DIPLOMARBEIT

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String-localized Quantum Fields for Fermions

Verfasser

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Previous results by Mund, Schroer, and Yngvason [5] regarding string-localized quantum fields are extended to include fermions by considering spinor representations of SU(2).

Die Ergebnisse einer jüngeren Arbeit von Mund, Schroer und Yngvason [5] betreffend string-lokalisierter Quantenfelder werden um Fermionen ergänzt, indem Spinor-Darstellungen der SU(2) betrachtet werden.

Contents

1 Introduction

In their paper [5], Mund, Schroer, and Yngvason (MSY) use findings from research concerning what has become known as *modular localization* [3] to construct quantum fields $\Phi(x, e)$ which are distinguished from "conventional", i. e. point-like quantum fields (as rigorously defined by the Wightman axioms [9]), in that they not only have a point-like coordinate x (in Minkowski space) as a parameter, but also a space-like direction e. Due to their localization which can mathematically be described by a ray, these fields are consequently named *string fields*.

After reviewing some general properties and aspects of these string fields, MSY then proceed to classify free string fields transforming covariantly with respect to irreducible representations of the Poincaré group [11]. The representations treated in [5] include, among others, the massive bosonic, the massless vectorial (photonic), and the massless infinite spin cases, the latter of which is inaccessible from the usual point-field account. Moreover, string fields have some other useful properties: for example, their short distance dimension (sdd) can be independent of their spin [5].

The present paper aims to complement the work of MSY [5] by treating the *massive fermionic*, i. e. half-integer spin case. This will require the use of *spinorial* string fields, whereas in the case of bosons, scalar string fields are sufficient. These scalar string fields are constructed by considering functions on a de Sitter space that carry representations belonging to all possible integer angular momenta [5, (50)], and then projecting onto a given (integer) spin representation. In the present paper, the corresponding L^2 representation function space of the little group SO(3) of the massive bosonic case is "doubled", i. e. is tensor multiplied by $\mathbb{C}^2$, in order to allow for representations of the little group SU(2) of the massive fermionic case.

This paper is structured as follows: Chapter 2 outlines the required terminology and introduces elements needed for subsequent chapters. In Chapter 3, we first define and then construct spinorial string fields out of their underlying intertwiners between representations of SU(2) and SL(2, $\mathbb{C}$), analyzing some of their properties along the way, such as anti-locality, uniqueness, and relation to point-localized fields. Chapter 4 shows an alternative way to construct intertwiners for spinorial string fields via representations on function spaces which have to be projected onto the required space. Chapter 5 sums up relevant results.

2 Single and many particle states

We give a brief preparatory overview of some required terminology. Most of the material presented in this section can be found in QFT textbooks such as [10] (including detailed derivations of results) or [1], mathematical physics primers such as [9], as well as in the comprehensive paper [5]. For a particularly easily accessible treatment, see [6, 7]. For a more in-depth account of representations of $\mathrm{SL}(2,\mathbb{C})$, see [8, ch. 8].

2.1 The Lorentz group and its universal cover $\mathrm{SL}(2,\mathbb{C})$

The *Lorentz group* $\mathcal{L}$ is the group of real 4×4 matrices Λ that leave the Minkowski dot product invariant, i. e. for which for arbitrary values of $x, y \in \mathbb{R}^4$,

$$x \cdot y = (\Lambda x) \cdot (\Lambda y),$$

where $x \cdot y := \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu y^\nu$, or equivalently,

$$\Lambda^{\mathrm{T}} \eta \Lambda = \eta,$$

with $\eta = \mathrm{diag}(1, -1, -1, -1)$ the Minkowski metric. It is composed of four disconnected components,

$$\mathcal{L} = \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow \cup \mathcal{L}_-^\uparrow \cup \mathcal{L}_-^\downarrow,$$

where the plus (minus) sign stands for the (*im*)*proper* components with $\det\Lambda = \pm 1$ which preserve (reverse) orientation, and the up (down) arrow stands for the positive (negative) sign of $\Lambda^0{}_0$ and denotes that time direction is preserved (reversed).

In the following, we will mainly be dealing with the *proper orthochronous* (or *restricted*) component $\mathcal{L}_+^\uparrow$ which contains the identity (and is sometimes denoted as $\mathrm{SO}(1,3)$). Let $\Lambda(A) \in \mathcal{L}_+^\uparrow$ be the proper orthochronous Lorentz transformation corresponding to $A \in \mathrm{SL}(2,\mathbb{C})$ [9], given by

$$\underset{\sim}{\Lambda(A)p} = A\underset{\sim}{p}A^*, \tag{2.1.1}$$

where A^* means the (Hermitian) adjoint of A, and the "undertilde" denotes the bijection $x \mapsto \underset{\sim}{x} := \sum_\mu x^\mu \sigma_\mu$ (with σ_0 the identity, and σ_i the Pauli matrices) from $\mathbb{R}^4$ to the space of Hermitian 2×2 matrices, and its inverse is given by $x^\mu = \frac{1}{2}\mathrm{Tr}(\underset{\sim}{x}\sigma_\mu)$; all this amounts to the $2:1$, $\mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$, $\pm A \mapsto \Lambda(A)$ homomorphism [8]

$$\Lambda^\lambda{}_\mu(A) = \frac{1}{2}\mathrm{Tr}(A\sigma_\mu A^* \sigma_\lambda),$$

by which one writes $\mathrm{SL}(2,\mathbb{C}) = \tilde{\mathcal{L}}_+^\uparrow$, meaning that it is the *double* (and, actually, the *universal*) *cover* of $\mathcal{L}_+^\uparrow$.

Note that the above formalism can be restricted to a homomorphism from SU(2) (the group of unitary 2×2 matrices with determinant 1) to the rotation group SO(3), $\pm A \mapsto \mathcal{R}(A)$, with

$$\underset{\sim}{\mathcal{R}(A)p} = A\underset{\sim}{p}A^{-1}, \tag{2.1.2}$$

and a bijection $\mathbb{R}^3 \to \mathrm{i}\mathfrak{su}(2)$, $x \mapsto \underset{\sim}{x} := \sum_i x^i \sigma_i$, by only considering the spatial components x^i and "dropping" the temporal component x^0.

Note the following useful property of $\mathrm{SL}(2,\mathbb{C})$ elements: let

$$\zeta := \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\zeta^{-1}.$$

Then for all $A \in \mathrm{SL}(2,\mathbb{C})$, $A\zeta A^{\mathrm{T}} = \zeta$, or equivalently, $\zeta(A^*)^{-1}\zeta^{-1} = \bar{A}$ (where $\bar{A}$ denotes the complex conjugate of A). If $R \in \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$, then $\zeta R \zeta^{-1} = \bar{R}$, because $RR^* = \mathbf{1}$.

2.2 Irreducible, unitary Poincaré group representations

We follow Wigner's approach [10, 11] of classifying single particle states by considering positive energy representations of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow = \mathbb{R}^4 \rtimes \mathcal{L}_+^\uparrow$ (or again, its universal cover $\tilde{\mathcal{P}}_+^\uparrow = \mathbb{R}^4 \rtimes \tilde{\mathcal{L}}_+^\uparrow$), parametrized by their mass and spin (or helicity, respectively).

For a given mass $m > 0$, define the *mass shell*

$$H_m^+ := \{p \in \mathbb{R}^4 : p \cdot p = m^2, p^0 > 0\},$$

where the dot $(\cdot)$ again denotes the Minkowski scalar product. Now fix a *standard momentum* $\hat{p} \in H_m^+$ such that its stabilizer subgroup (or *little group*) $G_{\hat{p}}$ is

$$G_{\hat{p}} := \{A \in \mathrm{SL}(2,\mathbb{C}) : \Lambda(A)\hat{p} = \hat{p}\}. \tag{2.2.1}$$

For $A_p \in \mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$, the customary choice is to set $\hat{p} = (m, 0, 0, 0)$, which is the momentum of the particle in its rest frame. [10].

A unitary irreducible representation D and corresponding little Hilbert space $\mathfrak{h}$ of a given little group $G_{\hat{p}}$ induce the full representation U and corresponding Hilbert space $\mathcal{H} \ni \psi$,

$$\mathcal{H} := L^2(H_m^+, \mathrm{d}\mu_m) \otimes \mathfrak{h}, \tag{2.2.2}$$

$$(U(a,A)\psi)_k(p) := \mathrm{e}^{ipa} \sum_{k'} D_{kk'}(R(A,p))\psi_{k'}(\Lambda(A)^{-1}p), \tag{2.2.3}$$

where $\mathrm{d}\mu_m(p) = \delta(p^2 - m^2)\Theta(p^0)\,\mathrm{d}^4p$, and k and k' are *spin* indices (or *helicity* indices for mass $m = 0$). The *Wigner rotation* is defined as

$$R(A,p) := A_p^{-1} A A_{\Lambda(A)^{-1}p}, \tag{2.2.4}$$

where $A_p \in \mathrm{SL}(2,\mathbb{C})$ is the boost that takes the resting particle to momentum p, i. e.

$$\Lambda(A_p)\hat{p} = p \tag{2.2.5}$$

for $p \in H_m^+$. The representation U can be extended to a representation of the proper Lorentz group $\mathcal{L}_+$ (and thus to the proper Poincaré group $\mathcal{P}_+$) by writing the reflection

$$j_0 := \mathrm{diag}(-1,1,1,-1) \in \mathcal{L}_+^{\downarrow}$$

along the edge $\{x \in \mathbb{R}^4 | x^3 = x^0 = 0\}$ of the standard wedge

$$W_0 := \{x \in \mathbb{R}^4 | x^3 > |x^0|\}$$

as a product of $\boldsymbol{PT}$ with an element of $\mathcal{L}_+^{\uparrow}$,

$$j_0 = (-\mathbf{1}) \cdot (-j_0) = \boldsymbol{PT} \cdot (-j_0),$$

yielding

$$(U(j_0)\psi)_k(p) = (U(\boldsymbol{PT})U(-j_0)\psi)_k(p).$$

In the massive case ($m > 0$), the representation of $\boldsymbol{PT}$ is given by [10, (2.6.16) and (2.6.18)][1]

$$U(\boldsymbol{PT})\psi_k(p) = \xi\chi(-1)^{j-\sigma}\overline{\psi_{-k}(p)},$$

where χ is the intrinsic parity and ξ is an arbitrary phase factor ($|\chi| = |\xi| = 1$); we will

[1]Note that [10, (2.6.16) and (2.6.18)] are for basis vectors. For linear combinations, antilinearity has to be observed, which results in complex conjugation of $\psi_{-k}(p)$.

assume both to be equal to 1. Thus

$$(U(j_0)\psi)_k(p) = \Big(D(j_0)\overline{\psi(-j_0p)}\Big)_{-k}.$$

2.3 Representations of $\mathrm{SU}(2)$

So far, we have treated the representation D of the little group $G_{\hat{p}}$ rather abstractly. For the case we are interested in, that is, positive mass m and half integer spin j, we have to consider SU(2) as little group $G_{\hat{p}}$, and its $(2j+1)$ dimensional unitary irreducible representations $D = D^{(j)}$, which are given by

$$D^{(j)}(A) = A^{\otimes 2j}. \tag{2.3.1}$$

These representations act on elements $\xi_{\rho_1 \dots \rho_j}$ $(\rho_i = 1, 2)$ of $\mathbb{C}^{2j+1}$, which are understood as the components of the totally symmetrized tensor powers of $\xi_\rho \in \mathbb{C}^2$ on which A acts naturally.

Note that apart from the representation (2.3.1), there is also an equivalent *complex conjugate* representation

$$\bar{D}^{(j)}(A) = \bar{A}^{\otimes 2j},$$

with $\bar{A}$ the complex conjugate of A.

2.4 Representations of $\mathrm{SL}(2,\mathbb{C})$

In the following chapters, we will also need representations of the double cover of the Lorentz group, i. e., of $\mathrm{SL}(2,\mathbb{C})$. For non-negative integers $2j$ and $2j'$, the irreducible representation $D^{(j,j')}(A)$ is defined [7] as $A^{\otimes_s 2j} \otimes \bar{A}^{\otimes_s 2j'}$ on $(\mathbb{C}^2)^{\otimes_s 2j} \otimes (\mathbb{C}^2)^{\otimes_s 2j'}$, and a spinor $\xi_{\alpha_1,\dots,\alpha_{2j},\dot{\beta}_1,\dots,\dot{\beta}_{2j'}}$ transforms according to

$$\xi_{\alpha_1,\dots,\alpha_{2j},\dot{\beta}_1,\dots,\dot{\beta}_{2j'}} \mapsto \sum_{(\rho)(\dot{\sigma})} A_{\alpha_1\rho_1} \dots A_{\alpha_{2j}\rho_{2j}} \bar{A}_{\dot{\beta}_1\dot{\sigma}_1} \dots \bar{A}_{\dot{\beta}_{2j'}\dot{\sigma}_{2j'}} \xi_{\rho_1,\dots,\rho_{2j},\dot{\sigma}_1,\dots,\dot{\sigma}_{2j'}}. \tag{2.4.1}$$

Note that the present paper only deals with string fields that correspond to the spin-$1/2$ representation of $\mathrm{SL}(2,\mathbb{C})$, $D^{(\frac{1}{2},0)}(A) = A$, and only in Section 3.6 relates them to point fields that correspond to a different $\mathrm{SL}(2,\mathbb{C})$ representation D'.

2.5 Second quantization

Given the single particle Hilbert space $\mathcal{H}$ (2.2.2) and corresponding representation U (2.2.3), we can use the symmetrized (or antisymmetrized) tensor product $\otimes_s$ and corresponding (anti-) symmetrized tensor power to define the bosonic (or fermionic) Fock space

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes_s n},$$

where $\mathcal{H}^{\otimes_s 0} := \mathbb{C}$, and the corresponding *creation* and *annihilation operators* acting on $\phi_1 \otimes_s \ldots \otimes_s \phi_n \in \mathcal{H}^{\otimes_s n}$,

$$\begin{aligned} a^*(\psi)(\phi_1 \otimes_s \ldots \otimes_s \phi_n) &:= \sqrt{n+1}(\psi \otimes_s \phi_1 \otimes_s \ldots \otimes_s \phi_n), \\ a(\psi)(\phi_1 \otimes_s \ldots \otimes_s \phi_n) &:= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\pm 1)^{k+1} \langle \psi, \phi_k \rangle (\phi_1 \otimes_s \ldots \otimes_s \hat{\phi_k} \otimes_s \ldots \otimes_s \phi_n), \end{aligned} \tag{2.5.1}$$

(where the hat over $\hat{\phi_k}$ means that ϕ_k is omitted in the tensor product on the right hand side of the second equation,) with $\psi \in \mathcal{H}$ and $a(\psi)\Omega = 0$, where Ω is the Fock vacuum. Note that the factor $(-1)^{k+1}$ is required in the fermionic case in order to ensure the correct transformation behavior under particle exchange. For simplicity's sake, we limit ourselves to the Majorana case $a_c^{(*)} = a^{(*)}$ (with particles identical to their anti-particles) instead of doubling the Fock space.

Direct computation shows that the creation and annihilation operators fulfill the *canonical (anti)commutation relations*

$$\begin{aligned} [a(\psi), a(\phi)]_\mp = [a^*(\psi), a^*(\phi)]_\mp &= 0, \\ [a(\psi), a^*(\phi)]_\mp &= \langle \psi, \phi \rangle \, \mathrm{id}_{\mathcal{F}(\mathcal{H})}. \end{aligned}$$

For the definition of the fields, we need operator valued distributions $a_k^*(p)$ and $a_k(p)$, which are symbolically defined via

$$\begin{aligned} a^*(\psi) &=: \sum_k \int a_k^*(p) \psi_k(p) \, \mathrm{d}\mu_m(p), \\ a(\psi) &=: \sum_k \int a_k(p) \bar{\psi}_k(p) \, \mathrm{d}\mu_m(p), \end{aligned} \tag{2.5.2}$$

yielding

$$\begin{aligned} [a_k(p), a_{k'}(p')]_\mp = [a_k^*(p), a_{k'}^*(p')]_\mp &= 0, \\ [a_k(p), a_{k'}^*(p')]_\mp &= \delta(p - p')\delta_{kk'}. \end{aligned}$$

Furthermore, we extend the single particle space representation U (2.2.3) naturally to the

Fock space by

$$U(a,A)(\phi_1 \otimes_s \ldots \otimes_s \phi_n) := (U(a,A)\phi_1 \otimes_s \ldots \otimes_s U(a,A)\phi_n) \tag{2.5.3}$$

and combine it with (2.5.1) to obtain the transformation behavior of the creation and annihilation operators,

$$U(a,A)a^{(*)}(\psi)U(a,A)^{-1} = a^{(*)}(U(a,A)\psi),$$

into which we insert (2.5.2) and (2.2.3), and arrive at the following transformation behavior of the creation and annihilation distributions:

$$\begin{aligned} U(a,A)a_k^*(p)U(a,A)^{-1} &= \mathrm{e}^{\mathrm{i}(\Lambda(\mathrm{A})p)a} \sum_{k'} a_{k'}^*(\Lambda(A)p) D_{k'k}^{(j)}(R(A,\Lambda(A)p)), \\ U(a,A)a_k(p)U(a,A)^{-1} &= \mathrm{e}^{-\mathrm{i}(\Lambda(A)p)a} \sum_{k'} a_{k'}(\Lambda(A)p) \overline{D_{k'k}^{(j)}(R(A,\Lambda(A)p))}. \end{aligned} \tag{2.5.4}$$

3 String Fields

Now, instead of introducing a conventional point-localized quantum field $\Phi_\rho(x)$, we proceed by defining a *string-localized quantum field* $\Phi_\rho(x, e)$.

3.1 Definition

In order to generalize work from [5] to account for half-integer spin, we apply the notion of string localization to a spinor field (instead of a scalar one as found in [5]). The "string" property of the field denotes what is geometrically a ray $S_{x,e} = x + \mathbb{R}^+ e$, which is defined by a point in Minkowski space, $x \in \mathbb{R}^4$, and a direction $e \in H^3$, with

$$H^3 := \{e \in \mathbb{R}^4 : e \cdot e = -1\}$$

a submanifold of space-like directions in Minkowski space, and H_c^3 its complexified counterpart. These strings are originally motivated as the cores of space-like cone-shaped localization regions in the context of *modular localization* [3].

Definition 1. A *free fermionic string-localized quantum field* in four-dimensional Minkowski space is an operator-valued distribution $\Phi_\rho(x, e)$ over $\mathbb{R}^4 \times H^3$ acting on $\mathcal{F}(\mathcal{H})$, satisfying the following properties:

1. *String-antilocality*: If the strings $x_1 + \mathbb{R}^+ e_1'$ and $x_2 + \mathbb{R}^+ e_2$ are space-like separated for all e_1' in an open neighborhood of e_1, then the field $\Phi_\rho(x_1, e_1)$ *anti-commutes* with both $\Phi_\sigma(x_2, e_2)$ and $\Phi_\sigma^*(x_2, e_2)$, i. e.

$$[\Phi_\rho(x_1, e_1), \Phi_\sigma(x_2, e_2)]_+ = [\Phi_\rho(x_1, e_1), \Phi_\sigma^*(x_2, e_2)]_+ = 0. \tag{3.1.1}$$

2. *Covariance*: The field transforms covariantly under a unitary representation U of the double cover of the Poincaré group $\tilde{\mathcal{P}}_+^\uparrow \ni (a, A)$ according to

$$U(a, A)\Phi_\rho(x, e)U(a, A)^{-1} = \sum_\sigma D_{\rho\sigma}^{(j,k)}(A^{-1})\Phi_\sigma(\Lambda(A)x + a, \Lambda(A)e), \tag{3.1.2}$$

while the conjugate field Φ_ρ^* transforms with $D^{(k,j)}$. Note that compared to [5], the scalar field is replaced by a spinorial one, and the transformation behavior has to be amended by the representation matrix $D^{(j,k)}_{\rho\sigma}$ of $\mathrm{SL}(2,\mathbb{C})$ (which in the case of a scalar field is trivially represented, i. e., by $\mathbf{1}$).

3. *Positivity of energy*: When restricting the representation U to the translation subgroup, the spectrum of its generators lies in the forward light cone $\overline{V}_+$.

4. *Free fields*: The field creates only single particle states when acting on the Fock vacuum $\Omega \in \mathcal{F}(\mathcal{H})$, i. e.
$$\Phi_\rho(f,g)\Omega \in \mathcal{H}, \tag{3.1.3}$$
where the field parameters x and e have been smeared with test functions $f : \mathbb{R}^4 \to \mathbb{R}$ and $g : H^3 \to \mathbb{R}$, respectively.

We will also require that the distributions $e \mapsto \Phi_\rho(f,e)\Omega$ and $e \mapsto \Phi^*_\rho(f,e)\Omega$ (where x has been smeared with a test function f) have an analytic continuation for $e \in \mathcal{T}_+ \cap H^3_c$, where $\mathcal{T}_+ := \mathbb{R}^4 + \mathrm{i}V_+$.

3.2 Construction

Note that the transformation behavior of the creation and annihilation distributions (2.5.4) is non-local because of the p-dependency of the Wigner rotation (2.2.4). Therefore, we need to construct fields which under Lorentz transformations are multiplied with position-independent matrices instead, see (3.1.2). To this end, we construct $\Phi_\rho(x,e)$ (essentially) as the Fourier transformed creation and annihilation distributions (2.5.2), multiplied with coefficients $u(p,e)$ and $\overline{u^c(p,e)}$ that ensure the desired transformation behavior (3.1.2),

$$\Phi_\rho(x,e) = \int_{H_m^+} \mu(p) \sum_{m=-j}^{+j} \left\{ \mathrm{e}^{\mathrm{i}px} u_{\rho m}(p,e) a^*_m(p) + \mathrm{e}^{-\mathrm{i}px} \overline{u^c_{\rho m}(p,e)} a_m(p) \right\}.$$

The criterion for the coefficients u and $\overline{u^c}$ is to intertwine this *covariant* with the *Wigner basis*. Their relation is expressed by the *intertwiner equations*

$$\begin{aligned} \sum_{m'} D_{mm'}(R(A,p)) u_{\rho m'}(\Lambda(A)^{-1}p,e) &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A^{-1}) u_{\sigma m}(p,\Lambda(A)e), \\ \sum_{m'} \bar{D}_{mm'}(R(A,p)) \overline{u^c_{\rho m'}(\Lambda(A)^{-1}p,e)} &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A^{-1}) \overline{u^c_{\sigma m}(p,\Lambda(A)e)}. \end{aligned} \tag{3.2.1}$$

Note that as D is a unitary representation, $D_{mm'}(R) = D^*_{mm'}(R^{-1}) = \bar{D}_{m'm}(R^{-1})$, which allows us to rewrite (3.2.1) in the somewhat more appealing form

$$\begin{aligned} u(\Lambda(A)^{-1}p, e)\bar{D}(R(A,p)^{-1}) &= D^{(j,k)}(A^{-1})u(p, \Lambda(A)e), \\ \overline{u^c(\Lambda(A)^{-1}p, e)}D(R(A,p)^{-1}) &= D^{(j,k)}(A^{-1})\overline{u^c(p, \Lambda(A)e)}. \end{aligned}$$

3.3 Anti-locality

We employ the results from the previous section to show that for an appropriate definition of the conjugate intertwiners $\overline{u^c}$, the two-point (or *two-string*) functions $\mathcal{W}_{\rho\rho'}$ and $\mathcal{V}^{(c)}_{\rho\rho'}$ really fulfill the desired relation, namely anti-locality (or *anti-commutativity*). We largely follow the approach layed out in [4, 7]. We start by defining the two-string functions $\mathcal{W}_{\rho\rho'}$ and $\mathcal{V}^{(c)}_{\rho\rho'}$:

$$\begin{aligned} \mathcal{W}_{\rho\rho'}(x-x', e, e') = \left\langle \Omega, \Phi_\rho(x,e)\Phi_{\rho'}(x',e')\Omega \right\rangle &= \int \mathrm{d}\mu(p)\, \mathrm{e}^{-\mathrm{i}p(x-x')} M_{\rho\rho'}(p, e, e'), \\ M_{\rho\rho'}(p,e,e') = \overline{u^c_\rho(p,e)} \circ u_{\rho'}(p,e') &= \sum_m \overline{u^c{}_{\rho m}(p,e)} u_{\rho' m}(p,e'), \end{aligned} \tag{3.3.1}$$

$$\mathcal{V}^c_{\rho\rho'}(x-x', e, e') = \left\langle \Omega, \Phi_\rho(x,e)\Phi^*_{\rho'}(x',e')\Omega \right\rangle, \mathcal{V}_{\rho\rho'}(x-x', e, e') = \left\langle \Omega, \Phi^*_\rho(x,e)\Phi_{\rho'}(x',e')\Omega \right\rangle,$$

$$\mathcal{V}^{(c)}_{\rho\rho'}(x-x', e, e') = \int \mathrm{d}\mu(p)\, \mathrm{e}^{-\mathrm{i}p(x-x')} N^{(c)}_{\rho\rho'}(p,e,e'), \tag{3.3.2}$$

$$N^{(c)}_{\rho\rho'}(p,e,e') = \overline{u^{(c)}_\rho(p,e)} \circ u^{(c)}_\rho(p,e') = \sum_m \overline{u^{(c)}_{\rho m}(p,e)} u^{(c)}_{\rho' m}(p,e').$$

The circle $\circ$ denotes the inner product in the space of intertwiners. We can now follow the (slightly modified) argumentation from [4, after (13)] as follows: by the analyticity requirement stated at the bottom of Section 3.1, $\mathcal{W}_{\rho\rho'}(x-x', e, e')$ and $\mathcal{V}^c_{\rho\rho'}(x-x', e, e')$ have an analytic continuation into the domain

$$x - x' \in \mathcal{T}_- := \mathbb{R}^4 - \mathrm{i}V_+, \qquad e \in \mathcal{T}_- \cap H^3, \qquad e' \in \mathcal{T}_+ \cap H^3,$$

while $\mathcal{W}_{\rho'\rho}(x'-x, e', e)$ and $\mathcal{V}_{\rho'\rho}(x'-x, e', e)$ have an analytic continuation into that domain with $\mathcal{T}_+$ and $\mathcal{T}_-$ interchanged. The covariance properties then result from (3.2.1),

$$
\begin{aligned}
M_{\rho\rho'}(p,\Lambda(A)e,\Lambda(A)e') &= \sum_{\sigma,\sigma'} D^{(j,k)}_{\rho\sigma}(A) D^{(j,k)}_{\rho'\sigma'}(A) M_{\sigma\sigma'}(\Lambda(A)^{-1}p,e,e'),\\
N^c_{\rho\rho'}(p,\Lambda(A)e,\Lambda(A)e') &= \sum_{\sigma,\sigma'} D^{(j,k)}_{\rho\sigma}(A) D^{(k,j)}_{\rho'\sigma'}(A) N^c_{\rho\rho'}(\Lambda(A)^{-1}p,e,e'),\\
N_{\rho\rho'}(p,\Lambda(A)e,\Lambda(A)e') &= \sum_{\sigma,\sigma'} D^{(k,j)}_{\rho\sigma}(A) D^{(j,k)}_{\rho'\sigma'}(A) N_{\sigma\sigma'}(\Lambda(A)^{-1}p,e,e').
\end{aligned} \tag{3.3.3}
$$

In order for space-like separated strings to anti-commute (3.1.1), we will show that the following TCP relations are sufficient,

$$
\begin{aligned}
M_{\rho\rho'}(p,-e,-e') &= -M_{\rho'\rho}(p,e',e),\\
N^c_{\rho\rho'}(p,-e,-e') &= N_{\rho'\rho}(p,e',e),
\end{aligned} \tag{3.3.4}
$$

by which criteria we define

$$
\overline{u^c_{\rho m}(p,e)} := \sum_{m'} D_{mm'}(\zeta) u_{\rho m'}(p,-e). \tag{3.3.5}
$$

This is compatible with the second line of (3.2.1) because of the properties of ζ noted at the bottom of Section 2.1.

The covariance properties (3.3.3), together with the TCP relations (3.3.4), then yield

$$
\begin{aligned}
&\mathcal{W}_{\rho\rho'}(x-j_0\Lambda(t)x',e,j_0\Lambda(t)e')\\
&\qquad = -\sum_{\sigma,\sigma'} D^{(j,k)}_{\rho\sigma}(\mathrm{i}\sigma_3 A(t)) D^{(j,k)}_{\rho'\sigma'}(\mathrm{i}\sigma_3 A(t)) \mathcal{W}_{\sigma'\sigma}(x'-j_0\Lambda(-t)x,e',j_0\Lambda(-t)e),
\end{aligned} \tag{3.3.6}
$$

$$
\begin{aligned}
&\mathcal{V}^c_{\rho\rho'}(x-j_0\Lambda(t)x',e,j_0\Lambda(t)e')\\
&\qquad = \sum_{\sigma,\sigma'} D^{(j,k)}_{\rho\sigma}(\mathrm{i}\sigma_3 A(t)) D^{(k,j)}_{\rho'\sigma'}(\mathrm{i}\sigma_3 A(t)) \mathcal{V}_{\sigma'\sigma}(x'-j_0\Lambda(-t)x,e',j_0\Lambda(-t)e).
\end{aligned} \tag{3.3.7}
$$

Note that we have used $-j_0 = -j_0^{-1} = \Lambda(\mathrm{i}\sigma_3)$ and $\Lambda(t) = \Lambda(-t)^{-1}$, and the fact that j_0 and $\Lambda(t)$ commute. The matrix-valued function $\Lambda(t) := \Lambda(A(t))$, $A(t) = \mathrm{diag}(\mathrm{e}^{t/2}, \mathrm{e}^{-t/2})$ is the one-parameter group of Lorentz boosts that leave the wedge W invariant; it is entire analytic in the boost parameter t.

If two strings, $x+\mathbb{R}^+e$ and $x'+\mathbb{R}^+e'$, are space-like separated, then there is a wedge W with causal complement $W' = j_0 W$ such that $x+\mathbb{R}^+e \in W$ and $x'+\mathbb{R}^+e' \in W'$ [5, Appendix A]. Due to the translational invariance of the two-string functions (3.3.1) and (3.3.2), the origin can be assumed to be contained in W, such that $x, e \in W$ and

$x', e' \in W'$. For any $\tau \in \mathbb{R}+\mathrm{i}(0,\pi)$, the imaginary parts of $j_0\Lambda(-\tau)x$, $j_0\Lambda(-\tau)e$, $j_0\Lambda(\tau)x'$, and $j_0\Lambda(\tau)e'$ all lie in V_+. Due to the analyticity properties of the two-string functions mentioned above, we can analytically continue (3.3.6) and (3.3.7) to $t \mapsto t+\mathrm{i}\pi$. We now employ the facts that $j_0\Lambda(\pm\mathrm{i}\pi) = \mathbf{1}$ and $A(\mathrm{i}\pi) = \mathrm{i}\sigma_3$, and that in the spin-1/2 case ($j=\frac{1}{2}$, $k=0$), $D^{(j,k)}(-\mathbf{1}) = -\mathbf{1}$, and $D^{(k,j)}(-\mathbf{1}) = \mathbf{1}$, and thus

$$\begin{aligned}\mathcal{W}_{\rho\rho'}(x-x',e,e') &= -\mathcal{W}_{\rho'\rho}(x'-x,e',e),\\ \mathcal{V}^c_{\rho\rho'}(x-x',e,e') &= -\mathcal{V}_{\rho'\rho}(x'-x,e',e).\end{aligned}$$

Note that because of the fermionic nature of the fields and corresponding anti-locality, $\mathcal{W}_{\rho\rho'}(x-x',e,e')$ vanishes. Also note that the converse of the above reasoning holds as well: anti-commuting fields imply the TCP relations (3.3.4).

3.4 The intertwiner equation

In order to solve the intertwiner equation, we first consider (3.2.1) for standard momentum $\hat{p}$. To this end, we insert $A = A_p$ into (2.2.4) which yields $R(A_p, p) = \mathbf{1}$. Observing (2.2.5), we can then easily evaluate (3.2.1) for $A = A_p$,

$$\begin{aligned}u_{\rho m}(p,e) &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A_p)u_{\sigma m}(\hat{p},\Lambda(A_p)^{-1}e),\\ \overline{u^c_{\rho m}(p,e)} &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A_p)\overline{u^c_{\sigma m}(\hat{p},\Lambda(A_p)^{-1}e)}.\end{aligned} \tag{3.4.1}$$

This means that $u^{(c)}(p,e)$ is fixed by $u^{(c)}(\hat{p},e) =: u^{(c)}(\underset{\sim}{e})$. Now, setting $p=\hat{p}$ in equations (3.2.1), and restricting them to $A \in G_{\hat{p}}$ (2.2.1) yields $\Lambda(A)^{-1}\hat{p} = \hat{p}$ and $R(A,\hat{p}) = A$, by which we obtain the intertwining equations for $u(\underset{\sim}{e})$ and $\overline{u^c(\underset{\sim}{e})}$:

$$\begin{aligned}\sum_{m'} D_{mm'}(A)u_{\rho m'}(\underset{\sim}{e}) &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A^{-1})u_{\sigma m}(A\underset{\sim}{e}A^*),\\ \sum_{m'} \bar{D}_{mm'}(A)\overline{u^c_{\rho m'}(\underset{\sim}{e})} &= \sum_\sigma D^{(j,k)}_{\rho\sigma}(A^{-1})\overline{u^c_{\sigma m}(A\underset{\sim}{e}A^*)}.\end{aligned} \tag{3.4.2}$$

So once we have found solutions to these (simpler) intertwiner equations for standard momentum (3.4.2), we can construct solutions for general momentum (3.2.1) by using relations (3.4.1).

3.5 Uniqueness

We now want to show that the intertwiners $u(\underset{\sim}{e})$ (and thus, also $u^c(\underset{\sim}{e})$) determined by the above relations (3.4.1) and (3.4.2) are unique, up to multiplication by functions $f(\hat{p} \cdot e)$. First note that any element $e \in H^3$ with $e \neq (\mathrm{i}, 0, 0, 0)$ can be written as

$$e = \begin{pmatrix} \sinh t \\ \cos\varphi \sin\theta \cosh t \\ \sin\varphi \sin\theta \cosh t \\ \cos\theta \cosh t \end{pmatrix},$$

and thus as

$$e = \Lambda(A_{\theta,\varphi})^{-1} e_t, \tag{3.5.1}$$

where

$$e_t = (\sinh t, 0, 0, \cosh t)^{\mathrm{T}}, \qquad \underset{\sim}{e_t} = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & -\mathrm{e}^{-t} \end{pmatrix}, \tag{3.5.2}$$

and

$$A_{\theta,\varphi} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \exp(\mathrm{i}\varphi/2) & 0 \\ 0 & \exp(-\mathrm{i}\varphi/2) \end{pmatrix}.$$

From (3.4.2) and (3.5.1), we obtain

$$\begin{aligned} \sum_{m'} D_{mm'}(A_{\theta,\varphi}) u_{\rho m'}(\underset{\sim}{e}) &= \sum_{\sigma} D^{(l,l')}_{\rho\sigma}(A^{-1}_{\theta,\varphi}) u_{\sigma m}(\underset{\sim}{e_t}), \\ \sum_{m'} \bar{D}_{mm'}(A_{\theta,\varphi}) \overline{u^c_{\rho m'}(\underset{\sim}{e})} &= \sum_{\sigma} D^{(l,l')}_{\rho\sigma}(A^{-1}_{\theta,\varphi}) \overline{u^c_{\sigma m}(\underset{\sim}{e_t})}, \end{aligned} \tag{3.5.3}$$

which means that the intertwiners for e are determined by the ones for the string direction e_t. We will prove their uniqueness by constructing an eigenbasis for those latter intertwiners $u^{(c)}_{\rho m}(\underset{\sim}{e_t})$.

To this end, note that e_t is invariant under rotations around the x^3-axis, so by (3.4.2), $u^{(c)}_{\rho m}(\underset{\sim}{e_t})$ must fulfill

$$\begin{aligned} \sum_{m'} D_{mm'}(A_{0,\varphi}) u_{\rho m'}(\underset{\sim}{e_t}) &= \sum_{\sigma} D^{(l,l')}_{\rho\sigma}(A^{-1}_{0,\varphi}) u_{\sigma m}(\underset{\sim}{e_t}), \\ \sum_{m'} \bar{D}_{mm'}(A_{0,\varphi}) \overline{u^c_{\rho m'}(\underset{\sim}{e_t})} &= \sum_{\sigma} D^{(l,l')}_{\rho\sigma}(A^{-1}_{0,\varphi}) \overline{u^c_{\sigma m}(\underset{\sim}{e_t})}. \end{aligned} \tag{3.5.4}$$

Note that in contrast to (3.4.2), we denote the $\mathrm{SL}(2,\mathbb{C})$ representation on the right-hand side by $D^{(l,l')}$ instead of $D^{(j,k)}$, as we need the parameter j to label the $\mathrm{SU}(2)$ representation $D = D^{(j)}$ that acts on $\mathbb{C}^{2j+1} \simeq \mathbb{C}^{\otimes_s 2j}$. The representation $D(A_{0,\varphi})$ can be realized as $A_{0,\varphi}^{\otimes 2j}$, thus acting on tensor products of $\mathbb{C}^2$. From here on, we will consider only the spin-1/2 representation of $\mathrm{SL}(2,\mathbb{C})$, i. e., we will set $D^{(l,l')}(A) = D^{(\frac{1}{2},0)}(A) = A$. An eigenbasis of (3.5.4) is then given by

$$u_\pm \otimes u_+^{\otimes k} \otimes_s u_-^{\otimes(2j-k)},$$

where $u_+ = \binom{1}{0}$ and $u_- = \binom{0}{1}$, and $0 \le k \le 2j$. Inserting into (3.5.4), this yields the following criterion for k:

$$\begin{aligned} u(\underset{\sim}{e_t}): &\quad k = j \mp \frac{1}{2}, \\ \overline{u^c(\underset{\sim}{e_t})}: &\quad k = j \pm \frac{1}{2}. \end{aligned}$$

Thus, the space of the intertwiners is two-dimensional, and its basis is given by $\hat{u}_+(\underset{\sim}{e_t})$ and $\hat{u}_-(\underset{\sim}{e_t})$, which are linearly independent because u_+ and u_- are (and an analog statement holds for the conjugated intertwiners),

$$\begin{aligned} \hat{u}_\pm(\underset{\sim}{e_t}) &= u_\pm \otimes u_+^{\otimes(j\mp\frac{1}{2})} \otimes_s u_-^{\otimes(j\pm\frac{1}{2})}, \\ \overline{\hat{u}^c_\pm(\underset{\sim}{e_t})} &= u_\pm \otimes u_+^{\otimes(j\pm\frac{1}{2})} \otimes_s u_-^{\otimes(j\mp\frac{1}{2})}. \end{aligned} \tag{3.5.5}$$

Now the intertwiner $u^{(c)}(\underset{\sim}{e_t})$ can be written as a linear combination of those basis elements $\hat{u}_\pm^{(c)}(\underset{\sim}{e_t})$, with coefficients $f_\pm^{(c)}(\hat{p}\cdot e)$, where $\hat{p}\cdot e_t = \hat{p}\cdot e = \sinh t$. In order for the intertwiners to expose the required analyticity and distributional properties, the $f_\pm^{(c)}$ have to be analytic in the upper half plane and their growth has to be at most polynomial at infinity, and an inverse power of the imaginary part when approaching the real axis. More generally,

$$u^{(c)}(p,e) = \sum_\pm f_\pm^{(c)}(p\cdot e)\hat{u}_\pm^{(c)}(p,e),$$

where the $\hat{u}_\pm(p,e)$ are determined by rewriting (3.5.3) as

$$\begin{aligned} u_{\rho m}(\underset{\sim}{e}) &= \sum_{\sigma,m'} D_{\rho\sigma}^{(l,l')}(A_{\theta,\varphi}^{-1}) u_{\sigma m'}(\underset{\sim}{e_t}) \bar{D}_{m'm}(A_{\theta,\varphi}), \\ \overline{u^c_{\rho m}(\underset{\sim}{e})} &= \sum_{\sigma,m'} D_{\rho\sigma}^{(l,l')}(A_{\theta,\varphi}^{-1}) \overline{u^c_{\sigma m'}(\underset{\sim}{e_t})} D_{m'm}(A_{\theta,\varphi}), \end{aligned}$$

and applying them to $\hat{u}_\pm(\underset{\sim}{e_t})$ and $\overline{\hat{u}^c_\pm(\underset{\sim}{e_t})}$ as given by (3.5.5), and then using (3.4.1) on the results.

In the next section, we shall discuss another way to construct intertwiners and relate the corresponding fields to integrals over point fields. We will construct string fields in yet another way in in Chapter 4, namely by projecting out a definite spin from $\mathbb{C}^2$-valued functions on the 2-sphere by means of spinor spherical harmonics.

3.6 Construction of intertwiners and their relation to point fields

We will now use the results from Section 3.5 to show that the string fields constructed so far can be written as a line integral over a point field [5, sec. 4.2]

$$\Psi_\sigma(x) = (2\pi)^{-\frac{3}{2}} \int \mathrm{d}\mu(p) \sum_k \left\{ \mathrm{e}^{\mathrm{i}px} v_{\sigma k}(p) a_k^*(p) + \mathrm{e}^{-\mathrm{i}px} \overline{v_{\sigma k}^c(p)} a_k(p) \right\} \tag{3.6.1}$$

that transforms covariantly as

$$U(a, A)\Psi_\sigma(x)U(a, A)^* = \sum_{\sigma'} D'_{\sigma\sigma'}(A^{-1})\Psi_{\sigma'}(\Lambda(A)x) + a),$$

with a certain $\mathrm{SL}(2, \mathbb{C})$ representation D' which will be defined below.

Together with (2.5.4), this yields the following criteria for the point-like intertwiners $v_{\sigma m}(p)$ and $\overline{v_{\sigma m}^c(p)} := \sum_{m'} D_{mm'}(\zeta) v_{\sigma m'}(p)$:

$$\begin{aligned} \sum_{m'} D_{mm'}(R(A, p)) v_{\sigma m'}(\Lambda(A)^{-1}p) &= \sum_{\sigma'} D'_{\sigma\sigma'}(A^{-1}) v_{\sigma' m}(p), \\ \sum_{m'} \bar{D}_{mm'}(R(A, p)) \overline{v_{\sigma m'}^c(\Lambda(A)^{-1}p)} &= \sum_{\sigma'} D'_{\sigma\sigma'}(A^{-1}) \overline{v_{\sigma' m}^c(p)}, \end{aligned} \tag{3.6.2}$$

by which, in analogy to (3.4.1), we can express $v(p) = D'(A_p)v^{(j)}$, where $v^{(j)} := v(\hat{p})$. We now rewrite the first equation of (3.6.2), and then insert that expression for $v(p)$,

$$\begin{aligned} v(\Lambda(A)^{-1}p)\bar{D}^{(j)}(R(A, p)^{-1}) &= D'(A^{-1})v(p) \\ D'(A_{\Lambda(A)^{-1}p})v^{(j)}\bar{D}^{(j)}(R(A, p)^{-1}) &= D'(A^{-1})D'(A_p)v^{(j)} \end{aligned}$$

Proceeding in further analogy to Section 3.4, we set $p = \hat{p}$. Using $A_{\Lambda(A)^{-1}\hat{p}} = A_{\hat{p}} = \mathbf{1}$ and $R(A, \hat{p}) = A$, and substituting A^{-1} by R, we finally obtain

$$v^{(j)}\bar{D}^{(j)}(R) = D'(R)v^{(j)}. \tag{3.6.3}$$

Comparing this to the criterion for a spin-1/2 string field intertwiner (3.4.2) gives rise to expressing

$$u(\underset{\sim}{e}) = w(\underset{\sim}{e})v^{(j)},$$

where the following relation must hold for $w(\underset{\sim}{e})$ (which is yet to be defined) in order to comply with (3.4.2):

$$w(\underset{\sim}{e})D'(R^{-1}) = R^{-1}w(R\underset{\sim}{e}R^{-1}), \tag{3.6.4}$$

where we have inserted the spin-1/2 case $D^{(\frac{1}{2},0)}(R^{-1}) = R^{-1}$ on the right hand side.

This is fulfilled by the following definitions of $w(\underset{\sim}{e}) \in \mathbb{C}^2 \otimes (\mathbb{C}^4)^{\otimes_s(j-\frac{1}{2})}$, and $D'(A)$:

$$\begin{aligned} w^{\mathrm{T}}(\underset{\sim}{e}) &= \mathbf{1}_{\mathbb{C}^2} \otimes \underset{\sim}{e}^{\otimes_s(j-\frac{1}{2})}, \\ D'^{\mathrm{T}}(A) &= A^{\mathrm{T}} \otimes \left[D^{(\frac{1}{2},\frac{1}{2})}(A^*)\right]^{\otimes_s(j-\frac{1}{2})}. \end{aligned}$$

Note that $D'|\mathrm{SU}(2)$ is reducible; it contains $D^{(j)}$ as a subrepresentation, which can be calculated by the usual repeated application of the ladder operator on a highest weight vector, reducing the 3-component of angular momentum by one each time applied.

We can now use $w(e)$ to construct a string field

$$\Phi_\rho(x,e) = \int_0^\infty \mathrm{d}t\, f(t) \sum_\sigma w_{\rho\sigma}(\underset{\sim}{e})\Psi_\sigma(x+te) \tag{3.6.5}$$

from the point field (3.6.1) with corresponding string-like intertwiner

$$\tilde{f}(p\cdot e)w(\underset{\sim}{e})v(p), \tag{3.6.6}$$

where $\tilde{f}(\tau)$ is the Fourier transform of $f(t)$, which in turn is supported in the interval $[0,\infty)$.

Note that while there seems to be a discrepancy between the conjugate intertwiner

$$\tilde{f}(-p\cdot e)w(\underset{\sim}{e})\overline{v^c(p)} \tag{3.6.7}$$

resulting directly from (3.6.5), and the one constructed from (3.6.6) according to (3.3.5),

$$\tilde{f}(-p\cdot e)w(-\underset{\sim}{e})\overline{v^c(p)} = (-1)^{(j-\frac{1}{2})}\tilde{f}(-p\cdot e)w(\underset{\sim}{e})\overline{v^c(p)}, \tag{3.6.8}$$

actually the sign vanishes in the quadratic expressions of (3.3.4), and thus, (3.6.7) and (3.6.8) are equivalent.

3.7 Solving the intertwiner equation for $D^{(\frac{1}{2})}$

Inserting $D^{(\frac{1}{2})}(A) = A$ and $\bar{D}^{(\frac{1}{2})}(A) = \bar{A}$ (2.3.1) in (3.4.2), we arrive at

$$\begin{aligned} u^{(\frac{1}{2})}(\underset{\sim}{e})\bar{A}^{-1} &= A^{-1}u^{(\frac{1}{2})}(A\underset{\sim}{e}A^{-1}), \\ \overline{u^{c(\frac{1}{2})}(\underset{\sim}{e})}A^{-1} &= A^{-1}\overline{u^{c(\frac{1}{2})}(A\underset{\sim}{e}A^{-1})}. \end{aligned}$$

These equations are solved by bases consisting of elements labelled with n,

$$\begin{aligned} u^{(\frac{1}{2})}_{(n)}(\underset{\sim}{e}) &= \underset{\sim}{e}^n\zeta, \\ \overline{u^{c(\frac{1}{2})}_{(n)}(\underset{\sim}{e})} &= u^{(\frac{1}{2})}_{(n)}(-\underset{\sim}{e})\zeta^{-1} = (-1)^n\underset{\sim}{e}^n. \end{aligned} \tag{3.7.1}$$

Note that because of (3.5.1) and (3.5.2),

$$\begin{aligned} \underset{\sim}{e}^{2n+1} &= A^{-1}_{\theta,\varphi}\underset{\sim}{e_t^{2n+1}}A_{\theta,\varphi} = A^{-1}_{\theta,\varphi}\underset{\sim}{e_{(2n+1)t}}A_{\theta,\varphi}, \\ \underset{\sim}{e}^{2n} &= A^{-1}_{\theta,\varphi}\underset{\sim}{e_t^{2n}}A_{\theta,\varphi} = A^{-1}_{\theta,\varphi}\underset{\sim}{e_{nt}}^2A_{\theta,\varphi}. \end{aligned}$$

Thus, if we set $t = 0$, it turns out that there are only two distinct basis elements for each $u^{(\frac{1}{2})}(\underset{\sim}{e})$ and $\overline{u^{c(\frac{1}{2})}(\underset{\sim}{e})}$, i. e. the space of intertwiners is two-dimensional,

$$\begin{aligned} u^{(\frac{1}{2})}(\underset{\sim}{e}) &= \frac{1}{2}A^{-1}_{\theta,\varphi}\left\{(f_+ + f_-)(\hat{p}\cdot e)\underset{\sim}{e_0} + (f_+ - f_-)(\hat{p}\cdot e)\underset{\sim}{e_0}^2\right\}A_{\theta,\varphi}\zeta, \\ \overline{u^{c(\frac{1}{2})}(\underset{\sim}{e})} &= \frac{1}{2}A^{-1}_{\theta,\varphi}\left\{-(\overline{f^c_+} - \overline{f^c_-})(\hat{p}\cdot e)\underset{\sim}{e_0} + (\overline{f^c_+} + \overline{f^c_-})(p\cdot\hat{e})\underset{\sim}{e_0}^2\right\}A_{\theta,\varphi}, \end{aligned} \tag{3.7.2}$$

where we have chosen coefficients $(f_+ \pm f_-)(\hat{p}\cdot e)$ and $\pm(\overline{f^c_+} \pm \overline{f^c_-})(\hat{p}\cdot e)$ as to agree with findings from Section 3.5. We can then easily insert (3.7.2) into (3.4.1) and check that the result really solves (3.2.1).

Alternatively, direct insertion of (3.7.1) into (3.4.1) yields

$$\begin{aligned} u^{(\frac{1}{2})}_{(n)}(p,\underset{\sim}{e}) &= \underset{\sim}{e}^nA_p\zeta, \\ \overline{u^{c(\frac{1}{2})}_{(n)}(p,\underset{\sim}{e})} &= (-1)^n\underset{\sim}{e}^nA_p. \end{aligned} \tag{3.7.3}$$

4 Intertwiners from representations on function spaces

4.1 Spinor-valued representations on function spaces

We now consider representations of $\mathrm{SU}(2) \ni A$ that act on the Hilbert space $L^2(\Gamma, \mathrm{d}\nu) \otimes \mathbb{C}^2 \ni \varphi = \binom{\varphi^+}{\varphi^-}$, with $\Gamma = \{q \in H_0^+ : q \cdot \hat{p} = 1\}$ isometric to the 2-sphere S^2 and $\mathrm{d}\nu$ the G-invariant measure on Γ [5]; we write them down as

$$\left(\tilde{D} \otimes D^{(\frac{1}{2})}\right)(A),$$

where $D^{(\frac{1}{2})}$ acts on $\mathbb{C}^2$ via

$$D^{(\frac{1}{2})}(A) = A,$$

and $\tilde{D}$ is the unitary representation of G defined in [5] acting on $L^2(\Gamma, \mathrm{d}\nu)$ via

$$(\tilde{D}(A)\varphi^{\pm})(q) := \varphi^{\pm}(\Lambda(A)^{-1}q).$$

For the tensor product of both representations, we get

$$\left(\left(\tilde{D} \otimes D^{(\frac{1}{2})}\right)(A)\varphi\right)(q) = A\varphi(\Lambda(A)^{-1}q). \tag{4.1.1}$$

The basis of $D^{(\frac{1}{2})}$ is obviously given by $u_+ = \binom{1}{0} =: \left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and $u_- = \binom{0}{1} =: \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$, whereas the basis of $\tilde{D}$ is given by the spherical harmonics $Y_{l,k}$. Now, according to the *Peter–Weyl theorem*,

$$\tilde{D}(A) = \bigoplus_{l \in \mathbb{N}_0} \tilde{D}^{(l)}(A),$$

with the (integer) spin-l irreducible representation of SO(3) (and thus, of its double cover SU(2)) given by [5, (56)]

$$(\tilde{D}^{(l)}(A)Y_{l,k})(n) := Y_{l,k}(\mathcal{R}(A)^{-1}n) = \sum_{k'=-l}^{+l} \tilde{D}^{(l)}_{kk'}(A)Y_{l,k'}(n), \quad k = -l, \ldots, +l,$$

where $n \in \mathbb{R}^3$, $|n| = 1$, and $\mathcal{R}(A)$ is given by (2.1.2). Thus

$$\begin{aligned}\tilde{D}(A) \otimes D^{(\frac{1}{2})}(A) &= \bigoplus_{l\in\mathbb{N}_0} \tilde{D}^{(l)}(A) \otimes D^{(\frac{1}{2})}(A) \\ &\cong D^{(\frac{1}{2})}(A) + \bigoplus_{l\in\mathbb{N}} \left(\tilde{D}^{(l-\frac{1}{2})}(A) \oplus \tilde{D}^{(l+\frac{1}{2})}(A)\right),\end{aligned} \tag{4.1.2}$$

where we have extended the notion of $\tilde{D}^{(j)}$ to half-integer values $j = l \pm \frac{1}{2}$, and which we will define in Section 4.3.

4.2 Solving the intertwiner equation

We now formulate an analog of (3.2.1) for $D^{(\frac{1}{2},0)}(A) = A$, with $u(p,e)$ replaced by a $L^2(\Gamma, \mathrm{d}\nu) \otimes \mathbb{C}^2$-valued intertwiner $\tilde{u}(p,e)$, and with D replaced by $\tilde{D} \otimes D^{(\frac{1}{2})}$ (4.1.1).

$$\begin{gathered}\left(\tilde{u}(\Lambda(A)^{-1}p, \Lambda(A)^{-1}e)\left(\overline{\tilde{D} \otimes D^{(\frac{1}{2})}}\right)(R^{-1}(A,p))\right)(q) = D^{(\frac{1}{2},0)}(A^{-1})\tilde{u}(p,e)(q) \\ \left(\tilde{u}(\Lambda(A)^{-1}p, \Lambda(A)^{-1}e)\overline{R^{-1}(A,p)}\right)(\Lambda^{-1}(R(A,p))q) = A^{-1}\tilde{u}(p,e)(q).\end{gathered}$$

This equation is solved by

$$\tilde{u}(p,e)(q) = \underset{\sim}{e} A_p \underset{\sim}{q} \zeta, \tag{4.2.1}$$

and more generally by

$$\tilde{u}(p,e)(q) = c(p \cdot e) F(e \cdot \Lambda(A_p) q) A_p \zeta. \tag{4.2.2}$$

We will later project this intertwiner $\tilde{u}(p,e)$ onto $\mathbb{C}^{2j+1}$ in order to obtain the desired $\mathbb{C}^{2lj+1}$-valued intertwiner $u(p,e)$ via the isometric projection $V : L^2(\Gamma, \mathrm{d}\nu) \otimes \mathbb{C}^2 \to \mathbb{C}^{2l+1}$,

$$u_{\rho m}(p,e) = \left(V_j \tilde{u}_{\rho\bullet}(p,e)\right)_m, \tag{4.2.3}$$

where $\tilde{u}_{\rho\bullet}$ is the ρth row vector of $\tilde{u}$.

4.3 Spinor spherical harmonics

The bases of the $2(2l+1)$-dimensional representations $\tilde{D}^{(l)} \otimes D^{(\frac{1}{2})}$ (4.1.1) and $\tilde{D}^{(l\pm\frac{1}{2})}$ are related via Clebsch-Gordan coefficients

$$\begin{gathered}\left\langle l \pm \frac{1}{2}, m | l, m - \frac{1}{2}; \frac{1}{2}, \frac{1}{2} \right\rangle =: C^{l,\frac{1}{2},l\pm\frac{1}{2}}_{m-\frac{1}{2},+\frac{1}{2},m}, \\ \left\langle l \pm \frac{1}{2}, m | l, m + \frac{1}{2}; \frac{1}{2}, -\frac{1}{2} \right\rangle =: C^{l,\frac{1}{2},l\pm\frac{1}{2}}_{m+\frac{1}{2},-\frac{1}{2},m},\end{gathered}$$

and are given by the equations

$$\begin{aligned}\left|l+\frac{1}{2},m\right\rangle &= \sqrt{\frac{l+1/2+m}{2l+1}}\left|l,m-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle+\sqrt{\frac{l+1/2-m}{2l+1}}\left|l,m+\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\\ \left|l-\frac{1}{2},m\right\rangle &= \sqrt{\frac{l+1/2-m}{2l+1}}\left|l,m-\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},\frac{1}{2}\right\rangle-\sqrt{\frac{l+1/2+m}{2l+1}}\left|l,m+\frac{1}{2}\right\rangle\otimes\left|\frac{1}{2},-\frac{1}{2}\right\rangle,\end{aligned} \tag{4.3.1}$$

where for integer values of l and $m \pm \frac{1}{2}$, $\left|l, m \pm \frac{1}{2}\right\rangle = Y_{l,m\pm\frac{1}{2}}$ are the spherical harmonics, which vanish for $|m \pm \frac{1}{2}| > l$. Note that the first line of (4.3.1) holds for $l \geq 0$, while the second line only holds for $l > 0$. The *total angular momentum* is obviously either $j = l - \frac{1}{2}$ or $j = l + \frac{1}{2}$ (cf. the indvidual representation terms on the right hand side of (4.1.2)), and therefore half-integer valued. As this is the quantity that is conserved in physical problems, we will consider expressions in $j = l \mp \frac{1}{2}$ and in $m = -j, \ldots, +j$. These two possible choices of sign correspond to the two-dimensional basis given by (3.5.5). Note that from here on, it would be sufficient to proceed with either choice out of the two possible signs. Wherever feasible, we are still going to state results for both choices.

Equations (4.3.1) amount to the construction of *spinor spherical harmonics*[1] [2, sec. 6.10]

$$\begin{aligned}\left|j=l+\frac{1}{2},m\right\rangle &=: Y^{(j-\frac{1}{2},\frac{1}{2}),j,m} = \begin{pmatrix}\sqrt{\frac{j+m}{2j}}Y_{j-\frac{1}{2},m-\frac{1}{2}}\\ \sqrt{\frac{j-m}{2j}}Y_{j-\frac{1}{2},m+\frac{1}{2}}\end{pmatrix},\\ \left|j=l-\frac{1}{2},m\right\rangle &=: Y^{(j+\frac{1}{2},\frac{1}{2}),j,m} = \begin{pmatrix}\sqrt{\frac{j-m+1}{2j+2}}Y_{j+\frac{1}{2},m-\frac{1}{2}}\\ -\sqrt{\frac{j+m+1}{2j+2}}Y_{j+\frac{1}{2},m+\frac{1}{2}}\end{pmatrix},\end{aligned}$$

with their inner product given by

$$\begin{aligned}\left\langle Y^{(l',\frac{1}{2}),j',m'}, Y^{(l,\frac{1}{2}),j,m}\right\rangle &= \sum_{\nu',\nu=\pm\frac{1}{2}} C^{l',\frac{1}{2},j'}_{m'-\nu',\nu',m'} C^{l,\frac{1}{2},j}_{m-\nu,\nu,m} (Y_{l',m'-\nu'}, Y_{l,m-\nu})\\ &= \sum_{\pm} C^{l',\frac{1}{2},j'}_{m'\mp\frac{1}{2},\pm\frac{1}{2},m'} C^{l,\frac{1}{2},j}_{m\mp\frac{1}{2},\pm\frac{1}{2},m} (Y_{l',m'\mp\frac{1}{2}}, Y_{l,m\mp\frac{1}{2}})\\ &= \delta_{j'j}\delta_{l'l}\delta_{m'm},\end{aligned}$$

where $l \in \{j - \frac{1}{2}, j + \frac{1}{2}\}$ and $l' \in \{j' - \frac{1}{2}, j' + \frac{1}{2}\}$. Now the representation $\tilde{D}^{(j)}$ can be

[1] sometimes also called *spin spherical harmonics* or *spherical spinors*; but not to be mistaken with *spin-weighted spherical harmonics* as e. g. described in [8, sec. 7.8]

written as (cf. [2, (6.56)])

$$\begin{aligned}\left(\tilde{D}^{(j)}(A)Y^{(j\pm\frac{1}{2},\frac{1}{2}),j,m}\right)(n) &= AY^{(j\pm\frac{1}{2},\frac{1}{2}),j,m}(\mathcal{R}(A)^{-1}n)\\ &= \sum_{m'=-j}^{+j}\tilde{D}^{(j)}_{mm'}(A)Y^{(j\pm\frac{1}{2},\frac{1}{2}),j,m'}(n).\end{aligned} \tag{4.3.2}$$

Given a function $\varphi:\mathbb{R}^3\to\mathbb{C}$, $x\mapsto\varphi(x)$, we define

$$\varphi_{(a,A)}(x) := \varphi(\mathcal{R}(A)^{-1}(x-a)),$$

by which the second equation in (4.3.2) implies

$$\tilde{D}^{(j)}_{mm'}(A) = \left(Y^{(j\pm\frac{1}{2},\frac{1}{2}),j,m'}, AY^{(j\pm\frac{1}{2},\frac{1}{2}),j,m}_{(0,A)}\right). \tag{4.3.3}$$

4.4 Projecting function space intertwiners onto the little Hilbert space

Like in [5], there is a partial isometry V, which in the fermionic case intertwines the representations $\tilde{D}^{(l)}\otimes D^{(\frac{1}{2})}$ and $\tilde{D}^{(l\pm\frac{1}{2})}$,

$$\tilde{D}^{(l\pm\frac{1}{2})}(A)V = V\left(\tilde{D}^{(l)}\otimes D^{(\frac{1}{2})}\right)(A), \quad A\in\mathrm{SU}(2).$$

The isometric projectors V_j thus read

$$\begin{aligned}(V_j\varphi)_m &= \left\langle Y^{(l,\frac{1}{2}),j,m},\varphi\right\rangle\\ &= \sum_{\pm} C^{l,\frac{1}{2},j}_{m\mp\frac{1}{2},\pm\frac{1}{2},m}(Y_{l,m\mp\frac{1}{2}},\varphi^{\pm})\\ &= \sum_{\pm} C^{l,\frac{1}{2},j}_{m\mp\frac{1}{2},\pm\frac{1}{2},m}\int_{S^2}\mathrm{d}\sigma(n)\,\overline{Y_{l,m\mp\frac{1}{2}}(n)}\varphi^{\pm}(q(n)),\end{aligned} \tag{4.4.1}$$

where $l\in\{j-\frac{1}{2},j+\frac{1}{2}\}$, and $q(n)=(1,n_1,n_2,n_3)/m$. By (4.2.3), this yields[2] [5]

$$u^{\alpha}_{(l)}(p,e)_{\rho m} = \mathrm{e}^{-\mathrm{i}\pi\alpha/2}\sum_{\pm} C^{l,\frac{1}{2},j}_{m\mp\frac{1}{2},\pm\frac{1}{2},m}\int_{S^2}\mathrm{d}\sigma(n)\,\overline{Y_{l,m\mp\frac{1}{2}}(n)}\tilde{u}_{\rho\pm}(q(n)). \tag{4.4.2}$$

[2] Note that for the sake of legibility, we did not attach the label α to the intertwiners anywhere else in this paper but only in this section.

Inserting $\tilde{u}_{\rho\pm}$ from (4.2.2) in (4.4.2), and using $F(w) = w^\alpha$, we arrive at

$$\begin{aligned} u^\alpha_{(l)}(p,e)_{\rho m} &= \mathrm{e}^{-\mathrm{i}\pi\alpha/2} \sum_\pm C^{l,\frac12,j}_{m\mp\frac12,\pm\frac12,m} \int_{S^2} \mathrm{d}\sigma(n)\, \overline{Y_{l,m\mp\frac12}(n)} (e\cdot\Lambda(A_p)q(n))^\alpha (A_p\zeta)_{\rho\pm} \\ &= \sum_\pm C^{l,\frac12,j}_{m\mp\frac12,\pm\frac12,m} u^\alpha(p,e)_{m\mp\frac12} (A_p\zeta)_{\rho\pm}, \end{aligned} \tag{4.4.3}$$

where $u^\alpha(p,e)_{m\mp\frac12}$ (with only one index) is the intertwiner from the scalar (i. e., the massive bosonic) case as found in [5, (63)],

$$u^\alpha(p,e)_k = \mathrm{e}^{-\mathrm{i}\pi\alpha/2} \int_{S^2} \mathrm{d}\sigma(n)\, \overline{Y_{l,k}(n)} (e\cdot\Lambda(A_p)q(n))^\alpha.$$

4.5 Intertwiners for $D^{(\frac12)}$

We check that our result for $D^{(j=\frac12)}$ (and thus, $l = 0 = j - \frac12$) agrees with what we found in (3.7.3). To that end, we use (4.2.3) and (4.4.1) to express $u^{(\frac12)}_{\rho,\pm\frac12}(p,e)$ as

$$u^{(\frac12)}_{\rho,\pm\frac12}(p,e) = \left(V_{\frac12}\tilde{u}_{\rho\bullet}(p,e)\right)_{\pm\frac12} = \left\langle Y^{(0,\frac12),\frac12,\pm\frac12}, \tilde{u}_{\rho\bullet}(p,e)\right\rangle.$$

Only components $u^{(\frac12)}_{\pm,\pm\frac12}(p,e)$ are non-vanishing; they are given by

$$u^{(\frac12)}_{\pm,\pm\frac12}(p,e) = (Y_{0,0}, \tilde{u}_{\pm,\pm\frac12}(p,e)) = \frac12 \frac{1}{\sqrt{\pi}} \int \mathrm{d}\sigma(n)\, \tilde{u}_{\pm,\pm\frac12}(p,e)(q(n)).$$

We now insert (4.2.1), which is linear in q, and thus in n,

$$\begin{aligned} u^{(\frac12)}_{\pm,\pm\frac12}(p,e) &= \frac12\frac{1}{\sqrt{\pi}} \int \mathrm{d}\sigma(n) \left(\underset{\sim}{e} A_p \underset{\sim}{q(n)} \zeta \right)_{\pm,\pm\frac12} \\ &= \frac12\frac{1}{\sqrt{\pi}} \left(\underset{\sim}{e} A_p \left(\int \mathrm{d}\sigma(n)\, \underset{\sim}{q(n)} \right) \zeta \right)_{\pm,\pm\frac12} \end{aligned} \tag{4.5.1}$$

Furthermore, we express the unit vector n as $n = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\varphi)^{\mathrm{T}}$, and $\mathrm{d}\sigma(n) = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. This yields

$$\underset{\sim}{q(\theta,\varphi)} = \frac{1}{m} \begin{pmatrix} 1+\cos\theta & \mathrm{e}^{-\mathrm{i}\varphi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\varphi}\sin\theta & 1-\cos\theta \end{pmatrix},$$

by which we can compute the integral

$$\int \underset{\sim}{q(\theta,\varphi)} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \frac{4\pi}{m}\mathbf{1}_2.$$

Inserting into (4.5.1), we obtain

$$u^{(\frac{1}{2})}_{\pm,\pm\frac{1}{2}}(p,e) = \frac{2}{m}\sqrt{\pi}\left(\underset{\sim}{e}A_p\zeta\right)_{\pm,\pm\frac{1}{2}},$$

which is in accordance with (3.7.3).

5 Conclusion

We have extended the notion of string-localized fields to the massive fermionic case, and have shown three different ways of constructing such fields, all of which involved a solution to the intertwiner equation that connects the Wigner basis of fields with the desired covariant ones. One of these ways was used to display the uniqueness properties of these fields, another one related them to corresponding point-localized fields, while the third one generated them by a projection of function space representations of SU(2). In contrast to the bosonic case, these fields, like their point-like counterparts, need to be spinor fields, and expose *anti*-locality when space-like separated.

This work can be seen in a broader context of efforts complementing [5] by other cases not treated therein, such as [6, 7].

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Curriculum Vitae

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Education

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2009–present: Designed and developed various web sites
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2010: Held workshop *Democracy and Participation in the Web 2.0*,
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2008: Created and illustrated a lecture script for Prof. Helmuth Hüffel's *Theoretical Methods of Physics 2* course, University of Vienna, Austria

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