DIPLOMA THESIS

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"Cobordism and fixed point sets of involutions"

Author

Manuel Radler

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Abstract

The concept of cobordism is presented. In particular, we demonstrate how the computation of cobordism groups can be reduced to a homotopy theoretical problem. This idea is due to René Thom. By proving the axioms of Eilenberg and Steenrod we point out that cobordism may be understood as a generalized homology theory. Equivariant cobordism is used to study involutions on closed manifolds. We are especially interested in the fixed point sets of such periodic maps. Information about these fixed sets can be deduced from the normal bundle. Our work culminates in a proof of the five-halves theorem which was given by J.M. Boardman. The theorem states that the fixed point set of a non-bounding involution cannot be too low dimensional.

Contents

1 Miscellaneous about manifolds

The purpose of this section is to provide necessary prerequisites. Throughout this work any space is assumed to be a smooth manifold (see below), if not stated otherwise. Mainly without proofs we recall concepts and well known facts from the field of differential topology and vector bundles which will be of importance for this thesis. Apart from this it is assumed that the reader is familiar with singular (co)homology. A concise summary can be found in the appendix. The following definitions and statements are mainly taken from [2], [3], [4], [5] and [15].

1.1 Differential topology

Let $H^n := \{(x_1, \dots, x_n) : x_n \geq 0\} \subset \mathbb{R}^n$ denote the halfspace. A topological n-manifold M is a separable metric space such that for any $x \in M$ there is an open neighbourhood of x which is homeomorphic to an open subset of H^n. A chart (ϕ, U) consists of an open subset $U \subset M$ and a homeomorphism $\phi : U \to \phi(U) \subset H^n$. If we speak of a *manifold* M^n we mean a topological n-manifold together with a smooth structure. This means that there is a maximal atlas of C^∞-compatible charts. Recall that two charts (U, ϕ), (V, ψ) of a topological manifold are called C^∞-compatible if the transition functions

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞. The *boundary* ∂M of M consists of all points which are mapped to the boundary of the halfspace by a chart (ϕ, U). It follows from the inverse function theorem that this is well-defined, i.e. independent of the chart, see for example [13, chapter 21]. Any map $f : M \to N$ between manifolds is assumed smooth, if not stated otherwise. $S \subset M$ is called a *submanifold of dimension* k if for any $x \in S$ there exists a chart (U, ϕ) with $x \in U$ such that

$$\phi(U \cap S) = \phi(U) \cap \left(\{0\}^{n-k} \times \mathbb{R}^k\right).$$

The integer $n-k$ is called *codimension* of S. The tangent space at a point $x \in M$ is denoted $T_x M$, the *tangent bundle* $\tau_M = (\pi, TM, M)$ of M is the collection of all tangential spaces. If $S \subseteq M$ is a submanifold we write $T^\perp S := TM|_S / TS$ for the *normal bundle* of S in M. A map $f : M \to N$ induces the *tangential map* or *differential* $Tf : TM \to TN$. We call $f : M \to N$ an *immersion* (*submersion*) if the differential $T_x f : T_x M \to T_{f(x)} N$ is injective (surjective) for any $x \in M$. If additionally f is a homeomorphism onto its image $f(M)$, f is called an *embedding.*

We will need the notion of a submanifold $S \subseteq M$ whose boundary is placed in ∂M such that S is nowhere tangent to the boundary of M, [2, chapter 1.4].

Definition 1.1 (neat submanifold)**.** *A submanifold $S \subseteq M$ is called **neat** if $\partial S = S \cap \partial M$ and $T_x S \nsubseteq T_x(\partial M)$ for any point $x \in S$.*

Every manifold can be embedded into Euclidean space (see [5, I.1.1]):

Theorem 1.2 (Whitney Embedding Theorem)**.** *Let $\epsilon : M^n \to \mathbb{R}$ be a positive map, $p > 2n$ and $f : M^n \to \mathbb{R}^p$ a map which is an embedding for a neighbourhood* *of the closed set $A \subseteq M^n$. There is an ϵ-approximation g of f with $g|_A = f|_A$ which is an embedding as a map $M^n \xrightarrow{g} \mathbb{R}^p$. In particular, there is an embedding $g : M \to \mathbb{R}^p$ so that $g(M)$ is closed in $\mathbb{R}^p$. If $p \geq 2n+2$ two embeddings i, j are isotopic (i can be deformed into j by a 1-parameter family of embeddings).*

Here ϵ-approximation means that the distance of $f(x)$ and $g(x)$ is always less than $\epsilon(x)$.

Theorem 1.3 (Whitney Approximation Theorem)**.** *Let $f : M \to N$ be a continuous map between smooth manifolds. Then there is a smooth map $g : M \to N$ which is homotopic to f ([21, 10.21]).*

Definition 1.4 (Transversality)**.** *Two smooth maps $f : M \to N$ und $g : P \to N$ are called **transversal**, in symbols $f \pitchfork g$, if for any $x \in M$ and $y \in P$ with $f(x) = g(y) =: z$ the images of the tangential maps span the tangential space at z.*

$$\operatorname{img}(T_xM \xrightarrow{T_xf} T_zN) + \operatorname{img}(T_yP \xrightarrow{T_yg} T_zN) = T_zN.$$

*Let $A \subseteq M$ and $S \subseteq N$ a submanifold. A smooth map $f : M \to N$ is called transverse to S along A if f and the embedding $\iota : S \to N$ are transverse for any point in A. We write $f \pitchfork_A S$. If $A = M$, we call f transerval to S and we write $f \pitchfork S$. In the case of S consisting of a single point $S = \{y\}$, we call y a **regular value** of f. Finally, two submanifolds S_1 and S_2 intersect transversally if the corresponding embeddings are transverse. S_1 and S_2 are then called transverse.*

Proposition 1.5. *Two smooth maps $f : M \to P$ and $g : N \to P$ are transverse if and only if the map $f \times g : M \times N \to P \times P$ is transverse to the diagonal $\Lambda \subseteq P \times P$ ([4, 15.9.3]).*

Remark 1.6. Transversality can be understood as the opposite of tangency. Suppose $x \in M$, $f(x) \in S$. In a more visual way f being transverse to S means that the image of the tangential space T_xM under the tangential mapping T_xf lies as 'cross' as possible to $T_{f(x)}S$. This is because $T_xf(T_xM)+T_{f(x)}S = T_{f(x)}N$ is equivalent to the map $T_xf : T_xM \to T_{f(x)}N/T_{f(x)}S$ being surjective.

Theorem 1.7. *Let $S \subseteq N$ be a submanifold without boundary and $f : M \to N$ such that both f and $f|_{\partial M}$ are transverse to S. Then $f^{-1}(S)$ is a submanifold of M with boundary $\partial(f^{-1}(S)) = f^{-1}(S) \cap \partial M$. Furthermore,*

$$\dim(M) - \dim(f^{-1}(S)) = \dim(N) - \dim(S),$$

i.e. the codimension of $f^{-1}(S)$ in M equals the codimension of S in N ([4, 15.9.2]).

Corollary 1.8. *Let $f : M \to N$. If $y \in N$ is a regular value of f, then the differential T_xf is surjective for any $x \in f^{-1}(y)$. Thus, $f^{-1}(y)$ is a submanifold of M and*

$$\dim(f^{-1}(y)) = \dim(M) - \dim(N)$$

The next theorem states that transversality is a generic property.

Theorem 1.9 (Thom's transversality theorem)**.** *Let $R \subseteq M$, $S \subseteq N$ be closed submanifolds and $f : M \to N$ transverse to S along R. Moreover, let $\epsilon : M \to \mathbb{R}$ be positive and N be given a metric. There is then an ϵ-approximation $g : M \to N$ of f such that $g|_R = f|_R$ and g is everywhere transverse to S ([5, I.2.5]).*

Theorem 1.10. *Let $f : M \to N$ be a map and choose a metric on N. For any positive map $\epsilon : M \to \mathbb{R}$ we find a positive map $\delta : M \to \mathbb{R}$ such that: If g is δ-approximation of f, there is a homotopy $H : M \times I \to N$ connecting f and g satisfying*

i) $H(x,t) = f(x)$ *if* $g(x) = f(x)$,

ii) H_t *is an* ϵ*-approximation of* f *for all* t.

([5, I.2.6]).

Definition 1.11. *An open neighborhood U of ∂M in M is called a* ***collar*** *of ∂M provided that there is a diffeomorphism*

$$\phi : \partial M \times [0,1) \cong U \quad \text{with} \quad \phi(x,0) = x \quad \forall x \in \partial M.$$

Theorem 1.12 (Collaring theorem)**.** *A smooth manifold with boundary has a collar ([4, 15.7.8]).*

Corollary 1.13. *Let $k > n+1$. Any embedding $j : \partial M^{n+1} \hookrightarrow S^{n+k}$ extends to an embedding $M^{n+1} \hookrightarrow D^{n+k+1}$.*

Proof. Choose a collar $\partial M^{n+1} \times [0,1) \overset{\phi}{\cong} U$ of $\partial M^{n+1} \subset M^{n+1}$. Define a map $\iota : M^{n+1} \to D^{n+k+1}$ by

$$\iota(x) = \begin{cases} (1-t)j(y) & x = \phi(y,t) \in U \cong \partial M^{n+1} \times [0,1) \\ 0 & x \notin U \end{cases}$$

Note $\iota|_{\partial M^{n+1}} = j$. Since $\iota|_U$ is an embedding there is an embedding $\tilde{\iota} : M^{n+1} \hookrightarrow D^{n+k+1}$ approximating ι by Whitney's theorem. Moreover, $\tilde{\iota}$ can be choosen such that $\tilde{\iota}|_{\partial M^{n+1}} = \iota|_{\partial M^{n+1}} = j$. □

Using collars we can glue manifolds with boundaries along common pieces of the boundary.

Proposition 1.14. *Let M_1 and M_2 be manifolds with boundary and $N \subset \partial M_i$ be a common component of the boundaries, i.e. there are subsets $N_i \subset \partial M_i$ and a diffeomorphism $\phi : N_1 \cong N_2$. Let $M = M_1 \cup_\phi M_2$ be the space which is obtained from $M_1 \cup M_2$ by identifying $x \in M_1$ with $\phi(x) \in M_2$. Then M can be given a smooth structure.*

Proof. For the sake of notation we will suppress the diffeomorphism ϕ and assume $M_1 \cap M_2 = N$. Hence, $M = M_1 \cup M_2$. Choose a collar $\phi_1 : N \times [0,1) \to U_1$ for N in M_1 and a collar $\phi_2 : N \times [0,1) \to U_2$ for N in M_2 and define the embedding $\Phi : (-1,1) \times N \to M$,

$$\Phi(t,x) = \begin{cases} \phi_1(-t,x) & \text{if} \quad t \leq 0 \\ \phi_2(t,x) & \text{if} \quad t \geq 0 \end{cases}$$

with image $U = U_1 \cup U_2$. Note $U_1 \cap U_2 = N$ and $\phi_1(x,0) = x = \phi_2(x,0)$. Now M is the union of open subsets all of which carry a smooth structure, $M = (M_1 - N) \cup \Phi((-1,1) \times N) \cup (M_2 - N)$. These structures coincide on the intersections $(M_i - N) \cap U$ since $\phi_i : N \times (0,1) \cong (U_i - N)$. Thus, there is a unique smooth structure on M such that $M_i - N$ and $\Phi((-1,1) \times N)$ are submanifolds. □

As an immediate consequence we obtain a very helpful construction in cobordism theory:

Corollary 1.15 (Connected sum). *Let M_1 and M_2 be n-manifolds. Choose embeddings $\iota_i : D^n \to M_i$ into the interior of the manifolds. Consider the union $\Big(M_1 - \iota_1(D^n - S^{n-1})\Big) \cup S^{n-1} \times I \cup \Big(M_2 - \iota_2(D^n - S^{n-1})\Big)$ and identify $\iota_1(S^{n-1})$ with $S^{n-1} \times \{0\}$ and $\iota_2(S^{n-1})$ with $S^{n-1} \times \{1\}$. The resulting space is again a smooth manifold called the **connected sum** $M_1 \# M_2$. One has to check that the smooth structure does not depend on the choice of embeddings, see [17, chapter 10] for details.*

Proposition 1.16. *Let T be a fixed point free involution on a manifold M. There is a unique canonical smooth structure on the quotient space M/T such that the projection $p : M \to M/T$ is a local diffeomorphism.*

Proof. Because the involution has no fixed points, we find for any point x in M a chart (U, ϕ) around x such that $U \cap T(U) = \emptyset$. The restriction $p|_U : U \to p(U)$ is then a homeomorphism and $(p(U), \phi \circ (p|_U)^{-1})$ is a chart for M/T. For more details see [16, 1.6]. □

Theorem 1.17 (Facts about the Euler characteristic). *Let M be an n-manifold.*

i) If M is closed and $\dim(M)$ is odd, then $\chi(M) = 0$.

ii) If M is compact with boundary ∂M, then $\chi(\partial M) = (1 - (-1)^n)\chi(M)$. In particular, one obtains $\chi(\partial M) = 2\chi(M)$ if $\dim(M)$ is odd.

([15, VI.8.9, VI.8.10])

As a corollary we state a theorem which will prove useful.

Theorem 1.18. *A closed manifold with odd Euler characteristic cannot be boundary of a compact manifold.*

1.2 Vector bundles

A vector bundle over a manifold B consists of a smooth map $p : E \to B$ together with a real vector space structure on any fibre $E_x := p^{-1}(x)$, $x \in M$, which is locally trivial. This means that for any point $x \in M$ there exists an open neighborhood U of x, a finite-dimensional real vector space V and a diffeomorphism $\phi : U \overset{\cong}{\to} U \times V$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[dr, "p"'] & U \times V \arrow[d, "\mathrm{pr}_1"] \\ & U \end{tikzcd}$$

commutes. Furthermore, ϕ is assumed to be fibrewise linear, i.e. $\phi_y = \phi|_{E_y} : E_y \to \{y\} \times V = V$ is a linear isomorphism for any $y \in U$. So, locally the projection p looks like the projection $U \times V \to U$. A concise introduction can be found in [3]. For more details on vector bundles see [4]. We will often use the notation $\xi = (p, E(\xi), B(\xi)) = (p, E, B)$ for the vector bundle ξ with total space

E, base space B and projection map p. Given a manifold M we denote the trivial vector bundle of rank k over M by $\theta^k = (\mathrm{pr}_1, M \times \mathbb{R}^k, M)$. A vector bundle ξ of rank k is called trivial if it is isomorphic to θ^k. A manifold is called *parallelizable* if its tangent bundle is trivial. There is a very useful theorem stating that a vector bundle homomorphism which is fibrewise an isomorphism is already a vector bundle isomorphism. A *bundle map* between two vector bundles ξ and η is a smooth function $E(\xi) \to E(\eta)$ that is fibrewise an isomorphism.

Theorem 1.19. *Let $\Phi : E \to F$ be a vector bundle homomorphism over M which induces linear isomorphisms $\Phi_x : E_x \cong F_x$ in all fibres. Then Φ is a vector bundle isomorphism ([15, II.1.5]).*

Proposition 1.20. *An n-plane bundle ξ is trivial if and only if there are n nowhere linear dependent cross-sections ([3, 2.2.2]).*

Let G be a topological group, $\xi = (p, E, B)$ a principal G-bundle and F a left G-space. Via $(e, f)g = (eg, g^{-1}f)$ a right G-space structure is defined on the product $E \times F$. We denote the quotient space by $E \times_G F$. The projection p factors through to a map $p_F : E \times_G F \to B$. The bundle $\xi[F] = (p_F, E \times_G F, B)$ is called the *associated fibre bundle* with fibre F. For more details we refer to [6, chapter 4.5].

Example 1.21. The antipodal map induces a principal $\mathbb{Z}_2$-bundle $S^r \to \mathbb{R}\mathrm{P}^r$. The associated fibre bundle with fibre $\mathbb{R}$ is called the *twisted line bundle.*

Theorem 1.22 (Normal bundle of the preimage)**.** *Let $S \subseteq N$ be a submanifold and $f : M \to N$ transverse to S. The tangential map $Tf : TM \to TN$ induces an isomorphism of vector bundles over $f^{-1}(S)$*

$$T^{\perp}(f^{-1}(S)) \cong (f|_{f^{-1}(S)})^*(T^{\perp}S).$$

([15, II.1.13]).

Let $f : M \to N$ smooth and $\xi = (\pi, E, N)$ a vector bundle. π is submersive being a vector bundle projection. Thus f and π are transverse. By proposition 1.5 and theorem 1.7 $f^*E := \{(x, e) : f(x) = \pi(e)\}$ is a submanifold of $M \times E$. Let $\tilde{f} : f^*E \to E$ and $f^*\pi : f^*(E) \to M$ be the restrictions of the canonical projections. We have the following commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle f^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

and obtain the *pullback bundle* $f^*\xi$ over M *induced by* f. The fibres are $(f^*E)_x = f^*(\pi)^{-1}(x) = x \times E_{f(x)} = E_{f(x)}$ by definition of f^*E.

Theorem 1.23 (pullback bundle)**.** *Let ξ and f be as above.*

i) (universal property) Let $\pi' : E' \to M$ be a vector bundle over M and $g : E' \to E$ a vector bundle homomorphism over f. There exists a unique vector bundle homomorphism $\tilde{g} : E' \to f^*E$ over M, such that g factors to $g = \tilde{f} \circ \tilde{g}$. $\tilde{g}$ is given by $(f^*\pi, g)$. If g is fibrewise a linear isomorphism, then $E' \overset{\tilde{g}}{\cong} f^*E$ as vector bundles.

$$\begin{tikzcd}
E' \arrow[r, "\tilde{g}"] \arrow[rr, bend left, "g"] \arrow[d, "\pi'"'] & f^*E \arrow[r, "\tilde{f}"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "\mathrm{id}"'] & M \arrow[r, "f"'] & N
\end{tikzcd}$$

ii) If $f_1 \simeq f_2 : M \to N$ are homotopic, then $f_1^*\xi$ and $f_2^*\xi$ are isomorphic vector bundles ([5, I.3.1]).

Let $\xi = (p, E, M)$ be a vector bundle. It is often convenient to have a description of the tangent bundle of the total space E. The tangential map $Tp : TE \to TM$ of the bundle projection p is a vector bundle epimorphism over $p : E \to M$ and thus induces a fibrewise surjective bundle homomorphism $TE \to p^*TM$ over M. The kernel of this map is a subbundle of TE called the *vertical bundle* of E and is denoted VE. In other words the vertical bundle consists of all vectors tangent to the fibres. Note that there is a canonical vector bundle isomorphism $VE \cong p^*E$, $(v, w) \mapsto \frac{d}{dt}|_{t=0}(v + tw)$. We obtain a short exact sequence

$$0 \longrightarrow VE = p^*E \longrightarrow TE \xrightarrow{Tp} p^*TM \longrightarrow 0.$$

If we pullback by the zero section $o : M \to E$, we get a short exact sequence of vector bundles over M which is split by To.

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] & o^*TE \arrow[r, "Tp"] & TM \arrow[r] \arrow[l, bend left, "To"] & 0
\end{tikzcd}$$

since $p \circ o = \mathrm{id}$. Hence we have isomorphisms

$$TE|_M = TM \oplus E \quad \text{and} \quad T^\perp M = E. \tag{1.1}$$

The normal bundle $T^\perp S$ of a submanifold $S \subseteq M$ gives us information about the way S is placed in M:

Theorem 1.24 (Tubular neighbourhood)**.** *Let $S \subseteq M$ be a closed submanifold. There is an open neighbourhood U of S in M and a diffeomorphism*

$$\phi : T^\perp S \xrightarrow{\cong} U \quad \textit{such that} \quad \phi|_S = \mathrm{id}_S \quad \textit{and} \quad T\phi|_S = \mathrm{id}_{T^\perp S} : T^\perp S \to T^\perp S.$$

Here, we identify S with the image of the zero section in M. Note that by 1.1

$$T(T^\perp S)|_S = TS \oplus T^\perp S$$

and with respect to this decomposition $T\phi|_S$ takes the form

$$TS \oplus T^{\perp}S = T(T^{\perp}S)|_S \longrightarrow TU|_S = TS \oplus T^{\perp}S$$

$$T\phi|_S = \begin{pmatrix} \mathrm{id}_{TS} & * \\ 0 & \mathrm{id}_{T^{\perp}S} \end{pmatrix}$$

U is called a ***tubular neighbourhood*** *of S in M. Furthermore, we may choose U arbitrarily small: If V is an open neighbourhood of S in M, there is a tubular neighbourhood U such that $U \subseteq V$ ([15, II.1.16]).*

One application of tubular neighborhoods is to make maps look like vector bundle maps, [2, Theorem 4.6.7].

Theorem 1.25. *Suppose we have a commutative diagram of manifolds*

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ D \subseteq U \arrow[u, "\cup\!\mid"'] \arrow[r, "f|_D"] & \tilde{U} \arrow[u, "\cup\!\mid"] \\ f^{-1}(S) \arrow[u, "\cup\!\mid"'] \arrow[r, "f"] & S \arrow[u, "\cup\!\mid"] \end{tikzcd}$$

where $S \subseteq M$ is a compact neat submanifold, f and $f|_{\partial M}$ are both transverse to S, U and $\tilde{U}$ are tubular neighbourhoods and $D \subset U$ is a disk bundle. Then there exists a map $g : M \to N$ such that g is homotopic to f and $g|_D$ is the restriction of a vector bundle map $U \to \tilde{U}$ over f.

Theorem 1.26 (Extension of tubular neighbourhoods)**.** *Let $S \subseteq M$ be a neat submanifold and U a tubular neighbourhood of ∂S in ∂M. Then there is a tubular neighbourhood V of S in M such that $U = \partial M \cap V$ ([2, 4.6.4]).*

2 Cobordism groups and reduction to homotopy theory

We give the basic definitions and outline the strategy of computing cobordism groups. Most of the following definitions and statements are taken from [5] and [4, chapter 21].

2.1 Singular manifolds

Definition 2.1 (Singular manifold). *Let X be a topological space. An n-dimensional* ***singular manifold*** *in X is a closed n-manifold M together with a continuous map $f : M \to X$. We write (M, f).*

Poincaré had the idea to identify two n-dimensional manifolds if their disjoint union is the boundary of an $(n+1)$-dimensional manifold.

Definition 2.2. *Let (M, f) be a singular manifold. By a* ***null bordism*** *we mean a triple (N, F, α) consisting of a compact manifold N together with a map $F : N \to X$ as well as a diffeomorphism $\alpha : M \to \partial N$ such that $(F|_{\partial N}) \circ \alpha = f$. We will suppress the diffeomorphism α in notation for the sake of simplicity and write $(\partial N, F|_{\partial N}) = (M, f)$. In this case (M, f) is called null bordant. We also write (M, f) bounds.*

Let (M_1, f_1), (M_2, f_2) be two singular manifolds in X. The disjoint union $M_1 \sqcup M_2$ gives us a new singular manifold $(M_1, f_1) + (M_2, f_2)$ with induced map $(f_1, f_2) : M_1 \sqcup M_2 \to X$.

Definition 2.3. *(M_1, f_1), (M_2, f_2) are called* ***cobordant****, if $(M_1, f_1) + (M_2, f_2)$ is null bordant. A null bordism of $(M_1, f_1) + (M_2, f_2)$ is called* ***cobordism*** *between (M_1, f_1) and (M_2, f_2).*

Example 2.4. A concrete example is given by a pair of pants which defines a cobordism between S^1 and $S^1 \sqcup S^1$.

Example 2.5. Let M be a manifold of dimension n and consider an embedding $\phi : S^k \times D^{n-k} \to M$. Cut out the interior of $S^k \times D^{n-k}$ and glue in $D^{k+1} \times S^{n-k-1}$ along the common boundary $S^k \times S^{n-k-1}$ to obtain the new manifold

$$\Big(M - \mathrm{int}(S^k \times D^{n-k})\Big) \cup_{S^k \times S^{n-k-1}} D^{k+1} \times S^{n-k-1}.$$

One says that N is produced from M by **k-surgery**. The **trace**

$$W = M \times I \cup_{S^k \times D^{n-k} \times \{1\}} D^{k+1} \times D^{n-k}$$

defines a cobordism between M and N. We refer to [22] for more details. A special case is the connected sum $M_1 \# M_2$ from example 1.15 which is the result of a 0-surgery on the disjoint union $M_1 \sqcup M_2$. Hence, we see that $M_1 \sqcup M_2$ and $M_1 \# M_2$ are cobordant.

Lemma 2.6 (Cobordism defines an equivalence relation)**.**

Proof. Symmetry is clear. Consider the cylinder $M \times [0,1]$ whose boundary is $M \sqcup M$. $(M \times [0,1], f \circ \mathrm{pr}_1)$ defines a null bordism of $(M,f) + (M,f)$.
Transitivity: Let (M,F) be a cobordism beween (M_1,f_1) and (M_2,f_2) und (N,G) a cobordism between (M_2,f_2) and (M_3,f_3). We glue M and N along their common boundary M_2 and obtain a new manifold $P := M \cup_{M_2} N$ with boundary $M_1 \sqcup M_3$ by proposition 1.14. Since $F|_{M_2} = G|_{M_2} = f_2$, F and G induce a map $H : P \to X$ with $H|_{M_1} = f_1$ and $H|_{M_3} = f_3$. Thus, (P,H) is the required cobordism. $\square$

We write $[M,f]$ for the equivalence class represented by (M,f). $[M,f]$ is called *cobordism class.* The set of all cobordism classes of n-dimensional singular manifolds is denoted by $MO_n(X)$. If one carries out this construction without taking maps $f : M \to X$ into account, that is $X = \{*\}$, we obtain the classic unoriented cobordism group MO_n.

Theorem 2.7. *$MO_n(X)$ becomes an abelian group by*

$$[M_1,f_1] + [M_2,f_2] = [M_1 \sqcup M_2, (f_1,f_2)].$$

Any element has order at most 2.

Proof. The operation is well-defined as is easily seen: Let $[M_1,f_1] = [M_2,f_2]$ and $[N_1,g_1] = [N_2,g_2]$. By definition, there exist compact $(n+1)$-manifolds B_M, B_N and maps $F : B_M \to X$, $G : B_N \to X$ such that

$$\partial B_M = M_1 \sqcup M_2 \quad \text{and} \quad F|_{M_i} = f_i,$$

$$\partial B_N = N_1 \sqcup N_2 \quad \text{and} \quad G|_{N_i} = g_i.$$

Thus, the compact $(n+1)$-manifold $B_M \sqcup B_N$ together with the induced map $(F,G) : B_M \sqcup B_N \to X$ acts as a cobordism between $[M_1 \sqcup M_2, (f_1,f_2)]$ and $[N_1 \sqcup N_2, (g_1,g_2)]$. It is clear that the addition is associative. The class of null bordant manifolds serves as a neutral element (it is often convenient to consider the empty set an n-manifold which serves as the neutral element): Let (M,f) a singular manifold and (N,g) null bordant. By definition there is a singular $(n+1)$-manifold (B,F) with $\partial B = N$ and $F|_{\partial B} = g$. We have to find a manifold whose boundary consists of two copies of M and one copy of N. For this consider $P := M \times I \sqcup B$ and the map $H : P \to X$ induced by $f \circ \mathrm{pr}_1 : M \times I \to X$ and $F : B \to X$. We have $\partial P = (M \times \{0\} \sqcup N) \sqcup M \times \{1\}$. By construction H restricts to (f,g) and f respectively. Every element is its own inverse, compare the proof of reflexivity in the previous lemma. Commutativity is clear. $\square$

The cartesian product defines a bilinear, associative map

$$MO_m(X) \times MO_n(Y) \to MO_{m+n}(X \times Y).$$

Let M be a singular m-manifold in X and N a singular n-manifold in Y. A singular $(m+n)$-manifold in $X \times Y$ is defined via

$$([M,f],[N,g]) \mapsto [M \times N, f \times g].$$

To see that this construction is well-defined, consider a cobordism (B,F) between (M_1,f_1) and (M_2,f_2). Then $(B \times N, F \times g)$ is a cobordism between

$(M_1, f_1) \times (N, g)$ and $(M_1, f_1) \times (N, g)$. Note that the differentiable structures on B and N induce a natural smooth structure on the cartesian product $B \times N$ the boundary of which is $\partial B \times N$.

Remark 2.8. As above consider the bilinear map

$$MO_m \times MO_n \to MO_{m+n},$$

$$([M], [N]) \to [M \times N] =: [M] \cdot [N].$$

$MO_* = \sum_{n=0}^{\infty} MO_n$ becomes a graded commutative algebra over the field $\mathbb{Z}_2$ with respect to $+$ and $\cdot$. A cobordism class $[M]$ is called **decomposable** if it is a sum of products of lower dimensional cobordism classes. Otherwise, we say $[M]$ is **indecomposable**. We set $MO_n = 0$ for $n < 0$. Thom showed that the graded $\mathbb{Z}_2$-algebra MO_* is isomorphic to the polynomial algebra $\mathbb{Z}_2[x_2, x_4, x_5, \ldots]$ in generators x_i for all i not of the form $2^k - 1$. An indecomposable cobordism class can be used as a generator. We will at least give a strategy to tackle this and show how the computation of MO_* may be reduced to a homotopy theoretical problem.

Proposition 2.9. *The graded group $MO_*(X) = \sum_{n=0}^{\infty} MO_n(X)$ becomes a graded MO_*-module by*

$$[M] \cdot [N, f] = [M \times N, f \circ \mathrm{pr}_2].$$

Remark 2.10. There is a completely analogous construction with respect to oriented manifolds. Two closed, oriented n-manifolds (M_1, ω_1), (M_2, ω_2) are cobordant, if there is an oriented, compact $(n+1)$-manifold (N, ω) with oriented boundary ∂N and a orientation preserving diffeomorphism

$$(\partial N, \partial \omega) \cong (M_0, -\omega_0) \sqcup (M_1, \omega_1).$$

The set of equivalence classes is denoted by MSO_n. There is also the notion of an oriented singular manifold when maps are taken into account. We arrive at the set $MSO_n(X)$. As above $MSO_*(X)$ is made into a graded module over the graded ring MSO_*. We omit details since we will mainly deal with $MO_*(X)$. See for example the book of Conner and Floyd [1] for further information. Rohlin showed that every oriented compact 3-manifold is an oriented boundary, that is $MSO_3 = 0$. There is a list of the first twelve oriented cobordism groups in [3, chapter 17]. Thom proved that $MSO_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$

$$MSO_* \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, \ldots]$$

with generators x_k for any $k \equiv 0 \pmod 4$. One may choose $x_{4k} = [\mathbb{C}\mathrm{P}^{2k}]$. Furthermore, there is a theorem by Thom stating that the oriented cobordism group MSO_n is finite if $n > 0$ is not divisible by 4. If $n = 4k$, then MSO_n is finitely generated of rank $\pi(k)$, where $\pi(k)$ denotes the number of partitions of k.

Proposition 2.11. *The Euler characteristic defines a surjective homomorphism $\chi_2 : MO_n \to \mathbb{Z}_2$ for $n \geq 0$ an even integer. Thus, $MO_n/ker(\chi_2) \cong \mathbb{Z}_2$.*

Proof. We have to check that this map is well-defined. Let $[M_1] = [M_2]$ in MO_n. Then there is a compact $(n+1)$-manifold B such that $\partial B = M_1 \sqcup M_2$. By theorem 1.17 we have $\chi(M_1)+\chi(M_2) = \chi(M_1 \sqcup M_2) = \chi(\partial B) \equiv 0 \pmod 2$. Thus, $\chi(M_1) \equiv \chi(M_2) \pmod 2$. Surjectivity follows from $\chi(\mathbb{RP}^n) = 1$ since n is even. $\square$

Remark 2.12. The homomorphism defined above cannot be surjective if n is odd since the Euler characteristic of any closed odd-dimensional manifold is zero by theorem 1.17.

We give some easy examples.

Theorem 2.13.

$$MO_0 = \mathbb{Z}_2 \quad MO_1 = 0 \quad MO_2 = \mathbb{Z}_2$$

$$MSO_0 = \mathbb{Z} \quad MSO_1 = 0 \quad MSO_2 = 0.$$

Proof. We first deal with 0-dimensional compact manifolds which are just finite discrete sets. Up to diffeomorphism there are only four distinct connected 1-manifolds:

$$S^1,\ [0,1],\ (0,1],\ (0,1).$$

In the unoriented case we see that any manifold consisting of an even number of points represent the zero element of MO_0. (just imagine the manifold embedded into real space and connect any two points by a smooth curve). Similarly, manifolds with an odd number of points are cobordant to $M = \{*\}$ which is no boundary of a compact 1-manifold. Thus, $MO_0 = \mathbb{Z}_2$.

Turn to the oriented case MSO_0. Denote the orientation of a point by ± 1. We define the boundary orientation at 0 of the 1-dimensional half-space $\mathbb{H}^1 = [0, \infty)$ to be -1. As a consequence the boundary orientation of an interval $[a, b]$ is -1 at a and $+1$ at b. We claim that $[(\{x\}, +1)]$ generates MSO_0. Since $-[(\{x\}, +1)] = [(\{x\}, -1)]$ it is clear that we can produce every compact oriented 0-manifold. Moreover, $k[(\{x\}, +1)] = l[(\{x\}, +1)]$ for k, $l \in \mathbb{Z}$ if and only if $k = l$. The only possibility of creating a cobordism between $k[(\{x\}, +1)]$ and $l[(\{x\}, +1)]$ is to connect any two points by a smooth curve (interval).The boundary of this cobordism should induce an orientation preserving diffeomorphism with $k[(\{x\}, -1)] \cup l[(\{x\}, +1)]$. Thus we must have $k = l$ and therefore $MSO_0 = \mathbb{Z}$.

Every closed 1-manifold is diffeomorphic to S^1 which is the (oriented) boundary of D^2. Hence, $MO_1 = 0 = MSO_1$.

To prove the claim about MSO_2 one has to know surfaces. From the classification of surfaces we obtain: Any 2-dimensional closed manifold is diffeomorphic to

- the sphere S^2,
- the connected sum of tori $S^1 \times S^1$ (called *surface of genus g* and denoted Σ_g. These cover the oriented case),
- the connected sum of projective spaces $\mathbb{RP}^2$ (non-orientable surfaces).

A surface of genus g is obtained by removing the interiors of $2g$ disjoint disks from S^2 and attaching g disjoint cylinders (so-called handles) to their boundaries. Embed such a surface into $\mathbb{R}^3$. The 'interior' can now be considered as a compact 3-dimensional manifold with boundary Σ_g. Thus, $MSO_2 = 0$.

From the above classification of compact 2-manifolds we also obtain the structure of MO_2. Any unoriented compact 2-manifold is diffeomorphic to the connected sum of of real projective spaces. We can now use example 2.5 and get

$$[\mathbb{R}\mathrm{P}^2 \# \cdots \# \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2 \sqcup \cdots \sqcup \mathbb{R}\mathrm{P}^2] = 0 \quad \text{for the } (2n)\text{-connected sum}$$

as well as

$$[\mathbb{R}\mathrm{P}^2 \# \cdots \# \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2 \sqcup \cdots \sqcup \mathbb{R}\mathrm{P}^2] = [\mathbb{R}\mathrm{P}^2] \quad \text{for the } (2n+1)\text{-connected sum,}$$

since any element in MO_2 has order at most 2. Note that $\mathbb{R}\mathrm{P}^2$ does not bord by theorem 1.18 since $\chi(\mathbb{R}\mathrm{P}^2) = 1$ (compare theorem 4.7). Together with the above discussion of oriented compact 2-manifolds we have shown $MO_2 = \{0, \ [\mathbb{R}\mathrm{P}^2]\} = \mathbb{Z}_2$ which completes the list above. □

Remark 2.14. In the terminology of proposition 2.11 χ_2 has trivial kernel in dimensions 0 and 2 and is thus an isomorphism. A closed 0-dimensional manifold with Euler characterstic 0 modulo 2 must have even cardinality and hence bords. A closed 2-manifold M with even Euler characteristic is either an orientable surface of genus g or the connected sum of an even number of projective spaces $\mathbb{R}\mathrm{P}^2$. In both cases we have shown above that M bords.

2.2 Cobordism groups as homotopy groups

The method above only works because we know the classification of manifolds in these dimensions. For higher dimensions this approach will fail. Therefore, one needs another idea to compute these groups. In this section we will show how the computation of cobordism groups can be reduced to a homotopy problem. The following is mainly based on [2, chapter 7]. The key concept here is *transversality*. Let $f, g : M \to N$ be transverse to $S \subseteq N$. Then $f^{-1}(S)$ and $g^{-1}(S)$ are submanifolds of M with the same codimension. How are these related?

Lemma 2.15. *Let M and N be closed manifolds and $S \subseteq N$ a closed submanifold. Furthermore, suppose $f, g : M \to N$ are homotopic maps which are transverse to S. Then $f^{-1}(S)$ and $g^{-1}(S)$ define the same cobordism class.*

Proof. By assumption there is a homotopy $H : M \times I \to N$ between f and g. Without loss of generality we may assume H to be transverse to S. This follows from theorem 1.9 since we can replace H by a homotopy $H^{'}$ which is everywhere transverse to S. Thus, $H^{-1}(S)$ is a submanifold of $M \times I$. $H^{-1}(S)$ defines a cobordism between $f^{-1}(S)$ and $g^{-1}(S)$ because theorem 1.7 implies $\partial H^{-1}(S) = H^{-1}(S) \cap \partial(M \times I) = H^{-1}(S) \cap (M \times 0 \sqcup M \times 1) = f^{-1}(S) \sqcup g^{-1}(S)$, since $H_0 = f$, $H_1 = g$. □

We fix M, N and S. Lemma 2.15 yields a well-defined map

$$[M,N] \to MO_{\dim M - \dim N + \dim S}.$$

Put $n = \dim M - \dim N + \dim S$. As we eventually wish to arrive at homotopy groups we put $M = S^{n+k}$, with large $k = \dim N - \dim S$ so that we may embed any n-manifold in M. Hence, we have a map $[S^{n+k}, N] \to MO_n$. How shall we choose N and S? The goal of this construction is to catch as many manifolds as possible. Theorem 1.22 gives information about the normal bundle of $f^{-1}(S)$ in S^{n+k}:

$$T^{\perp}(f^{-1}(S)) \cong f^*(T^{\perp}S).$$

Therefore S must have the following property: The normal bundle of any $P \subseteq S^{n+k}$ can be pulled back from a bundle over S by some map $f : P \to S$. It is here that Grassmann manifolds come into play.

2.2.1 Grassmannian manifolds

Definition 2.16. *The **Grassmann manifold** (also called the Grassmannian) $G_{k,n}(\mathbb{R})$ is the set of all k-dimensional linear subspaces of $\mathbb{R}^n$. A k-frame in $\mathbb{R}^n$ is a k-tuple of linear independent vectors in $\mathbb{R}^n$. The union of all k-frames constitutes an open subset in $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$ (k-fold product), called the **Stiefel manifold** $V_{k,n}(\mathbb{R})$.*

We give $G_{k,n}(\mathbb{R})$ the quotient topology with respect to the map $q : V_{k,n}(\mathbb{R}) \to G_{k,n}(\mathbb{R})$ which assigns to every k-frame the subspace that it spans, i.e. $U \subseteq G_{k,n}(\mathbb{R})$ is open if and only if $q^{-1}(U)$ is open in $V_{k,n}(\mathbb{R})$. An atlas is obtained as follows. Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace with $V^{\perp}$ the orthogonal complement and consider $\mathbb{R}^n$ a direct sum $V \oplus V^{\perp}$ with projection $p : V \oplus V^{\perp} \to V$. Define an open subset of $G_{k,n}(\mathbb{R})$ by $U := \{W : p \text{ maps } W \text{ onto } V\}$. In this way W may be considered the graph of a unique linear map $V \to V^{\perp}$. We obtain a homeomorphism $U \cong \mathrm{Hom}(V, V^{\perp}) \cong \mathbb{R}^{k(n-k)}$. One has to check that the transitions functions are smooth, see for example [21, 1.24]. A quicker way to arrive at this differentiable structure is as follows. Given two subspaces $V, W \subset \mathbb{R}^n$ we choose bases and extend them to bases b_i of $\mathbb{R}^n$. The linear transformation which takes b_1 to b_2 maps V into W. Thus, the general linear group acts transitively on $G_{k,n}(\mathbb{R})$. It is easy to see that the isotropy group of $\mathbb{R}^k \subset \mathbb{R}^n$ is a closed subgroup of the Lie group $GL_n(\mathbb{R})$. Hence, there is a smooth structure on $G_{k,n}(\mathbb{R})$ and we may view the Grassmannian as a homogeneous space (compare [21, 9.31]). We note (see [3, 5.5.1] for a proof)

Lemma 2.17. *The Grassmannian $G_{k,n}(\mathbb{R})$ is a compact smooth manifold of dimension $k(n-k)$. The map $V \mapsto V^{\perp}$ defines a diffeomorphism between $G_{k,n}(\mathbb{R})$ and $G_{n-k,n}(\mathbb{R})$.*

Definition 2.18. *By the **universal vector bundle** over $G_{k,n}(\mathbb{R})$ we understand the bundle $\gamma_{k,n}$ with total space*

$$E_{k,n} := \{(V, v) \in G_{k,n}(\mathbb{R}) \times \mathbb{R}^n : v \in V\} \subseteq G_{k,n}(\mathbb{R}) \times \mathbb{R}^n$$

*This is also called **Grassmann bundle** oder **tautological bundle**.*

Remark 2.19. Note that $G_{1,n+1} = \mathbb{R}\mathrm{P}^n$. The bundle $\gamma_{1,n+1} = (p, E_{1,n+1}, \mathbb{R}\mathrm{P}^n)$ is often referred to as the **canonical line bundle** over $\mathbb{R}\mathrm{P}^n$.

Theorem 2.20. *Let $\xi : E \to M$ a vector bundle of rank k over the n-manifold M.*

i) There is a number s and a map $f : M \to G_{k,s}(\mathbb{R})$ such that $\xi \cong f^*(\gamma_{k,s})$. It is enough to assume $s \geq k+n$. f is called ***classifying map*** *for ξ. Any classifying map $\partial M \to G_{k,s}$ for $\xi|_{\partial M}$ extends to a classifying map for ξ.*

ii) If even $s > k+n$, then any two classifying maps are homotopic. If η is another vector bundle over M, then $f_\xi \simeq f_\eta$ if and only if $\xi \cong \eta$.

The inclusion $\mathbb{R}^s \to \mathbb{R}^{s+1}$ induces closed inclusions

$$\begin{tikzcd} E_{k,s} \arrow[r] \arrow[d] & E_{k,s+1} \arrow[d] \\ G_{k,s} \arrow[r] & G_{k,s+1} \end{tikzcd}$$

Passing to direct limits with respect to these inclusions

$$E_{k,\infty} := \lim_{\to} E_{k,s}, \quad G_{k,\infty} := \lim_{\to} G_{k,s}, \quad \gamma_{k,\infty} := \lim_{\to} \gamma_{k,s}$$

we obtain the following version of theorem 2.20

Theorem 2.21. *Any vector bundle of rank k over the paracompact base M has a classifying map $M \to G_{k,\infty}$. Any two classifying maps are homotopic.*

As a consequence of the theorems 2.20 and 1.23 we obtain a fundamental property of Grassmann manifolds. There is a one-to-one correspondence between the isomorphy classes of vector bundles of rank k over M and the homotopy classes of maps $M \to G_{k,s}$, provided s is large enough.

Theorem 2.22. *There is a bijection between the isomorphy classes of vector bundles of rank k over a manifold M and the homotopy classes $[M, G_{k,s}(\mathbb{R})]$ of maps $M \to G_{k,s}(\mathbb{R})$ for s large enough.*

Proof. Denote by $\mathcal{B}^k(M)$ the isomorphy classes of vector bundles of rank k over M. Consider the map

$$[M, G_{k,s}(\mathbb{R})] \to \mathcal{B}^k(M),$$
$$[f] \mapsto f^*\gamma_{k,s}.$$

By theorem 1.23 this map is well-defined and onto by theorem 2.20(i). Let f_1, $f_2 : M \to G_{k,s}(\mathbb{R})$ be two maps such that $f_1^* E_{k,s} \cong f_2^* E_{k,s}$. Of course, f_1 and f_2 are classifying maps for the vector bundles $f_1^* E_{k,s}$ and $f_2^* E_{k,s}$. Thus, $[f_1] = [f_2]$ by theorem 2.20(ii). □

We write $G_{k,s} = G_{k,s}(\mathbb{R})$. By theorem 2.20 the pair $(N, S) = (E_{k,s}, G_{k,s})$ is a suitable candidate for our purpose, provided $s \geq k+n$. As usual wie identify $G_{k,s}$ with the zero section in $E_{k,s}$. Suppose $M^n \subseteq S^{n+k}$. We choose a tubular neighbourhood U of M^n. By 2.20 there is a commutative diagram

$$\begin{tikzcd} T^{\perp}M \arrow[r] \arrow[d] & E_{k,s} \arrow[d] \\ M^n \arrow[r, "f"'] & G_{k,s} \end{tikzcd}$$

where f is a classifying map for the normal bundle of M. Since $T^{\perp}M \cong U$ we have

$$\begin{tikzcd} U \subseteq S^{n+k} \arrow[r] \arrow[d] & E_{k,s} \arrow[d] \\ M^n \arrow[r, "f"'] & G_{k,s} \end{tikzcd}$$

Hence, given an arbitrary manifold $M^n \subseteq S^{n+k}$ we find a map defined on an open subset U of the sphere whose image lies in the total space of the Grassmann bundle and such that the preimage of $G_{k,s}$ is precisely M^n. In order to get a map which is defined on the whole of the sphere, Thom considers the one-point compactification $E^*_{k,s} := E_{k,s} \cup \infty$, the so-called *Thom space*, and maps $S^{n+k} - U$ to ∞. We can also understand the Thom space as a quotient space of $E_{k,s}$ where all vectors v with $|v| \geq 1$ are collapsed to a point. Let $\pi_{n+k}(E^*_{k,s})$ be the $(n+k)$-th homotopy group of $E^*_{k,s}$ with base point ∞. The collection of the preceding ideas will result in an isomorphism $\pi_{n+k}(E^*_{k,s}) \xrightarrow{\cong} MO_n$, provided k and s are large enough.

2.2.2 The Thom homomorphism

Let $[f] \in \pi_{n+k}(E^*_{k,s})$. We want to approximate f by a smooth map. However, $E^*_{k,s}$ is no longer a manifold (it still is a CW complex, see [3, 18.1]). Therefore we consider the restriction of f to $f^{-1}(E_{k,s})$ which can be deformed by a homotopy into a smooth map being transverse to the zero section $G_{k,s} \subseteq E_{k,s}$ (compare theorems 1.3, 1.9 and 1.10). Hence, given a homotopy class $\alpha \in \pi_{n+k}(E^*_{k,s})$ there is a map $f \in \alpha$ which is smooth as a map $f^{-1}(E_{k,s}) \to E_{k,s}$ and transverse to $G_{k,s}$. Furthermore, by lemma 2.15 the cobordism class of $f^{-1}(G_{k,s})$ only depends on the homotopy class α.

Definition 2.23. *The well-defined map*

$$\tau : \pi_{n+k}(E^*_{k,s}) \to MO_n,$$

$$\alpha \mapsto [f^{-1}(G_{k,s})]$$

is called ***Thom homomorphism****.*

Let us briefly recall the definition of the group structure in $\pi_n(X, x)$. Let $\psi : S^n \to S^n/\sim$ be the map that collapses the equator to a single point. Then we have $(S^n/\sim) = S^n \vee S^n$, where $\vee$ denotes the wedge sum. Let $\alpha, \beta \in \pi_n(X, x)$ and choose representatives $f, g : (S^n, y) \to (X, x)$. We define $\alpha + \beta := [(f \vee g) \circ \psi]$.

Lemma 2.24. *τ is a homomorphism.*

Proof. Let $[f]$, $[g] \in \pi_{n+k}(E^*_{k,s})$. By definition $[f] + [g]$ is then the class of the map $h : S^{n+k} \to E^*_{k,s}$ which is f on the upper hemisphere and g on the lower hemisphere. Applying again theorem 1.9 and 1.10 we may assume that f, g and

h are transverse to $G_{k,s}$. From the definition of the group structure it is obvious that $h^{-1}(G_{k,s}) = f^{-1}(G_{k,s}) \sqcup g^{-1}(G_{k,s})$. Hence

$$\begin{aligned}\tau([f]+[g]) &= [h^{-1}(G_{k,s})] \\ &= [f^{-1}(G_{k,s}) \sqcup g^{-1}(G_{k,s})] \\ &= [f^{-1}(G_{k,s})] + [g^{-1}(G_{k,s})] \\ &= \tau([f]) + \tau([g])\end{aligned}$$

□

2.2.3 The Pontrjagin-Thom construction

Suppose $[M] \in MO_n$. In the above discussion we already indicated how M can be related to a map $S^{n+k} \to E^*_{k,s}$. The first step is to apply Theorem 1.2 and embed the manifold M in $\mathbb{R}^{n+k}$ via $j : M \to \mathbb{R}^{n+k}$ where $k > n$. Denote by $\xi_j : T^\perp M \to M$ the normal bundle which is induced by this embedding. The fibre $E(\xi_j)_x$ over $x \in M^n$ can be understood as the linear space consisting of vectors $v \in \mathbb{R}^{n+k}$ orthogonal to M^n (with respect to the standard inner product). Along with the embedding comes a commutative diagram

$$\begin{tikzcd} T^\perp M \arrow[r, "g"] \arrow[d] & E_{k,s} \arrow[d] \\ M \arrow[r, "f"'] & G_{k,s} \end{tikzcd}$$

where f is a classifying map for the normal bundle and $s \geq n+k$. By theorem 1.24 we obtain a tubular neighbourhood $U \subseteq \mathbb{R}^{n+k}$ of M diffeomorphic to $T^\perp M$. Consider $S^{n+k} = \mathbb{R}^{n+k}_c$ where $\mathbb{R}^{n+k}_c = \mathbb{R}^{n+k} \cup \infty$ denotes the one-point compactification. By collapsing the complement of U to a point we obtain

$$\begin{tikzcd} S^{n+k} \arrow[r] \arrow[d, dotted, "h_M"'] & S^{n+k}/(S^{n+k} - U) = U_c \arrow[d, "\cong"] \\ E^*_{k,s} & (T^\perp M)_c \arrow[l, "g_c"'] \end{tikzcd}$$

Here the map g_c is induced by the bundle map g from above. Note that $(h_M)^{-1}(G_{k,s}) = M$ and h_M is transverse to $G_{k,s}$. Since g is proper, continuity of this map is guaranteed: Let V be an open neighbourhood of ∞. By definition of the topology of $E^*_{k,s}$, $V = (E_{k,s} - K) \cup \infty$ where K is a compact subset of $E_{k,s}$. Therefore,

$$g_c^{-1}(V) = g_c^{-1}\big((E_{k,s} - K) \cup \infty\big) = \big(T^\perp M - g^{-1}(K)\big) \cup \infty.$$

But $g^{-1}(K)$ is compact, hence $g_c^{-1}(V)$ is open. This construction immediately implies

Theorem 2.25. *τ is surjective if $k > n$ and $s \geq k+n$,*

We go on to show that τ is injective if $k > n+1$ and $s \geq k+n+1$. For the proof we need

Definition 2.26. *Let M be a manifold, $\xi = (p, E, B)$ a vector bundle over a compact manifold B and $g : M \to E^*$ a map. We say g is in* ***standard form*** *if there is a submanifold $S \subset M$, a tubular neighbourhood $U \subset M$ for S such that $U = g^{-1}(E)$, $S = g^{-1}(B)$ and the commutative diagram*

$$\begin{tikzcd} U \arrow[r, "g|_U"] \arrow[d] & E \arrow[d] \\ S \arrow[r, "g|_S"'] & B \end{tikzcd}$$

is a vector bundle map.

Lemma 2.27. *If $g : M \to E^*$ is in standard form, then $g(M - U) = \infty$ and $g \pitchfork B$.*

Proof. The first assertion is clear. The second one follows from the canonical isomorphism $T^{\perp}B = E$ and the fact that g induces linear isomorphisms in the fibres. $\square$

Theorem 2.28. *τ is injective if $k > n + 1$ and $s \geq k + n + 1$.*

Proof. Let $\alpha \in \pi_{n+k}(E^*_{k,s})$ such that $\tau(\alpha) = 0$ in MO_n. Choose $g : S^{n+k} \to E^*_{k,s}$ representing α. We want to construct an extension h of g to the disk D^{n+k+1}. This will do: D^{n+k+1} is contractible. Hence, there is a homotopy $H : D^{n+k+1} \times I \to D^{n+k+1}$ from a constant map to the identity. $h \circ H$ shows that g is null-homotopic.

Now α is mapped to 0, that is $\tau(\alpha) = \tau([g]) = [g^{-1}(G_{k,s})] = 0$. Set $M^n := g^{-1}(G_{k,s})$. Recall that we may assume g to be transverse to $G_{k,s}$ so that M^n is a submanifold of S^{n+k}. Choose a tubular neighbourhood $U \subseteq g^{-1}(E_{k,s})$ for M^n.

Claim 1. g is homotopic to a map in standard form.
Theorem 1.25 is tailormade for this purpose. We obtain a diagram

$$\begin{tikzcd} S^{n+k} \arrow[r, "g"] & E_{k,s} \\ D \subseteq U \arrow[u, "\cup\mid"] & \tilde{U} \arrow[u, "\cup\mid"'] \\ M^n \arrow[u, "\cup\mid"] \arrow[r, "g|_M"] & G_{k,s} \arrow[u, "\cup\mid"'] \end{tikzcd}$$

and a vector bundle map $\Phi : U \to E_{k,s}$ such that (up to a homotopy) $\Phi|_D = g|_D$ where $D \subset U$ denotes the disk bundle. Put

$$h = \begin{cases} \Phi & x \in U \\ \infty & x \in S^{n+k} - U \end{cases}$$

h agrees with g on ∂D. Furthermore $M^n \subseteq \mathrm{int}(D)$, hence g and h both map $S^{n+k} - \mathrm{int}(D)$ into the contractible space $E^*_{k,s} - G_{k,s}$. A homotopy from the identity map to ∞ is given by $H : (E^*_{k,s} - G_{k,s}) \times I \to (E^*_{k,s} - G_{k,s})$

$$H(x,t) = \begin{cases} \Big(\dfrac{1+t}{1-t}\Big)x & \text{if} \quad 0 \le t < 1, x \neq \infty \\ \infty & \text{if} \quad t = 1 \vee x = \infty \end{cases}$$

Lemma 2.31 below implies that g and h are homotopic.

Claim 2. h is still transverse to $G_{k,s}$.
This follows from lemma 2.27 because h is in standard form.

Since $[M^n] = \tau([g]) = 0$, M^n bounds a compact manifold W^{n+1}. We apply Whitney's theorem and embed M^n into S^{n+k}. By corollary 1.13 this inclusion $M^n \hookrightarrow S^{n+k}$ extends to a neat embedding $W^{n+1} \hookrightarrow D^{n+k+1}$.

Claim 3. The bundle map $h : U \to E_{k,s}$ extends to a bundle map $H : V \to E_{k,s}$ where V is a tubular neighbourhood of W^{n+1}.
U is a tubular neighbourhood of $M^n = \partial W^{n+1} = W^{n+1} \cap S^{n+k}$. We apply the extension theorem for tubular neighbourhoods, see 1.26, with $S = W^{n+1}$, $M = D^{n+k+1}$. Thus, there is a tubular neighbourhood V of W^{n+1} in D^{n+k+1} such that $U = S^{n+k} \cap V$. Note that we can understand U and V as vector bundles. Furthermore, we can interpret H as a classifying map for $V|_{\partial W^{n+1}}$ since $s \ge k + (n+1)$ by assumption Therefore we may apply theorem 2.20 which states that h extends to a vector bundle map $H : V \to E_{k,s}$.

All that remains to do is to extend H to all of the disk D^{n+k+1} by mapping $D^{n+k+1} - V$ to ∞ and to show

Claim 4. $H|_{S^{n+k}} = h$.
Let $x \in S^{n+k}$. Since H is an extension of $h : U \to E$, we clearly have $H(x) = h(x)$ for $x \in U$. If $x \notin U$, then h maps x to ∞ by claim 1. However, $U = S^{n+k} \cap V$, hence $x \notin V$. Therefore $H(x) = \infty$ by construction. □

Corollary 2.29 (Thom)**.** *The Thom homomorphism τ constitutes an isomorphism*

$$\pi_{n+k}(E^*_{k,s}) \xrightarrow{\cong} MO_n$$

if $k > n+1$ and $s \ge k+n+1$.

Remark 2.30. The Pontrjagin-Thom construction yields a map

$$\pi : MO_n \to \pi_{n+k}(E^*_{k,s}),$$

$$[M] \mapsto [h_M]$$

which obviously satisfies $\tau \circ \pi = \mathrm{id}$. Note that for k large enough, two embeddings of M into $\mathbb{R}^{k+n}$ are isotopic (homotopic through embeddings) by Whitney's embedding theorem and thus, the homotopy class $[h_M]$ does not depend on the choice of the embedding of M into $\mathbb{R}^{k+n}$. It is a formal consequence that π is a well-defined homomorphism which is inverse to τ.

Lemma 2.31. *Let M be a manifold, $A \subseteq M$ a closed subset and N a contractible CW-complex. If two maps $f, g : M \to N$ agree on A, there exists a homotopy relative A connecting them.*

In order to prove this, we need some terminology (compare [9, chapter 1.5]). By space we mean a separable, metrizable topological space.

Definition 2.32. *A space X is called an **absolute neighbourhood extensor (ANE)** if, whenever Y is a space and A is a closed subset of Y, any continuous map $f : A \to X$ can be extended to some neighbourhood U of A. X is called an **absolute extensor (AE)** if for any f U can be taken to be all of Y.*
*A space X is called an **absolute (neighbourhood) rectract A(N)R** if, whenever X is a closed subset of a space Y, then X is a retract of Y (respectively of some neighbourhood of X in Y).*

The notions of absolute extensors and absolute retracts coincide:

Theorem 2.33. *X is an A(N)E if and only if X is an A(N)R.*

Proof. See [9, theorem 1.5.2]. □

Example 2.34. Any convex subset of a normable linear space is an A(N)R (e.g. $\mathbb{R}^n$), compare [9, 1.5.1]. Clearly, any open subspace of an ANR is again an ANR.

Theorem 2.35. *A space X is an AR if and only if it is a contractible ANR.*

Proof. [9, corollary 1.6.7] □

Theorem 2.36. *If each $x \in X$ admits a neighbourhood which is an ANR, then X is an ANR.*

Proof. [9, theorem 5.4.5] □

We now proceed to prove lemma 2.31 and thus complete the proof of Thom's theorem above.

Proof of lemma 2.31. Consider the closed subset $X := (A \times I) \cup (M \times \{0, 1\})$ of $M \times I$. Define a continuous map $H : X \to N$ by $H_0 = f$, $H_1 = g$ and $H(x, t) = f(x) = g(x)$ for all x in A. Since N is a CW complex it is an absolute neighbourhood retract. But N is assumed to be contractible, so by theorem 2.35 N is an AR. Thus, N is an absolute extensor (compare 2.33) and therefore there is an extension of H to $M \times I$ which acts as a homotopy relative A from f to g. □

It remains to solve the homotopy problem. A thorough treatment of this topic is given in [5]. We will not go into the computation of $\pi_{n+k}(E^*_{k,s})$, but merely give the results. As already mentioned, Thom showed that the graded $\mathbb{Z}_2$-algebra MO_* is isomorphic to the polynomial algebra $\mathbb{Z}[x_2, x_4, x_5, \ldots]$ in variables x_k for $k \neq 2^i - 1$. In even dimensions the generators can be chosen to be the unoriented cobordism classes of $\mathbb{R}\mathrm{P}^{2k}$.

3 Cobordism as a homology theory

In this section we show that the concept of cobordism constitutes a generalized homology theory on the category TOP^2 of pairs of spaces. Moreover, we cite some important theorems. Most of the following may be found in [1] and [4, chapter 21]. We recall

Definition 3.1 (homology theory). *A homology theory for the category TOP^2 consists of*

i) a family $(H_n : n \in \mathbb{Z})$ of covariant functors $H_n : TOP^2 \to R$-Mod,

ii) a family $(\partial_n : n \in \mathbb{Z})$ of natural transformations $\partial_n : H_n(X, A) \to H_{n-1}(A)$ such that the Eilenberg-Steenrod axioms are satisfied.

The axioms of Eilenberg and Steenrod are contained in [4, chapter 10]. We need the notion of relative cobordism groups.

Definition 3.2. *Fix a topological pair (X, A). A **singular n-manifold** in (X, A) is a pair (M, f) consisting of a compact n-dimensional manifold M and a map of pairs $f : (M, \partial M) \to (X, A)$. A **cobordism** between two singular n-manifolds (M_0, f_0) and (M_1, f_1) is a pair (B, F) satisfying the following properties:*

i) B is a compact $(n + 1)$-manifold with boundary,

ii) there exists a submanifold N such that $\partial B = M_0 \sqcup M_1 \cup N$, $\partial N = \partial M_0 \sqcup \partial M_1$ and $M_i \cap N = \partial M_i$,

iii) $F : B \to X$ restricts to f_i at M_i, $F|_{M_i} = f_i$ and

iv) $F(N) \subset A$.

(M_0, f_0) and (M_1, f_1) are called cobordant, it there is a cobordism connecting them. If $A = \emptyset$, then $\partial M = \emptyset$ and we get back the previous definition.

This defines an equivalence relation. However, as the complexity of the definition suggests, the proof is more delicate now. We impose a differentiable structure on the product of two manifolds with boundary $M_1 \times M_2$ by *straightening the angles*, compare [1, chapter 3], [17, 13.12] and [4, 15.10.2]. If $\partial M_1 = \emptyset$ or $\partial M_2 = \emptyset$ we simply use product charts. However, if both boundaries are non-empty, we have to be take care of corners.

Example 3.3. Consider $\mathbb{R}_+ \times \mathbb{R}_+ \subset \mathbb{R}^2$. In this case, corners can be straightened by the homeomorphism Θ

$$\mathbb{R}_+ \times \mathbb{R}_+ \to H^2$$

$$(r, \theta) \mapsto (r, 2\theta)$$

which is a diffeomorphism except in 0.

There is a natural structure on $M_1 \times M_2 - (\partial M_1 \times \partial M_2)$ by using products of M_i-charts. Choose collars $\phi_i : \mathbb{R}_+ \times \partial M_i \overset{\cong}{\to} U_i \subset M_i$ and consider the composition ψ

$$\begin{tikzcd} H^2 \times \partial M_1 \times \partial M_2 \arrow[r, "\Theta\times\mathrm{id}"] \arrow[d, "\psi"'] & \mathbb{R}_+ \times \mathbb{R}_+ \times \partial M_1 \times \partial M_2 \arrow[d] \\ U_1 \times U_2 & (\mathbb{R}_+ \times \partial M_1) \times (\mathbb{R}_+ \times \partial M_2) \arrow[l, "\phi_1\times\phi_2"] \end{tikzcd}$$

where the vertical map interchanges factors two and three. The product differentiable structure on $H^2 \times \partial M_1 \times \partial M_2$ induces a differentiable structure on $U_1 \times U_2$ such that ψ is a diffeomorphism. Now $M_1 \times M_2 - (\partial M_1 \times \partial M_2)$ and $U_1 \times U_2$ have smooth structures which agree on their intersection. Thus, there is a smooth structure on $M_1 \times M_2$ with boundary $\partial M_1 \times M_2 \cup_{\mathrm{id}_{\partial M_1 \times \partial M_2}} M_1 \times \partial M_2$. Reflexivity of the relation defined in 3.2 is now seen by considering the product $M \times I$. Symmetry follows immediately. To check transitivity we glue cobordisms along their common boundary piece M and define a smooth structure as in proposition 1.14. However, we have to be careful at points in ∂M. Here, a similar device as described above is needed, see [4, 15.10.3].

The resulting set of equivalence classes is denoted by $MO_n(X, A)$. Again a commutative group structure is imposed on $MO_n(X, A)$ by disjoint union where any element has order at most 2. Given (M^n, f) and a closed manifold N^m we define a new singular $(m+n)$-manifold $(M^n \times N^m, g)$ by $g(x, y) = f(x)$. In this way the direct sum $MO_*(X, A) = \sum_{n=0}^{\infty} MO_n(X, A)$ is given the structure of a graded module over MO_*. Furthermore, a map $\phi : (X_0, A_0) \to (X_1, A_1)$ induces a natural homomorphism

$$\phi_* : MO_n(X_0, A_0) \to MO_n(X_1, A_1),\ \phi_*[M^n, f] = [M^n, \phi \circ f].$$

The boundary operators ∂_n are given by

$$\partial_n : MO_n(X, A) \to MO_{n-1}(A),$$

$$\partial_n[M^n, f] = [\partial M^n, f|_{\partial M^n}].$$

This is well-defined and additive. It is obvious that the assignment $MO_* : (X, A) \mapsto MO_*(X, A)$ is functorial. From the definition of the boundary operators it is also clear that for any map $\phi : (X_0, A_0) \to (X_1, A_1)$ the diagram

$$\begin{tikzcd} MO_n(X_0, A_0) \arrow[r, "\partial_n^0"] \arrow[d, "\phi_*"'] & MO_{n-1}(A_0) \arrow[d, "(\phi|_{A_0})_*"] \\ MO_n(X_1, A_1) \arrow[r, "\partial_n^1"'] & MO_{n-1}(A_1) \end{tikzcd}$$

commutes. This means that ∂ is a natural transformation. We now prove that $\{MO_*(X, A), \partial\}$ satisfies the Eilenberg-Steenrod axioms.

Proposition 3.4 (Homotopy invariance)**.** *Homotopic maps $\phi, \psi : (X_0, A_0) \to (X_1, A_1)$ induce the same homomorphisms $\phi_* = \psi_*$.*

Proof. Choose a homotopy $H : (X_0 \times I, A_0 \times I) \to (X_1, A_1)$ between ϕ and ψ. Let $[M^n, f] \in MO_n(X_0, A_0)$. We have to show $[M^n, \phi \circ f] = [M^n, \psi \circ f]$. Define $F : M^n \times I \to X_1$ by $F(x,t) = H(f(x), t)$. Then $F_0 = H_0 \circ f = \phi \circ f$ and $F_1 = H_1 \circ f = \psi \circ f$. From the foregoing discussion we know that $M^n \times I$ can be given the structure of a smooth manifold whose boundary consists of $M^n \times 0$, $M^n \times 1$ and $\partial M^n \times I$. Since $f(\partial M^n) \subset A_0$ and H is a homotopy of pairs, it follows $F(\partial M^n \times I) \subset A_1$. Thus, $(M^n \times I, F)$ defines a cobordism between $[M^n, \phi \circ f]$ and $[M^n, \psi \circ f]$ which means $\phi_*[M^n, f] = \psi_*[M^n, f]$. □

In order to prove exactness we need the following lemma.

Lemma 3.5. *Suppose M^n is a closed manifold and $S^n \subset M^n$ a compact submanifold of the same dimension. If $f : M^n \to X$ maps $M^n - \mathrm{int}(S^n)$ into A, then $[M^n, f] = [S^n, f|_{S^n}]$ in $MO_n(X, A)$.*

Proof. Define $B^{n+1} := M^n \times I$ and $F : B^{n+1} \to X$ by $F(x,t) = f(x)$. Since M^n is closed, $\partial B^{n+1} = M^n \times 0 \sqcup M^n \times 1$. Hence, $M^n \sqcup S^n \cong M^n \times 0 \sqcup S^n \times 1$ is a submanifold of ∂B^{n+1}. By assumption F maps $M^n - \mathrm{int}(S^n)$ into A and hence (B^{n+1}, F) is the required cobordism between $[M^n, f]$ and $[S^n, f|_{S^n}]$. □

Proposition 3.6 (Exact sequence)**.** *For each pair (X, A) the sequence*

$$\ldots \to MO_{n+1}(X, A) \xrightarrow{\partial} MO_n(A) \xrightarrow{i_*} MO_n(X) \xrightarrow{j_*} MO_n(X, A) \xrightarrow{\partial} MO_{n-1}(A) \to \ldots$$

is exact. Here i and j denote the inclusions $i : A \to X$, $j : (X, \emptyset) \to (X, A)$.

Proof. $\mathrm{img}(\partial) = \ker(i_*)$.
Let $[M, f] \in MO_{n+1}(X, A)$, then $i_* \circ \partial[M, f] = [\partial M, i \circ f|_{\partial M}] = 0$ by definition. Now suppose $[M, f] \in MO_n(A)$ and $i_*[M, f] = 0$. Then M is null bordant via a cobordism (B, F) where $F(\partial B) = f(M) \subset A$. Hence, $[B, F] \in MO_{n+1}(X, A)$ and $\partial[B, F] = [\partial B, F|_{\partial B}] = [M, f]$.

$\mathrm{img}(i_*) = \ker(j_*)$.
Let $[M, f] \in MO_n(A)$. We apply the preceding lemma with $S = \emptyset$ to conclude $j_* \circ i_* = 0$. Conversely, suppose $j_*[M, f] = 0$ in $MO_n(X, A)$. By definition, we find a compact $(n+1)$-manifold B together with a map $F : B \to X$ and a closed submanifold N such that $\partial B = M \sqcup N$ and $F(N) \subset A$. Hence, $[N, F|_N]$ can be considered as an element in $MO_n(A)$ and $i_*[N, F|_N] = [M, f]$.

$\mathrm{img}(j_*) = \ker(\partial)$.
$\partial \circ j_* = 0$ follows immediately from the definition of the boundary map. If $\partial[M, f] = 0$ in $MO_{n-1}(A)$, there is a null bordism $[B, F]$ of $(\partial M, f|_{\partial M})$. Now identify B and M along their common boundary ∂M to obtain a singular manifold (N, g) in X such that $g|_B = F$ and $g|_M = f$. Since $g(B) = F(B) \subset A$ we may apply lemma 3.5 and find $j_*[N, g] = [M, f]$. □

Proposition 3.7 (Excision)**.** *If (X, A) is a pair and $\bar{U} \subset$ int(A). Then the inclusion $i : (X - U, A - U) \to (X, A)$ induces an isomorphism*

$$i_* : MO_n(X - U, A - U) \cong MO_n(X, A).$$

Proof. *i_* is surjective.* Let (M^n, f) be a singular manifold in (X, A) and set $P = f^{-1}(X - \mathrm{int}(A))$, $Q = f^{-1}(\bar{U})$. Since $\bar{U} \subset \mathrm{int}(A)$, P and Q are closed disjoint subsets of M^n. By [1, lemma 3.1] there exists a closed n-dimensional manifold $N \subset M^n$ such that $P \subset N$ and $N \cap Q = \emptyset$. Hence, if we restrict f to N we obtain a map $f|_N : N \to X - U$ and therefore a singular manifold $(N, f|_N)$ in $(X - U, A - U)$. Moreover, $f(M^n - \mathrm{int}(N)) \subset A$ so that we may apply lemma 3.5 which shows $i_*[N, f|_N] = [M^n, f]$.

i_* is injective. Suppose there is a singular manifold (M^n, f) in $(X-U, A-U)$ with $i_*[M^n, f] = 0$. By definition we find a compact manifold B^{n+1} and a map $F : B^{n+1} \to X$ such that

- $\partial B = M^n \cup N$,
- $F|_{M^n} = f$ and
- $F(N) \subset A$.

Again use the lemma cited above to get rid of the elements which F maps into U and finally arrive at a nullbordism of (M^n, f) in $MO_n(X - U, A - U)$. □

It is not hard to prove

Theorem 3.8 (Additivity)**.** *Let $(X_i, A_i)_{i \in I}$ be topological pairs and $X = \bigsqcup_{i \in I} X_i$, $A = \bigsqcup_{i \in I} A_i$ the disjoint unions. The inclusions $(X_i, A_i) \hookrightarrow (X, A)$ induce an isomorphism*

$$\bigoplus_{i \in I} MO_*(X_i, A_i) \cong MO_*(X, A).$$

Remark 3.9. The dimension axiom is not satisfied, $MO_n(*) = MO_n$.

Suppose X is a finite CW-complex. Let (M^n, f) be a singular manifold in (X, A) with $\mathbb{Z}_2$-fundamental class $[M^n]_{\mathbb{Z}_2}$. The assignment

$$[M^n, f] \mapsto f_*[M^n]_{\mathbb{Z}_2}$$

defines a well-defined homomorphism

$$\mu : MO_n(X, A) \to H_n(X, A; \mathbb{Z}_2)$$

which constitutes a natural transformation of homology theories. It is a fundamental result of cobordism theory that this map is surjective, see [8].
We wish to state another crucial result, compare [1, chapter 17]. Consider the free MO_*-module $H_*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} MO_*$. For each $n \geq 0$ choose a basis $\{c_{n,i}\}$ of $H_n(X, \mathbb{Z}_2)$ as a vector space over $\mathbb{Z}_2$. By the preceding remark, for each $c_{n,i}$ we find a singular manifold (M_i^n, f_i^n) such that $(f_i^n)_*[M_i^n]_{\mathbb{Z}_2} = c_{n,i}$. Define an MO_*-module homomorphism

$$h : H_*(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} MO_* \to MO_*(X)$$

by $h(c_{n,i} \otimes 1) = [M_i^n, f_i^n]$. With the help of spectral sequences and Stiefel-Whitney numbers one proves

Theorem 3.10. *h is an isomorphism of MO_*-modules.*

Proof. This is [1, theorem 17.2]. □

At the end of this section we briefly describe a Mayer-Vietoris sequence for $MO_n(X)$. Since this will not be particular important for us we omit proofs. This may may be found in [5, chapter II]. Suppose $X = U \cup V$ is the union of two open subsets. We want to define a homomorphism

$$\partial : MO_n(X) \to MO_{n-1}(U \cap V).$$

Let (M, f) be a singular n-manifold in X. Then $M_0 = f^{-1}(X - U)$ and $M_1 = f^{-1}(X - V)$ are disjoint closed subsets of M. [5, II.3.3] shows that there is a smooth map $\phi : M \to [0, 1]$ (called **separating function**) such that

- $M_0 \subset \phi^{-1}(0)$,
- $M_1 \subset \phi^{-1}(1)$ and
- $\frac{1}{2}$ is a regular value of ϕ.

Then $M_\phi = \phi^{-1}\left(\frac{1}{2}\right)$ is an $(n-1)$-dimensional submanifold of M and by construction the restriction of f to M_ϕ takes values in $U \cap V$. Given cobordant singular manifolds $[M, f]$, $[N, g]$ in $MO_n(X)$ and two separating functions ϕ, ψ, one checks $[M_\phi, f|_{M_\phi}] = [N_\psi, g|_{N_\psi}]$. Thus, the map

$$\partial : MO_n(X) \to MO_n(U \cap V),$$

$$[M, f] \mapsto [M_\phi, f|_{M_\phi}]$$

is well-defined.

Theorem 3.11. *The sequence induced by the canonical inclusions*

$$\ldots \to MO_{n+1}(X) \xrightarrow{\partial} MO_n(U \cap V) \to MO_n(U) \oplus MO_n(V) \to MO_n(X) \xrightarrow{\partial} \ldots$$

is exact.

Proof. This is [5, II.3.8]. □

4 Stiefel-Whitney numbers

The Euler characteristic gives us a possibility to decide if a given manifold can be null bordant. However, up to now we do not have a sufficient condition that a manifold bounds. It is here that Stiefel-Whitney numbers enter the stage. A recommendable reference is [3]. A short summary is given in the appendix.

Theorem 4.1. *[Pontrjagin] If M is a manifold which bords, then all Stiefel-Whitney numbers of M are zero.*

Proof. Let M be an n-manifold. Since M is null bordant there is a compact $(n+1)$-manifold B such that $\partial B = M$. We have to relate the Stiefel-Whitney classes of τ_B to those associated with τ_M. The key fact is

$$TB|_M \cong TM \oplus \mathbb{R}.$$

In order to establish this decomposition, choose a smooth outward-pointing normal vector field along $\partial B = M$. Thus, the Stiefel-Whitney classes of τ_B restricted to M are equal to the Stiefel-Whitney classes of τ_M, $\iota^*(w_j(\tau_B)) = w_j(\tau_M)$. There is the natural homomorphism

$$\partial : H_{n+1}(B, M) \to H_n(M)$$

which maps $[B]_{\mathbb{Z}_2}$ to $[M]_{\mathbb{Z}_2}$, compare theorem A.8. Furthermore, there is the exact sequence associated with the pair (B, M):

$$H^n(B) \xrightarrow{\iota^*} H^n(M) \xrightarrow{\delta} H^{n+1}(B, M)$$

Hence,

$$\begin{aligned}
\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}, [M]_{\mathbb{Z}_2}\rangle &= \langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}, \partial([B]_{\mathbb{Z}_2})\rangle \\
&= \langle \delta(w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}), [B]_{\mathbb{Z}_2}\rangle \\
&= \langle (\delta \circ \iota^*)(w_1(\tau_B)^{r_1} \cdots w_n(\tau_B)^{r_n}), [B]_{\mathbb{Z}_2}\rangle \\
&= 0.
\end{aligned}$$

by exactness of the above sequence and $\langle a, \partial([B]_{\mathbb{Z}_2})\rangle = \langle \delta a, [B]_{\mathbb{Z}_2}\rangle$ for any $a \in H^n(M)$. □

The converse is also true (due to Thom) but much harder to show.

Theorem 4.2 (Thom)**.** *If all Stiefel Whitney numbers associated with M are zero, then M bords.*

Proof. We refer to [7]. □

Combining the preceding theorems we obtain a very valuable theorem:

Theorem 4.3. *Two closed manifolds of the same dimension are cobordant if and only if all of their corresponding Stiefel-Whitney numbers are equal.*

Corollary 4.4. *The unoriented cobordism group MO_n is finite.*

As an application we want to determine in which case the real projective space $\mathbb{R}\mathrm{P}^n$ bords. In order to apply Thom's theorem we need to compute Stiefel-Whitney numbers. This first of all requires knowledge about the cohomology.

Lemma 4.5.
$$H^i(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } 0 \le i \le n \\ 0 & \text{for } i > n \end{cases}$$

If a_n denotes the non-zero element of $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$, then each $H^i(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$ is generated by a_n^i. Furthermore, the cohomology ring $H^*(\mathbb{R}\mathrm{P}^\infty)$ is the polynomial ring $\mathbb{Z}_2[w_1]$ freely generated by the first Stiefel-Whitney class w_1. The inclusion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ induces a surjective homomorphism

$$H^*(\mathbb{R}\mathrm{P}^\infty, \mathbb{Z}_2) \twoheadrightarrow H^*(\mathbb{R}\mathrm{P}^n, \mathbb{Z}_2)$$

which maps w_1 to the generator a_n.

Proof. See [15]. □

Lemma 4.6 (Stiefel-Whitney classes of $\mathbb{R}\mathrm{P}^n$)**.** *The total Stiefel-Whitney class of $\mathbb{R}\mathrm{P}^n$ equals*

$$w(\mathbb{R}\mathrm{P}^n) = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \ldots + \binom{n+1}{n} a^n = (1+a)^{n+1},$$

where a denotes the non-zero element of $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Proof. We follow [3, chapter 4].

Claim 1. The total Stiefel-Whitney class of the canonical line bundle $w(\gamma_{1,n+1})$ over $\mathbb{R}\mathrm{P}^n$ equals $1 + a$.
The inclusion $j : \mathbb{R}\mathrm{P}^1 \hookrightarrow \mathbb{R}\mathrm{P}^n$, induced by $(x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0)$ is covered by the bundle map $J : \gamma_{1,2} \to \gamma_{1,n+1}$ which fibrewise maps 1-dimensional linear subspaces of $\mathbb{R}^2$ to the corresponding 1-dimensional linear subspace in $\mathbb{R}^{n+1}$. By naturality of Stiefel-Whitney classes and SW4, we obtain $0 \neq w_1(\gamma_{1,2}) = j^*(w_1(\gamma_{1,n+1}))$. Thus, $w_1(\gamma_{1,n+1}) \neq 0 \in H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \cong \mathbb{Z}_2$ which implies $w_1(\gamma_{1,n+1}) = a$. Furthermore, $w_j(\gamma_{1,n+1}) = 0$ for all $j \geq 2$ by SW1, compare the appendix. The key step is the following description of the tangent bundle of $\mathbb{R}\mathrm{P}^n$.

Claim 2. $\tau_{\mathbb{R}\mathrm{P}^n} \cong \mathrm{Hom}(\gamma_{1,n+1}, \gamma_{1,n+1}^\perp)$. We need to understand the tangential space of $\mathbb{R}\mathrm{P}^n = S^n/\mathbb{Z}_2$. Let l be a 1-dimensional subspace in $\mathbb{R}^{n+1}$ intersecting the sphere at $\pm x$. Note that

$$T\,\mathbb{R}\mathrm{P}^n = TS^n/\mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts via the differential of the antipodal map $A : S^n \to S^n$, $x \mapsto -x$, compare proposition [4, 15.6.4]. This action takes the form $S^n \times \mathbb{R}^{n+1} \to S^n \times \mathbb{R}^{n+1}$, $(x, v) \mapsto (-x, -v)$. Hence we can understand $T\,\mathbb{R}\mathrm{P}^n$ as the set $\{(x, v) : x \in S^n, v \in x^\perp\}$ under the identification $(x, v) \equiv (-x, -v)$. We can now define a well-defined fibrewise linear map $T_{[x]}\,\mathbb{R}\mathrm{P}^n \to \mathrm{Hom}(l, l^\perp)$, $[x, v] \mapsto (\phi : x \mapsto v)$. An inverse is of course given by $\phi \mapsto [\frac{x}{|x|}, \phi(\frac{x}{|x|})]$ where $x \neq 0 \in l$. Theorem 1.19 completes the argument.

Claim 3. If ξ is a vector bundle that admits an Euclidean metric, then $\xi \cong \xi'$ where $\xi' = Hom(\xi, \theta^1)$.

By assumption we have an inner product $\langle -, - \rangle$ on any fibre $E(\xi)_x$. Given $v \in E(\xi)_x$ the usual isomorphism $E(\xi)_x \cong E(\xi)'_x$, $v \mapsto (w \mapsto \langle v, w \rangle)$ between a vector space and its dual establishes the claim (see 1.19).

Claim 4. Hom($\gamma_{1,n+1}, \gamma_{1,n+1}$) $\cong \theta^1$
Given a line l through the origin in $\mathbb{R}^{n+1}$, there is the identity map on l. This defines a nowhere zero section of the line bundle $\mathrm{Hom}(\gamma_{1,n+1}, \gamma_{1,n+1})$. The result now follows from proposition 1.20.

Claim 5. $\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1 \cong (n+1)\gamma_{1,n+1}$.

$$\begin{aligned}
\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1 &= \mathrm{Hom}(\gamma_{1,n+1}, \gamma_{1,n+1}^\perp) \oplus \mathrm{Hom}(\gamma_{1,n+1}, \gamma_{1,n+1}) \\
&= \mathrm{Hom}(\gamma_{1,n+1}, \gamma_{1,n+1}^\perp \oplus \gamma_{1,n+1}) \\
&= \mathrm{Hom}(\gamma_{1,n+1}, \theta^{n+1}) \\
&= \mathrm{Hom}(\gamma_{1,n+1}, \theta^1 \oplus \ldots \oplus \theta^1) \\
&= (n+1)\mathrm{Hom}(\gamma_{1,n+1}, \theta^1) \\
&= (n+1)\gamma_{1,n+1} \quad \text{by claim 3.}
\end{aligned}$$

Since $w(\theta^1) = 1$ we obtain from the Whitney product theorem

$$\begin{aligned}
w(\tau_{\mathbb{R}\mathrm{P}^n}) &= w(\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1) \\
&= \prod_{i=1}^{n+1} w(\gamma_{1,n+1}) \\
&= (1+a)^{n+1} \\
&= 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \ldots + \binom{n+1}{n} a^n \quad \text{as } a^{n+1} = 0.
\end{aligned}$$

□

Corollary 4.7. $\mathbb{R}\mathrm{P}^n$ *bords if and only if* n *is odd.*

Proof. If n is even, then the Euler characteristic of $\mathbb{R}\mathrm{P}^n$ equals 1 and hence $\mathbb{R}\mathrm{P}^n$ cannot be a boundary by theorem 1.18. We use Thom's theorem 4.2 to prove the other implication. Let $n = 2k-1$ be odd. From lemma 4.6 we obtain the total Stiefel-Whitney class of $\tau_{\mathbb{R}\mathrm{P}^n}$

$$\begin{aligned}
w(\mathbb{R}\mathrm{P}^n) &= (1+a)^{2k} \\
&= (1+a^2)^k \quad \text{since we work with } \mathbb{Z}_2.
\end{aligned}$$

Here a denotes the non-zero element of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$, compare lemma 4.5 above. Therefore all Stiefel-Whitney classes in odd dimensions are zero. However, any monomial of dimension $2k-1$ must contain a class w_j of odd dimension and thus, all Stiefel-Whitney numbers are zero. □

As another consequence of lemma 4.6 we find

Corollary 4.8.

$$w(\mathbb{R}\mathrm{P}^n) = 1 \Leftrightarrow n+1 = 2^k,\ k \in \mathbb{N}.$$

Proof. Let $n+1 = 2^k$. Then

$$w(\mathbb{R}\mathrm{P}^n) = (1+a)^{2^k} = 1 + a^{2^k} = 1,$$

since we are working modulo 2 and $2^k > n$.

Conversely, if $n+1$ is not a power of 2, there occurs an odd number in the prime decomposition of $n+1$. Hence we can write $n+1 = 2^k \cdot m$ with $m > 1$ odd. Thus,

$$w(\mathbb{R}\mathrm{P}^n) = (1+a)^{2^k m} = (1+a^{2^k})^m = 1 + ma^{2^k} + \dots a^{2^k m} \neq 1$$

since $2^k < n+1$. □

Remark 4.9. It follows from the above corollary that $\mathbb{R}\mathrm{P}^n$ can only be parallelizable if $n = 1, 3, 7, 15, 31, 63, \dots$ is of the form $2^j - 1$. In fact $\mathbb{R}\mathrm{P}^1$, $\mathbb{R}\mathrm{P}^3$, and $\mathbb{R}\mathrm{P}^7$ are parallelizable. The others are not.

Corollary 4.10. *Any vector field on* $\mathbb{R}\mathrm{P}^{2n}$ *has at least one zero.*

Proof. Suppose to the contrary that there exists a vector field which is nowhere zero. Then the tangent bundle splits as $\tau_{\mathbb{R}\mathrm{P}^{2n}} = \eta \oplus \theta^1$ where η is a $(2n-1)$-bundle. Thus, $w_{2n}(\mathbb{R}\mathrm{P}^{2n}) = w_{2n}(\eta) = 0$. However, we know from lemma 4.6

$$w_{2n}(\mathbb{R}\mathrm{P}^{2n}) = (2n+1)a^{2n} = a^{2n} \neq 0,$$

a contradiction. □

5 Involutions

5.1 Smooth group actions and equivariant cobordism

We now turn to an application of cobordism theory, namely involutions on manifolds. What can we say about the fixed point set of such a map and how can we relate it to the cobordism class of the manifold on which it is defined? Since an involution is nothing else than an action of $\mathbb{Z}_2$ on the manifold we start with a study of group actions. Our approach is similar to [1, chapter II].

Definition 5.1 (Smooth actions)**.** *Let G be a compact Lie group and M a compact manifold. A **smooth action** of G on M, denoted (G,M) is a smooth map $G \times M \to M$, $(g, x) \mapsto g \cdot x$ satisfying*

i) $1 \cdot x = x$,

ii) $g \cdot (h \cdot x) = (gh) \cdot x$.

M is then called a G-manifold.

Definition 5.2. *Let (G, M^n) be a smooth action on a closed manifold M^n. We say (G, M^n) **bounds** (or **bords**) if there is a smooth action (G, B^{n+1}) on a compact $(n+1)$-manifold B^{n+1} for which the induced action $(G, \partial B^{n+1})$ is G-equivariantly diffeomorphic (in short G-diffeomorphic) to (G, M^n). More precisely, there is a diffeomorphism $\phi : M^n \to \partial B^{n+1}$ such that $g \cdot \phi(x) = \phi(g \cdot x)$. In this case (G, M^n) is also called **null bordant**. Two actions (G, M_1^n), (G, M_2^n) on closed G-manifolds M_1^n, M_2^n are called **cobordant** if $(G, M_1^n \sqcup M_2^n)$ bords.*

In order to show that this defines an equivalence relation we need an equivariant version of the collaring theorem, see [1, theorem 20.3]. G acts smoothly on $\partial M \times [0, 1]$ as $g \cdot (x, t) = (g \cdot x, t)$.

Theorem 5.3 (equivariant collaring theorem)**.** *Let M be a compact G-manifold. There is an open set U containing the boundary ∂M and an equivariant diffeomorphism $\phi : \partial M \times [0, 1) \to U$ such that $\phi(x, 0) = x$ for all $x \in \partial M$.*

Lemma 5.4. *Being cobordant defines an equivalence relation.*

Proof. By definition the relation is symmetric. Reflexivity follows by looking at $M \times [0, 1]$. The proof of transitivity is an application of the equivariant collaring theorem, compare proposition 1.14 and lemma 2.6. Let B_1 be a cobordism between M_1, M_2 and B_2 one connecting M_2, M_3. We glue B_1 and B_2 along their common boundary M_2 and obtain a new compact manifold B, compare proposition 1.14. Since M_2 is G-diffeomorphic to part of the boundary of B_1 as well as of B_2 we may suppose $B_1 \cap B_2 = M_2$ and the actions of G on B_i coincide on M_2. Now choose G-equivariant collars $\phi_1 : M_2 \times [0, 1) \to U_1$ for M_2 in B_1 and $\phi_2 : M_2 \times [0, 1) \to U_2$ for M_2 in B_2 and define $\Phi : (-1, 1) \times M_2 \to B$,

$$\Phi(t, x) = \begin{cases} \phi_1(-t, x) & \text{for } t \leq 0 \\ \phi_2(t, x) & \text{for } t \geq 0 \end{cases}$$

with image $U = U_1 \cup U_2$. Note $B = (B_1 - M_2) \cup U \cup (B_2 - M_2)$ as a union of open subsets. Since the collars are choosen equivariantly, there is induced a well-defined action $y \mapsto g \cdot y$ of G on B. Denote the action of G on M_2 by λ. Let $y = \Phi(t, x)$ in U. Then $g \cdot y = g \cdot \Phi(t, x) = \Phi(t, g \cdot x)$ since the collars were chosen to be G-equivariant. Thus, the action of G on U takes the form $\Phi \circ (\mathrm{id} \times \lambda) \circ \Phi^{-1}$ and therefore the induced action of G on B is smooth. $\square$

Definition 5.5. *The resulting set of equivalence classes of G-n-manifolds is denoted by $I_n(G)$. A group structure is induced by*

$$[M_1] + [M_2] := [M_1 \sqcup M_2]$$

Remark 5.6. If we put $G = \{1\}$, then we end up with the usual unoriented cobordism groups MO_n from chapter 2.

As in chapter 2 one shows

Proposition 5.7. *This addition is well-defined and makes $I_n(G)$ into an abelian group in which any element has order at most 2.*

Definition 5.8. *The direct sum $I_*(G) = \sum_{n=0}^{\infty} I_n(G)$ is a graded commutative ring with identity. Multiplication is induced by*

$$[G, M_1^n] \cdot [G, M_2^m] := [G, M_1^n \times M_2^m],$$

where G acts on $M_1 \times M_2$ as $g \cdot (x, y) = (g \cdot x, g \cdot y)$. As in chapter 2 one shows that this multiplication is well-defined. The identity element is represented by $(G, \{*\}) \in I_0(G)$. There is a map $MO_n \to I_n(G)$ by taking the action of G to be trivial. In this way we can view $I_*(G)$ as a graded commutative algebra over MO_*. We refer to $I_*(G)$ as the ***unrestricted unoriented G-cobordism algebra****.*

Remark 5.9. If we take orientations into account, we arrive at the **unrestricted oriented G-cobordism algebra** $O_*(G)$.

There is an analogue construction under the restriction that G acts free on n-manifolds. We call the resulting group $MO_n(G)$ the *principal unoriented G-cobordism group*. The direct sum $MO_*(G)$ can be made into a graded module over MO_* as follows. Given $[G, M^n] \in MO_n(G)$ and $[N^m] \in MO_m$, define G to act on $M^n \times N^m$ by $g \cdot (x, y) = (g \cdot x, y)$. This action is still free, hence $[G, M^n \times N^m]$ is an element in $MO_{m+n}(G)$. The relation between $MO_*(G)$ and MO_* is the following theorem which ensures that ordinary cobordism theory will help us in the study of $MO(G)$ for G a finite group. We denoty by BG the classifying space of G, compare [4, chapter 14.4].

Theorem 5.10. *Let G be a finite group. Then there is an MO_*-module isomorphism*

$$MO_*(G) \cong MO_*(BG).$$

Sketch of proof. We only define inverse maps $MO_n(G) \rightleftharpoons MO_n(BG)$. Let $\xi = (\pi_G, EG, BG)$ be a universal principal G-bundle. Consider $[M, f] \in MO_n(BG)$, $f : M \to BG$. We pullback ξ by the map f to obtain the bundle $f^*(\xi) = (p, X, M)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & EG \arrow[d, "\pi_G"] \\ M \arrow[r, "f"'] & BG \end{tikzcd}$$

There is an induced action on $X \subset M \times EG$: $g(x,y) = (x, gy)$. One shows that $[M,f] \mapsto [G,X]$ is well-defined.

Conversely let $(G,M) \in MO_n(G)$. Since ξ is universal, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d] & EG \arrow[d, "\pi_G"] \\ M/G \arrow[r, "f"'] & BG \end{tikzcd}$$

We thus arrive at an assignment $[G,M] \mapsto [M/G, f]$. If (G,M) bounds, that is $(G,M) = (G, \partial B)$, then we obtain an 'extended' diagram

$$\begin{tikzcd} B \arrow[r, "\tilde{F}"] \arrow[d] & EG \arrow[d, "\pi_G"] \\ B/G \arrow[r, "\tilde{f}"'] & BG \end{tikzcd}$$

where $\tilde{F}$ and $\tilde{f}$ restrict to F and g at the boundary. M/G is the boundary of B/G and hence $[M/G, f] = 0$ shows that this assignment is well-defined. $\square$

We now introduce Stiefel-Whitney numbers associated to maps. Consider a closed manifold M^n and a map $f : M^n \to X$. Suppose we are given a cohomology class $a \in H^m(X; \mathbb{Z}_2)$ and a partition ω of $n - m$. Then $f^*(a) \in H^m(M^n; \mathbb{Z}_2)$ and $W_\omega \in H^{n-m}(M^n; \mathbb{Z}_2)$, compare the appendix. The numbers modulo two

$$\langle W_\omega f^*(a), [M]_{\mathbb{Z}_2} \rangle$$

are called **Stiefel-Whitney** (or **characteristic**) **numbers** of the map f. Similar to 4.1 and 4.2 the following theorem holds (see [1, theorem 17.3]).

Theorem 5.11. *Let $f : M^n \to X$ be a singular manifold in a finite CW complex X. Then (M^n, f) bords if and only if all characteristic numbers of f vanish.*

Let (T, M) be a fixed point free involution. Since $B\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty$ there is a map $f : M/T \to \mathbb{R}\mathrm{P}^\infty$. $c := f^*(w_1)$ is called the *characteristic class of the involution* (here w_1 is the first Stiefel-Whitney class of $\mathbb{R}\mathrm{P}^\infty$ and hence c is the first Stiefel-Whitney class of M/T). By theorem 5.10 $[T, M^n] = 0$ in $MO_n(\mathbb{Z}_2)$ if and only if $[M/T, f] = 0$ in $MO_n(\mathbb{R}\mathrm{P}^\infty)$. Thus, we may state

Theorem 5.12. *A fixed point free involution (T, M^n) bords if and only if all characteristic numbers of the form $\langle W_\omega c^m, [M/T]_{\mathbb{Z}_2} \rangle$ vanish, $0 \leq m \leq n$ where ω is a partition of $n - m$.*

Furthermore, we obtain from theorem 3.10.

Theorem 5.13. *For each $n \geq 0$ let (T, M^n) be a fixed point free involution on a closed n-manifold such that $\langle c^n, [M^n/T]_{\mathbb{Z}_2} \rangle \neq 0$ in $\mathbb{Z}_2$. Then the elements $[T, M^n]$ form a homogeneous basis for the MO_*-module $MO_*(\mathbb{Z}_2)$.*

Proof. Again consider the map $f : M^n/T \to \mathbb{R}\mathrm{P}^\infty$ associated to the involution (T, M^n). By assumption $\langle w_1^n, f_*([M^n/T]_{\mathbb{Z}_2})\rangle = \langle c^n, [M^n/T]_{\mathbb{Z}_2}\rangle \neq 0$. Thus, $f_*([M^n/T]_{\mathbb{Z}_2}) \neq 0$ which shows that the elements $[M^n/T, f]$ constitute a basis for the MO_*-module $MO_*(\mathbb{R}\mathrm{P}^\infty)$. The result follows now from theorem 5.10. □

Note that $\mathbb{R}\mathrm{P}^n = S^n/A$.

Corollary 5.14. *Denote by (A, S^n) the antipodal involution on the n-sphere. A basis for the MO_*-module $MO_*(\mathbb{Z}_2)$ is given by the elements $[A, S^n]$, $n \geq 0$.*

5.2 The normal bundle to the fixed point set

We are particularly interested in the case $G = \mathbb{Z}_2$ and are concerned with the structure of $I_*(\mathbb{Z}_2)$. Our goal is to prove a theorem of Boardman which states that the fixed point set of a non-bounding involution cannot be too low dimensional, see theorem 5.65.

A taste of things to come is given by

Theorem 5.15. *Let M be a closed manifold M. If M admits a fixed point free involution, then M bords.*

Proof. Denote the involution by T. We consider the cylinder $M \times I$ and identify (x, t) with $(T(x), 1 - t)$. Since T has no fixed points the resulting space is a manifold of dimension $n + 1$ whose boundary is M (compare 1.16). □

Let (G, M) be a smooth action on a closed n-manifold M and denote by $F = \{x \in M : gx = x \quad \forall g \in G\}$ the fixed point set. Consider $M^g := \{x \in M : gx = x\}$ which is the preimage of the diagonal under the smooth map $M \to M \times M$, $x \mapsto (x, gx)$. The set of stationary points F is $\bigcap_{g\in G} M^g$, thus closed as the intersection of closed subsets. It follows from the local linearization theorem below that each component of F is a smooth closed submanifold of M, compare [1, chapter 20]. Note that there are only finitely many components. Denote by F^k the union of k-dimensional components of F, thus $F = \bigsqcup_{k=0}^{n} F^k$. We denote the normal bundle of F^k by $\left(\eta^{n-k} : T^\perp F^k \to F^k\right)$ and allow the possibility of 0-plane bundles if $n = k$. $\big(\eta \to F\big) = \big(\bigsqcup_{k=0}^{n} \eta^{n-k} \to F^k\big)$ is the normal bundle to the set of all stationary points F. By $\dim(F)$ we denote the maximum of the dimensions of the non-empty components of the fixed point set.

Remark 5.16 (Equivariant tubular neighbourhood)**.** G acts on the tangent bundle via the differential maps. Consider a G-invariant submanifold N (such as the stationary sets F^k) and let $D(\eta) \to N$ denote the normal disk bundle to N. Then G acts as a group of bundle maps on $D(\eta) \to N$ covering the action of G on N. There is then an equivariant tubular neighbourhood U of N that is a G-diffeomorphism of $(G, D(\eta))$ onto (G, U) which is the identity along N.

Theorem 5.17 (Local linearization theorem)**.** *Let G be a compact Lie-group acting on a manifold M. Given a fixed point x in M there are neighbourhoods U of x and V of 0 in T_xM and a G-equivariant diffeomorphism $U \cong V$.*

Proof. See [14, theorem 6.17]. □

5.2.1 The bundle involution and projective bundles

Definition 5.18 (Bundle involution)**.** *Let ξ be a smooth k-plane vector bundle over the closed n-manifold M and $S(\xi)$ the associated sphere bundle. The antipodal maps on the fibres induce a fibre-preserving fixed point free involution $(T, S(\xi))$ on the closed $(n+k-1)$-dimensional manifold $E(S(\xi))$. $(T, S(\xi))$ is called the* ***bundle involution****.*

Passing to real projective spaces in the fibres, we obtain a diagram

$$\begin{tikzcd} S(\xi) \arrow[rr] \arrow[dr] & & S(\xi)/T \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

We denote the induced bundle $S(\xi)/T$ with $\mathbb{R}\mathrm{P}^{k-1}$ as fibres by $P(\xi)$ and call it the **projective bundle**. We sometimes write $P(\xi)$ to mean the total space of this bundle.

Example 5.19. Let $r \geq 0$. Consider the twisted real line bundle ξ over $\mathbb{R}\mathrm{P}^r$ associated with the principal $\mathbb{Z}_2$-bundle $S^r \to S^r/A$. For $s \geq 0$ we add a trivial bundle of rank s, to get the $(s+1)$-bundle $\xi \oplus \theta^s$ over $\mathbb{R}\mathrm{P}^r$. Passing to the projective bundle we obtain $\mathbb{R}\mathrm{P}(r,s) := E(P(\xi \oplus \theta^s))$ which is a closed $(r+s)$-dimensional manifold. Another possibility to arrive at this manifold is as follows. The total space of ξ is $S^r \times_{\mathbb{Z}_2} \mathbb{R}$ which is the quotient of $S^r \times \mathbb{R}$ under the identification $(z,x) \equiv (-z,-x)$, compare the appendix. Therefore we may regard $\mathbb{R}\mathrm{P}(r,s)$ as the quotient of $S^r \times \mathbb{R}\mathrm{P}^s$ with respect to the fixed point free involution

$$\big(x_1, \ldots, x_{r+1}, [y_1, \ldots, y_{s+1}]\big) \mapsto \big(-x_1, \ldots, -x_{r+1}, [-y_1, \ldots, y_{s+1}]\big).$$

This construction will be of great importance in the course of proving Boardman's five-halves theorem, compare lemma 5.56.

We need to understand the tangent bundle τ of $E(P(\xi))$ and its Stiefel-Whitney classes. For this purpose we need the **Leray-Hirsch theorem**, compare [6, 17.1.1].

Theorem 5.20 (Leray-Hirsch)**.** *Let $p : E \to M$ be a vector bundle which is trivial over a finite covering and let F be such that for each $x \in M$ there is a homeomorphism*

$$j_x : F \to p^{-1}(x) \subset E.$$

If $a_1, \ldots, a_r \in H^*(E;\Lambda)$ are homogeneous elements such that $j_x^*(a_1), \ldots, j_x^*(a_r)$ is a Λ-base of $H^*(F;\Lambda)$ for each $x \in M$, then $H^*(E;\Lambda)$ is a free $H^*(M;\Lambda)$-module with base $a_1, \ldots, a_r$. Here Λ denotes a principal ring and the module structure is induced by $p^* : H^*(M;\Lambda) \to H^*(E;\Lambda)$.

There is a canonical line bundle $\lambda_\xi = \{(L, x) : x \in L\} \subset E(P(\xi)) \times E$ over $E(P(\xi))$. Choose a classifying map $f : E(P(\xi)) \to \mathbb{RP}^\infty$, i.e. $f^*(\gamma_{1,\infty}) \cong \lambda_\xi$. Let w_1 be as in lemma 4.5 and $c := f^*(w_1)$. Thus, c is the first Stiefel-Whitney class of the bundle λ_ξ, compare the appendix. Applying the Leray-Hirsch theorem with $\Lambda = \mathbb{Z}_2$ it follows (for a more detailed proof see [6, chapter 17 2.5]).

Corollary 5.21. *Let $\xi = (p, E, M)$ be a vector bundle of rank k over the closed manifold M. The classes $1, c, \ldots, c^{k-1}$ form a base of the $H^*(M; \mathbb{Z}_2)$-module $H^*\big(E(P(\xi)); \mathbb{Z}_2\big)$. Furthermore, the induced homomorphism*

$$q^* : H^*(M; \mathbb{Z}_2) \to H^*\big(E(P(\xi)); \mathbb{Z}_2\big)$$

is injective.

As mentioned in chapter one, there is a short exact sequence

$$0 \to VP(\xi) \longrightarrow TE(P(\xi)) \xrightarrow{Tq} q^*TM \longrightarrow 0.$$

Thus, $\tau = VP(\xi) \oplus q^*(\tau_M)$ splits into a direct Whitney sum of the tangent bundle along the fibre and the normal bundle to the fibre. By the Whitney product theorem it suffices to compute the Stiefel-Whitney classes of $VP(\xi)$ and $q^*(\tau_M)$. Let w_j denote the Stiefel-Whitney classes of τ_M and v_i denote those of ξ. Then, by naturality

$$w(q^*(\tau_M)) = 1 + q^*(w_1) + \ldots + q^*(w_n).$$

Theorem 5.22 (Borel-Hirzebruch)**.** *The total Stiefel-Whitney class of $VP(\xi)$ is given by*

$$w(VP(\xi)) = \sum_{j=0}^{k} (1 + c)^j q^*(v_{k-j}).$$

Furthermore, since $VP(\xi)$ has rank $k - 1$

$$\sum_{i=0}^{k} c^i q^*(v_{k-i}) = 0 \tag{5.1}$$

Before we prove this theorem we introduce a useful technique which reduces questions about vector bundles to the easier case of line bundles, compare [6, chapter 17.5]. This is often referred to as *splitting principle.*

Definition 5.23. *Let $\xi = (p, E, B)$ be a vector bundle. A* ***splitting map*** *of ξ is a map $f : X \to B$ such that*

1. *the induced homomorphism in cohomology $f^* : H^*(B; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)$ is injective,*
2. *the pullback bundle $f^*(\xi)$ breaks up into the sum of line bundles.*

It is clear that a splitting map exists for line bundles (just take the identity on the base). For the general case consider the projective bundle $P(\xi)$ associated with the k-bundle $\xi = (p, E, B)$. We pullback ξ be the induced projection and obtain the diagram

$$\begin{tikzcd} q^*E \arrow[r] \arrow[d] & E \arrow[d] \\ E(P(\xi)) \arrow[r, "q"] & B \end{tikzcd}$$

From 5.21 we know that $q^* : H^*(B;\mathbb{Z}_2) \to H^*\big(E(P(\xi));\mathbb{Z}_2\big)$ is injective. Furthermore, there is the line bundle λ_ξ and hence a decomposition $q^*(\xi) = \lambda_\xi \oplus \sigma_\xi$. By induction we may assume that there is a splitting map $g : X \to E(P(\xi))$ for σ_ξ. Consider the composition $f : X \xrightarrow{g} E(P(\xi)) \xrightarrow{q} B$. Since g^* and q^* induce monomorphisms in cohomology, so does $f^* = g^* \circ q^* : H^*(B;\mathbb{Z}_2) \to H^*(X;\mathbb{Z}_2)$. Moreover,

$$f^*(\xi) = g^*(q^*(\xi)) = g^*(\lambda_\xi \oplus \sigma_\xi) = g^*(\lambda_\xi) \oplus g^*(\sigma_\xi) = g^*(\lambda_\xi) \oplus (L_1 \oplus \cdots \oplus L_{k-1}).$$

We have shown

Theorem 5.24 (Splitting principle)**.** *Suppose ξ is a vector bundle over B. There exists a space X and a map $f : X \to B$ such that f is a splitting map for ξ.*

We are now able to compute the Stiefel-Whitney classes of $VP(\xi)$.

Proof of 5.22. We use a similar argument as in the proof of theorem 4.6. The task is to find a nice description of the tangent bundle along the fibre. There is the line bundle λ_ξ over $P(\xi)$. The total space of λ_ξ consists of pairs (L, w) such that $L \subseteq E_x$ and $w \in L$. Let L be a 1-dimensional subspace of E_x intersecting the sphere at $\pm v$. Moreover, consider the bundle $\eta \to S(\xi)$ with total space

$$\{(v, v') \in S(\xi) \times E(\xi) : v \perp v'\} \subset S(\xi) \times E(\xi).$$

This bundle may be identified with the tangent bundle along the fibre in $S(\xi)$. The differential of the bundle involution T induces an involution T' on η which takes the form $(v, v') \mapsto (-v, -v')$. The bundle $E(\eta)/T' \to P(\xi)$ is then isomorphic to $VP(\xi)$. As in the proof of 4.6 we conclude that $[v, v'] \mapsto (\phi : v \mapsto v')$ induces a vector bundle isomorphism $VP(\xi) \cong \mathrm{Hom}(\lambda_\xi, \lambda_\xi^\perp)$. Thus, we obtain an isomorphism

$$VP(\xi) \oplus \theta^1 = \lambda'_\xi \otimes q^*(\xi) = \lambda_\xi \otimes q^*(\xi).$$

To complete the argument we need to compute Stiefel-Whitney numbers of tensor products. For this purpose let $\xi \to M$ be a k-bundle and $\eta \to M$ a line bundle over M. Then

$$w(\xi \otimes \eta) = \sum_{j=0}^{k} (1 + w_1(\eta))^j w_{k-j}(\xi).$$

First, assume that ξ splits as a sum of line bundles $\xi = \xi_1 \oplus \cdots \oplus \xi_k$. We already know $w(\xi_j \otimes \eta) = 1 + w_1(\xi_j) + w_1(\eta)$, see the appendix. Thus,

$$\begin{aligned}
w(\xi \otimes \eta) &= w\Big(\sum_{j=0}^{k} \xi_j \otimes \eta\Big) \\
&= \prod_{j=0}^{k} \big(1 + w_1(\eta) + w_1(\xi_j)\big) \\
&= \sum_{j=0}^{k} \Big((1 + w_1(\eta))^j \sum_{i_1 < \ldots < i_{k-j}} w_1(\xi_{i_1}) \cdots w_1(\xi_{i_{k-j}})\Big) \\
&= \sum_{j=0}^{k} (1 + w_1(\eta))^j w_{k-j}(\xi)
\end{aligned}$$

The general case follows from the splitting principle. We now find,

$$w(VP(\xi)) = w(VP(\xi) \oplus \theta^1) = w(q^*(\xi) \otimes \lambda_\xi) = \sum_{j=0}^{k} (1 + c)^j q^*(v_{k-j}).$$

□

Corollary 5.25. *The Stiefel-Whitney classes W_m of the tangent bundle to $P(\xi)$ are given by*

$$W_m = \sum_{i+j+l=m} \binom{k-i}{l} c^l q^*(v_i w_j)$$

Proof. The total Whitney classes of $VP(\xi)$ and $q^*(\tau_M)$ are $v = \sum_{i=0}^{k} (1+c)^{k-i} q^*(v_i)$ and $w = \sum_{j=0}^{n} q^*(w_j)$. By the Whitney product theorem we compute

$$\begin{aligned}
W &= vw \\
&= \left(\sum_{i=0}^{k} (1 + c)^{k-i} q^*(v_i)\right) \left(\sum_{j=0}^{n} q^*(w_j)\right) \\
&= \sum_{i=0}^{k} \sum_{j=0}^{n} (1 + c)^{k-i} q^*(v_i) q^*(w_j) \\
&= \sum_{i=0}^{k} \sum_{j=0}^{n} \sum_{l=0}^{k-i} \binom{k-i}{l} c^l q^*(v_i) q^*(w_j)
\end{aligned}$$

□

5.2.2 Wall's theorem

Definition 5.26 (Normal sphere bundle)**.** *Let (T, M^n) be a closed involution. We apply definition 5.18 to the normal bundles of the components of the fixed point sets and obtain bundle involutions $\big(T, S(\eta^{n-k})\big)$. We refer to $S(\eta) = \bigsqcup_{0 \le k < n} S(\eta^{n-k})$ as the* ***normal sphere bundle*** *of the fixed point set. Note that the total space has dimension $n - 1$.*

We want to use information about the normal bundle to the fixed point set of an involution to arrive at conclusions about the cobordism class of M. Any involution may then be used to get information about the cobordism behaviour of M.

Lemma 5.27. *For the bundle involution on the normal sphere bundle the identity*

$$[T, S(\eta)] = \sum_{k=0}^{n-1} [T, S(\eta^{n-k})] = 0 \in MO_{n-1}(\mathbb{Z}_2)$$

is valid.

Proof. We may assume $F^n = \emptyset$. Choose a T-invariant closed equivariant tubular neighbourhood N around F diffeomorphic to the disk bundle $D(\eta)$. Then $P := M - \mathrm{int}(N)$ is a compact T-invariant n-dimensional submanifold of M and the involution T on M induces a $\mathbb{Z}_2$-action on the fibres leaving only the 0-vector fixed. The only such linear involution on a vector space is the antipodal map. By construction, $T|_P$ has no fixed points. Thus,

$$[T, S(\eta)] = [T, \partial N] = [T, \partial P] = 0.$$

□

Theorem 5.28. *Let (T, M^n) be a smooth involution on the closed manifold M^n and $\eta \to F$ the normal bundle to the fixed point set. Then $P(\eta \oplus \theta^1)$ is a closed n-manifold and $[M^n] = [P(\eta \oplus \theta^1)]$ in MO_n.*

Proof. Consider the involutions T_1, T_2 on $M^n \times I$,

$$T_1(x, t) := (x, 1 - t), \quad T_2(x, t) := (T(x), 1 - t).$$

The set of stationary points F_1 of T_1 is $M^n \times \{\frac{1}{2}\}$. Therefore the normal bundle to F_1 is just θ^1. By definition of T_2, the fixed point set F_2 is $F \times \{\frac{1}{2}\} \cong F$. Hence, the normal bundle to F_2 can be identified with $\eta \oplus \theta^1$.
We glue $(T_1, M^n \times I)$ and $(T_2, M^n \times I)$ along their boundaries by the diffeomorphism

$$\begin{aligned} \phi : (T_1, M^n \times \partial I) &\to (T_2, M^n \times \partial I), \\ (x, 0) \mapsto (x, 0), &\quad (x, 1) \mapsto (T(x), 1). \end{aligned}$$

Since

$$\phi(T_1(x, 0)) = \phi(x, 1) = (T(x), 1) = T_2(x, 0) = T_2(\phi(x, 0))$$

and

$$\phi(T_1(x, 1)) = \phi(x, 0) = (x, 0) = (T(T(x)), 0) = T_2(T(x), 1) = T_2(\phi(x, 1))$$

ϕ defines an equivariant diffeomorphism along the boundaries. Put $N^{n+1} := (M^n \times I) \cup_\phi (M^n \times I)$. There is an involution T_3 on N^{n+1} induced by T_1, T_2. We apply lemma 5.27 to (N^{n+1}, T_3) to find $[A, M^n \times S^0] + [T, S(\eta \oplus \theta^1)] = 0$ in $MO_n(\mathbb{Z}_2)$. Thus,

$$\begin{aligned} [A, S^0][M^n] &= [A, S^0 \times M^n] \\ &= -[T, S(\eta \oplus \theta^1)] \\ &= [T, S(\eta \oplus \theta^1)] \end{aligned}$$

Passing to projective bundles we obtain

$$[M^n] = [P(\eta \oplus \theta^1)]$$

as claimed. $\square$

We illustrate the use of 5.28 by proving Wall's theorem

Theorem 5.29 (Wall)**.** *For any closed manifold M, the product $M \times M$ is unoriented cobordant to an orientable manifold*

Proof. As mentioned at the end of chapter 2 Thom showed that MO_* is a polynomial algebra over $\mathbb{Z}_2$ and for even dimensional generators $[\mathbb{R}\mathrm{P}^{2k}]$ can be used. Given a manifold M of dimension n, there are generators $[x_k] \in MO_k$ and $n_i \in \mathbb{Z}_2$ such that

$$[M] = \sum_{i_1+\ldots+i_k=n} n_{i_1\ldots i_k}[x_{i_1}][x_{i_2}]\cdots[x_{i_k}].$$

As coefficients are in $\mathbb{Z}_2$ we find

$$[M \times M] = [M]^2 = \sum_{i_1+\ldots+i_k=n} n^2_{i_1\ldots i_k}[x_{i_1}]^2[x_{i_2}]^2\cdots[x_{i_k}]^2.$$

Since products and unions of orientable manifolds are orientable, it is enough to check the theorem on a set of generators. First, we take care of the odd-dimensional case.

Claim 1. The theorem is true if n is odd.
We define an involution T on $M \times M$ by swapping the coordinates $(x, y) \mapsto (x, y)$ and apply the theorem. The fixed point set of T is the diagonal Δ. The diffeomorphism $\psi : M \to M \times M$, $x \mapsto (x, x)$ is covered by the vector bundle homomorphism $\Psi : TM \to T(M \times M)|_\Delta$, $X \mapsto (0, X)$

$$\begin{tikzcd} TM \arrow[r, "\Psi"] \arrow[d] & T(M \times M)|_\Delta \arrow[d] \\ M \arrow[r, "\psi"'] & \Delta \end{tikzcd}$$

which induces a bijective homomorphism $TM \to T^\perp\Delta$ over ψ. Hence, the normal bundle to the diagonal is equivalent to the tangent bundle of M. By theorem 5.28 we find $[M]^2 = [M \times M] = [P(\tau_M \oplus \theta^1)]$. We compute the first Stiefel-Whitney number W_1 of $P(\tau_M \oplus \theta^1)$ with the help of corollary 5.25. Note $v_1 = w_1$. Therefore

$$W_1 = q^*(w_1) + q^*(w_1) + \binom{n+1}{1}c = \underbrace{2q^*(w_1)}_{0} + \underbrace{(n+1)c}_{0} = 0.$$

since n is odd. The claim now follows from theorem A.17.

Claim 2. $[\mathbb{C}\mathrm{P}^n] = [\mathbb{R}\mathrm{P}^n]^2$.
The conjugation $T : [z_1, \ldots, z_{n+1}] \mapsto [\overline{z}_1, \ldots, \overline{z}_{n+1}]$ gives us an involution on the

complex projective space which we understand very well. Its fixed point set is $\mathbb{R}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^n$. At a point in $\mathbb{R}\mathrm{P}^n$ the tangent space to $\mathbb{R}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^n$ consists of all vectors that are fixed by the tangential map dT and the set $\{v : dT(v) = -v\}$ makes up the normal space. Hence $T^\perp \mathbb{R}\mathrm{P}^n$ consists of purely imaginary vectors and multiplication by i provides an equivalence of the normal bundle to $\mathbb{R}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^n$ with the tangent bundle $\tau_{\mathbb{R}\mathrm{P}^n}$. From theorem 5.28 we get

$$[\mathbb{C}\mathrm{P}^n] = [P(\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1)]. \tag{5.2}$$

On the other hand consider the involution $(x, y) \mapsto (y, x)$ on the product $\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n$ whose fixed point set is the diagonal. As above, we see that the normal bundle is equivalent to the tangent bundle of real projective space $\mathbb{R}\mathrm{P}^n$. Thus again by theorem 5.28,

$$[P(\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1)] = [\mathbb{R}\mathrm{P}^n \times \mathbb{R}\mathrm{P}^n]$$

Combination with (5.2) completes the proof. □

5.3 Cobordism classes of vector bundles

We consider $MO_*(BO(k))$ where we understand $BO(k)$ as the Grassmann manifold $G_{k,s}$ with $s > n + k$. An element of $MO_n(BO(k))$ is given by a closed n-manifold M^n and a map $f : M^n \to BO(k)$. Homotopic maps induce the same cobordism class (the homotopy $M^n \times I \to BO(k)$ acts as a cobordism). Therefore we can understand a cobordism class as given by a closed manifold M^n together with a homotopy class of maps into the Grassmann manifold $BO(k)$. However, these homotopy classes are in bijective correspondence to k-plane vector bundles over M^n by theorem 2.20. Hence, a cobordism class in $MO_n(BO(k))$ is represented by a k-plane bundle $\xi = f^*(\gamma_{k,s})$ over a closed manifold M^n. We will write $[\xi]$ or $[\xi \to M^n]$ if we want to emphasize the base space. The characteristic numbers of f are sometimes referred to as Stiefel-Whitney numbers of ξ. If $[\xi]$ is null bordant, then the corresponding (M^n, f) bords. Hence, there is a compact manifold B^{n+1} and a map $F : B^{n+1} \to BO(k)$ such that $\partial B^{n+1} = M^n$ and $F|_{\partial B^{n+1}} = f$. Therefore the bundle $F^*(\gamma_{k,s})$ restricts to $f^*(\gamma_{k,s})$ over M^n. Hence, a bundle ξ over M^n bords if and only if there is a k-plane bundle ξ' over a compact manifold B^{n+1} with $\partial B^{n+1} = M^n$ which restricts to ξ over M^n. As a consequence of theorem 5.11 we find: If two k-plane bundles ξ_1 and ξ_2 over M^n have identical Stiefel-Whitney classes, they are cobordant as bundles. We put $MO_n(BO(0)) = MO_n$.

Definition 5.30. *Let ξ be a smooth vector bundle of rank k. We define a MO_*-homomorphism $\partial : MO_*(BO(k)) \to MO_*(\mathbb{Z}_2)$ of degree $k-1$ via*

$$\partial : MO_n(BO(k)) \to MO_{n+k-1}(\mathbb{Z}_2),$$

$$[\xi] \mapsto [T, S(\xi)].$$

We put $\partial(MO_n) = 0$.

Remark 5.31. From the construction of the bundle involution it is clear that ∂ is well-defined. What happens if $k = 1$? $\mathbb{R}\mathrm{P}^\infty = BO(1)$ is the universal space for $O(1) \cong \mathbb{Z}_2$. From theorem 5.10 we obtain an isomorphism $MO_n(\mathbb{Z}_2) \cong$

$MO_n(BO(1))$. Hence, we can consider ∂ as a map $\partial : MO_n(\mathbb{Z}_2) \to MO_n(\mathbb{Z}_2)$. Let $[T, M]$ be an element in $MO_n(\mathbb{Z}_2)$. Then $\phi([T, M]) = [M/T, f]$ by definition of ϕ. Here $f : M/T \to \mathbb{R}\mathrm{P}^\infty$ is a classifying map for $M \to M/T$. Pullback the canonical line bundle over the real projective space by f to obtain the corresponding element ξ in $MO_n(BO(1))$. Since ξ is a line bundle we have $\partial[\xi] = [M/T][A, S^0] = [T, M]$. Hence ∂ is the identity modulo the identification $MO_n(\mathbb{Z}_2) = MO_n(BO(1))$.

Theorem 5.32. *Let (T, M^n) be an involution on a closed manifold. If for each m, $0 \leq m < n$, all Stiefel-Whitney classes of $\eta^{n-m} \to F^m$ vanish, then $[F^m] = 0$ for $0 \leq m < n$ and $[M^n] = [F^n]$.*

Proof. This is another application of lemma 5.27. By assumption we have that η^{n-m} is cobordant to the trivial bundle θ^{n-m} if $0 \leq m < n$. From 5.27 we conclude

$$\begin{aligned}
0 &= [T, S(\eta)] \\
&= \sum_{m=0}^{n-1} [T, S(\eta^{n-k})] \\
&= \sum_{m=0}^{n-1} \partial[\eta^{n-m}] \\
&= \sum_{m=0}^{n-1} \partial[\theta^{n-m}] \\
&= \sum_{m=0}^{n-1} [T, S(\theta^{n-m})] \\
&= \sum_{m=0}^{n-1} [A, S^{n-m-1}][F^m].
\end{aligned}$$

We already know that the antipodal maps form a basis of $MO_*(\mathbb{Z}_2)$ from theorem 5.14. Thus, $[F^m] = 0$ for $0 \leq m < n$. Hence, by applying theorem 5.28, we find

$$[M^n] = [P(\eta \oplus \theta^1)] = [F^n] + \sum_{m=0}^{n-1} [\mathbb{R}\mathrm{P}^{n-m}][F^m] = [F^n]$$

which establishes the proof. $\square$

5.4 The Smith homomorphism

We introduce some important maps before we dedicate our attention to a thorough analysis of $I_*(\mathbb{Z}_2)$.

Definition 5.33. *Given a fixed point free involution $(T, M^n) \in MO_n(\mathbb{Z}_2)$, there is, for large enough N, an equivariant map $g : (T, M^n) \to (A, S^N)$ which is transverse to the submanifold $S^{N-1} \subset S^N$. Hence, $g^{-1}(S^{N-1})$ is a closed T-invariant $(n-1)$-dimensional submanifold of M^n. We define the Smith homomorphism Δ by*

$$\begin{aligned}
\Delta : MO_n(\mathbb{Z}_2) &\to MO_{n-1}(\mathbb{Z}_2), \\
[T, M^n] &\mapsto [T, g^{-1}(S^{N-1})].
\end{aligned}$$

It remains to take care of the details mentioned in this definition.

- *Existence of the map g.* The fixed point free involution defines a principal $\mathbb{Z}_2$-action on M^n. But we know that $B\mathbb{Z}_2 = \mathbb{R}\mathrm{P}^\infty$. Hence, there is a classifying map $f : M^n/T \to \mathbb{R}\mathrm{P}^N$ covered by an equivariant map $g : (T, M^n) \to (A, S^N)$. By the transversality theorem we may assume that g is transverse to S^{N-1}.

- $g^{-1}(S^{N-1})$ *is T-invariant.* Suppose $g(x) \in S^{N-1}$. Since g can be chosen to be equivariant we have $g(T(x)) = A(g(x)) = -g(x) \in S^{N-1}$.

- Δ *is well-defined.* Suppose $(T, M^n) = (T, \partial B^{n+1})$ is null bordant in $MO_n(\mathbb{Z}_2)$. We must show $[T, g^{-1}(S^{N-1})] = 0 \in MO_{n-1}(\mathbb{Z}_2)$. Choose a classifying equivariant map $g_1 : B^{n+1} \to S^N$ such that $g_1|_M = g$. By theorem 1.9 we may assume g_1 to be transverse to S^{N-1}. Therefore $g_1^{-1}(S^{N-1})$ is a n-dimensional submanifold of B^{n+1}. Morever,

  $$\partial(g_1^{-1}(S^{N-1})) = g_1^{-1}(S^{N-1}) \cap \partial B^{n+1} = g_1^{-1}(S^{N-1}) \cap M^n = g^{-1}(S^{N-1})$$

  and hence $[T, g^{-1}(S^{N-1})] = 0$ in $MO_{n-1}(\mathbb{Z}_2)$.

We want to make use of a homomorphism

$$I_* : MO_n(BO(k)) \to MO_n(BO(k+1)).$$

I_* assigns to a k-plane bundle ξ over a closed manifold M, the $(k+1)$-plane bundle $\xi \oplus \theta^1$ over M, i.e. $I_*([\xi]) = [\xi \oplus \theta^1]$. If ξ_1 and ξ_2 are cobordant bundles, then so are $\xi_1 \oplus \theta^1$ and $\xi_2 \oplus \theta^1$. Hence, I_* is well-defined.

Lemma 5.34. *$I_* : MO_n(BO(k)) \to MO_n(BO(k+1))$ is a monomorphism for all $n \geq 0$. A cobordism class $[\xi]$ of a $(k+1)$-bundle is in the image of I_* if and only if each Stiefel-Whitney number involving the class w_{k+1} vanishes. If $n \leq k$ then I_* is an isomorphism.*

Proof. The homomorphism I_* is induced by the natural inclusion $I : BO(k) \to BO(k+1)$ which assigns to every k-plane V the subspace $V \times \mathbb{R}$. Suppose $f : M^n \to BO(k)$ is a classifying map for ξ. Since $I^*(\gamma_{k+1,s}) = \gamma_{k,s} \oplus \theta^1$ we see that $I \circ f$ is a classifying map for $\xi \oplus \theta^1$. We conclude from theorem 5.11 that $[\xi \oplus \theta^1] = 0$ implies $[\xi]$=0 and thus, I_* is injective. For the next claim note $w_{k+1}(\xi \oplus \theta^1) = w_{k+1}(\xi) = 0$ if ξ is a k-plane bundle. We refer to [1, 24.2] for the other implication. To prove the last claim, recall $MO_*(BO(k)) \cong H_*(BO(k);\mathbb{Z}_2) \otimes MO_*$. The result follows now from the fact that $I_* : H_n(BO(k);\mathbb{Z}_2) \stackrel{\cong}{\to} H_n(BO(k+1), \mathbb{Z}_2)$ constitutes an isomorphism if $n \leq k$. □

What is the relation between the maps ∂, Δ and I_*? In order to answer this question, we need more information about the Smith homomorphism.

Lemma 5.35. *Suppose we have a fixed point free involution (T, M) on an n-dimensional manifold and $W \subset M$ a compact submanifold with codimension 0 such that*

- $W \cup T(W) = M$ *and*
- $W \cap T(W) = \partial W$.

Then $\Delta[T, M] = [T, \partial W]$.

Proof. The second assumption ensures that ∂W is T-invariant and therefore $[T, \partial W]$ defines an element in $MO_n(\mathbb{Z}_2)$. Choose a smooth equivariant map $f : \partial W \to S^{N-1}$. The normal bundle to ∂W is $\partial W \times \mathbb{R}$. Let U be an open T-invariant tubular neighbourhood of ∂W. We may assume

$$U \cong \partial W \times (-1, 1), \quad \partial W \times (-1, 0] \subset T(W) \quad \text{and} \quad \partial W \times [0, 1) \subset W.$$

The involution induces the map $(x, t) \mapsto (T(x), -t)$ on the normal bundle. Denote by N, S the north and south pole of the sphere S^N. $S^N - \{N, S\}$ is equivariantly diffeomorphic to $S^{N-1} \times (-1, 1)$ under the action $(x, t) \mapsto (-x, -t)$ by the diffeomorphism

$$S^{N-1} \times (-1, 1) \to S^N - \{N, S\}, \quad (x, t) \mapsto (\sqrt{1 - t^2}x, t).$$

We can now define a map $g : M \to S^N = \big(S^{N-1} \times (-1, 1)\big) \cup \{N, S\}$ by

$$g(x) = \begin{cases} (f(y), t) & x = (y, t) \in U \cong \partial W \times (-1, 1) \\ N & x \in W - U \\ S & s \in T(W) - U \end{cases}$$

By construction g is equivariant, transverse to S^{N-1} and $g^{-1}(S^{N-1}) = \partial W$. Hence, we conclude

$$\Delta[T, M] = [T, \partial W]$$

□

Corollary 5.36. $\Delta[A, S^n] = [A, S^{n-1}]$.

Proof. This follows from the definition of Δ. Alternatively, take W the upper or lower hemisphere in the preceding lemma. □

Theorem 5.37. *The following diagram is commutative*

$$\begin{tikzcd}
MO_n(BO(k+1)) \arrow[r, "\partial"] & MO_{n+k}(\mathbb{Z}_2) \arrow[d, "\Delta"] \\
MO_n(BO(k)) \arrow[u, "I_*"] \arrow[r, "\partial"] & MO_{n+k-1}(\mathbb{Z}_2)
\end{tikzcd}$$

Proof. Let ξ be a k-plane bundle over a closed manifold M^n. We have to show

$$\Delta[T, S(\xi \oplus \theta^1)]\Big(= (\Delta \circ \partial)[\xi \oplus \theta^1] = (\Delta \circ \partial \circ I_*)[\xi]\Big) = \partial[\xi] = [T, S(\xi)].$$

But this follows from lemma 5.35. □

Proposition 5.38. *The Smith homomorphism $\Delta : MO_{n+1}(\mathbb{Z}_2) \to MO_n(\mathbb{Z}_2)$ is surjective.*

Proof. We define a right inverse to Δ. Along with a fixed point free involution (T, M^n) comes a real line bundle ξ over the quotient M^n/T, compare theorem 5.10 and note $MO_n(\mathbb{Z}_2) \cong MO_n(O(1)) \cong MO_n(BO(1))$. $\xi \oplus \theta^1$ is then a 2-bundle over the n-manifold M^n/T and we may pass to the sphere bundle $S(\xi \oplus \theta^1)$. Together with the bundle involution this represents an element in $MO_{n+1}(\mathbb{Z}_2)$. Summing up, we define γ to be the composition

$$MO_n(\mathbb{Z}_2) \overset{\phi}{\cong} MO_n(BO(1)) \xrightarrow{I_*} MO_n(BO(2)) \xrightarrow{\partial} MO_{n+1}(\mathbb{Z}_2).$$

It follows from theorem 5.37 that

$$\Delta \circ \gamma = \Delta \circ \partial \circ I_* \circ \phi = \partial \circ \phi = \mathrm{id},$$

compare remark 5.31. □

Remark 5.39. A more geometric interpretion of the map γ is as follows. Given $[T, M^n] \in MO_n(\mathbb{Z}_2)$, consider the involutions $\mathrm{id} \times A$ and $T \times c$ on $M^n \times S^1$, where $A : S^1 \to S^1$ is the antipodal map on the sphere and c is the conjugation map. These involutions obviously commute and hence $\mathrm{id} \times A$ induces an involution $\big(t, (M^n \times S^1)/(T \times c)\big)$. Note that the quotient $V^{n+1} := (M^n \times S^1)/(T \times c)$ is a closed $(n+1)$-manifold by proposition 1.16 and t is again fixed point free. Hence, we arrive at the map

$$MO_n(\mathbb{Z}_2) \to MO_{n+1}(\mathbb{Z}_2),$$
$$[T, M^n] \mapsto [t, V^{n+1}]$$

which is seen to coincide with γ.

5.5 The $\mathbb{Z}_2$-cobordism algebra $I_*(\mathbb{Z}_2)$

We define $\mathcal{M}_n = \sum\limits_{m=0}^{n} MO_m(BO(n-m))$. $\mathcal{M}_* = \sum_0^\infty \mathcal{M}_n$ can be given the structure of a graded commutative algebra over MO_* with identity as follows. Let $[\xi_1 \to M_1^r] \in \mathcal{M}_n$ be an $(n-r)$-bundle over M_1^r and $[\xi_2 \to M_2^s] \in \mathcal{M}_m$ be an $(m-s)$-bundle over M_2^s. Define

$$[\xi_1 \to M_1^r] \cdot [\xi_2 \to M_2^s] := [\xi_1 \times \xi_2 \to M_1^r \times M_2^s] \in \mathcal{M}_{m+n}$$

to get an $(m+n-(r+s))$-bundle over $M_1^r \times M_2^s$. The identity element is given by the 0-bundle over a point. Recall that since $MO_* = MO_*(BO(0))$, we can view $MO_* \subset \mathcal{M}_*$ as a subring. As is shown in [1, 25.1] $\mathcal{M}_*$ is a polynomial algebra with a generator in each $\mathcal{M}_n$.
A multiplicative homomorphism $j_* : I_*(\mathbb{Z}_2) \to \mathcal{M}_*$ is given by the sum of the maps

$$j_n : I_n(\mathbb{Z}_2) \to \mathcal{M}_n$$
$$[T, M^n] \mapsto \sum_{m=0}^{n} [\eta^{n-m} \to F^m].$$

Proposition 5.40. *j_* is well defined.*

Proof. Suppose (T, M^n) bords. There is then an action (T_1, B^{n+1}) such that $\partial B^{n+1} = M^n$ and $T_1|_{M^n} = T$. Denote by E^{m+1} the union of the $(m+1)$-dimensional components of the fixed point set of T_1 and by η_1^{n-m} the corresponding normal bundle. E^{m+1} is a compact manifold with boundary $\partial E^{m+1} = E^{m+1} \cap B^{n+1} = F^m$. If we restrict η_1^{n-m} to the boundary, we obtain the normal bundle η^{n-m} of F^m. Hence, $j_*[T, M^n] = \sum_{m=0}^{n} [\eta^{n-m} \to F^m] = 0$, if $[T, M^n] = 0$. This shows that j_* is well defined. □

Theorem 5.41. *The sequence*

$$0 \longrightarrow I_n(\mathbb{Z}_2) \xrightarrow{j_n} \mathcal{M}_n \xrightarrow{\partial} MO_{n-1}(\mathbb{Z}_2) \longrightarrow 0$$

is exact and splits.

Proof. Here $\mathcal{M}_n \xrightarrow{\partial} MO_{n-1}(\mathbb{Z}_2)$ is the sum of the homomorphisms $MO_m(BO(n-m)) \to MO_{n-1}(\mathbb{Z}_2)$ from definition 5.30.

Exactness at $\mathcal{M}_n$.
$\partial(j_n[T, M^n]) = \sum_{m=0}^{n} \partial[\eta^{n-m}] = \sum_{m=0}^{n-1} [T, S(\eta^{n-m})] \overset{5.27}{=} 0$ shows $\mathrm{img}(j_n) \subseteq \ker(\partial)$. For the other inclusion suppose we have $\sum_{m=0}^{n} [\xi^{n-m} \to M^m]$ in $\mathcal{M}_n$ such that $0 = \sum_{m=0}^{n} \partial[\xi^{n-m}] = \sum_{m=0}^{n-1} [T, S(\xi^{n-m})]$ in $MO_{n-1}(\mathbb{Z}_2)$. There is the disk bundle $\big(T, D(\xi)\big) = \bigsqcup_{m=0}^{n} \big(T, D(\xi^{n-m})\big)$ which can be considered as a compact n-manifold with boundary $\big(T, \partial D(\xi)\big) = \big(T, S(\xi)\big)$. But since $\sum_{m=0}^{n} [\xi^{n-m} \to M^m]$ is contained in the kernel of ∂ there is a fixed point free involution (T_1, B^n) such that $(T_1, \partial B^n) = \big(T, S(\xi)\big)$. Now we glue $D(\xi)$ and B^n along their common boundary $S(\xi)$ to get a closed n-manifold N^n. T and T_1 induce an involution on N^n which we denote by T_2. Since T_1 is fixed point free and T is the antipodal map on each fibre, the fixed point set of T is just the union of all $M^m \subset D(\xi^{n-m})$ and the normal bundle to M^m is ξ^{n-m}. Thus, $j_n([T_2, N^n]) = \sum_{m=0}^{n} [\xi^{n-m} \to M^m]$.

∂ is an epimorphism.
This follows immediately from theorem 5.14.

j_n is a monomorphism.
We show that the kernel is trivial. Suppose $j_n([T, M^n]) = \sum_{m=0}^{n} [\eta^{n-m} \to M^m] = 0$. By definition we find compact manifolds E^{m+1} and corresponding normal bundles $\eta_1^{n-m} \to E^{m+1}$ such that $\eta_1^{n-m}|_{\partial E^{m+1}} = \eta^{n-m}$. Choose a closed invariant tube $(T, N) \cong \big(T, D(\eta)\big)$ around the fixed point set where $D(\eta)$ denotes the disk bundle $\bigsqcup_{m=0}^{n} \big(T, D(\eta^{n-m})\big)$. Consider $(T_1, M^n \times I)$ with the involution $T_1(x, t) = (T(x), t)$. We think of $(T, N) \subset (T, M^n \times \{0\}) \subset (T_1, M^n \times I)$. By

straightening the angles we can view $D(\eta_1^{n-m})$ as a compact $(n+1)$-manifold such that

$$\partial D(\eta_1^{n-m}) = S(\eta_1^{n-m}) \cup D(\eta^{n-m})$$

and

$$S(\eta_1^{n-m}) \cap D(\eta^{n-m}) = S(\eta^{n-m}).$$

Define a compact $(n+1)$-manifold by $(T_2, B^{n+1}) = \big(T, D(\eta_1)\big) \cup (T_1, M^n \times I)$ where we identify $(T, D(\eta)) \subset \big(T, \partial D(\eta_1)\big)$ with $(T, N) \subset (T_1, M^n \times \{0\}) \subset (T_1, M^n \times I)$. Then (T_2, B^{n+1}) constitutes a cobordism between the involution (T, M^n) we started with and a bounding fixed point free involution on $S(\eta_1) \cup (M^n - \operatorname{int}(N))$.

The sequence splits.
Again theorem 5.14 proves useful in the construction of a homomorphism $K : MO_{n-1}(\mathbb{Z}_2) \to \mathcal{M}_n$. Let (T, M^{n-1}) be a fixed point free involution on a closed manifold. Since the antipodal maps on the spheres generate $MO_{n-1}(\mathbb{Z}_2)$ there are cobordism classes $[N^k]$ such that

$$[T, M^{n-1}] = \sum_{k=0}^{n-1} [A, S^{n-1-k}][N^k]. \tag{5.3}$$

We define

$$K[T, M^{n-1}] := \sum_{k=0}^{n-1} [\theta^{n-k} \to *][N^k],$$

which is well defined since the decomposition (5.3) is unique. By definition of ∂ we conclude $\partial \circ K = \operatorname{id}$. □

The injectivity of j_* tells us that the unrestricted cobordism class of an involution is uniquely determined by the normal bundle to the fixed point set.

We now construct a homomorphism $\Gamma : I_n(\mathbb{Z}_2) \to I_{n+1}(\mathbb{Z}_2)$ similar to the right inverse of the Smith homomorphism. But now fixed points are allowed. Consider the closed $(n+1)$-manifold $M^n \times S^1$ with the involutions $T_1 = \operatorname{id} \times c$ and $T_2 = T \times A$. As usual c and A denote the conjugation and antipodal map on S^1. Since $A \circ c = c \circ A$, these involutions commute. Note that T_2 is fixed point free because the antipodal map is. Hence, we can form the quotient $N^{n+1} = (M^n \times S^1)/T_2$ which is a closed $(n+1)$-manifold. T_1 induces an involution τ on N^{n+1}. Hence, we arrive at an assignment

$$[T, M^n] \mapsto [\tau, N^{n+1}]$$

which defines a well-defined homomorphism of degree 1

$$\Gamma : I_*(\mathbb{Z}_2) \to I_*(\mathbb{Z}_2)$$

We need to determine the fixed set F_τ of the involution τ and its normal bundle.

Remark 5.42. Suppose (T_2, M) is a fixed point free involution. Any involution T_1 that commutes with T_2 induces an involution $(T, M/T_2)$. By construction

$$[x] \in F_T \Leftrightarrow T_1(x) \equiv x \Leftrightarrow T_1(x) = x \vee T_1(x) = T_2(x).$$

Hence, to determine the fixed point set of T, one first has to detect the fixed point set of T_1 as well as the set of coincidence points of T_1 and T_2. Then pass to the quotient.

Denote by F the fixed point set of T and by η the associated normal bundle to F.

Lemma 5.43. *The fixed point set of (τ, N^{n+1}) is $F \sqcup M^n$. The corresponding normal bundle is given by $(\eta \oplus \theta^1 \to F) \sqcup (\theta^1 \to M^n)$.*

Proof. By remark 5.42 the fixed point set F_τ is given by the quotient of the disjoint union $F_\tau = F_{T_1} \sqcup \{(x,z) : T_1(x,z) = T_2(x,z)\}$ under T_2 (disjoint since T_2 is fixed point free). Obviously $F_{T_1} = M^n \times \{-1\} \sqcup M^n \times \{1\}$ and therefore F_{T_1}/T_2 is just a copy of M^n. The normal bundle of $M^n \times \{1\} \subset M^n \times S^1$ is the trivial line bundle. Since the projection $M^n \times S^1 \to N^{n+1}$ is a local diffeomorphism we conclude that the normal bundle to $M^n \subset N^{n+1}$ is $\theta^1 \to M^n$.

Turn now to the set of coincidences of T_1 and T_2. $T_1(x,z) = T_2(x,z)$ if and only if $x \in F$ and z is purely imaginary. Thus, $\{(x,z) : T_1(x,z) = T_2(x,z)\} = F \times \{i\} \sqcup F \times \{-i\}$ which becomes a single copy of F in N^{n+1}. Of course, the normal bundle to $F \times \{i\} \subset M^n \times S^1$ is $\eta \oplus \theta^1 \to F$. As above, this implies that $\eta \oplus \theta^1 \to F$ is the normal bundle to $F \subset N^{n+1}$. $\square$

As a generalization of theorem 5.28 we find

Proposition 5.44. *In MO_{n+k} the formula*

$$\epsilon(\Gamma^k[T, M^n]) = [P(\eta \oplus \theta^{k+1})] + \sum_{j=0}^{k-1} [\mathbb{R}\mathrm{P}^{k-j}] \epsilon(\Gamma^j[T, M^n])$$

holds. Here $\epsilon : I_*(\mathbb{Z}_2) \to MO_*$ is the map that forgets the involution.

Proof. We have to show that the normal bundle to the fixed point set of $\Gamma^k([T, M^n])$ is

$$\big(\eta \oplus \theta^k \to F_T\big) \sqcup \left(\bigsqcup_{j=0}^{k-1} \theta^{k-j} \to \epsilon(\Gamma^j[T, M^n]) \right). \tag{5.4}$$

The proposition will then follow immediately from theorem 5.28.
We use induction on k. The case $k = 1$ is lemma 5.43. Set $[\tau, N] := \Gamma^{k-1}[T, M^n]$. We once more apply lemma 5.43 to find that the normal bundle of $\Gamma^k[T, M^n] = \Gamma[\tau, N]$ is

$$\big(\eta_\tau \oplus \theta^1 \to F_\tau\big) \sqcup \big(\theta^1 \to N\big). \tag{5.5}$$

By lemma 5.43

$$F_\tau = F_T \sqcup \left(\bigsqcup_{j=0}^{k-2} \epsilon(\Gamma^j[T, M^n]) \right)$$

and by induction hypothesis the normal bundle to F_τ is

$$\eta_\tau = \big(\eta \oplus \theta^{k-1} \to F_T\big) \sqcup \Big(\bigsqcup_{j=0}^{k-2} \theta^{k-1-j} \to \epsilon(\Gamma^j[T, M^n]) \Big).$$

Combining this with (5.5) gives (5.4). $\square$

As a corollary of theorem 5.37 and lemma 5.43 above we state

Theorem 5.45. *The formula*

$$j_*(\Gamma[T,M]) = I_*\big(j_*([T,M]+[\mathrm{id},M])\big)$$

holds.

Proof. From the above description of the fixed point set of $\Gamma[T,M]$ we have

$$j_*(\Gamma[T,M^n]) = \underbrace{\sum_{m=0}^{n}[\eta^{n-m}\oplus\theta^1 \to F^m]}_{I_*(j_*[T,M^n])} + [\theta^1 \to M^n].$$

The fixed point set of the identity is of course M^n. Hence $j_*[T,M^n]$ is just the 0-plane bundle over M^n. Hence $I_*(j_*[T,M^n]) = [\theta^1 \to M^n]$ which completes the proof. □

Proposition 5.46. (T,M) *represents an element in the kernel of* Γ *if and only if* (T,M) *is cobordant to* (id,M).

Proof. Since j_* is injective we find $\Gamma[T,M]=0$ if and only if $j_*(\Gamma[T,M])=0$. By the preceding theorem this means $I_*(j_*([T,M]+[\mathrm{id},M]))=0$. But I_* is a monomorphism, hence $j_*\big([T,M]+[\mathrm{id},M]\big)=0$. Therefore $[T,M]=-[\mathrm{id},M]=[\mathrm{id},M]$. □

5.6 The quotient algebra $\Lambda(\mathbb{Z}_2)$

We factor out a suitable ideal of $I_*(\mathbb{Z}_2)$ to obtain a quotient algebra $\Lambda(\mathbb{Z}_2)$ which we can understand better. Two filtrations of this quotient will lead the way to Boardman's theorem. In order to obtain this ideal we prove

Lemma 5.47. *Let x, y in $I_*(\mathbb{Z}_2)$. Then*

$$\Gamma(x\cdot y) = \Gamma(x)\cdot y + \epsilon(x)\cdot\Gamma(y) = x\cdot\Gamma(y) + \epsilon(y)\cdot\Gamma(x).$$

Proof. We compute the images of both sides of this equation under the homomorphism j_*. Let M^n (N^m) represent x (y), $\epsilon : I_*(\mathbb{Z}_2) \to MO_*$ be the homomorphism that forgets the involution and $[\xi_1]$ be the cobordism class of the trivial line bundle over a point. Note that $MO_* \subset I_*(\mathbb{Z}_2)$ by taking the trivial action and therefore $[\mathrm{id},M^n] = \epsilon([T,M^n])$. Recall that we identify $MO_n(BO(0)) = MO_n$.

$$\begin{aligned} j_*(\Gamma(x)\cdot y) &= j_*(\Gamma(x))\cdot j_*(y) \\ &= I_*\big(j_*(x+\epsilon(x))\big)\cdot j_*(y) \text{ by theorem 5.45} \\ &= \big(j_*(x)\cdot[\xi_1] + [\theta^1 \to M^n]\big)j_*(y) \\ &= \big(j_*(x)\cdot[\xi_1] + \epsilon(x)\cdot[\xi_1]\big)j_*(y) \end{aligned} \tag{5.6}$$

$$\begin{aligned} j_*(\epsilon(x)\cdot\Gamma(y)) &= j_*(\epsilon(x))\cdot j_*(\Gamma(y)) \\ &= [\theta^0 \to M^n]\cdot I_*\big(j_*(y+\epsilon(y))\big) \text{ by theorem 5.45} \\ &= \epsilon(x)\cdot\big(j_*(y)[\xi_1] + \epsilon(y)[\xi_1]\big) \end{aligned} \tag{5.7}$$

Furthermore

$$\begin{aligned}
j_*(\Gamma(x\cdot y)) &= I_*\big(j_*(x\cdot y+\epsilon(x\cdot y))\big)\\
&= I_*\big(j_*(x)\cdot j_*(y)\big) + I_*\big(j_*(\epsilon(x))\cdot j_*(\epsilon(y))\big)\\
&= j_*(x)\cdot j_*(y)\cdot[\xi_1] + [\theta^0\to M^n]\cdot[\theta^0\to N^m]\cdot[\xi_1]\\
&= (j_*(x)\cdot j_*(y)+\epsilon(x)\cdot\epsilon(y))\cdot[\xi_1] \qquad (5.8)
\end{aligned}$$

The sum of (5.6) and (5.7) gives (5.8) since we are working modulo 2. From the injectivity of j_* we may conclude

$$\Gamma(xy) = \Gamma(x)y + \epsilon(x)\Gamma(y)$$

To establish the second formula we compute

$$\begin{aligned}
j_*(x\cdot\Gamma(y)) &= j_*(x)\cdot I_*\big(j_*(y+\epsilon(y))\big)\\
&= j_*(x)\cdot\big(j_*(y)\cdot[\xi_1]+\epsilon(y)\cdot[\xi_1]\big) \qquad (5.9)
\end{aligned}$$

Interchange the roles of x and y in (5.7) to obtain

$$j_*(\epsilon(y)\cdot\Gamma(x)) = \epsilon(y)\cdot(j_*(x)[\xi_1]+\epsilon(x)[\xi_1]). \qquad (5.10)$$

Summation of (5.9) and (5.10) gives (5.8). Hence we have shown

$$\Gamma(x\cdot y) = x\cdot\Gamma(y)+\epsilon(y)\cdot\Gamma(x)$$

□

Lemma 5.48. *$S = \{x+\Gamma(x) : \epsilon(x)=0\}$ is an ideal in $I_*(\mathbb{Z}_2)$. Here $\epsilon : I_*(\mathbb{Z}_2)\to MO_*$ denotes the map which forgets the involution.*

Proof. S is clearly a subgroup of $I_*(\mathbb{Z}_2)$. Let $x+\Gamma(x)\in S$ and y be any element in $I_*(\mathbb{Z}_2)$. We have to show $(x+\Gamma(x))y\in S$. If we are given a null bordant manifold M and an arbitrary one N, then $M\times N$ bounds since $\partial(B\times N) = (\partial B)\times N = M\times N$ where B is a null bordism of M. Therefore $\epsilon(xy)=0$ because $\epsilon(x)=0$. Furthermore,

$$\Gamma(xy) = \Gamma(x)y+\epsilon(x)\Gamma(y) = \Gamma(x)y.$$

Hence $\big(x+\Gamma(x)\big)y = xy+\Gamma(xy)$ which means $xy\in S$. □

Definition 5.49. *The quotient algebra $I_*(\mathbb{Z}_2)/S$ is denoted by $\Lambda(\mathbb{Z}_2)$.*

The only homogeneous element in S is 0.

Lemma 5.50.

$$S\cap I_k(\mathbb{Z}_2) = 0$$

for any $k\geq 0$. In other words, $I_k(\mathbb{Z}_2)\to\Lambda(\mathbb{Z}_2)$ is a monomorphism.

Proof. Suppose $x=(x_l,\dots,x_k,\dots,x_m)\in I_*(\mathbb{Z}_2)$ such that $\epsilon(x)=0$ and $x+\Gamma(x)\in I_k(\mathbb{Z}_2)$. Hence, $x_i+\Gamma(x_{i-1})=0$ for $i\neq k$, $\Gamma(x_m)=0$ and $x_l=0$. Since $x_l=0$, we also have $\Gamma(x_l)=0$. But then $0=x_{l+1}+\Gamma(x_l)=x_{l+1}$. We proceed inductively and conclude $x_i=0$ for $l\leq i<k$. We now use $\Gamma(x_m)=0$. By lemma 5.46 this is the case if and only if $\epsilon(x_m)=x_m$. Since $\epsilon(x)=0$, we find $x_m=0$ and thus, $0=x_m+\Gamma(x_{m-1})=\Gamma(x_{m-1})$. Again we iterate this reasoning and obtain $x_i=0$ for $k<i\leq m$. From $x_{k+1}=0$ we know $\Gamma(x_k)=0$. Hence, $0=\epsilon(x_k)=x_k$ by lemma 5.46 and therefore $x=0$. □

We want j_* to factor through the quotient map $I_*(\mathbb{Z}_2) \to \Lambda(\mathbb{Z}_2)$ to a homomorphism $\Lambda(\mathbb{Z}_2) \to \mathcal{M}_*$. Therefore $j_*(x + \Gamma(x))$ must be zero in $\mathcal{M}_*$. If $\epsilon(x) = 0$ we have $j_*(x + \Gamma(x)) = j_*(x) + j_*(x)[\xi_1] = j_*(x)(1 + [\xi_1])$. Thus, we should factor $\mathcal{M}_*$ by the principal ideal $(1 + [\xi_1])$. In this way, we obtain a well-defined homomorphism

$$j_* : \Lambda(\mathbb{Z}_2) \to \mathcal{M}_*/(1 + [\xi_1])$$

Recall $[\xi_1] = [\theta^1 \to *]$ and 1 is just the 0-plane bundle over a point. Hence, for any vector bundle $\xi \to M$ is identified with $\xi \oplus \theta^1 \to M$ in $\mathcal{M}_*/(1 + [\xi_1])$. In other words, we pass to stable vector bundles, i.e. $\mathcal{M}_*/(1 + [\xi_1]) = MO_*(BO)$. $MO_*(BO)$ is still a polynomial algebra over MO_*, compare theorem 5.54 below.

Theorem 5.51. *$j_* : \Lambda(\mathbb{Z}_2) \to MO_*(BO)$ is an isomorphism*

Proof. Injectivity.
We show that the kernel of j_* is trivial. Suppose there is $x = (x_r, \ldots, x_s) \in I_*(\mathbb{Z}_2)$ with $x_j \in I_j(\mathbb{Z}_2)$ such that $j_*([x]) = 0$ in $MO_*(BO)$. This means that $j_*(x)$ lies in $(1 + [\xi_1])$ and hence we can write

$$j_*(x) = \alpha(1 + [\xi_1]) = \alpha + I_*(\alpha)$$

for $\alpha = (\alpha_m, \ldots, \alpha_n) \in \mathcal{M}_*$, $\alpha_k \in \mathcal{M}_k$. We have to show that x is in fact an element in the ideal S. Observe first that $x = (x_m, \ldots, x_{n+1})$. If $k < m$, then $j_*(x_k) = \underbrace{\alpha_k}_{0} + I_*(\underbrace{\alpha_{k-1}}_{0}) = 0$. Thus, $x_k = 0$ since j_* is injective. If $k > n+1$, we similarly have $j_*(x_k) = \underbrace{\alpha_k}_{0} + I_*(\underbrace{\alpha_{k-1}}_{0}) = 0$. From theorem 5.41 we know $\partial \circ j_* = 0$ and therefore in our situation $\partial(\alpha + I_*(\alpha)) = 0$ or equivalently

$$\partial\big(\alpha_k + I_*(\alpha_{k-1})\big) = 0 \in MO_{k-1}(\mathbb{Z}_2)$$

for all k. Thus,

$$0 = \partial(I_*(\alpha_n)).$$

Hence, we conclude from theorem 5.37

$$\partial(\alpha_n) = \Delta\partial I_*(\alpha_n) = 0.$$

We apply this to the equation $\partial(\alpha_n + I_*(\alpha_{n-1})) = 0$ and find $\partial I_*(\alpha_{n-1}) = 0$. Iteration of the argument gives $\partial(\alpha_k) = 0$ as well as $\partial I_*(\alpha_k) = 0$ for all k. But, $\ker(\partial) = \mathrm{img}(j_*)$. Hence, there exist $y_k \in I_k(\mathbb{Z}_2)$ such that $j_*(y_k) = \alpha_k$. We use theorem 5.45 to compute

$$\begin{aligned}
0 &= \partial j_* \Gamma(y_k) \\
&= \partial\Big(I_*\big(j_*(y_k + \epsilon(y_k))\big)\Big) \\
&= \partial\Big(I_*(\alpha_k) + \epsilon(y_k)[\xi_1]\Big) \\
&= \underbrace{\partial I_*(\alpha_k)}_{0} + \partial(\epsilon(y_k)[\xi_1]) \\
&= [A, S^0]\epsilon(y_k)
\end{aligned}$$

Thus, $\epsilon(y_k) = 0$. This means that for $y = (y_m, \dots, y_n)$, $y + \Gamma(y)$ lies in S. Moreover, we find

$$j_*\big(y_k + \Gamma(y_k)\big) = \alpha_k + j_*(\Gamma(y_k)) = \alpha_k + I_*(\alpha_k) + \underbrace{\epsilon(y_k)}_{0}[\xi_1] = j_*(x_k)$$

and therefore $y_k + \Gamma(y_k) = x_k$ for all k, since j_* is injective. This shows that $x = y + \Gamma(y) \in S$.

Surjectivity.
From theorem 5.41 we see that we need only take care of elements in the image of K. These are of the form $\sum_{k=0}^{n-1}[\xi_1]^{n-k}[N^k]$, compare theorem 5.41. Since we are working modulo 2 we find

$$\begin{aligned}(1+[\xi_1])(1+[\xi_1]+[\xi_1]^2+\ldots+[\xi_1]^{n-k-1}) = \\ &1+[\xi_1]+[\xi_1]^2+\ldots+[\xi_1]^{n-k-1} \\ &\quad+[\xi_1]+[\xi_1]^2+\ldots+[\xi_1]^{n-k-1}+[\xi_1]^{n-k} \\ &= 1+[\xi_1]^{n-k}.\end{aligned}$$

Thus, $1 + [\xi_1]^{n-k} = 0$ in $MO_*(BO)$. It follows that

$$\sum_{k=0}^{n-1}[\xi_1]^{n-k}[N^k] = \sum_{k=0}^{n-1}[N^k]$$

Note that we identify $MO_* = MO_*(BO(0)) \subset \mathcal{M}_*$. Of course, $[N^k] = j_*([\mathrm{id}, N^k])$, which completes the proof. $\square$

5.6.1 A generating set for $\Lambda(\mathbb{Z}_2)$

Lemma 5.56 below is the crucial point in the course of proving Boardman's theorem. We need a system of generators of $\Lambda(\mathbb{Z}_2)$ whose fixed point sets have nice properties, compare [1, 27.1]. Let ξ be a vector bundle of rank k over the n-manifold M. By the splitting principle we may assume that the total Stiefel-Whitney class factors to $(1+t_1)\cdots(1+t_k)$. Hence, we see that the classes $w_j(\xi)$ correspond to the j-th elementary symmetric polynomial in the indeterminates t_i. Since the elementary symmetric functions generate the symmetric polynomials there is a polynomial expression in the Stiefel-Whitney classes which corresponds to the polynomial $\sum_{i=0}^{k} t_i^n$. Denote the resulting cohomology class by $\sigma_n(\xi) \in H^n(M;\mathbb{Z}_2)$. This construction is

i) *natural*, $\sigma_n(f^*(\xi)) = f^*(\sigma_n(\xi))$,

ii) *additive*, $\sigma_n(\xi \oplus \eta) = \sigma_n(\xi) + \sigma_n(\eta)$ and

iii) $\sigma_n(\xi) = w_1(\xi)^n$, provided ξ is a line bundle.

Furthermore, we define

$$s_n(\xi) := \langle \sigma_n(\xi), [M]_{\mathbb{Z}_2} \rangle \in \mathbb{Z}_2.$$

Clearly, we have $s_n(\xi \oplus \theta^1) = s_n(\xi)$. It turns out that s_n only depends on the cobordism class of the bundle. Thus, we arrive at a homomorphism

$$s_n : MO_n(BO) \to \mathbb{Z}_2.$$

We also define $\sigma_n(M) = \sigma_n(\tau_M)$ and $s_n(M) = s_n(\tau_M)$. These numbers provide a very helpful tool in the analysis of MO_* and $MO_*(BO)$.

Theorem 5.52. *Let M be a closed n-manifold. The cobordism class $[M]$ represents a generator for the polynomial algebra MO_* in degree n if and only if $s_n(M) = 1$.*

Proof. See [8, IV.12] □

Corollary 5.53. *The real projective space $\mathbb{R}\mathrm{P}^n$ serves as a generator in degree n if and only if n is even.*

Proof. Denote the non-zero element of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ by a. From the proof of lemma 4.6 we see $\tau_{\mathbb{R}\mathrm{P}^n} \oplus \theta^1 \cong (n+1)\gamma_{1,n+1}$. This implies $\sigma_n(\mathbb{R}\mathrm{P}^n) = (n+1)a^n$, and $s_n(\mathbb{R}\mathrm{P}^n) \equiv n+1 \pmod 2$. □

Theorem 5.54. *$MO_*(BO)$ is a polynomial algebra over MO_* with one generator for each $n \geq 1$. The class of a vector bundle ξ over an n-manifold represents a polynomial generator if and only if $s_n(\xi) = 1$.*

Proof. See [19, lemma 4]. □

Since $MO_*(BO)$ is a polynomial algebra over MO_* and MO_* is a polynomial algebra over $\mathbb{Z}_2$ we can view $MO_*(BO)$ as a polynomial algebra over $\mathbb{Z}_2$. The next lemma tells us how to recognize a system of generators over $\mathbb{Z}_2$. Put $F_n = \sum_{k=0}^{n} MO_k(BO)$.

Lemma 5.55. *Let $\alpha_n \in F_n$ be a generating set of $MO_*(BO)$ as a polynomial algebra over MO_* such that $p_*(\alpha_n) \in \sum_{k=0}^{n} MO_k$ is $\mathbb{Z}_2$-decomposable. Here $p_* : MO_*(BO) \to MO_*$ denotes the homomorphism induced by the map p from BO to a point. Furthermore, let $\beta_n \in F_n$ be another sequence of elements in $MO_*(BO)$ defined for all $n \neq 2^j - 1$ with the property that $p_*(\beta_n)$ generate MO_* as a $\mathbb{Z}_2$-polynomial algebra. Then all the elements α_n, β_n together generate $MO_*(BO)$ as a polynomial algebra over $\mathbb{Z}_2$.*

Proof. Obviously, the elements α_n, $p_*(\beta_n)$ generate $MO_*(BO)$ as a polynomial algebra over $\mathbb{Z}_2$. Thus, there are $x, y \in \mathbb{Z}_2$ such that

$$\beta_n = x\alpha_n + yp_*(\beta_n) + q, \tag{5.11}$$

where q is a sum of products of the generators, hence $\mathbb{Z}_2$-decomposable. Applying p_* to (5.11) results in the relation

$$p_*(\beta_n) = xp_*(\alpha_n) + yp_*(\beta_n) + p_*(q).$$

$p_*(\alpha_n)$ is decomposable by assumption. Since $p_*(\beta_n)$ is indecomposable we conclude $y = 1$ and therefore $p_*(\beta_n) = \beta_n + x\alpha_n + q$. This shows that $p_*(\beta_n)$ can be replaced by β_n. □

Given $\alpha_n = (\alpha_n^0, \dots, \alpha_n^n) \in F_n$, we define $s_n(\alpha_n) := s_n(\alpha_n^n)$. We can also evaluate the homomorphism s_n on the stable tangent bundle of the base manifold $p_*(\alpha_n^n)$. This will be denoted by $s_n(p_*(\alpha_n))$.

Lemma 5.56. *For every $n \neq 2^j - 1$ there exists an involution on a closed manifold $(T, Y(n))$ with the following properties*

i) $[Y(n)]$ is indecomposable in MO_n,

ii) for even $n = 2k$, any component of the fixed point set has dimension less or equal to k, $s_k(\eta \to F) = 1$ and $[F^k]$ is decomposable,

iii) for odd $n = 2k+1$, any component of the fixed point set has dimension less or equal to k and $[F^k]$ is indecomposable.

By $y(n) \in \Lambda(\mathbb{Z}_2)$ we denote the cobordism classes represented by $(T, Y(n))$. These elements generate $\Lambda(\mathbb{Z}_2)$ as a polynomial algebra over $\mathbb{Z}_2$.

Proof. The proof is split into three parts. Denote by ξ_r^1 the line bundle associated to the principal $\mathbb{Z}_2$-bundle $S^r \to \mathbb{R}\mathrm{P}^r$, i.e. $\xi_r^1 = y_{1,r+1}$.

1. $\mathbf{n = 4m+2}$

In this case $k = 2m+1$. We take $Y(n) = \mathbb{R}\mathrm{P}^{4m+2}$ and define the involution T by

$$T([x_1, \dots, x_{4m+3}]) = [x_1, \dots, x_{2m+2}, -x_{2m+3}, \dots, -x_{4m+3}]$$

The fixed point set of T is of course the disjoint sum $F = \mathbb{R}\mathrm{P}^{2m+1} \sqcup \mathbb{R}\mathrm{P}^{2m}$ and hence every component of F has dimension less or equal to $2m+1 = k$. From corollary 4.7 we find $[F^k] = [\mathbb{R}\mathrm{P}^{2m+1}] = 0$. $[\mathbb{R}\mathrm{P}^{4m+2}]$ is indecomposable by corollary 5.53 because $n = 4m+2$ is even. Since the normal bundle to $F^k = \mathbb{R}\mathrm{P}^{2m+1}$ is the $(2m+1)$-fold Whitney sum $\xi_{2m+1}^1 \oplus \dots \oplus \xi_{2m+1}^1$ we find that the total Whitney class factors to $(1+a)^{2m+1}$. We compute

$$\begin{aligned} \sigma_k(\eta \to F) &= \sigma_{2m+1}(\xi_{2m+1}^1 \oplus \dots \oplus \xi_{2m+1}^1) \\ &= (2m+1)\sigma_{2m+1}(\xi_{2m+1}^1) \\ &= (2m+1)a^{2m+1} \end{aligned}$$

Therefore, $s_k(\eta \to F) = 2m+1 \equiv 1 \pmod 2$.

2. $\mathbf{n = 4m}$

Now $k = 2m$. First of all, there is the involution $(T_1, \mathbb{R}\mathrm{P}^{2m} \times \mathbb{R}\mathrm{P}^{2m})$ which simply interchanges coordinates

$$T_1[x_1, \dots, x_{2m+1}, y_1, \dots, y_{2m+1}] = [y_1, \dots, y_{2m+1}, x_1, \dots x_{2m+1}]. \tag{5.12}$$

The fixed point set of T_1 is the diagonal $\Delta \subset \mathbb{R}\mathrm{P}^{2m} \times \mathbb{R}\mathrm{P}^{2m}$. The normal bundle to the diagonal is equivalent to the tangent bundle to $\mathbb{R}\mathrm{P}^{2m}$, compare the proof of theorem 5.29. Therefore, $s_k(\eta \to F) = s_k(\tau_{\mathbb{R}\mathrm{P}^k}) \overset{5.53}{=} k+1 \equiv 1 \pmod 2$. Define another involution $(T_2, \mathbb{R}\mathrm{P}^{4m})$ by

$$T_2[x_1, \dots, x_{4m+1}] = [x_1, \dots, x_{2m+1}, -x_{2m+2}, \dots, -x_{4m+1}]. \tag{5.13}$$

Now the fixed point set is $\mathbb{R}\mathrm{P}^{2m} \sqcup \mathbb{R}\mathrm{P}^{2m-1}$. The normal bundle to $\mathbb{R}\mathrm{P}^{2m}$ is the $(2m)$-fold sum $\xi^1_{2m} \oplus \ldots \oplus \xi^1_{2m}$. As above we obtain $\sigma_k(\eta \to F) = 2ma^{2m}$ and hence $s_k(\eta \to F) = 0$. The involutions (5.12) and (5.13) induce an involution

$$(T, Y(4m)) = (T_1, \mathbb{R}\mathrm{P}^{2m} \times \mathbb{R}\mathrm{P}^{2m}) \sqcup (T_2, \mathbb{R}\mathrm{P}^{4m})$$

which satisfies the properties in (ii).

3. n odd, $\mathbf{n \neq 2^j - 1}$.

This is the most tricky part. We make use of the construction mentioned in example 5.19. Since $n+1$ is even and not a power of 2, we can write $n+1 = 2^l(2m+1)$, $l > 0$, $m > 0$. Hence $n = 2^{l+1}m + 2^l - 1$. We apply 5.19 with $r = 2^l$, $s = 2^{l+1}m - 1$. By lemma 5.57 below $\mathbb{R}\mathrm{P}(2^l, 2^{l+1}m-1)$ is indecomposable in MO_n. We may regard $\mathbb{R}\mathrm{P}(2^l, 2^{l+1}m-1)$ as the quotient of $S^{2^l} \times \mathbb{R}\mathrm{P}^{2^{l+1}m-1}$ with respect to the fixed point free involution

$$T_2\big(x_1, \ldots, x_{2^l+1}, [y_1, \ldots, y_{2^{l+1}m}]\big) = \big(-x_1, \ldots, -x_{2^l+1}, [-y_1, y_2, \ldots, y_{2^{l+1}m}]\big).$$

Another involution is defined as follows,

$$\begin{aligned} &T_1\big(x_1, \ldots, x_{2^l+1}, [y_1, \ldots, y_{2^{l+1}m}]\big) = \\ &\big(x_1, \ldots, x_{2^{l-1}+1}, -x_{2^{l-1}+2}, \ldots, -x_{2^l+1}, [y_1, \ldots, y_{2^l m}, -y_{2^l m+1}, \ldots, -y_{2^{l+1}m}]\big). \end{aligned}$$

These involutions obviously commute so that T_1 induces a closed involution $(T, \mathbb{R}\mathrm{P}(2^l, 2^{l+1}m-1))$ which will be our choice for $(T, Y(n))$. According to remark 5.42 we have to find the fixed point set F_{T_1} of T_1 and the set of coincidence points. Since in projective spaces $[z] = [z']$ if and only if $z' = \lambda z$, we see

$$F_{T_1} = \Big(S^{2^{l-1}} \times \mathbb{R}\mathrm{P}^{2^l m-1}\Big) \sqcup \Big(S^{2^{l-1}} \times \mathbb{R}\mathrm{P}^{2^l m-1}\Big).$$

Passing to the quotients under the action of T_2 these become $\mathbb{R}\mathrm{P}(2^{l-1}, 2^l m - 1)$ and $\mathbb{R}\mathrm{P}^{2^{l-1}} \times \mathbb{R}\mathrm{P}^{2^l m-1}$. Furthermore, $T_1(x) = T_2(x)$ if and only if

$$x = (0, \ldots, 0, x_{2^{l-1}+2}, \ldots, x_{2^l+1}, [0, y_2, \ldots, y_{2^l m}, 0, \ldots, 0] \quad \text{or} \tag{5.14}$$

$$x = (0, \ldots, 0, x_{2^{l-1}+2}, \ldots, x_{2^l+1}, [y_1, 0, \ldots, 0, y_{2^l m+1}, \ldots, y_{2^{l+1}m}]. \tag{5.15}$$

(5.14) corresponds to $S^{2^{l-1}-1} \times \mathbb{R}\mathrm{P}^{2^l m-2}$ whereas (5.15) equates to $S^{2^{l-1}-1} \times \mathbb{R}\mathrm{P}^{2^l m}$. Modulo the action of T_2 the set of coincidences is therefore $\mathbb{R}\mathrm{P}^{2^{l-1}-1} \times \mathbb{R}\mathrm{P}^{2^l m-2}$ and $\mathbb{R}\mathrm{P}(2^{l-1}-1, 2^l m)$ respectively. Summing up, the fixed point set of the involution $(T, Y(n))$ consists of four components:

1. $F_1 = \mathbb{R}\mathrm{P}(2^{l-1}, 2^l m - 1)$
2. $F_2 = \mathbb{R}\mathrm{P}^{2^{l-1}} \times \mathbb{R}\mathrm{P}^{2^l m-1}$
3. $F_3 = \mathbb{R}\mathrm{P}^{2^{l-1}-1} \times \mathbb{R}\mathrm{P}^{2^l m-2}$
4. $F_4 = \mathbb{R}\mathrm{P}(2^{l-1} - 1, 2^l m)$

Note that $n = 2^{l+1}m + 2^l - 1 = 2\underbrace{(2^l m + 2^{l-1} - 1)}_{k} + 1$. We immediately see that each component of the fixed point set has dimension less or equal to $k = 2^l m + 2^{l-1} - 1$. From corollary 4.7 we directly conclude $[F_2] = 0$ in MO_k and $[F_3] = 0$ in MO_{k-2}. We have to deal with the remaining components. Suppose for a moment $l > 1$. Then $[F_1]$ is indecomposable by lemma 5.57 whereas $[F_4]$ is decomposable because

$$2^l m + \sum_{j=0}^{2^{l-1}-1} \binom{2^{l-1} - 1 + 2^l m}{j} \equiv 0 \pmod 2.$$

This follows from $\binom{2^{l-1}-1+2^l m}{j} \equiv 0 \pmod 2$ for $0 \le j < 2^{l-1}$ since $2^l m + 2^{l-1} - 1 = 2^l m + 2^{l-2} + \ldots + 1$, compare the proof of the lemma below. If $l = 1$, we find $F_4 = \mathbb{R}\mathrm{P}(0, 2m) = \mathbb{R}\mathrm{P}^{2m}$ which represents a generator and thus is indecomposable in MO_{2m}, compare corollary 5.53. To complete the first part of the lemma we have to show that $[\mathbb{R}\mathrm{P}(1, 2m-1)]$ is decomposable. Let ξ_1^1 denote the twisted line bundle over the circle $\mathbb{R}\mathrm{P}^1 = S^1$ (the total space is seen to be the Möbius strip) and let $\xi = \xi_1^1 \oplus \theta^{2m-1}$. We use the notation of theorem 5.22. The total Stiefel-Whitney classes are $v(\xi) = v(\xi_1^1) = 1 + \underbrace{v_1}_{\neq 0}$ and $w(S^1) = 1$ since S^1 is orientable. By theorem 5.22

$$\begin{aligned} w(P(\xi)) &= (1+c)^{2m} + (1+c)^{2m-1} q^*(v_1) \\ &= (1+c)^{2m-1}(1 + c + q^*(v_1)) \end{aligned}$$

We compute

$$\begin{aligned} \sigma_{2m} &= (2m-1)c^{2m} + (c + q^*(v_1))^{2m} \\ &= c^{2m} + c^{2m} + 2m \cdot q^*(v_1) c^{2m-1} \quad \text{since } v_1^2 = 0 \\ &= 0 \end{aligned}$$

Hence, $s_{2m}(\mathbb{R}\mathrm{P}(1, 2m-1)) = s_{2m}(P(\xi)) = 0$ and $\mathbb{R}\mathrm{P}(1, 2m-1)$ is decomposable by theorem 5.52.

The involutions $(T, Y(n))$ generate two sequences of elements in $MO_*(BO)$

1. $\alpha_k = j_*\big(y(2k)\big) \in F_k$,
2. $\beta_k = j_*\big(y(2k+1)\big) \in F_k$

which satisfy the conditions of lemma 5.55 by construction: Since any component of the fixed point set has dimension less or equal to k, we see α_k, $\beta_k \in F_k$. Furthermore, α_k generate $MO_*(BO)$ over MO_* since $s_k(\alpha_k) = s_k(\eta \to F) = 1$ (see theorem 5.54) and $p_*(\alpha_k)$ is $\mathbb{Z}_2$-decomposable because $[F^k]$ is decomposable. Finally, in the odd case we have shown that $[F^k]$ is indecomposable and hence $p_*(\beta_k)$ generate MO_* as a polynomial algebra over $\mathbb{Z}_2$. By theorem 5.51 we conclude that the elements $y(n)$ constitute a set of polynomial generators of $\Lambda(\mathbb{Z}_2)$ over $\mathbb{Z}_2$. This completes the proof. □

Lemma 5.57. *Suppose $s > 0$. $[\mathbb{R}\mathrm{P}(r,s)]$ is indecomposable if and only if*

$$s + \sum_{j=0}^{r} \binom{r+s}{j} \equiv 1 \pmod 2.$$

Moreover, if $l > 0$, $m > 0$ then $[\mathbb{R}\mathrm{P}(2^l, 2^{l+1}m - 1]$ is indecomposable.

Proof. We take up the notation used in theorem 5.22. The total Stiefel-Whitney class of $\xi_r^1 \oplus \theta^s$ is $v(\xi_r^1 \oplus \theta^s) = v(\xi_r^1) = 1 + a$ where $a \in H^1(\mathbb{R}\mathrm{P}^r; \mathbb{Z}_2)$ denotes the non-zero element. The tangential class of $\mathbb{R}\mathrm{P}^r$ is $(1+a)^{r+1}$ by theorem 4.6. Thus,

$$\begin{aligned} w(\mathbb{R}\mathrm{P}(r,s)) &= \Big((1+c)^{s+1} + (1+c)^s q^*(a)\Big)\big(1+q^*(a)\big)^{r+1} \\ &= \big(1+q^*(a)\big)^{r+1}(1+c)^s\big(1+c+q^*(a)\big). \end{aligned}$$

Since $\xi \oplus \theta^s$ has rank $s+1$, we apply (5.1) to find $c^{s+1} + q^*(a)c^s = 0$. We compute

$$\begin{aligned} \sigma_{r+s}\big(\mathbb{R}\mathrm{P}(r,s)\big) &= (r+1) \underbrace{q^*(a^{r+s})}_{=0 \text{ since } s>0} + sc^{r+s} + (c+q^*(a))^{r+s} \\ &= sq^*(a^r)c^s + \sum_{j=0}^{r} \binom{r+s}{j} \underbrace{q^*(a)^j c^{r+s-j}}_{q^*(a^r)c^s} \\ &= \left(s + \sum_{j=0}^{r} \binom{r+s}{j}\right) q^*(a^r)c^s \end{aligned}$$

The first claim follows now from theorem 5.52 since $\langle q^*(a^r)c^s, [\mathbb{R}\mathrm{P}(r,s)]_{\mathbb{Z}_2}\rangle = 1$. Now suppose $r + s = 2^l + 2^{l+1}m - 1 = 2^{l+1}m + \sum_{i=0}^{l-1} 2^i$. It follows from Lucas' theorem that

$$\binom{r+s}{j} \equiv \binom{m}{j_{l+1}}\binom{0}{j_l}\binom{1}{j_{l-1}}\cdots\binom{1}{j_0} \pmod 2$$

where $j = \sum_{i=0}^{l+1} j_i 2^i$, see [20]. Therefore as long as $0 \le j < 2^l$, $\binom{r+s}{j} \equiv 1 \pmod 2$. However, $\binom{r+s}{2^l} \equiv 0 \pmod 2$. Hence,

$$\begin{aligned} s + \sum_{j=0}^{r} \binom{r+s}{j} &= 2^{l+1}m - 1 + \sum_{j=0}^{2^l} \binom{r+s}{j} \\ &\equiv 1 + \sum_{j=0}^{2^l-1} \binom{r+s}{j} \pmod 2 \\ &\equiv 1 + 2^l \cdot 1 \pmod 2 \\ &\equiv 1 \pmod 2. \end{aligned}$$

□

5.6.2 Two filtrations of $\Lambda(\mathbb{Z}_2)$

As mentioned at the beginning of this chapter, we now introduce two filtrations of $\Lambda(\mathbb{Z}_2)$. We first take care of the increasing one. Recall $F_n = \sum_{k=0}^{n} MO_k(BO)$.

Definition 5.58. *Suppose $x \in \Lambda(\mathbb{Z}_2)$. We define the* ***fixed point filtration*** *of the quotient algebra by*

$$\mathrm{Fil}_{\mathrm{FP}}(x) \leq n \quad \textit{if and only if} \quad j_*(x) \in F_n$$

Clearly, $F_m \cdot F_n \subset F_{m+n}$. $x \in F_n$ is called $\mathbb{Z}_2$-decomposable if and only if x can be expressed as a sum of products of elements of lower filtration. Otherwise, x is $\mathbb{Z}_2$-indecomposable.

We introduce

$$MO_*[[t]] := \{\sum_{k=0}^{\infty} [V^k]t^k \,:\, [V^k] \in MO_k\}$$

the ring of homogeneous formal power series over MO_* which will be important for the construction of a decreasing filtration. Define a map

$$\phi : I_*(\mathbb{Z}_2) \to MO_*[[t]]$$

by

$$\phi([T, M^n]) := \sum_{k=0}^{\infty} \epsilon(\Gamma^k[T, M^n])t^{n+k}$$

where $\epsilon : I_*(\mathbb{Z}_2) \to MO_*$ is the map that forgets the involution.

Lemma 5.59. *$\phi : I_*(\mathbb{Z}_2) \to MO_*[[t]]$ is a ring homomorphism.*

Proof. ϕ is clearly additive. Let $x \in I_m(\mathbb{Z}_2)$, $y \in I_n(\mathbb{Z}_2)$. We show by induction that

$$\epsilon(\Gamma^k(xy)) = \sum_{j=0}^{k} \epsilon(\Gamma^j(x))\epsilon(\Gamma^{k-j}(y))$$

This is clear for $k = 0$, since ϵ is multiplicative. Suppose $k > 0$. We compute

$$\begin{aligned}
\epsilon(\Gamma^k(xy)) &= \epsilon\big(\Gamma^{k-1}(\Gamma(xy))\big) \\
&= \epsilon\Big(\Gamma^{k-1}\big(x\Gamma(y) + \epsilon(y)\Gamma(x)\big)\Big) \quad \text{by theorem 5.45} \\
&= \sum_{j=0}^{k-1} \epsilon(\Gamma^j(x))\epsilon(\Gamma^{k-j}(y)) + \underbrace{\sum_{j=0}^{k-1} \epsilon\big(\Gamma^j(\epsilon(y))\big)\epsilon(\Gamma^{k-j}(x))}_{\epsilon(y)\epsilon(\Gamma^k(x))} \\
&= \sum_{j=0}^{k} \epsilon(\Gamma^j(x))\epsilon(\Gamma^{k-j}(y))
\end{aligned}$$

Note $\Gamma|_{MO_*} = 0$ and $\epsilon|_{MO_*} = \text{id}$. Now

$$\begin{aligned}
\phi(x)\phi(y) &= \Big(\sum_{i=0}^{\infty} \epsilon(\Gamma^i(x))t^{m+i}\Big)\Big(\sum_{j=0}^{\infty} \epsilon(\Gamma^j(y))t^{n+j}\Big) \\
&= \sum_{k=0}^{\infty} \underbrace{\Big(\sum_{i+j=k} \epsilon(\Gamma^i(x))\epsilon(\Gamma^j(y))\Big)}_{\epsilon(\Gamma^k(xy))} t^{k+m+n} \\
&= \phi(xy)
\end{aligned}$$

□

Lemma 5.60. *$S \subseteq ker(\phi)$.*

Proof. Suppose $x + \Gamma(x) \in S$, $x \in I_*(\mathbb{Z}_2)$ and recall $\epsilon(x) = 0$ by definition of S. Then

$$\begin{aligned}
\phi(x + \Gamma(x)) &= \phi(x) + \phi(\Gamma(x)) \\
&= \sum_{k=0}^{\infty} \epsilon(\Gamma^k(x))t^{n+k} + \sum_{l=0}^{\infty} \epsilon\Big(\Gamma^l\big(\Gamma(x)\big)\Big)t^{(n+1)+l} \\
&= \underbrace{\epsilon(x)t^n}_{=0} + \sum_{k=1}^{\infty} \epsilon(\Gamma^k(x))t^{n+k} + \sum_{k=1}^{\infty} \epsilon(\Gamma^k(x))t^{n+k} \\
&= 0
\end{aligned}$$

since any element in MO_* has order at most 2. □

Hence, we see that ϕ induces a homomorphism

$$\Lambda(\mathbb{Z}_2) \to MO_*[[t]]$$

which we again denote by ϕ. As $\phi(y(n)) = \underbrace{\epsilon([T, Y(n)])}_{[Y(n)]} t^n +$ higher terms in t, lemma 5.56 shows

Proposition 5.61. *The induced homomorphism $\phi : \Lambda(\mathbb{Z}_2) \to MO_*[[t]]$ is injective.*

We introduce a decreasing filtration on $MO_*[[t]]$. Consider the ideals

$$J(n) = \{\sum_{k=0}^{\infty} [V^k]t^k : [V^0] = \ldots = [V^{n-1}] = 0\}.$$

We write $\text{Fil}(x) \geq n$ if and only if $x \in J(n)$. Thus, $\text{Fil}(x) = n$ if we find the first nonzero coefficient in the power series expression of x at t^n. Note that $J(n) \supset J(n+1) \supset \ldots$ Clearly, this induces a decreasing filtration of $MO_*[[t]]$.

Definition 5.62. *Let $x \in \Lambda(\mathbb{Z}_2)$. We write*

$$\text{Fil}_\phi(x) \geq n \quad \textit{if and only if} \quad \text{Fil}(\phi(x)) \geq n.$$

The following properties are immediate from the construction of the generators in lemma 5.56.

Proposition 5.63. *Let $y(n)$ be the generators of $\Lambda(\mathbb{Z}_2)$.*

i) $\mathrm{Fil}_{\mathrm{FP}}(y(2k)) = k$*,*

ii) $\mathrm{Fil}_{\mathrm{FP}}(y(2k+1)) = k$*,*

iii) $\mathrm{Fil}_{\phi}(y(n)) = n$*.*

Proof. Since all components of the fixed set have dimension less or equal to k we have $\mathrm{Fil}_{\mathrm{FP}}(y(2k)) \leq k$ as well as $\mathrm{Fil}_{\mathrm{FP}}(y(2k+1)) \leq k$. But in the even case $s_k(\eta \to F) = 1$. Hence, $j_*(y(2k))$ is $\mathbb{Z}_2$-indecomposable which implies $\mathrm{Fil}_{\mathrm{FP}}(y(2k)) = k$. If n is odd we know that $[F^k]$ is indecomposable and therefore $\mathrm{Fil}_{\mathrm{FP}}(y(2k+1)) = k$. The third claim follows immediately since $[Y(n)]$ is indecomposable for all n. □

The filtrations $\mathrm{Fil}_{\mathrm{FP}}$ and Fil_{ϕ} satisfy

Proposition 5.64. *Let $x \neq 0, y \neq 0 \in \Lambda(\mathbb{Z}_2)$. Then*

i) $\mathrm{Fil}_{\mathrm{FP}}(xy) = \mathrm{Fil}_{\mathrm{FP}}(x) + \mathrm{Fil}_{\mathrm{FP}}(y)$*,*

ii) $\mathrm{Fil}_{\phi}(xy) = \mathrm{Fil}_{\phi}(x) + \mathrm{Fil}_{\phi}(y)$*.*

If x and y have no common monomials, we find

iii) $\mathrm{Fil}_{\mathrm{FP}}(x+y) = \max\Big(\mathrm{Fil}_{\mathrm{FP}}(x), \mathrm{Fil}_{\mathrm{FP}}(y)\Big)$*,*

iv) $\mathrm{Fil}_{\phi}(x+y) = \min\Big(\mathrm{Fil}_{\phi}(x), \mathrm{Fil}_{\phi}(y)\Big)$*.*

Proof. (i) and *(iii)* are obvious from the definition and $j_*(xy) = j_*(x)j_*(y)$. The assumption that x and y have no monomials in common ensures that the monomial responsible for the resulting filtration does not vanish in the polynomial expression for $x+y$. *(ii)* and *(iv)* immediately follow from the definition of the ϕ-filtration. □

5.7 Boardman's five-halves theorem

We now have all ingredients at our disposal to prove Boardman's theorem.

Theorem 5.65 (Boardman's five-halves theorem)**.** *Let (T, M^n) be an involution on a closed manifold M^n. If $[M^n] \neq 0$ in MO_n, then $n \leq \frac{5}{2}\dim(F)$.*

Proof. We consider $[T, M^n]$ as an element in $I_n(\mathbb{Z}_2) \subset \Lambda(\mathbb{Z}_2)$ and use the filtrations of the quotient algebra. Since M^n does not bord, we find $\mathrm{Fil}_{\phi}([T, M^n]) = n$. Let p be the polynomial which expresses $[T, M^n]$ as a sum of monomials in the generators $y(n)$. From proposition 5.63 we conclude

$$\mathrm{Fil}_{\phi}(y(n)) \leq \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}(y(n)). \tag{5.16}$$

Suppose x is a monomial in the generators $y(n)$. By proposition 5.64 and (5.16) we see that $\mathrm{Fil}_\phi(x) \le \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}(x)$. Finally, suppose x is a sum of monomials in the $y(n)$. Again by 5.64 we find: $\mathrm{Fil}_\phi(x)$ is the filtration of the monomial which minimizes Fil_ϕ and the fixed point filtration $\mathrm{Fil}_{\mathrm{FP}}(x)$ is the maximum of the FP-filtrations of its monomials. Thus, $\mathrm{Fil}_\phi(x) \le \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}(x)$. We obtain

$$\begin{aligned} n &= \mathrm{Fil}_\phi([T, M^n]) \\ &= \mathrm{Fil}_\phi(p) \\ &\le \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}(p) \\ &= \frac{5}{2}\,\underbrace{\mathrm{Fil}_{\mathrm{FP}}([T, M^n])}_{\le \dim(F)} \end{aligned}$$

as claimed. □

We even have (compare [23, 4.8])

Corollary 5.66. *Let (T, M^n) be an involution on a closed manifold M^n. If $[T, M^n] \neq 0$ in $I_*(\mathbb{Z}_2)$, then $n \le \frac{5}{2}\dim(F)$.*

Proof. Suppose $n > \frac{5}{2}\dim(F)$. Our task is to show that this already implies $[T, M^n] = 0$. Consider the composition

$$I_n(\mathbb{Z}_2) \hookrightarrow \Lambda(\mathbb{Z}_2) \xrightarrow{\phi} MO_*[[t]]$$

which maps $[T, M^n]$ to $\sum_{k=0}^{\infty} \epsilon(\Gamma^k[T, M^n])t^{n+k}$. Since $n > \frac{5}{2}\dim(F)$ we conclude from theorem 5.65 that M^n must bord, $[M^n] = 0$. Suppose that $k \ge 1$ is the least integer such that $\epsilon(\Gamma^k[T, M^n]) \neq 0$, that is $\mathrm{Fil}_\phi(\Gamma^k[T, M^n]) = n + k$ by definition of the ϕ-filtration. By (5.4) the normal bundle to the fixed set of $\Gamma^k([T, M^n])$ is

$$\left(\eta \oplus \theta^k \to F_T\right) \sqcup \left(\bigsqcup_{j=0}^{k-1} \theta^{k-j} \to \epsilon(\Gamma^j[T, M^n])\right).$$

But $\epsilon(\Gamma^j[T, M^n])) = 0$ for all $j < k$ which implies $\mathrm{Fil}_{\mathrm{FP}}(\Gamma^k([T, M^n])) =$ $\mathrm{Fil}_{\mathrm{FP}}([T, M^n])$. As above we have

$$\begin{aligned} n + k &= \mathrm{Fil}_\phi(\Gamma^k[T, M^n]) \\ &\le \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}(\Gamma^k[T, M^n]) \\ &= \frac{5}{2}\,\mathrm{Fil}_{\mathrm{FP}}([T, M^n]) \\ &\le \frac{5}{2}\dim(F) \end{aligned}$$

which contradicts $n > \frac{5}{2}\dim(F)$. Thus $\epsilon(\Gamma^k[T, M^n]) = 0$ for all k and therefore $\phi([T, M^n]) = 0$ in $MO_*[[t]]$. This implies $[T, M^n] = 0$ in $\Lambda(\mathbb{Z}_2)$ because ϕ is a monomorphism. Since $I_n(\mathbb{Z}_2)$ injects into $\Lambda(\mathbb{Z}_2)$ by lemma 5.50, $[T, M^n] = 0$ in $I_n(\mathbb{Z}_2)$ which completes the proof. □

If the fixed point set is assumed to have constant dimension, this can be improved to

Theorem 5.67. *Let (T, M^n) be an involution on a closed manifold such that the fixed point set is of constant dimension. If $[T, M^n] \neq 0$, then $n \leq 2\dim(F)$.*

Proof. We refer to [12, theorem C]. $\square$

Corollary 5.68. *Let (T, M^n) be an involution on a closed manifold M^n. If $[M^n]$ is indecomposable in MO_*, then*

$$n \leq \begin{cases} 2\dim(F) + 1 & \text{if } n \text{ is odd} \\ 2\dim(F) & \text{if } n \text{ is even} \end{cases}$$

Proof. Let p be the polynomial expression of $[T, M^n]$ in $\Lambda(\mathbb{Z}_2)$. Since $[M^n]$ is indecomposable we see that p takes the form $y(n) + q$ where q stands for $\mathbb{Z}_2$-decomposables. The fixed point filtration of a polynomial is the maximum of the filtrations of its monomials. Hence,

$$\begin{aligned} \dim(F) \geq \mathrm{Fil}_{\mathrm{FP}}([T, M^n]) = \mathrm{Fil}_{\mathrm{FP}}\big(q + y(n)\big) &\geq \mathrm{Fil}_{\mathrm{FP}}(y(n)) \\ &= \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

$\square$

We use theorem 1.17 to improve the estimate in Boardman's five-halves theorem in the case of odd Euler characterstic. From the product property of the Euler characteristic we obtain a formula for $\chi\big(P(\xi)\big)$.

Lemma 5.69. *Let $\xi = (p, E, B)$ be a k-plane bundle over B. Then*

$$\chi\big(P(\xi)\big) = \chi(B) \cdot \chi(\mathbb{R}\mathrm{P}^{k-1}).$$

Lemma 5.70. *Let (T, M^n) be an involution on a closed manifold M^n. The Euler characteristic of M^n satisfies the congruence*

$$\chi(M^n) \equiv \chi(F) \pmod 2.$$

Proof. Note $\chi(F) = \sum_{k=0}^{n} \chi(F^k)$. We start with the case of n being odd. By theorem 1.17 $\chi(F^n) = 0 = \chi(M^n)$. Therefore we have to show that $\chi(F)$ is even. Since the normal sphere bundle bords by lemma 5.27 it follows from theorem 5.10 that $[P(\eta)] = 0$ and thus $P(\eta)$ has even Euler characteristic. As usual η denotes the normal bundle to the fixed point set. $S^r \to \mathbb{R}\mathrm{P}^r$ is a 2-fold covering. Thus,

$$\chi(\mathbb{R}\mathrm{P}^r) = \frac{1}{2} \cdot \chi(S^r) = \frac{1}{2} \cdot (1 + (-1)^r) = \begin{cases} 0 & \text{if } r \text{ is odd} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

We use the lemma above to compute modulo 2

$$0 \equiv \chi(P(\eta)) = \sum_{k=0}^{n-1} \chi(F^k) \cdot \chi(\mathbb{R}\mathrm{P}^{n-k-1}) = \sum_{\text{k even}} \chi(F^k) \cdot \underbrace{\chi(\mathbb{R}\mathrm{P}^{n-k-1})}_{1} = \chi(F)$$

We turn to the case n even. Here, we find $[M^n] = [P(\eta \oplus \theta^1)]$ from theorem 5.28. Hence, we again apply the lemma to conclude

$$\chi\big(P(\eta \oplus \theta^1)\big) = \sum_{k=0}^{n} \chi(F^k) \cdot \chi(\mathbb{R}\mathrm{P}^{n-k}) = \sum_{\text{k even}} \chi(F^k) \cdot \underbrace{\chi(\mathbb{R}\mathrm{P}^{n-k})}_{1} = \chi(F).$$

Since the Euler characteristic is a cobordism invariant modulo 2 this completes the proof. □

Theorem 5.71. *Suppose (T, M^n) is an involution on a closed manifold M^n of odd Euler characteristic. Then n is even and the estimate $\dim(F) \geq \frac{n}{2}$ holds.*

Proof. It is part of theorem 1.17 that n is even in this case. Write $n = 2m$. The Euler characteristic induces a homomorphism $I_*(\mathbb{Z}_2) \to \mathbb{Z}_2$, compare proposition 2.11. We abuse notation and again denote this map by χ. This homomorphism maps the ideal S is to zero: Suppose

$$[\tau, N] + \Gamma[\tau, N] \in S,$$

$$[N] = \epsilon([\tau, N]) = 0.$$

Lemma 5.70 implies $\chi(F_\tau) \equiv \chi(N) \equiv 0 \pmod 2$ since N bords. By lemma 5.43 the fixed point set of $\Gamma[\tau, N]$ is the union $F_\tau \sqcup N$. Thus,

$$\chi\big(\epsilon(\Gamma[N, \tau])\big) \overset{5.70}{\equiv} \chi(F_\tau) + \chi(N) \equiv 0 \pmod 2.$$

Let p be the polynomial expression for $[T, M^n] \in I_n(\mathbb{Z}_2) \subset \Lambda(\mathbb{Z}_2)$ in the generators $y(k)$. From lemma 5.56 we find that $\chi(y(k)) = 1$ implies $k \equiv 2 \pmod 4$. But since M^n has odd Euler characteristic by assumption there must be at least one monomial y in the $y(4l+2)$. We have $2m = \mathrm{Fil}_\phi([T, M^n]) = \mathrm{Fil}_\phi(p) \leq \mathrm{Fil}_\phi(y)$ since $[M^n] \neq 0$. Thus,

$$\dim(F) \geq \mathrm{Fil}_{\mathrm{FP}}([T, M^n]) = \mathrm{Fil}_{\mathrm{FP}}(p) \geq \mathrm{Fil}_{\mathrm{FP}}(y) = \frac{1}{2}\,\mathrm{Fil}_\phi(y) \geq m,$$

compare lemma 5.63. □

What can we deduce about the cobordism behaviour of M^n in the case $n \leq \frac{5}{2}\dim(F)$? We give necessary and sufficient conditions that (T, M^n) with $n \leq \frac{5}{2}\dim(F)$ bords. This is due to Zhi Lü and Chun-Lian Zhou and can be found in [12]. The proof uses Boardman's theorem and relies heavily on the fact that the unrestricted cobordism class of an involution is uniquely determined by its fixed point data, compare theorem 5.41.

Theorem 5.72. *Let (T, M^n) be an involution on a closed manifold such that $n \leq \frac{5}{2}\dim(F)$. Set $a = \lfloor \frac{5}{2}\dim(F) - n \rfloor$. Then $[T, M^n] = 0 \in I_n(\mathbb{Z}_2)$ if and only if $\eta \oplus \theta^{a+1}$ is still the normal bundle to the fixed point set of a suitable involution acting on an (n+a+1)-manifold.*

Proof. Suppose $\big(\eta \oplus \theta^{a+1} \to F\big) = \bigsqcup_{k=0}^{n} \big(\eta^{n-k} \oplus (a+1)\mathbb{R} \to F^k\big)$ is the normal bundle to the fixed point set of an involution (τ, N^{n+a+1}). The rank of the bundle is chosen in such a way that $n+a+1 = n+\lfloor \frac{5}{2}\dim(F)-n\rfloor+1 > \frac{5}{2}\dim(F)$ holds. Hence, $[\tau, N^{n+a+1}] = 0$ by corollary 5.66. This implies

$$0 = j_*([\tau, N^{n+a+1}]) = \sum_{k=0}^{n}\Big(\eta^{n-k} \oplus (a+1)\mathbb{R} \to F^k\Big).$$

Thus, for each k, $[\eta^{n-k} \oplus (a+1)\mathbb{R} \to F^k] = 0$ in $MO_k\big(BO(n-k+a+1)\big)$. By lemma 5.34

$$I_*^{a+1} : MO_k\big(BO(n-k)\big) \to MO_k\big(BO(n-k+a+1)\big)$$

is a monomorphism and we obtain $[\eta^{n-k} \to F^k] = 0$ in $MO_k\big(BO(n-k)\big)$. Now theorem 5.41 states that j_* is injective which results in $[T, M^n] = 0$ in $I_n(\mathbb{Z}_2)$.

On the other hand suppose $[T, M^n] = 0$. We obtain $[\eta^{n-k} \to F^k] = 0$ and hence $I_*^i([\eta^{n-k} \to F^k]) = [\eta^{n-k} \oplus i\mathbb{R} \to F^k] = 0 \in MO_k\big(BO(n-k+i)\big)$ for all k and i. This shows that for all i there is an involution whose normal bundle to the fixed point set is $\coprod\limits_{k=0}^{n+i} (\eta^{n-k} \oplus i\mathbb{R} \to F^k)$. □

The map $\Gamma : I_*(\mathbb{Z}_2) \to I_{*+1}(\mathbb{Z}_2)$ provides a criterion that $I_*(\eta)$ still is the fixed data of an involution.

Lemma 5.73. *Let (T, M^n) be an involution on a closed manifold with η the normal bundle to the fixed point set. $I_*^k(\eta)$ is the fixed data of an involution if and only if*

$$[\epsilon(\Gamma^j[T, M^n])] = 0 \quad \textit{if}\, j \le k-1.$$

Proof. This is lemma 2.1 in [12]. □

Lemma 5.74. *Let (T, M^n) be as above. Then*

$$[\epsilon(\Gamma^j[T, M^n])] = 0\ \forall j \le k \quad \textit{if and only if} \quad [P(\eta \oplus \theta^{j+1})] = 0\ \forall j \le k.$$

Proof. We use induction on k and write $A[k]$ for the assertion in the lemma. $A[0]$ follows from $[M^n] = [P(\eta \oplus \theta^1)]$, see theorem 5.28. Now suppose $k \ge 1$. Proposition 5.44 gives us

$$\epsilon(\Gamma^k[T, M^n]) = [P(\eta \oplus \theta^{k+1})] + \sum_{j=0}^{k-1}[\mathbb{R}\mathrm{P}^{k-j}]\epsilon(\Gamma^j[T, M^n]). \tag{5.17}$$

Suppose that we have already shown $A[k-1]$. If $\epsilon(\Gamma^j[T, M^n]) = 0$ for all $j \le k$, then $[P(\eta \oplus \theta^{k+1})] = 0$ by (5.17) and $[P(\eta \oplus \theta^{j+1})] = 0$ for all $j \le k-1$ by induction hypothesis.

Conversely, assume $[P(\eta \oplus \theta^{j+1})] = 0$ for all $j \le k$. Again by (5.17) we find $\epsilon(\Gamma^k[T, M^n]) = \sum\limits_{j=0}^{k-1}[\mathbb{R}\mathrm{P}^{k-j}] \underbrace{\epsilon(\Gamma^j[T, M^n])}_{=0 \text{ by induction hypothesis}} = 0.$ □

Finally, 5.72, 5.73 and 5.74 result in two necessary and sufficient conditions that (T, M^n) bords if $n \le \frac{5}{2}\dim(F)$.

Theorem 5.75. *Again let (T, M^n) be as above and assume $n \leq \frac{5}{2}\dim(F)$. The following are equivalent:*

i) $[T, M^n] = 0$,

ii) $\epsilon(\Gamma^j[T, M^n]) = 0$ *for all* $j \leq \lfloor \frac{5}{2}\dim(F) - n \rfloor$,

iii) $[P(\eta \oplus \theta^{j+1})] = 0$ *for all* $j \leq \lfloor \frac{5}{2}\dim(F) - n \rfloor$.

Proof. . □

If we use the assumptions made in theorem 5.67, we arrive at the following theorem in a completely analogous manner. We only have to replace the constant $\frac{5}{2}$ by 2.

Theorem 5.76. *Let (T, M^n) be an involution on a closed manifold such that the fixed point set is of constant dimension k. If $n \leq 2k$, then $[T, M^n] = 0$ if and only if one (all) of the following conditions holds*

- $\eta^k \oplus (2k - n + 1)\mathbb{R} \to F^k$ *is still the fixed data of an involution,*
- $\epsilon(\Gamma^j[T, M^n]) = 0$ *for all* $j \leq 2k - n$,
- $[P(\eta \oplus \theta^{j+1})] = 0$ *for all* $j \leq 2k - n$.

If one has already determined the fixed point set of an involution, one can deduce facts about the cobordism behaviour of the manifold on which the involution acts. We list several of these theorems without proof. Theorem 5.71 can be used to prove

Theorem 5.77. *Suppose (T, M^n), $n \geq 1$, is an involution on a closed manifold M^n whose fixed point set is diffeomorphic to the disjoint union of a point and a k-sphere. Then $k \in \{1, 2, 4, 8\}$, $n = 2k$ and M^n is cobordant to the real, complex, quaternionic or octionic projective space, depending on the value of k.*

Proof. See [1, theorem 28.4]. □

Robert E. Stong gives a generalization of this.

Theorem 5.78 (Stong)**.** *Suppose (T, M^n) is an involution on a closed manifold whose fixed point set is the union of products of circles $(S^1)^k$, $0 \leq k \leq n$. Then either the involution bords or $n = 2m$ and $[T, M^n] = [(\mathbb{R}\mathrm{P}^2)^m, T \times \cdots \times T]$ where $T : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$, $[x, y, z] \mapsto [-x, y, z]$.*

Proof. See [10]. □

The proof of the following theorem, again due to Robert E. Stong, can be found in [1, theorem 29.1].

Theorem 5.79 (Stong)**.** *Let M^n be a closed manifold. Suppose there is a non-trivial involution on M^n whose fixed point set is $\mathbb{R}\mathrm{P}^{2r}$. Then $n = 4r$ and $[M^n] = [\tau, \mathbb{R}\mathrm{P}^{2r}]^2$ where τ is the involution which interchanges coordinates.*

Bruce F. Torrence states another theorem in this direction.

Theorem 5.80. *If the fixed point set of an involution (T, M^n) on a closed manifold is the disjoint union of odd dimensional projective spaces of constant dimension, then (T, M^n) bounds.*

Proof. See [11]. □

A Appendix: (Co)homology

We briefly explain the boundary homomorphisms, fundamental class of a manifold, cup product, cap product and poincaré duality following [3, Appendix A].

A.1 (Co)Boundary homomorphism

The *standard n-simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ is the convex hull of the standard unit vectors $e_0, \ldots, e_n \in \mathbb{R}^{n+1}$,

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\}$$

In $\mathbb{R}^3$, for example, Δ^2 is just the surface of the triangle with vertices $(1,0,0), (0,1,0)$ and $(0,0,1)$.

Let X be a topological space. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$. Define a linear embedding $\beta_i : \Delta^{n-1} \to \Delta^n$ by

$$\beta_i(t_0, \ldots, t_n) = \beta_i(t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{n-1}).$$

β_i parametrizes the edge opposite the corner e_i. Using these embeddings we can restrict a singular n-simplex σ to Δ^{n-1}, $\sigma \circ \beta_i : \Delta^{n-1} \to X$. The resulting singular $(n-1)$-simplex is called the i-th face of σ.

Definition A.1 (singular homology)**.** *The* ***singular chain group*** $C_n(X; \Lambda)$ *with coefficients in a commutative ring* Λ *is the free module over* Λ *generated by the singular n-simplexes in X. For* $n < 0$ *we put* $C_n(X; \Lambda) = 0$ *(we often write group although module is meant). The* ***boundary homomorphism***

$$\partial_n : C_n(X; \Lambda) \to C_{n-1}(X; \Lambda)$$

is defined by

$$\partial_n(\sigma) := \sum_{i=0}^{n} (-1)^i (\sigma \circ \beta_i).$$

Moreover, we define

$$Z_n(X; \Lambda) := \ker\big(C_n(X; \Lambda) \xrightarrow{\partial_n} C_{n-1}(X; \Lambda)\big)$$

called the group of ***n-cycles*** *and*

$$B_n(X; \Lambda) := \operatorname{img}\big(C_{n+1}(X; \Lambda) \xrightarrow{\partial_n} C_n(X; \Lambda)\big)$$

called the group of ***n-boundaries****. One checks* $\partial_{n-1} \circ \partial_n = 0$*, i.e.* $B_n(X; \Lambda) \subseteq Z_n(X; \Lambda)$*. Hence, we have a chain complex with differential* ∂ *and obtain the* ***n-th singular homology group with coefficients in*** Λ

$$H_n(X; \Lambda) = Z_n(X; \Lambda)/B_n(X; \Lambda).$$

Definition A.2 (singular cohomology)**.** *The* ***singular cochain group*** $C^n(X;\Lambda)$ *is the* Λ*-module Hom(*$C_n(X;\Lambda);\Lambda$*). Again we put* $C^n(X;\Lambda) = 0$ *for* $n < 0$*. We use* ∂ *to define the* ***coboundary homomorphism***

$$\delta^n : C^n(X;\Lambda) \to C^{n+1}(X;\Lambda),\ (\delta^n a)(\alpha) := a(\partial_{n+1}\alpha),$$

where $a \in C^n(X;\Lambda)$ *and* $\alpha \in C_{n+1}(X;\Lambda)$*. We obtain corresponding submodules*

$$Z^n(X;\Lambda) := \ker\big(C^n(X;\Lambda) \xrightarrow{\delta^n} C^{n+1}(X;\Lambda)\big)$$

called the group of ***n-cocycles*** *and*

$$B^n(X;\Lambda) := \operatorname{img}\big(C^{n-1}(X;\Lambda) \xrightarrow{\delta^n} C^n(X;\Lambda)\big)$$

called the group of ***n-coboundaries****. Clearly* $\delta^n \circ \delta^{n-1} = 0$*. Thus, we can define the* ***n-th singular cohomology group with coefficients in*** Λ

$$H^n(X;\Lambda) = Z^n(X;\Lambda)/B^n(X;\Lambda).$$

We denote the value of a cochain a *on a chain* α *by* $\langle a, \alpha\rangle$*.*

A.2 Cup product and Cap product

We omit reference to the coefficient ring Λ and just write $H^i(X)$ for $H^i(X;\Lambda)$. $H^*(X) = (H^1(X), H^2(X), \ldots)$ can be given the structure of a graded commutative ring as follows.
Let $a \in C^m(X)$, $b \in C^n(X)$ be cochains. Consider the maps $\alpha_m : \Delta^m \to \Delta^{m+n}$, $(t_0,\ldots,t_m) \mapsto (t_0,\ldots,t_m,0,\ldots,0)$ and $\beta_n : \Delta^n \to \Delta^{m+n}$, $(t_m,\ldots,t_{m+n}) \mapsto (0,\ldots,0,t_m,\ldots,t_{m+n})$. We define an associative and bilinear multiplication $ab = a \cup b$ by

$$\langle a \cup b, \sigma\rangle = \langle a, \sigma \circ \alpha_m\rangle \cdot \langle b, \sigma \circ \beta_n\rangle \in \Lambda$$

where $\sigma : \Delta^{m+n} \to X$. One checks the identity

$$\delta^{m+n}(a \cup b) = \delta_m a \cup b + (-1)^{|a|} a \cup \delta_n b,$$

which implies the existence of a graded commutative multiplication

$$H^m(X) \otimes H^n(X) \to H^{m+n}(X),$$
$$[a] \otimes [b] \mapsto [a \cup b].$$

The product operation $\cup$ on $H^*(X)$ is called **cup product**.

There is a bilinear pairing

$$\cap : C^i(X) \times C_n(X) \to C_{n-i}(X).$$

Let $b \in C^i(X)$ a cochain and σ a singular n-simplex. Define $\cap$ by

$$b \cap \sigma := (-1)^{i(n-i)}\langle b, \sigma \circ \beta_i\rangle(\sigma \circ \alpha_{n-i}).$$

One could also describe $\cap$ by the following property. For each cochain $b \in C^i(X)$ and each chain $\beta \in C_n(X)$ $b \cap \beta$ is the unique element in $C_{n-i}(X)$ satisfying

$$\langle a, b \cap \beta\rangle = \langle a \cup b, \beta\rangle.$$

for all $a \in C^{n-i}(X)$. From the identity

$$\partial_{n-i}(b \cap \beta) = (\delta^i b) \cap \beta + (-1)^i b \cap (\partial_n \beta)$$

it follows that we can pass to (co)homology and obtain the **cap product**

$$H^i(X) \otimes H_n(X) \to H_{n-i}(X),$$

$$[b] \otimes [\beta] \mapsto [b \cap \beta].$$

A.3 Fundamental homology class and Poincaré duality

Let M be a compact n-manifold. We write $H_n(M)$ for homology with coefficients in $\mathbb{Z}$. Recall

$$H_i(M, M - \{x\}) \cong \begin{cases} \mathbb{Z} & \text{for } i = n \\ 0 & \text{otherwise} \end{cases}$$

Definition A.3. *Given a point x in M there are two possible generators of $H_n(M, M - \{x\}) = \mathbb{Z}$. A* ***local orientation*** *$\mathfrak{o}_x$ for M at x is a choice of one of them.*

Definition A.4. *An orientation for M is a map which assigns to each $x \in M$ a local orientation $\mathfrak{o}_x$ satisfying the following property: for each x there exists a compact neighbourhood N and a class $\mathfrak{o}_N \in H_n(M, M - N)$ such that*

$$(\iota_y^N)_*(\mathfrak{o}_N) = \mathfrak{o}_y$$

for all $y \in N$. Here $(\iota_y^N)_*$ $H_i(M, M - N) \to H_i(M, M - \{y\})$ is the homomorphism induced by the inclusion $\iota_y^N : (M, M - N) \to (M, M - \{y\})$. If an orientation exists, then M is called an orientable manifold.

Theorem A.5 (fundamental homology class)**.** *Let M be an oriented closed n-manifold. There is a unique class $[M]_{\mathbb{Z}} \in H_n(M)$ such that*

$$(\iota_x^M)_*[M]_{\mathbb{Z}} = \mathfrak{o}_x$$

for each $x \in M$. $[M]_{\mathbb{Z}}$ is called the ***fundamental homology class*** *of M.*

There is a similar construction without any assumption about orientability.

Theorem A.6 ($\mathbb{Z}_2$-fundamental homology class)**.** *Let M be a closed n-manifold. There exists a unique class $[M]_{\mathbb{Z}_2} \in H_n(M; \mathbb{Z}_2)$ such that for any $x \in M$ the homomorphism*

$$(\iota_x^M)_* : H_n(M; \mathbb{Z}_2) \to H_n(M, M - \{x\}; \mathbb{Z}_2)$$

*induced by the natural inclusion $\iota_x^M : (M, \emptyset) \to (M, M - \{x\})$ maps $[M]_{\mathbb{Z}_2}$ to the unique non-zero element in $H_n(M, M - \{x\}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. $[M]_{\mathbb{Z}}$ is called the $\mathbb{Z}_2$-**fundamental homology class** of M.*

We also state relative versions with respect to manifolds with boundary. Note that an orientation on M induces an orientation on the boundary ∂M.

Theorem A.7 (fundamental homology class)**.** *Let M be a compact n-manifold with boundary ∂M. There exists a unique homology class $[M]_{\mathbb{Z}} \in H_n(M, \partial M)$ such that the homomorphism*

$$(\iota_x^{\partial M})_* : H_n(M, \partial M) \to H_n(M, M - \{x\})$$

induced by the natural inclusion $(M, \partial M) \to (M, M - \{x\})$ maps $[M]_{\mathbb{Z}}$ to the local orientation $\mathfrak{o}_x$ in $H_n(M, M - \{x\})$ for any x in the interior of M. This class relates to the fundamental homology class of the boundary $[\partial M]_{\mathbb{Z}}$ as follows.

$$\partial([M]_{\mathbb{Z}}) = [\partial M]_{\mathbb{Z}},$$

where $\partial : H_n(M, \partial M) \to H_{n-1}(\partial M)$ denotes the boundary map associated with the pair $(M, \partial M)$.

Theorem A.8 ($\mathbb{Z}_2$-fundamental homology class)**.** *Let M be a compact n-manifold with boundary ∂M. There exists a unique homology class $[M]_{\mathbb{Z}_2} \in H_n(M, \partial M; \mathbb{Z}_2)$ such that the homomorphism*

$$(\iota_x^{\partial M})_* : H_n(M, \partial M; \mathbb{Z}_2) \to H_n(M, M - \{x\}; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

induced by the natural inclusion $(M, \partial M) \to (M, M - \{x\})$ maps $[M]_{\mathbb{Z}_2}$ to the non-trivial element in $H_n(M, M - \{x\}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ for any x in the interior of M. Again we have the relation

$$\partial([M]_{\mathbb{Z}_2}) = [\partial M]_{\mathbb{Z}_2},$$

where $\partial : H_n(M, \partial M; \mathbb{Z}_2) \to H_{n-1}(\partial M; \mathbb{Z}_2)$ denotes the boundary map associated with the pair $(M, \partial M)$.

Theorem A.9 (Poincaré duality)**.** *Let M be a closed oriented manifold. Then $H^i(M)$ is isomorphic to $H_{n-i}(M)$ under the assignment $a \mapsto a \cap [M]_{\mathbb{Z}}$.*

For a non-orientable manifold this theorem remains true if one replaces $\mathbb{Z}$ by $\mathbb{Z}_2$.

A.4 Stiefel-Whitney classes

Stiefel-Whitney classes originated as obstructions to the existence of linearly independent sections. We introduce four axioms SW1 to SW4 characterizing Stiefel-Whitney cohomology classes. For proofs of existence and uniqueness of these classes we refer to [3].

SW1. To each vector bundle $\xi = (p, E, B)$ there is a sequence of cohomology classes

$$w_i(\xi) \in H^i(B; \mathbb{Z}_2),\ i = 0, 1, 2, \ldots$$

called the *Stiefel-Whitney classes* of ξ. The class $w_0(\xi)$ equals $1 \in H^0(B; \mathbb{Z}_2)$ and $w_i(\xi)$ is zero for i larger than the rank of ξ.

SW2 (Naturality). Let $\xi' = (p', E', B')$ be another vector bundle. If $f : B \to B'$ is covered by a bundle map from ξ to ξ'

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

then

$$f^* w_i(\xi') = w_i(\xi) \text{ for all } i.$$

SW3 (Whitney product theorem). If ξ and ξ' are vector bundles over the same base space, then

$$w_n(\xi \oplus \xi') = \sum_{i=0}^{n} w_i(\xi) \cup w_{n-i}(\xi').$$

SW4. For the line bundle $\gamma_{1,2}$ over $G_{1,2}(\mathbb{R}) = \mathbb{R}\mathrm{P}^1$, $w_1(\gamma_{1,2})$ is non-zero.

We immediately obtain

Proposition A.10. *If ξ is isomorphic to ξ', then $w_i(\xi) = w_i(\xi')$ for all i.*

Proposition A.11. *If ξ is a trivial vector bundle, the Stiefel-Whitney numbers $w_i(\xi)$ vanish for $i > 0$. Thus, $w_i(\xi \oplus \eta) = w_i(\eta)$ for any vector bundle η.*

Proof. Clearly, this is true if ξ is a vector bundle over a point because then $H^n(B(\xi)) = 0$ for $n > 0$. The general case follows since, if ξ is trivial, there exists a bundle map from ξ to a vector bundle over a point. The second statement is immediate from SW3. □

Proposition A.12. *If ξ is a vector bundle of rank n which admits k everywhere linearly independent cross sections, then the top k Stiefel-Whitney classes $w_{n-k+1}(\xi), w_{n-k+2}(\xi), \ldots, w_n(\xi)$ vanish.*

Proof. By proposition 1.20 there is a k-dimensional trivial subbundle η and ξ splits as a Whitney sum $\xi = \eta \oplus \eta^\perp$. Hence, $w_i(\xi) = w_i(\eta \oplus \eta^\perp) \overset{A.4}{=} w_i(\eta^\perp)$ and $\operatorname{rank}(\eta^\perp) = n - k$. □

Definition A.13 (total Stiefel-Whitney class)**.** *Let ξ be a vector bundle. We denote by $H^{\prod}(B(\xi);\mathbb{Z}_2)$ the ring of all formal infinite series*

$$a = a_0 + a_1 + a_2 + \ldots$$

where $a_i \in H^i(B(\xi);\mathbb{Z}_2)$. The product operation in this ring is given by

$$(a_0 + a_1 + \ldots) \cdot (b_0 + b_1 + \ldots) = (a_0 b_0) + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \ldots$$

*The **total Stiefel-Whitney class** of an n-plane bundle ξ is the element*

$$w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \ldots + w_n(\xi) + 0 + \ldots$$

in this ring. We write $w(M)$ for the total Stiefel-Whitney class of the tangent bundle of a manifold M.

Remark A.14. We can now express the Whitney product theorem by the simple formula

$$w(\xi \oplus \xi') = w(\xi) \cdot w(\xi').$$

There are several possibilities to construct Stiefel-Whitney classes. In the case of line bundles, this may be done as follows. Since the infinite projective space $\mathbb{R}\mathrm{P}^\infty$ is the Eilenberg-Maclane space $K(\mathbb{Z}_2, 1)$ (it is double covered by the contractible space S^∞), for any space X there is a bijection between the homotopy classes $[X, \mathbb{R}\mathrm{P}^\infty]$ and the first cohomology $H^1(X;\mathbb{Z}_2)$ given by $[f] \mapsto f^*(a)$. Here a denotes the non-zero element in $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$. From theorem 2.22 we obtain another bijection between $[X, \mathbb{R}\mathrm{P}^\infty]$ and $\mathcal{B}^1(X)$ the isomorphism classes of line bundles over X. Thus, there is a bijection

$$w_1 : \mathcal{B}^1(X) \to H^1(X;\mathbb{Z}_2)$$

which defines the first Stiefel-Whitney class for line bundles. Note that since $H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) = \mathbb{Z}_2$ there are only two line bundles over $\mathbb{R}\mathrm{P}^n$ up to isomorphism. These are the trivial bundle and the twisted line bundle (which is isomorphic to the tautological line bundle). Furthermore, the tensor product of line bundles endows $\mathcal{B}^1(X)$ with a group structure and the bijection w_1 becomes an isomorphism, that is (compare [6, 17.3.4])

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta). \tag{A.1}$$

Now turn to the general case. We briefly describe the approach which uses classifying spaces, see for example [18, chapter 23]. Let $R[x_1, \dots, x_n]$ denote the ring of polynomials in n variables, where R is a commutative ring with 1. Recall that a polynomial $p(x_1, \dots, x_n)$ is called *symmetric* if $p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = p(x_1, \dots, x_n)$ for any permutation $\sigma \in \Sigma_n$. The k-th *elementary symmetric function* $\sigma_k(x_1, \dots, x_n)$ is the unique homogeneous polynomial of degree k such that

$$\sum_{0 \le k \le n} \sigma_k(x_1, \dots, x_n) = \prod_{1 \le k \le n} (1 + x_i)$$

Note that σ_k is again symmetric. A fundamental theorem states that the polynomials σ_k are algebraically independent and the subring of symmetric functions is the polynomial ring $R[\sigma_1, \dots, \sigma_n]$.

Example A.15. There exist for each k symmetric polynomials s_k such that

$$\sum_{i=1}^{k} x_i^k = s_k(\sigma_1, \dots, \sigma_n)$$

Consider now the orthogonal group $O(n)$. Since the eigenvalues of an orthogonal map are ± 1, the subgroup of diagonal matrices may be identified with $(\mathbb{Z}_2)^n = O(1)^n$. The classifying space $BO(1)$ is $\mathbb{R}\mathrm{P}^\infty$. The inclusion $O(1)^n \to O(n)$ induces a map (note that $B(G_1 \times G_2) = BG_1 \times BG_2$)

$$\omega : (\mathbb{R}\mathrm{P}^\infty)^n \cong B(O(1)^n) \longrightarrow BO(n)$$

and hence a homomorphism

$$\omega^* : H^*(BO(n);\mathbb{Z}_2) \longrightarrow H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2).$$

It follows from the Künneth theorem that $H^*((\mathbb{R}\mathrm{P}^\infty)^n;\mathbb{Z}_2) = \mathbb{Z}_2[a_1, \dots, a_n]$ is a polynomial algebra on n generators of degree one. As it turns out ω^* is a monomorphism with image the subring of symmetric polynomials in the variables a_i. Hence, there are unique cohomology classes $w_k \in H^k(BO(n);\mathbb{Z}_2)$, $0 \leq k \leq n$ such that $\omega^*(w_k)$ is the k-th elementary symmetric function. These are the Stiefel-Whitney classes of $BO(n)$. Given an n-plane bundle $\xi = (p, E, B)$, there is a classifying map $f : B \to BO(n)$. The image of w_k under the induced homomorphism $f^* : H^*(BO(n);\mathbb{Z}_2) \to H^*(B;\mathbb{Z}_2)$ is then the k-th Stiefel-Whitney class of ξ. This is well-defined since classifying maps are unique up to homotopy and homotopic maps induce the same homomorphism in cohomology. From the theorem cited above, we conclude

$$H^*(BO(n);\mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$$

How can we compare Stiefel-Whitney classes of two different manifolds? We associate numbers to them as follows. Let $[a] \in H^n(M;\mathbb{Z}_2)$ and $[\alpha] \in H_n(M;\mathbb{Z}_2)$. Define the *Kronecker index* $\langle [a], [\alpha] \rangle := \langle a, \alpha \rangle \in \mathbb{Z}_2$. This is easily seen to be well-defined.

Let $r_1, \ldots, r_n$ be non-negative integers with $r_1 + 2r_2 + \ldots + nr_n = n$. Then for the tangent bundle $\tau_M : TM \to M$ of a manifold M we can form the element

$$w_1(\tau_M)^{r_1} w_2(\tau_M)^{r_2} \cdots w_n(\tau_M)^{r_n} \in H^n(\tau_M;\mathbb{Z}_2).$$

If $\omega = (i_1 \leq i_2 \leq \ldots \leq i_k)$ is a sequence of non-negative integers, we put $|\omega| = \sum_{l=1}^{k} i_l$. To each such sequence there is associated a product of Stiefel-Whitney classes

$$W_\omega = w_{i_1} \cdots w_{i_k} \in H^{|\omega|}(M;\mathbb{Z}_2).$$

Definition A.16. *The integer modulo 2*

$$\langle w_1(\tau_M)^{r_1} w_2(\tau_M)^{r_2} \cdots w_n(\tau_M)^{r_n}, [M]_{\mathbb{Z}_2} \rangle$$

is called the ***Stiefel-Whitney number*** *of M associated with the partition $r_1 + 2r_2 + \ldots + nr_n = n$.*

Theorem A.17. *A vector bundle ξ is orientable if and only if $w_1(\xi) = 0$.*

Summary

This thesis mainly consists of two parts. The first one deals with the general concept of cobordism. It is due to René Thom that cobordism groups can be computed with the help of homotopy theory. The main topic of the second chapter are involutions on closed manifolds. Our aim is to present a theorem of J.M. Boardman which provides a lower bound for the dimension of the fixed point set of a non-bounding involution. Here, our exposition mainly follows the book of Conner and Floyd [1].

In the first chapter we recall various definitions and facts about manifolds and vector bundles which are fundamental for this thesis.

Chapter two is devoted to the study of unoriented cobordism groups. Closed manifolds are partitioned into equivalence classes. Two n-manifolds are called cobordant if their disjoint union is the boundary of a compact $(n+1)$-manifold. The set of all equivalence classes can be made into an abelian group, denoted MO_n. We also mention oriented cobordism groups MSO_n but concentrate on the unoriented case. As an illustration we compute MO_n and MSO_n for $n \leq 2$. The main part of this chapter tries to relate homotopy theory and the theory of cobordism. Grassmann manifolds are introduced in order to construct the so-called Thom homomorphism $\pi_{n+k}(E^*_{k,s}) \to MO_n$. Here, $E^*_{k,s}$ denotes the Thom space of the universal vector bundle over the Grassmannian $G_{k,s}$. The Pontrjagin-Thom construction is used to define an inverse to this map resulting in an isomorphism $\pi_{n+k}(E^*_{k,s}) \cong MO_n$ provided k and s are large enough. Thus, the task of computing cobordism groups is reduced to a homotopy theoretical problem.

In chapter three we consider cobordism theory from the homological point of view. We adopt the notion of singular manifolds. By proving the Eilenberg-Steenrod axioms we show that cobordism constitutes a generalized homology theory.

In chapter four we consider Stiefel-Whitney numbers. We prove that all Stiefel-Whitney numbers of a bounding manifold are zero. It is a theorem of Thom that the converse holds too. We hence arrive at a necessary and sufficient condition that two manifolds are cobordant. As a corollary we find that the unoriented cobordism group MO_n is finite. The computation of Stiefel-Whitney classes of $\mathbb{R}\mathrm{P}^n$ allows to apply the results of this section to the real projective space. We prove that $\mathbb{R}\mathrm{P}^n$ bords if and only if n is odd.

Chapter five deals with involutions on closed manifolds. We apply the concept of equivariant cobordism to the study of the fixed point data of such periodic maps. As an application we prove a theorem by Wall stating that for any closed manifold M the product $M \times M$ is cobordant to an orientable manifold.

A special case is $[\mathbb{R}\mathrm{P}^n]^2 = [\mathbb{C}\mathrm{P}^n]$. We develop tools to analyse the unoriented cobordism algebra $I_*(\mathbb{Z}_2)$. The most important one is a map $j_* : I_*(\mathbb{Z}_2) \to \mathcal{M}_*$ which assigns to an involution (T, M) the (stable) normal bundle over its fixed point set. It turns out that j_* is a monomorphism which means that up to a cobordism an involution is uniquely determined by the cobordism class of the normal bundle to its fixed point set. Another very important map of degree 1 $\Gamma : I_*(\mathbb{Z}_2) \to I_*(\mathbb{Z}_2)$ is constructed. Γ is used to factor out a suitable ideal of $I_*(\mathbb{Z}_2)$. In doing so we arrive at the quotient algebra $\Lambda(\mathbb{Z}_2)$. Two filtrations are introduced on $\Lambda(\mathbb{Z}_2)$. The first one is an increasing filtration depending on the normal bundle to the fixed point set of an involution. The other comes from a filtration of $MO_*[[t]] = \{\sum_{k=0}^{\infty}[M^k]t^k,\ [M^k] \in MO_k\}$ the ring of homogeneous power series over MO_*. It is given by $\mathrm{Fil}_\phi(x) = n$ if the first nonzero coefficient in the power series for x is the coefficient of t^n. A monomorphism $\phi : \Lambda(\mathbb{Z}_2) \to MO_*[[t]]$ is presented. After exhibiting a set of generators for $\Lambda(\mathbb{Z}_2)$ as a polynomial algebra over $\mathbb{Z}_2$ we prove two versions of Boardman's theorem: If a closed involution (T, M^n) does not bound, then $n \leq \frac{5}{2}\dim(F)$. If the Euler characteristic of M is odd, this may be improved to $n \leq 2\dim(F)$ for any involution. We use Boardman's theorem to give three necessary and sufficient conditions that a closed involution with $n \leq \frac{5}{2}\dim(F)$ bords.

Zusammenfassung (Deutsch)

Die vorliegende Arbeit besteht hauptsächlich aus zwei Teilen. Der erste behandelt allgemein die Theorie der Kobordismen. Es ist der Arbeit von René Thom zu verdanken, dass man Kobordismengruppen mit Hilfe von Homotopietheorie berechnen kann. Der zweite Teil beschäftigt sich mit Involutionen auf geschlossenen Mannigfaltigkeiten. Unser Ziel ist ein Theorem von J.M. Boardman, welches eine untere Schranke für die Dimension der Fixpunktmenge einer nicht nullbordanten Involution liefert. Hier folgt unsere Darstellung dem Buch von Conner und Floyd [1].

Das erste Kapitel dient zur Wiederholung von Definitionen und Sätzen aus dem Gebiet der Mannigfaltigkeiten und Vektorbündel, die für diese Arbeit wichtig sind.

Kapitel zwei widmet sich dem Studium der unorientierten Kobordismengruppen. Wir führen eine Äquivalenzrelation auf den kompakten Mannigfaltigkeiten ohne Rand ein. Zwei Mannigfaltigkeiten der Dimension n heißen kobordant, falls ihre Vereinigung den Rand einer kompakten $(n+1)$-dimensionalen Mannigfaltigkeit bildet. Wir versehen die Menge der Äquivalenzklassen mit der Struktur einer abelschen Gruppe, die mit MO_n bezeichnet wird. Nebenbei erwähnen wir auch orientierte Kobordismengruppen MSO_n, konzentrieren uns aber auf den unorientierten Fall. Als Beispiel soll die Berechnung der Gruppen MO_n bzw. MSO_n für $n \leq 2$ dienen. Den größten Teil dieses Kapitels verwenden wir, um eine Verbindung von Homotopietheorie und Kobordismentheorie aufzuzeigen. Wir führen Grassmann Mannigfaltigkeiten ein, um den sogenannten Thom-Homomorphismus $\pi_{n+k}(E^*_{k,s}) \to MO_n$ zu konstruieren. $E^*_{k,s}$ bezeichnet hier den Thom-Raum des universellen Vektorbündels über der Grassmann-Mannigfaltigkeit $G_{k,s}$. Die Pontrjagin-Thom Konstruktion führt zu einer zum Thom-Homomorphismus inversen Abbildung. Wir erhalten einen Isomorphismus $\pi_{n+k}(E^*_{k,s}) \cong MO_n$, vorausgesetzt k und s sind groß genug. Die Berechnung der Kobordismengruppen kann also auf ein homotopietheoretisches Problem zurückgeführt werden.

In Kapitel drei wird gezeigt, dass dieses Konzept auch als verallgemeinerte Homologietheorie verstanden werden kann. Wir definieren den Begriff der singulären Mannigfaltigkeit und beweisen die Eilenberg-Steenrod Axiome.

Kapitel vier soll die Verwendung von Stiefel-Whitney Zahlen illustrieren, um die Kobordismenklasse von Mannigfaltigkeiten zu berechnen. Wir zeigen, dass die Stiefel-Whitney Zahlen einer nullbordanten Mannigfaltigkeit alle null sind. Thom hat bewiesen, dass auch die Umkehrung stimmt. Wir erhalten also eine notwendige und hinreichende Bedingung, wann zwei Mannigfaltigkeiten kobordant sind. Als Korollar schließen wir, dass die unorientierte Kobordismengruppe

MO_n endlich ist. Die Berechnung der Stiefel-Whitney Klassen des $\mathbb{R}\mathrm{P}^n$ erlaubt uns, die Resultate dieses Abschnitts auf den reell projektiven Raum anzuwenden. Wir zeigen, dass $\mathbb{R}\mathrm{P}^n$ genau für ungerade n nullbordant ist.

Kapitel fünf beschäftigt sich schließlich mit Involutionen auf geschlossenen Mannigfaltigkeiten. Wir verwenden äquivariante Kobordismengruppen, um Fixpunktmengen von Involutionen zu studieren. Als Anwendung beweisen wir einen Satz von Wall. Dieser besagt, dass für jede geschlossene Mannigfaltigkeit M das Produkt $M \times M$ kobordant zu einer orientierten geschlossenen Mannigfaltigkeit ist. Ein Spezialfall davon ist $[\mathbb{R}\mathrm{P}^n]^2 = [\mathbb{C}\mathrm{P}^n]$. Weiters widmen wir uns der Analyse der unorientierten Kobordismenalgebra $I_*(\mathbb{Z}_2)$. Wir definieren eine Abbildung $j_* : I_*(\mathbb{Z}_2) \to \mathcal{M}_*$, die jeder Involution (T, M) im Wesentlichen das Normalenbündel der Fixpunktmenge zuordnet. Es stellt sich heraus, dass j_* injektiv ist. Somit ist jede Involution (bis auf einen Kobordismus) eindeutig durch die Kobordismenklasse des Normalenbündels der Fixpunktmenge bestimmt. Eine weitere Abbildung $\Gamma : I_*(\mathbb{Z}_2) \to I_*(\mathbb{Z}_2)$ vom Grad 1 wird konstruiert. Wir verwenden Γ um die Algebra $I_*(\mathbb{Z}_2)$ nach einem Ideal zu faktorisieren. Im folgenden filtrieren wir den resultierenden Quotienten auf zwei unterschiedliche Arten. Die erste ist eine aufsteigende Filtrierung, die vom Normalenbündel der Fixpunktmenge einer Involution abhängt. Die zweite wird durch eine Filtrierung von $MO_*[[t]] = \{\sum_{k=0}^{\infty} [M^k]t^k,\ [M^k] \in MO_k\}$ induziert. Genauer ist $\mathrm{Fil}_\phi(x) = n$, falls sich der erste nichttriviale Koeffizient in der Potenzreihendarstellung von x bei t^n befindet. Wir beweisen die Existenz einer injektiven Abbildung $\phi : \Lambda(\mathbb{Z}_2) \to MO_*[[t]]$. Mit Hilfe von Erzeugern der Polynomalgebra $\Lambda(\mathbb{Z}_2)$ über $\mathbb{Z}_2$ beweisen wir zwei Versionen des Satzes von Boardman: Ist eine geschlossene Involution (T, M^n) nicht nullbordant, so gilt die Abschätzung $n \leq \frac{5}{2}\dim(F)$. Falls die Euler-Charakteristik von M ungerade ist, kann dies sogar zu $n \leq 2\dim(F)$ verbessert werden. Schlussendlich verwenden wir diesen Satz, um drei notwendige und hinreichende Bedingungen dafür zu geben, dass eine geschlossene Involution mit $n \leq \frac{5}{2}\dim(F)$ nullbordant ist.

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Curriculum vitae

Personal information

Name	Manuel Radler
Date of birth	September 19, 1985
Place of birth	Linz, Austria
Address	Lienfeldergasse 4/5/16, 1160 Vienna
Email	radlermanuel@gmail.com

Education

1/2009 – 3/2012	University of Vienna second part of mathematical studies specialization: geometry and topology, diploma thesis: 'Cobordism and fixed point sets of involutions' advisor: Prof. Stefan Haller
10/2006 – 1/2009	University of Vienna first part of mathematical studies with distinction
9/2005 – 4/2006	Military service
9/2000 – 6/2005	BHAK Auhof Matura (A-levels) with distinction.
9/1996 – 7/2000	Stiftsgymnasium Wilhering

Experience

10/2008 – 2/2012	undergraduate teaching assistant at University of Vienna
7/2010 – 8/2010	insurance company UNIQA (risk management) paper about actuarial reserves

Scholarships

Performance scholarships awarded by the University of Vienna for the academic years 2007 – 2008, 2008 – 2009 and 2009 – 2010.