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**Entanglement under Global Unitary Operations**

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Verfasser:	Philipp Köhler
Matrikel-Nummer:	0447676
Studienrichtung (lt. Studienblatt):	A 411 Diplomstudium Physik UniStG
Betreuerin / Betreuer:	Ao.Univ.-Prof. Dr. Reinhold A. Bertlmann

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# 1 Introduction

Quantum mechanics and its phenomena had a high impact on our view of the world. One phenomenon in particular has even puzzled the greatest physicians of their time, such as Planck, Schrödinger or Einstein, *Entanglement*. Or to use the original term introduced by Schrödinger, *Verschränkung* [1].

At the time it was discovered even the physicists involved had difficulties to foresee the fundamental changes quantum mechanics would bring. Planck himself said about his finding of the planck constant that it was "an act of desparation" to describe the spectrum of black body radiation. He had to quantize the energy of the photons to find a satisfying formula for this phenomenon. Einstein used this idea to describe the photo effect, for which he became the Nobel prize. But even Einstein had troubles with quantum mechanics, namely with the above mentioned entanglement. Entanglement means the connection between two or more particles, so that these particles act like one system. If you disturb one particle the other one reacts instantaneously. Einstein, Podolski and Rosen [23] argued that this feature of instantaneous reaction (or, as Einstein called it "spooky action at a distance") could not, among other things, be combined with the assumption of information not travelling faster than light. Thus they concluded, that quantum mechanics is not a complete theory. For about 30 years this discussion could not be decided until John Bell showed with his famous Bell inequality, that indeed quantum mechanics is a complete theory and that we instead have to change our view of notions like reality and locality.

Entanglement and related ideas have been investigated very thoroughly since then. Numerous ways to exploit the quantum behaviour have been found. Ideas like quantum computers, quantum cryptography and information processing with the help of entanglement led to the field of quantum information. Thus there is a great need in understanding when a system can be considered as entangled and therefore cannot be considered as a classical system.

This work will try to give an overview about entanglement and ways to detect and quantify it. In chapter 2 we will introduce the mathematical tools we will need throughout this work. In chapter 3 we will discuss entropy and its relation to the quantum world. Chapter 4 will deal with entanglement and entanglement detection. In chapter 5 we will introduce a selection of entanglement measures and how to calculate them if possible. In chapter 6 Bell's inequality will be investigated and a method to calculate it explicitly will be introduced. We will then discuss in chapter 7 measurement operations on quantum systems to show that we can change the entanglement therein through interactions. In chapter 8 we will give ways to visualize quantum systems for a more intuitive understanding of their behaviour. Chapter 9 will introduce the notion of unitary operations and factorizations to manipulate quantum systems. This will show

## *1 Introduction*

the connection of entanglement and the viewpoint we need to detect it. In chapter 10 we will give examples for factorizations and how they change quantum systems.

## 2 Mathematical Formalism

### 2.1 Mathematical Description of Quantum Mechanics

In classical physics, every object can be represented as a point in phase space and therefore as a set of variables  $(\vec{x}, \vec{p})$  which give a complete description of this physical entity. However in quantum mechanics, one cannot determine all properties of a physical object at once due to Heisenbergs Uncertainty Relation.

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \quad (2.1)$$

Thus, in quantum mechanics we can only describe probabilistic properties of a physical object.

The starting point for an approach to quantum mechanics is the Schrödinger Equation.

$$-i\hbar \frac{\partial}{\partial t} \psi(\vec{x}, t) = H\psi(\vec{x}, t) \quad (2.2)$$

This differential equation describes the time evolution of the wave function  $\psi(\vec{x}, t)$ . The solution of the equation can be written in the form

$$\psi(\vec{x}, t) = e^{\frac{iHt}{\hbar}} \psi(\vec{x}, 0) = U(t, 0)\psi(\vec{x}, t) \quad (2.3)$$

Thus this equation implies a unitary evolution of our considered system. The function  $\psi(\vec{x}, t)$  determines the properties of the underlying physical object. However, this function is not accessible through measurement. The physical properties can be obtained by taking the absolute value squared of  $\psi(\vec{x}, t)$ . The result can be interpreted as a probability distribution of possible results. Within this description lies one major difference between quantum mechanics and classical mechanics, which is, that measurement results can only be predicted with a certain probability.

We will now outline the mathematical foundations we will mostly use in this work. We start off, with defining a Hilbert space  $\mathcal{H}$

**Definition 2.2.** *A Hilbert Space is a complete function space with a scalar product in  $\mathbb{C}$*

Every wave function satisfying the Schrödinger equation is an element of the Hilbert Space  $\mathcal{H}$ . These wave functions have to be normalizable in order to be

## 2 Mathematical Formalism

interpreted as probabilistic entities. The inner product of our Hilbert space is defined as

$$\langle \psi | \phi \rangle := \int_{-\infty}^{\infty} d^3x \psi^*(\vec{x}, t) \phi(\vec{x}, t) \quad (2.4)$$

Because our Hilbert space is a complex vector space, we can associate a vector, called state vector, with every wave function.

$$\begin{aligned} \langle x | \psi \rangle &= \psi(\vec{x}) \\ \langle \psi | x \rangle &= \psi^*(\vec{x}) \end{aligned} \quad (2.5)$$

The usual notation for state vectors in quantum mechanics is the Dirac notation, where  $\psi(\vec{x})$  corresponds to the vector  $|\psi\rangle$ , which is called "ket" (2.5). Also this way of describing a state and its properties allows for associating the dual vector of  $|\psi\rangle$ , which is written as  $\langle\psi|$  ("bra"), with the complex conjugate of a wave function. Thus, the inner product can be interpreted as the multiplication of vectors in our Hilbert space.

Now in order to describe operations on our physical objects, like measurements, we introduce linear operators acting on our Hilbert space.

**Definition 2.3.** *A is called a linear operator, if for  $A\psi_1(\vec{x}) = \phi_1(\vec{x})$  and  $A\psi_2(\vec{x}) = \phi_2(\vec{x})$ , where  $\psi_1, \psi_2, \phi_1, \phi_2 \in \mathcal{L}_2$ , follows that  $A(c_1\psi_1 + c_2\psi_2) = c_1\phi_1 + c_2\phi_2$   $c_1, c_2 \in \mathbb{C}$ .*

Such linear operators map states  $\psi$  onto states  $\phi$ .

$$A|\psi\rangle = |\phi\rangle \quad (2.6)$$

For every operator on our Hilbert Space  $\mathcal{H}$ , there corresponds an *adjoint* operator, which is defined as follows:

**Definition 2.4.** *An operator  $A^\dagger$  is called adjoint to  $A$  if there exists a set of  $|\psi\rangle \in \mathcal{H}$ , such that  $|\bar{\psi}\rangle \in \mathcal{H}$  with*

$$\langle \psi | A^\dagger | \alpha \rangle = \langle \bar{\psi} | \alpha \rangle$$

Another subclass of linear operators, the hermitian operators are exceedingly important. These operators are connected to the observables of a system, because of there property to only have real valued eigenvalues. Thus there eigenvalues can be linked to measurement results.

**Definition 2.5.** *Linear operator  $A$ :*

$$A|\alpha\rangle = A^\dagger|\alpha\rangle, \quad \forall |\alpha\rangle \in \mathcal{H}$$



## 2.1 Mathematical Description of Quantum Mechanics

We will now take a closer look at the measurement process [2]. A general form to define the measurement process is to introduce a set of operators  $\{M_m\}$  where the index  $m$  labels the possible measurement outcomes. If we now measure a state  $|\psi\rangle$  then the probability to measure outcome  $m$  is

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \quad (2.7)$$

And the resulting state after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad (2.8)$$

In order to be a complete measurement the operators have to fulfill that

$$\sum_m M_m^\dagger M_m = \mathbb{1} \quad (2.9)$$

There are two special cases of measurements we want to point out. First there are projective measurements (PVM = projective valued measurements). Projective measurements are defined by an observable which has the spectral decomposition

$$M = \sum_m m P_m \quad (2.10)$$

where  $P_m$  denotes projectors on the eigenspace of  $M$  with property  $P^2 = P$ . The probability to measure result  $m$  is given by

$$p(m) = \langle\psi|P_m|\psi\rangle \quad (2.11)$$

From this properties it can be seen, that projective measurement are a special case of the general definition of measurement. Projective measurements also fulfill the completeness relation

$$\sum_m P_m = \mathbb{1}. \quad (2.12)$$

Furthermore projectors are orthogonal.

$$P_m P_{m'} = \delta_{mm'} P_m \quad (2.13)$$

Another way of defining measurements are positive operator valued measurements or short POVM. In the general case you do have a description of the state

## 2 Mathematical Formalism

after the measurement. However, there are cases where the experimenter is not interested in the state after it has been measured. For situations like this, the POVM formalism is quite useful. We already know the probabilities of outcome  $m$  (2.7). We will now define an operator  $E_m$  such that

$$E_m = M_m^\dagger M_m. \quad (2.14)$$

$E_m$  is a positive operator and fulfills the relation  $\sum_m E_m = \mathbb{1}$ . These operators are the POVM elements. The projective measurements are a special case of POVM with the additional properties defined above. Now, why do we need the POVM formalism? Projective measurements have the property to be repeatable. That means that after a state is measured, it is in the eigenstate of the measurement result.

$$\frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}} \quad (2.15)$$

If we now measure again, we do not change the state anymore, since applying a projector twice because of the property  $P^2 = P$ . But we know that in nature there are measurement processes that can not be repeated. Thus the projective measurements cannot be a complete description of the measurement process. Let us look at an example that emphasizes the difference between POVM and PVM the distinguishability of quantum states. Quantum states can be reliably distinguished only if the considered states are orthogonal. That means if one party gets a set of states that are not orthogonal it cannot find a set of projectors that can reliably distinguish between those states, since a projector that measures one state, also has a parallel overlap with one or more of the other states. Thus there is a nonzero probability to measure the wrong state.

In the POVM formalism there is a way to distinguish nonorthogonal states, but not with efficiency one. An example of this would be the following. Suppose you have two states  $|0\rangle$  and  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . The following POVM elements can now be used

$$\begin{aligned} E_1 &= \frac{\sqrt{2}}{1 + \sqrt{2}}|1\rangle\langle 1| \\ E_2 &= \frac{\sqrt{2}}{2(1 + \sqrt{2})}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) \\ E_3 &= \mathbb{1} - E_1 - E_2 \end{aligned} \quad (2.16)$$

to distinguish with a certain probability if we have the state  $|0\rangle$  or the state  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . The advantage of this measurement is that there will be no errors in distinguishing a state. However if we get the measurement result connected to  $E_3$  we will have no information about what state we have.

Another important point about general measurements is that they can be carried out by introducing another system and only using projective measurements and unitary transformation on this composite system. This artificially introduced system is often called ancilla system. We will consider a unitary acting on the following state

$$U|\psi\rangle|0\rangle = \sum_m M_m|\psi\rangle|m\rangle \quad (2.17)$$

where the state  $|0\rangle$  is a fixed state of our ancillary system. This definition preserves the inner product of two states.

$$\langle\phi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_m \langle\phi|M_m^\dagger M_m|\psi\rangle = \langle\phi|\psi\rangle \quad (2.18)$$

Now if we consider projectors of the form  $P_m = \mathbb{1} \otimes \langle m||m\rangle$  we get

$$\begin{aligned} p(m) &= \langle\psi|\langle 0|U^\dagger P_m U|\psi\rangle|0\rangle = \\ &= \sum_{m',m''} \langle\psi|M_{m'}^\dagger \langle m'|(\mathbb{1} \otimes \langle m||m\rangle)M_{m''}|\psi\rangle|m''\rangle = \\ &= \langle\psi|M_m^\dagger M_m|\psi\rangle \end{aligned} \quad (2.19)$$

Thus, every general measurement can be done by introducing an ancillary system and performing a projective valued measurement and unitary transformations. This result is very important for carrying out experiments, since it gives a method to perform such general measurements in the lab.

## 2.6 Pure States

If we write a state vector for a system with  $n$  degrees of freedom, the corresponding Hilbert space  $\mathcal{H}$  is  $n$ -dimensional (we restrict ourselves to the finite dimensional case). Every pure state can be described by a vector of this Hilbert space.

Now if we want to consider composite systems of more than one particle with different degrees of freedom, we use the Kronecker product to obtain a Hilbert space  $\mathcal{H}_{n \times m} = \mathcal{H}_n \otimes \mathcal{H}_m$ . A state in this vector space can then be written as

$$|\psi\rangle_{(n \times m)} = \sum_i c_{ij} |\phi_1\rangle_{i(n)} \otimes |\phi_2\rangle_{j(m)} \quad (2.20)$$

where  $|\phi_1\rangle$  and  $|\phi_2\rangle$  are a CONS of their Hilbert space. There also exists a special decomposition, if the dimensions of the subsystems are equal, the Schmidt decomposition, which states that

$$|\psi\rangle_{(n \times m)} = \sum_i c_i |\chi_1\rangle_{i(n)} \otimes |\chi_2\rangle_{i(m)}. \quad (2.21)$$

Thus only the dimension of the subsystem determines the number of basisvectors needed. The proof uses the singular value theorem which states that

$$C = UC^{\text{diag}}V \quad (2.22)$$

for unitary matrices  $U, V$ . Thus

$$\sum_i c_{ij} |\phi_1\rangle_i \otimes |\phi_2\rangle_j = \sum_i u_{ik} c_{kk} v_{kj} |\phi_1\rangle_i \otimes |\phi_2\rangle_j = c_i |\chi_1\rangle_i \otimes |\chi_2\rangle_i \quad (2.23)$$

For spin- $\frac{1}{2}$  or photon polarization systems, we have a composite system of two dimensional Hilbert spaces  $\mathcal{H}_2$ .

Thus, the following states are also physically realizable

$$\begin{aligned} |\psi^-\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle) \\ |\psi^+\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \\ |\phi^-\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle - |\downarrow\rangle \otimes |\downarrow\rangle) \\ |\phi^+\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\uparrow\rangle + |\downarrow\rangle \otimes |\downarrow\rangle) \end{aligned} \quad (2.24)$$

These states are known as Bell states and are the first evidence of "quantum" behaviour, because those state vectors cannot be decomposed into a tensor product of two pure states in the subsystems and thus can only be considered as one quantum object. We will have a go at this in a later chapter. As already mentioned, the state vector notation does not suffice to describe ensembles of states because linear combinations of pure states again lead to pure states.

## 2.7 Mixed States

So far we have established a description of a physical state, but it turns out that such a description cannot be used to ensembles of states, where we do not have every information about the system. Such systems are known as mixed states. We will therefore introduce the density matrix formalism as a way of describing such physical systems. First our new formalism must be able to describe pure states as well. Thus, a density matrix for pure states is defined as follows

$$\rho := |\psi\rangle\langle\psi| \quad (2.25)$$

In order to be a physical state, a density matrix has to fulfill the following relations

$$\begin{aligned}\rho^\dagger &= \rho \\ \text{Tr}(\rho) &= 1 \\ \rho^2 &= \rho \Leftrightarrow \text{Pure state} \\ \rho &\geq 0\end{aligned}\tag{2.26}$$

To achieve an ensemble of values we take the convex sum of density matrices of pure states

$$\rho_{\text{mixed}} = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad p_i \geq 0, \quad \sum_i p_i = 1\tag{2.27}$$

As can be easily seen, all properties for density matrices (Eq. (2.26)) are fulfilled. It is important to note that the factors  $p_i$  are the probability to find this mixed state in a certain pure state after a measurement. So if you try to measure a mixed state, you cannot predict with certainty the outcome of your measurement, no matter how you adjust your measuring apparatus. This is the key difference to a pure state, where there is always one measurement direction, in which you will get a certain measurement result with a probability of 1. This can be easily seen by the fact that a pure state is also a projecting operator.

Because this formalism of quantum mechanics is not bound to certain properties of a state, they can be used for every kind of degree of freedom. Examples would be spin, polarization, energy, momentum or spatial degrees of freedom. The description of spin systems or polarizations in this formalism becomes quite straight forward, because of their finite number of degrees of freedom. Therefore, the Hilbert Schmidt Space becomes finite dimensional.

## 2.8 Bloch Decomposition

Density matrices can be decomposed into different basis. We like to introduce one decomposition for later use, the Bloch decomposition. Because of the hermiticity of density matrices, there exists a basis of hermitian operators which reproduce a density matrix through linear combination with real coefficients. In the case of qubits there exists a unique basis, the Pauli matrices.

$$\begin{aligned}\sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\end{aligned}\tag{2.28}$$

where  $\sigma_0$  is the unity matrix. Thus a state can be written as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{a}\vec{\sigma}) \quad (2.29)$$

The vector  $\vec{a}$  is called the Bloch vector. The length of the Bloch vector determines whether the state is a pure state or not. Since for a pure state the trace squared has to be one, it follows that

$$\begin{aligned} \text{Tr}\rho^2 &= \text{Tr} \left[ \frac{1}{4}(\mathbb{1}_2 + \vec{a} \cdot \vec{\sigma})(\mathbb{1}_2 + \vec{a} \cdot \vec{\sigma}) \right] = \\ &= \text{Tr} \left[ \frac{1}{4}(\mathbb{1}_2 + 2\vec{a} \cdot \vec{\sigma} + \vec{a} \cdot \vec{\sigma} \vec{a} \cdot \vec{\sigma}) \right] = \\ &= \frac{1}{2} \left( 1 + |\vec{a}|^2 \right) \end{aligned} \quad (2.30)$$

If  $|\vec{a}| = 1$  the state is a pure state and if  $|\vec{a}| < 1$  it is a mixed state. The property  $\text{Tr}(\rho) = 1$  is automatically fulfilled through the use of the unity matrix in Eq. (2.29), because the Pauli matrices are traceless. A nice feature of this is

We can also extend this to two particles using the tensor product.

$$\rho = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{u} \cdot \vec{\sigma} + \sum_{i,j=1}^3 t_{ij} \sigma_i \otimes \sigma_j) \quad (2.31)$$

Note that we have now fifteen degrees of freedom for a general two qubit state. Also a separable state (see chapter: "Entanglement") can be written as a Bloch decomposition

$$\rho = \sum_k p_k \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{u} \cdot \vec{\sigma} + \sum_{i,j=1}^3 r_i u_j \sigma_i \otimes \sigma_j) \quad (2.32)$$

where the weighted sum comes from Eq. (2.27). The tensor product terms can be achieved by taking the tensor product of two states of the form (2.29).

## 3 Quantum Entropies

### 3.1 Shannon Entropy

The Shannon Entropy is the basic concept of classical information theory [3, 4]. There are two interpretation of the Shannon Entropy. On the one hand, it is the uncertainty we have before measuring a quantity  $X$ . On the other hand, it is the amount of information we gather after measuring the quantity  $X$ . This information content is stored in the probability of the quantity  $X$  to occur. Thus the definition of the Shannon Entropy is

$$H(X) = H(p_1, \dots, p_n) = \sum_i p_i \log p_i \quad (3.1)$$

This definition can be justified by the following assumptions on the Shannon entropy.

- The Information measure for an event  $E$  must only depend on its probability to occur. Thus  $I(E) = I(p)$  with  $p$  being the probability.
- $I$  is a smooth function of probability.
- $I(pq) = I(p) + I(q)$

Especially the last point is important since it ensures the independence of entropy of two outcomes. This means that if we have two or more independent event with probabilities  $p_1, p_2, \dots$  then the entropy is the sum of the entropy of every single event. Thus the logarithmic dependence occurs.

### 3.2 Renyi Entropy

The Renyi entropy [5] is a generalization of the Shannon entropy. By relaxing the assumption of additivity, we can define the Renyi entropy as

$$R_q = \frac{1}{1-q} \log \sum_i p_i^q. \quad (3.2)$$

For  $q \rightarrow 1$  this definition creates the form  $\frac{0}{0}$ . So we have to take the limit of the entropy. In order to do this, we apply l'Hopital's theorem

$$\begin{aligned}
 \lim_{q \rightarrow 1} R_q &= \lim_{q \rightarrow 1} \frac{\log \sum_i p_i^q}{(1-q)} = \lim_{q \rightarrow 1} \frac{\frac{d}{dq} \log \sum_i p_i^q}{\frac{d}{dq} (1-q)} = \lim_{q \rightarrow 1} - \frac{\sum_i \frac{d}{dq} p_i^q}{\sum_i p_i^q} = \\
 &= \lim_{q \rightarrow 1} - \frac{\sum_i p_i^q \log p_i}{\sum_i p_i^q} = - \sum_i p_i \log p_i.
 \end{aligned} \tag{3.3}$$

### 3.3 Von Neumann Entropy

The quantum analogue to the Shannon entropy is the von Neumann entropy [6]. It is defined as follows

$$S(\rho) = -\text{Tr} \rho \log \rho = - \sum_i^D \lambda_i \log \lambda_i \tag{3.4}$$

where  $\lambda_i$  denotes the  $i$ -th eigenvalue of  $\rho$  and  $D$  denotes the dimension of the system. This definition ensures that every pure state has the entropy zero, since the only eigenvalue that would contribute is 1.

$$S(\rho_{\text{pure}}) = -1 \log 1 - \sum_i^{D-1} 0 \log 0 = 0 \tag{3.5}$$

On the other end we have the maximally mixed state  $\frac{1}{D} \mathbb{1}_D$ . The entropy for this state is

$$S(\rho_{\text{mixed}}) = - \sum_i^D \frac{1}{D} \log \frac{1}{D} = \sum_i^D \frac{1}{D} \log D = \log D \tag{3.6}$$

So the von Neumann entropy lies between  $0 \leq S(\rho) \leq \log D$ . It is important to note that the von Neumann entropy is concave. So the inequality

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i) \tag{3.7}$$

holds. Furthermore, the difference between the left and the right part of inequality (3.7) is the Shannon entropy of our probabilities  $p_i$ .

$$S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i) \tag{3.8}$$

The von Neumann entropy of a state is a measure of the mixedness of the state. However, entropies can also be used to measure entanglement if applied to the subsystems. We will look at this in the chapter about entanglement.



### 3.4 Quantum Relative Entropy

The relative entropy is a measure of the closeness of two quantum states. The definition goes as follows

$$S(\rho||\sigma) = \text{Tr} \rho \log \rho - \text{Tr} \rho \log \sigma \quad (3.9)$$

The quantum relative entropy is always non-negative. This is known as the Klein inequality.

$$S(\rho||\sigma) \geq 0 \quad (3.10)$$

Also the entropy of a density matrix is smaller or equal the added entropy of the subsystems. This can be seen from the relative entropy if we consider two states  $\rho = \rho^{AB}$  and  $\sigma = \rho^A \otimes \rho^B$ .

$$S(\rho^{AB}||\rho^A \otimes \rho^B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}) \geq 0 \quad (3.11)$$

Also, the entropy of a system has a lower bound.

$$|S(\rho^A) - S(\rho^B)| \leq S(\rho^{AB}) \quad (3.12)$$

This bound is called the Araki-Lieb bound.

The linear entropy seems to be a distance measure of two density matrices. However it lacks certain properties, we would expect from a distance function. First and most importantly, it is not independent of the ordering of  $\rho$  and  $\sigma$ , namely

$$S(\rho||\sigma) \neq S(\sigma||\rho). \quad (3.13)$$

Thus it does not define an appropriate metric. Furthermore, it does not fulfill the triangle inequality.

### 3.5 Linear Entropy

The linear entropy is an approximation of the von Neumann entropy. The term  $\log \rho$  is replaced by the first order term of the Taylor series  $(\rho - 1)$ .

$$S_{lin} = -\text{Tr}(\rho(\rho - 1)) = \text{Tr}(\rho - \rho^2) = 1 - \text{Tr} \rho^2 = 1 - \sum_i \lambda_i \quad (3.14)$$

### 3 Quantum Entropies

It can be interpreted as a measure for the mixture of a state. Because of the properties of density matrices, for a pure state the linear entropy is zero. For the maximally mixed state it yields

$$S_{lin}(\rho_{mix}) = 1 - \frac{1}{D}. \quad (3.15)$$

Also the linear entropy can be connected to the Renyi entropy. For the case of  $q = 2$  the Renyi entropy takes the form

$$R_2 = -\log \sum_i \lambda_i^2 \quad (3.16)$$

therefore the linear entropy can be written as

$$S_{lin} = 1 - e^{R_2} \quad (3.17)$$

## 4 Entanglement

### 4.1 Entanglement of Pure States

A pure state is called separable iff it can be written as a tensor product of state vectors

$$\begin{aligned} |\psi\rangle_{AB} &= |\psi\rangle_A \otimes |\psi\rangle_B \Leftrightarrow \text{separable} \\ \rho_{AB} &= \rho_A \otimes \rho_B \end{aligned} \quad (4.1)$$

where  $|\psi\rangle_{AB}$  and  $\rho_{AB} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ . Entanglement is then defined as every state that cannot be written in such a way. To test whether a pure state is entangled or not is straightforward.

We just have to test if one of the subsystems is a mixed state. To determine the mixedness of a subsystem we introduce the following relations. First we need the partial trace of a density matrix, which is defined as follows

$$\begin{aligned} \rho_{kl,ij} &= (\langle k|_A \otimes \langle i|_B) \rho (|l\rangle_A \otimes |j\rangle_B) \\ \text{Tr}(\rho)_A &= \sum_i \rho_{kl,ii} \end{aligned} \quad (4.2)$$

where two sets of indices are used for the two subsystems and the ket vectors are CONS of the respective Hilbert spaces. Now in order to determine if the subsystem is mixed, we have to check whether  $\text{Tr}(\rho_A^2) < 1$ . If this is the case, then the subsystem is mixed therefore the composite system is entangled.

### 4.2 Entanglement of Mixed States

We can also define entanglement for mixed states. However in this case we can only define it via density matrices. It is quite natural to assume, that if a pure state is separable, iff it can be written as a tensor product of pure states of the subsystems, a separable mixed state must be a convex sum of pure separable states [7].

$$\rho_{\text{sep}} = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad (4.3)$$

As with pure states, an entangled mixed state cannot be written in that way. However, in the case of mixed states, the detection of an entangled state is far

more complicated. The criterion to check if the subsystem is mixed or not, does not work for mixed composite systems.

An easy example for this is the unity matrix which can be interpreted as a maximally mixed state. (Compare also to the definition of a Bloch decomposition, where the maximally mixed state occurs for  $|\vec{a}| = 0$ ) The partial trace of such a state is also a maximally mixed state in the subsystem. But such a state can clearly not be entangled.

To detect entanglement for mixed states, we need to introduce better criteria.

### 4.3 Positive Partial Transpose Criterion

Another criterion for entanglement detection is the PPT criterion (Positive Partial Transpose). It was first discovered by Asher Peres [8] and proven by Horodecki [9, 10]. This criterion uses properties of positive and completely positive maps which we will hereby introduce.

**Definition 4.4.** A map  $\Gamma : \mathcal{H} \rightarrow \mathcal{H}$  is called positive iff it maps positive operators onto positive operators.

$$A \geq 0 \Rightarrow \Gamma(A) \geq 0 \quad \forall A \in \mathcal{H}$$

Now to complete positivity:

**Definition 4.5.** A map  $\Gamma$  is called completely positive iff it is positive under every extension to higher dimensions.

$$A \geq 0 \Rightarrow (A \otimes \mathbb{1}_k) \geq 0 \quad \forall A \in \mathcal{H} \quad \forall k \in \mathbb{N}^+$$

The property of positivity can be used to create a criterion for entanglement detection because not all maps that are positive are also completely positive. In fact, a positive map that is extended to higher dimensions always preserves positivity for a separable state.

$$(\Gamma \otimes \mathbb{1})\rho_{sep} = \sum_i p_i \Gamma \rho_i^A \otimes \rho_i^B \geq 0 \quad (4.4)$$

But in the case of entangled states this need not be true. In fact one can always find a positive but not completely positive map, such that an entangled state becomes non positive.

$$(\Gamma \otimes \mathbb{1})\rho_{ent} < 0 \quad (4.5)$$

Now what kind of map is useful for detecting entanglement? This question now leads back to the discovery of Asher Peres, that the transposition, which is a positive map (but not completely positive) fulfills this task, at least for  $\mathcal{H}_2 \otimes \mathcal{H}_2$  and  $\mathcal{H}_2 \otimes \mathcal{H}_3$  in a sufficient way. For higher dimensions it is only a necessary

criterion for entanglement, but not a sufficient one. The reason why this is also a sufficient criterion in lower dimensions lies in the fact, that a positive map can be decomposed in the following way

$$\Gamma = \Gamma_1^{CP} + \Gamma_2^{CP}T \quad (4.6)$$

which leads to

$$(\Gamma \otimes \mathbb{1})\rho = (\Gamma_1^{CP} \otimes \mathbb{1})\rho + (\Gamma_2^{CP} \otimes \mathbb{1})(T \otimes \mathbb{1})\rho. \quad (4.7)$$

This relation can only be non positive, if  $(T \otimes \mathbb{1})\rho < 0$ . We will rely on this criterion for our analysis of entanglement in the later chapters. One last point to mention is, that for higher dimensions the PPT criterion only detects so-called *bound entangled states*.

## 4.6 Reduction Criterion

The reduction criterion is another example of a positive but not completely positive map, that can be used to detect entanglement [11]. As the PPT criterion it is a necessary and sufficient criterion for entanglement in the dimensions  $\mathcal{H}_2 \otimes \mathcal{H}_2$  and  $\mathcal{H}_2 \otimes \mathcal{H}_3$ . The corresponding map looks the following

$$\Gamma(\rho) = \mathbb{1}Tr(\rho) - \rho \quad (4.8)$$

and is applied to one of the subsystems. To be more precise

$$\begin{aligned} \rho^A \otimes \mathbb{1} - \rho &\geq 0 \Leftrightarrow \rho \text{ is sep.} \\ \mathbb{1} \otimes \rho^B - \rho &\geq 0 \Leftrightarrow \rho \text{ is sep.} \end{aligned} \quad (4.9)$$

This criterion however weaker than the PPT criterion, has the special property to find states that are for certain *distillable*. We will investigate distillability later on.

## 4.7 Entanglement Witnesses

A geometric way of detecting entanglement are entanglement witnesses. They first were considered in [12]. An entanglement witness defines a hyperplane in the Hilbert Schmidt space that has the following properties

$$\langle \rho_{ent}, A \rangle = Tr \rho_{ent} A < 0 \quad (4.10)$$

and

$$\langle \rho_{sep}, A \rangle = \text{Tr} \rho_{sep} A \geq 0 \quad \forall \rho_{sep}. \quad (4.11)$$

These properties arise from the Hahn Banach Theorem which states that any convex compact set  $\mathcal{A}$  can be separated via a hyper-plane from an element  $b \notin \mathcal{A}$ . An easy example of this is a plane in euclidean geometry. We can now use the scalar product to distinguish between states on the upper side of the plane and states on the lower side of the plane. For example if we choose our defining orthogonal vector of the plane to point down, than any scalar product with this vector and another vector with direction down is positive, whereas products with vectors with direction up are negative. This concept was applied to the above properties of an entanglement witness (4.10), (4.11).

We can optimize this witness by letting it touch the convex set of separable states. This is called an optimal entanglement witness or tangent functional. So such a witness has to fulfill

$$\langle \rho_s, A_{opt} \rangle = 0 \quad (4.12)$$

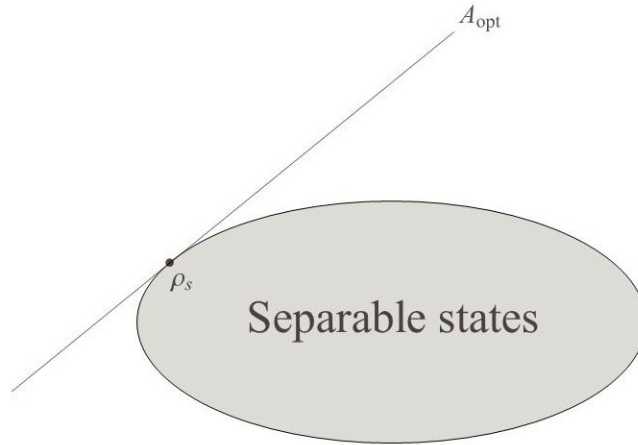


Figure 4.1: The optimal entanglement witness touches the set of separable states. There exists one state  $\rho_s$  which is called the nearest separable state to a specific entangled state.

where  $\rho_s$  is a separable state on the hull of the convex set (see Fig. 4.1). We need to be careful however, since the entangled states are not a convex set. Thus there exist entangled states for which (4.10) is still positive. Therefore we have to check if our witness is suited for the considered entangled state. We will look into this in the next chapter.

# 5 Entanglement Measures

## 5.1 What is an Entanglement Measure?

For now we only introduced ways to determine whether a state is entangled or not, but it may also be useful to be able to quantify the amount of entanglement within a state. For this purpose a lot of ideas were introduced to do so. Before we look into specific methods to measure entanglement, we will pursue the question of what an entanglement measure should fulfill in order to be rightfully called as such [13].

1.  $\rho$  is sep.  $\Leftrightarrow E(\rho) = 0$   
*If  $\rho$  is a separable state the entanglement measure needs to be zero.*  
The first and very natural condition ensures that only entangled states are shown by the measure.
2.  $E(\rho) \geq E(\Gamma_{LOCC}(\rho))$   
*The entanglement measure is non-increasing under LOCC*  
This statement ensures, that entanglement cannot be increased by actions only performed in the subsystems respectively. Moreover entanglement can also not be created by such local operations.
3.  $E(x\rho + (1-x)\sigma) \leq xE(\rho) + (1-x)E(\sigma)$  with  $x \in [0, 1]$   
*The entanglement of two states mixed together can not be greater than the entanglement of both states added separately.*  
This statement comes from the fact that the set of separable states is convex and therefore the mixing of two separable states cannot be entangled.
4.  $\|\rho_2 - \rho_1\| \rightarrow 0 \Rightarrow E(\rho_1) - E(\rho_2) \rightarrow 0$   
*If the norm difference of  $\rho_1$  and  $\rho_2$  goes to zero, the entanglement of those states also has to go to zero.*  
This condition ensures that the entanglement measure changes only infinitesimal if the state is infinitesimally shifted.
5.  $E(\rho \otimes \sigma) = E(\rho) + E(\sigma)$   
*The entanglement of the composite system of  $\rho$  and  $\sigma$  is equal to the entanglement of the two systems respectively.*  
This property is sometimes considered to be too strong. It can be replaced by subadditivity ( $E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma)$ ).
6.  $E(\rho^-) = 1$   
*The entanglement of a maximally entangled state is 1*

## 5 Entanglement Measures

This condition is quite easy to fulfill for most measures by introducing a proper normalization.

7. *An optimal entanglement measure is efficiently computable for every state in a reasonable time.*

So far no measure is known, that could fulfill all of these properties. Especially the computability is a main issue in higher dimensions. But also other conditions prove to be hard to embed in certain entanglement measures.

### 5.2 Entanglement Measures for Pure States

We already introduced a separability criterion for pure states, the mixedness of the subsystems. This can also be used to quantify the entanglement of a state. The more the subsystems are mixed, the more entangled the state must be.

Another way of quantifying entanglement for pure states is to look at the following way to write down a general separable pure state

$$|\psi\rangle = \sum_{i,j=1}^n (a_i|i\rangle) \otimes (b_j|j\rangle) \quad (5.1)$$

where  $n$  denotes the dimension of the subsystems. Thus the two coefficients  $a_i, b_j$  can be considered as a matrix  $\Lambda_{ij}$  which has to satisfy that

$$\det\Lambda = \det \begin{pmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{pmatrix} = a_1b_1a_2b_2 - a_2b_1a_1b_2 = 0 \quad (5.2)$$

A general state can be written as

$$|\psi\rangle = \frac{1}{n} \sum_{i,j=1}^n \Lambda_{ij}|i\rangle \otimes |j\rangle \quad (5.3)$$

Now in the case of two qubits one can show, that the determinant of  $\Lambda$  is related to the concurrence [45].

$$C = 2|\det\Lambda| \quad (5.4)$$

A thorough description of the concurrence will be done later on.

### 5.3 Entanglement of Formation

The first measure of entanglement for all states we want to look at, is the entanglement of formation [14]. As we have already discussed the entanglement



of pure states can be measured by looking at the entropy of the subsystems. The entanglement of formation is a generalization of this idea.

$$E_{EOF}(\rho) = \min_{\kappa} \sum_i p_i \mathcal{S}(|\psi_i\rangle\langle\psi_i|) \quad (5.5)$$

The minimization of  $\kappa$  means a minimization taken over all possible decompositions of the density matrix  $\rho$ . For example the maximally mixed state for two qubits can be decomposed into the four Bell states.

$$\mathbb{1}_4 = \frac{1}{4}(|\psi^-\rangle\langle\psi^-| + |\psi^+\rangle\langle\psi^+| + |\phi^-\rangle\langle\phi^-| + |\phi^+\rangle\langle\phi^+|) \quad (5.6)$$

This would clearly give the wrong notion for entanglement. A minimizing decomposition would be

$$\mathbb{1}_4 = \frac{1}{4}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|) \quad (5.7)$$

where we chose the spin notation for the standard basis. In general there exist infinitely many decompositions of a density matrix, which makes the entanglement of formation not computeable in every case. Thus, though it may be a very elegant way of measuring and describing entanglement, it is not a practical solution.

## 5.4 Entanglement of Distillation

Entanglement of distillation [14] is defined as follows

$$E_{EOD}(\rho) = \lim_{n \rightarrow \infty} \frac{m}{n}. \quad (5.8)$$

$m$  denotes the number of maximally entangled states that can be extracted from a given number of copies  $n$  of state  $\rho$  via a LOCC protocol. This measure is especially important for quantum cryptography, since it gives the number of states out of a certain number of copies that can be used for further tasks. This measure is hard to compute, but there exist bounds that can be calculated as well as an algorithmic way of deriving it.

## 5.5 Entanglement cost

Another operational way to measure entanglement is the entanglement cost. It is in some way the reverse of the entanglement of distillation.

$$E_{EC}(\rho) = \lim_{n \rightarrow \infty} \frac{m}{n} \quad (5.9)$$

## 5 Entanglement Measures

where the number  $m$  denotes the states  $|\psi^-\rangle$  that are needed to produce the state  $\rho$ . An interesting fact about the entanglement cost is, that it coincides with the *regularized* entanglement of formation.

$$E_{REOF}(\rho) = \lim_{n \rightarrow \infty} \frac{E_{EOF}(\rho^{\otimes n})}{n} \quad (5.10)$$

However, it is not known if entanglement of formation is additive, so it is not clear if entanglement cost and entanglement of formation itself coincide.

### 5.6 EOF: Concurrence

Another very important measure of entanglement, especially for two qubit states is the concurrence [15]. The concurrence will be one of the main entanglement measures used in this work, since it is one of the easiest to calculate. Therefore we will give a more thorough explanation of how it works.

The concurrence is directly related to the entanglement of formation. We have already introduced the connection for concurrence and the coefficient matrix  $\Lambda$  in Eq. (5.4). This equation can be rewritten in terms of the Schmidt vector components (2.21) of the pure state

$$|\psi\rangle = \sum_{i,j=1}^2 \Lambda_{ij} |i\rangle \otimes |j\rangle \quad (5.11)$$

$$|\psi\rangle = \sum_{i=1}^2 \nu_i |i\rangle \otimes |i\rangle \quad (5.12)$$

with  $\nu_i$  real coefficients. Thus Eq. (5.4) becomes

$$C = 2\nu_1\nu_2 \quad \Rightarrow \quad C^2 = 4\nu_1^2\nu_2^2 \quad (5.13)$$

for normalization reasons it has to hold that  $\nu_1^2 + \nu_2^2 = 1$  and Eq. (5.13) can be written as

$$C^2 = 4\nu_1^2(1 - \nu_1^2). \quad (5.14)$$

Now inverting relation (5.14) leads to

$$\nu_1^2 = \frac{1 - \sqrt{1 - C^2}}{2}. \quad (5.15)$$

So for the entanglement of a pure state we calculate the entropy of a subsystem

$$E(|\psi\rangle) = - \sum_{i=1}^2 \nu_i^2 \ln \nu_i^2 \quad (5.16)$$

which is just the definition of the entanglement of formation for pure states.

We will now look at a special basis in which a two qubit state can be written, the *magic basis*. This basis consists of all four bell states.

$$|\psi\rangle = (\mu_1|\phi^+\rangle + i\mu_2|\phi^-\rangle + i\mu_3|\psi^+\rangle + \mu_4|\psi^-\rangle) \quad (5.17)$$

This is equal to

$$|\psi\rangle = \frac{1}{\sqrt{2}} [(\mu_1 + i\mu_2)|\uparrow\uparrow\rangle + (i\mu_3 + \mu_4)|\uparrow\downarrow\rangle + (i\mu_3 - \mu_4)|\downarrow\uparrow\rangle + (\mu_1 - i\mu_2)|\downarrow\downarrow\rangle]. \quad (5.18)$$

Now according to Eq. (5.4) the concurrence reads

$$C(|\psi\rangle) = |\sum_{i=1}^4 \mu_i^2| \quad (5.19)$$

Note that if all the coefficients in (5.18) are real (i.e.  $\mu_2, \mu_3 = 0$ ), the concurrence is one and therefore the state is maximally entangled.

We can apply the following transformation to our state.

$$|\psi\rangle \rightarrow |\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle \quad (5.20)$$

One feature of the magic basis is, that under  $\sigma_y \otimes \sigma_y$  transformations it only becomes complex conjugated. This leads to the notation of concurrence

$$C|\psi\rangle = |\langle\psi|\tilde{\psi}\rangle|. \quad (5.21)$$

This special transformation also only complex conjugates density matrices (i.e. mixed states) if they are written in the magic basis. Thus, the concurrence for mixed states is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4) \quad (5.22)$$

where  $\lambda_i$  denote the squareroots of the eigenvalues of the matrix  $\rho\tilde{\rho}$  in decreasing order. Because for pure states there is only one positive eigenvalue of this matrix, this definition is equal to Eq. (5.21).

## 5 Entanglement Measures

Now we want to show, that this definition of concurrence is indeed a measure for entanglement. Let us first define the decomposition of  $\rho$

$$\rho = \sum_{i=1}^4 |\omega_i\rangle\langle\omega_i| \quad (5.23)$$

where  $|\omega\rangle$  is a subnormalized eigenstate of  $\rho$ . Subnormalization means that  $||\omega_i\rangle|^2|$  is equal to the  $i$ -th eigenvalue of  $\rho$ . Also the flipped states  $|\tilde{\omega}_i\rangle$  are eigenstates of  $\tilde{\rho}$ . Now the matrix  $\Omega\tilde{\Omega}$  ( $\Omega_{ij} = \langle\omega_i|\omega_j\rangle$ ) has the same spectrum as  $\rho\tilde{\rho}$ .

We can also find a unitary transformation such that we can diagonalize  $\Omega\tilde{\Omega}$  and with this define another decomposition

$$|x_i\rangle = \sum_{j=1}^4 U_{ij} |\omega_j\rangle \quad (5.24)$$

The matrix created by those vectors has diagonal form. Because the matrix  $\Omega$  is symmetric we can make sure that the eigenvalues of  $X_{ij} = \langle x_i|x_j\rangle$  are the squareroots eigenvalues of  $\Omega\tilde{\Omega}$  by adding relative phases. We will now show, that if  $C=0$  (Eq. (5.22)), it has to hold that  $\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \leq 0$  and we can find phases such that

$$\sum_{j=1}^4 e^{2i\zeta_j} \lambda_j = 0 \quad (5.25)$$

The next decomposition we want to introduce now is therefore as follows

$$\begin{aligned} |z_1\rangle &= \frac{1}{2} \left( e^{i\zeta_1} |x_1\rangle + e^{i\zeta_2} |x_2\rangle + e^{i\zeta_3} |x_3\rangle + e^{i\zeta_4} |x_4\rangle \right) \\ |z_1\rangle &= \frac{1}{2} \left( e^{i\zeta_1} |x_1\rangle + e^{i\zeta_2} |x_2\rangle - e^{i\zeta_3} |x_3\rangle - e^{i\zeta_4} |x_4\rangle \right) \\ |z_1\rangle &= \frac{1}{2} \left( e^{i\zeta_1} |x_1\rangle - e^{i\zeta_2} |x_2\rangle + e^{i\zeta_3} |x_3\rangle - e^{i\zeta_4} |x_4\rangle \right) \\ |z_1\rangle &= \frac{1}{2} \left( e^{i\zeta_1} |x_1\rangle - e^{i\zeta_2} |x_2\rangle - e^{i\zeta_3} |x_3\rangle + e^{i\zeta_4} |x_4\rangle \right). \end{aligned} \quad (5.26)$$

If the state is decomposed in such a way, then not only has it the same spectrum as  $\rho$  but also every pure state  $|z_i\rangle$  is separable, because  $\langle z_i|\tilde{z}_i\rangle = 0$ . Thus the state  $\rho$  has to be separable as well.

If we now look at the second case, where  $\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 > 0$  We have to look at a different decomposition, namely

$$\begin{aligned} |y_1\rangle &= |x_1\rangle \\ |y_2\rangle &= i|x_2\rangle \end{aligned}$$

$$\begin{aligned} |y_3\rangle &= i|x_3\rangle \\ |y_3\rangle &= i|x_3\rangle. \end{aligned} \quad (5.27)$$

The trace of the corresponding matrix takes the form

$$TrY = \sum_{i=1}^4 \langle y_i | \tilde{y}_i \rangle = \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \quad (5.28)$$

Now we are only left with the task to transform our vectors  $|y_i\rangle$  in such a way, that they all have the same concurrence. Thus, we again use unitary operations.

$$|z_i\rangle = \sum_{j=1}^4 U_{ij} |y_j\rangle \quad (5.29)$$

However in order to preserve the form of Eq. (5.28), we have to restrict ourselves to real unitary matrices.

$$TrZ = \sum_{i=1}^4 \langle z_i | \tilde{z}_i \rangle = \sum_{i=1}^4 (UYU^T)_{ii} = TrUYU^T \quad (5.30)$$

With such real unitary  $U$ 's we can create states  $|z_i\rangle$  with the same concurrence as  $C(\rho)$  for all  $i$ 's. This can be done by rotating only the two extremal states of  $\{|z_i\rangle\}$  to give one of them the same concurrence as  $C(\rho)$ . Now repeat this process by picking out again two of the states that are not yet at concurrence  $C(\rho)$ . Thus at the end, all the states share the same concurrence and therefore the whole state has the same concurrence.

## 5.7 Distance Measures

### 5.7.1 Hilbert Schmidt Measure

Another straight forward way to define an entanglement measure, is the Hilbert Schmidt measure [16]. Basically it uses the inner product of the Hilbert Schmidt space. We already know, that the set of separable states is convex. Thus if we have an entangled state  $\rho_{ent}$ , we can find a unique nearest separable state for this particular state.

Now the Hilbert Schmidt distance between these two states becomes an entanglement measure.

$$E(\rho_{ent}) = \min_{\rho_S \in S} ||\rho_{ent} - \rho_S|| \quad (5.31)$$

This measure is also strongly related to entanglement witnesses which will be discussed in the next section.

### 5.7.2 Quantum Relative Entropy

We have already introduced the relative entropy (3.9) [13] which is a measure to distinguish between two given states in the sense of statistical distinguishability.

Now in analogy to the Hilbert Schmidt distance, if  $\sigma$  is a nearest separable state, this distance measure can be used as an entanglement measure.

$$E(\rho_{ent}) = \min_{\rho_S \in S} \text{Tr}(\rho \log \rho - \rho \log \rho_S) \quad (5.32)$$

However this measure is called a distance, it does not satisfy the metric properties (e.g.  $d(\rho, \sigma) \neq d(\sigma, \rho)$ ).

A very nice feature of this measure is that for pure states it reduces to the von Neumann entropy of the subsystems which was proven by Vedral and Plenio [17]. We want to show now a special case of this property, namely the case of maximally entangled states for spin- $\frac{1}{2}$  systems. We start with the state

$$\rho_{ent} = |\phi^+\rangle\langle\phi^+| \quad (5.33)$$

where  $|\phi^+\rangle$  is the maximally entangled state defined in (2.24). The relative entropy measure now becomes

$$E(\rho_{ent}) = \min_{\rho_S \in S} \text{Tr}(\rho_{ent} \log \rho_{ent} - \rho_{ent} \log \rho_S) = \min_{\rho_S \in S} (-\text{Tr}(\rho_{ent} \log \rho_S)) \quad (5.34)$$

Now the trace  $\text{Tr}(\rho_{ent} \log \rho_S)$  can be rewritten as

$$\text{Tr}(\rho_{ent} \log \rho_S) = \langle\phi^+| \ln \rho_S |\phi^+\rangle \geq \ln \langle\phi^+| \rho_S |\phi^+\rangle. \quad (5.35)$$

The last part of (5.35) follows from the convexity of the logarithm. It is known [18] that the quantity  $\langle\phi^+| \rho_S |\phi^+\rangle$  can not exceed  $\frac{1}{2}$  for any  $\rho_S$  and reaches this value only for maximally entangled states. Thus the entanglement measure becomes

$$E(\rho_{ent}) = \ln 2 \quad (5.36)$$

which coincides with the von Neumann entropy of the subsystem.

## 5.8 Entanglement Witnesses as an Entanglement Measure

We have already introduced entanglement witnesses as an entanglement criterion. However they can also be used to measure entanglement because they are related to the Hilbert Schmidt distance [19][20][21]. One way to do this is to define an operator

$$C = \frac{\rho_1 - \rho_2 - \langle \rho_1, \rho_1 - \rho_2 \rangle \mathbb{1}}{\|\rho_1 - \rho_2\|} \quad (5.37)$$

With the help of this operator we can rewrite the Hilbert Schmidt distance of two density matrices as

$$d(\rho_1, \rho_2) = \|\rho_1 - \rho_2\| = \langle \rho_1 - \rho_2, C \rangle. \quad (5.38)$$

On the other hand the product of any density matrix with  $C$  is

$$\begin{aligned} \langle \rho, C \rangle &= \frac{1}{\|\rho_1 - \rho_2\|} \langle \rho, \rho_1 - \rho_2 \rangle - \langle \rho, \rho_1 - \rho_2 \rangle \langle \rho, \mathbb{1} \rangle = \\ &= \frac{1}{\|\rho_1 - \rho_2\|} \langle \rho - \rho_1, \rho_1 - \rho_2 \rangle. \end{aligned} \quad (5.39)$$

Now for a fixed  $\rho_P$  this is just a hyperplane in our Hilbert Schmidt space.

$$\langle \rho_P, C \rangle = 0 \quad (5.40)$$

and in analogy to the euclidean case there exist states that fulfill

$$\langle \rho_a, C \rangle < 0 \quad \text{and} \quad \langle \rho_b, C \rangle > 0 \quad (5.41)$$

We already know that those conditions apply to entanglement witnesses (4.10)(4.11) which is equal to

$$\langle \rho_{sep}, A \rangle - \langle \rho_{ent}, A \rangle \geq 0 \quad \forall \rho_{sep} \quad (5.42)$$

This inequality holds for any entangled state as long as we choose the right entanglement witness. The maximal violation of the inequality can be achieved with

$$B(\rho_{ent}) = \max_{A, \|A - a\mathbb{1}\| \leq 1} \left( \min_{\rho_{sep}} \langle \rho_{sep}, A \rangle - \langle \rho_{ent}, A \rangle \right). \quad (5.43)$$

This definition for an entanglement measure coincides with the Hilbert Schmidt measure (5.31) (see Fig. 5.1).

$$E_{HS}(\rho_{ent}) = B(\rho_{ent}) \quad (5.44)$$

This is also called the Bertlmann-Narnhofer-Thirring Theorem [19]. The proof is quite straight forward. First we can use (5.38) to rewrite the Hilbert Schmidt distance.

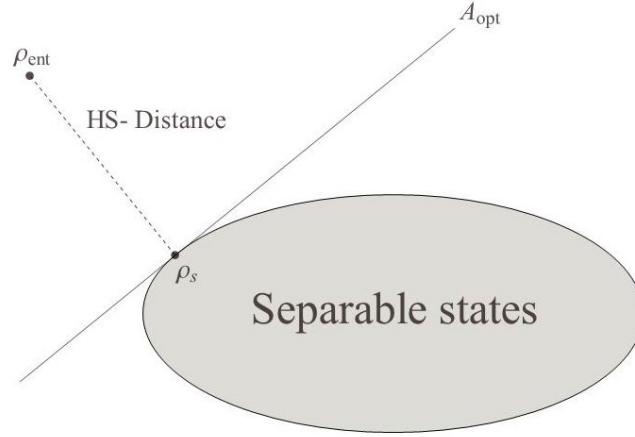


Figure 5.1: The distance between the nearest separable state and the entangled state can be considered as an entanglement measure. It coincides with the entanglement witness theorem of Bertlmann, Narnhofer and Thirring [19].

$$E_{HS}(\rho_{ent}) = \min_{\rho_{sep}} \langle \rho_{sep}, C \rangle - \langle \rho_{ent}, C \rangle \quad (5.45)$$

Since the minimum in the above equation leads to the nearest separable state, the operator  $C$  has to be an optimal entanglement witness. Thus we can write

$$\max_{A, \|A - a\mathbb{1}\| \leq 1} (\langle \rho_{ent}, A \rangle) = \langle \rho_{ent}, C \rangle \quad (5.46)$$

and therefore

$$\begin{aligned} E_{HS}(\rho_{ent}) &= \min_{\rho_{sep}} \langle \rho_{sep}, C \rangle - \langle \rho_{ent}, C \rangle = \\ &= \max_{A, \|A - a\mathbb{1}\| \leq 1} \left( \min_{\rho_{sep}} \langle \rho_{sep}, A \rangle - \langle \rho_{ent}, A \rangle \right) = B(\rho_{ent}) \end{aligned} \quad (5.47)$$

Let us now look at some examples for calculating witnesses. As stated in [22] we first have to guess the nearest separable state to our entangled state. In some cases this guess occurs naturally, like in the case of Werner states [7] where the nearest separable state is achieved for the parameter  $\alpha = \frac{1}{3}$ . However it is not always the case that the nearest separable state lies within the parametrized state we are looking at. So how can we be sure that the considered state is indeed the nearest separable state? We just have to calculate Eq. (5.37). If the operator  $C$  is an entanglement witness, then our guess has to be the nearest separable state. To check if we have indeed an entanglement witness, we have to calculate (4.10) and (4.11). Let us do that for the example of the Werner state.



The Werner state is defined as follows

$$\rho_W = \alpha |\psi^-\rangle\langle\psi^-| + \frac{1+\alpha}{4} \mathbb{1}. \quad (5.48)$$

Now in order to show for which parameter  $\alpha$  this state is entangled (i.e. not a convex combination of product states) we will use the PPT criterion (4.7).

The eigenvalues of the Werner state are

$$\lambda_{1,2,3} = \frac{1-\alpha}{4} \quad , \quad \lambda_4 = \frac{1+3\alpha}{4} \quad (5.49)$$

The Eigenvalues of the PPT Werner state are

$$\lambda_{1,2,3} = \frac{1+\alpha}{4} \quad , \quad \lambda_4 = \frac{1-3\alpha}{4} \quad (5.50)$$

So the Werner state is entangled for  $\alpha > \frac{1}{3}$ . Thus our guess for the nearest separable state will be the Werner state with  $\alpha = \frac{1}{3}$ .

$$\rho_W = \begin{pmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix} \quad (5.51)$$

We can now calculate the entanglement witness (5.37), which gives us

$$C = \frac{1}{2\sqrt{3}} \left( \mathbb{1} + \sum_{i=1}^3 \sigma_i \otimes \sigma_i \right) \quad (5.52)$$

Now we have to check (4.10) which gives

$$\text{Tr}(\rho_W C) < 0 \quad , \quad \alpha > \frac{1}{3} \Leftrightarrow \text{Tr}(\rho_W C) = \frac{1}{2\sqrt{3}}(1 - 3\alpha) \quad (5.53)$$

And last, we have to check (4.11) which gives

$$\text{Tr}(\rho_{sep} C) \geq 0 \Leftrightarrow \frac{1}{2\sqrt{3}} \left( 1 + \vec{n} \cdot \vec{m} \right) \geq 0. \quad (5.54)$$

The right hand side follows from Eq. (2.32) where  $\vec{n}$  and  $\vec{m}$  are the Bloch vectors. This is clearly fulfilled since  $-1 \leq \vec{n} \cdot \vec{m} \leq 1$ . Thus  $C$  is an optimal entanglement witness and our guess for the nearest separable state was correct. We can also calculate that in a more general way. Let us assume a  $d \times d$  dimensional Werner state.

$$\rho_W = \alpha P + \frac{1-\alpha}{d^2} \mathbb{1}_{d^2} \quad (5.55)$$

where  $P$  is a projector onto a maximally entangled state. Now let us look at the following decomposition of the state.

$$\rho_W = \beta P + (1-\beta)\sigma \quad (5.56)$$

The term  $\sigma$  denotes the orthogonal part to the maximally entangled state.

$$\langle \sigma | P \rangle = 0 \quad (5.57)$$

The Werner state thus takes the form

$$\rho_W = \beta \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (1-\beta) \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \quad (5.58)$$

This leads to the following observation: if  $\beta > \frac{1}{d}$  then the state is entangled. We want to prove this now. Let us introduce the following entanglement witness.

$$A = \mathbb{1}_{d^2} - dP \quad (5.59)$$

Again we will use Eq. (4.11) to check if it is indeed a witness. Thus we have to define all separable states in the following way and calculate explicitly

$$\begin{aligned} \langle \psi \otimes \phi | \mathbb{1}_{d^2} - dP | \psi \otimes \phi \rangle &= 1 - d \langle \psi \otimes \phi | P | \psi \otimes \phi \rangle = \\ &= \sum_{i,j=1}^d \frac{1}{d} \phi_i^* \psi_i^* \phi_j \psi_j = 1 - \langle \phi | \psi \rangle \langle \psi | \phi \rangle = 1 - d |\langle \phi | \psi \rangle|^2 \geq 0 \end{aligned} \quad (5.60)$$

If we now apply this witness to the state we get

$$0 > \langle \beta P + (1-\beta)\sigma | \mathbb{1}_{d^2} - dP \rangle = 1 - d\beta \Rightarrow \beta > \frac{1}{d} \quad (5.61)$$

And thus arrive at the expected result. For the special case of  $2 \times 2$  dimensions we get  $\beta > \frac{1}{2}$  which coincides with  $\alpha > \frac{1}{3}$  (compare with (5.58))

## 6 Bell Inequality

### 6.1 Nonlocality and the Absence of Reality

Since the beginning of the investigation of quantum mechanics, many very fundamental questions arised. We have already talked about entanglement, now we will look at the notion of nonlocality and realism within the quantum mechanical framework. The first mentioning of strange behaviour was found in a paper by Einstein, Podolsky and Rosen [23], where they investigated correlated states in order to find a proof of their assumption that quantum mechanics is not complete. Therefore they introduced the following definition of realism:

*If, without in any way disturbing the system, we can predict with certainty (i.e. with the probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*

Now following the description of Aharonov and Bohm [24], we will consider a system of two qubit states. The state we are interested in, is a Bell state  $\psi^-$  (Eq. (2.24)). Now if we measure such a state in a certain direction  $\vec{x}$  for both particles then the measurement outcome corresponds to the expectation value

$$E(\vec{x}) = \langle \psi^- | \vec{\sigma} \vec{x} \otimes \vec{\sigma} \vec{x} | \psi^- \rangle = (+1) \cdot (-1) \quad (6.1)$$

and always gives perfect anticorrelations. Now because of the correlation between the two particles Einstein et. al. concluded that if someone measures particle one, he can predict the outcome of measurement two, and thus there must be an element of reality associated with it. But this element does not occur in quantum theory, therefore it must be incomplete. Other assumptions that are embedded in this argument are

Locality: *Any physical theory should not allow faster than light communication between two parties to share information.*

Completeness: *Every element of physical reality must have a counterpart in the physical theory.*

For the time being it was not possible to decide wether the quantum mechanical theory is complete or not, until John S. Bell solved this problem in 1964 [25]. He showed that these assumptions of locality and realism are not compatible with quantum mechanics, but indeed have to be abandoned.

## 6.2 Derivation of the Bell Inequality

We will now introduce one version of a Bell inequality [25] which proves the difference of quantum mechanics compared to theories that obey the above mentioned assumptions of locality and reality. Let us start with two observables  $A(m, n, \lambda)$  and  $B(m, n, \lambda)$  for two parties which describe the definite values of physical measurement outcomes. The parameter  $\lambda$  denotes the possible hidden variables that are not contained within the theory. The parameters  $n, m$  denote the measurement settings for both parties respectively. We are now interested in the expectation value of measuring the observables  $A$  and  $B$ . Because of the assumption of locality,  $A$  can not depend on the measurement setting of  $B$ , therefore  $A = A(n, \lambda)$  and  $B = B(m, \lambda)$ . Also, the average measurement result of  $|A|$  and  $|B|$  is bounded by 1. Thus the expectation value takes the form

$$E(m, n) = \int d\lambda \rho(\lambda) A(n, \lambda) B(m, \lambda) \quad (6.2)$$

This corresponds to the quantum mechanical expectation value

$$E(n, m) = \langle \psi | A(n) \otimes B(m) | \psi \rangle \quad (6.3)$$

$\rho(\lambda)$  is a distribution function of the hidden variables which is normalized.

$$\int d\lambda \rho(\lambda) = 1 \quad (6.4)$$

We can now look at the difference of two such expectation values.

$$\begin{aligned} E(n, m) - E(n, m') &= \int d\lambda \rho(\lambda) \{ A(n, \lambda) B(m, \lambda) - A(n, \lambda) B(m', \lambda) \} = \\ &= \int d\lambda \rho(\lambda) A(n, \lambda) B(m, \lambda) \{ 1 \pm A(n', \lambda) B(m', \lambda) \} - \\ &\quad \int d\lambda \rho(\lambda) A(n, \lambda) B(m', \lambda) \{ 1 \pm A(n', \lambda) B(m, \lambda) \} \end{aligned} \quad (6.5)$$

From there we can conclude

$$\begin{aligned} |E(n, m) - E(n, m')| &\leq \int d\lambda \rho(\lambda) \{ 1 \pm A(n', \lambda) B(m', \lambda) \} - \\ &\quad \int d\lambda \rho(\lambda) \{ 1 \pm A(n', \lambda) B(m, \lambda) \} \end{aligned} \quad (6.6)$$

Because of the normalization of the distribution function Eq. (6.4) we arrive at

$$|E(n, m) - E(n, m')| \leq 2 \pm |E(n', m') + E(n', m)| \quad (6.7)$$

### 6.3 Calculating the Bell Inequality for Arbitrary Two Qubit States

or written in a more well known manner

$$|E(n, m) - E(n, m')| + |E(n', m') + E(n', m)| \leq 2. \quad (6.8)$$

This is the CHSH inequality named after Clauser, Horne, Shimony and Holt [26]. There are numerous different forms of Bell inequalities. The first one that was derived by John Bell has more restricting assumptions. Bell also used perfect anticorrelations between the measurement results ( $E(n, m) = -1$ ) and perfect measurement devices. If we plug in these assumptions we end up with the following inequality

$$|E(n, m) - E(n, m')| \leq 1 + E(m, m'). \quad (6.9)$$

Another form can be found by not looking at expectation values, but probabilities of measurement results. This is called the Wigner inequality. It makes use of the fact, that the expectation value can be rewritten through probabilities.

$$\begin{aligned} E(n, m) &= P(\uparrow n, \uparrow m) + P(\downarrow n, \downarrow m) - P(\uparrow n, \downarrow m) - P(\downarrow n, \uparrow m) = \\ &= 2P(\uparrow n, \uparrow m) - 2P(\uparrow n, \downarrow m) = \\ &= -1 + 4P(\uparrow n, \uparrow m) \end{aligned} \quad (6.10)$$

We have used that  $\sum P = 1$ . If we insert this result into (6.9) we get

$$P(\uparrow n, \uparrow m) \leq P(\uparrow n, \uparrow m') + P(\uparrow n', \uparrow m). \quad (6.11)$$

We have seen, that for any local realistic theory, there is a bound to the expectation value (6.8). Now if we insert a Bell state into (6.3) and adjust the measurement directions to have a relative angle of 0, 45, 90 and 135 degrees respectively, then the inequality will be violated by a value of  $2\sqrt{2}$ . Thus quantum mechanics is not compatible with the above mentioned assumptions. This forces us to conclude that quantum mechanics is indeed complete and no further hidden variables as defined above are needed.

### 6.3 Calculating the Bell Inequality for Arbitrary Two Qubit States

The main issue when looking at general mixed states in the two qubit case is to find the appropriate angles for maximizing the left side of Eq. (6.8). We will now give a method of calculating this value along the lines of [27].

First, every density matrix can be decomposed into a Bloch basis.

$$\rho = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{u} \cdot \vec{\sigma} + \sum_{i,j=1}^3 t_{ij} \sigma_i \otimes \sigma_j \right) \quad (6.12)$$

## 6 Bell Inequality

Then we use the operator form of Eq. (6.8) which has the form

$$\mathcal{B}_{\text{CHSH}} = \vec{a} \cdot \vec{\sigma} \otimes (\vec{b} + \vec{b}') \cdot \vec{\sigma} + \vec{a}' \cdot \vec{\sigma} \otimes (\vec{b} - \vec{b}') \cdot \vec{\sigma}. \quad (6.13)$$

From there we achieve the original inequality through the expectation value.

$$\text{Tr}(\rho \mathcal{B}_{\text{CHSH}}) \leq 2 \quad (6.14)$$

If we now use this operator form and the general decomposition of Eq. (6.12) we arrive at the following expectation value

$$\langle \mathcal{B}_{\text{CHSH}} \rangle_\rho = \vec{a} \cdot T(\vec{b} + \vec{b}') + \vec{a}' \cdot T(\vec{b} - \vec{b}') \quad (6.15)$$

where  $T$  corresponds to  $t_{ij}$  in Eq. (6.12).

The next step is to introduce new variables.

$$\vec{b} + \vec{b}' = 2 \cos \theta \vec{c} \quad , \quad \vec{b} - \vec{b}' = 2 \sin \theta \vec{c}' \quad , \quad \theta \in \left[0, \frac{\pi}{2}\right] \quad (6.16)$$

We have to rewrite Eq. (6.15) with these new variables and maximize over all of them.

$$\begin{aligned} \max_{\mathcal{B}_{\text{CHSH}}} \langle \mathcal{B}_{\text{CHSH}} \rangle_\rho &= \max_{\theta, \vec{a}, \vec{a}', \vec{c}, \vec{c}'} 2 \left[ (\vec{a} \cdot T\vec{c}) \cos \theta + (\vec{a}' \cdot T\vec{c}') \sin \theta \right] = \\ &= \max_{\theta, \vec{c}, \vec{c}'} 2 \left[ \|T\vec{c}\| \cos \theta + \|T\vec{c}'\| \sin \theta \right] = \\ &= \max_{\vec{c}, \vec{c}'} 2 \sqrt{\|T\vec{c}\|^2 + \|T\vec{c}'\|^2} \Rightarrow \\ &\Rightarrow 2\sqrt{u + u'} \end{aligned} \quad (6.17)$$

In the last step we used the relation

$$\|T\vec{c}\| = T\vec{c} \cdot T\vec{c} = \vec{c} \cdot T^T T \vec{c} \quad (6.18)$$

and the fact, that this scalar product is maximized if  $\vec{c}$  is an eigenvector of  $T^T T$ .  $u$  and  $u'$  denote the two bigger eigenvalues of  $T^T T$ . Thus, the expectation value for the operator  $\mathcal{B}$  is maximized and we arrive at the optimal form. However we need not find the right measurement directions  $\vec{a}, \vec{b}$ , because they are already embedded in this maximization. This creates an easy way to calculate the violation for any arbitrary two qubit state.

## 6.4 Bell Inequality as an Entanglement Measure

If one considers only pure states, then product states always admit a LHV theory. So it seemed natural to conjecture, that only a convex combination of product states could also be described by a LHV theory. However, it was shown by Werner [7] that there indeed exist states that are entangled, but do not admit a LHV theory. The state for which he showed this, is the Werner state (5.48).

The Bell violation of this state can now be calculated with the help of the Horodecki theorem (6.17). The Bloch decomposition of the Werner state reads

$$\rho_W = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} - \sum_{i=1}^3 \alpha_i \sigma_i \otimes \sigma_i \right) \quad (6.19)$$

and therefore the  $T$  matrix has the form

$$T = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\alpha \end{pmatrix}. \quad (6.20)$$

This gives the eigenvalues  $u, u' = \alpha^2$  for  $T^T T$ . Thus the violation for a Bell inequality is

$$\langle \mathcal{B}_{\text{CHSH}} \rangle_{\rho_W} \leq 2\sqrt{2\alpha^2} \Rightarrow \alpha > \frac{1}{\sqrt{2}}. \quad (6.21)$$

We can conclude from this, that the notion of entanglement does not coincide with the violation of the assumptions for a Bell inequality. Or differently put, a state that is entangled does not necessarily violate a Bell inequality. However they do coincide for the special case of pure states.





# 7 Quantum Measurement Operations on Composite Systems

## 7.1 LOCC Operations

We will now look at a certain class of measurement operations, the so called "Local operations and classical communication" and how they can change the entanglement of a state. An LOCC operation is basically a measurement performed by one party (Alice) who communicates the result to the other party (Bob). Depending on the outcome, Bob will perform a measurement and communicates the result to Alice and so on. Note that also general measurements are allowed (see section: Measurement in Quantum Mechanics) but they are only carried out on the subsystems respectively.

These operations are closely linked to the mathematical formalism of majorization. [28] The idea of majorization is the following. Suppose two vectors  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  with real values. Then  $y$  majorizes  $x$  ( $x \prec y$ ) if

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow \quad (7.1)$$

for every  $k$  between  $1, \dots, n$ . The arrow in (7.1) means that the vectors  $x, y$  are considered to be in descending order. Now how is this linked to the LOCC operations? Let us consider two states  $|\psi\rangle, |\phi\rangle$ , the reduced density matrices for one subsystem of these vectors are

$$\rho_\psi = \text{Tr}_B |\psi\rangle\langle\psi|, \quad \rho_\phi = \text{Tr}_B |\phi\rangle\langle\phi|. \quad (7.2)$$

The state  $|\psi\rangle$  can now be transformed to the state  $|\phi\rangle$  via LOCC operations iff the eigenvalues of  $\rho_\phi \rightarrow \lambda_\phi$  majorize the eigenvalues of  $\rho_\psi \rightarrow \lambda_\psi$  ( $\lambda_\psi \prec \lambda_\phi$ ).

This is a very important feature, since the eigenvalues of the subsystems determine the entanglement of the pure state. Due to this condition of majorization we can see that LOCC operations cannot increase the entanglement of a pure state! Indeed one can show that it is in general impossible to increase entanglement via LOCC operations. This also leads to the conclusion, that LOCC operations cannot create entanglement, which means that a separable state remains separable under LOCC operations.

We can also simplify the procedure of carrying out LOCC operations. We have stated above that those protocols involve back and forth communication between

the two parties. However we can show that all the measurements on Bob's side can be simulated by a measurement on Alice's side, a single communication from Alice to Bob and a unitary transformation carried out by Bob. The proof is as follows. Consider a general set of measurement operators for Bob that he wants to apply to a pure composite state  $|\psi\rangle$ . This state can be written in its Schmidt decomposition.

$$|\psi\rangle = \sum_l \sqrt{\lambda_l} |l_A\rangle |l_B\rangle \quad (7.3)$$

The measurement operators of Bob can also be written in this Bloch basis.

$$B_j = \sum_{kl} B_{jkl} |k_B\rangle \langle l_B| \quad (7.4)$$

We can do the same for measurement operators on Alice's side.

$$A_j = \sum_{kl} A_{jkl} |k_A\rangle \langle l_A| \quad (7.5)$$

If Bob now carries out his measurement, we get

$$|\psi_j\rangle \propto B_j |\psi\rangle = |\psi\rangle = \sum_{kl} B_{jkl} \sqrt{\lambda_l} |l_A\rangle |k_B\rangle \quad (7.6)$$

with probability  $p(j) = \sum_{kl} \lambda_l |B_{jkl}|^2$ . On the other hand, if Alice carries out her measurement she gets the state

$$|\phi_j\rangle \propto A_j |\psi\rangle = |\psi\rangle = \sum_{kl} A_{jkl} \sqrt{\lambda_l} |k_A\rangle |l_B\rangle \quad (7.7)$$

with the same probabilities as Bob. The only difference between the measurement result on Alice's side and Bob's side is an interchange of the subsystems. Therefore the two resulting states must have the same Schmidt components. But from this property follows immediately that these two outcome states can be transformed into each other by a unitary operation carried out on Alice's and Bob's system respectively. Thus a measurement  $B_j$  done by Bob can be replaced by a measurement  $U_j A_j$  of Alice and a unitary  $V_j$  carried out by Bob. This shows that to achieve LOCC operations, we only need communications in one direction.

We have stated above that LOCC operations do not increase entanglement of one state. However, they can be used to purify entanglement if applied to a number of copies of a state. This principle leads to entanglement distillation which was mentioned earlier on.

## 7.2 Filtering Operations

A special case of LOCC operations were studied by Linden, Massar and Popescu [29]. Their considered transformation is a combination of local unitary and local filtering operations.

$$\rho_f = \frac{A \otimes B \rho A^\dagger \otimes B^\dagger}{\text{Tr}(A \otimes B \rho A^\dagger \otimes B^\dagger)} \quad (7.8)$$

where  $A$  and  $B$  can be decomposed into

$$A = U_A F_A \quad B = U_B F_B \quad (7.9)$$

$U_A$  and  $U_B$  are unitary operators and the operator  $F$  denotes the filtering operation. Note that because of the nonunitary operations applied, we need to renormalize the state, which is done in Eq. (7.8) by the trace. Remember that LOCC operations cannot increase the entanglement. However, because in a normalized state there is no indication of the number of copies, we cannot see that this filtering operation reduces the number of available states. Therefore we trade the number of states for the entanglement within the states. If we take that into consideration, the entanglement decreases.

The filtering operation can be written as

$$F_A(\mu, a, \vec{m}) = \mu(\mathbb{1} + a\vec{m} \cdot \vec{\sigma}) \quad F_B(\nu, b, \vec{n}) = \nu(\mathbb{1} + b\vec{n} \cdot \vec{\sigma}) \quad (7.10)$$

Or even more simple  $A = U_A F_A U'_A, B = U_B F_B U'_B$  where the filtering operation reduces to

$$F_A = \mu(\mathbb{1} + a\sigma_z) \quad F_B = \nu(\mathbb{1} + b\sigma_z) \quad (7.11)$$

with  $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ . However the factors  $\gamma_1, \gamma_2$  are not explicitly needed, since they can be absorbed by the normalization. Furthermore, the parameters  $a, b$  need not be between 0 and 1, because

$$\begin{aligned} f_A = \mu(\mathbb{1} + a\sigma_z) &\rightarrow (\mathbb{1} + a\sigma_z) \rightarrow \begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix} = \\ &= (1-a) \begin{pmatrix} \frac{1+a}{1-a} & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (7.12)$$

The factor  $(1-a)$  can always be absorbed into the normalization. And also

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \rightarrow a \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \quad (7.13)$$

Thus it is sufficient to consider filtering operations of the form (7.13) with arbitrary  $a$  as long as the fact that  $a$  can be bigger than one gets compensated in the normalization.

We can now ask for the change of entanglement for such a transformation. In order to compute the entanglement we will use the concurrence (5.22). A filtered state will change in the following way

$$\rho_f \rightarrow \frac{U_A^1 f^A U_A^2 \otimes U_B^1 f^B U_B^2 \rho U_A^{2\dagger} f^A U_A^{1\dagger} \otimes U_B^{2\dagger} f^B U_B^{1\dagger}}{\text{Tr}[(f_A f_A \otimes f_B f_B)(U_A^2 \otimes U_B^2 \rho U_A^{2\dagger} \otimes U_B^{2\dagger})]} \quad (7.14)$$

In order to calculate the concurrence of this state we need to find the state  $\tilde{\rho}$ , which is defined via the transformation (5.20). The filtering changes under this transformation to

$$\tilde{f}^A = \begin{pmatrix} -a & 0 \\ 0 & -1 \end{pmatrix} \quad \tilde{f}^B = \begin{pmatrix} -b & 0 \\ 0 & -1 \end{pmatrix}. \quad (7.15)$$

Therefore, the matrix  $\rho\tilde{\rho}$  becomes

$$\rho_f \tilde{\rho}_f = \frac{(ab) U_A^1 f^A U_A^2 \otimes U_B^1 f^B U_B^2 \rho \tilde{\rho} U_A^{2\dagger} \tilde{f}^A U_A^{1\dagger} \otimes U_B^{2\dagger} \tilde{f}^B U_B^{1\dagger}}{\text{Tr}[(f_A f_A \otimes f_B f_B) \rho^U]} \quad (7.16)$$

where  $\rho^U$  is the unitary transformed density matrix for the normalization of Eq. (7.14). We now need to know the eigenvalues of this expression. In order to get them, we define the eigenvector  $|v_i\rangle$  of the matrix  $\rho_f \tilde{\rho}_f$  with the eigenvalue  $\lambda_i^2$  and the eigenvector  $|\omega_i\rangle$  which is defined as

$$|\omega_i\rangle = U_A^1 f_A U_A^2 \otimes U_B^1 f_B U_B^2 |v_i\rangle. \quad (7.17)$$

We can therefore write

$$\rho_f \tilde{\rho}_f |\omega_i\rangle = \frac{a^2 b^2}{\text{Tr}[(f_A f_A \otimes f_B f_B) \rho^U]} \lambda_i^2 |\omega_i\rangle. \quad (7.18)$$

Since the concurrence only depends on the eigenvalues of this matrix  $\rho_f \tilde{\rho}_f$  we can conclude that

$$C(\rho_f) = \frac{ab}{\text{Tr}[(f_A f_A \otimes f_B f_B) \rho^U]} C(\rho) \quad (7.19)$$

Now to calculate the normalization explicitly we need to introduce a parametrization of the density matrix. Since the filtering operation was chosen in the  $z$ -direction ( $\sigma_z$ ), we only need to consider density matrices of the form

$$\rho = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} + \alpha \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \beta \sigma_z + \sum_{i=1}^3 R_{ij} \sigma_i \otimes \sigma_j \right) \quad (7.20)$$

We now choose a generalized Werner state (5.48) ( $\alpha = 0, \beta = 0$ ), limit our parameters to  $0 \leq a \leq 1, 0 \leq b \leq 1, -1 \leq R_{33} \leq 1$  and look at the factor

$$\frac{ab}{\text{Tr}[(f_A f_A \otimes f_B f_B) \rho^U]}. \quad (7.21)$$

This factor can, under this constraints, never exceed 1 and therefore if we start with a Werner-like state we cannot increase the entanglement via these filtering operations. Or to put it differently, to optimize the entanglement of an arbitrary state with filtering operations, we need to find a filtering that creates a Werner-like state.

We have mentioned in the previous section, that entanglement cannot be created with LOCC operations. However if we look at the violation of a Bell inequality, the story is a little different. It was shown in [30] that filtering operations can change the nonlocality of a number of copies of states. We will look at the explicit example later on, but we want to introduce a way of calculating the change in nonlocality for a certain class of states without the need to specify the filtering operation explicitly.

We start off with the following class of states.

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23} & \rho_{33} & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix} \quad (7.22)$$

This state has the following Bloch decomposition

$$\rho = \frac{1}{4} \left( \mathbf{1} \otimes \mathbf{1} + r_z \mathbf{1} \otimes \sigma_z + u_z \sigma_z \otimes \mathbf{1} + \sum_{i=1}^3 t_{ii} \sigma_i \otimes \sigma_i \right) \quad (7.23)$$

or in matrix form

$$\rho = \frac{1}{4} \begin{pmatrix} 1 + r_z + u_z + t_{zz} & 0 & 0 & t_{xx} - t_{yy} \\ 0 & 1 - r_z + u_z - t_{zz} & t_{xx} + t_{yy} & 0 \\ 0 & t_{xx} + t_{yy} & 1 + r_z - u_z - t_{zz} & 0 \\ t_{xx} - t_{yy} & 0 & 0 & 1 - r_z - u_z + t_{zz} \end{pmatrix} \quad (7.24)$$

We are now interested in the violation of a Bell inequality. Therefore we use the Horodecki theorem (6.17) where our eigenvalues  $u, \tilde{u}$  are just the two bigger elements  $t_{ii}^2$ . We can reexpress those values with the elements  $\rho_{ij}$  of our matrix.

$$\begin{aligned}
 t_{zz} &= 1 - 2(\rho_{22} + \rho_{33}) \\
 t_{xx} &= 2(\rho_{23} + \rho_{14}) \\
 t_{yy} &= 2(\rho_{23} - \rho_{14})
 \end{aligned} \tag{7.25}$$

We will again apply our filtering operation (7.13) with a slight change for convenience.

$$f^A \otimes f^B = \begin{pmatrix} \sqrt[4]{a} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \sqrt[4]{b} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^{1/4}b^{1/4} & 0 & 0 & 0 \\ 0 & a^{1/4} & 0 & 0 \\ 0 & 0 & b^{1/4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.26}$$

This filtering operation changes the state to

$$f^A \otimes f^B \rho f^A \otimes f^B = \begin{pmatrix} \sqrt{a}\sqrt{b}\rho_{11} & 0 & 0 & a^{1/4}b^{1/4}\rho_{14} \\ 0 & \sqrt{a}\rho_{22} & a^{1/4}b^{1/4}\rho_{23} & 0 \\ 0 & a^{1/4}b^{1/4}\rho_{23} & \sqrt{b}\rho_{33} & 0 \\ a^{1/4}b^{1/4}\rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{7.27}$$

Again we try to eliminate the terms  $(\sigma_z \otimes \mathbb{1}, \mathbb{1} \otimes \sigma_z)$ . Thus, we have to compare the matrix elements of (7.27) matrix elements of (7.22). These terms vanish, if entry 22 of a density matrix is equal to entry 33 and entry 11 is equal to entry 44 (compare with (7.23)). From there we can calculate the parameters  $a, b$ .

$$a = \frac{\rho_{33}\rho_{44}}{\rho_{11}\rho_{22}}, \quad b = \frac{\rho_{22}\rho_{44}}{\rho_{11}\rho_{33}} \tag{7.28}$$

If we now enter these parameters into Eq. (7.27) and look at the Bloch decomposition of this state, we arrive at

$$\begin{aligned}
 t_{xx} &= \frac{\rho_{14} + \rho_{23}}{\sqrt{\rho_{22}\rho_{33}} + \sqrt{\rho_{11}\rho_{44}}} \\
 t_{yy} &= \frac{-\rho_{14} + \rho_{23}}{\sqrt{\rho_{22}\rho_{33}} + \sqrt{\rho_{11}\rho_{44}}} \\
 t_{zz} &= \frac{-1 + \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}}{1 + \sqrt{\frac{\rho_{11}\rho_{44}}{\rho_{22}\rho_{33}}}}.
 \end{aligned} \tag{7.29}$$

So to put it together, before the filtering procedure we had (7.25), and afterwards we have (7.29). The advantage of this way of writing it down is that there is no need to specify the exact filtering procedure anymore. Thus we can give the Bell inequality violation of a state of the form (7.22) after an optimal filtering operation.

## 8 Visualization of Quantum States

In this chapter we want to introduce possibilities to visualize quantum states. This is especially important to develop a certain intuition for entanglement and its properties. With visualizations it is possible to literally take a look at entanglement measures such as concurrence or entanglement witnesses, which are so to say, perfectly suited for visualizations.

For pure states, we can represent states as members of the Complex Projective Space ( $CP^n$ ) [31, 32]. Especially in the cases of three and four dimensions (two qubits), we are thus able to draw these spaces in a two and three dimensional picture. However only for two qubits can we investigate phenomena such as entanglement, thus we will concentrate on this system sizes. For higher dimensions we could only draw projections of our visualization methods. So it is a nice coincidence that for the smallest case (considering dimensions) of quantum states, it is possible to find a full visualization.

It is also possible to create a picture for certain subclasses of mixed states, which can be done by looking at the Bloch decomposition of such states. This picture was first introduced by Bertlmann, Thirring and Narnhofer [19] and is a convenient way to investigate the behaviour of entanglement and separability and also nonlocality of states. We will also introduce a possibility to extend this picture to states with more degrees of freedom than three real ones in the case of two qubits.

### 8.1 Visualization for Pure States

In order to connect our state space with the complex projective space, let us look at a pure quantum state.

$$|\psi\rangle = \sum_{i=1}^N \alpha_i |i\rangle \quad (8.1)$$

$N$  denotes the dimension of our quantum state. We can always add a global phase to our state without changing it. Thus we can write a state vector as

$$(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N) \sim \beta(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N) \quad \beta \in \mathbb{C} \quad (8.2)$$

This is by definition the  $CP^n$ . So for the two qubit case a state vector can be represented by

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \sim (n_1, e^{i\nu_1}n_2, e^{i\nu_2}n_3, e^{i\nu_3}n_4) \quad (8.3)$$

where  $0 \leq \nu_i \leq 2\pi$  and the real values  $n_i$  lie between 0 and 1 and are normalized.

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1 \quad (8.4)$$

The numbers  $n_i$  can be interpreted as the coordinates of an octant of the 3-sphere as well as the numbers  $\nu_i$  correspond to the coordinates of a torus.

If we consider the dimension  $N = 3$  the numbers  $n_i$  would correspond to the 2-sphere.

There is a natural way of defining a distance on  $CP^n$ , the Fubini-Study distance

$$\cos^2(s) = \frac{|\langle \psi_1 | \psi_2 \rangle|^2}{\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle} \quad (8.5)$$

or written in our coordinates  $\alpha$

$$\cos^2(s) = \frac{|\vec{\alpha}_1 \cdot \vec{\alpha}_2^*|}{\vec{\alpha}_1 \cdot \vec{\alpha}_1^* \vec{\alpha}_2 \cdot \vec{\alpha}_2^*}. \quad (8.6)$$

From this definition we can calculate the metric for our  $CP^n$ .

$$\begin{aligned} \cos(s) &= \sqrt{\frac{|\vec{\alpha} \cdot (\vec{\alpha} + d\vec{\alpha})|^2}{\vec{\alpha} \cdot \vec{\alpha}^* (\vec{\alpha} + d\vec{\alpha}) \cdot (\vec{\alpha}^* + d\vec{\alpha}^*)}} = \\ &= 1 - \frac{ds^2}{2} + \dots = \sqrt{1 + \left[ \frac{(\vec{\alpha} \cdot (\vec{\alpha}^* + d\vec{\alpha}^*)) ((\vec{\alpha} + d\vec{\alpha}) \cdot \vec{\alpha}^*)}{\vec{\alpha} \cdot \vec{\alpha}^* (\vec{\alpha} + d\vec{\alpha}) \cdot (\vec{\alpha}^* + d\vec{\alpha}^*)} - 1 \right]} \end{aligned} \quad (8.7)$$

If we now approximate the square root we get

$$ds^2 = \frac{\vec{\alpha} \cdot \vec{\alpha}^* d\vec{\alpha} \cdot d\vec{\alpha}^* - \vec{\alpha} \cdot d\vec{\alpha}^* \vec{\alpha}^* \cdot d\vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}^* (\vec{\alpha} + d\vec{\alpha}) \cdot (\vec{\alpha}^* + d\vec{\alpha}^*)}. \quad (8.8)$$

This is our metric for  $CP^n$ . We will now look at the explicit form for the two qubit case by inserting Eq. (8.3).

$$\begin{aligned} ds^2 &= dn_1^2 + dn_2^2 + dn_3^2 + dn_4^2 + n_2^2(1 - n_2^2)d\nu_1^2 + n_3^2(1 - n_3^2)d\nu_2^2 \\ &\quad + n_4^2(1 - n_4^2)d\nu_3^2 - 2n_2^2n_3^2d\nu_1d\nu_2 - 2n_2^2n_4^2d\nu_1d\nu_3 - 2n_3^2n_4^2d\nu_2d\nu_3 \end{aligned} \quad (8.9)$$



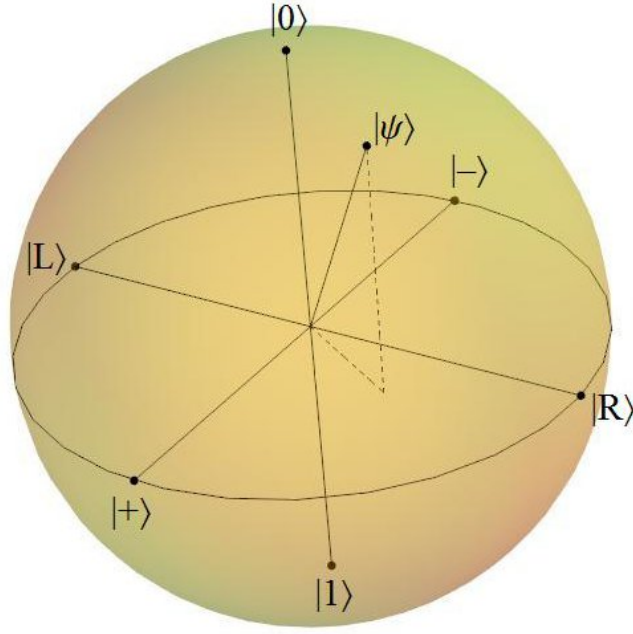


Figure 8.1: The Bloch sphere is a visualization of one qubit and spanned by the Bloch vector (see also (2.29)).

The first part of this metric represents just a sphere for  $n_i$ , whereas the second part describes a torus, whose shape changes based on the point where we are at the sphere.

We want to add here a certain special case, namely the one for  $CP^1$ . In this case the description reduces to the well known Bloch sphere (see Fig. 8.1). The metric reduces to

$$ds^2 = dn_1^2 + dn_2^2 + n_1^2 n_2^2 d\nu_1^2. \quad (8.10)$$

In the general case the metric takes the following form:

$$ds^2 = \sum_{i=0}^N \sum_{j=0}^N (dn_j^2 + n_j^2 d\nu_j^2 - n_i^2 n_j^2 d\nu_i d\nu_j) \quad (8.11)$$

with  $\nu_0 = 0$ .

Because of the property (8.4) and the fact that all  $n_i \geq 0$  we only cover an octant of our four dimensional sphere. this surface we are looking at, can be displayed via a gnomonic projection where it appears as a three dimensional

tetraeder. On every point of this tetraeder "sticks" a torus which embeds the  $\nu_i$  degrees of freedom.

So what do the separable and entangled states look like in this kind of picture? First of all let us recall the definition for separability for pure states from Eq. (5.1). We already concluded that separability is connected to the determinant of the matrix  $\Lambda$  of Eq. (5.2). Now we can relate our coordinates of the space  $CP^3$  with the components of  $\Lambda$ .

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\Lambda_{00}, \Lambda_{01}, \Lambda_{10}, \Lambda_{11}) \quad (8.12)$$

This gives us a condition for our coordinates  $\vec{\alpha}$  and in our special case of two qubits leads to the following relations for  $n_i$  and  $\nu_i$ .

$$\begin{aligned} n_1 n_4 - n_2 n_3 &= 0 \\ \nu_1 + \nu_2 - \nu_3 &= 0 \end{aligned} \quad (8.13)$$

Thus we see that this condition separates into two equations. One for the coordinates of our tetraeder and one for our phases which are the coordinates of the torus. To draw this in our picture, we fix one subsystem of our separable state (see Eq. (5.1)) to

$$\frac{b_0}{b_1} = k e^{i\phi}. \quad (8.14)$$

This leads to

$$\begin{aligned} n_1 &= k n_2 \\ n_3 &= k n_4. \end{aligned} \quad (8.15)$$

Now this is just a parametrization of a straight line in our gnomonic projection. In order to see, that it is indeed a flat space embedded in our octant picture, we will parametrize our coordinates  $n_i$  with Euler angles.

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} \sin \frac{\tau-\phi}{2} \sin \frac{\theta}{2} \\ \sin \frac{\tau+\phi}{2} \cos \frac{\theta}{2} \\ \cos \frac{\tau-\phi}{2} \sin \frac{\theta}{2} \\ \cos \frac{\tau+\phi}{2} \cos \frac{\theta}{2} \end{pmatrix} \quad (8.16)$$

From our condition (8.15) we find that  $\phi$  has to be zero. Thus the metric reduces to

$$ds^2 = \frac{1}{4}(d\tau^2 + d\theta^2) \quad (8.17)$$

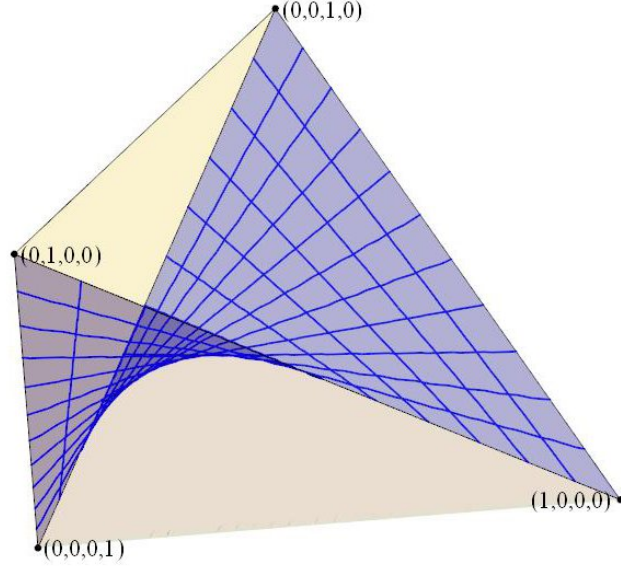


Figure 8.2: The gnomonic projection of the octant. The blue surface depicts the separable states. The surface is a parabolic hyperboloid embedded in the tetraeder. The edge points are the separable states  $|\uparrow\uparrow\rangle$ ,  $|\uparrow\downarrow\rangle$ ,  $|\downarrow\uparrow\rangle$  and  $|\downarrow\downarrow\rangle$

So the shape of separable states in this picture can be easily drawn and are shown in picture 8.3.

We also want to take a look at maximally entangled states in this picture. In order to get a maximally entangled state the subsystem of this state has to be maximally mixed. Therefore, we get the following condition on our  $\Lambda$

$$\sum_{j=1}^2 \Lambda_{ij} \Lambda_{jk} = \delta_{ik} \quad (8.18)$$

and thus  $\Lambda$  has to be

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}. \quad (8.19)$$

which is a unitary matrix. From there we can again calculate conditions for our parameters  $n_i$  and  $\nu_i$ .

$$\begin{aligned} n_1 &= n_4 \\ n_2 &= n_3 \\ \nu_1 + \nu_2 - \nu_3 &= \pi \end{aligned} \quad (8.20)$$

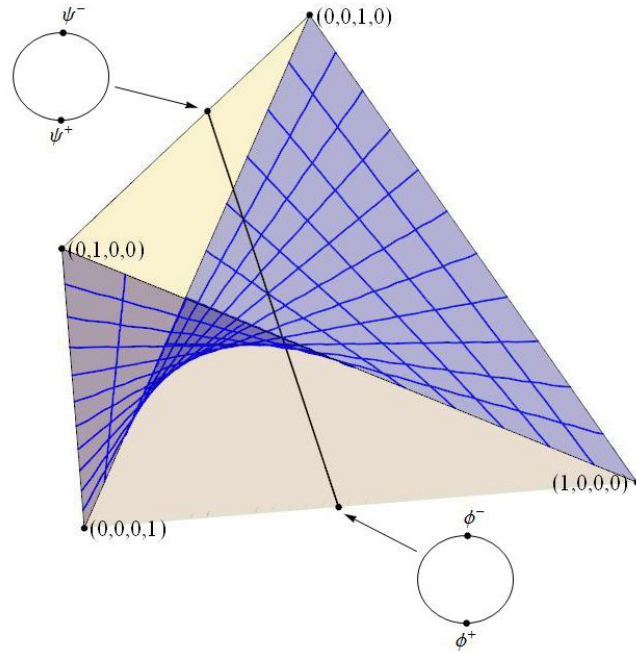


Figure 8.3: The entangled states lie on the black line through the centre. However, the picture is slightly misleading, since every point in the octant is associated with a torus. Thus there is no intersection between entangled and separable states. On the edges of the octant the 3-torus collapses to a circle. The maximally entangled states are located at the opposite sides of that circle.

This parametrization just leads to a straight line through the tetraeder which depicts every maximally entangled state. This can be seen in Fig. 8.3. Note that the line of entangled states crosses the surface of the separable states. However, it only seems that way, because at the center of our tetraeder the embedded 3-torus reaches its maximal size and thus "avoids" an intersection of separable and maximally entangled states. Also the distance between separable and entangled states stays the same for every state. In that sense the shown picture can be misleading and one has to be very careful when considering distances. On the other hand this visualization has the advantage, that it can really contain every considered state. In the case of mixed states this is no longer possible.

## 8.2 Visualization for Mixed States

Another method of visualizing states can be found by looking at the before-mentioned Bloch decomposition of the state. The usual Bloch decomposition for two qubits however has 15 degrees of freedom. Thus even for the smallest possible case there can be no full visualization for all states. However it turns out that we can draw a picture of a very important subclass of states, which are spanned by the four Bell states [19, 33]

$$|\psi\rangle = c_1|\psi^-\rangle\langle\psi^-| + c_2|\psi^+\rangle\langle\psi^+| + c_3|\phi^-\rangle\langle\phi^-| + (1 - c_1 - c_2 - c_3)|\phi^+\rangle\langle\phi^+| \quad (8.21)$$

where every coefficient has to be bigger than zero. This coincides with a special Bloch decomposition namely

$$\rho = \frac{1}{4} \left( \mathbb{1} + \sum_{i=1}^3 c_{ii} \sigma_i \otimes \sigma_i \right) \quad (8.22)$$

where the  $c_{ii}$  correspond to the  $c_i$  from Eq. (8.21) via

$$c_{11} = \frac{1}{2} - c_1 - c_3 \quad c_{22} = -\frac{1}{2} + c_2 + c_3 \quad c_{33} = -c_1 - c_2 \quad (8.23)$$

Therefore we are only left with three coordinates. In order to get the boundaries of our state space we look at the conditions arising from the positivity of density matrices. Thus, the following density matrix must be positive.

$$\rho = \begin{pmatrix} \frac{1}{4}(1 + c_{33}) & 0 & 0 & \frac{c_{11}+c_{22}}{4} \\ 0 & \frac{1-c_{33}}{4} & \frac{c_{11}-c_{22}}{4} & 0 \\ 0 & \frac{c_{11}-c_{22}}{4} & \frac{1-c_{33}}{4} & 0 \\ \frac{c_{11}+c_{22}}{4} & 0 & 0 & \frac{1}{4}(1 + c_{33}) \end{pmatrix} \quad (8.24)$$

The eigenvalues of this matrix determine four planes that form a tetraeder in three dimensions. (see Fig. 8.4)

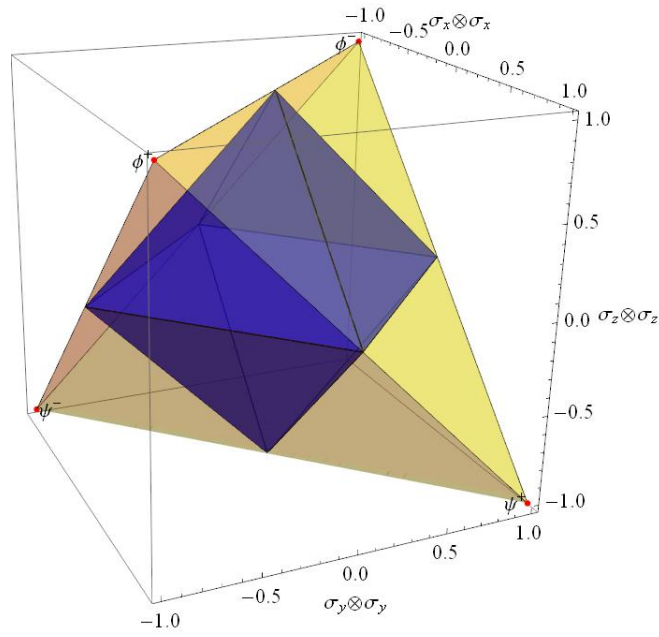


Figure 8.4: The shape of physical states is created by the positivity of the state which in turn gives conditions for the coefficients of the Bloch decomposition (yellow). The separable states can be found through the PPT criterion and form a double pyramid (blue) and lie within the physical states. Also they form a convex set.

$$\begin{aligned}\lambda_1 &= \frac{1}{4}(1 - c_{11} - c_{22} - c_{33}), \quad \lambda_2 = \frac{1}{4}(1 + c_{11} + c_{22} - c_{33}) \\ \lambda_3 &= \frac{1}{4}(1 + c_{11} - c_{22} + c_{33}), \quad \lambda_4 = \frac{1}{4}(1 - c_{11} + c_{22} + c_{33})\end{aligned}\tag{8.25}$$

The distance from the origin also gives the mixedness of the state. The state in the center is therefore the maximally mixed state (which can also be seen from the Bloch decomposition (8.22)).

It is also quite easy to embed the separable states in this picture. We just have to employ the PPT criterion (4.4) which gives us

$$\rho = \frac{1}{4} \left( \mathbb{1} + \sum_{i=1}^3 d_{ii} \sigma_i \otimes \sigma_i \right) \tag{8.26}$$

where

$$d_{ii} = c_{ii} \quad \forall i = 1, 3 \text{ and } d_{ii} = -c_{ii} \quad \forall i = 2 \tag{8.27}$$

Again we can look at the eigenvalues of the PPT density matrix and from there we get again eigenvalues which determine four planes that, combined with the positivity criterion for states, gives an octaeder, which is embedded in the tetraeder (Fig. 8.4). This way of depicting states gives immediately a pretty good intuition about entanglement of states.

Another feature that can be embedded in this picture is the nonlocality of states. We have already introduced a way of calculating the violation of a Bell inequality for a general two qubit density matrix (6.17). In our special case this reduces to

$$\begin{aligned}\max_{\mathcal{B}_{\text{CHSH}}} \langle \mathcal{B}_{\text{CHSH}} \rangle_\rho &= 2\sqrt{u_1 + u_2} \\ u_i &= c_{ii}^2 \text{ for the two bigger eigenvalues}\end{aligned}\tag{8.28}$$

From this condition we can already see that the violation of a bell inequality is not the same as entanglement for mixed states and this feature nicely translates into our picture (see Fig. 8.6).

In particular, this way of visualizing quantum states allows to draw the Werner state, since it has the decomposition (10.28). Therefore, the state is a straight line starting at a Bell state and ending at the maximally mixed state (see Fig. 8.6). From the picture we immediately see the difference between entanglement and nonlocality in this special case. Also for states in this picture we can see the entanglement witnesses which are just the surfaces of the separable octaeder.

However, this picture only depicts a subclass of states, so for states not lying within this picture we still have no intuition about their behaviour (at least geometrically). We therefore want to introduce an extension to this approach

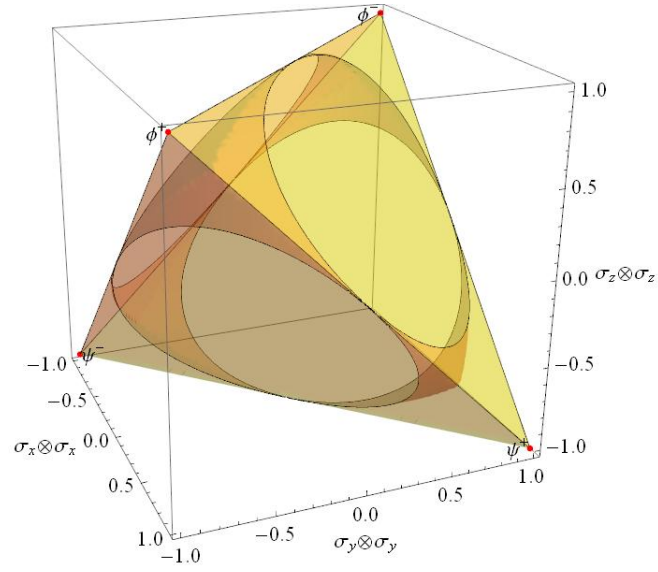


Figure 8.5: The set of physical states is bounded by the yellow surface, the region for locality is bounded by the orange surfaces.

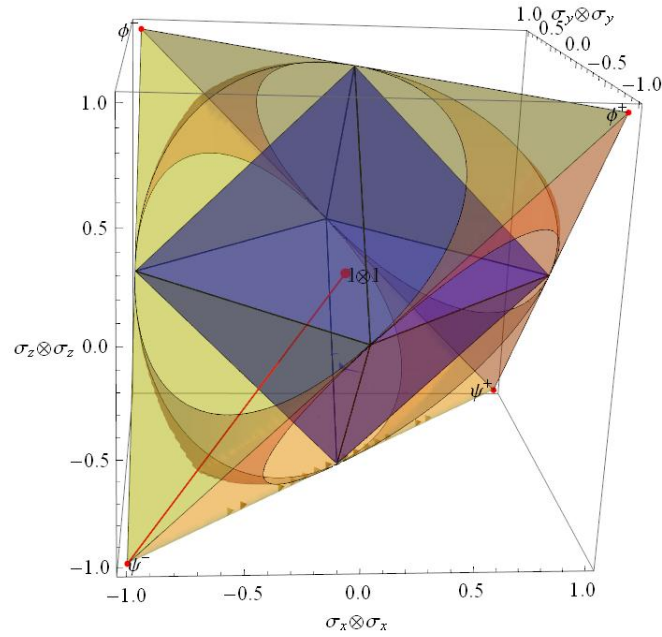


Figure 8.6: The Werner state, depicted by the red line. The difference between violation of a Bell inequality and separability can clearly be seen.



[34] which gives us the possibility to create visualizations for more degrees of freedom. This will be especially helpful when we will analyze a certain class of states, the Gisin states, which will be introduced in a later chapter. The key idea is the following. We will extend our class of states to states with the following Bloch decomposition.

$$\rho = \frac{1}{4} \left( \mathbf{1} \otimes \mathbf{1} + a (\sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z) + \sum_{i=1}^3 c_{ii} \sigma_i \otimes \sigma_i \right) \quad (8.29)$$

We therefore have more than three parameters. So in order to create a picture we fix the value of  $a$  to a satisfying value. The introduction of another component in the Bloch decomposition leads to different eigenvalues of the density matrix and thus to different conditions for positivity.

$$\begin{aligned} \lambda_1 &= \frac{1}{4}(1 - c_{11} - c_{22} - c_{33}), \quad \lambda_2 = \frac{1}{4}(1 + c_{11} + c_{22} - c_{33}) \\ \lambda_3 &= \frac{1}{4} \left( 1 - \sqrt{4a^2 + c_{11}^2 - 2c_{11}c_{22} + c_{22}^2 + c_{33}} \right) \\ \lambda_4 &= \frac{1}{4} \left( 1 + \sqrt{4a^2 + c_{11}^2 - 2c_{11}c_{22} + c_{22}^2 + c_{33}} \right) \end{aligned} \quad (8.30)$$

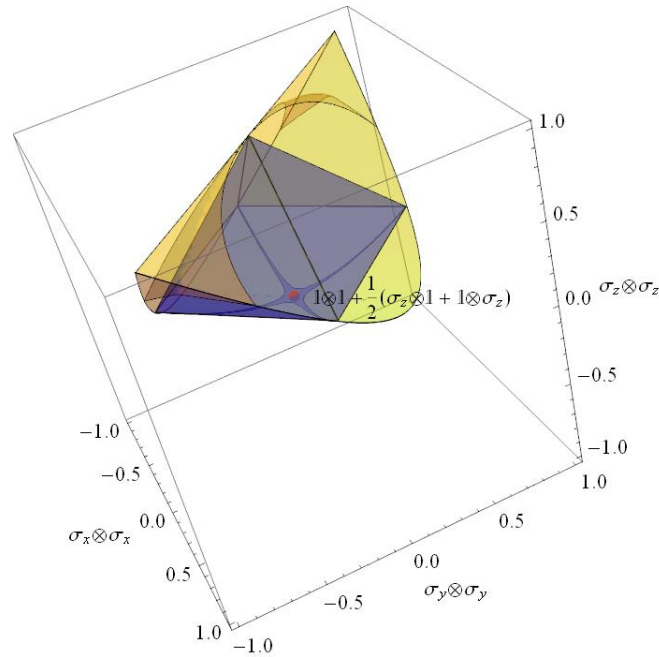


Figure 8.7: The set of physical states with an additional parameter  $a = \frac{1}{2}$ . The state  $\frac{1}{4} (\mathbf{1} \otimes \mathbf{1} + \frac{1}{2} (\sigma_z \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_z))$  lies in the origin.

As can be seen by those eigenvalues the shape of physical states is deformed (see Fig. 8.7). The same applies to the separability bounds, which again can

be found by looking at the eigenvalues of the PPT state. Because of the fixed choice of  $a$  this picture contains all the information about the entanglement (and nonlocality) of the state. Also the Bell bounds do not change because they only depend on the correlation matrix, which does not change due to the new parameter  $a$ . By changing  $a$  we change the deformation and thus can gather information about all states in our extended class.

This method can in principle be applied to any additional number of parameters added to the state. However, for the noncorrelation parts of the state (e.g.  $\vec{\sigma} \otimes \mathbb{1}, \mathbb{1} \otimes \vec{\sigma}$ ) we can always find a local unitary operation (which does not change the entanglement, see chapter: "Local unitary operations") that transforms this terms to the form  $\sigma_z \otimes \mathbb{1}, \mathbb{1} \otimes \sigma_z$ . So it is sufficient to consider such terms. But it is still a difficulty to reduce additional parameters arising from the correlation matrix (e.g.  $\sigma_x \otimes \sigma_y, \sigma_x \otimes \sigma_z, \dots$ ).

## 9 Entanglement under Unitary Operations

### 9.1 Local Unitary Operations

Local unitary operations play a crucial role in manipulating states. In photon experiments for example, the manipulation of the state through beam splitters and polarizers can be described with unitary operations. However they only give restricted possibilities to change the property of a state. A local unitary operation cannot change the entanglement of a state. We have already shown, how filtering operations (LOCC) can change a state. If we now look at (7.19) then a local unitary operation is equal to a filtering operation with  $a, b = 0$ . Thus the concurrence does not change under local unitary operations. Furthermore, under the entanglement of a state cannot be increased by operations of the form

$$G = \sum_i p_i U_i \otimes V_i \quad (9.1)$$

with  $U, V$  unitary operations. Let us give an example. If we start off with a maximally entangled state like  $\rho^-$  we can find a unitary operation that changes the state to another maximally entangled state e.g.  $\rho^+$ . The transformation in this case would be  $G = \mathbb{1} \otimes \sigma_z$ . Now if we find all the transformations such that the resulting states are all the maximally entangled states  $\rho^-, \rho^+, \omega^-, \omega^+$  then the final state after the transformation would be the unity matrix.

In general, this can be explained with the convexity of the separable states. Since every state with the same amount of entanglement has the same Hilbert Schmidt distance from its nearest separable state, a linear combination of two such states can only be closer or as close to the set of separable states.

### 9.2 Factorization of Density Matrices

We have seen before, that entanglement can not be increased by local unitary operations. Now we want to turn to global unitary transformations which act on the whole system of the state [34, 35, 36]. This can also be interpreted as changing the algebra of the state, namely if we work on a Hilbert Schmidt space  $\mathcal{H}_A \otimes \mathcal{H}_B$  we define an algebra  $M^D$  of our space where  $D$  denotes the dimension of the total space. A state is called separable with respect to the factorization  $M^{d_1} \otimes M^{d_2}$  iff a density matrix  $\rho$  can be written as

$$\rho = \sum_i \rho_{i1} \otimes \rho_{i2} \quad (9.2)$$

as defined before (Eq. (4.3)). We can now change the algebra via global unitary transformations, so that our algebra becomes  $U(M^{d_1} \otimes M^{d_2})U^\dagger$ . A state can become entangled through such a transformation, or vice versa, an entangled state becomes separable. We will investigate this behaviour further. First we will consider pure states.

**Theorem 1** (Factorization Algebra). *For any pure state  $\rho$  one can find a factorization  $M^D = \mathcal{A}_1 \otimes \mathcal{A}_2$  such that  $\rho$  is separable with respect to this factorization and another factorization  $M^D = \mathcal{B}_1 \otimes \mathcal{B}_2$  where  $\rho$  appears to be maximally entangled.*

**Proof 9.3.** *We can transform any state vector to any state vector by unitary transformations. Thus a state  $|\psi\rangle$  can be transformed to  $U|\psi\rangle = |\psi\rangle_1 \otimes |\psi\rangle_2$ . It is also possible to transform the state to a maximally entangled state, i.e.  $V|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |\psi\rangle_{i1} \otimes |\psi\rangle_{i2}$ . Or translated to density matrices  $\rho_{ent} = V\rho V^\dagger$  and  $\rho_{sep} = U\rho U^\dagger$ . But we can also transform the factorization algebras of the state with the same unitary operations  $M^D = U(M^{d_1} \otimes M^{d_2})U^\dagger$ , and therefore a state that originally was entangled can appear separable under this choice.  $\square$*

Now we will move on to the case of mixed states. There it is not that easy to determine the action of unitary operations. For example there is no way of changing the maximally mixed state  $\frac{1}{d}\mathbb{1}$  via a unitary transformation to an entangled state. However it is at least always possible to transform an arbitrary state to a separable one. This can be easily seen from the fact that if we consider a density matrix

$$\rho = \sum_i \rho_i |\chi_i\rangle\langle\chi_i| \quad (9.3)$$

we can always find a unitary transformation  $U$  that transforms the set of states  $\{|\chi_i\rangle\}$  to  $U|\chi_i\rangle = |\phi_{\alpha i}\rangle \otimes |\psi_{\beta i}\rangle$  where  $\{|\phi_{\alpha}\rangle\}$  and  $\{|\psi_{\beta}\rangle\}$  are a basis of the subsystems of the considered state. Thus the state becomes

$$U\rho U^\dagger = \sum_i \rho_i |\phi_{\alpha i}\rangle\langle\phi_{\alpha i}| \otimes |\psi_{\beta i}\rangle\langle\psi_{\beta i}| \quad (9.4)$$

and is clearly separable. On the other hand one might be tempted to assume that a decomposition into bell states (or more generally into maximally entangled states) could be the optimal choice for a factorization such that the resulting states is as entangled as possible. Such a decomposition can be written as

$$|\chi_{kl}\rangle = \sum_j e^{\frac{2\pi i}{d}jl} |\phi_j\rangle \langle \psi_{j+k}|. \quad (9.5)$$

With this unitary operation we can achieve a so-called Weyl state which is spanned by the maximally entangled states. The entanglement and separability of Weyl states is well known [37, 38, 39, 40, 41, 42, 43], thus it would be convenient to reduce the problem to this class of states. However because the set of entangled states is not convex, we cannot conclude that the resulting state carries the optimal entanglement that is achievable.

If we again restrict ourselves to  $2 \times 2$  dimensions, entanglement measures are well known as defined earlier. There exist better factorizations than (9.5) that enable us to achieve more entanglement out of a given state. One example would be the Werner state (5.48), where our decomposition is not favourable. We will therefore give another decomposition. Let us consider a mixed state  $\rho$  with the ordered spectrum  $\{\rho_1 \geq \rho_2 \geq \dots \geq \rho_{d^2-2} \geq \rho_{d^2-1} \rho_{d^2}\}$ . Thus we can decompose this state into its projectors.

$$\rho = \rho_1 P_1 + \rho_2 P_2 + \dots + \rho_{d^2-2} P_{d^2-2} \rho_{d^2-1} P_{d^2-1} \rho_{d^2} P_{d^2} \quad (9.6)$$

We will choose our projectors in such a way that one two-qubit subspace will be maximally entangled.

$$\begin{aligned} P_1 &= \frac{\rho_1}{2} |11 + 22\rangle \langle 11 + 22| \\ P_2 &= \rho_2 |12\rangle \langle 12| \\ P_3 &= \frac{\rho_3}{2} |11 - 22\rangle \langle 11 - 22| \\ P_4 &= \rho_4 |21\rangle \langle 21| \end{aligned} \quad (9.7)$$

From this factorization we can conclude

**Theorem 2.** *If  $\rho_1 > \frac{3}{d^2}$  i.e. the largest eigenvalue is bounded below by  $\frac{3}{d^2}$  then there is always a choice of factorization possible such that the partial algebras are entangled.*

**Proof 9.4.** *To find entanglement we will apply the PPT criterion to our density matrix  $\rho$  with the choice (9.7). The partially transposed density matrix will contain the following structure*

$$\rho^{PT} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \rho_2 & \frac{1}{2}(\rho_1 - \rho_3) & \cdot \\ \cdot & \frac{1}{2}(\rho_1 - \rho_3) & \rho_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (9.8)$$

## 9 Entanglement under Unitary Operations

The state  $\rho$  will be entangled if the density matrix contains a negative eigenvalue. Thus we get the following condition for the eigenvalues

$$\rho_1 - \rho_3 - 2\sqrt{\rho_2\rho_4} < 0 \quad (9.9)$$

which is a geometric mean. Instead of the geometric mean we will consider a weaker condition, the arithmetic mean value. Thus we get

$$\begin{aligned} \frac{\rho_2 + \rho_4}{2} &< \frac{1}{2}(\rho_1 - \rho_3) \\ \Rightarrow \rho_2 + \rho_3 + \rho_4 &< \rho_1 \end{aligned} \quad (9.10)$$

But we also know that

$$\rho_2 + \rho_3 + \rho_4 < \frac{1 - \rho_1}{d^2 - 3} \quad (9.11)$$

which leads us to

$$3\frac{1 - \rho_1}{d^2 - 3} < \rho_1 \quad \text{or} \quad \rho_1 > \frac{3}{d^2} \quad \square \quad (9.12)$$

Thus there are states which obey the constraints of Theorem 3 that are entangled with respect to certain factorizations and separable with respect to others. We will now try to find a criterion that allows us to characterize those states that remain separable under any unitary transformation. Or to put it differently, where we cannot find a factorization of our system into subsystems  $\mathcal{A}_1 \otimes \mathcal{A}_2$ . Such states are called *absolutely separable states* [44, 45, 46]. A first idea of how to determine these states is given by the fact that unitary operations do not change the mixedness of a given system.

$$\text{Tr}\rho^2 = \text{Tr}U\rho U^\dagger U\rho U^\dagger = \text{Tr}U\rho^2 U^\dagger \quad (9.13)$$

This can be used to define a Hilbert Schmidt distance

$$d(\rho, \mathbb{1}_D) = \left\| \rho - \frac{1}{D} \mathbb{1}_D \right\| = \sqrt{\text{Tr}(\rho - \frac{1}{D} \mathbb{1}_D)^2} \quad (9.14)$$

from the maximally mixed state  $\frac{1}{D} \mathbb{1}_D$  which in turn implies a sphere that has the same mixedness on all points of its surface [47]. Thus we can inscribe such a sphere into the separable states and get the *absolutely separable ball*. (see Fig. 9.1)

**Theorem 3.** *All states belonging to the maximal ball which can be inserted into the set of mixed states for a bipartite system are not only separable but also absolutely separable.*

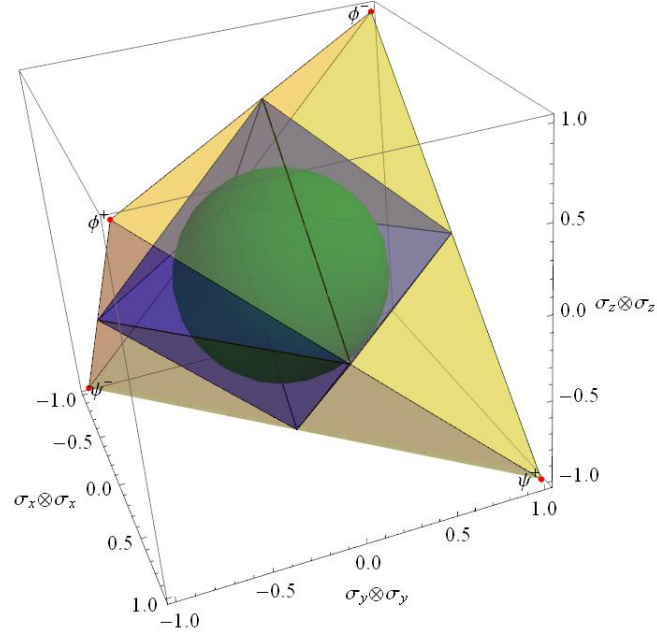


Figure 9.1: The absolutely separable ball (green) is inscribed in the set of separable states (blue).

But it turns out that this absolutely separable ball does not contain all the states that are absolutely separable. The space of these states is even a bit larger. In  $2 \times 2$  dimensions an even better description is known, which was found by Ishizaka and Hiroshima [48] and proven by Verstraete, Audenaard and DeMoor [49]. If a state fulfills the following constraints on its ordered spectrum it is absolutely separable.

$$\rho_1 - \rho_3 - 2\sqrt{\rho_2\rho_4} \leq 0 \quad (9.15)$$

One example was found in [44]. If a state has the spectrum  $\{0.47, 0.30, 0.13, 0.10\}$ , it does not belong to the absolutely separable ball, but satisfies (9.15), and is therefore absolutely separable. The convex set that is defined by (9.15) can be seen in Fig. 9.2.

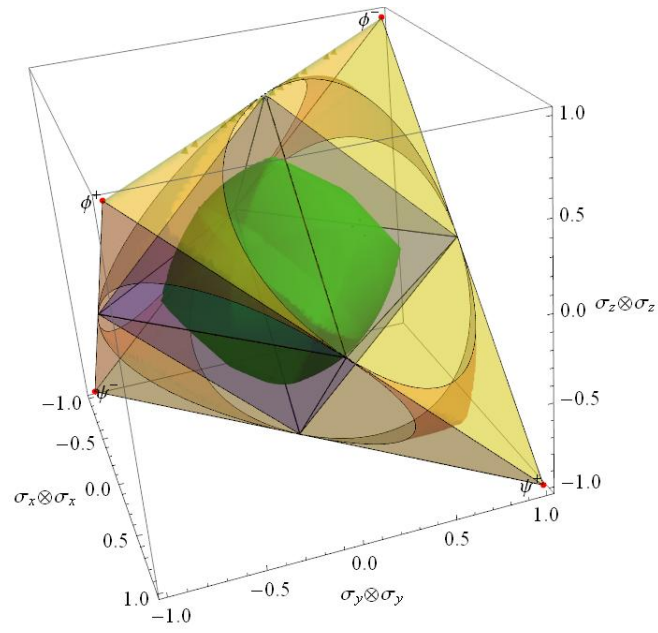


Figure 9.2: The set of absolutely separable states is deformed to a "Lobel" (green).



## 10 Examples of Global Unitary Operations

### 10.1 Separable but not Absolutely Separable States

Let us now turn to some examples to visualize the effects of global unitary operations [34]. We will consider the following state.

$$\rho_N = \frac{1}{2}(\rho^+ + \omega^+) = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad (10.1)$$

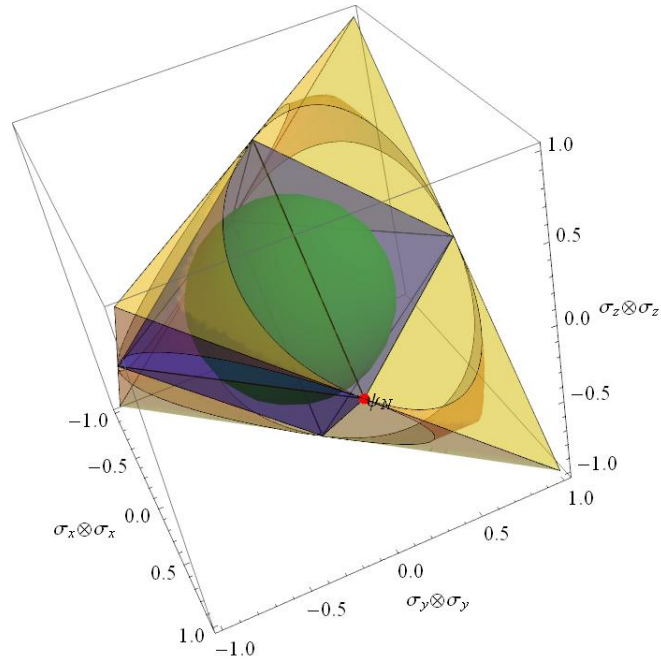


Figure 10.1: The state  $\rho_N$  is located at the corner of the separable states. However it lies outside the absolutely separable states.

This state is located at the edge of the separable states (see Fig. 10.1). However this state is not absolutely separable since the transformation

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (10.2)$$

transforms the state to

$$U\rho_N U^\dagger = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10.3)$$

or written in its Bloch decomposition

$$U\rho_N U^\dagger = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} + \frac{1}{2} (\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \frac{1}{2} (\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z) \right) \quad (10.4)$$

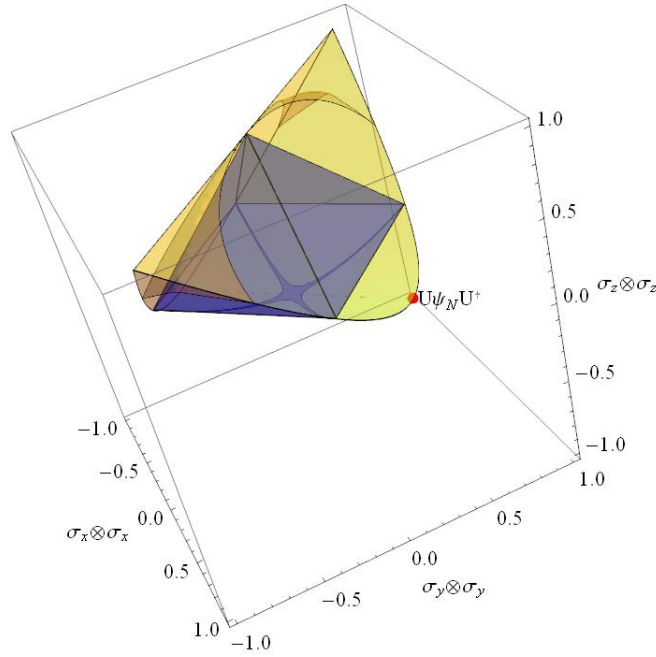


Figure 10.2: The state  $U\rho_N U^\dagger$  lies outside the separable states, which proves that the initial could not be an absolutely separable state.

Here we encounter terms of the form  $(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z)$  which can be visualized by the method we developed in chapter 8 (Eq. (8.30)). As can be seen in the such generated picture (Fig. 10.2), the state has become entangled. Note, that this state also yields the maximal entanglement for its mixedness, and

therefore belongs to the class of MEMS (maximally entangled mixed states) [48, 49, 50]. The entanglement of the transformed state can be calculated with the concurrence (5.22) and gives

$$C(U\rho_N U^\dagger) = \frac{1}{2}. \quad (10.5)$$

## 10.2 Alice and Bob

We will consider now two qubits of Alice and Bob. The standard basis for two qubits is

$$\begin{aligned} \rho_{\uparrow\uparrow} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{\uparrow\downarrow} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \rho_{\downarrow\uparrow} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_{\downarrow\downarrow} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \quad (10.6)$$

Whereas the Bell states have the following matrix form

$$\omega^\pm = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^\pm = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm 1 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10.7)$$

The Bell states can be transformed to the states  $|\uparrow\rangle \otimes |\uparrow\rangle, |\uparrow\rangle \otimes |\downarrow\rangle, |\downarrow\rangle \otimes |\uparrow\rangle, |\downarrow\rangle \otimes |\downarrow\rangle$  by the unitary transformation

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad (10.8)$$

To show the effect on the subalgebras we will also take a look on the Bloch decomposition of the case  $\rho^-, \rho_{\uparrow\downarrow}$ .

$$\begin{aligned} \rho^- &= \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} - \vec{\sigma} \otimes \vec{\sigma}) \\ \rho_{\uparrow\downarrow} &= \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z - \sigma_z \otimes \sigma_z) \end{aligned} \quad (10.9)$$

## 10 Examples of Global Unitary Operations

If we now apply our unitary operation  $U$ , we have the following changes of our algebra for Alice and Bob

$$\begin{aligned}\sigma_x \otimes \mathbb{1} &\xrightarrow{U} \sigma_x \otimes \mathbb{1}, \quad \mathbb{1} \otimes \sigma_x \xrightarrow{U} \sigma_x \otimes \sigma_z \\ \sigma_y \otimes \mathbb{1} &\xrightarrow{U} -\sigma_z \otimes \sigma_y, \quad \mathbb{1} \otimes \sigma_y \xrightarrow{U} \mathbb{1} \otimes \sigma_y \\ \sigma_z \otimes \mathbb{1} &\xrightarrow{U} \sigma_y \otimes \sigma_y, \quad \mathbb{1} \otimes \sigma_z \xrightarrow{U} -\sigma_x \otimes \sigma_x\end{aligned}\tag{10.10}$$

and in turn implies a change in the correlation terms as well

$$\begin{aligned}\sigma_x \otimes \sigma_x &\xrightarrow{U} \mathbb{1} \otimes \sigma_z \\ \sigma_y \otimes \sigma_y &\xrightarrow{U} -\sigma_z \otimes \mathbb{1} \\ \sigma_z \otimes \sigma_z &\xrightarrow{U} \sigma_z \otimes \sigma_z\end{aligned}\tag{10.11}$$

Now in order to stress the difference between changing the algebra and simply perform a unitary transformation lets have a look at the means of detecting entanglement of a state. If we consider an optimal entanglement witness (4.10), the in our case we have

$$A_{opt}^{\rho^-} = \frac{1}{2\sqrt{3}} \left( \mathbb{1} \otimes \mathbb{1} + \vec{\sigma} \otimes \vec{\sigma} \right).\tag{10.12}$$

From Eq. (4.10) and (4.11) we get

$$\begin{aligned}\langle \rho^- | A_{opt}^{\rho^-} \rangle &= -\frac{1}{\sqrt{3}} < 0 \\ \langle \rho_{sep} | A_{opt}^{\rho^-} \rangle &= \frac{1}{2\sqrt{3}} \left( 1 + \vec{n} \cdot \vec{m} \right) \geq 0\end{aligned}\tag{10.13}$$

where  $\vec{n}, \vec{m}$  are the bloch vectors of the separable states. If we now transform our state unitarily, we cannot detect entanglement with our witness anymore

$$\langle U \rho^- U^\dagger | A_{opt}^{\rho^-} \rangle = 0\tag{10.14}$$

since the resulting state is no longer entangled. But if we change the algebra of our system we have to take into account the change of our witness as well. Thus we end up with

$$\langle U \rho^- U^\dagger | U A_{opt}^{\rho^-} U^\dagger \rangle = -\frac{1}{\sqrt{3}} < 0\tag{10.15}$$

So even a separable state can act as if entangled given the right factorization of the algebra.

If we apply our unitary transformation to a nonmaximally entangled state

$$|\psi\rangle = \sin(\theta)|\uparrow\downarrow\rangle - \cos(\theta)|\downarrow\uparrow\rangle \quad (10.16)$$

or written as a density matrix

$$\rho_\theta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2(\theta) & -\cos(\theta)\sin(\theta) & 0 \\ 0 & -\cos(\theta)\sin(\theta) & \cos^2(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10.17)$$

Transforming this state with the unitary operation (10.8) leads to the state

$$U\rho_\theta U^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \cos(\theta)\sin(\theta) & -\frac{1}{2}\cos(2\theta) & 0 \\ 0 & -\frac{1}{2}\cos(2\theta) & \frac{1}{2} - \cos(\theta)\sin(\theta) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10.18)$$

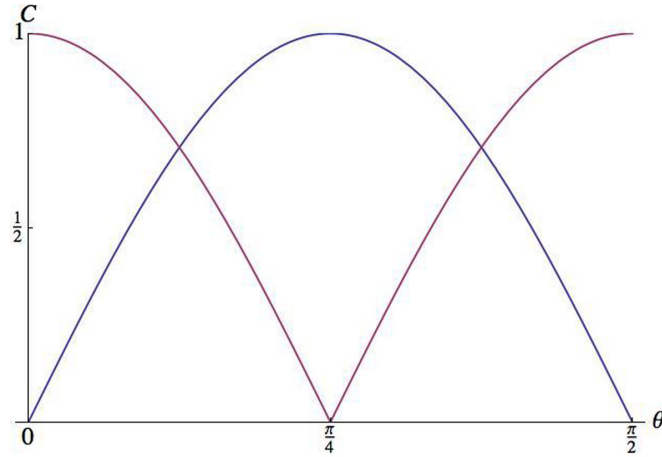


Figure 10.3: The state  $\rho_\theta$  is maximally entangled for  $\theta = \frac{\pi}{4}$  (blue). The state  $U\rho_\theta U^\dagger$  is maximally entangled for  $\theta = 0, \frac{\pi}{2}$  (purple).

We now want to look at the concurrence of the state and its transformed counterpart. (see Fig. 10.3) The concurrences of the state  $\rho_\theta, U\rho_\theta U^\dagger$  are

$$C(\rho_\theta) = \sin(2\theta), \quad C(U\rho_\theta U^\dagger) = \cos(2\theta). \quad (10.19)$$

So only for certain values of  $\theta$  ( $\theta = 0, \frac{\pi}{4}$ ) can this transformation be considered optimal. An optimal choice would be the unitary transformation  $V$ .

$$V = \begin{pmatrix} \frac{\cos(\theta)-\sin(\theta)}{\sqrt{2}} & 0 & 0 & -\frac{\cos(\theta)+\sin(\theta)}{\sqrt{2}} \\ 0 & \frac{\cos(\theta)-\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)+\sin(\theta)}{\sqrt{2}} & 0 \\ 0 & -\frac{\cos(\theta)+\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)-\sin(\theta)}{\sqrt{2}} & 0 \\ \frac{\cos(\theta)+\sin(\theta)}{\sqrt{2}} & 0 & 0 & \frac{\cos(\theta)-\sin(\theta)}{\sqrt{2}} \end{pmatrix} \quad (10.20)$$

Then the resulting state will be always the state  $\rho^+$ . Also if we apply  $U$  to the resulting state, we will always end up with a separable state.

### 10.3 GHZ State

We will now look at the GHZ state [51, 52] first introduced by Greenberger, Horne and Zeilinger.

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \quad (10.21)$$

Or for our purposes a more generalized version of it

$$|\psi\rangle_{GHZ} = \sin(\theta)|\uparrow\uparrow\uparrow\rangle - \cos(\theta)|\downarrow\downarrow\downarrow\rangle. \quad (10.22)$$

Next we want to trace out one subsystem, so that we only consider the part that can be seen from Alice's and Bob's point of view.

$$\rho_{\theta}^{GHZ} = \begin{pmatrix} \sin^2(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2(\theta) \end{pmatrix} \quad (10.23)$$

This state is a mixed state and separable for all  $\theta$ . We can now look for unitary operations that maximize the entanglement within this state. The optimal unitary transformation for  $0 \leq \frac{\pi}{4}$

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (10.24)$$

and for  $\frac{\pi}{4} \leq \frac{\pi}{2}$

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (10.25)$$

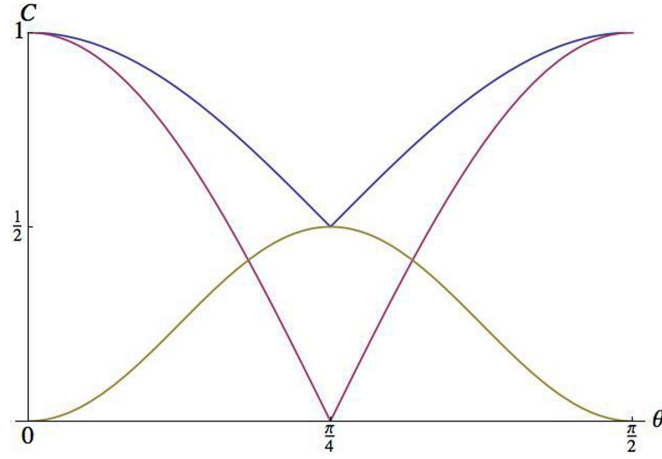


Figure 10.4: The unitary transformations  $U_1$  and  $U_2$  optimize the entanglement of the traced generalized GHZ state (blue). The unitary transformation  $U$  is not optimal (purple) and even fails to create entanglement for  $\theta = \frac{\pi}{4}$ . The brown line depicts the mixedness of the state. The minimal entanglement is reached for the maximal mixedness.

is the optimal choice. This is illustrated in Fig. 10.4. The such transformed states are

$$U_1 \rho_\theta^{GHZ} U_1^\dagger = \begin{pmatrix} \cos^2(\theta) & 0 & 0 & 0 \\ 0 & \frac{\sin^2(\theta)}{2} & \frac{\sin^2(\theta)}{2} & 0 \\ 0 & \frac{\sin^2(\theta)}{2} & \frac{\sin^2(\theta)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10.26)$$

$$U_2 \rho_\theta^{GHZ} U_2^\dagger = \begin{pmatrix} \sin^2(\theta) & 0 & 0 & 0 \\ 0 & \frac{\cos^2(\theta)}{2} & \frac{\cos^2(\theta)}{2} & 0 \\ 0 & \frac{\cos^2(\theta)}{2} & \frac{\cos^2(\theta)}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (10.27)$$

Note that the unitary transformation  $U$  (10.8) is not optimal. It even fails to create entanglement for  $\theta = \frac{\pi}{4}$ .

## 10.4 Werner States

We have already looked at the Werner state to give an example for entanglement witnesses (5.53), (5.54) and the difference between the violation of a Bell inequality and entanglement (6.21), Fig. 8.6. Now we want to emphasize a property of the choice of the factorization algebra.

We already know the Werner state (5.48). Its Bloch decomposition reads

$$\rho_W = \frac{1}{4} \left( \mathbb{1} \otimes \mathbb{1} - \alpha \vec{\sigma} \otimes \vec{\sigma} \right). \quad (10.28)$$

Transforming  $\rho_W$  with our unitary transformation (10.8) we obtain

$$\begin{aligned} U\rho_W U^\dagger &= \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} + \alpha(\sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z) - \sigma_z \otimes \sigma_z) \\ &= \frac{1}{4} \begin{pmatrix} 1-\alpha & 0 & 0 & 0 \\ 0 & 1+3\alpha & 0 & 0 \\ 0 & 0 & 1-\alpha & 0 \\ 0 & 0 & 0 & 1-\alpha \end{pmatrix} \end{aligned} \quad (10.29)$$

This state is separable for all  $\alpha$  since the entanglement witness (5.52) gives

$$\langle U\rho_W U^\dagger | C \rangle = \text{Tr}(U\rho_W U^\dagger C) = \frac{1}{2\sqrt{3}}(1-\alpha) \geq 0. \quad (10.30)$$

But if we choose a different factorization of our algebra, we can again detect entanglement

$$\langle U\rho_W U^\dagger | UCU^\dagger \rangle = \text{Tr}(U\rho_W U^\dagger UCU^\dagger) = \frac{1}{2\sqrt{3}}(1-3\alpha) < 0 \quad (10.31)$$

for  $\alpha > \frac{1}{3}$ . Note that even with an optimal factorization it is not possible to find all the states as entangled, since we need a certain amount of purity.

## 10.5 Gisin States

Another example we want to study is the Gisin state, first introduced by Nicolas Gisin [30]. His goal was to show that certain states that do not violate a Bell inequality, can do so after a LOCC operation (see sections "LOCC operations" and "Filtering operations" and also [53]). He used a special form of LOCC, the filtering operation. The state he used was the following

$$\rho_G(\lambda, \theta) = \lambda \rho_\theta + \frac{1}{2}(1-\lambda)(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}) \quad (10.32)$$

with  $0 \leq \lambda \leq 1$ . The Bloch form of this state is

$$\begin{aligned} \rho_G(\lambda, \theta) &= \frac{1}{4} (\mathbb{1} \otimes \mathbb{1} - \lambda \cos(2\theta)(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) \\ &\quad - \lambda \sin(2\theta)(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) + (1-2\lambda)\sigma_z \otimes \sigma_z) \end{aligned} \quad (10.33)$$

We have already discussed the Horodecki theorem, to calculate the violation of a Bell inequality through the correlation matrix of the Bloch decomposition (6.17). Thus there is a violation if the following condition holds.



$$\max_{\mathcal{B}_{\text{CHSH}}} \langle \mathcal{B}_{\text{CHSH}} \rangle_\rho = \max\{(2\lambda - 1)^2 + \lambda^2 \sin^2(2\theta), 2\lambda^2 \sin^2(2\theta)\} > 1 \quad (10.34)$$

This leads to no violation for

$$\lambda \leq \frac{1}{\sqrt{2} \sin(2\theta)} \quad (10.35)$$

if we assume that

$$\lambda \leq \frac{1}{2 - \sin(2\theta)} \quad (10.36)$$

Now Gisin applied the following filtering operation to improve the range for which the Gisin state  $\rho_G$  violates the Bell inequality.

$$F = \begin{pmatrix} \sqrt{\frac{\cos(\theta)}{\sin(\theta)}} & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{\cos(\theta)}{\sin(\theta)}} \end{pmatrix} \quad (10.37)$$

The corresponding filtered state becomes

$$\rho_G^f = \frac{\lambda \sin(2\theta) \rho^- + \frac{1}{2}(1 - \lambda)(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow})}{\lambda \sin(2\theta) + (1 - \lambda)} \quad (10.38)$$

The Bloch decomposition takes the form

$$\begin{aligned} \rho_G^f = \frac{1}{4} & \left( \mathbb{1} \otimes \mathbb{1} + \frac{\lambda \sin(2\theta)}{1 + \lambda \sin(2\theta)} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) \right. \\ & \left. - \frac{1 - \lambda - \lambda \sin(2\theta)}{1 - \lambda + \lambda \sin(2\theta)} \sigma_z \otimes \sigma_z \right) \end{aligned} \quad (10.39)$$

and we do not get a violation of the Bell inequality for

$$\lambda \leq \frac{1}{1 - \sin(2\theta)(\sqrt{2}) - 1}. \quad (10.40)$$

We could have also used the formalism we derived earlier to predict the maximal violation of a Bell inequality by using the results (7.29). We can plot the parameter  $\lambda$  versus the concurrence of the Gisin state (10.32) (green line) and the filtered Gisin state (10.38) (purple line) for a chosen  $\theta = 0.35$  (Fig. 10.5). We also included the violation of a Bell inequality for those states as vertical lines, which depict the parameter  $\lambda$  of which a violation can be detected. We can see that the parameter range that admits a violation increases for the filtered

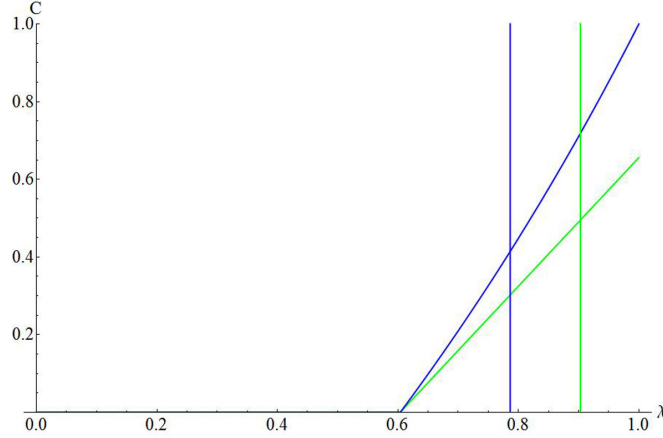


Figure 10.5: The concurrence of the Gisin state before (green) and after (purple) the filtering operation. The parameter  $\theta$  is fixed to 0.35. The vertical lines in the picture denote the bound for violation of a Bell inequality.

state. Thus the conclusion of Gisin was, that certain states that are considered local, can be nonlocal after an applied filtering operation.

The approach of Gisin is certainly different to what we do. A filtering operation is a LOCC operation and thus does not need to preserve unitarity, but it has to be a *local* operation. In contrast, our goal was to show the behaviour of *nonlocal unitary* operations.

If we apply the unitary operation (10.20) to our state

$$\rho_G^U(\lambda) = U\rho_G U^\dagger = \lambda\rho^+ + \frac{1}{2}(1-\lambda)(\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}) \quad (10.41)$$

we obtain a state that is not dependent on  $\theta$  anymore. To compare our two results we will plot the concurrence of the states versus the purity  $P(\rho) = \text{Tr}\rho^2$ . The results can be seen in Fig. 10.6. The initial state is depicted by the green dots. It has the least amount of concurrence. The blue dots symbolize the Gisin states after a filtering operation, whereas the red dots denote the unitary transformed Gisin states. It can be seen that the filtering operation increases the mixedness of the state but increases the concurrence, thus the Bell inequality can be violated. The unitary operation also increases the entanglement of the state but without touching the purity. Every dot in Fig. 10.6 has a certain value of  $\lambda$  between  $0 \leq \lambda \leq 1$ . We can also look at the Bell violation versus the concurrence. There exist bounds for the ratio between nonlocality and concurrence, which were found by Verstraete and Wolf [54]. The upper bound is

$$B = \sqrt{1 - C^2} \quad (10.42)$$

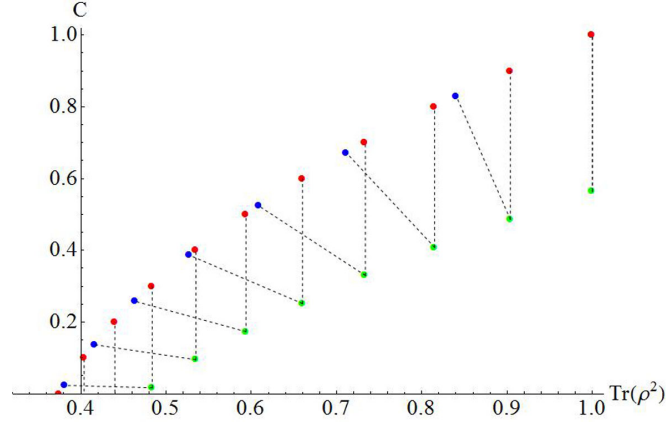


Figure 10.6: The green dots describe the concurrence and purity of the unfiltered Gisin state. After the filtering the purity reduces but the entanglement increases (blue dots). For the unitary transformation, the purity does not change, however the entanglement increases (red dots).

and the lower bound is

$$B = \max\{1, \sqrt{2}C\} \quad (10.43)$$

All states are plotted in Fig. 10.7. The two black lines depict the Verstraete-Wolf bounds. The blue and purple lines are the Gisin and Werner states. The green line depicts the filtered and unitary transformed states. For the values  $\lambda = 0.86, \theta = 0.4$  the blue, green and red dots symbolize the Gisin state, the filtered state and the unitary transformed state respectively. In this example, again the Gisin state does not violate the Bell inequality (blue dot, below  $B = 1$ ) whereas the transformed states do (green, red dots).

We can also visualize these states since the Bloch decomposition is compatible with our method (8.30). The yellow surface is again the bound for positivity, the blue surface describes the set of separable states and the orange surface gives the bound for locality. For the parameters  $\lambda = 0.8$  and  $\theta = 0.35$  the Gisin state lies within the set of local states, but is already entangled (see Fig. 10.8). If we now carry out our operations we arrive at the following picture (Fig. 10.9). All filtered or unitarily transformed states lie on a line between a maximally entangled state and the separable state  $\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}$ . Both the states (the filtered and the unitary transformed) are nonlocal. The filtered state lies closer to the origin of the graph since it has a higher mixedness. This is also connected to (5.61) since the transformed states are of this form. And indeed the state becomes separable for  $\beta \leq \frac{1}{2}$ .

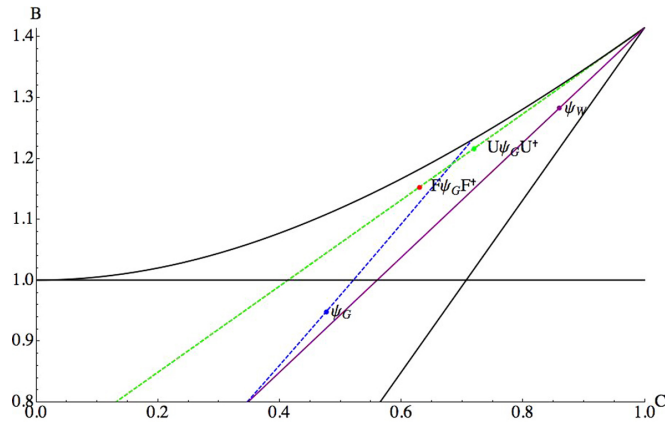


Figure 10.7: The Verstraete-Wolf bounds are depicted by the black lines. The Gisin state (blue), the Werner state (purple) and the filtered state and unitary transformed state (green) lie within these boundaries. The dots are states with values of  $\lambda = 0.86$  and  $\theta = 0.4$ , where the blue dot represents the Gisin state, the green dot denotes the filtered state and the blue dot describes the unitary transformed state.

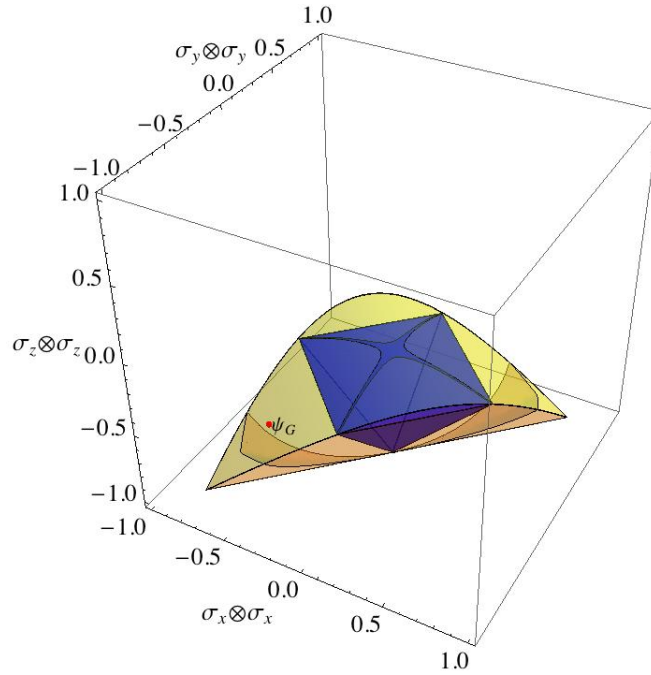


Figure 10.8: The Gisin state with parameters  $\lambda = 0.8$  and  $\theta = 0.35$  lies within the region of local states but is already entangled.

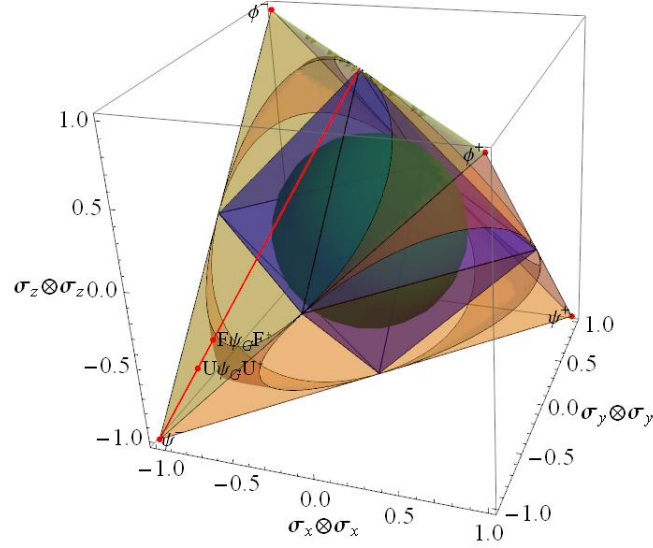


Figure 10.9: The filtered state and the unitarily transformed state lie on a line between the maximally entangled state and the separable state  $\rho_{\uparrow\uparrow} + \rho_{\downarrow\downarrow}$ . For the chosen parameters  $\lambda = 0.8$  and  $\theta = 0.35$  both states violate a Bell inequality.

## 10.6 Quantum Teleportation

A more general example of the impact of entanglement can be found in the process of quantum teleportation [55, 56, 57]. If we consider three qubits, e.g. photons, we can transfer the information of one qubit to another qubit with the help of entanglement. For example, If Bob has an entangled pair of photons, he can transfer one of his photons to Alice, who has the photon to teleport. She then applies a joint measurement on her and Bob's qubit which results in the transfer of the information to Bob's second qubit, that he didn't send over to Alice. These experiments were carried out and rely on a measurement procedure, the Bell measurement which basically changes the algebra we are considering from a joint system of Bob and a single qubit of Alice to a joint system of Alice combined with a single qubit of Bob. The procedure for qubits is well known, but we want to give a more general description of this phenomenon.

We study the tensor product of three matrix algebras  $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ , which all have the same dimension.  $\mathcal{A}_1$  is the algebra of the initial state Alice wants to teleport, while  $\mathcal{A}_2$  and  $\mathcal{A}_3$  are the algebras that describe the maximally entangled state of Bob. Alice now has the state  $|\phi\rangle$  she wants to teleport defined on  $\mathcal{A}_1$ . The goal is to transfer this state to  $\mathcal{A}_3$  without direct contact between Alice and Bob. Because of the maximal entanglement between Bob's states, there exists an *isometry* (a bijective map that preserves the distances) on  $\mathcal{A}_2$  and  $\mathcal{A}_3$  between the vectors of one factor and the other. Now in order to be able to teleport the state from Alice to Bob we have to assume that there also exists an isometry on  $\mathcal{A}_1$  and  $\mathcal{A}_3$ . Now Alice chooses an isometry on  $\mathcal{A}_1, \mathcal{A}_2$  which

is the Bell measurement in the experiment. We can now write down a relation between those isometries.

$$I_{12} \cdot I_{23} = I_{13} \quad (10.44)$$

If we restrict ourselves to an orthonormal basis we can write

$$|\psi\rangle_{12} = \sum_{i=1}^d |\phi_i\rangle_1 \otimes |I_{12}\phi_i\rangle_2. \quad (10.45)$$

If a measurement is carried out in that way, then we automatically teleport the initial state to Bob if he had this particular maximally entangled state in the first place. If Alice receives another maximally entangled state, the result on Bob's side only defers by a unitary operation which has to be carried out on his end. Thus Alice just has to communicate her measurement result to Bob for him to find the appropriate unitary operation. The measurement process can be denoted as follows

$$(|\psi\rangle\langle\psi|_{12} \otimes \mathbb{1}_3)|\phi\rangle_1 \otimes |\psi\rangle_{23} = \frac{1}{d^2} |\psi\rangle_{12} \otimes |\phi\rangle_3 \quad (10.46)$$

$$(U_{12}|\psi\rangle\langle\psi|_{12}U_{12}^\dagger \otimes \mathbb{1}_3)|\phi\rangle_1 \otimes |\psi\rangle_{23} = \frac{1}{d^2} U_{12}|\psi\rangle_{12} \otimes U_3|\phi\rangle_3. \quad (10.47)$$

One way to make this more usable in practical examples, is to fix the isometry  $I_{12}$  to  $I_{12} = \mathbb{1}$ . Then the states read

$$|\psi\rangle_{23} = \sum_{i=1}^{d^2} |\phi_i\rangle_2 \otimes |\phi_i\rangle_3 \quad (10.48)$$

$$|\phi\rangle_{1 \text{ or } 3} = \sum_{i=1}^d \alpha |\phi_i\rangle_{1 \text{ or } 3} \quad (10.49)$$

We can see that the usual result for teleportation lies within this description. However, this result is completely independent of the dimension of the states or the chosen basis used.

## 10.7 Entanglement Swapping

Another phenomenon, closely related to quantum teleportation, is entanglement swapping [58, 59]. The setup is the following. We again consider an entangled

state on Bob's side, but this time also on Alice's side. This gives us four states to consider with corresponding algebras  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ . So at the beginning we have the following situation

$$|\omega\rangle = |\psi\rangle_{12} \otimes |\psi\rangle_{34}. \quad (10.50)$$

Again we have maximally entangled states in these two subsystems. We apply a Bell measurement, this time between the states of  $\mathcal{A}_2, \mathcal{A}_3$  which results in the following result for our system.

$$(|\psi\rangle\langle\psi|_{23} \otimes \mathbb{1}_{14})|\psi\rangle_{12} \otimes |\psi\rangle_{34} = \frac{1}{d^2} |\psi\rangle_{23} \otimes |\psi\rangle_{14} \quad (10.51)$$

The state  $|\psi\rangle_{14}$  is a maximally entangled state corresponding to the isometry  $I_{14}$  which satisfies

$$I_{14} = I_{12} \cdot I_{23} \cdot I_{34}. \quad (10.52)$$

Thus after the Bell measurement we have swapped the entanglement from  $\mathcal{A}_1, \mathcal{A}_2$  and  $\mathcal{A}_3, \mathcal{A}_4$  to  $\mathcal{A}_2, \mathcal{A}_3$  and  $\mathcal{A}_1, \mathcal{A}_4$ . The remarkable feature of entanglement swapping is that the now entangled states of  $\mathcal{A}_1$  and  $\mathcal{A}_4$  do not originate from the same source and moreover their sources did not interact.





# 11 Conclusion

The notion of entanglement is still not fully understood. Although there are a lot of entanglement measures that claim to quantify entanglement, for systems with dimensions equal or higher than needed to describe two qutrits they all fail to be computable in general. Only in special cases can they be calculated. One entanglement measure that gives a very easy way to calculate the entanglement, at least for two qubits, is the concurrence, which was used in this work for finding the entanglement of the states introduced as examples. Also, the explicit calculation of the expectation values of a Bell inequality is not straightforward. We thus introduced a way to do this, found by the Horodecki family, which was used to calculate the Bell violation of our example states.

But not only the difficulties in describing entanglement and nonlocality properties were addressed, but also how measurement operations and unitary operations can change the property of a state. LOCC operations and the special case of filtering operations were discussed and a way of calculating the change of the concurrence through filtering operations was shown. Also the change of the violation of a Bell inequality through such operations was investigated and a method was found to calculate this change for a certain subclass of two qubit states.

Another important point to understand entanglement and to create an intuition for it, is to visualize the state space and to find the borders between entanglement and separability and also between local and nonlocal states. Especially the difference between entanglement and nonlocality can be appreciated through these pictures. With the above mentioned methods we were able to give such visualizations and also to extend the idea behind it to include more degrees of freedom. We were thus able to give visualizations for Gisin-like states, which enhances the understanding of their properties.

In the last part of this work we investigated the behaviour of quantum states under different factorizations of their algebra. We have shown that, depending on the algebra, a state can appear entangled or separable, as long as it does not exceed a certain bound of mixedness. The question of how to factorize a state seems to be a rather theoretical problem, since an experimentalist normally has a fixed factorization through his experimental setup. However, there exist physical problems where the choice of factorization becomes important. For example, the Hanbury-Brown-Twiss effect [60] where photons are created so far apart, that they cannot be entangled. But they can still give nonlocal correlations in joint measurement experiments. Another example for different algebras are the systems of strange  $K$ -mesons, or kaons [61]. The subalgebras in this case depend on our choice to look at the strangeness eigenstates, or the decay eigenstates. Also in quantum field theory the problem of factorization arises. Local subalge-

## 11 Conclusion

bras (i.e. double cones) are always entangled due to the Reeh-Schlieder theorem [62, 63]. Thus the vacuum state has to be entangled [64]. However, because of the large dimensions of the local subalgebras, this effect can hardly be detected. But we can restrict ourselves to some field modes, as long as they are accessible for the experimentalist. In particular, acceleration of the observer can change the entanglement [65, 66, 67, 68, 69].

We investigated the behaviour of factorizations with a number of examples taken from different parts of the field of quantum information. We analyzed the GHZ state, Werner state and Gisin state and showed explicitly how the unitary operation changed the properties of the state. The behaviour under this unitary transformation is especially important for the cases of quantum teleportation and entanglement swapping. There we showed that the factorization indeed determines the experimental result. Therefore, if we talk about entanglement of a state, we have to give a corresponding algebra in order to be precise.

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# Abstract

Quantum mechanics and its phenomena, like entanglement or nonlocality, are still not fully understood. The aim of this work is to give an introduction to the notion of entanglement. Entanglement detection criteria like the PPT criterion are introduced, as well as entanglement measures, such as the Entanglement of Formation and related measures. Also, Bell inequalities are investigated and a useful criterion to detect nonlocal states with respect to the CHSH criterion, at least for two qubit states, is introduced.

Another aspect of this work is the transformation of states under LOCC- and global unitary operations. Examples, like the GHZ-, Werner-, and Gisin states are studied and the change of entanglement and nonlocality due to this operations is discussed.

Another main focus of this work is to investigate the possibility to visualize quantum states in order to get a better understanding of their behaviour. This is done for pure two qubit states, by using their connection to the complex projective space. Thus a full visualization of these states is possible. For the case of mixed two qubit states, a restriction to subclasses of states is necessary to reduce the number of dimensions. A superposition of Bell states reduces the number of degrees of freedom to three and is therefor visualizable. But also states with more degrees of freedom can be drawn, using a method introduced in this work, by transforming the region of physically realizable states.

# Kurzfassung

Die Quantenmechanik und ihre Phänomene, wie zum Beispiel Verschränkung oder Nichtlokalität, sind bis heute nicht vollständig erschlossen. Das Ziel dieser Arbeit ist es, eine Einführung in den Begriff der Verschränkung und seine Auswirkungen zu geben. Insbesondere Kriterien zur Detektion und Maße zur Quantifizierung von Verschränkung werden vorgestellt. Desweiteren wird der Begriff der Nichtlokalität näher untersucht und eine Möglichkeit, die Bell Ungleichung, vorgestellt nichtlokale Quantenzustände zu detektieren.

Ein weiterer Teil dieser Arbeit ist der Frage gewidmet, wie sich Quantenzustände unter LOCC Transformationen und global unitären Transformationen verändern. Hierfür werden mehrere Beispiele behandelt. Unter anderem GHZ-, Werner-, Gisin Zustände werden genauer beleuchtet und deren Veränderung der Verschränkung und Nichtlokalität untersucht.

Ein weiterer wichtiger Teil dieser Arbeit widmet sich der Frage der Visualisierung von Quantenzuständen um ein besseres Verständnis für deren Verhalten zu erlangen. Explizit werden reine zwei qubit Systeme untersucht. Die Verknüpfung solcher Zustände mit dem komplex projektiven Raum ermöglicht eine vollständige geometrische Darstellung. Im Fall von gemischten Zuständen ist die Dimensionszahl zu hoch um eine vernünftige Darstellung aller Zustände zu ermöglichen. Allerdings zeigt sich, dass eine Superposition von Bell Zuständen nur drei Freiheitsgrade besitzt und dadurch darstellbar ist. Doch auch Zustände mit mehr als drei Freiheitsgraden können, durch eine in dieser Arbeit vorgestellte Methode, visualisiert werden, indem der Bereich der physikalischen Zustände innerhalb der darstellung angepasst wird.

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# Curriculum Vitae

<b>Name:</b>	Philipp Martin Köhler
<b>Geburtsdatum:</b>	31.01.1985
<b>Geburtsort:</b>	Wien
<b>Wohnort:</b>	1030 Wien, Schlachthausgasse 30/6/9
<b>Staatsbürgerschaft:</b>	ÖSTERREICH
<b>Familienstand:</b>	Ledig
<b>E-Mail:</b>	philipp.martin.koehler@univie.ac.at
<b>Ausbildungsweg:</b>	
1991-1995	Volksschule Eslarngasse, 1030 Wien
1995-2003	Gymnasium GRG3 Hagenmüllergasse, Abschluss mit Matura
seit SS2005	Studium der Physik an der Universität Wien
2007	Abschluss des 1. Abschnitts
2009	Abschluss des 2. Abschnitts
Juni 2011	Teilnahme an der CEQIP, Präsentation eines Posters
September 2011	Teilnahme am International Workshop Relativistic Quantum Information North, Präsentation eines Posters
September 2011	Teilnahme am Workshop Relativity meets Entanglement and High Energy mit Vortrag
<b>Sonstige Tätigkeiten:</b>	
seit 2005	Diverse Nachhilfetätigkeiten für Schüler und Studenten
WS2010	Tutor für die Übungen zur Vorlesung „Nichtlokalität, Verschränkung und Geometrie von Quantensystemen“
SS2011	Tutor für das Seminar „Verschränkung und Relativistik, Dekohärenz und Geometrie der Quantenphysik II“

**Publikationen:**

- 2011 „Residual entanglement of accelerated fermions is not nonlocal ”  
arXiv:1107.3235[quant-ph]
- 2011 „Entanglement or separability:  
The choice of how to factorize  
the algebra of a density matrix”  
arXiv:1106.3047[quant-ph]

**Besondere Kenntnisse:**

- Sprachen Deutsch  
Englisch in Wort und Schrift  
Französisch Grundkenntnisse
- EDV Programmieren mit HTML,  
PHP, JavaScript  
Erfahrung mit LaTeX, Mathematica