$ and $\Gamma^{l,p}(E)$ are Banach spaces. $\mathcal{C}^l(M, N)$ is equipped with the Whitney $\mathcal{C}^l$ topology. The topology on $H^{l,p}(M, N)$ is defined by specifying basic neighbourhoods of maps f, which consist of Sobolev maps with local representations having Sobolev distance at most ε to local representations of f. The condition $lp > m$, already assumed for the definition of the set of $H^{l,p}$ maps, is sufficient. The topology is well defined as composition with chart changes is continuous between spaces of local representations then. Both, $\mathcal{C}^l(M, N)$ and $H^{l,p}(M, N)$, are completely metrizable spaces and have a countable basis of their topology.
We extend the Sobolev embedding theorems to the Sobolev spaces of sections and maps in a natural way.
Spaces of smooth sections or maps are the intersection of $\mathcal{C}^l$ or $H^{l,p}$ spaces.

While the first part of this chapter introduces the topology of function spaces between manifolds, the second part treats the infinite dimensional geometry of these spaces.

We observe that composition with a smooth map ϕ between finite dimensional vector bundles is $\mathcal{C}^\infty$ between the spaces of vector bundle sections introduced before. The derivative is given by composition with the vertical derivative of ϕ. For this to hold in spaces of Sobolev class sections we need the Sobolev embedding theorems, or the general Hölder inequality which is implied by the latter. Therefore the assumption $lp > m$ is necessary.

We introduce canonical infinite dimensional $\mathcal{C}^\infty$ manifold structures on spaces of sections in fibre bundles. Let P be a fibre bundle over M. $\Gamma(P)$ is equipped with the trace topology as a closed subspace in $C(M, P)$. We construct charts for $\Gamma(P)$ centred at smooth sections $u \in \Gamma^\infty(P)$. We can locally think of a manifold as a of a vector space. Charts are for example given by the exponential mapping. Similarly we can locally think of fibre bundles as of vector bundles. The fibre exponential delivers a family of diffeomorphisms from u^*VP to open neighbourhoods V_u of $u(M)$ in P. This is possible in a way that V_u is a tubular neighbourhood of $u(M)$ and V_u does not depend on u. Composition with these diffeomorphisms are homeomorphisms $\Gamma(u^*VP) \to \Gamma(V_u)$. The chart changes are $\mathcal{C}^\infty$ by what has been said before. The collection of these maps form a $\mathcal{C}^\infty$ atlas for $\Gamma(P)$. It is not a priori obvious that $\Gamma(V_u)_{u \in \Gamma^\infty(P)}$ covers $\Gamma(P)$.

In the last section of this chapter we construct vector bundles of Banach spaces over manifolds of fibre bundle sections. If E is a vector bundle over P, we show that $\Gamma(E)$ is a vector bundle of Banach spaces over $\Gamma(P)$. Let $u \in \Gamma^\infty(P)$. Since $u(M)$ is a deformation retract in $V_{u,\lambda}$, $E|_{V_{u,\lambda}}$ is fibre diffeomorphic to $V_{u,\lambda} \times u^*E$. Hence the composition with this diffeomorphism may serve as vector bundle chart $\Gamma(V_{u,\lambda}) \times \Gamma(u^*E) \to \Gamma(E)|_{\Gamma(u^*VP)}$.
We treat $\Gamma(VP)$ as example. It turns out that it is canonically isomorphic to the tangent bundle of $\Gamma(P)$.

2.1 Spaces of $\mathcal{C}^k$ Maps

Let $0 \leq l < \infty$ and let M be a closed manifold. We introduce a separable Banach space structure on $\mathcal{C}^l$ vector bundle sections over M as a closed subspace of the finite product $\times_j \mathcal{C}^l(\overline{U_j}, \mathbb{R}^e)$, and give some equivalent descriptions. Here the collection (U_j) is an atlas of precompact sets for M and E.

We equip the space of $\mathcal{C}^l$ maps between the closed manifold M and a manifold N with the Whitney $\mathcal{C}^l$ topology.
Sections $\Gamma^l(P)$ in fibre bundles may be understood as a closed subspace in $\mathcal{C}^l(M, P)$ (2.1.11).

Γ^l **Sections** Let U be an open set in $\mathbb{R}^m$. $\mathcal{C}^l\left(\overline{U}, \mathbb{R}^n\right)$ denotes the space of $\mathcal{C}^l$ functions $U \to \mathbb{R}^n$, whose iterated derivatives $D^\alpha f$ are bounded and uniformly continuous on U for all $0 \leq |\alpha| \leq l$. Equipped with the norm $\|f\|_{\mathcal{C}^l, \overline{U}} = \max_{1 \leq |\alpha| \leq l} \sup_{x \in U} |D^\alpha f(x)|$, it is a separable Banach space.

Let E be a vector bundle over M. If $s \in \Gamma^l(E)$ we can choose an atlas $\mathfrak{U} = (U_i, \mathfrak{m}_i)_i$ for M which also trivializes the vector bundle E, so that there are vector bundle charts $\psi_i^E : E|_U \to U_i \times E_x$, and the local representations

$$s_i = pr_2 \psi_i^E \circ s \circ \mathfrak{m}_i{}^{-1} : \mathfrak{m}_i(U_i) \to \mathbb{R}^e$$

of vector bundle sections are $\mathcal{C}^l$.

2.1.1 (Topology on $\Gamma^l(E)$). If the open set $\widetilde{U}$ is relatively compact in U we write $\widetilde{U} \subset\subset U$. There is a finer atlas $\widetilde{\mathfrak{U}} \subseteq \mathfrak{U}$, such that $\widetilde{U}_j \subset\subset U_i$ for some i and the local representations

$$s_j = s_i|_{\mathfrak{m}_i(\widetilde{U}_j)} \in \mathcal{C}^l\left(\overline{\mathfrak{m}_i(\widetilde{U}_j)}, \mathbb{R}^e\right).$$

Choose an atlas $\mathfrak{U}$ for M which also trivializes the vector bundle and a finer covering $\widetilde{\mathfrak{U}}$ as above. Equip $\Gamma^l(E)$ with the initial topology with respect to the restrictions

$$\Gamma^l(E) \to \mathcal{C}^l\left(\overline{\widetilde{U}_j}, \mathbb{R}^e\right).$$

A norm then is defined by

$$\|s\|_l := \sum_{j=0}^N \|s_j\|_{\mathcal{C}^l, \mathfrak{m}_i(\overline{\widetilde{U}_j})}.$$

2.1.2 Remark. [33, IX §1] This topology is independent of the choice of the atlas. This follows from the equivalent description of the topology in 2.1.4(2) or by direct computation.

2.1.3 Proposition. $\Gamma^l(E)$ is a separable Banach space. It is not reflexive.

Proof. $\Gamma^l(E)$ is a closed subset in the finite product $\times_i \mathcal{C}^l(\widetilde{U}_i)$. A closed subspace in a separable Banach space is separable [12, 4 theorem 11], [1, 1.21]. The proposition holds as $\mathcal{C}^l(\widetilde{U}_i)$ has the stated properties . □

2.1.4 (Equivalent descriptions). The Banach space topology on the space of l-times continuously differentiable sections Γ^l can equally be defined in the following equivalent ways:

(1) We choose an euclidean metric g_E on E, a Riemannian metric on TM and covariant derivatives on E and TM. We can then define the norm

$$\|s\|_l := \max\{|\nabla^j s(x)| \, : \, x \in M, 1 \leq j \leq l\}$$

on $\Gamma^l(E)$, where $\nabla^j : \Gamma^l(E) \to \Gamma^l(\otimes^j T^*M \otimes E)$ is given by the composition

$$\big((\otimes^j \nabla^{T^*M}) \otimes \nabla^E\big) \circ \ldots \circ (\nabla^{T^*M} \otimes \nabla^E) \circ \nabla^E : \Gamma(E) \to \Gamma(\otimes^j T^*M \otimes E)$$

and the norm with respect to the usual tensor product metric.
This topology is independent of the choices, which can be seen similarly to 2.2.7(1).

(2) [33, IX §1] Let $J^l E$ denote the bundle of all l-jets of sections in E. This is the bundle of equivalence classes of sections whose local representations have the same Taylor expansion up to order l. ([23, 21.7], [33, IV])

We can use the initial topology with respect to the l-jet extension

$$\Gamma^l(E) \to \mathcal{C}^0(M, J^l(E)),$$

where $\mathcal{C}^0(M, J^l(E))$ carries the usual compact open topology.

2.1.5 Remark. Choose a vector bundle $E^{'}$ such that the Whitney sum $E^{'} \oplus E \cong M \times \mathbb{R}^N$. As

$$\Gamma^l(E) \times \Gamma^l(E') = \Gamma^l(E \oplus E') = \Gamma^l(M \times \mathbb{R}^N) = \mathcal{C}^l(M, \mathbb{R}^N),$$

$\Gamma^l(E)$ is a closed subspace in $\mathcal{C}^l(M, \mathbb{R}^N)$.

Space of Maps Let $\mathcal{C}^l(M,N)$ denote the space of l times continuously differentiable maps $M \to N$:

2.1.6 Definition ($\mathcal{C}^l(M,N)$). $\mathcal{C}^l(M,N)$ is the set of all continuous maps $f : M \to N$, such that for all x the local representations $\mathfrak{n}_i \circ f \circ {\mathfrak{m}_i}^{-1} : U_i \to V_i$ are $\mathcal{C}^l$.

2.1.7 Lemma. Let $0 \leq l < \infty$, U be a precompact subset and let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ be $\mathcal{C}^\infty$. Then the composition with ϕ induces a continuous map

$$\begin{gathered} \phi_* : \mathcal{C}^l(U, \mathbb{R}^m) \to \mathcal{C}^l(U, \mathbb{R}^n) \\ \phi_*(u) := \phi \circ u. \end{gathered}$$

2.1.8. We equip $\mathcal{C}^l(M,N)$ with equivalent topologies: We can think of this topology as a generalization of the topology of uniform convergence in each derivative.

(1) (Whitney $\mathcal{C}^k$ topology, cf. [10, p.35 and exercise 12 on p.41]). Let $f \in \mathcal{C}^l(M,N)$. Choose a family $\mathfrak{V}$ of charts for N, an atlas $\mathfrak{U}$ for M and a finer atlas $\widetilde{\mathfrak{U}}$, where for all j $\widetilde{U}_j \subset\subset U_i$ for some i and there is $V_j \in \mathfrak{V}$ such that $f\left(\overline{\widetilde{U}_j}\right) \subset V_j$. Let ε be a nonzero positive number. Consider the set $\mathcal{N}^l(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ defined as

$$\left\{g \in \mathcal{C}^l(M,N) \,:\, g\left(\overline{\widetilde{U}_j}\right) \subseteq V_j, \|f_j - g_j\|_{\mathcal{C}^l,\mathfrak{m}_i(\overline{\widetilde{U}_j})} < \varepsilon \text{ for all } j\right\}.$$

The collection of the sets $\mathcal{N}^l(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ for all possible choices $(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ forms a basis for the topology. To show this means to show that the topology is equivalent to

(2) Let $J^l(M,N)$ denote the bundle of l jets of maps from M to N over $M \times N$. This is the space of equivalence classes of maps whose local representations have the same Taylor expansion of order l. We equip $\mathcal{C}^l(M,N)$ with the initial topology with respect to the l jet extension $j^l : \mathcal{C}^l(M,N) \to \mathcal{C}^0(M,J^l(M,N))$, where $\mathcal{C}^0(M,J^l(M,N))$ carries the usual compact open topology. [10, p.58ff]

2.1.9 Remark. [10, 2.6] The set $\mathcal{C}^\infty(M,N)$ is dense in $\mathcal{C}^l(M,N)$.

2.1.10 Proposition. There is a number $\alpha(N)$ such that $\mathcal{C}^l(M,N)$ is a closed subset in $\mathcal{C}^l(M,\mathbb{R}^{\alpha(N)})$. The topology on $\mathcal{C}^l(M,N)$ is equivalent to the trace topology. Thus $\mathcal{C}^l(M,N)$ is a completely metrizable space, so especially Hausdorff and paracompact. Moreover $\mathcal{C}^l(M,N)$ is second countable, i.e. its topology is generated by a countable base.

Proof. We may suppose that N is a closed submanifold in $\mathbb{R}^{\alpha(N)}$ without restriction [10, chapter 2 2.14]. $\mathcal{C}^l(M,N)$ is a closed subset in $\mathcal{C}^l(M,\mathbb{R}^{\alpha(N)})$. We denote the inclusion of N by ι. The inclusion $\iota_* : \mathcal{C}^l(M,N) \to \mathcal{C}^l(M,\mathbb{R}^{\alpha(N)})$ is continuous. In fact, a basic neighbourhood set in the trace topology in $\mathcal{C}^l(M,\mathbb{R}^{\alpha(N)})$ at f contains a basic neighbourhood $\mathcal{N}^l(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ in $\mathcal{C}^l(M,N)$, where $\mathfrak{V}$ is a collection of submanifold charts for N.

Let (W,r) be a tubular neighbourhood. We show that $r_* : \mathcal{C}^l(M,W) \to \mathcal{C}^l(M,N)$ is continuous. Since for $V \subseteq \mathcal{C}^l(M,N)$ we have $V = (r_*)^{-1}(V) \cap \mathcal{C}^l(M,N)$, the topology on $\mathcal{C}^l(M,N)$ is equivalent to the trace topology then.

Use that $J^l(M,r)$ is again surjective and smooth and induces a continuous map by 2.1.7

$$J^l(M,r)_* : \mathcal{C}^0(M,J^l(M,W)) \to \mathcal{C}^0(M,J^l(M,N))$$

$$\begin{tikzcd}
\mathcal{C}^l(M,W) \arrow[r, "j^l"] \arrow[d, "r_*"] & \mathcal{C}^0(M,J^l(M,W)) \arrow[d, "J^l(M,r)_*"] \\
\mathcal{C}^l(M,N) \arrow[r, "j^l"] & \mathcal{C}^0(M,J^l(M,N))
\end{tikzcd}$$

such that $j^l \circ r_* = J^l(M,r)_* \circ j^l$.

Alternatively let $\mathcal{N}^l(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon) \subseteq \mathcal{C}^l(M,N)$ be given. Choose a map

$$g = (g_i) : M \to W$$

in the preimage of this set under r, i.e. $g_i : M \to W \cap \mathbb{R}^n \subseteq \times\mathbb{R}^n = \mathbb{R}^{\alpha(N)}$ such that

$$g_i\left(\overline{\widetilde{U}_i}\right) \subseteq r^{-1}(V_i)$$

and

$$\|r \circ g_i - f_i\|_{\mathcal{C}^l, \overline{\widetilde{U}_i}} \leq \varepsilon.$$

By lemma 2.1.7 choose δ such that

$$\|g_i - h_i\|_{\mathcal{C}^l, \overline{\widetilde{U}_i}} < \delta$$

implies

$$\|r_*(g_i) - r_*(h_i)\|_{\mathcal{C}^l, \overline{\widetilde{U}_i}} < \varepsilon/2$$

for all i. Let $h : M \to W$ be in the open set

$$\mathcal{N}^l(g, \mathfrak{U}, (r^{-1}(V_i))_i, \widetilde{\mathfrak{U}}, \delta) \subseteq \mathcal{C}^l(M, W).$$

Then $r \circ h \in \mathcal{N}^l(f, \mathfrak{U}, \mathfrak{V}, \widetilde{\mathfrak{U}}, \varepsilon)$ by the triangle inequality and

$$r_*\big(\mathcal{N}^l(g, \mathfrak{U}, (r^{-1}(V_i))_i, \widetilde{\mathfrak{U}}, \delta)\big) \subseteq \mathcal{N}^l(f, \mathfrak{U}, \mathfrak{V}, \widetilde{\mathfrak{U}}, \varepsilon)$$

in consequence. Thus $\mathcal{N}^l(g, \mathfrak{U}, (r^{-1}(V_i))_i, \widetilde{\mathfrak{U}}, \delta)$ is an open subset of $C(M, W)$ contained in $(r_*)^{-1}\big(\mathcal{N}^l(f, \mathfrak{U}, \mathfrak{V}, \widetilde{\mathfrak{U}}, \varepsilon)\big)$.
This means that the preimage of $\mathcal{N}^l(f, \mathfrak{U}, \mathfrak{V}, \widetilde{\mathfrak{U}}, \varepsilon)$ is open.

$\mathcal{C}^l(M, \mathbb{R}^{\alpha(N)})$ is completely metrizable and separable (2.1.3), thus second countable [12, 4 theorem 11] [20, 4.7]. A subset with trace topology in a second countable space is second countable. □

2.1.11 (Fibre bundle sections). The space of $\mathcal{C}^l$-sections in a fibre bundle (P, p) over M is equipped with the trace topology in $\mathcal{C}^l(M, P)$. $\Gamma^l(P)$ is closed since p_* is continuous $\mathcal{C}^l(M, P) \to \mathcal{C}^l(M, M)$.
The proof is similar to the proof of the last proposition.

We will soon see that $\mathcal{C}^k(M, N)$ is a Banach manifold modelled on Banach spaces $\Gamma^k(u^*TN)$ for a smooth maps $u : M \to N$.

2.2 Spaces of Sobolev Class Maps

Let E be a vector bundle over the closed manifold M. We will introduce a chain of Banach spaces $\Gamma^{l,p}(E)$, $0 \leq l < \infty$, $1 \leq p < \infty$. These are spaces describing regularity classes of vector bundle sections. We define them as generalization of the space $H^{l,p}(M, \mathbb{R}^n)$, the space of functions on M such that for all x there is a local representation of class $H^{l,p}_{loc}$ (2.2.2). Then every local representation is of class $H^{l,p}_{loc}$. Here we use that pull back with a diffeomorphism and composition with a linear map respects the Sobolev class and even induces a continuous map (2.2.1).
One common interpretation is that the weak derivatives up to order l of the local representations are L^p. We give more than those equivalent descriptions (2.2.7). We show the Sobolev embedding theorem (2.2.10) for these spaces.

Assuming that $lp > m$, the concept is generalized to $H^{l,p}$ maps between manifolds (2.2.12). The space $H^{l,p}(M, N)$ consists of those continuous maps with local representations of class $H^{l,p}_{loc}$. We define a family of sets serving as a neighbourhood basis by taking for each map $f : M \to N$, atlas of M, each family of charts V for N and subcovering of M consisting of precompact sets

$\tilde{U}$ with smooth boundary, such that $f\left(\overline{\tilde{U}}\right) \subseteq V$, and each positive nonzero number ε, the set of maps whose local representations have $H^{l,p}(\tilde{U}, V)$-distance at most ε to local representations of f. We have to show that this makes sense, i.e. the $H^{l,p}(\tilde{U}, V)$-distance does not change under chart changes of M and N. Here we use that pull back with a diffeomorphism and composition with a linear map respects the Sobolev class and even induces a continuous map. Moreover we need that composition with a smooth map induces a continuous map between $H^{l,p}$ spaces. Therefore the assumption $lp > m$ is necessary as the Sobolev embedding theorems for $H^{l,p}$ spaces on precompact domains with smooth boundary are used here (2.2.14). The proof of this result will be explained in the more general setting of vector bundle sections in section 3 of this chapter.
We get a chain of paracompact Hausdorff spaces $H^{l,p}(M, N)$ (2.2.16).
The Sobolev embedding theorem can be generalized for these spaces (2.2.20).
Let P be a fibre bundle over M. The space of sections $\Gamma^{l,p}(P)$ may be understood as closed subspace in $H^{l,p}(M, P)$ (2.2.19).

[1] serves as primary reference on Sobolev spaces. We refer to [8]/[9] and [2] for a detailed discussion on Sobolev spaces of functions $M \to \mathbb{R}^n$.

Sobolev VB Sections For all open sets $U \subseteq \mathbb{R}^m$, we define Sobolev spaces $H^{l,p}(U, \mathbb{R}^n)$ for all $0 \leq l$ and $1 \leq p < \infty$. $H^{l,p}(U, \mathbb{R}^n)$ is the completion of the set of all $f \in \mathcal{C}^l(U, \mathbb{R}^n)$ with $\|f\|_{l,p} < \infty$ with respect to the norm $\|f\|_{l,p} := \sum_{0 \leq |\alpha| \leq l} \|D^\alpha f\|_{L^p(U)}$.
For all $1 \leq p < \infty$, we may equivalently define $H^{l,p}(U, \mathbb{R}^n)$ as the space of functions whose weak derivatives up to order l are in $L^p(U)$, equipped with the norm $\|.\|_{l,p}$, by a result of Meyer-Serrin [29]. If we defined $H^{l,p}(U)$ as completion of the set of functions $\mathcal{C}^l(\overline{U}, \mathbb{R}^n)$, we would have to assume that the boundary of U is $\mathcal{C}^\infty$ for this result to hold.
In fact, if $U \subseteq \mathbb{R}^n$ is open and has smooth boundary, $\mathcal{C}^k\left(\overline{U}\right)$ is dense in $H^{l,p}(U)$ for all $l \leq k \leq \infty$.

$H^{l,p}(U)$ is a Banach space, which is separable for $1 \leq p < \infty$, and reflexive and uniformly convex for $1 < p < \infty$.

2.2.1 Proposition. Let U be an open set in $\mathbb{R}^m$. Then $H^{l,p}(U)$ is invariant under

(1) $\mathcal{C}^l$ diffeomorphisms [1, 3.35]: If ϕ is a $\mathcal{C}^l$ diffeomorphism between open sets U and V in $\mathbb{R}^m$, $\phi \in \mathcal{C}^l\left(\overline{U}, \mathbb{R}^m\right)$ and $\phi^{-1} \in \mathcal{C}^l\left(\overline{V}, \mathbb{R}^m\right)$, so that $0 < c \leq \det(d\phi(x)) \leq C < \infty$ for all $x \in U$, then $H^{l,p}(U)$ is homeomorphic to $H^{l,p}(V)$ via ϕ^*, i.e. $\phi^* : u \mapsto u \circ \phi$ is bounded and linear $H^{l,p}(V) \to H^{l,p}(U)$ and has a bounded inverse.

(2) and sections of $Gl(\mathbb{R}^n)$: If V and W are finite dimensional vector spaces and $\phi \in \mathcal{C}^1\left(\overline{U}, L(V, W)\right)$, $\phi_*(s)(x) = \phi_x s(x) = pr_2 \phi s(x)$, is a linear and bounded map $H^{l,p}(U, V) \to H^{l,p}(U, W)$. If $\phi_x \in Gl(V)$, this map defines a Banach space isomorphism.

If U is a relative compact open set with smooth boundary, the Sobolev embedding theorem [1, 5.4, 6.2] holds.

A function on an arbitrary open set $U \subseteq \mathbb{R}^m$ is said to be in $H^{l,p}_{loc}(U)$ if its restriction to any open relative compact subset with smooth boundary $\widetilde{U} \subseteq U$

is an element of $H^{l,p}(\widetilde{U})$, where functions which equal almost everywhere are identified in $H^{l,p}_{loc}$.
If $V \subseteq U$ is open, the restriction to V induces a linear and bounded map $H^{l,p}(U) \to H^{l,p}(V)$. Thus $H^{l,p}_{loc}(U)$ contains $H^{l,p}(U)$ as a subset.

2.2.2 Definition ($\Gamma^{l,p}(E)$). Let E be a vector bundle over M. Choose an atlas $\mathfrak{U}$ for M which also trivializes the vector bundle. $\Gamma^{l,p}(E)$ is the space of sections s in E, whose local representations

$$s_i = pr_2 \psi_i^E \circ s \circ \mathfrak{m}_i{}^{-1} : \mathfrak{m}_i(U_i) \to \mathbb{R}^e$$

are in $H^{l,p}_{loc}(\mathfrak{m}_i(U_i), \mathbb{R}^e)$, moduli the sections which are zero almost everywhere. The definition of $\Gamma^{l,p}(E)$ makes sense and does not depend on the choice of $\mathfrak{U}$ by 2.2.1.

Note that $\Gamma^{l+1,p}(E) \subseteq \Gamma^{l,p}(E)$ and $\Gamma^k(E) \subseteq \Gamma^{l,p}(E)$.

2.2.3 (Topology on $\Gamma^{l,p}(E)$). Choose $\mathfrak{U}$ and a finer covering $\widetilde{\mathfrak{U}}$, such that $\widetilde{U}_j \subset\subset U_i$ for some i and $\widetilde{U}_j$ has smooth boundary. Equip $\Gamma^{l,p}(E)$ with the initial topology with respect to the restrictions

$$\Gamma^{l,p}(E) \to H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j), \mathbb{R}^e).$$

A norm is then given by

$$\|s\|_{l,p} := \sum_{j=0}^{N} \|s_j\|_{H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j))}.$$

2.2.4 Remark. This topology is independent of the choice of the atlas $\widetilde{\mathfrak{U}}$ (and $\mathfrak{U}$).

2.2.5 Proposition. Let E be a vector bundle over M. $\Gamma^{l,p}(E)$ is a Banach space. It is separable if $\infty > p \geq 1$, reflexive and uniformly convex if $\infty > p > 1$ and a Hilbert space if $p = 2$.

Proof. Choose $\mathfrak{U}$ and $\widetilde{\mathfrak{U}}$. $\Gamma^{l,p}(E)$ is closed in the finite product $\times_j H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j), \mathbb{R}^e)$. In fact, let v_n be a convergent sequence in $\Gamma^{l,p}(E)$. The local representations $(v_n)_j : \mathfrak{m}_i(\widetilde{U}_j) \to \mathbb{R}^e$ are Cauchy sequences by definition of the topology, and converge to $(v_\infty)_j$ in $H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j), \mathbb{R}^e)$. The restriction to the intersection of two covering sets induces a continuous map between Sobolev spaces. Thus the restriction of a convergent sequence converges to the restriction of the limit. Then, by compactness of M, there is a subsequence of v_n, whose restriction to $H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j), \mathbb{R}^e)$ converges for all j. As the restrictions of v_n equal on the intersections $\widetilde{U}_j \cap \widetilde{U}_i$, the limits equal on the intersection and define a section in E.

The proposition follows as the statement holds for $H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j), \mathbb{R}^e)$ ([1, 3.5]) and subspaces of separable metrizable spaces are separable ([12, 4 theorem 11]), and closed subspaces of reflexive and uniformly convex Banach spaces are reflexive ([20, 7.3]) and uniformly convex ([1, 1.21]). $\square$

2.2.6 Proposition. $\Gamma^k(E)$ is dense in $\Gamma^{l,p}(E)$ for all $0 \leq l \leq k \leq \infty$ and $1 \leq p < \infty$.

Proof. Let f in $\Gamma^{l,p}(E)$. There is a sequence $(s_j)_n$ of functions in $\times \mathcal{C}^k\left(\overline{\tilde{U}_j}\right)$ converging to the restrictions of the local representations (f_i) in $\times H^{l,p}(\tilde{U}_j)$. Let $(\mathfrak{V}, (\phi_\alpha))$ be a $\mathcal{C}^\infty$ partition of unity subordinated to $\widetilde{\mathfrak{U}}$. Since

$$\begin{aligned}
\|\sum_{j,\alpha} \phi_\alpha (s_j)_n - f\|_{l,p} &\leq c \sum_j \|(\sum_\alpha \phi_\alpha (s_j)_n - \sum_\alpha \phi_\alpha f)|_{\tilde{U}_j}\|_{l,p} \\
&\leq \sum_{j,\alpha} \|\phi_\alpha ((s_j)_n - f_i)\|_{H^{l,p}(\tilde{U}_j)} \\
&\leq \sum_{j,\alpha} \|\phi_\alpha\|_{\mathcal{C}^{l+1}} \|((s_j)_n - f_i)\|_{H^{l,p}(\tilde{U}_j)},
\end{aligned}$$

there is a subsequence $(\sum_{j,\alpha} \phi_\alpha s_j)_n$ is a sequence of sections in $\Gamma^k(E)$, converging to f in $H^{l,p}(E)$. $\square$

2.2.7. Let $l \leq k$. We give the following equivalent descriptions of $\Gamma^{l,p}(E)$.

(1) We choose an euclidean metric g_E on E, a Riemannian metric on TM and covariant derivatives on E and TM. Define a norm on Γ^l sections by

$$\|s\|_{l,p} := \sum_{j=0}^{l} \left(\int_M |\nabla^j s|^p \right)^{1/p},$$

where $\nabla^j : \Gamma(E) \to \Gamma(\otimes^j T^*M \otimes E)$ is given by the composition

$$\left((\otimes^j \nabla^{T^*M}) \otimes \nabla^E\right) \circ \ldots \circ (\nabla^{T^*M} \otimes \nabla^E) \circ \nabla^E$$

and the norm with respect to the usual tensor product metric.
The Banach space $\Gamma^{l,p}(E)$ is the completion of $\Gamma^k(E)$ with respect to this norm. (Cf. [27, 10.2.4], [9, 2.1])
This description is independent of the choices.
A second covariant derivative ∇' differs from ∇ by a $\mathfrak{end}(E)$ valued one form. $(\nabla')^j$ differs from ∇^j by a $\mathcal{C}^\infty$ section ω in $\mathfrak{end}(\oplus_{i=0}^j \otimes^i T^*M \otimes E)$ then. Thus

$$\begin{aligned}
|(\nabla')^j s(x)| &\leq |\nabla^j s(x)| + |\omega(x) s(x)| \\
&\leq |\nabla^j s(x)| + C|\omega(x)||s(x)| \\
&\leq |\nabla^j s(x)| + C\|\omega\|_{\Gamma^1} |x||s(x)| \leq |\nabla s(x)| + C|s(x)|
\end{aligned}$$

by compactness of M. If g denotes a metric on $\oplus_{j=0}^l \otimes^j T^*M \otimes E$, a second metric $g'(X,Y)$ on $\oplus_{j=0}^l \otimes^j T^*M \otimes E$ is given by $g(AX, AY)$, where A is linear. Then $|g'| \leq \|A\|_{op}|g|$.
Of course, the claim equally follows if one shows that the description introduced here is equivalent to the original definition given before.

(2) For the same choices and notions as in (1)

$$\Gamma^{l,p}(E) = \left\{ s \in L^p(E) \,:\, \nabla^j s \in L^p(\otimes^j T^*M \otimes E), j \leq l \right\},$$

where the differentiation is to be understood in the weak sense, see [28, definition 1.2.12].

Working with local representations, remark 2.2.1, the triangle inequality, and the mean value theorem, shows that these descriptions are equivalent to the definition.

(3) It is sometimes helpful to think of $\Gamma^{l,p}(E)$ as the completion of $\Gamma^k(E)$ with respect to the norm induced by

$$j^l : \Gamma^k(E) \to \Gamma^{k-l}(J^l E) \subseteq \Gamma^{0,p}(J^l E).$$

This norm is given by

$$\|s\|_{l,p} = \|j^l s\|_{0,p}.$$

(Cf. [33, IX §3])

2.2.8 Remark. Let $E^{'}$ be another vector bundle such that the Whitney sum $E \oplus E^{'}$ is isomorphic to the trivial bundle $M \times \mathbb{R}^N$. Then

$$\Gamma^{l,p}(M \times \mathbb{R}^N) = \Gamma^{l,p}(E \oplus E^{'}) = \Gamma^{l,p}(E) \times \Gamma^{l,p}(E^{'}).$$

We now see that $\Gamma^{l,p}(E)$ is a closed subspace of $\Gamma^{l,p}(M, M \times \mathbb{R}^N) = H^{l,p}(M, \mathbb{R}^N)$.

2.2.9 Remark ($p = \infty$). Let $\Gamma^{l,\infty}(E)$ be the completion of $\Gamma^k(E)$ with respect to the norm $\max_{0 \le j \le l} \sup_{x \in M} |\nabla^j f|$, i.e. $\Gamma^{l,\infty}(E) = \Gamma^l(E)$. Define

$$W^{l,\infty} = \{s \in L^\infty \,:\, \nabla^\alpha s \in L^\infty \text{ for all } \alpha \text{, such that } 0 \le |\alpha| \le l\}$$

and note that $W^{l,\infty} \not\subset \Gamma^{l,\infty}$.

2.2.10 Theorem (Sobolev embedding theorem)**.** *[27, 10.2.36] (Cf. also [2, 2.10,2.20] and [8, 3.5,3.6].) Let M be a m-dimensional closed manifold.*

(1) If for real numbers $1 \le p \le q < \infty$ and nonnegative integers $l \ge k$ the inequality $l - m/p \ge k - m/q$ holds, the inclusion $\Gamma^{l,p}(E) \hookrightarrow \Gamma^{k,q}(E)$ is continuous.
If $l > k$ and $l - m/p > k - m/q$, the inclusion is compact.

(2) If $p \ge 1$ is real and l, k are nonnegativ integers such that $l - m/p \ge k$ the inclusion $\Gamma^{l,p}(E) \hookrightarrow \Gamma^k(E)$ is continuous.
If the inequality is sharp, the inclusion is compact.

Proof. (1) The idea we will catch here ([2]) is to globalize the respective embedding theorems [1, 5.4,6.2] for functions on open precompact domains with smooth boundary.
Let $s \in \Gamma^{l,p}(E)$, then

$$\|s\|_{k,q} \le c \sum_{j=1}^N \|s_j\|_{H^{k,q}(\mathfrak{m}_i(\widetilde{U}_j))} \le c \sum_{j=1}^N C_i \|s_j\|_{H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j))} \le C\|s\|_{l,p} \,,$$

where we used the classical Sobolev embedding theorem. This is possible as the covering $\widetilde{\mathfrak{U}}$ consists of open precompact sets with smooth boundary.
The compactness of this inclusion follows from the local case again (cf.[2, 2.34]), as

$$\|f_m - f_n\|_{k,q} \le c \sum_{j=0}^N \|(f_m)_j - (f_n)_j\|_{k,q}$$

for a sequence f_n in $H^{l,p}(E)$. A subsequence of the local representations $(f_m)_j$ are converging in $H^{k,q}$ by the classic Sobolev embedding theorem. Thus a subsequence of (f_m) is a Cauchy sequence in $H^{k,p}(E)$.

The proof of (2) is similar. Choose a covering $\widetilde{\mathfrak{U}}$ which consists of precompact sets with smooth boundary. In fact

$$\|s\|_{\mathcal{C}^l} \leq c \sum_{j=1}^{N} \|s_j\|_{\mathcal{C}^k, \overline{\widetilde{U}_j}} \leq C \sum_{j=1}^{N} \|s_j\|_{H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j))} \leq C\|s\|_{l,p}$$

holds by the classic Sobolev embedding theorems and implies the continuity of the inclusion. For the compactness observe again that

$$\|f_m - f_n\|_k \leq c \sum_{j=1}^{N} \|(f_m)_j - (f_n)_j\|_{\mathcal{C}^k, \overline{\widetilde{U}_j}},$$

where (f_m) is uniformly $\Gamma^{l,p}$ bounded. By the classic compactness result a subsequence of $(f_m)_j$ converges $\mathcal{C}^k$ to some f_j and is a Cauchy sequence. Thus a subsequence of (f_m) is a Cauchy sequence in $\Gamma^k(E)$. We conclude that f_m converges in $\Gamma^k(E)$. □

2.2.11 (Open sets of $\Gamma^{l,p}(E)$). We can use the Sobolev embedding theorems in an useful manner. Let U be an open, bounded subset of $\mathbb{R}^m$ with smooth boundary. If V is an open subset of $\mathbb{R}^n$,

$$\mathcal{C}^0(U,V) := \left\{ s \in \mathcal{C}^0(U) \,:\, s(x) \in V \text{ for all } x \in U \right\}$$

is open in $\mathcal{C}^0(U)$.
For all l such that $l - m/p \geq 0$,

$$H^{l,p}(U,V) := H^{l,p}(U) \cap \mathcal{C}^0(U,V) = \iota^{-1}(\mathcal{C}^0(U,V))$$

is open in $H^{l,p}(U)$, because the inclusion $\iota : H^{l,p} \hookrightarrow \mathcal{C}^0$ is continuous.

Let E be a $\mathcal{C}^k$ vector bundle over M and W an open set of the zero section in E. For all l such that $l - m/p \geq 0$, $\Gamma^0(W) := \left\{ s \in \Gamma^0(E) \,:\, s(M) \subseteq W \right\}$ is open in $\Gamma^0(E)$, and by the Sobolev embedding theorem $\Gamma^{l,p}(W) := \Gamma^0(E) \cap \Gamma^0(W)$ is open in $\Gamma^{l,p}(E)$.

Maps into Manifolds We now define the set of Sobolev maps from a closed m dimensional manifold M into a finite dimensional manifold N.

2.2.12 Definition ($H^{l,p}(M,N)$). If $l - m/p > 0$, let $H^{l,p}(M,N)$ be the set of continuous maps $f : M \to N$ such that for all $x \in M$ local representations $f_i = \mathfrak{n}_i f \mathfrak{m}_i{}^{-1}$ are in $H^{l,p}_{loc}(\mathfrak{m}_i(U_i), \mathfrak{n}_i(V_i)) \subseteq H^{l,p}_{loc}(\mathfrak{m}_i(U_i), \mathbb{R}^n)$.

2.2.13. This definition makes sense.
Use remark 2.2.1(1) for the independence of the choice of the atlas $\mathfrak{U}$ of M.
The assumption $l - m/p > 0$ is needed to show independence of the choice of the charts $\mathfrak{V} = (\mathfrak{n}_i, V_i)$ on N. Under this assumption we can use the next lemma 2.2.14 to conclude: If $U \subset\subset \mathbb{R}^m$

with smooth boundary and if $f : V \to W$ is a smooth map between two open subsets of $\mathbb{R}^m$, the lemma shows that f_* is a map $H^{l,p}(U,V) \to H^{l,p}(U,W)$.
If $f : \mathbb{R}^m \to W \subseteq \mathbb{R}^n$ is smooth, the diagram has to commute by definition of $H^{l,p}(U,V)$.

$$\begin{tikzcd}
\mathcal{C}^0(U) \arrow[r, "f_*"] & \mathcal{C}^0(U,W) \subseteq_{open} & \mathcal{C}^0(U) \\
 & H^{l,p}(U,W) \subseteq_{open} \arrow[u, hook] & H^{l,p}(U) \arrow[u, hook, "\iota"'] \\
H^{l,p}(U) \arrow[uu, hook, "\iota"] \arrow[ur, dashed, "f_*"] \arrow[urr, bend right=10, "f_*"'] & &
\end{tikzcd}$$

Moreover the lemma shows that f_* is continuous. If f is a diffeomorphism $V \to W$, the identity sits over the composition $f_* \circ f_*^{-1} : H^{l,p}(U,W) \to H^{l,p}(U,V) \to H^{l,p}(U,W)$, and thus f_* defines a homeomorphism.

2.2.14 Lemma. Assume $l - m/p > 0$. Let $U \subset\subset \mathbb{R}^m$ with smooth boundary and $\phi : \mathbb{R}^m \to \mathbb{R}^n$ smooth. Then $\phi_* : H^{l,p}(U,\mathbb{R}^n) \to H^{l,p}(U,\mathbb{R}^n)$, $\phi_*(u) = \phi \circ u$ is continuous.

Proof. See 2.3.11. □

We will in general implicitly assume that $lp > m$.

We equip the set of Sobolev maps with a topology defining a family of sets serving as a basis.

2.2.15 Definition (Topology on $H^{l,p}(M,N)$). Let f be in $H^{l,p}(M,N)$. Choose a family $\mathfrak{V}$ of charts for N, an atlas $\mathfrak{U}$ for M and a finer atlas $\widetilde{\mathfrak{U}}$, where for all j $\widetilde{U}_j \subset\subset U_i$ for some i, $\widetilde{U}_j$ has a smooth boundary and there is $V_j \in \mathfrak{V}$ such that $f\left(\overline{\widetilde{U}_j}\right) \subset V_j$. Let ε be a nonzero positive real number. Define a subset $\mathcal{N}^{l,p}(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ of $H^{l,p}(M,N)$ to be

$$\left\{ g \in H^{l,p}(M,N) \,:\, g\left(\overline{\widetilde{U}_j}\right) \subseteq V_j \text{ and } \|g_j - f_j\|_{H^{l,p}(\mathfrak{m}_i(\widetilde{U}_j))} < \varepsilon \text{ for all } j \right\}.$$

The collection of the sets $\mathcal{N}^{l,p}(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ for all possible choices $(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ forms a basis for the topology on $H^{l,p}(M,N)$.

2.2.16 Proposition. There is a number $\alpha(N)$ such that $H^{l,p}(M,N)$ is a closed subspace in $H^{l,p}(M,\mathbb{R}^{\alpha(N)})$. The topology of $H^{l,p}(M,N)$ is equivalent to the trace topology, so that $H^{l,p}(M,N)$ is completely metrizable. Especially, $H^{l,p}(M,N)$ is Hausdorff and paracompact. Moreover $H^{l,p}(M,N)$ is second countable.

Proof. Embed N in $\mathbb{R}^{\alpha(N)}$ as a closed submanifold. $H^{l,p}(M,N)$ is a closed subspace of $H^{l,p}(M,\mathbb{R}^{\alpha(N)})$ with trace topology. The proof is similar to the proof of 2.1.10.

If N is a submanifold in $\mathbb{R}^{\alpha(N)}$, there is a tubular neighbourhood $W \subseteq_{open} \mathbb{R}^{\alpha(N)}$ of N in $\mathbb{R}^{\alpha(N)}$. Associated to W, there is a retraction $r : W \to N$, i.e. $r\iota_N = id_N$. ([10, chapter4,5.]) $H^{l,p}(M,W)$ is ($\mathcal{C}^0$-)open in $H^{l,p}(M,\mathbb{R}^{\alpha(N)})$. The retraction induces a continuous map $r_* : H^{l,p}(M,W) \to H^{l,p}(M,N)$. The preimage of $\mathcal{N}(f,\mathfrak{U},\mathfrak{V},\widetilde{\mathfrak{U}},\varepsilon)$ consists of all maps $g = (g_i)$, where

$$g_i : U_i \to W_i = W \cap \mathbb{R}^n \subseteq \times\mathbb{R}^n = \mathbb{R}^{\alpha(N)}$$

such that $r \circ g_i : \widetilde{U}_j \to V_i$ is near f_i in the l,p norm. It is open because r_* is continuous $H^{l,p}(\widetilde{U}_j, W_j) \to H^{l,p}(\widetilde{U}_j, V_j)$ (2.2.14). Cf. the proof of 2.1.10. □

2.2.17 Proposition. The set $\mathcal{C}^\infty(M,N)$ is dense in $H^{l,p}(M,N)$.

Proof. Let $f \in H^{l,p}(M,N)$. Let (W,r) be a tubular neighbourhood of N in $\mathbb{R}^{\alpha(N)}$. Since $\mathcal{C}^\infty(M,\mathbb{R}^{\alpha(N)})$ is dense in $H^{l,p}(M,\mathbb{R}^{\alpha(N)})$, there is a sequence (g_n) of functions in $\mathcal{C}^\infty(M,W)$ converging to f in $H^{l,p}(M,W)$. $r_* : H^{l,p}(M,W) \to H^{l,p}(M,N)$ is continuous by the proof of the preceding proposition, so that $(r_*(g_n)) = (r \circ g_n)$ is a sequence of functions in $\mathcal{C}^\infty(M,N)$ converging to f in $H^{l,p}(M,N)$. □

We will see that $H^{l,p}(M,N)$ is a Banach manifold modelled on the Banach spaces $\Gamma^{l,p}(u^*TN)$.

2.2.18 Remark. It is sometimes helpful to view $H^{l,p}(M,N)$ with the initial topology along l-jet extension $H^{l,p}(M,N) \to \Gamma^{0,p}(J^l(M,N))$. For us, the case $p > m$ and $l = 1$ is especially interesting. As $J^1(M,N)$ is a vector bundle over $M \times N$, we will be able to view $\Gamma^{0,p}(J^1(M,N))$ as an infinite dimensional vector bundle over $H^{1,p}(M,N)$, whose fibres are Banach spaces.

2.2.19 (Fibre bundle sections). Let P be a smooth fibre bundle over M and $l - m/p > 0$. Define $\Gamma^{l,p}(P) := \{s \in H^{l,p}(M,P) : p_*(s) = p \circ s = id\}$. $p_* : H^{l,p}(M,P) \to H^{l,p}(M,M)$ is continuous and $\Gamma^{l,p}(P)$ is a closed subspace of $H^{l,p}(M,P)$. The proof uses lemma 2.2.14 similar to 2.2.16.

2.2.20 Theorem (Sobolev embedding theorem)**.** *Let M be a m-dimensional closed manifold and N a finite dimensional manifold.*

(1) If for real numbers $1 \le p \le q < \infty$ and nonnegative integers $l \ge k$ such that $kq > m$ and the inequality $l - m/p \ge k - m/q$ holds, the inclusion $H^{l,p}(M,N) \hookrightarrow H^{k,q}(M,N)$ is continuous.
If in addition $l > k$ and $l - m/p > k - m/q$, every set $B \subseteq H^{l,p}(M,N)$ which satisfies $B\left(\overline{\widetilde{U}}\right) \subseteq V$ and is uniformly bounded in all $H^{l,p}(\widetilde{U},V)$ for some choice of $\widetilde{\mathcal{U}}$ and $\mathcal{V}$, has an accumulation point in $H^{k,q}(M,N)$.

(2) If $p \ge 1$ is real and l,k are nonnegativ integers such that $lp > m$ and $l - m/p \ge k$ the inclusion $H^{l,p}(M,N) \hookrightarrow \mathcal{C}^k(M,N)$ is continuous.
If in addition the inequality is sharp, every set $B \subseteq H^{l,p}(M,N)$ which satisfies $B\left(\overline{\widetilde{U}}\right) \subseteq V$ and is uniformly bounded in all $H^{l,p}(\widetilde{U},V)$ for some choice of $\widetilde{\mathcal{U}}$ and $\mathcal{V}$, has an accumulation point in $\mathcal{C}^k(M,N)$.

$H^{l,p}(M,N) \subset C^1(M,N)$ is continuous and $H^{l,p}(M,N) \subset C^0(M,N)$ is always a inclusion, which satisfies the property explained above.

We say that v is locally $H^{l,p}$-bounded, if for a covering $\mathfrak{V}$ of N and a covering $\mathfrak{U}$ of M there is a finer covering $\widetilde{\mathfrak{U}}$ such that $\tilde{U}_j \subset\subset U_i$ for some i with smooth boundary, such that $v(\tilde{U}_j) \subseteq V_j$, and such that v_j is bounded in $H^{l,p}(\tilde{U}_j, V_j)$ for all j.

We should remark that $\{u_n\} \subseteq H^{1,p}(M,N)$ is uniformly locally $H^{1,p}$-bounded by K, if and only if $\{j^1_* u_n\}$ is uniformly bounded in $\Gamma^{0,p}(J^1(M,N))$, i.e. $\|j^1_* u_n\|_{0,p}$, is uniformly bounded by the number K. Here $\|.\|$ is meant to be a fibrewise norm on the fibres $\Gamma^{0,p}(T^*M \otimes u^*TN)$. This is by description (2) of the topology of $H^{l,p}$ maps.

Proof. Without restriction, let B be the set $\{u_n\} \subseteq H^{l,p}(M,N) \subseteq \mathcal{C}^0(M,N)$, which satisfies $(u_n)(\overline{U}) \subseteq V$ for all n and is uniformly locally $H^{l,p}$-bounded by a number K. The local representations $(u_n)_i$ lie in some $H^{l,p}_{loc}(U_i,V_i) \subseteq H^{k,q}_{loc}(U_i,V_i)$. By the classic Sobolev embedding theorem $H^{l,p}(\widetilde{U}_j,V_i) \subseteq H^{k,q}(\widetilde{U}_j,V_i)$ or $H^{l,p}(\widetilde{U}_j,V_i) \subseteq \mathcal{C}^k(\widetilde{U}_j,V_i)$ are continuous (compact) inclusions. Thus the preimage of a basis neighbourhood in $H^{k,q}(M,N)$, or $\mathcal{C}^k(M,N)$ respectively, is a basis neighbourhood and the inclusions are continuous.

For the second part of (1) we use the compactness of the local Sobolev embedding theorem. There is a subsequence (u_n) whose local representations $u_{n,i}$ converge $H^{k,q}$ to u_i say. As u_i equals u_j on $\widetilde{U}_i \cap \widetilde{U}_j$, determined by the local representations u_i determine a map $u \in \mathcal{C}^0(M,N)$. Since $u_{n,i}$ is converging in $H^{k,q}$ we conclude that $u \in H^{k,q}(M,N)$.

The second part of (2) is analogous. □

2.2.21 Remark. We have that

$$\Gamma^\infty(E) = \cap_{j=0}^\infty \Gamma^j(E) = \cap_{j=1}^\infty \Gamma^{l,p}(E)$$

and

$$\mathcal{C}^\infty(M,N) = \cap_{j=0}^\infty \mathcal{C}^j(M,N) = \cap_{j=1}^\infty H^{l,p}(M,N).$$

Thus $\Gamma^\infty(E)$ is a Fréchet space. In [15] and [7] these spaces are studied in great generality. Here also a notion of smoothness of maps between Fréchet spaces is discussed.

2.3 Differentiable Maps of Γ^l and $\Gamma^{l,p}$ Spaces

We need to know whether the composition with a smooth fibre preserving map ϕ over the identity is smooth on spaces of vector bundle sections $\Gamma(E \to M) \to \Gamma(F \to M)$. As mentioned these results are crucial for the development of the theory.

The composition of a smooth fibre preserving map $\phi : E \to F$ over the identity with a Γ^l map is again Γ^l, and is $\mathcal{C}^\infty$ as a map $\Gamma^l(E) \to \Gamma^l(F)$. This map is defined by $\phi_*(u) = \phi \circ u$. To proof this, one basically needs the chain rule, the mean value theorem, and the fundamental theorem of calculus. As the computations which have to be done are similar for $\Gamma^{l,p}$ sections we give a complete proof for Sobolev sections only.
For Sobolev class sections, the assumption $l/p > m$ is necessary to conclude that ϕ_* is smooth (2.3.9). Additional inputs are the Hölder inequality and the Sobolev embedding theorem.
The composition with ϕ is smooth $\Gamma^\infty(E) \to \Gamma^\infty(F)$ in the sense of [15] (or [7]). This statement can be found there.

It is helpful to think of fibre preserving maps ϕ between vector bundles as of maps depending on two variables. In fact, in the local picture $\phi : U \times \mathbb{R}^e \to U \times \mathbb{R}^f$. Composition of ϕ with a section $u : U \to U \times \mathbb{R}^e$ yields a section of $U \times \mathbb{R}^f$. In view of the function spaces we treat, we are interested in the full differential of this function. We have the chain rule $d(\phi \circ u) = d_1\phi + d_2\phi \circ du$ and Faà di Bruno's formula. These formulae play an important role in the proof of continuity and smoothness of ϕ_*.
In view of the definitions 2.1.4(1) for $\Gamma^l(F)$ and 2.2.7(1) for $\Gamma^{l,p}(F)$, it is more convenient to have these formulae in a global formulation for $\phi : E \to F$. To do this we introduce covariant and partial covariant derivatives of fibre preserving vector bundle maps (2.3.1) and formulate the chain rule (2.3.6) in terms of these.

Let (E, p, M) and (F, M) be vector bundles over M, W an open set in E. Let $\phi : W \to F$ be a fibre preserving $\mathcal{C}^l$ map over the identity; we write $\phi \in \mathcal{C}^l(W, F)$.

2.3.1. The covariant derivative of ϕ with respect to the covariant derivatives ∇ on F and E is the map $\nabla(\phi) : W \to L(TM \oplus E, F)$. We define it as in [5, p.176], using the connectors $K^E : TE \to E$ and $K^F : TF \to F$.

We recall what a connector is. Let $\alpha : TE \to VE$ be a connection on E. Then K is defined as

$$pr_2 \circ \mathrm{vl}^{-1} \circ \alpha : TE \to VE \to E \oplus E \to E.$$

Note that

$$(\pi_E, Tp, K) : TE \to E \oplus TM \oplus E$$

is a diffeomorphism with inverse given by

$$(\xi_1, \xi_2, \xi_3) \mapsto \mathrm{vl}(\xi_1, 0, \xi_3) + C_\alpha(\xi_1, \xi_2, 0),$$

where $C_\alpha := ((\pi_E, Tp)|_{HE})^{-1} : E \oplus TM \to HE$ denotes the horizontal lift with respect to the connection α. (see [23, 17.3]).

We have

$$\nabla_X u = K \circ Tu \circ X$$

for a section u in a vector bundle and a vector field X on M. The covariant derivative of ϕ is defined as

$$\nabla(\phi) = K \circ T\phi \circ (\pi_E, Tp, K)^{-1} \ .$$

We need the first partial covariant derivative

$$\nabla_1\phi : W \to L(TM, F),$$

and the second partial covariant derivative

$$\nabla_2\phi : W \to L(E, F).$$

2.3.2 Remark. [30, proof of 3° lemma 7] Let X be a vector field on M, c a smooth curve in M such that $c(0) = x$, $c'(0) = X$ and let $w \in W_{c(0)}$. Then

$$(\nabla_1)_X\phi(w)(x) = \frac{d}{dt}|_{t=0} Pt^F\big(-t, c, \phi(Pt^E(t, c, w))\big)(x) \ .$$

2.3.3 Definition. Let ϕ be a fibre preserving map over the identity. The vertical derivative of ϕ at $w \in W$ in direction $e \in VE$ is the smooth map

$$d^v\phi : W \to L(E, F) \ , \quad d^v\phi(w)(e)(x) := \frac{d}{dt}_{|t=0} \phi_x(w(x) + te(x)) \ .$$

2.3.4 Remark. The second partial derivative $\nabla_2\phi(w)(e)$ does not depend on the connections on E and F.
$\nabla_2\phi(w)(e)$ equals $d^v\phi(w)(e)$, cf. [5, p.176], [30, proof of 3° lemma 8].

Proof. Note that the fibre isomorphism

$$(\pi_E, Tp, K)^{-1} : E \oplus \{0\} \oplus E \to VE$$

equals the vertical lift $\mathrm{vl} : E \oplus E \to VE$. The vertical lift is the canonical isomorphism of $E \oplus_M E$ and VE given by

$$(w_x, e_x) \mapsto \frac{d}{dt}|_{t=0}(w_x + te_x).$$

We compute

$$\begin{aligned} \nabla_2\phi(w)(e) =& K \circ T\phi \circ (\pi_E, Tp, K)^{-1}(w, 0, e) \\ =& K \circ T\phi \circ \mathrm{vl}(w, e) \\ =& pr_2 \circ \mathrm{vl}^{-1} \circ \alpha^F \circ T\phi \circ \mathrm{vl}(w, e). \end{aligned}$$

$T\phi(VE) \subseteq VF$, because
$$\begin{tikzcd} T\phi:\ TE \arrow[r] \arrow[dr, "p_*"'] & TF \arrow[d, "p_*"] \\ & TM \end{tikzcd}$$
commutes. Thus $\nabla_2\phi(w)(e)$ is given by

$$pr_2 \circ \mathrm{vl}^{-1} \circ T\phi \circ \mathrm{vl}(w, e).$$

We observe that this formula depends neither on the connection on F nor on the connection on E. By the chain rule,

$$\begin{aligned}\nabla_2\phi(w)(e)(x) =& pr_2 \circ \mathrm{vl}^{-1} \circ T\phi \circ \frac{d}{dt}|_{t=0}(w_x + te_x)\\ =& pr_2 \circ \mathrm{vl}^{-1} \circ \frac{d}{dt}|_{t=0}\big(\phi\big((w_x, e_x) \mapsto w_x + te_x\big)\big)\\ =& pr_2\big(\phi(w_x), \frac{d}{dt}|_{t=0}\phi_x(w_x + te_x)\big)\\ =& \frac{d}{dt}|_{t=0}\phi_x\big(w_x + te_x\big) = d^v\phi(w)(e)(x).\end{aligned}$$

So we have shown that $\nabla_2\phi(w)(e)$ equals $d^v\phi(w)(e)$. □

2.3.5 Remark (Schwarz theorem). [30, 3° lemma 7] [5] If ϕ is $\mathcal{C}^2$ there is the Schwarz theorem

$$\nabla_1 d^v\phi = d^v\nabla_1\phi\ .$$

2.3.6 Remark (Chain Rule). [30] Let $u \in \Gamma^\infty(E)$. The chain rule takes the form

$$\nabla_X(\phi \circ u) = \nabla_{1X}\phi(u) + d^v\phi(u)(\nabla_X u).$$

Faà di Bruno's formula ([30, 3° Lemma 8]) for the chain rule of iterated derivatives says that $\nabla^l(\phi \circ u)(x)$ is a polynomial expression of the type

$$\sum_{r,t,\alpha} C_{r,t,|\alpha|=t}(\nabla_1)^r(d^v)^t\phi(u(x))\,(\nabla u(x))^{\alpha_1}\cdots(\nabla^l u(x))^{\alpha_l}\ ,$$

where α runs through the multi indices $(\alpha_1, \dots, \alpha_l)$ of length t and

$$l = r + \alpha_1 + 2\alpha_2 + \ldots + l\alpha_l$$

has to be satisfied. Note that the α_i can be zero.

We start by stating the following theorem for Γ^l sections.

2.3.7 Theorem (Differentiable maps of $\mathcal{C}^l$ sections)**.** *Let E, F be vector bundles over M.*

(1) Let W be an open,set in E. Let $\phi : E \supseteq_{open} W \to F$ be a $\mathcal{C}^{l+r+1}$ fibre preserving map over the identity .
Then $\phi_* : \Gamma^l(W) \to \Gamma^l(F)$ is $\mathcal{C}^r$.

(2) If $\phi \in \Gamma^{k+1}(L(E,F))$, $\phi_* : \Gamma^l(E) \to \Gamma^l(F)$ is $\mathcal{C}^k$.

The derivative of ϕ_* at $u \in \Gamma^l(E)$ is given by $(d^v\phi \circ u)_*$.

The chain rule and the mean value theorem imply that ϕ_* is continuous. The differential of ϕ_* at u is $(d^v\phi\circ u)_*$. Similar to the proof of the next theorem 2.3.9, this is seen using the continuity of $(d\phi)_*$ and the Taylor formula. We may then proceed by induction.

The aim of the remaining part of this section is to show such a statement for sections of Sobolev class. In the local picture, we have the following classical theorem.

2.3.8 Theorem. *[32, 9.9] Let $0 < l - m/p$, and let $U \subseteq \mathbb{R}^m$ an open relative compact set with smooth boundary. Let $\phi \in \mathcal{C}^\infty(U \times \mathbb{R}^e, U \times \mathbb{R}^f)$. Then $\phi_* : \Gamma^{l,p}(U, U \times \mathbb{R}^e) \to \Gamma^{l,p}(U, U \times \mathbb{R}^f), \phi(s)(x) = \phi(x, s(x))$ is $\mathcal{C}^\infty$. Especially, this theorem, applied to ϕ constant in U, implies lemma 2.2.14.*

The following is a formulation of this statement in global language.

2.3.9 Theorem (Differentiable maps of Sobolev sections)**.** *Let E, F be vector bundles over M .*

(1) Let $0 < l - m/p$, W an open set in E and $\phi : E \supseteq_{open} W \to F$ be a $\mathcal{C}^{l+r+1}$ fibre preserving map over the identity .
Then $\phi_* : \Gamma^{l,p}(W) \to \Gamma^{l,p}(F)$ is $\mathcal{C}^r$.

(2) If $\phi \in \Gamma^\infty(L(E, F))$, $\phi_* : \Gamma^{l,p}(E) \to \Gamma^{l,p}(F)$ is $\mathcal{C}^\infty$.

The derivative of ϕ_* at $u \in \Gamma^{l,p}(E)$ is given by $(d^v\phi \circ u)_*$.

We have to proof (1) only. The global derivatives, we introduced before, and the global description of Sobolev spaces of sections 2.2.7(1), can be used to give a proof which does not lead this statement back to the classical case using local representations. The arguments, however, follow those in the proof for the classical case, see [32].

The first step in the proof is the following generalization of Hölders inequality for multiplication of elements in $\Gamma^{l,p}$ spaces. It is an application of the Sobolev embedding theorem.

2.3.10 Lemma. Let $p \geq 1$. Let $E_1, E_2, ..., E_k$ be vector bundles over M. Let $\gamma = (\gamma_1, \ldots, \gamma_k)$ such that $\frac{m}{p} - \gamma_i \geq 0$ for all i and $\frac{m}{p} \geq \sum_{\gamma_i > 0} \gamma_i$. If moreover some $\gamma_i = 0$, we require $\frac{m}{p} > \sum_{\gamma_i > 0} \gamma_i$.
Then the evaluation

$$\begin{array}{ccc} E_1 \times L(E_1, E_2) \times \cdots \times L(E_{k-1}, E_k) & \to & E_k \\ (u, A_1, \cdots, A_{k-1}) & \mapsto & A_{k-1}A_{k-2}A_1u \end{array}$$

induces a continuous map from the set

$$\Gamma^{\frac{m}{p}-\gamma_1,p}(E_1) \oplus \Gamma^\infty(L(E_1, E_2)) \oplus \cdots \oplus \Gamma^\infty(L(E_{k-1}, E_k)) \subset \\ \Gamma^{\frac{m}{p}-\gamma_1,p}(E_1) \oplus \Gamma^{\frac{m}{p}-\gamma_2,p}(L(E_1, E_2)) \oplus \cdots \oplus \Gamma^{\frac{m}{p}-\gamma_k,p}(L(E_{k-1}, E_k))$$

to $\Gamma^{0,p}(E_k)$.

Proof. [32, 9.4] The proof relies on the Sobolev embedding theorems and the Hölder inequality. We choose q_i such that the inclusions $\Gamma^{\frac{m}{p}-\gamma_i,p} \subseteq \Gamma^{0,q_i}$ are continuous, i.e. $m/q_i \geq \gamma_i$ and $1 \leq p \leq q_i$. Moreover we need that the Hölder inequality holds, i.e. $\sum_i \frac{1}{q_i} \leq \frac{1}{p}$.
So if $\gamma_i > 0$ set $q_i = \frac{m}{\gamma_i}$.
If $\gamma_i < 0$ set $q_i = \infty$.
If $\gamma_i = 0$ choose q_i such that $\sum_i \frac{1}{q_i} < \frac{1}{p}$ is satisfied. This is possible as in this case we require that $m/p > \sum_{\gamma_i > 0} \gamma_i$.

It makes no sense to multiply distributions, but distributions can be multiplied with $\mathcal{C}^\infty$ functions (see [35, §2.2]). So the evaluation is defined, and continuous, on the subset given above. ☐

2.3.11 Lemma. Let $l - m/p > 0$. Let E, F and G be vector bundles over M.

(1) A bilinear, fibrewise non-degenerated map

$$\phi : E \times G \to F$$

induces a bilinear bounded map

$$\begin{array}{ccc} \Gamma^{l,p}(E) \times \Gamma^{l,p}(G) & \to & \Gamma^{l,p}(F) \\ (u,v) & \mapsto & \phi(u,v) \end{array} .$$

This means that the following inequality holds:

$$\|\phi(u,v)\|_{l,p} \leq C(l,p,m)\|u\|_{l,p}\|v\|_{l,p} \; .$$

Thus the evaluation map

$$\begin{array}{ccc} E \times L(E,F) & \to & F \\ (e,A) & \mapsto & Ae = ev_e A = ev_A e \end{array}$$

induces an injective, linear and continuous map

$$\begin{array}{cccc} (.)_* : & \Gamma^{l,p}(L(E,F)) & \to & \mathcal{L}_b(\Gamma^{l,p}(E), \Gamma^{l,p}(F)) \\ & R & \mapsto & ev_R = R_* \end{array} ,$$

where $\mathcal{L}_b$ carries the operator norm.

(2) Let W be an open precompact set in E. The evaluation map

$$\begin{array}{ccc} W \times \mathcal{C}^{l+1}(E,F) & \to & F \\ (e,\phi) & \mapsto & \phi(e) = ev_\phi e \end{array} ,$$

which is linear in $\mathcal{C}^{l+1}(E,F)$, induces an injective, linear map

$$\begin{array}{cccc} (.)_* : & \mathcal{C}^{l+1}(E,F) & \to & \mathcal{C}^0(\Gamma^{l,p}(W), \Gamma^{l,p}(F)) \\ & \phi & \mapsto & ev_\phi = \phi_* \end{array} .$$

The following inequality holds

$$\|\phi(v) - \phi(u)\|_{l,p} \leq C(l,p,m)\|\phi\|_{\mathcal{C}^{l+1},\overline{W}} P(\|v\|_{l,p}, \|v-u\|_{l,p})\|u-v\|_{l,p} \; ,$$

where P is a polynomial with positive coefficients of order at most l.

2.3.12 Remark. If $\Gamma(W)$ is empty, (2) holds trivially.

Proof. (1) As the identification of G with a sub vector bundle of $L(E,F)$ (by ϕ being non-degenerate) is a fibrewise linear injection, it suffices to show (1) for the evaluation map. By the chain rule 2.3.6, if R say is a $\mathcal{C}^\infty$ section,

$$\nabla^l ev(R,v) = \sum_{\substack{\alpha \leq l \\ \beta \leq \alpha}} \binom{\alpha}{\beta} \nabla^\beta R \nabla^{\alpha-\beta} v \; .$$

By the lemma, evaluation

$$\Gamma^{m/p-(m/p-l+\beta),p} \oplus \Gamma^{m/p-(m/p-l+\alpha-\beta),p} \to \Gamma^{0,p}$$

is continuous, because

$$(m/p - l + \beta) + (m/p - l + \alpha - \beta) < 0 + m/p - l + \alpha \leq m/p \,.$$

Thus $ev(R, v)$ is continuous for all $R \in \Gamma^\infty(G)$ and the stated inequality holds.

Let R_n be a sequence of smooth function converging to an arbitrary $R \in \Gamma^{l,p}(G)$. Since $ev(R_n, v)$ is continuous by what we said above, its limit in $\Gamma^{l,p}$ exists. $ev(R, v)$ coincides with this limit, as they coincide in $\Gamma^{0,p}$. Thus the inequality $\|ev(R, v)\| \leq C(l, p, m)\|R\|_{l,p}\|v\|_{l,p}$ holds. Compare [1, 5.23] or [32, chapter 9] for more details.

(2) The composition $\Gamma^{l,p}(W) \to \Gamma^0(W) \xrightarrow{\phi_*} \Gamma^0(F) \to L^{0,p}(F)$ is continuous, and

$$\|\phi(v) - \phi(u)\|_{0,p} \leq vol(M)\|\phi(v) - \phi(u)\|_{\mathcal{C}^0} \leq \|\phi\|_{\mathcal{C}^l,\overline{W}} C vol(M)\|u - v\|_{l,p} \,. \tag{*}$$

For the remaining terms we proceed as follows. Let u and v be smooth. Set $r_t(x) = v(x) + t(u(x) - v(x))$. We use the fundamental theorem of calculus, the product rule and Faá di Brunos formula (2.3.6) to compute

$$\begin{aligned}
\big|\nabla^j\big(\phi(u(x)) - \phi(v(x))\big)\big| &= \Big|\int_0^1 \partial_t \nabla^j \phi(r_t(x))dt\Big| \\
&= \Big|\int_0^1 \frac{d}{ds}|_{s=0} \nabla^j \phi(r_{t+s}(x))dt\Big| \\
&= \Big|\int_0^1 \nabla^j (d^v\phi)(r_t(x))(u(x) - v(x))dt\Big| \\
&\leq \big|\nabla^j\big((d^v\phi)(r_t(x))\big)(u(x) - v(x))\big| \\
&\leq C \sum_{\substack{\beta \leq j \\ \gamma \leq \beta}} \big|\nabla^\beta\big(d^v\phi(r_t(x))\big)\big(\nabla^{\beta-\gamma}u(x) - \nabla^{\beta-\gamma}v(x)\big)\big| \\
&\leq C \sum_{\substack{\beta \leq j \\ \gamma \leq \beta}} \sum_{\delta,s,\alpha} C_{\delta,s,|\alpha|=s} \big|(\nabla_1^\delta (d^v)^{s+1}\phi)(r_t(x)) \\
&\qquad (\nabla r_t(x))^{\alpha_1} \ldots (\nabla^\beta r_t(x))^{\alpha_\beta}\big|\big|\nabla^{\beta-\gamma}u(x) - \nabla^{\beta-\gamma}v(x)\big| \,,
\end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_\beta)$ and $\beta = \delta + \alpha_1 + 2\alpha_2 + \ldots + \beta\alpha_\beta$.
By Hölder's inequality lemma 2.3.10 (similar to [32, 9.8]) the map

$$(r, \ldots, r, u) \mapsto (\nabla r(x))^{\alpha_1} \cdots (\nabla^\beta r(x))^{\alpha_\beta} \nabla^{\beta-\gamma} u$$

induces a continuous map

$$\bigoplus^{|\alpha|+\beta-\gamma} \Gamma^{l,p} \to \Gamma^{0,p} \,.$$

Conclude that

$$\|\nabla^j(\phi(u) - \phi(v))\|_{0,p} \leq C(l, p, m)\|\phi\|_{\mathcal{C}^{l+1},\overline{W}} P(\|v\|_{l,p}, \|v - u\|_{l,p})\|u - v\|_{l,p} \,,$$

where P is a polynomial of degree at most l. Combined with (*), this shows that the stated inequality holds.

If v_n is a sequence of smooth functions converging $H^{l,p}$ to v, this formula says that $\phi(v_n)$ is a Cauchy sequence in $\Gamma^{l,p}(F)$ and converges, to $L \in \Gamma^{l,p}(F)$ say. L and $\phi(v)$ coincide, because

$$\|L - \phi(v)\|_{0,p} \leq \|L - \phi(v_n)\|_{0,p} + \|\phi(v_n) - \phi(v)\|_{0,p} \longrightarrow 0 \, .$$

Thus $\phi_* : \Gamma^{l,p}(W) \to \Gamma^{l,p}(F)$ is continuous. In fact

$$\begin{aligned} \|\phi(v) - \phi(u)\|_{l,p} = & \lim_{n\to\infty} \|\phi(v_n) - \phi(u_n)\|_{l,p} \\ \leq & \lim_{n\to\infty} C(l,p,m)\|\phi\|_{\mathcal{C}^{l+1}} P(\|v_n\|_{l,p}, \|v_n - u_n\|_{l,p})\|u_n - v_n\|_{l,p} \\ = & C(l,p,m)\|\phi\|_{\mathcal{C}^{l+1}} P(\|v\|_{l,p}, \|v - u\|_{l,p})\|u - v\|_{l,p}. \end{aligned}$$

□

Note the following corollary in going by

2.3.13 Corollary (Sobolev Extension of Partial Differential Operators)*.* Let $\mathrm{P} : \Gamma^\infty(E) \to \Gamma^\infty(F)$ be a linear partial differential operator of order k with $\mathcal{C}^r$ coefficients.
If $l - n/p > 0$ and $l \leq r$, P extends to a continuous map $\Gamma^{l+k,p}(E) \to \Gamma^{l,p}(F)$.

2.3.14 Remark. [17, III5.2] If P is an elliptic partial differential operator its Sobolev extension is a Fredholm operator whose index does not depend on l and p.

Now let $\phi : E \to F$ denote a fibre preserving smooth map, and let W be precompact and open in E. The vertical derivatives of the restriction of ϕ to W are smooth maps

$$d^v\phi : W \to L(E,F),\; d^v d^v\phi : W \to L(E \oplus E, F),\; \ldots \, .$$

Let $\mathcal{W} := \Gamma^{l,p}(W)$. The induced maps

$$\begin{aligned} \phi_* :& \mathcal{W} \to \Gamma^{l,p}(F) \, , \\ (d^v\phi)_* :& \mathcal{W} \to \Gamma^{l,p}(L(E,F)) \to \mathcal{L}_b(\Gamma^{l,p}(E), \Gamma^{l,p}(F)) \, , \\ (d^v d^v\phi)_* :& \mathcal{W} \to \Gamma^{l,p}(L(E\oplus E,F)) \to \mathcal{L}_b(\Gamma^{l,p}(E) \times \Gamma^{l,p}(E), \Gamma^{l,p}(F)) \, , \\ & \vdots \end{aligned}$$

are continuous by the lemma.

Proof of 2.3.9. The derivative of ϕ_* at $u \in \Gamma^{l,p}(E)$ is given by $(d^v\phi \circ u)_*$. Differentiability is a local question. So let W be a precompact open neighbourhood of u, so that ϕ is $\mathcal{C}^l$ in an open neighbourhood of $\overline{W}$. Then we have to compute the derivative of $\phi_* : \Gamma^{l,p}(W) \to \Gamma^{l,p}(F)$. We have to show that

$$\lim_{\|v\|_{l,p}\to 0} \frac{\|\phi(u+v) - \phi(u) - d^v\phi(u)(v)\|_{l,p}}{\|v\|_{l,p}} = 0 \, .$$

By the fundamental theorem of calculus we have

$$\|\phi(u+v) - \phi(u) - d^v\phi(u)(v)\|_{l,p} = \|\int_0^1 \big(d^v\phi(u+sv) - d^v\phi(u)\big) v \, ds\|_{l,p} \, .$$

Using the lemma for the map $d^v\phi : W \to L(E,F)$, we interpret $\big(d^v\phi(u+sv) - d^v\phi(u)\big)v$ as a continuous curve $[0,1] \to \Gamma^{l,p}(F)$. Choose $t \in [0,1]$, such that

$$\|\int_0^1 \big(d^v\phi(u+sv) - d^v\phi(u)\big)v\, ds\|_{l,p} \le \|\big(d^v\phi(u+tv) - d^v\phi(u)\big)(v)\|_{l,p} \ .$$

By the first part of the lemma this can be estimated by

$$C\|d^v\phi(u+tv) - d^v\phi(u)\|_{l,p}\|v\|_{l,p} \ .$$

Since $(d^v\phi)_* : \mathcal{W} \to \mathcal{L}_b(\Gamma^{l,p}(E), \Gamma^{l,p}(F))$ is continuous by the second part of the lemma, $\lim_{\|v\|_{l,p}\to 0}\|d^v\phi(u+tv) - d^v\phi(u)\|_{l,p}$ tends to zero. Thus $(d^v\phi)_*$ is the derivative of ϕ_*.
We now proceed inductively. $d^v\phi$ is a map of the same type ϕ is. Thus repeating the above arguments yields that $d(d\phi_*) = d(d^v\phi)_* = (d^v d^v\phi)_*$, which is continuous. If ϕ_* is $\mathcal{C}^k$ $(k < r)$ and $d^k\phi_* = ((d^v)^k\phi)_*$, as $(d^v)^l\phi$ is $\mathcal{C}^{l+r-k}$ and fibre preserving, $d^l\phi_*$ is $\mathcal{C}^1$ and $d^{l+1}\phi_* = d((d^v)^l\phi)_* = ((d^v)^{l+1}\phi)_*$.
Thus ϕ_* is $\mathcal{C}^r$, and if ϕ is $\mathcal{C}^\infty$, ϕ_* is $\mathcal{C}^\infty$ by induction.

$\square$

2.4 Manifolds of fibre bundle sections

The aim of this section is to describe the manifold structure on the space of fibre bundle sections $\Gamma(P \to M)$ (2.4.8). This manifold is modelled on its tangent space $\Gamma(u^*VP)$, the sections of class Γ in the vertical bundle of P lying over $u \in \Gamma^\infty(P)$.

We determine this canonical $\mathcal{C}^\infty$ Banach manifold structure on $\Gamma^l(P)$ and $\Gamma^{l,p}(P)$, whereas $\Gamma^\infty(P)$ is equipped with the canonical structure of a $\mathcal{C}^\infty$ manifold in the sense of [15] (or [7]).
There is a family of diffeomorphisms from vector bundles to open sets covering P, so that we can locally think of P as of a vector bundle. These diffeomorphisms induce homeomorphisms between sections in vector bundles and spaces of sections in open sets in P covering $\Gamma(P)$. It suffices to take charts centred at $u \in \Gamma^\infty(P)$. Then the chart changes are smooth $\Gamma(u_1^*VP) \to \Gamma(u_2^*VP)$ due to the results shown just before.

In [32] Palais introduces a category of function spaces for which these ideas apply and which include the classes of functions we discussed, see also [31]. [5], [21], [22] and [15] may also serve as a reference.

2.4.1 Lemma. Let (P, p, M) be a fibre bundle. Then, for all $u \in \Gamma^\infty(P)$, there are smooth diffeomorphisms λ mapping the vector bundle u^*VP onto an open set $V_{u,\lambda} \subseteq P$ of $u(M)$, such that $\lambda(0_x) = u(x)$ and $p(\lambda(\xi)) = (u^*\pi_P)(\xi)$. $V_{u,\lambda}$ is a sub fibre bundle of P that contains $u(M)$ as a strong deformation retract in a fibrewise fashion.

2.4.2 Definition. We denote the set of diffeomorphism from u^*VP to an open set of $u(M)$, with the properties stated in the lemma, by Λ_u.

Proof. Cf. the proof of [31, XB5]. We construct one such λ.

$u(M)$ is a compact submanifold in P. Take Riemannian structures g_x, $x \in M$, on the fibres of P. The fibre exponential

$$exp : VP \to P,$$

fibrewise given by

$$exp_{p(y)} : T(P_{p(y)}) \to P_{p(y)}\,, \qquad y \in P,$$

is a unique solution of the (fibrewise) geodesic equation with initial condition $exp(0_x) = x$ and $dexp(0_y) = id_{V_yP}$. Especially, $dexp(0_y) : V_yP \to V_yP$ is surjective.

Define the normal sub bundle $(Nu(M), u(M))$ in TP such that each fibre $N_{u(x)}u(M)$ is the orthogonal complement to $T_{u(x)}u(M) = T_xu(T_xM)$.

Since $Tp \circ Tu = id$, there is an isomorphism j, which for each x identifies the fibre $(T_{u(x)}u(M))^\perp$ of $Nu(M)$ with the fibre $(TP_x)_{u(x)} = V_{u(x)}P$ of VP. Thus there is a vector bundle isomorphism

$$j:\ u^*VP \xrightarrow{\cong} Nu(M)\,.$$

By the implicit function theorem for $exp \circ j^{-1}$ there is a local diffeomorphism $\tilde{\lambda} : Nu(M) \supseteq U \to V_{u,\lambda} \subseteq P$ of open neighbourhoods of $u(M)$.

Using the Riemannian structure construct a smooth fibre preserving retraction

$$r : N(u(M)) \to \tilde{U} \subseteq U\,.$$

Then $\lambda = \tilde{\lambda} \circ r \circ j$ has the properties stated in the lemma. □

2.4.3 Remark. Similarly to the above proof we see that

$$VP \xrightarrow{(\pi_P, exp)} V \subseteq P \times_M P$$

is a diffeomorphism onto an open neighbourhood V of the diagonal in $P \times_M P$. This is a fibred version of [23, 22.7(6)].
Define an open neighbourhood $V_u \subseteq P$ as

$$\{y \in P\,:\,(u(x), y_x) \in V_x \subseteq P_x \times P_x\} = pr_2(V|_{u(M)\times_M P}).$$

Then $exp_u : u^*VP \to VP \to V_u$ is in Λ_u.

We will soon need this special charts in the proof of theorem 2.4.8, where we use that the open set V does not depend on the section u.

2.4.4 Definition. Let $u \in \Gamma^\infty(P)$ and $\lambda \in \Lambda_u$. We then define $\Psi_{u,\lambda}$ as composition with λ, i.e.

$$\begin{aligned}\Psi_{u,\lambda} &: \Gamma(u^*VP) \to \mathcal{V}_{u,\lambda}\\ \Psi_{u,\lambda}(\xi) &= \lambda_*(\xi) = \lambda \circ \xi\end{aligned} \tag{1}$$

where the set $\mathcal{V}_{u,\lambda}$ is defined as

$$\Gamma(V_{u,\lambda}) = \left\{v \in \Gamma(P)\,:\,v(x) \in V_{u(x),\lambda} \text{ for all } x \in M\right\}\,.$$

2.4.5 Remark. $\mathcal{V}_{u,\lambda}$ is open in $\Gamma^0(P)$, because $V_{u,\lambda}$ is open in P.

2.4.6 Lemma. Let $u, v \in \Gamma^\infty(P)$, $\lambda \in \Lambda_u$ and $\lambda' \in \Lambda_v$. The composition with

$$\tau := \lambda^{-1} \circ \lambda' : v^*VP \supseteq U_{(v,\lambda),(u,\lambda')} \to U_{(u,\lambda'),(v,\lambda)} \subseteq u^*VP$$

is the smooth map

(1) $\tau_* = \Psi_{u,\lambda}^{-1} \circ \Psi_{v,\lambda'} : \Gamma^l(v^*VP) \supseteq \mathcal{U}_{(v,\lambda),(u,\lambda')} \to \mathcal{U}_{(u,\lambda'),(v,\lambda)} \subseteq \Gamma^l(u^*VP)$, if $0 \le l < \infty$.

(2) $\tau_* = \Psi_{u,\lambda}^{-1} \circ \Psi_{v,\lambda'} : \Gamma^{l,p}(v^*VP) \supseteq \mathcal{U}_{(v,\lambda),(u,\lambda')} \to \mathcal{U}_{(u,\lambda'),(v,\lambda)} \subseteq \Gamma^{l,p}(u^*VP)$, if $l - m/p > 0$.

(3) $\tau_* = \Psi_{u,\lambda}^{-1} \circ \Psi_{v,\lambda'} : \Gamma^\infty(v^*VP) \supseteq \mathcal{U}_{(v,\lambda),(u,\lambda')} \to \mathcal{U}_{(u,\lambda'),(v,\lambda)} \subseteq \Gamma^\infty(u^*VP)$ in the sense of [15].

Proof. τ is smooth, fibre preserving and sits over the identity. Thus τ_* is smooth in (1) by 2.3.7, in (2) by 2.3.9 and in (3) by [15, 3.13]. □

2.4.7 Remark. If $u \in \Gamma^0(P)$, we have the problem that we only know that u^*VP has $\mathcal{C}^0$ transition functions and the latter results are not applicable.

2.4.8 Theorem. *Let M be a closed manifold and $P \to M$ a fibre bundle.*

(1) $\Gamma^l(P)$ equipped with the trace topology 2.1.11 in $\mathcal{C}^l(M,P)$ admits a paracompact, Hausdorff, and second countable (canonical) $\mathcal{C}^\infty$ Banach manifold structure modelled on $\Gamma^l(u^*VP)$, where $u \in \Gamma^\infty(P)$. The $\mathcal{C}^\infty$ atlas is given by the collection $(\mathcal{V}^l_{u,\lambda}, \Psi_{u,\lambda})_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}$.

(2) If $l-m/p > 0$, $\Gamma^{l,p}(P)$ equipped with the trace topology 2.2.19 in $H^{l,p}(M,P)$ admits a paracompact, Hausdorff, and second countable (canonical) $\mathcal{C}^\infty$ Banach manifold structure modelled on $\Gamma^{l,p}(u^*VP)$, where $u \in \Gamma^\infty(P)$. The $\mathcal{C}^\infty$ atlas is given by the collection $(\mathcal{V}^{l,p}_{u,\lambda}, \Psi_{u,\lambda})_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}$.

(3) $\Gamma^\infty(P)$ equipped with the trace topology in $\mathcal{C}^\infty(M,P)$ admits a paracompact, Hausdorff (canonical) $\mathcal{C}^\infty$ Fréchet manifold structure modelled on $\Gamma^\infty(u^*VP)$. The $\mathcal{C}^\infty$ atlas is given by the collection $(\mathcal{V}^\infty_{u,\lambda}, \Psi_{u,\lambda})_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}$ ([15, 42.1]).

2.4.9 Remark (Functoriality). This construction is functorial: Let (P,p,M) and (P',p',M) fibre bundles. If $\phi : P \to Q$ is a smooth fibre bundle map over the identity ($p(y) = q(\phi(y))$, $y \in P$), $\phi_* : \Gamma(P) \to \Gamma(Q)$ is smooth.
A local representation of ϕ_* is given by $\big((\lambda^Q)^{-1} \circ \phi \circ \lambda^P\big)_*$. The tangent map of the local representation has the form

$$\big(K^{VQ} \circ T(\lambda^Q)^{-1} \circ T\phi \circ T\lambda^P \circ (\pi_{u^*VP}, 0, K^{VP})^{-1}\big)_* .$$

Proof of theorem 2.4.8. Let $u, v \in \Gamma^\infty(P)$.

The chart change is given by τ_*, which is smooth by the above lemma.

The collection of all sets $(\mathcal{V}_u)_{u\in\Gamma^\infty(P)}$ covers $\Gamma(P)$. To show this we use remark 2.4.3: Let $w \in \Gamma^0(P)$. Since $\Gamma^\infty(P)$ is dense in $\Gamma^0(P)$ we may choose $v \in \Gamma^\infty(P)$ such that $(w(x), v(x)) \in W \subseteq V$ for all x and for an arbitrary neighbourhood W of the diagonal. It follows that $(v(x), w(x)) \in V$ for all x, i.e. $w \in \Gamma^0(V_v)$. Then there is a section $\zeta \in \Gamma^0(v^*VP)$ such that $exp_v(\zeta) = w$.

(2) $(\mathcal{V}^{l,p}_{u,\lambda})_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}$ is a smooth atlas for $\Gamma^{l,p}(P)$.
There is $\zeta \in \Gamma^0(v^*VP)$ such that $exp_v(\zeta) = w$. ζ consequently is an element in $\Gamma^{l,p}(v^*VP)$. Thus $(\mathcal{V}^{l,p}_u)_{u\in\Gamma^\infty(P)}$ covers all of $\Gamma^{l,p}(P)$.
The topology of vector bundle sections for $\mathcal{V}^{l,p}_u$ and the topology as a subspace of $\Gamma^{l,p}(P)$ is equivalent, i.e. $\Psi_u : \Gamma(u^*VP) \to \mathcal{V}_u$ is a homeomorphism with inverse given by

$$\Psi_{u,\lambda_u}^{-1}(v) := (\lambda_u^{-1})_*(v) = \lambda_u^{-1}(v).$$

Here we have to use the local version 2.2.14 of 2.3.11. $\Gamma^{l,p}(P)$ is paracompact, Hausdorff, and second countable by 2.2.16.

(1)(3) The arguments are as for (2) using 2.3.7, 2.1.10, or the corresponding results of [15]. $\square$

2.4.10 Remark ($C(M,N) \cong \Gamma(M \times N)$). We will constantly use the identification $C(M,N) \cong \Gamma(M, M \times N)$. The mappings are given by $s \mapsto (id_M, s)$ and the projection, which are local inverses and smooth.

2.4.11 Example (Atlas for $\Gamma(M \times N)$)**.** Let

$$\Psi_u = \big(graph(exp_u)\big)_* = \big(u^*(\pi_N, exp)\big)_*$$

where $(u^*\pi_N, u^*exp) : u^*TN \to M \times N$ is a diffeomorphism from a neighbourhood of the zero section onto a neighbourhood of the graph of u in $M \times N$.
For $0 \le l \le \infty$, the family $(\mathcal{V}_u, \Psi_u)_{u \in \Gamma^l(P)}$ is a $\mathcal{C}^l$ atlas for $\Gamma^l(P)$ modelled on $\Gamma^l(u^*VP)$.
If $l - m/p > 0$, the family $(\mathcal{V}_u, \Psi_u)_{u \in \Gamma^\infty(P)}$ is a smooth atlas for $\Gamma^{l,p}(P)$ modelled on $\Gamma^{l,p}(u^*VP)$ $(u \in \Gamma^\infty(P))$.

2.4.12 Example (Atlas for $\Gamma(M \times TN)$.)**.** [22, 10.11] Let ψ_u be the local diffeomorphism

$$(0_N \circ u)^*\big(T(\pi_N, exp) \circ \kappa_N\big) : (0_N \circ u)^*TTN \to M \times u^*TN\ .$$

κ_N denotes the canonical conjugation $(TTN, T\pi_N) \to (TTN, \pi_{TN})$. In fact the map $T(\pi_N, exp) \circ \kappa_N : TTN \to TN \times TN$ is a local diffeomorphism of a zero neighbourhood of the diagonal.

2.5 Vector bundles over $\Gamma(P)$

Let $E \xrightarrow{\pi} P$ be a vector bundle of Banach spaces over P. We now want to show that $\Gamma(E) := \Gamma(E \to M)$ is a vector bundle over $\Gamma(P)$.
The projection π induces a projection

$$pr := \pi_* : \Gamma(E) \to \Gamma(P).$$

If $s \in \Gamma(E)$ and $pr(s) = v$, the set $pr^{-1}(v) \subseteq \Gamma(E)$ is a vector space. $\Gamma(E)$ is equipped with this fibred vector space structure over $\Gamma(P)$.
By the last theorem 2.4.8 $\Gamma(E)$ admits a canonical manifold structure with charts given there.

In 2.5.1 we show that $\Gamma(E)$ is a vector bundle over $\Gamma(P)$ in a canonical way. Let $u \in \Gamma^\infty(P)$ and $\lambda \in \Lambda_u$. We find vector bundle charts

$$\Upsilon_{u,\lambda} : \mathcal{V}_{u,\lambda} \times \Gamma(u^*VP) \to \Gamma(E)|_{\mathcal{V}_{u,\lambda}}$$

given by composition with the trivialization of E over $V_{u,\lambda}$ induced by homotopy equivalence.

$\Gamma(VP)$ consequently is a vector bundle of Banach spaces over $\Gamma(P)$. We show the intuitive statement that $\Gamma(VP)$ is canonically isomorphic to the tangent bundle $T\Gamma(P)$. In fact, if one differentiates smooth curves of sections starting at u one sees that $\Gamma^\infty(u^*VP) \cong T_u\Gamma^\infty(P)$. We give charts for $\Gamma(VP)$ which seem to be natural in this case and compare their chart changes with the chart changes of the definition of the tangent bundle (2.5.4).

2.5.1 Theorem. *Let (P,p,M) be a fibre bundle and (E,π,P) a vector bundle.*

(1) For $0 \le l < \infty$, $\Gamma^l(E \to M)$ is a $\mathcal{C}^\infty$ vector bundle of Banach spaces over $\Gamma^l(P)$ with typical fibres $pr^{-1}(u) = \{s \in \Gamma^l(E \to M) : \pi \circ s = u\} = \Gamma^l(u^*E)$, where $u \in \Gamma^\infty(P)$.

(2) If $l - m/p > 0$, $\Gamma^{l,p}(E \to M)$ is a $\mathcal{C}^\infty$ vector bundle of Banach spaces over $\Gamma^{l,p}(P)$ with typical fibres $pr^{-1}(u) = \{s \in \Gamma^{l,p}(E \to M) : \pi \circ s = u\} = \Gamma^{l,p}(u^*E)$, where $u \in \Gamma^\infty(P)$.

(3) $\Gamma^\infty(E \to M)$ is a $\mathcal{C}^\infty$ vector bundle of Fréchet spaces over $\Gamma^\infty(P)$ with typical fibres $pr^{-1}(u) = \{s \in \Gamma^\infty(E \to M) : \pi \circ s = u\} = \Gamma^\infty(u^*E)$, where $u \in \Gamma^\infty(P)$.

This means that there is a family of neighbourhoods $(\mathcal{W}_u)_{u \in \Gamma^\infty(P)}$ covering $\Gamma(P)$ and fibrewise linear and bounded diffeomorphisms Υ_u such that

$$\begin{tikzcd} \mathcal{W}_u \times \Gamma(u^*E) \arrow[rr, "\Upsilon_u", "\cong"'] \arrow[dr, "p_1"'] & & \Gamma(E)_{|\mathcal{W}_u} \arrow[dl, "pr"] \\ & \mathcal{W}_u & \end{tikzcd} ,$$

where $\Gamma(E)|_{\mathcal{W}_u} = pr^{-1}(\mathcal{W}_u) = \Gamma(E|_{W_u})$.

Proof. Let $u \in \Gamma^\infty(P)$, $\lambda^P \in \Lambda_u^P$ and W_{u,λ^P} as in 2.4.1. Then the zero section $u(M) \subseteq W_{u,\lambda^P}$ is a deformation retract of W_{u,λ^P}. Let

$$\upsilon_u : W_{u,\lambda^P} \times_M u^*E \to E|_{W_{u,\lambda^P}} \tag{2}$$

be the induced trivialization such that $(\upsilon_u)_x = \upsilon_{u(x)}$, which satisfies $\pi \circ \upsilon_u = pr_1$, $\upsilon_u(v, 0_x) = v$ and $\upsilon_u(u,\xi) = \xi$. A vector bundle chart of $\Gamma(E)$ at u is then given by

$$\Upsilon_{u,\lambda^P} := (\upsilon_u)_* : \mathcal{W}_{u,\lambda^P} \times \Gamma(u^*E) \to \Gamma(E)|_{\mathcal{W}_{u,\lambda^P}} .$$

Υ_{u,λ^P} is a smooth diffeomorphism $\mathcal{W}_{u,\lambda^P} \times \Gamma(u^*E) \to \Gamma(E)|_{\mathcal{W}_{u,\lambda^P}}$ by 2.4.9. The family $(\Gamma(E)|_{\mathcal{W}_{u,\lambda^P}})_{u \in \Gamma^\infty(P), \lambda^P \in \Lambda_u}$ covers $\Gamma(E)$ because $\mathcal{W}_{u,\lambda^P}$ covers $\Gamma(P)$. Thus $(\Upsilon_{u,\lambda^P}, \Gamma(E)|_{\mathcal{W}_{u,\lambda^P}})_{u \in \Gamma^\infty(P), \lambda^P \in \Lambda_u}$ is a $\mathcal{C}^\infty$ vector bundle atlas. □

2.5.2 Remark.

$$0 \to \pi^*E \to VE \to VP \to 0$$

is an exact sequence of vector bundles over E. In fact, there is a projection $VE \to \pi^*E$ by the universal property of the pullback. The sequence is exact as it is exact in each fibre. We can choose a vector bundle diffeomorphism

$$\begin{aligned} \varphi : 0_u^*VE &\to u^*VP \oplus_M u^*E \\ \xi &\mapsto (\xi^1, \xi^2) . \end{aligned}$$

The diffeomorphism $\lambda^E := \upsilon_u \circ (\lambda^P, id) \circ \varphi : 0_u^*VE \to E|_{W_{u,\lambda^P}}$ given by

$$\begin{array}{ccccccc} 0_u^*VE & \to & u^*VP \oplus_M u^*E & \to & W_{u,\lambda^P} \times_M u^*E & \to & E|_{W_{u,\lambda^P}} \\ \xi_x & \mapsto & (\xi_x^1, \xi_x^2) & \mapsto & (\lambda_x^P(\xi_x^1), \xi_x^2) & \mapsto & \upsilon_{u(x)}(\lambda_x^P(\xi_x^1), \xi_x^2) \end{array}$$

for each x is linear in the fibres of u^*E. Moreover

$$\lambda^E(0_x) = \upsilon_u(\lambda^P(0_x), 0_x) = \upsilon_u(u(x), 0_x)) = 0_{u(x)}$$

and

$$p \circ \pi(\lambda^E(\xi)) = p \circ \pi(\upsilon_u(\lambda^P(\xi^1), \xi^2)) = p(\lambda^P(\xi^1)) = u^*\pi_P(\xi^1) = 0_u^*\pi_E(\xi^1, \xi^2),$$

where $\pi_P : TP \to P$ and $\pi_E : TE \to E$. Thus composition with λ^E defines a chart for the manifold structure of $\Gamma(E \to M)$ at 0_u.

Example (The Tangent Bundle of $\Gamma(P)$). As motivation consider a smooth curve u_t with values in $\Gamma^\infty(P)$ and starting point $u \in \Gamma^\infty(P)$. u_t is a smooth map $[0,1] \times M \to P$. We differentiate u_t at zero and obtain a map $\dot{u}_t : \mathbb{R} \times M \to VP$ (linear in $\mathbb{R}$) as $Tp \circ \dot{u}_t = 0$, which we may consider as an element of $\Gamma^\infty(u^*VP)$. On the other hand, if $s \in \Gamma^\infty(u^*VP)$ is given, $Fl^s(t)$ is a smooth curve in $\Gamma^\infty(P)$ such that $Fl^s(0) = u$ and $\frac{d}{dt}|_{t=0}(Fl^s(t)) = s$. Thus $T_u\Gamma^\infty(P) = \Gamma^\infty(u^*VP)$.

2.5.3 (The Vector Bundle Charts for $\Gamma(VP \to M)$). We define a vector bundle chart

$$\Upsilon^{VP}_{u,\lambda} : \mathcal{V}_{u,\lambda} \times \Gamma(u^*VP) \to \Gamma(VP)|_{\mathcal{V}_{u,\lambda}}$$

as composition with

$$\begin{aligned}\upsilon_{u,\lambda} = T\lambda \circ \mathrm{vl} \circ (\lambda, id)^{-1} :& \\ & V_{u,\lambda} \times_M u^*VP \to u^*VP \oplus_M u^*VP \to Vu^*VP \to VP|_{V_{u,\lambda}} ,\end{aligned}$$

where $\mathrm{vl} : u^*VP \oplus_M u^*VP \to Vu^*VP$ is the vertical lift. The collection $\left(\Gamma(VP)|_{\mathcal{V}_{u,\lambda}}, \Upsilon_{u,\lambda}\right)_{u \in \Gamma^\infty(P), \lambda \in \Lambda_u}$ is a vector bundle atlas for $\Gamma(VP)$.

Proof. As
$$\begin{tikzcd} u^*VP \arrow[r, "\lambda", "\cong"'] \arrow[dr, "\pi_P"'] & V_{u,\lambda} \arrow[d, "p"] \\ & M \end{tikzcd}$$
, $T\lambda : Vu^*VP \to VP|_{V_{u,\lambda}}$ is a diffeomorphism.

Clearly $\pi_P \circ \upsilon_{u,\lambda} = pr_1$. We have to check that $\upsilon_{u,\lambda}(v, 0) = v$ and that $\upsilon_{u,\lambda}(v, .) : u^*VP \to v^*VP$ is a fibre diffeomorphism which preserves the linear structure. By the properties of λ we have

$$\begin{aligned} T\lambda \circ \mathrm{vl}((p,\zeta),(p,0)) =& T\lambda(0_{VP}(p,\zeta)) \\ =& (\lambda(p,\zeta), d\lambda(p,\zeta).0) = v(p) \end{aligned}$$

and that

$$\begin{aligned} T\lambda \circ \mathrm{vl}((p,\zeta),(p,\xi)) =& T\lambda((p,\zeta,\xi)) \\ =& (\lambda(p,\zeta), d\lambda(p,\zeta).\xi) = (v(p), d\lambda(p,\zeta).\xi) , \end{aligned}$$

where $\lambda(p,\zeta) = v(p)$ for all p. $(\upsilon_{u,\lambda})_*$ is a $\mathcal{C}^\infty$ vector bundle chart in consequence. □

2.5.4 Proposition (Tangent bundle of $\Gamma(P)$). Let (P, p, M) be a fibre bundle on a closed manifold. $\Gamma(VP \to M)$ is canonically isomorphic to the tangent bundle of $\Gamma(P)$.

Proof. Take the charts $\Psi_{u_i,\lambda_i} : \Gamma(u_i^* VP) \to \mathcal{V}_{u_i,\lambda_i}$, $i = 1, 2$ of $\Gamma(P)$. The tangent bundle is given by $\bigsqcup_{v\in\Gamma(P)} E_v$, where

$$E_v := \{[\mathcal{V}_{u,\lambda}, \xi] \,:\, v \in \mathcal{V}_{u,\lambda}, u \in \Gamma^\infty(P), \lambda \in \Lambda_u, \xi \in \Gamma^\infty(u^* VP)\}\,.$$

The equivalence relation on the atlas $\big(\Psi_{u,\lambda}, \mathcal{V}_{u,\lambda}\big)_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}$ is defined by

$$(\mathcal{V}_{u_1,\lambda_1}, \xi_1) \sim (\mathcal{V}_{u_2,\lambda_2}, \xi_2) \;\Leftrightarrow\; \xi_2 = \big(d(\Psi^{-1}_{u_2,\lambda_2} \circ \Psi_{u_1,\lambda_1})\,\big(\lambda_1^{-1}(v)\big)\,\big).\xi_1\,.$$

Let $\bar\alpha : TVP \to VVP$ be a connection. Let K be the connector of VP with respect to $\bar\alpha$. The vertical lift $\mathrm{vl} : u^*VP \oplus u^*VP \to Vu^*VP$ equals

$$(\pi_{u^*VP}, 0, K)^{-1} : u^*VP \oplus u^*VP \to Vu^*VP.$$

The transition functions

$$d(\Psi^{-1}_{u_2,\lambda_2} \circ \Psi_{u_1,\lambda_1})(\lambda_1^{-1}(v)) : \Gamma(u_1^*VP) \to \Gamma(u_2^*VP)$$

with of the tangent bundle are given by composition with

$$\begin{aligned}&\nabla_2\big((\lambda_2)^{-1}\lambda_1\big)(\lambda_1^{-1}(v))\\ &\qquad = K \circ T\big((\lambda_2)^{-1}\lambda_1\big) \circ (\pi_v^* VT, 0, K)^{-1}(\lambda_1^{-1}(v), .) : u_1^*VP \to u_2^*VP\end{aligned}$$

by 2.3.7 or 2.3.9. So let

$$\xi_2 = K \circ T\big((\lambda_2)^{-1}\lambda_1\big) \circ (\pi_v^* VT, 0, K)^{-1}(\lambda_1^{-1}(v), \xi_1).$$

We have to show that this is equivalent to $(pr_2 \circ \upsilon^{-1}_{u_2,\lambda_2} \circ \upsilon_{u_1,\lambda_1})(v)\xi_1$. But this equals

$$K \circ T\lambda_2^{-1} \circ T\lambda_1 \circ (\pi_{u^*VP}, 0, K)^{-1} \circ (\lambda_1, id)^{-1}(v, \xi_1) = \xi_2.$$

Thus there is a canonical isomorphism identifying $T\Gamma(P)$ with $\Gamma(VP)$. □

2.5.5. A $\mathcal{C}^\infty$ atlas for $T\Gamma(P)$ is given by

$$(T\mathcal{V}_{u,\lambda}, T\Psi_{u,\lambda})_{u\in\Gamma^\infty(P),\lambda\in\Lambda_u}.$$

$T\Psi_{u,\lambda}$ is given by composition with the partial derivative

$$\begin{aligned}&T\lambda \circ (\pi_{u^*VP}, 0, pr_2 \circ \mathrm{vl}^{-1} \circ \bar\phi)^{-1}\\ &\qquad = T\lambda \circ (\pi_{u^*VP}, 0, K)^{-1}.\end{aligned}$$

$$\begin{tikzcd} & \Gamma(VP)|_{\mathcal{V}_{u,\lambda}} \\ \Gamma(u^*VP) \times \Gamma(u^*VP) \arrow[ur, "T\Psi_{u,\lambda}", "\cong"'] \arrow[r, "\cong", "(\Psi_{u,\lambda}, id)"'] & \mathcal{V}_{u,\lambda} \times \Gamma(u^*VP) \arrow[u, "\Upsilon_{u,\lambda}", "\cong"'] \end{tikzcd}$$

Let Q be a second fibre bundle over M and let $\phi : P \to Q$ be a smooth fibre map over M. Let $u \in \Gamma^\infty(P)$. The tangent map $T_u(\phi_*)$ then equals the composition with $(T\phi)_*$.

3 Manifolds of J-holomorphic sections

Let Σ be a Riemannian surface with complex structure j and let N be an almost complex manifold with almost complex structure J, constant in $z \in \Sigma$. We can write down the Cauchy-Riemann equation

$$Tu + J(u)Tu \circ j = 0$$

for maps $u : \Sigma \to N$. We view this nonlinear partial differential equation as a section in $\Gamma((T^{0,1}\Sigma)^* \otimes TN) \to \Gamma(\Sigma \times N)$. In this interpretation Tu is the composition of the tangent of a section in $\Gamma(\Sigma \times N)$ with the canonical connection pr_2 on $T\Sigma \times TN$. The space of connections $\mathcal{A}(\Sigma \times N)$ is an affine space modelled on $\Gamma(T^*\Sigma \otimes TN \to \Sigma \times N)$. In contrast to [19], where the space of almost complex structures is taken as parameter space, we use the space of connections. For $A \in \Gamma(T^*\Sigma \otimes TN)$, we have the perturbed Cauchy-Riemann equation

$$Tu + J(u)Tu \circ j + A(u) + J(u)A(u) \circ j = 0.$$

We interpret the Cauchy-Riemann equation as a section in the vector bundle $\Gamma((T^{0,1}\Sigma)^* \otimes TN) \times \mathcal{A}(\Sigma \times N) \to \Gamma(\Sigma \times N) \times \mathcal{A}(\Sigma \times N)$. This section is transversal to the zero section.

Its restriction to a connection is a Fredholm section. At $u \in \Gamma^\infty(\Sigma \times N)$, the derivative of a local representation is a first order operator, whose first order part is a Cauchy-Riemann operator $\Gamma^\infty(u^*VP) \to \Gamma^\infty((T^{0,1}\Sigma)^* \otimes u^*TN)$. Riemann-Roch's theorem states that the index of this operator equals

$$n(1-g) + \langle c_1(u^*TN), [\Sigma] \rangle,$$

where g denotes the genus of Σ.

Then we can apply the general theory of the first chapter to this situation. The connected component of any $\mathcal{C}^\infty$ section u in the solution space of the Cauchy-Riemann equation is a finite dimensional manifold up to a set of first category in the space of connections on $\Sigma \times N$. The dimension of this manifold for a $\mathcal{C}^\infty$ connection coincides with the index of the Cauchy-Riemann operator given above. Note however, that we cannot presume the solution space at the connection pr_2 being a finite dimensional manifold, as it is not clear that pr_2 is a regular value for the projection onto the space of connections on $\Sigma \times N$. This depends on the choice of J. In [19] is shown, that the space of J-holomorphic curves is a finite dimensional manifold up to a set of first category in the space of $\mathcal{C}^\infty$ almost complex structures on N.

The class of solutions to the Cauchy-Riemann equation depends only on the class of the connection, because the Cauchy-Riemann operator is elliptic. We use ellipticity and the Sobolev embedding theorem to show that the space of $\mathcal{C}^\infty$ connections ϕ, such that the differential of the restriction of the Cauchy-Riemann equation is surjective at every section which is J-holomorphic with respect to ϕ, is the complement of a set of first category in $\mathcal{A}^\infty(\Sigma \times N)$.

We will however work in a more general setting; namely we consider a fibre bundle $P \to \Sigma$ with a vertical almost complex structure J in stead of $\Sigma \times N$. J restricts to an almost complex structure on each fibre. Moreover there is no need to require J constant on $z \in \Sigma$.

3.1 The covariant derivative

Let $N \hookrightarrow P \xrightarrow{p} M$ be a fibre bundle. Let ϕ be a connection on P. Let

$$D_\phi : \Gamma^{l+1}(P) \to \Gamma^l(p^*T^*M \otimes VP), D_\phi(u) := \phi_u(Tu).$$

At $u \in \Gamma^\infty(P)$ we have

$$D_\phi(u) = \phi_u(Tu) = \phi(j^1(u)) = u^*\phi,$$

which is a section of $T^*M \otimes u^*VP$. If we vary the connection, the resulting map is a section in the vector bundle $\Gamma(p^*T^*\Sigma \otimes VP) \times \mathcal{A}(P) \to \Gamma(P) \times \mathcal{A}(P)$, which is transversal to every section in $\Gamma(p^*T^*\Sigma \otimes VP)$ constant in $\mathcal{A}(P)$.

In order to view D_ϕ as a section, we note the following.

3.1.1 Proposition. Let $E \to P$ be a vector bundle.
The restriction of $\Gamma^l(E)$ to $\Gamma^{l+1}(P)$ is a vector bundle over $\Gamma^{l+1}(P)$.
The restriction of $\Gamma^{l,p}(E)$ to $\Gamma^{l+1,p}(P)$ is a vector bundle over $\Gamma^{l+1,p}(P)$.

Proof. We proof the second case. The first is similar. The inclusion

$$\iota : \Gamma^{l+1,p}(P) \hookrightarrow \Gamma^{l,p}(P)$$

is continuous. The local representation of the inclusion is the linear and bounded inclusion of $\Gamma^{l+1,p}(u^*VP)$ in $\Gamma^{l,p}(u^*VP)$. It is $\mathcal{C}^\infty$ as a linear and bounded map. Thus the inclusion ι is $\mathcal{C}^\infty$. However, $\Gamma^{l+1,p}(P)$ is no submanifold in $\Gamma^{l,p}(P)$.

Let $\Upsilon_{u,\lambda} : \Gamma^{l,p}(u^*E) \times \Gamma^{l,p}(V_{u,\lambda}) \to \Gamma^{l,p}(E)|_{\Gamma^{l,p}(V_u)}$ be the chart constructed in 2.5.1. The composition of $\Upsilon_{u,\lambda}$ with

$$(id, \iota) : \Gamma^{l,p}(u^*E) \times \Gamma^{l+1,p}(V_{u,\lambda}) \to \Gamma^{l,p}(u^*E) \times \Gamma^{l,p}(V_{u,\lambda})$$

is $\mathcal{C}^\infty$. The image of $\Upsilon_{u,\lambda} \circ (id, \iota)$ are exactly those sections in $\Gamma^{l,p}(E)$ over $\Gamma^{l+1,p}(V_u)$, i.e. $\Gamma^{l,p}(E)|_{\Gamma^{l+1,p}(V_u)}$. The transition functions

$$\Upsilon_{(u,\lambda_u),(v,\lambda_v)} \circ \iota : \Gamma^{l+1,p}(W) \hookrightarrow \Gamma^{l,p}(W) \to \mathcal{L}_b(\Gamma^{l,p}(u^*E), \Gamma^{l,p}(v^*E))$$

are $\mathcal{C}^\infty$ fibrewise isomorphisms. The transition function induce a vector bundle structure on $\Gamma^{l,p}(E) \to \Gamma^{l+1,p}(P)$, such that $\Upsilon_{u,\lambda} \circ (id, \iota)$ is a vector bundle chart. $\square$

3.1.2 Remark (Local representation of D_ϕ)*.* Since a fibre bundle is a local vector bundle, a connection on a fibre bundle P over M can locally be thought of as a linear connection ϕ' on a vector bundle E over M, which has been disturbed by a fibre preserving map A of E to $T^*M \otimes E$ over the identity. So we let ϕ be represented by $\phi' + A$. Thus a local representation $D_\phi : \Gamma(E) \to \Gamma(T^*M \otimes E)$ of D_ϕ is given by

$$D_\phi(\xi) = (\phi' + A)_* T\xi = \nabla^{\phi'}\xi + A(\xi), \text{ where } \xi \in \Gamma(E).$$

3.1.3 Proposition. D_ϕ defines a section $\Gamma(P) \to \Gamma(p^*T^*M \otimes VP)$, i.e.

(1) if $0 \leq l \leq \infty$, D_ϕ is a section of $\Gamma^l(p^*T^*M \otimes VP \to M) \to \Gamma^{l+1}(P)$.

(2) if $l - m/p > 0$, D_ϕ is a section of $\Gamma^{l,p}(p^*T^*M \otimes VP \to M) \to \Gamma^{l+1,p}(P)$.

The differentiability class of D_ϕ depends on the class of the connection. If ϕ is a $\mathcal{C}^{l+r+1}$ connection on (P), D_ϕ is $\mathcal{C}^r$.

Proof. Fix a $\mathcal{C}^{l+2}$ connection ϕ of P and a $\mathcal{C}^\infty$ section u. Then the local representation of Φ_ϕ at u is given by $\nabla + A$, where ∇ is a linear connection on u^*VP and A is a $\mathcal{C}^l$ fibre preserving map of u^*VP to $T^*M \otimes u^*VP$ over the identity. If $\phi : Tu^*VP \to u^*VP$ is $\mathcal{C}^{l+r+1}$, the local representations $\xi \mapsto \nabla^{\phi'}\xi + A(\xi)$, are $\mathcal{C}^r$ by 2.3.9. □

We would like to perturb D_ϕ in the connection ϕ.

3.1.4 Example (Space of connections)*.* The space $\mathcal{A}(P)$ of connections on P is an affine subspace in $\Omega^1(P; VP)$, modelled on the vector space $\Gamma(p^*T^*M \otimes VP \to P)$.
Fix a reference connection $\phi \in \Gamma(T^*P \otimes VP)$. Then every connection in $\mathcal{A}(P)$ can be written as $\phi + a$ in terms of a one form $a \in \Gamma(p^*T^*M \otimes VP)$.
The tangent bundle of $\mathcal{A}(P)$ is consequently given by the bundle $\mathcal{A}(P) \times \Gamma(p^*T^*M \otimes VP \to P)$. At ϕ in $\mathcal{A}(P)$ the tangent space is given by $\Gamma(p^*T^*M \otimes VP)$.

In contrast to D_ϕ, the bundle $\Gamma(p^*T^*M \otimes VP)$ does not depend on the choice of a connection. Thus we can view $\Phi(u,\phi) := D_\phi u$ as a section in the vector bundle

$$\Gamma(p^*T^*M \otimes VP \to M) \times \mathcal{A}(P) \to \Gamma(P \to M) \times \mathcal{A}(P).$$

3.1.5 Proposition. Let $l > 2$, $l > k$ and $kp > m$. Then Φ is a $\mathcal{C}^{l-k-1}$ section in the vector bundle

$$\Gamma^{k,p}(p^*T^*M \otimes VP \to M) \times \mathcal{A}^l(P) \to \Gamma^{k+1,p}(P \to M) \times \mathcal{A}^l(P).$$

In view of the assumptions on l and k, Φ is at least $\mathcal{C}^1$.

Proof. A local representation of Φ at $u \in \mathcal{C}^\infty(P)$ is $(\xi, \phi) \mapsto (\nabla^{\phi'}\xi + A(\xi), \phi)$, see 3.1.2. This map is $\mathcal{C}^\infty$ in the connection. Differentiability in $\Gamma(P)$ has been discussed in 3.1.3 above. □

3.2 The Cauchy-Riemann equations

Let (Σ, j) be a Riemannian surface. Let $N \hookrightarrow P \xrightarrow{p} \Sigma$ be a fibre bundle with a smooth vertical almost complex structure $J \in L(VP, VP)$. J restricts to an almost complex structure on each fibre N.

Let $pr^{0,1}$ be the projection of $p^*T^*\Sigma \otimes VP$ onto the vector bundle of fibrewise J antilinear maps $p^*(T^{0,1}\Sigma)^* \otimes_J VP$. If $\alpha \in p^*T^*\Sigma \otimes VP$, $pr^{0,1}(\alpha)(z) = \frac{1}{2}(\alpha_z + J_z\alpha_z \circ j_z)$. We show that the space of J-holomorphic sections is a $\mathcal{C}^\infty$ (Banach) manifold. Moreover, the space of solutions with respect to a generic connection is a finite dimensional manifold. We say that a connection is generic, if it is contained in the complement of a set of first category in $\mathcal{A}(P)$. It is sufficient to show that $\mathrm{CR} := pr^{0,1}_* \circ \Phi$ is transversal to the zero section and that its restriction to an arbitrary connection is a Fredholm section for we can apply theorem 1.0.25 then. For all $v \in \Gamma(P)$, which are J-holomorphic with respect to a connection ϕ, the tangential $T_v\,\mathrm{CR}(\phi)$ is a first order elliptic operator. For all generic connections ϕ, its index coincides with the dimension of the connected component of v in the space of all J-holomorphic sections with respect to ϕ.

3.2.1 (On the notation). Let F be a complex vector bundle with complex structure J. In the following superscripts $_J$ only indicate the chosen complex structure J on F. Let

$$T^*\Sigma \otimes_{\mathbb{C},J} F = L_{\mathbb{C},J}(T\Sigma, F)$$

be the complex tensor product of $T^*\Sigma$ and F. As complex vector bundle it carries a complex structure again denoted by J. J equals $1 \otimes J = j \otimes 1$.
As real vector bundle

$$T^*\Sigma \otimes F = L_{\mathbb{R}}(T\Sigma, F)$$

carries two different complex structures, $j \otimes 1$ and $1 \otimes J$.
The Eigenvalues of $j \otimes 1 : T\Sigma \otimes \mathbb{C} \to T\Sigma \otimes \mathbb{C}$ on the fibres are $\pm i$. $T^{1,0}\Sigma$ and $T^{0,1}\Sigma$ denote the respective Eigenspaces. We can identify $T^{1,0}\Sigma$ with the vector bundle $T\Sigma_j$, whose typical fibre is a complex vector space with scalar multiplication defined by $(a + ib)v = av + bjv$. $T^{0,1}\Sigma$ can be identified with $T\Sigma_{(-j)}$.
$T^*\Sigma \otimes F \otimes \mathbb{C}$ splits into (J, j) linear

$$\Lambda^{1,0}T^*\Sigma \otimes_J F = (T^{1,0}\Sigma)^* \otimes_J F$$

and antilinear part

$$\Lambda^{0,1}T^*\Sigma \otimes_J F = (T^{0,1}\Sigma)^* \otimes_J F.$$

The linear part may be identified with $T^*\Sigma \otimes_{\mathbb{C},J} F$.

Let J be a $\mathcal{C}^\infty$ vertical almost complex structure on P.

3.2.2 Proposition. Let $l > 2$, $l > k$ and $kp > 2$. The generalized Cauchy-Riemann equation

$$\begin{aligned}\mathrm{CR}(u, \phi, J) :=& (pr^{0,1})_* \circ \Phi(u, \phi)\\ =& \frac{1}{2}(\phi_u du + J(u)\phi_u du \circ j)\end{aligned}$$

is a $\mathcal{C}^{l-k-1}$ section in the vector bundle

$$\Gamma^{k,p}(p^*(T^{0,1}\Sigma)^* \otimes_J VP) \times \mathcal{A}^l(P) \to \Gamma^{k+1,p}(P) \times \mathcal{A}^l(P).$$

In view of the assumptions on l and k, CR is at least $\mathcal{C}^1$.

If CR_ϕ was a Fredholm section of

$$\Gamma(p^*(T^{0,1}\Sigma)^* \otimes_J VP) \to \Gamma(P),$$

the first assumption of theorem 1.0.25 would be fulfilled. In fact we will show more, namely that the derivative of CR_ϕ is elliptic of first order. Its index coincides with the dimension of the solution space at a generic connection then. Moreover, ellipticity allows to conclude that the Sobolev class of the solution space does only depend on the class of the connection.

3.2.3 Proposition. Let $l > 2$, $k < l$ and $kp > 2$. For all $\phi \in \mathcal{A}^l(P)$ and for all $v \in \Gamma^{k+1,p}(P)$, which are J-holomorphic with respect to ϕ, the first order part of the derivative of a local representation $\Gamma^{k+1,p}(u^*VP) \to \Gamma^{k,p}(T^*\Sigma \otimes u^*VP)$ of CR_ϕ at v, where $u \in \Gamma^\infty(P)$, is of the form $\frac{1}{2}(\nabla\xi + J(v)\nabla\xi \circ j)$ for a linear connection ∇ on u^*VP.

Proof. The local representation of $\mathrm{CR}_\phi : \Gamma(u^*VP) \to \Gamma(T^*\Sigma \otimes u^*VP)$ at $u \in \Gamma^\infty(P)$ is given by

$$\mathrm{CR}_\phi(\xi) = pr^{0,1} \circ \xi^*(\phi' + A) = pr^{0,1}(\xi) \circ (\nabla\xi + A(\xi)),$$

see 3.1.2. The derivative of CR_ϕ at a solution $\eta \in \Gamma(u^*VP)$ in direction of $\xi \in \Gamma(u^*VP)$ computes

$$\begin{aligned}\frac{d}{dt}|_{t=0}&(\nabla(\eta + t\xi) + A(\eta + t\xi) + J(\eta + t\xi)\nabla(\eta + t\xi) \circ j + J(\eta + t\xi)A(\eta + t\xi) \circ j)\\ &=\nabla\xi + J(\eta)\nabla\xi \circ j + dA(\eta)(\xi) + dJ(\eta)(\xi)A(\eta) \circ j + J(\eta)dA(\eta)(\xi) \circ j) + 0\\ &=\nabla\xi + J(\eta)\nabla\xi \circ j + \text{ zero order terms.}\end{aligned}$$

□

3.2.4 Theorem (CR_ϕ is a Fredholm section)**.** *Let $l > 2$, $k < l$, $p \geq 2$ and $kp > 2$. For all $v \in \Gamma^{k+1,p}(P)$, which are J-holomorphic with respect to a connection $\phi \in \mathcal{A}^l(P)$, the derivative of a local representation of* CR_ϕ *at v is a first order elliptic operator with $\mathcal{C}^k$ coefficients. Thus its Sobolev extension is a Fredholm operator $\Gamma^{k+1,p}(u^*VP) \to \Gamma^{k,p}(\Lambda^{0,1}T^*\Sigma \otimes_J u^*VP)$, where $u \in \Gamma^\infty(P)$. Consequently, the tangential $T_v\,\mathrm{CR}(\phi) : T_v\Gamma^{k+1,p}(P) \to \Gamma^{k,p}(\Lambda^{0,1}T^*\Sigma \otimes_J v^*VP)$ is Fredholm. The dimensions of kernel and cokernel do not depend on k or p.*

The reader may check that the definition of spaces of vector bundle sections of class $\Gamma^{l,p}$ extends to $\mathcal{C}^k$ vector bundles, whenever $l \leq k$; and note that v is at least $\mathcal{C}^k$.

We show

3.2.5 Proposition. Let $v \in \Gamma^{k+1,p}(P)$ be a J-holomorphic section with respect to a connection ϕ. The first order part of the differential of $\mathrm{CR}(\phi, J)$ at v, $\mathscr{C}_v := \nabla + J(v)\nabla \circ j : \Gamma^{k+1}(u^*VP) \to \Gamma^k((T^{0,1}\Sigma)^* \otimes u^*VP)$, where $u \in \Gamma^\infty(P)$, is an elliptic partial differential operator with $\mathcal{C}^k$ coefficients and symbol $\sigma(T_v\,\mathrm{CR}(\phi))(\eta) = \varepsilon(\eta^{0,1})$.

Then theorem 3.2.4 follows from the fact that the Sobolev extension of elliptic operators is Fredholm.

Proof. Let f be a function on Σ such that $df = \eta \in T\Sigma$ and m_f denote multiplication with f. We compute

$$\sigma(\nabla)(\eta) = [\nabla, \mathrm{m}_f] = \varepsilon(df) = \varepsilon(\eta)\,.$$

Then the symbol of the derivative of the local representation of CR equals

$$\sigma(pr^{0,1} \circ \nabla)(\eta) = pr^{0,1}\varepsilon(\eta) = \varepsilon(\eta^{0,1})\,.$$

Since Σ is a Riemannian surface (complex dimension 1),

$$\begin{tikzcd}
\Gamma^{k+1}(u^*VP) \arrow[r, "\mathscr{C}_v"] & \Gamma^k((T^{0,1}\Sigma)^* \otimes_J u^*VP) \arrow[r, "\mathscr{C}_v"] \arrow[d, "\sigma"] & 0\ . \\
u^*VP \arrow[r, "\varepsilon(\eta^{0,1})"] & (T^{0,1}\Sigma)^* \otimes_J u^*VP \arrow[r, "\varepsilon(\eta^{0,1})"] & 0
\end{tikzcd}$$

Thus $\varepsilon(\eta^{0,1}) : u^*VP \to (T^{0,1}\Sigma)^* \otimes_J u^*VP$ is an isomorphism whenever $\eta^{0,1} \not\equiv 0$, i.e. $\mathscr{C}_v$ is elliptic. □

3.2.6 Remark. Let $l > 2$, $k < l$, $p \geq 2$ and $kp > 2$. $\mathscr{C}_v : T_v\Gamma^{k+1,p}(P) \to \Gamma^{k,p}(\Lambda^{0,1}T^*\Sigma \otimes_J v^*VP)$ is a first order elliptic operator for all $v \in \Gamma^{k+1,p}(P)$, which are J holomorphic with respect to a $\mathcal{C}^l$ connection ϕ.

3.2.7 Theorem. *Let $l > 2$, $k < l$, $p \geq 2$ and $kp > 2$.*

$$\mathrm{CR} : \Gamma^{k+1,p}(P) \times \mathcal{A}^l(P) \to \Gamma^{k,p}(p^*(T^{0,1}\Sigma)^* \otimes_J VP) \times \mathcal{A}^l(P)$$

is transversal to the zero section.

Proof. [19, 3.4.1]. The differential of CR at pair (v, ϕ) in the solution space of CR is a Fredholm operator, as $T_{v,\phi}\,\mathrm{CR}$ is of the form

$$T_v\Gamma^{k+1,p}(P)\times\Gamma^l(p^*T^*\Sigma \otimes VP \to P) \to \Gamma^{k,p}((T^{0,1}\Sigma)^* \otimes v^*VP \to \Sigma)$$
$$(\xi,\omega) \mapsto \mathscr{C}_v\xi + \text{ zero order terms } + (pr^{0,1}\omega) \circ v.$$

(The reader may check that the definition of spaces of vector bundle sections of class $\Gamma^{l,p}$ extends to $\mathcal{C}^k$ vector bundles, whenever $l \leq k$; and note that v is at least $\mathcal{C}^k$.) To see this we need to compute the partial derivative of a local representation in direction $\omega \in \Gamma^l(u^*VP, T^*\Sigma \otimes u^*VP)$ at $\eta \in \Gamma^{k+1,p}(u^*VP)$, where $u \in \Gamma^\infty(P)$ and $v = \Psi_u(\eta)$. At $\phi \in \mathcal{A}^l(P)$ we have

$$\frac{d}{dt}|_{t=0}\,\mathrm{CR}(\phi + t\omega)(\eta) = \frac{d}{dt}|_{t=0} pr^{0,1}(\nabla\eta + A(\eta) + t\omega(\eta)) = pr^{0,1}\omega(\eta).$$

For a pair $(v, \phi) \in \Gamma^{k+1,p}(P)\times\Gamma^l(p^*T\Sigma\otimes VP)$ in the solution space, $(v, \phi) \in \Gamma^{t,p}(P)\times\Gamma^l(p^*T\Sigma\otimes VP)$ is a solution for the Cauchy-Riemann equation for all $1 \leq t \leq k$. Assume that $\mathrm{CR} : \Gamma^{1,p}(P) \times \mathcal{A}^l(P) \to \Gamma^{0,p}((T^{0,1}\Sigma)^* \otimes VP)$ is not transversal to the zero section, i.e. for $\frac{1}{p}+\frac{1}{q} = 1$, there are finitely many non zero functionals $a_1, \ldots, a_n$ in $\Gamma^{0,q}((T^{0,1}\Sigma)^* \otimes_J v^*VP)$ on $\Gamma^{0,p}((T^{0,1}\Sigma)^* \otimes_J v^*VP)$, such that

$$< \mathscr{C}_v\xi, a_i >= \int_\Sigma (\mathscr{C}_v\xi, a_i) = 0 \text{ and } < pr^{0,1}\omega \circ v, a_i >= \int_\Sigma (pr^{0,1}\omega \circ v, a_i) = 0,$$

for all $\xi \in T_v\Gamma^{1,p}(P)$, all $\omega \in \Gamma^l(p^*T^*\Sigma \otimes VP \to P)$, and all $1 \leq i \leq n$.
The first equation asserts that a_i is a (weak) solution to the adjoint of the elliptic operator $\mathscr{C}_v$ with $\mathcal{C}^k$ coefficients. By elliptic regularity for the adjoint $\mathscr{C}_v^*$, a_i are $\Gamma^{1,p}$, and thus continuous. Then the following argument shows that we may find $\omega \in \Gamma^l(p^*T^*\Sigma \otimes VP \to P)$ so that $< pr^{0,1}\omega \circ v, a_i > \neq 0$, a contradiction.
Assume that $a_i(x) \neq 0$. Then there is $\omega_x \in p^*T^*_x\Sigma \otimes V_{v(x)}P$, such that

$$(a_i(x), pr^{0,1}(x, v(x))\omega_x) \neq 0.$$

Then we use parallel transport and a $\mathcal{C}^l$ bump function to produce a $\mathcal{C}^l$ section ω in $p^*T^*\Sigma \otimes VP \to P$ such that $(a_i, pr^{0,1}\omega(v)) \not\equiv 0$ in a neighbourhood of $x \in \Sigma$.

If η is in $\Gamma^{k,p}((T^{0,1}\Sigma)^* \otimes v^*VP)$, by surjectivity for $k = 0$, there is $\xi \in \Gamma^{1,p}(v^*VP)$ so that $T_{v,\phi}\,\mathrm{CR}(\xi, \omega) = \mathscr{C}_v\xi + \text{ terms in } \Gamma^{1,p} = \eta$. Thus $\mathscr{C}_v\xi \in \Gamma^{1,p}$. By elliptic regularity $\xi \in \Gamma^{2,p}$. By induction, we have $\xi \in \Gamma^{k,p}$. □

Thus the solution space $\mathcal{M}^{l,k,p}(P \to \Sigma)$ is a Banach manifold. Ellipticity implies that the solution space does not depend on k (3.2.9).

3.2.8 Theorem (Elliptic regularity)**.** *Cf. [19, 3.2.2]. Let $l > 2$ and $p > 2$. Let $\phi \in \mathcal{A}^l(P)$.*
If u is $H^{k-1,p}$ and $\mathrm{CR}(u, \phi) = 0$, it follows that u also is $H^{k,p}$, for all $1 \leq k \leq l$. If $u \in \Gamma^{1,p}(P)$ such that $\mathrm{CR}(u, \phi) = 0$, then u is of class $H^{l,p}$ by induction. u has a $\mathcal{C}^{l-1}$ representation then. If u is locally $H^{1,p}$-bounded, then u is locally $H^{l,p}$-bounded.
In particular, if ϕ is smooth, so is u.

We say that v is locally $H^{l,p}$-bounded, if for a covering $\mathfrak{V}$ of N and a covering $\mathfrak{U}$ of M there is a finer covering $\widetilde{\mathfrak{U}}$ such that $\tilde{U}_j \subset\subset U_i$ for some i with smooth boundary, such that $v(\tilde{U}_j) \subseteq V_j$, and such that v_j is bounded in $H^{l,p}(\tilde{U}_j, V_j)$ for all j , see 2.2.20.

Proof. Let $v \in \Gamma^{k,p}(P)$ is J-holomorphic with respect to $\phi \in \mathcal{A}^l(P)$. Let $\eta \in \Gamma^{k,p}(u^*VP)$ for $u \in \Gamma^\infty(P)$, such that $v = \Psi_u(\eta)$. Then η is a solution of the nonlinear partial differential equation $\nabla(\eta)+J(\eta)\nabla(\eta)\circ j+A(\eta)+J(\eta)A(\eta)\circ j := \nabla(\eta)+J(\eta)\nabla(\eta)\circ j+H$, where H is a fibre preserving map of class $\Gamma^{k,p}$ whenever $k < l$. By [19, B.4], we have the elliptic estimate $\|\eta\|_{k+1,p} \leq C(\|\eta\|_{k,p}+\|H\|_{k,p})$ for solutions η of this equation, see [19, B4.7]. Thus, whenever $k < l$, $\|\eta\|_{k+1,p} < C < \infty$. Thus, we can conclude that $\eta \in \Gamma^{l,p}(u^*VP)$, and in $\Gamma^{l-1}(u^*VP)$. Consequently, we have that $v \in \Gamma^{l,p}(P)$ and thus in $\Gamma^{l-1}(P)$.

If a solution v has local representations which are $H^{1,p}$ bounded, then η has local representations which are $H^{1,p}$ bounded, i.e. η is bounded in $\Gamma^{1,p}$. By elliptic regularity, ξ is also $\Gamma^{l,p}$ bounded. Then v has local representations which are $H^{l,p}$ bounded. □

3.2.9 Corollary. The solution space $\mathcal{M}^{l,k,p}(P \to \Sigma)$ *does not depend on k.* Hence the space $\mathcal{M}^{l,p}(P \to \Sigma)$ is a $\mathcal{C}^{l-2}$, but at least $\mathcal{C}^1$ Banach submanifold in $\Gamma^{l,p}(P \to \Sigma) \times \mathcal{A}^l(P)$ for all $l > 2$ and $p > 0$.

3.2.10 Theorem. *Let $l > 2$ and $p > 0$.*
The solution space $\mathcal{M}^{l,p}(P)$ is a $\mathcal{C}^{l-2}$ Banach submanifold in $\Gamma^{l,p}(P) \times \mathcal{A}^l(P)$. For all sections $u \in \mathcal{M}^{l,p}(P)$, such that $l-2 > \max\{0, \mathrm{index}(\mathscr{C}_u)\}$, the connected component of u in $\mathcal{M}^{l,p}(P, \phi)$ is a finite dimensional $\mathcal{C}^{l-2}$ manifold up to a set of first category in $\mathcal{A}^l(P)$. Its dimension equals the index of the Fredholm operator $\mathscr{C}_u : T_u\Gamma^{l,p}(P) \to \Gamma^{l-1,p}((T^{0,1}\Sigma)^* \otimes u^*VP)$.

3.3 The case of $\mathcal{C}^\infty$ connections

The ellipticity has another consequence. Let $\phi \in \mathcal{A}^\infty(P)$. The solution space at ϕ contains only smooth sections. Moreover it is a finite dimensional $\mathcal{C}^\infty$ manifold up to a set of first category in $\mathcal{A}^\infty(P)$.

In this case, if v is J-holomorphic with respect to a $\mathcal{C}^\infty$ connection ϕ, we can compute the index of $\mathscr{C}_v : \Gamma^{k+1,p}(v^*VP) \to \Gamma^{k,p}((T^{0,1}\Sigma)^* \otimes v^*VP)$. Since the symbols of $\mathscr{C}_v$ and the Dolbeaut operator coincide, their indexes equal. The dimension of the connected components of the solution space with respect to a regular connection is thus given by the index of the Dolbeaut operator on v^*VP. We can give a formula for the index of this operator, which has been computed using the Riemann-Roch theorem. This formula then coincides with the dimension of the connected component of v in the solution space.

3.3.1 Theorem (Convergence Result). *Cf. [19, B.4.2] Let $l > 2$ and $p > 2$. Let ϕ_n be a convergent sequence in $\mathcal{A}^l(P)$ and u_n a sequence of J-holomorphic sections with respect to ϕ_n in $\Gamma^{1,p}(P \to \Sigma)$, such that $\{u_n\}(\overline{U}) \subseteq V$ for some atlas $\mathfrak{U}$ of precompact sets and a collection $\mathfrak{V}$ of charts for P, and $\|j^1(u_n)\|_{0,p} \leq K$ for all n.*
Then there exists a subsequence u_n converging $\mathcal{C}^{l-1}$ to a section u which is J-holomorphic with respect to ϕ.

Proof. There is a continuous section $u : \Sigma \to P$ such that a subsequence u_n converges $\mathcal{C}^0$ to u by the Sobolev embedding theorem 2.2.20.
Let ϕ_n be a sequence of $\mathcal{C}^l$ connections $\mathcal{C}^l$ converging to ϕ. u_n satisfies the partial differential equation $\mathrm{CR}(u_n, \phi_n) = 0$. Thus (u_n) is a sequence in $\Gamma^{k+1,p}(P \to \Sigma)$ and is uniformly locally $H^{k+1,p}$-bounded (in the sense needed for the application of Sobolev's embedding theorem 2.2.20) by elliptic regularity 3.2.8. By the Sobolev embedding theorem $H^{k+1,p} \subseteq \mathcal{C}^k$ 2.2.20(2), there is a $\mathcal{C}^k$ convergent subsequence. Thus $u \in \mathcal{C}^k$ for all $k < l$.
If ϕ_n is a convergent sequence of $\mathcal{C}^\infty$ connections, we use an inductive argument on the differentiability class. In fact, if u_n is $\mathcal{C}^{l-1}$ converging and satisfies $\mathrm{CR}(u_n, \phi_n) = 0$, then there is a subsequence which is converging $\mathcal{C}^l$ by the above argument.

u has to be J-holomorphic with respect to ϕ, because multiplication and composition with $\mathcal{C}^\infty$ maps is continuous for our choice of l and p. □

3.3.2 Proposition. The set of regular smooth connections $\mathcal{A}^\infty(P)_{reg}$ is the complement of a set of first category in $\mathcal{A}^\infty(P)$, i.e. $\mathcal{A}^\infty(P)_{reg}$ is a countable intersection of open and dense sets.

We say that $\phi \in \mathcal{A}^l(P)$ is regular if $T_u \,\mathrm{CR}(\phi)$ is surjective for all $\mathcal{C}^{l-1}$ sections u which are J-holomorphic with respect to ϕ.
By what we proved in the last section, the space of regular $\mathcal{C}^l$ connections $\mathcal{A}^l(P)_{reg}$ is an open and dense subset in $\mathcal{A}^l(P)$ whenever l is finite. In fact, $\mathcal{A}^l(P)_{reg}$ coincides with the set of regular values for the projection $\mathcal{M}^{l,p}(P) \to \mathcal{A}^l(P)$ then, because every section in $\mathcal{M}^{l,p}(P)$ is of class $\mathcal{C}^{l-1}$ by elliptic regularity 3.2.8.

It is however by no means a priori clear that $\mathcal{A}^\infty(P)_{reg}$ is a countable intersection of open and dense sets.

Proof. The proof is due to Taubes, cf. [19, pp.36/37]. Let $\mathcal{A}^l(P)_{K,reg}$ be the set of connections in $\mathcal{A}^l(P)$, which are regular for sections satisfying $\|j^1(u)\|_{0,p} < K$. Then $\mathcal{A}^l(P)_{reg} = \cap_{K>0}\mathcal{A}^l(P)_{K,reg}$.

For all $2 < l \leq \infty$, $\mathcal{A}^l(P)_{K,reg}$ is open in $\mathcal{A}^l(P)$. Let $p > 2$. We show that the complement of $\mathcal{A}^l(P)_{K,reg}$ is closed.
Let ϕ_n be a sequence of non regular connections in $\mathcal{A}^l(P)$, which converges to a connection ϕ. Let (u_n) be a sequence of $\mathcal{C}^{l-1}$ sections u_n, such that u_n is J-holomorphic with respect to ϕ_n and that $d\,\mathrm{CR}_{\phi_n}(u_n)$ is not surjective for all n. Moreover we assume that $\|j^1(u)\|_{0,p} < K$ and that $\{u_n\}$ is such that the Sobolev embedding theorem 2.2.20 and the convergence result 3.3.1 can be applied. This implies the existence of a J-holomorphic section u with respect to ϕ of class $H^{1,p}$, but even $\mathcal{C}^{l-1}$ by elliptic bootstrapping and the Sobolev embedding theorems. $d\,\mathrm{CR}_{\phi_n}(u_n)$ is continuous in u_n and ϕ_n, and not surjective for any n. $d\,\mathrm{CR}_\phi(u)$ cannot be surjective, because the space of surjective, linear and bounded maps is open in the space of linear and bounded maps. Thus ϕ is non-regular.

Next, we show that $\mathcal{A}^\infty(P)_{K,reg}$ is dense in $\mathcal{A}^\infty$.
Since $\mathcal{A}^l(P)_{reg}$ is dense in $\mathcal{A}^l(P)$, $\mathcal{A}^l(P)_{K,reg}$ is dense in $\mathcal{A}^l(P)$ for all $2 < l < \infty$. Note that $\mathcal{A}^\infty(P)_{K,reg}$ contains the intersection $\mathcal{A}^\infty(P) \cap A^l(P)_{K,reg}$ by elliptic regularity 3.2.8.
Let $\phi \in \mathcal{A}^\infty(P)$. For all $l < \infty$, there is a sequence of connections ϕ_n in $\mathcal{A}^l(P)_{K,reg}$ converging $\mathcal{C}^l$ to ϕ. Since $\mathcal{A}^\infty$ is dense in $\mathcal{A}^l$, there is a sequence $\tilde{\phi}_{m_n}$ in $\mathcal{A}^\infty$ converging $\mathcal{C}^l$ to ϕ_n. The argument in the last paragraph shows that almost all $\tilde{\phi}_{m_n}$ have to be in $\mathcal{A}^l(P)_{K,reg}$. Thus $\tilde{\phi}_m$ can be assumed to lie in $\mathcal{A}^\infty_{K,reg}$. So for all $l < \infty$, we find a sequence in $\mathcal{A}^\infty_{K,reg}$ converging $\mathcal{C}^l$ to ϕ. Thus a diagonal sequence converges $\mathcal{C}^\infty$ to ϕ.

As $\mathcal{A}^\infty(P)_{reg}$ is the countable intersection of the open and dense subsets $\mathcal{A}^\infty_{K,reg}$, $\mathcal{A}^\infty(P)_{reg}$ is of first category in $\mathcal{A}^\infty(P)$. □

3.3.3 Proposition. Let $u \in \Gamma^{k+1,p}(P)$ be a J-holomorphic section with respect to a $\mathcal{C}^\infty$ connection ϕ. Then u is in $\Gamma^\infty(P)$ and the differential $d\,\mathrm{CR}_\phi(u) : \Gamma^{k+1}(u^*VP) \to \Gamma^k((T^{0,1}\Sigma)^* \otimes u^*VP)$ at u is a Cauchy-Riemann operator and determines a holomorphic structure on (u^*VP, u^*J).

Proof. The form of the symbol implies that $\mathscr{C}$ is a Cauchy Riemann operator (A.3.1) of $u^*VP \to \Sigma$, i.e. an operator of degree one which satisfies the Leibniz rule $\mathscr{C}(fu) = (\bar{\partial} f)u + f\mathscr{C}u$ for $f \in \mathcal{C}^\infty(\Sigma)$ and $u \in \Gamma^\infty(VP)$. Moreover the curvature $\mathscr{C}^2 = 0$ since we are working on a Riemannian surface (complex dimension 1). Thus $\mathscr{C}_u$ determines a holomorphic structure on u^*VP. This is an important property of Cauchy-Riemann operators, see A.3.3. □

The following theorem of Riemann-Roch gives the index of $\mathscr{C}_u$.

3.3.4 Theorem (Riemann-Roch)**.** *[26, p. 152] Let E be a complex vector bundle over a Riemannian surface Σ of genus g. The index of the Dolbeaut operator $\bar{\partial} : \Gamma^\infty(E) \to \Gamma^\infty((T^{0,1}\Sigma)^* \otimes E)$ equals the number*

$$\mathrm{rk}_{\mathbb{R}}(E)(1-g) + 2\big\langle c_1(E), [\Sigma]\big\rangle.$$

This can be shown by applying the Atiyah-Singer index theorem to the Dolbeaut-Dirac operator on the Clifford module $\Lambda(T^{0,1}\Sigma)^* \otimes E$ over Σ. The interested reader may find more information in the appendix.

Since two operators with same symbol differ only by lower order perturbations, the index of $\mathscr{C}_u$ coincides with the index of the Dolbeaut operator given in the above theorem. Thus we can conclude the following.

3.3.5 Theorem. *Let Σ be a Riemannian surface of genus g and $N \hookrightarrow P \to \Sigma$ a fibre bundle with a smooth vertical almost structure. For all $\mathcal{C}^\infty$ sections u in $\mathcal{M}^\infty(P \to \Sigma)$, the connected component of u in the solution space $\mathcal{M}^\infty(P, \phi)$ of smooth sections, which are J-holomorphic with respect to a $\mathcal{C}^\infty$ connection ϕ, is a finite dimensional $\mathcal{C}^\infty$ manifold up to a set of first category in $\mathcal{A}^\infty(P)$. Its dimension equals*

$$n(1-g) + 2\big\langle c_1(u^*VP), [\Sigma]\big\rangle.$$

3.3.6 Remark. If u represents a homology class $a \in H_2(P)$ of the manifold P, the set of all sections in $\mathcal{M}^{l,p}(P \to \Sigma, \phi)$ representing a is a manifold of dimension

$$n(1-g) + 2\big\langle c_1(a), [\Sigma]\big\rangle$$

up to a set of first category in $\mathcal{A}^\infty(P)$, where $c_1(a)$ denotes the first Chern class of the bundle VP restricted to $u(\Sigma)$.

A Appendix

We discuss the Atiyah-Singer index theorem for Dirac operators. We sketch a proof via the heat kernel method.
We compute the index of the Dolbeaut-Dirac operator.
We discuss the Cauchy-Riemann operator, and show that the index of a Cauchy-Riemann operator over a Riemannian surface coincides with the index of the Dolbeaut-Dirac operator.

A.1 The Index theorem for Dirac operators

First, we collect everything we need for the formulation of Atiyah-Singer's theorem for Dirac operators.

A.1.1 (Clifford structures). [26, 103ff] Let V be a vector space equipped with a positive definite, quadratic form, denoted by $(.,.)$. The Clifford algebra $Cl(V)$ is the algebra over V satisfying the following universal property. If A is an algebra and $c : V \to A$ is a linear map which satisfies

$$c(v)c(w) + c(w)c(v) = -2(v, w),$$

there is an algebra morphism of $Cl(V)$ to A. $Cl(V)$ is unique up to isomorphism. $Cl(V)$ can be realized as the quotient of the tensor algebra $T(V) = \oplus_j \left(\otimes^j V\right)$ by the ideal generated by $\{v \otimes w + w \otimes v + 2(v, w) : v, w \in V\}$. The $\mathbb{Z}_2$ grading on $T(V)$ induces a $\mathbb{Z}_2$ grading on $Cl(V) = Cl(V)^+ \oplus Cl(V)^-$. The anti-automorphism $v \mapsto -v$ of $T(V)$ preserves the ideal, and thus induces a map $v \mapsto v^*$ on the Clifford algebra.
A Clifford module for $Cl(V)$ is a $\mathbb{Z}_2$ graded vector space $E = E^+ \oplus E^-$ and an action of $Cl(V)$ on E, which is even, i.e. $Cl(V)^+ \cdot E^\pm = E^\pm$ and $Cl(V)^- \cdot E^\pm = E^\mp$. We will moreover assume, that E is equipped with an inner-product, such that $c(a)$ is skew-adjoint, that is $-c(a) = c(-a) = c(a^*) = c(a)^*$.

The action of the orthogonal group $O(V)$ on $(V, (.,.))$ induces an action of $O(V)$ on $Cl(V)$, because it preserves the ideal used in the realization of $Cl(V)$.

The Clifford bundle on a Riemannian manifold M is the associated bundle

$$Cl(M) := O(M) \times_{O(m)} Cl(\mathbb{R}^m).$$

A Clifford module for $Cl(M)$ is a $\mathbb{Z}_2$ graded vector bundle $E = E^+ \oplus E^-$ over M and an action of $Cl(M)$ on E, which is even, i.e. $Cl(M)^+ \cdot E^\pm = E^\pm$ and $Cl(M)^- \cdot E^\pm = E^\mp$.
For example, ΛT^*M is a Clifford module. The Clifford action of $v \in T^*M$ on an element $\omega \in \Lambda T^*M$ is given by $\mathfrak{c}(v)\omega = (\varepsilon(v) - \iota(v))\,\omega$, where ε denotes exterior multiplication and ι contraction with v. It is in fact easy to check the relation $\mathfrak{c}(w)\mathfrak{c}(v)\omega + \mathfrak{c}(v)\mathfrak{c}(w)\omega = -2(w, v)\omega$ for all $\omega \in \Lambda T^*M$.

A.1.2 (Dirac operator). [26, 116ff] A Dirac operator is an odd first order elliptic operator determined by a Clifford module E and a Clifford connection ∇^E on the Clifford module. A Clifford connection is a covariant derivative on E which additionally satisfies

$$[c(\alpha), \nabla^E_X] = c(\nabla^{LeviCivita}_X \alpha)$$

for all $\alpha \in Cl(M)$ and $X \in \Gamma^\infty(TM)$.
A Dirac operator is defined by the composition of the Clifford action and the

Clifford connection such that

$$\mathrm{D} := c \circ \nabla^E.$$

It follows that a Dirac operator is a self adjoint odd operator whose square is a generalized Laplacian.

A.1.3 Definition (Index of Dirac operators). The index of a self adjoint Dirac operator is the number $\dim\ker(\mathrm{D}^+) - \dim\ker(\mathrm{D}^-)$, where $\mathrm{D}^\pm : \Gamma^\infty(E^\pm) \to \Gamma^\infty(E^\mp)$ denote the restrictions of D.

Recall some natural isomorphisms involving Clifford structures.

A.1.4 (Symbol and quantization map). [26, 3.4, 3.5] The *symbol* $\sigma : Cl(M) \to \Lambda T^*M$ is pointwise defined to be the linear isomorphism

$$\sigma_x : Cl(T_xM) \to \Lambda T_x^*M$$

$$\sigma_x(v) = \mathfrak{c}(v)1.$$

Its inverse is called quantization map q. q maps basis elements $e_1 \wedge ... \wedge e_n$ to the corresponding elements $c_1 \cdots c_n$ in the Clifford algebra.

We can therefore think about $Cl(M)$ as of the algebra ΛT^*M with the non-standard multiplication generated by $e_ie_i = -1$ and $e_ie_j = -e_je_i$.

A.1.5 (Adjoint action). [26, 3.7] The adjoint action ad induces an isomorphism from the space of Clifford elements of degree two, $Cl^2(T_xM)$, to the space of skew-adjoint linear maps $\mathfrak{so}(T_x^*M) \cong \mathfrak{so}(T_xM)$. If $A \in \mathfrak{so}(T_xM)$, the inverse is given by

$$\mathrm{ad}^{-1}(A) = \frac{1}{2}\sum_{i<j}(Ae_i, e_j)c^ic^j,$$

such that $[\mathrm{ad}^{-1}(A), v] = c(Av)$ for all $v \in T_xM$. Compare this with the isomorphism given by the quantization map which maps A to

$$\sum_{i<j}(Ae_i, e_j)c^ic^j.$$

A.1.6 Example. Using the isomorphism given by the adjoint map we regard the Riemannian curvature R on M as an element R^E of $\Omega^2(Cl(M))$, such that $[R^E, v] = c(Rv)$. For a local orthonormal frame (X_i) of the tangent bundle this means that

$$R^E = \frac{1}{2}\sum_{i<j}(RX_i, X_j)c(X^i)c(X^j).$$

We further compute

$$\begin{aligned} R^E(X_k, X_l) &= -\frac{1}{2}\sum_{i<j} R_{ijkl}c(X^i)c(X^j) \\ &= -\frac{1}{4}\sum_{\substack{i,j \\ i\neq j}} R_{ijkl}c(X^i)c(X^j)\,. \end{aligned}$$

Let E be a Clifford algebra. We can view $Cl(M)$ as a subalgebra in $\mathrm{End}(E)$. Then $Cl(M) \otimes \mathrm{End}_{Cl(M)}(E)$ is isomorphic to $\mathrm{End}(E)$.

A.1.7 Definition (Twisting curvature). The *twisting curvature* $\mathcal{F}^{E/S}$ of the Clifford module E is defined as $(\nabla^E)^2 - R^E$, where ∇^E is a connection on E.

A.1.8 Proposition. [26, 3.43] Let ∇^E be a Clifford connection on E. $\mathcal{F}^{E/S}$ is an element in $\Omega^2(\mathrm{End}_{Cl(M)}(E))$ and a invariant of ∇^E by definition. The splitting $(\nabla^E)^2 = R^E + \mathcal{F}^{E/S}$ reflects the isomorphism $Cl(M) \otimes \mathrm{End}_{Cl(M)}(E) \cong \mathrm{End}(E)$.

Proof. [26, 3.43] We have to show, that $\mathcal{F}^{E/S}$ supercommutes with $Cl(M)$. This follows from

$$[(\nabla^E)^2, c(\alpha)] = [\nabla^E, [\nabla^E, c(\alpha)]] = c(\nabla^2(\alpha)) = c(R\alpha) = [R^E, c(\alpha)],$$

where ∇ is the Levi-Civita connection and α is of degree 1. The second equality holds because we assume ∇^E to be a Clifford connection and the last equality is just the definition of R^E. □

A.1.9 Definition. We define the Chern character

$$\mathrm{ch}(E/S) := \mathrm{Str}_{E/S}(\exp(-\mathcal{F}^{E/S})).$$

A.1.10 Definition. [26, p. 51] The $\hat{A}$-genus of a closed manifold M is the differential form in $\Gamma^\infty(\oplus_{k=4*}\Lambda^k T^*M)$ defined as

$$\hat{A}(M) = \det\left(\frac{R/2}{\sinh(R/2)}\right)^{\frac{1}{2}}.$$

A.1.11 Theorem (Atiyah-Singer). *[26, 4.8] Let M be a closed oriented manifold of even dimension and E a Clifford module over M. If D is a self adjoint Dirac operator on E, then*

$$\mathrm{index}(\mathrm{D}) = (2\pi i)^{-m/2} \int_M \hat{A}(M)\,\mathrm{ch}(E/S).$$

A.1.12 Remark. [26, 3.51] The index is an invariant of the manifold and the Clifford module.

Sketch. [26] We sketch the proof of the Atiyah-Singer theorem for Dirac operators via the heat kernel method. The following arguments and all details may be found in the first chapters in the book [26] by Berline, Getzler and Vergne.

Let D be a Dirac operator associated to the Clifford connection ∇^E. The *heat kernel* for the Laplacian D^2 is the unique $\mathcal{C}^\infty$ section $p_t(x, y; \mathrm{D}^2)$ in the vector bundle $pr_1^*E \otimes pr_2^*E^* \to \mathbb{R}_+ \times M \times M$, which is a solution of the heat equation

$$(\partial_t + \mathrm{D}_x^2)p_t(x, y) = 0$$

with boundary condition $\lim_{t\to 0}\|P_t s - s\|_{\mathcal{C}^0} = 0$, $s \in \Gamma^\infty(E)$. Here $P_t s(x) = \int_M p_t(x, y)s(y)dy$ denotes the integral operator with respect to p_t. The boundary condition implies that the operators P_t form a semigroup.

We will formally write $P_t = e^{-t\,\mathrm{D}^2}$. This is motivated by formally differentiating $\partial_t e^{-t\,\mathrm{D}^2} = -\mathrm{D}^2\, e^{-t\,\mathrm{D}^2}$. However, let ϕ_i be the Eigenfunctions of the essentially self-adjoint operator D^2 with Eigenvalue λ_i. We can simultaneously diagonalize the semi-group of self-adjoint operators P_t. Since $P_t\phi_i$ is a solution of the heat equation,

$$P_t\phi_i = e^{-t\lambda_i}\phi_i.$$

The supertrace on a $\mathbb{Z}_2$ graded space is given by

$$\operatorname{Str}(A) = \begin{cases} \operatorname{Tr}_{E^+}(A) - \operatorname{Tr}_{E^-}(A) & \text{if } A \text{ is even} \\ 0 & \text{if } A \text{ is odd} \end{cases}.$$

The supertrace vanishes on graded commutators, i.e. $\operatorname{Str}([A, B]) = 0$.
The supertrace of $e^{-t\,\mathrm{D}^2}$ is given by

$$\operatorname{Str}(e^{-t\,\mathrm{D}^2}) = \int_M \operatorname{Str}\left(p_t(x,x)\right) dx,$$

where $\operatorname{Str}\left(p_t(x,x)\right)$ is the fibrewise supertrace of the $\mathcal{C}^\infty$ section $p_t(x,x)$ in $\operatorname{End}(E)$.
$\operatorname{Str}(e^{-t\,\mathrm{D}^2})$ does not depend on $t > 0$. To show this, we compute the time derivative

$$\begin{aligned} \partial_t \operatorname{Str}\left(e^{-t\,\mathrm{D}^2}\right) &= \operatorname{Str}\left(\partial_t e^{-t\,\mathrm{D}^2}\right) \\ &= -\operatorname{Str}\left(\mathrm{D}^2\, e^{-t\,\mathrm{D}^2}\right) \\ &= -\frac{1}{2} \operatorname{Str}\left([\mathrm{D}, \mathrm{D}\, e^{-t\,\mathrm{D}^2}]\right) = 0, \end{aligned}$$

where we used that $e^{-t\,\mathrm{D}^2}$ has a $\mathcal{C}^\infty$ kernel.

We compute two limits, $t \to \infty$, and $t \to 0$. Computing the first gives McKean-Singers formula which states

$$\operatorname{index}(\mathrm{D}) = \operatorname{Str}\left(e^{-t(\mathrm{D}^2)}\right) = \int_M \operatorname{Str}\left(p_t(x,x;\mathrm{D}^2)\right) dx.$$

The second limit yields the formula given in the theorem.

For letting $t \to \infty$, let $\lambda_i^\pm$ be the Eigenvalues for $(\mathrm{D}^2)^\pm$, and observe that

$$\operatorname{Str}(e^{-t\,\mathrm{D}^2}) = \sum_i e^{-t\lambda_i^+} - e^{-t\lambda_i^-}.$$

Letting $t \to \infty$, all what remains is $\dim\ker\left((\mathrm{D}^2)^+\right) - \dim\ker\left((\mathrm{D}^2)^-\right)$, which equals the index of D because D is self adjoint.

To study what happens if we let $t \to 0$, we work in local geodesic normal coordinates $\xi \in T_x M$ for $U \subseteq M$ around x. The heat kernel $p_t(x,\xi;\mathrm{D}^2)$ has an asymptotic expansion. Its coefficients $p_i(x,\xi)$ are local sections in $pr_1^* E \otimes pr_2^* E$ near the diagonal.
Letting $t \to 0$ shows that if we restrict to the diagonal, $p_i(x,x)$ is a section in $Cl_{2i}(M) \otimes \operatorname{End}_{Cl(M)}(TM)$, the Clifford elements of degree less or equal $2i$. It is possible to compute the highest order terms of the expansion of $\operatorname{Str}(k_t(x,x,\mathrm{D}^2))$ when letting $t \to 0$.

This claim can be proven using Getzler's rescaling argument. We introduce a rescaling of

$$\mathcal{C}^\infty(\mathbb{R}_+ \times U, \Lambda T_x^* M \otimes \operatorname{End}_{Cl(T_x^* M)}(E_x)).$$

For $0 < u \leq 1$, define the rescaling of $\mathbb{R}_+ \times U$ to map the time $t \to ut$ and $\xi \to u^{1/2}\xi$. The *rescaling operator* δ_u acts on $\mathcal{C}^\infty(\mathbb{R}_+ \times U, \Lambda T_x^* M \otimes \operatorname{End}_{Cl(T_x^* M)}(E_x))$. It maps a $\operatorname{End}_{Cl(T_x^* M)}(E_x)$ valued form $\alpha = \sum_{q=0}^n \alpha_q$ to the rescaled form

$\delta_u \alpha(t,\xi) = \sum_{q=0}^{n} u^{-q/2} \alpha(ut, u^{1/2}\xi)_q$, where the subscript q denotes the q-form component of α.
The rescaled heat kernel

$$p_u(t,\xi) := u^{n/2} \delta_u p_t(x,\xi;\mathrm{D}^2)$$

is the heat kernel of the rescaled Laplace operator $\mathrm{L}(u) = u\delta_u \, \mathrm{D}^2 \, \delta_u^{-1}$. If we could let u to zero in $p_u(t,\xi)|_{t=1,\xi=0}$, all what would remain is the term of highest order, which we would like to compute. We have to show that

$$\lim_{u\to 0} p_u(t,\xi)|_{t=1,\xi=0} = (2\pi i)^{-m/2} \hat{A}(M) \, \mathrm{ch}(E/S).$$

We have Lichnerowitz's formula

$$\mathrm{D}^2 = (\nabla^E)^* \nabla^E + c(\mathcal{F}^{E/S}) + 1/4 r_M$$

for the Laplacian D^2, where r_M denotes the scalar curvature of M. Lichnerowitz's formula implies that $\mathrm{L}(u)$ has the form $\mathrm{L}(0) + O(|u|^{1/2})$, where $\mathrm{L}(0)$ is the harmonic oscillator

$$\sum_i \left(\partial_i + \frac{1}{4} \sum_j R_{ij} \xi^j \right)^2 + \mathcal{F}^{E/S}.$$

The heat kernel of the harmonic oscillator can be computed. It equals

$$(4\pi t)^{-m/2} j^{-1/2}(tR) \exp(-1/(4t) \langle \xi | tR/2 \coth(tR/2) | \xi \rangle) \exp(-tF).$$

Evaluating at $t=1$ and $\xi = 0$ gives the formula for the index.
It is however not a priori clear that the rescaled heat kernel $p_u(t,\xi;\mathrm{L}(u))$ converges to the kernel of $\mathrm{L}(0)$ when letting u to zero. But the asymptotic expansion of the heat kernel $p_t(x,\xi;\mathrm{D}^2)$ transforms to a Laurent series in powers of $u^{1/2}$, which is an asymptotic expansion of the rescaled heat kernel $p_u(t,\xi;\mathrm{L}(u))$ in u. Moreover the Laurent series has no poles in powers of $u^{1/2}$.
Thus $p_0(t,\xi)$ exists and $p_0(t,\xi)$ is the leading term of the expansion. As leading term of the expansion it satisfies the heat equation for $\mathrm{L}(0)$.
The above argument also shows that the coefficients $p_i(x,x)$ are in $Cl(M)_{2i} \otimes \mathrm{End}_{Cl(M)}(E)$. $\square$

A.2 The index of the Dolbeaut-Dirac operator

We start fixing some notation. Let Σ be a complex manifold. Let F be a vector bundle with complex structure J.

A.2.1 (Grading on $\Lambda T^*\Sigma \otimes_J F$). There is a natural grading on the space of sections in $\Lambda T^*\Sigma \otimes_J F$. $\Gamma^\infty(\Lambda T^*\Sigma \otimes F)$ carries the bigrading of

$$\Omega^{p,q} = \Gamma^\infty(\Lambda^p (T^{1,0}\Sigma)^* \otimes \Lambda^q (T^{0,1}\Sigma)^* \otimes F), \qquad 0 \le p,q \le n.$$

A.2.2. The exterior differential

$$d : \Gamma^\infty(F) \to \Gamma^\infty(T^*\Sigma \otimes_J F)$$

splits into

$$\partial : \Gamma^\infty(F) \to \Gamma^\infty((T^{1,0}\Sigma)^* \otimes_J F)$$

and

$$\bar\partial : \Gamma^\infty(F) \to \Gamma^\infty((T^{0,1}\Sigma)^* \otimes_J F).$$

More general

$$\bar\partial : \Omega^{0,q} \to \Omega^{0,q+1}$$

is defined by composition of the projection $\Omega^{q+1} \to \Omega^{0,q+1}$ with $d : \Omega^q \to \Omega^{q+1}$ restricted to $\Omega^{0,q}$.
If F is holomorphic, we have the formula

$$\bar\partial = \sum_i \varepsilon(d\bar z^i)\partial_{\bar z^i}$$

in each system of complex coordinates z_i for Σ. [6, p.35ff][26, p.136]

A.2.3 (Clifford module). [26, p.135] Let Σ be a closed complex manifold. Let F be a hermitian complex vector bundle over Σ, i.e. F admits a fibrewise hermitian scalar product. The bundle

$$\Lambda(T^{0,1}\Sigma)^* \otimes_J F$$

is a Clifford module over Σ.
Let $\alpha \in \Gamma^\infty(T^*M)$. The Clifford action $c(\alpha)$ of α on $u \in \Lambda(T^{0,1}\Sigma)^* \otimes_J F$ is given by

$$\sqrt{2}(\varepsilon(\alpha^{0,1}) - \iota(\alpha^{1,0}))u,$$

where $\alpha^{0,1} \in \Gamma^\infty((T^{0,1}\Sigma)^*)$ and $\alpha^{1,0} \in \Gamma^\infty((T^{1,0}\Sigma)^*)$.

A.2.4 Remark. Let $\nabla : \Gamma^\infty(F) \to \Gamma^\infty(T^*\Sigma \otimes_J F)$ be a connection on F. ∇ restricts to

$$\nabla^{1,0} : \Gamma^\infty(F) \to \Gamma^\infty((T^{1,0}\Sigma)^* \otimes_J F)$$

and

$$\nabla^{0,1} : \Gamma^\infty(F) \to \Gamma^\infty((T^{0,1}\Sigma)^* \otimes_J F).$$

A.2.5 Proposition (Holomorphic structures on vector bundles I). [37, III, theorem 2.1] Let (F, J) be a complex vector bundle with hermitian structure h over a complex manifold Σ.
If (F, J) is a holomorphic vector bundle, it admits a canonical covariant derivative ∇, which is compatible with h and which satisfies

$$\bar\partial = \nabla^{0,1}.$$

Moreover the torsion of ∇ vanishes.

A.2.6 Theorem (Dolbeault-Dirac operator)**.** *[26, 3.67, 3.69] Let F be a holomorphic vector bundle with complex structure J over a closed Kähler manifold Σ. Then*

$$\bar\partial + \bar\partial^* : \Gamma^\infty\left((\Lambda(T^{0,1}\Sigma)^* \otimes_J F)^\pm\right) \to \Gamma^\infty\left((\Lambda(T^{0,1}\Sigma)^* \otimes_J F)^\mp\right)$$

is a Dirac operator on the Clifford module $\Lambda(T^{0,1}\Sigma)^* \otimes_J F$. Its index equals the Euler characteristic of the Dolbeaut-Dirac complex

$$\Gamma^\infty(F) \xrightarrow{\bar\partial} \Omega^{0,1}(F) \xrightarrow{\bar\partial} \Omega^{0,2}(F) \xrightarrow{\bar\partial} \dots \; .$$

Proof. The above is a complex as the vector bundle F is holomorphic. [6, p35ff] The Dolbeaut-Dirac operator has symbol $\sqrt{2}(\varepsilon(\eta^{0,1}) - \iota(\eta^{1,0}))$. Its square is a generalized Laplacian with symbol $2\left|\bar\partial(f)\right|^2$. The twisted connection of the Levi-Civita connection on $T\Sigma$ and the canonical connection on F is a Clifford connection on $(T^{0,1}\Sigma)^* \otimes F$. $\bar\partial + \bar\partial^*$ is the Dirac operator with respect to this Clifford connection. [26, 3.67]
By Hodge's theorem [37, IV 5.2] and because the Dolbeaut-Dirac operator is self adjoint, the index equals the Euler characteristic of the Dolbeaut complex [37, IV 5.7]. □

A.2.7 Definition. [26, p.51] The Todd genus of a complex manifold Σ is the complex valued differential form in $\Gamma^\infty(\oplus_{k=2*}\Lambda^k(T\Sigma)^* \otimes \mathbb{C})$ given by

$$\mathrm{Td}(\Sigma) = \det\left(\frac{R^+}{e^{R^+} - 1}\right) ,$$

where R^+ is the two form obtained from restricting the complexification of $R \in \Omega^2(\mathfrak{so}(T\Sigma))$ to $T^{1,0}\Sigma$. The restriction of R to $T^{0,1}\Sigma$ will be denoted by R^-, so that $R = R^+ \oplus R^-$.

A.2.8 Remark (Todd Genus). Let Σ be a closed complex manifold. The (complexification of the) Riemannian curvature on $T\Sigma$ restricts to R^+ on $T^{1,0}\Sigma$. Hence $\hat{A}(\Sigma)$ computes as $\det(\frac{R^+/2}{\sinh(R^+/2)})$, because $\det(R^+ \oplus R^-) = \det^2(R^+)$. Since $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$ for invertible matrices A, we have the following relation [26, p. 152]:

$$\hat{A}(\Sigma) = \mathrm{Td}(\Sigma)\exp(\mathrm{Tr}(R^+/2)).$$

Here we used that $\sinh(x) = 1/2\exp(x) - 1/2\exp(-x)$ by definition.

A.2.9 Theorem (Riemann-Roch-Hirzebruch). *[26, 4.9] Let Σ be a closed Kähler manifold of real dimension $2n$ and F a holomorphic vector bundle over Σ. The index of the Clifford module $\Lambda(T^{0,1}\Sigma) \otimes F$ over Σ equals*

$$\left(\frac{1}{2\pi i}\right)^n \int_\Sigma \mathrm{Td}(\Sigma)\,\mathrm{ch}(F). \tag{3}$$

This number coincides with the index of the Dolbeaut complex.

Proof. [26, p. 152f] The above number equals the index of the Dirac operator associated to the twisted connection $\nabla^E = \nabla \otimes \nabla^F$, where ∇^F is the canonical connection on F and ∇ the Levi-Civita connection.

We have to compute the twisting curvature $\mathcal{F}^{E/S}$.

Choose a local orthonormal frame of $T\Sigma$ and generate local orthonormal frames Z_i for the j Eigenspace $T^{1,0}\Sigma$ and $\bar{Z}_i$ for $T^{0,1}\Sigma$. $(Z_i, \bar{Z}_i)$ is a local

orthonormal frame for $T\Sigma \otimes \mathbb{C}$.
Then, using example A.1.6,

$$\begin{aligned}
R^E =& \frac{1}{4}\sum_{i,j}(RZ_j,\bar{Z}_i)c(Z^j)c(\bar{Z}^i) + (R\bar{Z}_j, Z_i)c(\bar{Z}^j)c(Z^i) \qquad \text{([26, p. 151f])}\\
=& \frac{1}{4}\sum_{i,j}(RZ_j,\bar{Z}_i)c(Z^j)c(\bar{Z}^i)\\
&\quad + \frac{1}{4}\Big(\sum_{i,j}(-1)(\bar{Z}_j, RZ_i)(-1)c(Z^i)c(\bar{Z}^j) - 2\sum_j(\bar{Z}_j, RZ_j)\Big)\\
=& \frac{1}{2}\sum_{i,j}(RZ_j,\bar{Z}_i)c(Z^j)c(\bar{Z}^i) - \frac{1}{2}\sum_j (RZ_j,\bar{Z}_j). \qquad (4)
\end{aligned}$$

The twisted connection $\nabla^E = \nabla \otimes \nabla^F$ of the Levi Civita connection ∇ on $T^{1,0}\Sigma$ and the canonical connection on F is a Clifford connection with curvature

$$\mathcal{F}^E = \sum_{k,l}((R+\mathcal{F}^F)Z_k, \bar{Z}_l)\varepsilon(\bar{Z}^l)\iota(Z^k).$$

Since $c(\bar{Z}^l)c(Z^k) = 2\varepsilon(\bar{Z}^l)\iota(Z^k)$, equation (4) gives

$$\mathcal{F}^{E/S} = \mathcal{F}^F + \frac{1}{2}\sum_k (RZ_k, \bar{Z}_k) = \mathcal{F}^F + \frac{1}{2}\operatorname{Tr}(R^+).$$

Then, if we take A.2.8 into account, Atiyah-Singer's theorem implies the result. □

A.2.10 Theorem (Riemann-Roch)**.** *[26, p. 152] Let Σ be a Riemannian surface of genus g and F a holomorphic vector bundle over Σ. The index of the Dolbeaut-Dirac operator on the Clifford module $\Lambda(T^{0,1}\Sigma)\otimes F$ over Σ equals*

$$\operatorname{rk}_{\mathbb{R}}(F)(1-g) + 2\langle c_1(F), [\Sigma]\rangle.$$

Proof. [26, p. 152] If Σ is a Riemannian surface, $\operatorname{ch}(F) = \operatorname{rk}_{\mathbb{R}}(F) + c_1(F)$, and $\operatorname{Td}(\Sigma) = 1 - \frac{R^+}{2}$. The latter is immediate from the Taylor expansion $\operatorname{Td}(\Sigma) = 1 + \frac{1}{2}\operatorname{Tr}(R^+) + \frac{1}{2}\operatorname{Tr}\left((R^+)^2\right) + \ldots$. Thus

$$\operatorname{Td}(\Sigma)\operatorname{ch}(F) = \operatorname{rk}_{\mathbb{R}}(F)\left(1 - \frac{R^+}{2}\right) + c_1(F) + \frac{R^+}{2}\wedge c_1(F)\,.$$

Integrating yields

$$\begin{aligned}
\operatorname{index}(\bar{\partial} + \bar{\partial}^*) &= -\frac{1}{4\pi i}\left(\int \operatorname{rk}_{\mathbb{R}}(F)R^+ + \int c_1(F)\right)\\
&= -\frac{\operatorname{rk}_{\mathbb{R}}(F)}{4\pi i}\left(\int R^+\right) + 2\langle c_1(F), [\Sigma]\rangle\,. \qquad (5)
\end{aligned}$$

Let g denote the genus of Σ. Then this formula equals

$$\operatorname{rk}_{\mathbb{R}}(F)(1-g) + 2\langle c_1(F), [\Sigma]\rangle.$$

In fact the Riemann-Roch-Hirzebruch theorem for the trivial bundle $M \times \mathbb{C}$ implies

$$1 - g = \dim(H^0(M, \mathcal{O}(\mathbb{C}))) - \dim(H^1(M, \mathcal{O}(\mathbb{C}))) = -\frac{1}{4\pi i}\int R^+.$$

□

A.3 The Cauchy-Riemann and the Dolbeaut-Dirac operator

The aim of this section is to compute the index of a Cauchy-Riemann operator on a Riemannian surface. It equals the index of a Dolbeaut-Dirac operator.

A.3.1 Definition (Cauchy-Riemann operator). Let (F, J) be a vector bundle with complex structure J. Let

$$\tilde{\nabla} : \Gamma^{\infty}(F) \to \Gamma^{\infty}((T^{0,1}\Sigma)^{*} \otimes_J F),$$

such that it satisfies the Leibniz rule

$$\tilde{\nabla}(f\, u) = \bar{\partial}(f)u + f\tilde{\nabla}(u),$$

where $f \in \mathcal{C}^{\infty}(\Sigma, \mathbb{C})$ and $u \in \Gamma^{\infty}(F)$. Then $\tilde{\nabla}$ is called partial covariant derivative. If $\mathscr{C}$ is a partial covariant derivative such that the $0, 2$ part of its curvature

$$F^{0,2}_{\mathscr{C}} \in \Omega^{0,2}(F) = \Gamma^{\infty}(\Lambda^2(T^{0,1}\Sigma)^{*} \otimes_J F)$$

vanishes, it is called Cauchy Riemann operator.

A.3.2 Remark. If Σ is a Riemannian surface, every partial covariant derivative is a Cauchy Riemann operator.

A.3.3 Proposition (Holomorphic structures on vector bundles II). [4, 2.1.59] Let (F, J) be a complex vector bundle with hermitian structure h over a complex manifold Σ.
If $\mathscr{C}$ is a Cauchy Riemann operator for (F, J), it determines a holomorphic structure on F. Especially, we have that $\bar{\partial} = \mathscr{C}$.
If Σ is a Riemannian surface, every partial covariant derivative in sense of the above definition determines a holomorphic structure.

Compare this result with A.2.5.

A.3.4 Remark. [26, 3.65] Let F be a holomorphic vector bundle. Then $\nabla^{0,1}$ already determines ∇. In fact

$$\bar{\partial}h(.,.) = h(\nabla^{1,0}.,.) + h(.,\nabla^{0,1}.),$$

which is seen by taking the component of $dh(.,.)$ lying in $\Omega^{0,1}$.

A.3.5 Theorem. *Let $\mathscr{C}$ be a Cauchy-Riemann operator on a hermitian vector bundle (F, J) over a Riemannian surface Σ.
The index of $\mathscr{C}$ coincides with the index of the self adjoint Dirac operator $\bar{\partial} + \bar{\partial}^{*}$ on the Clifford module $\Lambda(T^{0,1}\Sigma)^{*} \otimes_J F$.*

Proof. Since the symbol of $\mathscr{C}$ coincides with the symbol of $\bar{\partial}$, both have the same index. As we are working on a Riemannian surface (complex dimension one),

$$\begin{aligned}\operatorname{index}(\bar{\partial} + \bar{\partial}^{*}) &= \dim\ker((\bar{\partial} + \bar{\partial}^{*})^{+}) - \dim\ker((\bar{\partial} + \bar{\partial}^{*})^{-}) \\ &= \dim\ker(\bar{\partial}) - \dim\operatorname{coker}(\bar{\partial}).\end{aligned}$$

□

The following theorem is an application of Riemann-Roch's theorem A.2.10.

A.3.6 Theorem. *The index of a Cauchy-Riemann operator on a hermitian complex vector bundle (F, J) over a Riemannian surface Σ of genus g equals*

$$\operatorname{rk}_{\mathbb{R}}(F)(1 - g) + 2\langle c_1(F), [\Sigma]\rangle.$$

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Abstract

A J-holomorphic curve is a map from a Riemannian surface to an almost complex manifold, which satisfies the Cauchy-Riemann equation. A curve can be thought of as a section in a trivial fibre bundle. In this picture, differentiation of a curve is covariant differentiation of a section with respect to a connection on the fibre bundle. So the Cauchy-Riemann equation makes sense for fibre bundles over a Riemannian surface, which are equipped with a vertical almost complex structure, an endomorphism of the vertical bundle restricting to an almost complex structure on the fibres. In this diploma thesis it is shown that the solution space is a finite dimensional manifold for almost all connections, up to a set of first category. The dimension of the connected components of this manifolds can be computed using Riemann-Roch's theorem. Smale's Sard theorem is discussed. The canonical infinite dimensional manifold structure on $\mathcal{C}^l$ and Sobolev spaces of fibre bundle sections is introduced. Atiyah-Singer's Index theorem for Dirac operators and Riemann-Roch's theorem is explained in the appendix.

Zusammenfassung

Eine J-holomorphe Kurve ist eine Abbildung von einer Riemann Fläche in eine fast komplexe Mannigfaltigkeit, die die Cauchy-Riemann Gleichung erfüllt. Eine Kurve kann als Schnitt in ein triviales Faserbündel gedacht werden. In dieser Interpretation ist das Differential einer Kurve die kovariante Ableitung des Schnittes bezüglich einer Konnektion auf dem Faserbündel. Der Cauchy-Riemann Gleichung kann also auf Faserbündeln über einer Riemannfläche Sinn gegeben werden, die mit einer vertikalen fast komplexen Strukur ausgestattet sind, einem Vektorbündelendomorphismus des vertikalen Bündels, der sich auf den Fasern zu einer fast komplexen Struktur einschränkt. In dieser Diplomarbeit wird gezeigt, dass der Lösungsraum für fast alle Konnektionen, bis auf eine Menge von erster Kategorie, eine endlich dimensionale Mannigfaltigkeit ist. Die Dimension der Zusammenhangskomponenten dieser Mannigfaltigkeiten kann mit Hilfe des Satzes von Riemann-Roch berechnet werden. Der Satz von Smale und Sard wird diskutiert. Die kanonische unendlich dimensionale Mannigfaltigkeitsstruktur auf Räumen von Faserbündelschnitten von $\mathcal{C}^l$- oder Sobolevklasse wird eingeführt. Im Appendix wird der Satz von Atiyah-Singer für Dirac Operatoren und der Satz von Riemann-Roch erläutert.

Lebenslauf

Markus Oliver Steenbock, geboren am *07.05.1983* in *München, Deutschland.*

Staatsbürgerschaft	Deutsch
2002	Allgemeine Hochschulreife (Theodolinden Gymnasium München)
2002 - 2003	Zivildienst am Krankenhaus München Neuperlach
2003 - 2004	Tätigkeit als Verkäufer im Disneyland Resort Paris
2004 - 2005	Studium der Mathematik (Diplom) an der Ludwig Maximilians Universität München
ab 2005	Studium der Mathematik (Diplom) an der Universität Wien