DIPLOMARBEIT

Titel der Diplomarbeit

Functionalanalytic Methods for Paraxial Wave Equations

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Magister der Naturwissenschaften (Mag. rer. nat.)

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Abstract

This diploma thesis deals with functional analytic methods for paraxial wave equations of Schrödinger-type with non-smooth coefficients.

Main instrument is the theory of strongly continuous evolution systems, which is based on the theory of semigroups of bounded linear operators. The particular choice of results needed in the special situation, occupies a large part of this text. Getting started, some basics of functional analysis and distribution theory pave the way. Then basic results about semigroups and evolution systems are introduced. Conditions for the existence of a semigroup are well-known and are therefore taken from literature. Instead, the proof for existence of an evolution system in the hyperbolic case is given, since it contains techniques which are also relevant in the last chapter.

The last part of this thesis treats a certain paraxial wave equation of the above type. Thereto the geophysical benefit is outlined. Verifying the conditions for the existence of an evolution system, one additionally needs various technical results about Sobolev spaces. The difficulty is the low-coefficient regularity. Finally existence and uniqueness of a strong solution can be achieved.

Zusammenfassung

Diese Diplomarbeit behandelt funktionalanalytische Methoden für paraxiale Wellengleichungen vom Schrödinger-typ mit nicht-glatten Koeffizienten.

Wichtigstes Werkzeug ist die Theorie der stark stetigen Evolutionssysteme, welche wiederum auf der Theorie der stark stetigen Operatorhalbgruppen aufbaut. Die Abstimmung der beiden Theorien auf die spezielle Situation nimmt einen großen Teil dieser Arbeit in Anspruch. Um den Einstieg zu erleichtern, werden anfangs einige grundlegende Begriffe aus der Funktionalanalysis und der Distributionentheorie wiederholt. Danach werden Operatorhalbgruppen und Evolutionssysteme besprochen. Bedingungen für die Existenz von Halbgruppen sind wohlbekannt und werden aus der Literatur übernommen. Stattdessen wird die Existenz eines Evolutionssystems im hyperbolischen Fall genau bewiesen, da die Beweistechniken auch im letzten Kapitel relevant sind.

Im letzten Teil dieser Arbeit wird eine spezielle paraxiale Wellengleichung vom obigen Typ besprochen. Dazu wird einleitend ihr Nutzen in der Geophysik angedeutet. Danach werden die Bedingungen für die Existenz eines Evolutionssystems nachgewiesen. Hierfür wird, neben den oben genannten Werkzeugen, noch eine Vielzahl an technischen Resultaten über Sobolevräume benötigt. Die Schwierigkeit liegt bei den niedrigen Regularitätseigenschaften der Koeffizienten. Schließlich gelingt es, die Existenz und Eindeutigkeit einer starken Lösung im entsprechenden Setting zu beweisen.

Contents

Notation and Symbols

$\mathbb{R}$	Field of real numbers.
$\mathbb{R}_0^+$	Set of nonnegative real numbers.
$\mathbb{C}$	Field of complex numbers.
$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$	The partial derivative with respect to the i-th coordinate.
$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$,	The partial derivative of order α.
$\lvert\alpha\rvert = \alpha_1 + \dots + \alpha_n$	The order of a multi-index α.
$\Delta = \sum_{i=1}^n \partial_i^2$	The Laplace operator.
$\langle . \vert . \rangle$	The inner product on some Hilbert space.
$\langle . , . \rangle$	The distributional action.
X'	The topological dual of a Banach space X.
$L(X, Y)$	Set of bounded linear operators from X to Y.
$D(A)$	The domain of an operator A.
$\rho(A)$	The resolvent set of A.
$R(\lambda : A) = (\lambda - A)^{-1}$	The resolvent of A.
$G(M, \omega)$	Set of all generators of a semigroup satisfying $\lVert T(t)\rVert \leq M\mathrm{e}^{\omega t}$.
oE	The closed unit ball for any Banach space E.
$C_0^0(\mathbb{R}^n)$	Set of continuous functions on $\mathbb{R}^n$ satisfying $\lim_{\lvert x\rvert\to\infty} f(x) = 0$.
$C_c(0, T : X)$	Set of X-valued continuous functions with compact support.
$C([a, b] : X)$	Set of continuous, X-valued functions on $[a, b] \subseteq \mathbb{R}$. This interval can also be open, or half-open.
$C^1([a, b] : X)$	Set of functions which are continuously differentiable with respect to the norm of a Banach space X.
$L^1(0, T : X)$	The closure of $C_c(0, T : X)$ with respect to the norm $\lVert f\rVert_{L^1} = \int_0^T \lVert f(t)\rVert_X \, dt$.
$\mathcal{D}(\mathbb{R}^n)$	Space of test functions on $\mathbb{R}^n$, i.e. smooth functions with compact support in $\mathbb{R}^n$.
$\mathscr{S}(\mathbb{R}^d)$	Vector space of rapidly decreasing functions on $\mathbb{R}^d$, also called Schwartz functions.
$\mathscr{S}'(\mathbb{R}^d)$	Tempered distributions on $\mathbb{R}^d$, i.e. continuous linear functionals on $\mathscr{S}(\mathbb{R}^d)$.
$\mathscr{S}'(\mathbb{R}; E)$	Banach space-valued tempered distributions.
$H^s(\mathbb{R}^d)$	Set of all tempered distributions whose derivatives up to order s belong to $L^2(\mathbb{R}^d)$.
$C_*^r(\mathbb{R}^d)$	Set of Zygmund-Hölder continuous tempered distributions on $\mathbb{R}^d$ (cf. (4.1.2)).

Chapter 1

Introduction

Differential equations occur in a great number of problems, spread over lots of disciplines such as mathematics, physics, engineering, etc. Many methods have been developed to get solutions of the various equations, but still there are millions of unsolved problems.
It is well known, that the exponential-function is used to construct solutions of systems of linear first order ordinary differential equations with constant coefficients in $\mathbb{R}^n$. In this context the exponential function of a matrix is defined via the power series

$$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Generalising this concept to Banach spaces, one has the problem that linear operators on Banach spaces are not necessarily bounded. While the exponential of a bounded operator can still be defined via the exponential series, this is not the case for unbounded operators. This leads to the theory of semigroups of bounded linear operators.

After recalling some basic ideas of functional analysis and distribution theory, semigroups of bounded linear operators are the main purpose of the second chapter. The following properties of semigroups reproduce the features of the exponential function:

$$\begin{aligned} T(s+t) &= T(s) \cdot T(t) & \qquad \mathrm{e}^{a+b} &= \mathrm{e}^a \cdot \mathrm{e}^b \\ T(0) &= 1 & \mathrm{e}^0 &= 1. \end{aligned}$$

Since we want to apply the theory to a paraxial wave equation of Schrödinger-type, we concentrate on strongly continuous semigroups. There are necessary and sufficient conditions for an unbounded operator to be the generator of such a semigroup. Moreover self-adjointness plays a significant role if one wants to define the exponential of an unbounded operator. Closing Chapter 2, solutions of autonomous equations are constructed.

Since the method of semigroups is restricted to autonomous problems, we take the step to evolution systems in Chapter 3, which generalises the method to non-autonomous linear first order systems. The analogy to the exponential-function is not that clear, but one can still find a relation between their properties:

$$\begin{aligned} U(t,s) &= U(t,r) \cdot U(r,s) & \qquad \mathrm{e}^{t-s} &= \mathrm{e}^{t-r} \cdot \mathrm{e}^{r-s} \\ U(t,t) &= 1 & \mathrm{e}^{t-t} &= 1, \end{aligned}$$

where $0 \leq s \leq r \leq t \leq T$. It is not easy to establish existence of an evolution system. However in the hyperbolic case, as well as in the parabolic case, it is possible to state conditions which ensure existence of an evolution system. The major aim of Chapter 3 is to prove existence of an evolution system in the hyperbolic case. We start out with some motivation, then collect the most important technical tools and after proving existence, we use the evolution system to construct solutions of the evolution problem.

In Chapter 4 we discuss a paraxial wave equation of Schrödinger-type with non-smooth coefficients. At the beginning, the geophysical benefit is shortly outlined. The additional challenge is the mathematical setting of the formulated Cauchy problem. Due to low coefficient regularity, certain spaces of distributions are needed. Before coming to the application of semigroup theory, one has to establish self-adjointness of the corresponding operators, which involves a difficile result of elliptic regularity. After step by step checking the conditions for the existence of the evolution system in the hyperbolic case, we deal with the dependence from an additional parameter. In conclusion we give a solution in terms of the evolution system.

Working on paraxial wave equations, several fields of mathematics come into play. The diagram below gives an overview of the ingredients for this work.

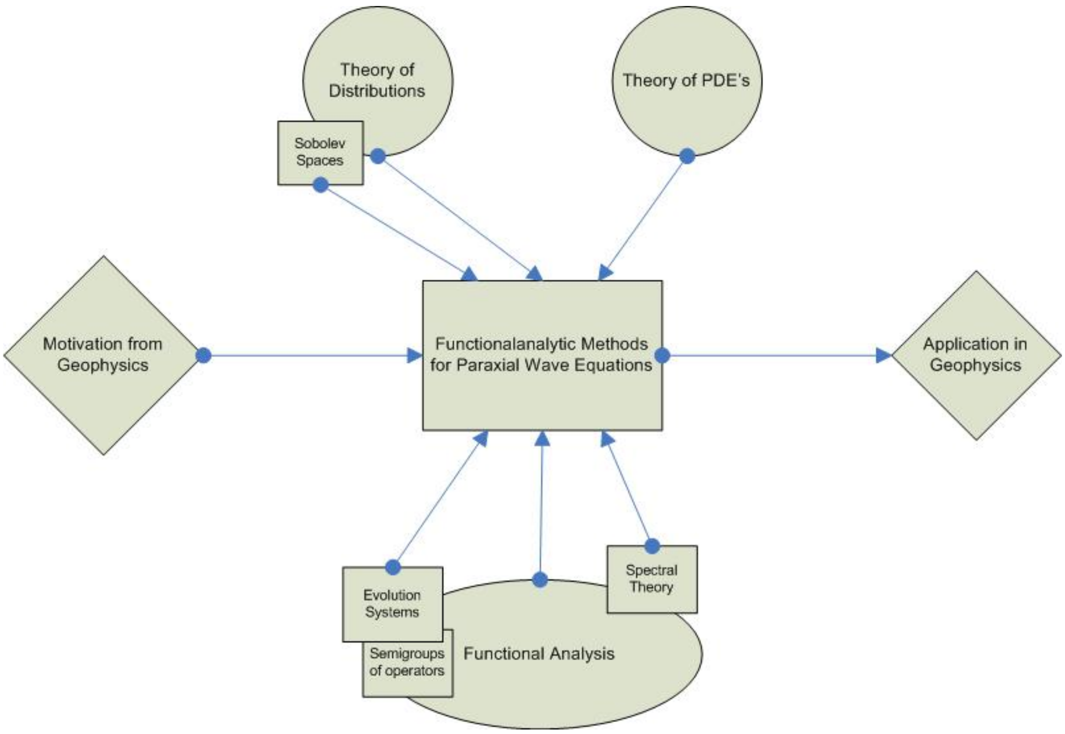

Chart1: Ingredients for this thesis.

In the part of semigroups and evolution systems we mainly follow the book of Pazy [14], the results of Chapter 4 were published by Maarten De Hoop, Günther Hörmann and Michael Oberguggenberger in 2008 [2]. Additional references for specific results are given in the text.

Chapter 2

Requisites from Functional Analysis

In this chapter we recall some basic ideas of functional analysis but take for granted that the reader has attended a course in functional analysis and one in distribution theory sometime. Therefore we reduce the sections about closed and unbounded operators as well as distributions and Sobolev spaces to the minimum (cf. 2.1, 2.2, resp. 2.3).
In section 2.4 we start with the basics of semigroups of bounded linear operators which are fundamental in Chapter 3.

2.1 Closed Operators

We know that differential operators are unbounded in general. But many of them are at least closed. We will see that this property implies qualities from bounded operators, i.e. that a closed operator is bounded on its domain equipped with a special norm.
In this section we follow the book of Werner [18, IV].

Definition 2.1.1. (Closed operator)
Let X and Y be normed vector spaces, $D \subseteq X$ a subspace and $T : D \to Y$ a linear operator (denoted by $T : X \supseteq D \to Y$). T is defined to be *closed* if the following holds:

- Let $(x_n)_n,\ x_n \in D$ be a sequence converging to $x \in X$ and let $y \in Y$ be the limit of $(Tx_n)_n$, then $x \in D$ and $Tx = y$.

We provide D with the so-called *graph-norm* $\|x\|_D = \|x\|_X + \|Tx\|_X$. With that said we state the following lemma.

Lemma 2.1.2. *Let X and Y be Banach spaces and let $T : X \supseteq D \to Y$ be closed. Then we have*

(i) $(D, \|.\|_D)$ *is a Banach space,*

(ii) $T : (D, \|.\|_D) \to (Y, \|.\|_Y)$ *is continuous.*

Proof. (cf. [18, Lemma IV.4.3]) □

A fundamental result of functional analysis is the closed-graph theorem. It says that any closed operator, defined on a Banach space is automatically bounded.

Theorem 2.1.3. *(The closed graph-theorem)*
Let X and Y be Banach spaces and let $T : X \to Y$ be linear and closed. Then T is continuous.

Proof. (cf. [18, Theorem IV.4.5]) □

2.2 Unbounded Operators on Hilbert Spaces

"It is a fact of life that many of the most important operators which occur in mathematical physics are not bounded" [15, p.249].

This is just a brief run through the facts we will need later on. If you need a detailed introduction, the chapter about unbounded operators in the book of Werner [18, Abschnitt VII.2] may please you.

For the following let H be a complex Hilbert space with inner product $\langle .|. \rangle$.

Definition 2.2.1. (Unbounded operator)

(i) An *(unbounded) operator* on H is a linear map $T : D(T) \to H$, where $D(T)$ is a subspace of H, called the *domain* of T.

(ii) T is said to be *densely defined* if $\overline{D(T)} = H$.

(iii) Let S, T be operators on H such that $D(T) \subseteq D(S)$ and $Su = Tu \ \forall u \in D(T)$. Then we call S an *extension* of T, denoted by $T \subset S$.

Example 2.2.2. Consider the differential-operator on $L^2[0,1]$ with domain $D(T) = C^1[0,1]$, defined via

$$Tf = f' \qquad \forall f \in C^1[0,1].$$

Then T is unbounded, since for $f_n(t) := \sin(nt\pi)$ we have that

$$\|f_n\|_{L^2}^2 = \int_0^1 |\sin(nt\pi)|^2 \, dt = \frac{1}{2} \qquad \forall n \in \mathbb{N},$$

but for $Tf_n(t) = f_n'(t) = n\pi \cos(nt\pi)$ we have that

$$\|f_n'\|_{L^2}^2 = (n\pi)^2 \int_0^1 |\cos(nt\pi)|^2 \, dt = \frac{(n\pi)^2}{2}$$

and hence $(f_n')_n$ is unbounded in $L^2[0,1]$.

Remark 2.2.3. Two operators S, T on H are equal, if and only if $T \subset S$ and $S \subset T$, or equivalently if $D(T) = D(S)$ and $Su = Tu \ \forall u \in D(T)$.

Definition 2.2.4. (The adjoint operator)
Let T be a densely defined operator on H. We define the *adjoint operator of* T by

$$\begin{aligned} D(T^*) &:= \{u \in H \mid \exists v \in H : \langle Tw|u\rangle = \langle w|v\rangle \ \forall w \in D(T)\} \\ \text{and} \qquad T^*u &:= v. \end{aligned} \tag{2.2.1}$$

Remark 2.2.5. (i) By the Riesz lemma [15, Theorem II.4] the mapping $w \mapsto \langle w|T^*u\rangle$ is a continuous linear functional. Therefore $w \mapsto \langle Tw|u\rangle$ is continuous on $D(T)$ and hence we can write the domain of T^* as

$$\{u \in H | w \mapsto \langle Tw|u\rangle \text{ continuous on } D(T)\}.$$

(cf. discussion below Definition VII.2.2 in [18]).

(ii) Note that for T^* to be well defined we need that $D(T)$ is dense, since then v in Definition 2.2.4 is unique.

(iii) The domain of T^* does not need to be dense (cf. [15, Section VIII.1, Example 4]).

Definition 2.2.6. (Symmetric and self-adjoint operators)

(i) An operator $T : D(T) \to H$ is *symmetric* if

$$\langle Tu|v\rangle = \langle u|Tv\rangle \ \forall u, v \in D(T).$$

(ii) If in addition T is densely defined and $T = T^*$, then we call T *self adjoint.*

Remark 2.2.7. Let T be a densely defined operator, then

(i) T is symmetric $\Leftrightarrow T \subset T^*$,

(ii) $T = T^* \Leftrightarrow T$ symmetric and $D(T) = D(T^*)$.

Spectral theory is a very important part of functional analysis. In this thesis we need the spectral theorem for unbounded, self-adjoint operators, which will be used in Section 2.4.2.

Theorem 2.2.8. *(The spectral theorem - functional calculus form)(cf. [15, Theorem VIII.5]) Let $\mathcal{B}(\mathbb{R})$ denote the bounded, Borel-measurable functions on $\mathbb{R}$ and let A be an unbounded, self-adjoint operator on H. Then there exists a unique map $\hat{\Phi} : \mathcal{B}(\mathbb{R}) \to L(H)$, such that for all $f, g, h, h_n \in \mathcal{B}(\mathbb{R})$ we have:*

(i) $\hat{\Phi}$ is an algebraic $*$-homomorphism, i.e.

$$\begin{aligned} &\hat{\Phi}(f \cdot g) = \hat{\Phi}(f) \cdot \hat{\Phi}(g), && \hat{\Phi}(\mathit{1}) = \mathrm{I}_H, \\ &\hat{\Phi}(\lambda f) = \lambda \hat{\Phi}(f) \text{ for all } \lambda \in \mathbb{C} \text{ and} && \hat{\Phi}(\bar{f}) = \hat{\Phi}(f)^*. \end{aligned}$$

(ii) $\hat{\Phi}$ is continuous in norm, i.e. $\left\|\hat{\Phi}(h)\right\|_{L(H)} \leq \|h\|_\infty$.

(iii) If $(h_n(x))_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(\mathbb{R})$ with

- *$h_n(x) \to x \ \forall x \in \mathbb{R}$ and*
- *$|h_n(x)| \leq |x| \ \ \forall x \in \mathbb{R} \ \forall n \in \mathbb{N}$,*

then

$$\lim_{n\to\infty} \hat{\Phi}(h_n)\psi = A\psi \qquad \forall \psi \in D(A),$$

(iv) $h_n(x) \to h(x)$ pointwise and $(\|h_n\|_\infty)_{n\in\mathbb{N}}$ bounded $\Rightarrow$ $\hat{\Phi}(h_n) \to \hat{\Phi}(h)$ strongly,

(v) $A\psi = \lambda\psi \Rightarrow \hat{\Phi}(h)\psi = h(\lambda)\psi$,

(vi) $h \geq 0 \Rightarrow \hat{\Phi}(h) \geq 0$ (i.e. $\hat{\Phi}(h)$ is a positive operator).

2.3 Distributions and Sobolev Spaces

We will need some results about distributions in Chapter 4. In this section we briefly recall the definition of scalar distributions and some properties of the Sobolev spaces $H^s(\mathbb{R}^n)$. More details of scalar distributions can be found in [10] or in [6]. For Sobolev spaces the book of Adams and Fournier [1] can be recommended in addition. We also recall regular distributions and discuss the case of a multi-dimensional distribution which is continuous in one variable, also called $t-$dependent distribution. Banach space-valued distributions are discussed in [17, Section

39]. They are needed in Chapter 4 and are treated analogous to scalar distributions.
In the following Ω resp. Ω_i, $i \in \mathbb{N}$ denotes an open subset of $\mathbb{R}^n$ resp. $\mathbb{R}^{n_i}$. Let $\mathcal{D}(\Omega)$ be the space of smooth functions with compact support in Ω, also called the space of test functions on Ω. By convergence of test functions we mean existence of a compact set in Ω which covers all involved supports and uniform convergence of all partial derivatives on this set.

Scalar Distributions: A scalar distribution on Ω is a linear functional on $\mathcal{D}(\Omega)$, i.e. a linear map $u : \mathcal{D}(\Omega) \mapsto \mathbb{C}$ satisfying the continuity condition

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) \Rightarrow \langle u, \varphi_n \rangle \to \langle u, \varphi \rangle \text{ in } \mathbb{C}. \tag{2.3.1}$$

The space of scalar distributions is denoted by $\mathcal{D}'(\Omega)$. The following result is a useful characterisation for the continuity condition of distributions.

Theorem 2.3.1. *Let $u : \mathcal{D}(\Omega) \to \mathbb{C}$ be linear. Then $u \in \mathcal{D}'(\Omega)$ if and only if for all $K \subset\subset \Omega$ there exists a $C > 0$ and $m \in \mathbb{N}$ such that*

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \qquad \forall \varphi \in \mathcal{D}(K).$$

Proof. (cf. [10, Theorem 1.26]) □

If $(u_n)_n$ is a sequence of scalar distributions, we say that $(u_n)_n$ converges in $\mathcal{D}'(\Omega)$ if there exists a distribution $u \in \mathcal{D}'(\Omega)$ such that

$$\langle u_n, \varphi \rangle \to \langle u, \varphi \rangle \text{ in } \mathbb{C} \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

This notion of convergence stems from a topology, generated by the family of seminorms $p_\varphi(u) := |\langle u, \varphi \rangle|$ $(\varphi \in \mathcal{D}(\Omega))$.

Sobolev Spaces: We denote the Fourier transform of some $u \in \mathscr{S}'(\mathbb{R}^n)$ by $\hat{u}$ and define for all $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : (1 + |.|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n)\}.$$

The spaces $H^s(\mathbb{R}^n)$ are Banach spaces with the Sobolev-norm

$$\|u\|_s^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |x|^2)^s \, |\hat{u}(x)|^2 \, dx$$

and Hilbert spaces with the inner product

$$\langle u|v \rangle_s = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |x|^2)^s \hat{u}(x) \overline{\hat{v}(x)} dx.$$

Example 2.3.2. For example we have that the delta-distribution δ belongs to $H^{-s}(\mathbb{R}^n)$ if and only if $s > \frac{n}{2}$. Indeed one observes that:

$$\begin{aligned} \delta \in H^{-s}(\mathbb{R}^n) &\Leftrightarrow (1 + |.|^2)^{\frac{-s}{2}} \cdot \hat{\delta} \in L^2(\mathbb{R}^n) \\ &\Leftrightarrow (1 + |.|^2)^{\frac{-s}{2}} \leq C(1 + |.|)^{-s} \in L^2(\mathbb{R}^n) \\ &\Leftrightarrow \int_{\mathbb{R}^n} \frac{1}{(1 + |x|)^{2s}} dx < \infty \\ &\Leftrightarrow 2s > n, \end{aligned}$$

where we used that the Fourier transform of the delta-distribution equals 1 and for $a, s \geq 0$ we have that $1 \leq (1 + a^2)^s \leq (1 + a)^{2s}$.

What will make it easier to handle such spaces are the following properties. The first one says that differentiation of some order m reduces the Sobolev order by m.

Lemma 2.3.3. *Let $s, t \in \mathbb{R}$. Then*

(i) for $s \geq t$ we have $H^s(\mathbb{R}^n) \hookrightarrow H^t(\mathbb{R}^n)$ continuously,

(ii) $u \in H^s(\mathbb{R}^n) \Leftrightarrow u, \partial_j u \in H^{s-1}(\mathbb{R}^n)\ \forall j = 1, \ldots, n$.

Proof. (cf. [10, Proposition 6.9 and Lemma 6.15]) □

If s is an integer, $H^s(\mathbb{R}^n)$ consists of those tempered distributions, whose derivatives up to order s lie in $L^2(\mathbb{R}^n)$.

Theorem 2.3.4. *Let $s \in \mathbb{N}$. Then*

$$H^s(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) : \frac{1}{i^{|\alpha|}}\partial^\alpha u \in L^2(\mathbb{R}^n)\ \forall \alpha \leq s\}.$$

Proof. (cf. [10, Theorem 6.16]) □

The last but most important property is that $H^s(\mathbb{R}^n)$ is an algebra if s is large enough.

Theorem 2.3.5. *If $s > \frac{n}{2}$ then $H^s(\mathbb{R}^n) \subseteq C_0^0(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ is an algebra.*

Proof. (cf. [10, Theorem 6.21] and [1, Theorem 4.39]) □

Regular Distributions: If $f \in C(\Omega)$ is a continuous function we receive a distribution u_f by setting

$$\langle u_f, \varphi \rangle := \int_\Omega f(x)\varphi(x)dx \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Since the assignment $f \mapsto u_f$ is linear and injective from $C(\Omega)$ to $\mathcal{D}'(\Omega)$, it is an embedding of $C(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. We simply write $C(\Omega) \subseteq \mathcal{D}'(\Omega)$.
But what if a function is continuous in one variable and 'only' distributional in the second? To answer this question we would like to define a generalisation of $C(\Omega_1) \otimes \mathcal{D}'(\Omega_2)$ which can be embedded in $C(\Omega_1, \mathcal{D}'(\Omega_2))$ which again can be embedded in $\mathcal{D}'(\Omega_1 \times \Omega_2)$:

$$C(\Omega_1) \otimes \mathcal{D}'(\Omega_2) \hookrightarrow C(\Omega_1, \mathcal{D}'(\Omega_2)) \hookrightarrow \mathcal{D}'(\Omega_1 \times \Omega_2).$$

But this leads too far. If distributions had pointvalues, the solution would be easy, but this is not the case. To get out of the affair we proceed as follows. We first justify to write $C(\Omega_1, \mathcal{D}'(\Omega_2)) \subseteq \mathcal{D}'(\Omega_1 \times \Omega_2)$ and then show that partial derivatives with respect to the second variable stay in $C(\Omega_1, \mathcal{D}'(\Omega_2))$.

Lemma 2.3.6. *Let $u \in C(\Omega_1, \mathcal{D}'(\Omega_2))$ then*

$$\langle u, \varphi \rangle := \int_{\Omega_1} \langle u(t), \varphi(t,.) \rangle dt \qquad \forall \varphi \in \mathcal{D}(\Omega_1 \times \Omega_2)$$

defines a scalar distribution u on $\Omega_1 \times \Omega_2$.

Proof. In this proof we use the semi norm-estimates from Theorem 2.3.1. Another method would be to establish the continuity condition (2.3.1) (cf. [3, 23.65.2(i)]).
Clearly u is a linear map from $\mathcal{D}(\Omega_1 \times \Omega_2) \mapsto \mathbb{C}$. We use Theorem 2.3.1 to show continuity of u. We have to establish

$$\forall K \subset\subset \Omega_1 \times \Omega_2\ \exists C > 0\ \exists m \in \mathbb{N} : |\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \qquad \forall \varphi \in \mathcal{D}(K).$$

So let $K \subset\subset \Omega_1 \times \Omega_2$. First we fix $t \in \Omega_1$ and set $\tilde{K_t} := (\{t\} \times \Omega_2) \cap K$ which is isomorphic to some set $K_t \subset\subset \Omega_2$. Since $u(t) \in \mathcal{D}'(\Omega_2)$ we have

$$\exists C_t > 0 \ \exists m_t \in \mathbb{N} : |\langle u(t), \psi \rangle| \leq C_t \sum_{|\alpha| \leq m_t} \|\partial^\alpha \psi\|_{L^\infty(K_t)} \qquad \forall \psi \in \mathcal{D}(K_t).$$

Consider the set $K_1 := \{t \in \Omega_1 : (t,x) \in K \ \text{ for some } x \in \Omega_2\}$ which is compact since K is. Since $t \mapsto u(t)$ is continuous, $C := \max\{C_t : t \in K_1\}$ and $m := \max\{m_t : t \in K_1\}$ are bounded.

Let now $\varphi \in \mathcal{D}(K)$ then $\varphi(t,.) \in \mathcal{D}(K_t)$ and we have that

$$|\langle u, \varphi \rangle| \leq \int_{\Omega_1} |\langle u(t), \varphi(t,.) \rangle| \, dt \leq \int_{\Omega_1} C_t \sum_{|\alpha| \leq m_t} \|\partial^\alpha \varphi(t,.)\|_{L^\infty(K_t)} \, dt.$$

Next we use that $\varphi(t,.) \equiv 0$ if $t \notin K_1$. From this fact and the above estimate follows that

$$|\langle u, \varphi \rangle| \leq \int_{K_1} C_t \sum_{|\alpha| \leq m_t} \|\partial^\alpha \varphi(t,.)\|_{L^\infty(K_t)} \, dt.$$

By setting $a := \max\{|t| : t \in K_1\}$ this yields

$$\begin{aligned} |\langle u, \varphi \rangle| &\leq \int_{K_1} C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi(t,.)\|_{L^\infty(K_t)} \, dt \\ &\leq \int_{K_1} C \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \, dt \\ &\leq 2aC \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^\infty(K)} \end{aligned}$$

where we used that $\|\partial^\alpha \varphi(t,.)\|_{L^\infty(K_t)} \leq \|\partial^\alpha \varphi\|_{L^\infty(K)}$. □

Lemma 2.3.7. *The space $C(\Omega_1, \mathcal{D}'(\Omega_2))$ is closed under taking partial derivatives with respect to the second variable, i.e.*

$$v \in C(\Omega_1, \mathcal{D}'(\Omega_2)) \Rightarrow \partial_x v \in C(\Omega_1, \mathcal{D}'(\Omega_2)).$$

Proof. The previous lemma ensures that $\partial_x v$ is well defined in $\mathcal{D}'(\Omega_1 \times \Omega_2)$ for all $v \in C(\Omega_1, \mathcal{D}'(\Omega_2))$. Since distributions have no pointvalues in general, we define a 'new' function $\tilde{\partial_x} v$ by

$$t \mapsto \partial_x(v(t)) : \Omega_1 \to \mathcal{D}'(\Omega_2).$$

This function is continuous since v is continuous from Ω_1 to $\mathcal{D}'(\Omega_2)$ and ∂_x is continuous on $\mathcal{D}'(\Omega_2)$. Hence we are allowed to write $\tilde{\partial_x} v \in C(\Omega_1, \mathcal{D}'(\Omega_2))$. Again by the previous lemma $\tilde{\partial_x} v$ corresponds to a distribution which is exactly $\partial_x v$. To see this take any $\psi \in \mathcal{D}(\Omega_1 \times \Omega_2)$, then

$$\begin{aligned} \langle \tilde{\partial_x} v, \psi \rangle &= \int_{\Omega_1} \langle \tilde{\partial_x} v(t), \psi(t,.) \rangle dt \\ &= \int_{\Omega_1} \langle \partial_x(v(t)), \psi(t,.) \rangle dt \\ &= -\int_{\Omega_1} \langle v(t), \partial_x \psi(t,.) \rangle dt \\ &= -\langle v, \partial_x \psi \rangle \\ &= \langle \partial_x v, \psi \rangle. \end{aligned}$$

This completes the proof. □

Remark 2.3.8. The same result can be shown for $C^k(\Omega_1, \mathcal{D}'(\Omega_2))$, $k \in \mathbb{N}$. Moreover if one replaces $\mathcal{D}'$ by a certain Sobolev space one can prove that

$$v \in C^k(\Omega, H^s(\mathbb{R})) \Rightarrow \partial_x v \in C^k(\Omega, H^{s-1}(\mathbb{R})).$$

2.4 Semigroups of Linear Operators

This part is devoted to the theory of semigroups of linear operators. Most of the time we follow the book of Pazy [14]. Additionally I recommend the book of Tanabe [16], which is more sophisticated and has a useful chapter of preliminaries.

2.4.1 Strong Continuity of Operator-Valued Functions

In this section we introduce a topology and hence a notion of convergence which turns out to be the proper regularity for semigroups. Things would be easier if we chose a stronger regularity, for example uniform continuity, but this would restrict our theory to differential equations with bounded operators (cf. Remark 2.4.12).

Let $(X, \|.\|_X)$ be a normed space (i.e. X a vector space and $\|.\|_X$ a norm on X) and let X' be its dual space. It is well known, that a norm induces a metric, which creates a topology. This topology is called the *norm topology* (or *strong topology*). Moreover there is the so called *weak topology* on X, induced by the family of seminorms $p_{x'}(x) = |x'(x)|\ \forall x \in X\ \forall x' \in X'$, which is in many examples different from the strong topology (cf. [18, p.397 Bsp(h)]).
Let $(Y, \|.\|_Y)$ be another normed space and $L(X, Y)$ be the space of bounded linear operators from X to Y. There are three interesting topologies on $L(X, Y)$ (cf. [18, p.398 Bsp(j)]):

- the *norm topology* (also *strong topology*, or *uniform operator topology* (cf. [15, Section VI.1])), induced by the operator norm,
- the *strong operator topology*, induced by the seminorms $p_x(A) = \|Ax\|_Y\ \forall x \in X,\ A \in L(X, Y)$ and
- the *weak operator topology*, induced by the seminorms $p_{x,y'}(A) = |y'(Ax)|\ \forall x \in X \forall y' \in Y'$.

For details consult some literature on locally convex vector spaces (e.g. [18, Kapitel VIII]). However at this point it gives some motivation for the definition of strong continuity for operator-valued functions.

Definition 2.4.1. (Strong continuity)
Let X be a normed space, Y a Banach space and $T : I \to L(X, Y)$, for $I \subseteq \mathbb{R}$ an interval. T is said to be *strongly continuous in* $t_0 \in I$ if it is continuous in t_0 with respect to the strong operator topology.
We say that T is *strongly continuous on* I if it has this property in every $t_0 \in I$.

Lemma 2.4.2. *Let X, Y, T and I be as above. The following are equivalent:*

(i) T *is strongly continuous in* $t_0 \in I$*,*

(ii) $\lim\limits_{t\to t_0} \|T(t)x - T(t_0)x\|_Y = 0\ \forall x \in X$.

Proof. We start out from $L(X,Y)$ equipped with the strong operator topology. A neighbourhood basis of $T(t_0)$ would be

$$\mathcal{U}_{T(t_0)} = \{U_{\varepsilon;x_1,\dots,x_m}(T(t_0)) \subseteq L(X;Y) : \varepsilon > 0,\ m \in \mathbb{N},\ x_1,\dots,x_m \in X\},$$

where

$$U_{\varepsilon;x_1,\dots,x_m}(T(t_0)) = \{S \in L(X,Y) : \|Sx_j - T(t_0)x_j\|_Y < \varepsilon,\ j = 1,\dots,m\}.$$

Now we use the neighbourhood-characterisation of continuity:
(i) ⇒ (ii):
Let $\varepsilon > 0$, $x_1,\dots,x_m$ be arbitrary and put $U_\varepsilon := U_{\varepsilon;x_1,\dots,x_m}(T(t_0))$. By strong continuity of T, $T^{-1}(U_\varepsilon)$ is a neighbourhood of t_0 in I. Therefore exists a $\delta > 0$ such that $B_\delta(t_0) \cap I \subseteq T^{-1}(U_\varepsilon)$. Hence $T(B_\delta(t_0) \cap I) \subseteq U_\varepsilon$ and this means that for all x_j

$$|t - t_0| < \delta \Rightarrow \|T(t)x_j - T(t_0)x_j\|_Y < \varepsilon.$$

Recalling that the x_j were arbitrary it follows that $\lim\limits_{t\to t_0} \|T(t)x - T(t_0)x\|_Y = 0\ \forall x \in X$.
(ii) ⇒ (i):
Let U_0 be a neighbourhood of $T(t_0)$. Then there exists an $\varepsilon_0 > 0$ and $x_1,\dots,x_m \in X$ such that $U_{\varepsilon_0;x_1,\dots,x_m} \subseteq U_0$. Now for all $0 < \varepsilon \le \varepsilon_0$ the set $U_{\varepsilon;x_1,\dots,x_m} \in \mathcal{U}_{T(t_0)}$ satisfies $U_{\varepsilon;x_1,\dots,x_m} \subseteq U_0$ and by *(ii)* there is a $\delta > 0$ with $|t - t_0| < \delta$ implies $T(t) \in U_{\varepsilon;x_1,\dots,x_m} \subseteq U_0$.
In other words $T(B_\delta(t_0) \cap I) \subseteq U_{\varepsilon;x_1,\dots,x_m} \subseteq U_0$, or $(B_\delta(t_0) \cap I) \subseteq T^{-1}(U_0)$ and hence $T^{-1}(U_0)$ is a neighbourhood of t_0. □

Remark 2.4.3. (i) In the book of Tanabe [16], strong continuity is defined via property (ii) of Lemma 2.4.2. In the same way Pazy [14] gives a definition for strongly continuous semigroups.

(ii) $T : I \to L(X,Y)$ is continuous in norm, if it is continuous with respect to the operator norm, i.e. if $\lim\limits_{t\to t_0} \|T(t) - T(t_0)\|_{op} = 0\ \forall t_0 \in I$. Therefore norm continuity is also called uniform continuity.

(iii) Note that for general Banach space-valued functions the term strong continuity is also used for the norm continuity. This is because the norm topology is called strong topology as well.

2.4.2 Strongly Continuous Semigroups of Bounded Linear Operators

In this section we define strongly continuous semigroups and their generators. We prove some basic properties and finally give a characterisation of the generator.

From now on X will be a Banach space.

Definition 2.4.4. (Strongly continuous semigroup)
Let X be a Banach space and for all $t \in [0,\infty)$ let $T(t)$ be a bounded linear operator on X. The one-parameter family $(T(t))_{t\ge 0}$ is called a *strongly continuous semigroup of bounded linear operators on* X (or simply a C_0*-semigroup*) if the following conditions hold:

(i) The corresponding map $T : \mathbb{R}_0^+ \to L(X)$ is strongly continuous,
i.e. $\lim\limits_{t\to t_0} T(t)x = T(t_0)x\ \forall x \in X\ \forall t_0 \ge 0$,

(ii) $T(0) = I$ (the identity on X),

(iii) $T(s+t) = T(s)T(t)\ \forall s,t \geq 0$ (the semigroup property).

Analogously we define a *strongly continuous group* by demanding that T is defined on $\mathbb{R}$ and conditions (i) and (iii) hold on all of $\mathbb{R}$.

Remark 2.4.5. (i) In 2.4.4(i) it suffices to claim that T is strongly continuous at 0 since by the semigroup property we have:
Let $s \in (0,\infty)$ and $0 \leq t \leq s$. Then

$$\lim_{s \downarrow t} \|T(s)x - T(t)x\| \leq \|T(t)\| \lim_{s \downarrow t} \|T(s-t)x - Ix\| = 0.$$

In the same way we find for $0 \leq s \leq t$:

$$\lim_{s \uparrow t} \|T(s)x - T(t)x\| \leq \lim_{s \uparrow t} \|T(s)\| \, \|Ix - T(t-s)x\| = 0,$$

since $\lim_{s\uparrow t} \|T(s)\| < \infty$, where we used Theorem 2.4.6 from below.

(ii) To get a better feeling for semigroups, one could read the chapter on uniformly continuous semigroups [14, Section 1.1] in the book of Pazy. Here we will only deal with C_0-semigroups resp. -groups.

(iii) This concept is similar to the flow of a vector field which also satisfies a semigroup property. Therefore one can see that the method of strongly continuous semigroups is just a special application of a bigger idea, which is widely spread in the theory of differential equations.

(iv) If $T(t)$ is a C_0-semigroup, then for all $x \in X$ the mapping $T_x : \mathbb{R}_0^+ \to X,\ T_x(t) := T(t)x$ is continuous.

Theorem 2.4.6. *Let $T(t)$ be a C_0-semigroup. Then there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\|_{op} \leq M\mathrm{e}^{\omega t} \qquad \forall t \geq 0. \tag{2.4.1}$$

Proof. That T is strongly continuous implies that $T_x : \mathbb{R}_0^+ \to X,\ T_x(t) := T(t)x$ is continuous for all $x \in X$ and hence we have that

$$\sup_{t\in[0,a]} \|T(t)x\|_X < \infty\ \forall x \in X\ \forall a > 0. \tag{2.4.2}$$

By the Banach-Steinhaus theorem, (2.4.2) implies uniform boundedness

$$\sup_{t\in[0,a]} \|T(t)\|_{op} < \infty\ \forall a > 0.$$

We set $\mathrm{e}^{\omega} := \|T(1)\|$ and $M := \begin{cases} \sup_{t\in[0,1]} \|T(t)\| & \text{if } \omega \geq 0 \\ \mathrm{e}^{-\omega} \cdot \sup_{t\in[0,1]} \|T(t)\| & \text{if } \omega < 0. \end{cases}$
For an arbitrary $t \geq 0$ we write $t = k + \alpha$ where $k \in \mathbb{N}$ and $\alpha \in [0,1)$. Then we have

$$\|T(t)\| \leq \|T(1)\|^k \|T(\alpha)\| \leq \begin{cases} \mathrm{e}^{\omega k} M \leq M\mathrm{e}^{\omega t} & \text{if } \omega \geq 0 \\ \mathrm{e}^{\omega k}\mathrm{e}^{\omega} M \leq M\mathrm{e}^{\omega t} & \text{if } \omega < 0. \end{cases}$$

□

Remark 2.4.7. (i) By continuity of T_x we mean continuity of $T_x : (\mathbb{R}_0^+, |.|) \to (X, \|.\|_X)$.

(ii) Below we will use the derivative of T_x, which is defined with respect to the Banach space norm (cf. [7, Section 175]).

Definition 2.4.8. (Semigroup of contractions)
If $T(t)$ is a C_0-semigroup with $M = 1$ and $\omega = 0$ then we call $T(t)$ a *C_0-semigroup of contractions.*

Definition 2.4.9. (Infinitesimal generator)
We define the *(infinitesimal) generator* of a C_0-semigroup $T(t)$ to be the operator with domain

$$D(A) := \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and action

$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \qquad \forall x \in D(A) \tag{2.4.3}$$

and say that A *generates* $T(t)$.

In the literature it is common to define spaces of generators.

Definition 2.4.10. We define $G(M, \omega)$ as the collection of all generators where the corresponding semigroup satisfies (2.4.1). If the underlying space is not clear one can also write $G(X, M, \omega)$.

In this view $T(t)$ is a semigroup of contractions if and only if its generator $A \in G(1, 0)$.

Theorem 2.4.11. *Let $T(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then:*

(i) $x \in D(A) \Rightarrow T(t)x \in D(A)$ $\forall t \geq 0$.

(ii) For $x \in D(A)$ the mapping T_x is continuously differentiable on $\mathbb{R}_0^+$ and we have

$$T_x'(t) = AT(t)x = T(t)Ax. \tag{2.4.4}$$

Proof. To prove the first part we establish the following limit

$$\lim_{h \downarrow 0} \frac{T(h)T(t)x - T(t)x}{h} = T(t) \lim_{h \downarrow 0} \frac{T(h)x - x}{h} = T(t)Ax.$$

The left hand side is per definition $AT(t)x$ which then proves the second equation in (2.4.4). Next we compare the right and left derivative of T_x. By definition of $D(A)$ we have

$$\frac{d^+}{dt} T_x(t) = \lim_{h \downarrow 0} \frac{T_x(t+h) - T_x(t)}{h} = T(t)Ax.$$

For the left derivative the case $t = 0$ is obvious. If $t > 0$ we have

$$\begin{aligned} \frac{d^-}{dt} T_x(t) - T(t)Ax \quad &= \lim_{h \downarrow 0} \left[\frac{T(t)x - T(t-h)x}{h} - T(t-h)Ax + T(t-h)Ax - T(t)Ax \right] \\ &= \lim_{h \downarrow 0} T(t-h) \left[\frac{T(h)x - x}{h} - Ax \right] + \lim_{h \downarrow 0} (T(t-h)Ax - T(t)Ax). \end{aligned}$$

By Theorem 2.4.6 and the definition of the generator A, both limits are zero and we are done. $\square$

Remark 2.4.12. (i) A linear operator A is the infinitesimal generator of a uniformly continuous semigroup $T(t)$, if and only if A is bounded (cf. [14, Chapter 1, Theorem 1.2]). In this case we have

$$T(t) = \mathrm{e}^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}.$$

Moreover the mapping $A \mapsto (T(t))_{t \geq 0}$ is bijective form $L(X)$ to the set of uniformly continuous semigroups and one can identify the uniformly continuous semigroups with the bounded operators on X (cf. [14, Chapter 1, Theorem 1.3 and Corollary 1.4]).

(ii) Hence every bounded operator generates a uniformly continuous semigroup and thus the generators of C_0-semigroups which are not uniformly continuous, have to be unbounded. The characterisation of such a generator A involves the resolvent set of A.

(iii) Although a theory of uniformly continuous semigroups would be more convenient, it would not be satisfying since some important examples like the Heat Equation would be excluded (cf. [17, Section 45]).

Definition 2.4.13. (The Resolvent)
Let A be a linear operator on X. We define the *resolvent set* of A as

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ is invertible with } (\lambda I - A)^{-1} \in L(X)\}$$

and call the family of bounded operators

$$R(\lambda : A) := (\lambda I - A)^{-1}, \qquad \lambda \in \rho(A)$$

the *resolvent* of A.

Theorem 2.4.14. *(Characterisation of generators of C_0-semigroups)*
A linear operator A is the infinitesimal generator of a C_0-semigroup $T(t)$ satisfying $\|T(t)\| \leq M\mathrm{e}^{\omega t}$, if and only if

(i) A is closed and $D(A)$ is dense in X,

(ii) $(\omega, \infty) \subseteq \rho(A)$ and

$$\|R(\lambda : A)^n\| \leq M(\lambda - \omega)^{-n} \quad \text{for } \lambda > \omega,\ n = 1, 2, \dots .$$

Proof. (cf. [14, Section 1.5]). □

We will also need the characterisation of generators of C_0-groups:

Theorem 2.4.15. *(Characterisation of generators of C_0-groups)*
A is the infinitesimal generator of a C_0-group of bounded linear operators $T(t)$ satisfying $\|T(t)\| \leq M\mathrm{e}^{\omega|t|}$ if and only if

(i) A is closed and $\overline{D(A)} = X$,

(ii) $\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ and

$$\|R(\lambda : A)^n\| \leq M(|\lambda| - \omega)^{-n},\ \forall \lambda \in \rho(A),\ n = 1, 2, \dots . \tag{2.4.5}$$

Proof. (cf. [14, Chapter 1, Theorem 6.3]). □

Stone's Theorem

As a short interlude we take a look at the question how one could define e^{itA} for an unbounded operator A (cf. [15, Section VIII.4]). We will use Stone's theorem in Chapter 4, where we want our self-adjoint operator to generate a strongly continuous group of unitary operators.
Recall that a bounded linear operator T on H is said to be a *unitary* operator if T is invertible and $TT^* = T^*T = I_H$.
For a bounded operator A the power series

$$\mathrm{e}^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n A^n}{n!}$$

converges in norm. In this way one can define the exponential of A. In case of an unbounded operator it is not that easy, but if A is additionally self-adjoint one can use the functional calculus to get something similar, namely a strongly continuous unitary group:

Theorem 2.4.16. *(cf. [15, Theorem VIII.7])*
Let A be a self-adjoint operator on H. By the spectral theorem [Theorem 2.2.8] we can define

$$U(t) := \mathrm{e}^{itA} := \hat{\Phi}(\lambda \mapsto \mathrm{e}^{it\lambda}).$$

Then we have

(i) $U(t)$ is a unitary operator for all $t \in \mathbb{R}$ and $U(t+s) = U(t)U(s)$ $\forall s, t \in \mathbb{R}$,

(ii) $\lim_{t\to t_0} U(t)\varphi = U(t_0)\varphi \qquad \forall \varphi \in H$,

(iii) $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t} = iA\psi \qquad \forall \psi \in D(A)$,

(iv) if $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ exists, then $\psi \in D(A)$.

Remark 2.4.17. (i) As in Section 2.4.2 we call a family of operators $(U(t))_{t\in\mathbb{R}}$ a *strongly continuous one-parameter unitary group* if it satisfies (i) and (ii) of Theorem 2.4.16.

(ii) By the following theorem, every strongly continuous unitary group possesses a self-adjoint operator on H such that $U(t) = \mathrm{e}^{itA}$ holds. Consequently we call iA the *infinitesimal generator* of $U(t)$.

Theorem 2.4.18. *(The theorem of Stone)(cf. [15, Theorem VIII.8])*
Let $U(t)$ be a strongly continuous one-parameter unitary group on H, then there exists a self-adjoint operator A on H, such that $U(t) = \mathrm{e}^{itA}$.

2.4.3 The Abstract Cauchy Problem - Autonomous Equations

Having some theory we turn immediately to a class of autonomous equations where we can apply the results derived so far. The abstract Cauchy problem consists of finding a solution to the following, in general inhomogeneous, initial value problem. First we discuss the *homogeneous initial value problem*:

$$AE(0,x) := \begin{cases} u'(t) = Au(t), & t > 0 \\ u(0) = x \end{cases}$$

where $A : X \supseteq D(A) \to X$ is a linear operator and $x \in X$ is called the initial value. A strong solution will always be of the following form (see also Definition 2.4.21 and Definition 3.1.1):

Definition 2.4.19. (The strong solution)
A *strong (or classical) solution* of the autonomous equation $AE(0,x)$ is a function $u : \mathbb{R}_0^+ \to X$, such that

(i) u is continuous on $\mathbb{R}_0^+$,

(ii) $u \in C^1(\mathbb{R}^+ : X)$,

(iii) $u(t) \in D(A)\ \forall t > 0$,

(iv) u satisfies $AE(0,x)$.

According to the preceding section we assume that A is a generator:

Proposition 2.4.20. *Let A be the infinitesimal generator of a C_0-semigroup $T(t)$. Then the homogeneous initial value problem $AE(0,x)$ has a unique strong solution for all initial values $x \in D(A)$, given by*

$$u(t) = T_x(t) = T(t)x \qquad \forall t \geq 0.$$

Proof. In order to see that u is a solution, one has to prove properties (i)-(iv) in Definition 2.4.19. This is established by Remark 2.4.5(iv) and Theorem 2.4.11. So only uniqueness needs some effort.
Uniqueness: Let u be a solution of $AE(0,x)$. For any fixed $x' \in X'$ and $t_1 > 0$ we set $F(t) := x'(T(t)u(t_1 - t))$. By doing this we get a continuous function $F : [0,t_1] \to \mathbb{R}$.

$$\begin{aligned}\frac{F(t+h)-F(t)}{h} &= x'\left(\frac{T(t+h)u(t_1-t-h)-T(t)u(t_1-t)}{h}\right)\\ &= x'\left(\frac{T(t+h)u(t_1-t)-T(t)u(t_1-t)}{h} - T(t+h)\frac{u(t_1-t-h)-u(t_1-t)}{-h}\right)\end{aligned}$$

Using Theorem 2.4.6 and the definition of A we get:

$$\lim_{h\to 0}\frac{F(t+h)-F(t)}{h} = x'(T(t)Au(t_1-t) - T(t)u'(t_1-t)) = 0.$$

Hence F is constant and by setting $t = 0$ respectively $t = t_1$ we get $x'(T(t_1)x) = x'(u(t_1))$ and since x' and t_1 were arbitrary this yields $T(t)x = u(t)$ for all $t \geq 0$. □

The Inhomogeneous Initial Value Problem:
To avoid confusions, a single T will always denote a positive real number. The operators of semigroups will always have a parameter, hence are denoted by $T(t)$.
We consider the *inhomogeneous initial value problem*:

$$\begin{cases} u'(t) = Au(t) + f(t), & 0 < t \leq T \\ u(0) = x \end{cases} \tag{2.4.6}$$

We assume that A is the infinitesimal generator of a C_0-semigroup $T(t)$ and $f : [0,T] \to X$. The above proposition implies that the corresponding homogeneous initial value problem has a unique solution for every initial value $x \in D(A)$.

Definition 2.4.21. (The strong solution)
A function $u : [0,T) \to X$ is called a *strong (or classical) solution* of (2.4.6) on $[0,T)$ if

(i) u is continuous on $[0,T)$,

(ii) $u \in C^1((0,T):X)$,

(iii) $u(t) \in D(A)$ $\forall t \in (0,T)$ and

(iv) u satisfies (2.4.6) on $[0,T)$.

In the next proposition we use the space $L^1(0,T:X)$, which is defined as follows. Take the X-valued continuous functions with compact support, $C_c(0,T:X)$, and complete this space with respect to the norm

$$\|f\|_{L^1} := \int_0^T \|f(t)\|_X \, dt.$$

Proposition 2.4.22. *If $f \in L^1(0,T:X)$ then for every $x \in X$ the initial value problem* (2.4.6) *has at most one strong solution. If a strong solution exists, it is given by*

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds. \tag{2.4.7}$$

Proof. Let u be a strong solution of (2.4.6). We consider the X-valued function $g(s) := T(t-s)u(s)$. To prove that g is differentiable on $(0,t)$ we write $g(s) = h(s,s)$, where $h(s_1,s_2) = T(t-s_1)u(s_2)$. h is continuously differentiable in both variables and thus g is differentiable:

$$\begin{aligned} \frac{\partial g(s)}{\partial s} &= \frac{\partial h(s,s)}{\partial s_1} + \frac{\partial h(s,s)}{\partial s_2} \\ &= -AT(t-s)u(s) + T(t-s)u'(s) \\ &= -AT(t-s)u(s) + T(t-s)(Au(s) + f(s)) \\ &= T(t-s)f(s), \end{aligned} \tag{2.4.8}$$

where we used that $T(t-s)$ is a bounded operator and that $T(t-s)$ and A commute on $D(A)$. Since $f \in L^1(0,T:X)$ it makes sense to integrate (2.4.8) from 0 to t. We obtain

$$g(t) - g(0) = \int_0^t T(t-s)f(s)ds$$

hence

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

Uniqueness is then implied by this equation. □

The right hand side of (2.4.7) is continuous but not necessary differentiable and hence we do not get a (classical) solution for all $f \in L^1(0,T:X)$. But still it would be natural to understand (2.4.7) as a generalised solution.

Definition 2.4.23. (The mild solution)
Let $x \in X$ and $f \in L^1(0,T:X)$. We call the function $u \in C([0,T]:X)$ with

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \tag{2.4.9}$$

the *mild solution* of the initial value problem (2.4.6) on $[0,T]$.

Remark 2.4.24. The initial value problem (2.4.6) has a mild solution for every $f \in L^1(0,T:X)$.

The next interesting question would be whether there are conditions on f such that the mild solution becomes a strong one. Continuity of f is in general not sufficient (cf. [14, p.106]).

Theorem 2.4.25. *(Existence of a classical solution)*
Let A be the infinitesimal generator of a C_0-semigroup $T(t)$, let $f \in L^1(0,T:X)$ be continuous on $(0,T]$ and define

$$v(t) := \int_0^t T(t-s)f(s)ds, \quad t \in [0,T].$$

The initial value problem (2.4.6) *has a strong solution u on $[0,T)$ for every $x \in D(A)$ if one of the following conditions is satisfied:*

(i) $v(t)$ is continuously differentiable on $(0,T)$,

(ii) $v(t) \in D(A)$ $\forall t \in (0,T)$ and $Av(t)$ is continuous on $(0,T)$.

If (2.4.6) *has a strong solution u on $[0,T)$ for some $x \in D(A)$ then v satisfies both conditions.*

Proof. (cf. [14, Chapter 4, Theorem 2.4]). □

Corollary 2.4.26. *If $f \in C^1([0,T]:X)$ then* (2.4.6) *has a strong solution u on $[0,T)$ for every $x \in D(A)$.*

Corollary 2.4.27. *Let $f \in L^1(0,T:X)$ be continuous on $(0,T)$. If $f(s) \in D(A)$ for all $s \in (0,T)$ and $s \mapsto Af(s) \in L^1(0,T:X)$ then for every $x \in D(A)$ the initial value problem* (2.4.6) *has a strong solution on $[0,T)$.*

Example 2.4.28. (A Schrödinger equation)(cf. [14, Section 7.5])
We consider the Schrödinger equation

$$\frac{1}{i}\frac{\partial u}{\partial t} = \Delta u - Vu$$

in the $L^2(\mathbb{R}^n)$-setting with potential $V \equiv 0$. Here u is a $L^2(\mathbb{R}^n)$-valued function on $\mathbb{R}$ and Δ is with respect to $x \in \mathbb{R}^n$.
By defining the operator A on $L^2(\mathbb{R}^n)$ via

$$Au = i\Delta u,$$

we obtain a linear first order ordinary differential equation on $L^2(\mathbb{R}^n)$

$$u'(t) = Au(t).$$

The operator iA is self-adjoint as an unbounded operator on $L^2(\mathbb{R}^n)$ with domain $D(iA) = H^2(\mathbb{R}^n)$:

- iA is well-defined on $H^2(\mathbb{R}^n)$ by Theorem 2.3.4,
- iA is symmetric (integration by parts),
- iA is self-adjoint by [4, Chapter VI, Theorem 5.18, Theorem 5.22].

As a consequence of Stone's theorem (cf. Theorem 2.4.16) A is the infinitesimal generator of a strongly continuous unitary group $U(t) = \mathrm{e}^{-tA}$ on $L^2(\mathbb{R}^n)$. Thus the Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t > 0 \\ u(0) = x_0 \end{cases}$$

has a unique strong solution for all $x_0 \in H^2(\mathbb{R}^n)$ given by

$$u(t) = U(t)x_0.$$

As a bridge to the next section we return to the homogeneous problem.

Remark 2.4.29. Since the coefficient operator A does not depend on t, the equations of this section are called autonomous equations. These equations are translation invariant in time such that the problem

$$AE(s,x) := \begin{cases} u'(t) = Au(t), & t > s \\ u(s) = x \end{cases}$$

is already solved if $AE(0,x)$ is solved. To be exact let u_x be the solution of $AE(0,x)$ then $u_{s,x}(t) := u_x(t-s)$ is the solution of $AE(s,x)$. Indeed $u_{s,x}$ satisfies $AE(s,x)$:

$$\begin{aligned} &u'_{s,x}(t) = u'_x(t-s) = Au_x(t-s) = Au_{s,x}(t) \\ &u_{s,x}(s) = u_x(0) = x. \end{aligned}$$

In this way one can write any solution of $AE(s,x)$ as $u_{s,x}(t) = T(t-s)x\ \forall t \geq s$. It is therefore reasonable to call $T(t-s)$ the *fundamental solution* of $AE(s,x)$.

Chapter 3

Evolution Systems in the Hyperbolic Case

In this chapter we introduce *evolution systems* which give a solution method for a class of non autonomous equations. In Section 3.1 we define the *evolution problem* and motivate the definition of the evolution system. Additionally we treat a special case, which gives an idea for the general case. Section 3.2 is devoted to the technical basics for the existence theorem in Section 3.3. This chapter is specially fitted to the application in Chapter 4. Since we need the hyperbolic case there, we concentrate on it here. Finally Section 3.4 answers how the evolution problem can be solved via evolution systems.
Again we follow the book of Pazy [14] and additionally the book of Tanabe [16].

3.1 The Evolution Problem - Nonautonomuous Equations

In this section we take the step to evolution equations. The last section in the previous chapter was about an equation with an operator A which was independent of t. Now we consider a more general initial value problem:

$$EP(s,x) := \begin{cases} u'(t) = A(t)u(t), & 0 \le s < t \le T \\ u(s) = x \end{cases}$$

referred to as the *evolution problem.* Here $x \in X$, $A(t) : X \supseteq D(A) \to X$ is a linear operator for every $t \in [0,T]$.

Definition 3.1.1. (The strong solution)
A function $u : [s,T] \to X$ is called a *strong (or classical) solution* of $EP(s,x)$ if

(i) u is continuous on $[s,T]$,

(ii) $u \in C^1((s,T] : X)$,

(iii) $u(t) \in D(A(t))$ for $t \in (s,T]$ and

(iv) u satisfies $EP(s,x)$.

As in Remark 2.4.29 we would like to have an object playing the role of a fundamental solution for $EP(s,x)$. While one parameter was sufficient for the autonomous problem, this is not the case for the non-autonomous equation. Since the coefficient operator A depends on time, the solution might have a different appearance if it starts at the same point but at a different time.

But there are still two properties, which the *fundamental solution* has to satisfy:
For $0 \leq s \leq t \leq T$ let $U(t,s)$ be a linear operator on X such that $t \mapsto U(t,s)x$ is a solution of $EP(s,x)$. Then we have

- $U(s,s)x = x$,
- $U(t,0)x = U(t,s)U(s,0)x$.

The first property is clear and the second one says that starting in x at time 0 and following the solution for a time t yields the same point as starting in $U(s,0)x$ at time s and following this solution for the time duration $t-s$.
This motivates the following definition:

Definition 3.1.2. (Evolution system)
Let $U : \{(t,s) \in [0,T]^2 : 0 \leq s \leq t \leq T\} \to L(X)$ be a strongly continuous function such that:

(i) $U(t,t) = I_X$,

(ii) $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$.

Then we call the two-parameter family of bounded linear operators $\{U(t,s)\}_{0 \leq s \leq t \leq T}$ an *(C_0-) evolution system* (or *fundamental solution* of $EP(s,x)$).

The strong continuity in this definition is continuity with respect to the Euclidean topology on $\{(t,s) \in [0,T]^2 : 0 \leq s \leq t \leq T\}$ and the strong operator topology on $L(X)$ (cf. Section 2.4.1). Unfortunately there is no convenient counterpart to the infinitesimal generator of a C_0-semigroup and its characterisation. But if we take a family of generators of C_0-semigroups under certain conditions we obtain at least the existence of an evolution system. In particular such sufficient conditions are known in the so-called hyperbolic case (cf. (H1)-(H3) below). In the parabolic case where the $A(t)$ are generators of analytic semigroups one can also prove existence of an unique evolution system (cf. [14, Section 5.6]). Those conditions are motivated by applications to partial differential equations.

A milestone in this chapter is to prove the existence of an evolution system under certain assumptions. But before we make a start on this we consider a special case, i.e. the case of evolution systems generated by bounded operators. We use that any bounded operator generates a uniformly continuous semigroup and additionally assume that $A(t)$ is uniformly continuous in t.

Theorem 3.1.3. *For every $t \in [0,T]$ let $A(t)$ be a bounded linear operator on X such that $t \mapsto A(t)$ is continuous with respect to the operator norm, i.e. continuous in the uniform operator topology. Then for every $x \in X$ the initial value problem $EP(s,x)$ has a unique classical solution.*

Proof. ([14, Chapter 5, Theorem 5.1]) □

Hence for each pair $(s,x) \in [0,T] \times X$ we get a solution $u_{s,x}$. With this notation we define:

Definition 3.1.4. (Solution operator)
We define the *solution operator* of $EP(s,x)$ with the assumptions of Theorem 3.1.3 as

$$U(t,s)x := u_{s,x}(t) \qquad \text{for } 0 \leq s \leq t \leq T,$$

where the domain of $U(t,s)$ is X.

In the next theorem we collect some properties of $U(t,s)$ which among others show that $U(t,s)$ forms an evolution system, but first we want to relate this concept to the previous section.
If $A(t)=A$ is independent of t we get that $u_{s,x}(t)=u_{0,x}(t-s)$ and therefore $U(t,s)=U(t-s,0)$ (cf. Remark 2.4.29). In this way the two-parameter family $U(t,s)$ reduces to a one-parameter family $T(t)$, the semigroup generated by A and hence we can write $U(t,s)=T(t-s)$.

Theorem 3.1.5. *For every $t\in[0,T]$ let $A(t)$ be a bounded linear operator on X such that $t\mapsto A(t)$ is continuous with respect to the operator norm, i.e. continuous in the uniform operator topology. Then the solution operator $U(t,s)$ is a bounded linear operator for all $0\le s\le t\le T$ and*

(i) $\|U(t,s)\|\le \mathrm{e}^{\int_s^t\|A(\tau)\|d\tau}$,

(ii) $U(t,t)=\mathrm{I}$, $U(t,s)=U(t,r)U(r,s)$ *for all* $0\le s\le r\le t\le T$,

(iii) $(t,s)\mapsto U(t,s)$ *is continuous in the operator norm for* $0\le s\le t\le T$,

(iv) $\frac{\partial}{\partial t}U(t,s)=A(t)U(t,s)$ *for* $0\le s\le t\le T$ *and*

(v) $\frac{\partial}{\partial s}U(t,s)=-U(t,s)A(s)$ *for* $0\le s\le t\le T$.

Proof. (cf. [14, Chapter 5, Theorem 5.2]) □

In the general case we want to proceed the other way round. We want to get solutions from the evolution system. So we have to prove its existence.

3.2 Technical Preliminaries for the Existence Theorem

This section lists the technical means that are fundamental for the construction of an evolution system. In the next section we will require the existence of a dense subspace of X, which is contained in the domain of each generator and satisfies certain conditions. These conditions are topic of this chapter. On the other hand we introduce a stability property, which is an estimate for the resolvents of the generators and gives us the so called *stability constants*.

3.2.1 Admissible Subsets

Admissibility is one of the main assumptions in the hyperbolic case and hence fundamental in the proof of the first existence theorem.
Let X be a Banach space, $Y\subseteq X$ a subspace and $S:D(S)\to X$ a linear operator on X.

Definition 3.2.1. (Invariant subspace)

(i) If $S(D(S)\cap Y)\subseteq Y$ then Y is called an *invariant subspace* of S.

(ii) Let $(T(t))_{t\ge0}$ be a C_0-semigroup on X. Y is said to be an *invariant subspace of* $(T(t))_{t\ge0}$ if Y is an invariant subspace of $T(t)$ for all $t\ge0$.

Theorem 3.2.2. *Let $(T(t))_{t\ge0}$ be a C_0-semigroup on X generated by A and let $Y\subseteq X$ be a closed subspace. Then:*

$$Y \textit{ is an invariant subspace of } (T(t))_{t\ge0}$$
$$\Updownarrow$$
$$\exists\omega\in\mathbb{R}: Y \textit{ is an invariant subspace of } R(\lambda:A)\ \forall\lambda>\omega.$$

Proof. (cf. [14, Section 4.5, Theorem 5.1]) □

Another definition is vital:

Definition 3.2.3. (The part of an operator)
Let $Y \subset X$ be a subspace. The operator $\tilde{S}$, defined by

$$\begin{aligned} D(\tilde{S}) &:= \{x \in D(S) \cap Y : Sx \in Y\} \\ \tilde{S}x &:= Sx \end{aligned} \tag{3.2.1}$$

is called the *part of* S *in* Y.

Remark 3.2.4. (i) For the restriction map $S|_Y$ we have $\tilde{S} \subset S|_Y$.

(ii) If Y is an invariant subspace of S then $\tilde{S} = S|_Y$.

(iii) Let S be invertible. If $S^{-1}(Y) \subseteq Y$ then $\tilde{S}$ is invertible and $\tilde{S}^{-1} = S^{-1}|_Y$.

For the remainder we assume that Y is a subspace of X which is complete with a norm $\|.\|_Y$. Moreover the Y-norm shall be stronger than the X-norm, i.e. $\exists C > 0 : \|y\|_X \leq C \|y\|_Y \;\; \forall y \in Y$. Note that Y does not need to be closed with respect to $\|.\|_X$.

Definition 3.2.5. (A-admissible)
Let A be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$. $Y \subseteq X$ is called A-*admissible* if

(i) Y is an invariant subspace of $(T(t))_{t\geq 0}$,

(ii) $(T(t)|_Y)_{t\geq 0}$ is a C_0-semigroup on Y, i.e. strongly continuous w.r.t. $\|.\|_Y$.

Theorem 3.2.6. *Let A be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ and let $\tilde{A}$ be the part of A in Y, subspace of X. Y is A-admissible if and only if the following two conditions hold:*

(i) $\exists \omega \in \mathbb{R} : Y$ is an invariant subspace of $R(\lambda : A)$ $\forall \lambda > \omega$,

(ii) $\tilde{A}$ is the generator of a C_0-semigroup on Y.

Moreover we have:

$$Y \textit{ is } A\textit{-admissible } \Rightarrow \tilde{A} \textit{ is the generator of } (T(t)|_Y)_{t\geq 0}.$$

Proof. (cf. [14, Section 4.5, Theorem 5.5]) □

3.2.2 Stable Families of Generators

Another important assumption in the hyperbolic case is the stability of the family of generators. This is a nontrivial assumption unless the semigroups are contractive.

Definition 3.2.7. A family $\{A(t)\}_{t\in[0,T]}$ of generators of C_0-semigroups on X is said do be *stable* if there exist $M \geq 1$ and $\omega \geq 0$ (the so called *stability constants*) such that:

$$(\omega, \infty) \subseteq \rho(A(t)) \qquad \forall t \in [0, T]$$

and

$$\left\| \prod_{j=1}^{k} R(\lambda : A(t_j)) \right\| \leq M(\lambda - \omega)^{-k} \qquad \forall \lambda > \omega,$$

for any finite sequence $0 \leq t_1 \leq \ldots \leq t_k \leq T$, $k = 1, 2, \ldots$.

In general it could take some effort to decide whether a given family of generators is stable. However one special case is easy, namely if $A(t) \in G(1, \omega)$ $\forall t \in [0, T]$:

Lemma 3.2.8. *Let $\{A(t)\}_{t\in[0,T]}$ be a family of generators of C_0-semigroups $(S_t(s))_{s\geq 0}$ on X and assume that there exists an $\omega \in \mathbb{R}$ such that*

$$\|S_t(s)\|_{op} \leq \mathrm{e}^{\omega s} \qquad \forall s \geq 0\ \forall t \in [0, T].$$

Then $\{A(t)\}_{t\in[0,T]}$ is stable with stability constants 1 and ω.

Proof. This result follows from the characterisation Theorem 2.4.14. □

To prove existence of an evolution system we need the following characterisation:

Theorem 3.2.9. *For $t \in [0, T]$ let $A(t)$ be the generator of a C_0-semigroup $(S_t(s))_{s\geq 0}$ on X. Then the following are equivalent:*

(i) The family of generators $\{A(t)\}_{t\in[0,T]}$ is stable.

(ii) There are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A(t))$ for all $t \in [0, T]$ and the following condition is satisfied:

$$\left\| \prod_{j=1}^{k} S_{t_j}(s_j) \right\| \leq M \mathrm{e}^{\omega \sum_{j=1}^{k} s_j} \qquad \text{for } s_j \geq 0\ (1 \leq j \leq k)$$

and any finite sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T,\ k \in \mathbb{N}$.

Proof. (cf. [14, Section 5.2, Theorem 2.2]). □

3.3 Existence of an Evolution System in the Hyperbolic Case

At this point we are able to state conditions which lead to the existence of a unique evolution system.

For the current section let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be Banach spaces such that Y is densely and continuously embedded in X, i.e. $Y \subseteq X$ is a dense subspace of X and $\exists C \in \mathbb{R} : \|y\|_X \leq C\|y\|_Y$ $\forall y \in Y$. We consider the hyperbolic case:

Assumption 3.3.1. For all $t \in [0, T]$ let $A(t)$ be a generator of a C_0-semigroups $(S_t(s))_{s\geq 0}$ on X. The following conditions are known as the so-called hyperbolic case.

(H1) $\{A(t)\}_{t\in[0,T]}$ is stable with stability constants M and ω.

(H2) Y is $A(t)$-admissible for all $t \in [0, T]$ and the family of parts of $A(t)$ in Y, namely $\{\tilde{A}(t)\}_{t\in[0,T]}$, is a stable family of generators in Y with stability constants $\tilde{M}$ and $\tilde{\omega}$.

(H3) For all $t \in [0, T]$ we have that

- $Y \subseteq D(A(t))$,
- $A(t) : Y \to X$ is bounded (with respect to $\|.\|_Y$ and $\|.\|_X$),
- $t \mapsto A(t)$ is continuous with respect to the norm of $L(Y, X)$.

In the following theorem we prove that we get a unique evolution system by assuming the hyperbolic conditions. It is a fundamental result which can be used for many existence results in the hyperbolic case.

Theorem 3.3.2. *For $t \in [0,T]$ let $A(t)$ be the infinitesimal generator of a C_0-semigroup $(S_t(s))_{s\geq 0}$ on X. If the family $\{A(t)\}_{t\in[0,T]}$ satisfies the conditions $(H1)-(H3)$, then there exists a unique evolution system $U(t,s), 0 \leq s \leq t \leq T$ in X satisfying:*

(E1) $\|U(t,s)\| \leq M\mathrm{e}^{\omega(t-s)}$ *for* $0 \leq s \leq t \leq T$.

(E2) $\frac{\partial^+}{\partial t}U(t,s)v|_{t=s} = A(s)v$ *for* $v \in Y, 0 \leq s \leq T$.

(E3) $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v$ *for* $v \in Y, 0 \leq s \leq t \leq T$.

The derivative from the right in $(E2)$ and the derivative in $(E3)$ are with respect to $\|.\|_X$.

Proof. The idea of the proof is to construct an evolution system $U(t,s)$ by approximation. For each $n \in \mathbb{N}$ we split $[0,T]$ into n intervals of length $\frac{T}{n}$. Next we approximate the operator-valued function $t \mapsto A(t)$ by piecewise constant functions $A_n(t)$. Recall that if $t \mapsto A(t)$ was constant one would get an evolution system by setting $U(t,s) := S_t(t-s)$ (cf. end of Section 3.1). Knowing this we set $U_n(t,s) := S_{t_j}(t-s)$ if s and t are in the same interval $[t_j, t_{j+1})$, where $t_j := j\frac{T}{n}$. If this is not the case, we use the second property of evolution systems, $U_n(t,s) = U_n(t,t_k)U_n(t_k,t_{k-1})\cdots U_n(t_l,s)$ for the finite sequence $0 \leq s \leq t_l \leq t_{l+1} \leq \ldots \leq t_k \leq t$ and define $U_n(t,s) := S_{t_k}(t-t_k)S_{t_{k-1}}(t_k-t_{k-1})\cdots S_{t_l}(t_{l-1}-s)$. We will prove that $(U_n(t,s))_{n\in\mathbb{N}}$ converges in the strong sense and the limit is exactly the evolution system we want.
We start the proof by approximating $A(t)$. For all $t \in [0,T]$ we set

$$A_n(t) := \begin{cases} A(t_k^n) & \text{if } t \in [t_k^n, t_{k+1}^n), \\ A(T) & \text{if } t = T, \end{cases} \tag{3.3.1}$$

where $t_k^n := \frac{kT}{n}$, $n \in \mathbb{N}$ and $k = 0,1,\ldots n$.
By (H3) the map $t \mapsto A(t)$ is continuous with respect to the norm of $L(Y,X)$, which we denote by $\|.\|_{L(Y,X)}$. Let $t \in [0,T]$ be arbitrary but fixed. For each $n \in \mathbb{N}$ we set $q_n := t_k^n$, where t_k^n is chosen such that $t \in [t_k^n, t_{k+1}^n)$ and $q_n := T$ if $t = T$. In this way we get a sequence $(q_n)_{n\in\mathbb{N}}$ converging to t. We have

$$\lim_{n\to\infty} A_n(t) = \lim_{n\to\infty} A(q_n) = A(t)$$

and therefore

$$\|A_n(t) - A(t)\|_{L(Y,X)} \to 0 \qquad \text{for } n \to \infty \tag{3.3.2}$$

uniformly in t because $t \mapsto A(t)$ is uniformly continuous on the compact set $[0,T]$.

If $0 \leq t_1 \leq \ldots \leq t_k \leq T$, $k \in \mathbb{N}$ is an arbitrary finite sequence, then we have

$$\left\| \prod_{j=1}^{k} R(\lambda : A_n(t_j)) \right\| \leq \left\| \prod_{i=1}^{k} R(\lambda : A(t_i)) \right\| \leq M(\lambda-\omega)^{-k} \qquad \forall \lambda > \omega,$$

where $(t_i)_{i=1,\ldots,k}$ is another finite sequence and M and ω are given in (H1). The same argument holds for $\{\tilde{A}_n(t)\}_{t\in[0,T]}$ with $\tilde{M}$, $\tilde{\omega}$ given in (H2). Consequently we have that for all $n \in \mathbb{N}$, $\{A_n(t)\}_{t\in[0,T]}$ resp. $\{\tilde{A}_n(t)\}_{t\in[0,T]}$ is a stable family of generators on X resp. Y with the stability constants M, ω resp. $\tilde{M}$, $\tilde{\omega}$, independent of n.
Next we formulate the construction of the evolution system. To this end let $n \in \mathbb{N}$ be arbitrary but fixed and define a two-parameter family of operators. For $0 \leq s \leq t \leq T$ we set

$$U_n(t,s) := \begin{cases} S_{t_j^n}(t-s) & \text{if } t_j^n \leq s \leq t \leq t_{j+1}^n, \\ S_{t_k^n}(t-t_k^n) \displaystyle\prod_{j=l+1}^{k-1} S_{t_j^n}\left(\frac{T}{n}\right) S_{t_l^n}(t_{l+1}^n - s) & \text{if } t_l^n \leq s \leq t_{l+1}^n \leq t_k^n \leq t \leq t_{k+1}^n. \end{cases}$$

Note that we have

$$t - t_k^n + (k-l-1)\frac{T}{n} + t_{l+1}^n - s = t - s. \tag{3.3.3}$$

It is not hard to see that $U_n(t,s),\ 0 \le s \le t \le T$ defines an evolution system:

(i) $U_n(s,s) = S_{t_j^n}(0) = I$,

(ii) $U_n(t,s) = U_n(t,r)U_n(r,s)$ is easy if one writes it down explicitly,

(iii) $(t,s) \mapsto U_n(t,s)$ is strongly continuous since $s \mapsto S_{t_j^n}(s)$ is strongly continuous for all $j = 1,\dots,n$.

Now it is crucial that $U_n(t,s)$ possesses at least properties (E1)-(E3).
Let $0 \le s \le t \le T$, then we have

$$\|U_n(t,s)\|_{L(X)} \le M\mathrm{e}^{\omega(t-s)} \tag{3.3.4}$$

uniformly in n, which follows from Theorem 3.2.9 and (3.3.3). Since Y is $A(t)$-admissible for all $t \in [0,T]$, Y is an invariant subspace of any $S_{t_j}(s)$ and we have

$$U_n(t,s)Y \subseteq Y. \tag{3.3.5}$$

If $v \in Y$ we have

$$\frac{\partial}{\partial t}U_n(t,s)v = A_n(t)U_n(t,s)v \qquad \text{for } t \ne t_j^n \tag{3.3.6}$$

and

$$\frac{\partial}{\partial s}U_n(t,s)v = -U_n(t,s)A_n(s)v \qquad \text{for } s \ne t_j^n, \tag{3.3.7}$$

$j \in \{1,2,\dots,n\}$, which follows from Theorem 2.4.11 and the construction above. Last but not least we get

$$\|U_n(t,s)\|_{L(Y)} \le \tilde{M}\mathrm{e}^{\tilde{\omega}(t-s)} \tag{3.3.8}$$

from (H2) and Theorem 3.2.9 again.
We come to the heart of the proof which is a method we will use four times (differentiate - integrate - estimate). So let $v \in Y$ and consider the map $r \mapsto U_n(t,r)U_m(r,s)v$. (3.3.6) and (3.3.7) imply differentiability of this map in $r \in [s,t]$ except for a finite number of values and we have

$$\frac{\partial}{\partial r}(U_n(t,r)U_m(r,s)v) = -U_n(t,r)A_n(r)U_m(r,s)v + U_n(t,r)A_m(r)U_m(r,s)v. \tag{3.3.9}$$

Integrating (3.3.9) from s to t yields

$$\begin{aligned} U_n(t,s)v - U_m(t,s)v &= -\int_s^t \frac{\partial}{\partial r}(U_n(t,r)U_m(r,s)v)dr \\ &= \int_s^t U_n(t,r)(A_n(r) - A_m(r))U_m(r,s)dr. \end{aligned} \tag{3.3.10}$$

We want $U_n(t,s)$ to be a Cauchy sequence in the strong sense and estimate

$$\begin{aligned}\|U_n(t,s)v - U_m(t,s)v\|_X &\le \int_s^t \|U_n(t,r)(A_n(r)-A_m(r))U_m(r,s)v\|_X \,dr \\ &\le \int_s^t \|U_n(t,r)\|_{L(X)} \|A_n(r)-A_m(r)\|_{L(Y,X)} \|U_m(r,s)\|_{L(Y)} \|v\|_Y \,dr \\ &\le M\tilde{M}\mathrm{e}^{\gamma(t-s)} \|v\|_Y \int_s^t \|A_n(r)-A_m(r)\|_{L(Y,X)} \,dr\end{aligned}$$

where $\gamma = \max\{\omega, \tilde{\omega}\}$. From (3.3.2) we obtain that $\|A_n(r) - A_m(r)\|_{L(Y,X)}$ gets small for large m, n uniformly in r. Consequently for all $v \in Y$, $\|U_n(t,s)v - U_m(t,s)v\|_X$ gets small uniformly in s and t. Hence $(U_n(t,s)v)_{n\in\mathbb{N}}$ is a Cauchy sequence in X uniformly in s,t and possesses a limit in X. Next we prove that $(U_n(t,s))_{n\in\mathbb{N}}$ converges strongly in X and that its limit defines an evolution system.
For strong convergence we need convergence of $(U_n(t,s)x)_{n\in\mathbb{N}}$ for all $x \in X$. The case $x \in Y$ is already proven. So let $x \in X \setminus Y$. Since Y is densely embedded in X there exists a sequence $(v_k)_{n\in\mathbb{N}}$ in Y with $v_k \to x$. Then

$$\begin{aligned}\|U_n(t,s)x - U_m(t,s)x\| &\le \|U_n(t,s)x - U_n(t,s)v_k\| + \|U_n(t,s)v_k - U_m(t,s)v_k\| \\ &\quad + \|U_m(t,s)v_k - U_m(t,s)x\| \\ &\le \|U_n(t,s)\|_{L(X)} \|x - v_k\| + \|U_n(t,s)v_k - U_m(t,s)v_k\| \\ &\quad + \|U_m(t,s)\|_{L(X)} \|v_k - x\| \\ &\le 2M\mathrm{e}^{\omega(t-s)} \|x - v_k\| + \|U_n(t,s)v_k - U_m(t,s)v_k\|.\end{aligned}$$

For any $\varepsilon > 0$ there exist $N_0 \in \mathbb{N}$ such that

$$\|x - v_k\| < \frac{\varepsilon}{4M\mathrm{e}^{\omega(t-s)}} \qquad \forall k \ge N_0.$$

If $k_0 > N_0$ is fixed, there exists a $N_1 \in \mathbb{N}$ such that

$$\|U_n(t,s)v_{k_0} - U_m(t,s)v_{k_0}\| < \frac{\varepsilon}{2} \qquad \forall m,n \ge N_1.$$

Summarising we have

$$\|U_n(t,s)x - U_m(t,s)x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall m,n \ge N_1.$$

Knowing that $(U_n(t,s)x)_{n\in\mathbb{N}}$ converges we set

$$U(t,s)x := \lim_{n\to\infty} U_n(t,s)x \qquad \forall x \in X.$$

Since $U_n(t,s)$ is an evolution system for all $n \in \mathbb{N}$, those properties transfer to $U(t,s)$ (e.g. the uniform limit of a sequence of strongly continuous functions is strongly continuous). It remains to prove properties (E1)-(E3) and uniqueness of the evolution system.
Using (3.3.4) we have for $0 \le s \le t \le T$

$$\|U(t,s)x\| = \left\|\lim_{n\to\infty} U_n(t,s)x\right\| \le \lim_{n\to\infty} \|U_n(t,s)x\| \le M\mathrm{e}^{\omega(t-s)} \|x\| \qquad \forall x \in X$$

and (E1) is proved. (E2) and (E3) need a bit more effort. First note that

$$\frac{\partial^+}{\partial t} S_s(t-s)v|_{t=s} = A(s)S_s(0)v = A(s)v \qquad \forall v \in Y$$

which is exactly what we want $\frac{\partial^+}{\partial t}U(t,s)v|_{s=t}$ to be. Comparing those derivatives yields

$$\frac{\partial^+}{\partial t}U(t,s)v\bigg|_{t=s} - \frac{\partial^+}{\partial t}S_s(t-s)v\bigg|_{t=s} = \lim_{h\downarrow 0}\frac{U(t+h,s)v - U(t,s)v - S_s(t+h-s)v + S_s(t-s)v}{h}.$$

Since $s=t$ this equals

$$\lim_{r\downarrow s}\frac{U(r,s)v - S_s(r-s)v}{r-s}. \tag{3.3.11}$$

It remains to show that (3.3.11) vanishes.
So consider the map $r \mapsto U_n(t,r)S_\tau(r-s)v$ where $v \in Y$ and $\tau \in [0,T]$. This function is differentiable except at a finite number of values of $r \in [s,T]$ and we have

$$\frac{\partial}{\partial r}U_n(t,r)S_\tau(r-s)v = -U_n(t,r)A_n(r)S_\tau(r-s)v + U_n(t,r)A(\tau)S_\tau(r-s)v \tag{3.3.12}$$

Integrating (3.3.12) from s to t yields

$$U_n(t,s)v - S_\tau(t-s)v = \int_s^t U_n(t,r)(A_n(r) - A(\tau))S_\tau(r-s)v dr$$

and therefore

$$\|U_n(t,s)v - S_\tau(t-s)v\|_X \le M\tilde{M}\mathrm{e}^{\gamma(t-s)}\|v\|_Y \int_s^t \|A_n(r) - A(\tau)\|_{L(Y,X)}\,dr$$

where we used Theorem 3.2.9 again. Passing to the limit as $n \to \infty$ yields

$$\|U(t,s)v - S_\tau(t-s)v\|_X \le M\tilde{M}\mathrm{e}^{\gamma(t-s)}\|v\|_Y \int_s^t \|A(r) - A(\tau)\|_{L(Y,X)}\,dr. \tag{3.3.13}$$

Choosing $\tau = s$ in (3.3.13), dividing by $t-s>0$ and letting $t \downarrow s$ we find

$$\lim_{t\downarrow s}\frac{1}{t-s}\|U(t,s)v - S_s(t-s)v\|_X = 0$$

where we used the continuity of $t \mapsto A(t)$ in the $L(Y,X)$-norm. Hence (E2) is shown.
For those who are out of breath now let us briefly outline the last arguments of this long proof. To establish (E3) we start similar to the proof of (E2) and then use the derivative at the boundary to get differentiability for all $s<t$. Last but not least we prove uniqueness of the evolution system by assuming another evolution system and as always we proceed by differentiating, integrating and estimating a certain differentiable map.
So by choosing $\tau = t$ in (3.3.13), dividing by $t-s>0$ and letting $s \uparrow t$ we find similar as above that

$$\lim_{s\uparrow t}\frac{\|S_t(t-s)v - U(t,s)v\|_X}{t-s} = 0$$

and hence

$$\frac{\partial^-}{\partial s}S_t(t-s)v\bigg|_{s=t} = \frac{\partial^-}{\partial s}U(t,s)v\bigg|_{s=t} = -A(t)v. \tag{3.3.14}$$

For $s<t$ we have

$$\begin{aligned}\frac{\partial^+}{\partial s}U(t,s)v &= \lim_{h\downarrow 0}\frac{U(t,s+h)v - U(t,s)v}{h}\\ &= \lim_{h\downarrow 0}\left(U(t,s+h)\frac{v - U(s+h,s)v}{h}\right)\\ &= -U(t,s)A(s)v\end{aligned} \tag{3.3.15}$$

where we used (E2) and the strong continuity of $U(t,s)$. For $s \le t$ we calculate the left derivative

$$\begin{aligned} \frac{\partial^-}{\partial s} U(t,s)v &= \lim_{h \downarrow 0} \frac{U(t,s)v - U(t,s-h)v}{h} \\ &= \lim_{h \downarrow 0} \left(U(t,s) \frac{v - U(s,s-h)v}{h} \right) \\ &= -U(t,s)A(s)v \end{aligned} \tag{3.3.16}$$

where we used (3.3.14). Combining (3.3.15) and (3.3.16) the proof of (E3) is complete.

To prove uniqueness of $U(t,s)$ we suppose that there is another evolution system $V(t,s)$, $0 \le s \le t \le T$ satisfying (E1)-(E3). For $v \in Y$ consider the map $r \mapsto V(t,r)U_n(r,s)v$. As above we have

$$V(t,s)v - U_n(t,s)v = \int_s^t V(t,r)(A(r) - A_n(r))U_n(r,s)dr$$

and therefore by (E1)

$$\|V(t,s)v - U_n(t,s)v\|_X \le M\tilde{M}\mathrm{e}^{\gamma(t-s)} \|v\|_Y \int_s^t \|A(r) - A_n(r)\|_{L(Y,X)} \, dr. \tag{3.3.17}$$

Letting $n \to \infty$ in (3.3.17) and using (3.3.2) implies $V(t,s)v = U(t,s)v$ for all $v \in Y$. Equality of $U(t,s)$ and $V(t,s)$ is now implied by the fact that Y is dense in X. □

We skip some intermediate issues and move on to a result, which is needed in Chapter 4. The proof of the following theorem relies on several technical lemmas (cf. [14, Section 5.4]) and is therefore omitted. It is not easy to figure out, how the following conditions (H1)′-(H3)′ are related to the conditions (H1)-(H3) of Theorem 3.3.2. However it can be shown that they imply the conditions (H1)-(H3). By Lemma 3.2.8 it is clear that (H1)′ implies that $\{A(t)\}_{t\in[0,T]}$ is a stable family of generators with constants $M = 1$ and $\omega = 0$. Moreover (H2)′ is easy to verify in applications and therefore an admissible subspace is not needed.
In the consequences we adjust M and ω, extend (E2) and get two new properties in addition.

Theorem 3.3.3. *For $t \in [0,T]$ let $A(t)$ be the generator of a C_0-semigroup $(S_t(s))_{s\ge0}$ on X and let D be densely and continuously embedded in X such that:*

(H1)′ $\|S_t(s)\| \le 1 \ \forall s \ge 0 \ \forall t \in [0,T]$,

(H2)′ $D(A(t)) = D$ *for all* $t \in [0,T]$,

(H3)′ $t \mapsto A(t)v$ *is continuously differentiable as a map* $: [0,T] \to X$ *for all* $v \in D$.

Then there exists a unique evolution system $U(t,s)$, $0 \le s \le t \le T$ satisfying

(E1)′ $\|U(t,s)\| \le 1$ *for* $0 \le s \le r \le t \le T$.

(E2)′ $\frac{\partial}{\partial t}U(t,s)v = A(t)U(t,s)v$ *for* $v \in D, 0 \le s \le t \le T$.

(E3)′ $\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v$ *for* $v \in D, 0 \le s \le t \le T$.

(E4)′ $U(t,s)D \subseteq D$ *for* $0 \le s \le t \le T$.

(E5)′ $U(t,s)v$ *is continuously differentiable in D, separately in s and t for all* $v \in D$.

Proof. The proof can be found in [14] or in [16, p.102 Corollary of Theorem 4.4.1]). One shows the existence of a family of isomorphisms $\{Q(t)\}_{t\in[0,T]}$ of D onto X such that

$$A(t) + B(t) = Q(t)A(t)Q(t)^{-1}$$

where $\{B(t)\}_{t\in[0,T]}$ is a strongly continuous family of bounded operators on X. For this purpose the clue is to set $Q(t) := \lambda_0 I - A(t)$ for $\lambda_0 > 0$. □

3.4 Solutions in the Hyperbolic Case

We conclude this chapter by solving the inhomogeneous evolution problem in the hyperbolic case. Consider

$$\begin{cases} u'(t) = A(t)u(t) + f(t), & 0 < s \le t \le T \\ u(s) = x. \end{cases} \tag{3.4.1}$$

As long as evolution system and strong solution exist, the solution is uniquely determined in terms of the evolution system.

Proposition 3.4.1. *Let $f \in L^1(0,T : X)$ and assume that there exists an evolution system $U(t,s)$, $0 \le s \le t \le T$ associated with* (3.4.1) *such that for all $v \in D(A(s))$ we have*

- *$U(t,s)v \in D(A(t))$*
- *$(t,s) \mapsto U(t,s)v : \{0 \le s \le t \le T\} \to X$ is differentiable.*

Then any strong solution of (3.4.1) *with $x \in D(A(s))$ (cf. Definition 3.1.1) satisfies*

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr. \tag{3.4.2}$$

If a strong solution exists, it is uniquely determined by (3.4.2).

Proof. Let u be a strong solution of (3.4.1). Consider the map $r \mapsto U(t,r)u(r)$ which is differentiable on $[s,t]$.

$$\begin{aligned} \frac{\partial}{\partial r}U(t,r)u(r) &= -U(t,r)A(r)u(r) + U(t,r)u'(r) \\ &= -U(t,r)A(r)u(r) + U(t,r)A(r)u(r) + U(t,r)f(r) \\ &= U(t,r)f(r) \end{aligned} \tag{3.4.3}$$

Integrating (3.4.3) from s to t yields

$$U(t,t)u(t) - U(t,s)u(s) = \int_s^t U(t,r)f(r)dr,$$

hence

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr$$

and the proof is complete. □

As in the autonomous case we may consider the right hand side of (3.4.2) as a generalised solution.

Definition 3.4.2. (The mild solution)
Let $\{A(t)\}_{t\in[0,T]}$ satisfy Assumption 3.3.1 and let $U(t,s)$ $0 \le s \le t \le T$ be the evolution system given by Theorem 3.3.2. If $f \in L^1(s,T : X)$ and $x \in X$ then we call the continuous function $u : [0,T] \to X$, given by

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr,$$

the *mild solution* of the initial value problem (3.4.1).

Having the mild solution, one can find conditions leading to the strong solution. We omit the proof of the following theorem since as in the proof of Theorem 3.3.3 one would have to deal with families of isomorphisms.

Theorem 3.4.3. *Let $\{A(t)\}_{t\in[0,T]}$ satisfy the conditions of Theorem 3.3.3. If $f \in C^1([s,T] : X)$ then for every $x \in D$ the initial value problem* (3.4.1) *has a unique strong solution u given by*

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr.$$

Proof. (cf. [14, Section 5.5, Theorem 5.3]) □

Chapter 4

Application to a Paraxial Wave Equation

4.1 Introduction

In this chapter we study an application of the theory introduced in the previous chapters, which was published in 2008 by Maarten De Hoop, Günther Hörmann and Michael Oberguggenberger (cf. [2]). The geophysical context is outlined below. Mathematically the problem consists of solving the following Cauchy problem:

$$\begin{cases} Pu := \partial_z u - i \cdot \operatorname{div}_x(C(z,x,D_t) \cdot \operatorname{grad}_x u) = f \in \mathcal{V} \\ u|_{z=0} = u_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)). \end{cases} \tag{4.1.1}$$

Here C is a pseudo differential operator in t with parameters z and x. If C is of order one, (4.1.1) is called a *classical paraxial wave equation*. In our case, C is of some order $m \in \mathbb{R}$. The prospective solution u should be an element of $\mathcal{V}$. The latter is defined by

$$\mathcal{V} := C([0,\infty), \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d))).$$

We will take a closer look at the Cauchy problem in Section 4.2, but first we try to get some idea of the geophysical model.

4.1.1 The Geophysical Problem

Since this is a mathematical thesis we will only give a brief introduction to the geophysical problem, such that one gets an imagination of the practical benefit. We will then assume a mathematical model and study the way of getting a solution.

The problem discussed in [2] leads us to a branch of geophysics called seismology, which is the scientific study of earthquakes. This introduction follows the book of Kertz [11, Chapter 4].
Although we will not research earthquakes, we cannot ignore them since they serve as a basis for seismology. The reasons for earthquakes are tectonic, volcanic and collapsing processes. Such a process causes extremely complex movements of the earth, seismic disturbances, which can also have man-made sources, such as controlled explosions. In or near an earthquake focus the deformation is inelastic, but some distance away, the disturbances can be modeled by elastic waves. For further details concerning elasticity theory see [12, Section 3.2]. We distinguish between longitudinal and transverse waves. Those seismic waves scatter to large parts of the earth and are measured on several places via so-called seismographs. In this way much data

is collected at various points worldwide. From this data much knowledge on earthquakes has been gained. But not only about earthquakes, also on the material the waves pass. Actually the whole structure of the earth's interior is deduced from seismic information. In the following we take a short look at how the scattering of seismic waves can be deduced from the earth's inner structure. For that reason we collect some physical facts.

Physical facts:

- Seismic waves can be described as longitudinal and transverse waves, which are characterised by their amplitude, velocity and wavelength.
- Seismic waves have similar properties as the waves in physical optics such as reflection, refraction, absorption etc.
- For the inner structure of our planet we have the following rough model: On the large scale, the earth consists of isotropic layers (cf. Table A), i.e. every layer consists of homogeneous matter in consistent phase. The density distribution is shown in Figure 1.
- Seismic waves get faster the deeper they get.
- In the (outer) core are no transverse waves.

Layer	Depth (km)	Sign
crust	0-33	A
upper mantle	33-1000	B and C
lower mantle	1000-2900	D' and D"
outer core	2900-4980	E
bridge layer	4980-5120	F
inner core	5120-6370	G

Table A

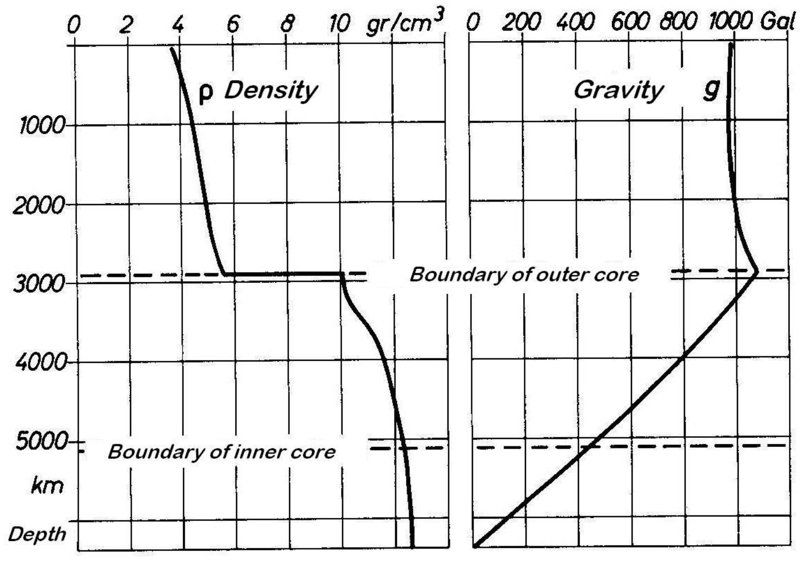

Figure 1: Density and Gravity Acceleration in Earth's Interior (1 Gal = 1cm/sec^2) [11, Section 12a), p.207]

Using these facts we get some properties for the wave scattering.

- By the refraction law a wave moving from medium of higher density into medium of lower density is refracted away from the normal line (similar for lower density medium into higher density medium).
- From the fourth physical fact follows that seismic waves are downwards convex.
- Since the medium in the earth's interior is not globally homogeneous, the velocity of seismic waves may change suddenly at certain depths.
- Longitudinal and transverse waves act essentially in the same way, but have different phase velocity.

These properties lead to the following model:
Longitudinal waves are faster and are therefore called primary waves (P-waves). Consequently transverse waves are called secondary waves (S-waves). A P-wave reflected by earth's crust is called a PP-wave. A P-wave reflected by the core is called a PcP-wave. A P-wave passing the core is called a PKP-wave and so on. This nomenclature yields the following picture.

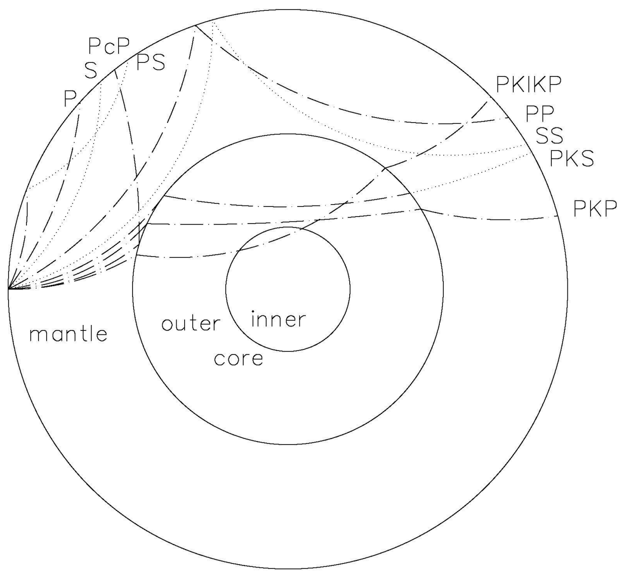

Figure 2: Nomenclature of Seismic Waves

The problem we will deal with, takes place in the D”-layer. Since the medium is heavily fluctuating on any scale at that depth, it is as inhomogeneous as possible and we need coefficients of low regularity. The question is what sound velocity the medium has. We would like to have a result of the following form: If the sound velocity of the medium is of some Zygmund-Hölder regularity ($c \in C^r_*$) then the solution u is in $H^{f(r)}$, where f is given as explicit as possible. Up to now such a result has not been proven, but in [2] at least a special version of this result is shown:

$$c \in C^r_*,\ r \in (0,1),\ u_0 \in H^2 \Rightarrow u \text{ is } H^2\text{-valued.}$$

Here u_0 represents the boundary data which can be prepared to be in H^2 in experiments.
In practice one can use such a result in so-called *inverse analysis of medium regularity*. There we have the following procedure: The solution is obtained by measurement. If this solution is not in H^2 then the sound velocity of the medium cannot be of Zygmund-Hölder regularity $C^r_*,\ r \in (0,1)$.

The Zygmund-Hölder spaces for any $r \in \mathbb{R}$ are defined in terms of a continuous Littlewood-Paley decomposition via

$$C^r_*(\mathbb{R}^d) := \left\{ u \in \mathscr{S}'(\mathbb{R}^d) : \|\varphi(D)u\|_{L^\infty} + \sup_{t>1} \left(t^s \left\| \psi\left(\frac{D}{t}\right) u \right\|_{L^\infty} \right) < \infty \right\}. \tag{4.1.2}$$

For details consult [9].

In Figure 3 we have a core-reflected ScS-wave and two precursors. Such precursors are caused by scattering of the wave, which is a physical phenomenon. The precursors in the neighbourhood of the top of the D"-layer are locally modeled by our equation. The ScS-wave is not included. Note that this model only works for large opening angles.

The model-equation in the form of (4.1.1) arises as an approximation of the elastic system in this special situation (for details see [2, Chapter 1]).

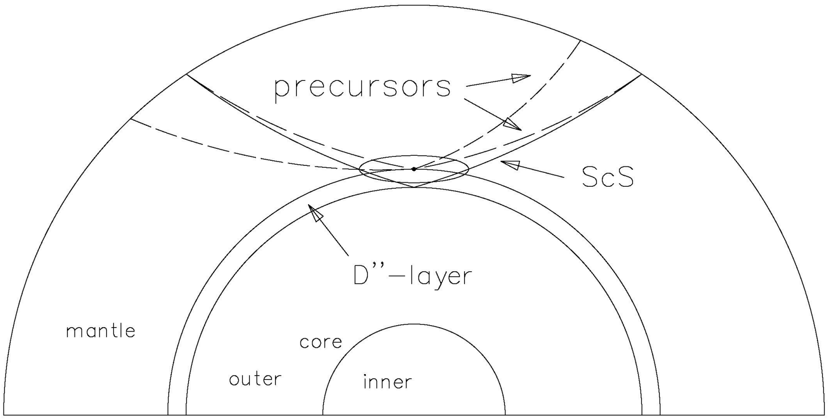

Figure 3: Core reflected ScS-wave

So the application in this chapter can be used for inverse analysis of medium regularity for such precursors.

4.2 The Cauchy Problem

We will not go into further details of the pseudo-differential operator since this would involve a lot more theory. In this thesis the focus lies on the regularity of the coefficients.

First we introduce coordinates $z \in [0, \infty)$ for the depth, $x \in \mathbb{R}^d$ for the lateral variation with $d \leq 2$ and a time variable $t \in \mathbb{R}$.

Motivated by the previous chapter we consider the Cauchy problem (4.1.1). The Fourier transform with respect to the time variable t forms an isomorphism $\mathcal{F}_t : \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)) \to \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d))$. We extend it to an isomorphism $\tilde{\mathcal{F}}_t : \mathcal{V} \to \mathcal{V}$ by setting

$$(\tilde{\mathcal{F}}_t u)(z) := \mathcal{F}_t(u(z, .)) \qquad \forall z \leq 0,\ \forall u \in \mathcal{V} \tag{4.2.1}$$

(cf. [2, Section 2, p.1418]). Applying $\tilde{\mathcal{F}}_t$ to (4.1.1), we obtain an equivalent formulation of the Cauchy problem in the frequency domain:

$$\begin{cases} \tilde{P}v := \partial_z v - i \cdot \operatorname{div}(c(z,x,\tau) \cdot \operatorname{grad} v) = g \in \mathcal{V} \\ v|_{z=0} = v_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)). \end{cases} \tag{4.2.2}$$

Note that the pseudo-differential operator reduces to a multiplication operator, which makes things a lot easier.
The following assumption is motivated by the geophysical application. It is not necessary to understand where it comes from in detail, but it is fundamental in the solution process.

Assumption 4.2.1. The coefficient c is a function on $[0,\infty) \times \mathbb{R}^d \times \mathbb{R}$, which can be written as $c(z,x,\tau) = c_0 + c_1(z,x,\tau)$ satisfying the following properties:

(i) $c_0 \in \mathbb{R}$, $c_0 > 0$,

(ii) $\exists r \in (0,1)$: $c_1 \in C^1([0,\infty) \times \mathbb{R}, H^{r+1}(\mathbb{R}^d))$ and we write $c_1(z,x,\tau) := c_1(z,\tau)(x)$.

(iii) $c_1(z,x,\tau) \geq 0 \ \forall (z,x,\tau) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}$.

Note that without assumption 4.2.1(ii) we would not know that c_1 and hence c is really a function. As a distribution it could have no pointvalues in x.

Remark 4.2.2. By [8, Proposition 8.6.10] we have

$$H^{r+1}(\mathbb{R}^d) \subseteq C_*^{r+1-\frac{d}{2}}(\mathbb{R}^d).$$

Therefore $c_1(z,.,\tau) \in C_*^{r+1-\frac{d}{2}}(\mathbb{R}^d)$ by Assumption 4.2.1. In the most relevant case from geophysics, namely $d = 2$, we have coefficients in $C_*^r(\mathbb{R}^2)$, but not necessarily Lipschitz continuous (cf. [2, Remark 2.1]).

Remark 4.2.3. In the next section we will fix z and τ as a first step. In so doing we get a map

$$\begin{aligned} c_{z,\tau} &: \mathbb{R}^d \to \mathbb{R} \\ c_{z,\tau} &:= c_0 + c_1(z,\tau), \end{aligned} \tag{4.2.3}$$

i.e. $c_{z,\tau}(x) = c_0 + c_1(z,x,\tau)$ for all $x \in \mathbb{R}^d$, where $c_1(z,\tau) \in H^{r+1}(\mathbb{R}^d)$ as above.

Moreover we will regard our problem as a functional analytical one. We therefore define a family of operators and rewrite the Cauchy problem (4.2.2) in order to see that we have an evolution problem in the sense of Chapter 3, but with z as the evolution parameter, instead of the time t.

Definition 4.2.4. Let $c_{z,\tau}$ be as in (4.2.3). We define a family of unbounded operators on $L^2(\mathbb{R}^d)$ via

$$\{A(z,\tau)\}_{z\in[0,\infty),\tau\in\mathbb{R}}$$
$$A(z,\tau)v = \operatorname{div}(c_{z,\tau} \cdot \operatorname{grad} v) \qquad \forall v \in D(A) \subseteq L^2(\mathbb{R}^d).$$

Since we deal with unbounded operators we have to find a proper domain for those operators. This will be done in the next section (cf. Lemma 4.3.7).

Remark 4.2.5. In view of the theory of differential operators we could also write

$$A(x,D_x)v = \operatorname{div}(c(x) \cdot \operatorname{grad} v)$$

to emphasize the (x,D_x)-dependence of the operator, or more general

$$A(\tau; z,x,D_x)v = \operatorname{div}(c(z,x,\tau) \cdot \operatorname{grad} v)$$

as in [2, p.1420 (2.8)]. Observe that in the last equation z and τ are not fixed and thus we get formally a different operator on a different space namely $\mathcal{V}$.

Inserting our new operator, the Cauchy problem takes the following shape

$$\begin{cases} \tilde{P}v = \partial_z v - iA(\tau; z, x, D_x)v = g \in \mathcal{V} \\ v|_{z=0} = v_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)) \end{cases}$$

and hence we get an evolution problem

$$\begin{cases} \partial_z v = iA(\tau; z, x, D_x)v + g \quad 0 \leq z < \infty \\ v|_{z=0} = v_0. \end{cases} \tag{4.2.4}$$

4.3 Evolution System

The issue of this section will be to construct an evolution system as in Chapter 3. As a first step we will show that for each pair (z, τ) the operator $iA(z, \tau)$ generates a unitary group. The clue will be to establish self-adjointness of A. In Section 4.3.2 we will see that the family $\{iA(z, \tau)\}_{z \in [0,Z]}$ generates an evolution system for each $\tau \in \mathbb{R}$ and additionally we will show that this evolution system depends strongly continuous on τ in Section 4.3.3.

4.3.1 Unitary Group at Frozen Values of τ and z

Let us fix z and τ. For simplicity we will use the following notations throughout this subsection.

Notation 4.3.1. (i) $A := A(z, \tau)$,

(ii) $c := c_{z,\tau}$,

(iii) $c_1 := c_1(z, \tau)$.

From Assumption 4.2.1 and Definition 4.2.4 we get new assumptions for this subsection:

Assumption 4.3.2. (i) $c(x) = c_0 + c_1(x)$ with $c_0 > 0$, $c_1 \geq 0$ and $c_1 \in H^{r+1}(\mathbb{R}^d)$ for some $r \in (0, 1)$.

(ii) A is a differential- and hence an unbounded operator on $L^2(\mathbb{R}^d)$, the domain is not yet specified and $Av = \operatorname{div}(c \cdot \operatorname{grad} v)$.

Remark 4.3.3. (i) A is formally self-adjoint: $\langle Au|v\rangle = \langle \operatorname{div}(c \cdot \operatorname{grad} u)|v\rangle$
$= \left\langle \sum_j \partial_j(c \cdot \partial_j u)|v \right\rangle = \left\langle u|\sum_j \partial_j(c \cdot \partial_j v) \right\rangle = \langle u|Av\rangle \; u, v \in \mathcal{D}(\mathbb{R}^d)$.

(ii) By the Sobolev embedding theorem, Assumption 4.3.2(i) implies that $c \in C(\mathbb{R}^d)$ (cf. [10, Theorem 6.21] and recall $d \leq 2$).

Throughout this chapter we will often need that the multiplication of various $H^s(\mathbb{R}^d)$- distributions is continuous and gives another $H^s(\mathbb{R}^d)$-distribution. This is formulated in [2, p.1421, Fact A] as a special case of Theorem 8.3.1 in [8].

Theorem 4.3.4. *(Continuous multiplication - the general case)*
Let w_1 and w_2 be elements of $H^{s_1}(\mathbb{R}^d)$ and $H^{s_2}(\mathbb{R}^d)$ such that $s_1 + s_2 \geq 0$. If

$$s \leq s_1, \; s \leq s_2, \; s \leq s_1 + s_2 - \frac{d}{2} \textit{ and } s_1, s_2, -s \neq \frac{d}{2},$$

or if

$$s \leq s_1,\ s \leq s_2,\ s < s_1 + s_2 - \frac{d}{2} \text{ and } s_1 \text{ or } s_2 \text{ or } -s \text{ is equal to } \frac{d}{2},$$

then $w_1 \cdot w_2 \in H^s(\mathbb{R}^d)$. Moreover those multiplications are continuous with respect to the corresponding Sobolev norms.

Proof. (cf. [8, Theorem 8.3.1]; Continuity of the multiplication is not explicitly assumed in the theorem, but can be seen from its proof.) □

Corollary 4.3.5. *(Continuous multiplication - $d = 2$)*
Let $w_1 \in H^{s_1}(\mathbb{R}^2)$ and $w_2 \in H^{s_2}(\mathbb{R}^2)$ such that $s_1 + s_2 \geq 0$ and let $\varepsilon > 0$ be arbitrary.
If

$$s_1 \neq \pm 1,\ s_2 \neq \pm 1 \text{ and } s_1 + s_2 \neq 0,$$

then

$$w_1 \cdot w_2 \in H^{\min(s_1, s_2, s_1+s_2-1)}(\mathbb{R}^2).$$

Otherwise

$$w_1 \cdot w_2 \in H^{\min(s_1, s_2, s_1+s_2-1-\varepsilon)}(\mathbb{R}^2).$$

Moreover those multiplications are continuous with respect to the corresponding Sobolev norms.

Proof. We want to apply Theorem 4.3.4 and hence we have to show that

$$s \leq s_1, s_2, s_1 + s_2 - 1.$$

If at least one of the three s_1, s_2, or $-s$ is equal to 1, we have to prove $s < s_1 + s_2 - 1$ in addition. Now if $s_1 \neq \pm 1$, $s_2 \neq \pm 1$ and $s_1 + s_2 \neq 0$, then $s := \min(s_1, s_2, s_1 + s_2 - 1) \neq -1$ and $s \leq s_1, s_2, s_1 + s_2 - 1$.
Otherwise we set $s := \min(s_1, s_2, s_1 + s_2 - 1 - \varepsilon)$ and get immediately that $s \leq s_1, s_2$ and $s < s_1 + s_2 - 1$ which finishes the proof. □

Example 4.3.6. (i) Let $s_1 = r + 1$, $r \in (0, 1)$ and $s_2 = 1$, then Corollary 4.3.5 provides a continuous multiplication $\cdot : H^{r+1}(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$.

(ii) Let $s_1 = r$, $r \in (0, 1)$ and $s_2 = 1$, then we have that for all $\varepsilon \in (0, r)$, $\cdot : H^r(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to H^{r-\varepsilon}(\mathbb{R}^2)$ is a continuous map.

The main part of this subsection will be to show that A is self-adjoint on the domain $D(A) := H^2(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. If the coefficients were smooth, or Lipschitz continuous, this would be a classical result (cf. [4, Chapter VI, Theorem 5.18, Theorem 5.22]). In case of L^∞-coefficients we got at least a self-adjoint extension of A, the so-called Friedrichs extension, on some unknown domain (cf. [18, Satz VII.2.11]). In this thesis we explicitly determine the domain for any z and τ.
We start with the following lemma, which additionally shows that A is well defined on its domain (just set $s = 0$).

Lemma 4.3.7. *(cf.[2, Lemma 3.2.])*
Let $0 \leq s < r < 1$ and $v \in H^{s+2}(\mathbb{R}^d)$. Then $Av \in H^s(\mathbb{R}^d)$.

Proof. First we collect some facts:

(a) $v \in H^{s+2}(\mathbb{R}^d) \Rightarrow \partial_j v \in H^{s+1}(\mathbb{R}^d)$ by Lemma 2.3.3(ii),

(b) $c_1 \in H^{r+1}(\mathbb{R}^d)$, $r > s \Rightarrow c_1 \in H^{s+1}(\mathbb{R}^d)$, since $H^a(\mathbb{R}^d) \subseteq H^b(\mathbb{R}^d)$ for $a \geq b$ by Lemma 2.3.3(i),

(c) $s > \frac{d}{2} \Rightarrow H^s(\mathbb{R}^d)$ is an algebra by Theorem 2.3.5.

In our case, $d \leq 2$ and $s+1 \geq 1$, so $H^{s+1}(\mathbb{R}^d)$ is an algebra if $s > 0,\ d = 2$ and if $d = 1$ by (c). If $s = 0,\ d = 2$, we use Corollary 4.3.5 and obtain

$$H^{r+1}(\mathbb{R}^2) \cdot H^{s+1}(\mathbb{R}^2) \subseteq H^{s+1}(\mathbb{R}^2), \quad r > 0.$$

Combining this with (a) and (b) yields $c_1 \cdot \partial_j v \in H^{s+1}(\mathbb{R}^d)$. Then

$$c \cdot \partial_j v = c_1 \cdot \partial_j v + c_0 \cdot \partial_j v \in H^{s+1}(\mathbb{R}^d) \tag{4.3.1}$$

Again by (b), $\partial_j(c \cdot \partial_j v) \in H^s(\mathbb{R}^d)$. In summary we have

$$Av = \operatorname{div}(c(x) \cdot \operatorname{grad} v) = \sum_j \partial_j(c \cdot \partial_j v) \in H^s(\mathbb{R}^d).$$

□

We show that A is symmetric, which means that $D(A) \subseteq D(A^*)$ and $A^*|_{D(A)} = A$ (cf. Remark 2.2.7). To begin with we observe that A is continuous on its domain.

Lemma 4.3.8. *The operator $A : (H^2(\mathbb{R}^d), \|.\|_{H^2}) \to (L^2(\mathbb{R}^d), \|.\|_{L^2})$ is continuous.*

Proof. Consider the following maps:

$$\alpha : u \mapsto \partial_j u : H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d), \text{ continuous (cf.[6, Corollary 9.3.2])},$$
$$\beta : v \mapsto c \cdot v : H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d), \text{ continuous by Corollary 4.3.5},$$
$$\gamma : w \mapsto \partial_j w : H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \text{ continuous (cf.[6, Corollary 9.3.2])}.$$

Thus $A = \gamma \circ \beta \circ \alpha$ is continuous as a combination of continuous operators. □

Lemma 4.3.9. *$A : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is symmetric (as an unbounded operator on $L^2(\mathbb{R}^d)$).*

Proof. Let $u \in H^2(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. A short calculation using integration by parts (for absolutely continuous functions) yields $\langle Au|\varphi\rangle = \langle u|A\varphi\rangle$:

$$\begin{aligned}
\langle Au|\varphi\rangle &= \int_{\mathbb{R}^d} Au(x) \cdot \varphi(x) dx \\
&= \sum_j \int_{\mathbb{R}^d} \partial_j(c \cdot \partial_j u)(x) \cdot \varphi(x) dx \\
&= -\sum_j \int_{\mathbb{R}^d} (c \cdot \partial_j u)(x) \cdot \partial_j \varphi(x) dx \\
&= -\sum_j \int_{\mathbb{R}^d} \partial_j u(x) \cdot c(x) \cdot \partial_j \varphi(x) dx \\
&= \sum_j \int_{\mathbb{R}^d} u(x) \cdot \partial_j(c \cdot \partial_j \varphi)(x) dx \\
&= \int_{\mathbb{R}^d} u(x) \cdot A\varphi(x) dx \\
&= \langle u|A\varphi\rangle.
\end{aligned} \tag{4.3.2}$$

Now let $v \in H^2(\mathbb{R}^d)$. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $H^2(\mathbb{R}^d)$ (cf. [19, Chapter 1, Corollary 3.1 and Definition 3.2]), we can choose a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R}^d)$ such that $\varphi_k \to v$ in $H^2(\mathbb{R}^d)$. Hence we obtain

$$\langle Au|v\rangle = \lim_{k\to\infty} \langle Au|\varphi_k\rangle \overset{(4.3.2)}{=} \lim_{k\to\infty} \langle u|A\varphi_k\rangle \overset{4.3.8}{=} \langle u|Av\rangle,$$

which completes the proof. □

It remains to show that $D(A^*) \subseteq D(A)$, to prove self-adjointness of A.

Remark 4.3.10. In the next proposition we will use the following two essential results.

(i) Consider the Sobolev spaces

$$W^{l,p}(\mathbb{R}^d) := \{u \in L^p(\mathbb{R}^d) : (-i)^{|\alpha|}\partial^\alpha u \in L^p(\mathbb{R}^d)\ \forall\, |\alpha| \leq l\},$$

where $l \in \mathbb{N}$, $1 \leq p \leq \infty$ and $\alpha \in \mathbb{N}^d$ is a multi-index. If $l \in \mathbb{Z},\ l \leq 0$, we define

$$W^{l,p}(\mathbb{R}^d) := \left\{ u \in \mathcal{D}'(\mathbb{R}^d) : u = \sum_{|\alpha|\leq|l|} (-1)^{|\alpha|}\partial^\alpha v_\alpha,\ v_\alpha \in L^p(\mathbb{R}^d) \right\},$$

(cf. [1, Chapter 3]). According to [13, §5, p.41], we set

$$W^{l,p}_{loc}(\mathbb{R}^d) := \{v \in L^p(\mathbb{R}^d) : \varphi v \in W^{l,p}(\mathbb{R}^d)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^d)\}.$$

Assume that $l, m \in \mathbb{Z}$ with $l + m \geq 0$ and $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} \leq 1$, then the duality method produces a continuous bilinear multiplication

$$\cdot : W^{m,q}_{loc}(\mathbb{R}^d) \times W^{l,p}_{loc}(\mathbb{R}^d) \to W^{k,r}_{loc}(\mathbb{R}^d),$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $k = \min(l, m)$ (cf. [13, Proposition 5.2]).

(ii) Elliptic regularity: more precisely we need an extension of elliptic regularity to $H^{r+1}(\mathbb{R}^d)$-coefficients, which is very hard to prove in the case $d = 2$ [2, Lemma 3.3] and more elementary in the case $d = 1$ [2, Remark 3.5]. It says that if $0 \leq s < r < 1$ and $d \leq 2$ then we have the following implication:

$$v \in H^s(\mathbb{R}^d),\ Av \in H^s(\mathbb{R}^d) \Rightarrow v \in H^{s+2}(\mathbb{R}^d).$$

Finally we are ready to complete the argumentation for self-adjointness of A.

Proposition 4.3.11. $D(A^*) \subseteq D(A)$.

Proof. We start by showing that

$$D(A^*) \subseteq \{v \in L^2(\mathbb{R}^d) | Av \in L^2(\mathbb{R}^d)\} =: B.$$

To this end we have to put many technical facts in the right order.

(a) $D(A^*) \subseteq L^2(\mathbb{R}^d)$ is clear by (2.2.1),

(b) $H^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ because $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ [5, Prop 8.17] and $\mathcal{D}(\mathbb{R}^d) \subseteq H^2(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$.

(c) $v \in D(A^*) \overset{\text{(a)+(b)}}{\Longrightarrow} \exists (v_k)_{k\in\mathbb{N}}$ sequence in $H^2(\mathbb{R}^d)$ such that $v_k \to v$ in $L^2(\mathbb{R}^d)$.

(d) $v \in D(A^*) \Rightarrow \exists w \in L^2(\mathbb{R}^d): \ \langle w|u\rangle = \langle v|Au\rangle \ \forall u \in \mathcal{D}(\mathbb{R}^d)$ (cf. (2.2.1)).

We calculate

$$\langle A^* v_k|u\rangle \overset{A \text{ symm.}}{=} \langle Av_k|u\rangle \overset{A \text{ symm.}}{=} \langle v_k|Au\rangle \overset{(c)}{\to} \langle v|Au\rangle \overset{(d)}{=} \langle w|u\rangle \ \forall u \in \mathcal{D}(\mathbb{R}^d). \tag{4.3.3}$$

Since the inner product on $L^2(\mathbb{R}^d)$ is compatible with the distributional action, (4.3.3) implies that $(Av_k)_{k\in\mathbb{N}}$ converges to $w \in L^2(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$ in the distributional sense.

(e) $c_1 \in H^{r+1}(\mathbb{R}^d) \subseteq H^1(\mathbb{R}^d)$ by Lemma 2.3.3(i),

(f) $v \mapsto \partial_j v : H^0(\mathbb{R}^d) \to H^{-1}(\mathbb{R}^d)$ is a continuous map by [6, Corollary 9.3.2].

(g) Form the duality product of distributions (cf. Remark 4.3.10(i)) we obtain a continuous multiplication : $W^{1,2}_{loc} \times W^{-1,2}_{loc} \to W^{-1,1}_{loc}$. Since for $v \in H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, we have that

$$v \cdot \varphi \in H^s(\mathbb{R}^d) \ \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \subseteq \mathscr{S}(\mathbb{R}^d)$$

(cf. [10, Remark 6.23]), we obtain that $H^{\pm 1}(\mathbb{R}^d) = W^{\pm 1,2}(\mathbb{R}^d) \subseteq W^{\pm 1,2}_{loc}$ and hence the multiplication : $H^1 \times H^{-1} \to W^{-1,1}_{loc}$ is continuous as a special case.

(h) As a last fact $\partial_j : W^{-1,1}_{loc} \to W^{-2,1}_{loc}$ is continuous.

Collecting these results, we find that

$$\begin{array}{rlll} & v_k \to v & & \text{in } L^2 \text{ by (c)} \\ \Rightarrow & \partial_j v_k \to \partial_j v & & \text{in } H^{-1} \text{ by (f)} \\ \Rightarrow & c \cdot \partial_j v_k \to c \cdot \partial_j v & & \text{in } W^{-1,1}_{loc} \text{ by (g) (cf. (4.3.1))} \\ \Rightarrow & Av_k = \sum_j \partial_j(c \cdot \partial_j v_k) \to \sum_j \partial_j (c \cdot \partial_j v) = Av & & \text{in } W^{-2,1}_{loc} \text{ by (h).} \end{array}$$

Due to the fact that $W^{-2,1}_{loc}$ is continuously embedded in $\mathcal{D}'$, the last convergence is also valid in the distributional sense. Together with (4.3.3) we get that $Av = w$ since the distributional limit is unique.
So let $v \in D(A^*)$ then (d) ensures the existence of a $w \in L^2(\mathbb{R}^d)$ such that $Av = w$ and hence $Av \in L^2(\mathbb{R}^d)$. We obtain

$$D(A^*) \subseteq \{v \in L^2(\mathbb{R}^d) | Av \in L^2(\mathbb{R}^d)\} = B.$$

To finish the proof we use elliptic regularity described in Remark 4.3.10(ii). In our case it says that

$$v \in L^2(\mathbb{R}^d), \ Av \in L^2(\mathbb{R}^d) \Rightarrow v \in H^2(\mathbb{R}^d).$$

Let $v \in D(A^*) \subseteq B$, then by the lemma of elliptic regularity, with $s = 0$, we get that $v \in H^2(\mathbb{R}^d) = D(A)$. □

Knowing that A is self-adjoint we can apply Stone's theorem to get a strongly continuous unitary group $(T(z))_{z\in\mathbb{R}}$.

Theorem 4.3.12. *Let $c \in C(\mathbb{R}^d)$ and A both be as in Assumption 4.3.2 with $D(A) = H^2(\mathbb{R}^d)$. Then A is self-adjoint and iA generates a strongly continuous unitary group $(T(z))_{z\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$ with $\|T(z)\| \le 1 \ \forall z \in \mathbb{R}$. Moreover we have the following resolvent estimate, valid for $\lambda \in \mathbb{R}\backslash\{0\}$:*

$$\left\|(\lambda \mathrm{I} - iA)^{-1} v\right\|_{L^2(\mathbb{R}^d)} \le \frac{\|v\|_{L^2(\mathbb{R}^d)}}{|\lambda|} \qquad \forall v \in L^2(\mathbb{R}^d). \tag{4.3.4}$$

Proof. From Lemma 4.3.9 we know that A is symmetric (i.e. $D(A) \subseteq D(A^*)$ and $A^*|_{D(A)} = A$) and together with Proposition 4.3.11 ($D(A^*) \subseteq D(A)$) this yields self-adjointness of A. Thus we are able to apply Theorem 2.4.16 to get a strongly continuous unitary group $(T(z))_{z\in\mathbb{R}} := (\mathrm{e}^{izA})_{z\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$. From the spectral theorem (Theorem 2.2.8(ii)) we get

$$\|T(z)\| \leq \left\|\lambda \mapsto \mathrm{e}^{i\lambda z}\right\|_\infty = 1 \ \forall z \in \mathbb{R}$$

and hence $(T(z))_{z\in\mathbb{R}}$ is a group of contractions. The characterisation theorem for generators of C_0-groups (cf. Theorem 2.4.15) gives us the resolvent estimate

$$\begin{aligned} &\left\|(\lambda I - iA)^{-1}\right\| \leq |\lambda|^{-1} && \forall \lambda \neq 0 \\ \Rightarrow\ &\left\|(\lambda I - iA)^{-1}v\right\| \leq \tfrac{\|v\|_{L^2(\mathbb{R}^d)}}{|\lambda|} && \forall v \in L^2(\mathbb{R}^d)\ \forall \lambda \neq 0. \end{aligned}$$

□

Finally we consider the initial value problem discussed in Section 2.4.3 and use $(T(z))_{z\in\mathbb{R}}$ to construct solutions of

$$\begin{cases} \frac{dv(z)}{dz} = iAv(z) + g(z), & z \in \mathbb{R} \\ v(0) = v_0 \end{cases} \tag{4.3.5}$$

on $\mathbb{R} \times \mathbb{R}^d$, where $v : \mathbb{R} \to L^2(\mathbb{R}^d)$ and $g : \mathbb{R} \to L^2(\mathbb{R}^d)$.

If $v_0 \in L^2(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}, L^2(\mathbb{R}^d))$ then the mild solution $v \in C(\mathbb{R}, L^2(\mathbb{R}^d))$ is given by

$$v(z) := T(z)v_0 + \int_0^z T(z-\rho)g(\rho)d\rho. \tag{4.3.6}$$

There are two possibilities to get even a unique classical solution. One could either take $g \in C^1(\mathbb{R}, L^2(\mathbb{R}^d))$ (cf. Corollary 2.4.26), or assume that $g : \mathbb{R} \to L^2(\mathbb{R}^d)$ is continuous with values in $H^2(\mathbb{R}^d)$ since this implies that $z \mapsto iAg(z) \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \subseteq L^1(\mathbb{R}, L^2(\mathbb{R}^d))$ (cf. Corollary 2.4.27). In both cases the initial value v_0 has to be in $H^2(\mathbb{R}^d)$. The classical solution is of the same form as (4.3.5) but is $C^1(\mathbb{R}, L^2(\mathbb{R}^d))$ and has pointvalues in $H^2(\mathbb{R}^d)$.

4.3.2 Evolution System at Fixed τ

From the previous section we know that for any fixed pair $(z, \tau) \in [0, \infty) \times \mathbb{R}$ we have an unbounded operator $A(z, \tau)$ on $L^2(\mathbb{R}^d)$ with domain $D(A(z, \tau)) = H^2(\mathbb{R}^d)$, which in addition is self-adjoint. Moreover $iA(z, \tau)$ generates a C_0-unitary group $(T_z(s))_{s\in\mathbb{R}}$ which is contractive. In this section we will apply the existence theorem of Section 3.3 to get an evolution system. For the next theorem it is useful to recall the properties $(H1)' - (H3)'$ and $(E1)' - (E5)'$.

Theorem 4.3.13. *Let $Z > 0$. Then for all $\tau \in \mathbb{R}$ the family $\{iA(z, \tau)\}_{z\in[0,Z]}$ generates a unique evolution system $(U_\tau(z_1, z_2))_{0\leq z_2\leq z_1\leq Z}$ on $L^2(\mathbb{R}^d)$ satisfying the properties (E1)′-(E5)′ of Theorem 3.3.3 where $D = H^2(\mathbb{R}^d)$.*

Proof. We have to check the conditions (H1)′, (H2)′ and (H3)′ of Theorem 3.3.3.
So let us fix $\tau \in \mathbb{R}$. (H1)′ is implied in Theorem 4.3.12 and (H2)′ is clear anyway. So let us prove (H3)′, i.e. that $z \mapsto A(z, \tau)v : [0, Z] \to L^2(\mathbb{R}^d)$ is continuously differentiable for all $v \in H^2(\mathbb{R}^d)$.

From Assumption 4.2.1 (ii) we obtain that

$$c_1(.,.,\tau) = c(.,.,\tau) - c_0 \in C^1([0, \infty), H^{r+1}(\mathbb{R}^d)), \qquad r \in (0, 1). \tag{4.3.7}$$

As in Remark 2.3.8 we deduce that

$$\partial_{x_j} c(.,.,\tau) \in C^1([0,\infty), H^r(\mathbb{R}^d)) \qquad r \in (0,1),\ j \leq d. \tag{4.3.8}$$

We treat the cases $d = 1$ and $d = 2$ separately. For $d = 2$ we will use Corollary 4.3.5, several times and the following property (cf. [2, p.1422, Fact C]):

$$A(z,\tau)v = \operatorname{grad} c(z,.,\tau) \cdot \operatorname{grad} v + c(z,.,\tau)\Delta v \qquad \forall v \in H^{r+1}(\mathbb{R}^2) \tag{4.3.9}$$

which holds in $\mathcal{D}'(\mathbb{R}^2)$ for all $(z,\tau) \in [0,\infty) \times \mathbb{R}$. The meaning of the products is explained by Corollary 4.3.5. Thus we have for $v \in H^2(\mathbb{R}^2)$ and $z \in [0, Z]$:

$$A(z,\tau)v = \sum_{j=1}^{2} \partial_{x_j} c(z,.,\tau) \cdot \partial_{x_j} v + c_0 \Delta v + c_1(z,.,\tau) \cdot \Delta v. \tag{4.3.10}$$

We show that (4.3.10) is well-defined in $L^2(\mathbb{R}^d)$. Note that this is already proven by Lemma 4.3.7, but here we give an alternative argument using Corollary 4.3.5, (4.3.7) and (4.3.8):

$$A(z,\tau)v = \sum_{j=1}^{2} \underbrace{\overbrace{\partial_{x_j} c(z,.,\tau)}^{\in H^r(\mathbb{R}^2)} \cdot \overbrace{\partial_{x_j} v}^{\in H^1(\mathbb{R}^2)}}_{\in H^{r-\varepsilon}(\mathbb{R}^2)} + \underbrace{\overbrace{c_0}^{\in \mathbb{R}} \overbrace{\Delta v}^{\in H^0(\mathbb{R}^2)}}_{\in H^0(\mathbb{R}^2)} + \underbrace{\overbrace{c_1(z,.,\tau)}^{\in H^{r+1}(\mathbb{R}^2)} \overbrace{\Delta v}^{\in H^0(\mathbb{R}^2)}}_{\in H^0(\mathbb{R}^2)} \in L^2(\mathbb{R}^2).$$

Observe that this already shows continuity of $z \mapsto A(z,\tau)v$ since all occurring products are continuous. Proceeding with the partial derivative of this mapping, we consider the following limit, taken in $L^2(\mathbb{R}^2)$:

$$\begin{aligned}
\partial_z A(z,\tau)v &= \lim_{h\to 0} \frac{A(z+h,\tau)v - A(z,\tau)v}{h} \\
&= \sum_{j=1}^{2} \lim_{h\to 0} \frac{1}{h} \left[\partial_{x_j} c(z+h,.,\tau) \cdot \partial_{x_j} v - \partial_{x_j} c(z,.,\tau) \cdot \partial_{x_j} v\right] \\
&\quad + \lim_{h\to 0} \frac{1}{h} \left[c_1(z+h,.,\tau) \cdot \Delta v - c_1(z,.,\tau) \cdot \Delta v\right].
\end{aligned}$$

The occurring products are continuous mappings and hence by pulling the factors out of the limits, we obtain

$$= \sum_{j=1}^{2} \partial_{x_j} v \lim_{h\to 0} \frac{1}{h} \left[\partial_{x_j} c(z+h,.,\tau) - \partial_{x_j} c(z,.,\tau)\right] + \Delta v \lim_{h\to 0} \frac{1}{h} \left[c_1(z+h,.,\tau) - c_1(z,.,\tau)\right].$$

The two limits are defined by (4.3.7) and (4.3.8), which finally yields

$$= \sum_{j=1}^{2} \partial_z \partial_{x_j} c(z,.,\tau) \cdot \partial_{x_j} v + \partial_z c_1(z,.,\tau) \cdot \Delta v.$$

Therefore existence of the partial derivative is established if the above term is $L^2(\mathbb{R}^2)$-valued. Indeed by choosing an $\varepsilon < r$ we obtain

$$\partial_z A(z,\tau)v = \sum_{j=1}^{2} \underbrace{\overbrace{\partial_z \partial_{x_j} c(z,.,\tau)}^{\in H^r(\mathbb{R}^2)} \cdot \overbrace{\partial_{x_j} v}^{\in H^1(\mathbb{R}^2)}}_{\in H^{r-\varepsilon}(\mathbb{R}^2)} + \underbrace{\overbrace{\partial_z c_1(z,.,\tau)}^{\in H^{r+1}(\mathbb{R}^2)} \overbrace{\Delta v}^{\in H^0(\mathbb{R}^2)}}_{\in H^0(\mathbb{R}^2)} \in L^2(\mathbb{R}^2),$$

where we used Corollary 4.3.5, (4.3.7) and (4.3.8) again. Continuity of the mapping $z \mapsto \partial_z A(z,\tau)v$ is justified by (4.3.7) and (4.3.8) once again and the case $d = 2$ is completed.

Let now $d = 1$. Since Corollary 4.3.5 and (4.3.9) are only valid for $d = 2$, we use Theorem 4.3.4 and obtain continuous multiplications

$$\begin{gathered} H^r(\mathbb{R}) \times H^1(\mathbb{R}) \to H^r(\mathbb{R}) \\ H^{r+1}(\mathbb{R}) \times H^0(\mathbb{R}) \to H^0(\mathbb{R}). \end{gathered}$$

Hence for every $v \in H^2(\mathbb{R})$ and $z \in [0, Z]$ we have

$$A(z,\tau)v = \underbrace{\overbrace{c_0}^{\in\mathbb{R}} \ \overbrace{\partial_x^2 v}^{\in H^0(\mathbb{R})}}_{\in H^0(\mathbb{R})} + \underbrace{\overbrace{\partial_x c(z,.,\tau)}^{\in H^r(\mathbb{R})} \ \overbrace{\partial_x v}^{\in H^1(\mathbb{R})}}_{\in H^r(\mathbb{R})} + \underbrace{\overbrace{c_1(z,.,\tau)}^{\in H^{r+1}(\mathbb{R})} \ \overbrace{\partial_x^2 v}^{\in H^0(\mathbb{R})}}_{\in H^0(\mathbb{R})} \in L^2(\mathbb{R})$$

and

$$\partial_z A(z,\tau)v = \underbrace{\overbrace{\partial_z\partial_x c(z,.,\tau)}^{\in H^r(\mathbb{R})} \cdot \ \overbrace{\partial_x v}^{\in H^1(\mathbb{R})}}_{\in H^r(\mathbb{R})} + \underbrace{\overbrace{\partial_z c_1(z,.,\tau)}^{\in H^{r+1}(\mathbb{R})} \cdot \ \overbrace{\partial_x^2 v}^{\in H^0(\mathbb{R})}}_{\in H^0(\mathbb{R})} \in L^2(\mathbb{R})$$

where existence of the partial derivative as well as continuity of the mappings $z \mapsto A(z,\tau)v$ and $z \mapsto \partial_z A(z,\tau)v$ are established as in the case $d = 2$ by using Theorem 4.3.4, (4.3.7) and (4.3.8) several times.
In summary $z \mapsto A(z,\tau)v \in C^1([0,Z] : L^2(\mathbb{R}^d)$ and the proof is complete. □

Remark 4.3.14. At this stage we obtain solutions of our problem at a fixed frequency τ (cf. Section 3.4). Consider the initial value problem

$$\begin{cases} \frac{dv(z)}{dz} = iA(z,\tau)v(z) + g(z), \quad z \in [0, Z] \\ v(0) = v_0 \end{cases} \tag{4.3.11}$$

where $v : \mathbb{R} \to L^2(\mathbb{R}^d)$, $g \in L^1([0,Z], L^2(\mathbb{R}^d))$ and $v_0 \in L^2(\mathbb{R}^d)$. The mild solution $v \in C([0,Z], L^2(\mathbb{R}^d))$ of (4.3.11) is defined by

$$v(z) := U_\tau(z,0)v_0 + \int_0^z U_\tau(z,\rho)g(\rho)d\rho.$$

Applying Theorem 3.4.3 we get even a strong solution by assuming that $g \in C^1([0,Z] : L^2(\mathbb{R}^d))$ and $v_0 \in H^2(\mathbb{R}^d)$. This strong solution belongs to $C^1([0,Z] : L^2(\mathbb{R}^d))$ and is $H^2(\mathbb{R}^d)$-valued.

4.3.3 Frequency Dependence of the Evolution System

In this subsection we work out a continuity statement for the evolution system $U_\tau(z_1, z_2)$, $0 \le z_2 \le z_1 \le Z$ of the previous section. It ensures strong continuity of U in the three parameters τ, z_1 and z_2 jointly. We will use this result in the next section to get solutions of the Cauchy problem (4.2.4).

The first ingredient is the following lemma, which ensures Lipschitz-continuity of $A(z,\tau)$ in (z,τ).

Lemma 4.3.15. *Let $A(z,\tau)$, $(z,\tau) \in [0,\infty) \times \mathbb{R}$ be as in Definition 4.2.4 and let the coefficient-function c satisfy Assumption 4.2.1. Then the map*

$$(z,\tau) \mapsto A(z,\tau) : [0,\infty) \to L(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$$

is Lipschitz-continuous with respect to the operator norm.

Proof. Let $M(z,\tau)$ denote the multiplication operator defined on $H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ by

$$M(z,\tau)(v_1,v_2) := (c(z,.,\tau) - c(z_0,.,\tau_0))(v_1,v_2) \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$$

where $(z_0,\tau_0) \in [0,\infty) \times \mathbb{R}$. This is well-defined since $H^1(\mathbb{R})$ is an algebra and by Corollary 4.3.5 the multiplication $H^1(\mathbb{R}^2) \cdot H^{r+1}(\mathbb{R}^2) \to H^1(\mathbb{R}^2)$ is continuous; recall that $c_1(z,.,\tau) = c(z,.,\tau) - c_0 \in H^{r+1}(\mathbb{R}^d)$. Using this definition we get

$$\begin{aligned} A(z,\tau) - A(z_0,\tau_0) &= \operatorname{div}(c_{z,\tau} \operatorname{grad}) - \operatorname{div}(c_{z_0,\tau_0} \operatorname{grad}) \\ &= \operatorname{div} \circ M(z,\tau) \circ \operatorname{grad}, \end{aligned}$$

which yields the estimate

$$\begin{aligned} &\|A(z,\tau) - A(z_0,\tau_0)\|_{L(H^2,L^2)} \\ &\quad \le \|\operatorname{div}\|_{L(H^1\times H^1,L^2)} \cdot \|M(z,\tau)\|_{L(H^1\times H^1)} \cdot \|\operatorname{grad}\|_{L(H^2,H^1\times H^1)} \\ &\quad \le \sqrt{2}\, \|M(z,\tau)\|_{L(H^1\times H^1)} \end{aligned}$$

where we used that $\|\operatorname{div}\|_{L(H^1\times H^1,L^2)} \le \sqrt{2}$ and $\|\operatorname{grad}\|_{L(H^2,H^1\times H^1)} \le 1$, shown at the end of the proof.
Next we estimate $\|M(z,\tau)(v_1,v_2)\|_{H^1\times H^1}$. To this end we estimate $\|(c(z,.,\tau) - c(z_0,.,\tau_0)v)\|_{H^1}$ for any $v \in H^1(\mathbb{R}^d)$. First we apply the mean value theorem [7, 175.3, p.337] to the function $(z,\tau) \mapsto c(z,x,\tau) - c_0 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ which is continuously differentiable and obtain

$$c(z,x,\tau) - c(z_0,x,\tau_0) = \begin{pmatrix} z - z_0 \\ \tau - \tau_0 \end{pmatrix} \cdot \int_0^1 \operatorname{grad}_{z,\tau} c(z_0 + t(z-z_0), x, \tau_0 + t(\tau - \tau_0))dt.$$

Multiplying by $v(x) \in \mathbb{R}$ and taking both sides of the equation as functions in $H^{r+1}(\mathbb{R}^d)$ yields

$$(c(z,.,\tau) - c(z_0,.,\tau_0))v = \begin{pmatrix} z - z_0 \\ \tau - \tau_0 \end{pmatrix} \cdot \int_0^1 \operatorname{grad}_{z,\tau} c(z_0 + t(z-z_0), ., \tau_0 + t(\tau - \tau_0))dt \cdot v.$$

Consequently we have

$$\begin{aligned} &\|(c(z,.,\tau) - c(z_0,.,\tau_0))v\|_{H^1} \\ &\quad \le \left\| \begin{pmatrix} z - z_0 \\ \tau - \tau_0 \end{pmatrix} \right\| \int_0^1 \left\| \operatorname{grad}_{z,\tau} c(z_0 + t(z-z_0), ., \tau_0 + t(\tau-\tau_0))v \right\|_{H^1\times H^1} dt \\ &\quad \le \left\| \begin{pmatrix} z - z_0 \\ \tau - \tau_0 \end{pmatrix} \right\| \sup_{t\in[0,1]} \left\| \operatorname{grad}_{z,\tau} c(z_0 + t(z-z_0), ., \tau_0 + t(\tau-\tau_0))v \right\|_{H^1\times H^1}. \end{aligned}$$

Now let us assume that $\gamma := \max(|z - z_0|, |\tau - \tau_0|) \le 1$. Then $t \in [0,1]$ implies that $z_0 + t(z - z_0) \in [z_0 - 1, z_0 + 1]$ and $\tau_0 + t(\tau - \tau_0) \in [\tau_0 - 1, \tau_0 + 1]$. In this way we obtain

$$\begin{aligned} \|(c(z,.,\tau) - c(z_0,.,\tau_0))v\|_{H^1} &\le \sqrt{2}\gamma \cdot \sup \sqrt{\|\partial_z c(z',.,\tau')v\|_{H^1}^2 + \|\partial_\tau c(z',.,\tau')v\|_{H^1}^2} \\ &\le \sqrt{2}\gamma \cdot \sup \left(\left\|\partial_z c(z',.,\tau')v\right\|_{H^1} + \left\|\partial_\tau c(z',.,\tau')v\right\|_{H^1} \right) \end{aligned}$$

where the supremum is taken over $(z', \tau') \in [z_0 - 1, z_0 + 1] \times [\tau_0 - 1, \tau_0 + 1]$.
From Assumption 4.2.1 we know that $\partial_z c$ and $\partial_\tau c$ are both in $C([0,\infty)\times\mathbb{R}, H^{r+1}(\mathbb{R}^d))$ and thus

$$\begin{aligned}&\left\|\partial_z c(z', ., \tau')v\right\|_{H^1} + \left\|\partial_\tau c(z', ., \tau')v\right\|_{H^1}\\ &\qquad \le C_1 \|v\|_{H^1} \left(\left\|\partial_z c(z', ., \tau')\right\|_{H^{r+1}} + \left\|\partial_\tau c(z', ., \tau')\right\|_{H^{r+1}}\right) \le C_2 \|v\|_{H^1}\end{aligned}$$

for all $(z', \tau') \in [z_0 - 1, z_0 + 1] \times [\tau_0 - 1, \tau_0 + 1]$.
Collecting the above estimates yields

$$\begin{aligned}&\|M(z,\tau)(v_1, v_2)\|_{H^1\times H^1}\\ &\qquad \le \sqrt{2}\max_{i=1,2} \|(c(z, ., \tau) - c(z_0, ., \tau_0))v_i\|_{H^1} \le \gamma \cdot C \max_{i=1,2} \|v_i\|_{H^1} \le \gamma \cdot C \|(v_1, v_2)\|_{H^1\times H^1}\end{aligned}$$

and therefore we have

$$\|M(z,\tau)\|_{L(H^1\times H^1)} \le \gamma \cdot C$$

and hence

$$\|A(z,\tau) - A(z_0,\tau_0)\|_{L(H^2,L^2)} \le \sqrt{2}C \max(|z - z_0|, |\tau - \tau_0|) \le \sqrt{2}C \left\| \begin{pmatrix} z - z_0 \\ \tau - \tau_0 \end{pmatrix} \right\|$$

which proves Lipschitz-continuity.

As already announced we estimate the norm of the divergence and the gradient. The supremum below is taken over the set $B_1 := \{(v_1, v_2) \in H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) : \|(v_1, v_2)\|_{H^1\times H^1} = \sqrt{\|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2} = 1\}$.

$$\begin{aligned}\|\mathrm{div}\|_{L(H^1\times H^1, L^2)} &= \sup_{(v_1,v_2)\in B_1} \|\partial_1 v_1 + \partial_2 v_2\|_{L^2}\\ &\le \sup_{(v_1,v_2)\in B_1} \underbrace{\|\partial_1 v_1\|_{L^2}}_{\le\|v_1\|_{H^1}} + \underbrace{\|\partial_2 v_2\|_{L^2}}_{\le\|v_2\|_{H^1}}\\ &\le \sup_{(v_1,v_2)\in B_1} \sqrt{2(\|v_1\|_{H^1}^2 + \|v_2\|_{H^1}^2)}\\ &\le \sqrt{2}\end{aligned}$$

where we used the elementary inequality $(a + b)^2 \le 2(a^2 + b^2),\ a, b \in \mathbb{R}$ and the Sobolev norm $\|v\|_{H^1}^2 = \sum_{|\alpha|\le 1} \|\partial^\alpha v\|_{L^2}^2$.
For the gradient we calculate:

$$\begin{aligned}\|\mathrm{grad}\|_{L(H^2,H^1\times H^1)} &= \sup_{v\in oH^2} \|(\partial_1 v, \partial_2 v)\|_{H^1\times H^1}\\ &\le \sup_{v\in oH^2} \sqrt{\|\partial_1 v\|_{H^1}^2 + \|\partial_2 v\|_{H^1}^2}\\ &\le \sup_{v\in oH^2} \sqrt{\|v\|_{H^2}^2}\\ &= 1,\end{aligned}$$

where we used that

$$\begin{aligned}\|\partial_1 v\|_{H^1}^2 + \|\partial_2 v\|_{H^1}^2 &= \sum_{0\le|\alpha|\le 1} \|\partial^\alpha \partial_1 v\|_{L^2}^2 + \sum_{0\le|\alpha|\le 1} \|\partial^\alpha \partial_2 v\|_{L^2}^2 \\ &\le \sum_{0\le|\alpha|\le 2} \|\partial^\alpha v\|_{L^2}^2 \\ &= \|v\|_{H^2}^2 .\end{aligned}$$

□

The next lemma shows strong continuity of the semigroup $(T_{z,\tau}(s))_{s\ge 0}$ in z and τ, strong continuity in s is clear anyway. Note that this continuity holds jointly in (z,τ,s). In the proof one uses the Kato-Trotter theorem on convergence of semigroups (cf. [20, Chapter IX, Section 12, Theorem 1]). Checking all assumptions it turns out that the hardest part is to show strong continuity of the resolvent map.

Lemma 4.3.16. *Let $A(z,\tau)$, $(z,\tau)\in[0,\infty)\times\mathbb{R}$ be as in Definition 4.2.4 and let the coefficient-function c satisfy Assumption 4.2.1. Moreover let $(T_{z,\tau}(s))_{s\ge 0}$ denote the semigroup generated by $A(z,\tau)$. Then the map*

$$(z,\tau,s)\mapsto T_{z,\tau}(s) : [0,\infty)\times\mathbb{R}\times[0,\infty)\to L(L^2(\mathbb{R}^d))$$

is strongly continuous.

Proof. (cf. [2, Lemma 3.12]) □

We go on to the main result of this subsection, which states strong continuity of the evolution system $U_\tau(z_1,z_2)$ in all parameters.

Theorem 4.3.17. *For arbitrary $Z>0$ and $\tau\in\mathbb{R}$ let $(U_\tau(z_1,z_2))_{0\le z_2\le z_1\le Z}$ be the evolution system generated by the family of operators $\{A(z,\tau)\}_{z\in[0,Z]}$, defined in Section 4.2 and satisfying Assumption 4.2.1. Then the map*

$$(\tau,z_1,z_2)\mapsto U_\tau(z_1,z_2) : \mathbb{R}\times\{(z_1,z_2): 0\le z_2\le z_1\le Z\} =: B \to L(L^2(\mathbb{R}^d)) \tag{4.3.12}$$

is strongly continuous.

Proof. To gain better readability we set $U_\tau(z_1,z_2) =: U(\tau;z_1,z_2)$ throughout this proof. Based on the proof of Theorem 3.3.2 we get the evolution system U as the strong limit of $U_n(\tau;z_1,z_2)$ defined as follows:

$$U_n(\tau;z,y) := \begin{cases} T_{\tau,z_j^n}(z-y) & \text{if } z_j^n\le y\le z\le z_{j+1}^n, \\ T_{\tau,z_k^n}(z-z_k^n)\displaystyle\prod_{j=l+1}^{k-1} T_{\tau,z_j^n}\left(\frac{Z}{n}\right) T_{\tau,z_l^n}(z_{l+1}^n - y) & \text{if } z_l^n\le y\le z_{l+1}^n\le z_k^n\le z\le z_{k+1}^n, \end{cases} \tag{4.3.13}$$

where $z_j^n = \frac{jZ}{n}$ for $n\in\mathbb{N}$ and $j=0,1,\dots,n$.
By Lemma 4.3.16, T is strongly continuous in (z,τ,s) and therefore U_n is strongly continuous in $(\tau;z,y)$. Moreover we have that $\|U_n(\tau;z,y)\|_{L^2(\mathbb{R}^d)}\le 1$ (cf. Theorem 4.3.12). In the same way as in (3.3.10) we have the following integral equation for $v\in H^2(\mathbb{R}^d)$:

$$U_n(\tau;z,y)v - U_m(\tau;z,y)v = \int_y^z U_n(\tau;z,r)(A_n(r,\tau)-A_m(r,\tau))U_m(\tau;r,y)v\,dr, \tag{4.3.14}$$

where $A_n(z,\tau)$ is the piecewise approximation of $A(z,\tau)$ (details can be found in the proof of Theorem 3.3.2).
By Lemma 4.3.15 we have

$$\|A_n(r,\tau) - A(r,\tau)\|_{L(H^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} \to 0 \qquad \text{for } n \to \infty \tag{4.3.15}$$

uniformly in (r,τ) on compact sets.
Passing to the limit $m \to \infty$ in (4.3.14) yields the estimate

$$\|U_n(\tau;z,y)v - U(\tau;z,y)v\|_{L^2(\mathbb{R}^d)} \leq \|v\|_{H^2(\mathbb{R}^d)} \int_y^z \|A_n(r,\tau) - A(r,\tau)\|_{L(H^2(\mathbb{R}^d),L^2(\mathbb{R}^d))}\, dr.$$

Below we will prove uniform convergence of $U_n(\tau;z,y)v$, which finishes the proof since $U(\tau;z,y)$ will then be the uniform limit of strongly continuous functions.
Due to (4.3.15) for any $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that

$$\|A_n(r,\tau) - A(r,\tau)\|_{L(H^2(\mathbb{R}^d),L^2(\mathbb{R}^d))} < \varepsilon \qquad \forall n \geq N_0\ \forall (r,\tau) \in [0,Z] \times K$$

for any compact $K \subset \mathbb{R}$. Consequently we have

$$\|U_n(\tau;z,y)v - U(\tau;z,y)v\|_{L^2(\mathbb{R}^d)} \leq \|v\|_{H^2(\mathbb{R}^d)}\,(z-y)\varepsilon$$

for all $n \geq N_0,\ \tau \in K$ and $0 \leq y \leq z \leq Z$, hence uniform convergence is established. □

4.4 Solution of the Cauchy Problem

We are now ready to solve the problem introduced in Section 4.1. There the Cauchy problem in the time domain was formulated as follows

$$\begin{cases} Pu := \partial_z u - i \cdot \operatorname{div}_x(C(z,x,D_t) \cdot \operatorname{grad}_x u) = f \in \mathcal{V} \\ u|_{z=0} = u_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)). \end{cases} \tag{4.4.1}$$

In Section 4.2 we applied the partial Fourier-transform $\tilde{\mathcal{F}}_t$ to get a Cauchy problem in the frequency domain:

$$\begin{cases} \tilde{P}v := \partial_z v - i \cdot \operatorname{div}(c(z,x,\tau) \cdot \operatorname{grad} v) = g \in \mathcal{V} \\ v|_{z=0} = v_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)). \end{cases}$$

Using Definition 4.2.4 we rewrote our problem as

$$\begin{cases} \partial_z v = iA(z,\tau)v + g \in \mathcal{V} \\ v|_{z=0} = v_0 \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)). \end{cases} \tag{4.4.2}$$

The next theorem shows that this problem has a unique strong solution for proper g and initial condition v_0. But first we state a proposition which ensures solvability for any fixed $\tau \in \mathbb{R}$. Such a result was already given in Remark 4.3.14.
Let us denote the space of bounded and continuous functions with values in a Banach space X by $C_b(\mathbb{R}, X)$. Note that we have $C_b(\mathbb{R}, X) \subseteq \mathscr{S}'(\mathbb{R}; X)$ and that $U_\tau(z_1,z_2)g(\tau)$ is well-defined for each triple $(\tau, z_1, z_2) \in \mathbb{R} \times \{(z_1,z_2) : 0 \leq z_2 \leq z_1 \leq Z\}$ and $g \in C_b(\mathbb{R}, L^2(\mathbb{R}^d))$.

Proposition 4.4.1. *Let $\tau \in \mathbb{R}$ and $Z > 0$ be fixed and let $g_\tau \in C^1([0,Z], L^2(\mathbb{R}^d))$ and $v_0(\tau) \in H^2(\mathbb{R}^d)$. Then the Cauchy problem*

$$\begin{cases} \partial_z v_\tau(z) = iA(z,\tau)v_\tau(z) + g_\tau \\ v_\tau|_{z=0} = v_0(\tau) \end{cases}$$

has a unique strong solution $v_\tau \in C^1([0,Z], L^2(\mathbb{R}^d))$ given by

$$v_\tau(z) = U_\tau(z,0)v_0(\tau) + \int_0^z U_\tau(z,r)g_\tau(r)dr. \tag{4.4.3}$$

Proof. This follows directly from Theorem 4.3.13 and Theorem 3.4.3. □

Our next subject is the solution of the Cauchy problem in the frequency domain:

Theorem 4.4.2. *Let $Z > 0$ and let $g \in C^1([0,Z], C_b(\mathbb{R}, L^2(\mathbb{R}^d)))$, $v_0 \in C_b(\mathbb{R}, H^2(\mathbb{R}^d))$. Define $v : [0,Z] \times \mathbb{R} \to L^2(\mathbb{R}^d)$ by*

$$v(z,\tau) = U_\tau(z,0)v_0(\tau) + \int_0^z U_\tau(z,r)g(r,\tau)dr. \tag{4.4.4}$$

Then v is the unique solution of the Cauchy problem (4.4.2) *and we have that*

$$v(z) \in C_b(\mathbb{R}, L^2(\mathbb{R}^d)) \qquad \forall z \in [0,Z] \tag{4.4.5}$$

and

$$v \in C^1([0,Z], C_b([-N,N], L^2(\mathbb{R}^d))) \qquad \forall N > 0. \tag{4.4.6}$$

Remark 4.4.3. Note that (4.4.5) and (4.4.6) together imply that v is a mapping from $[0,Z]$ to $C_b(\mathbb{R}, L^2(\mathbb{R}^d))$, but continuous differentiability is only valid with respect to the family of seminorms $(p_N)_{N>0}$ given by

$$p_N(h) := \sup_{|\tau|\leq N} \|h(\tau)\|_{L^2(\mathbb{R}^d)},$$

which generates a coarser topology than $\|.\|_\infty$.

Proof. For each $\tau \in \mathbb{R}$ and $z \in [0,Z]$ we set $g_\tau(z) := g(z,\tau)$. Then g_τ is an element of $C^1([0,Z], L^2(\mathbb{R}^d))$ and hence Proposition 4.4.1 implies solvability for each τ. We get the solution (4.4.4) by setting $v(z,\tau) := v_\tau(z)$ for all $(z,\tau) \in [0,Z] \times \mathbb{R}$. It remains to show the properties (4.4.5) and (4.4.6). We start by showing that $\tau \mapsto v(z,\tau)$ is continuous and bounded from $\mathbb{R}$ to $L^2(\mathbb{R}^d)$ for all fixed $z \in [0,Z]$:
Since v_0 and g are bounded and continuous in τ for all $z \in [0,Z]$, we have that

$$\sup_{\tau\in\mathbb{R}} \|v(z,\tau)\|_{L^2(\mathbb{R}^d)} \leq \sup_{\tau\in\mathbb{R}} \left[\|v_0(\tau)\|_{L^2(\mathbb{R}^d)} + z \cdot \sup_{0\leq r\leq z} \|g(r,\tau)\|_{L^2(\mathbb{R}^d)} \right] < \infty$$

where we used $(E1)'$. To prove continuity we estimate

$$\begin{aligned} &\|v(z,\tau_1) - v(z,\tau_2)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|U_{\tau_1}(z,0)v_0(\tau_1) - U_{\tau_2}(z,0)v_0(\tau_2)\|_{L^2(\mathbb{R}^d)} + \int_0^z \|U_{\tau_1}(z,r)g(r,\tau_1) - U_{\tau_2}(z,r)g(r,\tau_2)\|_{L^2(\mathbb{R}^d)} dr \\ &\leq \underbrace{\|U_{\tau_1}(z,0)v_0(\tau_1) - U_{\tau_2}(z,0)v_0(\tau_2)\|_{L^2(\mathbb{R}^d)}}_{(*)} + \underbrace{z \sup_{r\in(0,z)} \|U_{\tau_1}(z,r)g(r,\tau_1) - U_{\tau_2}(z,r)g(r,\tau_2)\|_{L^2(\mathbb{R}^d)}}_{(**)}. \end{aligned}$$

Continuity will be established by showing that $(*)$ as well as $(**)$ tends to zero if τ_1 tends to τ_2. So let $\tau_1 \to \tau_2$, then

$$\begin{aligned}(*) &\leq \|U_{\tau_1}(z,0)v_0(\tau_1) - U_{\tau_1}(z,0)v_0(\tau_2) + U_{\tau_1}(z,0)v_0(\tau_2) - U_{\tau_2}(z,0)v_0(\tau_2)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|U_{\tau_1}(z,0)v_0(\tau_1) - U_{\tau_1}(z,0)v_0(\tau_2)\|_{L^2(\mathbb{R}^d)} + \|U_{\tau_1}(z,0)v_0(\tau_2) - U_{\tau_2}(z,0)v_0(\tau_2)\|_{L^2(\mathbb{R}^d)} \\ &\leq \underbrace{\|U_{\tau_1}(z,0)\|_{L(L^2(\mathbb{R}^d))}}_{\text{bounded by } (E1)'} \underbrace{\|v_0(\tau_1) - v_0(\tau_2)\|_{L^2(\mathbb{R}^d)}}_{\to 0 \text{ since } v_0 \text{ is continuous in } \tau} + \underbrace{\|(U_{\tau_1}(z,0) - U_{\tau_2}(z,0))v_0(\tau_2)\|_{L^2(\mathbb{R}^d)}}_{\to 0 \text{ by (4.3.12)}}\end{aligned}$$

and

$$(**) \leq z \sup_{r\in(0,z)} \|U_{\tau_1}(z,r)g(r,\tau_1) - U_{\tau_1}(z,r)g(r,\tau_2) + U_{\tau_1}(z,r)g(r,\tau_2) - U_{\tau_2}(z,r)g(r,\tau_2)\|_{L^2(\mathbb{R}^d)}$$

$$\leq \underbrace{z \sup_{r\in(0,z)} [\underbrace{\|U_{\tau_1}(z,r)\|_{L(L^2(\mathbb{R}^d))}}_{\text{bounded by } (E1)'} \underbrace{\|g(r,\tau_1) - g(r,\tau_2)\|_{L^2(\mathbb{R}^d)}}_{\to 0 \text{ by assumption}}}_{\to 0 \text{ since } g \text{ is continuous in } r} + \underbrace{\|(U_{\tau_1}(z,r) - U_{\tau_2}(z,r))g(r,\tau_2)\|_{L^2(\mathbb{R}^d)}}_{\to 0 \text{ by (4.3.12)}}].$$

Next we show that $z \mapsto v(z,.)$ is continuous from $[0,Z]$ to $C_b([-N,N], L^2(\mathbb{R}^d))$ for all $N > 0$. We have that

$$\sup_{\tau\in[-N,N]} \|v_\tau(z_1) - v_\tau(z_2)\|_{L^2(\mathbb{R}^d)} \leq \alpha + \beta$$

where

$$\alpha = \sup_{\tau\in[-N,N]} \|U_\tau(z_1,0)v_0(\tau) - U_\tau(z_2,0)v_0(\tau)\|_{L^2(\mathbb{R}^d)}$$

and

$$\beta = \sup_{\tau\in[-N,N]} \left\| \int_0^{z_1} U_\tau(z_1,r)g(r,\tau)dr - \int_0^{z_2} U_\tau(z_2,r)g(r,\tau)dr \right\|_{L^2(\mathbb{R}^d)}.$$

Showing that α and β vanish if z_1 tends to z_2 will establish the asserted continuity. So let $N > 0$ and $\varepsilon > 0$ be arbitrary but fixed.

Estimate for α: Since v_0 is continuous, we have that $v_0([-N,N]) \subseteq L^2(\mathbb{R}^d)$ is compact. Consequently we may choose $\tau_1, \ldots, \tau_m \in [-N,N],\ m \in \mathbb{N}$ such that for all $\tau \in [-N,N]$ there exists a $j \in \{1,\ldots,m\}$ such that

$$\|v_0(\tau) - v_0(\tau_j)\|_{L^2(\mathbb{R}^d)} < \varepsilon. \tag{4.4.7}$$

By (4.3.12) we obtain uniform continuity of the mapping $(z,\tau) \mapsto U_\tau(z,0)v_0(\tau_0) : [0,Z] \times [-N,N] \to L^2(\mathbb{R}^d)$ for all $\tau_0 \in [-N,N]$ and hence for all $\tau_0 = \tau_j,\ j \in \{1,\ldots,m\}$. In this way we get $\delta_j,\ j \in \{1,\ldots,m\}$ such that

$$|z_1 - z_2| + |\tau - \tau'| < \delta_j \Rightarrow \|U_\tau(z_1,0)v_0(\tau_j) - U_{\tau'}(z_2,0)v_0(\tau_j)\|_{L^2(\mathbb{R}^d)} < \varepsilon. \tag{4.4.8}$$

So let us assume that $|z_1 - z_2| < \delta := \min(\delta_1,\ldots,\delta_m)$ and that for an arbitrary $\tau \in [-N,N],\ k \in \{1,\ldots,m\}$ is the proper index such that (4.4.7) holds. Then we have

$$\begin{aligned}\|U_\tau(z_1,0)v_0(\tau) - U_\tau(z_2,0)v_0(\tau)\|_{L^2(\mathbb{R}^d)} &\leq \|U_\tau(z_1,0)v_0(\tau) - U_\tau(z_1,0)v_0(\tau_k)\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|U_\tau(z_1,0)v_0(\tau_k) - U_\tau(z_2,0)v_0(\tau_k)\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|U_\tau(z_2,0)v_0(\tau_k) - U_\tau(z_2,0)v_0(\tau)\|_{L^2(\mathbb{R}^d)} \\ &\leq \|U_\tau(z_1,0)\|_{L(L^2(\mathbb{R}^d))} \cdot \varepsilon + \varepsilon + \|U_\tau(z_2,0)\|_{L(L^2(\mathbb{R}^d))} \cdot \varepsilon \\ &\leq 3\varepsilon\end{aligned}$$

where we used (4.4.7), (4.4.8) and $(E1)'$. Note that this δ depends on ε and N but is independent from τ. Therefore the above estimates stay valid by taking the supremum over $\tau \in [-N,N]$.
Estimate for β: We choose without loss of generality $z_1 \leq z_2$. Then

$$\begin{aligned}
\beta &\leq \sup_{\tau\in[-N,N]} \left[\int_0^{z_1} \|U_\tau(z_1,r)g(r,\tau) - U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr + \int_{z_1}^{z_2} \|U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr \right] \\
&\leq \sup_{\tau\in[-N,N]} \left[z_1 \cdot \sup_{r\in[0,z_1]} \|U_\tau(z_1,r)g(r,\tau) - U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)} \right] \\
&\quad + \sup_{\tau\in[-N,N]} \left[\int_{z_1}^{z_2} \|U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr \right].
\end{aligned}$$

First we take care of the term

$$\sup_{\tau\in[-N,N]} \int_{z_1}^{z_2} \|U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr \leq \int_{z_1}^{z_2} \sup_{\tau\in[-N,N]} \|U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr.$$

Similarly to the calculation for $(**)$ above, we get that the mapping

$$\tau \mapsto U_\tau(z_2,r)g(r,\tau) : \mathbb{R} \to L^2(\mathbb{R}^d)$$

is continuous. Thus its image of the compact set $[-N,N]$ is again compact, which ensures existence of an $M>0$ such that

$$\int_{z_1}^{z_2} \sup_{\tau\in[-N,N]} \|U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)}\, dr \leq M(z_2 - z_1).$$

The term with the double supremum is estimated by the following construction, similar to the estimate for α:
Since g is continuous, we have that $g([0,Z]\times[-N,N]) \subseteq L^2(\mathbb{R}^d)$ is compact. Again we may choose $\tau_1,\dots,\tau_m \in [-N,N],\ m\in\mathbb{N}$ and $r_1,\dots,r_p,\ p\in\mathbb{N}$ such that for all $(r,\tau)\in[0,Z]\times[-N,N]$ there exist $j\in\{1,\dots,m\}$ and $l\in\{1,\dots,p\}$ such that

$$\|g(r,\tau) - g(r_l,\tau_j)\|_{L^2(\mathbb{R}^d)} < \varepsilon. \tag{4.4.9}$$

Similarly as above we get the existence of $\delta_{jl},\ j\in\{1,\dots,m\},\ l\in\{1,\dots,p\}$ such that

$$|z_1 - z_2| + |\tau - \tau'| + |r - r'| < \delta_{jl} \Rightarrow \left\|U_\tau(z_1,r)g(r_l,\tau_j) - U_{\tau'}(z_2,r')g(r_l,\tau_j)\right\|_{L^2(\mathbb{R}^d)} < \varepsilon. \tag{4.4.10}$$

Let us assume that $|z_1 - z_2| < \delta := \min\{\varepsilon,\ \delta_{jl} : 1\leq j\leq m,\ 1\leq l\leq p\}$ and that for arbitrary $(r,\tau)\in[0,Z]\times[-N,N],\ k\in\{1,\dots,m\}$ and $q\in\{1,\dots,p\}$ are the proper indices such that (4.4.9) holds. Then we have

$$\begin{aligned}
\|U_\tau(z_1,r)g(r,\tau) - U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)} &\leq \|U_\tau(z_1,r)g(r,\tau) - U_\tau(z_1,r)g(r_q,\tau_k)\|_{L^2(\mathbb{R}^d)} \\
&\quad + \|U_\tau(z_1,r)g(r_q,\tau_k) - U_\tau(z_2,r)g(r_q,\tau_k)\|_{L^2(\mathbb{R}^d)} \\
&\quad + \|U_\tau(z_2,r)g(r_q,\tau_k) - U_\tau(z_2,r)g(r,\tau)\|_{L^2(\mathbb{R}^d)} \\
&\leq 3\varepsilon.
\end{aligned}$$

Note that this δ depends on ε and N but is independent from τ and r. Therefore and using the above integral-estimate we have

$$\beta \leq 3z_1\varepsilon + M\varepsilon.$$

In this way we have proven continuity of $z \mapsto v(z,.) : [0,Z] \to C_b([-N,N], L^2(\mathbb{R}^d))$ for all $N>0$. Similarly one proves that this map is even continuously differentiable and the proof is complete. $\square$

Remark 4.4.4. (i) If $g \in C_b([0,Z] \times \mathbb{R}, L^2(\mathbb{R}^d))$, then (4.4.4) defines a *mild solution* of our problem.

(ii) We could also assume that $g \in C_b([0,Z] \times \mathbb{R}, H^2(\mathbb{R}^d))$ to get a strong solution.

(iii) Here we used functions that are bounded and continuous with respect to τ. More generally one could consider elements in $\mathcal{V}$, whose distributional action (with respect to τ) is given by weak integration over a continuous function (times the test function).

Finally we discuss the solution of the Cauchy problem in the time domain. We wish to have a unique solution $u \in C^1([0,Z], \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)))$ of the problem (4.4.1). Since we want to apply Theorem 4.4.2, we need a function f and initial value u_0 such that $\mathcal{F}_t f \in C^1([0,Z], C_b(\mathbb{R}, L^2(\mathbb{R}^d)))$ and $\mathcal{F}_t u_0 \in C_b(\mathbb{R}, H^2(\mathbb{R}^d))$.
Recall that $f_1 \in L^1(\mathbb{R}^d)$ implies that $\mathcal{F} f_1 \in C_0(\mathbb{R}^n) \subseteq C_b(\mathbb{R}^n)$ (cf. [18, Satz V.2.2]). Moreover recall that $\mathcal{F}$ is an isomorphism on $\mathscr{S}'(\mathbb{R}^d)$ (cf. [10, Theorem 5.25]). Then by assuming that

$$f \in C^1([0,Z], L^1(\mathbb{R}, L^2(\mathbb{R}^d)))$$

and

$$u_0 \in L^1(\mathbb{R}, H^2(\mathbb{R}^d))$$

we obtain $v_0 := \mathcal{F}_t u_0 \in C_b(\mathbb{R}, H^2(\mathbb{R}^d))$ and $g := \mathcal{F}_t f \in C^1([0,Z], C_b(\mathbb{R}, L^2(\mathbb{R}^d)))$. From Theorem 4.4.2 we get a solution $v : [0,Z] \times \mathbb{R} \to L^2(\mathbb{R}^d)$ which satisfies (4.4.5) and (4.4.6).
Let us now additionally assume that $v \in C^1([0,Z], \mathscr{S}'(\mathbb{R}, L^2(\mathbb{R}^d))) \subseteq \mathcal{V}$, which is not necessarily the case, then we can apply the partial Fourier transform and obtain a unique solution of (4.4.1) by setting

$$u := \tilde{\mathcal{F}}_t^{-1} v.$$

Since $\tilde{\mathcal{F}}_t$ is an isomorphism on $\mathcal{V}$ (cf. (4.2.1)), we get that $u \in \mathcal{V} = C([0,\infty), \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)))$, which solves the equation (4.4.1). This solution belongs to $C^1([0,Z], \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d)))$, since we have that $u'(z) = \mathcal{F}_t^{-1}(v'(z)) \in \mathscr{S}'(\mathbb{R}; L^2(\mathbb{R}^d))$:

$$\begin{aligned}
\lim_{h \to 0} \frac{u(z+h) - u(z)}{h} &= \lim_{h \to 0} \frac{\mathcal{F}_t^{-1}(v(z+h)) - \mathcal{F}_t^{-1}(v(z))}{h} \\
&= \mathcal{F}_t^{-1} \left(\lim_{h \to 0} \frac{v(z+h) - v(z)}{h} \right).
\end{aligned}$$

Moreover the solution u is H^2-valued in the following sense:

$$\langle u(z), \varphi \rangle \in H^2(\mathbb{R}^d) \qquad \forall z \in [0,Z] \ \forall \varphi \in \mathscr{S}(\mathbb{R}).$$

This follows from the fact that v_τ from Proposition 4.4.1 is a strong solution, which implies that $v_\tau(z) \in D(A(z,\tau)) = H^2(\mathbb{R}^d)$ for all $z \in [0,Z]$ (cf. Definition 3.1.1).

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