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DIPLOMARBEIT

Titel der Diplomarbeit

Projective Descriptions of Inductive Limits of Locally Convex Vector Spaces

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angestrebter akademischer Grad

Magister der Naturwissenschaften (Mag. rer. nat.)

Wien, im Juni 2012

Studienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Diplomstudium Mathematik
Betreuer:	Univ.-Prof. Dr. Andreas Kriegl

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Abstract

An inductive limit of locally convex vector spaces is usually difficult to describe in a way that allows calculations in practice. In the present thesis the concept of *projective description* is introduced as a remedy. Furthermore, such descriptions are shown to exist for certain classes of weighted sequence spaces and weighted spaces of continuous functions. Among other things, using these descriptions one can find simple characterizations of the family of seminorms—and thus of the topology corresponding to such inductive limit spaces.

Zusammenfassung

Induktive Limiten von lokalkonvexen Vektorräumen sind für gewöhnlich schwierig auf eine Weise zu beschreiben, die es erlaubt konkrete Rechnungen durchzuführen. Als Hilfsmittel wird in vorliegender Arbeit das Konzept der *projektiven Darstellung* eingeführt. Außerdem wird gezeigt, dass solche Beschreibungen für gewisse Klassen gewichteter Folgenräume und gewichteter Räume stetiger Funktionen tatsächlich existieren. Unter anderem ermöglicht dies eine einfache Charakterisierung der Halbnormenfamilie – und damit der Topologie eines solchen induktiven Limiten.

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Introduction

1.1 Locally Convex Vector Spaces

We will give a short overview on the basic theory of locally convex vector spaces and state the most important facts. Proofs and more detailed information can be found in [Sch71, FW68, Jar81, Köt69, Bou81, MV97, Ste, Wer07]. The approach we are using here is mostly inspired by [Sch71], [Ste] and [FW68], while the details have been taken from [Jar81].

Every locally convex vector space is, as we will see shortly, a topological vector space:

1.1 Definition (Topological Vector Spaces). Let E denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then the pair (E, τ), where τ denotes a topology on E such that

$$\text{(TVS1)} \quad E \times E \to E,\ (x, y) \mapsto x + y \quad \text{is continuous and}$$

$$\text{(TVS2)} \quad \mathbb{K} \times E \to E,\ (\lambda, x) \mapsto \lambda x \quad \text{is continuous,}$$

is called a *topological vector space*.

Two topological vector spaces (E_1, τ_1) and (E_2, τ_2) are called *isomorphic*, in symbols $(E_1, \tau_1) \cong (E_2, \tau_2)$ or simply $E_1 \cong E_2$, if there exists an algebraic (linear) isomorphism $\varphi : E_1 \to E_2$ which is also a homeomorphism. We will simply call φ a *(topological) isomorphism* of E_1 and E_2.

To characterize the so-called *linear* topologies of topological vector spaces (and later those of locally convex vector spaces) we need some notions for special subsets of vector spaces:

1.2 Definition. Let V be a vector space over $\mathbb{K}$. A subset A of V is called *symmetric* (*circled*) if $\lambda A \subseteq A$ for each $|\lambda| \leqslant 1$. It is called *absorbing* if for every $x \in V$ there exists a $\lambda_0 \in \mathbb{K}$ such that $x \in \lambda A$ holds whenever $|\lambda| \geqslant |\lambda_0|$.

The set A is called *convex* whenever $x, y \in A$ implies $\lambda x + (1-\lambda) y \in A$ for any $0 < \lambda < 1$. Finally, we call A *absolutely convex* if for any $x, y \in A$ we have that $\lambda x + \mu y \in A$ whenever $|\lambda| + |\mu| \leqslant 1$ (this is equivalent to A being symmetric and convex [Jar81, sec. 6.1, Prop. 1]).

1.3 Proposition. (See [Sch71, ch. I, § 1.2, p. 14f].)
Let E denote a vector space over $\mathbb{K}$.

(i) A topology τ on E satisfies $(\mathrm{TVS1})$ and $(\mathrm{TVS2})$ (i. e., τ is a linear topology and the pair (E, τ) is a topological vector space) if and only if τ is translation invariant and possesses a 0-neighborhood base $\mathcal{U}$ with the following properties:

  (a) for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V + V \subseteq U$,

  (b) every $U \in \mathcal{U}$ is symmetric and absorbing.

(ii) If $\mathcal{U}$ is a filter base in E having the properties (a) and (b), there exists a unique topology τ on E turning (E, τ) into a topological vector space such that $\mathcal{U}$ is a neighborhood base of 0 for τ.

1.4 Definition (Duals). Let (E, τ) be a topological vector space. The *(topological) dual* E' is the vector space of all continuous linear functionals on E (with respect to τ and the usual topology on $\mathbb{K}$). It is a subspace of the *algebraic dual* E^* of E which is the vector space of *all* linear functionals on E.

We can turn E' into a topological vector space by equipping it with the *weak topology* $\sigma(E', E)$ of pointwise convergence on E, we then write E'_σ for $(E', \sigma(E', E))$ (but see also section 1.3→24).

1.5 Definition (Locally Convex Vector Spaces). Let (E, τ) be a topological vector space over $\mathbb{K}$. We call its (not necessarily separated) topology τ *locally convex* if every neighborhood of any $x \in E$ contains a convex neighborhood of x.

On the other hand, the pair (E, τ) will only be called a *locally convex (topological) vector space* if the topology τ, in addition to being locally convex, is also Hausdorff. This is equivalent to asking for the convex neighborhoods of 0 to form a base of neighborhoods with intersection $\{0\}$.

Similarly to Proposition 1.3 we obtain the following characterisation for locally convex topologies:

1.6 Proposition. (See [Sch71, p. 48], Proposition 1.3.)
Let E denote a vector space over $\mathbb{K}$.

(i) A topology τ on E is locally convex if and only if τ is translation invariant and possesses a 0-neighborhood base $\mathcal{U}$ with the following properties:

  (a) for each $U \in \mathcal{U}$ we also have $\frac{1}{2}U \in \mathcal{U}$,

  (b) every $U \in \mathcal{U}$ is absolutely convex and absorbing.

(ii) If $\mathcal{U}$ is a filter base in E having the properties (a) and (b), there exists a unique locally convex topology τ on E such that $\mathcal{U}$ is a neighborhood base of 0 for τ.

1.7 Example. Every normed vector space over $\mathbb{K}$, in particular any Banach or Hilbert space, is a locally convex vector space (the multiples of the closed unit balls form a base of neighborhoods of 0 which has all the needed properties).

The obvious question which now arises is, how far can the famous theorems of Banach and Hilbert spaces be generalized to the setting of locally convex vector spaces? A first step in this direction is to understand how locally convex vector spaces can be described analytically (i. e., by (semi)norms) as opposed to the "topological" description of Proposition 1.6↑12. The answer will also motivate why we looked at locally convex vector spaces in the first place instead of working with the more general topological vector spaces:

1.8 Definition (Vector Spaces with Seminorms). The pair $(E, \mathcal{P})$, where E is a vector space over $\mathbb{K}$ and $\mathcal{P}$ is a family of seminorms on E (non-negative, positive-homogeneous functionals on E that satisfy the triangle inequality), is called a *vector space with seminorms*.

There are many ways we could obtain a topology on a vector space with seminorms, luckily they all coincide, so that we can talk about *the* topology $\tau_{\mathcal{P}}$ of a vector space with seminorms:

1.9 Lemma. [Sch71, p. 48]
Let $(E, \mathcal{P})$ be a vector space with seminorms, then the following topologies coincide:

(i) the linear topology generated by the base of 0-neighborhoods

$$\mathcal{U} := \{U_{\varepsilon,P} \mid \varepsilon > 0,\ P \subseteq \mathcal{P} \text{ fi}\},$$

where $U_{\varepsilon,P} := \{x \in E \mid p(x) < \varepsilon\ \forall p \in P\}$,

(ii) the weakest topology on E such that all seminorms $p \in \mathcal{P}$ are continuous.

In this topology, we have the following convergence for a net $(x_\lambda)_{\lambda\in\Lambda}$ in E:

$$x_\lambda \xrightarrow{\tau_{\mathcal{P}}} x \iff \forall p \in \mathcal{P}\colon\ p(x_\lambda - x) \to 0.$$

So, every vector space with seminorms is a topological vector space, and because of the triangle inequality, the topology has to be locally convex (the sets in the base of neighborhoods $\mathcal{U}$ above are obviously convex). What makes locally convex vector spaces interesting is, that the converse holds as well:

1.10 Proposition. (See [Ste, Thm. 2.26].)
Let (E, τ) be a locally convex vector space. Then there exists a family of seminorms $\mathcal{P}$ on E such that the topology $\tau_{\mathcal{P}}$ of the vector space with seminorms $(E, \mathcal{P})$ equals τ. In fact, we may choose

$$\mathcal{P} := \operatorname{cs}(E) := \{\text{all seminorms on } E \text{ that are continuous with respect to } \tau\}.$$

Of course it would be nice to have a more explicit description of the family of seminorms, in order to be able to use the tools of analysis to derive facts about locally convex vector spaces. For example, using seminorms, one can easily define a notion of Cauchy-nets without having to resort to uniform structures. It is the goal of this thesis to find "analytic" descriptions of the families of seminorms for specific classes of locally convex vector spaces.

1.11 Remark. With the description of locally convex vector spaces using seminorms we can identify the objects of classical analysis. A locally convex vector space (E, τ) is *metrizable* if its topology can be described by a countable family of seminorms, it is *normable* if a single norm suffices. We will call a metrizable and complete locally convex vector space a *Fréchet space* (or *(F)-space*), if it is even normable we will call it a *Banach space* (or *(B)-space*).

Another motivation for working with locally convex vector spaces is the following result (which doesn't hold for general topological vector spaces, consider for example $(L^p)' = \{0\}$ where $0 < p < 1$):

1.12 Theorem (Hahn-Banach). (See [Sch71, ch. II, § 4.2, Thm.].)
Let (E, τ) be a topological vector space with locally convex topology. Then, for every continuous and linear functional $f : M \to \mathbb{K}$ defined on a subspace M of E, there exists a linear and *continuous* extension $f : E \to \mathbb{K}$ to the whole space E. In particular, for any $0 \neq x \in E$ there exists an $f \in E'$ such that $f(x) \neq 0$.

But we don't get everything for free, for example in classical functional analysis one often uses the fact that continuity and boundedness of operators is equivalent. Unfortunately this is not the case with general locally convex vector spaces.

1.13 Definition (Boundedness). Let (E, τ) denote a topological vector space. A set $B \subseteq E$ is called *(τ-)bounded* if it is absorbed by any 0-neighborhood of E, i. e., if for any 0-neighborhood $U \subseteq E$ there exists a scalar $\lambda_0 \in \mathbb{K}$ such that $B \subseteq \lambda U$ for any $|\lambda| \geqslant |\lambda_0|$.

Let (F, τ') be another topological vector space. A linear map $f : E \to F$ is called *(τ-τ'-) bounded* if it maps τ-bounded sets to τ'-bounded sets.

For locally convex vector spaces boundedness can be equivalently but more intuitively defined by calling a set bounded if all seminorms are bounded on it. When its bounded maps behave as in classical functional analysis, we call a locally convex vector space *bornological*:

1.14 Definition (Bornological Spaces). A locally convex vector space (E, τ) is called *bornological* if for any other locally convex vector space (F, τ'), a linear map $f : E \to F$ is continuous if and only if it is bounded.

In topological terms this can be formulated as follows.

1.15 Lemma. (See [Sch71, p. 61].)
A locally convex vector space is bornological if and only if every absolutely convex bornivorous set is a 0-neighborhood. (A set is called *bornivorous* if it absorbs every bounded set.)

1.16 Proposition. (See [MV97, Prop. 24.13].)
Every metrizable locally convex vector space is bornological.

1.2 Projective and Inductive Limits

Now that we combined the theory of vector spaces with general topology we have several possibilities of generating either new vector spaces or new topological spaces out of given topological vector spaces. In this section we will discuss the question whether this can be done in a way such that the resulting parts can be rejoined to again form a topological vector space.

In both cases, the operations at our disposal include products, subspaces, sums and quotients. But in general topology one can also transport the topological structure along maps between the spaces, generating the so-called final and initial topologies. This concept can be generalized such as to include the other operations mentioned above as special cases, therefore we will look into this version in more detail. First, we'll recall the definition of these topologies:

1.17 Definition. Let X be a set, given a family $(Y_i, \tau_i)_{i \in I}$ of topological spaces and corresponding maps $f_i : X \to Y_i$ we can define the *initial topology* on X to be the weakest (coarsest) topology such that all the mappings f_i are continuous.

For the dual concept let Y be a set and $(X_i, \tau_i)_{i \in I}$ be a family of topological spaces with maps $f_i : X_i \to Y$. Then the strongest (finest) topology on Y turning all f_i continuous is called the *final topology*.

It turns out, that in the setting of locally convex vector spaces, these two concepts are not really dual, in the sense that it is much easier to work with the one than the other. Let's do the easy one first:

1.2.1 Projective Limits

To generalize the concept of initial topologies to locally convex vector spaces we have to consider at least three possibly different topologies, luckily it is easy to choose the right one:

1.18 Lemma & Definition. (See [Ste, § 3.1] and [Jar81, sec. 2.4 and sec. 6.6, Prop. 2].) Let E denote a vector space, $(E_i, \tau_i)_{i\in I}$ a family of topological vector spaces E_i with locally convex topologies τ_i and $(f_i)_{i\in I}$ a family of linear maps $f_i : E \to E_i$. Then the following topologies on E (exist and) coincide:

(i) the weakest locally convex topology such that all f_i are continuous,

(ii) the weakest linear topology such that all f_i are continuous,

(iii) the initial topology with respect to $(E_i, \tau_i)_{i\in I}$ and $(f_i)_{i\in I}$.

This topology is called the *(locally convex) projective topology* on E with respect to $(E_i, \tau_i)_{i\in I}$ and $(f_i)_{i\in I}$. It is generated by the following base of neighborhoods of 0,

$$\mathcal{U} := \Big\{ \bigcap_{j=1}^{n} f_{i_j}^{-1}(U_j) \;\Big|\; U_j \in \mathcal{U}_{i_j},\ i_1, \dots, i_n \in I \Big\},$$

where $\mathcal{U}_i$ is a base of neighborhoods of 0 for τ_i, or alternatively by the family of seminorms

$$\mathcal{P} := \{ p \circ f_i \mid p \in \mathcal{P}_i,\ i \in I \},$$

where $\mathcal{P}_i$ is a family of seminorms generating τ_i.

Proof. Since the topology τ_i on E_i that is generated by the base of 0-neighborhoods $\mathcal{U}_i$ equals the one generated by the family of seminorms $\mathcal{P}_i$, also $\mathcal{U}$ and $\mathcal{P}$ give rise to the same locally convex topology τ on E. We have to show that this topology equals those of (i), (ii) and (iii) which we will denote by $\tau_{\text{(i)}}$, $\tau_{\text{(ii)}}$ and $\tau_{\text{(iii)}}$, respectively.

First we may observe that all f_i are continuous with respect to τ

$$\begin{aligned} x_\lambda \xrightarrow{\tau} x \overset{\text{(ii) } 1.9_{13}}{\Longleftrightarrow} & \ (p \circ f_i)(x_\lambda - x) \to 0 \quad \forall p \in \mathcal{P}_i,\ i \in I \\ \Longleftrightarrow & \quad f_i(x_\lambda - x) \xrightarrow{\tau_i} 0 \quad \forall i \in I \\ \Longleftrightarrow & \quad f_i(x_\lambda) \xrightarrow{\tau_i} f_i(x) \quad \forall i \in I, \end{aligned}$$

which implies that the topology τ must be stronger than $\tau_{\text{(i)}}$ (and consequently also stronger than $\tau_{\text{(ii)}}$ or $\tau_{\text{(iii)}}$).

On the other hand, if τ' is any topology on E such that all f_i are continuous, the sets $x + U$ (where $U \in \mathcal{U}$ and $x \in E$) have to be neighborhoods of x of the topology τ' as well:

$$x + U = x + \bigcap_{j=1}^{n} f_{i_j}^{-1}(U_j) = \bigcap_{j=1}^{n} f_{i_j}^{-1}(f_{i_j}(x) + U_{i_j}).$$

So we can conclude that τ is also weaker than the weakest of those topologies which is $\tau_{\text{(iii)}}$. Since we obviously have $\tau_{\text{(iii)}} \subseteq \tau_{\text{(ii)}} \subseteq \tau_{\text{(i)}}$, we are done. □

1.19 Examples.

(i) The induced topology on a subspace F of a topological vector space E with locally convex topology τ is the projective topology with respect to the embedding $\iota : F \to (E, \tau)$.

(ii) The product topology of a product of topological vector spaces with locally convex topologies is the projective topology with respect to the projections.

(iii) Given a family of linear (or locally convex) topologies $(\tau_i)_{i \in I}$ on a topological vector space E, the *supremum* of the topologies is defined to be the projective topology with respect to the identity mappings $\mathrm{id} : E \to (E, \tau_i)$ for $i \in I$.

In particular, considering the family of all linear (or locally convex) topologies on a topological vector space, we obtain that there always is a *strongest linear (respectively locally convex) topology* (which in both cases cannot be the discrete topology, see also Example 1.25(i)→20 below).

1.20 Definition (Projective Limits). (See also figure 1.1.) Let $(E_i, \tau_i)_{i \in I}$ be a family of topological vector spaces indexed by a directed set $(I, \leqslant)$. For any pair of indices $i \leqslant j \in I$ let $f_{ji} : E_j \to E_i$ be a linear and continuous map such that $f_{ki} \circ f_{lk} = f_{li}$ for $i \leqslant k \leqslant l \in I$. Then $(E_i, \tau_i, f_{ji})_{i \leqslant j \in I}$ is called a *projective system*, the f_{ji} are called *linking maps*.

A topological vector space $E = \varprojlim_j E_j$, together with (linear and continuous) *projections* $\pi_i : E \to E_i$ that satisfy $f_{ji} \circ \pi_j = \pi_i$ for all $i \leqslant j \in I$, is called a *projective limit* (*inverse limit*) of the projective system $(E_i, \tau_i, f_{ji})_{i \leqslant j}$ (its topology τ is called the *projective limit topology*), if the following universal property holds:

> For any other topological vector space (F, τ') and linear and continuous maps $\rho_i : F \to E_i$ satisfying $f_{ji} \circ \rho_j = \rho_i$ ($i \leqslant j \in I$), there exists a unique linear and continuous map $f : F \to E$ such that $\pi_i \circ f = \rho_i$ for all $i \in I$.

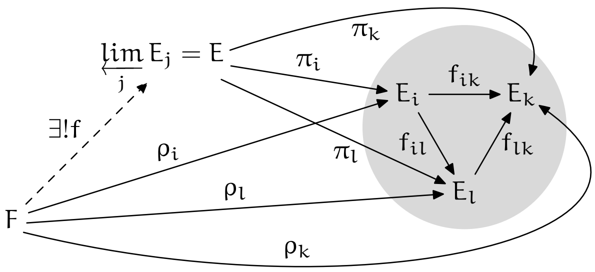

Figure 1.1 The projective limit $\varprojlim_j E_j$ of the projective system $(E_i, \tau_i, f_{ji})_{i \leqslant j \in I}$ which is represented by the grey disk.

1.21 Proposition. (See [Ste, Prop. 3.8] and [Jar81, sec. 2.6 and sec. 6.6, Cor. 3].)
The projective limit of a projective system $(E_i, \tau_i, f_{ji})_{i \leqslant j \in I}$ always exists and is, uniquely up to (topological) isomorphisms, given by

$$\varprojlim_j E_j = \left\{ x \in \prod_{i \in I} E_i \;\middle|\; f_{ji}(x_j) = x_i \right\},$$

equipped with the natural projections and the trace topology of the product topology.

In particular, if the topologies τ_i of the projective system are locally convex, then also the projective limit topology τ of $\varprojlim_j E_j$ is locally convex.

Proof. Let $E := \{x \in \prod_{i \in I} E_i \mid f_{ji}(x_j) = x_i\}$. Obviously we have $f_{ji} \circ \pi_j = \pi_i$. To show that the universal property holds for E, let (F, τ') be another topological vector space equipped with a family of projections $\rho_i : F \to E_i$. The relation $\pi_i \circ f = \rho_i$ already defines a unique and continuous linear map $f : F \to E$ by the universal property of the product.

To obtain the uniqueness of $\varprojlim_j E_j$, let $(E_1, \tau_1, (\pi_i^1)_i)$ and $(E_2, \tau_2, (\pi_i^2)_i)$ be two projective limits of (E_i, τ_i, f_{ji}). By the universal property, there exist unique continuous and linear maps $f : E_2 \to E_1$ and also $g : E_1 \to E_2$ such that $\pi_i^1 \circ f = \pi_i^2$ and $\pi_i^2 \circ g = \pi_i^1$, respectively. Together we obtain $\pi_i^1 \circ f \circ g = \pi_i^1$. Now the uniqueness statement of the universal property implies that $f \circ g = \mathrm{id}_{E_1}$. Similarly we obtain $g \circ f = \mathrm{id}_{E_2}$.

By Example 1.19[17] and the above, if all the topologies τ_i of the projective system are locally convex, the projective limit topology is locally convex as well. ☐

1.22 Remark.

(i) If all the spaces of a projective system are equipped with locally convex topologies, we can also look at the *locally convex projective limit*, where we only require the universal property of Definition 1.20[17] to hold for other topological vector spaces (F, τ') if they have locally convex topologies as well. But because of Lemma 1.18[16] we arrive at the same space as in Proposition 1.21[17], so it doesn't matter which version of the universal property we choose when working with projective limits.

(ii) Without loss of generality we can consider more simple projective systems. Two projective systems are called *equivalent* if they have isomorphic projective limits. A projective system is called *reduced* if the projections of its projective limit have dense range. Since every projective system is equivalent to a reduced one [Jar81, sec. 2.6, Prop. 2] we will from now on only consider such systems.

1.2.2 Inductive Limits

In this section we want to introduce the dual concept to projective limits, namely inductive limits, which are a generalization of the concept of *final* topologies to the setting of locally convex vector spaces.

Again there are several possible ways to obtain a topology τ on a vector space E, such that all members of a given family of maps $f_i : E_i \to E$, mapping topological vector spaces E_i with locally convex topology τ_i to E ($i \in I$), are continuous. But unfortunately this time we don't have an analogue of Lemma 1.18[16], i. e., we have to be more careful in which category we are working.

First, there is of course the final topology (or *inductive topology*) on E with respect to (E_i, τ_i, f_i) which we will denote by τ_f. But this need not be a linear topology (see Example 1.25(i)→20 below), so we also have to look at the strongest *linear* topology on E such that all f_i are continuous (see the Lemma below).

But again, there is a problem: even if all the topologies τ_i are locally convex, the linear inductive topology τ_l need not be locally convex (again, see Example 1.25(ii)→20 below). Therefore we will also need the concept of the strongest *locally convex* topology on E such that all f_i are continuous:

1.23 Lemma (Linear and Locally Convex Inductive Topologies). (See [Ste, § 3.2] and [Jar81, sec. 4.1 and p. 110].)
Given a vector space E and a family of linear maps $f_i : E_i \to E$ from topological vector spaces (E_i, τ_i) with linear (respectively locally convex) topologies τ_i to E, the strongest linear (locally convex) topology on E such that all f_i are continuous exists, and is uniquely defined by the following universal property (we will call this topology the *linear inductive topology* or *locally convex inductive topology* and denote it by τ_l or τ_o, respectively):

> A map $f : E \to (F, \tau')$, where (F, τ') is another topological vector space with linear (respectively locally convex) topology τ', is continuous, if and only if all the compositions $f \circ f_i : (E_i, \tau_i) \to (F, \tau')$ are continuous.

Proof. By Example 1.19(iii)←17 the linear (respectively locally convex) inductive topology τ on E exists, and for this topology the universal property obviously holds.

Now let τ' be another linear (locally convex) topology on E satisfying the universal property. Considering $f := \mathrm{id}_{(E,\tau)}$ or $f := \mathrm{id}_{(E,\tau')}$, respectively, we obtain that all $f_i : (E_i, \tau_i) \to E$ have to be continuous, no matter which of the two topologies τ and τ' we equip E with. On the other hand, the universal property now also implies that both $\mathrm{id}_E : (E, \tau) \to (E, \tau')$ and $\mathrm{id}_E : (E, \tau') \to (E, \tau)$ have to be continuous, i. e., that $\tau = \tau'$. □

1.24 Remark. By the universal property, a seminorm $p : E \to \mathbb{K}$ on a topological vector space E equipped with the locally convex inductive topology τ_o with respect to the maps $f_i : (E_i, \tau_i) \to E$, $i \in I$, is continuous (and thus, by Proposition 1.10←13, one of those describing the topology τ_o), if and only if each restriction $p \circ f_i : E_i \to \mathbb{K}$ is continuous. Unfortunately this characterization is too abstract for most applications—it is impossible to find a description as nice as the one for the projective topology we had in Lemma 1.18←16. Therefore, for now, we will have to content ourselves with an analogous description of the 0-neighborhoods of (E, τ_o):

An absorbing and absolutely convex set U in (E, τ_o) is a 0-neighborhood if and only if $f_i^{-1}(U)$ is a 0-neighborhood in (E_i, τ_i) for each $i \in I$. Thus we obtain a base of 0-neighborhoods for τ_o by collecting all such sets U.

A more useful description can be obtained when $E = \mathrm{span}\big(\cup_{i \in I} f_i(E_i)\big)$. Then the collection of all sets of the form

$$U = \Gamma\Big(\bigcup_{i\in I} f_i(U_i)\Big), \tag{1.1}$$

where U_i is a 0-neighborhood in (E_i, τ_i), is a base of 0-neighborhoods for the locally convex inductive topology $\tau_\circ$ on E (clearly all maps f_i are continuous with respect to the locally convex topology generated by this base of 0-neighborhoods, so it must be weaker than $\tau_\circ$, but on the other hand every 0-neighborhood of $\tau_\circ$ has to contain a set of the given form, which means that the topologies coincide, see [Jar81, § 6.6, Prop. 5(a)]).

1.25 Examples.

(i) The final topology on a vector space $E \neq \{0\}$ with respect to an empty family of mappings and topological vector spaces (or only 0-maps) is the discrete topology, but this is not a linear topology since for any $0 \neq x \in E$ we would have $\frac{1}{n}x \to 0$, although $\frac{1}{n}x$ can never reach the discrete 0-neighborhood $\{0\}$. This implies that, in general, the linear inductive topology τ_l must be strictly weaker than the final topology τ_f.

(ii) (See [Bou81, ch. II, p. 80, exerc. 15].)
Let I be an uncountable set and $\mathcal{F}(I)$ the family of all its finite subsets. Then define vector spaces $E := \mathbb{R}^{(I)} := \bigoplus_{i\in I}\mathbb{R}$ and $F_J := \mathbb{R}^J = \bigoplus_{j\in J}\mathbb{R}$ for $J \in \mathcal{F}(I)$. Let $g_J : F_J \to E$ be the canonical embeddings. Equipping each F_J with the product topology, we will see that the linear inductive topology τ_l on E is strictly stronger than the locally convex inductive topology $\tau_\circ$. To show this, let

$$U := \Big\{x = (\xi_i)_{i\in I} \in E \;\Big|\; p(x) := \sum_{i\in I} |\xi_i|^{\frac{1}{2}} \leqslant 1\Big\}.$$

This is obviously a 0-neighborhood for τ_l (it is symmetric, absorbing and the traces of U under the embeddings g_J are 0-neighborhoods of F_J, for all J). We will show that there is no absolutely convex and absorbing subset of U, which means that U cannot be a 0-neighborhood for $\tau_\circ$, i. e., that $\tau_\circ \subsetneq \tau_l$.

Suppose $V \subseteq U$ were such a set. Since it is absorbing, there exists an $\varepsilon_i > 0$ for every $e_i := (\delta_{ij})_{j\in I} \in U$ such that $\varepsilon_i e_i \in V \subseteq U$ for $i \in I$. Now using the convexity of V we obtain $x_J := \frac{1}{|J|}\sum_{j\in J}\varepsilon_j e_j \in V$ for any $J \in \mathcal{F}(I)$. On the other hand, since the uncountable set I equals the countable union $\cup_{n\in\mathbb{N}}\{i \in I \mid \varepsilon_i > \frac{1}{n}\}$, there must be an $n \in \mathbb{N}$ such that we have $\varepsilon_i > \frac{1}{n}$ for infinitely many indices $i \in I$. Let us denote this set of indices by I', so that we have

$$p(x_J) = \sum_{j\in J}\left|\frac{1}{|J|}\varepsilon_j\right|^{\frac{1}{2}} > |J|^{\frac{1}{2}}\frac{1}{\sqrt{n}} > 1,$$

for every finite $J \subseteq I'$ that is large enough, a contradiction to $x_J \in V \subseteq U$.

(iii) Let (E, τ) be a topological vector space with linear (respectively locally convex) topology τ and let $M \subseteq E$ be a subspace. The *quotient topology* on E/M is also the linear inductive topology with respect to the canonical projection $\Phi : E/M \to E$. If the topology τ is locally convex, the quotient topology is locally convex as well (i. e., the linear and locally convex inductive topologies coincide). The space E/M is separated if and only if M is closed in E. (See [Sch71, ch. 1, §2.2–2.3].)

(iv) Let (E_i, τ_i) be a family of topological vector space with linear (respectively locally convex) topologies τ_i indexed by $i \in I$. The *direct sum* $\bigoplus_{i \in I} E_i$ can be equipped with the linear (respectively locally convex) inductive topology with respect to the canonical embeddings. If all the summands are Hausdorff or complete, the direct sum has the same property. (See [Sch71, ch. 2, §6.1–6.2].)

1.26 Definition (Inductive Limits). (See also figure 1.2, Definition 1.2017.) Let $(E_i, \tau_i)_{i \in I}$ be a family of topological vector spaces indexed by a directed set $(I, \leqslant)$. For any pair of indices $i \leqslant j \in I$ let $f_{ij} : E_i \to E_j$ be a linear and continuous map such that $f_{kl} \circ f_{ik} = f_{il}$ for $i \leqslant k \leqslant l \in I$. Then $(E_i, \tau_i, f_{ij})_{i \leqslant j \in I}$ is called an *inductive system*, the f_{ij} are called *linking maps*.

A topological vector space $E = \varinjlim_j E_j$, together with linear and continuous *embeddings* $\iota_i : E_i \to E$ that satisfy $\iota_j \circ f_{ij} = \iota_i$ for all $i \leqslant j \in I$, is called a *(linear) inductive limit* (*direct limit*) of the inductive system $(E_i, \tau_i, f_{ij})_{i \leqslant j}$ (its topology τ is called the *linear inductive limit topology*), if the following universal property holds:

> For any other topological vector space (F, τ') and linear and continuous maps $\kappa_i : E_i \to F$ satisfying $\kappa_j \circ f_{ij} = \kappa_i$ $(i \leqslant j \in I)$, there exists a unique linear and continuous map $f : E \to F$ such that $\kappa_i \circ f = \iota_i$ for all $i \in I$.

If $(E_i, \tau_i)_{i \in I}$ is an inductive system of topological vector spaces with locally convex topologies τ_i, we call a topological vector space $E = \varinjlim_j E_j$ together with linear and continuous embeddings $\iota_i : E_i \to E$ a *(locally convex) inductive limit* of the inductive system (its topology τ is called the *locally convex inductive limit topology*), if the universal property holds for any other topological vector space (F, τ') with locally convex topology τ'.

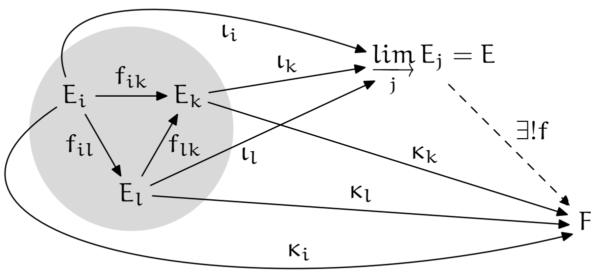

Figure 1.2 The inductive limit $\varinjlim_j E_j$ of the inductive system $(E_i, \tau_i, f_{ij})_{i \leqslant j \in I}$ which is represented by the grey disk.

1.27 Proposition. (See [Ste, Prop. 3.20] and [Jar81, sec. 4.5 and p. 110].)
The (linear) inductive limit of an inductive system $(E_i, \tau_i, f_{ij})_{i \leqslant j \in I}$ always exists and is, uniquely up to (topological) isomorphisms, given by

$$\varinjlim_j E_j = \bigoplus_{i \in I} E_i \Big/ \{x_i - f_{ij}(x_i) \mid x_i \in E_i,\ i \leqslant j \in I\},$$

equipped with the natural embeddings and the quotient topology with respect to the *linear* inductive direct sum topology.

If the topologies τ_i of the inductive system are all locally convex, the locally convex inductive limit exists as well and is given by the same space equipped with the quotient topology with respect to the *locally convex* inductive direct sum topology.

Proof. Let $E := \bigoplus_{i \in I} E_i / M$ where $M := \{x_i - f_{ij}(x_i) \mid x_i \in E_i,\ i \leqslant j \in I\}$. By definition, we have $\iota_j \circ f_{ij} = \iota_i$ for all $i \leqslant j \in I$. To prove that E/M satisfies the universal property of the inductive limit, let (F, τ') be another topological vector space together with linear and continuous maps $\kappa_i : E_i \to F$ such that $\kappa_j \circ f_{ij} = \kappa_i$. By the universal property of the direct sum, the relation $\kappa_i \circ f = \iota_i$ already defines a unique and continuous linear map $f : E \to F$.

The uniqueness of $\varinjlim_j E_j$ can be obtained similarly to Proposition 1.21. □

1.28 Remark.

(i) By looking at the proofs of Proposition 1.21 and Proposition 1.27, one can see that the concept of projective and inductive limits can be generalized to many other categories (then often called *inverse* resp. *direct limits*).

(ii) In particular, the "linear algebra" projective (or inductive) limit of a projective (inductive) system of topological vector spaces (ignoring the topologies) is the same as our projective (inductive) limit, just without the topology. (Compare with "linear inductive limits" and "locally convex inductive limits" of an inductive system of locally convex vector spaces—the spaces coincide as well, although in general the topologies will differ.)

1.2.3 Dualities

The following two results tell us more about the "duality" of projective and inductive limits.

1.29 Proposition. (See [FW68, § 26, Satz 1.2].)
Let $(E_i)_{i \in I}$ denote an inductive system of topological vector spaces. Then the corresponding family $((E_i)')_{i \in I}$ of duals is an algebraic projective system (see Remark 1.28), and we have the (algebraic) identity

$$\big(\varinjlim_j E_j\big)' = \varprojlim_j (E_j)'. \tag{1.2}$$

Proof. Let $f_{ij} : E_i \to E_j$ denote the linking maps of the given inductive system $(E_i)_{i\in I}$. We want to turn the family $((E_i)')_{i\in I}$ of duals into a projective system. The (algebraic) adjoint maps $f^*_{ij} : E^*_j \to E^*_i$ give us candidates for the new linking maps $g_{ji} : E'_j \to E'_i$ by setting $g_{ji} := f^*_{ij}\big|_{E'_j}$. To check that $g_{ji}(E'_j) \subseteq E'_i$, let $e'_j \in E'_j$, then $g_{ji}(e'_j) = e'_j \circ f_{ij} : (E_i, \tau_i) \to \mathbb{K}$ is continuous as a composition of continuous maps, and we obtain $g_{ji}(E'_j) \subseteq E'_i$.

Similarly, starting from the linear and continuous injections $\iota_i : E_i \to E := \varinjlim_j E_j$ of the (topological) inductive limit, we obtain linear projections $\rho_i : E' \to (E_i)'$. By the universal property of the (algebraic) projective limit $\varprojlim_j (E_j)'$ there exists a unique linear map $f : E' \to \varprojlim_j (E_j)'$ with $\pi_i \circ f = \rho_i$.

To show that f is injective, let $u \in E'$ such that $f(u) = 0$. It follows that $0 = (\pi_i \circ f)(u) = \rho_i(u) = u \circ \iota_i$ for all $i \in I$, i. e., $u = 0$. To prove surjectivity, let $y \in \varprojlim_j (E_j)'$ and consider the linear maps $\pi_i(y) : E_i \to \mathbb{K}$ for $i \in I$. Since $f_{ij} \circ \pi_i(y) = \pi_j(y)$, the universal property of the inductive limit tells us that there exists a (unique) linear and continuous map $v : E = \varinjlim_j E_j \to \mathbb{K}$, i. e., $v \in E'$ with $v \circ \iota_i = \pi_i(y)$. But since $v \circ \iota_i = \rho_i(v) = \pi_i(f(v))$, we can conclude that $f(v) = y$. □

1.30 Proposition. (See [FW68, § 26, Satz 1.6].)

Let $(E_i)_{i\in I}$ denote a *reduced* projective system of topological vector spaces. Then the corresponding family $((E_i)')_{i\in I}$ of duals is an (algebraic, see Remark 1.28) inductive system with injective linking maps, and we have the algebraic identity

$$\left(\varprojlim_j E_j\right)' = \varinjlim_j (E_j)'. \tag{1.3}$$

Proof. We can proceed as in the proof of Proposition 1.29 to obtain linking maps $g_{ij} := f^*_{ji}\big|_{E'_i} : (E_i)' \to (E_j)'$ and embeddings $\kappa_i := \pi^*_i : (E_i)' \to E' := (\varprojlim_j E_j)'$. By the reducedness of the projective system, the maps $f_{ji} : E_j \to E_i$ and $\pi_i : E \to E_i$ have dense images—the projections π_i by definition and the linking maps f_{ji} since $\pi_j(E) = f_{ji}(\pi_i(E)) \subseteq f_{ji}(E_i) \subseteq E_j$. Translating this to our new inductive system, we obtain that all g_{ij} and κ_i have to be injective.

The universal property of the inductive limit $\varinjlim_j (E_j)'$ gives us a unique linear map $f : \varinjlim_j (E_j)' \to E'$ such that $f \circ \iota_i = \kappa_i$ for all $i \in I$. This last property already implies the injectivity of f since all the maps ι_i and κ_i are injective (for the maps ι_i consider $\ker \iota_i = \cup_{i\leqslant j} \ker g_{ij} = \{0\}$).

To see the surjectivity of f consider $u \in E'$; as a continuous map from E to $\mathbb{K}$ and by Proposition 1.21 there exists an index $i \in I$ and a continuous seminorm $p : E_i \to \mathbb{K}$ such that $u(x) \leqslant p(\pi_i(x))$ for all $x \in E$. This means that u can be written as $u = u_i \circ \pi_i$ where $u_i \in (\pi_i(E))'$. By Hahn-Banach we may assume u_i to be the restriction of a functional on $E_i \supseteq \pi_i(E)$, i. e., $u_i \in (E_i)'$. Now we have

$$u(x) = u_i(\pi_i(x)) = \kappa_i(u_i)(x) = f(\iota_i(u_i))(x) \qquad \forall x \in E,$$

such that $f(\iota_i(u_i)) = u$, showing the surjectivity of f. □

The natural question that poses itself now is the one about the role of the topologies in the dualities above. We want the equations (1.2) and (1.3) to hold for locally convex limits as well. But as they would compare a limit topology with a topology on a dual it will prove itself worthwhile to first gain an understanding about all "possible" locally convex topologies on a dual space. These will be studied in the next section.

1.3 Dual Systems

1.31 Definition (Dual System). A *dual system* is a triple $(F, G, \langle\, .\,,\,.\,\rangle)$, written $\langle F, G\rangle$ for short, where F and G are vector spaces, and $\langle\, .\,,\,.\,\rangle : F \times G \to \mathbb{K}$ is a bilinear map such that

(i) for any $x \in F$, $x \neq 0$, there exists a $y \in G$ such that $\langle x, y\rangle \neq 0$.

(ii) for any $y \in G$, $y \neq 0$, there exists an $x \in F$ such that $\langle x, y\rangle \neq 0$.

Given a dual system $\langle F, G\rangle$ we want to construct locally convex topologies on its spaces, such that $F' \cong G$. One such topology that can be constructed "out of nothing" is the weak topology of pointwise convergence, via the bilinear map $\langle\, .\,,\,.\,\rangle$, on the respective dual space.

By remarking that this is also the topology of uniform convergence on finite sets of "functionals", we can motivate the general construction of topologies for dual systems.

1.32 Definition (Generating System). Let $\langle F, G\rangle$ denote a dual system. A family $\mathcal{S}$ of subsets of F is called a *generating system*, if it is directed and all its sets are $\sigma(F, G)$-bounded.

Such a system $\mathcal{S}$ is called *saturated*

(i) if any set $T \subseteq \lambda S$, where $\lambda > 0$ and $S \in \mathcal{S}$, is also part of the system $\mathcal{S}$,

(ii) if the system is closed under finite unions,

(iii) and if for any set $S \in \mathcal{S}$ the $\sigma(F, G)$-closure $\overline{\Gamma S}^{\sigma}$ of its absolutely convex hull is again part of $\mathcal{S}$.

The smallest saturated generating system containing a given system $\mathcal{S}$ is called its *saturated hull* and denoted by $\bar{\mathcal{S}}$.

1.33 Lemma. (See [Ste, Thm. 4.14].)
Let $\langle F, G\rangle$ denote a dual system and $\mathcal{S}$ a generating system in F. Then the sets $U_{\varepsilon,S} := \left\{y \in G \,\middle|\, |\langle S, y\rangle| < \varepsilon\right\}$, where $\varepsilon > 0$ and $S \in \mathcal{S}$, form a base of 0-neighborhoods for a locally convex topology $\tau_{\mathcal{S}}$ on G (called the topology of *uniform convergence* on all sets S of $\mathcal{S}$). Alternatively one can consider the family $\{p_S \mid S \in \mathcal{S}\}$ of seminorms $p_S(y) := |\langle S, y\rangle| := \sup_{s\in S} |\langle s, y\rangle|$, $y \in G$.

This topology is separated (i. e., $(G, \tau_{\mathcal{S}})$ is a locally convex vector space) if and only if the linear span $\operatorname{span}(\cup\mathcal{S})$ of $\mathcal{S}$ is $\sigma(F, G)$-dense in F.

If the generating system $\mathcal{S}$ is saturated, it suffices to consider the *polars* of its sets to obtain a base of 0-neighborhoods for the corresponding topology $\tau_{\mathcal{S}}$ of uniform convergence (by property (i) of Definition 1.32←24).

1.34 Definition (Polars). Let $\langle F, G\rangle$ denote a dual system. Given subsets $M \subseteq F$ and $N \subseteq G$ we define their *(absolute) polars* $M^\circ \subseteq G$ and $N^\circ \subseteq F$ as follows

$$M^\circ := \left\{ y \in G \,\middle|\, |\langle M, y\rangle| \leqslant 1 \right\} := \left\{ y \in G \,\middle|\, |\langle x, y\rangle| \leqslant 1 \ \forall x \in M \right\},$$
$$N^\circ := \left\{ x \in F \,\middle|\, |\langle x, N\rangle| \leqslant 1 \right\} := \left\{ x \in F \,\middle|\, |\langle x, y\rangle| \leqslant 1 \ \forall y \in N \right\}.$$

1.35 Remark. (See [Ste, Prop. 4.30] and [Jar81, § 8.2, Prop. 1].)
Let $\langle F, G\rangle$ be a dual system, $M, N, (M_i)_{i\in I} \subseteq F$ and $0 \neq \lambda \in \mathbb{K}$, then we have the following properties for polars taken in F:

(i) $\{0\}^\circ = G$, $F^\circ = \{0\}$,
(ii) $M \subseteq M^{\circ\circ} := (M^\circ)^\circ$,
(iii) $(\lambda M)^\circ = \frac{1}{\lambda} M^\circ$,
(iv) if $M \subseteq N$ then $M^\circ \supseteq N^\circ$,
(v) $(\cup_{i\in I} M_i)^\circ = \cap_{i\in I} M_i^\circ$,
(vi) if $0 \in M \cap N$ then $M^\circ \cap N^\circ \subseteq 2(M + N)^\circ \subseteq 2(M^\circ \cap N^\circ)$,
(vii) every polar M° is absolutely convex and $\sigma(G, F)$-closed.

1.36 Bipolar Theorem. (See [Sch71, ch. IV, Thm. 1.5].)
Let $\langle F, G\rangle$ denote a dual system. The bipolar $M^{\circ\circ}$ of any subset $M \subseteq F$ is the $\sigma(F, G)$-closed, absolutely convex hull of M:

$$M^{\circ\circ} = \overline{\Gamma M}^{\sigma(F,G)}. \tag{1.4}$$

1.37 Corollary.

(i) Let $\langle F, G\rangle$ be a dual system. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two saturated generating systems in F that give rise to the same topology, i. e., $\tau_{\mathcal{S}_1} = \tau_{\mathcal{S}_2}$, then they have to coincide.
(See [Ste, Cor. 4.33].)

(ii) Let (E, τ) be a topological vector space with a Hausdorff topology. Its topology τ is locally convex if (and only if) it is the topology of uniform convergence on the equicontinuous subsets of E'.
(See [Sch71, ch. IV, § 1.5, Cor. 4].)

This means that every separated locally convex topology can be obtained using a generating system. So what we have gained from this theory is another approach to cataloguing separated locally convex topologies. The weak topology, where the generating system consists of all finite subsets, is the weakest possible under the condition that it remains separated (see Lemma 1.33←24)—it might be interesting to also look at the strongest such topology:

1.38 Definition. Let (E, τ) be a topological vector space. The *strong topology* $\beta(E', E)$ on E' is the separated topology obtained using the generating system of all τ-bounded sets in E. We will write E'_b for $(E', \beta(E', E))$.

Can we also obtain the strong topology on a dual without using a given topology, i. e., can we define a strong topology for general dual systems (as we did with the weak topology)? To answer this question we will obviously have to return to our original motivation for dual systems—finding topologies on F such that $F' \cong G$, where $\langle F, G\rangle$ is a dual system.

1.39 Theorem (Mackey-Arens). (See [Sch71, ch. IV, Thm. 3.2 and Cor. 1].)
Let $\langle F, G\rangle$ be a dual system, and let τ be a locally convex topology on F. The topology τ is *compatible* with the duality (i. e., $(F, \tau)' \cong G$), if it can be obtained from a saturated generating system of $\sigma(G, F)$-relatively compact subsets of G whose union covers G.

In particular, there exists a unique strongest compatible topology on F, the *Mackey topology* $\mu(F, G)$, generated by the system of absolutely convex $\sigma(G, F)$-relatively compact subsets of G. A topological vector space (E,τ) where $\tau = \mu(E, E')$ is called a *Mackey space*.

1.40 Corollary. (See [Sch71, ch. IV, § 3.2, Cor. 2].)
All topologies that are compatible with a dual system $\langle F, G\rangle$ have the same bounded sets and the same closures of convex sets.

In particular, we can also use the weakly bounded sets to obtain a generating system for the strong topology, which allows us to define strong topologies $\beta(F, G)$ "out of nothing" for general dual systems $\langle F, G\rangle$.

Considering the duality $\langle E, E'\rangle$, where (E, τ) is a topological vector space with locally convex topology τ, we obtain that τ has to be weaker than the Mackey topology $\mu(E, E')$ (since τ is obviously compatible)—if (E, τ) is Mackey, the topologies coincide, which means that the corresponding saturated generating systems have to be equal as well.

This opens up the possibility to define many types of locally convex vector spaces (and at the same time keep an overview about the relations between them) by simply asking for the topologies corresponding to certain saturated generating families to coincide. For example this is one way to define *barrelled spaces*:

1.41 Definition. A locally convex vector space (E, τ) is called *barrelled*, if its topology is generated by the system of $\sigma(E', E)$-bounded sets, i. e., if τ is the strong topology $\beta(E, E')$.

By taking the polars of the sets of the generating system—and thus obtaining a base of 0-neighborhoods—we can arrive at the usual definition of barrelled spaces via *barrels* (closed, absolutely convex and absorbing sets): (E, τ) is barrelled if and only if every barrel is a 0-neighborhood. Our definition has the advantage that we immediately know that every barrelled space has to be a Mackey space.

Barrelled spaces are also of interest to us because of the following property.

1.42 Lemma. (See [Sch71, ch. II, § 7.2].)
Let (E, τ) carry the locally convex inductive topology with respect to a family $(E_i, \tau_i)_{i \in I}$ of barrelled locally convex vector space and linear maps $\iota_i : E_i \to E$, then (E, τ) is also barrelled. In particular, separated quotients, direct sums and separated inductive limits of barrelled spaces are barrelled.

Proof. Let A be a barrel of (E, τ). Then the preimages $\iota_i^{-1}(A)$ are still barrels in the spaces (E_i, τ_i), hence 0-neighborhoods, such that the original barrel A also has to be a 0-neighborhood of (E, τ) by Remark 1.24. □

Now that we've obtained a useful topology on E', we want to be able to return to the underlying locally convex vector space (E, τ), without knowing τ. This is of course possible if and only if $\beta(E', E)$ is compatible with $\langle E, E' \rangle$. Unfortunately, in general the dual of E'_b cannot be identified with E. We call the subspace $E'' := (E'_b)'$ of $E'^* := (E'_b)^*$ the *bidual* of the locally convex vector space (E, τ).

1.43 Definition. A locally convex vector space (E, τ) is called *semi-reflexive* if $E'' = E$, i. e., if the canonical embedding $E \to E''$ is surjective.

It is called *reflexive* if the *strong bidual* $E''_b := (E'_b)'_b$ is isomorphic to (E, τ).

Thus a semi-reflexive space E is reflexive, if and only if its topology is $\beta(E, E')$, i. e., if and only if it is barrelled. There are also many other useful characterisations of (semi-)reflexivity.

1.44 Proposition. (See [Sch71, ch. IV, 5.5].)
Let (E, τ) denote a locally convex vector space, then the following are equivalent:

(i) E is semi-reflexive,

(ii) every bounded subset of E is relatively weakly compact,

(iii) $(E, \sigma(E, E'))$ is *quasi-complete* (every bounded, closed subset is complete).

Furthermore, every semi-reflexive space (E, τ) is also quasi-complete under its original topology τ.

Returning to the question at the end of section 1.2.3, we can now at least answer when the dual system of an inductive (projective) system of topological vector spaces is again inductive (projective).

1.45 Lemma. Let $(E_i, \tau_i, f_{ij})_{i \leqslant j \in I}$ denote an inductive system of topological vector spaces with locally convex topologies. For each $i \in I$ let $\mathcal{S}_i$ denote a generating system in E_i for a locally convex topology $\tau_{\mathcal{S}_i}$ on E'_i, such that for every $S_i \in \mathcal{S}_i$ there exists a set $S_j \in \mathcal{S}_j$ where $i \leqslant j \in I$ with $f_{ij}(S_i) \subseteq S_j$. Then the corresponding dual family $(E'_i, \tau_{\mathcal{S}_i})_{i \in I}$ is a projective system of topological vector spaces with locally convex topologies.

The analogous result for projective systems $(E_i)_{i\in I}$ of topological vector spaces with locally convex topologies holds as well.

Proof. The algebraic adjoints give us candidates for the dual linking maps by setting $g_{ji} := f_{ij}^*\big|_{E_i'}$. Since for every $e_j' \in E_j'$ the functionals $g_{ji}(e_j') = e_j' \circ f_{ij} : (E_i, \tau_i) \to \mathbb{K}$ are continuous as compositions of continuous maps, we obtain $g_{ji}(E_j') \subseteq E_i'$. It is left to check that these maps are indeed continuous $g_{ji} : (E_j', \tau_{\mathcal{S}_j}) \to (E_i', \tau_{\mathcal{S}_i})$. Let $S_i \in \mathcal{S}_i$, then there is an $S_j \in \mathcal{S}_j$ with $f_{ij}(S_i) \subseteq S_j$ and we have

$$\begin{aligned}
g_{ji}^{-1}(S_i^\circ) &= \left\{ e_j' \in E_j' \;\middle|\; g_{ji}(e_j') \in S_i^\circ \right\} \\
&= \left\{ e_j' \in E_j' \;\middle|\; |\langle g_{ji}(e_j'), S_i\rangle| \leqslant 1 \right\} \\
&= \left\{ e_j' \in E_j' \;\middle|\; |\langle e_j', f_{ij}(S_i)\rangle| \leqslant 1 \right\} \\
&= \big(f_{ij}(S_i)\big)^\circ \supseteq S_j^\circ
\end{aligned}$$

which means that g_{ji} has to be continuous since the polars S_i° of the sets $S_i \in \mathcal{S}_i$ form a base of 0-neighborhoods for the topology $\tau_{\mathcal{S}_i}$ of E_i', $i \in I$. □

1.4 (DF)-Spaces

Our goal is to find alternative "descriptions" of an inductive limit space. Therefore, in view of Proposition 1.29 22 (which we will also show to hold with the respective strong topologies for certain inductive systems in Proposition 2.9 34 below), we should consider strong (pre)duals of metrizable (Fréchet) spaces.

1.46 Proposition. (See [MV97, Lem. 25.5 & 25.6] and [Jar81, § 12.4].)
Let E denote a metrizable locally convex vector space, then its strong dual has the following properties.

(i) E_b' has a fundamental sequence of bounded sets.

(ii) E_b' is σ-*quasi-barrelled*, i. e., if $(U_n)_{n\in\mathbb{N}}$ denotes a sequence of closed absolutely convex 0-neighborhoods in E_b' such that $U := \bigcap_{n\in\mathbb{N}} U_n$ is bornivorous (absorbs any bounded set), then U is also a 0-neighborhood in E_b'.

1.47 Definition. A locally convex vector space E is called a *(DF)-space* if it has the properties (i) and (ii) of Proposition 1.46 (applied to E itself instead of E_b').

Indeed, these conditions suffice to obtain further nice properties.

1.48 Proposition. (See [MV97, Prop. 25.7 & 25.9] and [Jar81, § 12.4, Thm. 8].)
The dual of a Fréchet space is a complete (DF)-space, and the dual of a (DF)-space is again a Fréchet space.

Furthermore, the class of (DF)-spaces is stable with respect to formation of Hausdorff quotients, countable direct sums (hence Hausdorff countable inductive limits), and completions.

2

Problems of General Inductive Limits

The main source of difficulties one is faced with when working with inductive limits is that the locally convex inductive topology is in general strictly weaker than the linear inductive topology which in turn is in general strictly weaker than the final topology.

By restricting ourselves to the study of countable locally convex inductive limits we can circumvent the first problem (see the lemma below). Since this is not a big restriction for most applications, we will from now on often only consider countable inductive systems (*inductive sequences*) of locally convex vector spaces.

2.1 Lemma. (See [Jar81, § 6.6, Prop. 9].)
Let $(E_n, \tau_n)_{n\in\{1,2,\ldots\}}$ denote a countable (possibly finite) inductive system of topological vector spaces with locally convex topologies τ_n. Then the linear and the locally convex inductive limits of $(E_n, \tau_n)_n$ coincide, i. e., the linear inductive limit topology is already locally convex.

If the final topology of an inductive sequence $(E_n, \tau_n)_n$ of locally convex vector spaces is already locally convex, i. e., if it coincides with the locally convex inductive topology, the corresponding inductive limit $E = \varinjlim_n E_n$ has to be separated as well (which is not clear in general, see also below), since in this case a set C in E is closed if and only if $\iota_n^{-1}(C)$ is closed in E_n for each $n \in \mathbb{N}$.

2.1 Regularity

Which conditions can we impose on an inductive sequence of locally convex vector spaces, such that its limit is necessarily separated? We need that the bounded subspace $\overline{\{0\}}$ equals $\{0\}$. Every set which is bounded in one of the spaces of the defining inductive sequence is of course also bounded in the limit space (by Remark 1.24↑19 any 0-neighborhood in the limit has to absorb it)—if these were all bounded sets of the limit we would be done (since $\overline{\{0\}} = \{0\}$ does hold in all of the “steps”). This leads us to the following condition of *regularity* which is also interesting in itself.

2.2 Definition (Regularity). An inductive system $(E_i, \tau_i)_{i\in I}$ of locally convex vector spaces or its limit $E := \varinjlim_j E_j$ is called *regular* if for any set B that is bounded in E there exists an index $i \in I$ such that B is already contained and bounded in (E_i, τ_i).

By the discussion preceding the definition we immediately obtain:

2.3 Lemma. (See [FW68, § 23, Satz 5.2].)
The inductive limit of a regular inductive sequence of (separated) locally convex vector spaces is again separated.

Historically, the study of inductive limits of locally convex vector spaces began with the analysis of countable *strict* inductive limits.

2.4 Definition (strict limits). An inductive system $(E_i, \tau_i)_{i\in I}$ of locally convex vector spaces or its limit $(E, \tau) := \varinjlim_j E_j$ is called

(i) *strict* if $f_{ij} : E_i \to f_{ij}(E_j) \subseteq E_j$ is a topological isomorphism for each $i < j \in I$ (i. e., the topologies of the steps "fit together", $\tau_i = \tau_j\big|_{E_i}$),

(ii) *hyperstrict* if $\iota_i : E_i \to \iota_i(E_i) \subseteq E$ is a topological isomorphism for each $i \in I$ (i. e., the inductive limit topology induces the topologies of its steps, $\tau_i = \tau|_{E_i}$).

Hyperstrict limits are obviously strict (since $\iota_i = \iota_j \circ f_{ij}$) and have to be separated, but the converses don't hold in general. On the other hand, if we again restrict ourselves to countable limits, everything is nice.

2.5 Proposition. (See [Bie86, Thm 0.3] and [Sch71, ch. II, 6.4–6.6].)
The inductive limit of a strict inductive sequence $(E_n, \tau_n)_{n\in\mathbb{N}}$ of locally convex vector spaces is even hyperstrict. If, additionally, each E_n is closed in (E_{n+1}, τ_{n+1}), the limit is also regular. Finally, a countable strict inductive limit of complete locally convex vector spaces is always hyperstrict, regular and complete. (See also Theorem 2.11→36 below for a proof of the completeness part.)

One advantage of *strictness* over *hyperstrictness* and *regularity* is that it is easy to check whether a given inductive system satisfies this property—one doesn't have to know anything about the limit topology.

Unfortunately, the class of spaces obtained as strict inductive limits is too small to contain all (or enough of) the inductive limit spaces occurring in practice. Therefore we will now introduce other restrictions, which can be formulated only in terms of the inductive system and still guarantee—as we will see—that the corresponding limit spaces have nice properties.

2.6 Definition (weakly compact, compact, nuclear). A (locally convex) inductive system $(E_i)_{i\in I}$ or (slightly inprecise) its limit $E = \varinjlim_i E_i$ is called *weakly compact*, *compact* or *nuclear*, respectively, if, for each $i \in I$ there exists $j \geqslant i$ such that the canonical injection $\iota_{ij} : E_i \to E_j$ is a weakly compact, compact or nuclear operator, respectively.

Although these conditions are only restrictions for the linking maps of an inductive system, if they are fulfilled, we may also assume the corresponding spaces to be of a special type:

2.7 Lemma. (See [Flo71, Satz 6.6 and § 16] and [FW68, § 19, Satz 1.9].)
If $(E_i)_{i\in I}$ is a weakly compact (compact, nuclear) inductive system of locally convex vector spaces, there exists an equivalent (i. e., it has the same limit) weakly compact (compact, nuclear) inductive system consisting of Banach spaces.

Proof. Choosing a subset I' of the index set I, we may assume—without changing the limit—that all linking maps are weakly compact. For every weakly compact $f_{ij} : E_i \to E_j$, $i \leqslant j \in I'$, there exists an absolutely convex 0-neighborhood U_{ij} in E_i such that $f_{ij}(U_{ij})$ is relatively weakly compact in E_j. Therefore we only have to consider the factorisations

$$f_{ij} : E_i \xrightarrow{\hat{f}_{ij}} F_{ij} \xrightarrow{e_{ij}} E_j,$$

where $F_{ij} := (E_j)_{\overline{f_{ij}(U_{ij})}^{\sigma}}$ is the Banach space generated by $\overline{f_{ij}(U_{ij})}^{\sigma} \subseteq E_j$, to obtain an equivalent weakly compact system $(F_p)_{p\in P}$ of Banach spaces indexed by $p \in P := \{(i,j) \in I' \times I' \mid i \leqslant j\}$ where $(i,j) \leqslant (k,l) :\Leftrightarrow i \leqslant k \leqslant l \geqslant j \geqslant i$. Its linking maps are given by maps of the form $g_{(i,j),(j,k)} := \hat{f}_{jk} \circ e_{ij} : F_{ij} \to F_{jk}$, where $i \leqslant j \leqslant k \in I'$, and compositions thereof. $\square$

This allows us to prove that all countable inductive systems of Definition 2.6 are actually regular.

2.8 Proposition. (See [Kom67, Lemma 3] and [FW68, § 25, Satz 2.2], or [Flo71, § 7.6].)
A weakly compact (compact, nuclear) inductive sequence $(E_n, \tau_n)_{n\in\mathbb{N}}$ with injective linking maps is regular.

Proof. By Lemma 2.7 there exists an equivalent inductive sequence $(F_n)_{n\in\mathbb{N}}$ of Banach spaces, such that all linking maps are weakly compact. Let B denote a bounded subset of the inductive limit $E := \varinjlim_n E_n = \varinjlim_n F_n$ (if it is contained and bounded in F_n it is also bounded in some E_m, so that it suffices to prove the regularity of $(F_n)_{n\in\mathbb{N}}$). Now assume indirectly that for each n either B isn't contained in $\iota_n(F_n)$ or $\iota_n^{-1}(B)$ is unbounded in F_n.

We will construct a sequence of absolutely convex 0-neighborhoods U_n in F_{k_n} (where $k_1 < k_2 < \dots$) and points $x_n \in B$, such that

(i) $f_{k_n k_m}(U_n) \subseteq U_m$ for all $m > n$,

(ii) $f_{k_n l}(U_n)$ is weakly compact in F_l for all $l > k_n$,

(iii) $x_1, \frac{1}{2}x_2, \dots, \frac{1}{n}x_n \notin \iota_{k_n}(U_n)$.

Once these are constructed we obtain a contradiction in the following way: Since B is bounded, we have that the sequence $\frac{1}{n}x_n$ converges to 0 in E as n goes to infinity. On the other hand, $U := \bigcap_{n=1}^{\infty} \iota_{k_n}(U_n)$ is a 0-neighborhood in E that, because of condition (iii), doesn't contain any of the points $\frac{1}{n}x_n$.

Construction of x_n and U_n: For $n = 1 = k_1$ let $0 \neq x_1$ be a point of B. By our assumption we may choose U_1 to be a multiple of the closed unit ball K_{k_1} in F_{k_1} such that $\iota_{k_1}(U_1)$ doesn't contain x_1. Of course $f_{k_1 l}(U_1)$ is weakly compact in each F_l for $l > k_1$.

Now let x_1, ..., x_n and U_1, ..., U_n be already constructed. Choose $k_{n+1} > k_n$ such that $x_1, ..., x_n \in \iota_{k_{n+1}}(F_{k_{n+1}})$. Then we have

$$\inf\left\{\left\|\tfrac{1}{m}\iota_{k_{n+1}}^{-1}(x_m) - u\right\|_{k_{n+1}} \,\middle|\, 1 \leqslant m \leqslant n,\ u \in f_{k_n k_{n+1}}(U_n)\right\} =: d > 0,$$

since $\frac{1}{m}x_m \notin \iota_{k_n}(U_n)$ for all $1 \leqslant m \leqslant n$ and $f_{k_n k_{n+1}}(U_n)$ is weakly compact and hence closed in the Banach space $F_{k_{n+1}}$. Now we can set

$$U_{n+1} := \Gamma\big(\tfrac{d}{2}K_{k_{n+1}} \cup f_{k_n k_{n+1}}(U_n)\big).$$

Obviously U_{n+1} is a 0-neighborhood in $F_{k_{n+1}}$ and satisfies condition (i) above. It is also weakly compact in F_l for $l > k_{n+1}$ because both $\frac{d}{2}K_{k_{n+1}}$ and $f_{k_n k_{n+1}}(U_n)$ are bounded, such that we have condition (ii). By assumption we can again find an $x_{n+1} \in B$ with $\frac{1}{n+1}x_{n+1} \notin \iota_{k_{n+1}}(U_{n+1})$.

Finally, to obtain condition (iii), it remains to be shown that also $\frac{1}{m}x_m \notin \iota_{k_{n+1}}(U_{n+1})$ for $m \leqslant n$. Let $y \in K_{k_{n+1}}$ and $u \in f_{k_n k_{n+1}}U_n$, then we have

$$\|\tfrac{1}{m}\iota_{k_{n+1}}^{-1}(x_m) - u - \tfrac{d}{2}y\|_{k_{n+1}} \geqslant \|\tfrac{1}{m}\iota_{k_{n+1}}^{-1}(x_m) - u\|_{k_{n+1}} - \|\tfrac{d}{2}y\|_{k_{n+1}} \geqslant d - \tfrac{d}{2} > 0,$$

such that we indeed obtain

$$\tfrac{1}{m}\iota_{k_{n+1}}^{-1}(x_m) \notin \tfrac{d}{2}K_{k_{n+1}} + f_{k_n k_{n+1}}(U_n) \supseteq U_{n+1}.$$

□

2.2 Dualities Revisited

Incidentally regularity of the inductive limit is also sufficient to finally obtain a topological version of Proposition 1.29 22.

2.9 Proposition. (See [FW68, § 26, Satz 2.1].)

Let $(E_i)_{i\in I}$ denote a regular inductive system, then we have the (topological) identity

$$\big(\varinjlim_j E_j\big)'_b \cong \varprojlim_j (E_j)'_b. \qquad (2.1)$$

Proof. By Proposition 1.29 we already have algebraic equality. The limit on the right hand side also makes sense topologically since the family $\big((E_j)'_b\big)_{j\in I}$ is a projective system by Lemma 1.45. The 0-neighborhoods of $\big(\varinjlim_j E_j\big)'_b$ are generated by the polars B° of bounded sets B of $E := \varinjlim_j E_j$. By the regularity of the inductive system, there exists a set $B_i \subseteq E_i$, bounded in E_i, such that $B \subseteq \iota_i(B_i)$. It follows that

$$\begin{aligned} B^\circ \supseteq \big(\iota_i(B_i)\big)^\circ &= \big\{u \in E'_b \;\big|\; |\langle \iota_i(B_i), u\rangle| \leqslant 1\big\} \\ &= \big\{u \in E'_b \;\big|\; |\langle B_i, \pi_i(u)\rangle| \leqslant 1\big\} \\ &= \big\{u \in E'_b \;\big|\; \pi_i(u) \in B_i^\circ\big\} = \pi_i^{-1}(B_i^\circ), \end{aligned}$$

which means that B° is also a 0-neighborhood of the projective topology of $\varprojlim_j (E_j)'_b = E'$. Therefore the projective topology on E' must be stronger than the strong topology of E'_b. On the other hand, all the maps $\pi_i : E'_b \to (E_i)'_b$ are continuous, and the projective topology is the weakest topology on E' with this property. Consequently the two topologies have to coincide. $\square$

To understand the topological analogue of the dual version, Proposition 1.30, we have to introduce the concept of *inductive duals* (see [Bie86, pp. 84f]). Given an arbitrary locally convex vector space (E, τ), we want to construct an inductive limit topology $\iota(E', E)$ on its dual E'.

The polar U° of a 0-neighborhood U of (E, τ) is an absolutely convex $\sigma(E', E)$-compact set in E', since it is equicontinuous, it is also $\beta(E', E)$-bounded. Therefore we may consider the associated Banach space E'_{U°. Taking all such spaces we obtain an inductive system whose limit algebraically coincides with E', although its topology $\iota(E', E)$ will in general be strictly stronger than the strong topology. We will write E'_ι for this *inductive dual* $(E', \iota(E', E))$ of (E, τ).

2.10 Proposition. (See [Kri, Lem. 3.27]).

Let $(E_i)_{i\in I}$ denote a reduced projective system of locally convex vector spaces, then we have the (topological) identity

$$\big(\varprojlim_j E_j\big)'_\iota \cong \varinjlim_j (E_j)'_b. \tag{2.2}$$

Proof. In Proposition 1.30 we already proved the algebraic equality. Furthermore, by Lemma 1.45, we obtain that the right hand side makes sense as a topological inductive limit with injective linking maps. Now recall that a base of 0-neighborhoods of $E := \varprojlim_j E_j$ is given by the the sets

$$\bigcap_{i\in J} \pi_i^{-1}(U_i)$$

where $J \in \mathcal{F}(I)$ is a finite index set and each U_i is a 0-neighborhood in (E_i, τ_i) for $i \in J$. Since it suffices to consider a base of 0-neighborhoods in the construction of the inductive dual we now only need to consider the polars of these sets, i. e., sets of the form $\cup_{i\in J}\big(\pi_i^{-1}(U_i)\big)^\circ$. By

Remark 1.24 we now obtain a base of 0-neighborhoods for the inductive dual topology of E'_ι by taking the absolutely convex hulls of arbitrary unions of such sets,

$$\Gamma\Big(\bigcup_{i\in I}(\pi_i^{-1}(U_i))^\circ\Big)$$

where each U_i is a 0-neighborhood in (E_i, τ_i) for $i \in I$.

On the other hand, Remark 1.24 also tells us that every 0-neighborhood of $\varinjlim_j (E_j)'_b$ contains a set of the form

$$\Gamma\Big(\bigcup_{i\in I}\iota_i(B_i^\circ)\Big).$$

for bounded sets B_i in (E_i, τ_i).

To see that the inductive topology on E' as induced by $\varinjlim_j (E_j)'_b$ must be weaker than the inductive dual topology of E'_ι, it therefore suffices to prove that

$$\big(\pi_i^{-1}(U_i)\big)^\circ \overset{!}{\subseteq} \pi_i^*(U_i^\circ) = \kappa_i(U_i^\circ) \overset{1.30}{=} \iota_i(U_i^\circ) \subseteq \iota_i(B_i^\circ)$$

for a 0-neighborhood $U_i \subseteq E_i$ with $B_i \subseteq U_i$.

So let $u \in \big(\pi_i^{-1}(U_i)\big)^\circ$, then $\left|\big\langle \pi_i^{-1}(U_i), u\big\rangle\right| \leqslant 1$ with respect to $\langle E, E'\rangle$ such that in particular $\big\langle \ker \pi_i|_E, u\big\rangle = 0$. Thus there exists $u_i : \pi_i(E) \to \mathbb{K}$ with $u_i \circ \pi_i = u$ and therefore $\left|\big\langle U_i \cap \pi_i(E), u_i\big\rangle\right| \leqslant 1$ with respect to $\langle \pi_i(E), \pi_i(E)'\rangle$, that is, $u_i \in \big(U_i \cap \pi_i(E)\big)^\circ$. Since $\pi_i(E)$ is dense in E_i we may extend u_i to $u_i \in U_i^\circ$ such that we indeed have $u \in \pi_i^*(U_i^\circ)$.

On the other hand, all the maps $\kappa_i : (E_i)'_b \to E'_\iota$ are continuous and the induced inductive topology on E' is the strongest topology with this property. Therefore the two inductive topologies have to coincide and the assertion follows. □

2.3 Completeness

In Proposition 2.5 we already saw that (separated) countable strict inductive limits of complete locally convex vector spaces are complete. In this section we will try to find further conditions for the completeness of an inductive limit, such as the following theorem by Raĭkov which is basically a slightly generalized version of the usual proof of the completeness part of Proposition 2.5.

2.11 Theorem (Raĭkov). (See [Flo71, § 4, Satz 1] and [Raĭ59, Thm. 1].)
Let (E, τ) denote a (not necessarily separated) topological vector space with locally convex topology τ. If there exists a sequence of absolutely convex sets $(K_n)_{n\in\mathbb{N}}$ such that

(a) $K_n \subseteq \frac{1}{2}K_{n+1}$ for each $n \in \mathbb{N}$,

(b) $E = \cup_{n=1}^\infty K_n$,

(c) an absolutely convex set $V \subseteq E$ is a 0-neighborhood of (E, τ) if, for each $n \in \mathbb{N}$, the set $V \cap K_n$ is a 0-neighborhood of $(K_n, \tau|_{K_n})$, then (E, τ) is complete if every Cauchy filter contained in a K_n converges in (E, τ) (in particular if all $(K_n, \tau|_{K_n})$ are complete).

Proof. Let $\mathcal{F}$ denote a Cauchy filter in (E, τ) and $\mathcal{U}$ the neighborhood filter of 0. Then $\mathcal{F} - \mathcal{U}$ is again a Cauchy filter base in (E, τ) which converges if and only if $\mathcal{F}$ converges (if U is a given 0-neighborhood of (E, τ) and $F \in \mathcal{F}$ such that $F - F \subseteq U$, then $(F + U) - (F + U) \subseteq 3U$). We show that there exists an $n \in \mathbb{N}$ such that $K_n \cap (F - U) \neq \emptyset$ for all $F - U \in \mathcal{F} - \mathcal{U}$.

Suppose that this were not the case. Then, for each $n \in \mathbb{N}$, we would obtain a set $F_n \in \mathcal{F}$ and a 0-neighborhood $U_n \in \mathcal{U}$ of (E, τ) such that $(F_n - U_n) \cap K_n = \emptyset$ or, equivalently, $(U_n + K_n) \cap F_n = \emptyset$. Consider the set

$$V := \bigcap_{k=1}^{\infty} (U_{k+1} + K_k).$$

Since $0 \in K_n$, property (a) implies $K_n \subseteq K_n + K_n \subseteq K_{n+1}$, so that we obtain

$$V \cap K_n = \Big(\bigcap_{k=1}^{n-1} (U_{k+1} + K_k \Big) \cap K_n,$$

which means that $V \cap K_n$ is a 0-neighborhood of K_n, equipped with the topology induced by (E, τ). Now, since this holds for each $n \in \mathbb{N}$, we can use assumption (c) to see that V has to be a 0-neighborhood of (E, τ) as well.

Using the Cauchy-property of $\mathcal{F}$, we obtain the existence of a set $F \in \mathcal{F}$ and a point $x \in E$ with $F \subseteq x + V$. By property (b) we can find an $n \in \mathbb{N}$ such that $x \in K_{n-1}$, thus

$$F \subseteq V + K_{n-1} \subseteq (U_n + K_{n-1}) + K_{n-1} \subseteq U_n + K_n.$$

Comparing this to our indirect assumption we arrive at a contradiction, since $\emptyset \neq F \cap F_{n_0} \subseteq (U_{n_0} + K_{n_0}) \cap F_{n_0} = \emptyset$.

Thus there really exists an $n \in \mathbb{N}$ such that K_n touches each set of the filter base $\mathcal{F} - \mathcal{U}$. But then also $(\mathcal{F} - \mathcal{U}) \cap K_n$ is a Cauchy filter base, since for $F_1, \ldots, F_k \in \mathcal{F}$ and $U_1, \ldots, U_k \in \mathcal{U}$ we have

$$\bigcap_{i=1}^{k} \big((F_i - U_i) \cap K_n \big) \supseteq (F - U) \cap K_n$$

where $F := \cap_{i=1}^{k} F_i$ and $U := \cap_{i=1}^{k} U_i$ form an element $F - U$ of $\mathcal{F} - \mathcal{U}$ such that $(F - U) \cap K_n \neq \emptyset$. But this means that the Cauchy filter $(\mathcal{F} - \mathcal{U}) \cap K_n$ converges in (E, τ), and therefore $\mathcal{F}$ has a limit in (E, τ) as well. $\square$

In Lemma 2.7↤33 we saw that weakly compact inductive limits are in fact inductive sequences of Banach spaces. Therefore we will from now on often consider general (LB)- and (LF)-spaces, i. e., separated inductive limits of sequences of Banach or Fréchet spaces with injective linking maps, respectively. In that setting, the following theorem of Grothendieck will be of great assistance.

2.12 Theorem (Grothendieck). (See [Köt69, § 19, 5.4] and [MV97, Satz 24.33].)
Let F denote a Fréchet space and $E = \varinjlim_n E_n$ a (LF)-space. Then every continuous and linear map $f : F \to E$ factors through a step of E, that is, there exists an $n \in \mathbb{N}$ and a continuous and linear map $g : F \to E_n$ such that $f = g \circ \iota_n$.

Proof. For each $k \in \mathbb{N}$ set

$$H_k := \{(x, y) \in F \times E_k \mid f(x) = y \in E_k\}.$$

Then H_k is a closed subspace of the Fréchet space $F \times E_k$, therefore also a Fréchet space. Consider the continuous projections $\pi_k : H_k \to F$, $\pi_k(x, y) := x$, such that $\pi_k(H_k) = f^{-1}(E_k)$. By assumption we have $F = \cup_{k=1}^{\infty} \pi_k(H_k)$, therefore Baire's category theorem implies the existence of an $n \in \mathbb{N}$ such that $\pi_n(H_n)$ is not meagre in F. Now we can apply the theorem of Banach-Schauder (see [Köt69, § 15, 12.1] or [MV97, Satz 8.4]) to obtain $\pi_n(H_n) = F$, i. e., $f(F) \subseteq E_n$. Furthermore, $f : F \to E_n$ is continuous since its graph H_n is closed. □

As a first application, this allows us to prove that—at least in the case of (LF)-spaces—regularity is necessary after all.

2.13 Proposition. (See [Flo71, § 5, Satz 4].)
If an injective inductive sequence of Fréchet spaces generates a sequentially complete (LF)-space, it has to be regular.

Proof. Let $B \subseteq E$ denote a bounded set of E which we may assume to be closed and hence sequentially complete. It generates a Banach space, which by Theorem 2.12 is continuously embedded in a step E_n of $E = \varinjlim_n E_n$. Thus B is also contained and bounded in E_n. □

Of course, we would like to have the converse as well. For (LB)-spaces we can use Raĭkov's Theorem 2.11←36 to obtain the following first step in this direction.

2.14 Proposition. (See [Flo71, § 4, Satz 3] and [Raĭ59, Thm. 3].)
An (LB)-space is complete if and only if it is *quasi-complete* (if every bounded, closed subset is complete).

Proof. ($\Leftarrow$) Let $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ denote an (LB)-space, and let B_n be the closed unit ball of the Banach space (E_n, τ_n). We will try to apply Theorem 2.11←36 for $K_n := 2^n B_n$.

Since, for each $n \in \mathbb{N}$, we may assume B_n to be continuously injected into B_{n+1}, we have the first two conditions (a) and (b) necessary for Theorem 2.11←36.

To prove (c), let $V \subseteq E$ be an absolutely convex set such that $V \cap K_n$ is a 0-neighborhood of $(K_n, \tau|_{K_n})$ for each $n \in \mathbb{N}$. By the injectivity of the inductive limit, $V \cap K_n$ already has to be a 0-neighborhood of $(K_n, \tau_n|_{K_n})$. Since K_n is a closed 0-neighborhood of (E_n, τ_n), we see that $V \cap K_n$ and hence $V \cap E_n$ are also 0-neighborhoods of (E_n, τ_n). This holds for all $n \in \mathbb{N}$, which means that V has to be a 0-neighborhood of the inductive limit (E, τ), exactly what condition (c) demanded.

Therefore, to apply Theorem 2.11 to see that (E, τ) has to be complete, we only have to show that τ-Cauchy nets contained in a K_n converge in (E, τ). But this is clear by the quasi-completeness of (E, τ) since the sets K_n and hence their Cauchy nets are bounded. $\square$

Unfortunately, Grothendieck's question whether there are (regular) (LB)-spaces which are not quasi-complete is still open (although there have been some not well-accepted claims, see for example [KM93]). Therefore we have to use additional assumptions to obtain completeness of (LB)-spaces. One way is to use the following notion of *boundedly retractive* inductive limits.

2.15 Definition (boundedly retractive). An inductive sequence $(E_n, \tau_n)_{n \in \mathbb{N}}$ of locally convex vector spaces with injective linking maps or its limit $(E, \tau) := \varinjlim_n E_n$ is called *boundedly retractive* if, for each bounded subset B of E (with respect to the inductive limit topology τ), there exists an $m \in \mathbb{N}$ such that B is not only bounded in (E_m, τ_m), but also both (E, τ) and (E_m, τ_m) induce the same topology on B.

This condition obviously implies quasi-completeness of the limit if each space of the inductive sequence is complete. Hence, by Proposition 2.14, boundedly retractive (LB)-spaces are complete.

Using the following similar condition for general locally convex vector spaces, we obtain that an inductive limit of normed spaces is boundedly retractive if and only if it is regular and satisfies the *strict Mackey convergence condition.*

2.16 Definition. A locally convex vector space (E, τ) satisfies the *strict Mackey convergence condition* if, for each bounded set A in (E, τ) there exists a closed and absolutely convex bounded subset B of (E, τ) which contains A such that both (E, τ) and the Banach space E_B associated to B induce the same topology on A.

Returning to the special case of weakly inductive limits (which had the advantage of not having to know about the inductive limit topology beforehand), we can now show that they are always complete.

2.17 Proposition. (See [Flo71, 7.6].)
A weakly compact (compact, nuclear) inductive limit of a sequence $(E_n, \tau_n)_{n \in \mathbb{N}}$ of locally convex vector spaces with injective linking maps is complete.

Proof. In Lemma 2.7 we already saw that this is actually an (LB)-space, therefore, by Proposition 2.14, it suffices to prove quasi-completeness of $(E, \tau) := \varinjlim_n E_n$.

Let B denote a bounded subset of E, then the regularity of the inductive sequence (which we obtained in Proposition 2.8) implies the existence of an $n \in \mathbb{N}$ such that B is contained and bounded in (E_n, τ_n).

Using the weak compactness of the inductive sequence we obtain another index $m \in \mathbb{N}$ such that the embedding of E_n into E_m is weakly compact, which means that B is actually relatively weakly compact in (E_m, τ_m), and therefore also in (E, τ). Thus (E, τ) is semi-reflexive and consequently quasi-complete (see Proposition 1.44). □

We actually proved that weakly compact inductive limits are semi-reflexive. Since these spaces are also barrelled (because Banach spaces are obviously barrelled this follows from Lemma 1.42 and Lemma 2.7), we even have reflexivity, which means that these spaces are the strong duals of Fréchet spaces (by Proposition 2.9 the strong duals of regular (LB)-spaces are Fréchet). Thus, summarizing the results on weakly compact inductive limits so far, we obtain the following.

2.18 Theorem. (See [Bie86, Thm. 2.4, p. 61f].)
A weakly compact injective inductive sequence $(E_n)_{n\in\mathbb{N}}$ of locally convex vector spaces is regular, and its limit $E := \varinjlim_n E_n$ is a complete reflexive (LB)-space. The dual projective sequence $((E_n)'_b)_{n\in\mathbb{N}}$ is again weakly compact, its limit is even a reflexive Fréchet space, and we have the duality

$$E'_b = \big(\varinjlim_n E_n\big)'_b = \varprojlim_n (E_n)'_b, \tag{2.3}$$

such that E is the strong dual of this reflexive Fréchet space.

What can be said about the completeness of an inductive limit if we know that every step is actually the strong dual of another locally convex vector space? In other words we now change from the setting of Proposition 2.9 to its dual version, Proposition 2.10. As it will turn out, all inductive duals of *metrizable* locally convex vector spaces are complete, but before showing this, we need to understand the concept of *bornologification.*

2.19 Lemma & Definition (bornologification). (See [KM97, Lem. 4.2] and [Jar81, § 13.3, Prop. 1].)
Let (E, τ) denote a locally convex vector space, then the following topologies on E coincide,

(i) the strongest locally convex topology having the same bounded sets as τ,

(ii) the inductive topology with respect to the inclusions $E_B \to E$, where $B \in \mathcal{B}$ runs through all bounded (closed) absolutely convex subsets of E.

This topology is called the *bornologification* τ^{bor} of τ, we also write E^{bor} for (E, τ^{bor}). It is always bornological and its continuous seminorms are exactly the bounded seminorms of (E, τ). An absolutely convex set is a 0-neighborhood in E^{bor} if and only if it is bornivorous (i. e., if it absorbs bounded sets).

Proof. Denote by $\tau_{(i)}$ and $\tau_{(ii)}$ the topologies of (i) and (ii), respectively. Then, by the universal property of the inductive topology, $\tau_{(ii)}$ must be stronger than $\tau_{(i)}$, since the inclusions of the normed spaces $E_B \to (E, \tau_{(i)})$ are all bounded and hence continuous.

On the other hand, since every bounded subset of (E, τ) is contained in some absolutely convex bounded set $B \in \mathcal{B}$, it also has to be bounded in the inductive topology $\tau_{(ii)}$ such that both topologies have the same bounded sets as (E, τ) and hence have to coincide.

A seminorm p on (E, τ) is bounded, if and only if $p(B)$ is bounded for all bounded subsets B of (E, τ), which is exactly the case if $p|_{E_B}$ is a bounded (or, equivalently, continous) seminorm on E_B for all B. Therefore, using (ii), the bounded seminorms of (E, τ) are exactly the continuous seminorms of the bornologification E^{bor}. Furthermore, this also means that every bounded seminorm on E^{bor} is actually continuous, such that the bornologification is indeed bornological. An absolutely convex subset U of E is a 0-neighborhood for $\tau^{\mathrm{bor}} = \tau_{(ii)}$ if and only if $U \cap E_B$ is a 0-neighborhood of E_B or, equivalently, U absorbs B for all $B \in \mathcal{B}$, i. e., if and only if U is bornivorous. $\square$

2.20 Proposition. (See [Jar81, § 8.5, proof of Prop. 6].)
Let (E, τ) denote a metrizable locally convex vector space, then the inductive dual topology $\iota(E', E)$ on E' equals the bornologification $\beta(E', E)^{\mathrm{bor}}$ of the strong topology.

Proof. We first remark that $\iota(E', E)$ is always stronger than $\beta(E', E)^{\mathrm{bor}}$ by definition of the inductive dual topology and 2.19(ii), since $\iota(E', E)$ is the inductive topology with respect to only some of the mappings of 2.19(ii). In particular, every bounded set of E'_ι is also bounded in $(E'_b)^{\mathrm{bor}}$ or E'_b.

Now, if we assume that (E, τ) is metrizable, then each bounded set B of E'_b has to be contained in the polar U° of a 0-neighborhood of (E, τ) and hence is also bounded in E'_ι. (Since (E, τ) is metrizable we can choose a countable, decreasing basis of 0-neighborhoods $(U_n)_{n \in \mathbb{N}}$. If we can not find a 0-neighborhood U with the desired property, then we also have $B \not\subseteq n U_n^\circ$ for all $n \in \mathbb{N}$. Therefore there exist $u_n \in B$ and $x_n \in U_n$ such that $|\langle u_n, x_n \rangle| > n$ for all $n \in \mathbb{N}$. On the one hand, $(x_n)_{n \in \mathbb{N}}$ is a 0-sequence and hence bounded in (E, τ), but on the other hand the $\beta(E', E)$-bounded sequence $(u_n)_{n \in \mathbb{N}}$ is not absorbed by the 0-neighborhood $\{x_n \mid n \in \mathbb{N}\}^\circ$ of E'_b, a contradiction.)

This means that if (E, τ) is metrizable, the spaces E'_b and E'_ι have the same bounded sets, such that we indeed obtain $\iota(E', E) = \beta(E', E)^{\mathrm{bor}}$ by 2.19(i). $\square$

To see that these spaces are actually complete, we will make use of the following simple fact.

2.21 Proposition. (See [Jar81, § 3.2, Thm. 4].)
Let τ_1 and τ_2 be two linear topologies on a vector space E such that τ_1 is stronger than τ_2 and such that (E, τ_1) has a base of 0-neighborhoods consisting of τ_2-closed sets only.

Then every τ_1-Cauchy net or filter which converges to $a \in E$ with respect to τ_2 also converges to a with respect to τ_1. In particular, every (sequentially) complete subset of (E, τ_2) is also (sequentially) complete in (E, τ_1).

2.22 Theorem. (See [Köt69, § 29.4] and [Jar81, § 13.4].)
Let (E, τ) denote a metrizable locally convex vector space. Then its inductive dual $E'_\iota = (E'_b)^{\mathrm{bor}}$ has a base of $\beta(E', E)$-closed neighborhoods of 0 such that it is always complete.

In particular, the inductive limit of an injective inductive sequence of strong duals, as in Proposition 2.10, is also always complete.

Proof. Since E'_b is complete (as the strong dual of a metrizable, hence bornological space: a given Cauchy net converges when restricted to an absolutely convex bounded set which means that its limit candidate is a locally bounded and therefore continuous linear form on E) we only have to find a base of 0-neighborhoods of $(E'_b)^{\mathrm{bor}}$ which consists of $\beta(E', E)$-closed sets in order to obtain the assertion using Proposition 2.21.

Since (E, τ) is metrizable we obtain a countable base of 0-neighborhoods $(U_n)_{n\in\mathbb{N}}$ which provides us with a fundamental sequence of absolutely convex bounded sets $(B_n := U_n^\circ)_{n\in\mathbb{N}}$ in E'_b (see for example [MV97, Lem. 25.5], also a consquence of Proposition 2.20: the bounded sets occurring in the construction of the inductive dual suffice to obtain all bounded sets).

So, if U is a 0-neighborhood in $(E'_b)^{\mathrm{bor}}$, it is actually an absolutely convex bornivorous set in E'_b, such that we can always find $\lambda_n > 0$ with $\lambda_n B_n \subseteq \frac{1}{2}U$. Now define V_k to be the absolutely convex hull of $\cup_{n=0}^{k} \lambda_n B_n$ such that we have $2V \subseteq U$ for $V := \cup_{k\in\mathbb{N}} V_k$. We are done if we can show that the algebraic hull $\cap_{\lambda>1} \lambda V$ of V contains the $\beta(E', E)$-closure $\bar{V}$ since then $\bar{V} \subseteq 2V \subseteq U$ such that all sets of the form of $\bar{V}$ (for arbitrary $\lambda_n > 0$) provide us with a base of 0-neighborhoods of $(E'_b)^{\mathrm{bor}}$ consisting of strongly closed sets.

For this purpose, assume $u \notin \cap_{\lambda>1} \lambda V$. Then there exists a $\lambda > 1$ such that $u \notin \lambda V$ and therefore $u \notin \lambda V_k$ for all $k \in \mathbb{N}$. Hence there exist $y_k \in V_k^\circ \subseteq E''$ with $\langle u, y_k\rangle > \lambda$. Since $(V_k)_{k\in\mathbb{N}}$ is still a fundamental sequence of bounded sets of E'_b, this means that the sequence $(y_k)_{k\in\mathbb{N}}$ is bounded in E''_b such that it is an equicontinuous and hence (by Alaoglu-Bourbaki) relatively $\sigma(E'', E'_b)$-compact subset of E''. Thus there exists a $y \in E''$ with $\langle u, y\rangle \geqslant \lambda > 1$ and $y \in \cap_{k=1}^{\infty} V_k^\circ = V^\circ$. This, on the other hand, means that u cannot belong to $V^{\circ\circ}$ which is (by the Bipolar Theorem 1.36) the weak closure of V, in particular u also can't be an element of the strong closure $\bar{V}$, which was to be shown. □

2.4 The Subspace Problem

Given an inductive limit $E = \varinjlim_{i\in I} E_i$ of locally convex vector space and subspaces $F_i \subseteq E_i$ of each step, $i \in I$, the inductive limit $F = \varinjlim_{i\in I} F_i$ is a linear subspace of E. Does it also have to be a topological subspace?

One might be led to believe that this follows directly from the definition of the inductive limit topology. But this only gives us a continuous injection $F \to E$, it could very well be the case that F carries a strictly stronger topology than the one induced by E, i. e., that F is a strict topological subspace of E. The explanation is simple: E carries the strongest locally convex topology which makes the injections $E_i \to E$ continuous while for the topology of F we just require the injections $F_i \to F \subseteq E$ (of much smaller spaces) to be continuous such that it can be strictly stronger. Actually the problem is even worse—the duals of F with respect to the considered topologies need not even coincide. Therefore the following properties are desirable.

2.23 Definition. (See [Bie86, § 3, Def. 16].)
Let $(E, \tau) = \varinjlim_{i \in I}(E_i, \tau_i)$ denote an injective inductive limit of locally convex spaces and let F be a linear subspace of E. We put $F_i := F \cap E_i$ and equip each of these spaces with the topology $\tilde{\tau}_i := \tau_i|_{F_i}$ induced by τ_i, $i \in I$. Then the space F is called

(i) *stepwise closed* if each $(F_i, \tilde{\tau}_i)$ is closed in (E_i, τ_i) for all $i \in I$,

(ii) a *limit subspace* of E if the inductive limit topology τ of $\varinjlim_{i \in I}(E_i, \tau_i)$ induces the inductive limit topology $\tilde{\tau}$ of $\varinjlim_{i \in I}(F_i, \tau_i)$ on F,

(iii) *well-located* in E if the dual of F is the same with respect to the restriction of the inductive limit topology τ of E and the topology of $\varinjlim_{i \in I}(F_i, \tilde{\tau}_i)$, i. e., if $(F, \tau|_F)' = (F, \tilde{\tau})'$.

2.24 Remark. Any closed linear subspace F of $E = \varinjlim_{i \in I} E_i$ is obviously stepwise closed, and each limit subspace is well-located. Under additional assumptions one can also obtain some results in the other direction (see [Bie86, § 3, Prop. 16] and [Flo80, § 5]):

Let $E = \varinjlim_n E_n$ denote a countable inductive limit of locally convex vector spaces.

(i) If the inductive limit E is weakly compact, then each stepwise closed linear subspace is closed and well-located.

(ii) If the inductive limit E is compact, then a stepwise closed subspace even has to be a (closed) limit subspace.

(iii) If all the steps E_n of E are metrizable Schwartz spaces, then every well-located subspace is also a limit subspace.

For our applications we will need a stronger version of 2.24(ii) which we will now prove using the following open-mapping lemma of Baernstein (which can be proven by applying Pták's open mapping theorem to the transposed mapping with respect to the strong duals).

2.25 Theorem (Baernstein). (See [Bae71, § 2, Lemma] and [Jar81, § 12.5, Thm. 10].)
Let E denote a separated locally convex vector space which is semi-Montel (i. e., every bounded set is relatively compact) and let F denote a (DF)-space. If $f : E \to F$ is a continuous linear map such that $f^{-1}(B)$ is bounded in E for each bounded set B of F, then f is open and hence an isomorphism $E \to f(E) \subseteq F$.

2.26 Corollary. (See [Bie86, § 3, Cor. 19].)
Let $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ denote a countable regular inductive limit of (DF)-spaces with injective linking maps. Let further F be a linear subspace of E and equip the spaces $F_n := F \cap E_n$ with the topology $\tilde{\tau}_n := \tau_n|_{F_n}$ induced by τ_n such that we may take the inductive limit topology $\tilde{\tau}$ of $\varinjlim_n (F_n, \tilde{\tau}_n)$ for F.

Now, if $(F, \tilde{\tau})$ is a semi-Montel space, then F is a limit subspace of E, i. e., $\tilde{\tau} = \tau|_F$. This condition is satisfied, in particular, if $(F_n, \tilde{\tau}_n)_{n \in \mathbb{N}}$ is a compact inductive sequence or if all the spaces $(F_n, \tilde{\tau}_n)$ are semi-Montel.

Proof. By definition of the inductive limit we obtain a continuous linear inclusion mapping $f : F \to E$. Also note that $(F_n, \tilde{\tau}_n)$ is again a regular inductive sequence. Moreover, regular inductive limits of semi-Montel spaces are clearly again semi-Montel just as separated countable inductive limits of (DF)-spaces are again (DF).

Hence, in order to apply Theorem 2.2543, we just have to consider a bounded subset B of $E = \varinjlim_n E_n$. Since E is regular there exists an index $n \in \mathbb{N}$ such that B is even contained and bounded in (E_n, τ_n). Therefore also $B \cap F$ is bounded in $(F_n, \tilde{\tau}_n)$, i. e., $f^{-1}(B)$ is bounded in $F = \varinjlim_n F_n$. The assertion now follows from the theorem. □

3

Projective Descriptions of Weighted Sequence Spaces

Our goal is to find systems of seminorms describing the inductive limit topology of some general types of countable inductive limits of locally convex vector spaces. In other words, we want to describe the limit space by trying to recognize it as a projective limit of other locally convex vector spaces, i. e., by giving a *projective description.*

We start our quest—guided by [Bie86]—by looking at the most simple general spaces, Köthe's sequence spaces, and then try to describe some inductive limits of such spaces projectively.

3.1 Definitions and Preliminaries

3.1 Definition (Köthe Sets). Let I be a (general index) set. A set A of real-valued functions on I (I-sequences) is called a *Köthe set*, if

$$\text{(K1)} \quad \forall i \in I\ \forall a = (a_i)_{i\in I} \in A\colon\ a_i \geqslant 0;$$

$$\text{(K2)} \quad \forall a, b \in A\ \exists c \in A\colon\ \max(a_i, b_i) \leqslant c_i, \forall i \in I;$$

$$\text{(K3)} \quad \forall i \in I\ \exists a \in A\colon\ a_i > 0.$$

3.2 Proposition (Köthe Sequence Spaces). Let A be a Köthe set of sequences over the general index set I. Then the space

$$\Lambda(A) := \{\lambda = (\lambda^i)_{i\in I} \in \mathbb{C}^I \mid \forall a \in A\colon\ (a_i\lambda^i)_{i\in I} \in \ell^1(I)\}$$

is a Hausdorff locally convex vector space, called the *Köthe sequence space* associated with A.

Proof. The family $(q_a)_{a\in A}$ obtained by setting $q_a(x) := \sup_{i\in I} a_i|x_i|$, $q_a : \mathbb{C}^I \to \mathbb{C} \cup \{\infty\}$, is a system of seminorms (by (K1) and (K2)) generating a locally convex topology on $\Lambda(A)$ which is Hausdorff by (K3). □

We will look at spaces similar to $\Lambda(A)$ for countable Köthe sets A, i. e., using families A of the form $((a_i^{(n)})_{i\in I})_{n\in\mathbb{N}}$ that satisfy (K1)–(K3). By going over from $a^{(n)}$ to $a^{(1)} + \ldots + a^{(n)}$, we can assume this sequence of sequences to be (pointwise) increasing. If we additionally require the sequences $a^{(n)}$ to be strictly positive (so we obtain norms instead of seminorms), the Köthe set A is called a *Köthe matrix*:

3.3 Definition (Köthe Matrix). Let I be a general index set. A (pointwise) increasing sequence $A = (a^{(n)})_{n\in\mathbb{N}}$ of strictly positive functions $a^{(n)} : I \to \mathbb{R}$ (I-sequences) is called a *Köthe matrix*.

3.4 Proposition (Köthe Echelon Spaces). Let $A = (a^{(n)})_{n\in\mathbb{N}}$ denote a Köthe matrix over the index set I, and let $1 \leqslant p < \infty$. Then we can define the *Köthe echelon spaces*

$$\lambda^p(A) := \Big\{x = (x_i)_{i\in I} \in \mathbb{C}^I \;\Big|\; \forall n \in \mathbb{N}\colon\ q_n^p(x) := \Big(\sum_{i\in I} \big(a_i^{(n)}|x_i|\big)^p\Big)^{\frac{1}{p}} < \infty\Big\},$$

$$\lambda^\infty(A) := \Big\{x = (x_i)_{i\in I} \in \mathbb{C}^I \;\Big|\; \forall n \in \mathbb{N}\colon\ q_n^\infty(x) := \sup_{i\in I} a_i^{(n)}|x_i| < \infty\Big\} \text{ and}$$

$$\lambda^0(A) := \Big\{x = (x_i)_{i\in I} \in \mathbb{C}^I \;\Big|\; \forall n \in \mathbb{N}\colon\ (a_i^{(n)}x_i)_{i\in I} \text{ tends to } 0 \text{ on } I\Big\}.$$

(An I-indexed sequence $(\alpha_i)_{i\in I}$, $\alpha_i \in \mathbb{C}$ is said to *tend to* 0 if nearly all its elements are arbitrarily small, i. e., if for each $\varepsilon > 0$ there exists a finite subset $J = J(\varepsilon)$ of I such that $|\alpha_i| < \varepsilon$ for all $i \in I\backslash J$.)

When considered together with the corresponding sequence $(q_n^p)_{n\in\mathbb{N}}$ of norms (and setting $q_n^0 := q_n^\infty$), each λ^p is a Fréchet space, $1 \leqslant p \leqslant \infty$ or $p = 0$.

Proof. Obviously all occurring q_n^p are norms. The completeness of the spaces follows from the completeness of the ℓ^p-spaces by the following remark. □

3.5 Remark. Simple examples of such Köthe echelon spaces are the “diagonal transforms” of the classical ℓ^p spaces. Let $a : I \to \mathbb{R}$ be a strictly positive function, then the constant sequence $(a)_{n\in\mathbb{N}}$ is a Köthe matrix. The associated spaces $\ell^p(a) := \lambda^p(a)$ are a-diagonal transforms of the spaces $\ell^p(I)$ (i. e., $\ell^p(I) \cong \ell^p(a)$ via $(x_i)_{i\in I} \mapsto (a_i \cdot x_i)_{i\in I}$), for $1 \leqslant p \leqslant \infty$ or $p = 0$ (where we set $\ell^0(I) := c_0(I)$).

By looking at the definition of the Köthe echelon spaces $\lambda^p(A)$ associated to a general Köthe matrix A, we obtain that they can always be identified both algebraically and topologically with projective limits of the Banach spaces $\ell^p(a^{(n)})$:

$$\lambda^p(A) \cong \varprojlim_n \ell^p(a^{(n)}).$$

So the Köthe matrix can be interpreted as a sequence of “weights” that are used in the construction of the Köthe echelon spaces out of ordinary ℓ^p-spaces.

This also motivates the following definition of *co-echelon spaces.*

3.6 Definition (Köthe co-echelon spaces). Let $V = (v^{(n)})$ denote a (pointwise) decreasing sequence of strictly positive functions $v^{(n)} : I \to \mathbb{R}$, and let $1 \leqslant p \leqslant \infty$ or $p = 0$. Then we can define the *Köthe co-echelon spaces*

$$k^p(V) := \varinjlim_n \ell^p(v^{(n)})$$

(again setting $\ell^0 := c_0$).

Obviously these spaces will be our spaces of interest—we'll try to find simple projective descriptions. But before, we still need to justify the name *co-echelon space*. For that purpose, fix a Köthe matrix $A = (a^{(n)})_{n\in\mathbb{N}}$ on some index set I. The simplest way to obtain a corresponding decreasing sequence of "weights" $V = (v^{(n)})_{n\in\mathbb{N}}$ is to set $v^{(n)} := \frac{1}{a^{(n)}}$ (this being another reason we wanted the sequences to be strictly positive).

Now let $1 \leqslant p < \infty$ or $p = 0$. Since $\lambda^p(A)$ is dense in each $\ell^p(a^{(n)})$, $n \in \mathbb{N}$ (the space c_{00} of all sequences that are eventually null is obviously contained in $\lambda^p(A)$ and dense in every $\ell^p(a^{(n)})$), the projective limit $\lambda^p(A) = \varprojlim_n \ell^p(a^{(n)})$ is reduced (cf. Remark 1.22(ii)18). Therefore we can apply the duality of projective and inductive limits (Proposition 1.3023) to obtain the algebraic identity

$$\lambda^p(A)' = \big(\varprojlim_n \ell^p(a^{(n)})\big)' = \varinjlim_n \ell^p(a^{(n)})' = \varinjlim_n \ell^q(v^{(n)}) = k^q(V), \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = 0$. Analogously, using the duality of inductive and projective limits (Proposition 1.2922), we obtain

$$k^p(V)' = \big(\varinjlim_n \ell^p(v^{(n)})\big)' = \varprojlim_n \ell^p(v^{(n)})' = \varprojlim_n \ell^q(a^{(n)}) = \lambda^q(A), \tag{3.2}$$

again for $1 \leqslant p < \infty$ or $p = 0$ where $\frac{1}{p} + \frac{1}{q} = 1$, $q = \infty$ or $q = 1$, respectively.

Are these algebraic isomorphisms also topological isomorphisms? The next Proposition is a first step towards an answer to this question:

3.7 Proposition. (See [Bie86, § 2, Prop. 9].)
Let $A = (a^{(n)})$ denote a Köthe matrix and $V = (v^{(n)})$ any decreasing sequence of strictly positive functions on an index set I, also let $1 < p < \infty$.

(i) The echelon space $\lambda^p(A)$ is a reflexive Fréchet space and the co-echelon space $k^p(V)$ is a (regular) complete reflexive (LB)-space.

(ii) With q such that $\frac{1}{p} + \frac{1}{q} = 1$, the following topological vector space dualities hold:

$$\big(\lambda^p(A)\big)_b' \cong k^q(V), \quad \text{where } V = (v^{(n)})_{n\in\mathbb{N}} \text{ with } v^{(n)} := \frac{1}{a^{(n)}}, \text{ and}$$
$$\big(k^p(V)\big)_b' \cong \lambda^q(A), \quad \text{where } A = (a^{(n)})_{n\in\mathbb{N}} \text{ with } a^{(n)} := \frac{1}{v^{(n)}}.$$

Proof. Since the building blocks $\ell^p(a^{(n)})$ and $\ell^p(v^{(n)})$ are all reflexive if $1 < p < \infty$, the projective sequence $(\ell^p(a^{(n)}))_{n\in\mathbb{N}}$ and the inductive sequence $(\ell^p(v^{(n)}))_{n\in\mathbb{N}}$ used in defining the echelon and co-echelon spaces, respectively, have to be weakly compact (see Proposition 1.44). Therefore the assertions follow easily from Theorem 2.18 and the discussion preceding this proposition. $\square$

Unfortunately our general theory of chapter 2 does not suffice to obtain similar results for the limiting cases $p = 0$, $p = 1$ and $p = \infty$, some of them actually don't hold at all (all the other statements which are not covered by the following counterexamples do hold and will be proven in the next sections).

3.8 Counterexample (Grothendieck-Köthe). (See [Bie86, § 2, Prop. 10(b)] and [Köt69, § 31.6–7].)

Let $A = (a^{(n)})_{n\in\mathbb{N}}$ denote a Köthe matrix on $I := \mathbb{N} \times \mathbb{N}$ defined by

$$a^{(n)}_{i,j} := \begin{cases} j & i \leqslant n \\ 1 & i \geqslant n+1, \end{cases}$$

and set $v^{(n)} := \frac{1}{a^{(n)}}$ to obtain the corresponding dual decreasing sequence $V = (v^{(n)})_{n\in\mathbb{N}}$ of weights. Then we have the following.

(i) Although $\lambda^1(A)' = k^\infty(V)$ holds algebraically, the strong topology of $\big(\lambda^1(A)\big)'_b$ is strictly weaker than the inductive topology of $k^\infty(V)$.

(ii) The inductive sequence $(c_0(v^{(n)}))_{n\in\mathbb{N}}$ is not regular and its limit $k^0(V) = \varinjlim_n c_0(v^{(n)})$ is (quasi-)incomplete.

Proof. (i) Let τ denote the usual inductive limit topology of $k^\infty(V)$ and let τ' denote the strong topology induced on $k^\infty(V)$ by $\big(\lambda^1(A)\big)'_b$. Now set ($n \in \mathbb{N}$)

$$B_n := \Big\{ x \in k^\infty(V) \,\Big|\, \sup_{(i,j)\in I} v^{(n)}_{i,j} |x_{i,j}| \leqslant 1 \Big\},$$

then, by Remark 1.24, the sets of the form $U = \Gamma\big(\cup_{n\in\mathbb{N}} c_n B_n\big)$, where $c_n > 0$, form a base of 0-neighborhoods in $(k^\infty(V), \tau)$ which we will denote by $\mathcal{U}$. To show that τ is strictly stronger than τ', we have to find a τ-0-neighborhood which contains no τ'-0-neighborhood. To this end, let

$$U_0 := \Gamma\Big(\bigcup_{n\in\mathbb{N}} \frac{1}{n} B_n \Big),$$

and note that U_0 contains no element of the set

$$A := \big\{ x \in k^\infty(V) \,\big|\, \forall i \in \mathbb{N}\ \exists j \in \mathbb{N}\colon |x_{i,j}| \geqslant 2 \big\}.$$

(If $u \in U_0 \cap A$, then there exists an $N \in \mathbb{N}$ such that $u \in \Gamma\big(\cup_{n=1}^N \frac{1}{n} B_n\big) \subseteq U_0$. Therefore, for $i > N$, we have $|u_{i,j}| \leqslant \sum_{n=1}^N \frac{|\alpha_n|}{n}$ where $\sum_{n=1}^N |\alpha_n| \leqslant 1$, which means that actually $|u_{i,j}| \leqslant 1$ for all $i > N$ and $j \in \mathbb{N}$, a contradiction to $u \in A$.)

A base of 0-neighborhoods for $(k^\infty(V), \tau')$ is given by the polars of $\lambda^1(A)$-bounded sets. But every bounded set of $\lambda^1(A)$ is contained in a bounded set of the form

$$\bigcap_{n\in\mathbb{N}} \{x \in \lambda^1(A) \bigm| \textstyle\sum_{(i,j)\in I} a_{i,j}^{(n)}|x_{i,j}| < \frac{1}{c_n}\} = \bigcap_{n\in\mathbb{N}} (c_n B_n)^\circ = \Big(\Gamma\Big(\bigcup_{n\in\mathbb{N}} c_n B_n\Big)\Big)^\circ$$

for suitably chosen $c_n > 0$, such that we only have to take the bipolars $U^{\circ\circ}$ of the sets $U \in \mathcal{U}$ to obtain a base of 0-neighborhoods for $(k^\infty(V), \tau')$.

Let V be a given 0-neighborhood of $(k^\infty(V), \tau')$, such there are $c_n > 0$ with $U^{\circ\circ} = \big(\Gamma(\cup_{n\in\mathbb{N}} c_n B_n)\big)^{\circ\circ} \subseteq V$. We want to show that U_0 cannot contain $U^{\circ\circ}$. Denote by $e^{(i,j)} \in \mathbb{C}^I$ the (i,j)-th unit vector, i. e., $e_{k,l}^{(i,j)} = \delta_{i,k} \cdot \delta_{j,l}$. We can always choose $k_n \in \mathbb{N}$ large enough such that $2^{n+1} e^{(n,k_n)} \in c_n B_n$. Consequently we also have

$$\sum_{n=1}^{N} \frac{1}{2^n} 2^{n+1} e^{(n,k_n)} \in \Gamma\Big(\bigcup_{n=1}^{N} c_n B_n\Big) \subseteq U.$$

Thus the weak limit $2\sum_{n\in\mathbb{N}} e^{(n,k_n)}$ of this sequence (as N tends to infinity) belongs to $U^{\circ\circ} \subseteq V$, but not to U_0 (since it is an element of A). Hence τ is indeed strictly stronger than τ'.

(ii) We first show that the (LB)-space $k^0(V) = \varinjlim_n c_0(v^{(n)})$ is not regular. Let B denote the set of all vectors $b^{(n)} \in k^0(V)$, defined as follows ($n \in \mathbb{N}$)

$$b_{i,j}^{(n)} := \begin{cases} 1 & i \leqslant n \\ 0 & i \geqslant n+1. \end{cases}$$

We obviously have $b^{(n)} \in c_0(v^{(n)})$ although $b^{(n)} \notin c_0(v^{(n-1)})$, such that B is not contained in any step $c_0(v^{(n)})$ of $k^0(V)$ even though $B \subseteq k^0(V)$. To see that B is bounded consider the base of 0-neighborhoods $\mathcal{U}$ of $k^0(V)$ consisting of the sets $U = \Gamma\big(\cup_{n\in\mathbb{N}} c_n B_n\big)$, defined as above. They all absorb the set B such that it is bounded in $k^0(V)$, hence the inductive limit cannot be regular.

Also, if $k^0(V)$ were (quasi-)complete, the closed absolutely convex hull $\Gamma(B)$ of the set B (and thus B itself) would have to be a subset of one of the steps $c_0(v^{(n)})$—which we already deemed impossible—since as it would be complete, we could apply Grothendieck's Theorem 2.12 38 to its associated Banach space $(k^0(V))_{\Gamma(B)}$. ☐

By viewing the situation from the perspective of Proposition 2.10 35 we obtain that

$$\big(\lambda^p(A)\big)'_\iota \cong k^q(V)$$

for $1 \leqslant p < \infty$ or $p = 0$ and corresponding $1 \leqslant q \leqslant \infty$. By Proposition 3.7 47 this means that for $1 < p < \infty$ the inductive dual of λ^p is the same as its strong dual,

$$\big(\lambda^p(A)\big)'_\iota \cong \big(\lambda^p(A)\big)'_b,$$

i. e., the strong dual $\big(\lambda^p(A)\big)'_b \cong k^q(V)$ is bornological. Furthermore, Counterexample 3.8(i)[48] is now also an example for a space where the inductive dual topology is indeed strictly stronger than the strong dual topology.

On the other hand, this perspective allows us to see that $k^\infty(V) \cong \big(\lambda^p(A)\big)'_\iota$ is at least always complete (by a simple application of Theorem 2.22[42]).

3.9 Corollary. Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive functions on an index set I, then $k^p(V)$ is complete for all $1 \leqslant p \leqslant \infty$.

3.2 Projective Descriptions

We arrived at our actual goal—finding projective descriptions of the Köthe co-echelon spaces $k^p(V)$, i. e., simple descriptions of their families of continuous seminorms. As a first step, we will try to find a bigger space containing $k^p(V)$ that can be described as a projective limit of ℓ^p-spaces.

Let $1 \leqslant p \leqslant \infty$ or $p = 0$ (and $\ell^0 := c_0$). For each building block $\ell^p(v^{(n)})$ of $k^p(V)$ to be continuously embedded in a space of the form $\ell^p(\check{v})$ it suffices to choose $\check{v}_i := \inf_{n\in\mathbb{N}} C_n v_i^{(n)}$ where $C_n > 0$ is any sequence of positive numbers. Therefore, if $\bar{v} = (\bar{v}_i)_{i\in I} : I \to [0,\infty)$ is a non-negative function on I such that for each $n \in \mathbb{N}$

$$\sup_{i\in I} \frac{\bar{v}_i}{v_i^{(n)}} < \infty,$$

then we obviously have that $k^p(V)$ is continuously embedded into $\ell^p(\bar{v})$, thus motivating the following definition.

3.10 Definition (Associated Köthe Sets). (See [Bie86, § 2, Def. 11] and [BMS82b, Def. 1.4].) Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive functions on an index set I, and let $A = (a^{(n)})_{n\in\mathbb{N}}$ denote the corresponding Köthe matrix where $a^{(n)} := \frac{1}{v^{(n)}}$. Then, by $\bar{V} = \bar{V}(V)$, we denote the system of non-negative functions given by

$$\begin{aligned}\bar{V}(V) &:= \lambda_+^\infty(A) := \lambda^\infty(A) \cap [0,\infty)^I \\ &= \Big\{\bar{v} = (\bar{v}_i)_{i\in I} \in [0,\infty)^I \;\Big|\; \forall n \in \mathbb{N}\colon \sup_{i\in I} \frac{\bar{v}_i}{v_i^{(n)}} = \sup_{i\in I} a_i^{(n)} \bar{v}_i < \infty\Big\}.\end{aligned}$$

The system $\bar{V}$ (which clearly satisfies (K1)–(K3) of Definition 3.1[45]) is called the *associated Köthe set* of V (respectively of A), the corresponding Köthe sequence spaces are called the *projective hulls* of the co-echelon spaces and will be denoted as follows

$$K_p(\bar{V}) := \lambda^p(\bar{V}) := \varprojlim_{\bar{v}\in\bar{V}} \ell^p(\bar{v}),$$

where $1 \leqslant p \leqslant \infty$ or $p = 0$ (and $\ell^0 := c_0$). Evidently they are always complete and separated locally convex vector spaces.

Unfortunately—although similar—these aren't always the Köthe echelon spaces we already discussed before, since $\bar{V}$ need not be countable nor does it have to contain any strictly positive functions (even though all $v^{(n)} \in V$ are strictly positive), as demonstrated in the following example.

3.11 Example. (See [BMS82b, Ex. 1.6].)
Let I denote the set of strictly decreasing sequences $i = (i_n)_{n\in\mathbb{N}}$ where $i_n \in (0, 1]$. Furthermore, consider the sequence $V = (v^{(n)})_{n\in\mathbb{N}}$ of functions $v^{(n)} : I \to (0, \infty)$ defined by $v_i^{(n)} := i_n$. Then V is apparently decreasing and strictly positive, such that we may consider the corresponding Köthe co-echelon spaces $k^p(V)$.

On the other hand, we claim that each element $\bar{v} \in \bar{V}$ must have a zero on I. By definition, there exists a sequence of positive numbers $C_n > 0$ such that $\bar{v} \leqslant \inf_{n\in\mathbb{N}} C_n v^{(n)}$. We may choose $(C_n)_{n\in\mathbb{N}}$ greater than one and increasing such that $\lim_{n\to\infty} C_n = \infty$. Then we obviously have that $j = (j_n)_{n\in\mathbb{N}} := \big(\frac{1}{C_n^2}\big)_{n\in\mathbb{N}}$ belongs to I, hence

$$\bar{v}_j \leqslant \inf_{n\in\mathbb{N}} C_n v_j^{(n)} = \inf_{n\in\mathbb{N}} C_n \frac{1}{C_n^2} = 0.$$

On the other hand, if $\bar{V}$ contains just one strictly positive function, we may restrict ourselves to strictly positive members of $\bar{V}$ when defining $K_p(\bar{V})$ as a projective limit, which explains the usefulness of the following.

3.12 Proposition. (See [BMS82b, Prop. 1.7].)
The associated Köthe set $\bar{V}$ of a decreasing sequence of strictly positive functions $V = (v^n)_{n\in\mathbb{N}}$ contains a strictly positive element, if and only if, there is an increasing sequence $(I_n)_{n\in\mathbb{N}}$ of subsets of I, such that $I = \cup_{n\in\mathbb{N}} I_n$ and $\inf_{i\in I_n} v_i^{(n)} > 0$ for each $n \in \mathbb{N}$.

In particular, $\bar{V}(V)$ always contains strictly positive elements if the index set I is countable.

Proof. $(\Rightarrow)$ Let $\bar{v} \in \bar{V}$ denote a strictly positive function and set $I_n := \big\{ i \in I \,\big|\, \bar{v}_i > \frac{1}{n} \big\}$ for $n \in \mathbb{N}$. Then $(I_n)_{n\in\mathbb{N}}$ is an increasing sequence of subsets of I and $I = \cup_{n\in\mathbb{N}} I_n$. And as there are $C_n > 0$ such that $\bar{v} \leqslant C_n v^{(n)}$ for each $n \in \mathbb{N}$, we also have

$$\inf_{i\in I_n} v_i^{(n)} \geqslant \frac{1}{C_n} \inf_{i\in I_n} \bar{v}_i \geqslant \frac{1}{nC_n} > 0.$$

$(\Leftarrow)$ For the other direction set $\frac{1}{C_n} := \inf_{I\in I_n} v_i^{(n)} > 0$ and define a function $\bar{v} := \inf_{n\in\mathbb{N}} C_n v^{(n)}$ which is obviously an element of $\bar{V}$. Since we have $C_n v_i^{(n)} \geqslant 1$ for all $i \in I_k$ where $k \leqslant n$, we obtain

$$\bar{v}_i = \inf_{n\in\mathbb{N}} C_n v_i^{(n)} \geqslant \min\big(1, \min_{n\leqslant k} C_n v_i^{(n)}\big) > 0,$$

i. e., that $\bar{v} \in \bar{V}$ is strictly positive on $I = \cup_{k\in\mathbb{N}} I_k$. □

Actually, with these projective hulls $K_p(\bar{V})$, we already obtained more than we had hoped for—an algebraic projective description of the co-echelon spaces $k^p(V)$.

3.13 Proposition. (See [BMS82b, Lemma 2.1].)
Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive I-functions and $\bar{V}$ its associated Köthe set.

Then $k^p(V)$ equals $K_p(\bar{V})$ algebraically for $1 \leqslant p \leqslant \infty$. Furthermore we have that the topologies of $k^\infty(V)$ and $K_\infty(\bar{V})$ lead to the same families of bounded sets.

Proof. We already know that $k^p(V) \subseteq K_p(\bar{V})$ hence only the other direction is left to be shown.

(a) First we will consider the case where $1 \leqslant p < \infty$. Let $x \in K_p(\bar{V})$, we have to find an $n \in \mathbb{N}$ with

$$q_n^p(x) = \Big(\sum_{i\in I} \big(v_i^{(n)}|x_i|\big)^p\Big)^{\frac{1}{p}} < \infty \tag{3.3}$$

Assume this were impossible. Then, for each $n \in \mathbb{N}$, the family $\big(\sum_{i\in J}(v_i^{(n)}|x_i|)^p\big)_{J\in\mathcal{F}(I)}$ where $\mathcal{F}(I)$ denotes all finite subsets of I, would be an unbounded subset of $\mathbb{R}$. Therefore we can find a strictly increasing sequence $(J_n)_{n\in\mathbb{N}}$ of finite subsets $J \in \mathcal{F}(I)$ such that $\sum_{i\in J_n}(v_i^{(n)}|x_i|)^p > n$.

Now set $I_1 := J_1$ and $I_k := J_k \backslash J_{k-1}$ for $k = 2, 3, \ldots$ to obtain a sequence $(I_k)_{k\in\mathbb{N}}$ of non-empty, finite and disjoint sets with $J_n = \cup_{k=1}^n I_k$. Define a function $\bar{v} = (\bar{v}_i)_{i\in I} : I \to [0, \infty)$ by setting

$$\bar{v}_i := \begin{cases} v_i^{(k)} & \text{for } i \in I_k \\ 0 & \text{if } i \notin \cup_{k\in\mathbb{N}} I_k \end{cases}$$

To prove $\bar{v} \in \bar{V}$ we remark that, given an arbitrary $n \in \mathbb{N}$, for all $i \in I_k$ where $k \geqslant n$, we obtain the inequality

$$\frac{\bar{v}_i}{v_i^{(n)}} \leqslant \frac{\bar{v}_i}{v_i^{(k)}} = 1 = \frac{\bar{v}}{v^{(n)}}\bigg|_{I_n}$$

since $V = (v^{(n)}))_{n\in\mathbb{N}}$ is decreasing. Therefore also

$$\sup_{i\in I} \frac{\bar{v}_i}{v_i^{(n)}} = \sup_{k\in\mathbb{N}} \max_{i\in I_k} \frac{\bar{v}_i}{v_i^{(n)}} = \max_{i\in J_n} \frac{\bar{v}_i}{v_i^{(n)}} < \infty$$

holds, which means that indeed $\bar{v} \in \bar{V}$. On the other hand, we clearly have $\bar{v} \geqslant v^{(n)}$ on J_n for each $n \in \mathbb{N}$ such that we obtain

$$\sum_{i\in I} \big(\bar{v}_i|x_i|\big)^p \geqslant \sum_{i\in J_n} \big(\bar{v}_i|x_i|\big)^p \geqslant \sum_{i\in J_n} \big(v_i^{(n)}|x_i|\big)^p > n,$$

a contradiction to our assumption $x \in K_p(\bar{V})$ such that also $k^p(V) \supseteq K_p(\bar{V})$ must hold.

(b) Using a similar strategy we will prove that in the case of $p = \infty$, the spaces $k^\infty(V)$ and $K_\infty(\bar{V})$ even have the same bounded sets (which obviously implies that they are algebraically equal). Again it suffices to prove that every bounded set of $K_\infty(\bar{V})$ is also contained and bounded in $k^\infty(V)$. Let B denote such a set and define a non-negative function $b = (b_i)_{i\in I} : I \to [0, \infty)$ by setting

$$b_i := \sup_{x\in B} |x_i|.$$

We have that b is an element of $K_\infty(\bar{V})$ since, for every $\bar{v} \in \bar{V}$,

$$q_{\bar{v}}^\infty(b) := \sup_{i\in I} \bar{v}_i|b_i| = \sup_{x\in B}\sup_{i\in I} \bar{v}_i|x_i| = \sup_{x\in B} q_{\bar{v}}^\infty(x) < \infty.$$

We claim that b is also an element of $k^\infty(V)$, i. e., that there exists an index $n \in \mathbb{N}$ such that

$$q_n^\infty(b) = \sup_{i\in I} v_i^{(n)} b_i < \infty \tag{3.4}$$

Again, let's assume this does not hold. This means that, for each $n \in \mathbb{N}$, the function $v^{(n)} \cdot b$ is unbounded on I. Thus we can choose indices $i_n \in I$ with $v_{i_n}^{(n)} b_{i_n} > n$ and such that $i_n \neq i_1, ..., i_{n-1}$ where $n \in \mathbb{N}$. Because of the last property we may define a non-negative function $\bar{v}$ on I by setting

$$\bar{v}_i := \begin{cases} v_{i_k}^{(k)} & \text{for } i = i_k, \\ 0 & \text{if } i \neq i_k \text{ for all } k \in \mathbb{N}. \end{cases}$$

To prove that $\bar{v}$ is actually an element of $\bar{V}$, consider as before (for $k \geqslant n$)

$$\frac{\bar{v}_{i_k}}{v_{i_k}^{(n)}} \leqslant \frac{\bar{v}_{i_k}}{v_{i_k}^{(k)}} = 1 = \frac{\bar{v}_{i_n}}{v_{i_n}^{(n)}}$$

which implies

$$\sup_{i\in I} \frac{\bar{v}_i}{v_i^{(n)}} = \sup_{k\in\mathbb{N}} \frac{\bar{v}_{i_k}}{v_{i_k}^{(n)}} \leqslant \max_{1\leqslant k\leqslant n} \frac{\bar{v}_{i_k}}{v_{i_k}^{(n)}} < \infty.$$

On the other hand, we have

$$q_{\bar{v}}^\infty(b) = \sup_{i\in I} \bar{v}_i b_i \geqslant \sup_{k\in\mathbb{N}} \bar{v}_{i_k} b_{i_k} = \sup_{k\in\mathbb{N}} v_{i_k}^{(k)} b_{i_k} \geqslant \sup_{n\in\mathbb{N}} n = \infty,$$

a contradiction to $b \in K_\infty(\bar{V})$, thus establishing our claim (3.4).

As a consequence of (3.4) we now obtain

$$\sup_{x\in B} q_n^\infty(x) = \sup_{x\in B}\sup_{i\in I} v_i^{(n)}|x_i| \leqslant \sup_{i\in I} v_i^{(n)} b_i = q_n^\infty(b) < \infty,$$

which means that B is a bounded subset of $\ell^\infty(v_n)$, and therefore also of $k^\infty(V)$, which was to be shown. $\square$

To also obtain a topological projective description we proceed as indicated by the following Lemma.

3.14 Lemma. (See [BMS82a, Lemma 1.2].)
Let (E_1, τ_1) denote a locally convex vector space, and let $E_2 \subseteq E_1$ be a linear subspace. If τ_2 is a locally convex topology on E_2 which is finer than the topology induced by τ_1 but such that both τ_1 and τ_2 induce the same topology on some dense linear subspace D of (E_2, τ_2), then we actually have $\tau_2 = \tau_1|_{E_2}$.

Proof. Let U be a closed 0-neighborhood in (E_2, τ_2). We have to find a 0-neighborhood in (E_1, τ_1) whose trace under E_2 is contained in U. Since $\tau_1|_D = \tau_2|_D$ there exists an open 0-neighborhood V in (E_1, τ_1) such that $V \cap D \subseteq U \cap D$.

Now let $x \in V \cap E_2$. For any neighborhood W of x in (E_2, τ_2) we have

$$(V \cap D) \cap W = (V \cap E_2) \cap W \cap D \neq \emptyset,$$

since D is dense in (E_2, τ_2) and $(V \cap E_2) \cap W$ is a τ_2-neighborhood of $x \in E_2$. Thus we have obtained

$$x \in \overline{V \cap D} \subseteq \overline{U \cap D} \subseteq \overline{U} = U,$$

where all closures are taken in (E_2, τ_2), such that $V \cap E_2 \subseteq U$ holds indeed. □

3.15 Lemma. (See [BMS82b, Lemma 2.2].)
If $1 \leqslant p < \infty$ or $p = 0$, then both $k^p(V)$ and $K_p(\bar{V})$ induce the same topology on their common dense subspace c_{00} of finite I-sequences.

Proof. Since, by construction, $k^p(V)$ is continuously embedded into $K_p(\bar{V})$, it suffices to show that, given an arbitrary 0-neighborhood U in $k^p(V)$, there exists a function $\bar{v} \in \bar{V}$ such that

$$B_{\bar{v}} := \{x \in c_{00} \mid q^p_{\bar{v}}(x) < 1\} \subseteq U, \tag{3.5}$$

where

$$q^p_{\bar{v}}(x) := \Big(\sum_{i \in I} \big(\bar{v}_i |x_i|\big)^p\Big)^{\frac{1}{p}} \quad \text{for } 1 \leqslant p < \infty, \quad \text{and} \quad q^0_{\bar{v}}(x) := \sup_{i \in I} \bar{v}_i |x_i|.$$

Let B_n denote the closed unit ball of $\ell^p(v^{(n)})$ or $c_0(v^{(n)})$, respectively. Then, by Remark 1.24[19], there exists a sequence of positive numbers $C_n > 0$ such that $\Gamma\big(\cup_{n \in \mathbb{N}} C_n B_n\big) \subseteq U$. Now we claim that the function $\bar{v} \in \bar{V}$,

$$\bar{v} := \inf_{n \in \mathbb{N}} \frac{2^n}{C_n} v^{(n)}$$

satisfies (3.5). To prove this claim, we will have to differentiate between the cases $1 \leqslant p < \infty$ and $p = 0$.

(a) First, let $p = 0$ and let x be an arbitrary element of $B_{\bar{v}}$. Since

$$\sup_{i \in I} \inf_{n \in \mathbb{N}} \frac{2^n}{C_n} v^{(n)}_i |x_i| = q^0_{\bar{v}}(x) < 1,$$

we obtain that for every $i \in I$ there has to be an $n \in \mathbb{N}$ with $i \in I_n$ where

$$I_n := \Big\{ i \in I \;\Big|\; \frac{2^n}{C_n} v_i^{(n)} |x_i| < 1 \Big\}.$$

In particular there exists an $m \in \mathbb{N}$ such that $\cup_{n=1}^m I_n$ contains the finite set $\operatorname{supp} x := \{i \in I \mid x_i \neq 0\}$. Let $(\varphi^{(n)})_{1 \leqslant n \leqslant m} \in c_{00}$ denote a finite partition of unity on $\operatorname{supp} x$ which is subordinate to the covering $\cup_{n=1}^m I_n$. By setting

$$a^{(n)} := \varphi^{(n)} \cdot 2^n x$$

we obtain a family $(a^{(n)})_{1 \leqslant n \leqslant m}$ of elements of $c_{00} \subseteq c_0(v^{(n)})$ with $\operatorname{supp} a^{(n)} \subseteq I_n$. We even obtain $a^{(n)} \in C_n B_n$, since for $i \in I_n$ we have

$$v_i^{(n)} \big| a_i^{(n)} \big| = \varphi_i^{(n)} 2^n v_i^{(n)} |x_i| < C_n.$$

Consequently we also have (3.5)54, since

$$x = \sum_{n=1}^m \varphi^{(n)} x = \sum_{n=1}^m \frac{1}{2^n} a^{(n)} \in \Gamma\Big(\bigcup_{n \in \mathbb{N}} C_n B_n \Big) \subseteq U.$$

(b) Now let $1 \leqslant p < \infty$ and $x \in B_{\bar{v}}$. Similarly to the previous case, given an $i \in I_0 := \operatorname{supp} x \subseteq I$ we can find an $m_i \in \mathbb{N}$ such that

$$\bar{v}_i = \inf_{n \in \mathbb{N}} \frac{2^n}{C_n} v_i^{(n)} \geqslant \frac{2^{m_i}}{C_{m_i}} v_i^{(m_i)} - \frac{\varepsilon}{|x_i| \cdot 2M^{\frac{1}{p}}}$$

where $\varepsilon := 1 - q_{\bar{v}}^p(x) > 0$ and $M := |I_0|$ is the number of elements of $I_0 = \operatorname{supp} x$. The numbers are chosen such that we obtain

$$\begin{aligned}
\Big(\sum_{i \in I_0} \Big(\frac{2^{m_i}}{C_{m_i}} v_i^{(m_i)} |x_i| \Big)^p \Big)^{\frac{1}{p}} &\leqslant \Big(\sum_{i \in I_0} \Big(\bar{v}_i |x_i| + \frac{\varepsilon}{2M^{\frac{1}{p}}} \Big)^p \Big)^{\frac{1}{p}} \\
&\leqslant \Big(\sum_{i \in I_0} \big(\bar{v}_i |x_i| \big)^p \Big)^{\frac{1}{p}} + \Big(\sum_{i \in I_0} \frac{\varepsilon^p}{2^p M} \Big)^{\frac{1}{p}} \\
&= q_{\bar{v}}^p(x) + \frac{\varepsilon}{2} = 1 - \varepsilon + \frac{\varepsilon}{2} < 1.
\end{aligned}$$

To apply this inequality we need to define a disjoint covering $(I_n)_{1 \leqslant n \leqslant N}$ of $I_0 = \operatorname{supp} x$ by setting

$$I_n := \big\{ i \in I_0 \;\big|\; m_i = n \big\} \quad \text{and} \quad N := \max_{i \in I_0} m_i.$$

The corresponding family $(a^{(n)})_{1 \leqslant n \leqslant N} \in c_{00} \subseteq \ell^p(v_n)$

$$a_i^{(n)} := \begin{cases} 2^n x_i & \text{for } i \in I_n \\ 0 & \text{if } i \notin \cup_{n=1}^N I_n, \end{cases}$$

obviously has the property that $x = \sum_{n=1}^N 2^{-n} a^{(n)}$, and its elements satisfy

$$q_n^p(a^{(n)}) = \Big(\sum_{i\in I}\big(v_i^{(n)}|a_i^{(n)}|\big)^p\Big)^{\frac{1}{p}} = C_n\left(\sum_{i\in I_n}\Big(\frac{2^n}{C_n}v_i^{(n)}|x_i|\Big)^p\right)^{\frac{1}{p}}$$
$$\leqslant C_n\left(\sum_{i\in I_0}\Big(\frac{2^{m_i}}{C_{m_i}}v_i^{(m_i)}|x_i|\Big)^p\right)^{\frac{1}{p}} < C_n.$$

Thus we even have $a^{(n)} \in C_n B_n$ which means—as in the conclusion of part (a) of this proof—that $x \in U$. $\square$

Combining the results so far, we obtain:

3.16 Theorem. (See [BMS82b, Thm. 2.3].)
Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive functions on an index set I and $\bar{V}$ its associated Köthe set. Then we have the following projective description results.

(i) For $1 \leqslant p < \infty$ the spaces $k^p(V)$ and $K_p(\bar{V})$ are topologically equal. In particular, the inductive limit topology of $k^p(V)$ is induced by the family of seminorms $(q_{\bar{v}}^p)_{\bar{v}\in\bar{V}}$ of $K_p(\bar{V})$ and $k^p(V)$ is always complete.

(ii) The projective hull $K_0(\bar{V})$ is the completion of the co-echelon space $k^0(V)$. Although $k^0(V)$ can be a proper subspace of $K_0(\bar{V})$, its topology is still induced by the projective hull.

(iii) If $p = \infty$ we only know that the complete space $k^\infty(V)$ is the bornologification of $K_\infty(\bar{V})$, since, in general, the inductive limit topology of $k^\infty(V)$ is strictly stronger than the topology induced by its projective hull $K_\infty(\bar{V})$.

Proof. By the previous Lemmas 3.14 ←54 and 3.15 ←54 we see that $K_p(\bar{V})$ always has to be the completion of $k^p(V)$ whenever $1 \leqslant p < \infty$ or $p = 0$.

If $1 \leqslant p < \infty$, then we really have $k^p(V) \cong K_p(\bar{V})$ since the spaces in question also coincide algebraically (Proposition 3.13 ←52). The counterexamples for the cases $p = \infty$ and $p = 0$ were already presented in Counterexample 3.8 ←48 (we will soon find out that $K_\infty(\bar{V}) \cong (\lambda^1(V))_b'$). $\square$

3.17 Corollary. The co-echelon space $k^0(V)$ is always a topological linear subspace of $k^\infty(V)$.

Proof. Since we have continuous injections $k^0(V) \to k^\infty(V) \to K_\infty(\bar{V})$, and $K_0(\bar{V})$ is a topological linear subspace of $K_\infty(\bar{V})$, the assertion follows immediately from statement (ii) of the theorem. $\square$

Returning to the topological duality of (co-)echelon spaces at the end of the previous section, using $\bar{V}(V)$ we can now improve upon the results we obtained there. The tool we'll be using is the following useful description of the bounded sets of the echelon spaces $\lambda^p(A)$.

3.18 Proposition. (See [BMS82b, Thm. 2.5].)
Let $A = (A_n)_{n\in\mathbb{N}}$ denote a Köthe matrix on the index set I and let $\bar{V}$ be its associated Köthe set.

A subset B of an echelon space $\lambda^p(A)$, where $1 \leqslant p \leqslant \infty$, is bounded if and only if there exists a $\bar{v} \in \bar{V}$ such that

$$B \subseteq B^p_{\bar{v}} := \left\{\bar{v}\cdot x \in \mathbb{K}^I \;\middle|\; x \in \ell^p \text{ with } \|x\|_p \leqslant 1\right\}.$$

In particular, if $\bar{V}$ contains a strictly positive function, $B \subseteq \lambda^p(A)$ is bounded if and only if there exists a strictly positive $\bar{v} \in \bar{V}$ such that B is contained and bounded in $\ell^p\left(\frac{1}{\bar{v}}\right)$.

Proof. First we consider the (easier) case of $p = \infty$. The sets $B^\infty_{\bar{v}}$ (and therefore also their subsets) are bounded in $\lambda^\infty(A)$ for each $\bar{v} \in \bar{V}$, since for all $\bar{v}\cdot x \in B^\infty_{\bar{v}}$ and $n \in \mathbb{N}$ we have

$$q^\infty_n(\bar{v}\cdot x) = \sup_{i\in I} a^{(n)}_i \bar{v}_i |x_i| \leqslant \sup_{i\in I} a^{(n)}_i \bar{v}_i =: C^{(n)}_{\bar{v}} < \infty,$$

by definition of $\bar{V}$. For the other direction let B denote any bounded subset of $\lambda^\infty(A)$. Define a function $\bar{v} = (\bar{v}_i)_{i\in I} \in [0,\infty)^I$ by setting $\bar{v}_i := \sup_{x\in B} |x_i|$. Because of

$$\sup_{i\in I} a^{(n)}_i \bar{v}_i = \sup_{i\in I}\sup_{x\in B} a^{(n)}_i |x_i| = \sup_{x\in B} q^\infty_n(x) < \infty,$$

which holds for all $n \in \mathbb{N}$, we really have $\bar{v} \in \bar{V}$. Furthermore, for each $x \in B$ we obviously have $|x| \leqslant \bar{v}$, such that indeed $B \subseteq B^\infty_{\bar{v}}$.

For the rest of the proof we will now assume that $1 \leqslant p < \infty$.

($\Leftarrow$) Again it is easy to see that $B^p_{\bar{v}}$ is bounded in $\lambda^p(A)$ for each $\bar{v} \in \bar{V}$. Let $\bar{v}\cdot x \in B^p_{\bar{v}}$ and $n \in \mathbb{N}$, then we have

$$q^p_n(\bar{v}\cdot x) = \left(\sum_{i\in I}\left(a^{(n)}_i \bar{v}_i |x_i|\right)^p\right)^{\frac{1}{p}} \leqslant C^{(n)}_{\bar{v}} \|x\|_p \leqslant C^{(n)}_{\bar{v}} < \infty$$

for $C^{(n)}_{\bar{v}} := \sup_{i\in I} a^{(n)}_i \bar{v}_i < \infty$ defined as above.

($\Rightarrow$) For the other direction it suffices to consider bounded sets $B \subseteq \lambda^p(A)$ of the form

$$B = \left\{x \in \lambda^p(A) \;\middle|\; q^p_n(x) = \left(\sum_{i\in I}\left(a^{(n)}_i |x_i|\right)^p\right)^{\frac{1}{p}} \leqslant C_n \text{ for all } n \in \mathbb{N}\right\},$$

where $C_n > 0$ are arbitrary, $n = 1, 2, \ldots$; let B denote such a set. Now we claim that, by setting $\bar{v}_i := \inf_{n\in\mathbb{N}}(2\cdot 2^n C_n v^{(n)}_i)$, which obviously defines an element $\bar{v} \in \bar{V}$, we obtain $B \subseteq B^p_{\bar{v}}$.

First we remark that we have $|x| \leqslant \bar{v}$, since for all $x \in B$ and $i_0 \in I$ we can estimate

$$|x_{i_0}| = a_{i_0}^{(n)}|x_{i_0}|v_{i_0}^{(n)} \leqslant \Big(\sum_{i\in I}\big(a_i^{(n)}|x_i|\big)^p\Big)^{\frac{1}{p}} v_{i_0}^{(n)} \leqslant C_n v_{i_0}^{(n)} < 2\cdot 2^n C_n v_{i_0}^{(n)}.$$

Thus we have also obtained that $\bar{v}_i = 0$ implies $x_i = 0$ for all $i \in I$ which will be useful later on.

Next we want to show that at least $B \cap c_{00}$ is a subset of $B_{\bar{v}}^p$. For this purpose fix an element $x \in B \cap c_{00}$ with only finitely many non-zero components and let $I_0 := \operatorname{supp}(x)$ denote its support. For each $i \in I_0$ we have (recall that by the above then also $\bar{v}_i \neq 0$)

$$\frac{1}{\bar{v}_i} = \sup_{n\in\mathbb{N}} \frac{1}{2\cdot 2^n C_n v_i^{(n)}} = \sup_{n\in\mathbb{N}} \frac{a_i^{(n)}}{2\cdot 2^n C_n}$$

which means that we can find an $m_i \in \mathbb{N}$ with

$$\frac{a_i^{(m_i)}}{2\cdot 2^{m_i} C_{m_i}} \geqslant \frac{1}{\bar{v}_i} - \frac{1}{|x_i|\cdot 2M^{\frac{1}{p}}}$$

where $M := |I_0|$ is the number of elements of I_0. Defining a disjoint covering $(I_n)_{1\leqslant n\leqslant N}$ of I_0 where $I_n := \{i \in I_0 \mid m_i = n\}$ and $N := \max_{i\in I_0} m_i$, as in the proof of Lemma 3.15[54], allows us to estimate $\frac{x}{\bar{v}}$ as follows

$$\begin{aligned}
\Big(\sum_{i\in I_0}\Big(\frac{|x_i|}{\bar{v}_i}\Big)^p\Big)^{\frac{1}{p}} &\leqslant \Big(\sum_{i\in I_0}\Big(\frac{a_i^{(m_i)}|x_i|}{2\cdot 2^{m_i}C_{m_i}} + \frac{1}{2M^{\frac{1}{p}}}\Big)^p\Big)^{\frac{1}{p}} \\
&\leqslant \Big(\sum_{n=1}^{N}\sum_{i\in I_n}\Big(\frac{a_i^{(n)}|x_i|}{2\cdot 2^n C_n}\Big)^p\Big)^{\frac{1}{p}} + \Big(\sum_{i\in I_0}\Big(\frac{1}{2M^{\frac{1}{p}}}\Big)^p\Big)^{\frac{1}{p}} \\
&= \frac{1}{2}\Big(\sum_{n=1}^{N}\Big(\frac{1}{2^n C_n}\Big)^p \underbrace{\sum_{i\in I_n}\big(a_i^{(n)}|x_i|\big)^p}_{\leqslant (C_n)^p}\Big)^{\frac{1}{p}} + \frac{1}{2} \\
&\leqslant \frac{1}{2}\Big(\sum_{n=1}^{\infty}\Big(\frac{1}{2^n}\Big)^p\Big)^{\frac{1}{p}} + \frac{1}{2} \leqslant 1.
\end{aligned}$$

Since $x_i = 0$ if $\bar{v}_i = 0$, this means that we have obtained $x \in B_{\bar{v}}^p$ and therefore $B \cap c_{00} \subseteq B_{\bar{v}}^p$ as claimed. Now it is only left to be shown that an arbitrary $x \in B$ is contained in $B_{\bar{v}}^p$ as well.

Let $x \in B$ and let $\mathcal{F}(I)$ denote the family of finite subsets of I. Define a family $(x^{(J)})_{J\in\mathcal{F}(I)} \in \mathbb{K}^I$ by setting $x_i^{(J)} := x_i$ for $i \in J$ and $x_i^{(J)} := 0$ otherwise. We clearly have that each $x^{(J)}$ is an element of $B \cap c_{00}$ and that the net $(x^{(J)})_{J\in\mathcal{F}(I)}$ converges pointwise to $x \in B$. Therefore, by the above, x also has to be contained in $B_{\bar{v}}^p$ since this set is closed under pointwise convergence on I. □

Using this description of bounded sets we can now prove the following stronger version of Proposition 3.7[47]. At the time we had only used our general theory of chapter 2[31], but this time we will present an elementary proof using the techniques introduced in this section. Since it doesn't complicate the proof much we will derive the corresponding results of Proposition 3.7[47] again, now without using the theory of weakly compact inductive limits.

3.19 Theorem. (See [BMS82b, Thm. 2.7].)
Let $A = (a^{(n)})$ denote a Köthe matrix and $V = (v^{(n)})$ any decreasing sequence of strictly positive functions on an index set I, also let $1 \leqslant p < \infty$ or $p = 0$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ (where we take $q = \infty$ for $p = 1$ and $q = 1$ for $p = 0$). Then we have the following topological dualities

$$\big(\lambda^p(A)\big)'_b \cong K_q(\bar{V}), \quad \text{where } V = (v^{(n)})_{n\in\mathbb{N}} \text{ with } v^{(n)} := \frac{1}{a^{(n)}}, \text{ and}$$

$$\big(k^p(V)\big)'_b \cong \lambda^q(A), \quad \text{where } A = (a^{(n)})_{n\in\mathbb{N}} \text{ with } a^{(n)} := \frac{1}{v^{(n)}}.$$

(For comparison with Proposition 3.7 note that we have $K_q(\bar{V}) \cong k^q(V)$ for $1 \leqslant q < \infty$ by Theorem 3.16.)

Proof. (a) We will start by proving the first duality. We already know that $(\lambda^q(A))' = k^p(V) = K_q(\bar{V})$ algebraically by the discussion in the previous section and by Proposition 3.13, thus we only have to show that the topologies of $(\lambda^p(V))'_b$ and $K_q(\bar{V})$ coincide.

For now, let's also assume $1 \leqslant p < \infty$, since in this case the previous Proposition 3.18 tells us that $(B^p_{\bar{v}})_{\bar{v}\in\bar{V}}$ is a fundamental system of bounded sets in $\lambda^p(A)$. Hence, by taking polars, we obtain a base of 0-neighborhoods in the strong dual $(\lambda^p(V))'_b$. Actually we will see that

$$\begin{aligned}
\big(B^p_{\bar{v}}\big)^\circ &= \Big\{ y \in \big(\lambda^p(A)\big)' \;\Big|\; \forall x \in \ell^p,\ \|x\|_p \leqslant 1\colon \Big|\sum_{i\in I} \bar{v}_i x_i y_i\Big| \leqslant 1 \Big\} \\
&= \begin{cases} \Big\{ y \in K_\infty(\bar{V}) \;\Big|\; q^\infty_{\bar{v}}(y) = \sup_{i\in I} \bar{v}_i |y_i| \leqslant 1 \Big\} & \text{for } p = 1, \\ \Big\{ y \in K_q(\bar{V}) \;\Big|\; q^q_{\bar{v}}(y) = \Big(\sum_{i\in I} \big(\bar{v}_i|y_i|\big)^q\Big)^{\frac{1}{q}} \leqslant 1 \Big\} & \text{for } 0 < p < \infty, \end{cases}
\end{aligned}$$

for each $\bar{v} \in \bar{V}$, such that $(\lambda^p(A))'_b \cong K_q(\bar{V})$. One of the inclusions follows using Hölder's inequality—formally we have $\|\bar{v}\cdot x \cdot y\|_1 \leqslant \|\bar{v}\cdot y\|_q \|x\|_p \leqslant \|\bar{v}\cdot y\|_q = q^q_{\bar{v}}(y)$. If $p = 1$ and $q = \infty$, the other direction is obtained by considering for $x \in \ell^p$ the unit vectors $(e^{(i)})_{i\in I}$, $e^{(i)}_j := \delta_{i,j}$. If $1 < p < \infty$ we have to choose

$$x_i := \begin{cases} \dfrac{|\bar{v}_i y_i|^q}{\bar{v}_i y_i} \big(q^q_{\bar{v}}(y)\big)^{1-q} & \text{if } \bar{v}_i y_i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In the case of $p = 0$ and $q = 1$, the above proof works equally well if we take the polars of the sets $B^\infty_{\bar{v}} \cap \lambda^0(A)$ for $\bar{v} \in \bar{V}$ to obtain a base of 0-neighborhoods in the strong dual $(\lambda^0(A))'_b$ since $\lambda^0(A)$ is a topological subspace of $\lambda^\infty(A)$. Indeed, we have

$$\big(B^\infty_{\bar{v}} \cap \lambda^0(A)\big)^\circ = \Big\{ y \in \big(\lambda^0(A)\big)' = K_1(\bar{V}) \;\Big|\; q^1_{\bar{v}}(y) = \sum_{i\in I} \bar{v}_i |y_i| \leqslant 1 \Big\},$$

which implies $(\lambda^0(A))'_b \cong K_1(\bar{V})$ as above.

(b) The second claimed identity follows by Proposition 2.9 (which doesn't make use of weakly compactness) if we can show that the inductive limit $k^p(V) = \varinjlim_n \ell^p(v^n)$ is regular. In fact, this follows easily for $1 \leqslant p < \infty$, since we already know (Theorem 3.16(i)) that in this case $k^p(V)$ has to be complete (which implies regularity by Proposition 2.13).

In the case of $p = 0$ and $q = 0$ we again have to deliver a special argument as the inductive limit $k^0(V) = \varinjlim_n c_0(v^n)$ need not necessarily be regular (Counterexample 3.8(ii)). But, proceeding as in the first part of this proof, we obtain

$$\left\{x \in c_0(v^{(n)}) \;\middle|\; q_n^\infty(x) = \sup_{i \in I} v_i^{(n)} |x_i| \leqslant 1\right\}^\circ = \left\{y \in \lambda^1(a^{(n)}) \;\middle|\; q_n^1(y) = \sum_{i \in I} a_i^{(n)} |y_i| \leqslant 1\right\}$$

with respect to the dual system $\langle k^0(V), (k^0(V))' \rangle = \langle k^0(V), \lambda^1(A) \rangle$. This means that each set of the corresponding base of 0-neighborhoods in $\lambda^1(A)$ is given by the polar of a bounded set of $k^0(V)$, such that the strong topology $\beta\big(\lambda^1(A), k^0(V)\big)$ on $\lambda^1(A)$ is stronger than its usual projective limit topology.

On the other hand, by the first part of this proof, we know that $(\lambda^1(A))'_b \cong K_\infty(\bar{V})$ which has the same bounded sets as $k^\infty(V)$ by Proposition 3.13. Furthermore, the space $k^0(V)$ is continuously embedded in $k^\infty(V)$, such that the topology $\beta\big(\lambda^1(A), k^0(V)\big)$ has to be weaker than $\beta\big(\lambda^1(A), k^\infty(V)\big) = \beta\big(\lambda^1(A), (\lambda^1(A))'\big)$, which is of course the canonical projective limit topology of the Fréchet space $\lambda^1(A)$.

Therefore we have obtained $\beta\big(\lambda^1(A), k^0(V)\big) = \beta\big(\lambda^1(A), (\lambda^1(A))'\big)$ and hence $(k^0(V))'_b \cong \lambda^1(A)$, as claimed. □

In view of Conterexample 3.8, we are now only left with the following two questions (see [Bie86, p. 71]).

(i) What is a necessary and sufficient condition (in terms of the Köthe matrix $A = (a_n)_{n \in \mathbb{N}}$) for the topological equality $\big(\lambda^1(A)\big)'_b \cong k^\infty(V)$ where $V = (v_n)_{n \in \mathbb{N}}$ with $v_n := \frac{1}{a_n}$?

(ii) What is a necessary and sufficient condition (in terms of the decreasing sequence of weights $V = (v_n)_{n \in \mathbb{N}}$) for $k^0(V)$ to be (regular or) complete?

In other words—via Theorem 3.16—, what is the exact relationship between the spaces $k^0(V)$ and $K_0(\bar{V})$ as well as between $k^\infty(V)$ and $K_\infty(\bar{V})$?

The answers to these questions will again be a byproduct of our actual goal—also obtaining projective descriptions of the inductive limit spaces $k^0(V)$ and $k^\infty(V)$.

3.3 The Regularly Decreasing Condition

In this section we want to find conditions in terms of the Köthe matrix such that we can also describe the spaces $k^\infty(V)$ and $k^0(V)$ projectively.

Actually, we already have a projective description of the topology of $k^0(V)$, namely $K_0(\bar{V})$, the only problem is that the space $k^0(V)$ itself could be smaller. For $k^\infty(V)$ it's the other way round—although we can describe the linear vector space projectively via $K_\infty(\bar{V})$, its topology might be strictly stronger than the one obtained from the projective description. In other words, by combining the properties of these spaces, there would be no more problems. Therefore the first obvious idea towards a solution is to find a condition which implies (topological) equality of the spaces $k^\infty(V)$ and $k^0(V)$. This will of course be quite a strong requirement—but maybe we can weaken it without destroying the positive effects.

Of course, if we even have $\ell^\infty(v^{(n)}) = c_0(v^n)$ for each $n \in \mathbb{N}$, then this equality follows immediately by definition of the inductive limits. A similar but less trivial condition would be one that only implies the equivalence of the inductive sequences $(\ell^\infty(v^{(n)}))_n$ and $(c_0(v^n))_n$ since this would lead us to the same conclusion.

For this to happen, for each $n \in \mathbb{N}$ we would need a continuous injection from $\ell^\infty(v^{(n)})$ into $c_0(v^{(m)})$ for some $m > n$. Surely we have such an injection if $v^{(m)}/v^{(n)}$ tends to zero on I (that is, if for every $\varepsilon > 0$ there is a finite set $J \subseteq I$ such that $v_i^{(m)} \leqslant \varepsilon \cdot v_i^{(n)}$ whenever $i \in I \setminus J$), since then

$$v_i^{(m)} |x_i| \leqslant \varepsilon \cdot v_i^{(n)} |x_i| \leqslant \varepsilon \cdot C \quad (i \in I \setminus J)$$

where the constant $C > 0$ only depends on $v^{(n)}$ and $x \in \ell^\infty(v^{(n)})$, such that $v^{(m)} \cdot |x|$ also tends to zero, which means that $x \in c_0(v^{(m)})$. Because $c_0(v^{(m)})$ is continuously embedded into $\ell^\infty(v^{(m)})$, we have obtained that the inductive sequences $(\ell^\infty(v^{(n)}))_n$ and $(c_0(v^{(n)}))_n$ are equivalent if

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}, m > n\colon\ \frac{v^{(m)}}{v^{(n)}} \text{ converges to } 0. \tag{S}$$

As we already suspected, this condition (S) (which is called (V) in [BMS82a]) really solves our problems:

3.20 Proposition. Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive functions on I. If V satisfies condition (S), then

$$k^0(V) \cong k^\infty(V) \cong K_0(\bar{V}) \cong K_\infty(\bar{V}),$$

such that $k^0(V)$ is complete and the topology of $k^\infty(V)$ is given by the projective limit topology of $K_\infty(\bar{V})$.

Proof. First, by the preceding discussion, we obviously have $k^0(V) \cong k^\infty(V)$. Furthermore condition (S) means that the inductive system $(\ell^p(v^{(n)}))_{n\in\mathbb{N}}$ is compact for all $1 \leqslant p \leqslant \infty$ and in particular also for $p = 0$ such that Theorem 2.18$_{\leftarrow 40}$ and Theorem 3.19$_{\leftarrow 59}$ imply

$$k^\infty(V) \cong k^0(V) \cong \left(\left(k^0(V)\right)_b'\right)_b' \cong \left(\lambda^1(V)\right)_b' \cong K_\infty(\bar{V}),$$

We also have $k^0(V) \cong K_0(\bar{V})$ by Theorem 3.16(ii)$_{\leftarrow 56}$ since $k^0(V)$ has to be complete by Proposition 2.17$_{\leftarrow 39}$. □

Put differently, condition (S) is sufficient to make the proof of Proposition 3.7 also work for the border cases of $p = 0$, $q = 1$ and $p = 1$, $q = \infty$. But we had proved this result also without using the properties of weakly compact inductive/projective systems in Theorem 3.19, so we can remain hopeful to find weaker conditions which still imply projective description of $k^0(V)$ or $k^\infty(V)$.

For example, if $V = (\nu)_{n\in\mathbb{N}}$ is the constant sequence consisting of a strictly positive function $\nu : I \to (0, \infty)$ then V does not satisfy condition (S), although we clearly have $k^0(V) \cong c_0(\nu) \cong K_0(\bar{V})$ and $k^\infty(V) \cong \ell^\infty(\nu) \cong K_\infty(\bar{V})$.

On the other hand, if $k^0(V)$ ought to be complete (such that $k^0(V) \cong K_0(\bar{V})$), then by Proposition 2.13 the corresponding inductive sequence has to be (at least) regular. In particular, we also have $k^0(V) \cong K_0(\bar{V})$ if the inductive sequence is boundedly retractive (see Definition 2.15). Since it will turn out that in this case regularity (and therefore completeness) of $k^0(V)$ is equivalent to being boundedly retractive, we will try to find a characterisation of boundedly retractiveness via the decreasing sequence V.

Condition (S) meant that V has to be "rapidly decreasing", so what about a less drastic "regularly decreasing" instead?

3.21 Definition (regularly decreasing). (See [BMS82b, Def. 3.1].)
A decreasing sequence $V = (\nu^{(n)})_{n\in\mathbb{N}}$ of strictly positive I-functions is called *regularly decreasing* if for each $n \in \mathbb{N}$ there exists an $m \geqslant n$ such that for all $\varepsilon > 0$ and $k \geqslant m$ we can find $\delta_{k,\varepsilon} > 0$ with $(i \in I)$

$$\nu_i^{(k)} < \delta_{k,\varepsilon} \cdot \nu_i^{(n)} \implies \nu_i^{(m)} < \varepsilon \cdot \nu_i^{(n)}. \tag{3.6}$$

(The quotient $\nu^{(m)}/\nu^{(n)}$ can be made arbitrarily small, as long as one of the following quotients $\nu^{(k)}/\nu^{(n)}$ is small enough.)

In other words, V is *regularly decreasing* if for each $n \in \mathbb{N}$ there exists an $m \geqslant n$ such that for all subsets $J \subseteq I$ and $k \geqslant m$

$$\inf_{i\in J} \frac{\nu_i^{(k)}}{\nu_i^{(n)}} = 0 \implies \inf_{i\in J} \frac{\nu_i^{(m)}}{\nu_i^{(n)}} = 0. \tag{3.7}$$

(This condition is clearly necessary, it is also sufficient: assuming indirectly that for each $\delta > 0$ we could choose an index $i \in I$ such that $\nu_i^{(m)} > \varepsilon \cdot \nu_i^{(n)}$ although $\nu_i^{(k)} < \delta \cdot \nu_i^{(n)}$ we would obtain a set of indices $J \subseteq I$ for which (3.7) does not hold either.)

To prove that $k^0(V) \cong K_0(\bar{V})$ is equivalent to V being regularly decreasing we will also need, among other things, the following characterisation of the regularly decreasing condition via the associated Köthe set.

3.22 Proposition. (See [BMS82b, Prop. 3.2].)
A decreasing sequence $V = (v^{(n)})_{n \in \mathbb{N}}$ is regularly decreasing if and only if for each $n \in \mathbb{N}$ there exists an $m > n$ such that for every $\varepsilon > 0$ there is a $\bar{v} \in \bar{V}$ with $(i \in I)$

$$\bar{v}_i < v_i^{(m)} \implies v_i^{(m)} \leqslant \varepsilon \cdot v_i^{(n)}, \tag{wS}$$

or, equivalently, if and only if, given $n \in \mathbb{N}$, there exists an $m > n$ such that for each subset $J \subseteq I$ and every $\bar{v} \in \bar{V}$ we have

$$\bar{v} < v^{(m)} \text{ on } J \implies \inf_{i \in J} \frac{v_i^{(m)}}{v_i^{(n)}} = 0. \tag{3.8}$$

Proof. ($\Leftarrow$) Let V satisfy condition (wS), and given $n \in \mathbb{N}$, take $m > n$ as in (wS). To show that V is regularly decreasing as in (3.6), fix an $\varepsilon > 0$ and $k \geqslant m$. By (wS) we can find a $\bar{v} \in \bar{V}$ such that $v_i^{(m)} \leqslant \frac{\varepsilon}{2} \cdot v_i^{(n)} < \varepsilon \cdot v_i^{(n)}$ whenever $\bar{v}_i < v_i^{(m)}$ (or, $\bar{v}_i \geqslant v_i^{(m)}$ whenever $v_i^{(m)} \geqslant \varepsilon \cdot v_i^{(n)}$). On the other hand, by construction of $\bar{V}$, there always exists $C_k > 0$ so that $\bar{v}_i \leqslant C_k v_i^{(k)}$ which means that by setting $\delta_{k,\varepsilon} := \frac{\varepsilon}{C_k}$ we obtain

$$v_i^{(m)} \geqslant \varepsilon \cdot v_i^{(n)} \implies v_i^{(k)} \geqslant \frac{1}{C_k} \bar{v}_i \geqslant \frac{1}{C_k} v_i^{(m)} \geqslant \delta_{k,\varepsilon} \cdot v_i^{(n)}$$

which is just (3.6) reversed.

($\Rightarrow$) For the other direction, given $n \in \mathbb{N}$ take $m > n$ as in (3.6) and, for a fixed $\varepsilon > 0$ and each $k > m$, choose $\delta_{k,\varepsilon} > 0$ such that $v_i^{(k)} \geqslant \delta_{k,\varepsilon} \cdot v_i^{(n)}$ whenever $v_i^{(m)} \geqslant \varepsilon \cdot v_i^{(n)}$. We claim that

$$\bar{v}_i := \inf_{l \in \mathbb{N}} \frac{1}{C_l} v_i^{(l)} \qquad \text{where} \qquad 0 < C_l := \begin{cases} \varepsilon & \text{for } 1 \leqslant l \leqslant m, \\ \delta_{l,\varepsilon} & \text{if } l > m, \end{cases}$$

which obviously defines an element $\bar{v} \in \bar{V}$, satisfies (wS). To prove this claim, assume indirectly that $v_i^{(m)} > \varepsilon \cdot v_i^{(n)}$, then by (3.6) also $v_i^{(k)} \geqslant \delta_{k,\varepsilon} \cdot v_i^{(n)}$ for all $k > m$ such that by construction of $\bar{v}$ and since V is decreasing $\bar{v}_i = \inf_{l \geqslant m} \frac{1}{C_l} v_i^{(l)} \geqslant v_i^{(n)} \geqslant v_i^{(m)}$ holds as well. □

This characterisation allows to prove that a regularly decreasing sequence V leads to a boundedly retractive inductive limit $k^0(V)$, as promised above.

3.23 Proposition. (See [BMS82b, Lemma 3.6, first part of the proof of Prop. 3.3].)
Let $V = (v^{(n)})_{n \in \mathbb{N}}$ denote a *regularly decreasing* sequence of strictly positive I-functions. Then $k^0(V) = \varinjlim_n c_0(v^{(n)})$ is a boundedly retractive inductive limit, such that $k^0(V)$ is complete and $k^0(V) \cong K_0(\bar{V})$.

Proof. We will prove that if the sequence $V = (v^{(n)})_{n \in \mathbb{N}}$ satisfies condition (wS), then $c_0(v^{(m)})$, $k^0(V)$ and $K_0(\bar{V})$ all induce the same topology on each bounded subset of $c_0(v^{(n)})$ where $m > n$ is chosen according to (wS). This then clearly implies that $k_0(V)$ is boundedly retractive and therefore also complete such that by Theorem 3.16(ii) we obtain $k^0(V) \cong K_0(\bar{V})$.

Let $n \in \mathbb{N}$ and let B denote any bounded subset of $c_0(v^{(n)})$, without loss of generality we may assume that

$$B = \left\{ x \in c_0(v^{(n)}) \;\middle|\; q_n^\infty(x) = \sup_{i \in I} v_i^{(n)} |x_i| \leqslant k \right\}$$

for some $k \in \mathbb{N}$. Chose $n < m \in \mathbb{N}$ according to (wS). Since we have continuous embeddings $c_0(v^{(n)}) \to c_0(v^{(m)}) \to k^0(V) \to K_0(\bar{V})$ it suffices to prove that for each $b \in B$ and all $\varepsilon > 0$ there exists a 0-neighborhood W in $K_0(\bar{V})$ such that

$$W \cap B \subseteq U_\varepsilon(b) := \left\{ x \in c_0(v^{(n)}) \;\middle|\; q_m^\infty(x - b) = \sup_{i \in I} v_i^{(m)} |x_i - b_i| \leqslant \varepsilon \right\},$$

because the family $(U_\varepsilon(x))_{\varepsilon > 0}$ is a basis of neighborhoods of x in $c_0(v^{(m)})$.

So, let $b \in B$ and $\varepsilon > 0$ be arbitrarily chosen. By (wS) there exists a function $\bar{v} \in \bar{V}$ such that

$$\bar{v}_i < v_i^{(m)} \implies v_i^{(m)} \leqslant \frac{\varepsilon}{2k} \cdot v_i^{(n)}.$$

Now set

$$W := \left\{ x \in k^0(V) \;\middle|\; q_{\bar{v}}^\infty(x - a) = \sup_{i \in I} \bar{v}_i |x_i - b_i| \leqslant \varepsilon \right\}.$$

Then we have $W \cap B \subseteq U_\varepsilon(b)$ since for $x \in W \cap B$ and if $\bar{v}_i \geqslant v_i^{(m)}$ we clearly obtain

$$v_i^{(m)} |x_i - b_i| \leqslant \bar{v}_i |x_i - b_i| \leqslant \varepsilon,$$

on the other hand, if $\bar{v}_i < v_i^{(m)}$ then $v_i^{(m)} \leqslant \frac{\varepsilon}{2k} \cdot v_i^{(n)}$ such that

$$v_i^{(m)} |x_i - b_i| \leqslant \frac{\varepsilon}{2k} v_i^{(n)} |x_i - b_i| \leqslant \frac{\varepsilon}{2k} \left(v_i^{(n)} |x_i| + v_i^{(n)} |b_i| \right) \leqslant \varepsilon$$

since $x, b \in B$. □

The regularly decreasing condition is not only sufficient to obtain projective description for $k^0(V)$, it is even equivalent. But before showing this we have to prove the following technical lemma.

3.24 Lemma. If a decreasing sequence $V = (v^{(n)})_{n \in \mathbb{N}}$ of strictly positive I-functions is *not* regularly decreasing, then there exists a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ of positive numbers and a pair of sequences $(\delta_{jl}, i_{jl})_{l \in \mathbb{N}}$ for each $j \in \mathbb{N}$ such that (for all $j, l, r, s \in \mathbb{N}$)

(i) $0 < \delta_{jl} \leqslant \frac{1}{l}$,

(ii) $v_{i_{jl}}^{(j)} \geqslant \varepsilon_j$ although $v_{i_{jl}}^{(j+1)} < \delta_{jl}$,

(iii) $i_{jl} \neq i_{rs}$ whenever $(j, l) \neq (r, s)$.

Proof. Since V is not regularly decreasing we can fix an $n \in \mathbb{N}$ such that for every $m \geqslant n$ there exist $k_m > m$ and $\varepsilon_m > 0$ for which, given $\delta > 0$, we can always find an index $i_\delta \in I$ such that

$$v_{i_\delta}^{(m)} \geqslant \varepsilon_m \cdot v_{i_\delta}^{(n)} \quad \text{although} \quad v_{i_\delta}^{(k_m)} < \delta \cdot v_{i_\delta}^{(n)}.$$

To simplify this situation we may assume, without loss of generality, that the sequence $(\varepsilon_m)_{m \geqslant n}$ is decreasing and $(k_m)_{m \geqslant n}$ is increasing (the condition above is clearly also satisfied for smaller ε or larger k). Furthermore, by passing to an equivalent subsequence of V, we may assume $n = 1$ and $k_m = m + 1$. Also, the resulting co-echelon space $k^0(V)$ does not change (and V is still *not* regularly decreasing) when dividing every $v^{(n)}$ by $v^{(1)}$ such that we may even assume $v^{(1)} \equiv 1$.

Thus, the simplified situation now tells us that for every $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ for which, given $\delta > 0$, we can always find $i_\delta \in I$ such that

$$v_{i_\delta}^{(m)} \geqslant \varepsilon_m \quad \text{although} \quad v_{i_\delta}^{(m+1)} < \delta. \tag{3.9}$$

For each $j \in \mathbb{N}$, we will now inductively construct a sequence of pairs $(\delta_{jl}, i_{jl})_{l \in \mathbb{N}}$ which satisfy the conditions (i)–(iii). If $j = 1$ we can simply use (3.9) with $\delta_{11} := 1$, and inductively choose $0 < \delta_{1l} < \min\{v_{i_{1,l-1}}^{(2)}, \frac{1}{l}\}$ for $l > 1$ to obtain the first desired sequence $(\delta_{1l}, i_{1l})_{l \in \mathbb{N}}$. (Because of $v_{i_{1l}}^{(2)} < \delta_{1l} < v_{i_{1,l-1}}^{(2)}$ for all $l > 1$ the sequence $(v_{i_{1l}}^{(2)})_{l \in \mathbb{N}}$ is strictly decreasing such that also (iii) holds.)

Assuming that $(\delta_{jl}, i_{jl})_{l \in \mathbb{N}}$ has already been constructed for $j = 1, \ldots, p-1$ such that (i)–(iii) hold, we will try to find the p-th sequence $(\delta_{pl}, i_{pl})_{l \in \mathbb{N}}$. By property (i) of the induction hypothesis we can find an index $l_p \in \mathbb{N}$ such that $\delta_{jl} < \varepsilon_p$ for all $l > l_p$ and $1 \leqslant j \leqslant p-1$. We will now construct δ_{pl} and i_{pl} by induction on $l \in \mathbb{N}$. For $l = 1$ choose

$$\delta_{p1} < \min\Big\{\beta_{p1}, \frac{1}{1}\Big\} \quad \text{where} \quad \beta_{p1} := \min_{\substack{1 \leqslant j \leqslant p-1 \\ 1 \leqslant \ell \leqslant l_p}} v_{i_{j\ell}}^{(p+1)},$$

(which clearly satisfies (i)), such that by (3.9) we can find $i_{p1} \in I$ with

$$v_{i_{p1}}^{(p)} \geqslant \varepsilon_p \quad \text{although} \quad v_{i_{p1}}^{(p+1)} < \delta_{p1},$$

i.e., we have (ii). By the selection of $\delta_{p1} < \beta_{p1}$ it follows as before that $i_{p1} \neq i_{jl}$ for all $1 \leqslant j \leqslant p-1$ and $1 \leqslant l \leqslant l_p$. If on the other hand $l > l_p$ and $1 \leqslant j \leqslant p-1$, we have

$$v^{(p)}(i_{jl}) \leqslant v^{(j+1)}(i_{jl}) < \delta_{jl} < \varepsilon_p,$$

such that $i_{p1} \neq i_{jl}$ for all $1 \leqslant j \leqslant p-1$ and $l \in \mathbb{N}$ as demanded by (iii). To obtain δ_{pl} and i_{pl} for $l > 1$ we only have to take

$$\delta_{pl} < \min\Big\{\beta_{pl}, \frac{1}{l}\Big\} \quad \text{where} \quad \beta_{pl} := \min\Big\{\beta_{p1}, v_{i_{p,1}}^{(p+1)}, \ldots, v_{i_{p,l-1}}^{(p+1)}\Big\},$$

then (ii) and (iii) follow as for $l = 1$. Thus we have completed both inductions and obtained the desired sequence of sequences. $\square$

3.25 Theorem. (See [BMS82b, Thm. 3.7].)
Let $V = (v^{(n)})_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive I-functions. Then the following assertions are equivalent:

(a) V is regularly decreasing,

(b) $k^0(V) = \varinjlim_n c_0(v^{(n)})$ is boundedly retractive,

(c) $k^0(V)$ is complete,

(d) $k^0(V)$ is algebraically (or topologically) equal to $K_0(\bar{V})$,

(e) $k^0(V) = \varinjlim_n c_0(v^{(n)})$ is a regular inductive limit.

Proof. We just proved in Proposition 3.23 that (a) $\Rightarrow$ (b), the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (e) are clear and (c) $\Leftrightarrow$ (d) follows from Theorem 3.16. Therefore it is only left to be shown that (e) $\Rightarrow$ (a) holds as well.

Assume indirectly that $V = (v^{(n)})_{n\in\mathbb{N}}$ is not regularly decreasing. Using the properties (i)–(iii) of the sequences $(\varepsilon_m)_{m\in\mathbb{N}}$ and $(\delta_{jl}, i_{jl})_{j,l\in\mathbb{N}}$ obtained in Lemma 3.24 we will now try to construct a bounded subset of $k^0(V)$ which is not contained in any $c_0(v^{(n)})$, thus demonstrating that in our situation $k^0(V)$ cannot be a regular inductive limit.

To obtain this bounded set we will first construct a sequence $(x^{(j)})_{j\in\mathbb{N}} \in \mathbb{C}^I$ such that $x^{(j)} \in c_0(v^{(j+1)})$ but $x^{(j)} \notin c_0(v^{(j)})$. Let $(\eta_j)_{j\in\mathbb{N}} \in \mathbb{R}$ denote a decreasing sequence with $\eta_j > 0$ and $\lim_{j\to\infty} \eta_j = 0$. For each $j \in \mathbb{N}$ and $i \in I$ set

$$x_i^{(j)} := \begin{cases} \eta_j & \text{if } i = i_{jl} \text{ for an } l \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

To prove that $x^{(j)} \in c_0(v^{(j+1)}) \setminus c_0(v^{(j)})$ for a fixed $j \in \mathbb{N}$ let $\varepsilon > 0$. Since (i) implies $\delta_{jl} \leqslant \frac{1}{l} \to 0$ as $l \to \infty$, we can find $L \in \mathbb{N}$ such that $\delta_{jl} \cdot \eta_j < \varepsilon$ for all $l > L$. Therefore, for all such $l > L$ we have

$$v_{i_{jl}}^{(j+1)} x_{i_{jl}}^{(j)} = v_{i_{jl}}^{(j+1)} \eta_j < \delta_{jl} \cdot \eta_j < \varepsilon.$$

Thus the set $\{ i \in I \mid v_i^{(j+1)} x_i^{(j)} \geqslant \varepsilon \}$ has to be finite since it is contained in $\{ i_{jl} \mid l \leqslant L \}$ which means that $x^{(j)} \in c_0(v^{(j+1)})$.

On the other hand, we have

$$v_{i_{jl}}^{(j)} x_{i_{jl}}^{(j)} = v_{i_{jl}}^{(j)} \eta_j \geqslant \varepsilon_j \cdot \eta_j$$

for all $l \in \mathbb{N}$, such that $x^{(j)} \notin c_0(v^{(j)})$ since infinitely many components of $x^{(j)}$ are bounded away from 0.

Obviously, $B := \{x^{(j)} \mid j \in \mathbb{N}\}$ will be our candidate for the bounded subset of $k^0(V)$ which is not contained in any of its steps $c_0(v^{(j)})$. To prove the boundedness of B we will show that $x := \sum_{j=1}^{\infty} x^{(j)}$ is an element of $K_0(\bar{V})$. Since the spaces $K_0(\bar{V})$ and $k^0(V)$ have the same bounded sets (Proposition 3.13) and $x^{(j)} \leqslant x$ for all $j \in \mathbb{N}$, we would then be done.

Firstly, we remark that $x := \sum_{j=1}^{\infty} x^{(j)}$ makes sense because of (iii). To see that $x \in K_0(\bar{V})$, fix $\bar{v} \in \bar{V}$ and let $\varepsilon > 0$. We have to show that

$$F := \left\{ i \in I \;\middle|\; \bar{v}_i x_i \geqslant \varepsilon \right\}$$

is a finite set. By definition, x_i can only be non-zero if $i = i_{jl}$ for some $j, l \in \mathbb{N}$, therefore F can only consist of such indices. Also, by definition of $\bar{V}$, there exist $\alpha_j > 0$ such that $\bar{v} \leqslant \inf_{j \in \mathbb{N}} \alpha_j \cdot v^{(j)}$. Now, if $\eta_j < \frac{\varepsilon}{\alpha_1}$ then

$$\bar{v}_{i_{jl}} x_{i_{jl}} \leqslant \alpha_1 v^{(1)}_{i_{jl}} x^{(j)}_{i_{jl}} = \alpha_1 \cdot \eta_j < \varepsilon.$$

Furthermore, for all $j \in \mathbb{N}$, if we have $\delta_{jl} < \frac{\varepsilon}{\alpha_{j+1} \cdot \eta_j}$ then also

$$\bar{v}_{i_{jl}} x_{i_{jl}} \leqslant \alpha_{j+1} v^{(j+1)}_{i_{jl}} x^{(j)}_{i_{jl}} < \alpha_{j+1} \cdot \delta_{jl} \cdot \eta_j < \varepsilon.$$

But both of these conditions are satisfied for infinitely many $j \in \mathbb{N}$ since $\eta_j \to 0$ for $j \to \infty$ and $\delta_{jl} \to 0$ for $l \to \infty$, such that F is indeed a finite set, which means that $x \in K_0(\bar{V})$ so that B is the desired counterexample to the regularity of $k^0(V)$. $\square$

Using similar ideas as in the proof of Proposition 3.23 one can also obtain projective description of $k^\infty(V) \cong K_\infty(\bar{V})$ as long as V is regularly decreasing. But in this case we do not have equivalence—a weaker condition than regularly decreasing which is equivalent to projective description of $k^\infty(V)$ will be presented in the next section, for now we only state that regularly decreasing would be sufficient (we give no proof since this will also follow from the results of the next section).

3.26 Corollary. (See [BMS82b, Cor. 3.5(b)].)
If $V = (v^{(n)})_{n \in \mathbb{N}}$ is regularly decreasing, then $(\lambda^1(A))'_b \cong K_\infty(\bar{V})$ is bornological such that $(\lambda^1(A))'_i \cong k^\infty(V) \cong K_\infty(\bar{V})$.

3.4 Heinrich's Density Condition

Before introducing the promised condition (D) which will characterize the projective description $k^\infty(V) \cong K_\infty(\bar{V})$ in terms of the decreasing sequence $V = (v_n)_{n \in \mathbb{N}}$ of weights, we will study several other related conditions (summarized in Figure 3.1) in order to be able to better understand (D) and its history.

We have already encountered the strong (i. e., sufficient for our goals) condition (S).

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}, m > n\colon\ \frac{v^{(m)}}{v^{(n)}} \text{ converges to } 0. \tag{S}$$

Its name is derived from the fact that it also characterizes the echelon spaces of *Schwartz* type (see [BMS82b, Thm. 4.9]).

Next, we studied the weaker but still sufficient *regularly decreasing condition.* We found out that it characterizes $k^0(V) \cong K_0(\bar{V})$ in terms of V and showed its equivalence to (wS) which was formulated in terms of the associated system of weights $\bar{V}$. It can be shown (see [BMS82b, Prop. 3.3]) that the regularly decreasing condition also characterizes *quasi-normable* echelon spaces which is why it is sometimes called (QN).

A condition (M), weaker than (S), suffices to return to (S) if the sequence V is regularly decreasing (in fact, (S) is equivalent to regularly decreasing together with (M) as we will see below).

$$\forall J \in \mathcal{F}(I)\ \forall n \in \mathbb{N}\ \exists m \in \mathbb{N}, m > n\colon\ \inf_{i \in I \setminus J} \frac{v_i^{(m)}}{v_i^{(n)}} = 0 \tag{M}$$

(The name of condition (M) was chosen because it characterizes the echelon spaces of *Montel* type, see [BMS82b, Thm. 4.7].)

3.27 Proposition. (See [BMS82b, Prop. 4.8].)
A decreasing sequence $V = (v_n)_{n \in \mathbb{N}}$ on an index set I satisfies condition (S) if and only if it is regularly decreasing and satisfies (M).

Proof. $(\Leftarrow)$ To show that condition (S) has to hold, let $n \in \mathbb{N}$. Choose a natural number $m \geqslant n$ according to version (3.7) of the regularly decreasing condition.

Assume indirectly that there exists a finite set $J \subseteq I$ such that $\inf_{i \in I \setminus J} v_i^{(m)}/v_i^{(n)} > 0$. Then, by the regularly decreasing condition (3.7), we also have $\inf_{i \in I \setminus J} v_i^{(k)}/v_i^{(n)} > 0$ for every $k \geqslant m$, and since V is decreasing, even for arbitrary $k \geqslant n$. But this is a contradiction to condition (M) which says that we should be able to find an $m_0 > n$ such that $\inf_{i \in I \setminus J} v_i^{(m_0)}/v_i^{(n)} = 0$.

Therefore, there can be no such set J, which means that $\frac{v^{(m)}}{v^{(n)}}$ indeed has to converge to 0. ☐

On the other hand, condition (M) is also sufficient to obtain the projective description $k^\infty(V) \cong K_\infty(\bar{V})$ (see Corollary 3.29 below).

3.28 Proposition. (See [BMS82b, Prop. 4.3 and Lemma 4.6].)
Let $V = (v_n)_{n \in \mathbb{N}}$ denote a decreasing sequence of strictly positive functions on the index set I. Then V satisfies condition (M) if and only if $K_0(\bar{V})$ equals $K_\infty(\bar{V})$ algebraically (and hence topologically).

Proof. $(\Leftarrow)$ Assume condition (M) does not hold. Then there exists a finite subset $J \subseteq I$ and an index $n_0 \in \mathbb{N}$ such that

$$\inf_{i \in I \setminus J} \frac{v_i^{(n)}}{v_i^{(n_0)}} =: \varepsilon_n > 0$$

for each $n > n_0$. Now define a function $x = (x_i)_{i \in I} : I \to [0, \infty)$ by setting $x_i := \frac{1}{v_i^{(n_0)}}$ whenever $i \in I \setminus J$ and $x_i := 0$ otherwise. We obviously have

$$\varepsilon_n \leqslant \frac{v_i^{(n)}}{v_i^{(n_0)}} = v_i^{(n)} |x_i| \leqslant 1$$

for all $i \in I \setminus J$ and every $n > n_0$ such that clearly $x \in K_\infty(\bar{V}) \setminus K_0(\bar{V})$.

$(\Rightarrow)$ Assume there exists an $x \in K_\infty(\bar{V}) \setminus K_0(\bar{V})$. Then we can find a weight $\bar{v} \in \bar{V}$ and a finite subset $J \subseteq I$ such that $\bar{v}_i |x_i| \geqslant \varepsilon$ for all $i \in I \setminus J$. On the other hand, since $x \in K_\infty(\bar{V}) = k^\infty(V)$, there has to be an $n_0 \in \mathbb{N}$ such that $\sup_{i \in I} v_i^{(n_0)} |x_i| =: C < \infty$. Furthermore, by definition of $\bar{V}$, for each $n > n_0$ we can find a constant $\alpha_n > 0$ with $\bar{v}_i \leqslant \alpha_n \cdot v_i^{(n)}$ for all $i \in I \setminus J$ such that

$$\varepsilon \leqslant \bar{v}_i |x_i| \leqslant \alpha_n \frac{v_i^{(n)}}{v_i^{(n_0)}} \sup_{i \in I} v_i^{(n_0)} |x_i| = C \cdot \alpha_n \frac{v_i^{(n)}}{v_i^{(n_0)}}.$$

Thus we have $\inf_{i \in I \setminus J} v_i^{(n)} / v_i^{(n_0)} \geqslant \frac{\varepsilon}{C \cdot \alpha_n} > 0$ which means that condition (M) cannot hold. □

3.29 Corollary. If V satisfies condition (M), then we have $k^\infty(V) \cong K_\infty(\bar{V})$.

Proof. By Theorem 3.16←56 and Corollary 3.17←56 we know that $K_0(\bar{V})$ is a topological linear subspace of the complete space $k^\infty(V) \cong \left(\lambda^1(A)\right)'_\iota$.

$$K_0(\bar{V}) \cong \widehat{k^0}(V) \cong \overline{k^0(V)}^{k^\infty(V)} \subseteq k^\infty(V)$$

Since we also have a continuous injection $k^\infty(V) \to K_\infty(\bar{V})$ (we even know that these two spaces have the same bounded sets) the result follows from the proposition above. □

Unfortunately (M) also cannot be the desired characterization of $k^\infty(V) \cong K_\infty(\bar{V})$ since there are sequences V which, although regularly decreasing (such that we have projective description), do not satisfy (M), see Counterexample 3.31→70 below.

Thus our desired characterizing condition (D) will have to be weaker than both regularly decreasing and condition (M). Therefore, in order to find a weaker form, we will now consider a reformulation of the regularly decreasing condition.

3.30 Proposition. (See [BMS82b, Prop. 3.9].)
A decreasing sequence $V = (v_n)_{n\in\mathbb{N}}$ on an index set I is regularly decreasing if and only if it satisfies *condition* (G), that is if for each $n \in \mathbb{N}$ there exists a natural number $m > n$ and an increasing sequence $(I_l^n)_{l\in\mathbb{N}}$ of subsets of I such that

$$\forall l \in \mathbb{N}\ \forall k \in \mathbb{N},\ k > n\colon\ \inf_{i\in I_l^n} \frac{v_i^{(k)}}{v_i^{(n)}} > 0$$

$$\text{while} \quad \lim_{l\to\infty} \sup_{i\in I\setminus I_l^n} \frac{v_i^{(m)}}{v_i^{(n)}} = 0.$$

(Condition (G) is named such since Grothendieck studied a weaker form of it which we will call (wG).)

Proof. ($\Rightarrow$) If V is regularly decreasing, given an $n \in \mathbb{N}$, we can choose $m > n$ as in (3.7)←62 such that for each subset $J \subseteq I$

$$\inf_{i\in J} \frac{v_i^{(m)}}{v_i^{(n)}} > 0 \implies \inf_{i\in J} \frac{v_i^{(k)}}{v_i^{(n)}} > 0$$

for all $k \geqslant m$ and since V is decreasing also for arbitrary $k \geqslant n$. For every $l \in \mathbb{N}$ define

$$I_l^n := \left\{ i \in I \;\middle|\; \frac{v_i^{(m)}}{v_i^{(n)}} \geqslant \frac{1}{l} \right\}.$$

The sequence of sets $(I_l^n)_{l\in\mathbb{N}}$ is obviously increasing, and, by our choice of $m > n$ we also have $\inf_{i\in I_l^n} v_i^{(k)}/v_i^{(n)} > 0$ for all $k > n$ and $l \in \mathbb{N}$ as claimed.

The second part of (G) follows again by definition of $(I_l^n)_{l\in\mathbb{N}}$,

$$\sup_{i\in I\setminus I_l^n} \frac{v_i^{(m)}}{v_i^{(n)}} \leqslant \frac{1}{l} \to 0 \qquad (l \to \infty).$$

($\Leftarrow$) Conversely, let V satisfy condition (G). For a given $n \in \mathbb{N}$ choose $m > n$ and an increasing sequence $(I_l^n)_{l\in\mathbb{N}}$ of subsets of I as in (G). Thus, for arbitrary $\varepsilon > 0$ we can find $l \in \mathbb{N}$ with $\sup_{i\in I\setminus I_l^n} v_i^{(m)}/v_i^{(n)} < \varepsilon$. Therefore, if $i \in I$ satisfies $\frac{v_i^{(m)}}{v_i^{(n)}} \geqslant \varepsilon$ we must have $i \in I_l^n$ such that

$$\frac{v_i^{(k)}}{v_i^{(n)}} \geqslant \inf_{i\in I_l^n} \frac{v_i^{(k)}}{v_i^{(n)}} > 0$$

for every $k > n$, which means that V is regularly decreasing. □

3.31 Counterexample (Grothendieck). (See [BMS82b, Ex. 4.11(1)].)
As in our previous Counterexample 3.8←48 we take $I := \mathbb{N} \times \mathbb{N}$. Define a Köthe matrix $A = (a^{(n)})_{n\in\mathbb{N}}$ on this index set by setting

$$a^{(n)}_{i,j} := \begin{cases} j^i & i \leqslant n \\ j^n & i \geqslant n+1, \end{cases}$$

then we obtain for the corresponding decreasing sequence $V = (v^{(n)})_{n\in\mathbb{N}} := \left(\frac{1}{a^{(n)}}\right)_{n\in\mathbb{N}}$ and all $k > n$ that

$$\frac{v^{(k)}_{i,j}}{v^{(n)}_{i,j}} = \frac{a^{(n)}_{i,j}}{a^{(k)}_{i,j}} = \begin{cases} 1 & i \leqslant n \\ j^{n-i} & n+1 \leqslant i \leqslant k \\ j^{n-k} & i \geqslant k+1. \end{cases}$$

This enables us to see that V cannot satisfy (M) since the expression above cannot become smaller than j^{n-k}.

On the other hand, V is still regularly decreasing: One simply has to take $I^n_l := \left\{(i,j) \in I \,\middle|\, i \leqslant n \text{ or } j \leqslant l\right\}$, then

$$\inf_{(i,j)\in I^n_l} \frac{v^{(k)}_{i,j}}{v^{(n)}_{i,j}} = l^{n-k} > 0 \qquad \text{and} \qquad \sup_{(i,j)\in I\setminus I^n_l} \frac{v^{(m)}_{i,j}}{v^{(n)}_{i,j}} = \frac{1}{l+1}$$

for all $k > n$ and $l \in \mathbb{N}$ such that condition (G) of Proposition 3.30 is satisfied.

We can weaken condition (G) by changing the order of quantifiers.

3.32 Corollary. (See [BMS82b, Cor. 3.10].)
If V is regularly decreasing, then also condition (wG) is satisfied, i. e., there exists an increasing sequence $\mathcal{I} = (I_l)_{l\in\mathbb{N}}$ of subsets of I such that

$$\forall l \in \mathbb{N}\ \exists n_l \in \mathbb{N}\ \forall k \in \mathbb{N},\ k > n_l\colon\ \inf_{i\in I_l} \frac{v^{(k)}_i}{v^{(n_l)}_i} > 0, \tag{$N,\mathcal{I}$}$$

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}, m > n\colon\ \lim_{l\to\infty} \sup_{i\in I\setminus I_l} \frac{v^{(m)}_i}{v^{(n)}_i} = 0. \tag{$S,\mathcal{I}$}$$

Proof. It suffices to prove that (wG) is implied by (G). By the latter condition, for each $n \in \mathbb{N}$, we may choose $m > n$ and an increasing sequence $(J^n_\ell)_{\ell\in\mathbb{N}}$ such that

$$\forall \ell \in \mathbb{N}\ \forall k \in \mathbb{N}, k > n\colon\ \inf_{i\in J^n_\ell} \frac{v^{(k)}_i}{v^{(n)}_i} > 0 \quad \text{while} \quad \lim_{\ell\to\infty} \sup_{i\in I\setminus J^n_\ell} \frac{v^{(m)}_i}{v^{(n)}_i} = 0.$$

Now set $I_l := \cup_{n+\ell\leqslant l} J^n_\ell$ to obtain an obviously increasing sequence $\mathcal{I} = (I_l)_{l\in\mathbb{N}}$ of subsets of I. Note that we have

$$\inf_{i\in J^n_\ell} \frac{v^{(k)}_i}{v^{(r)}_i} \geqslant \left(\inf_{i\in J^n_\ell} \frac{v^{(k)}_i}{v^{(n)}_i}\right) \underbrace{\left(\inf_{i\in J^n_\ell} \frac{v^{(n)}_i}{v^{(r)}_i}\right)}_{\geqslant 1} \geqslant \inf_{i\in J^n_\ell} \frac{v^{(k)}_i}{v^{(n)}_i} > 0$$

for all $\ell \in \mathbb{N}$ and $n \leqslant r < k$ since $V = (v^{(s)})_{s\in\mathbb{N}}$ is decreasing.

To see that $(N, \mathcal{I})$ holds, fix $l \in \mathbb{N}$. Then, by the above, we can find an $n_l \in \mathbb{N}$ such that $\inf_{i \in I_l} v_i^{(k)}/v_i^{(n_l)} > 0$ for each $k > n_l$ as claimed (take for example $n_l := l$).

Given $\varepsilon > 0$ (and $m > n$ chosen as in the beginning of this proof) our assumption implies that there exists an $\ell_0 \in \mathbb{N}$ such that we have $\sup_{i \in I \setminus J_\ell^n} v_i^{(m)}/v_i^{(n)} < \varepsilon$ for all $\ell \geqslant \ell_0$. If $L \in \mathbb{N}$ is chosen large enough (specifically $L \geqslant n + \ell_0$) then every I_l contains J_1^n, ..., $J_{\ell_0}^n$ if only $l \geqslant L$. Therefore, for such $l \geqslant L$, we have $\sup_{i \in I \setminus I_l} v_i^{(m)}/v_i^{(n)} < \varepsilon$ as well, hence also $(S, \mathcal{I})$ is satisfied. □

3.33 Remark. (See [BM86, Rem. 2.1].)
If a sequence $V = (v^{(n)})_{n \in \mathbb{N}}$ satisfies condition (M), then every set I_l with property $(N, \mathcal{I})$ of (wG) has to be finite. If this is the case, on the other hand, $(S, \mathcal{I})$ simply reduces to (S), such that we have

$$(S) \Leftrightarrow \big((M) \wedge (wG)\big).$$

Unfortunately, together with Corollary 3.32, this means that condition (wG) still cannot be weak enough for a characterization of $k^\infty(V) \cong K_\infty(\bar{V})$ (although (wG) is indeed strictly weaker than (G), see [BMS82b, Ex. 3.11]). On the positive side, it also means that we weakened (G) in the right way, we just didn't go far enough.

The obvious way to further weaken (wG) is to replace the Schwartz type condition $(S, \mathcal{I})$ by a Montel type one, $(M, \mathcal{I})$. As it turns out this is exactly what we were searching for.

3.34 Definition. Let $V = (v_n)_{n \in \mathbb{N}}$ denote a decreasing sequence of strictly positive functions on the index set I. We say that V *satisfies condition* (D) if there exists an increasing sequence $\mathcal{I} = (I_l)_{l \in \mathbb{N}}$ of subsets of I such that

$$\forall l \in \mathbb{N}\ \exists n_l \in \mathbb{N}\ \forall k \in \mathbb{N},\ k > n_l\colon\ \inf_{i \in I_l} \frac{v_i^{(k)}}{v_i^{(n_l)}} > 0, \qquad (N, \mathcal{I})$$

$$\forall n \in \mathbb{N}\ \forall J \subseteq I,\ J \cup \bigcup_{l \in \mathbb{N}} I_l \neq I\ \exists m \in \mathbb{N},\ m > n\colon\ \sup_{i \in I \setminus J} \frac{v_i^{(m)}}{v_i^{(n)}} = 0. \qquad (M, \mathcal{I})$$

We can now directly prove, as it was done in [BM86], that condition (D) still implies $k^\infty(V) \cong K_\infty(\bar{V})$ (see Proposition 3.36 below for a sketch of the proof). But as it turned out in [BB88], condition (D) actually characterizes those echelon spaces $\lambda^p(A)$ (of arbitrary order $1 \leqslant p < \infty$ or $p = 0$ and not just $p = 1$) which satisfy *Heinrich's density condition.* The equivalence of (D) with $k^\infty(V) \cong K_\infty(\bar{V})$ should rather be regarded as a "coincidence" [Bie86, p. 127].

3.35 Definition (S. Heinrich). (See [BB88, Def. 1.1].)
Let (E, τ) denote a general (Hausdorff) locally convex vector space, $\mathcal{U}(E)$ the system of all closed absolutely convex 0-neighborhoods in E, and $\mathcal{B}(E)$ the system of all closed absolutely convex and bounded subsets of E. Then E satisfies *Heinrich's density condition* if given any

function $\lambda : \mathcal{U}(E) \to (0, \infty)$ and an arbitrary element $V \in \mathcal{U}(E)$ there always exist finitely many $U_1, ..., U_n \in \mathcal{U}(E)$ and an element $B \in \mathcal{B}(E)$ such that

$$\bigcup_{k=1}^{n} \lambda(U_k) \cdot U_k \subseteq B + V. \tag{DC}$$

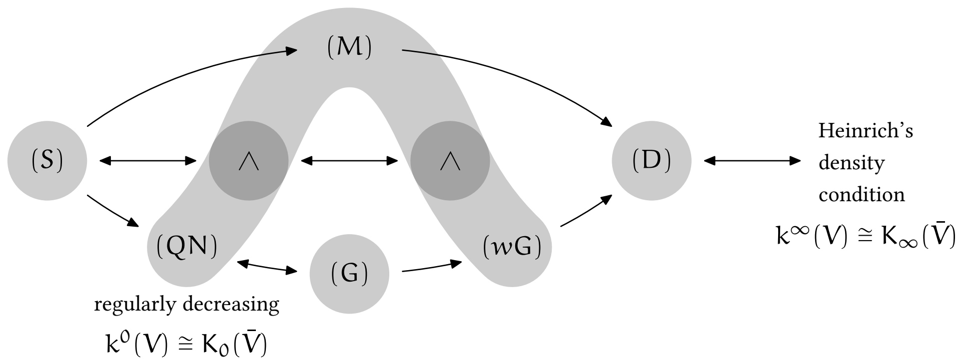

Figure 3.1 The relations between all mentioned conditions on the decreasing sequence of weights V (the arrows signify strict implications, i. e., there are counterexamples if an arrow points in only one direction).

3.36 Proposition. (See [BM86, in particular Thm. 2.3 and § 3].)
Let $V = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive functions on the index set I which satisfies condition (D). Then $k^\infty(V)$ can be described projectively, i. e., we have $k^\infty(V) \cong K_\infty(\bar{V})$.

Proof. *(Sketch)* We will essentially proceed as in Corollary 3.29[69]. There we used that (M) implies $K_0(\bar{V}) = K_\infty(\bar{V})$ which we now don't have at our disposal. To work around this problem we will introduce more general spaces $\lambda^0(A, \mathcal{S})$ and $k^0(V, \mathcal{S})$ which "interpolate" between $\lambda^0(A)$ and $\lambda^\infty(A)$ or $k^0(V)$ and $k^\infty(V)$, respectively, such that we may apply the idea of proof of Corollary 3.29[69]. Let $\mathcal{S}$ denote a non-empty family of subsets of the index set I, then we set

$$\begin{aligned} c_0(a, \mathcal{S}) &:= \Big\{ x \in \ell^\infty(a) \;\Big|\; \Big(\sup_{i\in I\setminus S} a_i|x_i|\Big)_{S\in\mathcal{S}} \text{ converges to } 0 \Big\} \\ \lambda^0(A, \mathcal{S}) &:= \varprojlim_{n\in\mathbb{N}} c_0(a^{(n)}, \mathcal{S}) \\ k^\infty(V, \mathcal{S}) &:= \varinjlim_{n\in\mathbb{N}} c_0(v^{(n)}, \mathcal{S}) \end{aligned}$$

($(\sup_{i\in I\setminus S} a_i|x_i|)_{S\in\mathcal{S}}$ converges to 0 if for each $\varepsilon > 0$, there is a set $S \in \mathcal{S}$ such that $a_i|x_i| < \varepsilon$ for all $i \in I \setminus S$). We obviously have $\lambda^0(A, \mathcal{F}) \cong \lambda^0(A)$ and $k^0(V, \mathcal{F}) \cong k^\infty(V)$ for the system $\mathcal{F}$ of finite subsets of I. On the other hand, if $\mathcal{S} = \{I\}$, we obtain $\lambda^0(A, \mathcal{S}) \cong \lambda^\infty(A)$ and $k^0(V, \mathcal{S}) \cong k^\infty(V)$.

Now, proceeding similarily as in the proof of Proposition 3.28, we can show that condition $(M, \mathcal{S})$ is equivalent to $K_0(\bar{V}, \mathcal{S}) = K_\infty(\bar{V})$, where $K_0(\bar{V}, \mathcal{S}) := \lambda^0(\bar{V}, \mathcal{S})$ and $\bar{V} := \lambda^\infty_+(A)$ are defined in the usual way.

As a next step we want to obtain that $K_0(\bar{V}, \mathcal{S})$ is the completion of $k_0(V, \mathcal{S})$. For this to hold we will have to assume that $\mathcal{S} = \mathcal{J}$ is actually an increasing sequence of subsets of I, as in condition (D). In this case we can show, similarily to the proof of Lemma 3.15 and using $(N, \mathcal{J})$ of condition (D), that both $k^0(V, \mathcal{J})$ and $K_0(\bar{V}, \mathcal{J})$ induce the same topology on their common dense subspace $c_{00}(V, \mathcal{J})$ of $\mathcal{J}$-finite sequences,

$$c_{00}(V, \mathcal{J}) := \left\{ x \in k^\infty(V) \;\middle|\; \exists I_0 \in \mathcal{J} \colon\; x|_{I \setminus I_0} \equiv 0 \right\}.$$

By an application of Lemma 3.14 we therefore indeed obtain that $K_0(\bar{V}, \mathcal{S})$ is the completion of $k_0(V, \mathcal{S})$. We also see that $k^0(V, \mathcal{J})$ is a topological linear subspace of $k^\infty(V)$ (as in Corollary 3.17).

Therefore we may know state

$$K_0(\bar{V}, \mathcal{J}) \cong \widehat{k^0}(V, \mathcal{J}) \cong \overline{k^0(V, \mathcal{J})}^{k^\infty(V)} \subseteq k^\infty(V)$$

such that we finally obtain $k^\infty(V) \cong K_\infty(\bar{V})$ as in Corollary 3.29. $\square$

4

Projective Descriptions of Weighted Spaces of Continuous Functions

One of the obvious generalisations of the situation in the previous chapter 3 would be to consider spaces of continuous functions instead of sequence spaces. This is what will be attempted in this chapter. By advancing similarly as previously we will try to find reasonably general spaces of continuous functions, so that we can still find simple projective descriptions for their inductive limits.

In this chapter, let X always denote a completely regular Hausdorff space (replacing the index set I), and E a complex (Hausdorff) locally convex vector space (replacing the range-space $\mathbb{C}$).

4.1 Definitions and Preliminaries

First we will need to find a concept that can replace the Köthe matrices of chapter 3—it will be that of *Nachbin families*.

4.1 Definition (Weights). (See [BMS82a, § 0.1].)
Let X be a completely regular Hausdorff space. A *weight* on X is a nonnegative, upper semi-continuous (i. e., $\nu^{-1}([\alpha,\infty))$ is closed for all $\alpha \in \mathbb{R}$) function $\nu : X \to [0,\infty) \subseteq \mathbb{R}$. Let U, V be two sets of weights, we write

$$U \preceq V :\iff \forall u \in U\ \exists \nu \in V,\ \lambda > 0\colon\ u(x) \leqslant \lambda\nu(x) \quad \forall x \in X,$$

i. e., if any weight of U can be dominated by some multiple of a weight of V. We also use the notation

$$V > 0 :\iff \forall x \in X\ \exists \nu \in V\colon\ \nu(x) > 0,$$

in this case we call the set of weights V *strictly positive* (this is not equivalent to $V \npreceq 0 = \{x \mapsto 0\}$ in the above notation!). Finally we call a set of weights V *directed upward* if

$$\forall \nu_1, \nu_2 \in V,\ \lambda > 0\ \exists \nu \in V\colon\ \lambda\nu_1 \leqslant \nu \text{ and } \lambda\nu_2 \leqslant \nu$$

(pointwise on X). If a set of weights on X is directed upward and strictly positive we refer to it as a *system of weights* or a *Nachbin family* on X.

4.2 Remark. If X carries the discrete topology, a (pointwise) increasing sequence of weights on X is simply a Köthe matrix indexed by the set X.

4.3 Definition (Weighted Spaces of Continuous Functions). (See [BMS82a, § 0.1].) Let V be a system of weights on X. Define a family of seminorms on $C(X, E)$ by

$$q_{v,p}(f) := \sup_{x \in X} v(x)p(f(x)) \in \mathbb{R} \cup \{+\infty\}$$

where $v \in V$ is a weight, $p \in cs(E)$ is a continuous seminorm on E and $f \in C(X, E)$. Then we can put

$$CV(X, E) := \{f \in C(X, E) \mid q_{v,p}(f) < \infty \ \forall v \in V,\ p \in cs(E)\},$$
$$CV_0(X, E) := \{f \in C(X, E) \mid v \cdot (p \circ f) \text{ vanishes at infinity } \forall v \in V,\ p \in cs(E)\}.$$

(A function $g : X \to \mathbb{C}$ is said to *vanish at infinity* if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ such that $|g(x)| \leqslant \varepsilon$ for all $x \in X \setminus K$.)

We equip both of these vector spaces with the Hausdorff (since $V > 0$) locally convex topology induced by the family of seminorms $\{q_{v,p} \mid v \in V,\ p \in cs(E)\}$. More generally, for a linear subspace $A(X, E)$ of $C(X, E)$, we put

$$AV(X, E) := A(X, E) \cap CV(X, E) \qquad \text{and} \qquad AV_0(X, E) := A(X, E) \cap CV_0(X, E).$$

If $E = \mathbb{C}$ we omit E from the notation, i. e., we write $CV(X)$, $CV_0(X)$, $AV(X)$ and $AV_0(X)$ for the above defined spaces. Also, if V consists only of the multiples of one single weight v on X (that is, if $V = \{\lambda v \mid \lambda > 0\}$), we write v instead of V so that we will use the notation $Cv(X, E)$, $Cv_0(X, E)$, $Av(X, E)$ and $Av_0(X, E)$.

4.4 Example. The set of weights $\mathcal{K} := \mathcal{K}(X) := \{\lambda 1_K \mid \lambda > 0,\ K \subseteq X,\ K \text{ compact}\}$ is the Nachbin family generated by the characteristic functions 1_K of the compact sets K of X.

Using this system of weights we obtain that the locally convex vector space $C\mathcal{K}(X, E)$ is equipped with the topology τ_c of uniform convergence on the compact sets of X. Thus $C\mathcal{K}(X, E)$ is complete whenever E is complete and X is a $k_{\mathbb{R}}$-space (i. e., X is such that a map $f : X \to \mathbb{R}$ already has to be continuous if all its restrictions to compact subsets of X are continuous, see [Jar81, § 3.6, Example E and § 16.6, Proposition 2]).

4.5 Remark.

(i) The spaces $CV(X, E)$ defined in Definition 4.3 are projective limits of the locally convex vector spaces $C(X, E)$ equipped with the topology induced by the seminorms $q_{v,p}$ for $v \in V$ and $p \in cs(X)$, respectively:

$$CV(X, E) = \varprojlim_{v} Cv(X, E) = \varprojlim_{v,p} (C(X, E), q_{v,p}),$$
$$CV_0(X, E) = \varprojlim_{v} Cv_0(X, E).$$

The spaces $Cv(X,E)$ and $Cv_0(X,E)$ correspond to the diagonal-transformed sequence spaces of Remark 3.5←46—then the building blocks of (co-)echelon spaces. (To generalize further, instead of single weights, we will this time consider whole systems of weights and use the spaces $CV(X,E)$ and $CV_0(X,E)$ as our building blocks.)

(ii) If we consider another Nachbin family W on X such that $W \preceq V$, we obtain that $CV(X,E)$ is continuously embedded into $CW(X,E)$, as is $CV_0(X,E)$ into $CW_0(X,E)$.

A useful consequence of Remark 4.5(ii)←76 is the following.

4.6 Lemma. (See [Bie73a, Satz 1.5].)
Let V be a system of weights on X and E be a complete locally convex vector space, then the space $CV(X,E)$ is complete if there exists a Nachbin family W on X with $W \preceq V$ such that $CW(X,E)$ is complete. An analogous statement holds for $CV_0(X,E)$.

Proof. Let $(f_i)_{i\in I}$ denote a Cauchy net in $CV(X,E)$. Since $CW(X,E)$ is complete, the net—embedded into this space—converges to an $f \in CW(X,E)$. On the other hand it also has to converge pointwise, since for a given $x \in X$ there exists a weight $w \in W$ with $w(x) > 0$ such that

$$p(f_i(x) - f_j(x)) \leqslant \frac{q_{w,p}(f_i - f_j)}{w(x)} \to 0$$

for all $p \in cs(E)$ as $i, j \to \infty$, implying that $(f_i(x))_{i\in I}$ is a Cauchy net of the complete space E.

Now let $\varepsilon > 0$ and $v \in V$, $p \in cs(E)$. We can choose $i, j \in I$ big enough such that $q_{v,p}(f_i - f_j) < \varepsilon$, then we also have

$$v(x)p(f_i(x) - f_j(x)) < \varepsilon \quad \xRightarrow{j\to\infty} \quad v(x)p(f_i(x) - f(x)) \leqslant \varepsilon$$

for all $x \in X$ such that $f \in CV(X,E)$, and $f_i \to f$ in $CV(X,E)$ as $i \to \infty$. □

Combining this result with Example 4.4←76 above, we obtain a simple condition for the completeness of the weighted spaces $CV(X,E)$ and $CV_0(X,E)$.

4.7 Proposition. (See [Bie73a, Korollar 1.6(2)].)
The spaces $CV(X,E)$ and $CV_0(X,E)$ are complete whenever E is a complete locally convex vector space, X is a completely regular $k_{\mathbb{R}}$-space and $\mathcal{K} \preceq V$.

(If $V = \{\lambda v \mid \lambda > 0\}$ consists of only one weight the condition $\mathcal{K} \preceq V$ simply means that $\inf_{x\in K} v(x) > 0$ for each compact set K of X.)

We won't look into the analogue of echelon spaces but instead jump directly to the counterpart of co-echelon spaces of order $p = 0$ and $p = \infty$, our current spaces of interest.

4.8 Definition. Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ denote a decreasing sequence (i. e., $V_{n+1} \preceq V_n$ for $n = 1$, 2, …) of systems of weights on X and $A(X, E)$ a linear subspace of $C(X, E)$. Then we may define the following *weighted locally convex inductive limits*:

$$\mathcal{V}A(X,E) := \varinjlim_n AV_n(X,E) \qquad \mathcal{V}_0 A(X,E) := \varinjlim_n A(V_n)_0(X,E)$$

(The spaces $\mathcal{V}C(X, E)$ and $\mathcal{V}_0 C(X, E)$ generalize $k^\infty(V)$ and $k^0(V)$, respectively, while $\mathcal{V}$ takes the role of the V of chapter 3[45].)

4.2 Projective Descriptions

4.9 Remark. (See [Bie86, page 109f].)
To find projective descriptions for weighted inductive limit spaces we should first consider the behaviour of seminorms on these spaces (cf. Proposition 1.10[13]). For now we simplify things by only considering single weights instead of whole systems of weights and letting $E = \mathbb{C}$.

Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of weights on X. A seminorm p on $\mathcal{V}C(X) = \varinjlim_n Cv_n(X)$ respectively $\mathcal{V}_0 C(X) = \varinjlim_n C(v_n)_0(X)$ is continuous if and only if its restriction to each of the normed spaces $Cv_n(X)$ respectively $C(v_n)_0(X)$ is continuous. That is, if and only if for each $n \in \mathbb{N}$, there is a constant $C_n > 0$ such that

$$p(f) \leqslant C_n \|f\|_n = C_n \sup_{x\in X} v_n(x)|f(x)|$$

for each $f \in Cv_n(X)$ or $f \in C(v_n)_0(X)$, respectively. Thus we obtain the condition

$$p(f) \leqslant \inf_{n\in\mathbb{N}} \sup_{x\in X} C_n v_n(x)|f(x)|$$

for the continuity of p. Now it would be very nice if we could simply exchange the order of the infimum and the supremum to obtain

$$p(f) \leqslant \sup_{x\in X} \big(\inf_{n\in\mathbb{N}} C_n v_n(x)\big)|f(x)|,$$

since then $\inf_{n\in\mathbb{N}} C_n v_n$ could be considered as a new nonnegative upper semi-continuous weight $\bar{v}$ on X, somehow *associated* with $\mathcal{V}$.

This motivates the following definition of associated systems of weights (also compare to Definition 3.10[50]):

4.10 Definition. (See [BMS82a, § 0.2].)
Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of systems of weights on X. We define two *associated systems of weights* on X:

(i) The system of weights $\check{V}_{\mathcal{V}}$ consisting of every weight $\check{v}$ on X which is of the form $\check{v}(x) = \inf_{n\in\mathbb{N}} C_n v_n(x)$ for $C_n > 0$ and $v_n \in V_n$, $n \in \mathbb{N}$.

(ii) The *maximal system of weights* $\bar{V}_{\mathcal{V}}$ associated with $\mathcal{V}$,

$$\bar{V}_{\mathcal{V}} := \left\{\bar{v} : X \to \mathbb{R} \text{ weight} \,\middle|\, \exists \check{v} \in \check{V}_{\mathcal{V}} \colon \bar{v}(x) \leqslant \check{v}(x)\ \forall x \in X\right\},$$

consisting of all weights on X which are pointwise majorized by some member of $\check{V}_{\mathcal{V}}$.

4.11 Remark. By definition we have $\bar{V}_{\mathcal{V}} \preceq \check{V}_{\mathcal{V}}$ as well as $\check{V}_{\mathcal{V}} \preceq \bar{V}_{\mathcal{V}}$ (since even $\check{V}_{\mathcal{V}} \subseteq \bar{V}_{\mathcal{V}}$). Furthermore $\bar{V}_{\mathcal{V}}$ contains all the systems of weights $V \preceq V_n$ on X that are already majorized by all the "steps" V_n for $n \in \mathbb{N}$ (i. e., $\bar{V}_{\mathcal{V}}$ contains all "lower bounds" of the decreasing sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ of Nachbin families).

In the special case that $\mathcal{V}$ is a decreasing sequence of weights v_n, i. e., $V_n = \{\lambda v_n \mid \lambda > 0\}$, $n \in \mathbb{N}$, we have that $\bar{V}_{\mathcal{V}}$ consists exactly of those weights $\bar{v}$ on X with the property that $\frac{\bar{v}}{v_n}$ is bounded for all $n \in \mathbb{N}$.

We now want to study the relationships between the weighted inductive limits $\mathcal{V}A(X, E)$ and $\mathcal{V}A_0(X, E)$ and their so-called *associated weighted hulls* $A\bar{V}(X, E)$ and $A\bar{V}_0(X, E)$.

4.12 Lemma. Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be a decreasing sequence of Nachbin families on X and $A(X, E)$ a subspace of $C(X, E)$. Then the spaces $\mathcal{V}A(X, E)$ and $\mathcal{V}A_0(X, E)$ can be continuously injected into $A\bar{V}(X, E)$ and $A\bar{V}_0(X, E)$, respectively.

Proof. By construction of the associated system $\bar{V}_{\mathcal{V}}$ every $AV_n(X, E)$ is continuously embedded into $A\bar{V}(X, E)$ (see Remark 4.9). Since $\mathcal{V}A(X, E)$ is the inductive limit of the spaces $AV_n(X, E)$, we can use the universal property of the inductive limit (see Definition 1.26) to also obtain a continuous injection $\mathcal{V}A(X, E) \to A\bar{V}(X, E)$. ☐

The problem of finding *projective descriptions* for the weighted inductive limits now boils down to the following questions (see [BMS82a, § 0.5]):

(i) When does $\mathcal{V}A(X, E)$ equal $A\bar{V}(X, E)$ and $\mathcal{V}A_0(X, E)$ equal $A\bar{V}_0(X, E)$ algebraically? (By the results on sequence spaces of chapter 3 we know that we can't, in general, hope for $\mathcal{V}A_0(X, E) = A\bar{V}_0(X, E)$.)

(ii) When do $A\bar{V}(X, E)$ or $A\bar{V}_0(X, E)$ induce the inductive limit topology on $\mathcal{V}A(X, E)$ or $\mathcal{V}A_0(X, E)$, respectively? (Again, we already know that there is no hope for the former case in full generality.)

In both cases, an algebraic equality means that via $\bar{V}$ we have obtained a complete weighted description of the functions in the inductive limit or, viewed from another perspective, that the associated weighted space is exhausted by an increasing sequence of weighted subspaces. If, on the other hand, the weighted inductive limit topology is induced by its associated weighted topology, we obtain an "analytic" description (i. e., via continuous seminorms) of the "abstract" inductive limit topology, something which is normally rather hard to obtain but still essential for direct estimates and computations in applications.

We will again try to find answers using an approach dictated by the Lemma 3.14. As it turns out—at least for a simplified case of $\mathcal{V}_0C(X,E)$ and $C\bar{V}_0(X,E)$—we can find a dense linear subspace as needed in this Lemma. Thus we may characterize the inductive limit topology of such a space $\mathcal{V}_0C(X,E)$ via the weighted topology of the corresponding associated weighted hull $C\bar{V}_0(X,E)$—a *projective description*.

4.13 Lemma. (See [BMS82a, Lemma 1.1] and [Bie86, Lemma 4.4].)
Let $(E, \|\,.\,\|)$ be a normed space and X be locally compact. Then the spaces $\mathcal{V}_0C(X,E)$ and $C\bar{V}_0(X,E)$ induce the same topology on the dense linear subspace $C_c(X,E)$ of all E-valued continuous functions on X which have compact support.

Proof. Since $\mathcal{V}_0C(X,E)$ is continuously injected into $C\bar{V}_0(X,E)$, given a 0-neighborhood U in $\mathcal{V}_0C(X,E)$ we have to show that we can find a 0-neighborhood in $C\bar{V}_0(X,E)$ whose intersection with $C_c(X,E)$ is contained in U. Therefore, we are done if we can prove that there exists a $\bar{v} \in \bar{V}$ such that

$$B_{\bar{v}} := \Big\{ f \in C_c(X,E) \;\Big|\; q_{\bar{v}}(f) = \sup_{x\in X} \bar{v}(x)\,\|f(x)\| < 1 \Big\} \subseteq U.$$

Since U is a 0-neighborhood in the locally convex inductive limit topology and each V_n is directed upward, for each $n \in \mathbb{N}$ there exists a $v_n \in V_n$ such that (cf. Remark 1.24)

$$\Gamma\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) \subseteq U, \quad \text{where} \quad B_n := \Big\{ f \in C(V_n)_0(X,E) \;\Big|\; q_{v_n}(f) \leqslant 1 \Big\}.$$

Now we claim that $\bar{v} := \inf_{n\in\mathbb{N}} 2^n v_n$ is the weight function we are searching for. First observe that indeed $\bar{v} \in \check{V} \subseteq \bar{V}$ (cf. Definition 4.10); fixing an $f \in B_{\bar{v}}$, i. e., $f \in C_c(X,E)$ with $q_{\bar{v}}(f) < 1$, consider the ("problematic") sets

$$F_n := \Big\{ x \in X \;\Big|\; 2^n v_n(x)\,\|f(x)\| \geqslant 1 \Big\} \quad (n \in \mathbb{N}).$$

Obviously every F_n is a closed subset of $\operatorname{supp} f$. If $x \in \bigcap_{n\in\mathbb{N}} F_n$, then by definition we have $2^n v_n(x)\,\|f(x)\| \geqslant 1$ such that we also obtain $\bar{v}(x)\,\|f(x)\| \geqslant 1$. But since this contradicts the assumption $q_{\bar{v}}(f) < 1$, we must conclude that the set $\bigcap_{n\in\mathbb{N}} F_n$ is void.

Thus, setting $U_n = X \setminus F_n$ for each $n \in \mathbb{N}$, we obtain an open covering of X. Since $\operatorname{supp} f$ is compact, we can find an $m \in \mathbb{N}$ such that $\operatorname{supp} f$ is already contained in $\bigcup_{n=1}^m U_n$. Therefore we may choose a finite partition of unity $(\varphi_n)_{1\leqslant n\leqslant m} \subseteq C_c(X)$ on $\operatorname{supp} f$ which is subordinate to the covering $(U_n)_{1\leqslant n\leqslant m}$.

Setting $g_n := 2^n \varphi_n f$ $(1 \leqslant n \leqslant m)$, we obtain $g_n \in C_c(X,E) \subseteq C(V_n)_0(X,E)$. Actually we even have $g_n \in B_n$ since either $g_n(x) = 0$ (for $x \in X \setminus U_n = F_n$) or (whenever $x \in U_n$)

$$v_n(x)\,\|g_n(x)\| = \varphi_n(x) 2^n v_n(x)\,\|f(x)\| < 1.$$

Consequently,

$$f = \sum_{n=1}^{m} \varphi_n f = \sum_{n=1}^{m} 2^{-n} g_n \in \Gamma\Big(\bigcup_{n\in\mathbb{N}} B_n\Big) \subseteq U.$$

□

Thus, by an application of Lemma 3.14[54], we obtain the following theorem.

4.14 Theorem. (See [BMS82a, Thm. 1.3].)
If X is a locally compact (Hausdorff) space, E is a normed space, and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a decreasing sequence of systems of weights on X, then $\mathcal{V}_0 C(X, E)$ is a topological linear subspace of $C\bar{V}_0(X, E)$, i. e., $C\bar{V}_0(X, E)$ induces the inductive limit topology on $\mathcal{V}_0 C(X, E)$ via the canonical injection.

If, additionally, E is a Banach space and $\mathcal{K} \preceq V_n$ for each $n \in \mathbb{N}$, then $C\bar{V}_0(X, E)$ is the completion of $\mathcal{V}_0 C(X, E)$.

Proof. The last statement follows, since by Remark 4.11[79] we then also have $\mathcal{K} \subseteq \bar{V}$, such that $C\bar{V}_0(X, E)$ is complete for any complete locally convex vector space E (see Proposition 4.7[77]). □

4.15 Corollary. Under the (original) conditions of Theorem 4.14, $\mathcal{V}_0 C(X, E)$ is a topological linear subspace of $\mathcal{V}C(X, E)$.

Proof. Since we have continuous injections $\mathcal{V}_0 C(X, E) \to \mathcal{V}C(X, E) \to C\bar{V}(X, E)$ and $C\bar{V}_0(X, E)$ is a topological linear subspace of $C\bar{V}(X, E)$ the assertion follows immediately from the theorem. □

In an even more restricted setting we can also obtain a generalized version of Proposition 3.13[52], i. e., algebraic projective description of $\mathcal{V}A(X, E)$. But first we need a special subsystem of the associated system of weights (which can be shown, under additional assumptions, to be equivalent to the maximal associated system $\bar{V}_{\mathcal{V}}$).

4.16 Definition. (See [BMS82a, Def. 1.7].)
Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of weights on a locally compact σ-compact space X such that $\inf_{x\in K} v_n(x) > 0$ for each compact set K of X and all $n \in \mathbb{N}$. Fix a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ of X such that $K_n \subseteq \mathring{K}_{n+1}$ and $X = \cup_{n\in\mathbb{N}} K_n$.

Since X is paracompact, given a strictly increasing sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{N}$, we may choose a continuous partition of unity $(\varphi_n^{(a)})_{n\in\mathbb{N}} \in C(X)$ subordinate to the covering $\big(\mathring{K}_{a_{n+1}} \setminus K_{a_{n-1}}\big)$ of X (where we set $a_0 := 0$ and $K_0 := \emptyset$). Now we can define a function $X \to [0, \infty)$

$$\bar{v}_{a,c}(x) := \sum_{k=1}^{\infty} v_{k-1}(x)\varphi_k^{(a)}(x), \tag{4.1}$$

where we set $v_0 \equiv c$ for a constant $c > 0$. We will denote by $\hat{V}_{\mathcal{V}}$ the family of all such functions where $a = (a_n)_{n\in\mathbb{N}}$ runs through all strictly increasing sequences of natural numbers and $c > 0$ is an arbitrary positive constant.

4.17 Lemma. (See [BMS82a, Lemma 1.8].)
The family $\hat{V}_{\mathcal{V}}$ of Definition 4.16 is a system of (strictly positive) weights and $\hat{V}_{\mathcal{V}} \subseteq \bar{V}_{\mathcal{V}}$. Furthermore, if all elements of $\mathcal{V}$ are continuous, then the weights $\bar{v}_{a,c} \in \hat{V}_{\mathcal{V}}$ are continuous as well.

Proof. The definition (4.1)81 of $\bar{v}_{a,c}$ makes sense since at most three terms of the sum do not vanish on $\mathring{K}_{a_{n+1}} \setminus K_{a_{n-1}}$ for each $n \in \mathbb{N}$. By the same argument $\bar{v}_{a,c}$ is a weight on X which is continuous whenever all $v_n \in \mathcal{V}$ are continuous.

To see that $\bar{v}_{a,c} \in \bar{V}_{\mathcal{V}}$ we have to show that $\frac{\bar{v}_{a,c}}{v_n}$ is bounded on X for each $n \in \mathbb{N}$ (see Remark 4.1179). Therefore, fix an $n \in \mathbb{N}$. If $x \notin K_{a_{n+1}}$ then we have $\varphi_k^{(a)}(x) = 0$ for $k \leqslant n$ such that

$$\frac{\bar{v}_{a,c}(x)}{v_n(x)} = \sum_{k=n+1}^{\infty} \varphi_k^{(a)}(x)\frac{v_{k-1}(x)}{v_n(x)} \leqslant \sum_{k=n+1}^{\infty} \varphi_k^{(a)}(x) \leqslant 1$$

since $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is decreasing. On the other hand, if $x \in K_{a_{n+1}}$, then $\varphi_k^{(a)}$ vanishes for $k \geqslant n+2$ such that

$$\frac{\bar{v}_{a,c}(x)}{v_n(x)} = \sum_{k=1}^{n+1} \varphi_k^{(a)}(x)\frac{v_{k-1}(x)}{v_n(x)} \leqslant \frac{C}{\inf_{x\in K_{a_{n+1}}} v_n(x)} < \infty.$$

for a constant $C > 0$ since the finitely many $v_1, \ldots, v_n$ are all upper semi-continuous and v_n can be bounded away from 0 on each compact set of X by assumption. $\square$

4.18 Theorem. (See [BM76, Satz 2.8] and [BMS82a, part (i) of *direct proof* of Thm. 1.3(d)], originally a theorem of B. A. Taylor.)
Let $(E, \|\,.\,\|)$ denote a normed space and let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive weights on the locally compact σ-compact space X such that $\inf_{x\in K} v_n(x) > 0$ for each compact set K of X and all $n \in \mathbb{N}$. Then $\mathcal{V}C(X, E)$ is algebraically equal to $C\bar{V}(X, E)$, both spaces even have the same families of bounded sets.

Proof. Since $\mathcal{V}C(X, E)$ is continuously injected in $C\bar{V}(X, E)$ it suffices to prove that every bounded set B of $C\bar{V}(X, E)$ is also contained and bounded in $\mathcal{V}C(X, E)$. Let B denote such a set, as in part (b) of the proof of Proposition 3.1352, define a map $b(x) := \sup_{f\in B} \|f(x)\|$. Then we have

$$\sup_{x\in X} \bar{v}(x)|b(x)| = \sup_{x\in X}\sup_{f\in B} \bar{v}(x)\|f(x)\| = \sup_{f\in B} q_{\bar{v}}(f) < \infty \tag{4.2}$$

for all $\bar{v} \in \bar{V}$ which means that b satisfies the weight conditions of $C\bar{V}(X, E)$ (although it need not be an element of this space). We are done if we can prove that it also satisfies the weight conditions of $\mathcal{V}C(X, E)$, i. e., if there is an $n \in \mathbb{N}$ such that

$$q_{v_n}(b) = \sup_{x \in X} v_n(x)b(x) < \infty, \tag{4.3}$$

since then we could estimate

$$\sup_{f \in B} q_{v_n}(f) = \sup_{f \in B} \sup_{x \in X} v_n(x)|f(x)| \leqslant \sup_{x \in X} v_n(x)b(x) < \infty$$

so that $B \subseteq Cv_n(X, E) \subseteq \mathcal{V}C(X, E)$ would indeed be bounded with respect to the inductive limit topology.

Assume indirectly that (4.3) does not hold. Then, for each $n \in \mathbb{N}$ and $C > 0$ there exists $y \in X$ such $v_n(y)b(y) > C$. Furthermore, B is uniformly bounded on compact sets such that also $v_n \cdot b$ is bounded on compact sets (by definition of b and since v_n is upper semi-continuous). Therefore, in the language of Definition 4.16, we can find a sequence $(a_n)_{n \in \mathbb{N}} \in \mathbb{N}$ and $x_n \in \mathring{K}_{a_{n+1}} \setminus K_{a_{n-1}}$ such that $v_n(x_n)b(x_n) > n$ for each $n \in \mathbb{N}$ (where we again set $a_0 := 0$ and $K_0 := \emptyset$).

For this sequence a and a constant $c \geqslant \max(v_1(x_1), v_2(x_2))$ construct a weight $\bar{v}_{a,c}$ as in (4.1), then we have

$$\begin{aligned}
\bar{v}_{a,c}(x_1) &= \sum_{k=1}^{\infty} v_{k-1}(x_1)\varphi_k^{(a)}(x_1) = \underbrace{v_0(x_1)}_{=c}\varphi_1^{(a)}(x_1) + v_1(x_1)\varphi_2^{(a)}(x_1) \\
&\geqslant v_1(x_1)\big(\varphi_1^{(a)}(x_1) + \varphi_2^{(a)}(x_1)\big) = v_1(x_1) \quad \text{and} \\
\bar{v}_{a,c}(x_n) &= \sum_{k=1}^{\infty} v_{k-1}(x_n)\varphi_k^{(a)}(x_n) = \sum_{k=n-1}^{n+1} v_{k-1}(x_n)\varphi_k^{(a)}(x_n) \\
&\geqslant \sum_{k=n-1}^{n+1} v_n(x_n)\varphi_k^{(a)}(x_n) = v_n(x_n),
\end{aligned}$$

for $n = 2, 3, \ldots$ (since $\mathcal{V}$ is decreasing), hence

$$q_{\bar{v}_{a,c}}(b) = \sup_{x \in X} \bar{v}_{a,c}(x)b(x) \geqslant \bar{v}_{a,c}(x_n)b(x_n) \geqslant v_n(x_n)b(x_n) > n$$

for each $n \in \mathbb{N}$, a contradiction to (4.2). Therefore we really must have (4.3) as claimed. $\square$

4.3 Completeness

In the case of weighted sequence spaces we know that the co-echelon space $k^\infty(V)$ is always complete since it is the inductive dual of the corresponding metrizable echelon space $\lambda^1(A)$. This time Theorem 4.18 only tells us that $\mathcal{V}C(X, E)$ is the bornologification of $C\bar{V}(X, E)$.

To obtain completeness of $\mathcal{V}C(X, E)$ we will restrict ourselves to a decreasing sequence of continuous weights $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ and $E = \mathbb{K}$ so that $\mathcal{V}C(X)$ is an (LB)-space. This allows us to use the completeness of the co-echelon space

$$k^\infty(\mathcal{V}) := \varinjlim_n \ell^\infty(v_n) \quad \text{where}$$

$$\ell^\infty(v_n) := \Big\{ u = \big(u(x)\big)_{x\in X} \in \mathbb{K}^X \;\Big|\; q_n(u) := \sup_{x\in X} v_n(x)\,|u(x)| < \infty \Big\}$$

by showing that $\mathcal{V}C(X)$ is a closed topological subspace of the former (since we obviously have a continuous injection $\mathcal{V}C(X) \to k^\infty(\mathcal{V})$).

4.19 Proposition. (See [BB91, Prop. 1].)
Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive continuous weights on X. Then $\mathcal{V}C(X)$ is a topological subspace of $k^\infty(\mathcal{V})$.

Proof. Since the topology of $\mathcal{V}C(X)$ is stronger than the one induced by $k^\infty(\mathcal{V})$, it suffices to show that each absolutely convex 0-neighborhood of the former space also contains the trace of a 0-neighborhood of the latter space. Thus, let U denote an absolutely convex 0-neighborhood of $\mathcal{V}C(X)$. Then, for each step $n \in \mathbb{N}$ of the inductive limit, we can find $\rho_n > 0$ such that U contains the closed ρ_n-ball B_n around 0 in $Cv_n(X)$,

$$B_n := \Big\{ f \in Cv_n(X) \;\Big|\; q_n(f) = \sup_{x\in X} v_n(x)\,|f(x)| \leqslant \rho_n \Big\} \subseteq U.$$

Now, for $\alpha_n := 2^{-n}\rho_n$, put

$$W := \bigcup_{m=1}^{\infty} \sum_{n=1}^{m} A_n \qquad \text{where} \qquad A_n := \big\{ u \in \ell^\infty(v_n) \;\big|\; q_n(u) \leqslant \alpha_n \big\}.$$

The set W is clearly an absolutely convex 0-neighborhood in $k^\infty(\mathcal{V})$, and we claim that we also have $W \cap \mathcal{V}C(X) \subseteq U$.

To see this, fix an $f \in W \cap \mathcal{V}C(X)$. By definition we can find $m \in \mathbb{N}$ with $f \in \sum_{n=1}^m A_n$ such that, for each $x \in X$,

$$|f(x)| \leqslant \sum_{n=1}^{m} \frac{\alpha_n}{v_n(x)}.$$

Now define functions f_n recursively for $n = 1, \ldots, m$ by the following relation,

$$f_n(x) := \begin{cases} h_{n-1}(x) & \text{if } |h_{n-1}(x)| \leqslant \frac{\alpha_n}{v_n(x)}, \\ \dfrac{\alpha_n}{v_n(x)}\,\dfrac{h_{n-1}(x)}{|h_{n-1}(x)|} & \text{otherwise,} \end{cases}$$

where $h_0 := f$ and $h_n := h_{n-1} - f_n = f - \sum_{k=1}^n f_k$. Then all these functions are continuous, furthermore since $|f_n(x)| \leqslant \frac{\alpha_n}{v_n(x)}$ we obtain $f_n \in Cv_n(x) \subseteq \mathcal{V}C(X)$ and therefore also $h_n \in \mathcal{V}C(X)$ for $n = 1, \ldots, m$. If $|h_{n-1}(x)| > \frac{\alpha_n}{v_n(x)}$, then

$$|h_n(x)| = |h_{n-1}(x) - f_n(x)| = |h_{n-1}(x)| \left| 1 - \underbrace{\frac{\alpha_n}{v_n(x)\,|h_{n-1}(x)|}}_{<1} \right| = |h_{n-1}(x)| - \frac{\alpha_n}{v_n(x)},$$

otherwise, if $|h_{n-1}(x)| \leqslant \frac{\alpha_n}{v_n(x)}$, we have $h_n(x) = h_{n-1}(x) - h_{n-1}(x) = 0$ by definition. Therefore we obtain inductively for $0 \leqslant k \leqslant m$ and all $x \in X$ that

$$|h_k(x)| \leqslant \sum_{n=k+1}^{m} \frac{\alpha_n}{v_n(x)},$$

i. e., $h_m = 0$ and hence $f = \sum_{n=1}^{m} f_n$. Finally, setting $g_n := 2^n f_n$, we obtain

$$q_n(g_n) = 2^n \sup_{x \in X} v_n(x)\,|f_n| \leqslant 2^n \alpha_n = \rho_n$$

by construction of f_n, and therefore $g_n \in B_n \subseteq U$ which means that $f = \sum_{n=1}^{m} 2^{-n} g_n$ is contained in the absolutely convex 0-neighborhood U as claimed. □

4.20 Theorem. (See [BB91, Thm. & Prop. 2].)
Let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ denote a decreasing sequence of strictly positive continuous weights on X. Then $\mathcal{V}C(X)$ is closed in $k^\infty(\mathcal{V})$ and hence always complete.

Proof. We will show that an arbitrary function f in the closure of $\mathcal{V}C(X)$ taken in $k^\infty(\mathcal{V})$ must be continuous at each point $x_0 \in X$, by Proposition 4.19 84 this suffices to establish our claim. For a given $\varepsilon > 0$ define a set

$$U := \bigcup_{n=1}^{\infty} \left\{ u \in k^\infty(\mathcal{V}) \;\middle|\; \forall x \in X\colon\ |u(x)| \leqslant \sum_{k=1}^{n} \frac{\alpha_k}{v_k(x)} \right\}$$

$$\text{where} \quad \alpha_k := \frac{\varepsilon}{3}\,\frac{v_k(x_0)}{2^{k+1}} > 0 \quad (k \in \mathbb{N}).$$

Then U is an absolutely convex 0-neighborhood in $k^\infty(\mathcal{V})$ since it is an increasing union of absolutely convex sets and it contains the closed α_n-ball around 0 of each step $\ell^\infty(v_n)$. Since f belongs to the closure of $\mathcal{V}C(X)$ in $k^\infty(\mathcal{V})$ we can therefore find an element $g \in \mathcal{V}C(X)$ with $f - g \in U$. Hence there is an index $n \in \mathbb{N}$ such that for all $x \in X$

$$|f(x) - g(x)| \leqslant \sum_{k=1}^{n} \frac{\alpha_k}{v_k(x)} = \frac{\varepsilon}{3} \sum_{k=1}^{n} \frac{v_k(x_0)}{2^{k+1}\, v_k(x)}.$$

Now, since the weights v_k are continuous and strictly positive, we can find a neighborhood W_k of $x_0 \in X$ for each $k = 1, \ldots, n$ such that $v_k(x) > \frac{v_k(x_0)}{2} > 0$ for all $x \in W_k$. By taking the intersection $W_0 := \bigcap_{k=1}^{n} W_k$ we can thus further estimate $|f(x) - g(x)|$ if $x \in W_0$,

$$|f(x) - g(x)| \leqslant \frac{\varepsilon}{3} \sum_{k=1}^{n} \frac{v_k(x_0)}{2^{k+1}\, v_k(x)} < \frac{\varepsilon}{3} \sum_{k=1}^{n} \frac{2}{2^{k+1}} < \frac{\varepsilon}{3}.$$

Finally we use the continuity of g at x_0 to obtain a neighborhood $W \subseteq W_0$ of $x_0 \in X$ with $|g(x) - g(x_0)| < \frac{\varepsilon}{3}$ for each $x \in W$. Combining these estimates we arrive at

$$\begin{aligned} |f(x) - f(x_0)| &\leqslant |f(x) - g(x)| + |g(x) - g(x_0)| + |g(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for each $x \in W$ whereby f must indeed be continuous at $x_0 \in X$. $\square$

4.4 The Regularly Decreasing Condition

We first present a version of Counterexample 3.8(ii)48 adapted to our present setting to illustrate the difficulties of finding general conditions such that $\mathcal{V}_0 C(X, E) = C\bar{V}_0(X, E)$ holds.

4.21 Example. (See [BMS82a, Rem. 1.5] and [BM76, Beispiel 2.3].)
Let $X := \{ z \in \mathbb{C} \mid |z| < 1 \}$ be the open unit disk of $E := \mathbb{C}$ and define a decreasing sequence of weights $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ on X as follows (see also Figure 4.1)

$$v_n(r, \varphi) := \begin{cases} 1, & \text{if } r = 0 \text{ or } 0 < r < 1,\ \varphi \in \left[-\frac{1}{n}, \frac{1}{n}\right] \\ 1 - r, & \text{if } 0 < r < 1 \text{ and } \varphi \notin \left(-\frac{2}{n}, \frac{2}{n}\right) \\ 1 + rn \cdot \left(\frac{1}{n} + \varphi\right), & \text{if } 0 < r < 1 \text{ and } \varphi \in \left(-\frac{2}{n}, -\frac{1}{n}\right) \\ 1 - rn \cdot \left(\frac{1}{n} - \varphi\right), & \text{if } 0 < r < 1 \text{ and } \varphi \in \left(\frac{1}{n}, \frac{2}{n}\right). \end{cases}$$

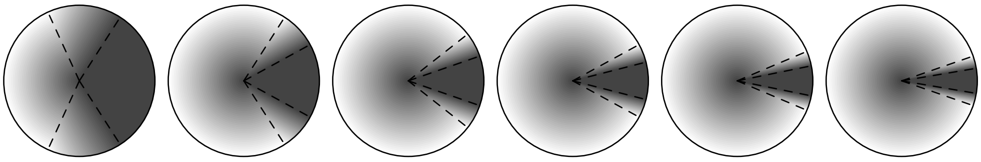

Figure 4.1 The weights v_n for $n = 1, \ldots, 6$.

First, we remark that in this situation $\mathcal{V}_0 C(X)$ is a proper subspace of $\mathcal{V}C(X)$, since although the function $f \equiv 1$ is an element of the latter space, it is not contained in the former.

Now we want to show that $\mathcal{V}_0 C(X) = \varinjlim_n C(v_n)_0(X)$ is not a regular inductive limit, i. e., that there exist bounded sets in $\mathcal{V}_0 C(X)$ which are not contained and bounded in any of the steps. For this purpose, consider the sequence of functions $(f_n)_{n \in \mathbb{N}}$ defined as follows (see also Figure 4.2)

$$f_n(r, \varphi) := \begin{cases} 0, & \text{if } r = 0 \text{ or } 0 < r < 1,\ \varphi \in \left[-\frac{2}{n}, \frac{2}{n}\right] \\ \frac{r}{n}, & \text{if } 0 < r < 1 \text{ and } \varphi \notin \left(-\frac{3}{n}, \frac{3}{n}\right) \\ r \cdot \left(\frac{-2}{n} - \varphi\right), & \text{if } 0 < r < 1 \text{ and } \varphi \in \left(-\frac{3}{n}, -\frac{2}{n}\right) \\ r \cdot \left(-\frac{2}{n} + \varphi\right), & \text{if } 0 < r < 1 \text{ and } \varphi \in \left(\frac{2}{n}, \frac{3}{n}\right). \end{cases}$$

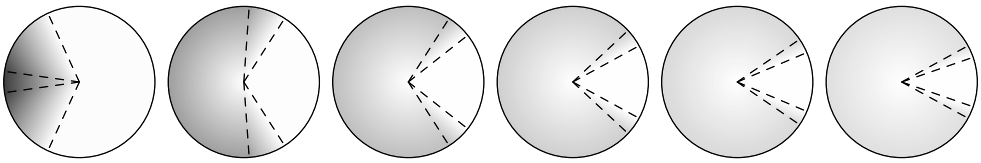

Figure 4.2 The functions f_n for $n = 1, \ldots, 6$.

We obtain that $f_n \in C(v_n)_0(X)$ and $f_n \notin C(v_m)_0(X)$ whenever $n > 3m$. Thus there is no step of the inductive limit $\mathcal{V}_0 C(X)$ which contains the whole sequence $(f_n)_{n \in \mathbb{N}}$. It is only left to be shown that it is bounded.

Let B_1 be the closed unit ball of the Banach space $C(v_1)_0(X)$; choose continuous functions $\rho_k : X \to [0,1]$ such that they satisfy $\rho_k \equiv 1$ on $X \setminus A_k$ where $A_k := \left\{ z \in X \;\middle|\; |z| \geqslant 1 - \frac{1}{k} \right\}$ and still vanish at infinity ($k \in \mathbb{N}$). Then, given an $\varepsilon > 0$, we have

$$\begin{aligned} \|\rho_k f_n - f_n\|_{C(v_n)_0(X)} &= \sup_{x \in X} v_n(x)\,|f_n(x)| \cdot |\rho_k(x) - 1| \\ &= \sup_{x \in A_k} v_n(x)\, f_n(x) \cdot |\rho_k(x) - 1| \leqslant \sup_{x \in A_k} v_n(x)\, f_n(x) < \varepsilon, \end{aligned}$$

for $k \in \mathbb{N}$ big enough, since $v_n \cdot f_n$ vanishes at infinity on X. On the other hand we have $\rho_k \cdot f_n \in B_1$ for all $k, n \in \mathbb{N}$ such that f_n belongs to the closure of B_1 in $C(v_n)_0(X)$. This means that we have obtained that the set $\{f_n \mid n \in \mathbb{N}\}$ is contained in the closure of B_1 taken in $\mathcal{V}_0 C(X)$ and therefore has to be bounded in $\mathcal{V}_0 C(X)$.

To conclude, we remark that in this example we cannot have $\mathcal{V}_0 C(X) = C\bar{V}_0(X)$ since the space $\mathcal{V}_0 C(X)$ cannot be complete (by Proposition 2.13) although $C\bar{V}_0(X)$ is (by Theorem 4.14). Thus $\mathcal{V}_0 C(X, E)$ can be a proper subspace of $C\bar{V}_0(X, E)$, even in the scalar case $E = \mathbb{C}$ and with continuous weights $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$.

A first (rather trivial) step to obtaining a projective description for the space $\mathcal{V}C(X, E)$ using Theorem 4.14 is to find conditions (in terms of X and $\mathcal{V}$) under which we already have $\mathcal{V}C(X, E) = \mathcal{V}_0 C(X, E)$ or, more generally,

$$\mathcal{V}A(X, E) = \mathcal{V}_0 A(X, E),$$

which might also remedy the problems of Example 4.21 (since there $\mathcal{V}_0 C(X)$ was a proper subspace of $\mathcal{V}C(X)$).

Therefore, as a first step, we will try to generalize condition (S) of section 3.3 to imply the equivalence of the inductive sequences $(AV_n(X, E))_n$ and $(A(V_n)_0(X, E))_n$ which would lead us to the desired conclusion.

In the special case where $\mathcal{V}$ is a sequence of weights (which is similar to the setting of section 3.3), i. e., $V_n = \{\lambda v_n \mid \lambda > 0\}$, we obtain a continuous injection from $Av_n(X, E)$ to $A(v_m)_0(X, E)$ for an $m > n$ if v_m / v_n vanishes at infinity (that is, if for every $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $v_m \leqslant \varepsilon \cdot v_n$ whenever $x \in X \setminus K$), since we then have

$$v_m(x)\, p(f(x)) \leqslant \varepsilon \cdot v_n(x)\, p(f(x)) \leqslant \varepsilon \cdot C \quad (x \in X \setminus K)$$

where the constant $C > 0$ only depends on v_n, $p \in cs(E)$ and $f \in Av_n(X, E)$, such that $v_m \cdot (p \circ f)$ also vanishes at infinity, which means that $f \in A(v_m)_0(X, E)$. Because $A(v_m)_0(X, E)$ is continuously embedded into $Av_m(X, E)$, we have obtained that the inductive sequences $(Av_n(X, E))_n$ and $(A(v_n)_0(X, E))_n$ are equivalent if, given an $n \in \mathbb{N}$, we can always find an $m > n$ such that v_m / v_n vanishes at infinity.

We can easily generalize this to also obtain a sufficient condition for $\mathcal{V}A(X, E) = \mathcal{V}_0A(X, E)$ where $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ is a decreasing sequence of general Nachbin families on X: Given an $n \in \mathbb{N}$ we have to be able to find an $m > n$ such that

$$\forall v_m \in V_m \ \exists v_n \in V_n\colon \ \frac{v_m}{v_n} \text{ vanishes at infinity} \tag{S}$$

(this is condition (V) of [BMS82a]).

Similarly to Proposition 3.23 of the sequence space case, we will now prove that this condition implies projective description, at least on bounded sets of the steps.

4.22 Lemma. (See [BM76, § 1, Satz 6].)
Let $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of systems of weights on X such that $\mathcal{K} \preceq \mathcal{V}$ and condition (S) is satisfied. Then, if $n < m$ is chosen according to (S), the spaces $AV_m(X, E)$, $\mathcal{V}A(X, E)$, $\mathcal{V}_0A(X, E)$ all induce the same topology on each bounded subset of $AV_n(X, E)$.

Proof. We have continuous embeddings $AV_n(X, E) \to AV_m(X, E) \to \mathcal{V}A(X, E)$ and by construction of (S) we also know that $\mathcal{V}A(X, E) \cong \mathcal{V}_0A(X, E)$. Furthermore the inductive limit topology of $\mathcal{V}A(X, E)$ is stronger than the compact-open topology τ_{co} such that it suffices to show that τ_{co} induces the natural topology on each bounded subset of $AV_n(X, E)$.

A subset B of $AV_n(X, E)$ is bounded if and only if for each continuous seminorm $p \in cs(E)$ and each weight $v_n \in V_n$, there exists a constant $C_{v_n,p} > 0$ such that

$$\sup_{f\in B} q_{v_n,p}(f) = \sup_{f\in B} \sup_{x\in X} v_n(x)\, p(f(x)) \leqslant C_{v_n,p}.$$

Hence, by condition (S), given a seminorm $p \in cs(E)$ and a weight $v_m \in V_m$ there is a corresponding weight $v_n \in V_n$ such that for any given $\varepsilon > 0$ we can find a compact set $K \subseteq X$ with

$$\sup_{x\in X\setminus K} v_m(x)\, p(f(x)) \leqslant \sup_{x\in X\setminus K} v_n(x)\, p(f(x)) \cdot \frac{\varepsilon}{2C_{v_n,p}} \leqslant \frac{\varepsilon}{2} \tag{$*$}$$

(where $f \in B$ is arbitrary). Of course we can also find a constant $D > 0$ with $v_m(x) < D$ for $x \in K$.

Every neighborhood of $f_0 \in B$ with respect to the topology of $AV_m(X, E)$ contains a set of the form

$$U_{v_m,p,\varepsilon}(f_0) := \Big\{ f \in AV_m(X, E) \ \Big|\ q_{v_m,p}(f - f_0) = \sup_{x\in X} v_m(x)\, p(f(x) - f_0(x)) < \varepsilon \Big\}$$

for a seminorm $p \in cs(E)$ and $\varepsilon > 0$. We are done if we can find a neighborhood of f_0 with respect to the compact-open topology on $C(X, E)$ whose trace under B is contained in $U_{v_m,p,\varepsilon} \cap B$. For this it suffices to consider

$$U_{K,p,\varepsilon}(f_0) := \Big\{ f \in C(X, E) \ \Big|\ \|f - f_0\|_K = \sup_{x\in K} p(f(x) - f_0(x)) < \tfrac{\varepsilon}{D} \Big\},$$

since then clearly $U_{K,p,\varepsilon} \cap B \subseteq U_{\nu_m,p,\varepsilon} \cap B$ such that all considered topologies have to coincide on B (given $f \in U_{K,p,\varepsilon}$, if the supremum of $q_{\nu_m,p}(f - f_0)$ is attained for an $x \in K$ we obtain $f \in U_{\nu_m,p,\varepsilon}$ by our choice of $D > 0$, otherwise we only need to apply the inequality $(*)$ proven above). □

In the simplified setting of Theorem 4.18 82 this can be used to show that condition (S) is indeed sufficient to obtain algebraic and topological projective descriptions of the whole spaces.

4.23 Proposition. (See [BMS82a, part (ii) of *direct proof* of Thm. 1.3(d)].)
Let $(E, \|\,.\,\|)$ denote a normed space and let $\mathcal{V} = (\nu_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive weights on the locally compact σ-compact space X such that $\inf_{x\in K} \nu_n(x) > 0$ for each compact set K of X and all $n \in \mathbb{N}$. If $\mathcal{V}$ satisfies condition (S), then we have

$$\mathcal{V}_0 C(X, E) \cong C\bar{V}_0(X, E) \cong \mathcal{V}C(X, E) \cong C\bar{V}(X, E).$$

Proof. Because of condition (S) we may assume without loss of generality that $\frac{\nu_{n+1}}{\nu_n}$ vanishes at infinity for each $n \in \mathbb{N}$ (by considering an equivalent subsequence of $\mathcal{V}$).

As a first step we want to show that, subject to condition (S), the inductive limit topology of $\mathcal{V}C(X, E)$ is not only stronger but also weaker than the weighted topology of $C\bar{V}(X, E)$. To this end, consider a 0-neighborhood of $\mathcal{V}C(X, E)$. By Remark 1.24 19 it has to contain a set U of the form

$$U = \overline{\Gamma\Big(\bigcup_{n=1}^{\infty} U_n\Big)}$$

where the closure is taken in $\mathcal{V}C(X, E)$ and with

$$U_n := \Big\{ f \in C\nu_n(X, E) \;\Big|\; q_{\nu_n}(f) = \sup_{x\in X} \nu_n(x)\|f(x)\| < \rho_n \Big\}$$

for a decreasing sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive numbers. Our claim is proven if we can find a 0-neighborhood $\bar{U}$ of $C\bar{V}(X, E)$ such that $\bar{U} \subseteq U$.

For a fixed sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets of X with $K_n \subseteq \mathring{K}_{n+1}$ and $X = \cup_{n=1}^{\infty} K_n$ we can find a strictly increasing sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{N}$ such that

$$\frac{\nu_{n+1}(x)}{\nu_n(x)} < \frac{\rho_{n+1}}{2^{n+1}} \quad \text{for all } x \notin K_{a_n}.$$

For this sequence a and $\nu_0 \equiv c := 1$ define the weight $\bar{\nu} \in \bar{V}$,

$$\bar{\nu} := \bar{\nu}_{a,c} := \sum_{k=1}^{\infty} \nu_{k-1}\varphi_k^{(a)} \in \hat{V} \subseteq \bar{V},$$

as in (4.1) 81 using a continuous partition of unity $(\varphi_n^{(a)})_{n\in\mathbb{N}}$ subordinate to the covering $\big(\mathring{K}_{a_{n+1}} \setminus K_{a_{n-1}}\big)_{n\in\mathbb{N}}$ of X (where, as usual, $a_0 := 0$ and $K_0 := \emptyset$). Now we can define

$$\bar{U} := \left\{ f \in C\bar{V}(X,E) \;\middle|\; q_{\bar{v}}(f) = \sup_{x\in X} \bar{v}(x)\|f(x)\| < \min\Big(1, \frac{\rho_1}{2\cdot \sup_{x\in K_{a_2}} v_1(x)}\Big) \right\}.$$

To see that we indeed have $\bar{U} \subseteq U$, consider that every $f \in C(X,E)$ can be represented in the following way,

$$f = \sum_{n=1}^{\infty} \varphi_n^{(a)} f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n \quad \text{with} \quad f_n := 2^n \varphi_n^{(a)} f,$$

where the series converges uniformly on compact subsets of X. Furthermore in Theorem 4.18 we proved that for each $f \in C\bar{V}(X,E)$ an $n_0 \in \mathbb{N}$ can be found with $f \in Cv_{n_0}(X,E)$ such that, in this case, the partial sums $\sum_{n=1}^{N} 2^{-n} f_n$ are all bounded by $q_{v_{n_0}}(f)$. Therefore, by Lemma 4.22, the series also converges with respect to the inductive limit topology of $\mathcal{V}C(X,E)$. Summarizing these observations we now know that to obtain $\bar{U} \subseteq U$ it suffices to see that $f \in \bar{U}$ implies $f_n \in U$ for each $n \in \mathbb{N}$. In other words we have to show

$$q_{\bar{v}}(f) < \min\Big(1, \frac{\rho_1}{2\cdot \sup_{x\in K_{a_2}} v_1(x)}\Big) \implies q_{v_n}(f_n) < \rho_n$$

for each $n \in \mathbb{N}$, which will be verified by the following calculation:

$$\begin{aligned}
q_{v_1}(f_1) &= \sup_{x\in X} v_1(x)\|f_1(x)\| = \sup_{x\in K_{a_2}} v_1(x) 2\varphi_1^{(a)}(x)\|f(x)\| \\
&\leqslant 2\Big(\sup_{x\in K_{a_2}} v_1(x)\Big)\Big(\sup_{x\in X} \underbrace{v_0(x)}_{=1} \varphi_1^{(a)}(x)\|f(x)\|\Big) \\
&\leqslant 2\Big(\sup_{x\in K_{a_2}} v_1(x)\Big)\Big(\sup_{x\in X} \sum_{k=1}^{\infty} v_{k-1}(x)\varphi_k^{(a)}(x)\|f(x)\|\Big) \\
&= 2\Big(\sup_{x\in K_{a_2}} v_1(x)\Big) q_{\bar{v}}(f) < \rho_1, \\
q_{v_{n+1}}(f_{n+1}) &= \sup_{x\in X} v_{n+1}(x)\|f_{n+1}(x)\| \\
&= \sup_{x\notin K_{a_n}} \frac{v_{n+1}(x)}{v_n(x)} v_n(x) 2^{n+1} \varphi_{n+1}^{(a)}(x)\|f(x)\| \\
&\leqslant \frac{\rho_{n+1}}{2^{n+1}} 2^{n+1} \sup_{x\in X} v_n(x)\varphi_{n+1}^{(a)}(x)\|f(x)\| \\
&\leqslant \rho_{n+1} \sup_{x\in X} \sum_{k=1}^{\infty} v_{k-1}(x)\varphi_k^{(a)}(x)\|f(x)\| \\
&= \rho_{n+1} q_{\bar{v}}(f) < \rho_{n+1}.
\end{aligned}$$

Thus, we have shown $\mathcal{V}C(X,E) \cong C\bar{V}(X,E)$. Since $\mathcal{V}_0C(X,E) \cong \mathcal{V}C(X,E)$ follows by construction of condition (S), we are only left with $C\bar{V}_0(X,E) \supseteq C\bar{V}(X,E)$. But this we obtain easily as follows: if $f \in C\bar{V}(X,E)$, then by Theorem 4.18 there exists an $n \in \mathbb{N}$ such that $f \in Cv_n(X,E)$. Given a weight $\bar{v} \in \bar{V}$ we know that $\frac{\bar{v}}{v_{n+1}}$ is bounded, and since by assumption $\frac{v_{n+1}}{v_n}$ vanishes at infinity also $\bar{v}f = \frac{\bar{v}}{v_{n+1}} \cdot \frac{v_{n+1}}{v_n} v_n f$ has to vanish at infinity such that $f \in C\bar{V}_0(X, E)$ as claimed. $\square$

On the other hand, condition (S) also implies projective description for the spaces $\mathcal{V}_0 A(X)$, i. e., that the topology of that space is induced by the weighted topology of $C\bar{V}_0(X)$.

4.24 Theorem. (See [BMS82a, Thm. 1.6].)
Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive weights on the locally compact σ-compact space X such that $\inf_{x\in K} v_n(x) > 0$ for each compact set K of X and all $n \in \mathbb{N}$. If $\mathcal{V}$ satisfies condition (S) and $A(X) \subseteq C(X)$ is a semi-Montel space with respect to the compact-open topology, then

$$\mathcal{V}A(X) \cong \mathcal{V}_0 A(X) \cong A\bar{V}_0(X) \subseteq C\bar{V}_0(X).$$

Proof. First, recall that condition (S) implies both $\mathcal{V}A(X) \cong \mathcal{V}_0 A(X)$ and $\mathcal{V}C(X) \cong \mathcal{V}_0 C(X)$. Since $\mathcal{V}A(X) \cong \mathcal{V}_0 A(X)$ is a compact inductive limit (see Lemma 4.22[88]) and $\mathcal{V}C(X) \cong \mathcal{V}_0 C(X)$ is always regular, the former space $\mathcal{V}A(X)$ is a topological subspace of the latter, $\mathcal{V}C(X)$, by Corollary 2.26[44] to Baernsteins open mapping lemma. Hence the assertion follows from Proposition 4.23[89] which stated that $\mathcal{V}_0 C(X) \cong C\bar{V}_0(X)$. $\square$

Next we want to look at the regularly decreasing condition which, in the setting of sequence spaces, characterized projective description for $p = 0$. By staying in the setting of Theorem 4.18[82] and the previous Proposition where we only considered *sequences* $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of weights we can copy the condition nearly literally from its sequence space version, Definition 3.21[62]. The only difference is that instead of "strictly positive functions on the index set I" we now only consider "strictly positive weights (i. e., upper semi-continuous functions) on X". So, a decreasing sequence of strictly positive weights $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is *regularly decreasing* if for each $n \in \mathbb{N}$ there exists an $m \geqslant n$ such that

$$\forall \varepsilon > 0,\ k \geqslant m\ \exists \delta_{k,\varepsilon} > 0: \quad v_k(x) < \delta_{k,\varepsilon} v_n(x) \implies v_m(x) < \varepsilon \cdot v_n(x). \tag{4.4}$$

Also, since in this setting the maximal system of weights $\bar{V}_{\mathcal{V}}$ consists of all weights contained in the associated Köthe set as defined in Definition 3.10[50], we obtain that Proposition 3.22[63] also holds analogously for sequences $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive weights, i. e., we have that (4.4) is equivalent to condition (wS): the sequence $\mathcal{V}$ is regularly decreasing if and only if for each $n \in \mathbb{N}$ there exists an $m > n$ such that

$$\forall \varepsilon > 0\ \exists \bar{v} \in \bar{V}_{\mathcal{V}}: \quad \bar{v}(x) < v_m(x) \implies v_m(x) \leqslant \varepsilon \cdot v_n(x) \tag{wS}$$

(the function $\bar{v}$ constructed in the proof of Proposition 3.22[63] is actually upper semi-continuous and therefore an element of $\check{V}_{\mathcal{V}} \subseteq \bar{V}_{\mathcal{V}}$).

We could now prove projective description in a similar way to Proposition 3.23 using condition (wS), but in order not to repeat ourselves we will give a different proof using Raĭkov's completeness criterion of Theorem 2.11.

4.25 Theorem. (See [BMS82a, Thm. 2.3].)
Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive weights on a locally compact space X such that $\inf_{x\in K} v_n(x) < \infty$ for each compact set $K \subseteq X$ and $n \in \mathbb{N}$. If $\mathcal{V}$ is regularly decreasing and $(E, \|\,.\,\|)$ denotes a Banach space, then $\mathcal{V}_0C(X,E)$ is also complete and we have the algebraic and topological identity

$$\mathcal{V}_0C(X,E) \cong C\bar{V}_0(X,E).$$

Proof. We want to apply Theorem 2.11 to the sets $K_n := 2^n B_n$, $n \in \mathbb{N}$, where

$$B_n := \left\{ f \in C(v_n)_0(X,E) \;\middle|\; q_{v_n}(f) = \sup_{x\in X} v_n(x)\,\|f(x)\| \leqslant 1 \right\}.$$

As in Proposition 2.14 we obtain the conditions (a), (b) and (c) necessary for Theorem 2.11. Therefore, to obtain completeness of $\mathcal{V}_0C(X,E) = \varinjlim_n C(v_n)_0(X,E)$, it suffices to show that for each $n \in \mathbb{N}$ there exists an $m \geqslant n$ such that the closure of B_n with respect to the (weaker) weighted topology of $C\bar{V}_0(X,E)$ is already contained in B_m.

To prove this claim, let $n \in \mathbb{N}$ be fixed and choose $m \geqslant n$ according to the regularly decreasing condition. Let f be an arbitrary element of the $C\bar{V}_0(X,E)$-closure of B_n. Since the weighted topology of $C\bar{V}_0(X,E)$ is stronger than pointwise convergence on X we still have $q_{v_n}(f) \leqslant 1$ and therefore also $q_{v_m}(f) \leqslant 1$. To see that $f \in C(v_m)_0(X,E)$ let $\varepsilon > 0$ and put $F := \left\{ x \in X \;\middle|\; v_m(x)\,\|f(x)\| \geqslant \varepsilon \right\}$. We are done if we can show that F is (relatively) compact in X.

If f were an element of B_n then F would be compact, therefore we will try to find a sufficiently close $g \in B_n$. So, define a weight $\bar{v} \in \check{V}_{\mathcal{V}} \subseteq \bar{V}_{\mathcal{V}}$ on X by setting

$$\bar{v}(x) := \inf_{k\in\mathbb{N}} \frac{1}{C_k} v_k(x) \qquad \text{for} \qquad 0 < C_k := \begin{cases} \frac{\varepsilon}{8} & k < m, \\ \delta_{k,\varepsilon/8} & k \geqslant m, \end{cases}$$

where $\delta_{k,\varepsilon/8} > 0$ is chosen according to the regularly decreasing condition (4.4). Now we can choose $g \in B_n$ such that $q_{\bar{v}}(f-g) < \frac{\varepsilon}{4}$ and $G := \left\{ x \in X \;\middle|\; v_m(x)\,\|g(x)\| \geqslant \varepsilon \right\}$ is compact. Obviously our goal is to prove $F \subseteq G$. But first assume indirectly that $v_m(y) > \bar{v}(y) + \frac{\varepsilon}{8} v_n(y)$ for some $y \in X$ (i. e., we can't estimate v_m using $\bar{v}$ and v_n). Then, for $k \geqslant m$, we obtain

$$\frac{1}{C_k} v_k(y) \geqslant \frac{1}{C_k} \delta_{k,\varepsilon/8}\, v_n(y) = v_n(y)$$

(by (4.4) since $v_m(y) > \frac{\varepsilon}{8} v_n(y)$). On the other hand, for $k < m$ this would imply

$$\frac{1}{C_k} v_k(y) \geqslant \frac{1}{C_k} v_m(y) > \frac{1}{C_k} \frac{\varepsilon}{8} v_n(y) = v_n(y),$$

(since $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is decreasing) such that in total $\bar{v}(y) \geqslant v_n(y)$. On the other hand, this now turns the indirect assumption into $v_m(y) > \left(1 + \frac{\varepsilon}{8}\right) v_n(y) > v_n(y)$, a contradiction to $\mathcal{V}$ being decreasing. Therefore we must have

$$v_m(x) \leqslant \bar{v}(x) + \frac{\varepsilon}{8} v_n(x)$$

for all $x \in X$. Finally, for $x \in F$, this implies

$$\begin{aligned}
v_m(x)\,\|g(x)\| &\geqslant v_m(x)\,\|f(x)\| - v_m(x)\,\|f(x) - g(x)\| \\
&\geqslant \varepsilon - \left(\bar{v}(x) + \frac{\varepsilon}{8} v_n(x)\right) \|f(x) - g(x)\| \\
&\geqslant \varepsilon - \bar{v}(x)\,\|f(x) - g(x)\| - \frac{\varepsilon}{8} v_n(x) \left(\|f(x)\| + \|g(x)\|\right) \\
&\geqslant \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8} - \frac{\varepsilon}{8} = \frac{\varepsilon}{2},
\end{aligned}$$

such that F is a subset of G and hence relatively compact as claimed. □

To obtain the necessity of the regularly decreasing condition for projective description we will proceed as in Theorem 3.25←66. Unfortunately we cannot reuse Lemma 3.24←64 of the sequence space setting—we will need property (iii) and the second part of (ii) for compact neighborhoods in X instead of only pointwise. Furthermore, this is also the reason why this time we need to assume *continuous* weights and take $E = \mathbb{K}$.

4.26 Lemma. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive continuous weights on a locally compact (Hausdorff) space X. If $\mathcal{V}$ is *not* regularly decreasing, then we may choose $n \in \mathbb{N}$ and a subsequence $(v_{\varphi(m)})_{m\in\mathbb{N}}$ of $\mathcal{V}$ for which there exists a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ of positive numbers, a sequence $(k_m)_{m\in\mathbb{N}}$ of indices and a sequence of triples $(\delta_{ij}, x_{ij}, U_{ij})_{j\in\mathbb{N}}$ for each $i \in \mathbb{N}$ such that (for all $i, j, r, s \in \mathbb{N}$)

(i) $0 < \delta_{ij} \leqslant \frac{1}{j}$,

(ii) $v_{\varphi(i)}(x_{ij}) \geqslant \varepsilon_i\, v_n(x_{ij})$ although $v_{k_i}(x) < \delta_{jl}\, v_n(x)$ for each $x \in U_{ij}$,

(iii) $U_{ij} \subseteq X$ is a compact neighborhood of $x_{ij} \in X$ such that $U_{ij} \cap U_{rs} = \emptyset$ whenever $(i, j) \neq (r, s)$. Moreover the sets

$$\bigcup_{j=1}^{\infty} U_{ij} \qquad \text{and} \qquad \bigcup \left\{ U_{ij} \,\middle|\, j \in \mathbb{N},\ j \neq j_0 \right\}$$

are closed in X for arbitrary $j_0 \in \mathbb{N}$.

Proof. Since $\mathcal{V}$ is not regularly decreasing we can fix an $n \in \mathbb{N}$ such that for every $m \geqslant n$ there exist $k_m > m$ and $\varepsilon_m > 0$ for which, given $\delta > 0$, we can always find $x_\delta \in X$ such that

$$v_m(x_\delta) \geqslant \varepsilon_m \cdot v_n(x_\delta) \quad \text{although} \quad v_{k_m}(x_\delta) < \delta \cdot v_n(x_\delta). \tag{4.5}$$

As in Lemma 3.24 we may assume, without loss of generality, that the sequence $(\varepsilon_m)_{m \geqslant n}$ is decreasing and $(k_m)_{m \geqslant n}$ is increasing. Thus we can choose a subsequence of $\mathcal{V}$ by setting $\varphi(1) := n+1$ and $\varphi(m) := k_{m-1} + 1$. We again denote $k_{\varphi(m)}$ by k_m and $\varepsilon_{\varphi(m)}$ by ε_m to obtain the required sequences.

We will now construct the sequences $(\delta_{ij}, x_{ij}, U_{ij})_{j \in \mathbb{N}}$ by induction on $i \in \mathbb{N}$ and $j \in \mathbb{N}$. If $i = j = 1$ we can simply set $\delta_{11} := 1 \leqslant \frac{1}{1}$ and choose $x_{11} \in X$ according to (4.5) for $m = \varphi(1)$ and $\delta = \delta_{11}$. Since X is locally compact and $\frac{v_{k_1}}{v_n} : X \to \mathbb{R}$ is continuous we can also choose a compact neighborhood U_{11} of $x_{11} \in X$ such that $v_{k_1}(x) < \delta_{11}\, v_n(x)$ even holds for all $x \in U_{11}$. Now assume that we have already found $(\delta_{1j}, x_{1j}, U_{1j})$ for $1 \leqslant j < s$ so that conditions (i)–(iii) are satisfied. Then choose

$$\delta_{1s} < \min\Big(\beta_{1s}, \frac{1}{s}\Big) \qquad \text{for} \qquad \beta_{1s} := \inf\left\{ \frac{v_{k_1}(x)}{v_n(x)} \;\middle|\; x \in \bigcup_{j=1}^{s} U_{1j} \right\}.$$

By (4.5) we again obtain an appropriate $x_{1s} \in X$. We cannot have $x_{1s} \in U_{1j}$ for some $j < s$ since then, by choice of δ_{1s}, we would have $\delta_{1s}\, v_n(x_{1s}) < v_{k_1}(x_{1s})$, a contradiction to (ii). Therefore we can again find a compact neighborhood U_{1s} of x_{1s} such that $v_{k_1}(x) < \delta_{1s}\, v_n(x_{1s})$ for $x \in U_{1s}$ and, in particular, $U_{1s} \cap U_{1j} = \emptyset$ for all $j < s$. Inductively we obtain the first sequence of triples $(\delta_{1j}, x_{1j}, U_{1j})_{j \in \mathbb{N}}$.

To see that the union of all compact neighborhoods defined so far is closed as was claimed, assume indirectly that there exists a $y \in \overline{\cup_{j=1}^{\infty} U_{1j}} \setminus \cup_{j=1}^{\infty} U_{1j}$. Then every neighborhood of $y \in X$ has to contain points of U_{1j} for some $j \in \mathbb{N}$ larger than an arbitrary $j_0 \in \mathbb{N}$. Together with $\delta_{1j} \to 0$ ($j \to \infty$), a consequence of (i), we therefore obtain $\frac{v_{k_1}(y)}{v_n(y)} = 0$, which is impossible for strictly positive weights. In the same way we can see that the other union of compact sets must be closed as well.

Now that we have constructed the first sequence, assume that $(\delta_{ij}, x_{ij}, U_{ij})_{j \in \mathbb{N}}$ has already been constructed for $1 \leqslant i < r$ such that (i), (ii) and (iii) hold, we will try to find the r-th sequence $(\delta_{rj}, x_{rj}, U_{rj})_{j \in \mathbb{N}}$ by induction on $j \in \mathbb{N}$. First we remark that by property (i) we can find an index $j_r \in \mathbb{N}$ such that $\delta_{ij} < \varepsilon_r$ for all $j > j_r$ and $1 \leqslant i < r$. Now, for $j = 1$, choose

$$\delta_{r1} < \min\Big(\beta_{r1}, \frac{1}{1}\Big) \qquad \text{for} \qquad \beta_{r1} := \inf\left\{ \frac{v_{k_r}(x)}{v_n(x)} \;\middle|\; x \in \bigcup_{i=1}^{r-1} \bigcup_{j=1}^{j_r} U_{ij} \right\},$$

(which clearly satisfies (i)), such that by (4.5) we can find an appropriate $x_{r1} \in I$. Similarly to above we obtain $x_{r1} \notin \cup_{i=1}^{r-1} \cup_{j=1}^{j_r} U_{ij}$, but since for $y \in U_{ij}$ where $j > j_r$ and $i < r$ we have

$$v_{\varphi(r)}(y) \leqslant v_{k_r}(y) < \delta_{ij}\, v_n(y) \leqslant \varepsilon_r\, v_n(y)$$

(by choice of $j_r \in \mathbb{N}$), we even know $x_{r1} \notin \cup_{i=1}^{r-1} \cup_{j=1}^{\infty} U_{ij}$. We already proved that this set is closed such that we can again find a compact neighborhood U_{r1} of $x_{r1} \in X$ with $v_{k_r}(x) < \delta_{rj}\, v_n(x)$ for $x \in U_{r1}$ and $U_{r1} \cap U_{ij} = \emptyset$ if $i < r$ and $j \in \mathbb{N}$.

Assuming we have also constructed $(\delta_{rj}, x_{rj}, U_{rj})_{1\leqslant j<s}$, to obtain δ_{rs}, x_{rs} and U_{rs} we only have to take

$$\delta_{rs} < \min\Big(\beta_{rs}, \frac{1}{s}\Big) \qquad \text{for} \qquad \beta_{rs} := \inf\left\{ \frac{v_{k_r}(x)}{v_n(x)} \;\middle|\; x \in \Big(\bigcup_{i=1}^{r-1}\bigcup_{j=1}^{j_r} U_{ij}\Big) \cup \Big(\bigcup_{j=1}^{s-1} U_{rj}\Big)\right\},$$

then (i), (ii) and (iii) follow as in the case of $i = 1$ and $j = s$. Thus we have completed both inductions and obtained the desired sequences. □

4.27 Theorem. (See [BMS82a, Thm. 2.5].)
Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ denote a decreasing sequence of strictly positive continuous weights on a locally compact (Hausdorff) space X. If $\mathcal{V}$ is *not* regularly decreasing, then $\mathcal{V}_0C(X) \neq C\bar{V}_0(X)$ as vector spaces.

Proof. Since $\mathcal{V}$ is not regularly decreasing we may use Lemma 4.26[93] to try to construct an element of $C\bar{V}_0(X)$ which is not contained in $\mathcal{V}_0C(X)$.

For each pair $(i, j) \in \mathbb{N}\times\mathbb{N}$ choose a continuous function $\varphi_{ij} : X \to [0, 1]$ such that $\varphi_{ij}(x_{ij}) = 1$ and $\operatorname{supp}(\varphi_{ij}) \subseteq U_{ij}$. Then for $i \in \mathbb{N}$ set $g_i := \sum_{j=1}^{\infty} \varphi_{ij}$. By Lemma 4.26[93] these are still continuous functions $X \to [0, 1]$ with $\operatorname{supp}(g_i) \subseteq \cup_{j=1}^{\infty} U_{ij}$. To be able to also sum over $i \in \mathbb{N}$, choose a decreasing sequence $(\eta_i)_{i\in\mathbb{N}}$ of positive numbers such that $\lim_{i\to\infty} \eta_i = 0$ and set $h_i := \frac{\eta_i}{2^i} g_i$. Indeed, given $\varepsilon > 0$, we can choose $r \in \mathbb{N}$ so that $\eta_r < \varepsilon$ which implies

$$\sum_{i=r}^{\infty} h_i = \sum_{i=r}^{\infty} \frac{\eta_i}{2^i} g_i \leqslant \eta_r \sum_{i=r}^{\infty} \frac{1}{2^i} \leqslant \eta_r < \varepsilon$$

(uniformly on X), i. e., we may define $h := \sum_{i=1}^{\infty} h_i$. Moreover, we obtain that h is a bounded continuous function on X which vanishes on the complement of $\cup_{i=1}^{\infty} \cup_{j=1}^{\infty} U_{ij}$.

Now we claim that, for $n \in \mathbb{N}$ as in Lemma 4.26[93], the continuous function $f := \frac{h}{v_n}$ belongs to $C\bar{V}_0(X)$ but not to $\mathcal{V}_0C(X)$. To prove the latter, fix $i \in \mathbb{N}$ and consider ($j \in \mathbb{N}$)

$$v_{\varphi(i)}(x_{ij})\, f(x_{ij}) = \frac{v_{\varphi(i)}(x_{ij})}{v_n(x_{ij})} h(x_{ij}) \overset{\text{(ii)}}{\geqslant} \varepsilon_i\, h(x_{ij}) \geqslant \varepsilon_i\, h_i(x_{ij}) = \frac{\varepsilon_i\,\eta_i}{2^i} g_i(x_{ij}) \geqslant \frac{\varepsilon_i\,\eta_i}{2^i}.$$

This means that at least on these constructed points $(x_{ij})_{j\in\mathbb{N}}$ we can bound $v_{\varphi(i)}\, f$ away from 0 by a constant which does not depend on $j \in \mathbb{N}$, i. e.,

$$\big\{x_{ij} \;\big|\; j \in \mathbb{N}\big\} \subseteq F_i := \left\{x \in X \;\middle|\; v_{\varphi(i)}(x)\, f(x) \geqslant \frac{\varepsilon_i\,\eta_i}{2^i}\right\}.$$

On the other hand, F_i cannot be compact, otherwise any cluster point x of $(x_{ij})_{j\in\mathbb{N}}$ would belong to F_i such that $f(x) \neq 0$ which means that $x \in U_{rs}$ for some indices $r, s \in \mathbb{N}$, a contradiction to x being a cluster point (since the sets $(U_{ij})_{i,j\in\mathbb{N}}$ are all disjoint neighborhoods of the points $x_{ij} \in X$). Therefore we obtain $f \notin C(v_{\varphi(i)})_0(X)$, and since $i \in \mathbb{N}$ was arbitrary and $\varphi(i) \to \infty$ for $i \to \infty$ we also have $f \notin \mathcal{V}_0C(X)$.

Finally, to see $f \in C\bar{V}_0(X)$, fix a weight $\bar{v} \in \bar{V}$ and let $\varepsilon > 0$. We are done if we can show that the following set is compact,

$$F := \left\{ x \in X \,\middle|\, \bar{v}(x)\, f(x) \geqslant \varepsilon \right\}.$$

Since f vanishes on the complement of $\cup_{i=1}^{\infty} \cup_{j=1}^{\infty} U_{ij}$ we again obtain that every $x \in F$ has to be contained in some U_{ij}. Furthermore, by definition of $\bar{V}$ we can find positive constants $(\alpha_\ell)_{\ell \in \mathbb{N}}$ such that $\bar{v} \leqslant \inf_{\ell \in \mathbb{N}} \alpha_\ell\, v_\ell$.

Now, if $x \in U_{ij}$ where $\frac{\eta_i}{2^i} < \frac{\varepsilon}{\alpha_n}$, then x cannot be an element of F since

$$\bar{v}(x)\, f(x) \leqslant \alpha_n\, v_n(x)\, f(x) = \alpha_n\, h(x) = \alpha_n\, h_i(x) = \alpha_n \frac{\eta_i}{2^i} g_i(x) \leqslant \alpha_n \frac{\eta_i}{2^i} < \varepsilon.$$

The same is true for $x \in U_{ij}$ where $\delta_{ij} < \frac{2^i\, \varepsilon}{\eta_i\, \alpha_{k_i}}$ because

$$\bar{v}(x)\, f(x) \leqslant \alpha_{k_i}\, v_{k_i}(x)\, f(x) = \alpha_{k_i} \frac{v_{k_i}(x)}{v_n(x)} h(x) \leqslant \alpha_{k_i}\, \delta_{ij}\, h(x) \leqslant \alpha_{k_i}\, \delta_{ij} \frac{\eta_i}{2^i} < \varepsilon.$$

But both $\frac{\eta_i}{2^i} \to 0$ for $i \to \infty$ and $\delta_{ij} \to 0$ for $j \to \infty$, such that F is actually already contained in a finite union of the sets $(U_{ij})_{i,j \in \mathbb{N}}$ and hence must be compact as claimed. $\square$

5

Conclusion

The topology of an inductive limit of locally convex vector spaces is generally difficult to describe in a way that is useful for calculations in practice. The goal of this thesis was to find simpler descriptions of such topologies. More precisely, we wanted to find a useful characterization of the associated family of seminorms. But knowing about the topology isn't always enough, in order to also characterize the elements of the underlying vector space, we were looking for *projective descriptions*—which is identifying an inductive limit of a given family of spaces as a projective limit of "similar" spaces. A common class of inductive limit spaces arrising in practice is the class of *weighted spaces*—and for such spaces one can indeed find projective descriptions.

First we considered weighted sequence spaces (*Köthe co-echelon spaces*, see Definition 3.6←47),

$$k^p(V) := \varinjlim_n \ell^p(v^{(n)}).$$

By a rather straightforward definition of an associated system of weights $\bar{V}$ (*associated Köthe set*, see Definition 3.10←50) one can quickly see that the corresponding *projective hulls* $K_p(\bar{V})$ are already algebraic projective descriptions in case $1 \leqslant p \leqslant \infty$ (see Proposition 3.13←52). Soon after we saw that also the topologies coincide if only $1 \leqslant p < \infty$,

$$k^p(V) \cong K_p(\bar{V}),$$

which was the first full projective description result (see Theorem 3.16(i)←56).

The border cases for $p = 0$ and $p = \infty$ turned out to be more difficult/interesting. In general, the topology of $k^0(V)$ can be described by the seminorms of $K_0(\bar{V})$ although the underlying vector spaces need not coincide. On the other hand, the inductive topology of $k^\infty(V)$ can be strictly stronger than the topology induced by $K_\infty(\bar{V})$ even though the two spaces always coincide algebraically. Therefore we searched for conditions in terms of the family of weights V for obtaining projective description even if $p = 0$ or $p = \infty$. A summary of these conditions can be found in figure 3.1←73. In particular it can be shown that the easily checked Schwartz-type condition (S) suffices to obtain projective description, although with the side-effect that all relevant spaces then have to coincide (see Proposition 3.20←61),

$$k^0(V) \cong k^\infty(V) \cong K_0(\bar{V}) \cong K_\infty(\bar{V}).$$

On the other hand, we discussed a weaker and slightly more involved *regularly decreasing condition* which completely characterises projective description in the case of $p = 0$, i. e., $k^0(V) \cong K_0(\bar{V})$ (see Theorem 3.25) but still implies $k^\infty(V) \cong K_\infty(\bar{V})$. Finally, the even more complicated *density condition* (D) can be shown to characterize projective description for $p = \infty$.

In chapter 4 we presented similar results for *weighted spaces of continuous functions* (see Definition 4.3), the spaces $\mathcal{V}C(X, E)$ and $\mathcal{V}_0C(X, E)$ generalize $k^\infty(V)$ and $k^0(V)$, respectively. Again we were quickly led to a definition of an *associated system of weights* $\bar{V}_\mathcal{V}$ (see Definition 4.10) enabling us to build projective hulls. As in the case of sequence spaces the topological projective description of $\mathcal{V}_0C(X, E)$ can be obtained by identifying it as a subspace of its projective hull $C\bar{V}_0(X, E)$ (see Theorem 4.14). With a bit more work and some restrictions one can also obtain an algebraic projective description of $\mathcal{V}C(X, E)$—see Theorem 4.18.

One difference to the sequence space case is, that this time the building blocks of the inductive limits aren't necesserily complete, and therefore also the completeness of $\mathcal{V}C(X)$ had to be specifically dealt with in Theorem 4.20 (the corresponding co-echelon spaces $k^\infty(V)$ of chapter 3 had turned out to be always complete, a fact which was actually used in the proof of the present theorem).

On the other hand, condition (S) again implied projective description as before (see Proposition 4.23),

$$\mathcal{V}_0C(X, E) \cong C\bar{V}_0(X, E) \cong \mathcal{V}C(X, E) \cong C\bar{V}(X, E).$$

Additionally, by applying Bernsteins open mapping lemma to this result, the important Theorem 4.24 can be obtained, which makes it possible to also derive projective description results for weighted spaces consisting only of functions of a semi-Montel subspace of the space of continuous functions.

Finally, we saw that also the regularly decreasing condition can be generalized to the setting of continuous function spaces to obtain a characterization of the projective description of $\mathcal{V}_0C(X, E)$, see Theorem 4.25 and Theorem 4.27.

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Danksagung

An dieser Stelle möchte ich mich bei allen bedanken die mich beim Erstellen dieser Arbeit unterstützt haben. Meinem Betreuer, Prof. Dr. Andreas Kriegl, danke ich für die vielen Tipps, Hinweise und Bemerkungen, aber auch für das Vertrauen und besonders die Geduld. Außerdem war er derjenige, der mich erst auf dieses interessante Thema gebracht hat.

Bei meine Eltern, Dr. Peter M. Schuster und Dr. Lily Wilmes, bedanke ich mich für die „naturwissenschaftlichen“ Gene, die sie mir vererbt haben, und für die finanzielle und seelische Unterstützung, ohne der mein Studium unmöglich gewesen wäre.

Aber besonders möchte ich mich bei meiner Frau Mag. Heidemarie Schuster bedanken. Sie hat mir jedesmal, wenn ich auf ein scheinbar unlösbares Problem gestoßen bin, geholfen die richtige Lösung zu finden – wenn nicht inhaltlich, dann als geduldige und aufmerksame Zuhörerin mit vielen kreativen Ideen. Außerdem hat sie mir immer dann, wenn meine vielen kleinen Projekte wieder einmal drohten mich von der Diplomarbeit abzuhalten, Arbeit abgenommen, sodass ich mich wieder auf die wichtigen Dinge konzentrieren konnte.

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