DISSERTATION

Titel der Dissertation

„Implications of finite family symmetry groups on the leptonic and scalar sector“

Verfasser

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I want to know God's thoughts, the rest are details.

Albert Einstein

Contents

Abstract

The problem of lepton masses and mixing is one of the most interesting topics in particle physics and withstood a solution for decades. One approach for an at least partial solution to this problem is the introduction of so-called finite family symmetry groups in the leptonic and scalar sector. This thesis concentrates on three questions arising from this approach.

The first question is which groups are eligible as finite family symmetry groups. Since there are three generations of fermions, the group must possess at least one non-trivial three-dimensional representation. Concentrating on those groups which possess faithful three-dimensional representations, we end up with the finite subgroups of $U(3)$. Using the so-called SmallGroups library, we could create a list of all finite subgroups of $U(3)$ of order smaller than 512 which possess a faithful three-dimensional irreducible representation and are not isomorphic to a direct product involving a cyclic group. By means of a theorem proven in this thesis we were also able to explicitly construct several infinite series of finite subgroups of $U(3)$. The smallest members of these series can also be found in our list of groups of order smaller than 512.

Since many finite subgroups of $SU(3)$ are frequently used as family symmetry groups, we put special emphasis on the study of this class of groups. Already at the beginning of the 20^{th} century H.F. Blichfeldt provided a classification of all finite subgroups of $SU(3)$ into five different classes (A)–(E). However, not all of these classes have been studied in detail. In particular the knowledge on the general structure of the groups of type (C) and (D), though they contain the well-known group series $\Delta(3n^2)$ and $\Delta(6n^2)$, was rather sparse. With the help of a theorem on the general structure of Abelian finite subgroups of $SU(3)$ we were able to unveil the general structure of the groups of type (C) and (D).

The second question that was discussed in this thesis is the question how symmetries restrict the mass and mixing matrices. Elaborating on the simplest case, which is the case of Abelian family symmetries, we encountered the intensively studied possibility of texture zeros in the lepton mass matrices. In this respect we concentrated on the seven experimentally allowed types of two texture zeros in the neutrino mass matrix in the framework of a diagonal charged-lepton mass matrix. We could show that two of these seven types of texture zeros lead to nearly maximal atmospheric neutrino mixing in the limit of a quasi-degenerate neutrino mass spectrum, irrespective of the values of the other two mixing angles. Another interesting aspect is the prediction of the seven types of two texture zeros in the light of the recent measurements of the reactor mixing angle. We found that, using the experimental ranges of the other oscillation parameters as an input, two of the seven textures enforce a non-zero reactor mixing angle at 3σ. In the course of

this analysis we also studied the correlation between the Dirac phase δ and the reactor mixing angle.

Finally, the third question treated in this thesis, was whether one could infer symmetries in the lepton sector from the presently available data on the observables of neutrino physics. Again choosing the framework of a diagonal charged-lepton mass matrix, we could analytically derive upper and lower bounds on the absolute values of the elements of the neutrino mass matrix. By means of a numerical analysis using the so-called Nelder-Mead algorithm we could furthermore create correlation plots of the absolute values of the elements of M_ν. These plots beautifully show the experimentally allowed cases of two texture zeros in the neutrino mass matrix and reveal strong correlations for some pairs of matrix elements. In this way our analysis allows to exclude models which produce textures which are in conflict with the correlation plots, and may thus serve as a helpful tool for model building.

The structure of the thesis is as follows. Part I collects introductory chapters explaining the basic knowledge needed in the course of the thesis. In chapter 1 we review the standard model of particle physics and its extensions accommodating massive neutrinos. The following chapter is devoted to the observables of neutrino physics and the corresponding experimental and observational status. Chapter 3 comprises a short review on the numerical methods used within this thesis. Finally, chapter 4 explains the fermion mass and mixing problem and motivates the contributions to its solution contained within this thesis.

The second part of the thesis collects five papers which have emerged in the course of this dissertation project. Each paper is contained in an own chapter, the first page always showing the names of all authors and the detailed reference to the journal where it is published.

The conclusions of the thesis are drawn in part III. Note that every chapter in this dissertation has its own bibliography at the end of the corresponding chapter.

Part I

Introduction

Chapter 1

Massive neutrinos in extensions of the standard model

1.1 The standard model of particle physics

Before we explain how one can extend the standard model in order to incorporate massive neutrinos, we briefly review the basics of the standard model needed in the course of this thesis.

The standard model of particle physics is a gauge theory based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The group $SU(3)_c$, acting on the quark and gluon fields, describes the strong interactions; the remaining gauge group $SU(2)_L \times U(1)_Y$ is the basis of the theory of electroweak interactions [1–3]. Since leptons (apart from gravity) interact via electroweak interactions only, in the following we will concentrate on this type of interactions. The standard model fermion fields and their associated charges are listed in table 1.1. The scalar doublet (Higgs doublet) is presented in table 1.2 and for the gauge bosons of electroweak interactions the reader may consult table 1.3.

Under the $SU(2)_L$ gauge symmetry the fields transform as

$$\psi \longmapsto U\psi \qquad \text{(fermion doublets)}, \tag{1.1a}$$
$$\psi \longmapsto \psi \qquad \text{(fermion singlets)}, \tag{1.1b}$$
$$\phi \longmapsto U\phi \qquad \text{(Higgs doublet)}, \tag{1.1c}$$
$$W_\mu \longmapsto UW_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1} \qquad (SU(2)_L \text{ gauge field}), \tag{1.1d}$$
$$B_\mu \longmapsto B_\mu \qquad (U(1)_Y \text{ gauge field}), \tag{1.1e}$$

where $U \in SU(2)$ and

$$W_\mu = W_\mu^a T^a \quad \text{with} \quad T^a = \frac{\tau^a}{2}. \tag{1.2}$$

The matrices τ^a $(a = 1, 2, 3)$ are the Pauli-matrices. The $U(1)_Y$ gauge transformations

Field multiplet	T	T^3	Y	Q
$D_{\alpha L} = \begin{pmatrix} \nu_{\alpha L} \\ \ell_{\alpha L} \end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$
$\ell_{\alpha R}$	0	0	-2	-1
$Q_{iL} = \begin{pmatrix} u_{iL} \\ d_{iL} \end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$
u_{iR}	0	0	$4/3$	$2/3$
d_{iR}	0	0	$-2/3$	$-1/3$

Table 1.1: Fermionic field content and associated charges of the standard model ($\alpha = e, \mu, \tau$; $i = 1, 2, 3$; $u_1, u_2, u_3 = u, c, t$; $d_1, d_2, d_3 = d, s, b$). Note that—following common notation—for the flavour indices of the lepton fields we use Greek letters which must not be confused with Lorentz indices. T denotes the weak isospin, i.e. the representation of $SU(2)_L$ according to which the fields transform. T^3 is the third generator of $SU(2)_L$ in this representation. Y denotes the weak hypercharge and Q the electric charge.

Field multiplet	T	T^3	Y	Q
$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
$\widetilde{\phi} = i\tau^2\phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

Table 1.2: The charges of the standard model Higgs doublet. τ^2 is the second Pauli matrix. For further information *cf.* table 1.1.

are given by

$$\psi \longmapsto \exp(-i\alpha Y_\psi/2)\psi, \tag{1.3a}$$

$$\phi \longmapsto \exp(-i\alpha Y_\phi/2)\phi, \tag{1.3b}$$

$$W_\mu \longmapsto W_\mu, \tag{1.3c}$$

$$B_\mu \longmapsto B_\mu + \frac{1}{g'}\partial_\mu\alpha, \tag{1.3d}$$

with $\alpha \in \mathbb{R}$.

Taking gauge invariance into account, we can write down the kinetic and gauge parts of the Lagrangian of electroweak interactions:

$$\mathcal{L} = \mathcal{L}_W + \mathcal{L}_B + \sum_\psi \mathcal{L}_\psi + \mathcal{L}_\phi, \tag{1.4}$$

Field multiplet	T	T^3	Y	Q
W^a_μ	1	$-i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	$-i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
B_μ	0	0	0	0
$\begin{pmatrix} W^+_\mu \\ W^-_\mu \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} W^1_\mu \\ W^2_\mu \end{pmatrix}$	—	—	0	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_w & -s_w \\ s_w & c_w \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$	—	—	0	0

Table 1.3: The gauge fields of the $SU(2)_L \times U(1)_Y$ gauge theory and their associated charges ($a = 1, 2, 3$; $c_w = \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}}$, $s_w = \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}$). If a collection of fields does not form a multiplet of $SU(2)_L \times U(1)_Y$, this is indicated by — in the corresponding column. For further information *cf.* table 1.1.

where

$$\mathcal{L}_W = -\frac{1}{2}\mathrm{Tr}(W_{\mu\nu}W^{\mu\nu}) = -\frac{1}{4}W^a_{\mu\nu}W^{\mu\nu a}, \tag{1.5a}$$

$$\mathcal{L}_B = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{1.5b}$$

$$\mathcal{L}_\psi = \overline{\psi}i\gamma^\mu D_\mu\psi, \tag{1.5c}$$

$$\mathcal{L}_\phi = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi). \tag{1.5d}$$

Here we have introduced the notations

$$W_{\mu\nu} = W^a_{\mu\nu}T^a = \partial_\mu W_\nu - \partial_\nu W_\mu + ig\,[W_\mu, W_\nu] \tag{1.6}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.7}$$

for the field strength tensors of the $SU(2)_L$ and $U(1)_Y$ gauge fields, respectively. The covariant derivative is given by

$$D_\mu = \partial_\mu + igW^a_\mu T^a + ig'\frac{Y}{2}B_\mu, \tag{1.8}$$

where

$$T^a = \begin{cases} \frac{\tau^a}{2} & \text{for } SU(2)_L\text{-doublets,} \\ 0 & \text{for } SU(2)_L\text{-singlets} \end{cases} \tag{1.9}$$

and Y is the hypercharge of the field the covariant derivative acts on. Due to the scalar potential

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 \qquad (\mu^2 < 0,\ \ \lambda > 0) \tag{1.10}$$

the gauge symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken [4–9] to the gauge group $U(1)_{EM}$ of electromagnetic interactions. Indeed it is possible to choose a gauge (the so-called *unitary gauge*) in which the Higgs doublet takes the form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \qquad v = \sqrt{-\frac{\mu^2}{\lambda}}, \tag{1.11}$$

where h is a real field. From equations (1.10) and (1.11) we can readily deduce that

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{1.12}$$

is a minimum of the scalar potential. The generator Q of the unbroken gauge symmetry $U(1)_{EM}$ can be determined from the vacuum expectation value (1.12) by means of the invariance condition[1]

$$Q\langle 0|\phi|0\rangle = 0, \tag{1.14}$$

which leads to the well-known expression for the electric charge

$$Q = T^3 + \frac{1}{2}Y. \tag{1.15}$$

Equation (1.15) is used to determine the a priori unknown hypercharges in tables 1.1, 1.2 and 1.3. Diagonalizing $(D_\mu\phi)^\dagger(D^\mu\phi)$ in unitary gauge, i.e. using equation (1.11), one obtains the mass eigenfields W^+_μ, W^-_μ, Z_μ and A_μ. The relations between the mass eigenfields and the gauge bosons W^a_μ and B_μ can be found in table 1.3. The masses of the new massive bosons are given by

$$M_W = \frac{gv}{2}, \qquad M_Z = \sqrt{\frac{g^2 + g'^2}{4}}v, \tag{1.16}$$

leading to the famous relation

$$\frac{M_W}{M_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\theta_w. \tag{1.17}$$

By introducing Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} = -(\overline{Q_{iL}}\phi\Gamma^{(d)}_{ij}d_{jR} + \overline{Q_{iL}}\widetilde{\phi}\Gamma^{(u)}_{ij}u_{jR} + \overline{D_{\alpha L}}\phi\Gamma^{(\ell)}_{\alpha\beta}\ell_{\beta R} + \text{H.c.}), \tag{1.18}$$

also the fermion masses can be accommodated within the standard model. The matrices $\Gamma^{(d)}, \Gamma^{(u)}$ and $\Gamma^{(\ell)}$ are complex 3×3-matrices of Yukawa coupling constants. Spontaneous symmetry breaking then gives rise to the fermion mass terms

$$\mathcal{L}_M = -(\overline{d_L}M_d d_R + \overline{u_L}M_u u_R + \overline{\ell_L}M_\ell \ell_R + \text{H.c.}), \tag{1.19}$$

[1] From $Q\langle 0|\phi|0\rangle = 0$ follows

$$e^{i\alpha Q}\langle 0|\phi|0\rangle = \langle 0|\phi|0\rangle \tag{1.13}$$

and thus invariance of $\langle 0|\phi|0\rangle$ under $U(1)_{EM}$.

where
$$M_d = \frac{v}{\sqrt{2}}\Gamma^{(d)}, \quad M_u = \frac{v}{\sqrt{2}}\Gamma^{(u)}, \quad M_\ell = \frac{v}{\sqrt{2}}\Gamma^{(\ell)} \tag{1.20}$$
are the mass matrices. At this point we have collected the members of a fermion family into one vector, i.e. $d = (d, s, b)^T$ etc. Since the standard model does not contain right-handed neutrino singlets, there is no neutrino mass matrix. Hence neutrinos are massless in the standard model. The physical (i.e. massive) fermion fields can be obtained by diagonalizing the mass matrices. The following theorem ensures that this is always possible.

Theorem 1. Let M be a complex $n \times n$-matrix. Then there exist unitary $n \times n$-matrices U_L and U_R such that

$$U_L^\dagger M U_R = \hat{M} = \mathrm{diag}(m_1, \ldots, m_n), \quad m_i \geq 0. \tag{1.21}$$

Proof. [2] The matrix $M^\dagger M$ is normal, hence diagonalizable, i.e. there exists an orthonormal basis $|x_i\rangle$ of eigenvectors of $M^\dagger M$:

$$M^\dagger M|x_i\rangle = \lambda_i |x_i\rangle. \tag{1.22}$$

For all non-zero eigenvalues $\lambda_i \neq 0$ we define[3]

$$|y_i\rangle \equiv \frac{1}{\sqrt{\lambda_i}} M|x_i\rangle. \tag{1.23}$$

Since
$$\langle y_i|y_j\rangle = \frac{\langle x_i|M^\dagger M|x_j\rangle}{\sqrt{\lambda_i \lambda_j}} = \frac{\lambda_j \delta_{ij}}{\sqrt{\lambda_i \lambda_j}} = \delta_{ij}, \tag{1.24}$$
the vectors $|y_i\rangle$ form an orthonormal system, which—by adding vectors $|z_j\rangle$—we can extend to an orthonormal basis of $\mathbb{C}^n$. For those vectors $|x_i\rangle$ corresponding to eigenvalues $\lambda_i \neq 0$, we have
$$M|x_i\rangle = \sqrt{\lambda_i}|y_i\rangle. \tag{1.25}$$
However, for the eigenvectors $|x_j\rangle$ with $\lambda_j = 0$ we have

$$0 = M|x_j\rangle = \underbrace{\sqrt{\lambda_j}}_{0}|z_j\rangle, \tag{1.26}$$

and we define $|y_j\rangle \equiv |z_j\rangle$. In this way we have obtained an orthonormal basis $|y_i\rangle$, such that
$$M|x_i\rangle = \sqrt{\lambda_i}|y_i\rangle \quad \forall\, i = 1, \ldots, n. \tag{1.27}$$
Using the completeness relation for the orthonormal basis $|x_i\rangle$, we find

$$M = M\mathbb{1} = \sum_i \underbrace{M|x_i\rangle}_{\sqrt{\lambda_i}|y_i\rangle}\langle x_i|. \tag{1.28}$$

[2] In this proof we do not use Einstein's summation convention.

[3] Note that the λ_i are non-negative since $\lambda_i = \langle x_i|M^\dagger M|x_i\rangle = ||Mx_i||^2 \geq 0$.

Defining the unitary matrices

$$U_L \equiv \sum_i |y_i\rangle\langle e_i| \qquad \text{and} \qquad U_R \equiv \sum_i |x_i\rangle\langle e_i|, \tag{1.29}$$

where $|e_i\rangle$ is the standard basis of $\mathbb{C}^n$, we obtain

$$U_L^\dagger M U_R = \sum_{i,j} |e_i\rangle \underbrace{\langle y_i|M|x_j\rangle}_{\sqrt{\lambda_j}\delta_{ij}} \langle e_j| = \sum_i \sqrt{\lambda_i}|e_i\rangle\langle e_i| = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}), \tag{1.30}$$

which proves the theorem. □

The mass eigenfields (indicated by a prime) are given by

$$u_{L,R} \equiv U^{(u)}_{L,R} u'_{L,R}, \quad d_{L,R} \equiv U^{(d)}_{L,R} d'_{L,R}, \quad \ell_{L,R} \equiv U^{(\ell)}_{L,R} \ell'_{L,R}, \tag{1.31}$$

and inserting this into the expression for the mass terms (1.19) leads to the usual Dirac mass terms

$$\begin{aligned} -\mathcal{L}_M &= \overline{d'_L} \hat{M}_d d'_R + \overline{u'_L} \hat{M}_u u'_R + \overline{\ell'_L} \hat{M}_\ell \ell'_R + \text{H.c.} \\ &= \overline{d'} \hat{M}_d d' + \overline{u'} \hat{M}_u u' + \overline{\ell'} \hat{M}_\ell \ell', \end{aligned} \tag{1.32}$$

where $\hat{M}_u = \mathrm{diag}(m_u, m_c, m_t), \ldots$ are the diagonalized mass matrices, and

$$d' \equiv d'_L + d'_R, \quad u' \equiv u'_L + u'_R, \quad \ell' \equiv \ell'_L + \ell'_R \tag{1.33}$$

are the Dirac fields describing the massive standard model fermions.

1.2 Neutrino mass terms and lepton number conservation

Since, due to the particular structure of the standard model, there is no neutrino mass term[4] in the Lagrangian, we have to extend the field content in order to allow massive neutrinos [10, 11].

The simplest possibility to accommodate a neutrino mass term is the introduction of N_R right-handed neutrinos possessing Yukawa couplings analogous to those of the right-handed quark fields

$$\mathcal{L}_{\text{Yukawa,D}} = -\overline{\nu_R} \widetilde{\phi}^\dagger \Gamma^{(\nu)} D_L + \text{H.c.}, \tag{1.34}$$

which, after spontaneous symmetry breaking, give rise to the neutrino mass term[5]

$$\mathcal{L}_D = -\overline{\nu_R} M_D \nu_L + \text{H.c.} \tag{1.35}$$

[4]A mass term is a Lorentz-invariant term bilinear in the fermion fields.

[5]Note that in the common convention for neutrino mass terms, the mass matrix M appears as $-\overline{\psi_R} M \psi_L + \text{H.c.}$, while in the quark and charged-lepton sector we have defined the mass matrix via $-\overline{\psi_L} M \psi_R + \text{H.c.}$

The mass term (1.35) is called *Dirac mass term*. As usual the vector D_L collects the lepton doublets D_{eL}, $D_{\mu L}$ and $D_{\tau L}$. ν_R is a vector containing the N_R right-handed neutrino fields $\nu_{1R}, \ldots, \nu_{N_R R}$ and $\Gamma^{(\nu)}$ is a complex $N_R \times 3$-matrix of Yukawa coupling constants. The generalization of theorem 1 to non-square matrices allows to diagonalize the resulting non-square mass matrix $M_D = \frac{v}{\sqrt{2}}\Gamma^{(\nu)}$ via

$$U_R^\dagger M_D U_L = \hat{M}_D = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.36}$$

where U_R and U_L are unitary $N_R \times N_R$ and 3×3-matrices, respectively. Obviously, the number of massive neutrino states is at most three, even if $N_R > 3$.

In order not to spoil gauge invariance, the right-handed neutrinos must have the following charges under the $SU(2)_L \times U(1)_Y$ gauge symmetry:

	T	T^3	Y	Q
ν_{jR}	0	0	0	0

Consequently, the right-handed neutrino fields do not couple to the gauge fields, i.e. they are so-called *sterile* neutrinos. At first glance everything looks fine now, however, there is still a subtle point of great importance, which has to do with lepton number conservation. When constructing the standard model following the rules of Lorentz and gauge invariance, an *accidental* (i.e. not imposed a priori) symmetry arises, namely the global $U(1)$ symmetry

$$\begin{aligned} L:\ \ell_{\alpha L,R} &\longmapsto e^{i\varphi}\ell_{\alpha L,R} \\ \nu_{\alpha L} &\longmapsto e^{i\varphi}\nu_{\alpha L}, \end{aligned} \tag{1.37}$$

which implies conservation of lepton number. This symmetry may also be extended to our new right-handed fields ν_R:

$$L:\ \ \nu_{jR} \longmapsto e^{i\varphi}\nu_{jR}. \tag{1.38}$$

If we promote the previously accidental symmetry L to a symmetry of the full Lagrangian, the Yukawa coupling (1.34) is the only allowed term generating neutrino masses. In this case the neutrino mass eigenfields are usual Dirac fields. On the contrary, if we allow for lepton number violation—which is of course a question to be answered by experiment—there is another neutrino mass term compatible with gauge invariance, namely

$$\mathcal{L}_R = -\frac{1}{2}\overline{\nu_R}M_R(\nu_R)^c + \text{H.c.} \tag{1.39}$$

where $\psi^c \equiv C\overline{\psi}^T$ denotes the charge conjugate field, and C is the charge conjugation matrix—see appendix C. From the antisymmetry of C and the anticommutativity of fermion fields, one can deduce that the $N_R \times N_R$-matrix M_R must be symmetric. In addition to the explicit breaking of lepton number introduced by the mass term (1.39), lepton number could also be violated spontaneously, leading to the left-handed neutrino mass term[6]

$$\mathcal{L}_L = -\frac{1}{2}\overline{(\nu_L)^c}M_L\nu_L + \text{H.c.} = \frac{1}{2}\nu_L^T C^{-1} M_L \nu_L + \text{H.c.} \tag{1.40}$$

Again, the $N_L \times N_L$ mass matrix M_L must be symmetric. As long as we do not introduce additional fermion generations, which we will not do in this thesis, N_L will always be three. Combining the three different neutrino mass terms, one obtains the so-called *Dirac-Majorana mass term*

$$\mathcal{L}_{\text{Dirac-Majorana}} = \mathcal{L}_D + \mathcal{L}_L + \mathcal{L}_R = -\frac{1}{2}\overline{(n_L)^c}Mn_L + \text{H.c.}, \tag{1.41}$$

where

$$n_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \quad (n_L)^c = \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{1.42}$$

Using the helpful relation

$$\overline{\psi_1^c}\psi_2^c = \overline{\psi_2}\psi_1 \tag{1.43}$$

for fermion fields, one can check the correctness of equation (1.41). The symmetric mass matrix M can now be diagonalized using the following theorem.

Theorem 2. Let M be a complex symmetric $n \times n$-matrix. Then there exists a unitary $n \times n$-matrix U, such that

$$U^T M U = \hat{M} = \text{diag}(m_1, \ldots, m_n), \quad m_i \geq 0. \tag{1.44}$$

The proof of this theorem can be found in [12]. Application of theorem 2 to the Dirac-Majorana mass term leads to

$$\begin{aligned}
\mathcal{L}_{\text{Dirac-Majorana}} &= -\frac{1}{2}\overline{(n_L)^c}Mn_L + \text{H.c.} \\
&= -\frac{1}{2}\overline{(n_L)^c}\,U^*\hat{M}U^\dagger n_L + \text{H.c.} \\
&= -\frac{1}{2}\sum_{j=1}^{N_L+N_R} m_j\overline{(n'_{jL})^c}\,n'_{jL} + \text{H.c.} \\
&= \ \ \frac{1}{2}\sum_{j=1}^{N_L+N_R} m_j n'^T_{jL}\,C^{-1}n'_{jL} + \text{H.c.},
\end{aligned} \tag{1.45}$$

where we have defined

$$n'_L \equiv U^\dagger n_L \qquad (\Rightarrow (n'_L)^c = (U^\dagger n_L)^c = U^T(n_L)^c). \tag{1.46}$$

[6]In order to generate a gauge-invariant Yukawa-coupling leading to this type of mass term, one could use scalar $SU(2)_L$-triplets—see section 1.3, equation (1.75).

Taking into account that n'_{jL} and n'^c_{jL} have opposite chiralities, we can rewrite equation (1.45) to finally obtain the mass term

$$\mathcal{L}_{\text{Dirac-Majorana}} = -\frac{1}{2}\sum_{j=1}^{N_L+N_R} m_j(\overline{(n'_{jL})^c}\, n'_{jL} + \overline{n'_{jL}}(n'_{jL})^c) = -\frac{1}{2}\sum_{j=1}^{N_L+N_R} m_j \overline{\nu'_j}\nu'_j, \tag{1.47}$$

where

$$\nu'_j \equiv n'_{jL} + (n'_{jL})^c \tag{1.48}$$

are the physical neutrino fields. Since these fields obey

$$(\nu'_j)^c = \nu'_j, \tag{1.49}$$

they are *Majorana fields*. Thus in extensions of the standard model with massive neutrinos and lepton number violation, neutrinos are Majorana particles.[7]

1.3 The seesaw mechanism

From the Dirac-Majorana mass term (1.41) emerges a mechanism, which provides an explanation of the smallness of the neutrino masses compared to the masses of the charged leptons. Since the right-handed neutrinos are sterile, the elements of their mass matrix M_R can be much higher than the electroweak scale, which roughly determines the masses of the quarks and charged leptons. To formulate this in a more precise way, we define, for a mass matrix M_X, the scale m_X to be of the order of magnitude[8] of the eigenvalues of

$$\sqrt{M_X^\dagger M_X}. \tag{1.50}$$

In the following we will study a situation, in which the scale m_R of M_R is much larger than the scales of M_D and M_L, i.e.

$$m_D, m_L \ll m_R. \tag{1.51}$$

Since, according to theorem 2, the symmetric mass matrix M in the Dirac-Majorana mass term

$$\mathcal{L}_{\text{Dirac-Majorana}} = -\frac{1}{2}\overline{(n_L)^c}\underbrace{\begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}}_{M} n_L + \text{H.c.} \tag{1.52}$$

---

[7] If the model exhibits no conserved lepton number, all neutrinos are Majorana particles. However, there may be a conserved lepton number (i.e. a $U(1)$-symmetry) which is not identical to the usual lepton number. In extensions of the standard model with such a conserved lepton number there will in general be both Dirac *and* Majorana neutrinos [13–17].

[8] At this point we implicitly assume all eigenvalues to be of roughly the same order of magnitude. However, in the framework of the so-called *singular seesaw mechanism* [18–20] $M_R^\dagger M_R$ can even have eigenvalues zero.

is diagonalizable via $U^T M U = \hat{M}$, there also exists a unitary matrix W which only block-diagonalizes M. Under the assumption (1.51) there will be a unitary matrix W block-diagonalizing M as

$$W^T \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix} W = \begin{pmatrix} M_{\text{light}} & 0 \\ 0 & M_{\text{heavy}} \end{pmatrix}, \tag{1.53}$$

where the scale m_{light} of the elements of M_{light} is much smaller than the scale m_{heavy} of the elements of M_{heavy}. Defining a new set of fields $\widetilde{n}_L$ by

$$\widetilde{n}_L = \begin{pmatrix} \nu_L^{\text{light}} \\ (\nu_R^{\text{heavy}})^c \end{pmatrix} \equiv W^\dagger n_L, \tag{1.54}$$

we can rewrite the Dirac-Majorana mass term as

$$\mathcal{L}_{\text{Dirac-Majorana}} = -\frac{1}{2}\overline{(\widetilde{n}_L)^c} \begin{pmatrix} M_{\text{light}} & 0 \\ 0 & M_{\text{heavy}} \end{pmatrix} \widetilde{n}_L + \text{H.c.} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}}, \tag{1.55}$$

where

$$\mathcal{L}_{\text{light}} = -\frac{1}{2}\overline{(\nu_L^{\text{light}})^c} M_{\text{light}} \nu_L^{\text{light}} + \text{H.c.} \tag{1.56a}$$

$$\mathcal{L}_{\text{heavy}} = -\frac{1}{2}\overline{\nu_R^{\text{heavy}}} M_{\text{heavy}} (\nu_R^{\text{heavy}})^c + \text{H.c.} \tag{1.56b}$$

are effective mass terms of the light and heavy neutrino fields[9] ν_L^{light} and ν_R^{heavy}, respectively.

Equation (1.53) can be solved using the ansatz [21]

$$W = \begin{pmatrix} \sqrt{\mathbb{1} - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbb{1} - B^\dagger B} \end{pmatrix} \tag{1.57}$$

for the unitary matrix W. Here the square-root of matrices is always understood in the sense of the Taylor expansion

$$\sqrt{\mathbb{1} - A} = \mathbb{1} - \frac{1}{2}A - \frac{1}{8}A^2 + \dots\ . \tag{1.58}$$

[9] M_{light} and M_{heavy} are not yet diagonal. Thus ν_L^{light} and ν_R^{heavy} are not mass eigenfields. The mass eigenfields are obtained by diagonalization of the effective mass matrices by means of theorem 2.

Inserting the ansatz (1.57) into equation (1.53) yields the following set of equations:

$$\begin{aligned} M_{\text{light}} = &\sqrt{\mathbb{1} - BB^\dagger}^T M_L \sqrt{\mathbb{1} - BB^\dagger} - B^* M_D \sqrt{\mathbb{1} - BB^\dagger} + \\ &- \sqrt{\mathbb{1} - BB^\dagger}^T M_D^T B^\dagger + B^* M_R B^\dagger, \end{aligned} \tag{1.59a}$$

$$\begin{aligned} 0 = &\sqrt{\mathbb{1} - BB^\dagger}^T M_L B - B^* M_D B + \\ &+ \sqrt{\mathbb{1} - BB^\dagger}^T M_D^T \sqrt{\mathbb{1} - B^\dagger B} - B^* M_R \sqrt{\mathbb{1} - B^\dagger B}, \end{aligned} \tag{1.59b}$$

$$\begin{aligned} M_{\text{heavy}} = &B^T M_L B + \sqrt{\mathbb{1} - B^\dagger B}^T M_D B + B^T M_D^T \sqrt{\mathbb{1} - B^\dagger B} + \\ &+ \sqrt{\mathbb{1} - B^\dagger B}^T M_R \sqrt{\mathbb{1} - B^\dagger B}. \end{aligned} \tag{1.59c}$$

Expanding B in powers of[10] m_R^{-1}, assuming the lowest order to be m_R^{-1} [21], i.e.

$$B = B_1 + \mathcal{O}(m_R^{-2}), \quad B_1 = \mathcal{O}(m_R^{-1}), \tag{1.60}$$

from equation (1.59b) we find

$$B_1^* = M_D^T M_R^{-1}. \tag{1.61}$$

Having obtained the expansion of B to lowest order in m_R^{-1}, we find the following expressions for M_{light} and M_{heavy} at lowest order:

$$M_{\text{light}} = M_L - M_D^T M_R^{-1} M_D, \tag{1.62a}$$

$$M_{\text{heavy}} = M_R. \tag{1.62b}$$

In agreement with our assumptions we have

$$m_{\text{light}} \sim m_L, \frac{m_D^2}{m_R} \ll m_R \sim m_{\text{heavy}}. \tag{1.63}$$

In the absence of the mass term $\mathcal{L}_L$ equation (1.62a) becomes

$$M_{\text{light}} = -M_D^T M_R^{-1} M_D. \tag{1.64}$$

The mechanism based on the usual Dirac mass term $\mathcal{L}_D$ and a right-handed Majorana mass term $\mathcal{L}_R$, leading to the effective Majorana mass matrix (1.64) of the light neutrino mass eigenfields, is the famous *type-I seesaw mechanism* [22–26]. Since $B = \mathcal{O}(m_R^{-1})$, we find $W = \mathbb{1} + \mathcal{O}(m_R^{-1})$ and thus from equation (1.54)

$$\nu_L^{\text{light}} = \nu_L + \mathcal{O}(m_R^{-1}). \tag{1.65}$$

Inserting this into the expression (1.56a) for the mass term of the light neutrinos, we obtain, up to orders $\mathcal{O}(m_R^{-1})$, the effective Majorana mass term

$$-\frac{1}{2}\overline{(\nu_L)^c} M_{\text{light}} \nu_L + \text{H.c.} \tag{1.66}$$

[10] To be exact, since B is dimensionless, it must be expanded in terms of m_D/m_R and m_L/m_R. In the following we will suppress the scales m_D and m_L in our notation.

for the left-handed neutrino fields ν_L.

As the name type-I seesaw mechanism suggests, there are other types of seesaw mechanisms, which we will discuss in the following.

The *type-II seesaw mechanism* [26–28] is based on a scenario without right-handed neutrinos. Consequently $M_D = M_R = 0$ in the Dirac-Majorana mass term (1.52), and the mass term

$$\mathcal{L}_L = -\frac{1}{2}\overline{(\nu_L)^c} M_L \nu_L + \text{H.c.} = \frac{1}{2}\nu_L^T C^{-1} M_L \nu_L + \text{H.c.} \tag{1.67}$$

is the only source of neutrino masses. However, since this mass term breaks gauge invariance, it has to be generated by means of a suitable Yukawa coupling. The only non-zero bilinear term formed from D_L has—considering only the fermion part—the structure

$$\overline{(D_L)^c} D_L. \tag{1.68}$$

In order to fathom the possibilities for building a gauge invariant Yukawa coupling, we have to consider the possible invariants which can be built from two $SU(2)_L$-doublets. From the tensor product

$$(D_L)^{c*} \otimes D_L \sim \underline{\frac{1}{2}} \otimes \underline{\frac{1}{2}} = \underline{0} \oplus \underline{1} \tag{1.69}$$

we see, that we need an additional scalar $SU(2)_L$-singlet ($\sim \underline{0}$) or an $SU(2)_L$-triplet ($\sim \underline{1}$) for this purpose [29]. Let us first investigate the possibility of an additional scalar $SU(2)_L$-singlet η. This singlet can couple to the $SU(2)_L$-invariant bilinear

$$\overline{(D_{\alpha L})^c}\, \varepsilon D_{\beta L}, \tag{1.70}$$

where we have defined

$$\varepsilon \equiv i\tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.71}$$

$SU(2)_L$-invariance follows from

$$U^T \varepsilon U = (\det U)\varepsilon \tag{1.72}$$

which holds for all 2×2-matrices U. For $U \in SU(2)$ we have $\det U = 1$ and thus invariance of (1.70). The Yukawa coupling with an $SU(2)_L$-singlet η is then given by

$$\mathcal{L}_{\text{Yukawa},\eta} = -F_{\alpha\beta}\overline{(D_{\alpha L})^c}\, \varepsilon D_{\beta L}\eta + \text{H.c.}, \tag{1.73}$$

where F is an antisymmetric 3×3-matrix of Yukawa coupling constants. The invariance under $U(1)_Y$ requires $Y_\eta = +2$ and thus, applying relation (1.15), $Q_\eta = +1$. Thus any VEV of η would break $U(1)_{EM}$ and is therefore forbidden. Consequently, the Yukawa coupling (1.73) is not capable of generating tree-level neutrino masses.[11]

[11] Extending the scalar sector with further additional scalars, however, can give rise to radiatively generated neutrino masses. Adding, apart from the $SU(2)_L$-singlet η, a second Higgs doublet, leads to the Zee model [30, 31] which generates the mass term (1.67) at one-loop order. Enriching the standard model with two scalar singlets η and k^{++} gives rise to the Zee-Babu model [31, 32], which accommodates neutrino masses at the two-loop level. In this thesis we will study tree-level neutrino masses only. For further literature on radiative neutrino mass generation see e.g. [33–35] and citations therein.

Field multiplet	T	T^3	Y	Q
$\vec{\delta} = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix}$	1	$-i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} H^{++} \\ H^0 \\ H^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \vec{\delta}$	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Table 1.4: The charges of the $SU(2)_L$-triplet Δ. For further information *cf.* table 1.1.

In contrast to the $SU(2)_L$-singlet η, the extension of the standard model field content by an $SU(2)_L$-triplet $\vec{\delta} = (\delta_1, \delta_2, \delta_3)^T$ can give rise to a tree-level neutrino mass term (1.67). Since the representation corresponding to isospin-1 is the adjoint representation of $SU(2)_L$, it is convenient to formulate the Lagrangian in terms of

$$\Delta \equiv \vec{\tau} \cdot \vec{\delta} = \begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix} \equiv \begin{pmatrix} H^+ & \sqrt{2}H^{++} \\ \sqrt{2}H^0 & -H^+ \end{pmatrix}, \tag{1.74}$$

just as we did in the case of the gauge boson triplet $\vec{W}_\mu$—see equation (1.2). The gauge invariant Yukawa coupling of Δ to the left-handed lepton doublets is given by

$$\mathcal{L}_{\text{Yukawa},\Delta} = -\frac{1}{2} G_{\alpha\beta} \overline{(D_{\alpha L})^c}\, \varepsilon \Delta D_{\beta L} + \text{H.c.}, \tag{1.75}$$

and the charges of the triplet can be found in table 1.4. Since the matrices $\varepsilon\tau^i$ are symmetric, and taking into account the antisymmetry of the charge conjugation matrix C as well as the antisymmetric property of fermion fields, we find that the matrix of Yukawa coupling constants fulfils $G_{\alpha\beta} = G_{\beta\alpha}$. The natural homomorphism $U \mapsto R_U$ from $SU(2)$ onto $SO(3)$, which is just the spin-1 representation, is defined via

$$U \vec{\tau} U^\dagger = R_U^T \vec{\tau}. \tag{1.76}$$

Thus Δ transforms as

$$\Delta = \vec{\tau} \cdot \vec{\delta} \longmapsto \vec{\tau} \cdot (R_U \vec{\delta}) = (R_U^T \vec{\tau}) \cdot \vec{\delta} = (U \vec{\tau} U^\dagger) \cdot \vec{\delta} = U \Delta U^\dagger, \tag{1.77}$$

which shows the $SU(2)_L$-invariance of the coupling (1.75). The expansion of (1.75) in terms of electric charge eigenfields reads

$$\begin{aligned} \mathcal{L}_{\text{Yukawa},\Delta} = \frac{1}{2} G_{\alpha\beta} [&\sqrt{2} \nu_{\alpha L}^T C^{-1} H^0 \nu_{\beta L} - \nu_{\alpha L}^T C^{-1} H^+ \ell_{\beta L} + \\ &- \ell_{\alpha L}^T C^{-1} H^+ \nu_{\beta L} - \sqrt{2} \ell_{\alpha L}^T C^{-1} H^{++} \ell_{\beta L}] + \text{H.c.}, \end{aligned} \tag{1.78}$$

showing that, if H^0 acquires the VEV

$$\langle 0 | H^0 | 0 \rangle = \frac{1}{\sqrt{2}} v_T, \tag{1.79}$$

the neutrinos obtain a mass term with the mass matrix

$$M_L = v_T G. \tag{1.80}$$

As required for Majorana neutrinos, the resulting mass matrix is symmetric. Taking Lorentz and gauge invariance into account we find the most general renormalizable Lagrangian of the triplet to be [36]

$$\mathcal{L}_\Delta = \frac{1}{2}\mathrm{Tr}\left[(D_\mu\Delta)^\dagger\,(D^\mu\Delta)\right] + \left(-\frac{1}{2}G_{\alpha\beta}\overline{(D_{\alpha L})^c}\,\varepsilon\Delta D_{\beta L} + \mathrm{H.c.}\right) - \mathcal{V}(\phi,\Delta), \tag{1.81}$$

where the scalar potential [36, 37]

$$\begin{aligned}\mathcal{V}(\phi,\Delta) = \;& \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 + \\ & + \frac{1}{2}M^2\mathrm{Tr}(\Delta^\dagger\Delta) + \Lambda[\mathrm{Tr}(\Delta^\dagger\Delta)]^2 + \Lambda'\,\mathrm{Tr}(\Delta^\dagger\Delta^\dagger)\,\mathrm{Tr}(\Delta\Delta) + \\ & + \left(\alpha_\Delta\phi^\dagger\Delta\widetilde{\phi} + \mathrm{H.c.}\right) + \beta_\Delta\phi^\dagger\phi\,\mathrm{Tr}(\Delta^\dagger\Delta) + \gamma_\Delta\phi^\dagger\Delta^\dagger\Delta\phi\end{aligned} \tag{1.82}$$

replaces the original scalar potential $V(\phi)$ of the standard model—see equation (1.10)—and $\alpha_\Delta \in \mathbb{C}$ and $\mu^2, \lambda, M^2, \Lambda, \Lambda', \beta_\Delta, \gamma_\Delta \in \mathbb{R}$ are coupling constants.

In the following we will discuss the essence of the type-II seesaw mechanism, which is the suppression of the triplet VEV v_T compared to the VEV v of the Higgs doublet, which is at the electroweak scale. In order for the triplet model to be successful, $U(1)_{EM}$-invariance must not be broken by the VEVs of the scalars. Therefore we will assume that there exists a choice of the free parameters of $\mathcal{V}(\phi,\Delta)$, such that the VEVs have the form

$$\langle 0|\Delta|0\rangle = \begin{pmatrix} 0 & 0 \\ v_T & 0 \end{pmatrix}, \quad \langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.83}$$

Inserting these VEVs into the scalar potential (1.82) gives

$$\mathcal{V}_0 = \frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^4 + \frac{1}{2}M^2|v_T|^2 + \Lambda|v_T|^4 + \frac{1}{2}(\alpha_\Delta v_T + \alpha_\Delta^* v_T^*)v^2 + \frac{1}{2}\beta_\Delta v^2|v_T|^2, \tag{1.84}$$

where we have, due to gauge freedom, assumed that v is real. Now we want to find a minimum of $\mathcal{V}_0$ which satisfies our requirement

$$|v_T| \ll v. \tag{1.85}$$

Under this assumption we find the conditions

$$\begin{aligned} 0 = \frac{\partial\mathcal{V}_0}{\partial v} &= \mu^2 v + \lambda v^3 + 2\mathrm{Re}(\alpha_\Delta v_T)v + \beta_\Delta v|v_T|^2 \approx \mu^2 v + \lambda v^3 \\ &\Rightarrow v^2 \approx -\frac{\mu^2}{\lambda}, \end{aligned} \tag{1.86}$$

and

$$\begin{aligned} 0 = \frac{\partial\mathcal{V}_0}{\partial v_T^*} &= \frac{1}{2}M^2 v_T + 2\Lambda|v_T|^2 v_T + \frac{1}{2}\alpha_\Delta^* v^2 + \frac{1}{2}\beta_\Delta v^2 v_T \approx \\ &\approx \frac{1}{2}\left[\left(M^2 + \beta_\Delta v^2\right)v_T + \alpha_\Delta^* v^2\right] \\ &\Rightarrow v_T \approx -\frac{\alpha_\Delta^* v^2}{M^2 + \beta_\Delta v^2}. \end{aligned} \tag{1.87}$$

Field multiplet	T	T^3	Y	Q
$\vec{\psi}_R = \begin{pmatrix} \psi_{R1} \\ \psi_{R2} \\ \psi_{R3} \end{pmatrix}$	1	$-i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	$-i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \psi_R^+ \\ \psi_R^- \\ \psi_R^0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\vec{\psi}_R$	1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Table 1.5: The charges of the $SU(2)_L$-triplet fermion Ψ_R. For further information *cf.* table 1.1.

Assuming now that $M^2 \gg \beta_\Delta v^2$ [38], we obtain

$$|v_T| \simeq \frac{|\alpha_\Delta| v^2}{M^2}. \tag{1.88}$$

Thus, if both α_Δ and M are of a large scale m_S in the scalar sector, then v_T is of the small scale

$$|v_T| \sim \frac{v^2}{m_S}. \tag{1.89}$$

Since in the type-II seesaw mechanism the smallness of the neutrino masses is induced by a very high scale in the scalar sector, it is also called *scalar seesaw mechanism.*

For the sake of completeness, let us also mention the *type-III seesaw mechanism* [39] which is similar to the type-I seesaw mechanism, but instead of right-handed neutrinos—which are $SU(2)_L$-singlets—it uses right-handed fermion $SU(2)_L$-triplets. The Yukawa interaction term giving rise to the Dirac mass matrix M_D is given by

$$\mathcal{L}_{\text{Yukawa},D} = -\overline{\nu_R}\widetilde{\phi}^\dagger \Gamma^{(\nu)} D_L + \text{H.c.} \tag{1.90}$$

and is allowed, because it corresponds to the singlet representation $\underline{0}$ in the tensor product

$$\widetilde{\phi}^* \otimes D_L \sim \underline{\frac{1}{2}} \otimes \underline{\frac{1}{2}} = \underline{0} \oplus \underline{1}. \tag{1.91}$$

In order to use the isospin-1 representation $\underline{1}$, we have to extend the standard model with n_ψ fermionic $SU(2)_L$-triplets

$$\vec{\psi}_{jR} = (\psi_{jR1}, \psi_{jR2}, \psi_{jR3})^T, \quad (j = 1, \ldots, n_\psi). \tag{1.92}$$

The charges of such a fermionic triplet under the standard model gauge group can be found in table 1.5. In the same way as we did for the scalar triplet—see equation (1.74)—we can write the fermionic triplets as

$$\Psi_{jR} \equiv \vec{\tau} \cdot \vec{\psi}_{jR} = \begin{pmatrix} \psi_{jR3} & \psi_{jR1} - i\psi_{jR2} \\ \psi_{jR1} + i\psi_{jR2} & -\psi_{jR3} \end{pmatrix} \equiv \begin{pmatrix} \psi^0_{jR} & \sqrt{2}\psi^+_{jR} \\ \sqrt{2}\psi^-_{jR} & -\psi^0_{jR} \end{pmatrix}, \tag{1.93}$$

and the Yukawa coupling of the triplet is given by

$$\mathcal{L}_{\text{Yukawa},\Psi} = -\lambda_{j\alpha}\widetilde{\phi}^\dagger\overline{\Psi_{jR}}D_{\alpha L} + \text{H.c.}, \tag{1.94}$$

where λ is a complex $n_\psi \times 3$-matrix of Yukawa coupling constants. Allowing for lepton number violation, also the mass term

$$\begin{aligned}\mathcal{L}_{M_\Psi} &= -\frac{1}{2}(M_\Psi)_{jk}\sum_{i=1}^{3}\overline{\psi_{jRi}}(\psi_{kRi})^c + \text{H.c.} = \\ &= -\frac{1}{2}(M_\Psi)_{jk}\left(2\,\overline{\psi^+_{jR}}(\psi^-_{kR})^c + \overline{\psi^0_{jR}}(\psi^0_{kR})^c\right) + \text{H.c.}\end{aligned} \tag{1.95}$$

is allowed. After spontaneous symmetry breaking, the Yukawa coupling gives rise to the Dirac mass term

$$-\frac{v}{\sqrt{2}}\lambda_{j\alpha}\overline{\psi^0_{jR}}\nu_{\alpha L} + \text{H.c.}, \tag{1.96}$$

i.e. we obtain the Dirac mass matrix

$$M_D = \frac{v}{\sqrt{2}}\lambda. \tag{1.97}$$

We can rewrite the Dirac mass term stemming from (1.94) together with the neutrino contribution of the mass term (1.95) as

$$-\frac{1}{2}\overline{(N_L)^c}\begin{pmatrix} 0 & M_D^T \\ M_D & M_\Psi \end{pmatrix} N_L + \text{H.c.}, \tag{1.98}$$

where

$$N_L \equiv \begin{pmatrix} \nu_L \\ (\psi^0_R)^c \end{pmatrix} \tag{1.99}$$

and—as usual—we have collected the fields $\nu_{\alpha L}$ and ψ^0_{jR} in the vectors ν_L and ψ^0_R, respectively. The mass term (1.98) is identical to the Dirac-Majorana mass term of the type-I seesaw mechanism, and thus generates the effective light neutrino mass matrix

$$M_{\text{light}} = -M_D^T M_\Psi^{-1} M_D. \tag{1.100}$$

In addition to the three discussed seesaw mechanisms, all varieties of "mixed" seesaw mechanisms exist, a prominent example being the type-I+II-seesaw mechanism, where all four entries of the block matrix M in the Dirac-Majorana mass term (1.52) are non-zero. It is interesting to note that, restricting oneself to dimension-five effective operators[12] for neutrino mass generation [40], the three discussed seesaw mechanisms (including the mixed forms) are the only possible mechanisms allowing Majorana neutrino masses at tree-level in renormalizable extensions of the standard model [41].

In this thesis we will make use of the type-I and type-II seesaw mechanism only.

[12] In order to generate Majorana neutrino masses one needs operators which violate lepton number. As discussed earlier, there are no operators of dimension ≤ 4 in the standard model which violate lepton number conservation. Thus any Lorentz- and $SU(2)_L \times U(1)_Y$-invariant operator that violates lepton number must be of dimension ≥ 5. In fact we can easily write down the lepton number violating operator

$$\frac{1}{M}\overline{(D_L)^c}\widetilde{\phi}^*\widetilde{\phi}^\dagger D_L, \tag{1.101}$$

which is of dimension five [40]. If ϕ acquires a VEV, the operator (1.101) gives rise to the neutrino mass term

$$-\frac{v^2}{2M}\nu_L^T C^{-1}\nu_L. \tag{1.102}$$

Analysing the three types of seesaw mechanism within the framework of effective field theories, integrating out the new heavy particles (right-handed neutrinos, scalar triplets or fermionic triplets) leads to operators of the form (1.101), where M is of the mass scale of the heavy particles [41]. Indeed equation (1.102) resembles the relations (1.63) and (1.89) derived for the three types of seesaw mechanism.

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Chapter 2

Observables of neutrino physics

In this chapter we will describe the observables of neutrino physics and briefly outline the current experimental and observational knowledge on their values.

2.1 The lepton mixing matrix

Neutrino oscillations—which will be discussed in section 2.2—are intimately connected with the phenomenon of lepton mixing, which we will discuss in the following.

Theorem 1 enables us to diagonalize the charged-lepton mass term

$$\mathcal{L}_{M_\ell} = -\overline{\ell_L} M_\ell \ell_R + \text{H.c.} \tag{2.1}$$

by means of the biunitary transformation

$$U_L^{(\ell)\dagger} M_\ell U_R^{(\ell)} = \operatorname{diag}(m_{\ell_1}, m_{\ell_2}, m_{\ell_3}) \equiv \hat{M}_\ell. \tag{2.2}$$

However, by a rearrangement of the columns of the unitary matrices $U_L^{(\ell)}$ and $U_R^{(\ell)}$ we can, since we know from experiment that the three charged leptons have different masses, always achieve

$$m_{\ell_1} < m_{\ell_2} < m_{\ell_3}, \tag{2.3}$$

i.e. there exists a biunitary transformation such that

$$U_L^{(\ell)\dagger} M_\ell U_R^{(\ell)} = \operatorname{diag}(m_e, m_\mu, m_\tau). \tag{2.4}$$

Thus we can rewrite the charged-lepton mass term as

$$\mathcal{L}_{M_\ell} = -\underbrace{\overline{\ell_L} U_L^{(\ell)}}_{\equiv \overline{\ell'_L}} \operatorname{diag}(m_e, m_\mu, m_\tau) \underbrace{U_R^{(\ell)\dagger} \ell_R}_{\equiv \ell'_R} + \text{H.c.} = -\overline{\ell'} \operatorname{diag}(m_e, m_\mu, m_\tau)\, \ell', \tag{2.5}$$

where we have defined

$$\ell' \equiv \ell'_L + \ell'_R \equiv \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}. \tag{2.6}$$

The mass eigenfields e, μ and τ are said to be the *flavour eigenfields* of the charged leptons, i.e. *the flavour of a charged lepton is defined by its mass.*

Since neutrino masses are so tiny, and thus play almost no role in production and detection processes,[1] it is not suitable to define the neutrino flavour via the neutrino mass. A much better definition is the following:

> *A neutrino ν is said to be of flavour α, if it is produced or detected in a charged current interaction process involving the charged lepton of flavour α.*

The charged current (CC) interactions are the interactions of the fermions with the charged vector bosons W^+ and W^-. Starting from the full interaction Lagrangian (1.5c) of the standard model fermions with the gauge bosons

$$\mathcal{L}_{\text{int}} = \sum_{\psi} \overline{\psi} i\gamma^\mu D_\mu \psi \tag{2.7}$$

and considering only the part involving leptons we find

$$\begin{aligned}\mathcal{L}_{\text{int, leptons}} &= -\overline{D_L}\gamma^\mu \left(g\vec{W}_\mu \cdot \frac{\vec{\tau}}{2} + g' B_\mu \frac{Y_{D_L}}{2} \mathbb{1}_2 \right) D_L = \\ &= \underbrace{-\frac{g}{2}\overline{D_L}\gamma^\mu \begin{pmatrix} 0 & \sqrt{2}W^+_\mu \\ \sqrt{2}W^-_\mu & 0 \end{pmatrix} D_L}_{\equiv \mathcal{L}_{\text{CC, leptons}}} + \dots ,\end{aligned} \tag{2.8}$$

where the dots indicate terms involving W^3_μ and B_μ (or equivalently Z_μ and A_μ). The charged current interactions for leptons are thus given by

$$\mathcal{L}_{\text{CC, leptons}} = -\frac{g}{\sqrt{2}} W^-_\mu \overline{\ell_L} \gamma^\mu \nu_L + \text{H.c.} \tag{2.9}$$

Next we want to rewrite the CC interaction Lagrangian (2.9) in terms of mass eigenfields. We already know how to do this for the charged leptons—see equation (2.5). In the case of Dirac neutrinos the diagonalization of the Dirac neutrino mass term

$$\mathcal{L}_D = -\overline{\nu_R} M_D \nu_L + \text{H.c.} \tag{2.10}$$

proceeds exactly the same way as for the charged leptons, i.e. we obtain

$$U_R^{(\nu)\dagger} M_D U_L^{(\nu)} = \text{diag}(m_1, m_2, m_3) \equiv \hat{M}_D \tag{2.11}$$

and the chiral neutrino fields in the mass eigenbasis are given by

$$\nu'_L \equiv U_L^{(\nu)\dagger} \nu_L, \qquad \nu'_R \equiv U_R^{(\nu)\dagger} \nu_R. \tag{2.12}$$

[1] A notable exception is beta decay of tritium

$${}^3\text{H} \to {}^3\text{He} + e^- + \overline{\nu}_e,$$

where there is a chance to see an effect of the neutrino mass on the endpoint of the energy spectrum of the produced electrons [1–3]. Thus precise investigation of the electron energy spectrum in tritium beta decay may allow to determine the absolute neutrino mass scale—see section 2.4.

In the case of Majorana neutrinos we will have a mass term of the form[2]

$$\mathcal{L}_{M_\nu} = -\frac{1}{2}\overline{(\nu_L)^c} M_{\text{light}} \nu_L + \text{H.c.} \tag{2.13}$$

By means of theorem 2 this mass term can be diagonalized via

$$U_L^{(\nu)T} M_{\text{light}} U_L^{(\nu)} = \hat{M}_{\text{light}} = \operatorname{diag}(m_1, m_2, m_3) \tag{2.14}$$

leading to the left-handed neutrino mass eigenfields

$$\nu'_L = U_L^{(\nu)\dagger} \nu_L. \tag{2.15}$$

Thus the first relation in equation (2.12) remains true also for Majorana neutrinos, and we can rewrite the charged current interactions in terms of mass eigenfields:

$$\mathcal{L}_{\text{CC, leptons}} = -\frac{g}{\sqrt{2}} W_\mu^- \overline{\ell'_L} \gamma^\mu \underbrace{U_L^{(\ell)\dagger} U_L^{(\nu)}}_{\equiv U_{\text{PMNS}}} \nu'_L + \text{H.c.} \tag{2.16}$$

The matrix

$$U_{\text{PMNS}} \equiv U_L^{(\ell)\dagger} U_L^{(\nu)} \tag{2.17}$$

is called *lepton mixing matrix* or PMNS-matrix [4–7].[3] The neutrino flavour eigenfields $\widetilde{\nu}_L$ are the fields interacting with the charged-lepton flavour eigenfields ℓ' in the charged current interactions, i.e.

$$\widetilde{\nu}_L \equiv \begin{pmatrix} \widetilde{\nu}_{eL} \\ \widetilde{\nu}_{\mu L} \\ \widetilde{\nu}_{\tau L} \end{pmatrix} \equiv U_{\text{PMNS}}\, \nu'_L. \tag{2.18}$$

Thus we have arrived at the important result that, if neutrinos are massive, the neutrino flavour eigenfields are rotated against the neutrino mass eigenfields by the unitary matrix U_{PMNS}. This feature is called *lepton mixing* and can lead to neutrino oscillations—see section 2.2.

The lepton mixing matrix, as any unitary 3×3-matrix, can be parameterized by nine parameters. We will, throughout the whole dissertation, use the parameterization suggested by [8]

$$U_{\text{PMNS}} = D_1 V D_2, \tag{2.19}$$

where

$$D_1 = \operatorname{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) \tag{2.20}$$

and

$$D_2 = \operatorname{diag}(e^{i\rho}, e^{i\sigma}, 1) \tag{2.21}$$

are diagonal phase matrices and

$$V = V_{23} V_{13} V_{12} \tag{2.22}$$

[2] In the case of the type-I and type-III seesaw mechanism the effective mass term for the ν_L has precisely this form—see equation (1.66). The type-II seesaw mechanism directly yields the mass term (2.13)—see equation (1.67).

[3] PMNS = Pontecorvo, Maki, Nakagawa, Sakata.

with

$$V_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad V_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix},$$
$$V_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.23}$$

Here $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$ are the sines and cosines of the so-called mixing angles θ_{12}, θ_{23} and θ_{13}, respectively. The mixing angles can take values in $[0, \pi/2]$ and the phase δ may assume values in $[0, 2\pi)$. Also the phases α, β, γ as well as ρ and σ can assume all values in $[0, 2\pi)$. The phase δ is called the *Dirac phase* and the two phases ρ and σ are called the *Majorana phases.* The phases α, β, γ do not affect charged-current interactions, since they may be absorbed through a redefinition of the charged lepton fields

$$\mathcal{L}_{\text{CC, leptons}} = -\frac{g}{\sqrt{2}} W_\mu^- \underbrace{\overline{\ell'_L} D_1}_{\equiv \overline{\ell''_L}} V D_2 \gamma^\mu \nu'_L + \text{H.c.} \tag{2.24}$$

leaving the charged-lepton mass term invariant:

$$-\overline{\ell'_L}\, \text{diag}(m_e, m_\mu, m_\tau)\, \ell'_L + \text{H.c.} = -\overline{\ell''_L} \underbrace{D_1^* \,\text{diag}(m_e, m_\mu, m_\tau)\, D_1}_{\text{diag}(m_e, m_\mu, m_\tau)} \ell''_L + \text{H.c.} \tag{2.25}$$

The same technique would apply to Dirac neutrinos, however, if neutrinos are Majorana particles, a similar redefinition does not work because in this case the Majorana mass term would not be invariant:

$$-\frac{1}{2}\overline{(\nu'_L)^c} \hat{M}_\nu \nu'_L + \text{H.c.} = -\frac{1}{2}\overline{(\nu''_L)^c} \underbrace{D_2^* \hat{M}_\nu D_2^*}_{\neq \hat{M}_\nu} \nu''_L + \text{H.c.} \tag{2.26}$$

Thus in the case of Majorana neutrinos the Majorana phases ρ and σ are *physical phases* and as such they are subject to experimental scrutiny—see section 2.5.

Note that the parameterization discussed above is not unique[4] for all elements of $U(3)$. However, in the range of the experimental values for U_{PMNS} it is unique and well-suited. The matrix V in this parameterization becomes

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}. \tag{2.27}$$

Since, as we will see in the next section, neutrino oscillations are sensitive to V only, it will play an important role in the following.

[4] Take the unit matrix as example. If we set all mixing angles and the phase γ to 0, we have infinitely many possibilities to choose the phases $\alpha = -\rho$ and $\beta = -\sigma$ to achieve $U_{\text{PMNS}} = \mathbb{1}_3$.

2.2 Neutrino oscillations

As we have found in the previous section, the neutrino flavour eigenfields $\widetilde{\nu}_\alpha$ are rotated against the neutrino mass eigenfields ν'_j by the PMNS-matrix:

$$\widetilde{\nu}_L = U_{\mathrm{PMNS}}\, \nu'_L. \tag{2.28}$$

In the following we will always use the shorthand notation U for the lepton mixing matrix U_{PMNS}. Since the involved quantum fields contain the annihilation operators b which annihilate the corresponding states, the neutrino *mass eigenstates*

$$|\nu'_j\rangle = (b'_j)^\dagger|0\rangle \tag{2.29}$$

are related to the flavour eigenstates via

$$|\widetilde{\nu}_\alpha\rangle \equiv U^*_{\alpha j}|\nu'_j\rangle. \tag{2.30}$$

Equation (2.30) contains the key to neutrino oscillations. Unfortunately, a rigorous derivation of the neutrino oscillation probabilities based on quantum field theoretical methods is quite involved and not suitable for a presentation in this thesis. For a rigorous discussion and a list of references to relevant papers in this context we refer the reader to [9, 10]. However, there is a "standard derivation" for the oscillation probabilities in vacuum[5] based on simple assumptions which produce the correct results. As a motivation for the oscillation probabilities let us repeat this simplified derivation here. Following [9] the assumptions can be summarized as:

- Neutrinos are produced as flavour eigenstates, but they propagate as a coherent superposition of mass eigenstates.
- The propagation of the mass eigenstates $|\nu'_j\rangle$ can be described by plane waves with the same energy E but different momenta p_j.

Furthermore we will consider plane waves in only one space-dimension (i.e. plane waves propagating along the x-axis). Suppose now that the neutrino flavour eigenstate $|\widetilde{\nu}_\alpha\rangle = U^*_{\alpha j}|\nu'_j\rangle$ is produced at $(t,x) = (0,0)$ and propagates, according to our simplifying assumptions, as a coherent superposition of mass eigenstates. Due to the plane wave approximation, at (t,x) the state will have evolved to

$$|\widetilde{\nu}_\alpha(t,x)\rangle = U^*_{\alpha j} e^{-i(Et - p_j x)}|\nu'_j\rangle, \tag{2.31}$$

where E is the energy and p_j the momentum corresponding to the mass eigenstate $|\nu'_j\rangle$. Thus the amplitude for a transition of the flavour eigenstate $|\widetilde{\nu}_\alpha\rangle$ produced at $(0,0)$ to the flavour eigenstate $|\widetilde{\nu}_\beta\rangle$ at (t,x) is given by

$$\mathcal{A}_{\nu_\alpha\to\nu_\beta}(t,x) = \langle\widetilde{\nu}_\beta|\widetilde{\nu}_\alpha(t,x)\rangle. \tag{2.32}$$

[5] In this thesis we will not consider neutrino oscillations in matter. However, matter affects neutrino oscillations, which has first been studied by Wolfenstein [11, 12] and Mikheev and Smirnov [13–15]. Especially for the explanation of the solar neutrino deficit the matter effect (also known as MSW effect after Mikheev, Smirnov and Wolfenstein) plays a crucial role—see for example [9].

Since one knows from experiment that neutrinos must be much lighter than electrons—see section 2.4, for all practical purposes one can assume them to be ultrarelativistic, allowing for the approximation

$$p_j = \sqrt{E^2 - m_j^2} \simeq E - \frac{m_j^2}{2E}. \tag{2.33}$$

Hence,

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t, L) = U_{\beta j} U_{\alpha j}^* \exp\left(-i\frac{m_j^2 L}{2E}\right) e^{-iE(t-L)}, \tag{2.34}$$

where we have replaced x by the distance L between the neutrino source and the detector. The transition probability from flavour α to flavour β is then given by

$$P_{\nu_\alpha \to \nu_\beta} = |\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t, L)|^2, \tag{2.35}$$

which leads to

$$P_{\nu_\alpha \to \nu_\beta}(L/E) = \left| U_{\beta j} U_{\alpha j}^* \exp\left(-i\frac{m_j^2 L}{2E}\right)\right|^2. \tag{2.36}$$

The corresponding transition probability for antineutrinos is obtained through replacing U by U^*—see equation (2.30). Thus

$$P_{\nu_\alpha \to \nu_\beta} = P_{\overline{\nu}_\beta \to \overline{\nu}_\alpha}. \tag{2.37}$$

The transition probability (2.36) has the following important properties:

1. It depends only on the mass squared differences

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2, \quad \Delta m_{31}^2 \equiv m_3^2 - m_1^2. \tag{2.38}$$

2. It shows an oscillatory behaviour in $L/E \Rightarrow$ neutrino oscillations.

3. It is invariant under the transformation

$$U \mapsto DUD', \tag{2.39}$$

where D and D' are diagonal phase matrices. Thus in expression (2.36) U can be replaced by V.

4. If $V = V^*$, i.e. if δ is 0 or π, we find

$$P_{\nu_\alpha \to \nu_\beta} = P_{\overline{\nu}_\alpha \to \overline{\nu}_\beta}, \tag{2.40}$$

which would imply CP-invariance in neutrino oscillations.

Depending on the neutrino source and energy as well as the experimental settings (especially the so-called baseline L) the oscillation probabilities will be governed by different mixing angles and mass-squared differences. From this fact there emerged the following commonly used names for the mixing angles and mass-squared differences:

- θ_{12} is also called the *solar mixing angle*,
- θ_{23} the *atmospheric mixing angle* and
- θ_{13} the *reactor mixing angle.*
- Δm_{21}^2 is often called *solar mass-squared difference* and
- Δm_{31}^2 (or sometimes Δm_{32}^2) is referred to as the *atmospheric mass-squared difference.*

2.3 Global fits of neutrino oscillation data

As we have learned in the previous section, neutrino oscillations are sensitive to V and the mass-squared differences Δm_{ij}^2 only. Thus there are six observables which can be determined by neutrino oscillations, namely

$$\Delta m_{21}^2, \ \Delta m_{31}^2, \ \theta_{12}, \ \theta_{23}, \ \theta_{13} \quad \text{and} \quad \delta. \tag{2.41}$$

At this point we have to add some important comments. When we had defined the charged lepton flavour eigenstates we required that the charged lepton masses must fulfil

$$m_{\ell_1} < m_{\ell_2} < m_{\ell_3}. \tag{2.42}$$

However, we did *not* explicitly require the analogous condition for the neutrino mass eigenfields, and we will also not do so now. We can always reorder the neutrino masses at the prize of a rearrangement of the columns of the mixing matrix U, which, in total, would leave the Lagrangian invariant. Instead of fixing the order of the neutrino masses, we fix the order of the columns of U, which (having enough experimental data) in principle determines the order of the neutrino masses. We can, by convention, always choose the order of the columns of U such that $m_1 < m_2$, i.e.

$$\Delta m_{21}^2 > 0, \tag{2.43}$$

but unfortunately the currently available data do not fix the order of the masses completely. Since we know from experiment that

$$|\Delta m_{31}^2| \gg \Delta m_{21}^2, \tag{2.44}$$

there are only two possibilities, namely[6]

1. the so-called *normal* mass spectrum

$$m_1 < m_2 < m_3, \tag{2.45}$$

2. and the so-called *inverted* mass spectrum

$$m_3 < m_1 < m_2. \tag{2.46}$$

Since experimental precision is by now not high enough to distinguish these two cases, in all global fits on neutrino oscillation data there will be two sets of fit results, namely one for the normal and one for the inverted mass spectrum.

As to be expected from the non-trivial expression (2.36) single oscillation experiments cannot determine all of the six oscillation parameters. Therefore different groups of physicists perform *global fits* of available neutrino oscillation data. Such fits are based on the minimization of a so-called χ^2-function, which is a "figure of merit"-function describing the agreement of the six oscillation parameters with the individual results of the different neutrino oscillation experiments. In chapter 3 we will discuss the "χ^2-method" for linking models with data in more detail. The latest global fits are the ones presented in [16–19]. As an example we pick the global fit [18] whose results are presented in table 2.1

parameter	best fit	1σ-range	2σ-range	3σ-range
Δm^2_{21} $[10^{-5}\,\mathrm{eV}^2]$	7.62	$7.43 - 7.81$	$7.27 - 8.01$	$7.12 - 8.20$
$\lvert\Delta m^2_{31}\rvert$ $[10^{-3}\,\mathrm{eV}^2]$	2.55	$2.46 - 2.61$	$2.38 - 2.68$	$2.31 - 2.74$
	2.43	$2.37 - 2.50$	$2.29 - 2.58$	$2.21 - 2.64$
$\sin^2\theta_{12}$	0.320	$0.303 - 0.366$	$0.29 - 0.35$	$0.27 - 0.37$
$\sin^2\theta_{23}$	$0.427 \oplus 0.613$	$(0.400 - 0.461) \oplus (0.573 - 0.635)$	$0.38 - 0.66$	$0.36 - 0.68$
	0.600	$0.569 - 0.626$	$0.39 - 0.65$	$0.37 - 0.67$
$\sin^2\theta_{13}$	0.0246	$0.0218 - 0.0275$	$0.019 - 0.030$	$0.017 - 0.033$
	0.0250	$0.0223 - 0.0276$	$0.020 - 0.030$	$0.017 - 0.033$
δ	0.80π	$0 - 2\pi$	$0 - 2\pi$	$0 - 2\pi$
	1.97π	$0 - 2\pi$	$0 - 2\pi$	$0 - 2\pi$

Table 2.1: The best fit values and $n\sigma$-ranges of the neutrino oscillation parameters according to the global fit [18]. If the fit results do not depend on the assumed spectrum, there is only one line. Else the upper line corresponds to the normal and the lower line to the inverted neutrino mass ordering, respectively. In the case of a normal spectrum in addition to the global minimum giving $\sin^2\theta_{23} = 0.613$, there is also a local minimum of the χ^2-function corresponding to $\sin^2\theta_{23} = 0.427$ with $\Delta\chi^2 = 0.02$ with respect to the global minimum (for further information see section 3.1). This is also the reason for the disconnected 1σ-range for $\sin^2\theta_{23}$ indicated with the symbol "$\oplus$". For further details and illustrative plots see [18].

and which describe the current situation of the global fits to oscillation data very well. The values of the mass-squared differences are well-determined, and also the values of the mixing angles θ_{12} and θ_{13} are quite settled now. While for a long time there have only been upper bounds on the reactor mixing angle θ_{13}, the reactor experiments Double Chooz [20, 21], Daya Bay [22] and RENO [23] impressively confirmed the earlier hints from the accelerator experiments T2K [24] and MINOS [25] for a non-zero reactor mixing angle. Consequently, in the new global fits including these novel results $\sin^2\theta_{13}$ is already determined with a precision at the level of $\approx 10\%$.

The global fit values for the atmospheric mixing angle θ_{23} have for a long time been compatible with maximal atmospheric neutrino mixing $\sin^2\theta_{23} = 1/2$, but increasing statistics and new results presented at the Neutrino-2012 conference [26], now indicate a deviation of θ_{23} from $45°$ [17–19]. However, from the global fits it is not clear in which octant θ_{23} lies—see table 2.1.

As can also be seen from table 2.1, the oscillation parameter which is least well determined, is the so-called Dirac phase δ, being completely undetermined at already 1σ.

Today's global fits include data of many neutrino experiments of different types. As an example we show the experiments included in the global fit of Forero *et al.* [18] in table 2.2.

[6] $|\Delta m^2_{31}| \gg \Delta m^2_{21}$ implies that m_3 cannot lie between m_1 and m_2.

Experiment	neutrino source	observable	references
Homestake	sun	solar ν_e-flux	[27]
Gallex/GNO	sun	solar ν_e-flux	[28]
SAGE	sun	solar ν_e-flux	[29]
Borexino	sun	solar ^{7}Be ν_e-flux	[30]
Super-Kamiokande (I, II, III)	sun	solar ^{8}B ν_e-flux	[31–33]
SNO (three phases)	sun	total solar ^{8}B ν-flux	[34, 35]
KamLAND	nuclear reactors	$P_{\overline{\nu}_e \to \overline{\nu}_e}$ (reactor $\overline{\nu}_e$-flux)	[36]
Super-Kamiokande (I, II, III)	atmosphere	atmospheric ν_e- and ν_μ-flux	[37]
T2K	accelerator	$P_{\nu_\mu \to \nu_\mu}$ (ν_μ disappearance)	[38]
		$P_{\nu_\mu \to \nu_e}$ (ν_e appearance)	
MINOS	accelerator	$P_{\nu_\mu \to \nu_\mu}$ (ν_μ disappearance)	[25, 39–41]
		$P_{\overline{\nu}_\mu \to \overline{\nu}_\mu}$ ($\overline{\nu}_\mu$ disappearance)	
		$P_{\nu_\mu \to \nu_e}$ (ν_e appearance)	
		$P_{\overline{\nu}_\mu \to \overline{\nu}_e}$ ($\overline{\nu}_e$ appearance)	
Double Chooz	nuclear reactors	$P_{\overline{\nu}_e \to \overline{\nu}_e}$ ($\overline{\nu}_e$ disappearance)	[20, 21]
Daya Bay	nuclear reactors	$P_{\overline{\nu}_e \to \overline{\nu}_e}$ ($\overline{\nu}_e$ disappearance)	[22]
RENO	nuclear reactors	$P_{\overline{\nu}_e \to \overline{\nu}_e}$ ($\overline{\nu}_e$ disappearance)	[23]

Table 2.2: Experiments included in the global fit of neutrino oscillation data by Forero *et al.* [18]. For the T2K and MINOS experiments, instead of the publications referred to in the right outermost column, newer—at the time of writing unpublished—data released at the “Neutrino 2012” conference [26] in Kyoto have been used [18]. The global fit furthermore includes new data from the Daya Bay experiment presented at the Kyoto conference [26].

2.4 Absolute neutrino masses

As outlined in section 2.2, neutrino oscillations are sensitive to the mass-squared differences of the neutrinos only. The experimental evidence for non-vanishing Δm_{21}^2 and Δm_{31}^2—see table 2.1—proves that *at least two neutrinos are massive.* The absolute neutrino mass scale, however, cannot be determined by means of oscillation experiments.[7] In the following we will discuss the present bounds on the absolute neutrino mass scale.

Bounds from tritium beta decay: As already mentioned earlier in footnote 1 on page 35, tritium beta decay offers the possibility to determine the absolute neutrino mass scale by investigation of the endpoint of the electron energy spectrum. A list of bounds obtained by different experiments is provided in [8], the best bounds being of the order of [1, 2]

$$m_\beta < 2.0\,\text{eV}. \tag{2.48}$$

The KATRIN experiment aims at a sensitivity of $m_\beta \sim 0.35\,\text{eV}$ [3]. Let us note here that—since the emitted electron neutrino ν_e is not a mass eigenstate—the bounds from the tritium decay experiments are bounds on an *effective electron antineutrino mass* m_β for which different expressions are given in the literature—for details see [9, 42] and references therein. However, in the regime of $m_\beta \gg \sqrt{|\Delta m_{31}^2|}$, the effective neutrino mass m_β is equivalent to the absolute neutrino mass scale [42]. Thus the bound (2.48) on m_β directly translates to an upper bound of 2.0 eV on the absolute neutrino mass scale.

Bounds from cosmology: The currently most stringent bound on the absolute neutrino mass scale comes from cosmology. In the standard model of cosmology the energy density[8] Ω_ν of the relic neutrinos[9] enters as a parameter and can thus be probed by

[7]From oscillation experiments we can only deduce the bound

$$m_{\max} \geq \sqrt{\Delta m_{31}^2} \tag{2.47}$$

on the largest neutrino mass $m_{\max}$.

[8]The dimensionless quantity

$$\Omega_\nu \equiv \frac{\rho_\nu}{\rho_c} \tag{2.49}$$

is defined as the ratio of the neutrino energy density ρ_ν over the critical energy density [8]

$$\rho_c \equiv \frac{3H_0^2}{8\pi G} = \left(\frac{H_0}{100\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}}\right)^2 \times 1.05375(13) \times 10^{-5}\,\text{GeV}\,\text{cm}^{-3}. \tag{2.50}$$

Here H_0 denotes today's Hubble constant and G is Newton's gravitational constant.

[9]The cosmological analyses do not assume the number of light neutrino species to be three. Thus the bounds coming from cosmology are bounds on the sum of the masses of *all* light neutrinos. In fact in fits to cosmological data one can also fit the number of light neutrinos. The current best fit value for the effective number of light neutrino species obtained from WMAP is given by [43]

$$N_{\text{eff}} = 4.34^{+0.86}_{-0.88} \quad \text{(68\% confidence level)}. \tag{2.51}$$

Bounds from big bang nucleosynthesis are of the same order of magnitude ($2 \leq N_{\text{eff}} \leq 4$) [8, 44–46]. Unfortunately at the present level of precision one cannot deduce the existence/non-existence of a further light neutrino species in addition to the known three neutrinos.

astrophysical observations [8]. Ω_ν is related to the sum of neutrino masses via [47]

$$\Omega_\nu h_0^2 \approx \frac{\sum_\nu m_\nu}{94\,\mathrm{eV}^2}, \tag{2.52}$$

where h_0 is today's Hubble constant[10] in units of $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. In particular investigation of the angular power spectrum of the cosmic microwave background (CMB) as well as structure formation in the early universe yield constraints on the sum of the neutrino masses. There is no unique consensus on the cosmological bounds on the neutrino masses as can be seen from the list of bounds presented in [8]. However, most bounds are of the order of magnitude

$$\sum_\nu m_\nu < \mathcal{O}(1\,\mathrm{eV}) \tag{2.54}$$

leading to a bound of roughly $0.3\,\mathrm{eV}$ on the absolute neutrino mass scale. The recent bound on the sum of the neutrino masses obtained by the WMAP Collaboration is [43]

$$\sum_\nu m_\nu < 0.58\,\mathrm{eV} \quad (95\% \text{ confidence level}). \tag{2.55}$$

However, in this thesis we will use the value $0.3\,\mathrm{eV}$ for the upper bound on the absolute neutrino mass scale.

Let us finally note that the bound (2.55) would imply that

$$\Omega_\nu \approx \frac{\sum_\nu m_\nu}{h_0^2 \times 94\,\mathrm{eV}} \lesssim 0.01. \tag{2.56}$$

Since the total energy density ρ of the universe is approximately the critical density, i.e.

$$\Omega \equiv \frac{\rho}{\rho_c} \approx 1, \tag{2.57}$$

we can deduce that neutrinos presently contribute at most one percent to the total energy density of the universe.

Bounds from neutrinoless double beta decay: If neutrinos are Majorana particles, also the experimental investigation of the so-called neutrinoless double beta decay allows for the derivation of upper bounds on the absolute neutrino mass scale. We will discuss these bounds in section 2.5.

2.5 Neutrinoless double beta decay and lepton number violation

We have not yet discussed the question of whether neutrinos are Dirac or Majorana particles from the experimental point of view. As outlined in chapter 1, the question of Dirac

[10] According to the results of the seven-year evaluation of WMAP data [48] the current value of the Hubble constant is given by

$$h_0 \equiv \frac{H_0}{100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}} = 0.710 \pm 0.025. \tag{2.53}$$

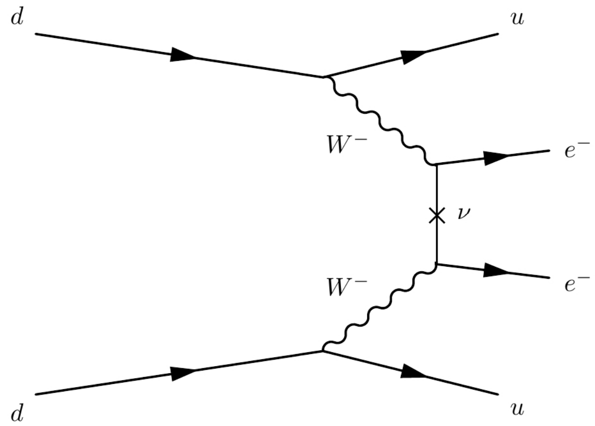

Figure 2.1: The Feynman graph for neutrinoless double beta decay at the quark level.

or Majorana nature of neutrinos is intimately connected to the question of lepton number conservation. The reason for this is the fact that Majorana mass terms violate lepton number conservation explicitly. Thus, if neutrinos are Majorana particles, one should be able to observe lepton number violating processes. The most promising candidate for such a process in reach of current experimental scrutiny is the so-called neutrinoless double beta decay

$$(Z, A) \to (Z+2, A) + 2e^-. \tag{2.58}$$

Here Z denotes the proton and A the total nucleon number of the nucleus. In contrast to the neutrinoless double beta decay (2.58) the two-neutrino double beta decay

$$(Z, A) \to (Z+2, A) + 2e^- + 2\overline{\nu}_e, \tag{2.59}$$

has been directly observed[11] for the nine nuclides [49–52]

$$^{48}\mathrm{Ca},\ ^{76}\mathrm{Ge},\ ^{82}\mathrm{Se},\ ^{96}\mathrm{Zr},\ ^{100}\mathrm{Mo},\ ^{116}\mathrm{Cd},\ ^{130}\mathrm{Te},\ ^{136}\mathrm{Xe} \quad \text{and} \quad ^{150}\mathrm{Nd}.$$

Up to now there is no unequivocal evidence for neutrinoless double beta decay.[12]

Majorana nature of neutrinos would enable neutrinoless double beta decay via the mechanism shown in figure 2.1. The cross at the inner neutrino line indicates the Wick-contraction of the two neutrino fields. If neutrinos are Majorana-particles (and thus their own antiparticles), this contraction is non-zero:

$$\langle 0|T\nu_{eL}(x_1)\nu_{eL}(x_2)^T|0\rangle = -\sum_{j,k} U_{ej}U_{ek}P_L \underbrace{\langle 0|T\nu_j(x_1)\overline{\nu}_k(x_2)|0\rangle}_{\neq 0} P_L C. \tag{2.60}$$

Inserting the fermion propagator

$$\langle 0|T\nu_j(x_1)\overline{\nu}_k(x_2)|0\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x_1-x_2)} \frac{p_\mu\gamma^\mu + m_j \mathbb{1}_4}{p^2 - m_j^2 + i\epsilon}\delta_{jk}, \tag{2.61}$$

[11]Double beta decay has furthermore been observed for the isotopes ^{128}Te and ^{238}U by means of geochemical and radiochemical techniques, respectively [51].

[12]There are claims for an observation of neutrinoless double beta decay in ^{76}Ge [53]. Up to now other independent experiments could not confirm this observation, and its validity is under debate.

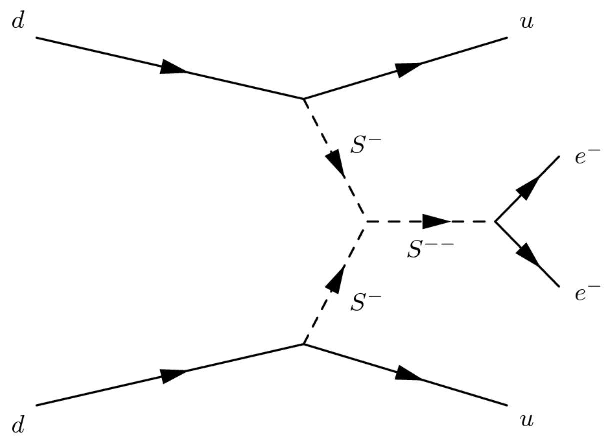

Figure 2.2: The Feynman graph for a mechanism of neutrinoless double beta decay involving charged scalars.

after some manipulations one obtains

$$\langle 0|T\nu_{eL}(x_1)\nu_{eL}(x_2)^T|0\rangle = -\sum_k U_{ek}^2 m_k\, i \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x_1-x_2)}}{p^2 - m_k^2 + i\epsilon} P_L C. \tag{2.62}$$

According to [49] one can neglect the neutrino masses m_k in the denominator of the above propagator and thus finds that the amplitude for neutrinoless double beta decay via the mechanism shown in figure 2.1 will be proportional to

$$m_{\beta\beta} \equiv \left|\sum_{k=1}^{3} U_{ek}^2 m_k\right|. \tag{2.63}$$

Thus an observation of neutrinoless double beta decay (and a measurement of its decay rate) would put an upper bound on the effective mass $m_{\beta\beta}$. The reason one would only obtain an upper bound is that also mechanisms other than the one of figure 2.1 could contribute to the decay rate. One could, for example, think of intermediate charged scalars—see the Feynman graph in figure 2.2. For a list of mechanisms contributing to neutrinoless double beta decay we refer the reader to [54].

However, if neutrinoless double beta decay is found, irrespective of the mechanism it is based on, a Majorana mass term cannot be forbidden by a symmetry of the Lagrangian [55, 56]. Thus an observation of neutrinoless double beta decay would imply Majorana nature of neutrinos.

Let us dwell a bit on the effective mass $m_{\beta\beta}$. Using equation (2.19) we can rewrite the effective mass as

$$m_{\beta\beta} = \left|\sum_{k=1}^{3} V_{ek}^2\, \mu_k\right| \tag{2.64}$$

with

$$\mu_1 = m_1 e^{2i\rho}, \;\; \mu_2 = m_2 e^{2i\sigma} \quad \text{and} \quad \mu_3 = m_3. \tag{2.65}$$

Thus, apart from the six oscillation parameters, $m_{\beta\beta}$ depends on the absolute neutrino mass scale, the two unknown Majorana phases and the type of spectrum (normal or

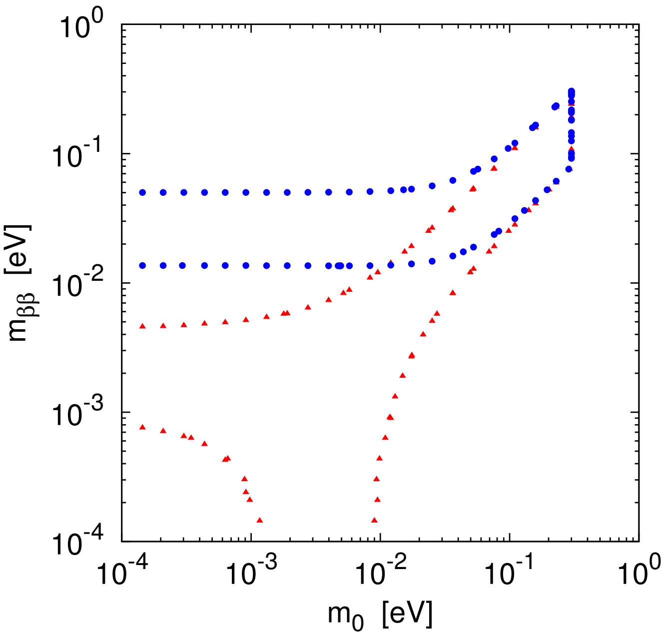

Figure 2.3: The allowed regions for $m_{\beta\beta}$ and the smallest neutrino mass m_0. The plot has been produced using the technique described in [57] using the 2σ-ranges for the oscillation parameters provided in [18]. The allowed area for the normal spectrum is bounded by triangles ▲, and the boundary of the region for the inverted spectrum is indicated by points •.

inverted). Using the 2σ-ranges of the global fit of Forero *et al.* (see table 2.1) for the six oscillation parameters, allowing the Majorana phases ρ and σ to vary in $[0, 2\pi)$ and varying the smallest neutrino mass between 0 and 0.3 eV, which is the cosmological bound we stipulated in section 2.4, we can determine the allowed regions in a plot of $m_{\beta\beta}$ versus the smallest neutrino mass[13] m_0. The resulting plot, obtained by means of the technique developed in [57], is shown as figure 2.3. For similar plots for different $n\sigma$-regions based on other global fits we refer the reader to [57–59].

If neutrinoless double beta decay was observed, the decay rate would set an upper bound on $m_{\beta\beta}$. From figure 2.3 we deduce that this bound on $m_{\beta\beta}$ could, provided there is an appropriate bound on the smallest neutrino mass, help to determine the neutrino mass spectrum. On the contrary, if an inverted neutrino mass spectrum were confirmed by a different experiment, an upper bound on $m_{\beta\beta}$ could rule out Majorana nature of neutrinos. Note that these considerations only hold true if there is no sensible interference between the "standard mechanism" (see Feynman graph in figure 2.1) and other mechanisms of neutrinoless double beta decay.

[13]The smallest neutrino mass is

$$m_0 = \begin{cases} m_1 & \text{for the normal mass spectrum,} \\ m_3 & \text{in the case of an inverted neutrino mass spectrum.} \end{cases} \tag{2.66}$$

For a list of the current experimental bounds on $m_{\beta\beta}$ we refer the reader to [59]. The presently most stringent bound is [59, 60]

$$m_{\beta\beta} \lesssim 0.4\,\mathrm{eV} \tag{2.67}$$

obtained by the EXO-Collaboration.

Let us finally discuss the upper bound on the absolute neutrino mass scale which an observation of neutrinoless double beta decay would imply. Suppose neutrinoless double beta decay was verified with $m_{\beta\beta}$ near the upper bound (2.67) and let us furthermore assume that the standard mechanism provides the main contribution to $m_{\beta\beta}$. Then neutrinos would have to be quasi-degenerate:

$$m \equiv m_1 \simeq m_2 \simeq m_3, \tag{2.68}$$

and $m_{\beta\beta}$ could well be approximated as

$$m_{\beta\beta} = \left| \sum_{k=1}^{3} U_{ek}^2 m_k \right| \approx m \left| \sum_{k=1}^{3} U_{ek}^2 \right|. \tag{2.69}$$

This would lead to the bound

$$m \lesssim \frac{0.4\,\mathrm{eV}}{\left|\sum_{k=1}^{3} U_{ek}^2\right|}. \tag{2.70}$$

Using the order of magnitude approximation $s_{12}^2 \approx 1/3$, $s_{23}^2 \approx 1/2$ and $s_{13}^2 \approx 0$ (see table 2.1) we find

$$|U_{e1}|^2 \approx \frac{2}{3}, \ \ |U_{e2}|^2 \approx \frac{1}{3} \quad \text{and} \quad |U_{e3}|^2 \approx 0. \tag{2.71}$$

Consequently

$$\left| \sum_{k=1}^{3} U_{ek}^2 \right| \lesssim |U_{e1}|^2 - |U_{e2}|^2 \approx \frac{1}{3} \tag{2.72}$$

and thus from equation (2.70)

$$m \lesssim 1.2\,\mathrm{eV}. \tag{2.73}$$

Doing an exact calculation rather than our order of magnitude estimation leads to the bound [59]

$$m \lesssim 1.9\,\mathrm{eV} \tag{2.74}$$

based on the 3σ-range for the oscillation parameters. Thus the bound on the absolute neutrino mass scale from neutrinoless double beta decay would be of the order of magnitude of the bounds from tritium decay.

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Chapter 3

Numerical methods

In this chapter we want to present the numerical methods which were used in the course of this thesis. Among these techniques the so-called *Nelder-Mead algorithm* will play a central role. Before we focus on the Nelder-Mead algorithm, we will shortly present the idea of the so-called "χ^2-analysis" which is a method to link models with data.

3.1 Linking models with data: The χ^2-method

Let us study the following common situation in physics. We are given a set of m observables $\mathbf{O}_i$ in the form

$$\mathbf{O}_i = \overline{\mathbf{O}}_i \pm \sigma_i, \tag{3.1}$$

where $\overline{\mathbf{O}}_i$ are the central values and σ_i the corresponding errors. As an attempt to understand the physics behind the experimental/observational results we have built a model providing predictions $P_i(x)$ for the observables $\mathbf{O}_i$. Here

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{3.2}$$

denotes a vector collecting the n free parameters of the model. An obvious question is now how well the model agrees with the data. In the χ^2-approach one defines a "figure-of-merit function", namely the so-called χ^2-function

$$\chi^2(x) \equiv \sum_{i=1}^{m} \left(\frac{P_i(x) - \overline{\mathbf{O}}_i}{\sigma_i} \right)^2 \tag{3.3}$$

as a measure for the agreement between a model with parameters x and the observables $\mathbf{O}_i = \overline{\mathbf{O}}_i \pm \sigma_i$.

Clearly the χ^2-function is non-negative:

$$\chi^2 \geq 0, \tag{3.4}$$

and its global minimum can become zero if and only if

$$\exists\, x_{\min}: \quad P_i(x_{\min}) = \overline{\mathbf{O}}_i \;\; \forall i. \tag{3.5}$$

Thus, for a given model the global minimum

$$\chi^2_{\min} \equiv \min_x \chi^2(x) = \chi^2(x_{\min}) \tag{3.6}$$

of the corresponding χ^2-function is a measure for the agreement between the model predictions and the data: the smaller $\chi^2_{\min}$, the better the agreement. An important quantity is the deviation of $\chi^2(x)$ from the global minimum:

$$\Delta\chi^2(x) \equiv \chi^2(x) - \chi^2(x_{\min}). \tag{3.7}$$

Note that the global minimum $x_{\min}$ is not necessarily unique. Even if it is unique one has to take into account that there may be several local minima x with $\chi^2(x)$ close to the global minimum $\chi^2_{\min}$. This is also the case for the global fit of neutrino oscillation data [1], see table 2.1, where there is a local minimum with $\Delta\chi^2 = 0.02$ with respect to the global minimum.

Frequently the error distribution of the observables is not Gaussian, resulting in asymmetric errors

$$\mathbf{O}_i = \overline{\mathbf{O}}_i{}^{+\sigma_i^{\text{right}}}_{-\sigma_i^{\text{left}}}\,. \tag{3.8}$$

This can easily be accommodated within the χ^2-function by defining

$$\chi^2(x) \equiv \sum_{i=1}^{m} \chi_i^2(x) \tag{3.9}$$

with

$$\chi_i^2(x) = \begin{cases} [(P_i(x) - \overline{\mathbf{O}}_i)/\sigma_i^{\text{right}}]^2 & \text{for} \quad P_i(x) \geq \overline{\mathbf{O}}_i, \\ [(P_i(x) - \overline{\mathbf{O}}_i)/\sigma_i^{\text{left}}]^2 & \text{for} \quad P_i(x) < \overline{\mathbf{O}}_i. \end{cases} \tag{3.10}$$

The χ^2-function defined before may be refined to incorporate additional constraints. Defining for example the characteristic function[1]

$$\Pi_D(x) = \begin{cases} 0 & \text{for} \quad x \in D, \\ \infty & \text{for} \quad x \notin D \end{cases} \tag{3.11}$$

and replacing $\chi^2(x)$ by $\chi^2(x) + \Pi_D(x)$ allows to resrict x to the domain D. Another commonly used technique is to pin an observable $\mathbf{O}$ to a value $\overline{\mathbf{O}}$ by adding the so-called *pinning term*

$$\Pi_{\mathbf{O}}(x) = \left(\frac{P(x) - \overline{\mathbf{O}}}{0.01\,\overline{\mathbf{O}}}\right)^2 \tag{3.12}$$

which pins the prediction P for the observable $\mathbf{O}$ to the value $\overline{\mathbf{O}}$ by artificially introducing a very small error of only $0.01\,\overline{\mathbf{O}}$.

[1]Usually the minimization of χ^2 is possible only numerically. Thus ∞ in Π_D has to be replaced with a number much higher than the typical value of $\chi^2(x)$. If $\chi^2(x)$ is continuous (or even differentiable) and one wants to preserve this property, one has to suitably "smear out" the characteristic function.

Before we close this section we finally want to come back to the global fit of neutrino oscillation data. There the "model" is the assumption of three-neutrino oscillations as outlined in section 2.2. Let us note here that though at first glance the χ^2-function looks very simple, the task of properly defining the functions $P_i(x)$ may take a lot of effort, since these functions have to take into account the details of the experimental setup, systematic uncertainties, specifications of the detectors and so on, or to be short "the physics of neutrino oscillation experiments".[2] Once the $P_i(x)$ are defined, one can minimize the χ^2-function and obtain the best-fit values $x_{\rm min}$ for the oscillation parameters x. The $n\sigma$-range for x is then defined by [1]

$$\sqrt{\Delta\chi^2(x)} \leq n. \tag{3.13}$$

3.2 The Nelder-Mead algorithm

In the previous section we have described the idea of the so-called χ^2-analysis which involves the minimization of the χ^2-function which is a real positive function of n parameters. Therefore, in order to use the χ^2-method one needs an algorithm for the minimization of real-valued functions. Since we will frequently make use of such a numerical method for function minimization not only for the χ^2-method but also for other purposes in this dissertation, we will briefly present our algorithm of choice—the *Nelder-Mead algorithm*.

The Nelder-Mead algorithm [3] (downhill simplex algorithm) is a so-called direct search method[3] allowing for the minimization of a function

$$f: \ \mathbb{R}^n \to \mathbb{R} \tag{3.14}$$

of n real variables. The basic element of the Nelder-Mead algorithm for application to a function of n variables is the so-called *simplex*, which is the complex hull of $n+1$ points in $\mathbb{R}^n$ (i.e. a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3, \ldots$). In the following we will describe the single steps of the Nelder-Mead algorithm, for the sake of clarity always accompanied by graphical illustrations for the two-dimensional case.

The first step of the Nelder-Mead algorithm is the creation of a *random start simplex*

$$\{x_1, \ldots, x_{n+1}\} \qquad (x_i \in \mathbb{R}^n). \tag{3.15}$$

The points $x_1, ..., x_{n+1}$ are the *vertices* of the simplex. Calculating the function values

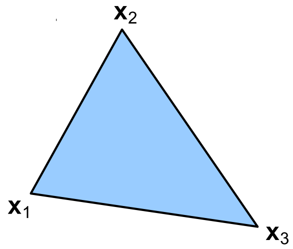
$$f_i \equiv f(x_i) \tag{3.16}$$

at the vertices, we can sort the vertices such that

$$f_1 \leq f_2 \leq \ldots \leq f_{n+1}. \tag{3.17}$$

[2] For a global fit including a thorough discussion of such issues see [2].

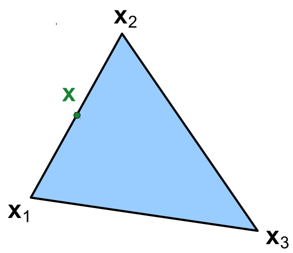
[3] A direct search method is an algorithm which is based on comparison of function values only (e.g. $f_1 < f_2, ...$) [4]. Direct search methods do not need any information on the derivatives of the function, neither analytical nor numerical.



We now define the following points in $\mathbb{R}^n$:

1. The *centroid*, which is the mean value of the n "best"[4] points:

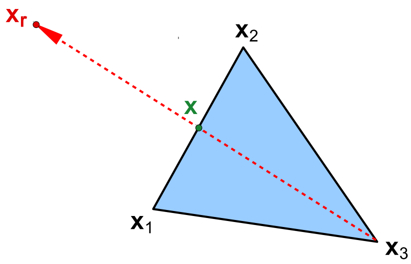
$$x \equiv \frac{1}{n}\sum_{i=1}^{n} x_i, \tag{3.18}$$



2. the *reflection point*

$$x_r \equiv x + \rho(x - x_{n+1}), \tag{3.19}$$

where $\rho > 0$ is the *reflection parameter* (standard choice: $\rho = 1$),

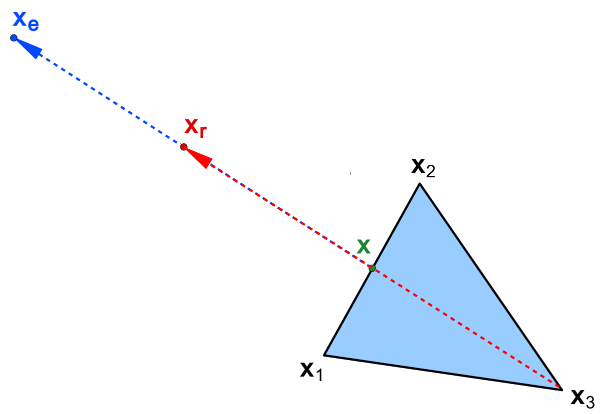


3. the *expansion point*

$$x_e \equiv x + \chi(x_r - x), \tag{3.20}$$

where $\chi > 1$ is the expansion parameter ($\chi > \rho$, standard choice: $\chi = 2$),

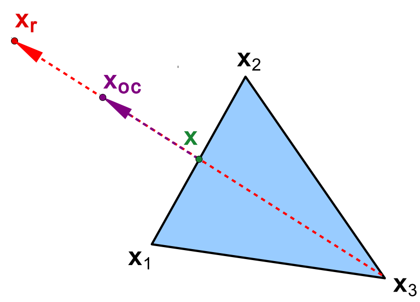
[4]Since the aim of the Nelder-Mead algorithm is the minimization of f, a point x_i is "better" than a point x_j if $f_i < f_j$.



4. the *outside contraction point*

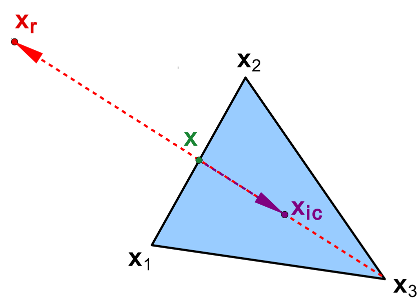
$$x_{oc} \equiv x + \gamma(x_r - x), \tag{3.21}$$

where $0 < \gamma < 1$ is the contraction parameter (standard choice: $\gamma = 1/2$),



5. and finally the *inside contraction point*

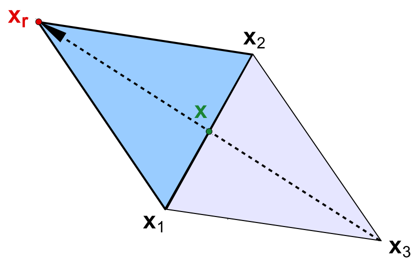
$$x_{ic} \equiv x - \gamma(x - x_{n+1}). \tag{3.22}$$



In one iteration of the Nelder-Mead algorithm the initial simplex (either the random start simplex or the simplex resulting from the previous iteration) may be transformed in the following ways:

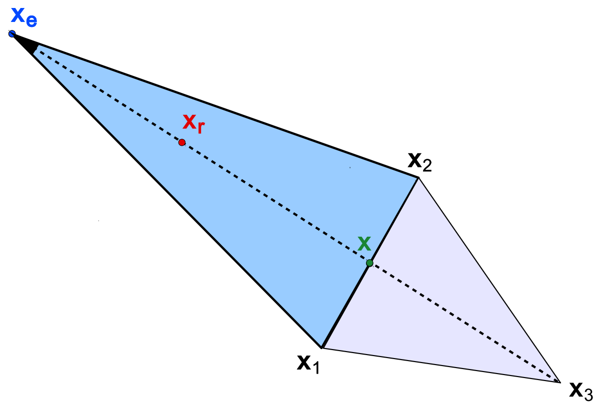
(A) *Reflection* of the simplex:

$$\{x_1, \ldots, x_{n+1}\} \longmapsto \{x_1, \ldots, x_n, x_r\}, \tag{3.23}$$



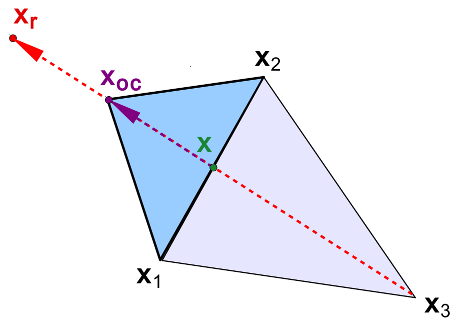
(B) *expansion* of the simplex:

$$\{x_1, \ldots, x_{n+1}\} \longmapsto \{x_1, \ldots, x_n, x_e\}, \tag{3.24}$$



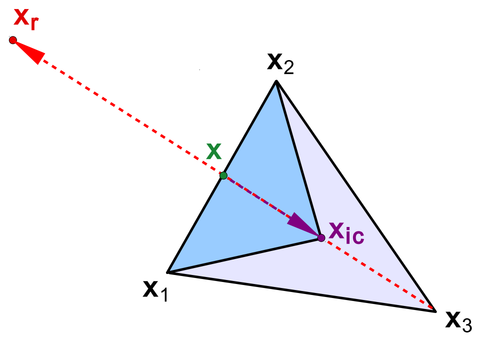
(C) *outside contraction* of the simplex:

$$\{x_1, \ldots, x_{n+1}\} \longmapsto \{x_1, \ldots, x_n, x_{oc}\}, \tag{3.25}$$



(D) *inside contraction* of the simplex:

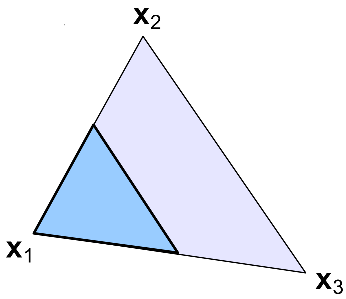
$$\{x_1, \ldots, x_{n+1}\} \longmapsto \{x_1, \ldots, x_n, x_{ic}\}, \tag{3.26}$$



(E) and the *shrinkage* of the simplex towards the best point x_1:

$$x_i \longmapsto x_1 + \sigma(x_i - x_1), \tag{3.27}$$

where $\sigma \in (0, 1)$ is the *shrinkage parameter* (standard choice: $\sigma = 1/2$).



The heart of the Nelder-Mead algorithm is the decision which of the five possible transformations (A)–(E) is applied to the simplex. The mechanism of this decision is shown as a flowchart in figure 3.1. Successively performing transformations of the simplex, the algorithm would never come to an end. Hence a suitable stopping criterion is required. For the implementation of the algorithm in this thesis we will use a stopping criterion similar to the one proposed by Nelder and Mead in [3]. Defining

$$\bar{f} \equiv \frac{1}{n+1}\sum_{i=1}^{n+1} f_i, \tag{3.28}$$

the expression

$$\Delta f^2 \equiv \frac{1}{n+1}\sum_{i=1}^{n+1}(f_i - \bar{f})^2 \tag{3.29}$$

provides a measure for the convergence of the values of f at the vertices to a common value.[5] If

$$\Delta f^2 \le \varepsilon, \tag{3.32}$$

[5]This does not imply convergence of the vertices themselves to some point in $\mathbb{R}^{n+1}$, which could be measured by the quantity

$$\frac{1}{n+1}\sum_{i=1}^{n+1} ||x_i - \bar{x}||^2, \tag{3.30}$$

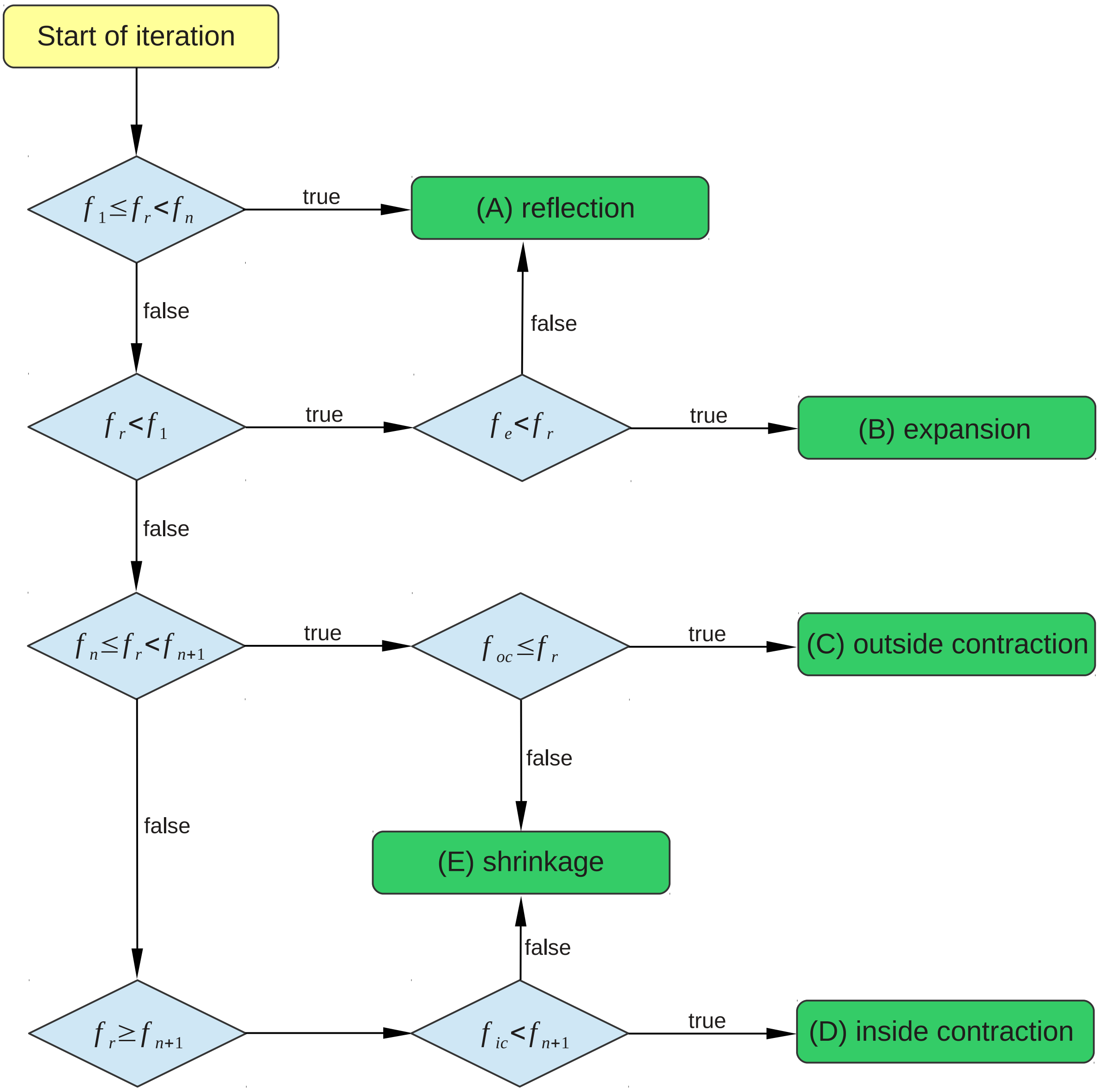


Figure 3.1: The flowchart for one iteration of the Nelder-Mead algorithm [3, 5, 6].

where ε is the so-called *accuracy parameter*, we stop the algorithm and store the obtained simplex. Furthermore we will restart the algorithm with a different random start simplex if after a number of $N_{\max}$ iterations the stopping criterion (3.32) has not been reached. This also provides an "emergency exit" in the case of numerical instabilities. The flowchart of the Nelder-Mead algorithm as discussed before can be found in the left-hand side of figure 3.2.

Going through the algorithm one finds that in the course of successive transformations of the simplex, the simplex strictly "moves downhill", i.e. the value of f at the best vertex of the transformed simplex is equal or smaller than the value at the best vertex of the previous simplex. Thus, nothing prevents the algorithm from getting stuck in a local minimum. Therefore, the Nelder-Mead algorithm as we have discussed it by now is suitable for finding *local minima* only. However, as we usually search for the global minimum, we have to improve the algorithm. In order to do so, we follow the strategy successfully used in [5, 6]. This strategy consists of two ideas:

- We start the Nelder-Mead algorithm with a large number of different random start simplices in order to increase the chance to find a good minimum.
- We modify the algorithm such that also "uphill moves" are allowed.

Concerning the uphill moves reference [5] suggests two possibilities, namely "Nelder-Mead plus perturbations" and "Nelder-Mead plus simulated annealing". The "Nelder-Mead plus simulated annealing" technique allows occasional uphill moves according to a probability distribution described by a parameter similar to the temperature for a thermal distribution. By slowly decreasing the "temperature" (*annealing*) one reduces the number of uphill moves to finally end up in a good (or even the global) minimum. For details see [5] and references therein. For this thesis we choose the other possibility, namely "Nelder-Mead plus perturbations". The idea is simple: once a (local) minimum is found, we perturb this minimum and restart the algorithm. If the minimum improves, we continue with a further perturbation[6], else we use the found minimum as our final result. The flowchart for the "Nelder-Mead plus perturbations" algorithm can be found in the right-hand side of figure 3.2.

At the end of this section let us discuss the virtues and drawbacks of the Nelder-Mead algorithm for our purposes. Since the χ^2-function (and also other functions we will deal with) may contain very involved calculations as for example the diagonalization of mass matrices (see theorems 1 and 2), it will in general not be differentiable (nor continuous). *Direct search methods* as the Nelder-Mead algorithm are well-suited to deal with such functions. Another advantage of the Nelder-Mead algorithm lies in its great simplicity. Though there are only five different possible transformations of the simplex (see the list (A)–(E) on page 58), the simplex can adapt to the "local landscape" on its search for a

where $\bar{x}$ is given by

$$\bar{x} \equiv \frac{1}{n+1} \sum_{i=1}^{n+1} x_i. \tag{3.31}$$

[6]We could either perform perturbations of constant size, or decrease the size of the perturbation in every subsequent perturbation step in order to create a situation similar to the annealing process discussed before. In this thesis both approaches have been applied.

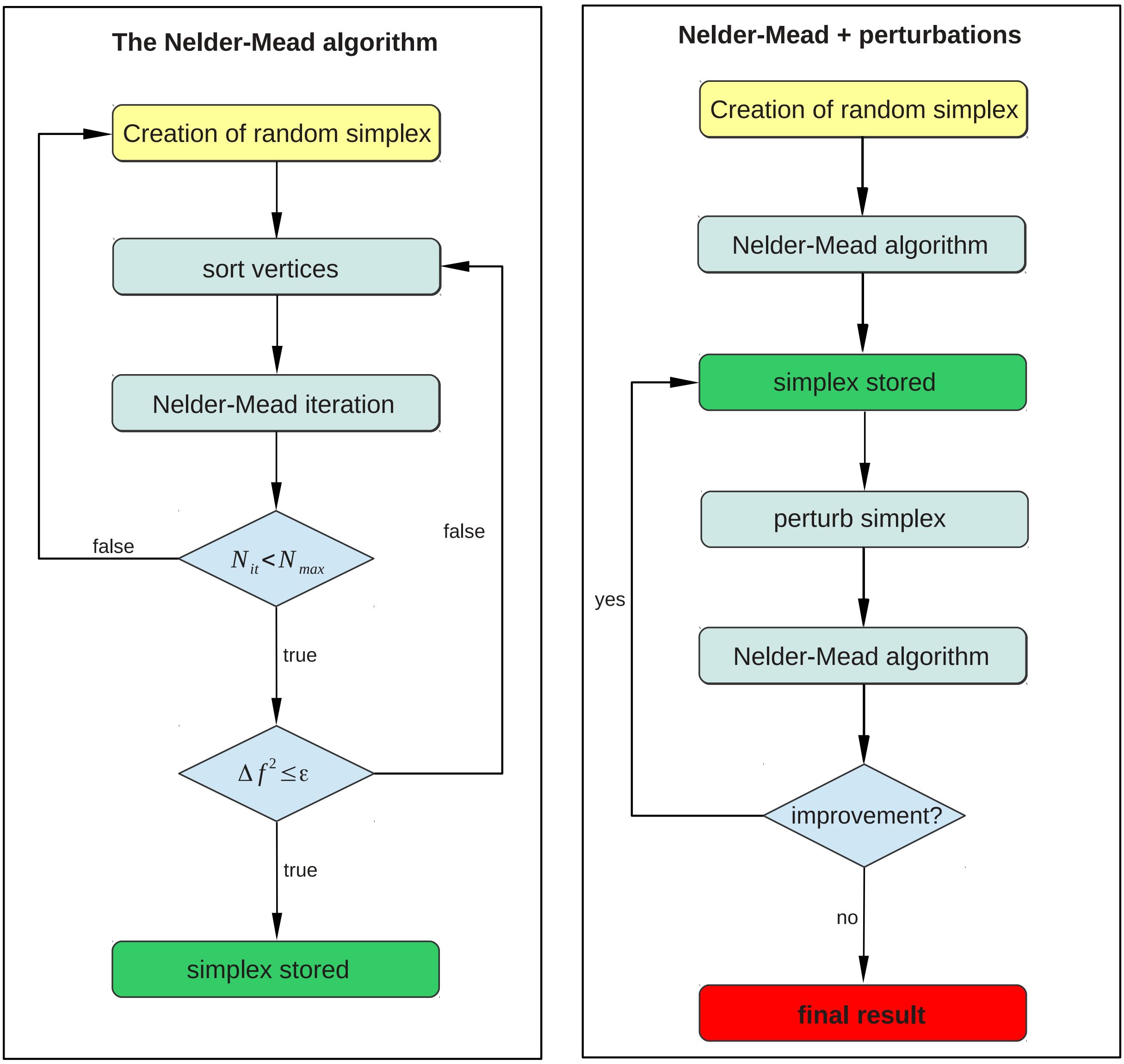


Figure 3.2: The flowchart of the Nelder-Mead algorithm (left-hand side) and a typical realization of the "Nelder-Mead + perturbations" algorithm (right-hand side).

minimum and is thus well suited to minimize functions showing a complicated topology in $\mathbb{R}^n$.

Unfortunately we have to pay a prize for all these virtues. One major drawback of the Nelder-Mead algorithm is that it can be very slow compared to other minimization techniques based on the estimation of gradients. However, with usually only one or two function evaluations per iteration[7] it is quite efficient for a direct search method. The greatest disadvantage of the Nelder-Mead algorithm are its convergence properties. Though convergence of the Nelder-Mead algorithm in one dimension could be proven under some circumstances [7], little is known on the convergence properties for $n > 1$. It is even unknown whether the two-dimensional Nelder-Mead algorithm converges on the function $f(x, y) = x^2 + y^2$ for any start simplex [7]. To make things even worse, there are situations where the algorithm is known to fail or converge extremely slowly [8]. However, despite the discussed theoretical problems, the Nelder-Mead algorithm has proven to work very well in practice—see for example [5, 6] where the Nelder-Mead algorithm has been successfully used on problems having similar structures as in the present thesis. In particular the problems with convergence are in practice circumvented by the "emergency exit" (maximal allowed number of steps) and the fact that we start the algorithm with a large number of random start simplices.

Further details on our implementation of the Nelder-Mead algorithm and other used numerical techniques are presented in appendix A.

[7]As can be deduced from figure 3.1, the reflection needs only the calculation of f_r, expansion and contraction need one more function evaluation (f_e, f_{oc}, f_{ic}). Only the shrinkage step needs more than two (namely $n + 2$) function evaluations. However, the shrinkage usually occurs near the end of the algorithm, where the simplex is already close to the minimum, thus not affecting the efficiency of the algorithm too much.

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Chapter 4

Finite family symmetry groups and their implications on the leptonic and scalar sector

4.1 Model building with finite family symmetry groups

In chapter 1 we discussed extensions of the standard model incorporating massive neutrinos. Following one of the mechanisms presented there, we first encounter the question *how many additional fields can/should be introduced.* Choosing for example the type-II seesaw mechanism—see section 1.3—a priori we could introduce an arbitrary number of scalar $SU(2)_L$-triplets.

Suppose we have fixed the field content of our extension of the standard model. Then we can write down the most general *renormalizable*[1] Lagrangian which is *Lorentz invariant*, *gauge invariant* and invariant under all other imposed *global symmetries* (e.g. lepton number conservation, CP-invariance, SUSY, ...). The obtained Lagrangian $\mathcal{L}$ will in general contain a lot of free parameters, in particular in the flavour sector. Since namely the gauge interactions are flavour-blind, the Yukawa-couplings and the parameters of the scalar potential are not restricted by the gauge symmetries. As an example let us briefly discuss the electroweak part of the standard model itself. In the gauge and scalar sector we have the four free parameters

$$g,\, g',\, \mu^2 \quad \text{and} \quad \lambda. \tag{4.1}$$

The remaining interactions are the Yukawa-interactions with three complex 3×3-matrices of Yukawa coupling constants. Thus the Yukawa interactions contribute no less than 54 free parameters to the standard model.[2] In extensions of the standard model, the number of free parameters will in general be even higher.

[1]In this thesis we will confine ourselves to the study of renormalizable theories. This is, however, not necessary. Based on an effective field theory approach, also non-renormalizable Lagrangians are physically meaningful.

[2]However, only 13 of these 54 parameters are physical, namely the nine masses (six quarks and three charged leptons) and the four physical parameters

$$\theta_{12}^{\mathrm{CKM}},\, \theta_{23}^{\mathrm{CKM}},\, \theta_{13}^{\mathrm{CKM}},\, \delta^{\mathrm{CKM}} \tag{4.2}$$

of the quark mixing matrix. The quark mixing matrix has the following origin: The charged current

particle	mass
e	0.510998928(11) MeV
μ	105.6583715(35) MeV
τ	1776.82 ± 0.16 MeV
u	$2.3^{+0.7}_{-0.5}$ MeV
d	$4.8^{+0.7}_{-0.3}$ MeV
s	95 ± 5 MeV
c	1.275 ± 0.025 GeV
b	4.18 ± 0.03 GeV
t	$173.5 \pm 0.6\,(\text{stat.}) \pm 0.8\,(\text{syst.})$ GeV

Table 4.1: The masses of the charged standard model fermions as given by the particle data group collaboration in [3].

Being faced with such a large number of free parameters, we arrive at the question whether there exists a fundamental principle, comparable to the gauge principle, which would reduce the number of free parameters. In this respect, imposing additional global symmetries in the flavour-dependent sector seems to be a promising idea. Such *flavour symmetries* would constrain the Yukawa and scalar sector and thus may offer possibilities to "explain" some of the peculiarities of the fermion mass spectrum and fermion mixing.

At this point let us review the experimental data on the fermion masses and the mixing matrices. The masses of the nine known charged fermion species are listed in table 4.1. As one can see, there is a strong hierarchy among the masses of the three generations of fermions:

$$\frac{m_e}{m_\tau} \sim 10^{-3},\ \frac{m_d}{m_b} \sim 10^{-3} \quad \text{and} \quad \frac{m_u}{m_t} \sim 10^{-5}. \tag{4.5}$$

With masses of the order of at most $\sim$ eV the neutrino mass scale is totally different from

---

interactions of the quarks are described by the Lagrangian

$$-\frac{g}{\sqrt{2}}\overline{Q_L}\gamma^\mu \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix} Q_L = -\frac{g}{\sqrt{2}} W^-_\mu\, \overline{d_L}\gamma^\mu u_L + \text{H.c.} \tag{4.3}$$

in the same way as for the charged leptons—see equation (2.8). Expressing the CC-interactions in terms of mass eigenfields we obtain, in full analogy to the charged lepton sector, the expression

$$-\frac{g}{\sqrt{2}} W^-_\mu\, \overline{d'_L}\gamma^\mu U^\dagger_{\text{CKM}} u'_L + \text{H.c.}, \tag{4.4}$$

where U_{CKM} is the unitary *quark mixing matrix* (Cabibbo-Kobayashi-Maskawa matrix) [1, 2]. The parametrization of U_{CKM} suggested in [3] is the same as for the PMNS-matrix—see equations (2.19) to (2.23).

the masses of the charged fermions. Comparing to the electron mass we find a ratio of

$$\frac{m_\nu}{m_e} \lesssim 10^{-6}. \tag{4.6}$$

Finally, considering the ratio of the lightest to the heaviest particle yields

$$\frac{m_\nu}{m_t} \lesssim 10^{-11}, \tag{4.7}$$

strongly suggesting different mechanisms for the generation of neutrino and charged fermion masses. One possibility for such a mechanism for generation of small neutrino masses is the seesaw mechanism—see section 1.3.

Let us now turn to the fermion mixing matrices. The quark mixing matrix U_{CKM} can be parametrized in the same way as the lepton mixing matrix U_{PMNS}, i.e.

$$U_{\mathrm{CKM}} = D_{1,\mathrm{CKM}} V_{\mathrm{CKM}} D_{2,\mathrm{CKM}}. \tag{4.8}$$

In stark contrast to the lepton mixing matrix, V_{CKM} is quite close to the unit matrix [3]:

$$V_{\mathrm{CKM}} = \mathbb{1}_3 + \begin{pmatrix} \mathcal{O}(\lambda^2) & \mathcal{O}(\lambda) & \mathcal{O}(\lambda^3) \\ \mathcal{O}(\lambda) & \mathcal{O}(\lambda^2) & \mathcal{O}(\lambda^2) \\ \mathcal{O}(\lambda^3) & \mathcal{O}(\lambda^2) & \mathcal{O}(\lambda^4) \end{pmatrix} \tag{4.9}$$

with [3]

$$\lambda = \sin\theta_{12}^{\mathrm{CKM}} = 0.22535 \pm 0.00065. \tag{4.10}$$

On the contrary V_{PMNS} is close to the so-called Harrison-Perkins-Scott mixing matrix [4]

$$V_{\mathrm{HPS}} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \tag{4.11}$$

which is far from the unit matrix. In particular, all three mixing angles of U_{PMNS} are large, while in the quark sector, there is only one large mixing angle, namely $\theta_{12}^{\mathrm{CKM}} \approx 13°$.

The particular form of the Harrison-Perkins-Scott mixing matrix (4.11) has strongly promoted the idea of *family symmetries* (flavour symmetries, horizontal symmetries) in the lepton sector.[3] In contrast to gauge symmetries, family symmetries act among fields of different flavours (i.e. on the members of a fermion family, hence the name "family symmetry"), e.g.

$$\begin{pmatrix} D_{eL} \\ D_{\mu L} \\ D_{\tau L} \end{pmatrix} \mapsto S_L \begin{pmatrix} D_{eL} \\ D_{\mu L} \\ D_{\tau L} \end{pmatrix}, \quad \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix} \mapsto S_R \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}, \ \ldots \tag{4.12}$$

[3]For a recent review on the status of Harrison-Perkins-Scott mixing in the light of flavour symmetries see [5].

with unitary 3×3-matrices S_L and S_R.[4] All extensions of the standard model we will consider contain more scalars than the usual standard model Higgs doublet.[5] Thus, in order to build a Lagrangian invariant under the family symmetries one also has to impose family symmetries in the scalar sector, i.e.

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \mapsto S_\varphi \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix} \tag{4.13}$$

with a unitary matrix S_φ. Note that if some of the φ_i form multiplets of $SU(2)_L \times U(1)_Y$, gauge invariance may restrict the form of S_φ.

Family symmetries provide a tool to restrict the Lagrangian in such a way that one can possibly "explain" some of the features of lepton masses and mixing. However, there are several important issues one has to take into account when one constructs a model based on flavour symmetries:

- Since the flavour symmetries act also in the scalar sector, they will be broken together with the gauge symmetries. Therefore *continuous flavour symmetries* will usually lead to massless *Goldstone bosons* [15–17]. Since there are no experimental hints for the existence of massless bosons, neither from investigations of long-range forces [18], nor from direct searches (see [3]), one frequently uses *discrete flavour symmetries* to avoid the emergence of Goldstone bosons. However, the decision for discrete flavour symmetries does not completely rule out Goldstone bosons. If the chosen discrete symmetry is too strong, a continuous global symmetry may arise accidentally, possibly leading to Goldstone bosons after spontaneous symmetry breaking.

- Choosing discrete family symmetries one has to consider that spontaneous breaking of discrete symmetries possibly leads to the cosmological problem of domain walls [19]. In the following considerations we will always assume that there is a solution to this problem and hence will allow discrete flavour symmetries.

- Apart from the fermion and gauge boson masses and the fermion mixing also other experimental results may severely constrain our freedom in model building. In particular the non-observation of so-called *flavour changing neutral currents* (FCNCs) puts constraints on the Yukawa couplings and the scalar sector. Let us illustrate this issue with an example. Consider the leptonic Yukawa coupling in an n-Higgs-doublet model:

  $$-\overline{D_L} \sum_{i=1}^{n} \Gamma_i^{(\ell)} \phi_i \ell_R + \text{H.c.} = -\overline{\nu_L} \sum_{i=1}^{n} \Gamma_i^{(\ell)} \phi_i^+ \ell_R - \overline{\ell_L} \sum_{i=1}^{n} \Gamma_i^{(\ell)} \phi_i^0 \ell_R + \text{H.c.} \tag{4.14}$$

  The $\Gamma_i^{(\ell)}$ are complex 3×3-matrices of Yukawa coupling constants. We can express the pure charged-lepton contribution to the Yukawa coupling in terms of the mass

[4]Flavour symmetries can also be combined with CP transformations leading to so-called *generalized CP-transformations* [6–14].

[5]The additional scalars don't need to be $SU(2)_L$-doublets. They could also be singlets, triplets, ...

eigenfields[6] $\ell'_{L,R} = U^{(\ell)\dagger}_{L,R}\ell_{L,R}$ (see equation (1.31)):

$$-\overline{\ell'_L}\sum_{i=1}^{n}\underbrace{(U_L^{(\ell)\dagger}\Gamma_i^{(\ell)}U_R^{(\ell)})}_{\equiv\Gamma_i^{(\ell)\prime}}\phi_i^0\ell'_R + \text{H.c.} \tag{4.15}$$

If the coupling matrices $\Gamma_i^{(\ell)\prime}$ are non-diagonal, the Yukawa couplings allow rare decays like $\mu^- \to e^-e^+e^-$ or $\mu^- \to e^-\gamma$ which are strongly constrained by experiment. The branching ratio of the decay $\mu^- \to e^-e^+e^-$ for example is bounded from above by [3]

$$\frac{\Gamma(\mu^- \to e^-e^+e^-)}{\Gamma_{\text{total}}} < 10^{-12}. \tag{4.16}$$

The Feynman graphs for $\mu^- \to e^-e^+e^-$ and $\mu^- \to e^-\gamma$ are shown in figure 4.1.

Thus one has to take care of these couplings by either choosing a symmetry which leads to diagonal coupling matrices $\Gamma_i^{(\ell)\prime}$, or by choosing the model parameters in such a way that the rare decays are suitably suppressed.

Being aware of the possible "pitfalls" discussed before, we can now think of building a model based on discrete flavour symmetries. A very frequently used realization of discrete symmetries is the imposition of invariance of the Lagrangian under a *finite family symmetry group* [20–23]. The idea of finite family symmetry groups is simple. Suppose we are given a finite group G with unitary representations $\mathcal{D}_L$, $\mathcal{D}_R$ and $\mathcal{D}_\Phi$. If the Lagrangian is invariant under the action

$$D_L \mapsto \mathcal{D}_L(g)D_L,\ \ \ell_R \mapsto \mathcal{D}_R(g)\ell_R,\ \ \Phi \mapsto \mathcal{D}_\Phi(g)\Phi \quad \forall\, g \in G \tag{4.17}$$

of G on the fields, G is called a finite family symmetry group. Here Φ denotes a vector containing all scalars. The generalization to a model also containing other fields (right-handed neutrinos, fermion-triplets, ...) is obvious. Also the quarks can be included, the symmetry thereby restricting the quark masses and the CKM-matrix. However, in this thesis we will concentrate on the lepton sector.

Once having decided to implement a finite family symmetry we are confronted with the question: "How to proceed?" Unfortunately, there is no simple answer to this question. In fact model building with discrete symmetries is very difficult, since a lot of issues act together. We have tried to illustrate the interdependence and the interplay of the different aspects one has to take into account in figure 4.2. Due to the complexity of the problem, not all of its aspects can be treated in this thesis. In particular we concentrate on three aspects of the wide field of research on finite family symmetries, namely:

- *Which possibilities do we have for choosing a finite family symmetry group?* This question will be addressed in section 4.2.

- *How do symmetries restrict the mass and mixing matrices?* Elaborating on the simplest case, which is the case of Abelian symmetries, we will discuss the restrictions on the fermion mass and mixing matrices in section 4.3, thereby encountering the intensively studied possibility of texture zeros in the neutrino mass matrix.

[6] Remember that for the charged leptons the mass eigenfields are by definition identical to the flavour eigenfields.

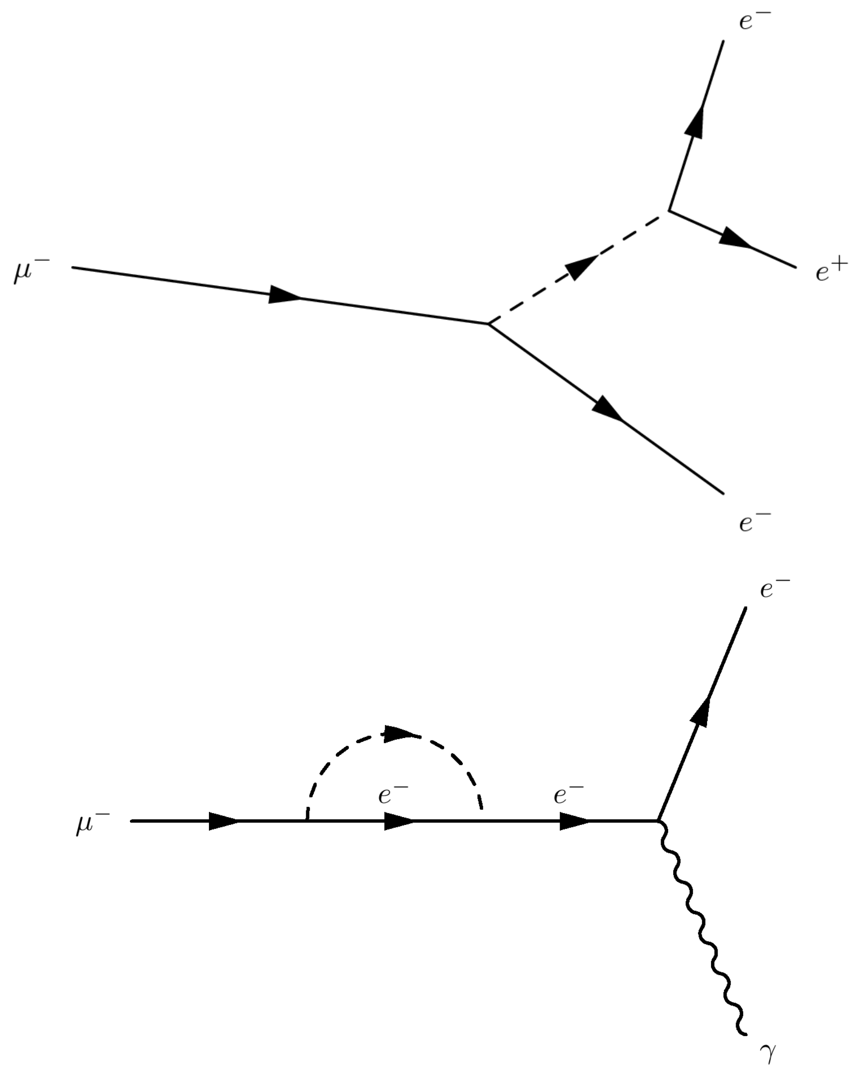

Figure 4.1: Feynman graphs allowing for the flavour changing rare decays $\mu^- \to e^- e^+ e^-$ and $\mu^- \to e^- \gamma$ in a multi-Higgs-doublet model. Not all contributing graphs are shown.

- *Can we infer symmetries of the neutrino mass matrix from the experimental data?* Based on the global fits of neutrino oscillation data and the other observables of neutrino physics, see chapter 2, we can perform a numerical analysis unveiling the allowed ranges of the elements of the neutrino mass matrix. This topic will be treated in section 4.4.

4.2 Finite groups with three-dimensional representations

In this section we want to elaborate on the possible candidates for finite family symmetry groups. Consequently, this section will mainly contain group theoretical considerations. Summaries of all needed group-theoretical basics can be found for example in [22, 24]

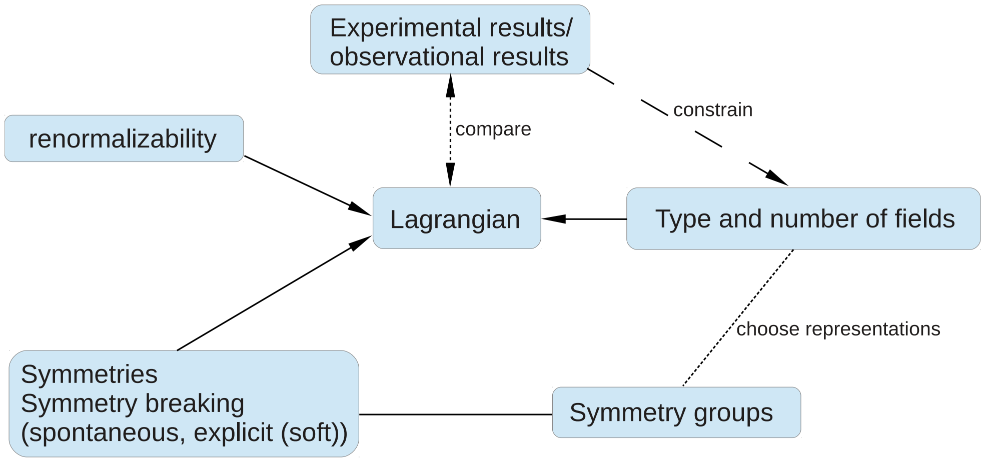

Figure 4.2: Some of the issues one has to take into account when one constructs an extension of the standard model including new fields and symmetries.

and the excellent textbook [25]. For reviews concentrating in particular on the group-theoretical aspects of finite family symmetry groups we refer the reader to [22, 26].

Since there are three known generations of fermions,[7] the group must possess at least one *non-trivial three-dimensional* representation. Unfortunately, this restriction is not very strong, and we have to specialize our investigations to some subset of all eligible groups.

This is a general drawback of the approach of first imposing a group, then comparing the corresponding model with the data: The number of eligible groups is just too high to allow a systematic investigation. However, there are promising attempts of model building with very large classes of finite family symmetry groups. As an example, in [27] Parattu and Wingerter study all finite groups of order smaller than 100 for their ability of enforcing the so-called Harrison-Perkins-Scott mixing matrix V_{HPS} [4] in the lepton sector. An important goal of the research field of finite family symmetry groups would be to find a powerful way to deduce the possible symmetries from the available data on fermion masses and mixing. Though this task seems to be very hard, there have been a lot of important developments in the recent years. A non-exhaustive list of references to works on this topic is [28–33].

The contributions to the flavour symmetry problem in this thesis follow the other approach. Starting from the set of all groups which possess a non-trivial three-dimensional representation, we specialize to those finite groups G which possess a *faithful three-dimensional representation* $\mathcal{D}$. This restriction implies that G is isomorphic to the matrix group $\mathcal{D}(G)$. Since every representation of a finite group is equivalent to a unitary representation, we can without loss of generality assume that $\mathcal{D}(G)$ is a representation of G in terms of *unitary* 3×3*-matrices*. Thus, any finite group possessing a faithful three-dimensional representation is isomorphic to a *finite subgroup of* $U(3)$. Unfortunately, to our knowledge there is no complete classification of the finite subgroups of $U(3)$.[8]

[7]We do not consider extensions of the standard model including a fourth generation of fermions in this thesis.

[8]Though the classification of all *finite simple groups* is complete, see [34] and references therein, we

The situation becomes much better if we turn to the finite subgroups of $SU(3)$. The different types of finite subgroups of $SU(3)$ have been classified already at the beginning of the twentieth century by H.F. Blichfeldt. The results of this classification are published in Blichfeldt's part of the book [35]. According to this analysis the finite subgroups of $SU(3)$ can be cast into five different classes, namely [22, 24, 35, 36]:

(A) *Abelian groups*,

(B) finite subgroups of $SU(3)$ with *faithful two-dimensional representations* (these groups are isomorphic to the finite subgroups of $U(2)$ [22]),

(C) the groups $C(n, a, b)$ generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n, a, b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}), \tag{4.18}$$

where $\eta = \exp(2\pi i/n)$, $n \in \mathbb{N}\backslash\{0, 1\}$ and $a, b \in \{0, ..., n-1\}$,

(D) the groups $D(n, a, b; d, r, s)$ generated by E, $F(n, a, b)$ and

$$\widetilde{G}(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix}, \tag{4.19}$$

where $\delta = \exp(2\pi i/d)$, $d \in \mathbb{N}\backslash\{0\}$ and $r, s \in \{0, ..., d-1\}$,

(E) and six exceptional finite subgroups of $SU(3)$:

- $\Sigma(60) \cong A_5$, $\Sigma(168) \cong \mathrm{PSL}(2, 7)$,
- $\Sigma(36 \times 3)$, $\Sigma(72 \times 3)$, $\Sigma(216 \times 3)$ and $\Sigma(360 \times 3)$,

as well as the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$.

Figure 4.3 shows the five classes and their relations with each other. Furthermore, for some important and well-known finite subgroups of $SU(3)$ the class they belong to is indicated. For a detailed description of the five classes of finite $SU(3)$-subgroups we refer the reader to [36], which is actually part of this thesis (see chapter 6), and to the review article [22]. Having mused about the finite subgroups of $SU(3)$, we now would like to know how the finite subgroups of $SU(3)$ are related to the finite subgroups of $U(3)$. The following theorem answers this question.

Theorem 3. Let $G \subset U(3)$ be a finite subgroup of $U(3)$, then the subset

$$N \equiv \{g \in G | \det g = 1\} \tag{4.20}$$

of all matrices of determinant one is a normal subgroup of G, and the corresponding factor group G/N is cyclic. Thus every finite subgroup of $U(3)$ is a so-called *cyclic extension* of a finite subgroup of $SU(3)$.

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are far from knowing all finite groups.

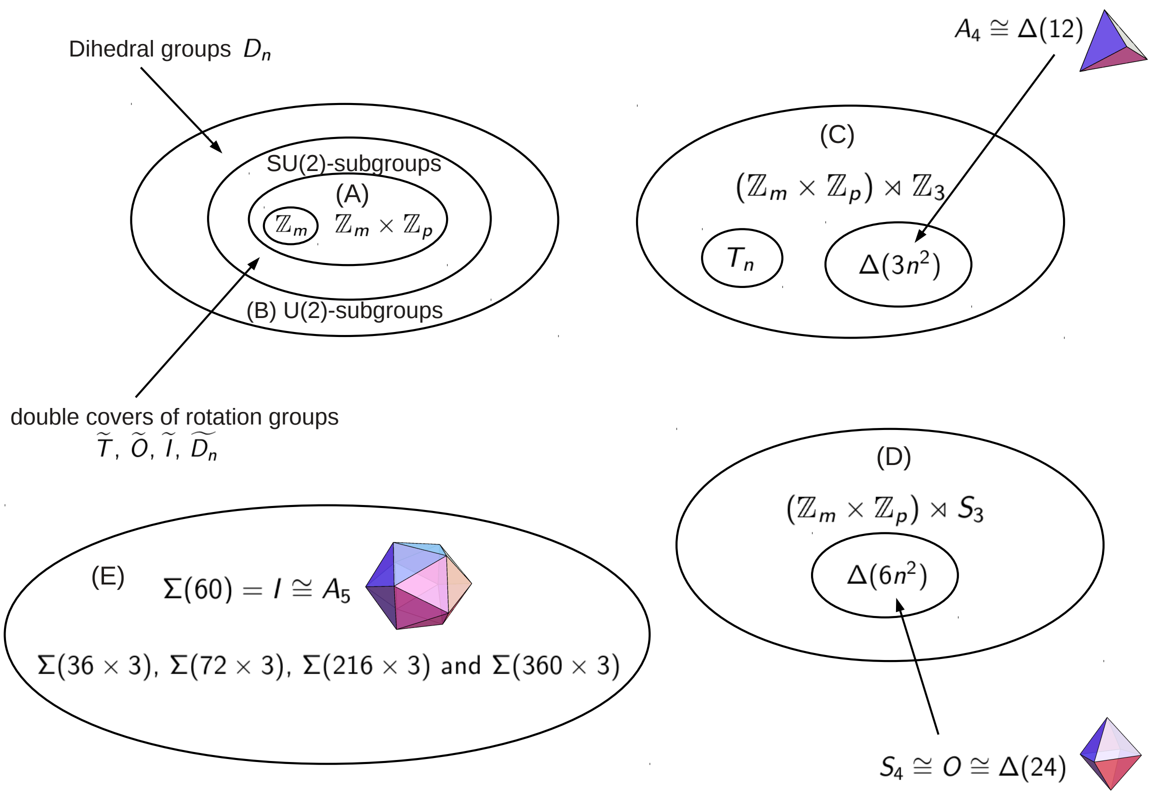

Figure 4.3: Graphical illustration of the five different classes (A)–(E) of finite subgroups of $SU(3)$ [22, 24, 35, 36]. Along with the five classes some of their important members are shown. This graphic has been copied from the author's own publication [37].

Also the generalization of this theorem with $U(3)$ replaced by $U(n)$ is true, the proof being provided in [22]. What does this theorem mean in practice? To answer this question, we have to study the properties of the factor group G/N. Since it is isomorphic to the cyclic group $\mathbb{Z}_m$, it can be generated by a single element $xN \in G/N$ fulfilling[9]

$$(xN)^m = N \quad \text{and} \quad (xN)^j \neq N \ \text{ for } \ j \in \{1, \ldots, m-1\}. \tag{4.25}$$

[9] The factor group G/N consists of all cosets

$$aN \equiv \{an | n \in N\} \tag{4.21}$$

with the multiplication law

$$(aN)(bN) \equiv \{an_1bn_2 | n_1, n_2 \in N\}. \tag{4.22}$$

Due to

$$an_1bn_2 = ab(\underbrace{b^{-1}n_1b}_{\in N}\,n_2) \tag{4.23}$$

we find

$$(aN)(bN) = (ab)N. \tag{4.24}$$

In particular, due to $N(bN) = bN$, N is the unit element of G/N.

Due to the multiplication rules for factor groups (see footnote 9 on page 74) we deduce that there exists a 3×3-matrix $x \in G \subset U(3)$ such that

$$x^j \notin N \quad \forall\, j \in \{1, \ldots, m-1\}, \tag{4.26a}$$
$$x^m \in N, \tag{4.26b}$$
$$x n x^{-1} \in N \quad \forall\, n \in N. \tag{4.26c}$$

Therefore we can formulate the following theorem [22].

Theorem 4. Let $N \subset SU(3)$ be a finite subgroup of $SU(3)$, then there exists a finite subgroup $G \subset U(3)$ with $G/N \cong \mathbb{Z}_m$ if and only if there exists a unitary 3×3-matrix x fulfilling the conditions (4.26a)–(4.26c). The $U(3)$-subgroup G is then given by

$$G = N \cup xN \cup x^2 N \cup \ldots \cup x^{m-1} N. \tag{4.27}$$

Thus, by finding all matrices x, we can construct all finite subgroups of $U(3)$ from the finite subgroups of $SU(3)$.

Unfortunately, though it looks much simpler than the general extension problem, also for the problem of finding all possible matrices x for a given subgroup N of $SU(3)$ we do not know a general solution. Nonetheless, we can at least partly study the finite subgroups of $U(3)$ relying on another classification of finite groups. The so-called SmallGroups library [38] contains information on all non-isomorphic finite groups of order smaller than 2000, except for the groups of order 1024.[10] In order to access the library, one can use the computer algebra system GAP [39]. In particular, GAP can construct generators for all representations of all groups contained in the library. In this way we could construct all finite subgroups G of $U(3)$ with the following properties:

- The order of G is smaller than 512,
- G possesses a faithful *irreducible* three-dimensional representation and
- there is no non-trivial cyclic group $\mathbb{Z}_m$, such that

$$G \cong H \times \mathbb{Z}_m. \tag{4.28}$$

The restriction to groups of order smaller than 512 finds its reason in the high number of 10494213 non-isomorphic finite groups of order 512 (in general there are a lot of finite groups of orders which are powers of a single prime number, like here $512 = 2^9$ and $1024 = 2^{10}$—see e.g. [40] and references therein). The restriction that the faithful three-dimensional representation should also be *irreducible* simplifies the procedure of extracting the desired groups from the library[11] and also reduces the number of groups to be studied

[10] Since there is the impressively high number of 49487365422 non-isomorphic groups of order 1024, these groups could not be included in the library.

[11] The number of faithful three-dimensional irreducible representations of a group can be read off from the character table directly. In order to find also the number of faithful *reducible* three-dimensional representations, one has to calculate the characters of all three-dimensional representations from the character table. This has for example been done in [27], where all finite groups of order 100 have been studied.

significantly. Finally, since the construction of direct product groups is straightforward, in order to further reduce the number of groups to be studied, we do not consider groups G of the form

$$G \cong H \times \mathbb{Z}_m, \tag{4.29}$$

where H is a finite group and $m > 1$. The results of this analysis of finite subgroups of $U(3)$ of order smaller than 512 are presented in [41], which is also part of this thesis—see chapter 5.

4.3 Texture zeros in the neutrino mass matrix

In this section we want to answer the question how flavour symmetries restrict the mass and mixing matrices. Since we consider flavour symmetries only, it is enough to investigate the Yukawa couplings and the scalar potential.

The best way to explain the restrictions implied by finite family symmetries, is by means of an example. For this purpose let us consider the Yukawa couplings of the charged leptons in a multi-Higgs-doublet model:

$$\mathcal{L} = -\overline{D_{\alpha L}} \Gamma_{j\alpha\beta} \phi_j \ell_{\beta R} + \text{H.c.} \tag{4.30}$$

Collecting the n Higgs doublets in a vector

$$\Phi \equiv \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}, \tag{4.31}$$

we can describe the action of the finite family symmetry group on the Lagrangian through the transformations

$$D_L \mapsto \mathcal{D}_L(g) D_L, \;\; \ell_R \mapsto \mathcal{D}_R(g) \ell_R, \;\; \Phi \mapsto \mathcal{D}_\Phi(g) \Phi \quad \forall\, g \in G. \tag{4.32}$$

The question under which circumstances there exists a non-trivial Yukawa coupling of the form (4.30) invariant under the transformations (4.32), is answered by the following theorem.

Theorem 5. There exists a non-trivial Yukawa coupling

$$\mathcal{L} = -\overline{D_{\alpha L}} \Gamma_{j\alpha\beta} \phi_j \ell_{\beta R} + \text{H.c.} \tag{4.33}$$

invariant under the transformations

$$D_L \mapsto \mathcal{D}_L(g) D_L, \;\; \ell_R \mapsto \mathcal{D}_R(g) \ell_R, \;\; \Phi \mapsto \mathcal{D}_\Phi(g) \Phi \quad \forall\, g \in G, \tag{4.34}$$

if and only if the tensor product $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$ contains the trivial representation $\mathbf{1}$, i.e.

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R = \mathbf{1} \oplus \;\ldots\; . \tag{4.35}$$

The proof of this theorem can be found in appendix B. Theorem 5 also tells us the number of linearly independent invariant Yukawa couplings. If namely $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$ contains the trivial representation $\mathbf{1}$ m times, i.e.

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{m \text{ times}} \oplus \ldots, \tag{4.36}$$

there are m linearly independent invariant Yukawa couplings of the form (4.30). The most general invariant Yukawa coupling would then have the form

$$\mathcal{L} = -\sum_{k=1}^{m} y_k \overline{D_{\alpha L}}\, \Gamma^{(k)}_{j\alpha\beta} \phi_j \ell_{\beta R} + \text{H.c.}, \tag{4.37}$$

where each of the m linearly independent couplings

$$-\overline{D_{\alpha L}} \Gamma^{(k)}_{j\alpha\beta} \phi_j \ell_{\beta R} + \text{H.c.} \tag{4.38}$$

is itself invariant, and the y_k are free complex parameters.

The scalar potential has to be treated similarly. Again restricting ourselves to the example of a multi-Higgs-doublet model, the scalar potential has the form

$$V(\phi_1, \ldots, \phi_n) = \mu^2_{ij} \phi_i^\dagger \phi_j + \lambda_{ijkl} (\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l). \tag{4.39}$$

The analogue of theorem 5 tells us that the number of linearly independent quadratic and quartic invariants can be found by considering the tensor product decompositions

$$\mathcal{D}_\Phi^* \otimes \mathcal{D}_\Phi = \mathbf{1} \oplus \cdots \oplus \mathbf{1} \oplus \ldots \tag{4.40}$$

and

$$\mathcal{D}_\Phi^* \otimes \mathcal{D}_\Phi \otimes \mathcal{D}_\Phi^* \otimes \mathcal{D}_\Phi = \mathbf{1} \oplus \cdots \oplus \mathbf{1} \oplus \ldots, \tag{4.41}$$

respectively. The most general invariant scalar potential will then have the form

$$V(\phi_1, \ldots, \phi_n) = \left(\sum_a c_a \mu^{2(a)}_{ij} \phi_i^\dagger \phi_j \right) + \left(\sum_b d_b \lambda^{(b)}_{ijkl} (\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l) \right), \tag{4.42}$$

where the indices a and b indicate the linearly independent invariants and the c_a and d_b are free real parameters. It is one of the hardest problems in model building to choose (or show that there exists a choice) the free parameters of the scalar potential in such a way that its global minimum $\mathbf{v}$ does not break the gauge group of electromagnetism [42–45], i.e. that

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} 0 \\ v_n \end{pmatrix} \end{pmatrix}. \tag{4.43}$$

To summarize, we have learned that the finite family symmetry group determines the Yukawa couplings and the scalar potential up to the free parameters y_k, c_a and d_b. However, other symmetries may even further restrict the Lagrangian.

A question important in practice is how the coefficients $\Gamma^{(k)}_{j\alpha\beta}$, ... can be constructed from the group. There are two possibilities to solve this problem. One way is to use the connection between the coefficients of the invariants and the Clebsch-Gordan coefficients. In appendix B we show that the coefficients $\Gamma^{(k)}_{j\alpha\beta}$ of the invariants are the complex conjugates of the Clebsch-Gordan coefficients for the decomposition

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R \to \mathbf{1}. \tag{4.44}$$

(These are the same as the Clebsch-Gordan coefficients for the decomposition $\mathcal{D}_L^* \otimes \mathcal{D}_R \to \mathcal{D}_\Phi^*$.) Thus all techniques for the construction of Clebsch-Gordan coefficients of finite groups can also be applied for the construction of invariants. A systematic method has for example been given in [46]. However, if all elements and all involved representations of the finite group are known, one can construct the invariants directly—see for example [47–49]. At this point we have to remark that, though the Clebsch-Gordan coefficients are basis-dependent, the physical predictions (i.e. the masses and the mixing matrix) are independent of the choice of basis [24].

As an explicit example we will now study the restrictions of Abelian groups on the mass and mixing matrices. Since in Abelian groups, all elements commute with each other, also all representation matrices commute. Therefore, the representation matrices are simultaneously diagonalizable. Hence, Abelian groups possess only one-dimensional irreducible representations. Turning back to our example of the charged-lepton Yukawa interactions, assuming an Abelian finite family symmetry group we can write the involved representations as sums of one-dimensional irreducible representations:

$$\begin{aligned}
\mathcal{D}_\Phi &= \mathcal{D}_\Phi^{(1)} \oplus \cdots \oplus \mathcal{D}_\Phi^{(n)}, \\
\mathcal{D}_L &= \mathcal{D}_L^{(1)} \oplus \mathcal{D}_L^{(2)} \oplus \mathcal{D}_L^{(3)}, \\
\mathcal{D}_R &= \mathcal{D}_R^{(1)} \oplus \mathcal{D}_R^{(2)} \oplus \mathcal{D}_R^{(3)}.
\end{aligned} \tag{4.45}$$

Thus, as it must be, also the tensor product $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$ completely decays into one-dimensional representations:

$$\begin{aligned}
&\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R = \\
&= (\mathcal{D}_\Phi^{(1)} \oplus \cdots \oplus \mathcal{D}_\Phi^{(n)}) \otimes (\mathcal{D}_L^{(1)} \oplus \mathcal{D}_L^{(2)} \oplus \mathcal{D}_L^{(3)})^* \otimes (\mathcal{D}_R^{(1)} \oplus \mathcal{D}_R^{(2)} \oplus \mathcal{D}_R^{(3)}) = \\
&= \mathcal{D}_\Phi^{(1)} \otimes \mathcal{D}_L^{(1)*} \otimes \mathcal{D}_R^{(1)} \oplus \cdots \oplus \mathcal{D}_\Phi^{(j)} \otimes \mathcal{D}_L^{(\alpha)*} \otimes \mathcal{D}_R^{(\beta)} \oplus \cdots \oplus \mathcal{D}_\Phi^{(n)} \otimes \mathcal{D}_L^{(3)*} \otimes \mathcal{D}_R^{(3)}.
\end{aligned} \tag{4.46}$$

In the form given in the last line of equation (4.46) the tensor product is already completely reduced to its one-dimensional constituents. In order for a non-trivial invariant to exist, the reduced tensor product must contain at least one trivial representation $\mathbf{1}$. In fact, the Yukawa coupling

$$-\overline{D_{\alpha L}}\, \phi_j \ell_{\beta R} + \text{H.c.} \tag{4.47}$$

is invariant if and only if

$$\mathcal{D}_\Phi^{(j)} \otimes \mathcal{D}_L^{(\alpha)*} \otimes \mathcal{D}_R^{(\beta)} = \mathbf{1}. \tag{4.48}$$

Thus we find the following restriction on the coefficients $\Gamma_{j\alpha\beta}$:

$$\Gamma_{j\alpha\beta} = 0 \quad \text{for} \quad \mathcal{D}_\Phi^{(j)} \otimes \mathcal{D}_L^{(\alpha)*} \otimes \mathcal{D}_R^{(\beta)} \neq \mathbf{1}. \tag{4.49}$$

If $\mathcal{D}_\Phi^{(j)} \otimes \mathcal{D}_L^{(\alpha)*} \otimes \mathcal{D}_R^{(\beta)} = \mathbf{1}$, the coefficients $\Gamma_{j\alpha\beta}$ are unrestricted and are therefore free parameters of the model. Thus the essence of the action of Abelian finite flavour symmetries is that they can set some of the elements of the matrices Γ_j of Yukawa coupling constants to zero, leaving the other elements unrestricted. This property also translates to the mass matrix. In our example the elements of the charged-lepton mass matrix generated through spontaneous symmetry breaking are given by

$$(M_\ell)_{\alpha\beta} = \sum_{j=1}^{n} \frac{v_j}{\sqrt{2}} \Gamma_{j\alpha\beta}. \tag{4.50}$$

Since the $\Gamma_{j\alpha\beta}$ are either zero or free parameters, we find that the only restriction an Abelian finite family symmetry group can impose on the mass matrices are so-called *texture zeros*, i.e. zero elements of the mass matrix. Note that these texture zeros do not necessarily arise from the restrictions on the $\Gamma_{j\alpha\beta}$ alone, since the family symmetry may also set some of the VEVs v_j to zero.

At this point we want to remark that the way texture zeros in the mass matrices are generated as described above works only for the mass matrices directly emerging from the matrices of Yukawa coupling constants. However, Abelian symmetries can also be used to generate texture zeros in the *effective* light neutrino mass matrix of the type-I seesaw mechanism by suitably placing texture zeros in M_D and M_R, though this method will not allow all possible patterns of texture zeros in M_ν.[12] For a general discussion of how to realize texture zeros by means of symmetries we refer the reader to [51].

The reason why we chose the simplest example (namely the one of Abelian finite family symmetry groups) is that their consequences, texture zeros, are intensively studied as viable restrictions on the mass matrices. One of the most studied subcases of texture zeros is the case of a *diagonal* charged-lepton mass matrix (which corresponds to six texture zeros) and one or more *texture zeros in the neutrino mass matrix.*[13] While there is some literature on texture zeros in the mass matrix of Dirac neutrinos, see e.g. [54, 55], there are much more papers on texture zeros in the light Majorana neutrino mass matrix—for selected contributions see [56–59]. The appealing feature of such textures is that, since the charged-lepton mass matrix is diagonal,[14] the light neutrino mass matrix is given by

$$M_\nu = U_{\text{PMNS}}^* \operatorname{diag}(m_1, m_2, m_3) U_{\text{PMNS}}^\dagger. \tag{4.52}$$

[12] However, for a diagonal M_D, placing texture zeros in M_R directly leads to texture zeros in the *inverse* M_ν^{-1} of the effective light neutrino mass matrix. This interesting possibility has for example been studied in [50].

[13] Also texture zeros in M_ℓ and M_ν with M_ℓ non-diagonal have been studied—see e.g. [52] and references therein. Note that in the most general framework (where M_ℓ is non-diagonal) not all possible types of texture zeros necessarily imply physical constraints, since some types of texture zeros may always be achieved through a suitable weak basis transformation [52, 53].

[14] To be more precise, we assume M_ℓ to be given by

$$M_\ell = \operatorname{diag}(m_e, m_\mu, m_\tau), \tag{4.51}$$

i.e. the charged lepton masses shall already be in the correct ordering.

Thus the neutrino mass matrix is completely determined by the neutrino masses and the lepton mixing matrix. Hence, imposing texture zeros in the above matrix will in general have strong consequences on the observables of neutrino physics. Since M_ν is symmetric, there can be at most six independent texture zeros, and if one speaks of n texture zeros in the Majorana neutrino mass matrix, one usually means that there are n *independent* zeros. Texture zeros are powerful restrictions. The maximum number of independent texture zeros in M_ν compatible with the experimental data is two, and among the fifteen possibilities for two texture zeros in M_ν only seven cases are viable [56, 60].

It is noteworthy that the analysis of the viability of texture zeros also has implications on models based on non-Abelian groups. Also in such models there may be texture zeros, usually accompanied by additional relations among the matrix elements. Let us illustrate this by means of an example. Reference [24] lists the Clebsch-Gordan coefficients for the reduction of $3 \otimes 3$-tensor products for many finite non-Abelian subgroups of $SU(3)$. Browsing through this list one finds that many of the mass matrices resulting from models based on these tensor products will also show texture zeros. Choosing for example the group S_4 as a family symmetry group and assigning the leptons and scalars of our "standard example" (charged leptons in a multi-Higgs-doublet model) to irreducible representations of S_4 as

$$D_L \sim \mathbf{3}, \quad \ell_R \sim \mathbf{3} \quad \text{and} \quad \Phi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \sim \mathbf{2}, \tag{4.53}$$

would imply the following structure of the charged lepton mass matrix [24]

$$M_\ell \propto \frac{v_1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{v_2}{2\sqrt{3}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.54}$$

Thus we would obtain a charged-lepton mass matrix with six texture zeros fulfilling the additional relation $\mathrm{Tr}\, M_\ell = 0$. (Note that the charged-lepton mass matrix given above is not viable because it cannot reproduce the correct mass hierarchies in the charged lepton sector.)

Due to their interesting features, texture zeros still comprise an interesting field of study in the research area of finite family symmetries. Turning to the content of this thesis in particular, two papers treat texture zeros of the light Majorana neutrino mass matrix in the basis where the charged lepton mass matrix is diagonal—see chapters 7 and 8. While the first paper (chapter 7) discusses the possibility of nearly maximal atmospheric neutrino mixing, the second paper (chapter 8) focuses on the question of the implication of the recently established high value of the reactor mixing angle θ_{13} on predictions for the CP-phase δ in the case of Majorana neutrino mass matrices with two texture zeros.

4.4 Beyond texture zeros: Relations among the elements of the neutrino mass matrix

In the previous section we discussed the implications of finite family symmetries on the Yukawa couplings and the scalar potential which lead to restrictions on the fermion mass and mixing matrices. In general a finite family symmetry group will always impose relations among the elements of the mass matrices, be it vanishing of elements (texture zeros) or other relations. Therefore, detailed investigation of the experimental data on the mass matrices may allow to infer symmetries in the lepton sector.

In the basis where the charged-lepton mass matrix is given by

$$M_\ell = \mathrm{diag}(m_e,\, m_\mu,\, m_\tau), \tag{4.55}$$

the neutrino mass matrix

$$M_\nu = U^*_{\mathrm{PMNS}}\, \mathrm{diag}(m_1, m_2, m_3) U^\dagger_{\mathrm{PMNS}} \tag{4.56}$$

depends only on the neutrino masses and the parameters of the PMNS-matrix. Considering the absolute values of the elements of M_ν, which are independent of the phases α, β and γ (see equation (2.20)), we are left with the nine parameters

$$m_1,\, m_2,\, m_3,\, \theta_{12},\, \theta_{23},\, \theta_{13},\, \delta,\, \rho \quad \text{and} \quad \sigma \tag{4.57}$$

presented in chapter 2.

Thus the constraints on these nine parameters[15] can be translated to *constraints on the absolute values of the elements of the Majorana neutrino mass matrix* (4.56). With the help of the numerical methods discussed in chapter 3 it is then possible to create correlation plots of the absolute values of the elements of M_ν. Though such plots do not contain any information on the relative phases of the elements of M_ν, they still reveal some important properties of the neutrino mass matrix. For example, the correlation plots would directly show the allowed cases for two texture zeros in the neutrino mass matrix. Moreover, such plots could even allow to deduce (or refute) relations among the elements of M_ν. In this way one could infer (exclude) symmetries of M_ν, which would be very helpful for model building with finite family symmetry groups.

A numerical analysis as described above has been performed in [61], which is part of this thesis—see chapter 9.

Alternatively, one can also use the χ^2-method—see chapter 3—in order to obtain relations among the elements of M_ν from the experimental data. Fitting neutrino mass matrices with two texture zeros to the available data, we found two promising *hybrid textures* (texture zeros + additional relations) which are in very good agreement with experiment at the the current level of accuracy, namely [60]

$$\bar{A}1 : M_\nu = \begin{pmatrix} 0 & 0 & a \\ 0 & b & 2a \\ a & 2a & b \end{pmatrix} \quad \text{and} \quad \bar{A}2 : M_\nu = \begin{pmatrix} 0 & a & 0 \\ a & b & 2a \\ 0 & 2a & b \end{pmatrix}, \tag{4.58}$$

[15] The constraints on ρ and σ are trivial, i.e. $\rho, \sigma \in [0, 2\pi)$.

where a and b are real parameters and the charged-lepton mass matrix is again assumed to be diagonal. Note that the two textures (4.58) are special cases of the two texture zeros A_1 and A_2 of [56] and as such they are compatible only with a normal neutrino mass spectrum.

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Part II

Papers

Chapter 5

On the finite subgroups of $U(3)$ of order smaller than 512

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On the finite subgroups of U(3) of order smaller than 512

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Abstract

We use the SmallGroups Library to find the finite subgroups of $U(3)$ of order smaller than 512 which possess a faithful three-dimensional irreducible representation. From the resulting list of groups we extract those groups that can not be written as direct products with cyclic groups. These groups are important building blocks for models based on finite subgroups of $U(3)$.
All resulting finite subgroups of $SU(3)$ can be identified using the well known list of finite subgroups of $SU(3)$ derived by Miller, Blichfeldt and Dickson at the beginning of the 20^{th} century. Furthermore we prove a theorem which allows to construct infinite series of finite subgroups of $U(3)$ from a special type of finite subgroups of $U(3)$. This theorem is used to construct some new series of finite subgroups of $U(3)$. The first members of these series can be found in the derived list of finite subgroups of $U(3)$ of order smaller than 512.
In the last part of this work we analyse some interesting finite subgroups of $U(3)$, especially the group $S_4(2) \cong A_4 \rtimes \mathcal{Z}_4$, which is closely related to the important $SU(3)$-subgroup S_4.

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I Introduction

The problem of lepton masses and mixing (more generally the fermion mass and mixing problem) is one of the most interesting current research topics of particle physics and withstood a solution for decades. Invariance of the Lagrangian under finite family symmetry groups constitutes an interesting possibility, at least for a partial solution of this problem.

In 2002 Harrison, Perkins and Scott suggested the tribimaximal lepton mixing matrix [1]

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{1}$$

which is in agreement with the current experimental bounds on the lepton mixing matrix [2]. The nice appearance of this matrix induced the idea of an underlying symmetry in the Lagrangian of the lepton and scalar sector. A large number of models involving especially discrete symmetries followed. For a review of today's state of the art in model building see [3].

Due to the fact that there are three known families of leptons (and quarks) models based on finite subgroups of $U(3)$ have become very popular, so a systematic analysis of the finite subgroups of $U(3)$ would provide an invaluably helpful tool for model building with finite family symmetry groups.
Unfortunately the finite subgroups of $U(3)$ have, to our knowledge, not been classified by now, while the finite subgroups of $SU(3)$ have been classified already at the beginning of the 20$^{\text{th}}$ century by Miller, Blichfeldt and Dickson [4].

The idea of a systematic analysis of finite subgroups of $SU(3)$ in the context of particle physics is not new [5, 6, 7, 8], and research on the application of finite groups in particle physics (especially finite subgroups of $SU(3)$) continues unabated [9, 10, 11, 12, 13, 14, 15, 16].

II The finite subgroups of U(3) of order smaller than 512

In this work we want to concentrate on the finite subgroups of $U(3)$. We will use both the analytical tools of group theory as well as the modern tool of computer algebra to investigate the finite subgroups of $U(3)$ up to order 511.

II.1 Classification of finite subgroups of $U(3)$

At first let us consider the different types of finite subgroups of $U(3)$ we will distinguish. Knowing that every representation of a finite group is equivalent to a unitary representation we find:

> A finite group G is isomorphic to a finite subgroup of $U(3)$ if and only if it possesses a faithful three-dimensional representation.

Thus, if we would search for all finite groups which fulfil the above properties, we would obtain all finite subgroups of $U(3)$, especially we would obtain all finite subgroups of $U(1)$ and $U(2)$ too. When we usually speak of "finite subgroups of $U(3)$" we primarily mean those finite subgroups of $U(3)$ which are not finite subgroups of $U(1)$ and $U(2)$.
At this point it is important to notice that the possession of a faithful 3-dimensional *irreducible* representation is sufficient but not necessary for a group to be a finite subgroup of $U(3)$[1]. In fact there are many finite subgroups of $U(3)$ which possess a faithful 3-dimensional *reducible* representation but do not possess any faithful 1- or 2-dimensional representations. According to [17] these groups correspond to finite subgroups of $U(2) \times U(1)$.
Though doing so we will miss the $U(2) \times U(1)$-subgroups mentioned above, we will in this work specialise to the finite subgroups of $U(3)$ which possess a faithful 3-dimensional irreducible representation. Among these groups we can differentiate

- groups which have a faithful irreducible representation of determinant[2] 1 ($SU(3)$-subgroups) and
- groups which don't have a faithful irreducible representation of determinant 1.

An analysis similar to the one performed in this work can be found in [17], where all finite groups up to order 100 are listed. The list given in [17] especially indicates which groups of order up to 100 possess 3-dimensional faithful representations (both reducible and irreducible).

II.2 The computer algebra system GAP and the SmallGroups Library

The task of classifying all finite groups which have a faithful three-dimensional irreducible representation would need an analysis using the techniques described in [4] applied to $U(3)$. Instead of doing that, we want to get a first impression of the finite subgroups of $U(3)$ by searching for $U(3)$-subgroups of small orders with the help of the computer algebra system GAP [18]. Through the SmallGroups package [19] GAP allows access to the SmallGroups Library [19, 20] which contains, among other finite groups, all finite groups up to order 2000, except for the groups of order 1024, up to isomorphism.

The way finite groups are labeled in the SmallGroups Library is the following: Let there be n non-isomorphic groups of order g, then these n groups are labeled by their order g and a number $j \in \{1, ..., n\}$:

> $[\![g, j]\!]$ denotes the $j-$th finite group of order g ($j \in \{1, ..., n\}$) listed in the SmallGroups Library.

[1] The author wants to thank K.M. Parattu and A. Wingerter for pointing this out in their paper [17].
[2] Let $D : G \to D(G)$ be a representation of a finite group G. We say D has determinant 1, if all matrices in $D(G)$ have determinant 1.

The way in which the n non-isomorphic groups of a given order g are arranged in the SmallGroups Library depends on g. For a detailed description of the SmallGroups Library we refer the reader to chapter 48.7 of the GAP reference manual [18]. For our purpose we only need to know that $[\![g,j]\!]$ is not isomorphic to $[\![g,k]\!]$ if $j \neq k$. To all "small groups" $[\![g,j]\!]$ listed in this paper the reader can find the common name or a list of generators in tables 4 and 5, respectively.
Let us now get a picture of the number of finite groups of some given order. Figure 1 shows the total number $N(g)$ of non-Abelian groups of order $\leq g$.

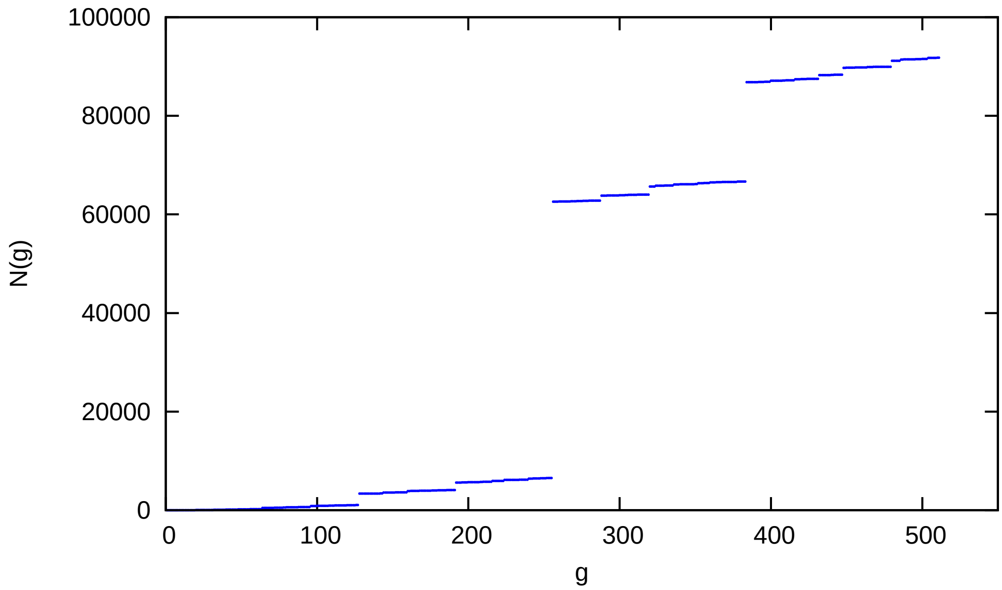

Figure 1: Total number $N(g)$ of non-Abelian groups up to order g.

From figure 1 one can immediately deduce that the number of finite groups of order g is usually very high if g contains high powers of 2. Therefore there are high "jumps" in $N(g)$ at

$$g = 2^8 = 256 \quad \text{and} \quad g = 3 \times 2^7 = 384.$$

Indeed a much larger "jump" occurs at $g = 512$: $N(511) = 91774$, while there are 10494213 groups of order 512 [19] of which only 30 are Abelian[3]. If we want to analyse groups with faithful 3-dimensional irreducible representations only, we don't need to consider groups of order 512 due to the following theorem:

II.1 Theorem. Let D be an irreducible representation of a finite group G, then the dimension $\dim(D)$ of D is a divisor of the order $\mathrm{ord}(G)$ of G.

The proof of this theorem can be found in textbooks on finite group theory, see for example [22] p. 176f. or [23] p. 288f. Note that theorem II.1 tells us that the order of any

[3]The number of non-isomorphic Abelian groups of a given order can be calculated explicitly. See for example the article "Abelian Group" in [21].

group which possesses a 3-dimensional irreducible representation must be divisible by 3. This implies that the groups of order 512 do not possess 3-dimensional irreducible representations (but there could be groups of order 512 which possess faithful 3-dimensional reducible representations).
In this work we will analyse all groups of order up to 511. From tables 4 and 5, which show our results, one can find that the orders of all groups we have found are indeed divisible by 3.

II.3 Extraction of finite subgroups of $U(3)$ from the SmallGroups Library

Using GAP the determination of the finite subgroups of $U(3)$ from the SmallGroups Library is not difficult. GAP offers the opportunity to calculate the character tables[4] as well as all irreducible representations of a given "small group" $[\![g, j]\!]$. Using criterion II.2 one can immediately deduce the dimensions of the faithful irreducible representations of a "small group" using its character table. If the analysed group has a three-dimensional faithful irreducible representation it is a finite subgroup of $U(3)$[5]. By explicit construction of the irreducible representations[6] one can determine the $U(3)$-subgroups which have a faithful three-dimensional irreducible representation of determinant 1 ($SU(3)$-subgroups).

II.2 Criterion. Let D be a d-dimensional representation of a finite group G. Then D is non-faithful if and only if D has more than one character d in the character table.

The proof of criterion II.2 can be found in appendix A.1.

Let us, in this paper, choose the following convention: Let G be a finite group. We say that "G can not be written as a direct product with a cyclic group" if there does not exist a group F and an $m > 1$ such that

$$G \cong F \times \mathcal{Z}_m. \tag{2}$$

Before we list the results let us finally divide the obtained groups into another two sets, namely

- groups that can be written as direct products with cyclic groups and
- groups that can not be written as direct products with cyclic groups.

[4] We use the GAP command `CharacterTable(.)` to calculate the character table of a group.

[5] Please note that this is sufficient, but not necessary [17]. Here we specialise onto the finite subgroups of $U(3)$ which possess a faithful 3-dimensional irreducible representation.

[6] We use the GAP command `IrreducibleRepresentations(.)` to calculate the irreducible representations of a group. Since the labeling of the irreducible representations computed with `IrreducibleRepresentations(.)` does not necessarily agree with the labeling of `CharacterTable(.)`, we use the commands `Image(.)` and `Order(.)` to find the faithful irreducible representations of the group under consideration.

How can we determine whether a "small group" can be written as a direct product with a cyclic group? The GAP command `StructureDescription(.)` gives the basic structure of a group, especially it tells us whether a group can be written as a direct product with a cyclic group. Let us clarify this with two examples:

```
gap>StructureDescription(SmallGroup([12,3]));
"A4"
gap>StructureDescription(SmallGroup([24,13]));
"C2 x A4"
```

So GAP tells us that the group $[\![12,3]\!]$ is isomorphic to A_4, and that $[\![24,13]\!]$ is isomorphic to $\mathcal{Z}_2 \times A_4$, i.e. it is a direct product with a cyclic group.

How are the groups that can be written as direct products with cyclic groups related to the groups which can not be written as direct products with cyclic groups? The answer is provided by theorem II.3.

II.3 Theorem. Let G be a finite group with an m-dimensional faithful irreducible representation D. Let C be the center of G, $\mathrm{ord}(C) = c$ and let $\gcd(n,c)$ be the greatest common divisor of $n, c \in \mathbb{N}\backslash\{0\}$.

Then $\mathcal{Z}_n \times G$ has a faithful m-dimensional irreducible representation if and only if $\gcd(n,c) = 1$.

The proof of this theorem can be found in appendix A.2. Theorem II.3 implies that we can construct all finite groups which have a faithful three-dimensional irreducible representation from all finite groups which have a faithful three-dimensional irreducible representation and can not be written as direct products with cyclic groups.

Let us consider the following examples:

1. The group $A_4 \cong [\![12,3]\!]$ has center $C = \{e\}$. $\Rightarrow c = \mathrm{ord}(C) = 1$, thus $n \in \mathbb{N}\backslash\{0\}$ and $c = 1$ have no common divisor $d \neq 1$. Therefore all direct products
$$\mathcal{Z}_n \times A_4, \quad n \in \mathbb{N}\backslash\{0,1\}$$
have three-dimensional faithful irreducible representations. Among these direct products only $\mathcal{Z}_3 \times A_4$ will have a faithful three-dimensional irreducible representation of determinant 1 ($\det(\omega^k \mathbb{1}_3) = 1,\ \omega = e^{\frac{2\pi i}{3}},\ k = 0,1,2$).

2. The group $\Delta(27) \cong [\![27,3]\!]$ has center $C \cong \mathcal{Z}_3$. $\Rightarrow c = \mathrm{ord}(C) = 3$, thus $n \in \mathbb{N}\backslash\{3k|k \in \mathbb{N}\}$ and $c = 3$ have no common divisor $d \neq 1$. Therefore all direct products
$$\mathcal{Z}_n \times \Delta(27), \quad n \in \mathbb{N}\backslash(\{3k|k \in \mathbb{N}\} \cup \{1\})$$
have faithful three-dimensional irreducible representations. None of these groups has a three-dimensional faithful irreducible representation of determinant 1.

In the results we will only list groups that can not be written as direct products with cyclic groups. From these groups all other groups can be constructed using theorem II.3. The results obtained from the SmallGroups Library are in perfect agreement with theorem II.3.

II.4 Results

II.4.1 Generators

In tables 1 and 2 we list all matrices needed to generate the finite subgroups of $U(3)$ of order smaller than 512.

Generators of determinant 1	
$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$F(n,a,b) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}$
$G(d,r,s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix}$	$H = \frac{1}{2}\begin{pmatrix} -1 & \mu_- & \mu_+ \\ \mu_- & \mu_+ & -1 \\ \mu_+ & -1 & \mu_- \end{pmatrix}$
$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$	$K = \frac{1}{\sqrt{3}\,i}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$
$L = \frac{1}{\sqrt{3}\,i}\begin{pmatrix} 1 & 1 & \omega^2 \\ 1 & \omega & \omega \\ \omega & 1 & \omega \end{pmatrix}$	$M = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}$
$N = \frac{i}{\sqrt{7}}\begin{pmatrix} \beta^4-\beta^3 & \beta^2-\beta^5 & \beta-\beta^6 \\ \beta^2-\beta^5 & \beta-\beta^6 & \beta^4-\beta^3 \\ \beta-\beta^6 & \beta^4-\beta^3 & \beta^2-\beta^5 \end{pmatrix}$	$P = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon\omega \end{pmatrix}$
$Q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\omega \\ 0 & -\omega^2 & 0 \end{pmatrix}$	

Table 1: Generators of finite subgroups of $SU(3)$.
$\eta := e^{2\pi i/n}$, $\delta := e^{2\pi i/d}$, $\mu_\pm = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$, $\omega = e^{2\pi i/3}$, $\beta = e^{2\pi i/7}$, $\epsilon = e^{4\pi i/9}$.

Generators for groups of determinant unequal 1	
$R(n,a,b,c) = \begin{pmatrix} 0 & 0 & \eta^a \\ \eta^b & 0 & 0 \\ 0 & \eta^c & 0 \end{pmatrix}$	$S(n,a,b,c) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & 0 & \eta^b \\ 0 & \eta^c & 0 \end{pmatrix}$
$T(n,a,b,c) = \begin{pmatrix} 0 & 0 & \eta^a \\ 0 & \eta^b & 0 \\ \eta^c & 0 & 0 \end{pmatrix}$	$U(n,a,b,c) = \begin{pmatrix} 0 & \eta^a & 0 \\ \eta^b & 0 & 0 \\ 0 & 0 & \eta^c \end{pmatrix}$
$V(n,a,b,c) = \begin{pmatrix} 0 & \eta^a & 0 \\ 0 & 0 & \eta^b \\ \eta^c & 0 & 0 \end{pmatrix}$	$W(n,a,b,c) = \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^c \end{pmatrix}$

Generators for groups of determinant unequal 1	
$X_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\gamma^{11} & \frac{1}{\sqrt{2}}\gamma^{14} \\ \frac{1}{\sqrt{2}}\gamma^{5} & \frac{1}{2}\gamma^{20} & \frac{1}{2}\gamma^{11} \\ \frac{1}{\sqrt{2}}\gamma^{14} & \frac{1}{2}\gamma^{17} & \frac{1}{2}\gamma^{8} \end{pmatrix}$	$X_2 = \begin{pmatrix} \frac{1}{\sqrt{3}}\gamma^{21} & \frac{1}{\sqrt{6}}\gamma^{16} & \frac{1}{\sqrt{2}}\gamma^{13} \\ \sqrt{\frac{2}{3}}\gamma^{14} & \frac{1}{2\sqrt{3}}\gamma^{21} & \frac{1}{2}\gamma^{18} \\ 0 & \frac{\sqrt{3}}{2}\gamma^{18} & \frac{1}{2}\gamma^{3} \end{pmatrix}$
$X_3 = \begin{pmatrix} \frac{1}{\sqrt{3}}\vartheta^{31} & \frac{1}{\sqrt{6}}\vartheta^{14} & \frac{1}{\sqrt{2}}\vartheta^{4} \\ \sqrt{\frac{2}{3}}\vartheta^{30} & \frac{1}{2\sqrt{3}}\vartheta^{31} & \frac{1}{2}\vartheta^{21} \\ 0 & \frac{\sqrt{3}}{2}\vartheta^{32} & \frac{1}{2}\vartheta^{4} \end{pmatrix}$	$X_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\vartheta^{13} & \frac{1}{\sqrt{2}}\vartheta^{12} \\ \frac{1}{\sqrt{2}}\vartheta^{35} & \frac{1}{2}\vartheta^{24} & \frac{1}{2}\vartheta^{5} \\ \frac{1}{\sqrt{2}}\vartheta^{18} & \frac{1}{2}\vartheta^{25} & \frac{1}{2}\vartheta^{6} \end{pmatrix}$
$X_5 = \begin{pmatrix} \frac{1}{\sqrt{3}}\phi^{9} & \sqrt{\frac{2}{3}} & 0 \\ \sqrt{\frac{2}{3}}\phi^{2} & \frac{1}{\sqrt{3}}\phi & 0 \\ 0 & 0 & \phi^{5} \end{pmatrix}$	$X_6 = \begin{pmatrix} \gamma^{22} & 0 & 0 \\ 0 & \frac{1}{2}\gamma^{10} & \frac{\sqrt{3}}{2}\gamma^{11} \\ 0 & \frac{\sqrt{3}}{2}\gamma^{21} & \frac{1}{2}\gamma^{10} \end{pmatrix}$
$X_7 = \begin{pmatrix} \frac{1}{\sqrt{3}}\psi^{9} & \frac{1}{\sqrt{6}}\psi^{2} & \frac{1}{\sqrt{2}}\psi^{7} \\ \frac{1}{\sqrt{6}}\psi^{4} & \frac{9+\sqrt{3}i}{12} & \frac{1}{2}\psi^{10} \\ \frac{1}{\sqrt{2}}\psi^{11} & \frac{1}{2} & \frac{1}{2}\psi^{2} \end{pmatrix}$	$X_8 = \begin{pmatrix} \frac{1}{\sqrt{3}}\psi^{6} & \sqrt{\frac{2}{3}}\psi & 0 \\ \sqrt{\frac{2}{3}}\psi^{11} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \psi^{3} \end{pmatrix}$
$X_9 = \begin{pmatrix} \frac{1}{\sqrt{3}}\gamma^{13} & \sqrt{\frac{2}{3}}\gamma^{14} & 0 \\ \sqrt{\frac{2}{3}}\gamma^{12} & \frac{1}{\sqrt{3}}\gamma & 0 \\ 0 & 0 & \gamma^{19} \end{pmatrix}$	$X_{10} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\gamma^{3} & \frac{1}{\sqrt{2}}\gamma^{19} \\ \frac{1}{\sqrt{2}}\gamma & \frac{1}{2}\gamma^{2} & \frac{1}{2}\gamma^{6} \\ \frac{1}{\sqrt{2}}\gamma^{21} & \frac{1}{2}\gamma^{10} & \frac{1}{2}\gamma^{14} \end{pmatrix}$

Table 2: Generators of finite subgroups of $U(3)$.
$\gamma := e^{2\pi i/24}$, $\vartheta = e^{2\pi i/36}$, $\phi = e^{2\pi i/16}$, $\psi = e^{2\pi i/12}$.

Following [4, 5] all non-Abelian finite subgroups of $SU(3)$ which have a faithful three-dimensional irreducible representation can be cast into one of the types[7] listed in table 3.

To our knowledge only one series of finite subgroups of $U(3)$ (determinant $\neq 1$) is known by now, namely $\Sigma(3N^3)$, $(N \in \{3k \vert k \in \mathbb{N}\backslash\{0,1\}\})$, which has been published recently by Ishimori et al. in [16]. A well known member of $\Sigma(3N^3)$ is $\Sigma(81)$ [29, 35].

II.4.2 The finite subgroups of $SU(3)$ of order smaller than 512

In table 4 we list the finite groups of order smaller than 512 that can not be written as direct products with cyclic groups and that have a faithful three-dimensional irreducible representation of determinant 1.

[7] Note that for some choices of n, a, b, d, r, s the three-dimensional representations of $C(n, a, b)$, $D(n, a, b; d, r, s)$ given here could be reducible or lead to direct products with cyclic groups, so not all values of the parameters are allowed.
The allowed values for n in T_n are products of powers of primes of the form $3k + 1$, $k \in \mathbb{N}$. Please note furthermore that T_n is in general not unique. The equation $(1 + a + a^2) \bmod n = 0$ may have more than one solution, which can lead to non-isomorphic groups T_n with the same n. There are for example two non-isomorphic groups T_{91} in table 4.

Group	Generators	References
$C(n,a,b)$	$E,\ F(n,a,b)$	[4, 14]
$D(n,a,b;d,r,s)$	$E,\ F(n,a,b),\ G(d,r,s)$	[4, 14, 15]
$\Delta(3n^2)=C(n,0,1),\ n\geq 2$	$E,\ F(n,0,1)$	[5, 6, 10, 14]
$\Delta(6n^2)=D(n,0,1;2,1,1),\ n\geq 2$	$E,\ F(n,0,1),\ G(2,1,1)$	[5, 6, 12, 14]
$T_n=C(n,1,a),\ (1+a+a^2)\bmod n=0$	$E,\ F(n,1,a)$	[6, 7, 8, 14]
$A_5=\Sigma(60)$	$E,\ F(2,0,1),\ H$	[4, 5, 11, 14, 24]
$PSL(2,7)=\Sigma(168)$	$E,\ M,\ N$	[4, 5, 11, 14]
$\Sigma(36\phi)$	$E,\ J,\ K$	[4, 5, 14, 25]
$\Sigma(72\phi)$	$E,\ J,\ K,\ L$	[4, 5, 14, 25]
$\Sigma(216\phi)$	$E,\ J,\ K,\ P$	[4, 5, 14, 25]
$\Sigma(360\phi)$	$E,\ F(2,0,1),\ H,\ Q$	[4, 5, 14]

Table 3: Types of finite subgroups of $SU(3)$ [4, 5].

$[\![g,j]\!]$	Classification	Other names	ord(C)	References
$[\![12,3]\!]$	$\Delta(12)=\Delta(3\times 2^2)$	$A_4,\ T$	1	[4, 14, 26, 27]
$[\![21,1]\!]$	$C(7,1,2)$	T_7	1	[6, 7, 8, 14, 28, 29]
$[\![24,12]\!]$	$\Delta(24)=\Delta(6\times 2^2)$	$S_4,\ O$	1	[4, 14, 26, 30]
$[\![27,3]\!]$	$\Delta(27)=\Delta(3\times 3^2)$		3	[31, 32]
$[\![39,1]\!]$	$C(13,1,3)$	T_{13}	1	
$[\![48,3]\!]$	$\Delta(48)=\Delta(3\times 4^2)$		1	
$[\![54,8]\!]$	$\Delta(54)=\Delta(6\times 3^2)$		3	[33]
$[\![57,1]\!]$	$C(19,1,7)$	T_{19}	1	
$[\![60,5]\!]$	A_5	$\Sigma(60),\ I$	1	[4, 5, 11, 14, 24, 26]
$[\![75,2]\!]$	$\Delta(75)=\Delta(3\times 5^2)$		1	
$[\![81,9]\!]$	$C(9,1,1)$		3	
$[\![84,11]\!]$	$C(14,1,2)$		1	
$[\![93,1]\!]$	$C(31,1,5)$	T_{31}	1	
$[\![96,64]\!]$	$\Delta(96)=\Delta(6\times 4^2)$		1	
$[\![108,15]\!]$	$\Sigma(36\phi)$		3	[4, 5, 14, 25]
$[\![108,22]\!]$	$\Delta(108)=\Delta(3\times 6^2)$		3	
$[\![111,1]\!]$	$C(37,1,10)$	T_{37}	1	
$[\![129,1]\!]$	$C(43,1,6)$	T_{43}	1	
$[\![147,1]\!]$	$C(49,10,6)$	T_{49}	1	
$[\![147,5]\!]$	$\Delta(147)=\Delta(3\times 7^2)$		1	
$[\![150,5]\!]$	$\Delta(150)=\Delta(6\times 5^2)$		1	

$[\![g,j]\!]$	Classification	Other names	ord(C)	References
$[\![156,14]\!]$	$C(26,1,3)$		1	
$[\![162,14]\!]$	$D(9,1,1;2,1,1)$		3	
$[\![168,42]\!]$	$PSL(2,7)$	$\Sigma(168)$	1	[4, 11, 14, 34]
$[\![183,1]\!]$	$C(61,1,13)$	T_{61}	1	
$[\![189,8]\!]$	$C(21,1,2)$		3	
$[\![192,3]\!]$	$\Delta(192)=\Delta(3\times 8^2)$		1	
$[\![201,1]\!]$	$C(67,1,29)$	T_{67}	1	
$[\![216,88]\!]$	$\Sigma(72\phi)$		3	[4, 5, 14, 25]
$[\![216,95]\!]$	$\Delta(216)=\Delta(6\times 6^2)$		3	
$[\![219,1]\!]$	$C(73,1,8)$	T_{73}	1	
$[\![228,11]\!]$	$C(38,1,7)$		1	
$[\![237,1]\!]$	$C(79,1,23)$	T_{79}	1	
$[\![243,26]\!]$	$\Delta(243)=\Delta(3\times 9^2)$		3	
$[\![273,3]\!]$	$C(91,1,16)$	T_{91}	1	
$[\![273,4]\!]$	$C(91,1,9)$	T_{91}	1	
$[\![291,1]\!]$	$C(97,1,35)$	T_{97}	1	
$[\![294,7]\!]$	$\Delta(294)=\Delta(6\times 7^2)$		1	
$[\![300,43]\!]$	$\Delta(300)=\Delta(3\times 10^2)$		1	
$[\![309,1]\!]$	$C(103,1,46)$	T_{103}	1	
$[\![324,50]\!]$	$C(18,1,1)$		3	
$[\![327,1]\!]$	$C(109,1,45)$	T_{109}	1	
$[\![336,57]\!]$	$C(28,1,2)$		1	
$[\![351,8]\!]$	$C(39,1,3)$		3	
$[\![363,2]\!]$	$\Delta(363)=\Delta(3\times 11^2)$		1	
$[\![372,11]\!]$	$C(62,1,5)$		1	
$[\![381,1]\!]$	$C(127,1,19)$	T_{127}	1	
$[\![384,568]\!]$	$\Delta(384)=\Delta(6\times 8^2)$		1	
$[\![399,3]\!]$	$C(133,1,11)$	T_{133}	1	
$[\![399,4]\!]$	$C(133,1,30)$	T_{133}	1	
$[\![417,1]\!]$	$C(139,1,42)$	T_{139}	1	
$[\![432,103]\!]$	$\Delta(432)=\Delta(3\times 12^2)$		3	
$[\![444,14]\!]$	$C(74,1,10)$		1	
$[\![453,1]\!]$	$C(151,1,32)$	T_{151}	1	
$[\![471,1]\!]$	$C(157,1,12)$		1	
$[\![486,61]\!]$	$\Delta(486)=\Delta(6\times 9^2)$		3	
$[\![489,1]\!]$	$C(163,1,58)$	T_{163}	1	

$[\![g,j]\!]$	Classification	Other names	ord(C)	References
$[\![507,1]\!]$	$C(169,1,22)$	T_{169}	1	
$[\![507,5]\!]$	$\Delta(507)=\Delta(3\times 13^2)$		1	

Table 4: The finite subgroups of $SU(3)$ of order smaller than 512. $[\![g,j]\!]$ denotes the SmallGroups number and ord(C) denotes the order of the center of the group.

II.4.3 The finite subgroups of $U(3)$ of order smaller than 512

In table 5 we list the finite groups of order smaller than 512 that can not be written as direct products with cyclic groups and that have a faithful three-dimensional irreducible representation of determinant unequal 1.

The generators of these groups were taken from the list of faithful three-dimensional irreducible representations constructed with GAP (see footnote 6 on page 5). For most groups GAP constructed unitary representations. For the groups where GAP did not give a unitary representation we constructed a unitary representation in the following way: Let D be an n-dimensional representation of a group G and let $\{v_j|j=1,...,n\}$ be an orthonormal basis of $\mathbb{C}^n$ with respect to the scalar product

$$\langle x,y\rangle := \frac{1}{\text{ord}(G)}\sum_{a\in G}(D(a)x,D(a)y), \tag{3}$$

where $(x,y):=x^\dagger y$ is the standard scalar product on $\mathbb{C}^n$. Then if we define $T:=(v_1,...,v_n)$, the representation $T^{-1}DT$ is unitary with respect to $(.,.)$. Usually this construction is used to prove that every representation of a finite group is equivalent to a unitary representation. The power of modern computer algebra systems allows us to explicitly calculate the scalar product $\langle .,.\rangle$ and to construct T by Gram-Schmidt orthogonalisation. In this way the unitary generators $X_1,...,X_{10}$ were obtained.

$[\![g,j]\!]$	Classification	Generators	ord(C)
$[\![27,4]\!]$		$R(3,1,1,2),\ R(3,1,2,1)$	3
$[\![36,3]\!]$	$\Delta(3\times 2^2,2)$	$R(6,2,1,1),\ R(3,0,1,1)$	3
$[\![48,30]\!]$	$S_4(2)$	$S(4,1,3,1),\ T(4,3,3,1)$	2
$[\![63,1]\!]$	$T_7(2)$	$R(21,5,10,13),\ R(21,3,20,5)$	3
$[\![81,6]\!]$		$R(9,2,4,7),\ R(9,3,8,5)$	9
$[\![81,7]\!]$	$\Sigma(3\times 3^3)$	$R(3,2,2,0),\ R(3,1,1,0)$	3
$[\![81,8]\!]$		$R(9,2,2,8),\ R(9,4,4,7)$	3
$[\![81,10]\!]$		$R(9,4,7,4),\ R(9,2,5,8)$	3
$[\![81,14]\!]$		$R(9,4,7,1),\ R(9,8,5,2),\ R(9,6,3,0)$	9

$[\![g,j]\!]$	Classification	Generators	ord(C)
$[\![96,65]\!]$	$S_4(3)$	$S(8,1,5,1),\ \ T(8,3,3,7)$	4
$[\![108,3]\!]$	$\Delta(3\times 2^2,3)$	$R(18,4,5,5),\ \ R(9,0,5,5)$	9
$[\![108,11]\!]$	$\Delta(6\times 3^2,2)$	$S(12,5,9,1),\ \ T(12,3,7,11),\ \ U(12,1,9,5)$	6
$[\![108,19]\!]$		$R(18,4,7,1),\ \ R(9,4,7,1)$	3
$[\![108,21]\!]$		$R(6,0,3,5),\ \ R(3,2,0,2)$	3
$[\![117,1]\!]$	$T_{13}(2)$	$R(39,1,29,35),\ \ R(39,15,19,31)$	3
$[\![144,3]\!]$	$\Delta(3\times 4^2,2)$	$R(12,7,8,5),\ \ R(12,6,4,10)$	3
$[\![162,10]\!]$		$S(3,0,1,0),\ \ T(3,1,1,0)$	3
$[\![162,12]\!]$		$S(9,4,7,4),\ \ T(9,2,2,8)$	3
$[\![162,44]\!]$	$\Delta'(6\times 3^2,2,1)$	$S(9,2,8,5),\ \ T(9,4,1,7),\ \ U(9,5,8,2)$	9
$[\![171,1]\!]$	$T_{19}(2)$	$R(57,7,11,20),\ \ R(57,33,22,40)$	3
$[\![189,1]\!]$	$T_7(3)$	$R(63,22,23,32),\ \ R(63,9,46,1)$	9
$[\![189,4]\!]$		$R(63,1,58,25),\ \ R(63,2,53,50)$	3
$[\![189,5]\!]$		$R(63,1,16,4),\ \ R(63,2,32,8)$	3
$[\![189,7]\!]$		$R(21,5,17,6),\ \ R(21,3,13,12)$	3
$[\![192,182]\!]$	$\Delta(6\times 4^2,2)$	$T(4,0,2,1),\ \ U(4,3,0,2)$	2
$[\![192,186]\!]$	$S_4(4)$	$S(16,1,9,1),\ \ T(16,3,3,11)$	8
$[\![216,17]\!]$	$\Delta(6\times 3^2,3)$	$T(24,3,11,19),\ \ T(24,21,13,5),\ \ U(24,17,9,1)$	12
$[\![216,25]\!]$		$X_1,\ \ X_2$	6
$[\![225,3]\!]$	$\Delta(3\times 5^2,2)$	$R(15,1,11,8),\ \ R(15,12,7,1)$	3
$[\![243,16]\!]$		$R(9,2,4,7),\ \ R(9,6,8,5)$	9
$[\![243,19]\!]$		$V(27,5,14,5),\ \ W(27,2,2,11)$	9
$[\![243,20]\!]$		$V(27,5,23,5),\ \ W(27,2,2,20)$	9
$[\![243,24]\!]$		$R(27,5,13,22),\ \ R(27,9,26,17)$	27
$[\![243,25]\!]$		$R(9,0,2,4),\ \ R(9,0,4,8)$	3
$[\![243,27]\!]$		$R(9,2,0,4),\ \ R(9,7,0,8)$	3
$[\![243,50]\!]$		$R(27,5,23,14),\ \ V(27,11,20,2),\ \ V(27,17,17,17)$	27
$[\![243,55]\!]$		$R(9,2,2,8),\ \ R(9,4,4,7),\ \ V(3,1,0,0)$	9
$[\![252,11]\!]$		$R(42,23,25,22),\ \ R(21,9,4,1)$	3
$[\![279,1]\!]$	$T_{31}(2)$	$R(93,7,20,35),\ \ R(93,45,40,70)$	3
$[\![300,13]\!]$	$\Delta(6\times 5^2,2)$	$S(20,1,9,5),\ \ T(20,15,19,11)$	2
$[\![324,3]\!]$	$\Delta(3\times 2^2,4)$	$R(54,10,17,17),\ \ R(27,0,17,17)$	27
$[\![324,13]\!]$		$S(12,3,7,3),\ \ T(12,1,1,9)$	6
$[\![324,15]\!]$		$S(36,1,13,1),\ \ T(36,23,23,11)$	6
$[\![324,17]\!]$		$S(36,1,25,1),\ \ T(36,35,35,11)$	6
$[\![324,43]\!]$		$R(54,10,19,1),\ \ R(54,20,38,2)$	9

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$[\![g,j]\!]$	Classification	Generators	ord(C)
$[\![324,45]\!]$		$R(18,4,17,5),\ \ R(9,3,8,5)$	9
$[\![324,49]\!]$		$R(18,4,13,7),\ \ R(9,4,4,7)$	3
$[\![324,51]\!]$		$R(18,4,13,13),\ \ R(9,7,4,4)$	3
$[\![324,60]\!]$		$R(6,0,3,5),\ \ R(3,0,0,2)$	3
$[\![324,102]\!]$	$\Delta'(6\times 3^2,2,2)$	$T(36,1,25,13),\ \ T(36,29,5,17),\ \ U(36,35,11,23)$	18
$[\![324,111]\!]$		$X_3,\ \ X_4$	9
$[\![324,128]\!]$		$R(18,4,7,1),\ \ R(9,4,7,1),\ \ R(18,8,17,17)$	9
$[\![333,1]\!]$	$T_{37}(2)$	$R(111,1,26,47),\ \ R(111,39,52,94)$	3
$[\![351,1]\!]$	$T_{13}(3)$	$R(117,8,37,46),\ \ R(117,81,74,92)$	9
$[\![351,4]\!]$		$R(117,16,100,40),\ \ R(117,32,83,80)$	3
$[\![351,5]\!]$		$R(117,16,22,1),\ \ R(117,32,44,2)$	3
$[\![351,7]\!]$		$R(39,1,16,9),\ \ R(39,15,32,18)$	3
$[\![384,571]\!]$	$\Delta(6\times 4^2,3)$	$T(8,1,5,3),\ \ U(8,1,3,7)$	4
$[\![384,581]\!]$	$S_4(5)$	$S(32,1,17,1),\ \ T(32,3,3,19)$	16
$[\![387,1]\!]$	$T_{43}(2)$	$R(129,11,52,109),\ \ R(129,108,104,89)$	3
$[\![432,3]\!]$	$\Delta(3\times 4^2,3)$	$R(36,1,8,35),\ \ R(36,18,16,34)$	9
$[\![432,33]\!]$	$\Delta(6\times 3^2,4)$	$T(48,3,19,35),\ \ U(48,25,9,41),\ \ U(48,29,29,29)$	24
$[\![432,57]\!]$		$X_5,\ \ X_6$	12
$[\![432,100]\!]$		$R(36,1,22,25),\ \ R(18,1,4,7)$	3
$[\![432,102]\!]$		$R(12,3,0,1),\ \ R(6,1,0,1)$	3
$[\![432,239]\!]$		$X_7,\ \ X_8$	6
$[\![432,260]\!]$	$\Delta(6\times 4^2,2)$	$S(12,7,5,3),\ \ T(12,9,5,7),\ \ U(12,9,1,11)$	6
$[\![432,273]\!]$		$X_9,\ \ X_{10}$	12
$[\![441,1]\!]$		$R(147,94,125,26),\ \ R(147,90,103,52)$	3
$[\![441,7]\!]$	$\Delta(3\times 7^2,2)$	$R(21,1,8,5),\ \ R(21,9,16,10)$	3
$[\![468,14]\!]$		$R(78,2,19,31),\ \ R(39,15,19,31)$	3
$[\![486,26]\!]$		$S(27,5,14,5),\ \ T(27,19,19,10)$	9
$[\![486,28]\!]$		$S(27,5,23,5),\ \ T(27,1,1,10)$	9
$[\![486,125]\!]$		$S(9,2,5,2),\ \ T(9,7,7,4),\ \ U(3,0,1,1)$	9
$[\![486,164]\!]$	$\Delta'(6\times 3^2,3,1)$	$T(27,5,14,23),\ \ T(27,7,25,16),\ \ U(27,19,10,1)$	27

Table 5: The finite subgroups of $U(3)$ (which are not finite subgroups of $SU(3)$) of order smaller than 512.

II.4.4 Numerical consistency check of the obtained results.

The results listed in sections II.4.2 and II.4.3 are based on the computer algebra system GAP [18] and the SmallGroups Library [19, 20]. As already mentioned, our results are in perfect agreement with theorem II.3, which is the reason we did not list groups that can be written as direct products with cyclic groups.
Furthermore all finite subgroups of $SU(3)$ listed in table 4 could be cast into one of the types listed in [4, 5] (see table 3).
In order not to rely on GAP and the SmallGroups Library only we developed a program (in the programming language C) which performs the following tasks:

1. Given the generators (as 3×3-matrices) of a finite group G it numerically[8] constructs all group elements in the defining representation D. An example for an algorithm for this purpose can be found in [14]. The program uses the data type "double" for the real and imaginary parts of the matrix elements, respectively. An important subroutine of the program is to determine whether two matrices are equal. We decided to use the following criterion: Two matrices A and B are to be regarded as equal by the program if

$$|\mathrm{Re}(A_{ij} - B_{ij})| < 10^{-7} \text{ and } |\mathrm{Im}(A_{ij} - B_{ij})| < 10^{-7} \quad \forall i, j \in \{1, 2, 3\}. \tag{4}$$

   Using the program the orders of all groups listed in tables 4 and 5 were verified (more precise: not falsified) numerically. In addition the orders of these groups were checked analytically using GAP.

2. After the explicit construction of the defining representation D of the group its character χ_D can be calculated numerically. A scalar product of the characters χ_D and $\chi_{D'}$ of two representations D, D' of G can be defined as

$$(\chi_D, \chi_{D'})_G = \frac{1}{\mathrm{ord}(G)} \sum_{b \in G} \chi_D(b)^* \chi_{D'}(b) \tag{5}$$

   D is irreducible if and only if $(\chi_D, \chi_D)_G = 1$ [26], which can easily be tested numerically. Again we regard the representation D as irreducible if

$$|\mathrm{Re}(\chi_D, \chi_D) - 1| < 10^{-7} \text{ and } |\mathrm{Im}(\chi_D, \chi_D)| < 10^{-7}. \tag{6}$$

   In this sense the irreducibility of all defining representations of the groups listed in tables 4 and 5 was verified (more precise: not falsified) numerically.

Please note that the numerical analysis described above can of course not prove the correctness of the results listed in tables 4 and 5. Note furthermore that we do not, in any sense, claim that the lists 4 and 5 are complete.

[8]The reason why we decided to perform a numerical analysis was of course calculation time. For some of the larger groups more than 500000 matrix multiplications were needed to obtain all group elements.

III Construction of some series of finite subgroups of $U(3)$.

The following theorem will allow us to construct some new infinite series of finite subgroups of $U(3)$ that have a faithful 3-dimensional irreducible representation and can not be written as a direct product with a cyclic group.

III.1 Theorem. Let $G = H \rtimes \mathcal{Z}_n$ be a finite group with the following properties[9]

1. G has a faithful m-dimensional irreducible representation D.
2. n is prime.
3. The center of G is of order $c \neq n$ with c prime or $c = 1$.
4. G can not be written as a direct product with a cyclic group.

Let furthermore $A_1, ..., A_a$ be generators of $D(H)$ and let B be a generator of $D(\mathcal{Z}_n)$. Then the group[10]

$$G_b := \langle\langle A_1, ..., A_a, \beta B\rangle\rangle, \quad \beta = e^{2\pi i/b}, \quad b \in \mathbb{N}\backslash\{0\} \tag{7}$$

(which by construction has a faithful m-dimensional irreducible representation) can not be written as a direct product with a cyclic group if and only if

$$b = c^j n^k, \quad j, k \in \mathbb{N}. \tag{8}$$

III.2 Theorem. Let $G = H \rtimes \mathcal{Z}_n$ be a finite group fulfilling the properties 1.-4. of theorem III.1. Then the center of

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle, \quad \beta = e^{2\pi i/(c^j n^k)}, \quad j, k \in \mathbb{N} \tag{9}$$

is given by

- $\langle\langle e^{2\pi i/c} \mathbb{1}_m \rangle\rangle \cong \mathcal{Z}_c$ for $j = 0, \;\; k = 0$,
- $\langle\langle e^{2\pi i/c} \mathbb{1}_m, e^{2\pi i/n^{k-1}} \mathbb{1}_m \rangle\rangle \cong \mathcal{Z}_{cn^{k-1}}$ for $j = 0, \;\; k > 0$,
- $\langle\langle e^{2\pi i/c^j} \mathbb{1}_m \rangle\rangle \cong \mathcal{Z}_{c^j}$ for $j > 0, \;\; k = 0$ and
- $\langle\langle e^{2\pi i/c^j} \mathbb{1}_m, e^{2\pi i/n^{k-1}} \mathbb{1}_m \rangle\rangle \cong \mathcal{Z}_{c^j n^{k-1}}$ for $j > 0, \;\; k > 0$.

If $c = 1, \;\; k > 0$: G_{n^k} is isomorphic to $H \rtimes \mathcal{Z}_{n^k}$ and the center of G_{n^k} is isomorphic to $\mathcal{Z}_{n^{k-1}}$.

The proofs of these theorems can be found in appendix A.3. Let us now use theorem III.1 to construct some infinite series of finite subgroups of $U(3)$.

[9] We use the following notation for the semidirect product of two groups A and B: $G = A \rtimes B \Rightarrow A$ is a normal subgroup of G and there exists a homomorphism $\phi : B \rightarrow \mathrm{Aut}(A)$. The product is defined by $(a, b)(a', b') = (a\phi(b)a', bb')$.

[10] The symbol $\langle\langle ... \rangle\rangle$ means "generated by".

III.1 The group series $T_n(m)$

The groups T_n [6, 7, 8, 14] have the structure

$$T_n = \mathcal{Z}_n \rtimes \mathcal{Z}_3, \tag{10}$$

where n is a prime of the form $3k+1$. Since the center of T_n is trivial we can apply theorem III.1 to find (for every T_n) an infinite series of finite subgroups of $U(3)$

$$T_n(m) := \langle\langle e^{2\pi i/3^m} E,\ \ F(n,1,a)\rangle\rangle \cong \mathcal{Z}_n \rtimes \mathcal{Z}_{3^m}, \tag{11}$$

where $(1+a+a^2) \bmod n = 0,\ \ m \in \mathbb{N}\backslash\{0\}$. Theorem III.2 tells us that the center of $T_n(m)$ is isomorphic to $\mathcal{Z}_{3^{m-1}}$.

III.2 The group series $\Delta(3n^2, m)$

The group $\Delta(3n^2)$ has the structure [6, 10]

$$\Delta(3n^2) \cong (\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{Z}_3, \quad n \in \mathbb{N}\backslash\{0,1\} \tag{12}$$

and it has trivial center if $\gcd(n,3) = 1$ [10]. In the other cases one finds $c = 3$, and theorem III.1 can not be applied.

Thus we find the following series of finite subgroups of $U(3)$:

$$\Delta(3n^2, m) := \langle\langle e^{2\pi i/3^m} E,\ \ F(n,0,1)\rangle\rangle \cong (\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{Z}_{3^m}, \tag{13}$$

where $n \in \{k \in \mathbb{N} |\ \gcd(3,k) = 1, k > 1\},\ \ m \in \mathbb{N}\backslash\{0\}$.

III.3 The group series $S_4(m)$

The group S_4 is a semidirect product of A_4 (which is generated by the even permutations (14)(23) and (123)) and $\mathcal{Z}_2$ (generated by the odd permutation (23)). Since S_4 possesses a faithful three-dimensional irreducible representation and its center is trivial theorem III.1 leads to the following series of finite subgroups of $U(3)$:

$$A_4 \rtimes \mathcal{Z}_{2^m}, \quad m \in \mathbb{N}\backslash\{0\}.$$

A faithful three-dimensional irreducible representation of S_4 can be obtained by reduction of the four-dimensional representation

$$D(\sigma)(x_1, ..., x_4)^T = (x_{\sigma(1)}, ..., x_{\sigma(4)})^T \qquad \sigma \in S_4,$$

which leads to

$$(14)(23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A, \quad (123) \mapsto \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} =: B,$$

$$(23) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: C.$$

Thus we have found the following series of finite subgroups of $U(3)$:

$$S_4(m) := \langle\langle A, B, e^{2\pi i/2^m} C\rangle\rangle \cong A_4 \rtimes \mathcal{Z}_{2^m}, \quad m \in \mathbb{N}\backslash\{0\}. \tag{14}$$

III.4 The group series $\Delta(6n^2, m)$ and $\Delta'(6n^2, j, k)$

The group $\Delta(6n^2)$ has the structure [6, 12]

$$(\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes S_3 \tag{15}$$

and its presentation is given by [12]

$$\begin{aligned}
&a^3 = b^2 = (ab)^2 = \mathbb{1} \quad \text{(presentation of } S_3\text{)},\\
&c^n = d^n = \mathbb{1}, \quad cd = dc \quad \text{(presentation of } \mathcal{Z}_n \times \mathcal{Z}_n\text{)},\\
&aca^{-1} = c^{-1}d^{-1},\ bcb^{-1} = d^{-1},\ ada^{-1} = c,\ bdb^{-1} = c^{-1} \quad \text{(semidirect product)}.
\end{aligned}$$

This presentation can easily be rearranged to a presentation of

$$((\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{Z}_3) \rtimes \mathcal{Z}_2 \cong \Delta(3n^2) \rtimes \mathcal{Z}_2 \tag{16}$$

in the following way:

$$\begin{aligned}
&c^n = d^n = \mathbb{1},\ cd = dc \quad \text{(presentation of } \mathcal{Z}_n \times \mathcal{Z}_n\text{)},\\
&a^3 = \mathbb{1} \quad \text{(presentation of } \mathcal{Z}_3\text{)},\\
&aca^{-1} = c^{-1}d^{-1},\ ada^{-1} = c \quad \text{(semidirect product with } \mathcal{Z}_3\text{)},\\
&b^2 = \mathbb{1} \quad \text{(presentation of } \mathcal{Z}_2\text{)},\\
&(ab)^2 = \mathbb{1} \Rightarrow abab = \mathbb{1} \Rightarrow bab = bab^{-1} = a^{-1},\\
&bcb^{-1} = d^{-1},\ bdb^{-1} = c^{-1} \quad \text{(semidirect product with } \mathcal{Z}_2\text{)}.
\end{aligned}$$

The center of $\Delta(6n^2) \cong \Delta(3n^2) \rtimes \mathcal{Z}_2$ is given by the center of $\Delta(3n^2)$, which can be of order 1 or 3. Thus we can apply theorem III.1 to construct new series of finite subgroups of $U(3)$.

A faithful three-dimensional irreducible representation of $\Delta(6n^2)$ is given by [12]

$$a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} =: A, \quad b \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} =: B, \quad d \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^* \end{pmatrix} =: C,$$

where $\eta = e^{2\pi i/n}$ and $n \in \mathbb{N}\backslash\{0, 1\}$.

There are two possibilities:

- $\gcd(3, n) = 1 \Rightarrow$ The center of $\Delta(6n^2)$ is trivial, which leads to the group series

  $$\Delta(6n^2, m) := \langle\langle A, e^{2\pi i/2^m} B, C\rangle\rangle \cong \Delta(3n^2) \rtimes \mathcal{Z}_{2^m}, \tag{17}$$

  $n \in \{k \in \mathbb{N}|\ \gcd(3, k) = 1, k > 1\}$, $m \in \mathbb{N}\backslash\{0\}$. This series contains $S_4(m) = \Delta(6 \times 2^2, m)$ as a subseries.

- $\gcd(3, n) = 3 \Rightarrow$ The center of $\Delta(6n^2)$ is of order 3, which leads to the group series

  $$\Delta'(6n^2, j, k) := \langle\langle A, e^{2\pi i/(3^j 2^k)} B, C\rangle\rangle, \tag{18}$$

  $n \in \{k \in \mathbb{N}|\ \gcd(3, k) = 3\}$, $j, k \in \mathbb{N}\backslash\{0\}$.

IV Analysis of two interesting finite subgroups of $U(3)$

IV.1 The group $[\![27,4]\!]$

The group $[\![27,4]\!]$ is the smallest group listed in table 5. It is generated by

$$A := R(3,1,1,2) = \begin{pmatrix} 0 & 0 & \omega \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad B := R(3,1,2,1) = \begin{pmatrix} 0 & 0 & \omega \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}.$$

A much simpler set of generators is given by

$$R := BA = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad S := AB^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{19}$$

which is a generating set, because $(SR)^5 = A$ and $R(SR)^{-5} = B$. From the generators R and S given in equation (19) we find that

$$\langle\langle R\rangle\rangle \cong \mathcal{Z}_9, \quad \langle\langle S\rangle\rangle \cong \mathcal{Z}_3, \quad \langle\langle R\rangle\rangle \cap \langle\langle S\rangle\rangle = \{\mathbb{1}_3\}, \quad S^{-1}RS = R^4,$$

thus

$$[\![27,4]\!] := \langle\langle R,S\rangle\rangle \cong \mathcal{Z}_9 \rtimes \mathcal{Z}_3. \tag{20}$$

Due to the semidirect product structure (especially using the fact that every element of $[\![27,4]\!]$ can be written in the form $\omega^x R^y S^z$) the derivation of the conjugacy classes is straight forward. One finds eleven conjugacy classes

$$\begin{aligned}
C_1 &= \{\mathbb{1}_3\}, \\
C_2 &= \{\omega\mathbb{1}_3\}, \\
C_3 &= \{\omega^2\mathbb{1}_3\}, \\
C_4 &= \{R, \omega R, \omega^2 R\}, \\
C_5 &= \{R^2, \omega R^2, \omega^2 R^2\}, \\
C_6 &= \{S, \omega S, \omega^2 S\}, \\
C_7 &= \{S^2, \omega S^2, \omega^2 S^2\}, \\
C_8 &= \{RS, \omega RS, \omega^2 RS\}, \\
C_9 &= \{RS^2, \omega RS^2, \omega^2 RS^2\}, \\
C_{10} &= \{R^2S, \omega R^2S, \omega^2 R^2S\}, \\
C_{11} &= \{R^2S^2, \omega R^2S^2, \omega^2 R^2S^2\}.
\end{aligned} \tag{21}$$

The nontrivial normal subgroups are found to be

$$\begin{aligned}
&C_1 \cup C_2 \cup C_3 = \langle\langle \omega\mathbb{1}_3\rangle\rangle \cong \mathcal{Z}_3, \\
&C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5 = \langle\langle R\rangle\rangle \cong \mathcal{Z}_9, \\
&C_1 \cup C_2 \cup C_3 \cup C_6 \cup C_7 = \langle\langle \omega\mathbb{1}_m, S\rangle\rangle \cong \mathcal{Z}_3 \times \mathcal{Z}_3 \\
&C_1 \cup C_2 \cup C_3 \cup C_8 \cup C_{11} = \langle\langle RS, R^2S^2\rangle\rangle \cong \mathcal{Z}_9, \\
&C_1 \cup C_2 \cup C_3 \cup C_9 \cup C_{10} = \langle\langle RS^2, R^2S\rangle\rangle \cong \mathcal{Z}_9.
\end{aligned} \tag{22}$$

$[\![27,4]\!]$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
$(\# C_k)$	(1)	(1)	(1)	(3)	(3)	(3)	(3)	(3)	(3)	(3)	(3)
$\mathrm{ord}(C_k)$	1	3	3	9	9	3	3	9	9	9	9
$\mathbf{1}_{(0,0)}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_{(0,1)}$	1	1	1	1	1	ω	ω^2	ω	ω^2	ω	ω^2
$\mathbf{1}_{(0,2)}$	1	1	1	1	1	ω^2	ω	ω^2	ω	ω^2	ω
$\mathbf{1}_{(1,0)}$	1	1	1	ω	ω^2	1	1	ω	ω	ω^2	ω^2
$\mathbf{1}_{(1,1)}$	1	1	1	ω	ω^2	ω	ω^2	ω^2	1	1	ω
$\mathbf{1}_{(1,2)}$	1	1	1	ω	ω^2	ω^2	ω	1	ω^2	ω	1
$\mathbf{1}_{(2,0)}$	1	1	1	ω^2	ω	1	1	ω^2	ω^2	ω	ω
$\mathbf{1}_{(2,1)}$	1	1	1	ω^2	ω	ω	ω^2	1	ω	ω^2	ω^2
$\mathbf{1}_{(2,2)}$	1	1	1	ω^2	ω	ω^2	ω	ω	1	1	1
$\underline{\mathbf{3}}$	3	3ω	$3\omega^2$	0	0	0	0	0	0	0	0
$\underline{\mathbf{3}}^*$	3	$3\omega^2$	3ω	0	0	0	0	0	0	0	0

Table 6: Character table of $[\![27,4]\!]$. The number of elements in each class is given by the numbers in parentheses in the second line of the table.

Since there are eleven conjugacy classes there must be eleven inequivalent irreducible representations. These are the nine one-dimensional irreducible representations of the factor group

$$\langle\langle R, S\rangle\rangle / \langle\langle \omega \mathbb{1}_3 \rangle\rangle \cong \mathcal{Z}_3 \times \mathcal{Z}_3,$$

the defining representation and its complex conjugate:

$$\underline{\mathbf{1}}_{(i,j)}: \quad R \mapsto \omega^i, \;\; S \mapsto \omega^j, \quad (i,j = 0,1,2), \tag{23a}$$

$$\underline{\mathbf{3}}: \quad R \mapsto R, \;\; S \mapsto S, \tag{23b}$$

$$\underline{\mathbf{3}}^*: \quad R \mapsto R^*, \;\; S \mapsto S^*. \tag{23c}$$

The character table of $[\![27,4]\!]$ is shown in table 6.

The tensor products are given by

$$\underline{\mathbf{1}}_{(i,j)} \otimes \underline{\mathbf{1}}_{(k,l)} = \underline{\mathbf{1}}_{((i+k)\,\mathrm{mod}3,\,(j+l)\,\mathrm{mod}3)}, \quad (i,j,k,l = 0,1,2) \tag{24a}$$

$$\underline{\mathbf{3}} \otimes \underline{\mathbf{3}} = \underline{\mathbf{3}}^* \oplus \underline{\mathbf{3}}^* \oplus \underline{\mathbf{3}}^*, \tag{24b}$$

$$\underline{\mathbf{3}}^* \otimes \underline{\mathbf{3}}^* = \underline{\mathbf{3}} \oplus \underline{\mathbf{3}} \oplus \underline{\mathbf{3}}, \tag{24c}$$

$$\underline{\mathbf{3}} \otimes \underline{\mathbf{3}}^* = \bigoplus_{i,j=0}^{2} \underline{\mathbf{1}}_{(i,j)}. \tag{24d}$$

The corresponding invariant subspaces are given by

$$V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}\to\underline{\mathbf{3}}^*} = \mathrm{Span}\left(e_1 \otimes e_1, \;\; e_2 \otimes e_2, \;\; e_3 \otimes e_3\right), \tag{25a}$$

$$
\begin{aligned}
V'_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}\to\underline{\mathbf{3}}^*} &= \mathrm{Span}\left(e_2\otimes e_3,\ \ \omega e_3\otimes e_1,\ \ \omega^2 e_1\otimes e_2\right), && (25b)\\
V''_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}\to\underline{\mathbf{3}}^*} &= \mathrm{Span}\left(e_3\otimes e_2,\ \ \omega e_1\otimes e_3,\ \ \omega^2 e_2\otimes e_1\right), && (25c)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(0,0)}} &= \mathrm{Span}\left(e_1\otimes e_1+e_2\otimes e_2+e_3\otimes e_3\right), && (25d)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(0,1)}} &= \mathrm{Span}\left(e_1\otimes e_2+e_2\otimes e_3+\omega^2 e_3\otimes e_1\right), && (25e)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(0,2)}} &= \mathrm{Span}\left(e_1\otimes e_3+\omega^2 e_2\otimes e_1+\omega^2 e_3\otimes e_2\right), && (25f)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(1,0)}} &= \mathrm{Span}\left(e_1\otimes e_1+\omega e_2\otimes e_2+\omega^2 e_3\otimes e_3\right), && (25g)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(1,1)}} &= \mathrm{Span}\left(e_1\otimes e_2+\omega e_2\otimes e_3+\omega e_3\otimes e_1\right), && (25h)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(1,2)}} &= \mathrm{Span}\left(e_1\otimes e_3+e_2\otimes e_1+\omega e_3\otimes e_2\right), && (25i)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(2,0)}} &= \mathrm{Span}\left(e_1\otimes e_1+\omega^2 e_2\otimes e_2+\omega e_3\otimes e_3\right), && (25j)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(2,1)}} &= \mathrm{Span}\left(e_1\otimes e_2+\omega^2 e_2\otimes e_3+e_3\otimes e_1\right), && (25k)\\
V_{\underline{\mathbf{3}}\otimes\underline{\mathbf{3}}^*\to\underline{\mathbf{1}}_{(2,2)}} &= \mathrm{Span}\left(e_1\otimes e_3+\omega e_2\otimes e_1+e_3\otimes e_2\right). && (25l)
\end{aligned}
$$

Since the defining representations of $[\![27,4]\!]$ and $\Delta(27)$ differ by phase factors only,

$$\langle\langle e^{2\pi i/9}R, S\rangle\rangle \cong \Delta(27), \tag{26}$$

all Clebsch-Gordan coefficients for corresponding tensor product decompositions are equal. The structure of $[\![27,4]\!] \cong \mathcal{Z}_9 \rtimes \mathcal{Z}_3$ is very similar to the structure of $\Delta(27) \cong (\mathcal{Z}_3 \times \mathcal{Z}_3) \rtimes \mathcal{Z}_3$. Though these groups are not isomorphic they share the nine one-dimensional irreducible representations as well as the character table (except for the values of $\mathrm{ord}(C_k)$). The character table of $\Delta(27)$ can be found in [25, 32]. Since the character tables of $[\![27,4]\!]$ and $\Delta(27)$ are equal, all tensor products are equal. Since also the Clebsch-Gordan coefficients are equal we find that, from the point of view of model building, $\Delta(27)$ and $[\![27,4]\!]$ are equivalent.

IV.2 The group $S_4(2) \cong A_4 \rtimes \mathcal{Z}_4$

The group $S_4 \cong A_4 \rtimes \mathcal{Z}_2$ has been commonly used in model building, and especially in the last years interest in S_4 began to increase [30, 36, 37, 38]. Therefore the group $S_4(2)$, which is a relative of S_4, may be of interest. From subsection III.3 we know the structure

$$S_4(2) \cong A_4 \rtimes \mathcal{Z}_4 \tag{27}$$

and generators of a faithful three-dimensional irreducible representation of $S_4(2)$:

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C := i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{28}$$

S_4 possesses five conjugacy classes C_i. The classes of $S_4(2)$ differ from C_i by the element $C^2 = -\mathbb{1}_3$ only, thus one finds ten conjugacy classes of $S_4(2)$:

$$\begin{aligned}
&C^1_\pm = \{\pm\mathbb{1}_3\},\\
&C^2_\pm = \{\pm A, \pm BAB^2, \pm B^2AB\},\\
&C^3_\pm = \{\pm B, \pm AB, \pm BA, \pm ABA, \pm B^2, \pm BAB, \pm B^2A, \pm AB^2\},\\
&C^4_\pm = \{\pm C, \pm B^2C, \pm BABC, \pm BC, \pm AC, \pm ABAC\},\\
&C^5_\pm = \{\pm AB^2C, \pm B^2AC, \pm B^2ABC, \pm ABC, \pm BAB^2C, \pm BAC\}.
\end{aligned}\tag{29}$$

The nontrivial normal subgroups of $S_4(2)$ are given by

$$\begin{aligned}
&C^1_+ \cup C^1_- \cong \mathcal{Z}_2,\\
&C^1_+ \cup C^2_+ \cong \mathcal{Z}_2 \times \mathcal{Z}_2,\\
&C^1_+ \cup C^1_- \cup C^2_+ \cup C^2_- \cong \mathcal{Z}_2 \times \mathcal{Z}_2 \times \mathcal{Z}_2,\\
&C^1_+ \cup C^2_+ \cup C^3_+ \cong A_4,\\
&C^1_+ \cup C^1_- \cup C^2_+ \cup C^2_- \cup C^3_+ \cup C^3_- \cong A_4 \times \mathcal{Z}_2.
\end{aligned}\tag{30}$$

Since $S_4(2)/\{\mathbb{1}_3, -\mathbb{1}_3\} \cong (A_4 \rtimes \mathcal{Z}_4)/\mathcal{Z}_2 \cong A_4 \rtimes \mathcal{Z}_2 \cong S_4$ we find that all irreducible representations of S_4 are irreducible representations of $S_4(2)$ too. By construction A_4 is an invariant subgroup of $S_4(2)$, thus all irreducible representations of $\mathcal{Z}_4 \cong S_4(2)/A_4$ are irreducible representations of $S_4(2)$ too. They are given by:

$$\underline{\mathbf{1}}_1 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto 1, \tag{31a}$$
$$\underline{\mathbf{1}}_2 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto -1, \tag{31b}$$
$$\underline{\mathbf{1}}_3 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto i, \tag{31c}$$
$$\underline{\mathbf{1}}_4 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto -i. \tag{31d}$$

We will now construct the irreducible representations of S_4: Since $S_4/A_4 \cong \mathcal{Z}_2$ we obtain two one-dimensional irreducible representations

$$\underline{\mathbf{1}}_1 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto 1, \tag{32a}$$
$$\underline{\mathbf{1}}_2 : \ A \mapsto 1,\ B \mapsto 1,\ C \mapsto -1, \tag{32b}$$

which are irreducible representations of $\mathcal{Z}_4$ also. Multiplying these one-dimensional representations with the defining representation of S_4 we obtain the two three-dimensional irreducible representations of S_4:

$$\underline{\mathbf{3}}_1 : \ A \mapsto A,\ B \mapsto B,\ C \mapsto -iC, \tag{33a}$$
$$\underline{\mathbf{3}}_2 : \ A \mapsto A,\ B \mapsto B,\ C \mapsto iC. \tag{33b}$$

The missing two-dimensional irreducible representation can be obtained in the following way: The Klein four-group $\mathcal{Z}_2 \times \mathcal{Z}_2$ is an invariant subgroup of S_4:

$$C^1_+ \cup C^2_+ = \{\mathbb{1}_3, A, BAB^2, B^2AB\} \cong \mathcal{Z}_2 \times \mathcal{Z}_2. \tag{34}$$

Therefore all irreducible representations of $S_3 \cong S_4/(\mathcal{Z}_2 \times \mathcal{Z}_2)$ are irreducible representations of S_4 too. This has also been pointed out in [37]. Assuming that $\mathcal{Z}_2 \times \mathcal{Z}_2$ given in equation (34) is mapped onto $\mathbb{1}_3$ one can easily construct the three-dimensional *reducible* S_3-representation

$$\underline{\mathbf{3}}_r: \ A \mapsto \mathbb{1}_3, \ B \mapsto B, \ C \mapsto -iC. \tag{35}$$

$v = \frac{1}{\sqrt{3}}(1,-1,-1)^T$ is a common eigenvector of B and $-iC$ to the eigenvalue 1. This enables reduction via

$$U := \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$U^T B U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad U^T(-iC)U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus the two-dimensional irreducible representation of S_4 is given by

$$\underline{\mathbf{2}}: \ A \mapsto \mathbb{1}_2, \ B \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ -iC \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{36}$$

The irreducible representations of $S_4(2)$ are thus given by

$$\underline{\mathbf{1}}_1: \ A \mapsto 1, \ B \mapsto 1, \ C \mapsto 1, \tag{37a}$$
$$\underline{\mathbf{1}}_2: \ A \mapsto 1, \ B \mapsto 1, \ C \mapsto -1, \tag{37b}$$
$$\underline{\mathbf{1}}_3: \ A \mapsto 1, \ B \mapsto 1, \ C \mapsto i, \tag{37c}$$
$$\underline{\mathbf{1}}_4: \ A \mapsto 1, \ B \mapsto 1, \ C \mapsto -i, \tag{37d}$$
$$\underline{\mathbf{2}}_1: \ A \mapsto \mathbb{1}_2, \ B \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ C \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{37e}$$
$$\underline{\mathbf{2}}_2: \ A \mapsto \mathbb{1}_2, \ B \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ C \mapsto i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{37f}$$
$$\underline{\mathbf{3}}_1: \ A \mapsto A, \ B \mapsto B, \ C \mapsto -iC, \tag{37g}$$
$$\underline{\mathbf{3}}_2: \ A \mapsto A, \ B \mapsto B, \ C \mapsto iC, \tag{37h}$$
$$\underline{\mathbf{3}}_3: \ A \mapsto A, \ B \mapsto B, \ C \mapsto C, \tag{37i}$$
$$\underline{\mathbf{3}}_4: \ A \mapsto A, \ B \mapsto B, \ C \mapsto -C. \tag{37j}$$

The extension of this analysis to $S_4(m)$ is easy - one just needs to take the irreducible representations of $S_4(m)/A_4 \cong \mathcal{Z}_{2^m}$ and multiply them with the irreducible representations of S_4 to obtain all irreducible representations of $S_4(m)$.

From this it is clear that also all tensor product decompositions and corresponding invariant subspaces have the same structure as those of S_4. All $3 \otimes 3$-tensor products can be constructed from the $3 \otimes 3$-tensor product

$$\underline{\mathbf{3}}_1 \otimes \underline{\mathbf{3}}_1 = \underline{\mathbf{1}}_1 \oplus \underline{\mathbf{2}}_1 \oplus \underline{\mathbf{3}}_1 \oplus \underline{\mathbf{3}}_2 \tag{38}$$

of S_4 by multiplication with one-dimensional irreducible representations of $S_4(2)$. The corresponding invariant subspaces for the Clebsch-Gordan decomposition (38) are spanned by the following vectors [14]:

$$v(\underline{\mathbf{1}}_1) = \frac{1}{\sqrt{3}}\left(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3\right), \tag{39a}$$

$$v_1(\underline{\mathbf{2}}_1) = \frac{1}{\sqrt{6}}\left(-2e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3\right), \tag{39b}$$

$$v_2(\underline{\mathbf{2}}_1) = \frac{1}{\sqrt{2}}\left(e_2 \otimes e_2 - e_3 \otimes e_3\right), \tag{39c}$$

$$v_1(\underline{\mathbf{3}}_1) = \frac{1}{\sqrt{2}}\left(e_2 \otimes e_3 + e_3 \otimes e_2\right), \tag{39d}$$

$$v_2(\underline{\mathbf{3}}_1) = \frac{1}{\sqrt{2}}\left(e_1 \otimes e_3 + e_3 \otimes e_1\right), \tag{39e}$$

$$v_3(\underline{\mathbf{3}}_1) = \frac{1}{\sqrt{2}}\left(e_1 \otimes e_2 + e_2 \otimes e_1\right), \tag{39f}$$

$$v_1(\underline{\mathbf{3}}_2) = \frac{1}{\sqrt{2}}\left(e_2 \otimes e_3 - e_3 \otimes e_2\right), \tag{39g}$$

$$v_2(\underline{\mathbf{3}}_2) = \frac{1}{\sqrt{2}}\left(-e_1 \otimes e_3 + e_3 \otimes e_1\right), \tag{39h}$$

$$v_3(\underline{\mathbf{3}}_2) = \frac{1}{\sqrt{2}}\left(e_1 \otimes e_2 - e_2 \otimes e_1\right). \tag{39i}$$

Let us finally investigate the differences between the symmetry groups S_4 and $S_4(2)$ from the physical point of view.

Let us as an example consider a field theory describing seven real scalar fields arranged in the following $S_4(2)$-multiplets:

$$\underline{\mathbf{3}}_1 : \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \underline{\mathbf{3}}_3 : \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \underline{\mathbf{1}}_3 : \eta. \tag{40}$$

The Lagrangian

$$\mathcal{L}_1 = \phi^T \psi \eta + \eta^4 \tag{41}$$

is invariant under this transformation, while

$$\mathcal{L}_2 = \eta^2 \tag{42}$$

clearly is not. $\mathcal{L}_2$ can not be forbidden in a pure S_4-theory (allowing $\mathcal{L}_1$), because for this issue one needs the $\mathcal{Z}_4$-representation $\underline{\mathbf{1}}_3$ of $S_4(2)$, which is not contained in S_4. Another group based on S_4 containing the needed $\mathcal{Z}_4$-representation is $S_4 \times \mathcal{Z}_4$. The 20 irreducible representations of $S_4 \times \mathcal{Z}_4$ are given by ($j = 0, 1, 2, 3$):

$$\underline{\mathbf{1}}_{1j} : \quad a \mapsto 1, \;\; b \mapsto 1, \;\; c \mapsto 1, \;\; d \mapsto i^j, \tag{43a}$$

$$\underline{\mathbf{1}}_{2j}: \quad a \mapsto 1, \ b \mapsto 1, \ c \mapsto -1, \ d \mapsto i^j, \tag{43b}$$

$$\underline{\mathbf{2}}_{j}: \quad a \mapsto \mathbb{1}_2, \ b \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ d \mapsto i^j \mathbb{1}_2, \tag{43c}$$

$$\underline{\mathbf{3}}_{1j}: \quad a \mapsto A, \ b \mapsto B, \ c \mapsto -iC, \ d \mapsto i^j \mathbb{1}_3 \tag{43d}$$

$$\underline{\mathbf{3}}_{2j}: \quad a \mapsto A, \ b \mapsto B, \ c \mapsto iC, \ d \mapsto i^j \mathbb{1}_3. \tag{43e}$$

From equations (43) it is clear that all irreducible representations of $S_4 \times \mathcal{Z}_4$ can be interpreted as irreducible representations of $S_4(2)$ with an additional generator d, which acts as multiplication with a phase factor. Indeed $S_4(2)$ is a subgroup of $S_4 \times \mathcal{Z}_4$. It is thus clear that all Lagrangians based on a symmetry group $S_4 \times \mathcal{Z}_4$ are allowed in the corresponding $S_4(2)$-theory too. The question remains whether there are $S_4(2)$-models which do not fit to an appropriate $S_4 \times \mathcal{Z}_4$-model. The answer is no for the following reason: Consider a Lagrangian

$$\mathcal{L}(\phi_1, ..., \phi_m) = \sum_j \mathcal{L}_j(\phi_1, ..., \phi_m) \tag{44}$$

invariant under the action of a symmetry group G:

$$G \ni a: \quad \phi_i \mapsto D_i(a)\phi_i \quad i = 1, ..., m, \tag{45}$$

where D_j are representations of G. The Lagrangians $\mathcal{L}_j$ are assumed to fulfill the following properties:

- $\mathcal{L}_j$ is invariant under the action (45) of G.
- $\mathcal{L}_j$ can not be split up into two "smaller" Lagrangians being invariant under G themselves[11]. More precise: $\exists k \in \mathbb{N}$ such that $\forall \alpha \in U(1): \mathcal{L}_j(\alpha\phi_1, ..., \alpha\phi_m) = \alpha^k \mathcal{L}_j(\phi_1, ..., \phi_m)$.

The construction of an invariant Lagrangian (44) can then be split up into two steps:

1. $\mathcal{L}_j$ must transform as one-dimensional representations of G.
2. If possible, the chosen representations $D_1, ..., D_m$ have to be multiplied by one-dimensional representations in such a way that $\mathcal{L}_j$ are invariant under G. If this is not possible $\mathcal{L}_j$ is forbidden by the symmetry G.

In this language the problem of the relation between $S_4(2)$ and $S_4 \times \mathcal{Z}_4$ can be reformulated as follows: Suppose a Lagrangian $\mathcal{L}$ invariant under the action of $S_4(2)$ is given. Since the irreducible representations of $S_4(2)$ and $S_4 \times \mathcal{Z}_4$ differ by phase factors only, we find that point 1. stated above is fulfilled automatically. We can now replace any irreducible representation D_i of $S_4(2)$ containing elements of the form "real matrix times $\pm i$" by the corresponding irreducible representation of $S_4 \times \mathcal{Z}_4$ containing all four elements of the

[11] E.g. $\mathcal{L}(\eta) = \eta^2 + \eta^4$ is invariant under $\mathcal{Z}_2: \ \eta \mapsto -\eta$, but it can be split up into the two "smaller" Lagrangians $\mathcal{L}_1(\eta) = \eta^2$ and $\mathcal{L}_2(\eta) = \eta^4$. $\nexists k \in \mathbb{N}$ such that $\mathcal{L}(\alpha\eta) = \alpha^k \mathcal{L}(\eta) \ \forall \alpha \in U(1)$, while $\mathcal{L}_1(\alpha\eta) = \alpha^2 \mathcal{L}_1(\eta)$ and $\mathcal{L}_2(\alpha\eta) = \alpha^4 \mathcal{L}_2(\eta) \ \forall \alpha \in U(1)$.

center. In this case all Lagrangians $\mathcal{L}_j$ will remain invariant, because in order to construct $\mathcal{L}_j$ invariant under $S_4(2)$ one already had to take care of the phase factor i contained in the element C of $S_4(2)$. Thus from the point of view of invariant Lagrangians (which is the interesting point of view for physics), $S_4 \times \mathcal{Z}_4$ and $S_4(2)$ are equivalent.

In this section we have analysed two interesting finite subgroups of $U(3)$. It turned out that in the case of these two groups it is possible to find (in the case of $[\![27,4]\!]$) a finite subgroup of $SU(3)$ or (in the case of $S_4(2)$) a direct product of a finite subgroup of $SU(3)$ with a cyclic group which is equivalent to the $U(3)$-subgroup from the physical point of view, i.e. which allows the same Lagrangians. The question remains whether this is true in general.

V Conclusions

In this work we used the SmallGroups Library [19, 20] to find the finite subgroups of $U(3)$ of order smaller than 512. Using the computer algebra system GAP it was possible to construct generators for all these groups.

Inspired by the results (see tables 4 and 5) of this analysis we developed the two theorems III.1 and III.2 which led to the discovery of the series of finite subgroups of $U(3)$

$$T_n(m), \quad \Delta(3n^2,m), \quad S_4(m), \quad \Delta(6n^2,m) \quad \text{and} \quad \Delta'(6n^2,j,k).$$

In the last part of this work we analysed the groups $[\![27,4]\!] \cong \mathcal{Z}_9 \rtimes \mathcal{Z}_3$ and $S_4(2) \cong A_4 \rtimes \mathcal{Z}_4$ in more detail. It turned out that, from the physical point of view, $[\![27,4]\!]$ is equivalent to the $SU(3)$-subgroup $\Delta(27)$ and $S_4(2)$ is equivalent to $S_4 \times \mathcal{Z}_4$. We closed our discussion with the open question whether this scheme holds true for all finite subgroups of $U(3)$.

We hope that this work will shed some light onto the structures of the finite subgroups of $U(3)$, which may be as important in the context of particle physics as the well known finite subgroups of $SU(3)$.

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A Proofs

A.1 Proof of criterion II.2

Proof. Claim 1: The identity element is the only element mapped onto $\mathbb{1}_d$ by $D \Leftrightarrow D$ is faithful.

$\Leftarrow$: By definition of "faithful".

$\Rightarrow$: Suppose the identity e is the only element mapped on $\mathbb{1}_d$ and let $D(a) = D(b)$ for some $a, b \in G$.

$$D(a) = D(b) \Rightarrow \mathbb{1}_d = D(a)D(b)^{-1} = D(ab^{-1}) \Rightarrow ab^{-1} = e \Rightarrow a = b \Rightarrow D \text{ injective.}$$

D injective $\Rightarrow$ D faithful.

Claim 2: Let M be equivalent to a unitary $m \times m$-matrix. Then $\mathrm{Tr}M = m \Leftrightarrow M = \mathbb{1}_m$.

$\Leftarrow$: $\mathrm{Tr}\mathbb{1}_m = m$.

$\Rightarrow$: All eigenvalues e_j of M are elements of $U(1)$, thus

$$|\mathrm{Tr}M| = \left|\sum_{j=1}^{m} e_j\right| \leq \sum_{j=1}^{m} |e_j| = m.$$
$$|\mathrm{Tr}M| = m \Leftrightarrow e_j = \lambda \in U(1) \ \forall j \Leftrightarrow M = \lambda\mathbb{1}_m.$$
$$\mathrm{Tr}M = \lambda m = m \Rightarrow \lambda = 1 \Rightarrow M = \mathbb{1}_m.$$

After all we find: If D is non-faithful there must be more than one element mapped onto $\mathbb{1}_d$, which is equivalent to the fact that there is more than one character d of D in the character table. ☐

A.2 Proof of theorem II.3

A.1 Proposition. Let $a, b \in \mathbb{N}\backslash\{0\}$, and let $\gcd(a,b)$ be the greatest common divisor of a and b. Then
$$\mathcal{Z}_a \cap \mathcal{Z}_b = \{e\} \Leftrightarrow \ \gcd(a,b) = 1. \tag{46}$$

Proof. In this proof we represent $\mathcal{Z}_k$ as $\mathcal{Z}_k = \{1, \kappa, \kappa^2, ..., \kappa^{k-1}\}$, where $\kappa = e^{\frac{2\pi i}{k}}$.

$\Rightarrow$: $\mathcal{Z}_a \cap \mathcal{Z}_b = \{1\}$. Suppose $\gcd(a,b) > 1$, and let $\mathrm{lcm}(a,b)$ denote the lowest common multiple of a and b.
$$\Rightarrow \mathrm{lcm}(a,b) = \frac{ab}{\gcd(a,b)} < ab.$$
$\Rightarrow \exists\ (x,y) \in \{1, ..., a-1\} \times \{1, ..., b-1\}$ such that
$$\mathrm{lcm}(a,b) = ax = by \Rightarrow \frac{y}{a} = \frac{x}{b} = \frac{\mathrm{lcm}(a,b)}{ab} = \frac{1}{\gcd(a,b)} < 1 \Rightarrow$$
$$\Rightarrow e^{2\pi i \frac{y}{a}} = e^{2\pi i \frac{x}{b}} \Rightarrow$$
$$\Rightarrow \underbrace{\left(e^{2\pi i/a}\right)^y = \left(e^{2\pi i/b}\right)^x}_{\in\ \mathcal{Z}_a \cap \mathcal{Z}_b} \neq 1. \Rightarrow \ \text{contradiction!}$$

$\Leftarrow$: $\gcd(a,b) = 1$. Let $g \in \mathcal{Z}_a \cap \mathcal{Z}_b$.

$$\Rightarrow \exists (x,y) \in (\mathbb{N}\backslash\{0\}) \times (\mathbb{N}\backslash\{0\}) \ \text{such that}$$
$$g = \left(e^{2\pi i/a}\right)^y = \left(e^{2\pi i/b}\right)^x \Rightarrow$$
$$\Rightarrow \frac{y}{a} = \frac{x}{b} + k, \ \ k \in \mathbb{N} \Rightarrow yb = xa + kab = (x+kb)a \Rightarrow$$
$$\Rightarrow y \text{ is a multiple of } a \text{ (because } \gcd(a,b) = 1) \Rightarrow g = 1.$$

☐

A.2 Corollary. Let $a, b \in \mathbb{N}\backslash\{0\}$, then

$$\mathcal{Z}_a \times \mathcal{Z}_b \cong \mathcal{Z}_{ab} \Leftrightarrow \gcd(a, b) = 1. \tag{47}$$

Proof. $\alpha := e^{\frac{2\pi i}{a}}$, $\beta := e^{\frac{2\pi i}{b}}$, $\gamma := e^{\frac{2\pi i}{ab}}$.

$$\mathcal{Z}_{ab} \cong \{1, \gamma, ..., \gamma^{ab-1}\}.$$

$\mathcal{Z}_a \cong \{1, \alpha, ..., \alpha^{a-1}\}$ and $\mathcal{Z}_b \cong \{1, \beta, ..., \beta^{b-1}\}$ are normal subgroups of $\mathcal{Z}_{ab}$, thus

$$\mathcal{Z}_{ab} \cong \mathcal{Z}_a \times \mathcal{Z}_b \Leftrightarrow \mathcal{Z}_a \cap \mathcal{Z}_b = \{1\},$$

and from proposition A.1:

$$\mathcal{Z}_a \cap \mathcal{Z}_b = \{1\} \Leftrightarrow \quad \gcd(a, b) = 1.$$

□

Proof of theorem II.3. Let us represent $\mathcal{Z}_n$ as $\mathcal{Z}_n = \{1, a, a^2, ..., a^{n-1}\}$, where $a = e^{2\pi i/n}$. From proposition A.1 we know that

$$\begin{aligned} &\gcd(n, c) = 1 \Leftrightarrow \mathcal{Z}_n \cap \mathcal{Z}_c = \{e\} \Leftrightarrow \\ \mathcal{D} :\ &\mathcal{Z}_n \times G \to \mathcal{D}(\mathcal{Z}_n \times G) \\ &(a^k, g) \mapsto a^k D(g), \quad k \in \{0, ..., n-1\} \end{aligned} \tag{48}$$

is a faithful representation of $\mathcal{Z}_n \times G$. It remains to show the irreducibility of $\mathcal{D}$. Remember that a representation R of H is irreducible if and only if $(\chi_R, \chi_R)_H = 1$ (see equation (5) and explanations there).

$$\begin{aligned} (\chi_{\mathcal{D}}, \chi_{\mathcal{D}})_{\mathcal{Z}_n \times G} &= \frac{1}{\mathrm{ord}(\mathcal{Z}_n \times G)} \sum_{b \in \mathcal{Z}_n \times G} \chi_{\mathcal{D}}(b)^* \chi_{\mathcal{D}}(b) = \\ &= \frac{1}{\mathrm{ord}(G)} \times \frac{1}{n} \sum_{k=0}^{n-1} \sum_{b' \in G} \chi_{\mathcal{D}}(a^k b')^* \chi_{\mathcal{D}}(a^k b') = \\ &= \frac{1}{\mathrm{ord}(G)} \times \frac{1}{n} \times n \sum_{b' \in G} \chi_D(b')^* \chi_D(b') = (\chi_D, \chi_D)_G = 1. \end{aligned}$$

$\Rightarrow \mathcal{D}$ is a faithful m-dimensional irreducible representation of $\mathcal{Z}_n \times G$.

□

A.3 Proofs of theorem III.1 and theorem III.2

A.3 Lemma. Let $A := \langle\langle A_1, ..., A_a\rangle\rangle$ be a normal subgroup of $G := \langle\langle A_1, ..., A_a, B\rangle\rangle$, then

$$\mathrm{ord}(G) \leq \mathrm{ord}(A)\mathrm{ord}(B). \tag{49}$$

Proof. A is an invariant subgroup of G, thus every element of G can be written as an element of A times an element of $\langle\langle B\rangle\rangle$, i.e. a power of B. Thus there are at most $\mathrm{ord}(A)\mathrm{ord}(B)$ different elements in G. □

Proof of theorem III.2. Every element of $\langle\langle A_1, ..., A_a, B\rangle\rangle$ can be written as a product of generators of the group. Let

$$P(A_1, ..., A_a, B) = \mathbb{1}_m. \tag{50}$$

be a representation of the unit element in terms of generators. The number $x[P]$ of factors B contained in the product P can be defined by

$$P(A_1, ..., A_a, \delta B) = \delta^{x[P]} P(A_1, ..., A_a, B) \tag{51}$$

for some $\delta \in U(1)$, $\delta^n \neq 1\ \forall n \in \mathbb{N}\backslash\{0\}$. Let M be the set of all products P fulfilling equation (50). We can now define s to be the smallest positive number of factors B contained in a product (50), i.e.

$$s := \min_{P\in M}\{x[P] | x > 0\}. \tag{52}$$

Since $B^n = \mathbb{1}_m$ we find $s \leq n$. Let $\tilde{P}$ denote a product of generators fulfilling

$$\tilde{P}(A_1, ..., A_a, \delta B) = \delta^s \tilde{P}(A_1, ..., A_a, B). \tag{53}$$

Suppose $s < n$: The center of the group is generated by $e^{2\pi i/c}\mathbb{1}_m$, $(\beta B)^{n^k} = e^{2\pi i/c^j}\mathbb{1}_m$ and $\beta^s \mathbb{1}_m = \tilde{P}(A_1, ..., A_a, \beta\mathbb{1}_m)$.

$$(\beta^s)^{c^j} = \left(e^{2\pi i/n^k}\right)^s, \text{ which (if } s < n) \text{ generates } \mathcal{Z}_{n^k}.$$

Thus we find

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle = \langle\langle A_1, ..., A_a, B, e^{2\pi i/c^j}\mathbb{1}_m, e^{2\pi i/n^k}\mathbb{1}_m\rangle\rangle$$

and since $\langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle \subset \langle\langle A_1, ..., A_a\rangle\rangle$:

$$\mathrm{ord}(\langle\langle A_1, ..., A_a, \beta B\rangle\rangle) = \mathrm{ord}(\langle\langle A_1, ..., A_a, B\rangle\rangle) \times c^{j-1}n^k = c^{j-1}n^{k+1}\mathrm{ord}(H). \tag{54}$$

On the other hand we know that

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle = \langle\langle A_1, ..., A_a, e^{2\pi i/c^j}\mathbb{1}_m, XB\rangle\rangle$$

for some $X \in \langle\langle e^{2\pi i/n^k}\mathbb{1}_m\rangle\rangle$. Using lemma A.3 this leads to

$$\mathrm{ord}(\langle\langle A_1, ..., A_a, \beta B\rangle\rangle) \leq \underbrace{\mathrm{ord}(\langle\langle A_1, ..., A_a, e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle)}_{c^{j-1}\mathrm{ord}(H)} \times \mathrm{ord}(\langle\langle XB\rangle\rangle) \leq c^{j-1}n^k\mathrm{ord}(H),$$

which is a contradiction to equation (54) $\Rightarrow s = n$.

Since $s = n$ the center of the group is given by

$$\langle\langle e^{2\pi i/c^j}\mathbb{1}_m, e^{2\pi i/n^{k-1}}\mathbb{1}_m\rangle\rangle \cong \mathcal{Z}_{c^j n^{k-1}} \qquad \text{for } j>0,\ \ k>0.$$

The other cases follow immediately noticing that (by definition) $\langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle$ always is a subgroup of the center.

Let us finally consider the case $c = 1,\ \ k > 0$. In this case the group

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle, \qquad \beta = e^{2\pi i/n^k}$$

(by construction) is the semidirect product

$$\langle\langle A_1, ..., A_a\rangle\rangle \rtimes \langle\langle \beta B\rangle\rangle \cong H \rtimes \mathcal{Z}_{n^k}$$

with center $\langle\langle e^{2\pi i/n^{k-1}}\mathbb{1}_m\rangle\rangle \cong \mathcal{Z}_{n^{k-1}}$.

□

A.4 Lemma. Let $n, q \in \mathbb{N}\backslash\{0\}$ and $\gcd(n,q) = 1$. Then

$$\exists p \in \{1, ..., n-1\} : (pq) \bmod n = 1. \tag{55}$$

Proof. Consider the numbers

$$q \bmod n,\ \ (2q) \bmod n,\ \ ...\ \ ,\ \ [(n-1)q] \bmod n.$$

Suppose

$$(k_1 q) \bmod n = (k_2 q) \bmod n, \qquad k_1, k_2 \in \{1, ..., n-1\}, \quad k_1 > k_2.$$

This implies

$$[\underbrace{(k_1 - k_2)}_{\in\{1,...,n-2\}} q] \bmod n = 0 \Rightarrow q \bmod n = 0 \Rightarrow \text{ contradiction to } \gcd(n,q) = 1.$$

$\Rightarrow$ The $n-1$ numbers

$$q \bmod n,\ \ (2q) \bmod n,\ \ ...\ \ ,\ \ [(n-1)q] \bmod n$$

are different elements of $\{1, ..., n-1\}$. $\Rightarrow$ One of them must be 1. □

Proof of theorem III.1. Let $b = qc^j n^k;\ \ j, k \in \mathbb{N};\ \ \gcd(q,n) = \gcd(q,c) = 1$.

From lemma A.4 we know that

$$\exists p \in \{1, ..., n-1\} : (pq) \bmod n = 1.$$

Then

$$(\beta B)^{pq} = \beta^{pq} B^{pq} = \beta^{pq} B$$

and we can write βB as a product of the two group elements $(\beta^{pq}B)^{-1}\beta B = \beta^{1-pq}\mathbb{1}_m$ and $\beta^{pq}B$. Therefore

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle = \langle\langle A_1, ..., A_a, \beta^{pq}B, \beta^{1-pq}\mathbb{1}_m\rangle\rangle.$$

$(\beta^{1-pq})^{c^j n^k} = e^{2\pi i/q}$, thus $\beta^{1-pq}\mathbb{1}_m$ generates $\mathcal{Z}_{rq} \cong \mathcal{Z}_r \times \mathcal{Z}_q$, where r contains factors n and c only.

$$\begin{aligned}\Rightarrow \langle\langle A_1, ..., A_a, \beta B\rangle\rangle &= \langle\langle A_1, ..., A_a, \beta^{pq}B, e^{2\pi i/r}\mathbb{1}_m, e^{2\pi i/q}\mathbb{1}_m\rangle\rangle \cong \\ &\cong \langle\langle A_1, ..., A_a, \beta^{pq}B, e^{2\pi i/r}\mathbb{1}_m\rangle\rangle \times \langle\langle e^{2\pi i/q}\mathbb{1}_m\rangle\rangle \cong \\ &\cong \langle\langle A_1, ..., A_a, \beta^{pq}B, e^{2\pi i/r}\mathbb{1}_m\rangle\rangle \times \mathcal{Z}_q.\end{aligned}$$

$\Rightarrow$ If we want that $\langle\langle A_1, ..., A_a, \beta B\rangle\rangle$ can not be written as a direct product with a cyclic group we must impose $q = 1$, thus

$$b = c^j n^k, \quad j, k \in \mathbb{N}.$$

It remains to show that for $b = c^j n^k$ $\langle\langle A_1, ..., A_a, \beta B\rangle\rangle$ can not be written as a direct product with a cyclic group.

Let from now on $\beta := e^{2\pi i/(c^j n^k)}$. Suppose

$$\langle\langle A_1, ..., A_a, \beta B\rangle\rangle = X \times Y,$$

where Y is a cyclic group. Because of irreducibility Y must be a subgroup of the center C of the group. In the following we will frequently use the fact that $X \cap Y = \{\mathbb{1}_m\}$ in $X \times Y$.

Let us first consider the case $j > 0,\ \ k > 1$. From theorem III.2 we know that

$$C = \langle\langle e^{2\pi i/c^j}\mathbb{1}_m, e^{2\pi i/n^{k-1}}\mathbb{1}_m\rangle\rangle \cong \mathcal{Z}_{c^j} \times \mathcal{Z}_{n^{k-1}}.$$

Since every element of $X \times Y$ can be uniquely written as a product of an element of Y and an element of X it follows that

$$\exists\ \alpha\mathbb{1}_m \in Y: \quad \alpha\beta B \in X.$$

$$\Rightarrow (\alpha\beta B)^{n^{k-1}} = \alpha^{n^{k-1}} e^{2\pi i/(c^j n)}\mathbb{1}_m \in X.$$

Since $Y \subset C \cong \mathcal{Z}_{c^j} \times \mathcal{Z}_{n^{k-1}}$ and $\alpha\mathbb{1}_m \in Y$ we find that $\alpha^{n^{k-1}}\mathbb{1}_m \in \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$, which implies

$$(\alpha\beta B)^{c^j n^{k-1}} = (\alpha^{n^{k-1}} e^{2\pi i/(c^j n)})^{c^j}\mathbb{1}_m = e^{2\pi i/n}\mathbb{1}_m \in X,$$

thus $Y \cap \langle\langle e^{2\pi i/n}\mathbb{1}_m\rangle\rangle = \{\mathbb{1}_m\} \Rightarrow Y$ is a subgroup of $\langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$. In the cases of $j > 0,\ \ k \in \{0, 1\}$ we find $Y \subset \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$ too. The case of $c = 1$ directly leads to $Y = \{\mathbb{1}_m\}$.

Knowing that $Y \subset \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$ we can deduce that (if Y is nontrivial)

$$\langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle \subset Y \Rightarrow \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle \cap X = \{\mathbb{1}_m\},$$

because else we would find $X \cap Y \neq \{\mathbb{1}_m\}$. This leads to

$$Y = \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle.$$

Thus every element of $\langle\langle A_1, ..., A_a, \beta B\rangle\rangle$ can be uniquely written as a product of an element of X and an element of $Y = \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$. This implies that every element of $\langle\langle A_1, ..., A_a\rangle\rangle$ can be uniquely written as an element of some subgroup $S \subset X$ and an element of $Y = \langle\langle e^{2\pi i/c^j}\mathbb{1}_m\rangle\rangle$.

$$\begin{aligned}&\Rightarrow \langle\langle A_1, ..., A_a\rangle\rangle = S \times \langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle \Rightarrow\\ &\Rightarrow G = \langle\langle A_1, ..., A_a, B\rangle\rangle \cong (S \times \langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle) \rtimes \mathcal{Z}_n \cong (S \rtimes \mathcal{Z}_n) \times \mathcal{Z}_c,\end{aligned}$$

which is a contradiction to "$G \cong H \rtimes \mathcal{Z}_n$ can not be written as a direct product with a cyclic group".

The case $j = 0$, in a similar way, leads to $Y = \langle\langle e^{2\pi i/c}\mathbb{1}_m\rangle\rangle$ leading to the same contradiction as above.

□

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Chapter 6

Comments on the classification of the finite subgroups of $SU(3)$

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Comments on the classification of the finite subgroups of SU(3)

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Abstract

Many finite subgroups of SU(3) are commonly used in particle physics. The classification of the finite subgroups of SU(3) began with the work of H.F. Blichfeldt at the beginning of the 20th century. In Blichfeldt's work the two series (C) and (D) of finite subgroups of SU(3) are defined. While the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ (which are subseries of (C) and (D), respectively) have been intensively studied, there is not much knowledge about the group series (C) and (D). In this work we will show that (C) and (D) have the structures (C) $\cong (\mathbb{Z}_m \times \mathbb{Z}_{m'}) \rtimes \mathbb{Z}_3$ and (D) $\cong (\mathbb{Z}_n \times \mathbb{Z}_{n'}) \rtimes S_3$, respectively. Furthermore we will show that, while the (C)-groups can be interpreted as irreducible representations of $\Delta(3n^2)$, the (D)-groups can in general not be interpreted as irreducible representations of $\Delta(6n^2)$.

1 Introduction

Today finite groups are widely used in physics, and particle physics offers a wide range of applications for the theory of finite groups in particular. Especially the finite subgroups of SU(3) have been intensively studied in the past, and their investigation and application in various fields of particle physics continues unabated.

The systematic analysis of finite subgroups of SU(3) for application in particle physics started with the work of Fairbairn, Fulton and Klink [1] in 1964. Since then many contributions to a systematic analysis of these groups for application in particle physics have been published. The finite subgroups of SU(3) have been used for model building in hadron physics [1], as well as a computational tool in lattice QCD (see e.g. [2]). Today's most important application is flavour physics: On the one hand there is an enormous

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amount of models using finite subgroups of SU(3) (see e.g. [3, 4] and references therein) trying to solve the fermion mass and mixing problems in the lepton sector as well as the quark sector. On the other hand finite subgroups of SU(3) have also been used in the context of minimal flavour violation (see e.g. [5]).

The well-known group series T_n, $\Delta(3n^2)$ and $\Delta(6n^2)$ are sub-series of the series (C) and (D) [6] of finite subgroups of SU(3). However, (C) and (D) also contain other groups, which have been paid much less attention. The aim of this work is to study the structure of the finite subgroups of SU(3) of type (C) and (D) and their relation to other finite subgroups of SU(3).

The classification of the finite subgroups of SU(3)

Here we want to list the main efforts that have been put in the classification of the finite subgroups of SU(3) in chronological order.

- In 1916 G.A. Miller, H.F. Blichfeldt and L.E. Dickson published their book *Theory and Applications of Finite Groups* [6]. In the part written by H.F. Blichfeldt the finite subgroups of SU(3) are classified in terms of their generators. The series $\Delta(3n^2)$ and $\Delta(6n^2)$ are not explicitly defined but are contained in the series (C) and (D).

- In 1964 the article *Finite and Disconnected Subgroups of SU(3) and their Application to the Elementary-Particle Spectrum* by W.M. Fairbairn, T. Fulton and W.H. Klink was published [1]. It was the first paper which faced the task of analyzing a large set of finite subgroups of SU(3) for their use as symmetries in particle physics (hadron physics in this special case). In [1] the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ are already included.

- In their articles *Representations and Clebsch-Gordan coefficients of Z-metacyclic groups* [7] and *Finite subgroups of SU(3)* [8] (both published in 1981) A. Bovier, M. Lüling and D. Wyler defined and analyzed the SU(3)-subgroups T_n. Especially they constructed all irreducible representations and calculated all Clebsch-Gordan coefficients for these groups. Furthermore Bovier et al. investigated the group series $\Delta(3n^2)$ and $\Delta(6n^2)$ in detail giving not only the irreducible representations but also the Clebsch-Gordan coefficients for both series.

- Two years later, in their paper *Some comments on finite subgroups of SU(3)* [9], W.M. Fairbairn and T. Fulton proved that some groups of the type T_n given by Bovier et al. in [8] are not subgroups of SU(3).

- In 2007 C. Luhn, S. Nasri and P. Ramond published their work *The Flavor Group* $\Delta(3n^2)$ [10], giving all conjugacy classes, irreducible representations, character tables and Clebsch-Gordan coefficients of $\Delta(3n^2)$.

- In 2008 J.A. Escobar and C. Luhn published their analysis *The Flavor Group* $\Delta(6n^2)$ [11], giving all conjugacy classes, irreducible representations, character tables and Clebsch-Gordan coefficients of $\Delta(6n^2)$.

- In 2009 the work *Systematic analysis of finite family symmetry groups and their application to the lepton sector* [3] was published. It contains an analysis and summary of all finite subgroups of SU(3). With the help of [10] it could be shown that all SU(3)-subgroups of type (C) can be interpreted as three-dimensional irreducible representations of $\Delta(3n^2)$. The generators of the group series (D) were determined explicitly.
  In the same year R. Zwicky and T. Fischbacher showed that every (D)-group is a subgroup of $\Delta(6n^2)$ for a suitable n in their article *On discrete Minimal Flavour Violation* [5].

- In the work *On the finite subgroups of U(3) of order smaller than 512* [12] (published in 2010) all finite subgroups of U(3) of order smaller than 512 which possess a faithful three-dimensional irreducible representation are listed.[1] Among these groups there is no SU(3)-subgroup which does not fit into the classification scheme of Blichfeldt [6].
  In their article *Tribimaximal Mixing From Small Groups* [13] K.M. Parattu and A. Wingerter began to analyze also those finite subgroups of U(3) which possess a faithful three-dimensional *reducible* representation but which do not possess any faithful irreducible representation. Their analysis of all groups up to order 100 shows no finite subgroups of SU(3) which do not fit into Blichfeldt's classification scheme [6].

Table 1 shows the different types of finite subgroups of SU(3), as they are classified by now. Especially it contains all finite subgroups of SO(3), see [14, 19], as follows:

- The *uniaxial groups* (groups of rotations about one axis) are cyclic and thus isomorphic to $\mathbb{Z}_n = A(n, 1)$.
- The *dihedral groups* possess faithful two-dimensional representations and are thus contained in B.
- The *tetrahedral group* is isomorphic to $A_4 \cong \Delta(12)$.
- The *octahedral group* is isomorphic to $S_4 \cong \Delta(24)$.
- The *icosahedral group* is isomorphic to $A_5 \cong \Sigma(60)$.

Only two of the groups presented in table 1 are not contained in the list given by Blichfeldt in [6], namely the direct products $\Sigma(60) \times \mathbb{Z}_3$ and $\Sigma(168) \times \mathbb{Z}_3$.

2 Abelian subgroups of SU(3)

In the following sections we will frequently deal with Abelian subgroups of SU(3). The remarkably simple theorem 2.1 provides us with all necessary information we will need in our later analysis.

[1] The list in [12] contains only those groups which are not isomorphic to a group of the form $H \times \mathbb{Z}_n$ with $n > 1$.

Group	Order	References
$A(m,n) \cong \mathbb{Z}_m \times \mathbb{Z}_n$ (Abelian groups), n divides m	mn	
B (finite subgroups of U(2))	no general formula	[6, 14]
$C(n,a,b)$	no general formula	[3, 6]
$D(n,a,b;d,r,s)$	no general formula	[3, 5, 6]
$\Delta(3n^2) \cong C(n,0,1),\ \ n \geq 2$	$3n^2$	[1, 3, 7, 10]
$\Delta(6n^2) \cong D(n,0,1;2,1,1),\ \ n \geq 2$	$6n^2$	[1, 3, 7, 11]
$T_n \cong C(n,1,a),\ \ (1+a+a^2) \bmod n = 0$, or	$3n$	[3, 7, 8, 9]
$T_n \cong C(3p,1,a),\ \ (1+a+a^2) \bmod 3p = 0;\ n = 3p$	$3n = 9p$	[3, 7, 8, 9]
$\Sigma(60) \cong A_5$	60	[1, 3, 6, 15, 16]
$\Sigma(60) \times \mathbb{Z}_3$	180	
$\Sigma(168) \cong PSL(2,7)$	168	[1, 3, 6, 15]
$\Sigma(168) \times \mathbb{Z}_3$	504	
$\Sigma(36 \times 3)$	108	[1, 3, 6, 17, 18]
$\Sigma(72 \times 3)$	216	[1, 3, 6, 17, 18]
$\Sigma(216 \times 3)$	648	[1, 3, 6, 17, 18]
$\Sigma(360 \times 3)$	1080	[1, 3, 6]

Table 1: Types of finite subgroups of SU(3) [1, 6, 8, 9]. The allowed values for n and p in T_n are products of powers of primes of the form $3k+1$, $k \in \mathbb{N}$ [8, 9].

2.1 Theorem. Every finite Abelian subgroup $\mathcal{A}$ of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, where

$$m = \max_{a \in \mathcal{A}} \operatorname{ord}(a) \tag{1}$$

and n is a divisor of m. The proof of this theorem can be found in appendix A.

3 On the SU(3)-subgroups of type (C)

In this section we will investigate the structure of the SU(3)-subgroups of type (C). Knowing the structure of (C) we can easily show that there exist SU(3)-subgroups of type (C) which neither belong to the series $\Delta(3n^2)$, nor to the groups of type [7, 8, 9]

$$T_n = \mathbb{Z}_n \rtimes \mathbb{Z}_3.$$

In the following the symbol $\langle\langle ... \rangle\rangle$ means "generated by". The group series (C) is generated by the matrices

$$E := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F(n,a,b) := \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \tag{2}$$

with $\eta = \exp(2\pi i/n)$.

$$C(n,a,b) := \langle\langle E,\ \ F(n,a,b)\rangle\rangle, \quad n \in \mathbb{N}\backslash\{0\},\ \ a,b \in \{0,...,n-1\}. \tag{3}$$

Since the irreducible three-dimensional representations of $\Delta(3n^2)$ are [10]

$$\underline{\mathbf{3}}_{(a,b)}:\ \ G_1 \mapsto E,\ \ G_2 \mapsto F(n,b,a), \tag{4}$$

where G_1 and G_2 denote the generators of $\Delta(3n^2)$, we find

$$C(n,a,b) \cong \underline{\mathbf{3}}_{(b,a)}(\Delta(3n^2)). \tag{5}$$

There is no general formula for the order of $C(n,a,b)$, but we can give a prescription for the calculation of the order of $C(n,a,b)$ for given n,a,b.

Let us first think about the structure of $C(n,a,b)$. Defining

$$X := F(n,a,b), \quad Y := F(n,b,-a-b), \tag{6}$$

we find the commutation relations

$$XE = EX^{-1}Y^{-1}, \quad YE = EX. \tag{7}$$

Therefore

- The subgroup $\langle\langle X,\,Y\rangle\rangle$ of all diagonal matrices is a normal subgroup of $C(n,a,b)$.
- Every element of $C(n,a,b)$ can be written in the form
$$E^j X^k Y^l.$$

Furthermore

$$\langle\langle E\rangle\rangle \cap \langle\langle X,\,Y\rangle\rangle = \{\mathbb{1}_3\}, \tag{8}$$

thus

$$C(n,a,b) = \langle\langle X,\,Y\rangle\rangle \rtimes \langle\langle E\rangle\rangle. \tag{9}$$

From theorem 2.1 we find

$$\langle\langle X,\,Y\rangle\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p, \tag{10}$$

where

$$m = \max_{A\in\langle\langle X,\,Y\rangle\rangle} \mathrm{ord}(A) = \mathrm{ord}(X) = \mathrm{ord(Y)} = \mathrm{lcm}(\mathrm{ord}(\eta^a), \mathrm{ord}(\eta^b)). \tag{11}$$

$\mathrm{lcm}(r,s)$ denotes the lowest common multiple of $r,s \in \mathbb{N}$. Defining

$$p := \min\{k \in \{1,...,m\} | Y^k \in \langle\langle X\rangle\rangle\}, \tag{12}$$

we find

$$\langle\langle X,\,Y\rangle\rangle = \{X^i Y^j |\ \ i = 0,...,m-1;\ \ j = 0,...,p-1\} \Rightarrow \langle\langle X,\,Y\rangle\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p. \tag{13}$$

For the sake of completeness we also want to find one possible choice of generators of $\mathbb{Z}_m$ and $\mathbb{Z}_p$. Applying the same argumentation as above on the definition

$$q := \min\{k \in \{1, ..., m\} | \, X^k \in \langle\langle Y \rangle\rangle\},$$

we find $\langle\langle X, Y \rangle\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_q \Rightarrow q = p$, and therefore

$$\langle\langle X \rangle\rangle \cap \langle\langle Y \rangle\rangle = \langle\langle X^p \rangle\rangle = \langle\langle Y^p \rangle\rangle. \tag{14}$$

Since $\mathrm{ord}(X^p) = \frac{m}{p}$ this leads to

$$\exists t \in \left\{1, ..., \frac{m}{p} - 1\right\} : \;\; X^{pt} = Y^p \Rightarrow (YX^{-t})^p = \mathbb{1}_3, \tag{15}$$

but

$$(YX^{-t})^a \neq \mathbb{1}_3 \text{ for } a < p,$$

because otherwise $Y^a \in \langle\langle X \rangle\rangle$ for $a < p$, which would be a contradiction to the definition (12) of p. Therefore

$$\langle\langle YX^{-t} \rangle\rangle \cong \mathbb{Z}_p \tag{16}$$

Noticing furthermore that

$$\langle\langle X \rangle\rangle \cap \langle\langle YX^{-t} \rangle\rangle = \{\mathbb{1}_3\} \tag{17}$$

we finally find

$$\langle\langle X, Y \rangle\rangle = \langle\langle X \rangle\rangle \times \langle\langle YX^{-t} \rangle\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_p. \tag{18}$$

Now there are three cases:

1. $p = 1 \Rightarrow Y \in \langle\langle X \rangle\rangle \Rightarrow \langle\langle X, Y \rangle\rangle = \langle\langle X \rangle\rangle \Rightarrow C(n, a, b) \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$.
2. $p = m \Rightarrow \langle\langle X \rangle\rangle \cap \langle\langle Y \rangle\rangle = \{\mathbb{1}_3\}$
   $\Rightarrow \langle\langle X, Y \rangle\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m \Rightarrow C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3 \cong \Delta(3m^2)$.
3. $p \in \{2, ..., m-1\} \Rightarrow C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3$.

m is determined by equation (11) and from equations (12) and (15) we can determine p and t. One finds

$$Y^p = X^{pt} \Rightarrow \begin{cases} p(b - at) \bmod n = 0, \;\; p \in \{1, ..., m\}, \text{ smallest possible} \\ p(a + b(1+t)) \bmod n = 0, \;\; t \in \{1, ..., \frac{m}{p} - 1\} \end{cases} \tag{19}$$

Let us summarize our results on the structure of the groups of type (C):

- $C(n, a, b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3$, where
- $m = \mathrm{lcm}(\mathrm{ord}(\eta^a), \mathrm{ord}(\eta^b))$, and
- $\begin{cases} p(b - at) \bmod n = 0, \;\; p \in \{1, ..., m\}, \text{ smallest possible,} \\ p(a + b(1+t)) \bmod n = 0, \;\; t \in \{1, ..., \frac{m}{p} - 1\}. \end{cases}$

- In terms of generators: $C(n,a,b) = (\langle\langle X\rangle\rangle \times \langle\langle YX^{-t}\rangle\rangle) \rtimes \langle\langle E\rangle\rangle$.
- $\mathrm{ord}(C(n,a,b)) = 3mp$.
- $p = 1 \Rightarrow C(n,a,b) \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$ ($\to T_m$ for appropriate m (see [7, 8, 9])).
- $p = m \Rightarrow C(n,a,b) \cong \Delta(3m^3) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3$.

Note that (C)-groups can be direct products with $\mathbb{Z}_3$, i.e. the case

$$C(n,a,b) \cong (\mathbb{Z}_{3x} \times \mathbb{Z}_y) \rtimes \mathbb{Z}_3 \cong ((\mathbb{Z}_x \times \mathbb{Z}_y) \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3$$

is possible for some choices of n, a, b. Examples for this case are the groups

$$C(6,1,1) \cong (\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3 \cong A_4 \times \mathbb{Z}_3 \tag{20}$$

and the group T_{21}, which is described in [9]:

$$T_{21} \cong C(21,1,4) \cong \mathbb{Z}_{21} \rtimes \mathbb{Z}_3 \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_3. \tag{21}$$

As the last part of our investigation of the group series (C) we want to give an example for a (C)-group which neither belongs to the series $\Delta(3n^2)$, nor to the groups of type T_n. We already encountered an example in the group $C(6,1,1)$, but we also want to give an example for a "new" SU(3)-subgroup which is not just a direct product with an already well-known group. In [12] all groups of order smaller 512 which possess a faithful three-dimensional irreducible representation (and are not isomorphic to a direct product with a cyclic group) have been listed. Among these groups $C(9,1,1)$ appears as the smallest (C)-group which is not classified as T_n or $\Delta(3n^2)$. Using the tools we have developed in this section we immediately find

- $n = 9$, $a = b = 1$.
- $m = \mathrm{lcm}(\mathrm{ord}(\eta^a), \mathrm{ord}(\eta^b)) = \mathrm{lcm}(9,9) = 9$.
- The equations for p and t read $\begin{cases} p(1-t) \bmod 9 = 0, & p \in \{1,...,9\}, \text{ smallest possible,} \\ p(2+t) \bmod 9 = 0, & t \in \{1,..., \frac{9}{p} - 1\}. \end{cases}$
  with the solution $p = 3$, $t = 1$.
- Thus
  $$C(9,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3, \tag{22}$$
  which coincides with the structure description given in [13], where the group is named by its SmallGroup number[2] $[\![81,9]\!]$.

[2] See [12, 13] for an explanation of the SmallGroup number (also called GAP ID in [13]).

4 On the structure of the SU(3)-subgroups of type (D)

According to [6] the groups of type (D) are generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F = F(n,a,b) := \begin{pmatrix} \eta^a & 0 & 0 \\ 0 & \eta^b & 0 \\ 0 & 0 & \eta^{-a-b} \end{pmatrix}, \tag{23}$$

($\eta = \exp(2\pi i/n), \quad n \in \mathbb{N}\backslash\{0\}, \quad a,b \in \{0,...,n-1\}$) of (C) and an additional generator $\widetilde{G}$ of the form

$$\widetilde{G} = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & z & 0 \end{pmatrix}. \tag{24}$$

The conditions $\det \widetilde{G} = 1$ and $\mathrm{ord}(\widetilde{G}) < \infty$ lead to [3]

$$\widetilde{G} = \widetilde{G}(d,r,s) := \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix} \tag{25}$$

with

$$\delta = \exp(2\pi i/d), \quad d \in \mathbb{N}\backslash\{0\}, \quad r,s \in \{0,...,d-1\}. \tag{26}$$

Thus we define

$$D(n,a,b;d,r,s) := \langle\langle E,\, F(n,a,b),\, \widetilde{G}(d,r,s)\rangle\rangle. \tag{27}$$

For a better understanding of the structure of (D) it is helpful to reformulate the generators of the group. We define

$$A := \widetilde{G}^2 = \begin{pmatrix} \delta^{2r} & 0 & 0 \\ 0 & -\delta^{-r} & 0 \\ 0 & 0 & -\delta^{-r} \end{pmatrix}, \quad G' := E^2\widetilde{G}^2E\widetilde{G} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \delta^{2r+s} \\ 0 & \delta^{-(2r+s)} & 0 \end{pmatrix}, \tag{28}$$

which leads to

$$D(n,a,b;d,r,s) = \langle\langle A,\, E,\, F,\, G'\rangle\rangle. \tag{29}$$

Our first important observation is that, as in the case of (C), the subgroup $\mathcal{A}$ of all diagonal matrices is a normal subgroup of (D). This lets us hope to find a semidirect product structure as in the case of (C).

The action of the (non-diagonal) generators of (D) on any diagonal matrix is given by

$$\begin{aligned} &G'^{-1}\,\mathrm{diag}(a,b,c)\,G' = \mathrm{diag}(a,c,b), \\ &E^{-1}\,\mathrm{diag}(a,b,c)\,E = \mathrm{diag}(c,a,b), \\ &(EG')^{-1}\,\mathrm{diag}(a,b,c)\,EG' = \mathrm{diag}(c,b,a), \\ &E^{-2}\,\mathrm{diag}(a,b,c)\,E^2 = \mathrm{diag}(b,c,a), \\ &(E^2G')^{-1}\,\mathrm{diag}(a,b,c)E^2G' = \mathrm{diag}(b,a,c). \end{aligned} \tag{30}$$

This describes an S_3-action, which is well known from $\Delta(6n^2)$ [11], so the structure of (D) will be, though not identical in general, very similar to the structure

$$\Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3 \cong ((\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \tag{31}$$

of $\Delta(6n^2)$. Indeed

$$\langle\langle A,\, E,\, F,\, G'\rangle\rangle \cong (\mathcal{A} \rtimes \langle\langle E\rangle\rangle) \rtimes \langle\langle G'\rangle\rangle \cong (\mathcal{A} \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2. \tag{32}$$

In [14] it was shown that in fact

$$(\mathcal{A} \rtimes \langle\langle E\rangle\rangle) \rtimes \langle\langle G'\rangle\rangle \cong \mathcal{A} \rtimes S_3. \tag{33}$$

This can be illustrated by means of a similarity transformation [14]. We define

$$T := \mathrm{diag}(-\delta^{-2r-s},\, -\delta^{2r+s},\, 1) \tag{34}$$

and find

$$\begin{aligned}
T^{-1}G'T &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} =: G, \\
T^{-1}ET &= \underbrace{\begin{pmatrix} \delta^{4r+2s} & 0 & 0 \\ 0 & -\delta^{-2r-s} & 0 \\ 0 & 0 & -\delta^{-2r-s} \end{pmatrix}}_{(EG'E)^2 =: B \in \mathcal{A}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = BE, \\
T^{-1}\mathcal{A}T &= \mathcal{A}.
\end{aligned} \tag{35}$$

Thus

$$\langle\langle A,\, E,\, F,\, G'\rangle\rangle \cong \langle\langle A,\, B,\, E,\, F,\, G\rangle\rangle \cong \mathcal{A} \rtimes \langle\langle E,\, G\rangle\rangle, \tag{36}$$

and due to $\langle\langle E,\, G\rangle\rangle \cong S_3$

$$\langle\langle A,\, B,\, E,\, F,\, G\rangle\rangle \cong \mathcal{A} \rtimes S_3 \tag{37}$$

with $\langle\langle A,\, B,\, F\rangle\rangle \subset \mathcal{A}$. For the explicit construction of $\mathcal{A}$ we refer the reader to appendix B. Theorem 2.1 and equation (37) lead to

$$D(n,a,b;d,r,s) \cong \mathcal{A} \rtimes \langle\langle E,\, G\rangle\rangle \cong \mathcal{A} \rtimes S_3 \cong (\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes S_3, \tag{38}$$

where p and q are functions of n, a, b, d, r, s.

While every group of type (C) can be interpreted as an irreducible representation of a group of type $\Delta(3n^2)$ (see equation (5)), a similar statement does *not* hold for (D) and $\Delta(6n^2)$. In the following we will show that there is at least one SU(3)-subgroup of type (D), which cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$. In [12] the following (D)-group has been found:

$$D(9,1,1;2,1,1) \cong [\![162, 14]\!],$$

which is of order 162. Since $C(9,1,1)$ is invariant under the action of the $\mathbb{Z}_2$-generator G, we find

$$D(9,1,1;2,1,1) \cong \underbrace{C(9,1,1)}_{\mathcal{A}\rtimes\langle\langle E\rangle\rangle} \rtimes \langle\langle G\rangle\rangle \cong ((\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2 \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3. \quad (39)$$

Equation (39) suggests that there might be an irreducible three-dimensional representation $\mathcal{D}$ of $\Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ such that $\mathcal{D}(\Delta(6n^2)) \cong D(9,1,1;2,1,1)$. However, this is not the case, which we will show in the following.
If we can show, that there is no three-dimensional irreducible representation $\mathcal{D}$ of $\Delta(6n^2)$ with $\mathrm{ord}(\mathcal{D}(\Delta(6n^2))) = 162$, we have found an example for a (D)-group that cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$. This is indeed possible:

4.1 Proposition. Let $\mathcal{D}$ be a three-dimensional irreducible representation of $\Delta(6n^2)$, then

$$\exists m \in \{1,...,n\}: \ \mathcal{D}(\Delta(6n^2)) \cong \Delta(6m^2). \quad (40)$$

The proof of this proposition can be found in appendix C. Since $162/6 = 27$ is not a square number, we have proven that $D(9,1,1;2,1,1)$ cannot be interpreted as an irreducible representation of some $\Delta(6n^2)$.

5 Conclusions

In this work we tried to shine some light onto the hitherto not very well known series (C) and (D) of finite subgroups of SU(3). We were able to show that

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3$$

and we gave a method for the determination of m and p from the parameters n, a, b. We could also give a simple example for a (C)-group which is neither of the form $\Delta(3n^2)$ nor of the form T_n, thus showing that (C) contains some hitherto unclassified subgroups of SU(3).
For the SU(3)-subgroups of type (D) we could determine the structure

$$D(n,a,b;d,r,s) \cong (\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes S_3,$$

where p and q are functions of n, a, b, d, r, s. Since every (D)-group is a subgroup of some $\Delta(6m^2) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3$ it is tempting to assume that the (D)-groups can be interpreted as irreducible representations of $\Delta(6m^2)$. However, by giving a counterexample, we could show that this is not the case in general.

We hope that the analysis given here can lead us a small step closer towards the goal of the classification of all finite subgroups of SU(3). Furthermore we hope that some of the "new" types of finite subgroups of SU(3) discovered in this work can be useful for application in particle physics.

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A Proof of theorem 2.1

2.1 Theorem. Every finite Abelian subgroup $\mathcal{G}$ of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, where

$$m = \max_{a\in\mathcal{G}} \operatorname{ord}(a) \tag{41}$$

and n is a divisor of m.

Proof. Since $\mathcal{G}$ is an Abelian group of 3×3-matrices, we can choose a basis in which all group elements are diagonal. Then, due to $\det(a) = 1\ \forall a \in \mathcal{G}$, all elements of $\mathcal{G}$ are of the form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha^*\beta^* \end{pmatrix}, \quad \alpha, \beta \in \mathrm{U}(1). \tag{42}$$

Let

$$m := \max_{a\in\mathcal{G}} \operatorname{ord}(a). \tag{43}$$

Then $a^m = \mathbb{1}_3\ \forall a \in \mathcal{G}$, which we will prove by contradiction. Suppose $\exists a \in \mathcal{G} : \ a^m \neq \mathbb{1}_3$. $\Rightarrow \operatorname{ord}(a)$ does not divide m. Let

$$g := \gcd(\operatorname{ord}(a), m) < \operatorname{ord}(a) \tag{44}$$

denote the greatest common divisor of $\operatorname{ord}(a)$ and m. Then the group $\langle\langle a^g \rangle\rangle$ is a nontrivial cyclic group. Let now b be an element of $\mathcal{G}$ of order m. Since $\operatorname{ord}(a^g)$ and m have no common divisor larger than 1 we find:

$$\langle\langle a^g \rangle\rangle \cap \langle\langle b \rangle\rangle = \{\mathbb{1}_3\}. \tag{45}$$

$$\Rightarrow \operatorname{ord}(\underbrace{a^g b}_{\in\mathcal{G}}) = \operatorname{ord}(a^g)\operatorname{ord}(b) = \underbrace{\frac{\operatorname{ord}(a)}{\gcd(\operatorname{ord}(a), m)}}_{>1} \times m > m \Rightarrow \text{contradiction to (43).}$$

Defining

$$\mu := \exp(2\pi i/m), \tag{46}$$

every element of $\mathcal{G}$ has the form

$$\begin{pmatrix} \mu^c & 0 & 0 \\ 0 & \mu^d & 0 \\ 0 & 0 & \mu^{-c-d} \end{pmatrix}, \quad c, d \in \{0, ..., m-1\}. \tag{47}$$

Thus $\mathcal{G}$ is a subgroup of

$$\left\langle\left\langle\begin{pmatrix}\mu & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mu^*\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu^*\end{pmatrix}\right\rangle\right\rangle \cong \mathbb{Z}_m \times \mathbb{Z}_m.$$

Let

$$m = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j} \tag{48}$$

be the prime factorization of m ($p_1, \ldots, p_j$ are the distinct prime factors of m). Then every Abelian group of order m is the direct product of j Abelian groups A_i of order $p_i^{k_i}$ (see e.g. theorem 3.3.1 of [20]). Since every subgroup of a cyclic group is cyclic (see e.g. theorem 3.1.1 of [20]), in the case of $\mathbb{Z}_m$ all A_i are cyclic and we find

$$\mathbb{Z}_m \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \cdots \times \mathbb{Z}_{p_j^{k_j}}, \tag{49}$$

and thus

$$\mathbb{Z}_m \times \mathbb{Z}_m \cong (\mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_1^{k_1}}) \times \cdots \times (\mathbb{Z}_{p_j^{k_j}} \times \mathbb{Z}_{p_j^{k_j}}). \tag{50}$$

Next we use the fact that if p is prime, every subgroup of $\mathbb{Z}_{p^{e_1}} \times \cdots \times \mathbb{Z}_{p^{e_s}}$ is of the form $\mathbb{Z}_{p^{f_1}} \times \cdots \times \mathbb{Z}_{p^{f_s}}$ with $0 \leq f_i \leq e_i$ (for a proof of this statement see e.g. theorem 3.3.3 of [20]). Consequently, every subgroup $\mathcal{G}$ of $\mathbb{Z}_m \times \mathbb{Z}_m$ has the form

$$\mathcal{G} \cong (\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_1^{n_1}}) \times \cdots \times (\mathbb{Z}_{p_j^{r_j}} \times \mathbb{Z}_{p_j^{n_j}}) \cong \mathbb{Z}_r \times \mathbb{Z}_n, \tag{51}$$

where $0 \leq n_i \leq r_i \leq k_i$ and $r := \prod_j p_j^{r_j}, n := \prod_j p_j^{n_j}$. Without loss of generality we have assumed $n_i \leq r_i$, which implies $1 \leq n \leq r \leq m$.

$\mathbb{Z}_m$ is a subgroup of $\mathcal{G}$, because by definition (43) there exists at least one element of order m in $\mathcal{G}$. Therefore—from equation (51)—we conclude $r = m$. By construction n is a divisor of r, which completes the proof. □

B The group $\mathcal{A}$ of diagonal matrices in (D)

In this appendix we want to construct a generating set of the invariant subgroup $\mathcal{A}$ of all diagonal matrices in (D). The generators of $\mathcal{A}$ are A, B and F as well as the actions of E and G on A, B and F (see equation (30)).

$$\begin{aligned}\mathcal{A} = \langle\langle & A, B, F,\\ & G^{-1}AG, E^{-1}AE, (EG)^{-1}AEG, E^{-2}AE^2, (E^2G)^{-1}AE^2G,\\ & G^{-1}BG, E^{-1}BE, (EG)^{-1}BEG, E^{-2}BE^2, (E^2G)^{-1}BE^2G,\\ & G^{-1}FG, E^{-1}FE, (EG)^{-1}FEG, E^{-2}FE^2, (E^2G)^{-1}FE^2G\rangle\rangle.\end{aligned} \tag{52}$$

For any diagonal phase matrix D of determinant 1 (and thus for any element of $\mathcal{A}$) we have

$$\begin{aligned}& E^{-2}DE^2 = D^{-1}(E^{-1}DE)^{-1} \quad \text{and}\\ & (E^2G)^{-1}D(E^2G) = (G^{-1}DG)^{-1}[(EG)^{-1}D(EG)]^{-1},\end{aligned} \tag{53}$$

therefore generators of the form (53) with $D = A,\ B,\ F$ are redundant. Using furthermore

$$G^{-1}DG = D \quad \text{and} \quad (EG)^{-1}D(EG) = D^{-1}(E^{-1}DE)^{-1} \tag{54}$$

for $D = A,\ B$, we end up with

$$\begin{aligned}\mathcal{A} = \langle\langle A,\ & E^{-1}AE,\\ & B,\ E^{-1}BE,\\ & F,\ E^{-1}FE,\ G^{-1}FG,\ (EG)^{-1}FEG\rangle\rangle.\end{aligned} \tag{55}$$

C Proof of proposition 4.1

4.1 Proposition. Let $\mathcal{D}$ be a three-dimensional irreducible representation of $\Delta(6n^2)$, then

$$\exists m \in \{1, ..., n\}: \ \mathcal{D}(\Delta(6n^2)) \cong \Delta(6m^2). \tag{56}$$

Proof. Following [11] equation (57) comprises a presentation of $\Delta(6n^2)$ in terms of four generators:

$$\begin{array}{ll} P^3 = Q^2 = (PQ)^2 = \mathbb{1} & S_3 - \text{presentation} \\ R^n = S^n = \mathbb{1},\ \ RS = SR & \mathbb{Z}_n \times \mathbb{Z}_n - \text{presentation} \\ PRP^{-1} = R^{-1}S^{-1},\ \ PSP^{-1} = R & \text{action of } S_3 \\ QRQ^{-1} = S^{-1},\ \ QSQ^{-1} = R^{-1} & \text{on } \mathbb{Z}_n \times \mathbb{Z}_n \end{array} \tag{57}$$

There are only two types of three-dimensional irreducible representations of $\Delta(6n^2)$, namely [11]

$$\underline{\mathbf{3}}_{1(l)}: P \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}$$

$$\underline{\mathbf{3}}_{2(l)}: P \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}.$$

$\eta := \exp(2\pi i/n)$, $n \in \mathbb{N}\backslash\{0, 1\}$, $l \in \{1, ..., n-1\}$.

The matrix groups defined by the irreducible representations given above fulfill the presentation (57) with n replaced by

$$m := \operatorname{ord}(\eta^l), \tag{58}$$

and thus

$$\mathcal{D}(\Delta(6n^2)) \cong \Delta(6m^2) \tag{59}$$

for all three-dimensional irreducible representations of $\Delta(6n^2)$. $\square$

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Chapter 7

Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses

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Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses

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Abstract

It is well-known that, in the basis where the charged-lepton mass matrix is diagonal, there are seven cases of two texture zeros in Majorana neutrino mass matrices that are compatible with all experimental data. We show that two of these cases, namely B_3 and B_4 in the classification of Frampton, Glashow and Marfatia, are special in the sense that they automatically lead to near-maximal atmospheric neutrino mixing in the limit of a quasi-degenerate neutrino mass spectrum. This property holds true irrespective of the values of the solar and reactor mixing angles because, for these two cases, in the limit of a quasi-degenerate spectrum, the second and third row of the lepton mixing matrix are, up to signs, approximately complex-conjugate to each other. Moreover, in the same limit the aforementioned cases also develop a maximal CP-violating CKM-type phase, provided the reactor mixing angle is not too small.

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1 Introduction

It is by now well-established that at least two of the neutrino masses m_j ($j = 1, 2, 3$) are non-zero. The same applies to the angles in the lepton mixing matrix V. Its parameterization is usually chosen in analogy to the CKM matrix as [1]

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{1}$$

with $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$, the θ_{ij} being angles of the first quadrant. While the angles θ_{12} and θ_{23} are approximately 34° and 45°, respectively, the angle θ_{13} is compatible with zero [2, 3]. All data on lepton mixing are compatible with the tri-bimaximal matrix

$$V_{\mathrm{HPS}} \equiv \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \tag{2}$$

which has been put forward by Harrison, Perkins and Scott (HPS) [4] already in 2002.

Equation (2) has lead to the speculation that there is a *non-abelian* family symmetry behind the scenes,[1] enforcing $s_{23}^2 = 1/2$ in particular. This speculation is in accord with the finding of [6] that the only extremal angle which can be obtained by an *abelian* symmetry is $\theta_{13} = 0°$, *i.e.*, $\theta_{23} = 45°$ cannot be enforced by an abelian symmetry. A favourite non-abelian group in this context is A_4 [7]. For recent developments and other favourite groups see the reviews in [8] and references therein, for attempts on systematic studies see [9, 10, 11] (the latter paper refers to abelian symmetries).

However, there is an alternative to non-abelian groups. It is not necessary that, for instance, $\theta_{23} = 45°$ is *exactly* realized at some energy scale. It suffices that such a relation is fulfilled with reasonable accuracy. This could happen without need for a non-abelian symmetry. In order to pin down what we mean specifically we consider the structure of the mixing matrix V. It has two contributions, the unitary matrices U_ℓ and U_ν, stemming from the diagonalization of the charged-lepton mass matrix M_ℓ and of the neutrino mass matrix $\mathcal{M}_\nu$, respectively. Then the matrix

$$U \equiv U_\ell^\dagger U_\nu = e^{i\hat{\alpha}}\, V\, e^{i\hat{\sigma}} \tag{3}$$

occurs in the charged-current interaction and the lepton mixing matrix V defined above is obtained by removing the diagonal unitary matrices

$$e^{i\hat{\alpha}} = \mathrm{diag}\left(e^{i\alpha_1},\, e^{i\alpha_2},\, e^{i\alpha_3}\right) \quad \text{and} \quad e^{i\hat{\sigma}} = \mathrm{diag}\left(e^{i\sigma_1},\, e^{i\sigma_2},\, e^{i\sigma_3}\right) \tag{4}$$

from U. Without loss of generality we will use the convention $e^{i\sigma_3} = 1$ in the following. Suppose that we have a model in which U_ℓ and U_ν are functions of the charged-lepton and neutrino mass ratios, respectively, and that these mass ratios also parameterize the

[1] However, recently, it has been argued that tri-bimaximal mixing might nevertheless have an accidental origin [5].

deviations of U_ℓ and U_ν from a diagonal form. In U_ℓ these ratios are m_e/m_μ, m_e/m_τ and m_μ/m_τ. Since the mass hierarchy in the charged-lepton sector is rather strong, U_ℓ is approximately a diagonal matrix of phase factors, with the possible exception of the occurrence of m_μ/m_τ; if this ratio appears in a square root in analogy to the famous formula $\sin\theta_c \simeq \sqrt{m_d/m_s}$ for the Cabibbo angle [12], with quark masses m_d and m_s, then $\sqrt{m_\mu/m_\tau} \simeq 0.24$ is even larger than $\sin\theta_c$. The simplest way to avoid such a deviation of U_ℓ from a diagonal matrix is to have a model which, through its symmetries, enforces a diagonal M_ℓ. Switching to U_ν, we point out that up to now the type of neutrino mass spectrum is completely unknown [1]. A particularly exciting possibility would be a quasi-degenerate mass spectrum in which case the neutrino mass ratios could be close to one such that effectively U_ν is independent of the masses and could look like a matrix of pure numbers, potentially disturbed by phase factors. Thus, with U_ℓ sufficiently close to a diagonal matrix and a quasi-degenerate neutrino mass spectrum it might be possible to simulate a mixing matrix V consisting of "pure numbers," leading for instance to an atmospheric neutrino mixing angle θ_{23} which is in practice indistinguishable from 45°.

The advantage is that such a scenario could be achieved with texture zeros and that texture zeros in mass matrices may *always* be explained by abelian symmetries [13], at the expense of an extended scalar sector in renormalizable models.[2]

Let us summarize the assumptions of this paper:

- U_ℓ is sufficiently close to a diagonal matrix such that in good approximation it does not contribute to V.
- The neutrino mass spectrum is quasi-degenerate.
- The symmetry groups we have in mind are abelian, *i.e.*, we deal with texture zeros.

In the following we will show that these assumptions can indeed lead to a realization of maximal atmospheric neutrino mixing, in the framework which consists of Majorana neutrino mass matrices with two texture zeros and a diagonal mass matrix M_ℓ; two of the viable cases of neutrino mass matrices classified in [15] exhibit precisely the desired features.

In section 2 we review the viable textures presented in [15] and point out models in which they can be realized. Then, in section 3, we discuss the phenomenology of the cases B_3 and B_4 of [15] in the light of the philosophy specified above. The remaining cases are discussed in section 4. We summarize our findings in section 5.

2 The framework

Assuming the neutrinos to be Majorana particles, the neutrino mass term is given by

$$\mathcal{L}_{\nu\,\mathrm{mass}} = \frac{1}{2}\,\nu_\mathrm{L}^\mathrm{T} C^{-1} \mathcal{M}_\nu \nu_\mathrm{L} + \mathrm{H.c.}, \tag{5}$$

[2] We emphasize that our approach is different from that of [11] where the Froggatt–Nielsen mechanism [14] is used and, therefore, order-of-magnitude relations among the elements of mass matrices are assumed.

case	texture zeros
A_1	$(\mathcal{M}_\nu)_{ee} = (\mathcal{M}_\nu)_{e\mu} = 0$
A_2	$(\mathcal{M}_\nu)_{ee} = (\mathcal{M}_\nu)_{e\tau} = 0$
B_1	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{e\tau} = 0$
B_2	$(\mathcal{M}_\nu)_{\tau\tau} = (\mathcal{M}_\nu)_{e\mu} = 0$
B_3	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{e\mu} = 0$
B_4	$(\mathcal{M}_\nu)_{\tau\tau} = (\mathcal{M}_\nu)_{e\tau} = 0$
C	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{\tau\tau} = 0$

Table 1: The viable cases in the framework of two zeros in the Majorana neutrino mass matrix $\mathcal{M}_\nu$ and a diagonal charged-lepton mass matrix M_ℓ [15].

with a symmetric mass matrix $\mathcal{M}_\nu$. In the basis where the charged-lepton mass matrix is diagonal, there are seven possibilities for an $\mathcal{M}_\nu$ with two texture zeros which are compatible with all available neutrino data, as was shown in [15]. These seven viable cases are listed in table 1. The phenomenology of those seven mass matrices has been discussed in [15, 16, 17]. Moreover, case C has also been investigated in [18].

There are several ways to construct models where the cases of table 1 together with a diagonal charged-lepton mass matrix are realized by symmetries. Five of the seven mass matrices, but not B_1 and B_2, have various embeddings in the seesaw mechanism [19], by placing zeros in the Majorana mass matrix M_R of the right-handed neutrino singlets ν_R and in the Dirac mass matrix M_D connecting the ν_R with the ν_L [20]. With the methods described in [13], one can then construct models where the zeros in the various mass matrices, including the six off-diagonal zeros in M_ℓ, are enforced by abelian symmetries.

Four of the seven textures of table 1, namely A_1, A_2, B_3, B_4, have a realization in the seesaw mechanism with a *diagonal* M_D [20, 21]: by suitably placing two texture zeros in M_R or, equivalently, in $\mathcal{M}_\nu^{-1}$, these four textures can be obtained.[3] Actually, now we are dealing with 14 texture zeros, namely six in M_ℓ and M_D each and two in M_R. In order to construct models for these four cases, one Higgs doublet is sufficient, but one needs two scalar gauge singlets in order to implement the desired form of M_R [21].

All of the seven cases of table 1 can be realized as models in scalar-triplet extensions of the Standard Model [18], *i.e.*, in the type II seesaw mechanism [22] without any right-handed neutrino singlets.

[3]There are three more viable cases of texture zeros in $\mathcal{M}_\nu^{-1}$ which do not correspond to texture zeros in $\mathcal{M}_\nu$ [21].

3 Cases B_3 and B_4

In this section we discuss the cases B_3 and B_4 which correspond to the Majorana mass matrices

$$B_3: \quad \mathcal{M}_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}, \qquad B_4: \quad \mathcal{M}_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{pmatrix}. \tag{6}$$

The symbol $\times$ denotes non-zero matrix elements. The equations which follow from these cases have the form

$$\sum_{j=1}^{3} V_{\alpha j} V_{\alpha j} \mu_j = \sum_{j=1}^{3} V_{\alpha j} V_{\beta j} \mu_j = 0 \quad \text{with} \quad \mu_j \equiv m_j e^{2i\sigma_j} \tag{7}$$

and $\alpha \neq \beta$, where B_3 is given by $(\alpha, \beta) = (\mu, e)$ and B_4 by $(\alpha, \beta) = (\tau, e)$.

Equation (7) can be considered from a linear-algebra perspective. Defining line vectors

$$z_\alpha = (V_{\alpha j}), \quad z_\beta = (V_{\beta j}) \tag{8}$$

of V, equation (7) tells us that, because of the unitarity of V, the line vector

$$(V^*_{\alpha 1}\, \mu^*_1,\, V^*_{\alpha 2}\, \mu^*_2,\, V^*_{\alpha 3}\, \mu^*_3) \tag{9}$$

is orthogonal to both, z_α and z_β. Therefore, this vector must be proportional to the line vector z_γ with $\gamma \neq \alpha, \beta$, and we obtain

$$z_\gamma = (V_{\gamma j}) = \frac{e^{i\phi}}{N_\alpha} \left(V^*_{\alpha j}\, \mu^*_j \right) \quad \text{with} \quad N^2_\alpha = \sum_{k=1}^{3} |V_{\alpha k}|^2\, m^2_k. \tag{10}$$

We use equation (10) for the discussion of the physical consequences of cases B_3 and B_4. We begin with case B_3 where $\gamma = \tau$. Defining $\epsilon = s_{13} e^{i\delta}$, $t_{12} = s_{12}/c_{12}$ and $t_{23} = s_{23}/c_{23}$, from equations (1) and (10) we find the following relations:

$$B_3: \quad \frac{\mu_1}{m_3} = \frac{V_{\mu 3} V^*_{\tau 1}}{V_{\mu 1} V^*_{\tau 3}} = -\frac{t_{12} t_{23} - \epsilon^*}{t_{12} + t_{23}\epsilon}\, t_{23}, \quad \frac{\mu_2}{m_3} = \frac{V_{\mu 3} V^*_{\tau 2}}{V_{\mu 2} V^*_{\tau 3}} = -\frac{t_{23} + t_{12}\epsilon^*}{1 - t_{12} t_{23}\epsilon}\, t_{23}. \tag{11}$$

Alternatively, one can use the procedure of [16] to arrive at these expressions.

The analysis of equation (11) proceeds in the following way. We define

$$\rho_j = \left(\frac{m_j}{m_3} \right)^2 \quad (j = 1, 2), \tag{12}$$

take the absolute values of the two expressions in equation (11) and eliminate

$$\zeta \equiv 2 t_{12} t_{23} s_{13} \cos\delta = \frac{t^2_{12} t^2_{23} + s^2_{13} - \rho_1 (t^2_{12} + t^2_{23} s^2_{13})/t^2_{23}}{1 + \rho_1/t^2_{23}}. \tag{13}$$

Then we end up with a cubic equation for t^2_{23}:

$$t^6_{23} + t^4_{23} \left[s^2_{13} + c^2_{13} \left(c^2_{12}\rho_1 + s^2_{12}\rho_2 \right) \right] - t^2_{23} \left[s^2_{13}\rho_1\rho_2 + c^2_{13} \left(s^2_{12}\rho_1 + c^2_{12}\rho_2 \right) \right] - \rho_1\rho_2 = 0. \tag{14}$$

Inspection of this equation shows that it has a unique positive root. Thus, given the neutrino masses m_1, m_2, m_3 and the mixing angles θ_{12} and θ_{13}, equation (14) determines θ_{23}. Using equations (11) and (13), we adopt the following philosophy:

$$\text{input: } m_{1,2,3},\, \theta_{12},\, \theta_{13} \quad \Rightarrow \quad \text{predictions: } \theta_{23},\, \delta,\, 2\sigma_{1,2}. \tag{15}$$

Since the Majorana phases $2\sigma_{1,2}$ are not directly accessible to experimental scrutiny, we will later consider instead the observable $m_{\beta\beta}$, the effective mass in neutrinoless double-beta decay.

An approximate solution of equation (14) for a quasi-degenerate neutrino mass spectrum is given by

$$t_{23}^2 \simeq 1 - \frac{1}{2}\frac{\Delta m_{31}^2}{m_1^2}\left(1 + s_{13}^2\right) \quad \text{or} \quad s_{23}^2 \simeq \frac{1}{2} - \frac{1}{8}\frac{\Delta m_{31}^2}{m_1^2}\left(1 + s_{13}^2\right), \tag{16}$$

where corrections of order $(\Delta m_{31}^2/m_1^2)^2$ and $\Delta m_{21}^2/m_1^2$ have been neglected.[4] The latter term is small because we know from experiment that $\Delta m_{21}^2/|\Delta m_{31}^2| \sim 1/30$ [2, 3]. We observe that the leading correction to t_{23}^2 is independent of s_{12}^2.

Case B_4 is treated analogously. We obtain

$$\mathrm{B}_4: \quad \frac{\mu_1}{m_3} = \frac{V_{\mu 1}^* V_{\tau 3}}{V_{\mu 3}^* V_{\tau 1}} = -\frac{t_{12} + t_{23}\epsilon^*}{t_{12}t_{23} - \epsilon}\,\frac{1}{t_{23}}, \quad \frac{\mu_2}{m_3} = \frac{V_{\mu 2}^* V_{\tau 3}}{V_{\mu 3}^* V_{\tau 2}} = -\frac{1 - t_{12}t_{23}\epsilon^*}{t_{23} + t_{12}\epsilon}\,\frac{1}{t_{23}}. \tag{17}$$

Comparison with equation (11) shows that in the present case the cubic equation for t_{23}^2 is obtained from equation (14) by the replacement $\rho_1 \to 1/\rho_1$, $\rho_2 \to 1/\rho_2$. It is then easy to show that the atmospheric mixing angles in the cases B_3 and B_4 are related by

$$t_{23}^2\Big|_{\mathrm{B}_4} = \left(t_{23}^2\Big|_{\mathrm{B}_3}\right)^{-1} \quad \text{or} \quad s_{23}^2\Big|_{\mathrm{B}_4} = 1 - s_{23}^2\Big|_{\mathrm{B}_3}. \tag{18}$$

Accordingly, the curves for B_4 in figure 1 are obtained from those of B_3 by reflection at the dashed line.

In figures 1 and 2 we have plotted s_{23}^2 and $\cos\delta$ versus m_1, respectively, for cases B_3 and B_4. For definiteness, for the solar and reactor mixing angles and the mass-squared differences we have used the best-fit values listed in [3]: $s_{12}^2 = 0.316$, $\Delta m_{21}^2 = 7.64 \times 10^{-5}\,\mathrm{eV}^2$, which are the same values for both normal and inverted spectrum, and $s_{13}^2 = 0.017$, $\Delta m_{31}^2 = 2.45 \times 10^{-3}\,\mathrm{eV}^2$ for the normal and $s_{13}^2 = 0.020$, $\Delta m_{31}^2 = -2.34 \times 10^{-3}\,\mathrm{eV}^2$ for the inverted spectrum. The two figures illustrate nicely that in all four instances (cases B_3 and B_4 and both spectra) in the limit $m_1 \to \infty$ we find $s_{23}^2 \to 1/2$ and $\cos\delta \to 0$.

Some remarks are at order. First of all, by a numerical comparison it turns out that the approximate formula (16) works quite well. The deviation from the exact value of s_{23}^2 is less than 3% at $m_1 = 0.08\,\mathrm{eV}$ and the approximation rapidly improves at larger m_1. Secondly, from equations (11) and (17) we read off that cases B_3 and B_4 do not allow $s_{13} = 0$ because this would lead to $\mu_1 = \mu_2$. However, this observation does not give a strong restriction on s_{13}, as we find numerically. Thirdly, the lower bound on

[4]Note that $\Delta m_{21}^2 = m_2^2 - m_1^2 > 0$ whereas $\Delta m_{31}^2 = m_3^2 - m_1^2$ can have either sign: $\Delta m_{31}^2 > 0$ indicates the normal order of the neutrino mass spectrum and $\Delta m_{31}^2 < 0$ the inverted order.

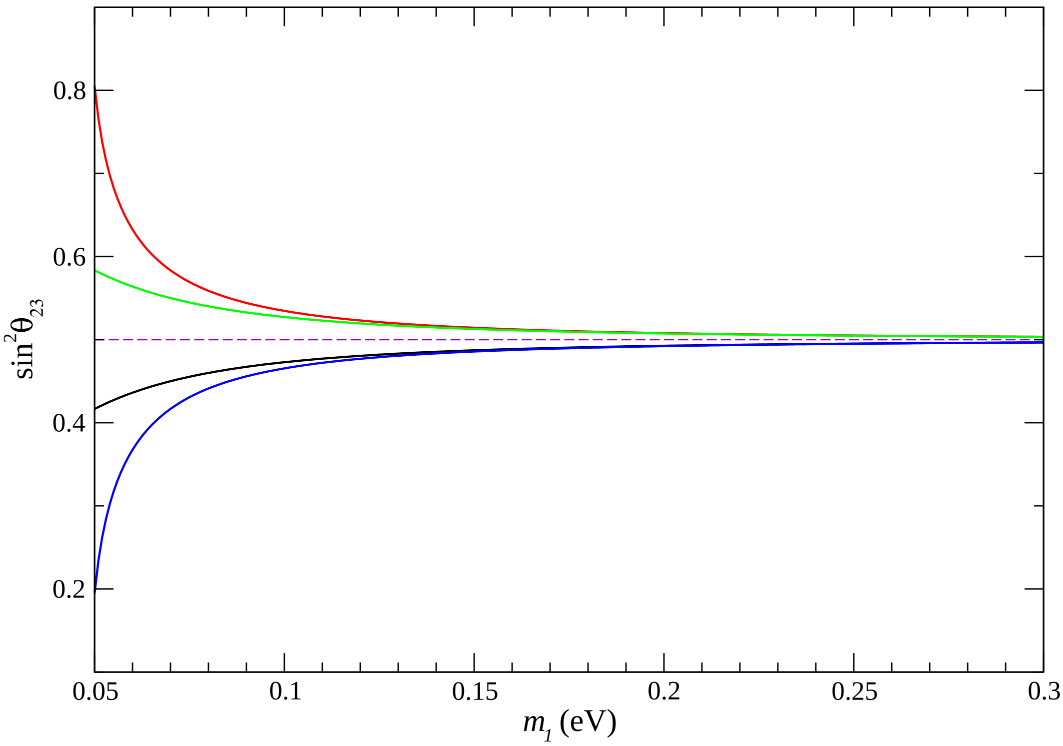

Figure 1: s_{23}^2 as a function of m_1. In descending order the full curves refer to case B_3 (inverted spectrum), case B_4 (normal spectrum), case B_3 (normal spectrum), and case B_4 (inverted spectrum). The dashed line indicates the value 0.5, *i.e.*, maximal atmospheric mixing. In this plot, for s_{12}^2, s_{13}^2, Δm_{21}^2 and Δm_{31}^2 the best-fit values of [3] have been used.

s_{13} is correlated with a lower bound on m_1. The reason is that, in our treatment of cases B_3 and B_4, $\cos\delta$ is computed via equation (13) after the determination of s_{23}^2 by equation (14); then the condition $|\cos\delta| \leq 1$ leads to the lower bound on m_1. For the inverted spectrum we obtain numerically that the lower bound $m_1 \gtrsim 0.05\,\mathrm{eV}$ is rather stable for $s_{13}^2 \gtrsim 0.0001$. The normal spectrum allows smaller values of m_1, for instance, $m_1 \gtrsim 0.03\,\mathrm{eV}$ at $s_{13}^2 \simeq 0.0001$ and $m_1 \gtrsim 0.01\,\mathrm{eV}$ at $s_{13}^2 \simeq 0.01$. Therefore, cases B_3 and B_4 do *not* automatically entail a quasi-degenerate spectrum which would require something like $m_1 > 0.1\,\mathrm{eV}$. In accord with the philosophy of this paper we really have to postulate such a spectrum and only for quasi-degeneracy we obtain an atmospheric mixing angle sufficiently close to 45°.

Computing an approximation for $\cos\delta$ is a bit laborious. It turns out that, due to the smallness of s_{13}^2, it is necessary to expand $\cos\delta$ to second order in both

$$\Delta_1 = \frac{\Delta m_{31}^2}{m_1^2} \quad \text{and} \quad \Delta_2 = \frac{\Delta m_{21}^2}{m_1^2}, \tag{19}$$

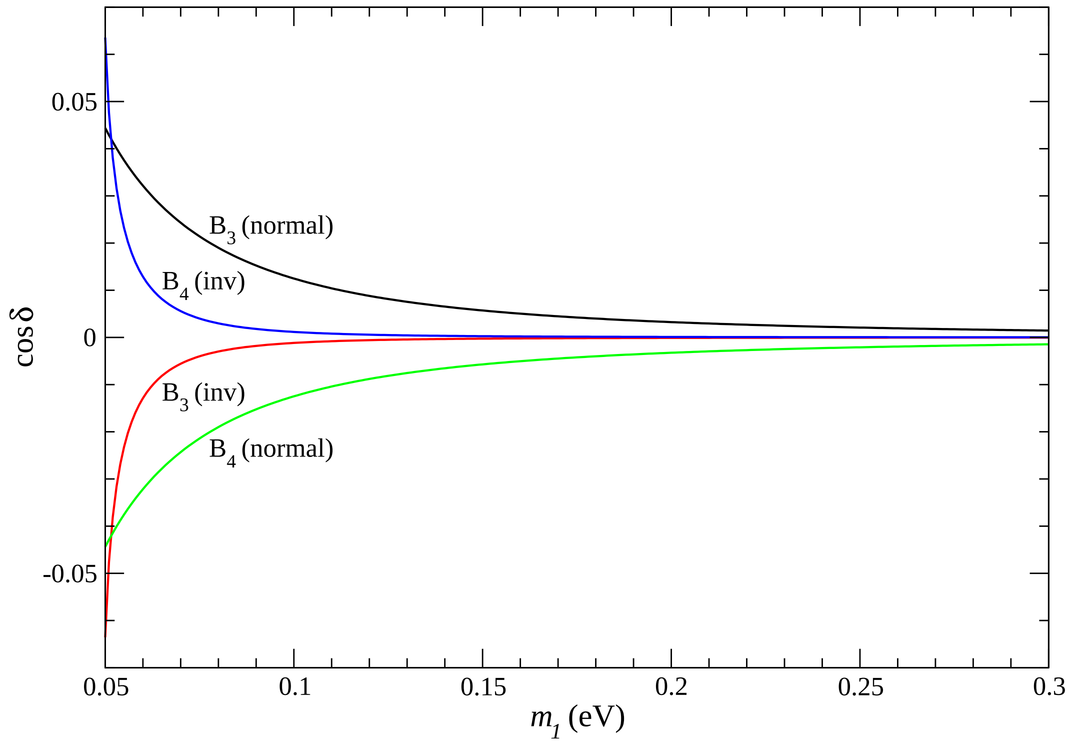

Figure 2: $\cos\delta$ as a function of m_1. For further details see the legend of figure 1.

in order to obtain a reasonable accuracy. The result is[5]

$$\cos\delta \simeq \mp \frac{s_{13}t_{12}}{4} \left\{ \left(1 - \frac{1}{t_{12}^2}\right)\left(\Delta_1 - \frac{1}{2}\Delta_1^2\right) + \frac{s_{12}^2 c_{13}^2 - 1}{s_{13}^2}\left(\Delta_2 - \frac{1}{2}\Delta_2^2\right)\right\}, \tag{20}$$

where the minus and plus signs correspond to B_3 and B_4, respectively. At $m_1 = 0.16\,\text{eV}$ the approximation (20) deviates from the exact value by less than 1% (5%) assuming a normal (inverted) spectrum. Actually, the sign difference in equation (20) between cases B_3 and B_4 holds to all orders; with equations (13) and (18) it is easy to show that

$$\cos\delta|_{B_4} = -\cos\delta|_{B_3}\,, \tag{21}$$

in perfect agreement with the numerical computation.

The general formula for the effective mass in neutrinoless double-beta decay (for reviews see for instance [23]) is given by the formula

$$m_{\beta\beta} = |(\mathcal{M}_\nu)_{ee}| = m_3 \left| c_{13}^2 \left(c_{12}^2 \frac{\mu_1}{m_3} + s_{12}^2 \frac{\mu_2}{m_3}\right) + (\epsilon^*)^2 \right|. \tag{22}$$

With μ_1 and μ_2 from equations (11) and (17) we specify it to cases B_3 and B_4, respectively. Inspection of the same equations reveals a simple procedure to switch from B_3 to B_4:

$$\left.\frac{\mu_j}{m_3}\right|_{B_4} = \left(\left.\frac{m_3}{\mu_j}\right|_{B_3} \right)^* \quad (j = 1, 2). \tag{23}$$

[5]Note that the coefficient of $\Delta_1\Delta_2$ is zero.

In μ_1 and μ_2 we need to insert the numerical values obtained for t_{23} and δ. Equation (13) determines $\cos\delta$, therefore, $\sin\delta$ is only determined up to a sign. However, since $\sin\delta \leftrightarrow -\sin\delta$ corresponds to $\epsilon \leftrightarrow \epsilon^*$, this sign has no effect on $m_{\beta\beta}$ because this observable is computed by an absolute value.

The effective mass $m_{\beta\beta}$ applied to the cases B_3 and B_4 has the property that, if we do not care that θ_{23} and $\cos\delta$ are actually determined by equations (14) and (13) and simply plug in $\theta_{23} = 45°$ and $\cos\delta = 0$, we obtain the equality $m_{\beta\beta} = m_3$. Since for a quasi-degenerate neutrino mass spectrum $m_1 \simeq m_3$ holds, this demonstrates that we should expect

$$m_{\beta\beta} \simeq m_1 \tag{24}$$

in the limit of quasi-degeneracy. Numerically it turns out that the deviation of $m_{\beta\beta}/m_1$ from one is very small—even at $m_1 = 0.05\,\mathrm{eV}$, for the inverted spectrum, the ratio $m_{\beta\beta}/m_1$ deviates from one by only -3.2%, at $m_1 = 0.1\,\mathrm{eV}$ the deviation is -1.5 per mill. For the normal spectrum this ratio is even closer to one. This renders a plot $m_{\beta\beta}$ vs. m_1 superfluous. The smallness of $m_{\beta\beta}/m_1 - 1$ is partially explained by the smallness of s_{13}^2 which brings the phases of both μ_1 and μ_2 close to π [16]. One can check that choosing a large (and thus unphysical) s_{13}^2 there is indeed a substantial deviation of $m_{\beta\beta}/m_1$ from one at the lower end of our range of m_1.

4 The remaining cases

Here we will show that the remaining five cases of two texture zeros in $\mathcal{M}_\nu$ are either such that the assumption of a quasi-degenerate spectrum is incompatible with the data or that they do not conform to the philosophy put forward in this paper.

Cases A_1 and A_2 are incompatible with quasi-degenerate neutrino masses, as was noticed in [15]. This can be seen in the following way. From $(\mathcal{M}_\nu)_{ee} = 0$, assuming a quasi-degenerate spectrum and using equation (1) we readily find

$$s_{13}^2 \gtrsim c_{13}^2 \left(c_{12}^2 - s_{12}^2 \right) = c_{13}^2 \cos(2\theta_{12}), \tag{25}$$

in contradiction to our experimental knowledge on θ_{13} and θ_{12}.

Next we consider cases B_1 and B_2. Taking into account that one knows from experiment that s_{13}^2 is small, in first order in s_{13} for B_1 one obtains [15]

$$\frac{\mu_1}{m_3} \simeq -\left[t_{23}^2 + s_{13}\left(e^{-i\delta} t_{23} + e^{i\delta}/t_{23} \right)/t_{12} \right], \quad \frac{\mu_2}{m_3} \simeq -\left[t_{23}^2 - s_{13}\left(e^{-i\delta} t_{23} + e^{i\delta}/t_{23} \right)/t_{12} \right]. \tag{26}$$

Now we ask the question if the assumption of a quasi-degenerate mass spectrum compellingly leads to $t_{23} \simeq 1$. The answer is "no" because we could choose $s_{13}/(t_{23}t_{12}) \simeq 1$ in order to achieve quasi-degeneracy; even with the experimentally allowed values for s_{13} and t_{12} we would obtain a rather small $t_{23} \simeq s_{13}/t_{12}$, far from maximal atmospheric neutrino mixing. Case B_2 can be discussed analogously.

Case C is a bit more involved—for details see [18]. In the case of the inverted spectrum, maximal atmospheric neutrino mixing is not compelling. For the normal ordering of the spectrum, using the experimental knowledge on the mass-squared differences and the mixing angles θ_{12} and θ_{13} it follows that t_{23} is extremely close to one and that the spectrum is quasi-degenerate. However, if we do not use the experimental information on s_{13}^2, we could assume c_{13}^2 being small instead which would then admit t_{23} being smaller than one. This is in contrast to cases B_3 and B_4 where, for quasi-degeneracy, t_{23} is always close to one, independently of the values s_{12}^2 and s_{13}^2 assume.

5 Conclusions

In this paper we have considered the possibility that neutrinos differ from charged fermions not only in their Majorana nature but also in a quasi-degenerate mass spectrum, in stark contrast to the hierarchical mass spectra of the charged fermions. The appealing aspect of this assumption is that it is already under scrutiny by present experiments, and more experiments will join in the near future [1]. Such experiments search for neutrinoless double-beta decay, whose decay amplitude is proportional to the effective mass $m_{\beta\beta}$, and for a deviation in the shape of the endpoint spectrum of the β-decay of ^{3}H which is, in essence, sensitive to the average of the squares of the neutrino masses if the spectrum is quasi-degenerate. Moreover, that neutrino mass effects in cosmology have not yet been observed puts already a stringent although model-dependent bound on the sum of the masses.

However, the aspect on which we elaborated in this paper was the possibility to obtain near maximal atmospheric neutrino mixing from a quasi-degenerate neutrino mass spectrum. The idea is quite simple: if we have a model with symmetries enforcing a diagonal charged-lepton mass matrix and the atmospheric neutrino mixing angle being a function of the neutrino mass ratios, then in the limit of quasi-degeneracy this mixing angle will become independent of the masses. We have found two instances in the framework of two texture zeros in the Majorana neutrino mass matrix where in this limit atmospheric neutrino mixing becomes maximal, namely the cases B_3 and B_4 of [15]. We have shown that these two textures have the following properties if the neutrino mass spectrum is quasi-degenerate:

1. Using the mass-squared differences as input, the value of s_{23}^2 tends to $1/2$ irrespective of the values of s_{12}^2 and s_{13}^2; therefore, maximal atmospheric neutrino mixing has to be considered a true prediction of the textures B_3 and B_4 in conjunction with quasi-degeneracy.

2. If s_{13}^2 is not exceedingly small, then CP violation in lepton mixing becomes maximal too, *i.e.*, $\cos\delta$ tends to zero.

3. Exact vanishing of s_{13}^2 is forbidden because this would entail $\Delta m_{21}^2 = 0$, however, values as small as $s_{13}^2 = 0.0001$ are nevertheless possible.

The results for the cases B_3 and B_4 can be understood in the following way. With the usual phase convention (1) for the mixing matrix, in equation (10) we have $e^{i\phi} = 1$ and, therefore, $V_{\mu j} \simeq -V_{\tau j}^*$ for $j = 1, 2$ and $V_{\mu 3} \simeq V_{\tau 3}^*$ for a quasi-degenerate spectrum. The signs we obtained here are convention-dependent and have no physical significance. That the exact relation $V_{\mu j} = V_{\tau j}^*$ for $j = 1, 2, 3$ is a viable and predictive restriction of V was already pointed out in [24], later on in [25] a model was constructed where this relation is enforced by a generalized CP transformation and softly broken lepton numbers, and it was also shown that such a mixing matrix leads to $\theta_{23} = 45°$ and $s_{13} \cos\delta = 0$ at the tree level. While in [25] the symmetry structure is non-abelian and a type of μ–τ interchange symmetry (see [26] for some early references, and [27] for a recent paper and references therein), the textures B_3 and B_4 can be enforced by abelian symmetries and, provided the neutrino mass spectrum is quasi-degenerate, we have the *approximate* relations $\theta_{23} \simeq 45°$ and $\cos\delta \simeq 0$. Therefore, we have shown that maximal atmospheric neutrino mixing could have an origin completely different from μ–τ interchange symmetry, it could simply be a consequence of texture zeros and quasi-degeneracy of the neutrino mass spectrum.

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Chapter 8

The reactor mixing angle and CP violation with two texture zeros in the light of T2K

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The reactor mixing angle and CP violation with two texture zeros in the light of T2K

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Abstract

We reconsider the phenomenological implications of two texture zeros in symmetric neutrino mass matrices in the light of the recent T2K results for the reactor angle and the new global analysis which gives also best fit values for the Dirac CP phase δ. The most important results of the analysis are: Among the viable cases classified by Frampton et al. only A_1 and A_2 predict θ_{13} to be different from zero at 3σ. Furthermore these two cases are compatible only with a normal mass spectrum in the allowed region for the reactor angle. At the best fit value A_1 and A_2 predict $0.024 \geq \sin^2\theta_{13} \geq 0.012$ and $0.014 \leq \sin^2\theta_{13} \leq 0.032$, respectively, where the bounds on the right and the left correspond to $\cos\delta = -1$ and $\cos\delta = 1$, respectively. The cases B_1, B_2, B_3 and B_4 predict nearly maximal CP violation, i.e. $\cos\delta \approx 0$.

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1 Introduction

Recently the T2K Collaboration [1] gave hints for a nonzero reactor angle, and also the results of the MINOS Collaboration [2] point towards the same direction. The global fits

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of neutrino oscillation experiments give[1]

$$
\begin{array}{lr}
\left.\begin{array}{ll} 0.001 \leq \sin^2\theta_{13} \leq 0.035 & \text{(NS)}, \\ 0.001 \leq \sin^2\theta_{13} \leq 0.039 & \text{(IS)}, \end{array}\right. \quad [3] & \\
& (1) \\
0.005 \leq \sin^2\theta_{13} \leq 0.050, \qquad\qquad [4] &
\end{array}
$$

and the best fit values are $\sin^2\theta_{13} = 0.013$ and $\sin^2\theta_{13} = 0.025$, respectively. In particular the global analysis by Schwetz, Tortola and Valle [3] gives a weak hint for a nonvanishing CP violating phase, namely (at the best fit point)

$$\sin^2\theta_{13} = 0.013, \qquad \delta = -0.61\,\pi \quad \text{(NS)}, \tag{2}$$

$$\sin^2\theta_{13} = 0.016, \qquad \delta = -0.41\,\pi \quad \text{(IS)}. \tag{3}$$

A lot of papers have been proposed recently in order to reproduce such a large value of the reactor mixing angle [5]. Already before the recent T2K data there have been models based on discrete flavor symmetries which predict a large reactor mixing angle—for an incomplete list see Ref. [6], and for a classification of models with flavor symmetries classified by their predictions for the reactor angle see [7].

Here we reconsider the interesting case of Majorana neutrino mass matrices with two texture zeros in the basis where the charged lepton mass matrix is diagonal, which has been extensively studied in the past. Our aim is to point out the phenomenological implication of such textures in the light of the T2K results[2].

It was shown in [10, 11] and [8] that, in the basis where the charged lepton mass matrix is diagonal, there are seven types of two texture zeros in symmetric neutrino mass matrices compatible with the experimental data on neutrino oscillations. In this work we want to analyze the correlation between the CP violating phase δ and the reactor mixing angle θ_{13} in the framework of these two texture zeros.

Another interesting possibility is to place texture zeros in the inverted neutrino mass matrix–see e.g. [12]. The implications of this type of two texture zeros on the reactor mixing angle and CP violation have been studied in [13].

2 Two texture zeros

In the basis where the charged lepton mass matrix is diagonal, the lepton mixing matrix U and the Majorana neutrino mass matrix $\mathcal{M}_\nu$ are related via

$$\mathcal{M}_\nu = U^* \,\mathrm{diag}(m_1, m_2, m_3)\, U^\dagger. \tag{4}$$

The standard parameterization [14] for U is given by[3]

$$U = e^{i\hat{\alpha}} V e^{i\hat{\sigma}}, \tag{5}$$

[1] Throughout this work the abbreviations NS and IS will stand for normal and inverted neutrino mass spectrum, respectively.

[2] While we were finishing this work two papers treating the same problem were published in [8, 9].

[3] The parameterization used here is a re-writing of the symmetrical parameterization proposed in [15].

where

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{6}$$

$\hat{\alpha} = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3)$ and $\hat{\sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3)$. $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with $\theta_{ij} \in [0, \pi/2]$. $\delta \in [0, 2\pi)$ is the CP violating phase and the $\sigma_i \in [0, 2\pi)$ are the Majorana phases, which are not measurable in oscillation experiments. The phases α_i are irrelevant for neutrino oscillations and will play no role in our analysis, as we will see in the following. Inserting (5) into (4) we obtain

$$(\mathcal{M}_\nu)^*_{ij} = \sum_k m_k U_{ik} U_{jk} = \sum_k m_k e^{2i\sigma_k} V_{ik} V_{jk} e^{i(\alpha_i + \alpha_j)}. \tag{7}$$

Placing a texture zero in the neutrino mass matrix corresponds to the condition

$$(\mathcal{M}_\nu)_{ij} = 0 \quad (\Leftrightarrow (\mathcal{M}_\nu)^*_{ij} = 0) \tag{8}$$

for some indices (i, j). Defining $\mu_k := m_k e^{2i\sigma_k}$ and dividing by $e^{i(\alpha_i+\alpha_j)}$ we arrive at

$$\sum_k \mu_k V_{ik} V_{jk} = 0. \tag{9}$$

The assumption of two texture zeros can thus be described by the two equations

$$\sum_k \mu_k V_{ak} V_{bk} = 0, \quad \sum_k \mu_k V_{ck} V_{dk} = 0. \tag{10}$$

The viable cases of two texture zeros given in [10] and the corresponding parameters (a, b, c, d) can be found in table 1.

case	texture zeros	(a,b,c,d)
A_1	$(\mathcal{M}_\nu)_{ee} = (\mathcal{M}_\nu)_{e\mu} = 0$	(1,1,1,2)
A_2	$(\mathcal{M}_\nu)_{ee} = (\mathcal{M}_\nu)_{e\tau} = 0$	(1,1,1,3)
B_1	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{e\tau} = 0$	(2,2,1,3)
B_2	$(\mathcal{M}_\nu)_{\tau\tau} = (\mathcal{M}_\nu)_{e\mu} = 0$	(3,3,1,2)
B_3	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{e\mu} = 0$	(2,2,1,2)
B_4	$(\mathcal{M}_\nu)_{\tau\tau} = (\mathcal{M}_\nu)_{e\tau} = 0$	(3,3,1,3)
C	$(\mathcal{M}_\nu)_{\mu\mu} = (\mathcal{M}_\nu)_{\tau\tau} = 0$	(2,2,3,3)

Table 1: The viable cases in the framework of two texture zeros in the Majorana neutrino mass matrix $\mathcal{M}_\nu$ and a diagonal charged-lepton mass matrix $\mathcal{M}_\ell$ [10].

3 General remarks

The system (10) is equivalent to

$$\begin{pmatrix} V_{a1}V_{b1} & V_{a2}V_{b2} \\ V_{c1}V_{d1} & V_{c2}V_{d2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = -\mu_3 \begin{pmatrix} V_{a3}V_{b3} \\ V_{c3}V_{d3} \end{pmatrix}. \tag{11}$$

The set of solutions of this system of linear equations depends on the determinant

$$D_{abcd} := \det \begin{pmatrix} V_{a1}V_{b1} & V_{a2}V_{b2} \\ V_{c1}V_{d1} & V_{c2}V_{d2} \end{pmatrix} = V_{a1}V_{b1}V_{c2}V_{d2} - V_{a2}V_{b2}V_{c1}V_{d1}. \tag{12}$$

For $D_{abcd} \neq 0$ we find

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = -\frac{\mu_3}{D_{abcd}} \begin{pmatrix} V_{c2}V_{d2} & -V_{a2}V_{b2} \\ -V_{c1}V_{d1} & V_{a1}V_{b1} \end{pmatrix} \begin{pmatrix} V_{a3}V_{b3} \\ V_{c3}V_{d3} \end{pmatrix}. \tag{13}$$

Since at least two neutrino masses must be nonzero, the above equation implies that the lightest neutrino mass is different from zero.[4] Thus we are allowed to divide by μ_3 and we can easily calculate

$$r := \frac{\Delta m^2_{21}}{\Delta m^2_{31}} = \frac{\frac{m_2^2}{m_3^2} - \frac{m_1^2}{m_3^2}}{1 - \frac{m_1^2}{m_3^2}} = \frac{\left|\frac{\mu_2}{\mu_3}\right|^2 - \left|\frac{\mu_1}{\mu_3}\right|^2}{1 - \left|\frac{\mu_1}{\mu_3}\right|^2}. \tag{14}$$

Inserting (13) into (14) we find an equation which relates the six quantities

$$\Delta m^2_{21},\ \Delta m^2_{31},\ \theta_{12},\ \theta_{23},\ \theta_{13} \text{ and } \delta.$$

Fixing the mass squared differences and the two mixing angles θ_{12} and θ_{23} (e.g. to their best fit values or their $n\sigma$-ranges), we obtain a relation between the reactor mixing angle θ_{13} and δ. Note that in this way one can eliminate the unknown absolute neutrino mass scale. This approach has been previously used in [16].

The main question we have to answer before beginning our analysis is whether the determinant D_{abcd} can become zero for the seven different cases within the experimental limits. The first issue we notice is that all entries of the $2 \times 2-$matrix

$$\begin{pmatrix} V_{a1}V_{b1} & V_{a2}V_{b2} \\ V_{c1}V_{d1} & V_{c2}V_{d2} \end{pmatrix} \tag{15}$$

are nonzero (by experiment). Thus $D_{abcd} = 0$ implies

$$\frac{V_{a1}V_{b1}}{V_{c1}V_{d1}} = \frac{V_{a2}V_{b2}}{V_{c2}V_{d2}}. \tag{16}$$

[4] In general a normal (inverted) neutrino mass spectrum allows $\mu_1 = 0$ ($\mu_3 = 0$). However, one can verify that within the experimental 3σ-range (13) implies $\mu_1 = 0 \Leftrightarrow \mu_3 = 0$ for all types of two texture zeros we will study in this work. Thus the lightest neutrino mass must be nonzero.

Using the experimentally known fact that the absolute values of all elements of the second column of V are of the same order of magnitude, we find

$$\left|\frac{V_{a1}V_{b1}}{V_{c1}V_{d1}}\right| = \left|\frac{V_{a2}V_{b2}}{V_{c2}V_{d2}}\right| \simeq 1 \Rightarrow |V_{a1}V_{b1}| \simeq |V_{c1}V_{d1}|. \tag{17}$$

From

$$|V_{i1}| \simeq (2/\sqrt{6},\, 1/\sqrt{6},\, 1/\sqrt{6})^{\mathrm{T}} \tag{18}$$

one easily finds that the only case allowing (17) is C. Therefore for the cases A_1 to B_4 we can assume $D_{abcd} \neq 0$ and use equation (14) for our analysis.

Let us now turn to case C. In [18] it has been shown that for $\theta_{13} = 0$ the determinant D_{2233} becomes zero and the system (10) is therefore singular in this case. Inversely assuming $D_{2233} = 0$, we can proceed as follows. Defining $\epsilon = s_{13}e^{i\delta}$ we find

$$D_{2233} = \epsilon\left(\frac{1}{2}\sin(2\theta_{12})\sin(2\theta_{23})(1+\epsilon^2) - \epsilon\cos(2\theta_{12})\cos(2\theta_{23})\right). \tag{19}$$

Thus D_{2233} can be zero only for $\epsilon = 0$ or

$$\tan 2\theta_{12}\, \tan 2\theta_{23} = \frac{2\epsilon}{1+\epsilon^2}. \tag{20}$$

For $0 \leq s_{13}^2 \leq 0.05$ we find

$$\left|\frac{2\epsilon}{1+\epsilon^2}\right| = \frac{2s_{13}}{|1+s_{13}^2\exp(2i\delta)|} \leq \frac{2s_{13}}{1-s_{13}^2} < 0.48\,. \tag{21}$$

Using the $3\sigma-$ranges provided in [3] one easily finds that at 3σ

$$\tan 2\theta_{12}\, \tan 2\theta_{23} > 8.55, \tag{22}$$

which implies that $s_{13} = 0$ is indeed the only possibility for D_{2233} to become 0 at 3σ. Since we are not interested in the limit $s_{13} \to 0$, we can use (10) and (14) also to analyze case C.

Analysis of the relation between $\sin^2\theta_{13}$ and $\cos\delta$

It turns out that for all texture zeros studied in this work r (see equation (14)) can be expressed as a rational function of at most cubic polynomials in $\cos\delta$, i.e.

$$r = \frac{p(\cos\delta)}{q(\cos\delta)}, \tag{23}$$

where p and q are polynomials of order at most 3. Thus we find

$$r\,q(\cos\delta) - p(\cos\delta) = 0, \tag{24}$$

which is an equation of at most third order in $\cos\delta$. Thus the dependence of $\cos\delta$ on the mixing angles can be computed *exactly.* Note that (24) will in general have more solutions than (23), because we have multiplied by $q(\cos\delta)$. In fact we have the additional solution

$$q(\cos\delta) = p(\cos\delta) = 0, \tag{25}$$

which corresponds to the limit $\Delta m^2_{ij}/m^2_3 \to 0$ (see equation (14)), i.e. a *quasi-degenerate* neutrino mass spectrum.

For the cases A_1 and A_2 (24) is linear in $\cos\delta$. B_1, B_2 and C lead to quadratic equations and B_3 and B_4 yield cubic equations for $\cos\delta$, respectively. We used Mathematica to obtain the coefficients of (24). The well-known formulae for the general solutions of quadratic and cubic equations were implemented in C-programs which, scanning over the experimentally allowed ranges for r, θ_{12} and θ_{23}, allowed us to plot $\sin^2\theta_{13}$ versus $\cos\delta$. $\sin^2\theta_{13}$ was varied between 0 and 0.05 and for all other experimentally accessible quantities we used the values obtained from the newest global fit [3] including already the new T2K data [1]. Our numerical analysis consists of the following steps:

- Input: $\sin^2\theta_{13}$, $\sin\theta_{12}$, $\sin\theta_{23}$, $r = \Delta m^2_{21}/\Delta m^2_{31}$ (best fit values or $n\sigma-$range ($n = 1, 2, 3$)). The range of $\sin^2\theta_{13}$ (0 to 0.05) is divided into 600 steps. The ranges of r, $\sin\theta_{23}$ and $\sin\theta_{12}$ are divided into steps of equal length in such a way that the corresponding $1\sigma-$ranges are divided into 40 steps. Thus e.g. the computation for the 1σ-range alone consists of $600 \times 40^3 = 3.84 \times 10^7$ individual calculation cycles.
- Equation (24) is solved for $\cos\delta$ (there can be up to three solutions).
- Only the real solutions $\in [-1, 1]$ are processed further, the others are discarded.
- Now we want to insert the remaining solutions for $\cos\delta$ into (13) to calculate the mass ratios
$$\frac{m_i}{m_j} = \left|\frac{\mu_i}{\mu_j}\right|. \tag{26}$$
In order to do that we have to calculate $e^{i\delta}$ from $\cos\delta$. There are two solutions to this problem, namely
$$e^{i\delta} := \cos\delta \pm i\sqrt{1-\cos^2\delta}, \tag{27}$$
but since they are complex conjugates of each other the mass ratios (26) do not depend on the choice of the solution.
- Finally the program checks whether the following inequalities are fulfilled.
$$\begin{aligned} &\frac{m_1}{m_3} < 1,\ \frac{m_2}{m_3} < 1,\ \frac{m_1}{m_2} < 1 \quad \text{(NS)},\\ &\frac{m_1}{m_3} > 1,\ \frac{m_2}{m_3} > 1,\ \frac{m_1}{m_2} < 1 \quad \text{(IS)}. \end{aligned} \tag{28}$$
If they are fulfilled the data point ($\cos\delta$, $\sin^2\theta_{13}$) is stored.
- Finally, when the whole parameter range has been scanned, all stored data points are plotted[5].

[5]In order to create plots of a suitable size (in terms of disk space) we constructed a lattice dividing the range of $\sin^2\theta_{13}$ into 600 and the range of $\cos\delta$ (-1 to 1) into 800 points. Data points falling into the same part of the lattice were plotted only once.

It turns out that a simple scan over the allowed $n\sigma$-ranges for the parameters as described above works very well for all cases of types A and B. However, for case C (NS) the sizes of the steps chosen in our systematic scan are just too large in order to obtain enough data points to produce good and reliable results. The reason for this issue is that C (NS) implies $\theta_{23} \approx 45°$ [18], so one would need an enormously high resolution in the scan over the $n\sigma$-ranges of $\sin\theta_{23}$ to produce reliable results. Thus we have to analyze C (NS) in a different way. A good method to deal with case C (NS) is to assign the input parameters $\sin^2\theta_{13}$, $\sin\theta_{12}$, $\sin\theta_{23}$, $r = \Delta m^2_{21}/\Delta m^2_{31}$ *random* values in their $n\sigma$-ranges, rather than varying them step by step. In this way one obtains a so-called *scatter plot*. Since also case C (IS) shows some hints of problems using a systematic scan, we also did a scatter plot for this case. The number of random points ($\sin^2\theta_{13}$, $\sin\theta_{12}$, $\sin\theta_{23}$, r) we used was 10^9 for each of the $n\sigma$-ranges ($n = 1, 2, 3$).

4 Results

We will now present the results of our numerical analysis. As already explained we have produced plots of $\sin^2\theta_{13}$ versus $\cos\delta$ (see figures 1–12). The color code is the same for all plots:

- The best fit value for the point $(\cos\delta, \sin^2\theta_{13})$ according to the global fit [3] is indicated by a black cross.
- The best fit values for $\sin^2\theta_{13}$ as a function of $\cos\delta$ according to our analysis are indicated by a black line.
- The $n\sigma-$regions are shown as colored areas in the plots (red=1σ, green=2σ and blue=3σ).

The cases A_1 and A_2 are incompatible with an inverted spectrum if the reactor angle is varied within the 3σ-range $0 \leq s^2_{13} \leq 0.05$. Assuming a normal neutrino mass spectrum A_1 and A_2 predict θ_{13} to be different from zero at 3σ. For the best fit values for the observables, namely θ_{12}, θ_{23} and r, we predict $0.024 \geq \sin^2\theta_{13} \geq 0.012$ corresponding to the bounds $-1 \leq \cos\delta \leq 1$ for the case A_1 and $0.014 \leq \sin^2\theta_{13} \leq 0.032$ corresponding to $-1 \leq \cos\delta \leq 1$ for the case A_2.

The cases B_1, B_2, B_3 and B_4 predict the Dirac CP phase to be close to maximal, i.e. $\cos\delta \approx 0$. Note furthermore that the cases B_1 (NS), B_2 (IS), B_3 (NS) and B_4 (IS) are incompatible with $\theta_{23} > 45°$.[6] Therefore, for these cases, the plots do not show the black "best fit line" (the best fit value for $\sin^2\theta_{23}$ given in [3] is 0.52 which corresponds to an atmospheric mixing angle larger than 45°). The generic predictions we found for the cases of type B concerning the atmospheric angle are shown in table 2.

For case C and an inverted spectrum we do not find a strong correlation between $\sin^2\theta_{13}$ and $\cos\delta$. However, from figure 12 we can see that the best fit value for the point $(\cos\delta, \sin^2\theta_{12})$ lies in the 1σ-region of the plot. For case C and a normal spectrum there is no correlation between the Dirac CP phase and the reactor angle (see figure 11), but,

[6]This is in accordance with the best fit results for $\sin^2\theta_{23}$ given in [17] for B_3 and B_4.

case	NS	IS
B_1	$\theta_{23} \leq 45°$	$\theta_{23} \geq 45°$
B_2	$\theta_{23} \geq 45°$	$\theta_{23} \leq 45°$
B_3	$\theta_{23} \leq 45°$	$\theta_{23} \geq 45°$
B_4	$\theta_{23} \geq 45°$	$\theta_{23} \leq 45°$

Table 2: Inequalities for the atmospheric mixing angle for the cases of type B.

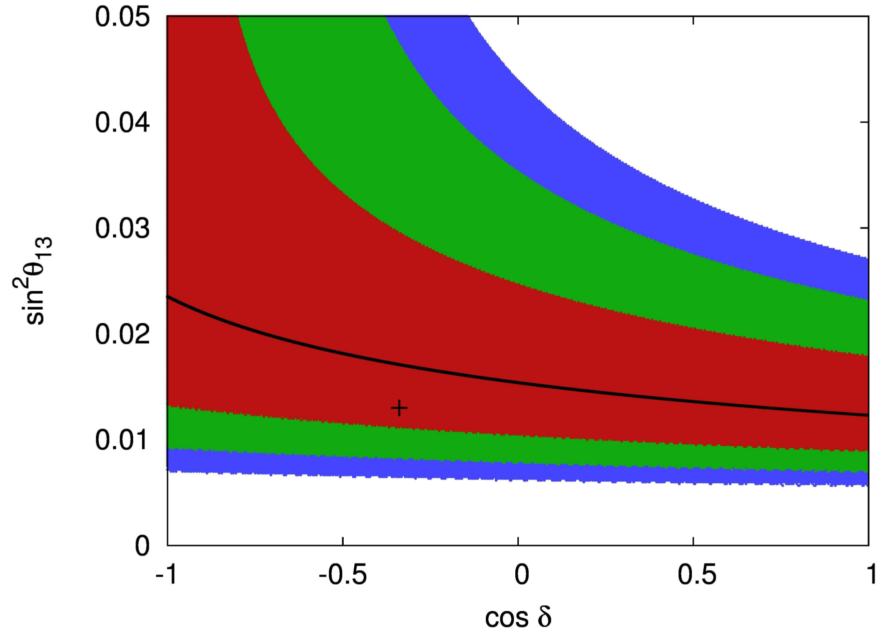

Figure 1: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case A_1 (normal spectrum). For description of the colors see the text.

as was pointed out by Grimus and Lavoura [18], atmospheric neutrino mixing is close to maximal. As shown in figure 13 there is a correlation between the reactor angle and the atmospheric angle, but the deviation from the maximal value of the atmospheric angle is negligible.

5 Conclusions

In the light of the recent T2K result which points towards a large reactor mixing angle θ_{13}, we reconsidered the interesting case of two texture zeros in the neutrino mass matrix. In particular we studied the correlation between the reactor mixing angle θ_{13} and the Dirac CP phase δ for the viable cases classified in [10] as A_1, A_2, B_1, B_2, B_3, B_4 and C. All of these cases are still compatible with the global fit of the neutrino data at 3σ, but only the cases A_1 and A_2 predict the reactor angle to be different from zero at 3σ. In

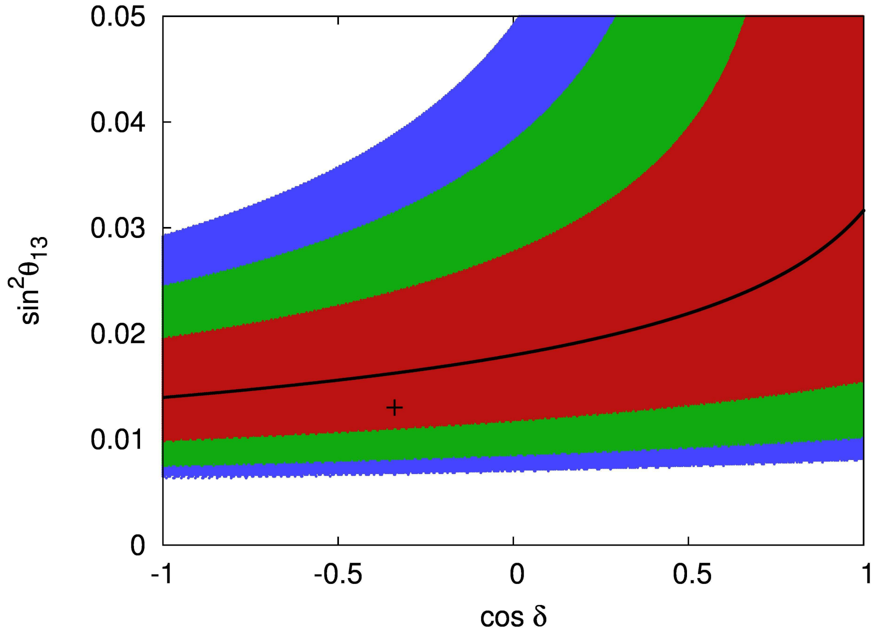

Figure 2: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case A_2 (normal spectrum).

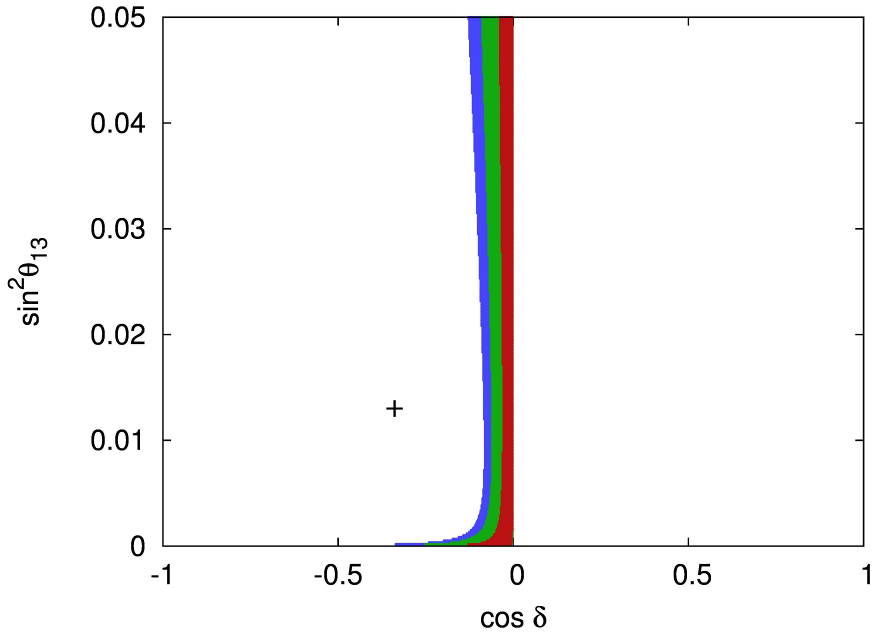

Figure 3: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_1 (normal spectrum).

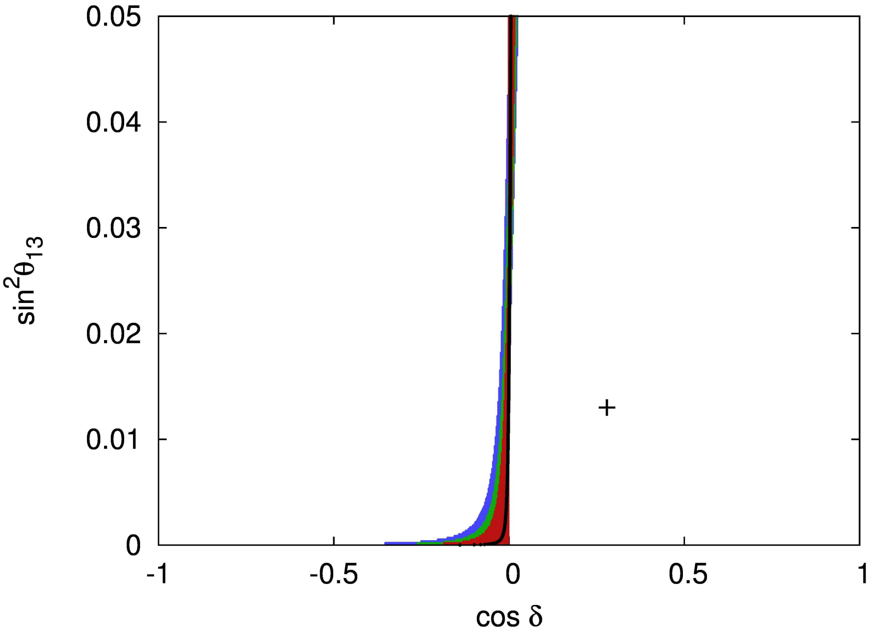

Figure 4: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_1 (inverted spectrum).

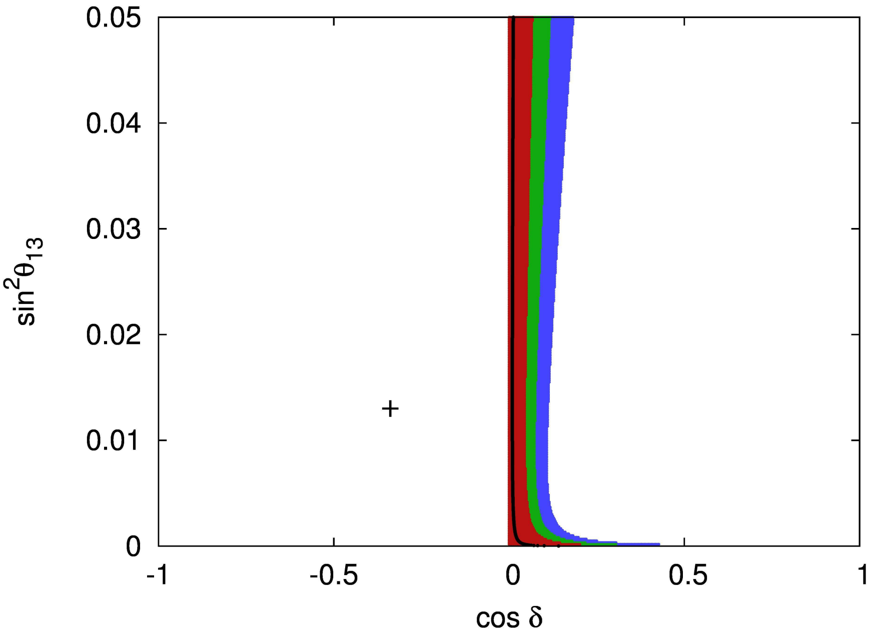

Figure 5: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_2 (normal spectrum).

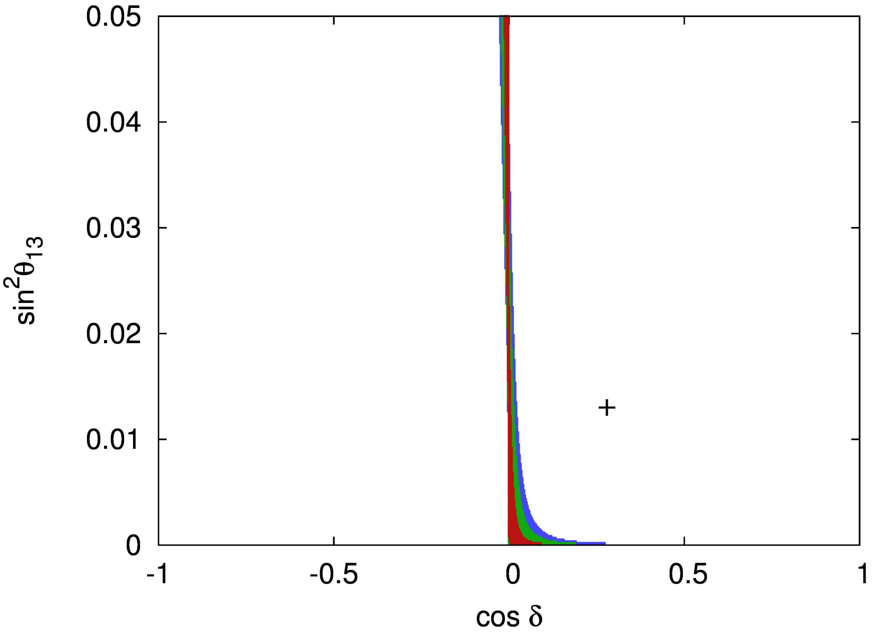

Figure 6: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_2 (inverted spectrum).

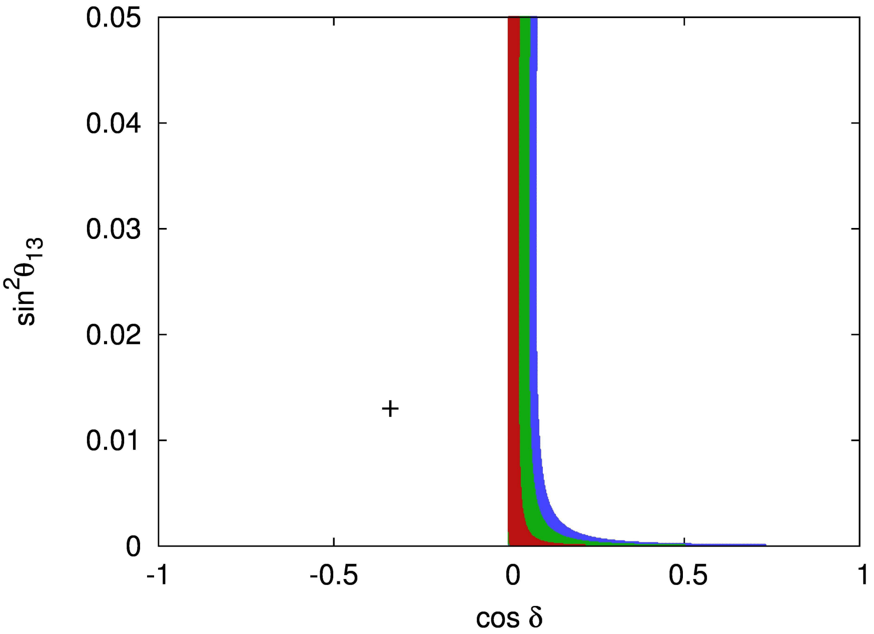

Figure 7: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_3 (normal spectrum).

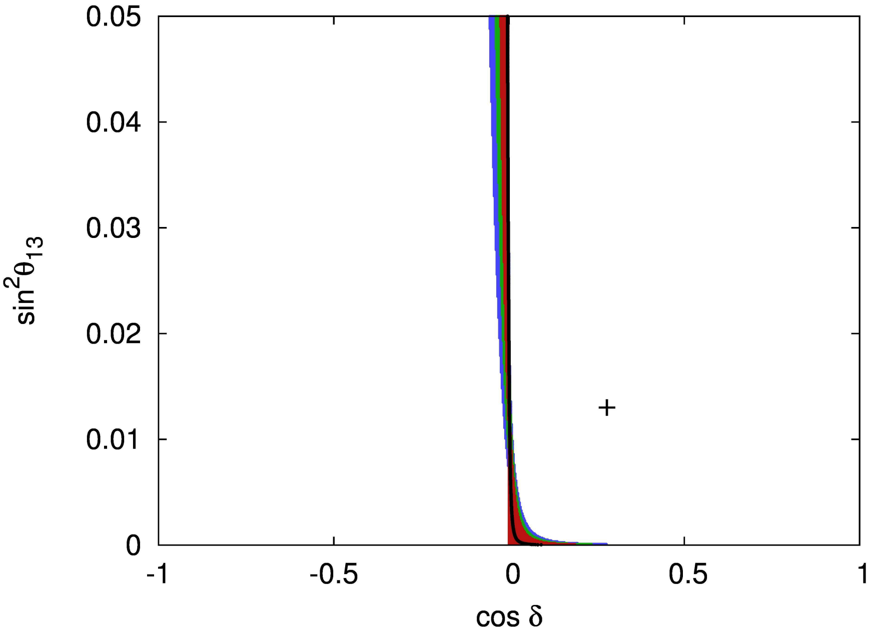

Figure 8: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_3 (inverted spectrum).

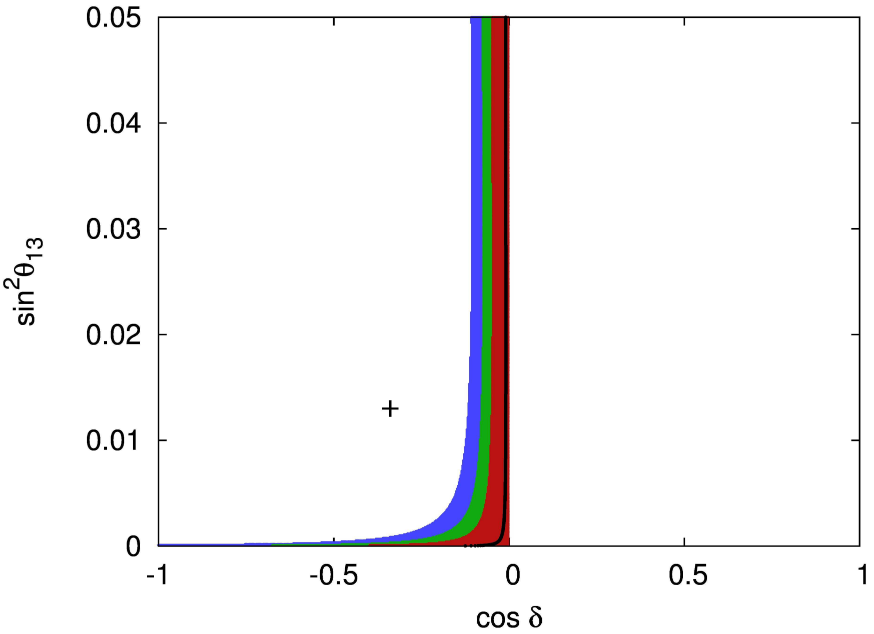

Figure 9: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_4 (normal spectrum).

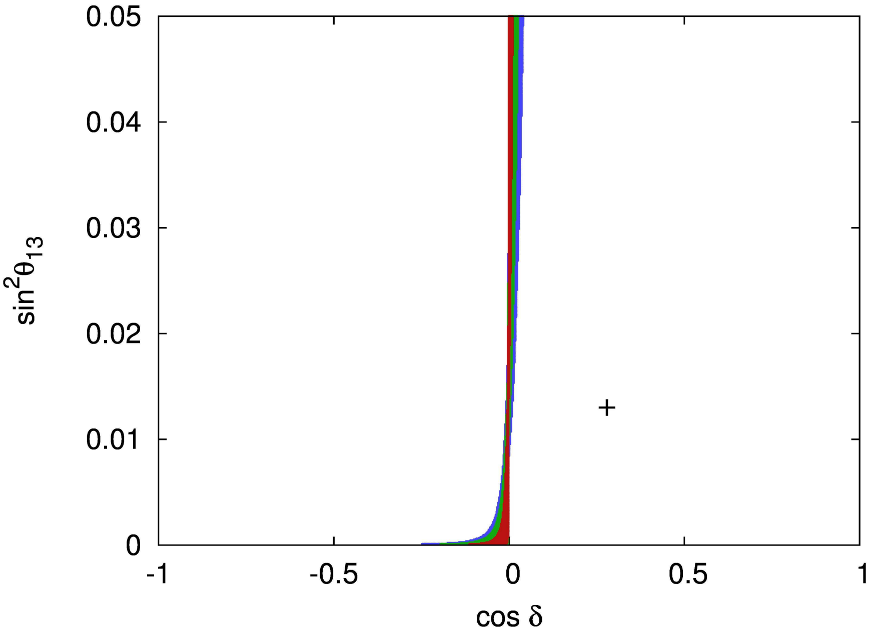

Figure 10: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case B_4 (inverted spectrum).

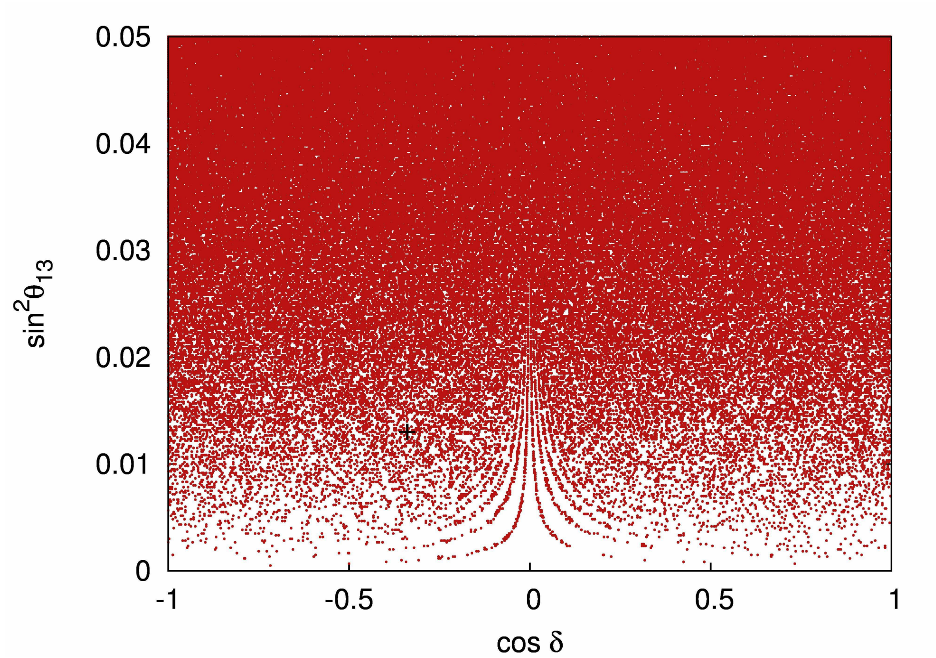

Figure 11: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case C (normal spectrum).

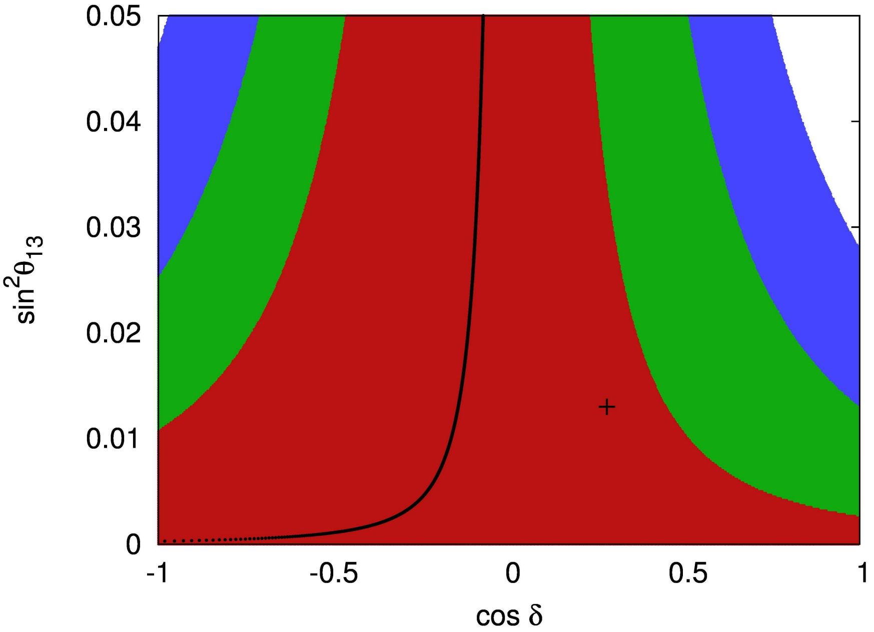

Figure 12: The relation between $\sin^2\theta_{13}$ and $\cos\delta$ for case C (inverted spectrum).

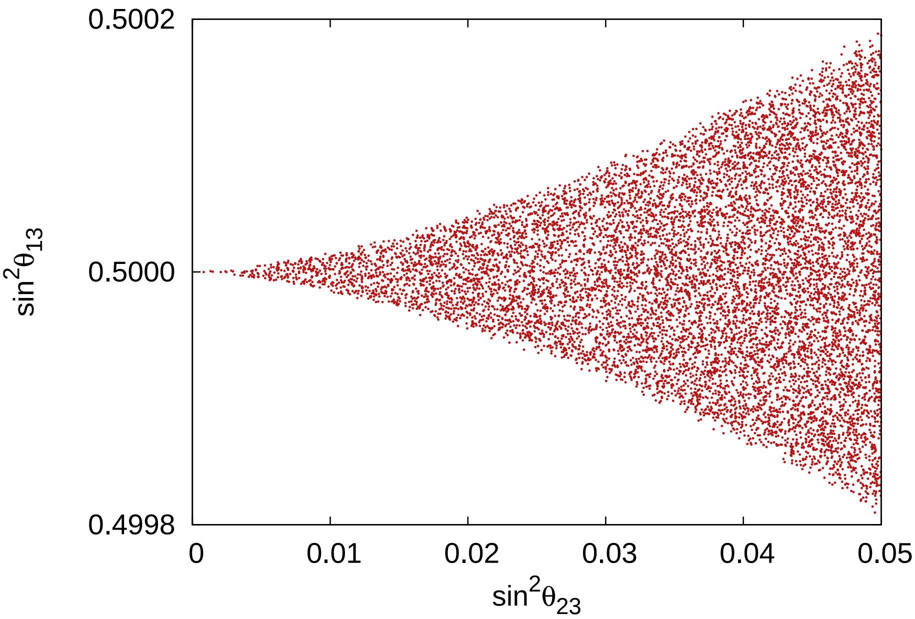

Figure 13: The relation between $\sin^2\theta_{13}$ and $\sin^2\theta_{23}$ for case C (NS) (scatter plot for the 1σ-range with 10^7 random points).

particular for the case A_1, asserting all the free parameters their best fit values, predicts $0.012 \leq \sin^2\theta_{13} \leq 0.024$ while for the case A_2 assuming the best fit values predicts $0.014 \leq \sin^2\theta_{13} \leq 0.032$.

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Chapter 9

Correlations of the elements of the neutrino mass matrix

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Correlations of the elements of the neutrino mass matrix

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Abstract

Assuming Majorana nature of neutrinos, we re-investigate, in the light of the recent measurement of the reactor mixing angle, the allowed ranges for the absolute values of the elements of the neutrino mass matrix in the basis where the charged-lepton mass matrix is diagonal. Apart from the derivation of upper and lower bounds on the values of the matrix elements, we also study their correlations. Moreover, we analyse the sensitivity of bounds and correlations to the global fit results of the neutrino oscillation parameters which are available in the literature.

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1 Introduction

Our knowledge of the neutrino oscillation parameters has enormously improved in the recent years. The experimental results of the Double Chooz, Daya Bay and RENO Collaborations [1] have impressively confirmed the earlier hints [2] for a non-zero reactor mixing angle. Taking these novel results into account, the recent global fits [3, 4] establish $\theta_{13} > 0$ at a confidence level of $\sim 10\sigma$. Also the values of the solar mixing angle and the mass-squared differences are now known with good accuracy. While for a long time $\sin^2\theta_{23} \approx 1/2$ has been a very good approximation for the best fit value, the recent global fits [3, 4] hint towards a deviation from maximal atmospheric neutrino mixing. However, even at 1σ it is not clear from the recent fits in which octant θ_{23} lies. The global fit of [3] allows $\theta_{23} = 45^\circ$ at 2σ and that of [4] allows maximal atmospheric mixing within the 3σ range. The least known mixing parameter is the CP phase δ: it is unconstrained at 2σ, but the best fit values of [4] hint towards $\delta \approx \pi$.

With all these improved results, it is worthwhile to perform an investigation of the allowed ranges for the elements of the neutrino mass matrix $\mathcal{M}_\nu$, as was done by Merle and Rodejohann in their seminal paper [5]. In the context of textures of $\mathcal{M}_\nu$, the correlations of the elements $(\mathcal{M}_\nu)_{\alpha\beta}$ $(\alpha, \beta = e, \mu, \tau)$ of the neutrino mass matrix are of particular interest. Therefore, the main goal of our paper is the construction of plots correlating the absolute values of the elements of the neutrino mass matrix with each other. We will assume Majorana nature of neutrinos in this paper.

The structure of our paper is as follows. In section 2 we investigate analytically upper and lower bounds on the absolute values of the elements of the neutrino mass matrix. In section 3 we will outline the numerical methods we will apply in our analysis, and the results will be presented in section 4. Finally we will draw conclusions in section 5.

2 The elements of the neutrino mass matrix

Assuming neutrinos to be of Majorana nature, the neutrino mass matrix is a complex symmetric 3×3 matrix, which can be diagonalized as

$$U^T \mathcal{M}_\nu U = \mathrm{diag}\,(m_1,\, m_2,\, m_3)\,, \tag{1}$$

where the m_i are the neutrino masses and U is a unitary matrix. In the following we will always assume to work in the basis in which the charged lepton mass matrix is given by

$$\mathcal{M}_\ell = \mathrm{diag}(m_e,\, m_\mu,\, m_\tau), \tag{2}$$

which implies $U = U_{\mathrm{PMNS}}$. Consequently, U can be decomposed as

$$U = D_1 V D_2, \tag{3}$$

where $D_1 = \mathrm{diag}\,(e^{i\varphi_e},\, e^{i\varphi_\mu},\, e^{i\varphi_\tau})$ is a diagonal phase matrix, V is the mixing matrix in the parameterization suggested in [8], and $D_2 = \mathrm{diag}(e^{i\sigma_1},\, e^{i\sigma_2},\, e^{i\sigma_3})$ is the matrix of Majorana phases. Without loss of generality, we assume $\sigma_3 = 0$. Inserting this into equation (1) leads to

$$(\mathcal{M}_\nu)_{\alpha\beta} = \left(U^* \mathrm{diag}(m_1,\, m_2,\, m_3) U^\dagger\right)_{\alpha\beta} = e^{-i(\varphi_\alpha+\varphi_\beta)} \sum_{k=1}^{3} m_k\, e^{-2i\sigma_k} V^*_{\alpha k} V^*_{\beta k}. \tag{4}$$

The absolute values of the elements of the neutrino mass matrix

$$|(\mathcal{M}_\nu)_{\alpha\beta}| = |(\mathcal{M}^*_\nu)_{\alpha\beta}| = \left|\sum_k m_k\, e^{2i\sigma_k} V_{\alpha k} V_{\beta k}\right| \tag{5}$$

depend on nine real parameters, namely the three neutrino masses m_i, the three mixing angles θ_{12}, θ_{23} and θ_{13}, the Dirac CP phase δ and the two Majorana phases σ_1 and σ_2.

From equation (5) we can deduce an upper bound on $|(\mathcal{M}_\nu)_{\alpha\beta}|$ as follows. Rewriting the absolute value as a scalar product

$$|(\mathcal{M}_\nu)_{\alpha\beta}| = \left|\sum_k m_k\, e^{2i\sigma_k} V_{\alpha k} V_{\beta k}\right| = \left|\sum_k \underbrace{\sqrt{m_k} e^{i\sigma_k} V_{\alpha k}}_{A^*_k} \underbrace{\sqrt{m_k} e^{i\sigma_k} V_{\beta k}}_{B_k}\right| \equiv |\langle A|B\rangle|, \tag{6}$$

we can use Cauchy-Schwarz's inequality to find

$$|(\mathcal{M}_\nu)_{\alpha\beta}| \leq |A|\,|B|. \tag{7}$$

Due to the unitarity of V, we have $\sum_k |V_{\alpha k}|^2 = 1$, and thus

$$|A| \leq \sqrt{\max_k m_k}, \quad |B| \leq \sqrt{\max_k m_k}, \tag{8}$$

which leads to the final result

$$|(\mathcal{M}_\nu)_{\alpha\beta}| \leq \max_k m_k, \tag{9}$$

i.e. the absolute value of an element of the neutrino mass matrix is smaller than the largest neutrino mass.

We can also construct a lower bound on $|(\mathcal{M}_\nu)_{\alpha\beta}|$. Defining

$$a_k \equiv m_k |V_{\alpha k}||V_{\beta k}| \tag{10}$$

and taking into account that the Majorana phases are not constrained by experiment up to now, we find

$$|(\mathcal{M}_\nu)_{\alpha\beta}| = \left|\sum_{k=1}^{3} e^{i\rho_k} a_k\right|, \tag{11}$$

where the ρ_k are unconstrained phases. Now we have to distinguish two cases. If the three numbers a_k are such that they can be conceived as the lengths of the sides of a triangle, then the right-hand side of equation (11) can become zero and $|(\mathcal{M}_\nu)_{\alpha\beta}|$ as well. It is easy to show that it is possible to construct a triangle with side lengths a_k, if and only if[1]

$$2 \max_k a_k - \sum_k a_k \leq 0. \tag{12}$$

[1]A different but equivalent inequality has been derived in [6].

Therefore, we end up with the inequality[2]

$$|(\mathcal{M}_\nu)_{\alpha\beta}| \geq 2 \max_k a_k - \sum_k a_k. \tag{13}$$

In this way, one gets rid of the Majorana phases and this inequality may be used to rule out single texture zeros in the neutrino mass matrix.

3 Numerical analysis

As already discussed in the previous section, the absolute values of the elements of $\mathcal{M}_\nu$ depend on nine variables, two of which—namely the two Majorana phases σ_1 and σ_2—are totally unconstrained. In [5] the then available experimental data were used to produce plots of the $|(\mathcal{M}_\nu)_{\alpha\beta}|$ versus the smallest neutrino mass m_0. The goal of the present paper is to produce—using the results of the latest global fits of neutrino oscillation experiments—plots of $|(\mathcal{M}_\nu)_{\alpha\beta}|$ versus $|(\mathcal{M}_\nu)_{\alpha'\beta'}|$. Since the knowledge of the neutrino oscillation parameters has improved considerably in the recent years, we also redo the numerical analysis of [5] and show the plots of $|(\mathcal{M}_\nu)_{\alpha\beta}|$ versus m_0.

Let us now turn to our numerical strategy. Concerning the desired plots, a first attempt of creating scatterplots was, unfortunately, doomed to failure because even random point numbers as high as 10^9 were not sufficient to fathom enough of the allowed parameter space to achieve appealing plots. Therefore we follow a different strategy. From the scatterplots we can guess the shapes of the areas which would be filled in the limit of infinitely many points. In particular, we find that the allowed areas have no "holes," from where it becomes clear that it is sufficient to construct their boundaries. We do this in the following way. Consider a plot showing $|(\mathcal{M}_\nu)_{\alpha\beta}|$ or m_0 on the x-axis and $|(\mathcal{M}_\nu)_{\alpha'\beta'}|$ on the y-axis. Then we start by pinning the quantity on the x-axis to some given value x_0. Then we minimize and maximize y for fixed $x = x_0$ and obtain two points (x_0, y_{min}) and (x_0, y_{max}) of the boundary. Afterwards we do the same for fixed $y = y_0$ which leads to the two points (x_{min}, y_0) and (x_{max}, y_0). Repeating this procedure with a suitable number of different values for x_0 and y_0 finally yields the desired allowed area.

As described above, we need an algorithm which allows to minimize and maximize real functions of nine variables.[3] For this purpose we choose the Nelder–Mead algorithm (downhill simplex method) [7], which is a direct search method for finding a local minimum of a given function. However, since all functions f we will consider are non-negative, by minimizing $-f$ we can use the Nelder–Mead algorithm to find a local maximum of f. Since we are interested in the global minima (maxima) of f, single runs of the algorithm are not sufficient. Thus for every minimization (or maximization) we start with 200000 different random start simplices and also perform perturbations of candidates for a good minimum (maximum).

Since there are nine parameters in $\mathcal{M}_\nu$, the domain of the Nelder–Mead algorithm is, by construction, the full parameter space $\mathbb{R}^9$. In order to restrict the search to some

[2]If the right-hand side of inequality (13) is negative, then the resulting bound is $|(\mathcal{M}_\nu)_{\alpha\beta}| \geq 0$.

[3]Analysing the predictions of the "best fit" results of the global fits would reduce the number of parameters. However, instead of adapting our program to a smaller number of parameters, we let the parameters which shall assume their best fit values b vary in the interval $(b - \epsilon|b|, b + \epsilon|b|)$, with $\epsilon = 10^{-6}$.

domain $D \subset \mathbb{R}^9$, we decided for the following procedure. In the physical region, i.e. for an absolute neutrino mass scale of the order of at most $\sim$ eV (see the discussion at the end of this section), also the values of the function f to be minimized will be of the order of at most $\sim$ eV. Therefore, instead of minimizing $f(p)$, we minimize $f(p)+\Pi_D(p)$, where $\Pi_D(p)$ is the characteristic function

$$\Pi_D(p) := \begin{cases} 0 \text{ eV} & \text{for } p \in D, \\ 10^6 \text{ eV} & \text{for } p \notin D. \end{cases} \tag{14}$$

Maximization of f is then equivalent to minimization of $\Pi_D - f$. The vector p collects the nine parameters of the $|(\mathcal{M}_\nu)_{\alpha\beta}|$. The pinning of x to x_0 is also achieved by adding a characteristic function, namely

$$\Pi_I(x(p)) := \begin{cases} 0 \text{ eV} & \text{for } x(p) \in I, \\ 10^6 \text{ eV} & \text{for } x(p) \notin I, \end{cases} \tag{15}$$

where I is an interval containing x_0. The pinning of y to y_0 is done in the same way.

We also want to evaluate the lower bound (13) derived in section 2. For this purpose, we will use the Nelder–Mead algorithm on the function

$$2 \max_k a_k - \sum_k a_k + \Pi_D(p') \tag{16}$$

of the seven real variables $p' = (m_0, \Delta m^2_{21}, \Delta m^2_{31}, \theta_{12}, \theta_{23}, \theta_{13}, \delta)$.

Next let us discuss the domain D of the parameters p. For the mass-squared differences and the sines squared of the mixing angles we use the best fit as well as the $n\sigma$-ranges ($n = 1, 2, 3$) provided by [3, 4]. The Majorana phases σ_1 and σ_2 are unconstrained and can thus vary between zero and 2π. The situation for δ is a bit ambiguous because the best fit values are very different in the two global fits [3] (version 3) and [4]; while for the normal ordering of the neutrino mass spectrum both fits favour a value of δ which is roughly π, the best fit values in the case of an inverted spectrum are ~ 0 and $\sim \pi$ for [3] (version 3) and [4], respectively. For this reason, in the "best fit"-plot we do not fix δ to its best fit value, but allow it to vary in its 1σ-range. Also for the 1σ-plots δ is allowed to vary in the respective 1σ-ranges; however, δ is unconstrained at the two and three sigma level.

The last parameter which remains to be discussed here, is the smallest neutrino mass m_0. The strongest constraints on the absolute values of the neutrino masses comes from cosmology, where the sum of the light neutrino masses—in the form of the relic neutrino energy density Ω_ν—is one of the parameters of the standard model of cosmology [8]. There is no unique consensus on the resulting upper bound on $\sum_\nu m_\nu$, however most constraints are of the order of $\sum_\nu m_\nu < \mathcal{O}(1 \text{ eV})$ [8]. Therefore, we allow m_0 to vary between zero and 0.3 eV. In the limit of $m_0 \to 0.3$ eV, this implies an upper bound on the largest neutrino mass of

$$\max_k m_k \lesssim m_0 + \frac{|\Delta m^2_{31}|}{2 m_0} \approx 0.304 \text{ eV}. \tag{17}$$

According to equation (9), this directly translates to an upper bound of ≈ 0.3 eV on the absolute values of the elements of the neutrino mass matrix.

		1σ	2σ	3σ
$\lvert(\mathcal{M}_\nu)_{ee}\rvert$ (inv. spectrum)	Forero *et al.*	1.52×10^{-2}	1.36×10^{-2}	1.14×10^{-2}
	Fogli *et al.*	1.62×10^{-2}	1.44×10^{-2}	1.24×10^{-2}
$\lvert(\mathcal{M}_\nu)_{\tau\tau}\rvert$ (norm. spectrum)	Forero *et al.*	0	0	0
	Fogli *et al.*	1.86×10^{-2}	1.27×10^{-2}	0

Table 1: Numerical results for the lower bounds on $|(\mathcal{M}_\nu)_{ee}|$ (inverted spectrum) and $|(\mathcal{M}_\nu)_{\tau\tau}|$ (normal spectrum) in units of eV using the global fits of Forero *et al.* (version 3) [3] and Fogli *et al.* [4]. The lower bounds for all other $|(\mathcal{M}_\nu)_{\alpha\beta}|$ are zero.

Assuming that the "standard mechanism" (induced by the Majorana mass term) dominates neutrinoless double beta decay, its non-observation gives upper bounds on $|(\mathcal{M}_\nu)_{ee}|$, the current bounds being of the order of [9, 10]

$$|(\mathcal{M}_\nu)_{ee}| \lesssim 0.4 \text{ eV}, \tag{18}$$

which is comparable to the cosmological bound. Thus, unfortunately, the bound (18) will be fulfilled automatically in our numerical analysis and will, therefore, give no additional constraint on $|(\mathcal{M}_\nu)_{ee}|$. However, the bound (18) implies an upper bound on the absolute neutrino mass scale [9]. Using the 3σ-ranges for the oscillation parameters one can estimate the bound to be [9]

$$m_0 \lesssim 1.9\,\text{eV}. \tag{19}$$

Here the constraints from cosmology are much stronger.

The bound on m_0 coming from tritium decay is of almost the same size as the one from neutrinoless double beta decay, namely $m_0 < 2\,\text{eV}$ [8], and will therefore not constrain the results of our analysis.

4 Results

Lower bounds on $|(\mathcal{M}_\nu)_{\alpha\beta}|$: Using inequality (13) and the global fits of [3, 4] shows that the only elements of $|(\mathcal{M}_\nu)_{\alpha\beta}|$ which have a non-trivial lower bound are $|(\mathcal{M}_\nu)_{ee}|$ in the case of an inverted neutrino mass spectrum and $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of normal ordering of the neutrino masses. The resulting bounds can be found in table 1.

Let us compare these bounds with the results of the analysis of Merle and Rodejohann [5]. From the plots presented there one can read off that the only non-trivial lower bound on an element of $\mathcal{M}_\nu$ is the bound

$$|(\mathcal{M}_\nu)_{ee}| \geq 7 \times 10^{-3}\,\text{eV} \tag{20}$$

in the case of an inverted neutrino mass spectrum.

How to read the plots: Before we discuss the resulting plots, we have to explain how they are to be read. This may best be done by means of an example. The properties we will outline in the following, hold for all plots in this paper.

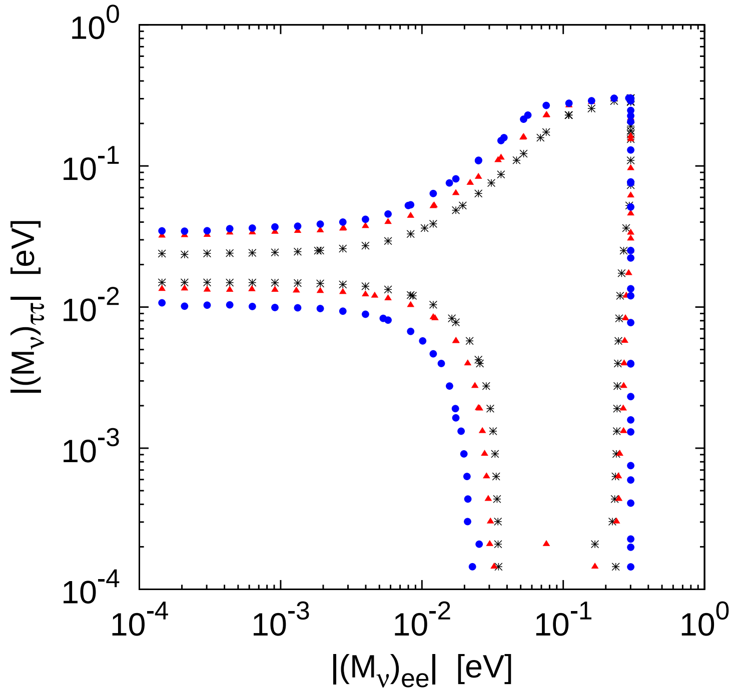

Figure 1: Allowed ranges for $|(\mathcal{M}_\nu)_{ee}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal mass spectrum, using the global fit of Forero *et al.* (version 3). The best fit area is bounded by black stars, the 1σ area by red triangles, and the 3σ area by blue points.

Figure 1 shows the allowed areas for $|(\mathcal{M}_\nu)_{ee}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal mass spectrum. The plot consists of three types of points, each one describing the boundary of an allowed area. The best fit area is bounded by black stars $*$, the 1σ area is bounded by red triangles ▲, and finally the 3σ area is bounded by blue points •. Note that also the plot axes comprise parts of the boundaries of the allowed areas, as can be seen in figure 1. This feature is very pronounced for instance in figure 2.

In appendix A we present the plots based on the global fit of Fogli *et al.* For the full set of plots (Fogli *et al.* and both versions of Forero *et al.*) we refer the reader to the arXiv version of this publication [11].

Plots of the smallest neutrino mass versus $|(\mathcal{M}_\nu)_{\alpha\beta}|$: Concerning the plots of $|(\mathcal{M}_\nu)_{\alpha\beta}|$ as a function of the smallest neutrino mass, an interesting point is whether they have changed since the original analysis of Merle and Rodejohann [5] in 2006. The main difference is that at the time when reference [5] was written, only an upper bound on the size of θ_{13} was known, which is the reason that [5] contains plots for different sizes of $\sin^2 2\theta_{13}$. Comparing our plots to the ones of [5] with $\sin^2 2\theta_{13} = 0.1$, we find that at the 3σ level the plots in [5] are still in good agreement also with the new results. However, for the best fit only the plots of the smallest neutrino mass versus $|(\mathcal{M}_\nu)_{ee}|$ and $|(\mathcal{M}_\nu)_{\mu\tau}|$ are in good agreement with the plots obtained from the new data. The 1σ regions are not indicated in [5].

Finally, from the plots we can readily read off the lower bounds on the $|(\mathcal{M}_\nu)_{\alpha\beta}|$ and

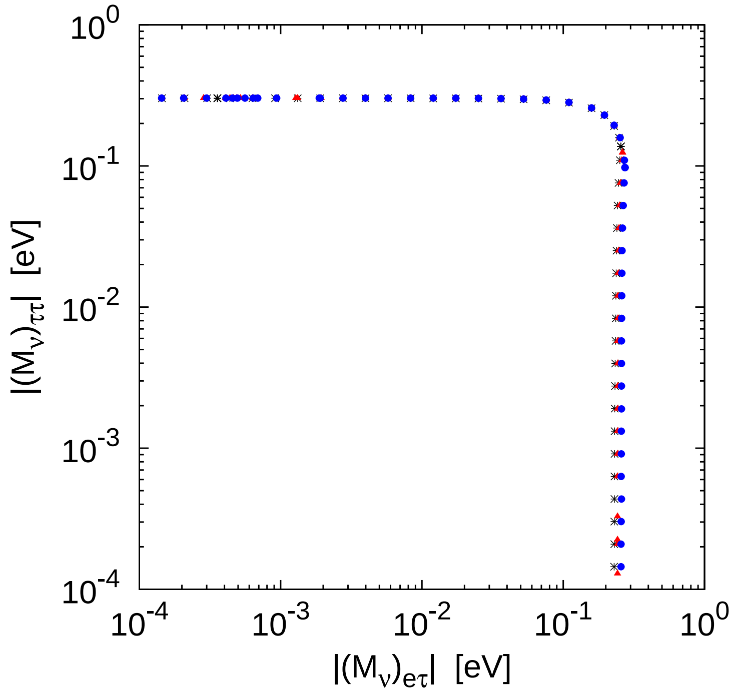

Figure 2: Allowed ranges for $|(\mathcal{M}_\nu)_{e\tau}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal mass spectrum, using the global fit of Forero *et al.* (version 3). For more information *cf.* figure 1 and section 4.

compare them with the bounds found from evaluating inequality (13)—see table 1. We find full agreement, which provides a successful consistency check of our results.

Plots of $|(\mathcal{M}_\nu)_{\alpha\beta}|$ **versus** $|(\mathcal{M}_\nu)_{\alpha'\beta'}|$**:** In this section we will discuss some of the conclusions one can draw from the correlation plots. These plots can be used to test the viability of two texture zeros in $\mathcal{M}_\nu$. The results obtained from the plots are in perfect agreement with the results of a recent numerical analysis provided in [12], and may—at 3σ—be condensed to the fact that the seven two texture zeros originally presented in [13] are still viable with all of the three global fits [3, 4]. For further details we refer the reader to [12].

Let us continue by discussing those correlations which appear manifest at the 3σ level. Going through all the plots, we find the correlations

$$\begin{array}{lcll}
|(\mathcal{M}_\nu)_{ee}| & \text{vs.} & |(\mathcal{M}_\nu)_{\mu\mu}| & \text{(normal spectrum)} \\
|(\mathcal{M}_\nu)_{ee}| & \text{vs.} & |(\mathcal{M}_\nu)_{\mu\tau}| & \text{(normal spectrum)} \\
|(\mathcal{M}_\nu)_{ee}| & \text{vs.} & |(\mathcal{M}_\nu)_{\tau\tau}| & \text{(normal spectrum)} \\
|(\mathcal{M}_\nu)_{\mu\mu}| & \text{vs.} & |(\mathcal{M}_\nu)_{\mu\tau}| & \text{(normal spectrum)} \\
|(\mathcal{M}_\nu)_{\mu\tau}| & \text{vs.} & |(\mathcal{M}_\nu)_{\tau\tau}| & \text{(normal spectrum)}
\end{array}$$

most stringent. All these five correlations may be subsumed as follows: “If one matrix element is small, the other one must be large,” as can, for example, be seen from figure 1. However, there are also matrix elements which appear to be totally uncorrelated. A good example for this case is $|(\mathcal{M}_\nu)_{e\tau}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal spectrum—see figure 2.

While at the 3σ level all plots produced using the three different global fits of Forero *et al.* and Fogli *et al.* agree very well, at the 1σ level this situation changes completely. Instead of presenting a confusing list showing all differences, let us just mention the most important point which is the fact that the fit results of Fogli *et al.* no longer allow a vanishing matrix element $|(\mathcal{M}_\nu)_{\tau\tau}|$ at 1σ in the case of a normal neutrino mass spectrum, while the fit results of Forero *et al.* still do. This evidently also has strong consequences on all correlation plots including $|(\mathcal{M}_\nu)_{\tau\tau}|$, a fine example being the plot of $|(\mathcal{M}_\nu)_{\mu\tau}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$, which is shown in figure 3.

Let us finally discuss the sensitivity of the correlation plots to the data. As already pointed out, correlations which contain $|(\mathcal{M}_\nu)_{\tau\tau}|$ are particularly sensitive, due to the strong constraint on $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal neutrino mass spectrum—see table 1. However, also for an inverted mass spectrum the plots involving $|(\mathcal{M}_\nu)_{\tau\tau}|$ differ visibly for the fits by Forero *et al.* (version 3) and Fogli *et al.* Other correlation plots which are quite sensitive to the data at 1σ are:

$\lvert(\mathcal{M}_\nu)_{ee}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{e\tau}\rvert$	(normal spectrum)
$\lvert(\mathcal{M}_\nu)_{ee}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	(inverted spectrum)
$\lvert(\mathcal{M}_\nu)_{e\mu}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	(normal and inv. spectrum)
$\lvert(\mathcal{M}_\nu)_{e\tau}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	(normal and inv. spectrum)
$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\tau}\rvert$	(inverted spectrum)

Thus in total 17 out of the 30 possible correlations are particularly sensitive to the data at the 1σ level. In appendix B we provide the 17 corresponding plots showing the allowed 1σ regions for both Forero *et al.* (version 3) and Fogli *et al.*

5 Conclusions

In this paper, assuming Majorana neutrinos, we re-investigated the allowed ranges for the elements of the neutrino mass matrix in the light of the results of the recent global fits. In particular our analysis could profit from the fact that also the reactor mixing angle is by now well determined. In contrast, at the time of the original analysis by Merle and Rodejohann [5] only an upper bound on $\sin^2\theta_{13}$ was known.

By means of Cauchy-Schwarz's inequality we could show that, in the basis where the charged lepton mass matrix is diagonal, the absolute value of an element of $\mathcal{M}_\nu$ cannot exceed the largest neutrino mass. The most stringent bound on the absolute neutrino mass scale coming from cosmology thus implies the bound

$$|(\mathcal{M}_\nu)_{\alpha\beta}| \lesssim 0.3\,\mathrm{eV}. \tag{21}$$

We could also derive lower bounds on the elements of the neutrino mass matrix. Numerically evaluating these bounds on the basis of the global fits of oscillation data revealed that only for $|(\mathcal{M}_\nu)_{ee}|$ (inverted spectrum) and $|(\mathcal{M}_\nu)_{\tau\tau}|$ (normal spectrum) non-trivial lower bounds exist—see table 1. While the fact that $|(\mathcal{M}_\nu)_{ee}|$ is non-zero at 3σ was already clear in [5], the lower bound on $|(\mathcal{M}_\nu)_{\tau\tau}|$ is a feature of the new fit of Fogli *et al.* [4].

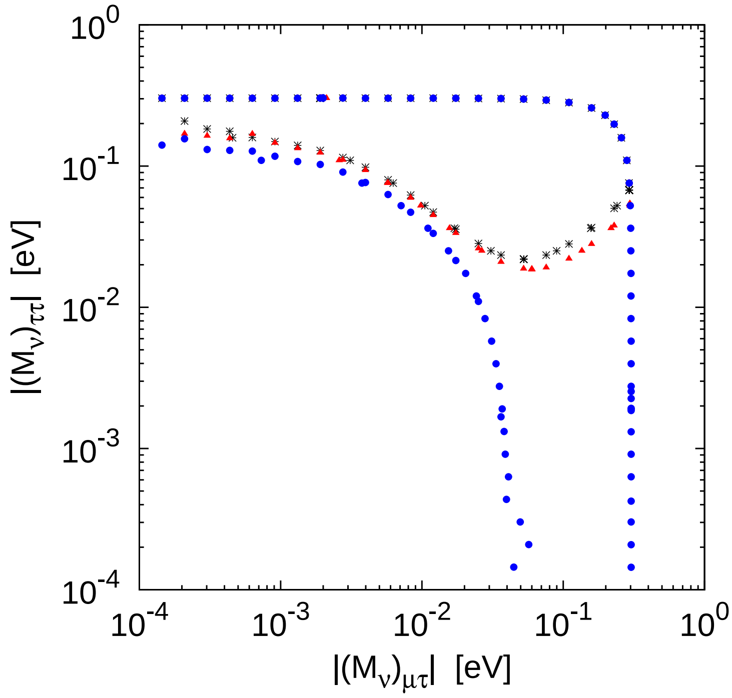

Figure 3: Allowed ranges for $|(\mathcal{M}_\nu)_{\mu\tau}|$ vs. $|(\mathcal{M}_\nu)_{\tau\tau}|$ in the case of a normal mass spectrum, using the fit results of Fogli *et al.* For more information *cf.* figure 1 and section 4.

However the non-trivial bound is only valid at 2σ and the fit of Forero *et al.* [3] still allows zero $|(\mathcal{M}_\nu)_{\tau\tau}|$ even at 1σ.

The second main point of our analysis was the creation of correlation plots of the absolute values of the elements of $\mathcal{M}_\nu$ against each other. We created plots based on three different global fits [3, 4] and found that at the 3σ level there is no discrepancy between the different global fits. For every global fit we obtained 30 correlation plots (15 correlations, two spectra). Among the 30 possibilities we found only five stringent correlations at the 3σ level, namely:

$\lvert(\mathcal{M}_\nu)_{ee}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	(normal spectrum)
$\lvert(\mathcal{M}_\nu)_{ee}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\tau}\rvert$	(normal spectrum)
$\lvert(\mathcal{M}_\nu)_{ee}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\tau\tau}\rvert$	(normal spectrum)
$\lvert(\mathcal{M}_\nu)_{\mu\mu}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\mu\tau}\rvert$	(normal spectrum)
$\lvert(\mathcal{M}_\nu)_{\mu\tau}\rvert$	vs.	$\lvert(\mathcal{M}_\nu)_{\tau\tau}\rvert$	(normal spectrum)

All these correlations may be subsumed as: "If one matrix element is small, the other one must be large."

While at the 3σ level the different global fits all agree, this is not so when one considers the 1σ level. There the most striking fact is that the fit of Fogli *et al.* [4] does no longer allow a vanishing matrix element $(\mathcal{M}_\nu)_{\tau\tau}$ at 1σ in the case of a normal mass spectrum.

In summary, we conclude that correlations evident at the 3σ level exist only for the normal mass spectrum. However, there are interesting features also at the 1σ level which may be corroborated (or refuted) by future experimental results increasing the accuracy of global fit data.

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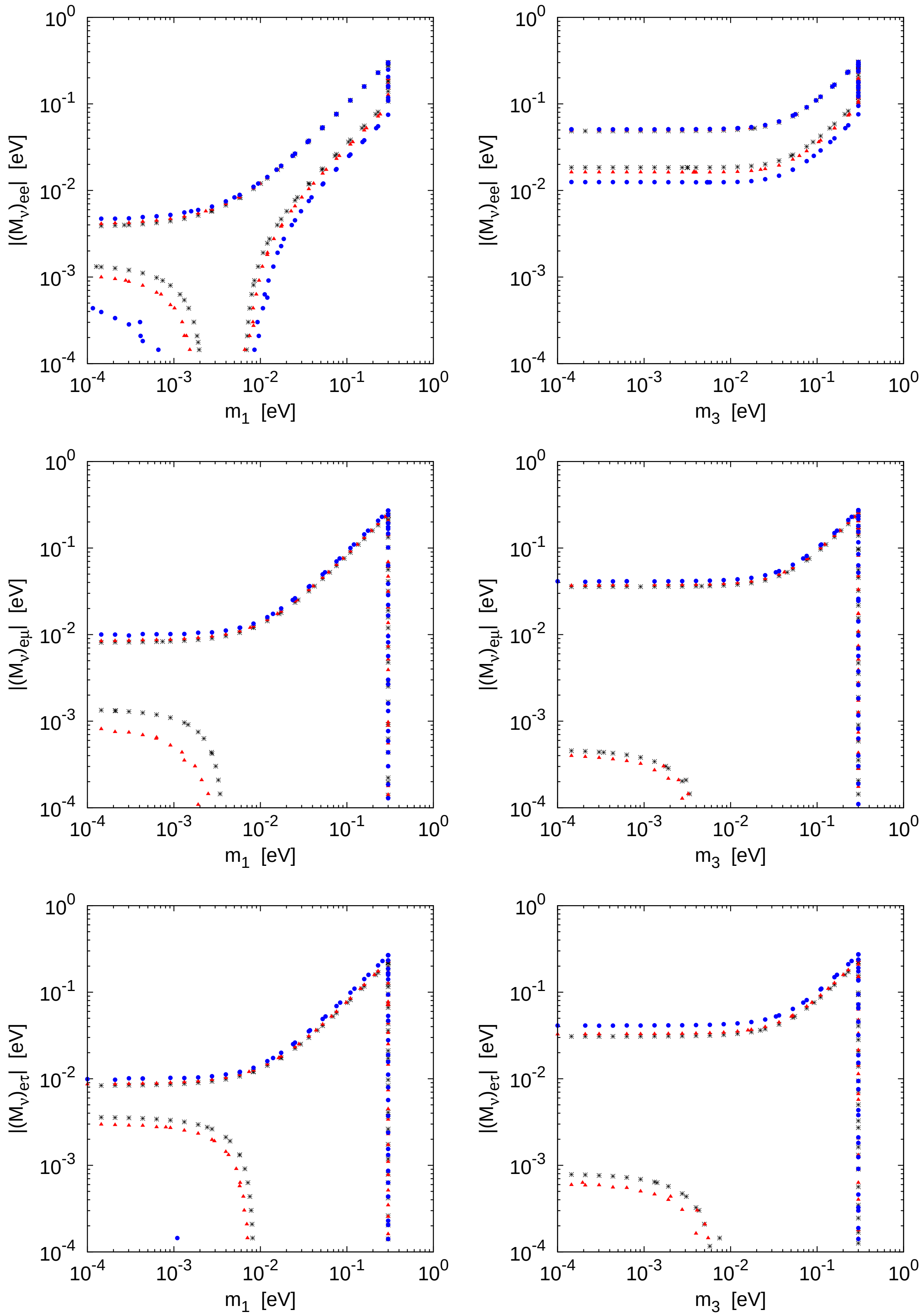
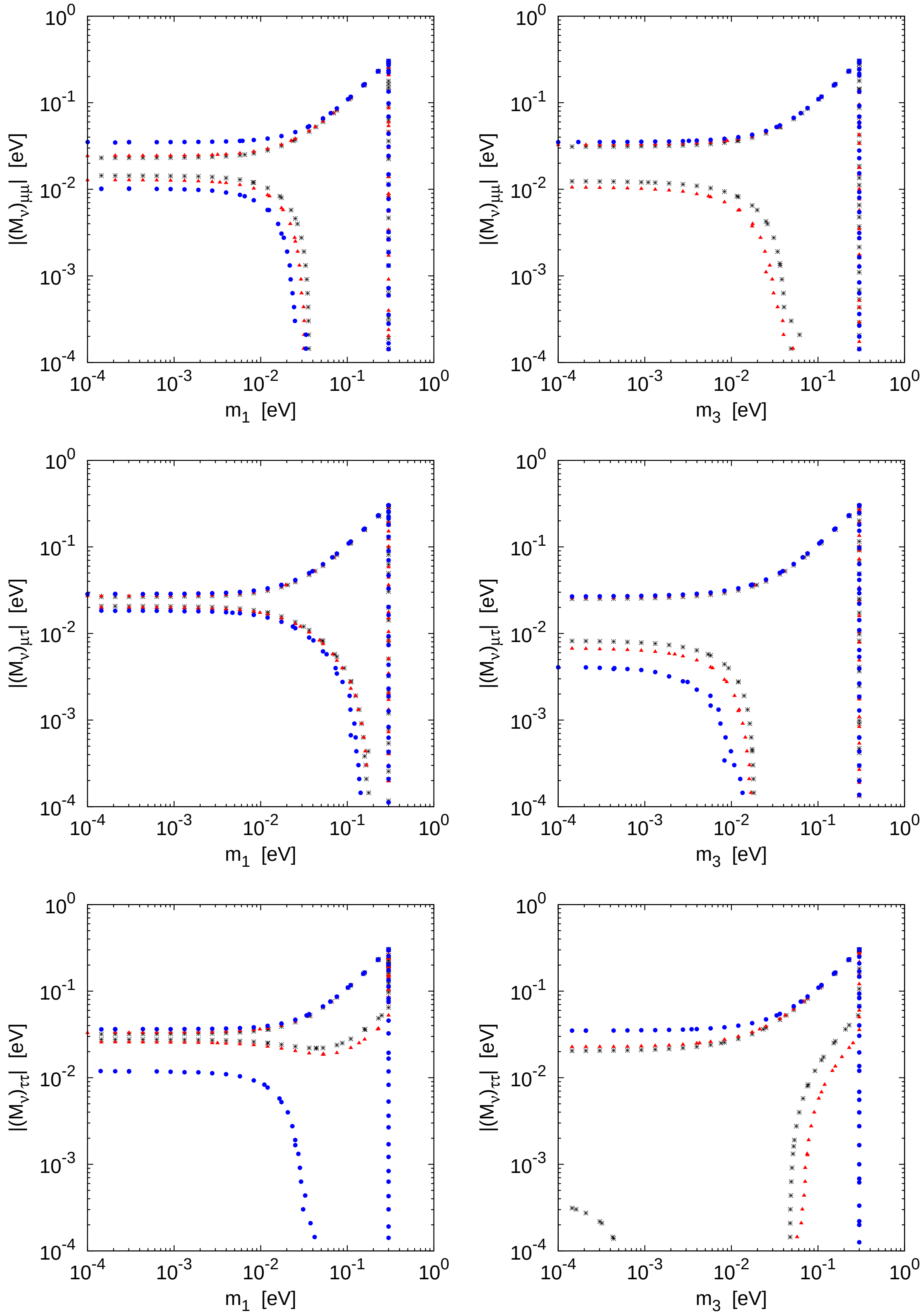
A Plots based on Fogli *et al.*

This appendix contains all 42 plots produced on the basis of the global fit by Fogli *et al.* [4].

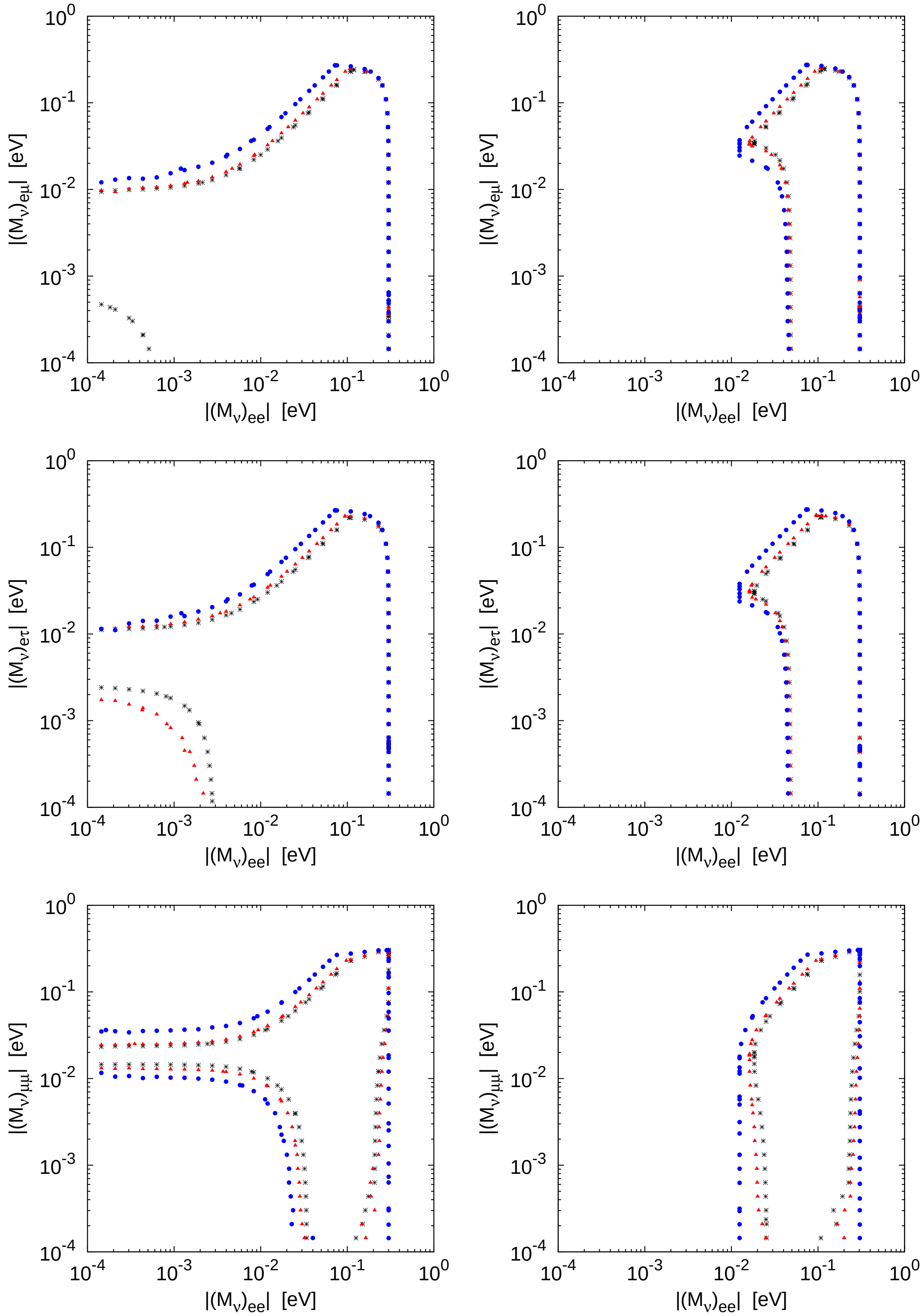
The figures in the left and right columns correspond to normal and inverted mass ordering, respectively. The plots consist of three types of points, each one describing the boundary of an allowed area. The best fit area is bounded by black stars $*$, the 1σ area is bounded by red triangles ▲, and finally the 3σ area is bounded by blue points •.

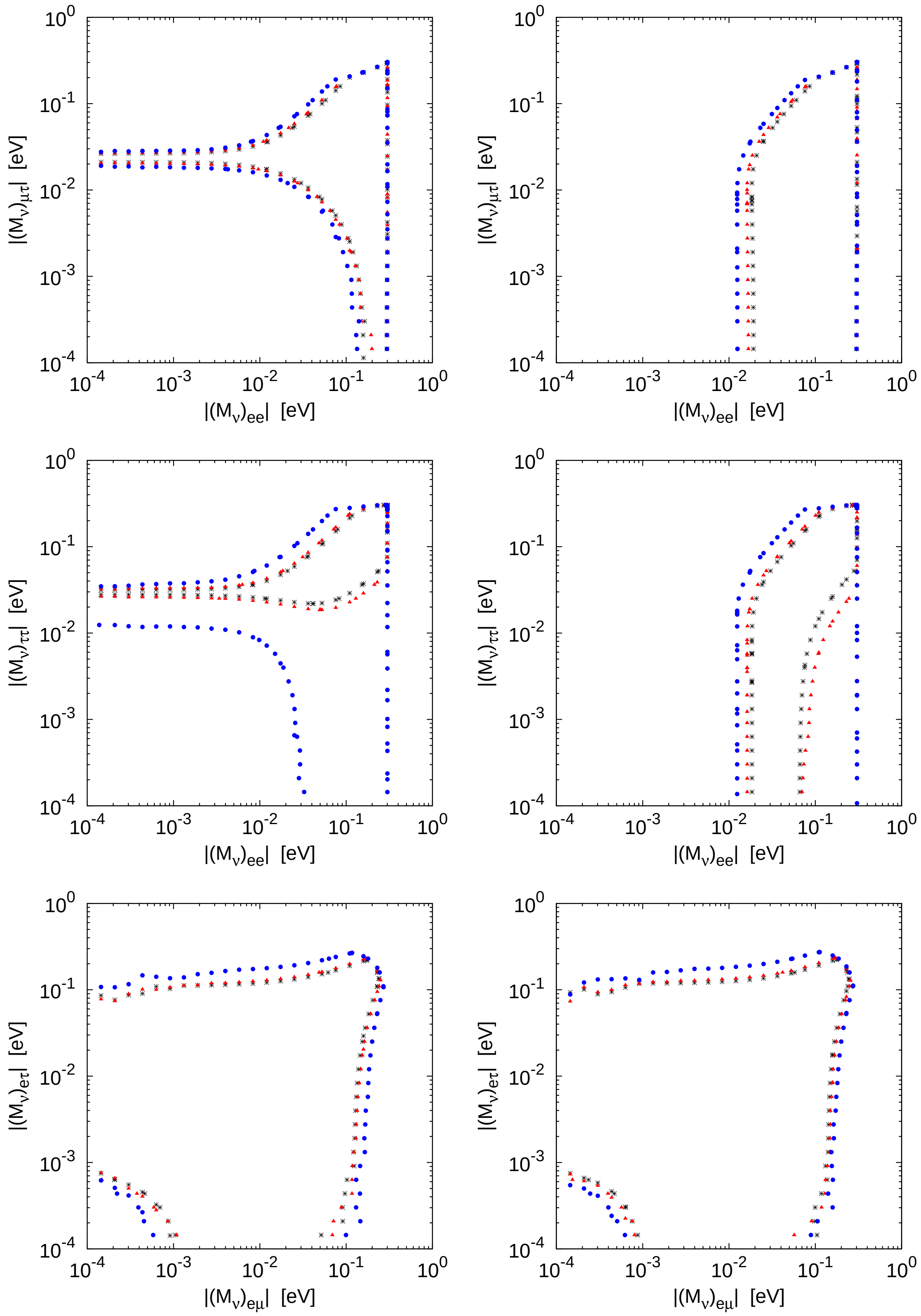


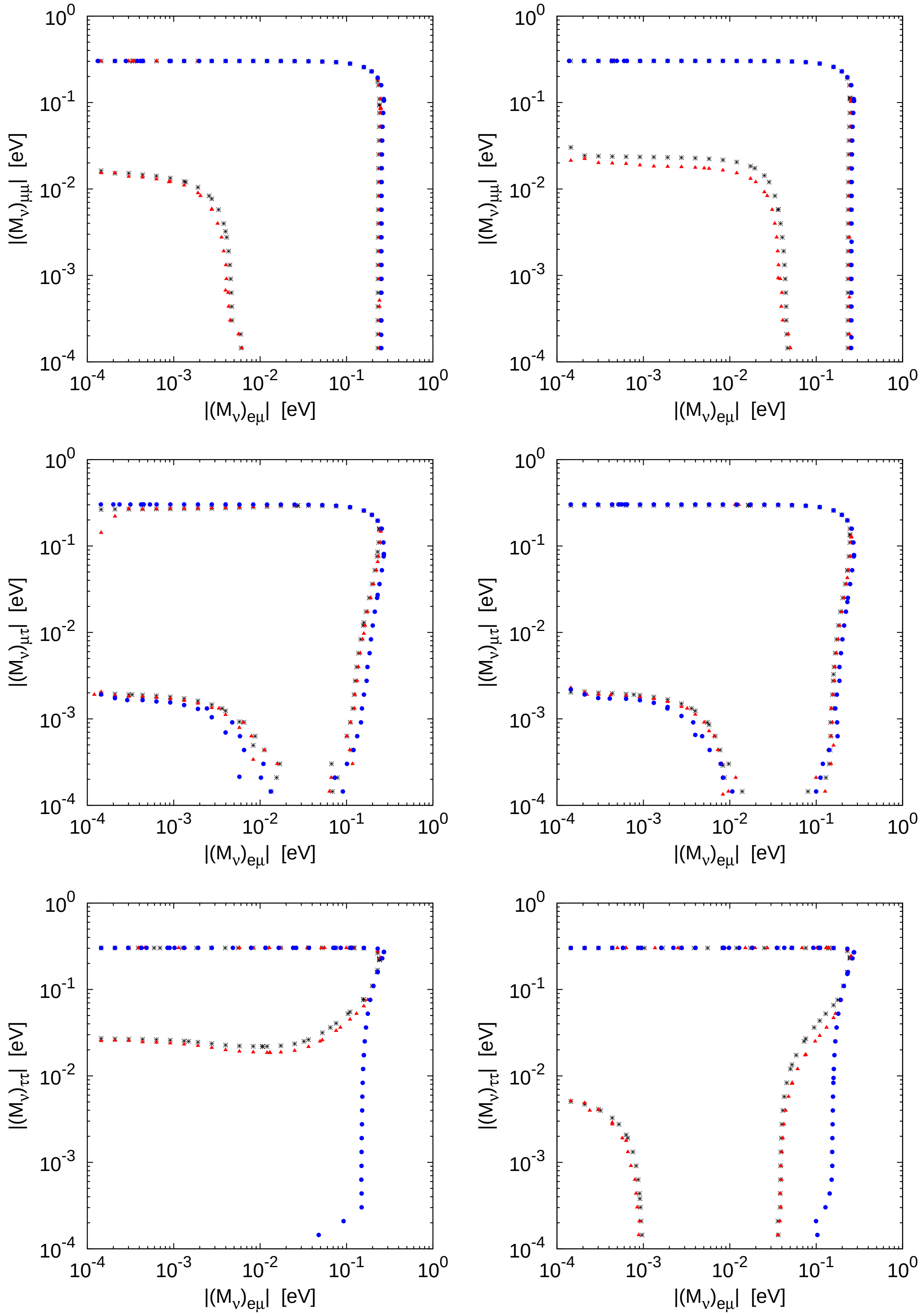
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|(M_\nu)_{ee}|$ [eV]
m_1 [eV]
m_3 [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{e\tau}|$ [eV]

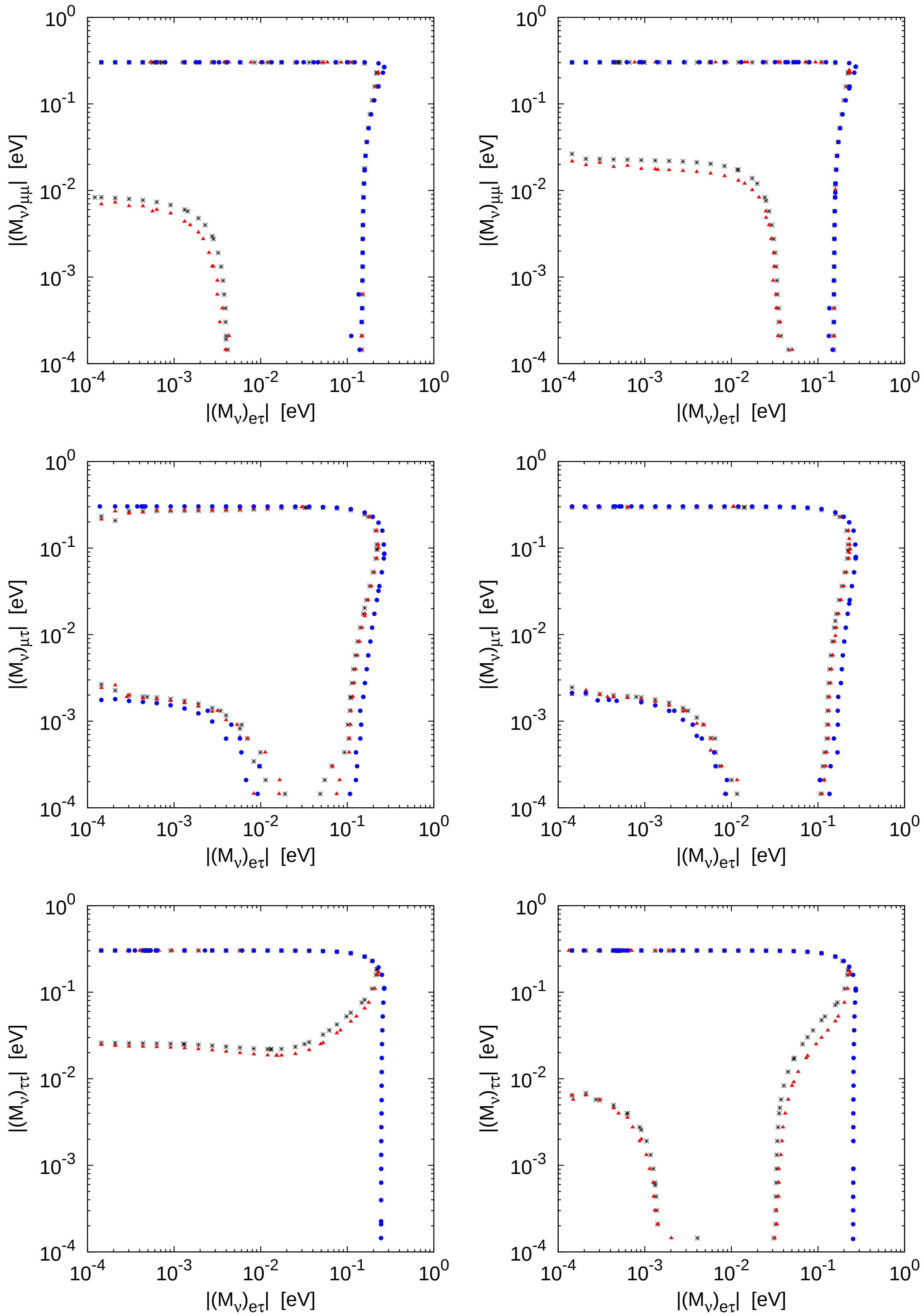


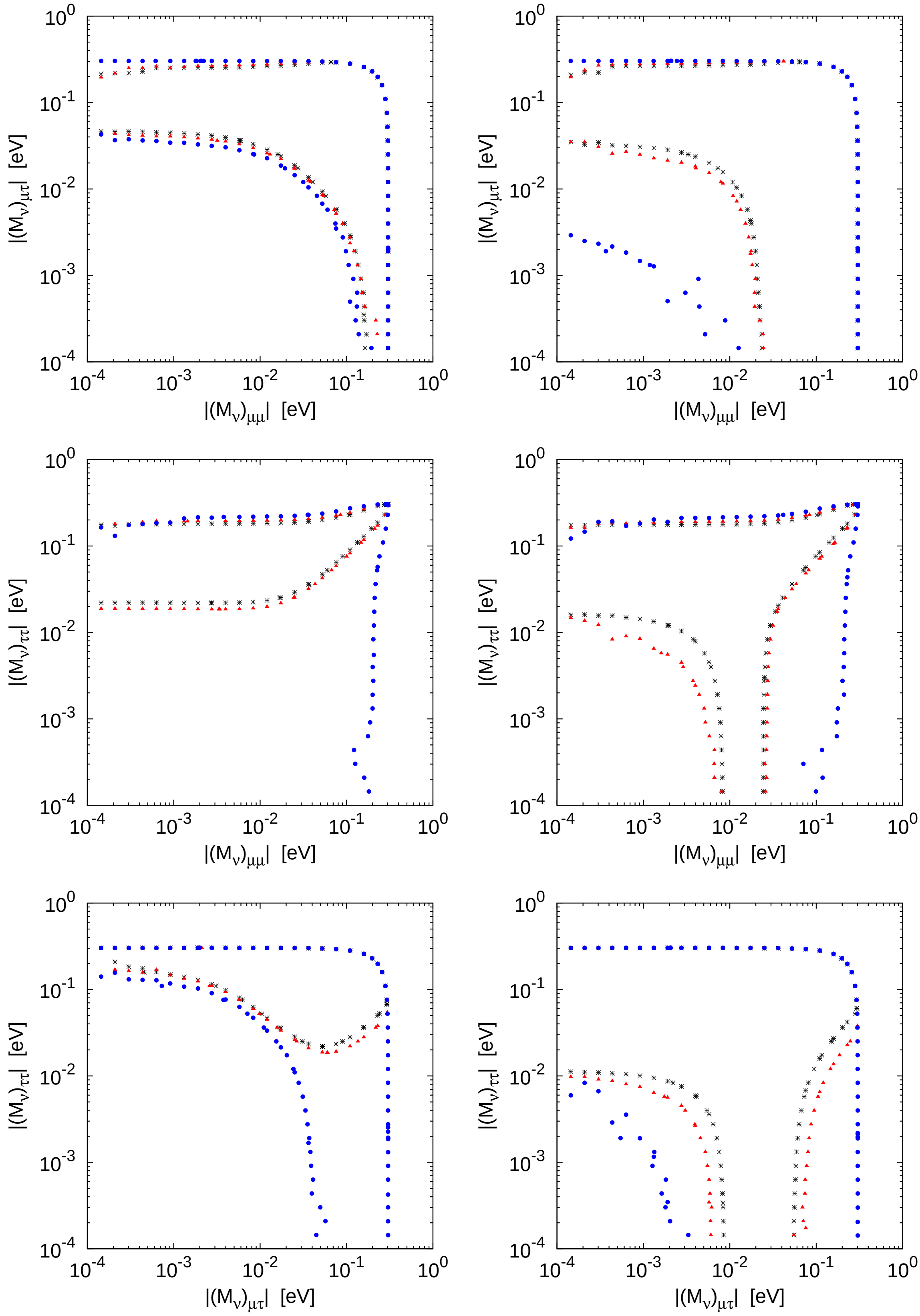
$|(M_\nu)_{\mu\mu}|$ [eV]
m_1 [eV]
m_3 [eV]
$|(M_\nu)_{\mu\tau}|$ [eV]
$|(M_\nu)_{\tau\tau}|$ [eV]


$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
$|(M_\nu)_{e\tau}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
$|(M_\nu)_{e\tau}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{ee}|$ [eV]
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}


|(M_ν)_μτ| [eV]
|(M_ν)_ee| [eV]
|(M_ν)_ττ| [eV]
|(M_ν)_eτ| [eV]
|(M_ν)_eμ| [eV]
10^0
10^-1
10^-2
10^-3
10^-4


$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{\mu\tau}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{\mu\tau}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{e\mu}|$ [eV]


|(M_ν)_μμ| [eV]
|(M_ν)_μτ| [eV]
|(M_ν)_ττ| [eV]
|(M_ν)_eτ| [eV]
10^0
10^-1
10^-2
10^-3
10^-4

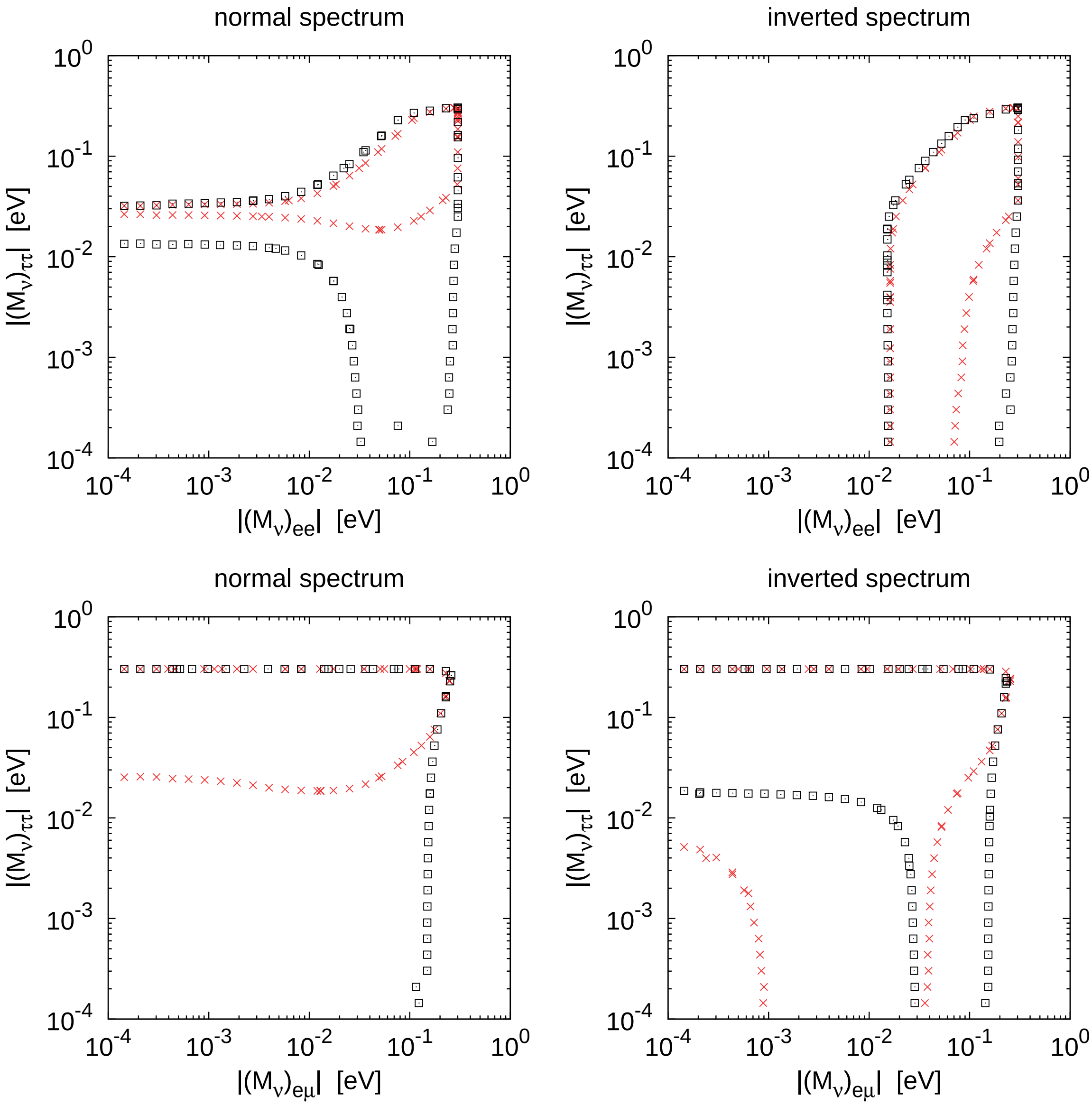


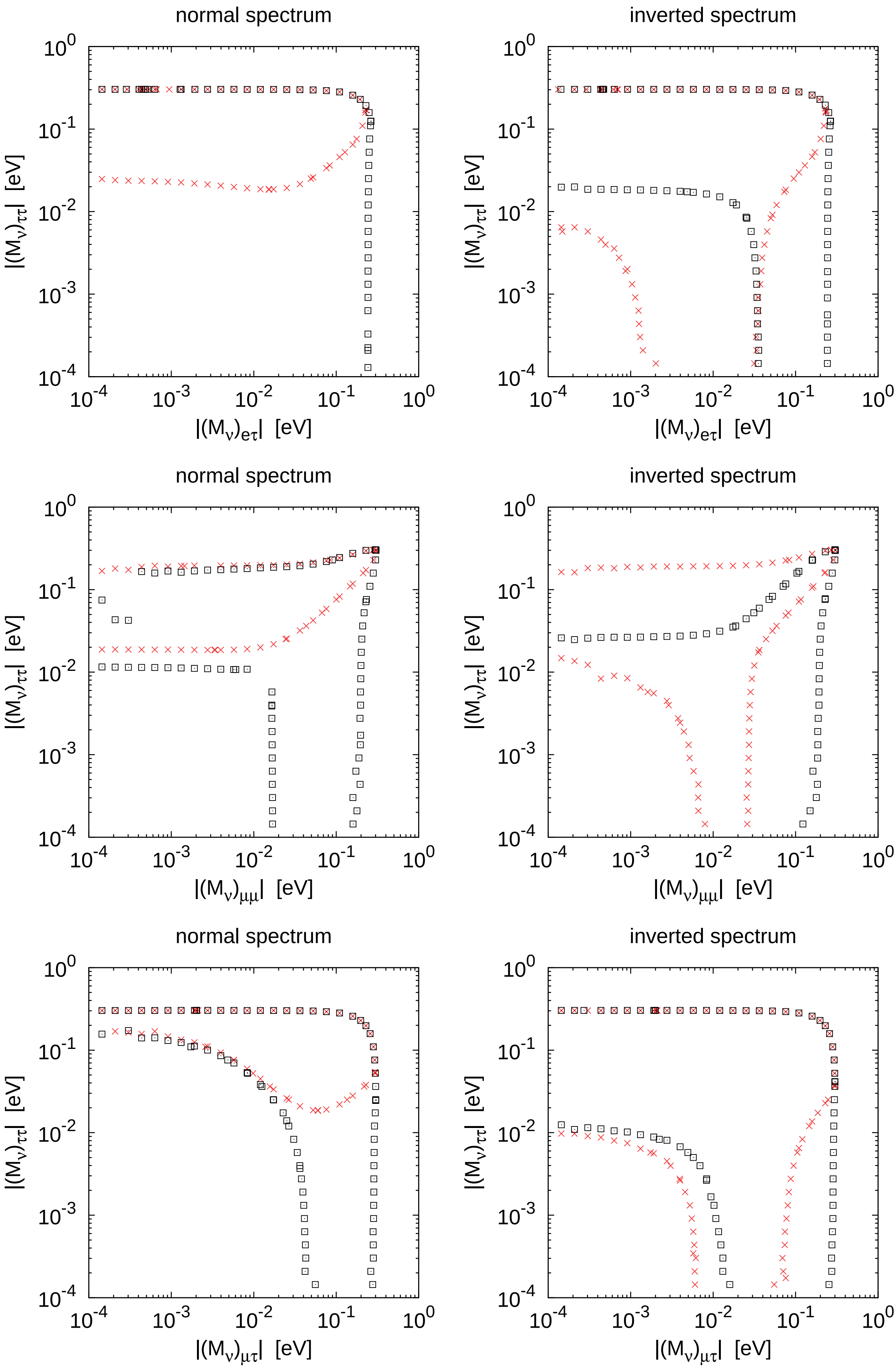
$|(M_\nu)_{\mu\tau}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{\mu\tau}|$ [eV]

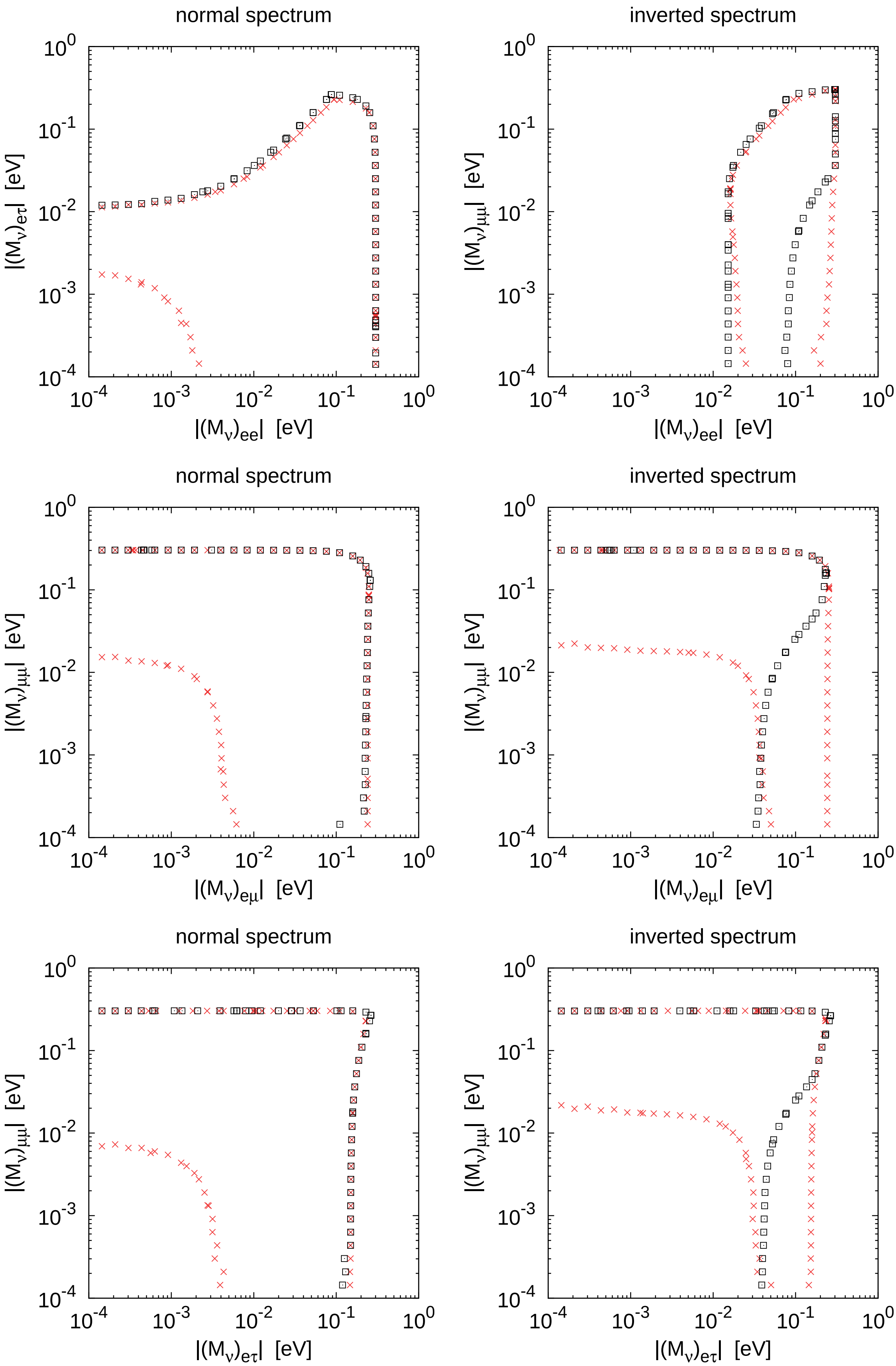
B Differences between the plots based on Fogli *et al.* and Forero *et al.* (version 3) at one sigma

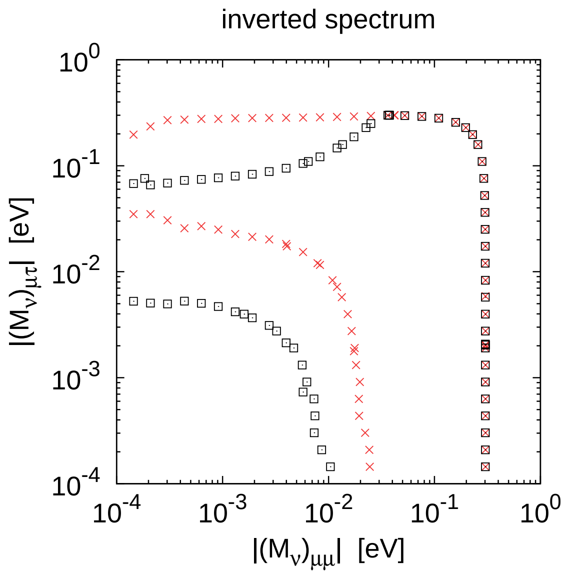
Here we present those correlations where there are notable differences between the plots based on Fogli *et al.* [4] and Forero *et al.* (version 3) [3] at the 1σ level. The following plots always contain the allowed 1σ areas for both Fogli *et al.* (bounded by red crosses $\times$) as well as Forero *et al.* (bounded by black boxes $\boxdot$). The plot title shows the neutrino mass spectrum.

We begin with the ten correlation plots involving $|(\mathcal{M}_\nu)_{\tau\tau}|$. Afterwards we show the seven remaining plots.




normal spectrum
inverted spectrum
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{e\tau}|$ [eV]
normal spectrum
inverted spectrum
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{\mu\mu}|$ [eV]
normal spectrum
inverted spectrum
$|(M_\nu)_{\tau\tau}|$ [eV]
$|(M_\nu)_{\mu\tau}|$ [eV]
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}


normal spectrum
inverted spectrum
|(Mν)eτ| [eV]
|(Mν)μμ| [eV]
|(Mν)ee| [eV]
normal spectrum
inverted spectrum
|(Mν)μμ| [eV]
|(Mν)eμ| [eV]
normal spectrum
inverted spectrum
|(Mν)μμ| [eV]
|(Mν)eτ| [eV]
10^0
10^-1
10^-2
10^-3
10^-4


inverted spectrum
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|(M_\nu)_{\mu\tau}|$ [eV]
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
$|(M_\nu)_{\mu\mu}|$ [eV]

Part III

Conclusions

This thesis deals with three main questions arising from the idea to use finite family symmetry groups as an approach to the lepton mass and mixing problem.

The first question is the one for the eligible symmetry groups which can be used as flavour symmetries in the lepton sector. Since there are three generations of fermions, the family symmetry group must possess a non-trivial three-dimensional representation. Clearly the task of a classification of all finite groups which possess such a three-dimensional representation would be a too ambitious goal for this thesis. Therefore we made a first restriction by considering only groups which possess a *faithful* three-dimensional representation, which leads us to the finite subgroups of $U(3)$. Since there is no systematic classification of the finite subgroups of $U(3)$ we decided to use the SmallGroups library [1] which contains information on all finite groups of order less than 2000. With the help of the computer algebra system GAP [2] we extracted all groups from the library which are of order smaller than 512, possess a faithful three-dimensional irreducible representation and are not isomorphic to direct product groups of the form $H \times \mathbb{Z}_n$ $(n > 1)$. We divided the resulting list of groups into two parts, namely those groups which possess a faithful three-dimensional irrep of determinant one ($SU(3)$-subgroups) and the remaining groups ($U(3)$-subgroups). Comparing the resulting list of finite subgroups of $SU(3)$ to the classification of finite $SU(3)$-subgroups given by Blichfeldt in [3], we find full agreement. Furthermore, for all obtained groups we constructed generators and verified the group order and the irreducibility of the defining three-dimensional representation.

In order to at least partly classify the finite subgroups of $U(3)$ of order smaller than 512, we proved a theorem (theorem III.1. in chapter 5) which allows the construction of finite subgroups of $U(3)$ from special finite subgroups of $SU(3)$. Since many well-known finite subgroups of $SU(3)$ fulfil the requirements of this theorem, we were able to construct several infinite series of finite subgroups of $U(3)$, namely

$$T_n(m) \cong \mathbb{Z}_n \rtimes \mathbb{Z}_{3^m}, \tag{9.1a}$$

$$\Delta(3n^2, m) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_{3^m}, \tag{9.1b}$$

$$\Delta(6n^2, m) \cong \Delta(3n^2) \rtimes \mathbb{Z}_{2^m} \quad \text{and} \tag{9.1c}$$

$$\Delta'(6n^2, j, k). \tag{9.1d}$$

By construction all these groups are subgroups of groups of the form $G \times \mathbb{Z}_a$ where G is a finite subgroup of $SU(3)$. However, they are not isomorphic to a direct product with a cyclic group themselves. All groups which belong to one of the series (9.1a)–(9.1d) and are of order smaller than 512 are contained in the list of $U(3)$-subgroups we extracted from the SmallGroups library.

We investigated two of the found $U(3)$-subgroups in detail. The smallest newly found $U(3)$-subgroup is a group of the form $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, which possesses the same tensor product decompositions and Clebsch-Gordan coefficients as the well-known $SU(3)$-subgroup $\Delta(27)$. The other group is $A_4 \rtimes \mathbb{Z}_4$, which is a subgroup of $S_4 \times \mathbb{Z}_4$.

Finally we cogitated about the question whether the newly found $U(3)$-subgroups enable the construction of models which cannot be built based on $SU(3)$-subgroups alone, and we found the general answer to be affirmative. However, as shown by the investigation of the group $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$, there are $U(3)$-subgroups for which there exists a corresponding $SU(3)$-subgroup allowing precisely the same Lagrangians.

Having elaborated on the finite subgroups of $U(3)$ we turned to the much more intensively studied finite subgroups of $SU(3)$. According to the classification of Blichfeldt [3] all finite subgroups of $SU(3)$ can be cast into one of five types called (A)–(E). Since many finite subgroups of $SU(3)$ are frequently used as family symmetry groups, a detailed investigation of these five classes is important in particular from the point of view of flavour physics. Though they contain the well-known group series $\Delta(3n^2)$ and $\Delta(6n^2)$ as subseries, the groups of type (C) and (D) are probably the least studied groups among the five classes. Therefore, the primary aim of the publication in chapter 6 was the investigation of the structure of the members of these two classes. It turned out that the key to the understanding of the groups of type (C) and (D) is the following theorem on Abelian finite subgroups of $SU(3)$:

> Every finite Abelian subgroup $\mathcal{A}$ of $SU(3)$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$, where
>
> $$m = \max_{a \in \mathcal{A}} \operatorname{ord} a$$
>
> and n is a divisor of m.

Using this theorem it is easy to show that the groups of type (C) are all of the form

$$(\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3, \tag{9.2}$$

where p is a divisor of m. In this way one can also see immediately that the $SU(3)$-subgroups $T_m \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$ and $\Delta(3n^2) \cong (\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3$ are special cases of groups of type (C). We may ask ourselves whether there are groups of type (C) which are not of the already known types T_m and $\Delta(3n^2)$. In fact, the answer is positive, and going through the list of finite subgroups of $SU(3)$ of order smaller than 512 derived in chapter 5 we could identify the smallest such group to be the group $C(9,1,1)$. Using the prescription derived in the publication included as chapter 6 we could even determine the detailed structure of $C(9,1,1)$ which is given by

$$C(9,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3. \tag{9.3}$$

The groups of type (D) are treated similarly yielding the result that every group of type (D) must be of the form

$$(\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes S_3. \tag{9.4}$$

Thus the well-known group series $\Delta(6n^2) \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ comprises a subseries of (D). Again we ask ourselves the question for "new" (i.e. previously unknown to the flavour physics community) groups of type (D). Indeed the group

$$D(9,1,1;2,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3, \tag{9.5}$$

which could be identified using the SmallGroups library, is such a group. The final question we answered was whether for every group G of type (D) there exists an irrep $\mathcal{D}$ of a suitable $\Delta(6n^2)$ such that

$$G \cong \mathcal{D}(\Delta(6n^2)). \tag{9.6}$$

While a similar relation holds for groups of type (C) and $\Delta(3n^2)$, the answer for the groups of type (D) is negative—we could explicitly show that the group $D(9,1,1;2,1,1)$ cannot be regarded as an irrep of some $\Delta(6n^2)$.

The second main question we wanted to address in the thesis concerns the possible restrictions finite family symmetries may introduce in the leptonic and scalar sector. Studying the simplest case, which is the case of Abelian finite family symmetry groups, we encountered the intensively studied possibility of texture zeros in the lepton mass matrices. Frequently a non-Abelian family symmetry is used to enforce maximal atmospheric neutrino mixing, i.e.

$$\theta_{23} = 45^\circ. \tag{9.7}$$

However, though Abelian symmetries cannot enforce the above exact relation [4], the approximate relation

$$\theta_{23} \approx 45^\circ \tag{9.8}$$

may be supported by means of an Abelian family symmetry. Indeed we could show that there are two cases of two texture zeros in the neutrino mass matrix (assuming a diagonal charged-lepton mass matrix) which together with the assumption of a *quasi-degenerate neutrino mass spectrum* lead to nearly maximal atmospheric neutrino mixing. An astonishing fact is that for these two cases (B_3 and B_4 in the terminology of [5]) this behaviour is independent of the values of the other two mixing angles. Indeed, for the two discussed texture zeros one finds the following approximate expressions for the sine squared of the atmospheric mixing angle:

$$\sin^2\theta_{23} \simeq \frac{1}{2} \mp \frac{1}{8}\frac{\Delta m_{31}^2}{m_1^2}(1+\sin^2\theta_{13}) + \ldots\,. \tag{9.9}$$

The different signs correspond to B_3 and B_4, respectively, and the dots indicate corrections of order $(\Delta m_{31}^2/m_1^2)^2$ and $\Delta m_{21}^2/m_1^2$. Moreover we find that within the discussed setting exact vanishing of the reactor mixing angle is not allowed, though values as small as $\sin^2\theta_{13} = 10^{-4}$ are still possible. Furthermore, provided that the reactor mixing angle is not too small, CP violation in neutrino oscillations is predicted to be almost maximal in the limit of quasi-degeneracy of the neutrino masses. To summarize, we have shown that close to maximal atmospheric neutrino mixing could have an origin different from non-Abelian symmetries (as for example the μ-τ interchange symmetry)—it could just be a consequence of texture zeros and quasi-degenerate neutrino masses.

Inspired by the earlier hints from T2K [6] and MINOS [7] for a non-zero reactor mixing angle (which have later been impressively confirmed by Double Chooz [8], Daya Bay [9] and RENO [10]) we also investigated the predictions of the seven viable types of two texture zeros [5] on the reactor mixing angle θ_{13}. As shown in chapter 8, in the framework of these seven types of two texture zeros, using the experimental ranges for

$$\Delta m_{21}^2,\, \Delta m_{31}^2,\, \theta_{12} \quad \text{and} \quad \theta_{23} \tag{9.10}$$

as an input, one can derive a relation between the reactor mixing angle θ_{13} and the Dirac phase δ. Using this relation we could create correlation plots of $\sin^2\theta_{13}$ versus $\cos\delta$. Two of these seven types of two texture zeros (A_1 and A_2 in the terminology of [5]) enforce a non-zero reactor mixing angle at 3σ. The four cases of type B all predict nearly maximal CP violation (i.e. $\cos\delta \approx 0$). Finally, case C does not impose strong constraints on θ_{13} and δ.

The last question we investigated within this thesis was whether one could infer symmetries in the lepton sector from the presently available observational and experimental data on the neutrino physics observables. Taking advantage of the enormous improvement in our knowledge of the neutrino oscillation parameters, we decided to redo an analysis of the elements of the neutrino mass matrix originally performed by Merle and Rodejohann [11]. Their analysis resulted in plots of the allowed ranges for the absolute values of the elements of the neutrino mass matrix. Apart from the repetition of this study in the light of the new data, since in the context of textures of M_ν the correlations of its elements are of particular interest, our main goal was the construction of corresponding correlation plots.

Assuming Majorana neutrinos and working in a basis where the charged-lepton mass matrix is given by

$$M_\ell = \mathrm{diag}(m_e,\, m_\mu,\, m_\tau), \tag{9.11}$$

we could analytically derive upper and lower bounds on the absolute values of the elements of M_ν. The generic upper bound on the elements of M_ν turns out to be equal to the largest neutrino mass. The lower bound is more complicated, but can easily be evaluated numerically. For a long time the only non-trivial lower bound on an element of M_ν has been the lower bound on $|(M_\nu)_{ee}|$ in the case of an inverted neutrino mass spectrum. However, the global fit results of the group of Fogli *et al.* [12] also imply a non-trivial lower bound on $|(M_\nu)_{\tau\tau}|$ in the case of a normal mass spectrum (the same bound derived from the fit results of Forero *et al.* [13] is still trivial).

The correlation plots were created using a numerical method based on the Nelder-Mead algorithm, which is outlined in chapter 3. Concerning the different global fits [12] and [13], we find that the correlation plots all agree at the 3σ level. However, at the 1σ level this is no longer true, the most striking fact being that the fit results of Fogli *et al.* [12] no longer allow a vanishing $(M_\nu)_{\tau\tau}$ at 1σ.

Among the 30 possible correlations (15 pairings, 2 possible neutrino mass spectra) we find only five to be stringent at the 3σ level, namely

$$\begin{array}{lll}
|(M_\nu)_{ee}| & \text{vs.} & |(M_\nu)_{\mu\mu}| \quad \text{(normal spectrum)} \\
|(M_\nu)_{ee}| & \text{vs.} & |(M_\nu)_{\mu\tau}| \quad \text{(normal spectrum)} \\
|(M_\nu)_{ee}| & \text{vs.} & |(M_\nu)_{\tau\tau}| \quad \text{(normal spectrum)} \\
|(M_\nu)_{\mu\mu}| & \text{vs.} & |(M_\nu)_{\mu\tau}| \quad \text{(normal spectrum)} \\
|(M_\nu)_{\mu\tau}| & \text{vs.} & |(M_\nu)_{\tau\tau}| \quad \text{(normal spectrum)}
\end{array}$$

All these correlations may be subsumed as: "If one matrix element is small, the other one must be large." Another gain from our analysis is that the correlation plots beautifully show the experimentally allowed cases of two texture zeros in the neutrino mass matrix. In summary, we have obtained interesting correlations among the elements of M_ν. Consequently our analysis allows to exclude models which produce textures which are in conflict with the correlation plots, and may thus serve as a helpful tool for model building.

I hope that the results of this thesis will be helpful for future research on the fermion mass and mixing problem.

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Appendix A

Further details on the used numerical techniques

A.1 Implementation of the Nelder-Mead method

In this section we want to provide some more detailed information on the implementation of the Nelder-Mead algorithm in this dissertation.

The Nelder-Mead algorithm as outlined in section 3.2 has been implemented in the programming language C [1]. For the compilation of the source code we used gcc [2]. The reason to develop the program "from scratch" was the possibility to adapt parts of the program easily to special purposes and to make sure that the program follows the original algorithm by Nelder and Mead [3]. The choice of the programming language went for C because it is much faster than most interpreted languages or computer algebra systems. The validity of the program was checked by application to a series of test functions, and also through application to numerical problems with known solution, for example taken from investigations of special neutrino mass matrices in the literature. In particular in combination with the χ^2-method we tested the algorithm on many occasions and it always produced the expected results.

Our choice for the parameters of the Nelder-Mead algorithm are as follows:

parameter	symbol	value
reflection parameter	ρ	1
expansion parameter	χ	2
contraction parameter	γ	1/2
shrinkage parameter	σ	1/2
accuracy parameter	ε	10^{-16}
maximal number of steps in one iteration	$N_{\max}$	50000

The values for ρ, χ, γ and σ are the standard values suggested in [3].

All floating point numbers were stored with "double precision" (eight bytes). Since the algorithm also needs a fast and reliable source of random numbers, we used the unix pseudorandom number generator /dev/urandom.

Finally we want to comment on the creation of the random start simplices and our implementation of the "Nelder-Mead plus perturbations" method. Consider the Nelder-Mead algorithm for the minimization of a function

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ y &\mapsto f(y). \end{aligned} \tag{A.1}$$

According to the structure of f and/or the physical interpretation of the variables y, there will be some domain D for y, which we want to take into account when we construct the start simplices. Since the Nelder-Mead algorithm "lives" on the whole space[1] $\mathbb{R}^n$, the precise form of D will not be relevant when used as a guideline for the range of the start simplices. Therefore, for the construction of the start simplices we can safely approximate the constraints imposed by D as

$$y \in [y_{\text{min},1}, y_{\text{max},1}] \times \ldots \times [y_{\text{min},n}, y_{\text{max},n}]. \tag{A.4}$$

Then a start simplex

$$\{y_1, \ldots, y_{n+1}\} \tag{A.5}$$

can be constructed by assigning the coordinates $(y_j)_k$ random values in $[y_{\text{min},k}, y_{\text{max},k}]$ $(k = 1, \ldots, n;\ j = 1, \ldots, n+1)$.

We implemented two different versions of perturbations of simplices. Suppose we have arrived at a simplex with best vertex x_{best}, which we want to perturb. The simplest possibility is to define

$$x_{\text{min},i} \equiv x_{\text{best},i} - p|x_{\text{best},i}| \quad \text{and} \quad x_{\text{max},i} \equiv x_{\text{best},i} + p|x_{\text{best},i}|, \tag{A.6}$$

where p is a real parameter measuring the size of the perturbation (*perturbation parameter*).[2] Then a perturbed simplex can be constructed in the same way as a random start simplex with x_{min} and x_{max} as ranges. We have used this type of perturbations in particular for application in the χ^2-analysis. However, we can refine the method by keeping the previous best vertex as one of the vertices of the perturbed simplex. For this purpose we define a vector L by

$$L_i \equiv p(y_{\text{max},i} - y_{\text{min},i}), \tag{A.7}$$

where p is again the perturbation parameter. We keep the best vertex x_{best} as one vertex of the perturbed simplex and the other vertices x_j $(j = 2, \ldots, n+1)$ are constructed by assigning their coordinates values of the form

$$(x_j)_k \equiv x_{\text{best,k}} + \text{random}(-L_k, L_k). \tag{A.8}$$

[1]The algorithm itself is not capable of respecting any boundaries, and all constraints have to be included in the function f. If for example y shall be restricted to the domain D, one has to replace $f(y)$ by

$$f(y) + \Pi_D(y), \tag{A.2}$$

where Π_D is a suitable numerical approximation to the characteristic function

$$\Pi_D(y) = \begin{cases} 0 & \text{for} \quad y \in D, \\ \infty & \text{for} \quad y \notin D. \end{cases} \tag{A.3}$$

[2]By successively decreasing the perturbation parameter after every perturbation, one can mimic the "simulated annealing" effect [4]; see also section 3.2.

The expression random$(-L_k, L_k)$ denotes a random value in the interval $[-L_k, L_k]$. This type of perturbations has, in the course of this thesis, been successfully used for the computation of lower bounds on functions depending on the oscillation parameters.

A.2 Diagonalization of mass matrices

As discussed in section 2.1, in order to compute the lepton mixing matrix we need to diagonalize the mass matrices of the charged leptons and the light neutrinos. As a general complex 3×3-matrix the charged lepton mass matrix can be diagonalized by means of theorem 1 (see page 18). The process of diagonalizing a matrix in this way is called *singular value decomposition (SVD)*. In order to do the SVD numerically, we rely on the library LAPACK [5, 6] which provides routines for the SVD for the programming language FORTRAN. Since for the Nelder-Mead algorithm we use the programming language C, we use the C-version CLAPACK [7] of LAPACK.

According to theorem 2 (see page 21) the diagonalization of a symmetric $n \times n$-matrix M can be done by a single unitary matrix W via

$$W^T M W = \mathrm{diag}(m_1, \ldots, m_n) \equiv \hat{M}. \tag{A.9}$$

Unfortunately, the LAPACK library does not provide a routine for the construction of W (which we need in the case of Majorana mass matrices), so we have to rely on the standard singular value decomposition

$$U^\dagger M V = \mathrm{diag}(m_1, \ldots, m_n) = \hat{M} \tag{A.10}$$

and have to find a way to relate W to the unitary matrices U and V (which the SVD routine of LAPACK tells us). From equations (A.9) and (A.10) we find that

$$M M^\dagger = U \hat{M}^2 U^\dagger = W^* \hat{M}^2 W^T \tag{A.11}$$

from which follows that

$$(W^T U) \hat{M}^2 (W^T U)^\dagger = \hat{M}^2. \tag{A.12}$$

If $\hat{M}^2$ is non-degenerate (which must be the case for the light neutrino mass matrix, since both independent mass-squared differences are non-zero) this implies that

$$W^T U = e^{i\hat{\alpha}}, \tag{A.13}$$

where

$$\hat{\alpha} = \mathrm{diag}(\alpha_1, \ldots \alpha_n) \qquad (0 \le \alpha_i < 2\pi) \tag{A.14}$$

is a diagonal $n \times n$-matrix of phases. Calculating $e^{i\hat{\alpha}}$ via

$$U^\dagger M U^* = e^{-2i\hat{\alpha}} \hat{M} \tag{A.15}$$

allows for the construction of

$$W = U^* e^{i\hat{\alpha}}. \tag{A.16}$$

A.3 The inverse of the parametrization of the mixing matrix

In section 2.1 we introduced the parametrization

$$U_{\mathrm{PMNS}} = D_1 V D_2 \tag{A.17}$$

with

$$D_1 = \mathrm{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma}), \tag{A.18}$$
$$D_2 = \mathrm{diag}(e^{i\rho}, e^{i\sigma}, 1), \tag{A.19}$$
$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{A.20}$$

for the lepton mixing matrix $U \equiv U_{PMNS}$—see equations (2.19) to (2.23). In the course of our numerical studies we will frequently need a method to extract the nine parameters

$$\alpha,\ \beta,\ \gamma,\ \rho,\ \sigma,\ \theta_{12},\ \theta_{23},\ \theta_{13} \quad \text{and} \quad \delta \tag{A.21}$$

from the PMNS-matrix $U = U_{\mathrm{PMNS}}$. For a long time the so-called Harrison-Perkins-Scott mixing matrix [8]

$$V_{\mathrm{HPS}} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \tag{A.22}$$

has been a good approximation to the mixing matrix V, showing that the only element of V which could be zero is V_{13}. However, due to our improved knowledge on the reactor mixing angle θ_{13} we know that also V_{13} must be non-zero. Therefore none of the elements of V and U can vanish, a fact which we will need for the calculation of the parameters of U.

We will now present our approach for the calculation of the mixing angles and phases from U. At this point one has to keep in mind that by definition $\theta_{ij} \in [0, \pi/2]$ and thus

$$s_{ij} \equiv \sin\theta_{ij} \geq 0 \quad \text{and} \quad c_{ij} \equiv \cos\theta_{ij} \geq 0. \tag{A.23}$$

Let us begin with the computation of the phases α, β, γ, ρ and σ. Knowing that V_{11}, V_{12}, V_{23} and V_{33} are real and positive, we find the equations

$$\alpha + \rho = \mathrm{Arg}\, U_{11}, \tag{A.24a}$$
$$\alpha + \sigma = \mathrm{Arg}\, U_{12}, \tag{A.24b}$$
$$\beta = \mathrm{Arg}\, U_{23}, \tag{A.24c}$$
$$\gamma = \mathrm{Arg}\, U_{33}. \tag{A.24d}$$

Thus we already know β and γ. With

$$\alpha + \beta + \gamma + \rho + \sigma = \mathrm{Arg}(\det U) \tag{A.25}$$

we can compute the remaining phases to

$$\sigma = \mathrm{Arg}(\det U) - \beta - \gamma - \mathrm{Arg}\, U_{11}, \tag{A.26a}$$
$$\alpha = \mathrm{Arg}\, U_{12} - \sigma, \tag{A.26b}$$
$$\rho = \mathrm{Arg}\, U_{11} - \alpha. \tag{A.26c}$$
$$\tag{A.26d}$$

Knowing all phases (except δ) allows for the computation of V. The Dirac phase δ is then given by

$$\delta = -\mathrm{Arg}\, V_{13} \tag{A.27}$$

and the same matrix element gives us

$$\sin^2\theta_{13} = |V_{13}|^2. \tag{A.28}$$

Since the extreme case $\sin^2\theta_{13} = 1$ is excluded by experiment we obtain the sines squared of the other two mixing angles as

$$\sin^2\theta_{12} = \frac{V_{12}^2}{1 - \sin^2\theta_{13}} \quad \text{and} \quad \sin^2\theta_{23} = \frac{V_{23}^2}{1 - \sin^2\theta_{13}}. \tag{A.29}$$

Due to the convention that the three mixing angles can assume values between zero and 90 degrees, the sines squared uniquely determine the mixing angles.

Let us finally note that in the course of a χ^2-minimization, or any other numerical investigation of a model, the physical constraints like $V_{ij} \neq 0$ or $\sin^2\theta_{13} \neq 1$ are not necessarily fulfilled. However, since the corresponding mixing matrices are excluded by experiment we can still use the method described above. In order to avoid possible problems, we implemented a series of checks and error messages which are displayed by our programs when problems with the inversion of the parametrization of the mixing matrix occur. Also if for a given mixing matrix the parameters are not unique our program immediately shows error messages describing the type of error and at which point it occurred.

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Appendix **B**

Proof of theorem 5

Theorem 5. There exists a non-trivial Yukawa coupling

$$\mathcal{L} = -\overline{D_{\alpha L}}\,\Gamma_{j\alpha\beta}\phi_j\ell_{\beta R} + \text{H.c.} \tag{B.1}$$

invariant under the transformations

$$D_L \mapsto \mathcal{D}_L(g)D_L,\;\; \ell_R \mapsto \mathcal{D}_R(g)\ell_R,\;\; \Phi \mapsto \mathcal{D}_\Phi(g)\Phi \quad \forall\, g \in G, \tag{B.2}$$

if and only if the tensor product $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$ contains the trivial representation $\mathbf{1}$, i.e.

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R = \mathbf{1} \oplus \;\ldots\;\; . \tag{B.3}$$

Proof. Transforming $\mathcal{L}$ according to the transformation (B.2) leads to

$$\mathcal{L} = -\overline{D_L}\Gamma_j\phi_j\ell_R + \text{H.c.} \longmapsto -\overline{D_L}\,(\mathcal{D}_L^\dagger\Gamma_j\mathcal{D}_R)(\mathcal{D}_\Phi)_{jk}\,\phi_k\ell_{\beta R} + \text{H.c.} \tag{B.4}$$

Invariance thus implies

$$(\mathcal{D}_L^\dagger\Gamma_j\mathcal{D}_R)(\mathcal{D}_\Phi)_{jk} = \Gamma_k \quad \forall g \in G \tag{B.5}$$

for the Yukawa-coupling matrices Γ_j. Clearly, a Yukawa-coupling invariant under the family symmetry group exits if and only if equation (B.5) possesses a non-trivial solution. Taking into account that we can assume all involved representations to be unitary, we can reformulate equation (B.5) as:

$$\begin{aligned}
&\Gamma_j(\mathcal{D}_\Phi)_{jk} = \mathcal{D}_L\Gamma_k\mathcal{D}_R^\dagger \quad \text{/rename indices}\\
&\Gamma_k(\mathcal{D}_\Phi)_{kj} = \mathcal{D}_L\Gamma_j\mathcal{D}_R^\dagger \quad /\times(\mathcal{D}_\Phi^\dagger)_{jl}\\
&\Gamma_l = \mathcal{D}_L\Gamma_j\mathcal{D}_R^\dagger(\mathcal{D}_\Phi^*)_{lj} \quad \text{/rename indices}\\
&\Gamma_k = \mathcal{D}_L\Gamma_j\mathcal{D}_R^\dagger(\mathcal{D}_\Phi^*)_{kj} \quad /^*\\
&\Gamma_k^* = \mathcal{D}_L^*\Gamma_j^*\mathcal{D}_R^T(\mathcal{D}_\Phi)_{kj}.
\end{aligned}$$

In components the result of the above calculation reads

$$(\mathcal{D}_\Phi)_{kj}(\mathcal{D}_L^*)_{\rho\alpha}(\mathcal{D}_R)_{\sigma\beta}\,\Gamma_{j\alpha\beta}^* = \Gamma_{k\rho\sigma}^*. \tag{B.6}$$

We will now show that there exists a common eigenvector v to the eigenvalue 1 to all matrices

$$\mathcal{D}_\Phi(g) \otimes \mathcal{D}_L^*(g) \otimes \mathcal{D}_R(g). \tag{B.7}$$

In order to define v, we have to introduce bases in the vector spaces the representations $\mathcal{D}_\Phi$, $\mathcal{D}_L$ and $\mathcal{D}_R$ act on. We denote the corresponding basis vectors by

$$e_j, \quad f_\alpha \quad \text{and} \quad g_\beta, \tag{B.8}$$

respectively. Using these bases we define

$$v \equiv \Gamma^*_{j\alpha\beta}\, e_j \otimes f_\alpha \otimes g_\beta. \tag{B.9}$$

Indeed, using equation (B.6), we find

$$(\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R)\, v = \underbrace{\Gamma^*_{j\alpha\beta}\,(\mathcal{D}_\Phi)_{kj}(\mathcal{D}_L^*)_{\rho\alpha}(\mathcal{D}_R)_{\sigma\beta}}_{\Gamma^*_{k\rho\sigma}}\, e_k \otimes f_\rho \otimes g_\sigma = v. \tag{B.10}$$

Thus v spans a one-dimensional invariant subspace and therefore $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$ is reducible. Since v is a common eigenvector to the eigenvalue 1, we arrive at the desired result

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R = \mathbf{1} \oplus \ldots \ . \tag{B.11}$$

□

Since

$$v = \Gamma^*_{j\alpha\beta}\, e_j \otimes f_\alpha \otimes g_\beta \tag{B.12}$$

spans the invariant subspace corresponding to $\mathbf{1}$ in the tensor product $\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R$, the numbers $\Gamma^*_{j\alpha\beta}$ are by definition proportional to the Clebsch-Gordan coefficients for the decomposition

$$\mathcal{D}_\Phi \otimes \mathcal{D}_L^* \otimes \mathcal{D}_R \to \mathbf{1}. \tag{B.13}$$

Thus we have found the general result:

> *The coefficients of the invariants are the complex conjugates of the Clebsch-Gordan coefficients.*

Appendix C

Conventions and Symbols

Units: We will use units in which $\hbar = c = 1$. Thus [1]

$$\hbar c = 197.3269718(44)\,\mathrm{MeV\,fm} = 1. \tag{C.1}$$

Indices and summation: If not stated otherwise, we will always use Einstein's summation convention, which means that it is always summed over equal indices.

- Lorentz indices: Greek letters, $\mu = 0, 1, 2, 3$.
- Spatial indices: Latin letters, $i = 1, 2, 3$.

Minkowski metric: We will use the following convention for the Minkowski metric in Cartesian coordinates:

$$(g_{\mu\nu}) = (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{C.2}$$

SU(2): The Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{C.3}$$

comprise a basis of the Lie algebra of $SU(2)$. They satisfy the commutation relations

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\varepsilon_{ijk}\frac{\sigma^k}{2}.$$

If we work with $SU(2)_L$ we will use the symbol τ^i instead of σ^i.

Dirac-γ-matrices If not stated otherwise we will always use the *chiral representation* (*Weyl basis*) of the Dirac-γ-matrices γ^μ:

$$\gamma^0 = \begin{pmatrix} \mathbf{0}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbf{0}_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} \mathbf{0}_2 & \sigma^i \\ -\sigma^i & \mathbf{0}_2 \end{pmatrix}. \tag{C.4}$$

In this basis γ^5 is given by

$$\gamma^5 = i\varepsilon_{\mu\nu\sigma\lambda}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{C.5}$$

Here $\mathbb{1}_2$ is the 2×2 identity-matrix, and $\mathbf{0}_2$ is the 2×2 zero-matrix.

Properties of the γ-matrices in the chiral representation: Apart from the general anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4, \tag{C.6}$$

in the Weyl basis the gamma matrices fulfil:

$$\gamma^{0*} = (\gamma^0)^T = \gamma^{0\dagger} = \gamma^0, \tag{C.7a}$$
$$\gamma^{i\dagger} = -\gamma^i, \gamma^{5\dagger} = \gamma^5, \tag{C.7b}$$
$$(\gamma^0)^2 = \mathbb{1}_4, (\gamma^i)^2 = -\mathbb{1}_4, (\gamma^5)^2 = \mathbb{1}_4, \tag{C.7c}$$
$$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu. \tag{C.7d}$$

The chiral projectors are given by:

$$P_L \equiv \frac{\mathbb{1}_4 - \gamma^5}{2} = \begin{pmatrix} \mathbb{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix}, \quad P_R \equiv \frac{\mathbb{1}_4 + \gamma^5}{2} = \begin{pmatrix} \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbb{1}_2 \end{pmatrix}. \tag{C.8}$$

The charge conjugation matrix: Charge conjugation is defined as an operation which maps a solution of the Dirac equation

$$(i\gamma^\mu(\partial_\mu + ieQA_\mu) - m)\psi(x) = 0 \tag{C.9}$$

into a solution $\psi^c(x)$ of the charge conjugate Dirac equation

$$(i\gamma^\mu(\partial_\mu - ieQA_\mu) - m)\psi^c(x) = 0. \tag{C.10}$$

A solution is given by

$$\psi^c(x) = C\overline{\psi(x)}^T, \tag{C.11}$$

where the *charge conjugation matrix* C must fulfil

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu. \tag{C.12}$$

In the chiral representation of the gamma matrices one finds that

$$C = e^{i\alpha}\gamma^2\gamma^0 \qquad \alpha \in \mathbb{R} \text{ (arbitrary)} \tag{C.13}$$

is a solution of equation (C.12). Some important properties of C are:

$$C^T = -C \tag{C.14a}$$

$$C(\gamma^5)^T C^{-1} = \gamma^5. \tag{C.14b}$$

In a basis in which γ^0 is Hermitian and the γ^i are anti-Hermitian we furthermore have

$$C^\dagger = C^{-1}. \tag{C.15}$$

Frequently used symbols:

ε_{ijk}	Totally antisymmetric ε-symbol in three dimensions, $\epsilon_{123} = 1$
$\varepsilon_{\alpha\beta\gamma\delta}$	Totally antisymmetric ε-symbol in four dimensions, $\epsilon_{0123} = 1$
σ^i, τ^i	The Pauli matrices
$\mathbb{1}_n$	The $n \times n$ identity-matrix
$\mathbf{0}_n$	The $n \times n$ zero-matrix
$\mathbb{N}$	The natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$
$\mathbb{Z}$	The integer numbers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{R}$	The real numbers
$\mathbb{C}$	The complex numbers
$\mathbb{Z}_n$	The cyclic group of order n
A_n	The group of even permutations of n symbols
S_n	The group of all permutations of n symbols
$G \rtimes H$	The semi-direct product of the groups G and H.
	(We use the convention where G is a normal subgroup of $G \rtimes H$.)
H.c.	Hermitian conjugate
VEV	Vacuum expectation value
$\mathrm{Tr}\, A$	Trace of the matrix A
$\mathbf{1}$	The trivial representation of a group
$\mathrm{diag}(a_1, \ldots, a_n)$	The $n \times n$ diagonal matrix with diagonal elements $a_1, \ldots, a_n$.

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Zusammenfassung

Das Problem der Leptonmassen und -mischung ist eines der interessantesten auf dem Gebiet der Teilchenphysik jenseits des Standardmodells. Ein möglicher Ansatz zur zumindest teilweisen Lösung dieses Problems ist die Einführung endlicher Familiensymmetrien im leptonischen und skalaren Sektor. Die vorliegende Dissertation befasst sich mit drei in diesem Zusammenhang auftretenden Fragen.

Die erste zu beantwortende Frage ist jene nach den möglichen endlichen Gruppen, welche als Familiensymmetrien herangezogen werden können. Da es drei Generationen von Fermionen gibt, muss die entsprechende Gruppe zumindest eine nicht-triviale dreidimensionale Darstellung besitzen. Beschränken wir uns weiters nur auf jene Gruppen, die treue dreidimensionale Darstellungen haben, gelangen wir zu den endlichen Untergruppen von $U(3)$. Mit Hilfe der sogenannten SmallGroups Library war es möglich eine Liste aller endlicher $U(3)$-Untergruppen von Ordnung kleiner als 512 zu erzeugen, die eine treue dreidimensionale irreduzible Darstellung besitzen und nicht in ein direktes Produkt mit einer zyklischen Gruppe zerfallen. Im Zuge dieser Untersuchung konnte ein Satz bewiesen werden, der die Konstruktion mehrerer unendlicher Serien von $U(3)$-Untergruppen erlaubt. Da insbesondere die endlichen Untergruppen von $SU(3)$ beliebte Familiensymmetrien darstellen, konzentrierten wir uns auch auf diese Gruppen. Bereits zu Beginn des 20. Jahrhunderts stellte H.F. Blichfeldt eine Klassifikation aller endlicher Untergruppen von $SU(3)$ in fünf Klassen (A)–(E) auf, jedoch wurden nicht alle dieser fünf Klassen gleich intensiv untersucht. Insbesondere das Wissen um die allgemeine Struktur der Gruppen der Klassen (C) und (D) war bisher, obwohl diese Klassen die wohlbekannten Gruppenserien $\Delta(3n^2)$ und $\Delta(6n^2)$ enthalten, eher spärlich. Mit Hilfe eines Theorems über die allgemeine Struktur der abelschen Untergruppen von $SU(3)$ gelang es uns, interessante Erkenntnisse über die allgemeine Struktur der $SU(3)$-Untergruppen der Typen (C) und (D) zu gewinnen.

Die zweite behandelte Frage ist jene, auf welche Weise Symmetrien die Massen- und Mischungsmatrizen einschränken. Den einfachsten Fall behandelnd (das ist der Fall von abelschen Familiensymmetrien), stießen wir auf den intensiv untersuchten Fall von sogenannten Texture Zeros in den Lepton-Massenmatrizen. In dieser Hinsicht konzentrierten wir uns auf jene sieben Fälle von zwei Texture Zeros in der Neutrinomassenmatrix (für den Fall einer diagonalen Massenmatrix der geladenen Leptonen), die mit dem Experiment verträglich sind. Wir konnten zeigen, dass in zwei dieser sieben Fälle die Annahme eines quasi-entarteten Neutrinomassenspektrums automatisch nahezu maximale atmosphärische Neutrinomischung bewirkt (unabhänging von den anderen beiden Mischungswinkeln). Des Weiteren untersuchten wir die Vorhersagen der sieben Arten

von zwei Texture Zeros im Lichte der neuen Messdaten zum Reaktormischungswinkel. Dabei fanden wir unter Verwendung der experimentell erlaubten Wertebereiche für die Oszillationsparameter, dass zwei dieser sieben Arten von Texture Zeros auf 3σ einen nichtverschwindenden Reaktormischungswinkel erzwingen. Im Zuge dieser Studie untersuchten wir auch die Korrelation zwischen der Dirac-Phase δ und dem Reaktormischungswinkel.

Zu guter Letzt untersuchten wir, ob es möglich ist, Symmetrien im Leptonsektor aus den derzeit vorhandenen experimentellen Daten abzuleiten. Unter der Annahme einer diagonalen Massenmatrix der geladenen Leptonen, konnten wir analytisch obere und untere Schranken an die Absolutbeträge der Elemente der Neutrinomassenmatrix gewinnen. Weiters konnten wir mit Hilfe einer numerischen Analyse, basierend auf dem sogenannten Nelder-Mead-Algorithmus, Korrelationsplots der Absolutbeträge der Elemente von M_ν erzeugen.

Die resultierenden Plots zeigen nicht nur die derzeitigen Einschränkungen an die Modelle mit zwei Texture Zeros, sondern zeigen auch starke Korrelationen zwischen einigen Paaren von Matrixelementen. In diesem Sinne erlaubt unsere Analyse den Ausschluss von Modellen, die mit den Plots unverträgliche Texturen der Neutrinomassenmatrix vorhersagen, und wird so zu einem nützlichen Werkzeug für die Konstruktion neuer Modelle.

Statement of the thesis advisor

I confirm that Patrick Otto Ludl has provided the main contribution to the following five papers, which are part of his thesis:

- P.O. Ludl, *On the finite subgroups of $U(3)$ of order smaller than 512.* J. Phys. **A 43** (2010) 395204 [Erratum-ibid. **A 44** (2011) 139501], [arXiv:1006.1479].
- P.O. Ludl, *Comments on the classification of the finite subgroups of SU(3).* J. Phys. **A 44** (2011) 255204 [Erratum-ibid. **A 45** (2012) 069502], [arXiv:1101.2308].
- W. Grimus and P.O. Ludl, *Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses.* Phys. Lett. **B 700** (2011) 356, [arXiv:1104.4340].
- P.O. Ludl, S. Morisi and E. Peinado, *The Reactor mixing angle and CP violation with two texture zeros in the light of T2K.* Nucl. Phys. **B 857** (2012) 411, [arXiv:1109.3393].
- W. Grimus and P.O. Ludl, *Correlations of the elements of the neutrino mass matrix*, arXiv:1209.2601.

According to the common practice in particle physics, the authors are listed in alphabetical order. Nevertheless Patrick Otto Ludl is the main author of the above publications. In particular he has written the major part of each paper himself.

Ao. Univ.-Prof. Dr. Walter Grimus

Curriculum vitae - Patrick Otto Ludl

Name: Mag. rer. nat. Patrick Otto Ludl

Born: March 22nd, 1985 in Wiener Neustadt, Austria

Education

School education

September 1991–June 1995	Attendance of the Volksschule Baumkirchnerring-West (primary school) in Wiener Neustadt.
September 1995–June 2003	Attendance of the Bundesrealgymnasium Gröhrmühlgasse mit naturwissenschaftlichem Schwerpunkt (secondary school) in Wiener Neustadt.
1999–2003	Participation in the Austrian Chemistry Olympiad (competition for Lower Austria).
2000–2003	Participation in the Austrian Chemistry Olympiad (national competition).
July 5th–July 14th, 2002	Silver medal at the 34th International Chemistry Olympiad in Groningen (The Netherlands).
School year 2002/2003	Writing of a “Fachbereichsarbeit” (short thesis for the final examination (Matura) of secondary school) in the field of chemistry with the title “Die Chemie des Stickstoffs und die Rolle seiner Verbindungen für die Chemie der Atmosphäre” (“The chemistry of nitrogen and its role in the chemistry of the atmosphere”). Advisor: Mag. Jana Ederova.
June 5th, 2003	Matura (final examination of secondary school) passed with distinction.
July 5th–July 14th, 2003	Silver medal at the 35th International Chemistry Olympiad in Athens (Greece).

Military service

Sept. 1st, 2003–April 30th, 2004	Military service.

University education

Since May 2004	Study at the University of Vienna.
October 2004–July 2009	Study of physics at the University of Vienna.
September 7th, 2005	1st part of the diploma examination passed with distinction.
July 11th, 2007	2nd part of the diploma examination passed with distinction.
August 2008–June 2009	Diploma thesis at the University of Vienna (thesis advisor: Ao. Univ.-Prof. Dr. Walter Grimus).
July 24th, 2009	3rd and final part of the diploma examination passed with distinction.
Since October 2009	PhD study in physics at the University of Vienna.
April 26th–Sept. 22nd, 2010	Position as university assistant ("prae doc") in the field of particle physics at the University of Vienna.
Nov. 1st, 2010–Mar. 23rd, 2011	Position as university assistant ("prae doc") in the field of particle physics at the University of Vienna.
Since February 1st, 2012	Position at the University of Vienna associated to the Austrian Science Fund (FWF) project No. P 24161-N16 "Dirac-Neutrinos und Familiensymmetrien" ("Dirac neutrinos and family symmetries").

Teaching at the University of Vienna

- *Exercises to "Mathematical methods in physics II" (Übungen zu "Mathematische Methoden der Physik II").*
  Course given at the University of Vienna in the winter semester 2009.

- *Exercises to "Mathematical methods in physics I" (Übungen zu "Mathematische Methoden der Physik I").*
  Course given at the University of Vienna in the summer semester 2010.

- *Seminar: "New developments in neutrino physics" (Seminar: "Neue Entwicklungen in der Neutrinophysik").*
  Seminar given (as co-organizer) at the University of Vienna in the summer semester 2010.

- *Exercises to "Theoretical Physics II: Quantum Mechanics I" (Übungen zu "Theoretische Physik II: Quantenmechanik I").*
  Course given at the University of Vienna in the summer semester 2012.

- *Exercises to "Theoretical Physics IV: Thermodynamics and Statistical Physics I" (Übungen zu "Theoretische Physik IV: Thermodynamik und Statistische Physik I").* Course given at the University of Vienna in the summer semester 2012.

List of publications

Research papers

- W. Grimus, L. Lavoura and P.O. Ludl, *Is S_4 the horizontal symmetry of tri-bimaximal lepton mixing?* J. Phys. **G 36** (2009) 115007, [arXiv:0906.2689].
- W. Grimus and P.O. Ludl, *Principal series of finite subgroups of $SU(3)$.* J. Phys. **A 43** (2010) 445209, [arXiv:1006.0098].
- P.O. Ludl, *On the finite subgroups of $U(3)$ of order smaller than 512.* J. Phys. **A 43** (2010) 395204 [Erratum-ibid. **A 44** (2011) 139501], [arXiv:1006.1479].
- P.O. Ludl, *Comments on the classification of the finite subgroups of SU(3).* J. Phys. **A 44** (2011) 255204 [Erratum-ibid. **A 45** (2012) 069502], [arXiv:1101.2308].
- W. Grimus and P.O. Ludl, *Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses.* Phys. Lett. **B 700** (2011) 356, [arXiv:1104.4340].
- P.O. Ludl, S. Morisi and E. Peinado, *The Reactor mixing angle and CP violation with two texture zeros in the light of T2K.* Nucl. Phys. **B 857** (2012) 411, [arXiv:1109.3393].
- P.M. Ferreira, W. Grimus, L. Lavoura and P.O. Ludl, *Maximal CP violation in lepton mixing from a model with $\Delta(27)$ flavour symmetry.* JHEP **1209** (2012) 128 [arXiv:1206.7072].
- W. Grimus and P.O. Ludl, *Two-parameter neutrino mass matrices with two texture zeros*, arXiv:1208.4515.
- W. Grimus and P.O. Ludl, *Correlations of the elements of the neutrino mass matrix*, arXiv:1209.2601.

Review article

- W. Grimus and P.O. Ludl, *Finite flavour groups of fermions.* J. Phys. **A 45** (2012) 233001 [arXiv:1110.6376].

Diploma thesis

- P.O. Ludl, *Systematic analysis of finite family symmetry groups and their application to the lepton sector.* Diploma thesis at the University of Vienna (2009) [arXiv:0907.5587].

Conference proceedings

- P.O. Ludl, *Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses* in proceedings:
  M. Hirsch, D. Meloni, S. Morisi, S. Pastor, E. Peinado, J.W.F. Valle, A. Adulpravitchai and D. Aristizabal Sierra *et al.*, *Proceedings of the first workshop on Flavor Symmetries and consequences in Accelerators and Cosmology (FLASY2011)*, arXiv:1201.5525.

- P.O. Ludl, *The finite subgroups of* $SU(3)$ in proceedings:
  I. de Medeiros Varzielas, C. Hambrock, G. Hiller, M. Jung, P. Leser, H. Pas, S. Schacht and M. Aoki *et al.*, *Proceedings of the 2nd Workshop on Flavor Symmetries and Consequences in Accelerators and Cosmology (FLASY12)*, arXiv:1210.6239.

Talks at international conferences

- *"Finite family symmetry groups."*
  Talk presented at the "48. Internationale Universitätswochen für theoretische Physik" ("Schladming winter school 2010"), March 2010, Schladming, Austria.

- *"Endliche Familiensymmetriegruppen im Leptonsektor."*
  Talk presented at the "42. Herbstschule für Hochenergiephysik, Maria Laach 2010", September 2010, Maria Laach, Germany.

- *"Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses."*
  Talk presented at the "First workshop on flavour symmetries (FLASY 2011)" July 2011, Valencia, Spain.

- *"Two texture zeros in neutrino mass matrices."*
  Talk presented at the "52$^{\text{nd}}$ Cracow school of theoretical physics" May 2012, Zakopane, Poland.

- *"The finite subgroups of* $SU(3)$*."*
  Talk presented at the "Second workshop on flavour symmetries (FLASY 2012)" July 2012, Dortmund, Germany.

Invited talks

- Talk on "Helpful tools in finite group theory" in the High Energy Physics Seminar at the University of Southampton on December 6$^{\text{th}}$, 2010.