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Connections on the infinite dimensional path bundle of finite dimensional principal fiber bundles

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0. Introduction

This thesis considers the so called path spaces of smooth mappings from a compact interval I into Riemannian manifolds M. They can be viewed as manifolds modelled on convenient vector spaces which were introduced in [10, *The Convenient Setting of Global Analysis*].

Following [2, *Connections on the path bundle of a principal fiber bundle*] we can construct fiber bundles which contain path spaces as base and total manifolds and consider connections on principal path bundles.

The aim is to change the calculus on infinite dimensional manifolds used in [2] to the convenient calculus introduced in [10] with the benefit of easier calculations showing the smoothness of the mappings under consideration.

Chapter 1 is devoted to the fundamentals of this calculus. It starts with the introduction of convenient vector spaces. Next the notion of smoothness is introduced which is based on smooth curves in locally convex spaces. Afterwards infinite dimensional manifolds modelled over convenient vector spaces are introduced together with their tangent spaces. Thereafter a short section about actions of Lie groups fixes notation and is stated for further reference. The first chapter ends with theorems concerning the extension of smooth maps by [15, *Extension of C^∞ Functions Defined in a Half Space*] and their application to the construction of the modelling space of the infinite dimensional manifolds in later chapters.

Chapter 2 deals with the construction and fundamental properties of the path bundle. Here the notion of the manifold of paths as the space of all mappings from a compact interval $[a, b]$ in $\mathbb{R}$ to a Riemannian manifold is introduced. This construction can be applied to Lie groups too. So by starting at a given principal fiber bundle of finite dimensional objects, the notion of a principal fiber bundle consisting of path manifolds and a path Lie group becomes available. A large variety of submanifolds and subbundles are constructed and some criteria for triviality are given.

After the introductions of the main objects under consideration Chapter 3 starts by stating the definitions of differential forms and connections on the path bundle. A smooth lift of curves in the base manifold is constructed which is mainly used in the proofs of triviality of special subbundles of the path bundle. Furthermore it is shown how connections on the finite dimensional principal fiber bundle induce connections on the path bundle and criteria are given to recognise induced connections.

Chapter 4 introduces the notion of curvature on the path bundle and shows how Lie algebra valued connection forms can be characterized by mappings from the tangent space of the base space into the Lie algebra of the structure group of the path bundle. In the end of the thesis a special type of connections, the so called uniform ones, are introduced. They contain the induced connections and further a special kind called Polyakov connections. The motivation of these constructions is the relation to a problem of quantum field theory which was discussed in [13, *Gauge fields as rings of glue*] and made mathematically more precise in [2].

Nevertheless some significant changes which go beyond simple changes of smoothness arguments took place and are stated in the following.

First of all there is a more precise formulation of the construction of the modelling space of the manifold of paths at the end of Chapter 1.

The definition of the charts of the path spaces in the beginning of Chapter 2 is now based on [10].

Section 3.2 generalizes the construction of a smooth lift in the path bundle. Where the construction in [2] is restricted to matrix Lie groups, the formulation in this thesis is applicable to general Lie groups.

Section 3.4 includes Lemma 3.25 which clarifies how vector subbundles of the tangent bundle of the finite dimensional manifold lead to vector subbundles of the corresponding path manifold. This leads to a more rigorous formulation and proof of a result in [2] which is stated now as Proposition 3.27.

The part of the proof of Theorem 3.36 concerning smoothness is totally different to the original one. It depends strongly on properties introduced in the calculus of convenient vector spaces. The same is true for Theorem 4.13.

I would like to thank my supervisor Andreas Kriegl for his very helpful, constructive support on this thesis and his time spent for discussing the problems which arose during the writing process. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly.

1. Preliminary results for infinite dimensional manifolds

This chapter consists mainly of preparation work for the following chapters. Its objective is to introduce convenient vector spaces and construct the modelling space for the path manifolds used later on.

1.1. Convenient vector spaces

We begin with the definitions required for the introduction of convenient vector spaces. First we state the definition of locally convex spaces.

Definition 1.1. [10, p. 575] A *locally convex space* E is a vector space together with a Hausdorff topology such that addition $E \times E \to E$ and scalar multiplication $\mathbb{R} \times E \to E$ (or $\mathbb{C} \times E \to E$) are continuous and 0 has a basis of neighbourhoods consisting of (absolutely) convex sets (see Definition A.7).

Equivalently: The topology on E can be described by a system of (continuous) seminorms (see Chapter A.1).

The next task is to introduce differentiability and other properties of curves in locally convex spaces.

Definition 1.2. [10, p. 8] Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *differentiable* if the derivative $c'(t) := \lim_{s\to\infty} \frac{1}{s}\left(c(t+s) - c(t)\right)$ at t exists for all t. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all iterated derivatives exist. It is called C^n for some finite n if its iterated derivatives up to order n exist and are continuous.

Likewise, a mapping $f : \mathbb{R}^n \to E$ is called smooth if all iterated partial derivatives $\partial_{i_1,\cdots,i_p} f := \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_p}} f$ exist for all $i_1, \cdots, i_p \in \{1, \cdots, n\}$.

Definition 1.3. [10, p. 9] A curve $c : \mathbb{R} \to E$ is called *locally Lipschitzian* if every point $r \in \mathbb{R}$ has a neighbourhood U such that the Lipschitz condition is satisfied on U , i.e., the set

$$\left\{ \frac{1}{t-s}\left(c(t) - c(s)\right) : t, s \in U, t \neq s \right\}$$

is bounded.

Also the usual notion of convergence has to be generalized to work for locally convex spaces.

Lemma 1.4. [10, p. 12] *Let B be a bounded and absolutely convex subset of a locally convex space E and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in the normed space E_B, the linear span of B in E equipped with the Minkowski functional (see Definition A.11). Then the following two conditions are equivalent:*

1. *x_γ converges to 0 in the normed space E_B*
2. *There exists a net $\mu_\gamma \to 0$ in $\mathbb{R}$, such that $x_\gamma \in \mu_\gamma \cdot B$.*

Definition 1.5. [10, p. 12] A net (x_γ) for which a bounded absolutely convex $B \subseteq E$ exists, such that x_γ converges to x in E_B is called *Mackey convergent* to x or for short *M-convergent.*

As a generalization of Cauchy sequences and completeness we introduce Mackey-Cauchy nets and Mackey completeness:

Definition 1.6. [10, p. 14] A net $(x_\gamma)_{\gamma\in\Gamma}$ in E is called *Mackey-Cauchy* provided that there exists a bounded (absolutely convex) set B and a net $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbb{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} \cdot B$.

Lemma 1.7. [10, p. 15] *For a locally convex space E the following conditions are equivalent:*

1. *Every Mackey-Cauchy net converges in E.*
2. *Every Mackey-Cauchy sequence converges in E.*
3. *For every absolutely convex closed bounded set B the space E_B is complete.*
4. *For every bounded set B there exists an absolutely convex bounded set $B' \supseteq B$ such that $E_{B'}$ is complete.*

Definition 1.8. [10, p. 15] A space satisfying the equivalent conditions of Lemma 1.7 is called *Mackey complete.* Note that a sequentially complete space is Mackey complete.

With the generalisations of convergence and completeness in mind one turns to the integration of curves.

Lemma 1.9. [10, p. 15] *For continuous curves $c : [0,1] \to E$ one can show analogously to 1-dimensional analysis that the Riemann sums $R(c, Z, \xi)$, defined by $\sum_k (t_k - t_{k-1}) c(\xi_k)$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition Z of $[0,1]$ and $\xi_k \in [t_{k-1}, t_k]$, form a Cauchy net with respect to the partial strict ordering given by the size $\max\{|t_k - t_{k-1}| : 0 < k < n\}$ of the mesh. So under the assumption of sequential completeness we have a Riemann integral of curves.*

Lemma 1.10. [10, p. 16] *Let $c : \mathbb{R} \to E$ be a continuous curve in a locally convex vector space. Then there is a unique differentiable curve $\int c : \mathbb{R} \to \hat{E}$ in the completion $\hat{E}$ of E such that $\left(\int c\right)(0) = 0$ and $\left(\int c\right)' = c$.*

Proposition 1.11. [10, p. 17] *Let $c : [0,1] \to E$ be a Lipschitz curve into a Mackey complete space. Then the Riemann integral exists in E as (Mackey)-limit of the Riemann sums.*

Via the definition of smooth curves in E we define the following topology on E:

Definition 1.12. [10, p. 19] The c^∞*-topology* on a locally convex space E is the final topology with respect to all smooth curves $c : \mathbb{R} \to E$. Its open sets will be called c^∞*-open*.

So after the introduction of the necessary terms we finally arrive at the definition of convenient vector spaces given by the following theorem.

Theorem 1.13. [10, p. 20] *Let E be a locally convex vector space. E is said to be c^∞-complete or convenient if one of the following equivalent (completeness) conditions is satisfied:*

1. *Any Lipschitz curve in E is locally Riemann integrable.*

2. *For any smooth $c_1 : \mathbb{R} \to E$ there exists a smooth $c_2 : \mathbb{R} \to E$ with $c_2' = c_1$ (existence of an anti-derivative).*

3. *E is c^∞-closed in any locally convex space.*

4. *If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.*

5. *Any Mackey-Cauchy sequence converges; i.e. E is Mackey complete.*

6. *If the set B is bounded closed absolutely convex, then E_B is a Banach space.*

7. *Any continuous linear mapping from a normed space into E has a continuous extension to the completion of the normed space.*

Various constructions of convenient vector spaces inherit the c^∞-completeness property.

Theorem 1.14. [10, p. 21] *The following constructions preserve c^∞-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings;*

The same is true for formation of $\ell^\infty(X, .)$: Let X be a set together with a family $\mathcal{B}$ of subsets of X containing all finite ones. We call the elements of $\mathcal{B}$ bounded. Now denote by $\ell^\infty(X, E)$ the space of all functions $f : X \to E$, which are bounded on all $B \in \mathcal{B}$, supplied with the topology of uniform convergence on the sets in $\mathcal{B}$.

Remark 1.15. [10, p. 21] The definition of the topology of uniform convergence as initial topology shows, that adding all subsets of finite unions of elements in $\mathcal{B}$ to $\mathcal{B}$ does not change this topology. Hence, we may always assume that $\mathcal{B}$ has this stability property; this is the concept of a *bornology on a set.*

1.2. Smooth mappings on locally convex spaces

Now we want to discuss smooth mappings between locally convex vector spaces E and F where the notion of smoothness is introduced via smooth curves.

Definition 1.16. [10, p. 28] Let $C^\infty(\mathbb{R}, E)$ be the locally convex vector space of all smooth curves in E, with the pointwise vector operations, and with the topology of uniform convergence on compact sets of each derivative separately.

Lemma 1.17. [10, p. 28] *The topology given by Definition 1.16 is the initial topology with respect to the set of linear mappings*

$$C^\infty(\mathbb{R}, E) \xrightarrow{d^k} C^\infty(\mathbb{R}, E) \to \ell^\infty(K, E),$$

for $k \in \mathbb{N}$ and $K \subset \mathbb{R}$ compact, where $\ell^\infty(K, E)$ carries the topology of uniform convergence.

The derivatives $d^k : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$, the point evaluation $\mathrm{ev_t} : C^\infty(\mathbb{R}, E) \to E$ and the pullbacks $g^* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, E)$ for $g \in C^\infty(\mathbb{R}, \mathbb{R})$ are continuous and linear.

Lemma 1.18. [10, p. 28] *A space E is c^∞-complete if and only if $C^\infty(\mathbb{R}, E)$ is.*

Example 1.19. As a first simple example for a convenient vector space we get $C^\infty(\mathbb{R}, \mathbb{R})$ since $\mathbb{R}$ is obviously locally convex and c^∞-complete.

Definition 1.20. [10, p. 30] A mapping $f : E \supseteq U \to F$ defined on a c^∞-open subset U is called *smooth* (or C^∞) if it maps smooth curves in U to smooth curves in F.

Let $C^\infty(U, F)$ denote the locally convex *space of all smooth mappings* $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$.

For $U = E = \mathbb{R}$ this coincides with our usual definition. Obviously, the composition of smooth mappings is also smooth.

As we defined smoothness for mappings between locally convex vector spaces, we get that some commonly used mappings are automatically smooth.

Theorem 1.21. [10, p. 30] *Let $U_i \subset E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to C^\infty(U_2, F)$ exists and is smooth.*

Corollary 1.22. [10, p. 31] *Let E, F, G, etc. be locally convex spaces, and let U, V be c^∞-open subsets of such. Then the following canonical mappings are smooth.*

1. $\mathrm{ev} : C^\infty(U, F) \times U \to F, \quad (f, x) \mapsto f(x)$
2. $\mathrm{ins} : E \to C^\infty(F, E \times F), \quad x \mapsto (y \mapsto (x, y))$
3. $(\quad)^\wedge : C^\infty(U, C^\infty(V, G)) \to C^\infty(U \times V, G)$

4. $(\quad)^\vee : C^\infty(U \times V, G) \to C^\infty(U, C^\infty(V, G))$

5. $\text{comp} : C^\infty(F, G) \times C^\infty(U, F) \to C^\infty(U, G), \quad (f, g) \mapsto f \circ g$

6. $C^\infty(\quad,\quad) : C^\infty(E_2, E_1) \times C^\infty(F_1, F_2) \to C^\infty(C^\infty(E_1, F_1), C^\infty(E_2, F_2)),$
   $(f, g) \mapsto (h \mapsto (g \circ h \circ f)$

7. $\prod : \prod C^\infty(E_i, F_i) \to C^\infty(\prod E_i, \prod F_i)$, *for any index set.*

The new definition of smoothness is a generalization of the finite dimensional case.

Corollary 1.23. [10, p. 31] *The smooth mappings on open subsets of $\mathbb{R}^n$ in the sense of Definition 1.20 are exactly the usual smooth mappings.*

Later on we need a differential operator not only for curves but for arbitrary smooth mappings between locally convex vector spaces.

Theorem 1.24. [10, p. 33] *Let E and F be locally convex spaces, and let $U \subseteq E$ be c^∞-open. Then the differentiation operator*

$$d : C^\infty(U, F) \to C^\infty(U, L(E, F)) \qquad df(x)(v) := lim_{t\to 0} \frac{f(x + tv) - f(x)}{t}$$

exists, is linear and bounded (smooth). It also satisfies the chain rule:

$$d(f \circ g)(x)(v) = (df(g(x)) \circ dg(x))\,(v)$$

Finally the following proposition gives some examples of spaces of smooth functions which are convenient vector spaces.

Proposition 1.25. [10, p. 66] *Let M be a smooth finite dimensional paracompact manifold. Then the space $C^\infty(M, \mathbb{R})$ of all smooth functions on M is a convenient vector space in any of the following (bornologically)(see Definition A.16) isomorphic descriptions, and it satisfies the uniform boundedness principle for the point evaluations.*

1. *The initial structure with respect to the cone*

$$C^\infty(M, \mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R}, \mathbb{R})$$

   for all $c \in C^\infty(\mathbb{R}, M)$.

2. *The initial structure with respect to the cone*

$$C^\infty(M, \mathbb{R}) \xrightarrow{(u_\alpha^{-1})^*} C^\infty(\mathbb{R}^n, \mathbb{R})$$

   where $(U_\alpha, u_\alpha)_{\alpha\in}$ is a smooth atlas with $u_\alpha(U_\alpha) = \mathbb{R}^n$.

3. The initial structure with respect to the cone

$$C^\infty(M,\mathbb{R}) \xrightarrow{j^k} C^\infty(J^k(M,\mathbb{R}))$$

for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M, where j^k is the jet prolongation, and where all the spaces of continuous sections are equipped with the compact open topology.

1.3. Infinite dimensional manifolds

The following definitions are generalizations of the standard definitions of manifolds:

Definition 1.26. [10, p. 264] A chart (U,u) on a set M is a bijection $u : U \to u(U) \subseteq E_U$ from a subset $U \subseteq M$ onto a c^∞-open subset of a convenient vector space E_U.

Definition 1.27. [10, p. 264] For two charts (U_α, u_α) and (U_β, u_β) on M the mapping $u_{\alpha\beta} := u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ for α, $\beta \in A$ is called the *chart changing*, where $U_{\alpha\beta} := U_\alpha \cap U_\beta$. A family $(U_\alpha, u_\alpha)_{\alpha\in A}$ of charts on M is called an atlas for M, if the $\{U_\alpha\}_{\alpha\in A}$ form a cover of M and all chart changings $u_{\alpha\beta}$ are defined on c^∞-open subsets.

Definition 1.28. [10, p. 264] An atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$ for M is said to be a C^∞*-atlas*, if all chart changings $u_{\alpha\beta} : u_\beta(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$ are smooth. Two C^∞-atlases are called C^∞*-equivalent*, if their union is again a C^∞-atlas for M.

An equivalence class of C^∞-atlases is sometimes called a C^∞*-structure* on M . The union of all atlases in an equivalence class is again an atlas, the *maximal atlas* for this C^∞-structure. A C^∞*-manifold* M is a set together with a C^∞-structure on it.

To study C^∞-manifolds we describe mappings between them.

Definition 1.29. [10, p. 264] A mapping $f : M \to N$ between manifolds is called *smooth* if for each $x \in M$ and each chart (V,v) on N with $f(x) \in V$ there is a chart (U,u) on M with $x \in U$, $f(U) \subseteq V$, such that $v \circ f \circ u^{-1}$ is smooth. This is the case if and only if $f \circ c$ is smooth for each smooth curve $c : \mathbb{R} \to M$. The space of all C^∞-mappings from M to N will be denoted by $C^\infty(M,N)$.

Definition 1.30. [10, p. 264] A smooth mapping $f : M \to N$ is called a *diffeomorphism* if f is bijective and its inverse is also smooth. Two manifolds are called *diffeomorphic* if there exists a diffeomorphism between them.

Next we mean to induce a topology on the manifold M.

Definition 1.31. [10, p. 265] The *natural topology* on a manifold M is the identification topology with respect to some (smooth) atlas $(u_\alpha : M \supseteq U_\alpha \to u_\alpha(U_\alpha) \subseteq E_\alpha)$, where a subset $W \subseteq M$ is open if and only if $u_\alpha(U_\alpha \cap W)$ is c^∞-open in E_α for all α. This topology depends only on the C^∞-structure of M, since diffeomorphisms are homeomorphisms for the c^∞-topologies. It is the final topology with respect to all inverses of chart mappings in one atlas and it is also the final topology with respect to all smooth curves.

To discuss metrizability of a manifold we require the following corollary and lemma.

Corollary 1.32. [10, p. 42] *Let E be a bornological (see Definition A.16) convenient vector space containing a nonempty c^∞-open subset which is either locally compact or metrizable in the c^∞- topology. Then the c^∞-topology on E is locally convex. In the first case E is finite dimensional, in the second case E is a Fréchet space (see Definition A.17).*

Lemma 1.33. [10, p. 267] *A manifold M is metrizable if and only if it is paracompact and modeled on Fréchet spaces.*

Proof. A topological space is metrizable if and only if it is paracompact and locally metrizable. The c^∞-open subsets of the modeling vector spaces are metrizable if and only if the spaces are Fréchet (see Corollary 1.32). □

Next we introduce C^∞-submanifolds.

Definition 1.34. [10, p. 268] A subset N of a manifold M is called a *submanifold*, if for each $x \in N$ there is a chart (U, u) of M such that $u(U \cap N) = u(U) \cap F_U$, where F_U is a closed linear subspace of the convenient modeling space E_U. Then clearly N is itself a manifold with $(U \cap N, u|_{U \cap N})$ as charts, where (U, u) runs through all these *submanifold charts* from above.

A submanifold N of M is called a *splitting submanifold* if there is a cover of N by submanifold charts (U, u) as above such that the $F_U \subseteq E_U$ are complemented (i.e. splitting) linear subspaces. Then every submanifold chart is splitting.

The following lemma states a generalization of Proposition 1.25 and brings up new examples of convenient vector spaces.

Lemma 1.35. [10, p. 273] *For a convenient vector space E and any smooth manifold M the set $C^\infty(M, E)$ of smooth E-valued functions on M is also a convenient vector space in any of the following isomorphic descriptions, and it satisfies the uniform boundedness principle for the point evaluations.*

1. *The initial structure with respect to the cone*

$$C^\infty(M, E) \xrightarrow{c^*} C^\infty(\mathbb{R}, E)$$

for all $c \in C^\infty(\mathbb{R}, M)$.

2. *The initial structure with respect to the cone*

$$C^\infty(M,E) \xrightarrow{(u_\alpha^{-1})^*} C^\infty(u_\alpha(U_\alpha),E)$$

where (U_α, u_α) *is a smooth atlas with* $u_\alpha(U_\alpha) \subseteq E_\alpha$.

1.4. The kinematic tangent bundle

We start this section by introducing the notion of a tangent vector at a given point of a convenient vector space.

Definition 1.36. [10, p. 276] Let E denote a convenient vector space and $p \in E$. A *kinematic tangent vector* with foot point p is simply a pair (p, X) with $X \in E$. Let $T_pE = E$ be the space of all kinematic tangent vectors with foot point p. It consists of all derivatives $c'(0)$ at 0 of smooth curves $c : \mathbb{R} \to E$ with $c(0) = p$, which explains the choice of the name kinematic.

For each open neighbourhood U of p in E the pair (p, X) induces a linear mapping

$$X_p : C^\infty(U,\mathbb{R}) \to \mathbb{R} \qquad X_p(f) := df(p)(X)$$

which is continuous for the convenient vector space topology on $C^\infty(U,\mathbb{R})$ and satisfies

$$X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g),$$

so it is a *continuous derivation over* $\mathrm{ev_p}$.

In contrast to the finite dimensional case there exists the possibility to define a more general notion of tangent vectors which contains the kinematic ones. They are bounded derivations and called *operational tangent vectors* [10, p. 276]. Because we will only use kinematic tangent vectors the general case is not discussed in more detail. Next we construct the kinematic tangent space of a manifold.

Definition 1.37. [10, p. 284] Let M be a C^∞-manifold with a smooth atlas $(U_\alpha, u_\alpha)_{\alpha\in A}$ where $u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq E_\alpha$ denote the charts to c^∞-open subsets of convenient vector spaces E_α. On the disjoint union

$$\bigsqcup_{\alpha\in A} \{U_\alpha\} \times \{E_\alpha\} \times \{\alpha\}$$

we define the equivalence relation

$$(x, v, \alpha) \sim (y, w, \beta) \iff x = y \text{ and } d(u_{\alpha\beta})(u_\beta(x))w = v$$

where $u_{\alpha\beta}$ denotes the chart changing mapping (see Definition 1.27). The quotient space to this equivalence relation is denoted by TM and called the *kinematic tangent space*.

Analogously to the finite dimensional case we want TM to become a manifold and $TM \to M$ a vector bundle.

Definition 1.38. [10, p. 284] With the notation from above we define the mappings:

$$\begin{aligned} \pi_M &: TM \to M & \pi_M([x,v,\alpha]) &= x \\ Tu_\alpha &: TU_\alpha \to u_\alpha(U_\alpha) \times E_\alpha & Tu_\alpha([x,v,\alpha]) &= (u_\alpha(x), v) \end{aligned}$$

where $TU_\alpha := \pi_M^{-1}(U_\alpha) \subseteq TM$. The pairs (TU_α, Tu_α) satisfy the the chart properties for TM and the chart changings are given by

$$\begin{aligned} Tu_\alpha \circ (Tu_\beta)^{-1} : u_\beta(U_{\alpha\beta}) \times E_\beta &\to u_\alpha(U_{\alpha\beta}) \times E_\alpha \\ (x,v) &\to (u_{\alpha\beta}(x), d(u_{\alpha\beta})(x)v) \end{aligned}$$

So the family $(TU_\alpha, Tu_\alpha)_{\alpha\in A}$ becomes a C^∞-atlas for TM. The chart changing formula also implies that the smooth structure on TM depends only on the equivalence class of the C^∞-atlas for M.

The mapping $\pi_M : TM \to M$ is obviously smooth. It is called the *(foot point) projection*. The triple (TM, M, π_M) is called the *kinematic tangent bundle*. The natural topology is automatically Hausdorff: (p,X), $(q,Y) \in TM$ can be separated by open sets of the form $\pi_M^{-1}(V)$ for $V \subset M$, if $p \neq q$, since M is Hausdorff, and by open subsets of the form $(Tu_\alpha)^{-1}(E_\alpha \times W)$ for W open in E_α, if $p = q$.

Definition 1.39. [10, p. 284] For $x \in M$ the set $T_xM := \pi_M(x)$ is called the *kinematic tangent space at x* or the fiber over x of the tangent bundle. It carries a canonical convenient vector space structure induced by $T_x(u_\alpha) := Tu_\alpha|_{T_xM} : T_xM \to \{x\} \times E_\alpha \cong E_\alpha$ for some (equivalently any) α with $x \in U_\alpha$.

Remark 1.40. [10, p. 286] From the construction of the tangent bundle (see Definition 1.38) it follows that

$$T(M \times N) \cong TM \times TN$$

in a canonical way.

The following paragraphs give a description of TM as the space of all velocity vectors of curves, which motivates the name *kinematic tangent bundle*:

Remark 1.41. [10, p. 285] We put on $C^\infty(\mathbb{R}, M)$ the equivalence relation: $c \sim e$ if and only if $c(0) = e(0)$ and in one (equivalently each) chart (U,u) with $c(0) = e(0) \in U$ we have:

$$\frac{d}{dt}\Big|_0 (u \circ c)(t) = \frac{d}{dt}\Big|_0 (u \circ e)(t)$$

To $c \in C^\infty(\mathbb{R}, M)$ we associate the tangent vector $\delta(c) := [c(0), \frac{\partial}{\partial t}\big|_0 (u_\alpha \circ c)(t), \alpha]$.

$$\begin{tikzcd}
C^\infty(\mathbb{R}, M)/\sim \arrow[d, "\cong"] & C^\infty(\mathbb{R}, M) \arrow[l] \arrow[d, "\mathrm{ev}_0"] \arrow[dl, "\delta"'] \\
TM \arrow[r, "\pi_M"] & M
\end{tikzcd}$$

This mapping factors to a bijection $C^\infty(\mathbb{R}, M)/\sim \to TM$, whose inverse associates to $[x, v, \alpha]$ the equivalence class of

$$t \to u_\alpha^{-1}(u_\alpha(x) + h(t)v)$$

for h an absolutely small function with $h(t) = t$ near 0.

Lemma 1.42. [10, p. 41] *Let E be a convenient vector space, U a c^∞-open subset of $E \times \mathbb{R}$ and $K \subseteq \mathbb{R}$ compact. Then the set*

$$U_0 := \big\{x \in E : \{x\} \times K \subseteq U\big\}$$

is c^∞-open in E.

Remark 1.43. [10, p. 285] With the equivalence relation defined in Remark 1.41 the c^∞-topology on $\mathbb{R} \times E_\alpha$ is the product topology by Lemma 1.42. So one can choose h uniformly for (x, v) in a piece of a smooth curve. Thus, a mapping $g : TM \to N$ into another C^∞-manifold is smooth if and only if

$$g \circ \delta : C^\infty(\mathbb{R}, M) \to N$$

maps smooth curves to smooth curves, by which is meant $C^\infty(\mathbb{R}^2, M)$ to $C^\infty(\mathbb{R}, N)$.

In Definition 1.37 we defined the kinematic tangent space and in Definition 1.38 the kinematic tangent bundle. Now we focus on the tangent mapping of a given mapping between C^∞-manifolds. We give a definition analogues to the finite dimensional case using local trivializations.

Definition 1.44. Let $f : M \to N$ be a smooth mapping between C^∞-manifolds. Then T_pM denotes the tangent space in the point $p \in M$ and $T_{f(p)}N$ the tangent space in the point $f(p) \in N$. Elements of T_pM can be denoted by equivalence classes $[(p, \nu, \alpha)]$ where α denotes a chart index and ν a tangent vector at p, which is an element of the modelling vector space.

Now let $u_\alpha : M \supseteq U_\alpha \to u_\alpha(U_\alpha) \subseteq E$ denote a chart of M satisfying $p \in U_\alpha$ and $v_\beta : N \supseteq V_\beta \to v_\beta(V_\beta) \subseteq F$ a chart of N satisfying $f(p) \in V_\beta$ where E and F denote the modelling convenient vector spaces of the manifolds M and N respectively. Then f induces a linear mapping T_pf by

$$T_pf : T_pM \to T_{f(p)}N \qquad [(p, \nu, \alpha)] \mapsto [(f(p), (v_\beta \circ f \circ u_\alpha^{-1})'(u_\alpha(p))(\nu), \beta)]$$

where we use the derivative defined for convenient vector spaces. The mapping T_pf is called the *tangent mapping of* f *at* p.

$$\begin{tikzcd}
N \supseteq V_\beta \arrow[r, "v_\beta"] & v_\beta(V_\beta) \subseteq F \\
M \supseteq U_\alpha \arrow[r, "u_\alpha"] \arrow[u, "f"] & u_\alpha(U_\alpha) \subseteq E \arrow[u, "v_\beta \circ f \circ u_\alpha^{-1}"']
\end{tikzcd}$$

1.5. Actions of Lie groups

We start with a review of the definition of Lie group to fix some notation.

Definition 1.45. [10, p. 369] A *Lie group* is a group G carrying the structure of a smooth manifold for which the following maps are smooth:

$$\mu : G \times G \to G \qquad (g,h) \mapsto g \cdot h \qquad (\textit{multiplication}) \quad \text{and}$$
$$\nu : G \to G \qquad g \mapsto g^{-1} \qquad (\textit{inverse})$$

If not stated otherwise, G may be infinite dimensional. Furthermore we write for the left and right translation by an element $g \in G$: $\mu_g(x) := g \cdot x$ and $\mu^g(x) = x \cdot g$.

A *Lie group homomorphism* is a smooth group homomorphism between two Lie groups.

The following lemma leads to a closer view of the tangent mappings of the just introduced mappings.

Lemma 1.46. [10, p. 370] *For $X_a \in T_aG$ and $Y_b \in T_bG$ the kinematic tangent mapping μ is given by*

$$T_{(a,b)}\mu : T_aG \times T_bG \to T_{a \cdot b}G \qquad T_{(a,b)}\mu(X_a, Y_b) = T_a\mu^b(X_a) + T_b\mu_a(Y_b)$$

and the tangent mapping of the inverse is given by

$$T_a\nu : T_aG \to T_{a^{-1}}G \qquad T_a\nu(X_a) = -(T_e\mu^{a^{-1}} \circ T_a\mu_{a^{-1}})(X_a) = -(T_e\mu_{a^{-1}} \circ T_a\mu^{a^{-1}})(X_a).$$

Next we introduce sections of the kinematic tangent bundle and view some of their properties.

Definition 1.47. [10, p. 321] Let M be a smooth manifold. A *kinematic vector field* X on M is a smooth section of the kinematic tangent bundle $TM \to M$. The space of all kinematic vector fields will be denoted by $\mathfrak{X}(M) := \Gamma(TM \to M)$.

Definition 1.48. Let M and N be smooth manifolds and $f : M \to N$ a smooth mapping. Two kinematic vector fields $\xi \in \Gamma(TM \to M)$ and $\zeta \in \Gamma(TN \to N)$ are called *f-related* if

$$Tf \circ \xi = \zeta \circ f,$$

making the following diagram commute:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "\xi"] \arrow[r, "f"] & N \arrow[u, "\zeta"'] \end{tikzcd}$$

Definition 1.49. [5, p. 46] A *Lie algebra* (over $\mathbb{R}$) is a real vector space V equipped with a bilinear map $[\cdot,\cdot] : V \times V \to V$, the *Lie bracket*, satisfying:

(i) $[X, X] = 0 \quad$ for all $X \in V$.

(ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad$ for all $X,\ Y,\ Z \in V$.

A *Lie algebra homomorphism* is a linear mapping between two Lie algebras which is compatible with the Lie brackets.

Definition 1.50. [10, p. 370] Let G be a (real) Lie group. A (kinematic) vector field ξ on G is called *left invariant*, if $\mu_a^*\xi = \xi$ for all $a \in G$, where $\mu_a^*\xi := T\mu_{a^{-1}} \circ \xi \circ \mu_a$.

The Lie bracket on the space of all kinematic vector fields $\mathfrak{X}(G)$ is defined in the following.

Definition 1.51. With the notation already introduced let U denote an open subset of G which is diffeomorphic to an open subset V of a convenient vector space. Furthermore let ξ denote a vector field on G which can be written locally in coordinates on V as $\bar{\xi}$.

$$\begin{tikzcd} TG \arrow[d, shift left, "\pi_G"] & \supseteq & TU \arrow[d, "\pi_U"] \arrow[r] & V \times E \\ G \arrow[u, shift left, "\xi"] & \supseteq & U \cong V \subseteq E \arrow[ur, "(\mathrm{Id},\bar{\xi})"'] & \end{tikzcd}$$

Then the Lie bracket of two vector fields ξ and η can be defined locally as:

$$[\xi, \eta] : V \ni x \mapsto (\bar{\xi}'(x) \cdot \bar{\eta}(x) - \bar{\eta}'(x) \cdot \bar{\xi}(x))$$

This is well defined because $\bar{\xi}(x) \in E$ and $\bar{\xi}'(x) \in L(E, E)$.

The mapping μ_a^* satisfies $\mu_a^*[\xi, \eta] = [\mu_a^*\xi, \mu_a^*\eta]$, so the space $\mathfrak{X}_L(G)$ of all left invariant vector fields on G is closed under the Lie bracket and is a sub Lie algebra of $\mathfrak{X}(G)$.

Remark 1.52. [10, p. 370] Any left invariant vector field ξ is uniquely determined by $\xi(e) \in T_eG$, since $\xi(a) = T_e\mu_a(\xi(e))$. Thus, the Lie algebra $\mathfrak{X}_L(G)$ of left invariant vector fields is linearly isomorphic to T_eG, and the Lie bracket on $\mathfrak{X}_L(G)$ induces a Lie algebra structure on T_eG, whose bracket is again denoted by $[\ ,\ ]$. This Lie algebra will be denoted as usual by $\mathfrak{g}$.

Definition 1.53. [10, p. 370] Analogously a vector field η on G is called right invariant, if $(\mu^a)^*\eta = \eta$ for all $a \in G$.

Remark 1.54. [10, p. 370] The right invariant vector fields form a sub Lie algebra $\mathfrak{X}_R(G)$ of $\mathfrak{X}(G)$, which also is linearly isomorphic to T_eG and induces a Lie algebra structure on T_eG. Since $\nu^* : \mathfrak{X}_L(G) \to \mathfrak{X}_R(G)$ is an isomorphism of Lie algebras, $T_e\nu = -\operatorname{Id} : T_eG \to T_eG$ is an isomorphism between the two Lie algebra structures.

This leads to the definition of further mappings.

Definition 1.55. [10, p. 373] Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $a \in G$ we define

$$\operatorname{conj}_a : G \to G \qquad \operatorname{conj}_a(x) = a \cdot x \cdot a^{-1}$$

as the *conjugation* or the *inner automorphism* by $a \in G$. This defines a smooth action of G on itself by automorphisms.

Next we define the *adjoint representation of the Lie group G*

$$\operatorname{Ad} : G \to GL(\mathfrak{g}) \subset L(\mathfrak{g}, \mathfrak{g}) \qquad \operatorname{Ad}(a) := (\operatorname{conj}_a)' = T_e\operatorname{conj}_a : \mathfrak{g} \to \mathfrak{g} \text{ for } a \in G.$$

The mapping $\operatorname{Ad}(a)$ is a Lie algebra homomorphism and satisfies:

$$\operatorname{Ad}(a) = T_e\operatorname{conj}_a = T_a\mu^{a^{-1}} \circ T_e\mu_a = T_{a^{-1}}\mu_a \circ T_e\mu^{a^{-1}}$$

The *adjoint representation of the Lie algebra* $\mathfrak{g}$ is given by:

$$\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) := L(\mathfrak{g}, \mathfrak{g}) \qquad \operatorname{ad} := \operatorname{Ad}' = T_e \operatorname{Ad}$$

Definition 1.56. Let $g \in G$ and $\mu^{g^{-1}}$ denote the right multiplication by g^{-1} in G. Then the mapping

$$\kappa_g^r := T_g(\mu^{g^{-1}}) : T_gG \to \mathfrak{g}$$

defines the *right Maurer-Cartan form* $\kappa^r \in \Omega^1(G, \mathfrak{g})$.

Analogously the left multiplication by g^{-1} denoted as $\mu_{g^{-1}}$ leads to the *left Maurer-Cartan form* $\kappa^l \in \Omega^1(G, \mathfrak{g})$.

As we will need principal fiber bundles later on, we introduce the action of Lie groups on manifolds together with the notation required.

Definition 1.57. [5, p. 50] [10, p. 375] One says that G *acts on a smooth manifold M from the left* if there is a smooth mapping

$$\ell : G \times M \to M$$
$$(g, x) \mapsto g \cdot x$$

that respects the Lie group structure of G in the sense that

$$g(hx) = (g \cdot h)x \qquad \forall g, h \in G, x \in M.$$

Then the mapping $\ell^\vee : G \to \mathrm{Diff}(M)$ is a group homomorphism and the mappings $\ell_g : M \to M$ and $\ell^x : G \to M$ satisfy $\ell_g(x) = \ell^x(g) = \ell(g,x) = g \cdot x$.

Analogously one defines the *action from the right* $r : (x,g) \mapsto x \cdot g$ with $(x \cdot g)h = x(g \cdot h)$. Then $r^\vee : G \to \mathrm{Diff}(M)$ is a group anti homomorphism and the mappings $r^g : M \to M$ and $r_x : G \to M$ satisfy $r^g(x) = r_x(g) = r(x,g) = x \cdot g$.

Definition 1.58. [10, p. 375] For any $X \in \mathfrak{g}$ we define the *fundamental vector field* $\zeta_X = \zeta_X^M \in \mathfrak{X}(M)$ by

$$\zeta_X(x) = T_e\ell^x(X) = T_{(e,x)}\ell(X, 0_x)$$

or alternatively for a right action by

$$\zeta_X(x) = T_e r_x(X) = T_{(x,e)} r(0_x, X).$$

1.6. Fiber bundles and vector bundles

After the introduction of manifolds and the action of Lie groups the next necessary definitions are related to fiber bundles.

Definition 1.59. [10, p. 375] A *(fiber) bundle* $(\pi : E \to M, S) = (E, M, \pi, S)$ consists of smooth (finite or infinite dimensional) manifolds E, M , S, and a smooth mapping $\pi : E \to M$. Furthermore, each $x \in M$ has an open neighbourhood U such that $E|_U := \pi^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diffeomorphism ψ:

$$\begin{tikzcd} E|_U \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times S \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

In this notation E is called *total space*, M is called *base space* or *basis*, π is a final surjective smooth mapping, called *projection*, and S is called *standard fiber*. The pair (U, ψ) is called a *fiber chart*.

Definition 1.60. [10, p. 376] A collection of fiber charts (U_α, ψ_α) such that the set (U_α) is an open cover of M, is called a *fiber bundle atlas*. If we fix such an atlas, then $\psi_\alpha \circ \psi_\beta^{-1}(x,s) = (x, \psi_{\alpha\beta}(x,s))$, where $\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \times S \to S$ is smooth and $\psi_{\alpha\beta}(x, .)$ is a diffeomorphism of S for each $x \in U_{\alpha\beta} := U_\alpha \cap U_\beta$. These mappings $\psi_{\alpha\beta}$ are called *transition functions* of the bundle. They satisfy the *cocycle condition*: $\check{\psi}_{\alpha\beta}(x) \circ \check{\psi}_{\beta\gamma}(x) = \check{\psi}_{\alpha\gamma}(x)$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $\psi_{\alpha\alpha}(x) = \mathrm{Id}_S$ for $x \in U_\alpha$. Therefore, the collection $(\psi_{\alpha\beta})$ is called a *cocycle of transition functions.*

For the special case where the fiber S is a vector space the fiber bundle becomes a vector bundle which is described in the following.

Definition 1.61. [10, p. 287] Let $p : E \to M$ be a smooth mapping between manifolds. By a *vector bundle chart* on (E, p, M) we mean a pair (U, ψ), where U is an open subset in M , and where $\psi : p^{-1}(U) \to U \times V$ is a fiber respecting diffeomorphism. In the finite dimensional case V is an $\mathbb{R}^n$, in the infinite dimensional case this can be generalized to V being a convenient vector space called the *standard fiber* or the *typical fiber*.

Two vector bundle charts (U_1, ψ_1) and (U_2, ψ_2) are called *compatible*, if $\psi_1 \circ \psi_2^{-1}$ is a fiber linear isomorphism, i.e., $(\psi_1 \circ \psi_2)(x, v) = (x, \psi_{1,2}(x)v)$ for some mapping $\psi_{1,2} : U_{1,2} := U_1 \cap U_2 \to GL(V)$. The mapping $\psi_{1,2}$ is then unique and smooth into $L(V, V)$, and it is called the transition function between the two vector bundle charts.

A *vector bundle atlas* $(U_\alpha, \psi_\alpha)_{\alpha\in A}$ for $p : E \to M$ is a set of pairwise compatible vector bundle charts (U_α, ψ_α) such that $(U_\alpha)_{\alpha\in A}$ is an open cover of M . Two vector bundle atlases are called *equivalent*, if their union is again a vector bundle atlas.

A *(smooth)* vector bundle $p : E \to M$ consists of manifolds E *(the total space)*, M *(the base)*, and a smooth mapping $p : E \to M$ *(the projection)* together with an equivalence class of vector bundle atlases.

Again we define morphisms between the just introduced structures.

Definition 1.62. [10, p. 289] Let $q : F \to N$ and $p : E \to M$ be vector bundles. A *vector bundle homomorphism* $\varphi : F \to E$ over Φ is a fiber respecting, fiber linear smooth mapping i.e., we require that $\varphi_x : F_x \to E_{\Phi(x)}$ is linear. We say that φ covers Φ, which turns out to be smooth. If φ is invertible, it is called a *vector bundle isomorphism*.

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{\Phi} & M \end{array}$$

Definition 1.63. [11, p. 93] A *vector subbundle* (F, p, M) of a vector bundle (E, p, M) is a vector bundle and a vector bundle homomorphism $\tau : F \to E$, which covers Id_M, such that $\tau_x : F_x \to E_x$ is a linear embedding for each $x \in M$.

Lemma 1.64. [11, p. 93] *Let $\varphi : (F, q, N) \to (E, p, M)$ be a vector bundle homomorphism between finite dimensional vector bundles such that* $\operatorname{rank}\left(\varphi_x : F_x \to E_{\Phi(x)}\right)$ *is locally constant in $x \in M$. Then* $\ker\varphi$, *given by* $(\ker\varphi)_x = \ker(\varphi_x)$, *is a vector subbundle of* (F, q, N).

Definition 1.65. [10, p. 377] Let (E, M, π, S) be a fiber bundle, and consider a smooth mapping $f : N \to M$. Let us consider the *pullback*

$$N \times_{(f,M,p)} E := \{(n, e) \in N \times E : f(n) = p(e)\};$$

we will denote it by f^*E. The following diagram sets up some further notation for it:

$$\begin{tikzcd} f^*E \arrow[r, "\pi^*f"] \arrow[d, "f^*\pi"] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Proposition 1.66. [10, p. 377] *The pullback construction introduced in Definition 1.65 satisfies that $(f^*E, f^*\pi, N, S)$ is a fiber bundle, and π^*f is a fiberwise diffeomorphism.*

Definition 1.67. [10, p. 293] For a fixed vector bundle $p : E \to M$ on each fiber $E_x := p^{-1}(x)$ for $x \in M$ there is a unique structure of a convenient vector space, induced by any vector bundle chart (U_α, ψ_α). So $0_x \in E_x$ is a special element, and $0 : M \to E \quad 0(x) := 0_x$, is a smooth mapping called the *zero section.*

Next we consider the kinematic tangent bundle of a vector bundle.

Lemma 1.68. [10, p. 292] *Let $p : E \to M$ be a vector bundle with fiber addition $+_E : E \times_M E \to E$ and fiber scalar multiplication $m_t^E : E \to E$. Then $\pi_E : TE \to E$, the tangent bundle of the manifold E, is itself a vector bundle, with fiber addition $+_{TE}$ and scalar multiplication m_t^{TE}.*

Proof. If $(U_\alpha, \psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V)_{\alpha \in A}$ is a vector bundle atlas for E, and if $(u_\alpha : U_\alpha \to u_\alpha(U_\alpha) \subseteq F)$ is a manifold atlas for M, then $(E|_{U_\alpha}, \psi'_\alpha)_{\alpha \in A}$ is an atlas for the manifold E, where

$$\psi'_\alpha := (u_\alpha \times \mathrm{Id}_V) \circ \psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V \to u_\alpha(U_\alpha) \times V \subseteq F \times V.$$

Hence, the family

$$(T(E|_{U_\alpha}), T\psi'_\alpha : T(E|_{U_\alpha}) \to T(u_\alpha(U_\alpha) \times V) = (u_\alpha(U_\alpha) \times V \times F \times V)_{\alpha \in A}$$

is the atlas describing the canonical vector bundle structure of $\pi_E : TE \to E$.

The transition functions are:

$$\begin{aligned} (\psi_\alpha \circ \psi_\beta^{-1})(x, v) &= (x, \psi_{\alpha\beta}(x)v) \\ (u_\alpha \circ u_\beta^{-1})(x) &= u_{\alpha\beta}(x) \\ (\psi'_\alpha \circ ((\psi'_\beta)^{-1})(x, v) &= (u_{\alpha\beta}(x), \psi_{\alpha\beta}(u_\beta^{-1}(x))v) \\ (T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(x, v, \xi, w) &= \\ = (u_{\alpha\beta}(x), \psi_{\alpha\beta}(u_\beta^{-1}(x))v, &d(u_{\alpha\beta})(x)\xi, (d(\psi_{\alpha\beta} \circ u_\beta^{-1})(x)\xi)v + \psi_{\alpha\beta}(u_\beta^{-1}(x))w). \end{aligned}$$

So we see that for fixed (x, v) the transition functions are linear in $(\xi, w) \in F \times V$. This describes the vector bundle structure of the tangent bundle $\pi_E : TE \to E$. $\square$

Lemma 1.69. [10, p. 292] *Let $p : E \to M$ be a vector bundle with the same mappings as defined in Lemma 1.68. Then $Tp : TE \to TM$ is a vector bundle too.*

Proof. Considering the proof of Lemma 1.68 it follows for fixed (x, ξ) that the transition functions of TE are also linear in $(v, w) \in V \times V$. This gives a vector bundle structure on $Tp : TE \to TM$. Its fiber addition will be denoted by

$$T(+_E) : T(E \times_M E) = TE \times_{TM} TE \to TE,$$

since it is the tangent mapping of $+_E$. Likewise, its scalar multiplication will be denoted by $T(m_t^E)$. One might say that the vector bundle structure on $Tp : TE \to TM$ is the derivative of the original one on E. $\square$

Definition 1.70. [10, p. 292] Let $p : E \to M$ be a vector bundle. The subbundle

$$\{X \in TE : Tp(X) = 0 \text{ in } TM\} = (Tp)^{-1}(0) \subseteq TE$$

is denoted by VE and is called the *vertical bundle over E*. With the notation introduced in Lemma 1.68, the local form of a vertical vector X is $T\psi'_\alpha(X) = (x, v, 0, w)$, so the transition functions look like

$$(T\psi'_\alpha \circ T(\psi'_\beta)^{-1})(x, v, 0, w) = (u_{\alpha\beta}(x), \psi_{\alpha\beta}(u_\beta^{-1}(x))v, 0, \psi_{\alpha\beta}(u_\beta^{-1}(x))w).$$

They are linear in $(v, w) \in V \times V$ for fixed x, so VE is a vector bundle over M. It coincides with $0^*(TE, Tp, TM)$, the pullback of the bundle $TE \to TM$ over the zero section.

Now we introduce sections of vector bundles and their properties.

Definition 1.71. [10, p. 294] A *section* u of $p : E \to M$ is a smooth mapping $u : M \to E$ with $p \circ u = \mathrm{Id}_M$. The support of the section u is the closure of the set $\{x \in M : u(x) \neq 0_x\}$ in M. The space of all smooth sections of the bundle $p : E \to M$ will be denoted as $\Gamma(E \to M)$ or $\Gamma(E)$ if there is no risk of confusion. Clearly this set is a vector space with fiber wise addition and scalar multiplication.

Remark 1.72. [10, p. 294] If $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is a vector bundle atlas for $p : E \to M$, then any smooth mapping

$$f_\alpha : U_\alpha \to V \qquad x \mapsto \psi_\alpha^{-1}(x, f_\alpha(x)),$$

where V is the standard fiber, defines a local section on U_α. If $(g_\alpha)_{\alpha \in A}$ is a partition of unity subordinated to (U_α), then a global section can be formed by:

$$x \mapsto \sum_\alpha g_\alpha(x) \cdot \psi_\alpha^{-1}(x, f_\alpha(x))$$

Next we want to show that $\Gamma(E \to M)$ has the structure of a convenient vector space. First we state a theorem that describes the embedding of a vector bundle into a trivial one.

Theorem 1.73. [10, p. 291] *For any vector bundle $p : E \to M$ with M smoothly regular there is a smooth vector bundle embedding into a trivial vector bundle over M with locally (over M) splitting image. If the fibers are Banach spaces, and M is smoothly paracompact then the fiber of the trivial bundle can be chosen as Banach space as well.*

A fiberwise short exact sequence of vector bundles over a smoothly paracompact manifold M which is locally splitting is even globally splitting.

With this theorem in mind, the convenient structure of $\Gamma(E \to M)$ can be defined via the following lemma.

Lemma 1.74. [10, p. 295] *If M is smoothly regular, choose a smooth closed embedding $E \to M \times F$ into a trivial vector bundle with fiber a convenient vector space F. Then $\Gamma(E \to M)$ can be considered as a closed linear subspace of $C^\infty(M, F)$, with the natural structure from Lemma 1.35.*

The space $\Gamma(E \to M)$ of sections of the vector bundle $p : E \to M$ with this structure satisfies the uniform boundedness principle with respect to the point evaluations $\mathrm{ev_x} : \Gamma(E \to M) \to E_x$ for all $x \in M$.

Remark 1.75. Analysing the special case $M = \mathbb{R}$, we see that the smooth sections of an arbitrary vector bundle with base space $\mathbb{R}$ form a convenient vector space. The next task is to show that by restricting to a compact interval $I \subset \mathbb{R}$ one gets a closed subspace.

1.7. Extension of smooth functions on compact intervals

This section is dedicated to the extension of smooth functions on the compact interval $I = [a, b]$ to smooth functions on $\mathbb{R}$ and the discussion of closed subspaces of $C^\infty(\mathbb{R}, \mathbb{R})$. For this the results in [15] are used together with the special case described in [10, p. 170].

Theorem 1.76. [15] *Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ then one defines the sets $S_+ = \mathbb{R}^n \times \{t > 0\}$ and $D_+ = \{f : f \in C^\infty(S_+, \mathbb{R}), f$ and all its derivatives have continuous limits as $t \to 0+\}$. The set D_+ has the topology of uniform convergence of each derivative on compact subsets of the closure of S_+ in $\mathbb{R}^{n+1}$ and $C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ has a corresponding topology.*

There is a continuous linear extension operator $E : D_+ \to C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ satisfying

$$Ef(x, t) = f(x, t) \text{ for } t > 0.$$

Lemma 1.77. [10, p. 170] *The subspace $\{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f(t) = 0 \ for \ t \leq 0\}$ of the Fréchet space $C^\infty(\mathbb{R}, \mathbb{R})$ is a direct summand.*

Because we want to show that $C^\infty(\mathbb{R},\mathbb{R})$ is the direct sum of $\{f \in C^\infty(\mathbb{R},\mathbb{R}) : f|_{\mathbb{R}_+} = 0\}$ and $C^\infty(\mathbb{R}_+,\mathbb{R})$ we discuss the following sequence:

$$\{f \in C^\infty(\mathbb{R},\mathbb{R}) : f|_{\mathbb{R}_+} = 0\} \hookrightarrow C^\infty(\mathbb{R},\mathbb{R}) \twoheadrightarrow C^\infty(\mathbb{R}_+,\mathbb{R})$$

By Theorem 1.76 for the case $n = 0$ we get a section $s : C^\infty(\mathbb{R}_+,\mathbb{R}) \to C^\infty(\mathbb{R},\mathbb{R})$. This sequence is exact in the category of locally convex spaces because the image of the embedding is the kernel of the quotient map to $C^\infty(\mathbb{R}_+,\mathbb{R})$.

Remark 1.78. [9, p. 101] Up to an isomorphism we have the following description of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\cong"] & B \arrow[r, "p"] \arrow[d, equal] & C \arrow[r] \arrow[d, "\cong"] & 0 \\
 & i(A) \arrow[r, hook] & B \arrow[r, two heads] & B/i(A) &
\end{tikzcd}$$

Lemma 1.79. [9, p. 101] *For a short exact sequence of locally convex vector spaces*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

with f a linear embedding and g a quotient map the following statements are equivalent:

1. *There is an isomorphism of locally convex vector spaces $\varphi : A \oplus C \to B$ such that the diagram below is commutative.*
2. *g has a continuous linear right inverse ρ.*
3. *f has a continuous linear left inverse λ.*

$$\begin{tikzcd}
 & A & & C \arrow[d, "\mathrm{Id}"] \arrow[dl, dotted, "\rho"'] & \\
0 \arrow[r] \arrow[d, equal] & A \arrow[r, "f"] \arrow[u, "\mathrm{Id}"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[ul, dotted, "\lambda"'] & C \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r, "\mathrm{inj}_1"] & A \oplus C \arrow[r, "\mathrm{pr}_2"] \arrow[u, dotted, "\varphi", "\cong"'] & C \arrow[r] & 0
\end{tikzcd}$$

Under these equivalent conditions the sequence is called splitting.

Proof. ($1 \implies 2$): For $c \in C$ define

$$\rho := \varphi \circ \mathrm{inj}_2 \qquad c \mapsto \varphi(0,c)$$

where inj_2 denotes the injection from C into the second component of $A \oplus C$. Of course this mapping is a right inverse of g: The application of g on $\varphi(0,c)$ gives again c because g is the same as $\mathrm{pr}_2 \circ \varphi^{-1}$.

($2 \implies 3$): Because $g \circ \rho = \mathrm{Id}_C$ we get that $g \circ \rho \circ g = g$. This implies that $g \circ (\mathrm{Id}_B - \rho \circ g) = g - g = 0$ and that $\mathrm{Id}_B - \rho \circ g$ has image in $\ker(g)$ which is by exactness the image of f. So $\mathrm{Id}_B - \rho \circ g$ factors to a morphism $\lambda : B \to A$ since f is an embedding. We get that

$$f \circ \lambda \circ f = (\mathrm{Id}_B - \rho \circ g) \circ f = f - 0 = f \circ \mathrm{Id}_A$$

which implies that $\lambda \circ f = \mathrm{Id}_A$. This shows that λ is the left inverse of f.

($3 \implies 1$): We define $\psi := (\lambda, g) : B \to A \oplus C$. Because $\mathrm{pr}_2 \circ \psi = g$ and $\psi \circ f = (\mathrm{Id}_A, 0) = \mathrm{inj}_1$ the diagram becomes commutative. By taking into account Remark 1.78 we get that ψ is an isomorphism which shows the first statement. □

Remark 1.80. If we consider a short exact sequence where

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

consist of continuous mappings f and g and we assume that one of the three statements in Lemma 1.79 is true for a continuous mapping, then we get that the other two mappings are continuous too. This follows because our constructions involved only compositions and factorizations of continuous mappings.

Remark 1.81. By Lemma 1.79 it follows that we get the direct sum:

$$C^\infty(\mathbb{R}, \mathbb{R}) \cong \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f|_{\mathbb{R}_+} = 0\} \oplus C^\infty(\mathbb{R}_+, \mathbb{R})$$

Because the sets $(-\infty, b]$ and $[a, \infty)$ are diffeomorphic to $\mathbb{R}_+$ we get the direct sums:

$$C^\infty(\mathbb{R}, \mathbb{R}) \cong \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f|_{(-\infty, b]} = 0\} \oplus C^\infty((-\infty, b], \mathbb{R})$$

$$C^\infty(\mathbb{R}, \mathbb{R}) \cong \{f \in C^\infty(\mathbb{R}, \mathbb{R}) : f|_{[a, \infty)} = 0\} \oplus C^\infty([a, \infty), \mathbb{R})$$

With that we get that $C^\infty([a, \infty), \mathbb{R})$ and $C^\infty((-\infty, b], \mathbb{R})$ are embedded as closed linear subspaces of $C^\infty(\mathbb{R}, \mathbb{R})$.

Because this construction can be done simultaneously at both ends of the interval we get that $C^\infty(I, \mathbb{R})$ is a closed linear subspace too.

1.8. Extensions of smooth sections of vector bundles over a compact interval

The next step is to generalize this result to vector bundles with finite dimensional fiber. Our aim is to show that the smooth sections of the vector bundle $E \to I = [a, b]$ with typical fiber $F = \mathbb{R}^n$ can be extended to smooth sections of $G \to \mathbb{R}$ again with typical fiber $\mathbb{R}^n$. We select an open subinterval $V_a = [a, c)$ of I which contains the left boundary a and there is a trivialization of the bundle over V_a of the form $E|_{V_a} \cong V_a \times F$. The same can be done for an open subinterval $V_b = (d, b]$ of I which contains the right boundary b and there is a trivialization of the bundle over V_b of the form $E|_{V_b} \cong V_b \times F$.

For the extension we define for points outside of the interval I as typical fiber again F and get the trivializations $G|_{(-\infty,c)} \cong (-\infty, c) \times F$ and $G|_{(d,\infty)} \cong (d, \infty) \times F$.

The results of the previous section are applicable for each component of the typical fiber. So the vector bundle $E \to I$ can be viewed as a restriction of a greater vector bundle $G \to \mathbb{R}$ with typical fiber F. By Corollary A.37 both bundles are trivial.

The space of sections of the bundle $G \to \mathbb{R}$ forms a convenient vector space by Lemma 1.74 and $\Gamma(E \to I)$ is a closed subspace which will be used as a modelling space for infinite dimensional manifolds later on.

2. Path bundles

2.1. Construction of the manifold of paths

This section describes the construction of the manifold of paths corresponding to a given Riemannian manifold. The main part of this construction is contained in the theorem about the infinite dimensional manifold of all smooth mappings between smooth finite dimensional manifolds. Because this theorem is essential for the whole thesis, in the following not only the result is stated, but also the proof, which gives a major insight in the way this infinite dimensional manifold is constructed.

We start with preliminary results reformulated for a compact interval I. We require the notion of smooth curves in spaces of smooth mappings, in particular $C^\infty(I, N)$.

Lemma 2.1. [10, p. 299] *For a smooth vector bundle $p : E \to M$ a curve $c : \mathbb{R} \to \Gamma(E \to M)$ is smooth if and only if $c^\wedge : \mathbb{R} \times M \to E$ is smooth.*

Corollary 2.2. [10, p. 300] *Let $p : E \to I$ and $p' : E' \to I$ be smooth vector bundles with a compact interval I as base manifold (with boundary). Let $W \subseteq E$ be an open subset, and let $f : W \to E$ be a fiber respecting smooth (nonlinear) mapping. Then*

$$\Gamma(W \to I) := \{s \in \Gamma(E \to I) : s(I) \subseteq W\},$$

is open in the convenient vector space $\Gamma(E \to I)$.

The mapping $f^* : \Gamma(W \to I) \to \Gamma(E \to I)$ is smooth.

The following lemma was stated for a more general case in [10, p. 442]. Together with Theorem 1.76 we get a simplified version of that lemma on the compact interval I.

Lemma 2.3. [10, p. 442] *Let I be a compact interval and N be smooth finite dimensional manifold. Then the smooth curves c in $C^\infty(I, N)$ correspond exactly to the smooth mappings $c^\wedge \in C^\infty(\mathbb{R} \times I, N)$*

The theorem about the infinite dimensional manifold of all smooth mappings between smooth finite dimensional manifolds is reformulated for the case of a compact interval I.

Theorem 2.4. [10, p. 439] *Let I be a compact interval and N a smooth finite dimensional manifold. Then the space $C^\infty(I, N)$ of all smooth mappings from I to N is a smooth manifold, modelled on spaces $\Gamma(\gamma^* TN \to I)$ of smooth sections of pullback bundles along $\gamma : I \to N$ over I. We will call $C^\infty(I, N)$ the path manifold or manifold of paths to N.*

Proof. First choose a smooth Riemannian metric on N. Let $\exp : TN \supseteq U \to N$ be the smooth exponential mapping of this Riemannian metric, defined on a suitable open neighbourhood of the zero section. We may assume that U is chosen such that $(\pi_N, \exp) : U \to N \times N$ is a smooth diffeomorphism onto an open neighbourhood V of the diagonal.

For $\gamma \in C^\infty(I, N)$ we consider the pullback vector bundle $I \times_N TN = \gamma^* TN$.

$$\begin{tikzcd}
\gamma^*TN \arrow[r, "\pi_N^*\gamma"] \arrow[d, "\gamma^*\pi_N"] & TN \arrow[d, "\pi_N"] \\
I \arrow[r, "\gamma"] & N
\end{tikzcd}$$

Now define
$$S(\gamma) := \{\mu \in C^\infty(I, N) \,:\, (\gamma(t), \mu(t)) \in V \text{ for all } t \in I\}$$
and consider the mappings
$$\Phi_\gamma : S(\gamma) \to \Gamma\left(\gamma^*TN \to I\right)$$
$$\Phi_\gamma(\mu)(t) = \left(t, \exp_{\gamma(t)}^{-1}(\mu(t))\right) = \left(t, ((\pi_N, \exp)^{-1} \circ (\gamma, \mu))(t)\right)$$
Then Φ_γ is a bijective mapping from $S(\gamma)$ onto the set
$$\{s \in \Gamma\left(\gamma^*TN \to I\right) \,:\, s(I) \subseteq \gamma^*U = (\pi_N^*\gamma)^{-1}(U)\},$$
whose inverse is given by
$$\Phi_\gamma^{-1}(s) = \exp \circ (\pi_N^*\gamma) \circ s,$$
where we view $U \to N$ as a fiber bundle. The set $\Phi_\gamma(S(\gamma))$ is open in $\Gamma\left(\gamma^*TN \to I\right)$ by Corollary 2.2.

The spaces $\Gamma\left(\gamma^*TN \to I\right)$ of smooth sections of pullback bundles along $\gamma : I \to N$ are convenient vector spaces by Section 1.8.

Now we consider the atlas $(S_\gamma, \Phi_\gamma)_{\gamma \in C^\infty(I,N)}$ for $C^\infty(I, N)$. Its chart change mappings are given for
$$s \in \Phi_\mu(S(\gamma) \cap S(\mu)) \subseteq \Gamma\left(\gamma^*TN \to I\right)$$
by
$$(\Phi_\gamma \circ \Phi_\mu^{-1})(s) = (\mathrm{Id}_I, (\pi_N, \exp)^{-1} \circ (\gamma, \exp \circ (\pi_N^*\mu) \circ s)) = (\tau_\gamma^{-1} \circ \tau_\mu)_*(s),$$
where τ_μ is the smooth diffeomorphism
$$\tau_\mu : \mu^*TN \supseteq \mu^*U \to (\mu \times \mathrm{Id}_N)^{-1}(V) \subseteq I \times N \qquad \tau_\mu(t, Y_{\mu(t)}) := (t, \exp_{\mu(t)}(Y_{\mu(t)}))$$
which is fiber respecting over I.

Smooth curves in $\Gamma\left(\gamma^*TN \to I\right)$ correspond by Lemma 2.1 to smooth mappings from

$\mathbb{R} \times I \to \gamma^*TN$ which correspond to smooth sections of the bundle $\mathrm{pr}_2^* \gamma^*TN \to \mathbb{R} \times I$.

$$\begin{tikzcd}
\mathrm{pr}_2^* \gamma^*TN \arrow[r, "(\gamma^*\pi_N)^* \mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_2^* \gamma^*\pi_N"] & \gamma^*TN \arrow[r, "\pi_N^*\gamma"] \arrow[d, "\gamma^*\pi_N"] & TN \arrow[d, "\pi_N"] \\
\mathbb{R} \times I \arrow[r, "\mathrm{pr}_2"] & I \arrow[r, "\gamma"] & N
\end{tikzcd}$$

The chart change $\Phi_\gamma \circ \Phi_\mu^{-1} = (\tau_\gamma \circ \tau_\mu)_*$ is defined on an open subset and it is also smooth by Corollary 2.2.

Finally, the natural topology on $C^\infty(I, N)$ is the identification topology from this atlas (with the c^∞-topologies on the modeling spaces), which is finer than the compact-open topology and thus Hausdorff.

The equation $\Phi_\gamma \circ \Phi_\mu^{-1} = (\tau_\gamma \circ \tau_\mu)_*$ shows that the smooth structure does not depend on the choice of the smooth Riemannian metric on N. □

Remark 2.5. [10, p. 442] For a compact interval I and a convenient vector space E the smooth manifold $C^\infty(I, E)$ is diffeomorphic to the convenient vector space $C^\infty(I, E)$, which is a special case for a trivial bundle with finite dimensional base. Throughout this thesis we will not distinguish between the two notions of $C^\infty(I, E)$.

2.2. The construction of the path bundle

The construction of the manifold of paths in the previous section can be accomplished for the base and total space of a given principal fiber bundle. The aim of this section is to show that the resulting manifolds together with the path Lie group constructed by the same procedure from the structure group of the finite dimensional principal fiber bundle form again a principal fiber bundle which we will call path bundle.

Lemma 2.6. *Let $f : M \to N$ be a smooth mapping between finite dimensional manifolds. Then the mapping*

$$f_* : C^\infty(I, M) \to C^\infty(I, N) \qquad \gamma \mapsto f \circ \gamma$$

is smooth.

Proof. The smoothness of this mapping is shown by the property that smooth curves are mapped to smooth curves (see Definition 1.20). This means that smooth curves $c : \mathbb{R} \to C^\infty(I, M)$ have to be mapped to smooth curves in $C^\infty(I, N)$.

Lemma 2.3 shows that such smooth curves c correspond to smooth mappings

$$c^\wedge \in C^\infty(\mathbb{R} \times I, M).$$

The composition with the finite dimensional smooth mapping $f : M \to N$ gives an element of $C^\infty(\mathbb{R} \times I, N)$.

$$\mathbb{R} \times I \xrightarrow{c^\wedge} M \xrightarrow{f} N$$

This shows the smoothness of $(f \circ c)^\wedge = f \circ c^\wedge$ and therefore the smoothness of f_*. $\square$

The following proposition was stated in [2, p. 204]. The proof has been changed and is given now in terms of convenient calculus.

Proposition 2.7. *Let $\pi : P \to M$ be a principal fiber bundle with structure group G and let $r : C^\infty(I, P) \times C^\infty(I, G) \to C^\infty(I, P)$ be defined by*

$$r(\mu_P, a)(t) := r_{\text{fin}}\,(\mu_P(t), a(t))$$

for $\mu_P \in C^\infty(I, P)$, $a \in C^\infty(I, G)$ and $t \in I$ where r_{fin} denotes the finite dimensional right action of G on P. Then r is smooth.

Proof. First observe that the domain of r may also be viewed as the space $C^\infty(I, P \times G)$. The mapping r is nothing else than $r_{\text{fin}\,*}$ and is therefore smooth by Lemma 2.6. $\square$

The final result of this section will be that for a given principal fiber bundle (P, M, π, G) the tuple $(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$ forms again a principal fiber bundle, we will call it the path bundle of (P, M, π, G).

We start by showing the smoothness of the projection where we keep the already introduced notation in mind. The following lemma, originally a part of the proof of the theorem in [2, p. 205], is now shown by arguments introduced in [10].

Lemma 2.8. *The mapping $\pi_* : C^\infty(I, P) \to C^\infty(I, M)$ defined by $\pi_*(\mu_P) = \pi \circ \mu_P$ is smooth.*

Proof. Again the smoothness of this mapping is shown by Lemma 2.6. $\square$

Theorem 2.9. [2, p. 205] *Let (P, M, π, G) be a principal fiber bundle. The tuple*

$$(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$$

is a principal fiber bundle consisting of C^∞-manifolds.

Proof. Lemma 2.8 shows that the mapping $\pi_* : \mu_P \mapsto \pi \circ \mu_P$ is smooth and from Proposition 2.7 it follows that the action of $C^\infty(I, G)$ on the total space $C^\infty(I, P)$ is smooth. The fibers of π_* are orbits under $C^\infty(I, G)$ because they are pointwise orbits under G.

To complete the proof it is necessary to show that $(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$ is smoothly locally trivial. This is done by showing that for each $\gamma \in C^\infty(I, M)$ there exists a neighbourhood $S(\gamma)$ such that $\pi_*^{-1}(S(\gamma))$ is smoothly trivial.

In Theorem 2.4 we used an open neighbourhood U of the zero section in the bundle $TM \to M$ for the definition of charts of the manifold $C^\infty(I, M)$ as bijective mappings to the set

$$\{s \in \Gamma(\gamma^* TM \to I) \,:\, s(I) \subseteq \gamma^* U = (\pi_M^* \gamma)^{-1}(U)\}.$$

It was also mentioned that $U \to M$ is a fiber bundle.

Now we want to consider again the bundle $\gamma^* U \to I$. By Corollary A.37 we know that this bundle is smoothly trivial and therefore diffeomorphic to a bundle $I \times O \to I$. Because we chose U to be an open neighbourhood of the zero section in $TM \to M$ we can restrict it to a smaller neighbourhood such that O becomes smoothly contractible. In [2, p. 203] the charts of $C^\infty(I, M)$ where already chosen in such a way that this condition was satisfied.

Now we apply the exponential mapping to this trivial bundle and we get a smoothly contractible open neighbourhood $N(\gamma)$ of the image of $I \ni t \mapsto (t, \gamma(t))$ in $I \times M$.

Taking into account the principal fiber bundle $(I \times P, I \times M, \mathrm{Id}_I \times \pi, G)$, the restriction to $N(\gamma)$ yields the principal fiber bundle $\big((\mathrm{Id}_I \times \pi)^{-1}(N(\gamma)), N(\gamma), \mathrm{Id}_I \times \pi, G\big)$ with a smoothly contractible base space.

So by Remark A.35 the bundle is trivial and there exists a smooth section $\bar{s} : N(\gamma) \to (\mathrm{Id}_I \times \pi)^{-1}(N(\gamma))$ which maps $N(\gamma)$ into a subset of $I \times P$ such that $(\mathrm{Id}_I \times \pi) \circ \bar{s} = \mathrm{Id}_{N(\gamma)}$.

This section can be rewritten as $\bar{s}(t, m) = (t, s_t(m))$ for $(t, m) \in N(\gamma)$ where the mapping s_t is defined as a local section $s_t : N(\gamma)_t \to P$ for each $t \in I$.

Together with the projection $\pi_P : I \times P \to P$ we define the mapping:

$$\grave{s} : C^\infty(I, M) \supseteq S(\gamma) \to \pi_*^{-1}(S(\gamma)) \subseteq C^\infty(I, P) \qquad \grave{s}(\mu) := \pi_P \circ \bar{s} \circ \tilde{\mu}$$

where $\tilde{\mu} : t \mapsto (t, \mu(t))$ denotes a smooth section of $I \times M \to I$.

Because forming $\tilde{\mu}$ is trivially smooth by the inclusion $C^\infty(I, P) \hookrightarrow C^\infty(I, I \times P)$, we have to focus on the composition with the finite dimensional smooth mappings $\bar{s}$ and π_P. But this is nothing else than forming $\bar{s}_*$ and π_{P*} which is smooth by Lemma 2.6.

The last step of this proof is to show that the trivialization is smooth and has a smooth inverse.

The mapping

$$S(\gamma) \times C^\infty(I, G) \to \pi_*^{-1}(S(\gamma)) \qquad (\mu, a) \mapsto r(\grave{s}(\mu), a)$$

is smooth because of the smoothness of $\grave{s}$, the smoothness of r given by Proposition 2.7 and the smoothness of their composition given by Corollary 1.22.

Let $f : (\mathrm{Id}_I \times \pi)^{-1}(N(\gamma)) \to G$ be the unique mapping defined by the implicit relation

$$u = \bar{s}\left((\mathrm{Id}_I \times \pi)(u)\right) \cdot f(u)$$

for $u \in (\mathrm{Id}_I \times \pi)^{-1}(N(\gamma))$. This gives the mapping:

$$\psi : (\mathrm{Id}_I \times \pi)^{-1}(N(\gamma)) \to N(\gamma) \times G \qquad \psi(u) := ((\mathrm{Id}_I \times \pi)(u), f(u))$$

It is smooth because of the smoothness of $(\mathrm{Id}_I \times \pi)$ and the smoothness of $\bar{s}$ which implies the smoothness of f.

Now the inverse of the trivialization is given as

$$\pi_*^{-1}(S(\gamma)) \to S(\gamma) \times C^\infty(I, G) \qquad \mu_P \mapsto (\pi_*(\mu_P), f \circ \tilde{\mu}_P)$$

where the smoothness of π_* is already shown in Lemma 2.8. The second component is the mapping:

$$C^\infty(I, P) \supset \pi_*^{-1}(S(\gamma)) \longrightarrow \Gamma(I \times P \to I) \subset C^\infty(I, I \times P) \xrightarrow{f_*} C^\infty(I, G)$$

The only non trivial step is to prove that the composition with the mapping f which is defined on a finite dimensional manifold is a smooth mapping. But this is just the formation of f_* which is smooth by Lemma 2.6.

So we get that the tuple

$$(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$$

is a principal fiber bundle. $\square$

2.3. Submanifolds of the total space of the path bundle

In the last section we discussed the construction of the principal fiber bundle $(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$. Now we want to give examples for submanifolds of its total space $C^\infty(I, P)$. To simplify notation let $I = [0, 1]$.

Definition 2.10. Let x_0 be an element of M, then the set $C^\infty(I, M)_{x_0}$ denotes the set of paths in M starting at the point x_0. As a further restriction the set $C^\infty(I, M)_{x_0 \to x_0}$ denotes the set of all closed paths beginning and ending at x_0, so called *loops at* x_0.

Now define $\mathrm{Imm}\,(I, M)$ as the set of all $\gamma \in C^\infty(I, M)$ such that $\dot{\gamma}(t) \neq 0$ for all $t \in I$. Combinations of these definitions give sets of the sort:

$$\mathrm{Imm}\,(I, M)_{x_0} := \{\gamma \in C^\infty(I, M) : \dot{\gamma}(t) \neq 0\, \forall t \in I, \gamma(0) = x_0\}$$

With the projection π of a given principal fiber bundle (P, M, π, G) one gets via the inverse image of the projection mapping in the path bundle subsets of the path space $C^\infty(I, P)$

where $u_0 \in \pi^{-1}(x_0)$, for example:

$$C^\infty(I, P)_{u_0} = \left(\pi_*^{-1}(C^\infty(I, M))\right)_{u_0} = \left(\pi_*^{-1}(C^\infty(I, M)_{x_0})\right)_{u_0}$$

First we prove a lemma concerning the subset $\mathrm{Imm}\,(I, M)$ using methods from [10].

Lemma 2.11. *The set* $\mathrm{Imm}\,(I, M)$ *is an open submanifold of* $C^\infty(I, M)$.

Proof. Let $c : \mathbb{R} \to C^\infty(I, M)$ denote a smooth curve in the path space of the finite dimensional manifold M. Then the corresponding mapping $c^\wedge : \mathbb{R} \times I \to M$ is smooth too (see Lemma 2.3).

Now let $c(0)$ be an element of $\mathrm{Imm}\,(I, M)$. This is equivalent to:

$$\frac{\partial}{\partial t} c^\wedge(0, t) \neq 0 \quad \forall t \in I$$

Because $c^\wedge$ is smooth we find for each $t_0 \in I$ a $\delta_{t_0} > 0$ such that for all $|s| < \delta_{t_0}$ and $|t - t_0| < \delta_{t_0}$ we get that

$$\frac{\partial}{\partial t} c^\wedge(s, t) \neq 0.$$

So by compactness it follows that there exists a neighbourhood of $c(0)$ in $C^\infty(I, M)$ such that all elements of this neighbourhood are contained in $\mathrm{Imm}\,(I, M)$. Because $c(0)$ was chosen arbitrary we get that $\mathrm{Imm}\,(I, M)$ is open in $C^\infty(I, M)$. It is an open submanifold because we get the charts by restriction of those who are given by $C^\infty(I, M)$. □

The proof of the following proposition was changed compared to [2, p. 206] and is now a simple consequence of Lemma 2.11.

Proposition 2.12. [2, p. 206] *The set* $\pi_*^{-1}(\mathrm{Imm}\,(I, M))$ *of all paths of* P *which project to elements of* $\mathrm{Imm}\,(I, M)$ *is an open submanifold of the manifold* $C^\infty(I, P)$.

Proof. We saw in Lemma 2.11 that $\mathrm{Imm}\,(I, M)$ is an open submanifold of $C^\infty(I, M)$. Because π_* is smooth by Lemma 2.8 we get that the inverse image $\pi_*^{-1}(\mathrm{Imm}\,(I, M))$ is open in $C^\infty(I, P)$. □

Now we turn to the discussion of special splitting submanifolds by fixing the starting point of the paths under consideration.

Lemma 2.13. [2, p. 208] *Let* Q *be any finite dimensional manifold and* $q_0 \in Q$. *Then the space* $C^\infty(I, Q)_{q_0}$ *is a splitting submanifold of* $C^\infty(I, Q)$.

Proof. In Theorem 2.4 we saw that the manifold $C^\infty(I, Q)$ is modelled on the convenient vector spaces $\Gamma\left(\gamma^*TQ \to I\right)$ where $\gamma : I \to Q$ and we have the following pullback diagram:

$$\begin{tikzcd} f^*TQ \arrow[r, "\pi^*\gamma"] \arrow[d, "\gamma^*\pi"] & TQ \arrow[d, "\pi"] \\ I \arrow[r, "\gamma"] & Q \end{tikzcd}$$

Let $\gamma_0 \in C^\infty(I,Q)_{q_0} \subseteq C^\infty(I,Q)$ and let $\Gamma\left(\gamma_0^*TQ \to I\right)_0 \ni v$ denote the space of sections of the pullback of the tangent bundle along γ_0 satisfying $v(0) = 0$. Then the inclusion ι of $\Gamma\left(\gamma_0^*TQ \to I\right)_0 \hookrightarrow \Gamma\left(\gamma_0^*TQ \to I\right)$ is clearly an embedding.

Next we consider the evaluation $\mathrm{ev}_0 : \Gamma\left(\gamma_0^*TQ \to I\right) \to T_{q_0}Q$ which assigns each element $\sigma \in \Gamma\left(\gamma_0^*TQ \to I\right)$ its value at the point $0 \in I$ by $\sigma \mapsto \sigma(0)$. By Corollary 1.22 the mapping ev_0 is smooth which implies continuity and it is clearly surjective.

The kernel $\ker(\mathrm{ev}_0)$ consists of the elements satisfying $\sigma(0) = 0$ which shows that it is equal to $\Gamma\left(\gamma_0^*TQ \to I\right)_0$. So we get the following short exact sequence:

$$0 \longrightarrow \Gamma\left(\gamma_0^*TQ \to I\right)_0 \xrightarrow{\quad\iota\quad} \Gamma\left(\gamma_0^*TQ \to I\right) \xrightarrow{\quad\mathrm{ev}_0\quad} T_{q_0}Q \longrightarrow 0$$

The evaluation ev_0 has a continuous right inverse. It is given by the extension of the value in the fiber over q_0 to a section over I. So it follows by Lemma 1.79 that

$$\Gamma\left(\gamma_0^*TQ \to I\right) \cong \Gamma\left(\gamma_0^*TQ \to I\right)_0 \oplus T_{q_0}Q$$

and $\Gamma\left(\gamma_0^*TQ \to I\right)_0$ is a closed subspace of $\Gamma\left(\gamma_0^*TQ \to I\right)$.

Now we restrict the charts of the manifold $C^\infty(I,Q)$ to the set $C^\infty(I,Q)_{q_0}$. Because their images lie in $\Gamma\left(\gamma_0^*TQ \to I\right)_0$ we get by Definition 1.34 that $C^\infty(I,Q)_{q_0}$ is a splitting submanifold modelled on the convenient vector space $\Gamma\left(\gamma_0^*TQ \to I\right)_0$. □

This lemma can be used to prove that special infinite dimensional manifolds of paths which start or end in a point are submanifolds of $C^\infty(I,P)$.

Lemma 2.14. [2, p. 208] *The set* $\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M))\right)_{u_0} = \left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0})\right)_{u_0}$ *is a splitting submanifold of* $\pi_*^{-1}(\mathrm{Imm}\,(I,M))$.

Proof. By Lemma 2.13 $C^\infty(I,P)_{u_0}$ is a splitting submanifold of $C^\infty(I,P)$. We can write $\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M))\right)_{u_0}$ as the intersection $\pi_*^{-1}(\mathrm{Imm}\,(I,M)) \cap C^\infty(I,P)_{u_0}$.

$$\begin{tikzcd} & C^\infty(I,P) & \\ \pi_*^{-1}(\mathrm{Imm}\,(I,M)) \arrow[ur, hook] & & C^\infty(I,P)_{u_0} \arrow[ul, hook'] \\ & \left(\pi_*^{-1}(\mathrm{Imm}\,(I,M))\right)_{u_0} \arrow[ul, hook'] \arrow[ur, hook] & \end{tikzcd}$$

□

The arguments used in Lemma 2.14 are applicable similarly to other subsets of $C^\infty(I,P)$. Examples are seen in the formulation of the following corollary where $x_0 \to x_0$ denotes paths starting and ending in the same point.

Corollary 2.15. [2, p. 207] *The following sets are open submanifolds of the corresponding path spaces:*

$$\begin{aligned}
\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0\to x_0})\right)_{u_0} &\subseteq \left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0} \\
\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0\to x_0}) &\subseteq \pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0}) \\
\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0})\right)_{u_0} &\subseteq \left(\pi_*^{-1}(C^\infty(I,M)_{x_0})\right)_{u_0} \\
\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0}) &\subseteq \pi_*^{-1}(C^\infty(I,M)_{x_0})
\end{aligned}$$

Proof. This follows by the same argument as in the proof of Proposition 2.12. We consider again the inverse images of π_*. $\square$

As a summary of the previous results we state the following proposition:

Proposition 2.16. [2, p. 208] *Each path space in the following diagram is a splitting submanifold of any other path space which contains it.*

$$\begin{tikzcd}
\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0\to x_0})\right)_{u_0} \arrow[r, hook] \arrow[d, hook] & \pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0\to x_0}) \arrow[d, hook] & \\
\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0})\right)_{u_0} \arrow[r, hook] & \pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0}) \arrow[r, hook] & \pi_*^{-1}(\mathrm{Imm}\,(I,M))
\end{tikzcd}$$

The same holds for the diagram:

$$\begin{tikzcd}
& \left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0} \arrow[r, hook] \arrow[d, hook] & \pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0}) \arrow[d, hook] & \\
C^\infty(I,P)_{u_0} = & \left(\pi_*^{-1}(C^\infty(I,M)_{x_0})\right)_{u_0} \arrow[r, hook] & \pi_*^{-1}(C^\infty(I,M)_{x_0}) \arrow[r, hook] & \pi_*^{-1}(C^\infty(I,M)) = C^\infty(I,P)
\end{tikzcd}$$

Each path space in the first diagram is an open submanifold of the corresponding path space in the second diagram.

Proof. This proposition follows by the application of the steps introduced in Proposition 2.12, Lemma 2.14 and Corollary 2.15. $\square$

2.4. Subbundles of the path bundle

Definition 2.17. Let $C^\infty(I,G)$ denote the Lie group of smooth mappings from I to G then $C^\infty(I,G)_0$ denotes the subgroup of all a such that $a(0)$ is the identity of G.

Proposition 2.18. [2, p. 209] *Let (P, M, π, G) be an arbitrary principal fiber bundle and $u_0 \in \pi^{-1}(x_0)$ for $x_0 \in M$. The bundle*

$$\left(\left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0}, \mathrm{Imm}\,(I, M)_{x_0}, \pi_{*0}, C^\infty(I, G)_0\right)$$

with $\pi_{*0} := \pi_*|_{\left(\pi_*^{-1}(\mathrm{Imm}(I,M))\right)_{u_0}}$ is a principal fiber bundle.

Proof. The sets $\mathrm{Imm}\,(I, M)$ and $\pi_*^{-1}(\mathrm{Imm}\,(I, M))$ are open submanifolds of $C^\infty(I, M)$ and $\pi_*^{-1}(C^\infty(I, M)) = C^\infty(I, P)$ respectively. This follows by Proposition 2.12 and Lemma 2.11 in the case of $\mathrm{Imm}\,(I, M)$. By Lemma 2.14 the sets $\left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0}$ and $\mathrm{Imm}\,(I, M)_{x_0}$ are splitting submanifolds. So with some slight modifications of the proof of Theorem 2.9 the tuple $\left(\pi_*^{-1}(\mathrm{Imm}\,(I, M)), \mathrm{Imm}\,(I, M), \pi_{*0}, C^\infty(I, G)\right)$ becomes a principal fiber bundle.

The mapping π_{*0} is smooth as the restriction of π_* which is smooth by Lemma 2.8.

By the restriction of the right action in Proposition 2.7 the mapping

$$r : \pi_*^{-1}(\mathrm{Imm}\,(I, M)) \times C^\infty(I, G) \to \pi_*^{-1}(\mathrm{Imm}\,(I, M))$$

is smooth and $r\left(\left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0} \times C^\infty(I, G)_0\right) \subseteq \left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0}$. This induces an action

$$r_0 : \left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0} \times C^\infty(I, G)_0 \to \left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0}$$

The last step is to show that $\pi_{*0} : \left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0} \to \mathrm{Imm}\,(I, M)_{x_0}$ is locally trivial. This follows analogues to the proof of Theorem 2.9 by restricting the sets and mappings under consideration to the ones in this theorem. □

Proposition 2.19. [2, p. 209] *Let (P, M, π, G) be an arbitrary principal fiber bundle and $u_0 \in \pi^{-1}(x_0)$ for $x_0 \in M$. Each of the path spaces of Proposition 2.16 defines a principal fiber bundle over the corresponding set of paths in M. The group of each bundle is either $C^\infty(I, G)$ or $C^\infty(I, G)_0$ depending on whether or not the bundle consists of paths which begin at u_0.*

Proof. The proofs for the different principal fiber bundles are similar to the proof of Proposition 2.18. They differ by the appropriate choice of restrictions of the bundle mapping and the group action. □

The following proposition shows how the triviality of the bundle $P \to M$ corresponds to the triviality of $C^\infty(I, P) \to C^\infty(I, M)$.

Proposition 2.20. [2, p. 210] *There is a smooth global section of the bundle $C^\infty(I, P) \to C^\infty(I, M)$ which takes constant mappings of $C^\infty(I, M)$ to constant mappings of $C^\infty(I, P)$ if and only if $P \to M$ is smoothly trivial.*

Proof. Any global section σ of $P \to M$ induces a global section s of $C^\infty(I,P) \to C^\infty(I,M)$ by:

$$s(f) := \sigma \circ f \qquad \text{for } f \in C^\infty(I,M)$$

Then s clearly carries constant mappings to constant mappings.

Any finite dimensional manifold Q can be identified with the constant mappings in $C^\infty(I,Q)$. It is even a submanifold of $C^\infty(I,Q)$: In the proof of Lemma 2.13 we saw the decomposition of the modelling space into $\mathbb{R}^{\dim(Q)}$ and the rest. Constant curves are uniquely defined by their starting point. The neighbourhoods S of constant curves in the set of constant curves are just the points. So we get the finite dimensional manifold as a restriction of the chart mappings of $C^\infty(I,Q)$.

It follows that the restriction of $s : C^\infty(I,M) \to C^\infty(I,P)$ to $M \subseteq C^\infty(I,M)$ is a global smooth section of the bundle $P \to M$. $\square$

3. Connections and horizontal lifts

3.1. Differential forms

In the following we introduce differential forms. They have a vital part in the discussion of connections and curvature.

Definition 3.1. [10, p. 352] Let M be a C^∞-manifold. The space of all *differential forms of order k* on a manifold M consists of the smooth sections of the bundle

$$L^k_{\text{alt}}\left(TM, M \times \mathbb{R}\right) \to M$$

and will be denoted by $\Omega^k(M)$. Here L^k_{alt} is the space of bounded k-linear alternating mappings. Then $\Omega^k(M)$ carries the structure of a convenient vector space induced by the embedding:

$$\Omega^k(M) \to \prod_\alpha C^\infty\left(U_\alpha, L^k_{\text{alt}}(E, \mathbb{R})\right) \qquad s \mapsto \mathrm{pr}_2 \circ \psi_\alpha \circ (s|_{U_\alpha})$$

where $(U_\alpha, u_\alpha : U_\alpha \to E)_{\alpha \in A}$ is a smooth atlas for the manifold M and ψ_α are the vector bundle charts of the bundle $L^k_{\text{alt}}\left(TM, M \times \mathbb{R}\right) \to M$ induced by the charts of M. This situation is illustrated in the following diagram.

$$\begin{tikzcd}
L^k_{\text{alt}}\left(TM, M \times \mathbb{R}\right)|_{U_\alpha} \arrow[r, "\psi_\alpha"] & L^k_{\text{alt}}\left(U_\alpha \times E, U_\alpha \times \mathbb{R}\right) \arrow[r, "\cong"] & U_\alpha \times C^\infty\left(U_\alpha, L^k_{\text{alt}}\left(E, \mathbb{R}\right)\right) \arrow[d, "\mathrm{pr}_2"] \\
M \supseteq U_\alpha \arrow[u, "s|_{U_\alpha}"] & & C^\infty\left(U_\alpha, L^k_{\text{alt}}\left(E, \mathbb{R}\right)\right)
\end{tikzcd}$$

In a similar way the set

$$\Omega^k\left(M, V\right) := \Gamma\left(L^k_{\text{alt}}(TM, M \times V) \to M\right)$$

denotes the *space of differential forms of order k with values in a convenient vector space V* and

$$\Omega^k\left(M, E\right) := \Gamma\left(L^k_{\text{alt}}(TM, E) \to M\right)$$

the *space of differential forms of order k with values in a vector bundle $p : E \to M$.*

By omitting the upper index k one arrives at the notation for the graded algebras of all differential forms in each of the cases above respectively.

Next we define common mappings on the spaces of differential forms. We choose a local description in charts.

Definition 3.2. [10, p. 342, 348, 352] Let $U \subseteq E$ be c^∞-open in a convenient vector space E and let $\omega \in C^\infty\left(U, L^k_{\text{alt}}(E, \mathbb{R})\right)$ be a kinematic k-form on $U \ni x$. We define (by abuse of notation) the exterior derivative $d\omega \in C^\infty(U, L^{k+1}_{\text{alt}}(E, \mathbb{R}))$ as the skew symmetrization of the derivative $d\omega(x) : E \to L^k_{\text{alt}}(E, \mathbb{R})$, i.e.

$$
\begin{aligned}
(d\omega)(x)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i d\omega(x)(X_i) \left(X_0, \ldots, \hat{X}_i, \ldots, X_k\right) + \\
+ \sum_{i<j} (-1)^i d(\omega(\ )(X_0, \ldots, \hat{X}_i, \ldots, X_k))(x)\,(X_i)
\end{aligned}
$$

where the $X_j \in E$ and $\hat{X}_i$ denotes the missing vector field.

So far this definition is given only for open subsets of convenient vector spaces. But taking into account that the manifolds under consideration are modelled on such spaces and therefore charts to open subsets of convenient vector spaces exist, we get a local definition of the mapping d by putting together the chart mappings and the above definition.

By the same arguments we define the insertion mapping and the Lie derivative locally via:

$$
\begin{aligned}
(i_X\omega)(x)(X_1, \ldots, X_{k-1}) &= \omega(x)(X, X_1, \ldots, X_{k-1}) \\
(\mathfrak{L}_X\omega)(x)(X_1, \ldots, X_k) &= ((i_X \circ d + d \circ i_X)\omega)\,(x)(X_1, \ldots, X_k)
\end{aligned}
$$

Corollary 3.3. [10, p. 352] *For a smooth mapping $f : N \to M$, the pullback mapping*

$$f^* : \Omega^k(M) \to \Omega^k(N)$$

is smooth.

Proof. The pullback mapping is smooth because it is induced by $Tf \times \ldots \times Tf$. □

3.2. Connections on principal fiber bundles

In Section 1.6 we introduced fiber and vector bundles and discussed vertical bundles. So again let $\ker(T\pi) =: VE$ be the vertical bundle of a fiber bundle (E, M, π, S). The aim of this section is to introduce connections on principal fiber bundles.

Definition 3.4. [10, p. 376] A *connection form* on the fiber bundle (E, M, π, S) is a vector valued 1-form $\Phi \in \Omega^1(E, VE)$ with values in the vertical bundle VE such that:

1. $\Phi \circ \Phi = \Phi$
2. $\operatorname{img}(\Phi) = VE$

So Φ is just a projection $TE \to VE$.

The kernel $\ker(\Phi)$ is a sub vector bundle of TE, it is called the *space of horizontal vectors* or the *horizontal bundle*, and it is denoted by HE. So it follows (for example with Lemma 1.79) that $TE = HE \oplus VE$ and $T_uE = H_uE \oplus V_uE$ for $u \in E$.

Definition 3.5. [10, p. 376] Now we consider the mapping:

$$(Tp, \pi_E) : TE \to TM \times_M E := \{(X, e) \in TM \times E \,:\, \mathrm{pr}_1(X) = \pi(e)\}$$

Then by definition $(Tp, \pi_E)^{-1}(0_{p(u)}, u) = V_uE$, so $(Tp, \pi_E)|_{HE} : HE \to TM \times_M E$ is a fiber linear isomorphism over E. Its inverse is denoted by $((Tp, \pi_E)|_{HE})^{-1} : TM \times_M E \to HE$. Together with the inclusion $HE \hookrightarrow TE$ one gets the mapping

$$C : TM \times_M E \to TE$$

which is fiber linear over E and a right inverse for (Tp, π_E). It is called the *horizontal lift* associated to the connection form Φ.

Remark 3.6. [10, p. 376] The formula $\Phi(\xi_u) = \xi_u - C(Tp(\xi_u), u)$ holds for $\xi_u \in T_uE$. So we can equally well describe a connection form Φ by specifying C. Then we call Φ *vertical projection* and $\chi := \mathrm{Id}_{TE} - \Phi = C \circ (Tp, \pi_E)$ will be called *horizontal projection*.

Theorem 3.7. [10, p. 386] *Let (P, M, π, G) be a principal fiber bundle with principal right action $r : P \times G \to P$. Then the following assertions hold:*

1. *$(TP, TM, T\pi, TG)$ is a principal fiber bundle with principal right action $Tr : TP \times TG \to TP$, where the structure group TG is the tangent group of G.*
2. *The vertical bundle $(VP, P, \mathrm{pr}_1, \mathfrak{g})$ of the principal bundle is trivial as a vector bundle over P, written as $VP \cong P \times \mathfrak{g}$.*
3. *The vertical bundle of the principal bundle as bundle over M is a principal bundle written as $(VP, M, \pi \circ \mathrm{pr}_1, TG)$.*

In the case of a principal fiber bundle (P, M, π, G) there is a refinement of the notion of a connection form. As introduced above a (general) connection form on P is a fiber projection $\Phi : TP \to VP$, viewed as a 1-form in $\Omega^1(P, VP) \subset \Omega^1(P, TP)$.

Definition 3.8. [10, p. 387] Such a connection form Φ is called a *principal connection form* if it is:

1. G-equivariant for the principal right action $r : P \times G \to P$, so that $Tr^g \circ \Phi = \Phi \circ Tr^g$,

2. r^g-related to itself, or $(r^g)^*\Phi = \Phi$, for all $g \in G$.

Definition 3.9. [10, p. 387] Because of Theorem 3.7 the bundle $VP \to P$ is trivial. So one can define the 1-form ω by

$$\omega(X_u) := ((T_e r_u)^{-1} \circ \Phi)(X_u) \in \mathfrak{g}$$

with $u \in P$ and $X_u \in T_u P$. So $\omega \in \Omega^1(P, \mathfrak{g})$ and is called the *Lie algebra valued connection form* of the connection form Φ. Recalling the definition of the fundamental vector field of a right action from Definition 1.58 one can reformulate the defining relation of ω as $\Phi(X_u) = \zeta_{\omega(X_u)}(u)$.

Lemma 3.10. [10, p. 387] *If $\Phi \in \Omega^1(P, VP)$ is a principal connection on the principal fiber bundle (P, M, π, G), then the Lie algebra valued connection form ω has the following three properties:*

1. *ω reproduces the generators of fundamental vector fields, so that we have $\omega(\zeta_X(u)) = X$ for all $X \in \mathfrak{g}$.*
2. *ω is G-equivariant, $((r^g)^*\omega)(X_u) = \omega(T_u r^g(X_u)) = (\mathrm{Ad}(g^{-1}) \circ \omega)(X_u)$ for all $g \in G$ and $X_u \in T_u P$.*
3. *We have for the Lie derivative $\mathfrak{L}_{\zeta_X}\omega = -ad(X) \circ \omega$.*

Conversely, a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying (1) *defines a connection form Φ on P by $\Phi(X_u) = (T_e r_u \circ \omega)(X_u)$, which is a principal connection form if and only if* (2) *is satisfied.*

For the construction of horizontal lifts on the path bundle later in the section we introduce and review some definitions and results for the finite dimensional case.

Definition 3.11. [1, p. 75] A *connection* on a finite dimensional principal fiber bundle (P, M, π, G) is a horizontal tangent bundle HP of P which satisfies

$$T_u r^g(H_u P) = H_{u \cdot g} P \text{ for } g \in G \text{ and } u \in P.$$

Theorem 3.12. [1, p. 75] *There is a bijection of the connections and Lie algebra valued connection forms on a finite dimensional principal fiber bundle (P, M, π, G) described by:*

1. *Let HP be a connection on P. Then the relation*

$$\omega(\tilde{X}(u) \oplus Y_u) := X \quad \forall u \in P,\ X \in \mathfrak{g},\ Y_u \in H_u(P)$$

defines a Lie algebra valued connection form ω on P.

2. *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a Lie algebra valued connection form on P. Then the mapping*

$$P \ni u \mapsto H_u(P) := \ker(\omega|_u)$$

defines a connection on P.

Definition 3.13. [1, p. 88] Let (P, M, π, G) be a finite dimensional principal fiber bundle and X a vector field on M. A vector field $\tilde{X}$ on P is called *horizontal lift of X* if for all $u \in P$:

1. $\tilde{X}(u) \in H_u P$
2. $T_u\pi(\tilde{X}(u)) = X(\pi(u))$

Theorem 3.14. [1, p. 88] *Let (P, M, π, G) be a finite dimensional principal fiber bundle.*

1. *For each vector field X on M there exists a unique horizontal lift $\tilde{X}$ on P. The vector field $\tilde{X}$ is right invariant.*
2. *If Z is a horizontal and right invariant vector field on P, then there exists a unique vector field X on M such that $\tilde{X} = Z$.*

Lemma 3.15. [1, p. 12] *Let G be a finite dimensional Lie group and $v : [0,1] \to T_eG$ a smooth path. Then there exist unique smooth paths $a, g : [0,1] \to G$ which satisfy the following differential equations:*

$$\dot{a}(t) = T_e l_{a(t)}(v(t)) \quad \text{with} \quad a(0) = e$$

$$\dot{g}(t) = T_e r^{g(t)}(v(t)) \quad \text{with} \quad g(0) = e$$

Compare the following lemma with [1, p. 38, modified]. We require only the first part and the notation is adapted to the one we use in this thesis.

Lemma 3.16. *The tangent mapping of the right action $r : P \times G \to P$ is given by*

$$T_{(u,g)}r(X_u, Y_g) = T_u r^g(X_u) + \zeta_{T_g l_{g^{-1}}(Y_g)}(u \cdot g)$$

where $X_u \in T_uP$ and $Y_g \in T_gG$.

Proof. By Remark A.20 the tangent space $T_{(u,g)}(P \times G)$ of $P \times G$ at (u, g) is isomorphic to $T_uP \oplus TG$. Because of linearity and $r(u, g) = r_u(g) = r^g(u) = u \cdot g$ the tangent mapping of the right action can be written as:

$$T_{(u,g)}r(X_u, Y_g) = T_{(u,g)}r(X_u, 0) + T_{(u,g)}r(0, Y_g) = T_u r^g(X_u) + T_g r_u(Y_g)$$

Now we write $Y_g = T_e l_g \circ T_g l_{g^{-1}}(Y_g)$ with $T_g l_{g^{-1}}(Y_g) \in T_eG \cong \mathfrak{g}$. So we get:

$$T_{(u,g)}r(X_u, Y_g) = T_u r^g(X_u) + (T_g r_u \circ T_e l_g \circ T_g l_{g^{-1}})(Y_g)$$

and with the application of the chain rule (see Theorem 1.24) for the tangent mapping this reads

$$T_{(u,g)}r(X_u, Y_g) = T_u r^g(X_u) + T_e r_{u \cdot g}(T_g l_{g^{-1}}(Y_g)).$$

With the definition of the fundamental vector field (see Definition 1.58) we get:

$$T_{(u,g)}r(X_u, Y_g) = T_u r^g(X_u) + \zeta_{T_g l_{g^{-1}}(Y_g)}(u \cdot g)$$

□

Definition 3.17. [1, p. 89] For M and P finite dimensional manifolds let $\gamma : I \to M$ and $\tilde{\gamma} : I \to P$ be paths, then $\tilde{\gamma}$ is called *horizontal lift* of γ if:

1. $\pi(\tilde{\gamma}(t)) = \gamma(t) \quad \forall t \in I$ and

2. the tangent vectors $\dot{\tilde{\gamma}}$ are horizontal $\forall t \in I$.

Theorem 3.18. [1, p. 89] *Let $\gamma : I \to M$ be a path in the finite dimensional manifold M, $t_0 \in I$ and $u \in \pi^{-1}(\gamma(t_0))$ a point in the finite dimensional fiber above $\gamma(t_0)$. Then there exists a unique horizontal lift $\tilde{\gamma}_u$ of γ with $\tilde{\gamma}_u(t_0) = u$.*

Proof. For simplicity let $I = [0,1]$ and $t_0 = 0$. As P is locally trivial, there exists a path $\delta : I \to P$ with $\delta(0) = u$ and $\pi \circ \delta = \gamma$. To prove the theorem one has to show that this path δ can be deformed into a horizontal path $\tilde{\gamma}_u : I \to P$ such that $\tilde{\gamma}_u(t) := \delta(t) \cdot g(t) = r(\delta(t), g(t))$ with $g(t) : I \to G$.

By Definition 3.9 the Lie algebra valued connection form ω on P and the tangent vector $\dot{\tilde{\gamma}}(t)$ satisfy $\omega\left(\dot{\tilde{\gamma}}(t)\right) = 0$ if and only if $\dot{\tilde{\gamma}}(t)$ is horizontal. Inserting the result from Lemma 3.16 this reads:

$$0 = \omega\left(T_{(\delta(t),g(t))} r(\dot{\delta}(t), \dot{g}(t))\right) = \omega\left(T_{\delta(t)} r^{g(t)}(\dot{\delta}(t)) + \zeta_{\left[T_{g(t)} l_{g^{-1}(t)}(\dot{g}(t))\right]}(\delta(t) \cdot g(t))\right)$$

Because of linearity and the properties of ω (see Lemma 3.10) this can be rewritten to:

$$0 = \left(\mathrm{Ad}(g^{-1}(t)) \circ \omega\right)(\dot{\delta}(t)) + T_{g(t)} l_{g^{-1}(t)}(\dot{g}(t))$$

After the multiplication from the left with $T_e l_{g(t)}$ this condition reads:

$$0 = \left(T_e r^{g(t)} \circ \omega\right)\left(\dot{\delta}(t)\right) + \dot{g}(t)$$

Now we define the smooth path

$$v : I \to \mathfrak{g} \qquad v(t) := -\omega\left(\dot{\delta}(t)\right).$$

Lemma 3.15 shows that there exists a unique smooth path $g : I \to G$ with $g(0) = e$ satisfying $-\left(T_e r^{g(t)} \circ \omega\right)\left(\dot{\delta}(t)\right) = \dot{g}(t)$. So we found the path $g(t)$ in G which deforms δ to the horizontal path $\tilde{\gamma}_u(t)$. □

Proposition 3.19. [10, p. 377] *The pullback construction introduced in Definition 1.65 satisfies that if $\Phi \in \Omega^1(E, TE)$ is a connection form on the bundle E, then the vector valued form $f^*\Phi$, given by*

$$(f^*\Phi)_u(X) := (T_u(\pi^* f)^{-1} \circ \Phi \circ T_u\pi^* f)(X)$$

for $X \in T_uE$, is a connection on the bundle f^*E. The forms $f^*\Phi$ and Φ are $\pi^* f$ -related.

Our next aim is to show that the mapping $s_\omega : C^\infty((I,0),(M,x_0)) \to C^\infty((I,0),(P,u_0))$, which assigns each path in M its horizontal lift, is a smooth section to π_*. For this we have to show that smooth curves in $C^\infty((I,0),(M,x_0))$ are mapped to smooth curves in $C^\infty((I,0),(P,u_0))$ as illustrated in the following diagram where $\tilde{c}$ is the image of c:

$$\begin{tikzcd} & C^\infty((I,0),(P,u_0)) \arrow[d, "\pi_*"] \\ \mathbb{R} \arrow[ur, dashed, "\tilde{c}"] \arrow[r, "c"] & C^\infty((I,0),(M,x_0)) \end{tikzcd}$$

By Lemma 2.3 we get that the smoothness of curves $c : \mathbb{R} \to C^\infty((I,0),(M,x_0))$ and $\tilde{c} : \mathbb{R} \to C^\infty((I,0),(P,u_0))$ is equivalent to the smoothness of mappings

$$c^\wedge : \mathbb{R} \times I \to M \quad \text{with} \quad (s,0) \mapsto x_0 \qquad \text{and} \qquad \tilde{c}^\wedge : \mathbb{R} \times I \to P \quad \text{with} \quad (s,0) \mapsto u_0.$$

So this problem can be reformulated: For a given smooth mapping $c^\wedge$ one has to show that its lift in the bundle $\pi : P \to M$ is smooth too, which is illustrated in the following diagram:

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ \mathbb{R} \times I \arrow[ur, dashed, "\tilde{c}^\wedge"] \arrow[r, "c^\wedge"] & M \end{tikzcd}$$

By introducing the pullback of P along $c^\wedge$, which is a trivial bundle over $\mathbb{R} \times I$ by Remark A.35, we get:

$$\begin{tikzcd} \mathbb{R} \times I \times G \arrow[r, equal] \arrow[dr] & (c^\wedge)^*(P) \arrow[r, "\pi^* c^\wedge"] \arrow[d, "{c^\wedge}^*\pi"] & P \arrow[d, "\pi"] \\ & \mathbb{R} \times I \arrow[ur, dashed, "\tilde{c}^\wedge"] \arrow[r, "c^\wedge"] & M \end{tikzcd}$$

With Theorem 3.18 we assign to the mapping $c(0) : I \to M$ a mapping $g(0) : I \to G$. This mapping $g(0)$ is given as the solution of an ordinary differential equation. Because we are considering a smooth variation of mappings $c^\wedge(s, \cdot)$ we get a smooth family of differential equations. So the solutions of these differential equations depend smoothly on the variation parameter. This gives a smooth curve g in the space $C^\infty(I,G)$ and we can define a section

$$\sec : \mathbb{R} \times I \to \mathbb{R} \times I \times G \qquad \sec(s,t) := (s,t,g^\wedge(s,t)).$$

Because of Proposition 1.66 and Proposition 3.19 we get a fiberwise diffeomorphism from the pullback bundle to P. So the composition with sec defines a smooth mapping $\tilde{c}^\wedge$. This shows that forming the horizontal lift of curves is a smooth mapping and the section s_ω is smooth.

The previous results are used to prove the following theorem stated in [2, p. 213]. We do not restrict ourself to matrix Lie groups in the construction of the global section s_ω.

Theorem 3.20. [2, p. 213] *Let (P,M,π,G) be a principal fiber bundle and $x_0 \in M$, $u_0 \in \pi^{-1}(x_0)$ then $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ is smoothly trivial. Moreover each Lie algebra valued connection form ω on P induces a smooth global section*

$$s_\omega : C^\infty(I,M)_{x_0} \to C^\infty(I,P)_{u_0}$$

where, for $\gamma \in C^\infty(I,M)_{x_0}$, the section $s_\omega(\gamma)$ is the ω-horizontal lift of γ to u_0.

Finally the restriction of s_ω to $C^\infty(I,M)_{x_0\to x_0}$ is a smooth section of the principal fiber bundle $\Big(\big(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\big)_{u_0}, C^\infty(I,M)_{x_0\to x_0}, \pi_{*0}, C^\infty(I,G)_0\Big)$.

Proof. From the previous results of this chapter it follows that s_ω is smooth. Now consider the mapping:

$$F^{-1} : C^\infty(I,M)_{x_0} \times C^\infty(I,G)_0 \to C^\infty(I,P)_{u_0} \qquad F^{-1}(\mu,a) \mapsto s_\omega(\mu)\cdot a = r(s_\omega(\mu),a)$$

It is smooth because s_ω and the right action r are smooth (see Proposition 2.7). Moreover it has an inverse denoted by

$$F : C^\infty(I,P)_{u_0} \to C^\infty(I,M)_{x_0} \times C^\infty(I,G)_0 \qquad F(\mu_P) = (\pi_*(\mu_P),a)$$

where a is the unique element of $C^\infty(I,G)_0$ satisfying $\mu_P = s_\omega(\pi_*(\mu_P))\cdot a$. Again by the results from above the assignment $k : \mu_P \mapsto a$ is smooth. So one can write $F = (\pi_*, k)$ which is smooth. We found a smooth trivialization of $C^\infty(I,P)_{u_0}$ over $C^\infty(I,M)_{x_0}$ so the bundle is trivial.

The restriction of s_ω to the closed submanifold $C^\infty(I,M)_{x_0\to x_0} \subseteq C^\infty(I,M)_{x_0}$ is smooth and consequently the bundle $\Big(\big(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\big)_{u_0}, C^\infty(I,M)_{x_0\to x_0}, \pi_{*0}, C^\infty(I,G)_0\Big)$ is smoothly trivial. $\square$

3.3. Existance of bump functions

Definition 3.21. [10, p. 153] We consider a Hausdorff topological space X with a subalgebra

$\mathcal{S} \subseteq C(X, \mathbb{R})$, whose elements will be called the *smooth* or $\mathcal{S}$*-functions* on X. We assume that for functions $h \in C^\infty(\mathbb{R}, \mathbb{R})$ (at least for those being constant off some compact set, in some cases) one has $h_*(\mathcal{S}) \subseteq \mathcal{S}$, and that $f \in \mathcal{S}$ provided it is locally in $\mathcal{S}$, i.e., there exists an open covering $\mathfrak{U}$ such that for every $U \in \mathfrak{U}$ there exists a $f_U \in \mathcal{S}$ with $f = f_U$ on U. In particular, we will use for $\mathcal{S}$ the classes of C^∞-mappings on c^∞-open subsets X of convenient vector spaces with the c^∞-topology.

Definition 3.22. [10, p. 153] For a (convenient) vector space F the *carrier* $\operatorname{carr}(f)$ of a mapping $f : X \to F$ is the set $\{x \in X : f(x) \neq 0\}$. The *zero set* of f is the set where f vanishes, $\{x \in X : f(x) = 0\}$. The *support* of f, written as $\operatorname{support}(f)$, is the closure of $\operatorname{carr}(f)$ in X.

We say that X is *smoothly regular* (with respect to $\mathcal{S}$) or $\mathcal{S}$-regular if for any neighbourhood U of a point x there exists a smooth function $f \in \mathcal{S}$ such that $f(x) = 1$ and $\operatorname{carr}(f) \subseteq U$. Such a function f is called a *bump function*.

Proposition 3.23. [10, p. 153] *Every Banach space with $\mathcal{S}$-norm is $\mathcal{S}$-regular. More general, a convenient vector space is smoothly regular if its c^∞-topology is generated by seminorms which are smooth on their respective carriers. For example, nuclear Fréchet spaces have this property.*

3.4. Connections on the path bundle of a principal fiber bundle

First of all we review some results from Section 2.1 and reconsider them under the view of the terms introduced in the previous section. For $\gamma \in C^\infty(I, P)$ we introduced the pullback of the tangent space TP along γ and considered its sections $\Gamma(\gamma^*TP \to I)$. Together with the exponential mapping on each fiber we defined charts for the manifold $C^\infty(I, P)$.

The set $\Gamma(\gamma^*TP \to I)$ may also be viewed as the set of all vector fields along γ and therefore as the set of vectors tangent to $C^\infty(I, P)$ at the point γ. This considerations show that $\Gamma(\gamma^*TP \to I)$ is the kinematic tangent space $T_\gamma(C^\infty(I, P))$ (see Definition 1.39) of the C^∞-manifold $C^\infty(I, P)$ in the point γ.

This means δ is an element of $T_\gamma(C^\infty(I, P))$ if and only if δ is a section of the pullback bundle $\gamma^*TP \to I$.

$$\begin{tikzcd}
\gamma^*TP \arrow[r, "\mathrm{pr}_P^*\gamma"] \arrow[d, "\gamma^*\mathrm{pr}_P"] & TP \arrow[d, "\mathrm{pr}_P"] \\
I \arrow[r, "\gamma"] & P
\end{tikzcd}$$

The projection $\pi_* : C^\infty(I, P) \to C^\infty(I, M)$ defined by $\pi_*(\gamma) := \pi \circ \gamma$ allows to define vertical vector fields on this bundle. So it follows that a vector field δ_γ along γ is called π_**-vertical* if and only if for each $t \in I$, $\delta_\gamma(t)$ is π-vertical.

We write VP for the vertical bundle introduced in Definition 1.70, so for $u \in P$ we define $V_uP := \{v \in T_uP \,:\, T_u\pi(v) = 0\}$. Then the vertical tangent space of $C^\infty(I,P)$ at the point γ is defined as:

$$V_\gamma(C^\infty(I,P)) = \{\delta \in \Gamma\left(\gamma^*(TP)\right) \,:\, \delta \in \ker(T\pi_*)\}$$

That means $\delta \in V_\gamma(C^\infty(I,P))$ if and only if it is a vector field along γ with $\delta(t)$ being a vertical vector of $T_{\gamma(t)}P$ for all $t \in I$.

With the considerations in Definition 1.38 we get the kinematic tangent bundle

$$T(C^\infty(I,P)) \to C^\infty(I,P)$$

and the kinematic vertical bundle

$$V(C^\infty(I,P)) \to C^\infty(I,P)$$

where we have to take into account Lemma 1.68, Lemma 1.69 and Definition 1.70.

For later use note that the space of sections $\Gamma\left(\gamma^*(TP)\right)$ is a $C^\infty(I,\mathbb{R})$-module. This can be seen easily by taking $f \in C^\infty(I,\mathbb{R})$ and $\delta \in \Gamma\left(\gamma^*(TP)\right)$. Then the multiplication is defined as $(f\delta)(t) := f(t)\delta(t)$ for all $t \in I$ and it follows that $f\delta \in \Gamma\left(\gamma^*(TP)\right)$. Moreover this implies that $V_\gamma(C^\infty(I,P))$ is a submodule of $T_\gamma(C^\infty(I,P))$.

Now that we have introduced the vertical bundle of the path bundle, we go on with the horizontal bundle and a generalization of Definition 3.11.

Definition 3.24. [2, p. 214] One says that $H(C^\infty(I,P))$ is a *(strong) connection* on $C^\infty(I,P)$ if $H(C^\infty(I,P))$ is a vector subbundle of $T(C^\infty(I,P))$ such that

1. for each $\gamma \in C^\infty(I,P)$, the space $H_\gamma(C^\infty(I,P))$ is a closed $C^\infty(I,\mathbb{R})$-submodule of $T_\gamma(C^\infty(I,P))$ such that

$$T_\gamma(C^\infty(I,P)) = H_\gamma(C^\infty(I,P)) \oplus V_\gamma(C^\infty(I,P)),$$

2. for $g \in C^\infty(I,G)$, one gets:

$$T_\gamma r^g(H_\gamma(C^\infty(I,P))) \subseteq H_{\gamma\cdot g}(C^\infty(I,P)).$$

The following lemma shows how vector subbundles of the tangent bundle of a manifold give rise to vector subbundles of the corresponding path space of this manifold.

Lemma 3.25. *Let P be smooth manifold and F a vector subbundle of the tangent bundle $TP \to P$ of fiber dimension k. Then the subset $\mathcal{F} \subseteq T(C^\infty(I,P))$ defined fiberwise as*

$$\mathcal{F}_\gamma := \{\mu \in \Gamma\left(\gamma^*(TP) \to I\right) \cong T_\gamma\left(C^\infty(I,P)\right) : \mu(t) \in F_{\gamma(t)} \subseteq T_{\gamma(t)}P \;\forall t \in I\}$$

is a vector subbundle.

Proof. We saw in Theorem 2.4 that the smooth manifold $C^\infty(I,P)$ is locally diffeomorphic to $\Gamma(\gamma^*(TP) \to I)$. We defined charts from open subsets of $C^\infty(I,P)$ into the modelling space via

$$\Phi_\gamma : C^\infty(I,P) \supseteq S(\gamma) \to \Gamma(\gamma^*(TP) \to I) \qquad \Phi_\gamma(\mu)(t) = (\pi_P^*\gamma)^{-1}\left(\exp^{-1}_{\gamma(t)}(\mu(t))\right)$$

with the inverse

$$\Phi_\gamma^{-1} : \Gamma(\gamma^*(TP) \to I) \supseteq \{v \in \Gamma(\gamma^*(TP) \to I) \,:\, v(I) \subseteq \gamma^* U = (\pi_P^*\gamma)^{-1}(U)\} \to C^\infty(I,P)$$

$$v \mapsto \left(t \mapsto \exp_{\gamma(t)}(\pi_P^*\gamma(v(t)))\right)$$

where U denotes a suitable open neighbourhood of the zero section in TP and V an open neighbourhood of the diagonal in $P \times P$.

$$\begin{tikzcd} \gamma^*TP \arrow[r, "\pi_P^*\gamma"] \arrow[d, "\gamma^*\pi_P"] & TP \arrow[d, "\pi_P"] \\ I \arrow[r, "\gamma"] & P \end{tikzcd}$$

Thus the tangent mapping $T\Phi_\gamma$ gives a local diffeomorphism

$$T(C^\infty(I,P)) \cong_{\text{loc}} \Gamma(\gamma^*(TP) \to I) \times \Gamma(\gamma^*(TP) \to I) \cong \Gamma(\gamma^*TP \times_I \gamma^*TP \to I)\,.$$

So the composition with $\Phi_\gamma^{-1} \times \mathrm{Id}$ gives vector bundle charts

$$T(C^\infty(I,P)) \cong_{\text{loc}} C^\infty(I,P) \times \Gamma(\gamma^*(TP) \to I)\,.$$

By Corollary A.37 we get that the bundle $\gamma^*TP \to I$ is trivial. So it follows that

$$\Gamma(\gamma^*(TP) \to I) \cong \Gamma(I \times \mathbb{R}^n \to I) \cong C^\infty(I, \mathbb{R}^n)$$

where $n = \dim(P)$ denotes the fiber dimension.

Let $\tilde{F}$ be the to F corresponding subset in the bundle $C^\infty(I,P) \times \Gamma(\gamma^*(TP) \to I)$. We have to find local vector bundle isomorphisms

$$C^\infty(I,P) \times \Gamma(\gamma^*(TP) \to I) \cong_{\text{loc}} C^\infty(I,P) \times C^\infty(I,\mathbb{R}^n)$$

which map $\tilde{F}$ to $C^\infty(I,P) \times C^\infty(I,\mathbb{R}^k)$.

For this consider the smooth mapping

$$\epsilon : \gamma^*TP \to P \qquad v_t \mapsto \exp_{\gamma(t)}(\pi_P^*\gamma(v_t))$$

and the pullback bundle $\epsilon^*TP \to \gamma^*TP$:

$$\begin{tikzcd}
\epsilon^*TP \arrow[r, "\pi_P^*\epsilon"] \arrow[d, "\epsilon^*\pi_P"] & TP \arrow[d, "\pi_P"] \\
\gamma^*TP \arrow[r, "\epsilon"] & P
\end{tikzcd}$$

Since F is a vector subbundle of $TP \to P$ we get that ϵ^*F is a vector subbundle of ϵ^*TP. The base manifold γ^*TP is a vector bundle over I and hence contractible by Corollary A.37. So ϵ^*TP is a trivial bundle by Corollary A.38 and the trivialization $\psi : \gamma^*TP \times \mathbb{R}^n \to \epsilon^*TP$ may be chosen such that $\psi^{-1}(\epsilon^*F) = \gamma^*TP \times \mathbb{R}^k$.

Furthermore, ϵ^*TP is isomorphic to $\gamma^*TP \times_I \gamma^*TP$ where ϵ^*F corresponds to the set

$$\{(u, w) \in \gamma^*TP \times_I \gamma^*TP : w \in \tilde{F}_{\epsilon(u)}\}.$$

The isomorphism in the converse direction is given by assigning

$$\frac{d}{ds}|_{s=0} \exp_{\gamma(t)}(u + s \cdot w) = T_u \exp_{\gamma(t)} \cdot w \in T_{\epsilon(u)}P$$

to $(u, w) \in \gamma^*TP \times_I \gamma^*TP$, i.e. u, $w \in T_{\gamma(t)}P$ for some $t \in I$.

Composing these two isomorphisms gives a vector bundle isomorphism

$$\chi : \gamma^*TP \times \mathbb{R}^n \to \epsilon^*TP \to \gamma^*TP \times_I \gamma^*TP,$$

which maps $\{v\} \times \mathbb{R}^k$ to $\{v\} \times \tilde{F}_{\exp_{\gamma(t)}(v(t))}$.

$$\begin{tikzcd}
\gamma^*TP \times \mathbb{R}^n \arrow[r, "\cong"] \arrow[rr, bend left, "\chi"] \arrow[dr] & \epsilon^*TP \arrow[r, "\cong"] \arrow[d, "\epsilon^*\pi_P"] & \gamma^*TP \times_I \gamma^*TP \arrow[dl] \\
& \gamma^*TP \arrow[d, "\gamma^*\pi_P"] & \\
& I &
\end{tikzcd}$$

Finally,

$$\chi_* : \Gamma\left(\gamma^*TP \times \mathbb{R}^n \to \gamma^*TP \to I\right) \to \Gamma\left(\gamma^*TP \times_I \gamma^*TP \to \gamma^*TP \to I\right)$$

is the required vector bundle isomorphism. □

How this fits to the discussion of connections in the previous section will become clear in the next definition.

Definition 3.26. [2, p. 214] Let ω be a Lie algebra valued connection form on P, then one can define the *connection* $H^\omega(C^\infty(I,P))$ *induced by* ω on the path bundle as

$$H^\omega_\gamma(C^\infty(I,P)) := \{\delta \in T_\gamma(C^\infty(I,P)) \,:\, \delta(t) \text{ is } \omega\text{-horizontal } \forall t \in I\},$$

where $\gamma \in C^\infty(I,P)$. Thus if HP is the subbundle (connection) of TP given by $H_uP = \ker(\omega|_u)$ for each $u \in P$, one can rewrite the definition in the following form:

$$H^\omega_\gamma(C^\infty(I,P)) := \{\delta \in T_\gamma(C^\infty(I,P)) \,:\, \delta(t) \in H_{\gamma(t)}P \quad \forall t \in I\}$$

Now that we defined induced connections we turn to the discussion of how special vector subbundles lead to connections on the path bundle.

Proposition 3.27. [2, p. 214] *Let* $\operatorname{Comp} P$ *denote a vector subbundle of* TP *which satisfies that for each* $\gamma \in C^\infty(I,P)$ *and for all* $t \in I$ *the space* $\operatorname{Comp}_{\gamma(t)} P$ *is a subspace of* $T_{\gamma(t)}P$ *which is complementary to* $V_{\gamma(t)}P$*. Now define*

$$H_\gamma(C^\infty(I,P)) := \{\delta \in T_\gamma(C^\infty(I,P)) \,:\, \delta(t) \in \operatorname{Comp}_{\gamma(t)} P \quad \forall t \in I\}.$$

Then $H(C^\infty(I,P)) = \bigsqcup_{\gamma \in C^\infty(I,P)} H_\gamma(C^\infty(I,P))$ *is a connection on* $C^\infty(I,P)$ *if the assignment* $\gamma \mapsto \operatorname{Comp}_\gamma P$ *satisfies the condition that if* $\delta(t) \in \operatorname{Comp}_{\gamma(t)} P$ *then* $T_{\gamma(t)} r^{g(t)}(\delta(t)) \in \operatorname{Comp}_{(\gamma\cdot g)(t)} P$ *for all* $t \in I$ *and* $g \in C^\infty(I,G)$.

Proof. By Lemma 3.25 $H(C^\infty(I,P))$ is a vector subbundle of $TC^\infty(I,P)$. For $H(C^\infty(I,P))$ to be a connection on $C^\infty(I,P)$ it has to satisfy

$$T_\gamma r^g(H_\gamma(C^\infty(I,P))) \subseteq H_{\gamma\cdot g}(C^\infty(I,P))$$

for arbitrary $\gamma \in C^\infty(I,P)$ and $g \in C^\infty(I,G)$. For a given element $\delta \in H_\gamma(C^\infty(I,P))$ and $t \in I$ this mapping reads

$$T_\gamma r^g(\delta)(t) = \mathrm{ev_t}(T_\gamma r^g(\delta)).$$

Because of the structure of the tangent space it is the same for the tangent mapping first to evaluate the variable with respect to t and then use the mapping at the so given points, or first to use the mapping and then evaluate the result. So the condition reads

$$T_\gamma r^g(\delta)(t) = \mathrm{ev_t}(T_\gamma r^g(\delta)) = T_{\gamma(t)} r^{g(t)}(\delta(t)) \subseteq \operatorname{Comp}_{(\gamma\cdot g)(t)} P.$$

Furthermore we require that the decomposition

$$T_\gamma(C^\infty(I,P)) = H_\gamma(C^\infty(I,P)) \oplus V_\gamma(C^\infty(I,P)) \quad \forall \gamma \in C^\infty(I,P)$$

exists. But this is trivial because the horizontal spaces were chosen complementary to the vertical ones for each $t \in I$ and for $f \in C^\infty(I, \mathbb{R})$ and $\delta \in H_\gamma(C^\infty(I,P))$ the product $(f\delta)(t) := f(t)\delta(t) \in \mathrm{Comp}_{\gamma(t)} P$ for each $t \in I$. $\square$

Next we see how a given connection on the path bundle induces a decomposition of the tangent spaces along curves.

Theorem 3.28. [2, p. 215] *If $H(C^\infty(I,P))$ is any connection on the principal fiber bundle $(C^\infty(I,P), C^\infty(I,M), \pi_*, C^\infty(I,G))$ and if, for $t \in I$ and $\gamma \in C^\infty(I,P)$ one defines*

$$\mathrm{Comp}_{\gamma(t)} P := \{\delta(t) \,:\, \delta \in H_\gamma(C^\infty(I,P)) \subseteq T_\gamma(C^\infty(I,P))\},$$

then $\mathrm{Comp}_{\gamma(t)} P$ is a subspace of $T_{\gamma(t)}P$ of dimension $\dim(M)$ which is complementary to $V_{\gamma(t)}P$. Moreover, if

$$\mathrm{Comp}_\gamma P := \bigsqcup_{t\in I} \mathrm{Comp}_{\gamma(t)} P \quad \text{and} \quad V_\gamma P := \bigsqcup_{t\in I} V_{\gamma(t)}P,$$

then $\mathrm{Comp}_\gamma P \to I$ is a vector subbundle of $\gamma^*TP \to I$ such that

$$\gamma^*TP \cong \mathrm{Comp}_\gamma P \oplus V_\gamma P,$$

and

$$\delta \in H_\gamma(C^\infty(I,P)) \quad \text{if and only if} \quad \delta(t) \in \mathrm{Comp}_{\gamma(t)} P \quad \forall t \in I.$$

Proof. **Step 1,** Identifications: Let $\gamma \in C^\infty(I,P)$. The bundle $\gamma^*TP \to I$ is trivial because of Corollary A.37. The same is true for the subbundle $V_\gamma P \to I$ of vertical spaces along γ. Let $\{e_{m+i}\}_{i=1}^d$ denote a set of sections of $V_\gamma P \to I$ such that $\{e_{m+i}(t)\}_{i=1}^d$ are linearly independent for all $t \in I$. This can be enlarged to an ordered set of sections $\{e_i\}_{i=1}^{m+d}$ of the bundle $\gamma^*TP \to I$ with $\{e_i(t)\}_{i=1}^{m+d}$ forming a basis of γ^*TP_t $\forall t \in I$. Because of triviality one can identify the bundle $\gamma^*TP \to I$ with $I \times \mathbb{R}^{m+d} \to I$ and the bundle $V_\gamma P \to I$ with $I \times \mathbb{R}^d \to I$.

The connection $H(C^\infty(I,P))$ on $C^\infty(I,P)$ defines a subspace

$$H_\gamma(C^\infty(I,P)) \subseteq T_\gamma(C^\infty(I,P)) \cong \Gamma\left(\gamma^*TP\right)$$

complementary to $V_\gamma(C^\infty(I,P)) \subseteq T_\gamma(C^\infty(I,P))$. Then the space $\Gamma\left(\gamma^*TP\right)$ can be identified with $\Gamma\left(I \times \mathbb{R}^{m+d}\right) \cong C^\infty(I, \mathbb{R}^{m+d})$ so that $V_\gamma(C^\infty(I,P))$ can be identified as the subspace $\Gamma\left(I \times \mathbb{R}^d\right) \cong C^\infty(I, \mathbb{R}^d)$. With these identifications $H_\gamma(C^\infty(I,P))$ is a $C^\infty(I,\mathbb{R})$-submodule of $C^\infty(I,\mathbb{R}^{m+d})$ complementary to $V_\gamma(C^\infty(I,P)) = C^\infty(I,\mathbb{R}^d)$.

Step 2, $\max_{t\in I}\left(\dim(\mathrm{Comp}_{\gamma(t)} P)\right) \geq m$: The space $\mathrm{Comp}_{\gamma(t)} P$ is a subspace of $\mathbb{R}^{m+d}$ for each $t \in I$ because $H_\gamma(C^\infty(I,P))$ is a $C^\infty(I,\mathbb{R})$ module. Define now

$$s := \max_I \left(\dim(\mathrm{Comp}_{\gamma(t)} P)\right).$$

Then there exists a $t_0 \in I$ so that $s = \dim(\operatorname{Comp}_{\gamma(t_0)} P)$. Now let $f_1, f_2, \cdots, f_s \in H_\gamma(C^\infty(I,P))$ such that $\{f_i(t_0)\}_{i=1}^s$ is a basis of $\operatorname{Comp}_{\gamma(t_0)} P$ and let $\{e_j\}_{j=1}^{m+d}$ be the standard basis of $\mathbb{R}^{m+d}$. Then the f_i can be written as

$$f_i(t) = \sum_{j=1}^{m+d} c_i^j(t) \cdot e_j$$

and the rank of the transformation matrix formed by the coefficients $c_i^j(t)$ is maximal at $t = t_0$. Since the rank can not fall locally there exists an open interval $J \subseteq I$ around t_0 such that the transformation matrix has rank s for each $t \in J$, which means that $\{f_i(t)\}_{i=1}^s$ is a basis of $\operatorname{Comp}_{\gamma(t)} P$ for each $t \in J$.

Every vector $w \in \mathbb{R}^{m+d}$ defines a constant map $\bar{w} \in C^\infty(I, \mathbb{R}^{m+d})$ by $I \ni t \mapsto w$. By the identifications above this can be written as $\bar{w} = \bar{w}_H + \bar{w}_V \in H_\gamma(C^\infty(I,P)) \oplus V_\gamma(C^\infty(I,P))$. So one gets for all $t \in I$ that $w = \bar{w}_H(t) + \bar{w}_V(t) \in \operatorname{Comp}_{\gamma(t)} P + V_{\gamma(t)}P$ which means that the whole space $\mathbb{R}^{m+d}$ is generated by the sum

$$\mathbb{R}^{m+d} = \operatorname{Comp}_{\gamma(t)} P + V_{\gamma(t)}P.$$

So the equation for the dimensions $m + d = s + d - \dim\left(\operatorname{Comp}_{\gamma(t)} P \cap V_{\gamma(t)}P\right)$ holds and it follows that $\dim\left(\operatorname{Comp}_{\gamma(t)} P \cap V_{\gamma(t)}P\right) = s - m$ is constant for $t \in J$ and $s \geq m$.

Step 3, $s = m$ on J: Assume $s > m$ and let $\operatorname{Comp}_\gamma P|_J \to J$ and $V_\gamma P|_J \to J$ be subbundles of $\gamma^*TP|_J \to J$. Now we consider the bundle mappings

$$\lambda_{\operatorname{Comp} P} : \gamma^*TP|_J \to \gamma^*TP|_J \Big/ \operatorname{Comp}_\gamma P|_J$$
$$\lambda_{VP} : \gamma^*TP|_J \to \gamma^*TP|_J \Big/ V_\gamma P|_J$$

and define for $v \in \gamma^*TP|_J$ the mapping:

$$\lambda :\ \gamma^*TP|_J \to \left(\gamma^*TP|_J \Big/ \operatorname{Comp}_\gamma P|_J\right) \oplus \left(\gamma^*TP|_J \Big/ V_\gamma P|_J\right) \qquad \lambda(v) := \left(\lambda_{\operatorname{Comp} P}(v), \lambda_{VP}(v)\right)$$

Then λ is a bundle homomorphism (see Definition 1.62) with

$$\ker(\lambda)_t = \operatorname{Comp}_{\gamma(t)} P|_J \cap V_{\gamma(t)}P|_J \quad \forall t \in J$$

because of the quotients defined above. Since $\dim(\ker(\lambda)_t) = s - m$ is locally constant it follows from Lemma 1.64 that $\ker(\lambda) \to J$ is a subbundle of $\gamma^*TP|_J \to J$. This bundle is trivial by Corollary A.37 because J is an interval. So there exists a smooth section $\sigma : J \to \gamma^*TP|_J$ such that $\sigma(t)$ is a nonzero element of $\operatorname{Comp}_{\gamma(t)} P|_J \cap V_{\gamma(t)}P|_J$ for all $t \in J$.

With the basis $f_1,\, f_2,\, \cdots,\, f_s$ of $H_\gamma(C^\infty(I,P))$ introduced above, one can write σ as

$$\sigma(t) = \sum_{j=1}^{s} a_j(t) \cdot f_j(t)$$

with $\sigma(t) \in \mathbb{R}^{s-m}$ for each $t \in J$. By definition σ is nonzero at least at one point $t_0 \in J$. So Proposition 3.23 is applicable because J is an open neighbourhood of $t_0 \in \mathbb{R}$ and of course $\mathbb{R}$ is a Banach space. It follows that there exists a bump function c such that $c(t_0) = 1$ and $c(I \setminus J) = 0$. So the function

$$c\sigma = \sum_{j=1}^{s} (ca_j) \cdot f_j$$

is smooth on I, not only on J, nonzero and in $H_\gamma(C^\infty(I,P)) \cap V_\gamma(C^\infty(I,P))$. This means we found a nonzero function lying in $H_\gamma(C^\infty(I,P))$ and $V_\gamma(C^\infty(I,P))$ which is a contradiction to the assumption that these two spaces are complementary and we get that s has to be equal to $m = \dim\left(\operatorname{Comp}_{\gamma(t)} P\right)$ for all $t \in J$.

Step 4, $s = m$ on I: Let $u \in I$ then $\mathbb{R}^{m+d} = \operatorname{Comp}_{\gamma(u)} P + V_{\gamma(u)}P$ and $m + d = \dim\left(\operatorname{Comp}_{\gamma(u)} P\right) + d - \dim\left(\operatorname{Comp}_{\gamma(u)} P \cap V_{\gamma(u)}P\right)$. Hence

$$m = \max_{t\in I} \dim\left(\operatorname{Comp}_{\gamma(t)} P\right) \geq \dim\left(\operatorname{Comp}_{\gamma(u)} P\right) = m + \dim\left(\operatorname{Comp}_{\gamma(u)} P \cap V_{\gamma(u)}P\right) \geq m$$

which means that m has to be equal to $\dim\left(\operatorname{Comp}_{\gamma(u)} P\right)$ for all $u \in I$ and so $s = m$.

Step 5, $\gamma^*TP = \operatorname{Comp}_\gamma P \oplus V_\gamma P$: From the equation for m in the step before it follows that the intersection $\operatorname{Comp}_{\gamma(u)} P \cap V_{\gamma(u)}P$ is zero, so it follows that $\gamma^*TP_t = \operatorname{Comp}_{\gamma(t)} P \oplus V_{\gamma(t)}P$ for each $t \in I$. We already introduced the subset $\{f_i\}_{i=1}^m$ of $H_\gamma(C^\infty(I,P))$ and discussed that it can be chosen depending on $t_0 \in I$ such that the set $\{f_i(t_0)\}_{i=1}^m$ forms a basis of $\operatorname{Comp}_{\gamma(t_0)} P$. As the set $\{f_i(t)\}_{i=1}^m$ is a basis for all t in an open neighbourhood of t_0 we get that

$$\operatorname{Comp}_\gamma P = \bigcup_{t\in I} \operatorname{Comp}_{\gamma(t)} P$$

is locally trivial as a vector bundle over I. Even more it is a trivial subbundle of γ^*TP such that:

$$\gamma^*TP \cong \operatorname{Comp}_\gamma P \oplus V_\gamma P$$

Step 6, $\delta \in H_\gamma(C^\infty(I,P))$ if and only if $\delta(t) \in \operatorname{Comp}_{\gamma(t)} P$ for all $t \in I$: The first direction is clear because for a $\delta \in H_\gamma(C^\infty(I,P))$ it follows that $\delta(t) \in \operatorname{Comp}_{\gamma(t)} P$ for all $t \in I$ by the definition of $\operatorname{Comp}_{\gamma(t)} P$.

Now let $\delta \in C^\infty(I, \mathbb{R}^{m+d})$ such that $\delta(t) \in \operatorname{Comp}_{\gamma(t)} P$ for each $t \in I$. Choose again $\{f_i\}_{i=1}^m$ in $H_\gamma(C^\infty(I,P))$ such that $\{f_i(t)\}_{i=1}^m$ forms a basis of $\operatorname{Comp}_{\gamma(t)} P$ for all t in an open interval $J \subseteq I$. Then δ can be written as

$$\delta(t) = \sum_{j=1}^{m} b_j(t) f_j(t)$$

for all $t \in J$. By modifications of the bump function in Proposition 3.23 there exists a bump function $c \in C^\infty(I, \mathbb{R})$ such that c is 1 on some relatively open interval $K \subseteq J$ and $c(I \setminus J) = 0$. Then it follows that

$$c\delta = \sum_{j=1}^{m} (cb_j) f_j$$

is an element of $H_\gamma(C^\infty(I, P))$ because of the choice of the basis $\{f_i\}_{i=1}^m$ and one gets that $c\delta|_K = \delta|_K$.

Define $\mathcal{K}$ as the set of all $\xi \in C^\infty(I, \mathbb{R}^{m+d})$ such that for each $t_0 \in I$ there exists a relatively open interval K such that $t_0 \in K \subseteq I$ and $\xi|_K = \eta|_K$ for some $\eta \in H_\gamma(C^\infty(I, P))$. So clearly $H_\gamma(C^\infty(I, P))$ is contained in $\mathcal{K}$. Since $H_\gamma(C^\infty(I, P))$ is a $C^\infty(I, \mathbb{R})$ submodule of $C^\infty(I, \mathbb{R}^{m+d})$ the same is true for $\mathcal{K}$. We get that δ lies in $\mathcal{K}$ because of the construction of $c\delta$ above.

Let now $\tau \in \mathcal{K} \cap V_\gamma(C^\infty(I, P))$ and $\tau \neq 0$. Then $\tau|_K = \eta|_K$ for some $\eta \in H_\gamma(C^\infty(I, P))$ and for some $K \subseteq I$ such that $\tau|_K \neq 0$. By Proposition 3.23 there exists a bump function $c \in C^\infty(I, \mathbb{R})$ with carrier on K and c equal to 1 at some point of K at which τ is nonzero. So one gets that $c\tau = c\eta$.

It is clear that $c\tau \in V_\gamma(C^\infty(I, P))$ because it had been in the intersection $\mathcal{K} \cap V_\gamma(C^\infty(I, P))$. On the other side $c\eta$ is in $H_\gamma(C^\infty(I, P))$ by the definition of η. So we found a nonzero element of $H_\gamma(C^\infty(I, P)) \cap V_\gamma(C^\infty(I, P))$ which is a contradiction to the choice of $H_\gamma(C^\infty(I, P))$. Thus we get that $\mathcal{K} \cap V_\gamma(C^\infty(I, P)) = \{0\}$. We had before that $H_\gamma(C^\infty(I, P)) \subseteq \mathcal{K}$ which means that $C^\infty(I, \mathbb{R}^{m+d})$ is the sum of $\mathcal{K}$ and $V_\gamma(C^\infty(I, P))$. Because we just proved that $\mathcal{K} \cap V_\gamma(C^\infty(I, P)) = \{0\}$ this is a direct sum and we get

$$H_\gamma(C^\infty(I, P)) \oplus V_\gamma(C^\infty(I, P)) \cong C^\infty(I, \mathbb{R}^{m+d}) \cong \mathcal{K} \oplus V_\gamma(C^\infty(I, P))$$

and $\mathcal{K} = H_\gamma(C^\infty(I, P))$. Because $\delta \in \mathcal{K}$, it is an element of $H_\gamma(C^\infty(I, P))$ too and the last step is proved. $\square$

3.5. Connection forms on the path bundle

Definition 3.29. [2, p. 218] Every connection $\gamma \mapsto H_\gamma(C^\infty(I, P)) \subseteq T_\gamma(C^\infty(I, P))$ on $C^\infty(I, P)$ defines a *Lie algebra valued connection form* $\bar{\omega} : T(C^\infty(I, P)) \to C^\infty(I, \mathfrak{g})$ by:

$$\bar{\omega}(H_\gamma(C^\infty(I, P))) = 0 \qquad \text{and} \qquad \bar{\omega}(\zeta_a(\gamma)) = a, \qquad \text{for } a \in C^\infty(I, \mathfrak{g})$$

where ζ_a is the corresponding fundamental vector field on $C^\infty(I,P)$.

Lemma 3.30. [2, p. 218] *The Lie algebra valued connection form $\bar{\omega}$ has the following properties:*

1. *$\bar{\omega}$ is smooth.*

2. *At each $\gamma \in C^\infty(I,P)$, the mapping $\bar{\omega}_\gamma$ is a $C^\infty(I,\mathbb{R})$-module homomorphism.*

3. *$(r^g)^*(\bar{\omega}) = \mathrm{Ad}(g^{-1})(\bar{\omega})$ for $g \in C^\infty(I,G)$, $\mathrm{Ad}_G : G \to \mathfrak{g}$ the usual adjoint representation of G and $\mathrm{Ad}(g) = \mathrm{Ad}_G \circ g$.*

Proof. This follows from the first section of this chapter and the finite dimensional case for each $t \in I$. □

Remark 3.31. On the other hand, if we have already a Lie algebra valued connection form $\bar{\omega}$, then it defines a connection. This is analogues to the finite dimensional case already introduced in Theorem 3.12.

The next paragraphs discuss the correspondence between Lie algebra valued connection forms on P and on $C^\infty(I,P)$. We start by describing how a Lie algebra valued connection form on P induces one on $C^\infty(I,P)$.

Proposition 3.32. [2, p. 218] *Let $\omega : TP \to \mathfrak{g}$ denote a Lie algebra valued connection form on P. Then the mapping*

$$\bar{\omega} : T(C^\infty(I,P)) \to C^\infty(I,\mathfrak{g}) \qquad \bar{\omega}_\gamma(\delta)(t) = \omega_{\gamma(t)}(\delta(t))$$

for $\gamma \in C^\infty(I,P)$, $\delta \in T_\gamma(C^\infty(I,P))$, $t \in I$ defines a Lie algebra valued connection form on $C^\infty(I,P)$.

Proof. For each $t \in I$ one gets for the tangent mapping of the right action

$$T_\gamma r^g(\delta)(t) = T_{\gamma(t)} r^{g(t)}(\delta(t))$$

for $g \in C^\infty(I,G)$. This leads together with the G-equivariance of ω to

$$\begin{aligned}((r^g)^*\bar{\omega})_\gamma(\delta)(t) &= \omega_{\gamma(t)}(Tr^g(\delta)(t)) = (r^{g(t)})^*\omega(\delta(t)) = \\ &= \mathrm{Ad}(g(t)^{-1})\omega_{\gamma(t)}(\delta(t)) = \left(\mathrm{Ad}(g^{-1})(\bar{\omega}_\gamma(\delta))\right)(t)\end{aligned}$$

which reads in short $(r^g)^*\bar{\omega} = \mathrm{Ad}(g^{-1})\bar{\omega}$.

The smoothness of $\bar{\omega}$ follows by adaptations of the arguments in Section 3.2 and the application of Corollary 1.22. See Definition 3.1 and the discussion followed for more details of the definition of smooth differential forms.

The mapping $\bar{\omega}$ is a $C^\infty(I,\mathbb{R})$-module homomorphism for each $\gamma \in C^\infty(I,P)$ by the properties of ω. For $a \in C^\infty(I,\mathfrak{g})$ it follows that $\bar{\omega}_\gamma(\zeta_a(\gamma)) = a$ because it is true for each $t \in I$ and ω. □

What we have seen in the last proposition leads to the following definition.

Definition 3.33. [2, p. 218] If $\omega : TP \to \mathfrak{g}$ is a Lie algebra valued connection form on the bundle (P, M, π, G), then the Lie algebra valued connection form $\bar{\omega} : T(C^\infty(I,P)) \to C^\infty(I,\mathfrak{g})$ defined by $\bar{\omega}_\gamma(\delta)(t) := \omega_{\gamma(t)}(\delta(t))$ for $\gamma \in C^\infty(I,P)$, $\delta \in T_\gamma(C^\infty(I,P))$, $t \in I$ is called the *Lie algebra valued connection form on $C^\infty(I,P)$ induced by ω.*

Naturally the question arises, whether or not the Lie algebra valued connection forms on $C^\infty(I,P)$ induced by connections of P can be characterized. The answer is given in the following proposition.

Proposition 3.34. [2, p. 218] *If (P, M, π, G) is a principal fiber bundle then a Lie algebra valued connection form $\bar{\omega}$ on $(C^\infty(I,P), C^\infty(I,M), \pi_*, C^\infty(I,G))$ is induced by a Lie algebra valued connection form ω of P if and only if $\bar{\omega}_\gamma(\delta)$ is a constant element of $C^\infty(I,\mathfrak{g})$ whenever γ and δ are constant elements of $C^\infty(I,P)$ and $T_\gamma(C^\infty(I,P))$, respectively.*

Proof. The first direction is clear, indeed assume $\bar{\omega}$ induced by ω on P. Let $\gamma \in C^\infty(I,P)$ and $\delta \in T_\gamma(C^\infty(I,P)) = C^\infty(I, T_{\gamma(0)}P)$ be constant. Since $\bar{\omega}_\gamma(\delta)(t) := \omega_{\gamma(t)}(\delta(t))$ for all $t \in I$ it follows that $\bar{\omega}_\gamma(\delta)$ is constant.

For the other direction assume $\bar{\omega}_\gamma(\delta)$ is constant whenever $\gamma \in C^\infty(I,P)$ and $\delta \in T_\gamma(C^\infty(I,P))$ are constant. Now define the mappings

$$J : P \to C^\infty(I,P) \qquad J(u)(t) = u$$
$$j : M \to C^\infty(I,M) \qquad j(x)(t) = x$$

for $x \in M$, $u \in P$ and $t \in I$. The following diagram commutes:

$$\begin{tikzcd} P \arrow[r, "J"] \arrow[d, "\pi"] & C^\infty(I,P) \arrow[d, "\pi_*"] \\ M \arrow[r, "j"] & C^\infty(I,M) \end{tikzcd}$$

If we define for $g \in G$ the mapping $\bar{g} : I \to G$ via $\bar{g}(t) = g$ for all $t \in I$ then the following identity is satisfied for all $u \in P$:

$$J(u \cdot g) = J(u) \cdot \bar{g} \qquad \text{or equivalently} \qquad J \circ r^g = r^{\bar{g}} \circ J \tag{3.1}$$

Next define ω on P by $\omega := J^*\bar{\omega}$. For $u \in P$ and $X \in T_uP$ the definition leads to

$$\omega_u(X)(t) = \bar{\omega}_{J(u)}\left((TJ)(X)\right)(t)$$

for all $t \in I$. Since $J(u)$ and $T_uJ(X)$ are constant for all $t \in I$ it follows that $\omega_u(X)$ is constant too and may be identified as an element of $\mathfrak{g} \subseteq C^\infty(I, \mathfrak{g})$, again by constant mappings. By Equation 3.1 and the property $(r^g)^*\bar{\omega} = \mathrm{Ad}(g^{-1})\bar{\omega}$ of $\bar{\omega}$ we get

$$(r^g)^*(J^*\bar{\omega}) = J^*\left((r^{\bar{g}})^*\bar{\omega}\right) = \mathrm{Ad}(g^{-1})(J^*\bar{\omega}).$$

For $a \in \mathfrak{g}$ and $u \in P$ we get again by constant mappings that

$$T_uJ(\zeta_a(u))(t) = \zeta_a(u) = \zeta_{\bar{a}}(J(u))(t) \quad \forall t \in I$$

which leads to

$$\omega(\zeta_a(u)) = (J^*\bar{\omega})(\zeta_a(u)) = \bar{\omega}(T_uJ(\zeta_a(u))) = \bar{\omega}(\zeta_{\bar{a}}(J(u))) = \bar{a} = a.$$

This property shows that ω is a Lie algebra valued connection form on P. Clearly $\bar{\omega}$ is induced by ω. □

3.6. Connections on subbundles of the path bundle

As far we discussed connections on the bundle $(C^\infty(I, P), C^\infty(I, M), \pi_*, C^\infty(I, G))$ as described in Definition 3.24. Now we want to specialize these results for the bundles

$$(C^\infty(I, P)_{u_0}, C^\infty(I, M)_{x_0}, \pi_*, C^\infty(I, G)_0) \text{ and}$$

$$\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}, C^\infty(I, M)_{x_0 \to x_0}, \pi_{*0}, C^\infty(I, G)_0\right).$$

For these bundles Definition 3.24 can be reformulated by replacing the total space and the group of the path bundle by the total spaces and groups of the subbundles respectively.

Theorem 3.35. [2, p. 219] *If $H\left(C^\infty(I, P)_{u_0}\right)$ and $H\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}\right)$ are connections on the principal fiber bundles $(C^\infty(I, P)_{u_0}, C^\infty(I, M)_{x_0}, \pi_*, C^\infty(I, G)_0)$ and $\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}, C^\infty(I, M)_{x_0 \to x_0}, \pi_{*0}, C^\infty(I, G)_0\right)$, then the conclusions of Theorem 3.28 hold on the tangent spaces of the total spaces at each point γ of the principal bundles under consideration.*

Proof. Let γ be an element of the total space of one of the principal bundles being considered. The argument in the proof of Theorem 3.28 is a local one. So it works for the two cases under consideration too. □

Lie algebra valued connection forms on $(C^\infty(I, P)_{u_0}, C^\infty(I, M)_{x_0}, \pi_*, C^\infty(I, G)_0)$ and $\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}, C^\infty(I, M)_{x_0 \to x_0}, \pi_{*0}, C^\infty(I, G)_0\right)$ are given by Definition 3.33 and appropriate restrictions of the spaces under consideration.

Theorem 3.36. [2, p. 220] *If (P, M, π, G) is a principal fiber bundle such that P and G are path connected and if $\bar{\omega}$ is a Lie algebra valued connection form on $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$, then $\bar{\omega}$ is induced by a Lie algebra valued connection form ω on P if and only if $\bar{\omega}$ has the property that whenever (γ_1, v_1) and (γ_2, v_2) are elements of $T(C^\infty(I,P)_{u_0})$ such that $\gamma_1(1) = u = \gamma_2(1)$ and $v_1(1) = v_2(1)$ it follows that $\bar{\omega}_{\gamma_1}(v_1)(1) = \bar{\omega}_{\gamma_2}(v_2)(1)$.*

Proof. **Step 1,** First direction: If ω is a Lie algebra valued connection form on P then ω induces a Lie algebra valued connection form $\bar{\omega}$ on $C^\infty(I,P)_{u_0}$ by $\bar{\omega}_\gamma(v)(t) := \omega_{\gamma(t)}(v(t))$. So it follows that

$$\bar{\omega}_{\gamma_1}(v_1)(1) = \omega_{\gamma_1(1)}(v_1(1)) = \omega_u(v_1(1)) = \omega_{\gamma_2(1)}(v_2(1)) = \bar{\omega}_{\gamma_2}(v_2)(1)$$

which shows the first direction of the theorem.

Step 2, Definition of ω: Let $\bar{\omega}$ be a Lie algebra valued connection form on $C^\infty(I,P)_{u_0}$ which satisfies the conditions stated in the theorem. For $u \in P$ and $X \in T_uP$ choose $\tilde{\gamma}_u \in C^\infty(I,P)_{u_0}$ such that $\tilde{\gamma}_u(1) = u$ and a vector field v_X along $\tilde{\gamma}_u$ with $v_X(1) = X$. Their existence is shown by their smooth construction in step 4. Now define ω by:

$$\omega_u(X) := \bar{\omega}_{\tilde{\gamma}_u}(v_X)(1).$$

Since the definition of $\omega_u(X)$ is independent of the choice of $\tilde{\gamma}_u$ and v_X we get a well-defined $\mathfrak{g}$-valued 1-form on P.

Step 3, Connection properties of ω: Let $g_1 \in G$ and $u \in P$ then we can choose $g \in C^\infty(I,G)_0$ such that $g(1) = g_1$ and $\tilde{\gamma}_u \in C^\infty(I,P)_{u_0}$ such that $\tilde{\gamma}_u(1) = u$ because of the path connectedness of G and P. So we get by the right actions of the principal bundles involved that $\tilde{\gamma}_u \cdot g \in C^\infty(I,P)_{u_0}$ and $(\tilde{\gamma}_u \cdot g)(1) = u \cdot g_1 \in P$. The tangent mapping of the right action gives $(Tr^g(v_X)(1) = T_u r^{g_1}(X)$ so that

$$((r^{g_1})^*\omega)_u(X) = \omega(T_u r^{g_1}(X)) = \bar{\omega}_{\tilde{\gamma}\cdot g}((Tr^g)(v_X))(1)$$

follows which is equal to $((r^g)^*\bar{\omega})_{\tilde{\gamma}_u}(v_X)(1)$. By the properties of $\bar{\omega}$ (see Lemma 3.30), it follows that

$$((r^g)^*\bar{\omega})_{\tilde{\gamma}_u}(v_X)(1) = [\mathrm{Ad}(g^{-1})\bar{\omega}_{\tilde{\gamma}_u}(v_X)](1) = \mathrm{Ad}(g^{-1})\omega_u(X).$$

So we found that ω satisfies $((r^{g_1})^*\omega)_u(X) = \mathrm{Ad}(g^{-1})\omega_u(X)$.

Let $a_1 \in \mathfrak{g}$ and $a \in C^\infty(I,\mathfrak{g})_0$ such that $a(1) = a_1$. Let ζ_a denote the fundamental vector field on $C^\infty(I,P)_{u_0}$ induced by a such that for $\mu \in C^\infty(I,P)_{u_0}$ one gets $\zeta_a(\mu)(t) = \zeta_{a(t)}(\mu(t))$. Then it follows that

$$\omega_u(\zeta_{a_1}(u)) = \bar{\omega}_{\tilde{\gamma}_u}(\zeta_a(\tilde{\gamma}_u))(1) = a(1) = a_1$$

by using Definition 3.29.

Step 4, Smoothness: To prove smoothness we first have to review some definitions and results already introduced. First let $c \in C^\infty(\mathbb{R}, P)$ denote a curve in P. Then by Remark 1.41 its associated tangent vector at the point $c(0)$ is defined via the mapping

$$\delta : C^\infty(\mathbb{R}, P) \to TP \qquad \delta(c) := [c(0), \frac{\partial}{\partial t}\big|_0 (u_\alpha \circ c)(t), \alpha]$$

where u_α denotes a chart.

By Remark 1.43 a mapping $g : TP \to N$ from the tangent space TP of a manifold P to a C^∞-manifold N is smooth if and only if the composition

$$g \circ \delta : C^\infty(\mathbb{R}, P) \to N$$

maps smooth curves to smooth curves i.e. induces a map from $C^\infty(\mathbb{R}^2, P)$ to $C^\infty(\mathbb{R}, N)$.

In our case we are interested in the mapping $\omega : TP \to \mathfrak{g}$ from the tangent space of P into the Lie algebra $\mathfrak{g}$ of G. By Remark 1.43 ω is smooth if and only if

$$\omega \circ \delta : C^\infty(\mathbb{R}, P) \to \mathfrak{g}$$

is smooth.

We defined ω in a point $u \in P$ and for a tangent vector $X \in T_u P$ as

$$\omega_u(X) := \bar{\omega}_{\tilde{\gamma}_u}(v_X)(1)$$

where $\tilde{\gamma}_u \in C^\infty(I, P)_{u_0}$ denotes an arbitrary smooth path starting in $u_0 \in P$ and ending in u and v_X an arbitrary vector field along $\tilde{\gamma}_u$ satisfying $v_X(1) = X$.

By the path connectedness of P it follows that for given u there exists a smooth curve $\tilde{\gamma}_u$. What we do not know up to now is, whether this choice depends smoothly on u.

First we want to know how $\omega \circ \delta$ is exactly defined. Therefore let $c \in C^\infty(\mathbb{R}, P)$. Then we have to insert $c(0) = pr_1(\delta(c))$ instead of u and $\mathrm{pr}_2(\delta(c))$ instead of X in the definition of ω. So we get:

$$\omega \circ \delta : C^\infty(\mathbb{R}, P) \to \mathfrak{g} \qquad (\omega \circ \delta)(c) := \bar{\omega}_{\tilde{\gamma}_{c(0)}}(v_{\mathrm{pr}_2(\delta(c))})(1)$$

To prove its smoothness we have to show that smooth curves in $C^\infty(\mathbb{R}, P)$ are mapped to smooth curves in $\mathfrak{g}$ or the space $C^\infty(\mathbb{R} \times \mathbb{R}, P)$ to the space $C^\infty(\mathbb{R}, \mathfrak{g})$.

Let therefore $d : \mathbb{R} \to C^\infty(\mathbb{R}, P)$ denote a smooth curve in $C^\infty(\mathbb{R}, P)$, which means that $d^\wedge \in C^\infty(\mathbb{R} \times \mathbb{R}, P)$ satisfying $d^\wedge(0, t) = c(t)$. Then it follows that $s \mapsto d^\wedge(s, 0)$ is a smooth curve in P and $c(0) = d^\wedge(0, 0)$.

By the path connectedness of P there exists a smooth curve $\tilde{\gamma}_{c(0)} \in C^\infty(I, P)$ with $\tilde{\gamma}_{c(0)}(0) = u_0$ and $\tilde{\gamma}_{c(0)}(1) = c(0)$. W.l.o.g. we may assume that $I = [0, 1]$. Then we can reparametrize $\tilde{\gamma}_{c(0)}$ smoothly such that $\tilde{\gamma}_{c(0)}(t) = c(0)\ \forall t \in \left[\frac{1}{2}, 1\right]$.

Next we require a smooth bump function (see Definition 3.22) $h : \mathbb{R} \to \mathbb{R}$ satisfying $h = 0$ for $t \leq \frac{1}{2}$ and $h = 1$ for $t \geq 1$.

Now define a curve

$$b_s : \left[\frac{1}{2}, 1\right] \to P \qquad t \mapsto d^\wedge(h(t) \cdot s, 0)$$

which describes the smooth curve segment from $d^\wedge(0,0)$ to $d^\wedge(s,0)$.

So we can consider the concatenation of $\tilde{\gamma}_{c(0)}$ with b_s which is a piecewise smooth curve. It can be reparametrized to a smooth curve with starting point $u_0 \in P$ and endpoint $d^\wedge(s,0)$ which we will denote as $\tilde{\gamma}_{d^\wedge(s,0)} : I \to P$.

This shows that $\tilde{\gamma}_{d^\wedge(s,0)}$ depends smoothly on s which means that the assignment of $\tilde{\gamma}_{c(0)}$ to a curve c is a smooth mapping. We require furthermore that the vector field $v_{\tilde{\gamma}_{d^\wedge(s,0)}}$ along $\tilde{\gamma}_{d^\wedge(s,0)}$ depends on s smoothly too. To achieve that we define

$$v_{\tilde{\gamma}_{d^\wedge(s,0)}}(t) := h(t) \cdot \mathrm{pr}_2(\delta(d(h(t) \cdot s))).$$

It is zero for $t \leq \frac{1}{2}$ and then increases until it satisfies $v_{\tilde{\gamma}_{d^\wedge(s,0)}}(1) = \mathrm{pr}_2(\delta(d(s)))$. The factor $h(t)$ in the argument of d is necessary to guarantee that the vectors for each t lie in the right tangent space. So we get the smooth dependence of $v_{\mathrm{pr}_2(\delta(c))}$ on the curve c too.

$$\begin{tikzcd}
C^\infty(\mathbb{R}, P) \arrow[r, "\delta"] \arrow[d, "c \mapsto \left(\tilde{\gamma}_{c(0)}, v_{\mathrm{pr}_2(\delta(c))}\right)"'] & TP \arrow[r, "\omega"] & \mathfrak{g} \\
T(C^\infty(I, P)_{u_0}) \arrow[r, "\bar{\omega}"] & C^\infty(I, \mathfrak{g})_0 \arrow[ur, "\mathrm{ev}_1"] &
\end{tikzcd}$$

So the mapping $\omega \circ \delta$ is a composition of the smooth assignments of $c \mapsto \tilde{\gamma}_{c(0)}$ and $c \mapsto v_{\mathrm{pr}_2(\delta(c))}$, the smooth connection $\bar{\omega}$ and the by Corollary 1.22 smooth evaluation mapping. This shows the smoothness of ω. $\square$

Remark 3.37. [2, p. 222] The last theorem can also be formulated for the bundle $\left(\left(\pi_*^{-1}(\mathrm{Imm}\,(I, M))\right)_{u_0}, \mathrm{Imm}\,(I, M)_{x_0}, \hat{\pi}, C^\infty(I, G)_0\right)$, where only little modifications are required. The paths which connect the point u_0 with the points u can be chosen such that their projection $\dot{\gamma}(t) \neq 0$ for each $t \in I$. The idea is to avoid problematic points by modifying γ in a small neighbourhood of them such that this modification can take place in a local trivialization above this neighbourhood too.

The last bundle where we characterize Lie algebra valued connection forms induced by the ones on $P \to M$ is $\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}, C^\infty(I, M)_{x_0 \to x_0}, \pi_{*0}, C^\infty(I, G)_0\right)$.

Theorem 3.38. [2, p. 222] *A Lie algebra valued connection form $\acute{\omega}$ on $\left(\left(\pi_*^{-1}(C^\infty(I, M)_{x_0 \to x_0})\right)_{u_0}, C^\infty(I, M)_{x_0 \to x_0}, \pi_{*0}, C^\infty(I, G)_0\right)$ is induced by a Lie algebra valued connection form ω on P if and only if $\acute{\omega}$ has an extension $\bar{\omega}$ defined on*

$(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ such that $\bar{\omega}_{\gamma_1}(\delta_1)(1) = \bar{\omega}_{\gamma_2}(\delta_2)(1)$ for all (γ_1,δ_1) and (γ_2,δ_2) in $T(\big(\pi_*^{-1}(C^\infty(I,M))\big)_{u_0})$ such that $\gamma_1(1)=\gamma_2(1)$ and $\delta_1(1)=\delta_2(1)$.

Proof. If $\acute{\omega}$ is a Lie algebra valued connection form on $\big(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\big)_{u_0}$ which is induced by a Lie algebra valued connection form ω on P then ω also induces a Lie algebra valued connection form $\bar{\omega}$ on $\big(\pi_*^{-1}(C^\infty(I,M))\big)_{u_0}$. The restriction of $\bar{\omega}$ to $T(\big(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\big)_{u_0})$ is precisely $\acute{\omega}$.

Otherwise if $\acute{\omega}$ has an extension with the properties formulated above, then Theorem 3.36 gives the Lie algebra valued connection form ω on P. □

4. Curvature

4.1. Curvature on infinite dimensional manifolds

In the last chapter we introduced connections on principal fiber bundles. Now we discuss the curvature of the connections. To simplify the notation let w.l.o.g. I be the interval $[0,1]$ as in the previous chapter.

Definition 4.1. [10, p. 377] Let Φ be a connection form on the fiber bundle (E, M, π, S). We define the *curvature form* $\mathcal{R}$ of Φ as

$$\mathcal{R} := \frac{1}{2}[\Phi, \Phi] = \frac{1}{2}[\mathrm{Id} - \Phi, \mathrm{Id} - \Phi] \in \Omega^2(E, VE).$$

The curvature form $\mathcal{R}$ is an obstruction against involutivity of the horizontal subbundle in the following sense: If the curvature form $\mathcal{R}$ vanishes, then horizontal kinematic vector fields on E also have a horizontal Lie bracket.

Proposition 4.2. [10, p. 377] *Let $(f^*E, N, f^*\pi, S)$ be the pullback bundle of (E, M, π, S) over $f : N \to M$ and $\pi^* f : f^*E \to E$ the mapping between the total spaces of the bundles. The curvatures of $f^*\Phi$ and Φ are $\pi^* f$-related.*

For principal fiber bundles we get:

Proposition 4.3. [10, p. 388] *Let Φ be a principal connection form on the principal fiber bundle (P, M, π, G). The curvature form $\mathcal{R} = \frac{1}{2}[\Phi, \Phi]$ is G-equivariant, that means $(r^g)^*\mathcal{R} = \mathcal{R}$ for all $g \in G$.*

Definition 4.4. [10, p. 388] Let Φ be a principal connection form on the principal fiber bundle (P, M, π, G) and $\omega \in \Omega^1(P, \mathfrak{g})$ the corresponding Lie algebra valued connection form. Since $\mathcal{R}$ has vertical values we may define a $\mathfrak{g}$-valued 2-form $F \in \Omega^2(P, \mathfrak{g})$ by

$$F(X_u, Y_u) := -\left(T_e r_u\right)^{-1} \mathcal{R}(X_u, Y_u)$$

for all $u \in P$. We call it the *Lie algebra valued curvature form.*

Proposition 4.5. [10, p. 388] *The curvature form $\mathcal{R}$ and the Lie algebra valued curvature form F satisfy the following relation with the fundamental vector field ζ.*

$$\mathcal{R}(X_u, Y_u) = -\zeta_{F(X_u, Y_u)}(u)$$

where $u \in P$ and X_u, $Y_u \in T_uP$.

Definition 4.6. [10, p. 388] We consider the space $\Omega(P, \mathfrak{g})$ of all $\mathfrak{g}$-valued forms on P equipped with the structure of a graded Lie algebra in a canonical way by

$$[\Psi, \Theta]^{\mathfrak{g}}_{\wedge}(X_1, \dots, X_{p+q}) := \frac{1}{p!q!}\sum_{\sigma} \operatorname{sign}(\sigma)[\Psi(X_{\sigma(1)}, \dots, X_{\sigma(p)}), \Theta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})]_{\mathfrak{g}}.$$

Equivalently this can be written as

$$[\Psi \otimes X, \Theta \otimes Y]_{\wedge} := \Psi \wedge \Theta \otimes [X, Y]_{\mathfrak{g}}.$$

So it follows with the outer derivative d that

- $d[\Psi, \Theta]_{\wedge} = [d\Psi, \Theta]_{\wedge} + (-1)^{\deg \Psi}[\Psi, d\Theta]_{\wedge}$ and
- $[\omega, \omega]^{\mathfrak{g}}_{\wedge}(X, Y) = 2[\omega(X), \omega(Y)]_{\mathfrak{g}}$ for $\omega \in \Omega^1(P, \mathfrak{g})$.

Theorem 4.7. [10, p. 388] *The Lie algebra valued curvature form F of a principal connection with Lie algebra valued connection form ω has the following properties:*

1. *F is horizontal, i.e., it kills vertical vectors.*
2. *F is G-equivariant in the following sense: $(r^g)^*F = \mathrm{Ad}(g^{-1})F$*
   Consequently, $\mathfrak{L}_{\zeta_X} F = -\operatorname{ad}(X)F$.
3. *The Maurer-Cartan formula holds: $F = d\omega + \frac{1}{2}[\omega, \omega]_{\wedge}$.*

4.2. Curvature on the path bundle

The terms introduced in the previous section are used for the path bundle as shown in the following.

Theorem 4.8. [2, p. 224] *Let (P, M, π, G) be a principal fiber bundle and ω a Lie algebra valued connection form on P. Furthermore let $x_0 \in M$, $u_0 \in \pi^{-1}(x_0)$ and A an arbitrary smooth $C^\infty(I, \mathfrak{g})_0$-valued 1-form on $C^\infty(I, M)_{x_0}$.*

1. *There exists a unique Lie algebra valued connection form $\bar{\omega}$ on $C^\infty(I, P)_{u_0}$ such that $s^*_\omega(\bar{\omega}) = A$ where s_ω is the global section of $C^\infty(I, P)_{u_0} \to C^\infty(I, M)_{x_0}$ defined by requiring that $s_\omega(\gamma)$ be the unique ω-horizontal lift of γ to u_0.*
2. *Conversely every Lie algebra valued connection form $\bar{\omega}$ on $C^\infty(I, P)_{u_0} \to C^\infty(I, M)_{x_0}$ arises as the pullback of a smooth mapping $A : T(C^\infty(I, M)_{x_0}) \to C^\infty(I, \mathfrak{g})_0$ in this way.*
3. *If $\bar{F}$ is the Lie algebra valued curvature form of $\bar{\omega}$ then $s^*_\omega(\bar{F}) = dA + \frac{1}{2}[A, A]_{\wedge}$.*

4. *Similarly Lie algebra valued connection forms $\bar{\omega}$ on $\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}$ are characterized by smooth mappings $A : T(C^\infty(I,M)_{x_0\to x_0}) \to C^\infty(I,\mathfrak{g})_0$ with a corresponding characterization of the curvature of $\bar{\omega}$.*

Proof. This theorem is a consequence of the fact that

$$C^\infty(I,P)_{u_0} \cong s_\omega(C^\infty(I,M)_{x_0}) \times C^\infty(I,G)_0$$

is trivial in the sense that global sections s_ω exist as shown in Theorem 3.20. The following commutative diagram shows (1) and (2).

$$\begin{tikzcd}
 & T\left(C^\infty(I,P)_{u_0}\right) \arrow[r] \arrow[dl, "\bar{\omega}"'] \arrow[d, shift left, "T\pi_*"] & s_\omega(C^\infty(I,M)_{x_0}) \times C^\infty(I,G)_0 \arrow[d, shift left, "\pi_*"] \\
C^\infty(I,\mathfrak{g})_0 & T\left(C^\infty(I,M)_{x_0}\right) \arrow[l, "A"'] \arrow[u, shift left, "Ts_\omega"] \arrow[r] & C^\infty(I,M)_{x_0} \arrow[u, shift left, "s_\omega"]
\end{tikzcd}$$

(3) is true because of Theorem 4.7.

(4) follows through modifications of previous arguments and due to the fact that $\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}$ is trivial too by Theorem 3.20. □

The following example should help to avoid misinterpretations.

Example 4.9. Assume that A is the trivial mapping, which means that each element in $T\left(C^\infty(I,M)_{x_0}\right)$ is mapped to the curve $I \to \{0 \in \mathfrak{g}\}$. Then (and only in this case) $\bar{\omega}$ is the Lie algebra valued connection form induced by ω because all horizontal lifts are again mapped to the trivial path. In general the association between these Lie algebra valued connection forms is not that simple.

So this example should be a warning not to mistake the Lie algebra valued connection forms induced by the section s_ω for the *induced Lie algebra valued connection forms* by the finite dimensional ω.

The following theorem is a modification of the one stated in [2, p. 224]. There the theorem assumes that G is a matrix Lie group because the smooth horizontal lift s_ω was only defined for matrix Lie groups. This is different now because we showed the existence of a smooth horizontal lift for general Lie groups in Section 3.2.

Theorem 4.10. [2, p. 224] *Let (P,M,π,G) be a principal fiber bundle, ω a Lie algebra valued connection form on P, $x_0 \in M$ and $u_0 \in \pi^{-1}(x_0)$. Define the mapping*

$$\psi_\omega : C^\infty(I,P)_{u_0} \to C^\infty(I,G)_0$$

which assigns each $\gamma \in C^\infty(I,P)_{u_0}$ a path $\psi_\omega(\gamma) \in C^\infty(I,G)_0$ such that the horizontal lift $\bar{\gamma}$ of $\pi \circ \gamma$ is given by

$$\bar{\gamma}(t) = \gamma(t) \cdot \psi_\omega(\gamma)(t).$$

Then ψ_ω is a smooth equivariant mapping which maps all of $C^\infty(I,P)_{u_0}$ onto $C^\infty(I,G)_0$.

The inverse image $\psi_\omega^{-1}(\bar{e})$ of the identity $\bar{e} \in C^\infty(I,G)_0$ is a subbundle of $C^\infty(I,P)_{u_0}$ and precisely the set of all $\gamma \in C^\infty(I,P)_{u_0}$ which are ω-horizontal.

In a similar way the mapping $\psi_\omega|_{\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}}$ has analogous properties on $\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}$.

Proof. The smoothness of ψ_ω is discussed in the proof of Theorem 3.20 and follows there by the projection to the second component of the trivialization mapping.

Because the bundle $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ is smoothly trivial by Theorem 3.20, we can factorize γ into the horizontal lift $\bar{\gamma}$ to u_0 of its projection $\pi \circ \gamma$ and an element $a \in C^\infty(I,G)_0$. So for each $t \in I$ one can write $\gamma(t) = \bar{\gamma}(t) \cdot a(t)$. Now take $g \in C^\infty(I,G)_0$, so one gets $\gamma(t) \cdot g(t) = \bar{\gamma}(t) \cdot a(t) \cdot g(t)$.

Using the trivialization the mapping ψ_ω reads $\gamma \mapsto a^{-1}$. Inserting the product $\gamma \cdot g = \bar{\gamma} \cdot a \cdot g$ one gets for all $t \in I$ that ψ_ω satisfies

$$\psi_\omega : \gamma \mapsto (a \cdot g)^{-1} = g^{-1} \cdot a^{-1}.$$

With the definition of ψ_ω this reads

$$\psi_\omega(\gamma \cdot g)(t) = g^{-1} \cdot \psi_\omega(\gamma)(t)$$

which shows the equivariance of ψ_ω.

With the help of the global section s_ω we get that $\psi_\omega(s_\omega(\lambda)) = \bar{e}$ for each $\lambda \in C^\infty(I,M)_{x_0}$.

Conversely, if we take $\gamma \in C^\infty(I,P)_{u_0}$ which satisfies $\psi_\omega(\gamma) = \bar{e}$ then $\psi_\omega(\gamma)(t) = e$ for all $t \in I$ where e is the unit element of G. In this case $\gamma = \bar{\gamma} = s_\omega(\pi \circ \gamma)$. So we get that $\psi_\omega^{-1}(\bar{e}) = s_\omega(C^\infty(I,M)_{x_0})$ and hence is a submanifold of $C^\infty(I,P)_{u_0}$.

By restriction to $\psi_\omega|_{\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}}$ one gets the similar result for the bundle $\left(\left(\pi_*^{-1}(C^\infty(I,M)_{x_0\to x_0})\right)_{u_0}, C^\infty(I,M)_{x_0\to x_0}, \pi_{*0}, C^\infty(I,G)_0\right)$. □

Remark 4.11. [2, p. 225] A slight modification of the last two theorems leads to analogous results for the bundle $\left(\left(\pi_*^{-1}(\mathrm{Imm}\,(I,M)_{x_0})\right)_{u_0}, \mathrm{Imm}\,(I,M)_{x_0}, \hat{\pi}, C^\infty(I,G)_0\right)$.

4.3. Uniform connections on the path bundle

Definition 4.12. [2, p. 230] A Lie algebra valued connection form $\bar{\omega}$ on $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ is *uniform* if and only if it satisfies the property that $\bar{\omega}_\gamma(v)(t) = \bar{\omega}_{\gamma_t}(v_t)(1)$ for every $(\gamma, v) \in T(C^\infty(I,P)_{u_0})$ and $t \in I$ where $\gamma_t(s) := \gamma(ts)$ and $v_t(s) := v(ts)$ for each $s \in I$

Theorem 4.13. [2, p. 231] *There is a bijection between the set of all uniform connection forms on $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ and the set of all mappings*

$$f : T(C^\infty(I,P)_{u_0}) \to \mathfrak{g}$$

satisfying the following conditions:

1. *f is smooth,*

2. *f is equivariant, in the sense that $(r^g)^* f = \mathrm{Ad}(g(1)^{-1}) f$ for $g \in C^\infty(I,G)_0$, and*

3. *$f(\gamma, \zeta_a(\gamma)) = a(1)$ for $a \in C^\infty(I,\mathfrak{g})_0$ with corresponding fundamental vector field ζ_a, $\gamma \in C^\infty(I,P)_{u_0}$.*

In the following we denote the set of all such f with $\mathcal{A}(C^\infty(I,P)_{u_0}, \mathfrak{g})$.

Proof. **Step 1,** First direction: Let $\bar\omega$ be a uniform connection form on $C^\infty(I,P)_{u_0}$. Define the mapping

$$f_{\bar\omega} : T(C^\infty(I,P)_{u_0}) \to \mathfrak{g} \qquad f_{\bar\omega}(\gamma, v) := \bar\omega_\gamma(v)(1) = \mathrm{ev}_1(\bar\omega_\gamma(v))$$

which is smooth by the smoothness of $\bar\omega$ and the smoothness of the evaluation mapping by Corollary 1.22. Applying the right action on $f_{\bar\omega}$ we get by the equivariance property of $\bar\omega$ that

$$(r^g)^* f_{\bar\omega}(\gamma, v) = \bar\omega_{r^g\gamma}(T_\gamma r^g(v))(1) = \left(\mathrm{Ad}(g^{-1})(\bar\omega_\gamma(v))\right)(1) = \mathrm{Ad}(g(1)^{-1}) f_{\bar\omega}(\gamma, v)$$

for arbitrary pairs $(\gamma, v) \in T(C^\infty(I,P)_{u_0})$. The use of the $\bar\omega$-property for fundamental vector fields gives

$$f_{\bar\omega}(\gamma, \zeta_a(\gamma)) = \bar\omega_\gamma(\zeta_a(\gamma))(1) = a(1)$$

for each $a \in C^\infty(I,\mathfrak{g})_0$. This finishes the first direction.

Step 2, Connection properties: For a given $f \in \mathcal{A}(C^\infty(I,P)_{u_0}, \mathfrak{g})$ define the Lie algebra valued one form

$$\bar\omega_f : T(C^\infty(I,P)_{u_0}) \to C^\infty(I,\mathfrak{g})_0 \qquad (\bar\omega_f)_\gamma(v)(t) := f(\gamma_t, v_t)$$

for arbitrary pairs $(\gamma, v) \in T(C^\infty(I,P)_{u_0})$ and $t \in I$.

First we show the uniformity condition. We get that $(\bar\omega_f)_{\gamma_t}(v_t)(1) = f((\gamma_t)_1, (v_t)_1)$ by the definition of $\bar\omega_f$. Then $f((\gamma_t)_1, (v_t)_1) = f(\gamma_t, v_t)$ follows because $(\gamma_t)_1 = \gamma_t$ and $(v_t)_1 = v_t$ as the maximal subpaths of γ_t and v_t respectively. Again using the definition of $\bar\omega_f$ leads to $(\bar\omega_f)_{\gamma_t}(v_t)(1) = (\bar\omega_f)_\gamma(v)(t)$ for each $t \in I$.

Next we discuss the equivariance property. For $t \in I$ we get by the application of $(r^g)^*$ on one forms and the definition of $\bar{\omega}_f$ that

$$((r^g)^*\bar{\omega}_f)_\gamma(v)(t) = (\bar{\omega}_f)_{\gamma\cdot g}(T_\gamma r^g(v))(t) = f(r^g(\gamma)_t, T_\gamma r^g(v)_t).$$

Because it is the same first to apply the right action and then to take only the paths up to t or the other way around, it follows that

$$f(r^g(\gamma)_t, T_\gamma r^g(v)_t) = f(r^{g_t}(\gamma_t), T_{\gamma_t} r^{g_t}(v_t)) = ((r^{g_t})^* f)(\gamma_t, v_t).$$

Using the equivariance property of f and having in mind the subpath constructions of g_t, γ_t and v_t, this leads to

$$((r^{g_t})^* f)(\gamma_t, v_t) = (\mathrm{Ad}(g_t(1)^{-1})f)(\gamma_t, v_t) = (\mathrm{Ad}(g(t)^{-1})(\bar{\omega}_f)_\gamma)(v)(t)$$

So we get

$$((r^g)^*\bar{\omega}_f)_\gamma(v)(t) = \mathrm{Ad}(g^{-1})((\bar{\omega}_f)_\gamma(v))(t).$$

Similar arguments lead to the fundamental vector field property. Taking $a \in C^\infty(I, \mathfrak{g})_0$ and $t \in I$ one gets that

$$(\bar{\omega}_f)_\gamma(\zeta_a(\gamma))(t) = f(\gamma_t, (\zeta_a)(\gamma)_t) = f(\gamma_t, \zeta_{a_t}(\gamma_t)) = a_t(1) = a(t).$$

Step 3, Smoothness: As in the situation of Theorem 3.36 we start by reviewing some definitions and notation. First let $c \in C^\infty(\mathbb{R}, C^\infty(I, P)_{u_0})$ denote a curve in $C^\infty(I, P)_{u_0}$. Then by Remark 1.41 its associated tangent vector at the point $c(0)$ is defined via the mapping

$$\delta : C^\infty(\mathbb{R}, C^\infty(I, P)_{u_0}) \to T(C^\infty(I, P)_{u_0}) \qquad \delta(c) := [c(0), \frac{\partial}{\partial t}\Big|_0 (u_\alpha \circ c)(t), \alpha]$$

where u_α denotes a chart.

This is similar to the situation in Theorem 3.36 with the difference that in the infinite dimensional case instead of the point $u \in P$ we consider the curve $\gamma \in C^\infty(I, P)_{u_0}$ and instead of a tangent vector X a tangent vector field v along γ.

As before we define $\gamma_\iota(t) := \gamma(\iota \cdot t)$ and $v_\iota(t) := v(\iota \cdot t)$ for each $t, \iota \in I$. In order to show the smoothness of

$$\bar{\omega}_f : T(C^\infty(I, P)_{u_0}) \to C^\infty(I, \mathfrak{g})_0 \qquad (\bar{\omega}_f)_\gamma(v)(\iota) := f(\gamma_\iota, v_\iota)$$

we have to show that smooth curves in $T(C^\infty(I, P)_{u_0})$ are mapped to smooth curves in $C^\infty(I, \mathfrak{g})_0$.

By Remark 1.43 it suffices to show that

$$\bar{\omega}_f \circ \delta : C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0}) \to C^\infty(I,\mathfrak{g})_0$$

maps smooth curves to smooth curves which means that $f(\gamma_\iota, v_\iota)$ defines a mapping from $C^\infty(\mathbb{R}^2, C^\infty(I,P)_{u_0})$ to $C^\infty(\mathbb{R}, C^\infty(I,\mathfrak{g})_0)$.

Let now $c^\wedge \in C^\infty(\mathbb{R}\times I, P)$ denote the smooth mapping corresponding (see Lemma 2.3) to a smooth curve $c : \mathbb{R} \to C^\infty(I,P)_{u_0}$. We want to show that the assignment of a smooth family of mappings to a curve c defined by

$$\Lambda : C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0}) \to C^\infty(I, C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0})) \qquad c \mapsto (\iota \mapsto (s \mapsto (t \mapsto c(s, \iota\cdot t))))$$

is smooth. To see this we have to show that elements d of $C^\infty(\mathbb{R}, C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0}))$ are mapped to elements $\Lambda(d)$ of $C^\infty(\mathbb{R}, C^\infty(I, C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0})))$.

The notation gets quite complicated here, so we discuss after applying Lemma 2.3 mappings $d^\wedge \in C^\infty(\mathbb{R}\times\mathbb{R}\times I, P)_{u_0}$ which are mapped to elements $\Lambda(d)^\wedge \in C^\infty(I\times\mathbb{R}\times\mathbb{R}\times I, P)_{u_0}$ where the index u_0 means that $d^\wedge(\kappa, s, 0) = \Lambda(d)^\wedge(\kappa,\iota,s,0) = u_0$.

In this setting we can define

$$\Lambda(d)^\wedge(\kappa,\iota,s,t) := d^\wedge(\kappa, s, \iota\cdot t).$$

So $\Lambda(d)^\wedge$ is obviously an element of $C^\infty(I\times\mathbb{R}\times\mathbb{R}\times I, P)_{u_0}$ by the smoothness of $d^\wedge$ and the smoothness of the multiplication with $\iota \in I$. This implies that Λ is a smooth mapping.

Next we consider the following diagram:

$$\begin{tikzcd}
C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0}) \arrow[r, "\delta"] \arrow[d, "\Lambda"'] & T(C^\infty(I,P)_{u_0}) \arrow[r, "\bar{\omega}_f"] & C^\infty(I,\mathfrak{g})_0 \\
C^\infty(I, C^\infty(\mathbb{R}, C^\infty(I,P)_{u_0})) \arrow[r, "\delta"] & C^\infty(I, T(C^\infty(I,P)_{u_0})) \arrow[ur, "f"] &
\end{tikzcd}$$

Because $f : T(C^\infty(I,P)_{u_0}) \to \mathfrak{g}$ is smooth, the same is true for the composition $f\circ\delta$. So the composition $f\circ\delta\circ\Lambda$ is smooth which proves the smoothness of $\bar{\omega}_f$. $\square$

Example 4.14. [2, p. 230] If (P, M, π, G) is a principal fiber bundle and ω is any Lie algebra valued connection form on P, then the induced connection form $\bar{\omega}$ on $(C^\infty(I,P)_{u_0}, C^\infty(I,M)_{x_0}, \pi_*, C^\infty(I,G)_0)$ is uniform. This follows because the equation

$$\bar{\omega}_\gamma(\delta)(t) = \bar{\omega}_{\gamma_t}(\delta_t)(1)$$

is trivially satisfied by an induced connection form.

The following (and last) section of this thesis gives as an example of a uniform connection which is not an induced connection. It is called Polyakov connection on $C^\infty(I,P)_{u_0}$.

4.4. Polyakov connection

In this section we will discuss a special uniform connection on the path bundle which arose through a discussion of a physical problem in gauge theories. A thorough motivation can be found in [2] where this construction is applied to a result given in [13]

Example 4.15. [2, p. 229] Let (P, M, π, G) be a principal fiber bundle, $x_0 \in M$, $u_0 \in \pi^{-1}(x_0)$ and ω an arbitrary Lie algebra valued connection form on P. As shown in Theorem 3.20 there exists a global section $s_\omega : C^\infty(I, M)_{x_0} \to C^\infty(I, P)_{u_0}$. Furthermore by Theorem 4.10 there is an equivariant mapping $\psi_\omega : C^\infty(I, P)_{u_0} \to C^\infty(I, G)_0$. Now let κ denote the Maurer-Cartan form of Definition 1.56 where we can choose the right or left one as we like.

The Maurer-Cartan form on G induces a Maurer-Cartan form $\varkappa$ on $C^\infty(I, G)_0$ by

$$\varkappa_g(v)(t) = \kappa_{g(t)}(v(t))$$

for $g \in C^\infty(I, G)_0$, $v \in T_g(C^\infty(I, G)_0)$ and $t \in I$. So it is possible to define a Lie algebra valued connection form

$$\bar{\omega} = (\psi_\omega^{-1})^* \varkappa$$

as the composition of the smooth mapping ψ_ω and the smooth Lie algebra valued one form $\varkappa$.

At this point it is easy to observe that

$$(s_\omega^* \bar{\omega})_\lambda(v_\lambda) = (\psi_\omega^{-1} \circ s_\omega)^*(\varkappa)_\lambda(v_\lambda) = 0$$

for $\lambda \in C^\infty(I, M)_{x_0}$ and $v_\lambda \in T_\lambda(C^\infty(I, M)_{x_0})$ because the composition $\psi_\omega^{-1} \circ s_\omega$ gives only the unit element $\bar{e} \in C^\infty(I, G)_0$. This implies that the curvature is 0 and so $\bar{\omega}$ is flat.

Proposition 4.16. [2, p. 230] *The Lie algebra valued connection form $\bar{\omega} = (\psi_\omega^{-1})^* \varkappa$ introduced in Example 4.15 is a uniform connection on $(C^\infty(I, P)_{u_0}, C^\infty(I, M)_{x_0}, \pi_*, C^\infty(I, G)_0)$.*

Proof. Let $\gamma \in C^\infty(I, P)_{u_0}$ and $t \in I$. Because of the construction of subpaths introduced in Definition 4.12, one gets that $\psi_\omega(\gamma_t) = \psi_\omega(\gamma)_t$. The same property follows for its inverse:

$$(\psi_\omega^{-1})(\gamma_t) = \psi_\omega^{-1}(\gamma)_t$$

Now let $v \in T_\gamma(C^\infty(I, P)_{u_0})$. We get for $t, s \in I$ that

$$T_{\gamma_t}(\psi_\omega^{-1})(v_t)(s) = T_\gamma(\psi_\omega^{-1})(v)(ts) = \left(T_\gamma(\psi_\omega^{-1})(v)\right)_t(s).$$

By inserting the definitions of $\bar{\omega}$ and $\varkappa$ we get for all $t, s \in I$:

$$((\psi_\omega^{-1})^* \varkappa)_{\gamma_t}(v_t)(s) = \varkappa_{\psi_\omega^{-1}(\gamma_t)}(T_{\gamma_t}(\psi_\omega^{-1}))(v_t)(s) = \kappa_{\psi_\omega^{-1}(\gamma_t)(s)}(T_{\gamma_t}(\psi_\omega^{-1})(v_t)(s)) =$$

$$= \kappa_{\psi_\omega^{-1}(\gamma)(ts)}(T_\gamma(\psi_\omega^{-1})(v)(ts)) = ((\psi_\omega^{-1})^* \varkappa)_\gamma(\delta)(ts).$$

Setting $s = 1$ the proposition follows. □

Remark 4.17. [2, p. 233] According to Theorem 4.13 there exists a mapping $f : T(C^\infty(I, P)_{u_0} \to \mathfrak{g}$ defined by

$$f(\gamma, v) = ((\psi_\omega^{-1})^* \varkappa)_\gamma(v)(1).$$

Although there are still some open problems concerning arbitrary connections or uniform ones in special, we close here our discussion of the principal path bundle and its connections.

A. Appendix

A.1. Functional Analysis

Definition A.1. [7, p, 9] A mapping $p : E \to \mathbb{R}$ where E is a vector space is called *sublinear* if the following properties are satisfied:

- $p(x + y) \leq p(x) + p(y)$ (subadditivity)
- $p(\lambda \cdot x) = \lambda \cdot p(x)$ for all $\lambda \in \mathbb{R}^+$. ($\mathbb{R}^+$-homogeneity)

Definition A.2. [10, p. 575] A *seminorm* $p : E \to \mathbb{R}$ is specified by the following properties:

- $p(x) \geq 0$
- $p(x + y) \leq p(x) + p(y)$
- $p(\lambda \cdot x) = |\lambda| \cdot p(x)$

Definition A.3. [7, p, 13] Let E be a vector space, p a mapping $E \to \mathbb{R}$ and $c \in \mathbb{R}$. Then we define

$$p_{<c} := \{x : p(x) < c\} \qquad p_{\leq c} := \{x : p(x) \leq c\}$$

as the *open and closed p-balls* around 0 with radius c.

Definition A.4. [7, p, 13] A subset A in a vector space E is called *absorbing* if

$$\forall x \in E \quad \exists \lambda > 0 : x \in \lambda \cdot A.$$

Equivalently [10, p. 575]: A subset A in a vector space is called absorbing if

$$\bigcup \{r \cdot A : r > 0\}$$

is the whole space.

Definition A.5. [7, p, 13] A subset A of a vector space E is called *convex* if for all $\lambda_i \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in A$ follows that $\sum_{i=1}^{n} \lambda_i x_i \in A$.

Lemma A.6. [7, p, 13] *For each sublinear mapping $0 \leq p : E \to \mathbb{R}$ and $c > 0$ the subsets $p_{\leq c}$ and $p_{<c}$ of E are convex and absorbing. The mapping p satisfies the following properties:*

- $p_{\leq c} = c \cdot p_{\leq 1}$

- $p_{<c} = c \cdot p_{<1}$
- $p(x) = c \cdot \inf\{\lambda > 0 : x \in \lambda \cdot p_{\leq c}\}$

Definition A.7. [7, p, 14] A subset A of a vectorspace is called *absolutly convex* if from $x_i \in A$ and $\lambda_i \in \mathbb{R}$ with $\sum_{i=1}^n |\lambda_i| = 1$ follows that $\sum_{i=1}^n \lambda_i x_i \in A$.

Definition A.8. [7, p, 14] A set A is called *balanced*, if for all $x \in A$ and $|\lambda| = 1$ it follows that $\lambda \cdot x \in A$, too.

Lemma A.9. [7, p, 14] *For each seminorm $p : E \to \mathbb{R}$ and $c > 0$ the sets $p_{\leq c}$ and $p_{<c}$ are absorbing, absolutely convex and they satisfy $p(x) = \inf\{\lambda > 0 : x \in \lambda p_{\leq 1} = p_{\leq \lambda}\}$.*

Lemma A.10. *A set A is absolutely convex if and only if it is convex and balanced.* [7, p, 14]

Definition A.11. [7, p, 14], [10, p. 575] The *Minkowski functional* is defined as

$$p_A(x) := \inf\{\lambda > 0 : x \in \lambda \cdot A\} \in \mathbb{R}$$

for all $x \in E$. It satisfies $p_A(x) < \infty$ if and only if x is in the cone generated by A which is the set $\{\lambda \in \mathbb{R} : \lambda > 0\} \cdot A$.

Lemma A.12. [7, p, 14] *Let A be a convex and absorbing set. Then the Minkowski functional is a well defined sublinear mapping $p := p_A \geq 0$ on E and it satisfies for $\lambda > 0$ that*

$$p_{<\lambda} \subseteq \lambda \cdot A \subseteq p_{\leq \lambda}.$$

If A is absolutely convex then p is a seminorm.

Example A.13. [7, p, 16] $C(\mathbb{R}, \mathbb{R})$: Space without reasonable norm. Define for each compact set $K \subseteq \mathbb{R}$ the mapping

$$p_K(f) := \sup\{|f(x)| : x \in K\}.$$

So one gets a family of seminorms p_K.

Definition A.14. [7, p, 16] Let P_0 be a family of seminorms on the vector space E. Then we call a subset $O \subseteq E$ open if

$$\forall a \in O \; \exists p_1, \cdots, p_n \in P_0, \; \exists \epsilon > 0 : a \in \{x : p_i(x - a) < \epsilon \, for \, i = 1, \cdots, n\} \subseteq O$$

The set $T := \{O : O \, \text{is open in} \, E\}$ is the *topology generated by* P_0.

Definition A.15. [10, p. 34] A radial subset U (i.e. $[0, 1]U \subseteq U$) of a locally convex space E is called *bornivorous* if it absorbs each bounded set, i.e. for every bounded B there exists $r > 0$ such that $[0, r]U \supseteq B$.

Definition A.16. [10, p. 34] A locally convex vector space E is called *bornological* if and only if the following equivalent conditions are satisfied:

1. For any locally convex vector space F any bounded linear mapping $T : E \to F$ is continuous; it is sufficient to know this for all Banach spaces F.

2. Every bounded seminorm on E is continuous.

3. Every absolutely convex bornivorous subset is a 0-neighborhood.

Definition A.17. [10, p. 577], [7, p, 46] A *Fréchet space* is a complete locally convex space with a metrizable topology, equivalently, with a countable base of seminorms.

Corollary A.18. [4, p. 59] *The projective limit of a projective system of complete (quasi-complete, sequentially complete) Hausdorff topological vector spaces is again a complete (quasi-complete, sequentially complete) topological vector space.*

A.2. Differential Geometry

A.2.1. Abstract Manifolds

Remark A.19. [11, p. 11] Let M and N be smooth finite dimensional manifolds described by smooth atlases $(U_\alpha, u_\alpha)_{\alpha \in A}$ and $(V_\beta, v_\beta)_{\beta \in B}$, respectively. Then the family $(U_\alpha \times V_\beta, u_\alpha \times v_\beta)_{(\alpha,\beta) \in A \times B}$ is a smooth atlas or the cartesian product $M \times N$. The projections from $M \times N$ to the factors M and N are smooth.

Remark A.20. [11, p. 11] From the construction of the tangent bundle for finite dimensional manifolds and A.19 it follows that the tangent space of a product is equal to the product of the tangent spaces in a canonical way. This generalizes to products of finitely many manifolds.

Definition A.21. [3, p. 29] A *halfspace of* $\mathbb{R}^n$, or a *n-halfspace*, is a subset of the form

$$H = \{x \in \mathbb{R}^n : \lambda(x) \geq 0\}$$

where $\lambda : \mathbb{R}^n \to \mathbb{R}$ is a linear map. If $\lambda \equiv 0$ then $H = \mathbb{R}^n$; otherwise H is called a proper halfspace. If H is proper, its boundary is the set $\partial H = \ker \lambda$. This is a linear subspace of dimension $n-1$. If $H = \mathbb{R}^n$ we set $\partial H = \{\}$.

We now extend the definition of chart on a space M to mean a mapping $\phi : U \to M$ which maps the open set $U \in M$ homeomorphically onto an open subset of a halfspace in $\mathbb{R}^n$. This includes all charts as defined earlier, since $\mathbb{R}^n$ is itself a halfspace, and many new charts as well. Using this definition of chart, we systematically extend the meaning of atlas, C^r-atlas, C^r-differential structure, and finally, C^r manifold for $r \in \{0, 1, 2, \ldots, \infty\}$. We call those *manifolds with boundary.*

A.2.2. Riemannian Geometry

Let M denote a finite dimensional Riemannian manifold.

Definition A.22. [8, p. 141] Let $\xi_x \in T_x M$. The *length of* ξ_x is defined as $\sqrt{g_x(\xi_x, \xi_x)}$.

Definition A.23. [8, p. 141] Let $c : [0,1] \to M$ be a smooth curve. The *length of* c is defined as

$$L(c) := \int_0^1 \sqrt{g_{c(t)}(c'(t), c'(t))}$$

Definition A.24. [8, p. 141] Given an Riemannian metric, the *distance function* d_g : $M \times M \to \mathbb{R}^+$ is defined as:

$$d_g(p,q) := \inf\{L(c) : c \in C^\infty(\mathbb{R}, M);\ c(0) = p,\ c(1) = q\}$$

This distance function defines the topology on M.

Example A.25. [14, p. 24] Let $(M_i, g_i)(i = 1, 2)$ be Riemannian manifolds. On the product manifold $M_1 \times M_2$ one may introduce the *product Riemannian metric* $g_1 \times g_2$ (or $g_1 \oplus g_2$) by

$$(g_1 \times g_2)_{(p_1,p_2)}((u_1, u_2), (v_1, v_2)) := (g_1)_{p_1}(u_1, v_1) + (g_2)_{p_2}(u_2, v_2),$$

where one uses the identification $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1} M_1 \oplus T_{p_2} M_2$.

Lemma A.26. [8, p. 201] *Let (M, g) be a Riemannian manifold. Then for all $p \in M$ and $\xi \in T_p M$ there exists a unique geodesic $c_\xi : I \to M$, defined on a maximal Interval $I \subseteq \mathbb{R}$, with constant scalar velocity and initial conditions $c_\xi(0) = p$ and $c'_\xi(0) = \xi$.*

Now define the mapping $exp : TM \supset U \to M$ by $(p, \xi) \mapsto c_\xi(1)$. The exponential map is defined on an open neighborhood U of the zero section of M in TM and is smooth there (see Lemma A.27).

The mapping $exp_p := exp\,|_{T_pM} : T_p M \to M$ satisfies $exp_p(0_p) = p$ and $T_{0_p}(exp_p) = id_{T_pM}$. The geodesic c_ξ is given by $c_\xi(t) = exp(t\xi)$.

Lemma A.27. [5, p. 18] *The exponential map exp_p maps a neighborhood of $0 \in T_p M$ diffeomorphically onto a neighborhood of $p \in M$.*
Or: There exists $\epsilon > 0$ such that the ϵ-ball $B_\epsilon(p) := \{q \in M : d_g(p,q) < \epsilon\}$ is precisely $\exp_p(U_\epsilon(p))$ where $U_\epsilon(p) := \{v \in T_p M : g_p(v,v) < \epsilon^2\}$

A.3. Topology

Lemma A.28. [12, p.175] *Let $\mathfrak{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having a diameter less than δ, there exists an element of $\mathfrak{A}$ containing it.*

The number δ is called the Lebesgue number of the covering $\mathfrak{A}$.

Definition A.29. [6, p. 37] Let X be a topological space. It is a $T1$-*space* if $\{x\}$ is a closed set of X for each $x \in X$. If any two disjoint closed sets are separated by open sets in X then X is a $T4$-*space.* The topological space X is called *normal space* if it is a $T1$- and a $T4$-space.

Definition A.30. [6, p. 170] A Hausdorff space X is called a *locally compact space* provided that every point of X has an open neighbourhood the closure of which is compact.

Definition A.31. [16, p. 50] Let the topological space X be normal, locally compact and such that any covering of X by open sets is reducible to a countable covering, then X is called a C_σ-space.

Theorem A.32. [16, p. 25] *Let $p : E \to X$ be a bundle over the differentiable manifold X such that E is a differentiable manifold, p is differentiable, and E, p, X, ϕ_j, and p_j (trivialization and restriction of projection) have class $\geq r$ ($r = 1, 2, \cdots, \infty$).*

Let $f : X \to E$ be a continuous cross-section. We shall suppose that f is of class $\geq r$ on a closed subset A of X. This means that f is of class $\geq r$ in some open set U of X containing A. (The case of A being vacuous is not excluded.) Finally, let ρ be a metric on E, and let ϵ be a positive number.

There exists a differentiable cross-section $f' : X \to E$ of class $\geq r$ such that $\rho(f(x), f'(x)) < \epsilon$ for each $x \in X$, and $f'(x) = f(x)$ for $x \in A$.

Theorem A.33. [16, p. 36] *A finite dimensional principal fiber bundle with group G is equivalent in G to the product bundle if and only if it admits a cross-section.*

Corollary A.34. [16, p. 53] *If X is a C_σ-space and is contractible on itself to a point, then any bundle over X is equivalent to a product bundle.*

Remark A.35. By Corollary A.34 a principal fiber bundle over a contractible base space is continuously trivial. This means there exists a global continuous section. So by Theorem A.33 there exists a continuous section. With Theorem A.32 it follows that a smooth section exists. So the continuously trivial principal fiber bundle is smoothly trivial too.

The trivialization for a principal bundle $X \to B$ with group G and section s is given by:

$$B \times G \to X \qquad (x, g) \mapsto s(x) \cdot g$$

Lemma A.36. [3, p. 89] *Let $\xi = (P, E, B \times I)$ be a C^r vector bundle and $0 \leq r \leq \infty$. Then each $b \in B$ has a neighborhood $V \subset B$ such that $\xi|_{V \times I}$ is trivial.*

Corollary A.37. [3, p. 89] *Every C^r vector bundle with $0 \leq r \leq \infty$ over an interval is trivial.*

Corollary A.38. [3, p. 97] *Every vector bundle over a contractible paracompact space is trivial.*

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Abstract

This thesis describes the construction of the so called path bundle and is mainly based on [2] and [10]. The starting point of the construction is a principal fiber bundle of finite dimensional manifolds. Next one considers the spaces of all mappings from a compact interval I into the base space and the total space of the bundle. These spaces are manifolds modelled over convenient vector spaces as described in Chapter 1 and Chapter 2. They are called manifolds of paths.

Together with the Lie group of all paths in the structure group of the finite dimensional principal fiber bundle, one constructs a principal fiber bundle where the base and total spaces are infinite dimensional manifolds of paths. This is called the path bundle. After that one can consider natural subbundles of the path bundle.

In Chapter 3 connections and connection forms on the path bundle are introduced and used to show the triviality of certain subbundles. Also it is discussed how connections on the finite dimensional principal fiber bundle induce connections on the path bundle and which properties they posses.

Chapter 4 defines curvature on the path bundle. Furthermore as a special type of connections, the uniform ones are introduced together with a non trivial example.

Zusammenfassung

Diese Arbeit beschreibt die Konstruktion des sogenannten Pfadbündels und basiert auf [2] und [10]. Ausgehend von einem Hauptfaserbündel bestehend aus endlich dimensionalen Mannigfaltigkeiten werden die Räume aller Abbildungen von einem kompakten Intervall I in diese Mannigfaltigkeiten betrachtet. Diese Abbildungsräume können als unendlich dimensionale Mannigfaltigkeiten modelliert über "convenient vector spaces" betrachtet werden was in Kapitel 1 und Kapitel 2 beschrieben wird. Diese Abbildungsräume werden als Pfadmannigfaltigkeiten bezeichnet.

Diese Konstruktion kann auch für die Lie Gruppe des endlich dimensionalen Hauptfaserbündels durchgeführt werden und man erhält eine unendlich dimensionale Lie Gruppe bestehend aus Pfaden in der endlich dimensionalen Lie Gruppe. Nun ist es möglich die so entstandenen Pfadmannigfaltigkeiten und die Pfad Lie Gruppe zu einem Hauptfaserbündel zusammenzufügen. Dieses aus unendlich dimensionalen Mannigfaltigkeiten bestehende Hauptfaserbündel wird als Pfadbündel bezeichnet. Anschließend können kanonische Teilbündel dieses Hauptfaserbündels betrachtet werden.

In Kapitel 3 werden Konnexionen und Konnexionsformen auf dem Pfadbündel definiert. Diese werden verwendet um die Trivialität bestimmter Teilbündel nachzuweisen. Weiters folgt eine Betrachtung wie Konnexionen auf dem endlich dimensionalen Hauptfaserbündel wiederum Konnexionen auf dem Pfadbündel induzieren und welche Eigenschaften diese besitzen.

Kapitel 4 definiert den Begriff der Krümmung auf dem Pfadbündel. Desweiteren werden spezielle Konnexionen definiert. Diese heißen uniforme Konnexionen. Für diese wird ein spezielles, nicht triviales Beispiel konstruiert.

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