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# MASTERARBEIT

Titel der Masterarbeit

## "Compactly Supported Cohomology Of Systolic Pseudomanifolds"

Verfasser

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# Abstract

We show that the second group of cohomology with compact supports is nontrivial for 3–dimensional systolic pseudomanifolds. As a prerequisite we review the basics on systolic complexes and compactly supported cohomology, and we explore basic examples.

# Zusammenfassung

Es wird gezeigt, dass die zweite Kohomologiegruppe mit kompaktem Träger für 3–dimensionale systolische Pseudomannigfaltigkeiten nichttrivial ist. Als Voraussetzung dafür werden die Grundlagen der systolischen Komplexe und der Kohomologie mit kompaktem Träger besprochen und einfache Beispiele angegeben.

# Contents

# 1 Introduction

Systolic complexes were introduced by T. Januszkiewicz and J. Świątkowski ([5]) and, independently, by F. Haglund([3]) and V. Chepoi ([1]) as combinatorial analogues of nonpositively curved spaces. They are simply connected simplicial complexes satisfying some local combinatorial conditions. Roughly speaking, there is a lower bound on the length of "essential" closed paths in the one-skeleton of every link.

This condition is an analogue of the Gromov condition implying nonpositive curvature for cubical complexes. However, systolic complexes equipped with the metric for which every simplex is isometric to the regular Euclidean simplex are not necessarily nonpositively curved. Conversely, there exist nonpositively curved spaces (e.g. manifolds of dimension at least three) that do not admit systolic triangulations. Nevertheless systolic spaces (sometimes referred to as complexes of *simplicial nonpositive curvature* - SNPC) possess many properties analogous to those of spaces of nonpositive curvature. They are contractible (this is an analogue of the Cartan-Hadamard theorem), with some additional assumptions they are Gromov hyperbolic or $CAT(0)$ ($CAT(-1)$), and complexes of groups with local developments satisfying the same conditions as links in systolic spaces are developable (see [5] for details).

The latter property allows one to construct many examples of systolic spaces and groups (i.e. groups acting geometrically on systolic complexes) with some additional properties and to answer some open questions (see [5]).

In this paper we study systolic complexes, focusing on pseudomanifolds and compactly supported cohomolgy groups. Our main result is the following.

**Main Theorem** (Theorem 4.3.2) *Let $X$ be a systolic $3$–pseudomanifold. Then the second group of cohomology with compact supports $H_c^2(X;\mathbb{Z})$ is nontrivial.*

The most important consequence of this result, and the initial motivation for this work, is the following. Main Theorem implies that groups acting geometrically on systolic 3–

pseudomanifolds are not duality groups. However, in the current paper we do not explore this issue — see e.g. [5, 7] for related discussions and results.

The paper is divided in three parts. In the first part we introduce systolic complexes (which are particular simplicial complexes) and pseudomanifolds. Basic definitions and examples are given, and we finish by proving that 2–dimensional spheres in a systolic 3–pseudomanifold are 2–pseudomanifolds. This result is central to prove the main theorem of this paper.

In the second part we define cohomology of simplicial complexes and cohomology with compact supports. Before that, we recall some basic results from homology theory.

The last chapter is devoted to the proof of Main Theorem. We start by computing the cohomology groups $H^1_c(\mathbb{R};\mathbb{Z})$ and $H^*_c(\mathbb{R}^n;\mathbb{Z})$, which use the tools presented in the previous sections and help us to understand the techniques used to solve our problem. Finally, we proceed to the proof of Main Theorem.

# 2 Systolic complexes

In this section we introduce systolic complexes (which are particular simplicial complexes) and pseudomanifolds. Basic definitions and examples are given, and we finish by proving that 2–dimensional spheres in a systolic 3–pseudomanifold are 2–pseudomanifolds. This result is central to prove the main theorem of this paper. Further details on systolic complexes can be found in [5,7].

## 2.1 Simplicial complexes

**Definition 2.1.1.** Given a set $V_S$ of vertices, an *abstract simplicial complex* on $V_S$ is a collection $S$ of subsets of $V_S$ satisfying the condition that if $\sigma$ is one of the subsets in $S$, then so is every subset of $\sigma$. The subsets $\sigma$ are called the *simplices* of $S$.

If $\sigma \in S$ has $n+1$ elements, we say that $\sigma$ is an *$n$–simplex*, or *dimension of $\sigma$ is $n$*, denoted $\dim(\sigma) = n$. *Vertices* are the 0–simplices of $X$, 1–simplices are called *edges* and 2–simplices are called *triangles*. We say that $S$ has dimension $n$ if it contains $n$–simplices but it does not contain simplices of dimension $n+1$. That means it has no simplices of dimension greater than $n+1$ either.

So far we have given an abstract description of a simplicial complex which is purely combinatorial. We can think as well in a geometric way, where a simplex is a generalization of the notion of a triangle to arbitrary dimensions. An $n$–simplex $\sigma = [v_0, \ldots, v_n]$ is the convex hull of $n+1$ points $v_0, \ldots, v_n$ (the vertices) in $\mathbb{R}^{n+1}$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

**Definition 2.1.2.** The *standard $n$–simplex* is

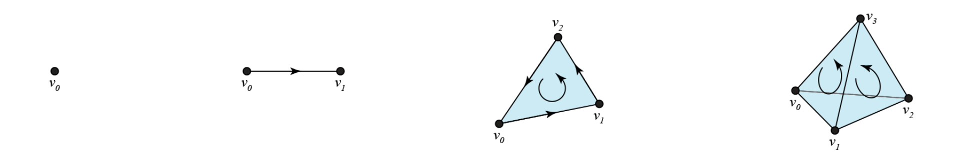
$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0\}$$

and there is a canonical map $\Delta^n \longrightarrow [v_0, \ldots, v_n]$ via

$$(t_0, \ldots, t_n) \mapsto \sum_i t_i v_i$$

called *barycentric coordinates* on $[v_0, \ldots, v_n]$.

The convex hull of any nonempty subset of $n+1$ points that define an $n$–simplex is called a *face* of the simplex. A *facet* of $[v_0, \ldots, v_n]$ is defined as the simplex obtained by deleting just one of the vertices $v_i$ and we denote it $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$. The union of all facets is called the *boundary* of the simplex.



a) Vertex $[v_0]$ b) Edge $[v_0, v_1]$ c) Triangle $[v_0, v_1, v_2]$ d) Tetrahedron $[v_0, v_1, v_2, v_3]$

Figure 1: *Oriented $n$–simplices, $0 \leq n \leq 3$. An oriented simplex induces orientation on its faces, as shown for the edges of the triangle and two faces of the tetrahedron.*

**Definition 2.1.3.** A *simplicial complex* $X$ is a finite set of simplices such that:

1. Any face of a simplex from $X$ is also in $X$.

2. The intersection of any two simplices $\sigma_1$, $\sigma_2 \in X$ is a face of both $\sigma_1$ and $\sigma_2$.

A *subcomplex* is a subset $L \subseteq X$ that is also a simplicial complex. The union of all the simplices of dimension lower or equal to $k$, denoted by $X^{(k)}$, is a subcomplex of the simplicial complex $X$, and it is called its *$k$–skeleton.*

**Definition 2.1.4.** The *geometric realization* $|X|$ of a simplicial complex $X$ is the topological space $|X| = \bigcup_{\sigma \in X} \sigma$, where we regard each simplex as a topological subspace.

**Definition 2.1.5.** A *triangulation* of a topological space $Y$ is a pair $(X, f)$ with $X$ simplicial complex and $f : |X| \to Y$ homeomorphism. We say $Y$ is *triangulable* when $X$ exists.

**Definition 2.1.6.** Suppose we fix an order on the set of vertices. An *orientation* of an $n$–simplex $\sigma = [v_0, \ldots, v_n] \in X$ is an equivalence class of orderings of the vertices of $\sigma$, where $(v_0, \ldots, v_n) \sim (v_{\tau(0)}, \ldots, v_{\tau(n)})$ are equivalent orderings if the parity of the permutation $\tau$ is even. An *oriented simplex* is a simplex with an equivalence class of orderings. We may show an orientation graphically using arrows, as in Figure 1. An oriented simplex induces orientations on its faces, where we drop the vertices not defining a face in the sequence to get the orientation. For example, triangle $[v_0, v_1, v_2]$ induces oriented edge $[v_0, v_1]$. Two $n$–simplices sharing an $(n-1)$–face $\tau$ are *consistently oriented* if they induce different orientations on $\tau$.

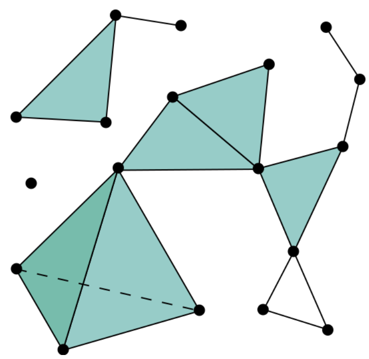


Figure 2: *3-dimensional simplicial complex*

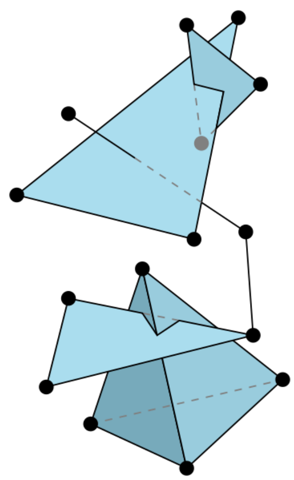


Figure 3: *Arrangement of simplices that is not a valid simplicial complex*

**Definition 2.1.7.** Let $\sigma = [v_0, \ldots, v_n]$ be an $n$–simplex of $X$. We define the *barycenter of* $\sigma$ as the point $b_\sigma \in \sigma$ given by

$$b_\sigma = \sum_{i=0}^{n} \frac{1}{n+1} v_i.$$

For example, the barycenter of a 0–simplex $\sigma = [v]$ is $b_\sigma = v$ and the barycenter of a 1–simplex $\tau = [v_0, v_1]$ is its middle point.

**Definition 2.1.8.** Given a simplicial complex $X$, we define its *barycentric subdivision* as the simplicial complex $X'$ whose vertices are all the barycenters of the simplices of $X$ and the simplices are all the finite ordered sets $[b_{\sigma_0}, \ldots, b_{\sigma_n}]$ with $\sigma_i$ a face of $\sigma_{i+1}$ for all $i$.

The map $|X'| \to |X|$ induced by the identity on the vertices is a homeomorphism, so we can identify the geometric realization of the barycentric subdivision with the one of $X$.

The following examples illustrate the notions we have introduced so far, and they are used as well in the rest of the paper:

**Example 2.1.9.**

a) The complex $X$ with vertices corresponding to the integer numbers and 1–simplices $[n, n+1]$ for $n \in \mathbb{Z}$ is a 1–dimensional simplicial complex and $|X| \simeq \mathbb{R}$.

b) The complex $X = (v_1, \ldots, v_6, v_1)$ with vertices $v_i$ and 1–simplices $[v_i, v_{i+1}]$ is a 1–dimensional simplicial complex and $|X| \simeq \mathbb{S}^1$. As we will see in Section 2.2, $X$ is called a cycle.

c) A triangulation $X$ as in Figure 4 is a 2–dimensional simplicial complex and $|X| \simeq \mathbb{T}^2$, where $\mathbb{T}^2$ is the 2–dimensional torus.

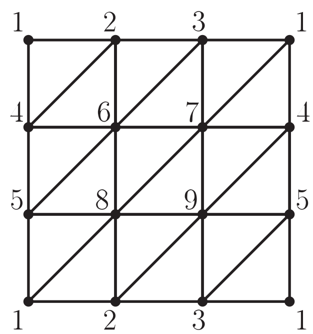



Figure 4: *Triangulation of a torus (vertices with the same labels are identified).*

d) A triangulation $X$ as in Figure 5 is a 2–dimensional simplicial complex and $|X| \simeq \mathbb{R}^2$.

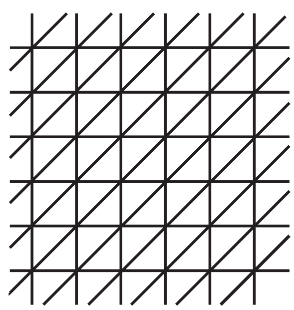


Figure 5: *Triangulation of the plane.*

**Remark 2.1.10.** There is a strong relationship between the geometric and abstract definitions. Every abstract simplicial complex $S$ is isomorphic to the vertex scheme of some simplicial complex $X$, which is its geometric realization. We usually compute simplicial complexes using geometric techniques, but discard the realization and focus on its topology as captured by the vertex scheme. As such, we refer to abstract simplicial complexes simply as simplicial complexes from now on, and we use $X$ to denote both the simplicial complex and the associated geometric realization.

## 2.2 k-large and k-systolic simplicial complexes

For the rest of this section we follow the notations of T. Januszkiewicz and J. Świątkowski from [5]. From now on, we are interested in locally finite simplicial complexes, i.e., every vertex belongs to finitely many edges.

Now that we know what simplicial complexes are, we want to define systolic complexes. To that end, we start with the following definitions.

**Definition 2.2.1.** A simplicial complex $X$ is *flag* if any finite set of vertices, which are pairwise connected by edges of $X$, *spans* a simplex of $X$ (i.e., it is contained in a simplex of $X$). A subcomplex $K$ in $X$ is called *full* (in $X$) if any simplex of $X$ spanned by a set of vertices in $K$ is a simplex of $K$.

**Remark 2.2.2.** Let $A, B \subseteq V_X$ be two sets of vertices. We denote $A * B := \text{span}\{A, B\}$, i.e., the minimal full subcomplex containing $A$ and $B$ (the subcomplex spanned by $A \cup B$). If two vertices $v, w$ are joined by an edge, then $\{v\} * \{w\}$ is denoted by $vw$. Let $\sigma, \tau$ be two disjoint simplices spanning a simplex in $X$. Then $\dim\,(\sigma * \tau) = \dim\,\sigma + \dim\,\tau + 1$.

**Definition 2.2.3.** Let $X$ be a simplicial complex, and let $\sigma$ be a simplex in $X$. The *link* of $\sigma$ in $X$, denoted by $X_\sigma$, is a subcomplex of $X$ consisting of all simplices that are disjoint from $\sigma$ and which span a simplex of $X$ together with $\sigma$; i.e., $X_\sigma = \{\gamma \in X | \gamma \cap \sigma = \emptyset \text{ and } \gamma * \sigma \in X\}$.

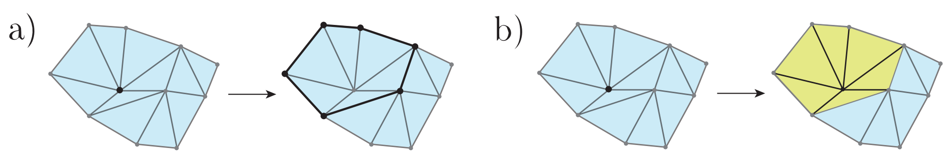



Figure 6: *a) A 0–simplex and its link. b) A 0–simplex and its residue.*

**Definition 2.2.4.** The *residue* of a simplex $\sigma$ in $X$, denoted by $Res(\sigma, X)$, is the union of all simplices of $X$ that contain $\sigma$; i.e., $Res(\sigma, X) = \{\gamma \in X | \sigma \subseteq \gamma\}$.

**Remark 2.2.5.** The residue $Res(\sigma, X)$ is naturally the *join* of $\sigma$ and the link $X_\sigma$. Notice that the link and residue describe the local structure of $X$.

**Definition 2.2.6.** A *cycle* in a simplicial complex $X$ is a subcomplex $\gamma$ of $X$ isomorphic to a triangulation of $\mathbb{S}^1$. We denote by $|\gamma|$ the length of $\gamma$, i.e., the number of 1–simplices in $\gamma$. A *full cycle* in $X$ is a cycle that is full as a subcomplex of $X$.

**Definition 2.2.7.** The *systole* of $X$ is

$$sys(X) = \min\{|\gamma| : \gamma \text{ is a full cycle in } X\}.$$

In particular, we have $sys(X) \geq 3$ for any simplicial complex $X$, and if there is no full cycle in $X$, then $sys(X) = \infty$.

**Definition 2.2.8.** Let $k \geq 4$ be a natural number and let $X$ be a flag simplicial complex. Then:

1. $X$ is *$k$–large* if $sys(X) \geq k$.
2. $X$ is *locally $k$–large* if the residue of every simplex of $X$ is $k$–large.
3. $X$ is *$k$–systolic* if it is locally $k$–large, connected and simply connected.

We abbreviate 6-systolic to *systolic*.

Some easy properties of the above introduced classes of simplicial complexes are gathered in Fact 2.2.9. The proofs are immediate, hence we omit them.

**Fact 2.2.9.**

1) A complex is locally $k$–large iff the link of every nonempty simplex has the systole at least $k$.

2) A (locally) $k$–large complex is (locally) $m$–large for $k \geq m$.

3) A full subcomplex in a (locally) $k$–large complex is (locally) $k$–large.

4) A simplicial complex is 4–large iff it is flag.

5) For $k > 4$, $X$ is $k$–large iff it is flag and $sys(X) \geq k$.

6) A $k$–large complex is locally $k$–large.

7) The universal cover $\widetilde{X}$ of a connected, locally $k$–large complex $X$ is $k$–systolic.

Here are some examples of $k$–large and $k$–systolic complexes:

**Example 2.2.10.**

a) Let $X = (v_i)$ for $i \in \mathbb{Z}$ be a triangulation of $\mathbb{R}$ as in Example 2.1.9(a), with vertices $v_i$ corresponding to the integer numbers and 1–simplices $v_i v_{i+1}$. Since there is no full cycle in $X$, then $sys(X) = \infty$. So in particular $X$ is 6–large. It is locally 6–large (by Fact 2.2.9(6)), connected and simply connected. Therefore $X$ is a systolic complex.

b) A tree is a systolic complex.

c) A triangulation of $\mathbb{R}^2$ as in Example 2.1.9(d) is a systolic complex.

d) Let $X = (v_1, \ldots, v_6, v_1)$ be a cycle. Then $X$ is 6–large.

e) A triangulation of the torus as in Example 2.1.9(c) is 6–large.

The following results are easy as well, but this time we prove them, so it helps us understand better the concepts we have introduced so far and get us acquainted with the techniques we use in the rest of the paper.

**Lemma 2.2.11.** Let $X$ be a simplicial complex. Then the intersection of a simplex $\gamma \in X$ with a full subcomplex $K \subseteq X$, if nonempty, is a simplex.

*Proof.* By definition of fullness, any simplex $\gamma \in X$ spanned by vertices of $K$ is a simplex of $K$. So in particular, $\gamma \cap K$, if is nonempty, is a subcomplex whose set of vertices is in $K$ and spans a simplex in $X$, and by fullness spans a simplex in $K$.

$\square$

**Lemma 2.2.12.** Given a simplex $\sigma$ in a flag simplicial complex $X$, its link $X_\sigma$ is a full subcomplex of $X$.

*Proof.* Let $A = \{v_1, \ldots, v_n\}$ be a set of vertices contained in $X_\sigma$, and spanning in $X$ a simplex $\tau$. We want to show that $\tau \in X_\sigma$.

Since by definition of the link $A \cap \sigma = \emptyset$, we have $\tau \cap \sigma = \emptyset$. Now we consider $\sigma^{(0)} \cup \tau^{(0)}$, which is a set of vertices pairwise connected by edges. By flagness we have that $\sigma * \tau$ is a simplex in $X$. Therefore $\tau \in X_\sigma$.

$\square$

**Lemma 2.2.13.** Let $X$ be a flag simplicial complex and let $\sigma \subseteq \tau$ be simplices in $X$. Then $X_\tau = (X_\sigma)_{\tau \cap X_\sigma}$.

*Proof.* By definition

$$X_\tau = \{\gamma \in X | \gamma \cap \tau = \emptyset \text{ and } \gamma * \tau \in X\}$$

$$(X_\sigma)_{\tau \cap X_\sigma} = \{\gamma' \in X_\sigma | \gamma' \cap (\tau \cap X_\sigma) = \emptyset \text{ and } \gamma' * (\tau \cap X_\sigma) \in X_\sigma\}$$

First we show that $X_\tau \subseteq (X_\sigma)_{\tau \cap X_\sigma}$:

Let $\gamma \in X_\tau$. We have that $\gamma \cap \tau = \emptyset$, therefore $\gamma \cap (\tau \cap X_\sigma) = \emptyset$. Now let $X_\sigma = \{\beta \in X | \beta \cap \sigma = \emptyset \text{ and } \beta * \sigma \in X\}$. Obviously, $\gamma \cap \sigma = \emptyset$ (since $\gamma \cap \tau = \emptyset$ and $\sigma \subset \tau$), and $\gamma * \sigma$ spans a simplex in $X$ (since $\gamma * \tau \in X$ and $\sigma \subset \tau$) $\Rightarrow \gamma \in X_\sigma$. Therefore $\gamma * (\tau \cap X_\sigma) \in X_\sigma$ (since $\gamma \in X_\sigma$ and $(\tau \cap X_\sigma) \in X_\sigma$ due to Lemmas 2.2.11 and 2.2.12 ).

Now let us see that $(X_\sigma)_{\tau \cap X_\sigma} \subseteq X_\tau$:

Let $\gamma' \subseteq (X_\sigma)_{\tau \cap X_\sigma}$. Then $\emptyset = \gamma' \cap (\tau \cap \ X_\sigma) = (\gamma' \cap \ X_\sigma) \cap \tau = \gamma' \cap \tau$. Now let $\gamma'' = \gamma' * (\tau \cap \ X_\sigma) \in X_\sigma$. Then, by Lemmas 2.2.11 and 2.2.12, $\gamma'' * \sigma \in X$, with $\gamma', \tau \in (\gamma'' * \sigma) \Rightarrow \gamma' * \tau \in X$.

$\square$

## 2.3 Convexity, Balls and Spheres

Here we introduce the notions of balls and spheres, and we state the so-called Projection Lemma, fundamental result for what follows. Unless otherwise stated, proofs of the results of this section can be found in [5].

A subcomplex $Q$ in a 6–large simplicial complex $X$ is 3*–convex* if $Q$ is full in $X$ and for every geodesic $(v_0, v_1, v_2)$ in $X$ with $v_0, v_2 \in Q$ we have $v_1 \in Q$. Note that here by *geodesic* we mean the shortest path that joins a given set of vertices.

A subcomplex $Q$ in a systolic simplicial complex $X$ is *locally* 3*–convex* if $\forall \sigma \subset Q$, $Q_\sigma$ is 3–convex in $X_\sigma$. We say that $Q$ is *convex* if it is connected, and it is locally 3–convex. Note that if $Q$ is a convex subcomplex, then it is full.

Now let $X$ be a systolic complex and let $\sigma$ be a simplex contained in it. One can define a *closed combinatorial ball of radius i around $\sigma$ in $X$*, $B_i(\sigma, X)$, inductively:

$$B_0(\sigma, X) = \sigma \text{ and } B_i(\sigma, X) = \cup\{\tau \subset X | \tau \cap B_{i-1}(\sigma, X) \neq \emptyset\}, \text{ for } i > 0.$$

Note that, more generally, one can define combinatorial balls for any convex subcomplex instead of just considering a simplex. But we are interested in this special case since is the one we use.

The *closed combinatorial sphere of radius i around $\sigma$ in $X$*, $S_i(\sigma, X)$, is the subcomplex of $B_i(\sigma, X)$ spanned by the vertices at combinatorial distance $i$ from $\sigma$, i.e., not belonging to $B_{i-1}(\sigma, X)$. By $\overset{\circ}{B_i}(\sigma, X)$ we denote the *interior* of the closed combinatorial i-ball around $\sigma$ in $X$; i.e., $\overset{\circ}{B_i}(\sigma, X) = B_i(\sigma, X) \backslash S_i(\sigma, X)$.

One can define closed combinatorial balls of small radii in $k$–large complexes so that they are isomorphic to ones in the corresponding universal covers (i.e. in systolic complexes):

**Lemma 2.3.1.** For a 6–large simplicial complex $X$ and a simplex $\tau \in X$, let $\widetilde{X} \xrightarrow{p} X$ with $p(\tilde{\tau}) = \tau$ be the universal cover of $X$. Then $p|_{B_1(\tilde{\tau}, \widetilde{X})} : B_1(\tilde{\tau}, \widetilde{X}) \longrightarrow B_1(\tau, X)$ is an isomorphism.

The following lemma gives us another way of viewing balls and spheres. For that, first we define the *distance between a vertex and a simplex* as the minimum distance between the vertex and any vertex contained in the simplex, i.e., $d(v, \sigma) = \min \{d(v, v')\}$ for $v' \in \sigma$.

**Lemma 2.3.2.** $B_i(\sigma, X)$ is the simplicial span of the vertex set $\{v \in X | \ d(v, \sigma) \leq i\}$. $S_i(\sigma, X)$ is the simplicial span of the vertex set $\{v \in X | \ d(v, \sigma) = i\}$.

Some properties of balls and spheres are gathered in the next lemma:

**Lemma 2.3.3.** The ball $B_i(\sigma, X)$ and the sphere $S_i(\sigma, X)$ are full subcomplexes of $X$ and by Fact 2.2.9(3) they are $k$–large. Moreover, balls are convex.

The next lemma is fundamental for the understanding of this paper. Its application plays an important role in most of the proofs.

**Lemma 2.3.4. [Projection Lemma]** For any $\tau \in S_i(\sigma, X)$, $\rho = S_{i-1}(\sigma, X) \cap X_\tau$ is a single (nonempty) simplex. Moreover, $X_\tau \cap B_i(\sigma, X) = B_1(\rho, X_\tau)$ and $X_\tau \cap S_i(\sigma, X) = S_1(\rho, X_\tau)$.

*Proof.* See Section 2 of [7]. □

In the rest of the paper we call the simplex $\rho$, as in the above lemma, the *projection of* $\tau$ *on* $B_{i-1}(\sigma, X)$.

Let $X$ be a simplicial complex, and let $\sigma \in X$ be a simplex. By Projection Lemma (Lemma 2.3.4) we can define an elementary contraction

$$\pi_{B_i(\sigma,X)} : B_{i+1}(\sigma, X)' \to B_i(\sigma, X)'$$

between barycentric subdivision of balls by putting

$$\pi_{B_i(\sigma,X)}(b_\nu) = \begin{cases} b_{\nu \cap B_i(\sigma,X)} & \text{if } \nu \cap B_i(\sigma, X) \neq \emptyset, \\ b_{X_\nu \cap B_i(\sigma,X)} & \text{if } \nu \cap B_i(\sigma, X) = \emptyset \end{cases}$$

and then extending simplicially. In Section 8 of [5] it is shown that $\pi_{B_i(\sigma,X)}$ is a deformation retraction and that $\pi_{B_i(\sigma,X)}(B_{i+1}(\sigma, X) \setminus \overset{\circ}{B_i} (\sigma, X)) \subset S_i(\sigma, X)$. Then we define a deformation retraction $P_{B_i(\sigma,X)} : X \to B_i(\sigma, X)$ as follows: if $x \in B_j(\sigma, X)$ then $P_{B_i(\sigma,X)}(x) = \pi_{B_i(\sigma,X)} \circ \pi_{B_{i+1}(\sigma,X)} \circ \cdots \circ \pi_{B_{i+j}(\sigma,X)}(x)$.

**Lemma 2.3.5.** For $j > i$, the projection $P_{B_i(\sigma,X)|B_j(\sigma,X)} : B_j(\sigma, X)X \to B_i(\sigma, X)$ provides a deformation retraction of $B_j(\sigma, X) \setminus \overset{\circ}{B_i} (\sigma, X)$ onto $S_i(\sigma, X)$ within $B_j(\sigma, X) \setminus \overset{\circ}{B_i} (\sigma, X)$.

The above lemma implies the following.

**Theorem 2.3.6.** Let $X$ be a finite-dimensional systolic complex. Then $X$ is contractible (see [5], Theorem 4.1).

## 2.4 Manifolds and Surfaces

Recall that an $n$–*manifold* is a Hausdorff space in which every point has a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$. Here are some examples of manifolds. The Euclidean space $\mathbb{R}^n$ is certainly an $n$–manifold. Also, the $n$–sphere $\mathbb{S}^n$ is an $n$–manifold. $\mathbb{D}^n$ is not a manifold, but there is a more general concept of 'manifold with boundary' of which $\mathbb{D}^n$ is an example. Since the product of an $n$–ball and an $m$–ball is homeomorphic to an $(n+m)$–ball, it follows that the product of an $n$–manifold with an $m$–manifold is an $(n+m)$–manifold. It follows that any $n$–torus is an $n$–manifold.

If $p : \tilde{X} \to X$ is a covering, then it is not hard to see that $X$ is an $n$–manifold if and only if $\tilde{X}$ is an $n$–manifold. It follows that $\mathbb{RP}^n$ is an $n$–manifold. Note that a manifold is certainly locally path connected.

A triangulable $n$–manifold is orientable if all $n$–simplices in any of its triangulations can be oriented consistently.

Now we classify all compact, connected 2-manifolds. Such manifolds are called *closed surfaces.*

**Theorem 2.4.1.** Let $S$ be a closed surface. Then $S$ is homeomorphic to one of the following:

1. the 2–dimensional sphere $\mathbb{S}^2$,
2. the connected sum of $g$ tori ($g \geq 1$),
3. the connected sum of $g$ projective planes ($g \geq 1$).

*Proof.* See Section 5 of [2] □

The surfaces in the first two families are orientable. It is convenient to combine the two families by regarding the sphere as the connected sum of 0 tori. The number $g$ of tori involved is called the genus of the surface. The surfaces in the third family are nonorientable.

To complete the classification, let us recall that the *Euler characteristic* of the geometric realization of a simplicial complex $X$ is defined as $\chi(|X|) = V - E + F$, where $V, E$ and $F$ are, respectively, the number of vertices, edges and faces of $|X|$. Then we have the following lemma:

**Lemma 2.4.2.** Let $S$ be a closed surface, and $\chi(S)$ the Euler characteristic of its triangulation. Then,

1. If $S$ is homeomorphic to a sphere, then $\chi(S) = 2$.
2. If $S$ is homeomorphic to the connected sum of $g$ tori, then $\chi(S) = 2 - 2g$.

3. If $S$ is homeomorphic to the connected sum of $g$ projective planes, then $\chi(S) = 2 - g$.

It follows that a closed surface is determined, up to homeomorphism, by two pieces of information: its Euler characteristic, and whether it is orientable or not. In other words, Euler characteristic and orientability completely characterize closed surfaces up to homeomorphism.

We note that every closed surface $S$ can be triangulated. This is actually a rather deep theorem not at all easy to prove. But accepting that fact, we are interested in showing which surfaces admit a 6–large triangulation. Suppose that we have a surface $S$ that admits a $k$–large triangulation. The link of a vertex is an $l$–cycle with $l \geq k$, and its residue consists of exactly $l$ triangles. Therefore, each vertex is contained in at least $k$ edges, and each edge contains 2 vertices, hence $2E \geq kV$. Analogously, each triangle contains 3 edges, and each edge is contained in 2 triangles, hence $3F = 2E$. Using this relations in Euler's characteristic we obtain

$$\chi(S) = V - E + F = V - \frac{1}{3}E \leq V(1 - \frac{k}{6}) \ .$$

By Lemma 2.4.2, $\chi(\mathbb{S}^2) = 2$ and $\chi(\mathbb{RP}^2) = 1$, hence

$$2 \leq V(1 - \frac{k}{6}) \Rightarrow k \leq 5 \ ,$$

and

$$1 \leq V(1 - \frac{k}{6}) \Rightarrow k \leq 5 \ .$$

We conclude that the sphere and projective plane do not admit 6–large triangulations. It is a fact that any other surface admits a 6–large triangulation (see [5], 1.8(3)). Thus we have:

**Lemma 2.4.3.** The only surfaces that admit a 6–large triangulation are:

1. The connected sum of $g$ tori ($g \geq 1$).
2. The connected sum of $g$ projective planes ($g > 1$).

**Remark 2.4.4.** The fact that there is no $k$–large triangulation of the 2–sphere for $k \geq 6$, implies that no triangulation of a manifold of dimension above 2 is 6–large, since 2–spheres would occur as links of some simplices in such triangulation (see [5], 1.8(5)).

## 2.5 Pseudomanifolds

A simplicial complex $X$ is called a *simplicial pseudomanifold* of dimension $n$ (or shortly *$n$–pseudomanifold*) if it is a union of $n$–simplices such that every $(n-1)$–simplex is contained in exactly two $n$–simplices. Let $\tau$ be a subsimplex of a maximal simplex $\sigma$ of dimension $n$. Then we say that $\tau$ has *codimension* $k$ if its dimension is $(n-k)$. Note that the notion of pseudomanifold is more general than the notion of manifold.

**Example 2.5.1.**

a) The only connected 1–pseudomanifolds are: a line, i.e., a triangulation of $\mathbb{R}$; a cycle, i.e., a triangulation of $\mathbb{S}^1$. Other 1–pseudomanifolds are unions of those.

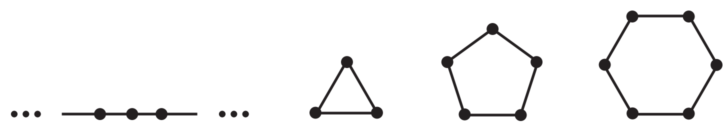


Figure 7: *Examples of 1–pseudomanifolds*

b) Links of vertices in 2–pseudomanifolds are 1–pseudomanifolds (see Lemma 2.5.2 below), and thus disjoint unions of circles. It follows that 2–pseudomanifolds are triangulated surfaces (possibly disconnected) with some vertices identified.

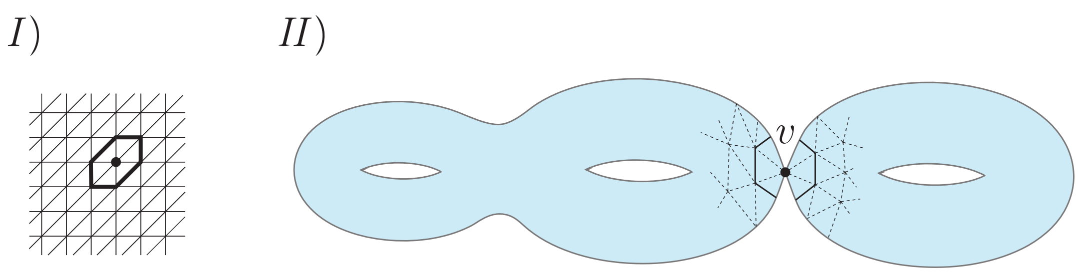



Figure 8: *I) Example of a 2–pseudomanifold in a 2 dimensional space and the link of a vertex. II) Example of a 2–pseudomanifold in a 3 dimensional space and the link of a singular vertex v*

**Lemma 2.5.2.** Let $X$ be an $n$–pseudomanifold and let $\sigma$ be a $k$–simplex of $X$. Then $X_\sigma$ is an $(n-k-1)$–pseudomanifold.

*Proof.* First we show that $X_\sigma$ is the union of $(n-k-1)$–simplices. By definition, $X_\sigma = \{\tau \in X | \tau \cap \sigma = \emptyset$ and $\sigma * \tau$ is a simplex in $X\}$. Take $\tau \in X_\sigma$. Since $X$ is an $n$–pseudomanifold, $\sigma * \tau \subseteq \rho$, where $\rho$ is an $n$–simplex of $X$. Let $\rho = (\sigma * \tau')$, where $\tau' \cap \sigma = \emptyset$. Since $n = \dim\, \rho = \dim\,(\sigma * \tau') = \dim\, \sigma + \dim\, \tau' + 1 = k + \dim\, \tau' + 1 \Rightarrow \dim\, \tau' = n-k-1$. Therefore every simplex $\tau$ in the link is contained in an $(n-k-1)$–simplex, and thus $X_\sigma$ is the union of $(n-k-1)$–simplices.

Now we show that every codimension 1 simplex in $X_\sigma$ is contained in exactly two $(n-k-1)$–simplices. Let $\omega$ be a codimension 1 simplex in $X_\sigma$. Therefore dim $\omega = n-k-2$ and dim $(\omega * \sigma) =$ dim $\omega +$ dim $\sigma + 1 = n-k-2+k+1 = n-1$. Thus $(\omega * \sigma)$ is an $(n-1)$–simplex in $X$, i.e., a codimension 1 simplex in $X$. Since $X$ is an $n$–pseudomanifold, there exists exactly two $n$–simplices $\alpha, \beta$ that contain $(\omega * \sigma)$. Now we take $\alpha'$ and $\beta'$ such that $\alpha' \cap (\omega \cup \sigma') = \emptyset$, $\beta' \cap (\omega \cup \sigma') = \emptyset$ and $\alpha = (\omega * \sigma) * \alpha'$, $\beta = (\omega * \sigma) * \beta'$. We show that $\alpha' * \omega$ and $\beta' * \omega$, that belong to $X_\sigma$, have dimension $(n-k-1)$:

$$n = \dim \alpha = \dim(\sigma * \omega * \alpha') = k + \dim(\omega * \alpha') + 1 \Rightarrow$$
$$\dim(\omega * \alpha') = n-k-1.$$

Similarly,

$$\dim(\omega * \beta') = n-k-1$$

Finally, remains to see that this two simplices are the only two $(n-k-1)$–simplices. If there exists $\gamma'$ with dim $\gamma' = n-k-1$, $\gamma' \supseteq \omega$ and $\gamma' \neq \alpha' * \omega$,$\gamma' \neq \beta' * \omega$, then $\gamma' * \sigma \neq \alpha' * \omega * \sigma$ and $\gamma' * \sigma \neq \beta' * \omega * \sigma$. But since all three have dimension $n$, that contradicts the fact that $\alpha$ and $\beta$ are the only $n$–simplices containing $\omega * \sigma$.

□

**Lemma 2.5.3.** Let $X$ be a systolic pseudomanifold of dimension 2. Then $S_k(\sigma, X)$ is a 1–dimensional pseudomanifold for all $k \geq 1$.

*Proof.* First we show that $S_k(\sigma, X)$ is at most 1–dimensional. Suppose there exists an $l$–simplex $\tau$ in $S_k(\sigma, X)$ with $l \geq 2$. By Projection Lemma (Lemma 2.3.4), the projection of $\tau$ on $B_{k-1}(\sigma, X)$ is a nonempty simplex $\rho = S_{k-1}(\sigma, X) \cap X_\tau$. That means $\rho \subset X_\tau$ and by definition of the link, $\tau * \rho$ must be a simplex of $X$. But $\dim(\tau * \rho) = l + 1 + \dim(\rho) > 2 = \dim(X)$, which is a contradiction.

Now we show that every $l$–simplex $\tau$ of $S_k(\sigma, X)$ is contained in some 1–simplex. We have seen that $\tau$ can just have dimensions 0 or 1. If it has dimension 1 we are done. Suppose it has dimension 0. By Projection Lemma (Lemma 2.3.4), $X_\tau \cap S_k(\sigma, X) = S_1(\rho, X_\tau)$, and $X_\tau$ is a 6–large (by Fact 2.2.9(3)) 1–pseudomanifold (by Lemma 2.5.2). Thus $X_\tau$ is a union of cycles. Take $\rho$ contained in one of those cycles. Then $S_1(\rho, X_\tau) = \{v, w\}$, with $v, w$ two distinct vertices. Therefore, $\tau * v$ is a 1–simplex of $S_k(\sigma, X)$ containing $\tau$ (by Lemma 2.3.3).

Moreover, $\tau * v$ and $\tau * w$ are the only two maximal simplices of $S_k(\sigma, X)$ that contain the codimension 1 simplex $\tau$ of $S_k(\sigma, X)$.

□

**Lemma 2.5.4.** Let $X$ be a systolic pseudomanifold of dimension 3. Then $S_k(\sigma, X)$ is a 2–dimensional pseudomanifold for all $k \geq 1$.

*Proof.* To see that $S_k(\sigma, X)$ is at most 2–dimensional, the proof is analogous as in the previous lemma.

We show that every $l$–simplex $\tau$ of $S_k(\sigma, X)$ is contained in some 2–simplex. Now $\tau$ can have dimensions $0, 1$ and 2. If it has dimension 2 we are done. Suppose it has dimension 0. By Projection Lemma (Lemma 2.3.4), $X_\tau \cap S_k(\sigma, X) = S_1(\rho, X_\tau)$. Now $X_\tau$ is a 6–large (by Fact 2.2.9(1)) 2–pseudomanifold (by Lemma 2.5.2). By Fact 2.2.9(7) its universal cover $\widetilde{X}_\tau$ is systolic, and by Lemma 2.3.1 $S_1(\rho, X_\tau)$ is isomorphic to $S_1(\rho, \widetilde{X}_\tau)$. Therefore by Lemma 2.5.3 $S_1(\rho, X_\tau)$ is a 1–pseudomanifold. The span of one of its 1–simplices, call it $\tau'$, with our vertex $\tau$, is a 2–simplex $\tau * \tau'$ of $S_k(\sigma, X)$.

Now suppose that $\tau$ has dimension 1. Then $X_\tau$ is a 6–large (by Fact 2.2.9(3)) 1–pseudomanifold (by Lemma 2.5.2). Thus $X_\tau$ is a union of cycles. Take $\rho$ contained in one of those cycles. Then $S_1(\rho, X_\tau) = \{v, w\}$, with $v, w$ two distinct vertices. Therefore, $\tau * v$ is a 1–simplex of $S_k(\sigma, X)$ containing $\tau$ (by Lemma 2.3.3).

Moreover, $\tau * v$ and $\tau * w$ are the only two maximal simplices of $S_k(\sigma, X)$ that contain the codimension 1 simplex $\tau$ of $S_k(\sigma, X)$.

□

**Remark 2.5.5.** The general version of this last lemma, with $X$ being $n$–dimensional and $S_k$ $(n-1)$–dimensional is also true. See [7], Lemma 4.1, for a proof.

# 3 Cohomology

In this section we define cohomology of simplicial complexes and cohomology with compact supports. Before that, we recall some basic results from homology theory. Explanations in more detail together with proofs of all the results presented can be found in [4].

We start with the notion of *homotopy*, which is any family of maps $f_t : X \to Y$, with $X, Y$ topological spaces and $t \in I = [0, 1]$, such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous.

Continuous functions $f$ and $g$ are said to be *homotopic*, denoted $f \simeq g$, if and only if there is a homotopy $F$ taking $f$ to $g$ as described above.

Given a topological space $X$ and a subspace $A$, we say that a *deformation retraction* of $X$ onto $A$ is a homotopy from the identity map of X to a *retraction* of X onto A, i.e., a map $r : X \to X$ such that $r(X) = A$ and $r|_A = \mathbb{1}_X$.

A map $f : X \to Y$ is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}_Y$ and $gf \simeq \mathbb{1}_X$. The spaces $X$ and $Y$ are said to be *homotopy equivalent.*

A *chain complex* $(A_\bullet, d_\bullet)$ is a sequence of abelian groups or modules

$$\ldots, A_2, A_1, A_0, A_{-1}, A_{-2}, \ldots$$

connected by homomorphisms $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is zero, i.e., $d_n d_{n+1} = 0$ for all $n$. They are usually written out as:

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \to \cdots \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \cdots$$

Let $X$ be a simplicial complex. A *simplicial $n$–chain* is a formal sum of $n$–simplices $\sum_{i=1}^{N} c_i \sigma^i$, where $c_i \in \mathbb{Z}$, $\sigma^i \in X$ is the $i$–th $n$–simplex of $X$. The *group of simplicial $n$–chains on* $X$, the free abelian group defined on the set of $n$–simplices in $X$, is denoted $C_n(X)$.

Consider a basis element of $C_n(X)$, i.e., an $n$–simplex $\sigma = [v_0, \ldots, v_n]$. The *boundary operator* $\partial_n : C_n(X) \to C_{n-1}(X)$ is a homomorphism defined by:

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

In $C_n(X)$, elements of the subgroup $Z_n(X) := \ker \partial_n$ are referred to as *cycles*, and the subgroup $B_n(X) := \mathrm{Im}\ \partial_{n+1}$ is said to consist of *boundaries*. Direct computation shows that $B_n(X) \subseteq Z_n(X)$, thus the boundary of a boundary must be zero. In other words, $(C_n(X), \partial_n)$ form a simplicial chain complex. The *$n$–th homology group $H_n$ of $X$* is defined to be the quotient $H_n(X) := Z_n(X)/B_n(X)$.

A sequence

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

of groups and group homomorphisms is said to be *exact* if Ker $\alpha_n$ = Im $\alpha_{n+1}$ for each $n$. The inclusions Im $\alpha_{n+1} \subset$ Ker $\alpha_n$ are equivalent to $\alpha_n \alpha_{n+1} = 0$, so the sequence is a chain complex, and the opposite inclusions Ker $\alpha_n \subset$ Im $\alpha_{n+1}$ say that the homology groups of this chain complex are trivial. A sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is exact iff $\alpha$ is injective, $\beta$ is surjective, and Ker $\beta$ = Im $\alpha$, so $\beta$ induces an isomorphism $C \simeq B/\mathrm{Im}\ \alpha$. An exact sequence like that is called a *short exact sequence*.

## 3.1 Cohomology of Simplicial Complexes

Cohomology is in many ways dual to homology, but not (always) literally so. To obtain the cohomology groups $H^n(X; \mathbb{Z})$ we replace the chain groups $C_n(X)$ by the dual groups $Hom(C_n(X), \mathbb{Z})$ and the boundary maps $\partial$ by their dual maps $\delta$, before forming the cohomology groups ker $\delta$/Im $\delta$. Now we explain that in detail.

Given a simplicial complex $X$, we define the group $C^n(X; \mathbb{Z})$ of *simplicial $n$–cochains with coefficients in* $\mathbb{Z}$ to be the dual group $Hom(C_n(X), \mathbb{Z})$ of the simplicial chain group $C_n(X)$. Thus an $n$–cochain $\varphi \in C^n(X; \mathbb{Z})$ assigns to each simplicial $n$–simplex $\sigma$ a value $\varphi(\sigma) \in \mathbb{Z}$.

The coboundary map $\delta^n : C^{n-1}(X; \mathbb{Z}) \to C^n(X; \mathbb{Z})$ is the dual from $\partial_n$, and has the following description: for $\varphi \in C^{n-1}(X; \mathbb{Z})$, the coboundary $\delta^n \varphi$ is the composition

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\varphi} \mathbb{Z}$$

so that for a simplicial $n$–simplex $\sigma = [v_0, \ldots, v_n]$,

$$\delta^n \varphi(\sigma) = \sum_i (-1)^i \varphi([v_0, \ldots, \widehat{v_i}, \ldots, v_n]).$$

In $C^n(X;\mathbb{Z})$, elements of the subgroup $Z^n(X;\mathbb{Z}) := \ker \delta^{n+1}$ are referred to as *cocycles*, and the subgroup $B^n(X;\mathbb{Z}) := \mathrm{Im}\ \delta^n$ is said to consist of *coboundaries*. For a cochain $\varphi$ to be a cocycle means that $\delta^n\varphi = \varphi\partial_n = 0$, or in other words, $\varphi$ vanishes on boundaries. Since $\delta^{n+1}\delta^n$ is the dual of $\partial_n\partial_{n+1} = 0$, then $\delta^{n+1}\delta^n = 0$. Thus the sequence

$$\cdots \longleftarrow C^{n+1}(X;\mathbb{Z}) \xleftarrow{\delta^{n+1}} C^n(X;\mathbb{Z}) \xleftarrow{\delta^n} C^{n-1}(X;\mathbb{Z}) \longleftarrow \cdots \longleftarrow C^0(X;\mathbb{Z}) \longleftarrow 0$$

forms a simplicial cochain complex. We define the *cohomology group $H^n$ of $X$ with coefficients in $\mathbb{Z}$* as $H^n(X;\mathbb{Z}) := Z^n(X;\mathbb{Z})/B^n(X;\mathbb{Z})$.

We know from homology theory that the maps $f : X \to Y$ induce chain maps $f_\# : C_n(X) \to C_n(Y)$. The *cochain maps* are the dual maps $f^\# : C^n(Y;\mathbb{Z}) \to C^n(X;\mathbb{Z})$, defined as $f^\#(\varphi)(\sigma) := \varphi(f_\#(\sigma))$ . The relation $f_\#\partial = \partial f_\#$ dualizes to $\delta f^\# = f^\#\delta$, so $f^\#$ induces homomorphisms $f^* : H^n(Y;\mathbb{Z}) \to H^n(X;\mathbb{Z})$.

Homotopy equivalent spaces have the same cohomology groups. More precisely, if $f \simeq g : X \to Y$, then $f^* = g^* : H^n(Y;\mathbb{Z}) \to H^n(X;\mathbb{Z})$. From this result immediately follows,

**Lemma 3.1.1.** If a space $X$ is contractible, then $H^n(X;\mathbb{Z}) \simeq \{0\}$ for $n \geq 1$.

Some other important results that we are going to use in the rest of the paper are the following:

**Lemma 3.1.2.** Given $n, m \in \mathbb{Z}$ with $n, m \geq 1$, let $\mathbb{S}^m$ be the $m$–dimensional sphere. Then

$$H^n(\mathbb{S}^m;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } n = m, \\ \{0\} & \text{otherwise.} \end{cases}$$

**Lemma 3.1.3.** Let $S_g$ be a closed orientable surface of genus $g$. Then

$$H^n(S_g;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } n = 0, 2 \\ \mathbb{Z}^{\oplus 2g} & \text{for } n = 1 \\ \{0\} & \text{for } n > 2 \end{cases}$$

**Lemma 3.1.4.** Let $S$ be a closed nonorientable surface different from $\mathbb{RP}^2$. Then $H^1(S;\mathbb{Z}) \not\simeq \{0\}$.

## 3.2 Relative Cohomology Groups and the Long Exact Sequence of a Pair

It sometimes happens that by ignoring a certain amount of data or structure one obtains a simpler, more flexible theory which, almost paradoxically, can give results not readily obtainable in the original setting. Relative homology is an example, where one ignores all simplicial chains in a subspace of the given space.

Relative homology groups are defined in the following way. Given a space $X$ and a subspace $A \subseteq X$, let $C_n(X,A)$ be the quotient group $C_n(X)/C_n(A)$. Thus chains in $A$ are trivial in $C_n(X,A)$. Since the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map $\partial_n : C_n(X,A) \to C_{n-1}(X,A)$. Letting $n$ vary, we have a sequence of boundary maps

$$\cdots \longrightarrow C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \longrightarrow \cdots$$

The relation $\partial_n \partial_{n+1} = 0$ holds for these boundary maps since it holds before passing to quotient groups. So we have a chain complex, and the homology groups Ker $\partial_n$ / Im $\partial_{n+1}$ of this chain complex are by definition the *relative homology groups* $H_n(X,A)$.

Now we want to define relative cohomology groups $H^n(X,A;\mathbb{Z})$. We first consider the short exact sequence

$$0 \longrightarrow C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j_\#} C_n(X,A) \longrightarrow 0$$

where $i_\#$ is the map induced by the inclusion $i : A \hookrightarrow X$, and $j_\#$ is the map induced by the quotient map $j : X \to X/A$. We dualize it by applying Hom(-,$\mathbb{Z}$), getting

$$0 \longleftarrow C^n(A;\mathbb{Z}) \xleftarrow{i^\#} C^n(X;\mathbb{Z}) \xleftarrow{j^\#} C^n(X,A;\mathbb{Z}) \longleftarrow 0$$

where by definition $C^n(X,A;\mathbb{Z}) = \mathrm{Hom}(C_n(X,A),\mathbb{Z})$. This sequence is exact by the following direct argument. The map $i^\#$ restricts a cochain on $X$ to a cochain on $A$. Thus for a function from $n$–simplices in $X$ to $\mathbb{Z}$, the image of this function under $i^\#$ is obtained by restricting the domain of the function to $n$–simplices in $A$. Every function from $n$–simplices in $A$ to $\mathbb{Z}$ can be extended to be defined on all $n$–simplices in $X$, for example by assigning the value 0 to all $n$–simplices not in $A$, so $i^\#$ is surjective. The kernel of $i^\#$ consists of cochains taking the value 0 on $n$–simplices in $A$. Such cochains are the same as homomorphisms $C_n(X,A) = C_n(X)/C_n(A) \to \mathbb{Z}$, so the kernel of $i^\#$ is exactly $C^n(X,A;\mathbb{Z}) = \mathrm{Hom}(C_n(X,A),\mathbb{Z})$, giving the desired exactness. Notice that we can view $C^n(X,A;\mathbb{Z})$ as the functions from $n$–simplices in $X$ to $\mathbb{Z}$ that vanish on simplices in $A$, since the basis for $C_n(X)$ consisting of $n$–simplices in $X$ is the disjoint union of the simplices with image contained in $A$ and the simplices with image not contained in $A$.

The coboundary map $\delta^n : C^{n-1}(X;\mathbb{Z}) \to C^n(X;\mathbb{Z})$ takes $C^{n-1}(A;\mathbb{Z})$ to $C^n(A;\mathbb{Z})$, thus it induces a quotient coboundary map $\delta^n : C^{n-1}(X,A;\mathbb{Z}) \to C^n(X,A;\mathbb{Z})$. The relation $\delta^{n+1}\delta^n = 0$ holds for these coboundary maps since it holds before passing to quotient groups. So we have a cochain complex, and the cohomology groups $\mathrm{Ker}\ \delta^{n+1} / \mathrm{Im}\ \delta^n$ of this cochain complex are by definition the *relative cohomology groups* $H^n(X,A;\mathbb{Z})$.

We do not prove it here, but one can see in Section 3.1 of [4], that the preceding displayed short exact sequence of cochain groups gives rise to an associated long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n(X,A;\mathbb{Z}) \xrightarrow{j^*} H^n(X;\mathbb{Z}) \xrightarrow{i^*} H^n(A;\mathbb{Z}) \xrightarrow{\delta} H^{n+1}(X,A;\mathbb{Z}) \longrightarrow \cdots$$

There are induced homomorphisms for relative (co)homology just as there are in the nonrelative case. A map $f : X \to Y$ with $f(A) \subset B$, or more concisely, $f : (X,A) \to (Y,B)$, induces homomorphisms $f_\# : C_n(X,A) \to C_n(Y,B)$ since the chain map $f_\# : C_n(X) \to C_n(Y)$ takes $C_n(A)$ to $C_n(B)$, so we get a well-defined map on quotients. The *relative cochain maps* are the dual maps $f^\# : C^n(Y,B;\mathbb{Z}) \to C^n(X,A;\mathbb{Z})$. The relation $f_\#\partial = \partial f_\#$ holds for relative chains since it holds for absolute chains, and it dualizes to $\delta f^\# = f^\#\delta$, thus $f^\#$ induces homomorphisms $f^* : H^n(Y,B;\mathbb{Z}) \to H^n(X,A;\mathbb{Z})$.

Analogously as for the nonrelative case, if $f \simeq g : (X,A) \to (Y,B)$, then $f^* = g^* : H^n(Y,B;\mathbb{Z}) \to H^n(X,A;\mathbb{Z})$.

## 3.3 Cohomology with Compact Supports

We start with a simplicial complex $X$ which is locally compact. This is equivalent to saying that every point has a neighbourhood which meets only finitely many simplices. Consider the subgroup $C^i_c(X;\mathbb{Z})$ of the simplicial group $C^i(X;\mathbb{Z})$ consisting of cochains which are *compactly supported* in the sense that they attain nonzero values only on finitely many simplices. The coboundary of such cochain $\varphi$ can have a nonzero value only on those $(i+1)$–simplices having a face on which $\varphi$ is nonzero, and there are only finitely many such simplices by the local compactness assumption, so $\delta\varphi$ lies in $C^{i+1}_c(X;\mathbb{Z})$. Thus we have a subcomplex of the simplicial cochain complex. The resulting cohomology groups for this subcomplex, denote by $H^i_c(X;\mathbb{Z})$, are called *compactly supported cohomology groups.*

However, we will use another definition of the cohomology groups with compact supports $H^i_c(X;\mathbb{Z})$ in terms of algebraic limits. The cochain group $C^i_c(X;\mathbb{Z})$ is the union of its subgroups $C^i(X,X-K;\mathbb{Z})$ as $K$ ranges over compact subsets of $X$. Each inclusion $K \hookrightarrow L$ induces inclusions $C^i(X,X-K;\mathbb{Z}) \hookrightarrow C^i(X,X-L;\mathbb{Z})$ for all $i$, so there are induced maps $H^i(X,X-K;\mathbb{Z}) \to H^i(X,X-L;\mathbb{Z})$.

In order to give this definition, we need to introduce first the notion of direct limit and show some basic properties. Let $I$ be a set with a partial order $\leq$ satisfying the property that for any $i, j \in I$ there is a $k \in I$ with $i \leq k$ and $j \leq k$. Such a set is called a *directed set*. Let $A_i$ be an abelian group for each $i$, and for each pair $i \leq j$ a map $\varphi_{ij} : A_i \to A_j$ with $\varphi_{ii} = \mathbb{1}_{A_i}$ for each $i$, and such that whenever $i \leq j \leq k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Then $\{A_i, \varphi_{ij}\}$ is called a *directed system of groups*. The *direct limit* $\varinjlim A_i$ is the unique up to isomorphism group $L$ satisfying the following universal mapping property. Consider maps $\varphi_i : A_i \to L$ such that $\varphi_i = \varphi_j \circ \varphi_{ij}$ for every pair $i \leq j$. If there is an abelian group $C$ together with maps $\tau_i : A_i \to C$ such that $\tau_i = \tau_j \circ \varphi_{ij}$ for each $i \leq j$, then there is a unique group homomorphism $\tau : L \to C$ with $\sigma = \tau \circ \varphi_i$.

$$\begin{tikzcd} A_i \arrow[r, "\tau_i"] \arrow[d, "\varphi_i"'] \arrow[dr, phantom, "\exists!"] & C \\ L \arrow[ur, dotted, "\tau"'] & \end{tikzcd}$$

Let $M$ be the direct sum of the $A_i$, and let $N$ be the subgroup generated by all elements of the form $a - \varphi_{ij}(a)$ for all $i \leq j$ and $a \in A_i$. Then $M/N$ together with $\varphi_i$ the compositions of the natural maps $A_i \to M \to M/N$, satisfy the mapping property for the direct limit.

An alternative description for the direct limits is the following. Let $\{A_i, \varphi_{ij}\}$ be a directed system of groups. Consider pairs $(A_i, a_i)$ with $a_i \in A_i$. Define a relation $\sim$ on such pairs by $(A_i, a_i) \sim (A_j, a_j)$ if there is a $k \geq i, j$ with $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. We denote $[A_i, a_i]$ the equivalence class of a pair $(A_i, a_i)$. Let $G$ be the set of equivalence classes. Then we can define an operation on $G$ by

$$[A_i, a_i] + [A_j, a_j] = [A_k, \varphi_{ik}(a_i) + \varphi_{jk}(a_j)],$$

where $k$ is any index with $k \geq i, j$. With this operation $G$ becomes a group. The map $\varphi_i : A_i \to G$ given by $\varphi_i(a) = [A_i, a]$ is a group homomorphism. Then, it is easy to prove (see [6]) that $G \simeq \varinjlim A_i$.

**Lemma 3.3.1.** Let $\varinjlim A_i$ be the direct limit of a directed system of groups. Then,

1. Every element of $\varinjlim A_i$ can be written in the form $\varphi_i(a)$ for some $a \in A_i$.
2. If $a \in A_i$ satisfies $\varphi_i(a) = 0$, then there is a $j \geq i$ with $\varphi_{ij}(a) = 0$ (see [6] for a proof).

*Proof.* see [6] $\square$

Now we can give the definition of *cohomology with compact supports in terms of direct limits*. For a space $X$, the compact subsets $K \subset X$ form a directed set under inclusion since the union of two compact sets is compact. To each compact $K \subset X$ we associate the group $H^i(X, X-K; \mathbb{Z})$, with a fixed $i$ and a coefficient group $\mathbb{Z}$. To each inclusion $K \subset L$ of compact sets, we have the inclusion $(X-L) \overset{i}{\hookrightarrow} (X-K)$. We associate the natural homomorphism $H^i(X, X-K; \mathbb{Z}) \xrightarrow{i^*} H^i(X, X-L; \mathbb{Z})$. The resulting limit group $\varinjlim H^i(X, X-K; \mathbb{Z})$ is then equal to $H^i_c(X; \mathbb{Z})$ since each element of this limit group is represented by a cocycle in $C^i(X, X-K; \mathbb{Z})$ for some compact $K$ (by Lemma 3.3.1(1)), and such a cocycle is zero in $\varinjlim H^i(X, X-K; \mathbb{Z})$ iff it is zero in $C^i(X, X-L; \mathbb{Z})$ for some compact $L \supset K$ (by Lemma 3.3.1(2)), which means is the coboundary of a cochain in $C^{i-1}(X, X-L; \mathbb{Z})$.

# 4 Computation of Compactly Supported Cohomology Groups of Systolic Pseudomanifolds

This last chapter is devoted to the proof of Main Theorem. We start by computing the cohomology groups $H^1_c(\mathbb{R};\mathbb{Z})$ and $H^*_c(\mathbb{R}^n;\mathbb{Z})$ , which use the tools presented in the previous sections and help us to understand the techniques used to solve our problem. Finally we proceed to the proof of Main Theorem by proving first a lemma that states that $H^1(Y;\mathbb{Z})$ is nontrivial for $Y$ a 6–large 2–pseudomanifold.

## 4.1 Computation of $H^1_c(\mathbb{R};\mathbb{Z})$

We start by computing $H^1_c(X;\mathbb{Z})$, where $X$ is a triangulation of $\mathbb{R}$ as in Example 2.1.9(a). As seen in Example 2.2.10(a), $X$ is a systolic complex, and by Example 2.5.1(a) it is a pseudomanifold.

**Example 4.1.1.** $H^1_c(X;\mathbb{Z}) \simeq \mathbb{Z}$.

*Proof.* Let $K_1 = (v_1, \ldots, v_n)$ be a closed interval in $X$, where $v_i$ are the vertices and $v_iv_{i+1}$ 1–simplices. Then $C^1(X, X - K_1;\mathbb{Z})$ is the set of functions $\varphi$ that assign some integer values to the 1–simplices inside $K_1$ and 0 outside. We define a map $\Sigma^C : C^1(X, X - K_1;\mathbb{Z}) \to \mathbb{Z}$ that sends each cochain $\varphi$ to the sum of its values on all the 1–simplices, i.e., $\Sigma^C\varphi := \sum_{i\in\mathbb{Z}} \varphi(v_iv_{i+1}) = \sum_{i=1}^{n-1} \varphi(v_iv_{i+1})$. We show that this induces an isomorphism $\Sigma^H : H^1(X, X - K_1;\mathbb{Z}) \to \mathbb{Z}$ with $\Sigma^H[\varphi] = \Sigma^C\varphi$.

We show that $\Sigma^H$ is well defined: $[\varphi_1] = [\varphi_2] \iff [\varphi_1 - \varphi_2] = 0 \iff (\varphi_1 - \varphi_2) \in \delta^1(C^0(X, X - K_1;\mathbb{Z})) \iff (\varphi_1 - \varphi_2) = \delta^1\alpha$ (for $\alpha \in C^0(X, X - K_1;\mathbb{Z})$). But $\Sigma^C$ vanishes on coboundaries, therefore $\Sigma^C(\delta^1\alpha) = 0 \Rightarrow \Sigma^C(\varphi_1 - \varphi_2) = 0 \Rightarrow \Sigma^C\varphi_1 = \Sigma^C\varphi_2 \Rightarrow \Sigma^H[\varphi_1] = \Sigma^H[\varphi_2]$.

Obviously $\Sigma^H$ is surjective since $\Sigma^C$ is surjective and every element of $C^1(X, X - K_1;\mathbb{Z})$ is a cocycle.

We show that $\Sigma^H$ is injective: Suppose $\Sigma^H[\varphi] = 0$. We show that in this case, $\varphi = \delta\alpha$ for some $\alpha \in C^0(X, X - K_1; \mathbb{Z})$. For that, we construct $\alpha$ such that given a vertex $v_k$, $\alpha := \sum_{i=-\infty}^{k} \varphi(v_{i-1}v_i) = \sum_{i=2}^{k} \varphi(v_{i-1}v_i)$. We check first that with this definition, $\varphi = \delta\alpha$:

$$\delta\alpha(v_{k-1}v_k) = \alpha\partial(v_{k-1}v_k) = \alpha(v_k) - \alpha(v_{k-1}) = \sum_{i=2}^{k} \varphi(v_{i-1}v_i) - \sum_{i=2}^{k-1} \varphi(v_{i-1}v_i) = \varphi(v_{k-1}v_k).$$

Now we need to check that $\alpha \in C^0(X, X - K_1; \mathbb{Z})$. For that it must assign some integer value to the vertices contained in $K_1$ and 0 outside. Obviously, for all $i \leq 0$ we have that $\alpha(v_i) = 0$, since $\varphi(v_{i-1}v_i) = 0$ for all $v_{i-1}v_i$. Now consider $k \geq n$. Then $\alpha(v_k) = \sum_{i=2}^{k} \varphi(v_{i-1}v_i) = \sum_{i=2}^{n} \varphi(v_{i-1}v_i) = \Sigma^C\varphi = \Sigma^H[\varphi] = 0$. Therefore $\alpha \in C^0(X, X - K_1; \mathbb{Z}) \Rightarrow \varphi \in \delta^1(C^0(X, X - K_1; \mathbb{Z})) \Rightarrow [\varphi] = 0$.

So we proved that $H^1(X, X - K_1; \mathbb{Z}) \simeq \mathbb{Z}$. Now, given an interval $K_2$ such that $K_1 \subseteq K_2$, the exact same proof gives that $H^1(X, X - K_2; \mathbb{Z}) \simeq \mathbb{Z}$. Given the inclusion $(X - K_2) \overset{i}{\hookrightarrow} (X - K_1)$, there is an induced map $H^1(X, X - K_1; \mathbb{Z}) \xrightarrow{i^*} H^1(X, X - K_2; \mathbb{Z})$. Since we can map an element $[\varphi_1] \in H^1(X, X - K_1; \mathbb{Z})$, such that $[\varphi_1] \xrightarrow{\Sigma^H} 1$ is a generator, as $i^*([\varphi_1]) = [\varphi_2]$, with $[\varphi_2] \in H^1(X, X - K_2; \mathbb{Z})$ and $[\varphi_2] \xrightarrow{\Sigma^H} 1$, we conclude that $i^*$ is an isomorphism. The compact subsets $K_1 \subset K_2 \subset K_3 \subset \ldots$ form a directed set under inclusion. But since the induced homomorphisms between cohomology groups associated to each inclusion turn to be isomorphisms, the resulting limit group is

$$\varinjlim H^1(X, X - K_i; \mathbb{Z}) = H^1(X, X - K_n; \mathbb{Z}) = \mathbb{Z}$$

Since this limit is equal to $H^1_c(X; \mathbb{Z})$, the calculation is done.

□

## 4.2 Computation of $H^*_c(\mathbb{R}^n; \mathbb{Z})$

The method used in the next example will be used to prove the main theorem. We want to compute $H^*_c(\mathbb{R}^n; \mathbb{Z})$. Note that $\mathbb{R}^2$, as well as $\mathbb{R}$ as seen in the prior example, has a systolic triangulation (see Example 2.2.10(c)). However, in general there is no systolic triangulation of $\mathbb{R}^n$ for $n > 2$ (see Remark 2.4.4).

**Example 4.2.1.** For $i, n \geq 1$, $H^i_c(\mathbb{R}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n, \\ \{0\} & \text{otherwise.} \end{cases}$

*Proof.* Let $B_k \subset \mathbb{R}^n$ be the $n$–dimensional ball centred in $(0, \ldots, 0) \in \mathbb{R}^n$ of radius $k$ with the Euclidean metric. Consider the long exact sequence of cohomology groups

$$\cdots \longrightarrow H^i(\mathbb{R}^n;\mathbb{Z}) \xrightarrow{i^*} H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \xrightarrow{\delta} H^{i+1}(\mathbb{R}^n, \mathbb{R}^n - B_k;\mathbb{Z}) \xrightarrow{j^*} H^{i+1}(\mathbb{R}^n;\mathbb{Z}) \longrightarrow \cdots$$

By Lemma 3.1.1, we have that $H^i(\mathbb{R}^n;\mathbb{Z}) = \{0\}$ and $H^{i+1}(\mathbb{R}^n;\mathbb{Z}) = \{0\}$, since $\mathbb{R}^n$ is contractible. Therefore we have the sequence

$$\cdots \longrightarrow 0 \xrightarrow{i^*} H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \xrightarrow{\delta} H^{i+1}(\mathbb{R}^n, \mathbb{R}^n - B_k;\mathbb{Z}) \xrightarrow{j^*} 0 \longrightarrow \cdots$$

In this situation, $\delta$ is an isomorphism, which means that $H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \simeq H^{i+1}(\mathbb{R}^n, \mathbb{R}^n - B_k;\mathbb{Z})$.

Now, let $i : \partial B_k \hookrightarrow (\mathbb{R}^n - B_k)$ be the inclusion map and $r : (\mathbb{R}^n - B_k) \to \partial B_k$ with $r(x) = k\frac{x}{|x|}$ a deformation retraction. Then $ri = \mathbb{1}$ and $ir \simeq \mathbb{1}$, so $\partial B_k$ and $(\mathbb{R}^n - B_k)$ are homotopy equivalent and thus $H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \simeq H^i(\partial B_k;\mathbb{Z})$. Since $\partial B_k \simeq \mathbb{S}^{n-1}$, we have $H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \simeq H^i(\mathbb{S}^{n-1};\mathbb{Z})$. By Lemma 3.1.2, we know

$$H^i(\mathbb{S}^{n-1};\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n-1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Therefore we obtain

$$H^{i+1}(\mathbb{R}^n, \mathbb{R}^n - B_k;\mathbb{Z}) \simeq H^i(\mathbb{R}^n - B_k;\mathbb{Z}) \simeq H^i(\mathbb{S}^{n-1};\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = n-1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Now we take $B_L$ such that $B_K \subset B_L$. Using a similar argument as before, the inclusion map $(\mathbb{R}^n - B_L) \overset{i}{\hookrightarrow} (\mathbb{R}^n - B_K)$ is a homotopy equivalence. Therefore, the induced map $H^i(\mathbb{R}^n, \mathbb{R}^n - B_K;\mathbb{Z}) \xrightarrow{i^*} H^i(\mathbb{R}^n, \mathbb{R}^n - B_L;\mathbb{Z})$ is an isomorphism. The compact subsets $B_1 \subset B_2 \subset B_3 \subset \ldots$ form a directed set under inclusion. But since the induced homomorphisms between cohomology groups associated to each inclusion turn to be isomorphisms, the resulting limit group is

$$\varinjlim H^i(\mathbb{R}^n, \mathbb{R}^n - B_j;\mathbb{Z}) = H^i(\mathbb{R}^n, \mathbb{R}^n - B_m;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

Since this limit is equal to $H^i_c(\mathbb{R}^n;\mathbb{Z})$, the calculation is done.

$\square$

## 4.3 Main Theorem

We come now to the main goal of this paper. But before that, we start by proving an important lemma:

**Lemma 4.3.1.** If $Y$ is a 6–large 2–pseudomanifold, then $H^1(Y;\mathbb{Z}) \neq \{0\}$ .

*Proof.* Let $S$ be the disjoint union of all the 2–simplices $\sigma$ in $Y$, i.e., $S := \sqcup\{\sigma \in Y \mid \dim \sigma = 2\}$. Consider the map $f : S \to Y$ such that $f(x) = x$ for $x \in \sigma$. Now let $\bar{S} = S/\sim$ be the quotient space where we identify 1–simplices $\tau$ and $\tau'$ of $S$ in the following way: for $x \in \tau \in \sigma^{(1)} \in S$ and $x' \in \tau' \in \sigma'^{(1)} \in S$, $x \sim x'$ if $f(x) = f(x')$ and $f(\tau) = f(\tau')$. Observe that $\sim$ is an equivalence relation since each edge in $Y$ belongs to exactly two triangles. Consider the map $i : \bar{S} \to Y$ such that $i([x]_\sim) = x$ for $x \in \sigma$. This is well defined: if two edges $\tau$ and $\tau'$ are identified in $Y$, then $i([x]_\sim) = x = f(x) = f(x') = x' = i([x']_\sim))$ for $x \in \tau$ and $x \in \tau'$.

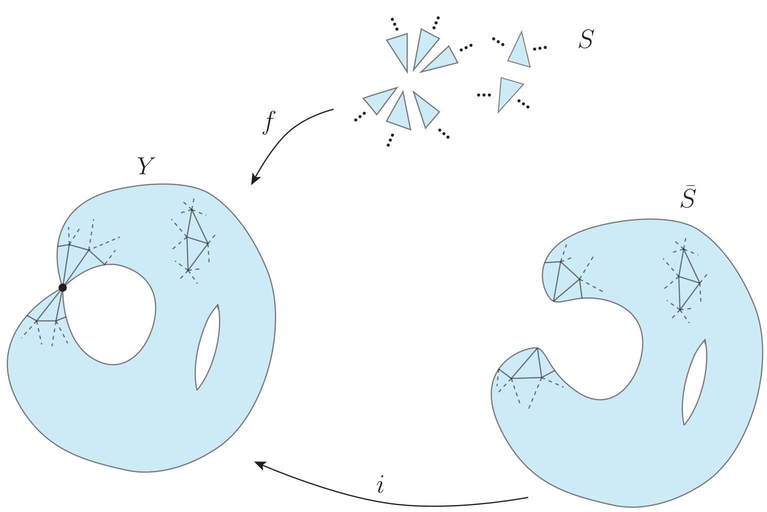



Figure 9: *Diagram of a 6–large 2–pseudomanifold $Y$ and the spaces $S$ and $\bar{S}$ formed from it, together with the maps between them.*

Now we show that $\bar{S}$ is a surface. The space $\bar{S}$ is a simplicial complex being a union of 2–simplices, and by construction, each 1–simplex is contained in exactly two 2–simplices. Therefore it is a pseudomanifold. Since $\bar{S}$ is finite, the link of a vertex can only be a union of circles. Again, by construction, all the triangles are glued together so that they have an edge in common, but it can not happen that two triangles have just one vertex in common. Therefore the link of any vertex is connected, so it is just one circle. Thus we conclude that $\bar{S}$ is a surface. Moreover, it is 6–large.

Now we see that the map $i : \bar{S} \to Y$ is locally injective. Take any vertex $v \in \bar{S}$ and consider its link. If $v$ and any of the vertices of its link were mapped to the same vertex in $Y$, that would mean that the triangles containing both vertices would be mapped either to a vertex or to an edge in $Y$, contradicting the fact that $i$ sends triangles to triangles. Therefore $i$ is locally injective.

Given a cocycle $\bar{\varphi} \in Z^1(\bar{S};\mathbb{Z})$, suppose $[\bar{\varphi}] \in H^1(\bar{S};\mathbb{Z})$ is different from 0. We define a cochain $\varphi$ in $C^1(Y;\mathbb{Z})$ as

$$\varphi(i([\tau]_\sim) := \bar{\varphi}([\tau]_\sim) \text{ for an edge } [\tau]_\sim \in \bar{S}.$$

This is well defined since $i$ is a bijection on the set of edges. We show that $\varphi$ is a cocycle. Consider a triangle $\sigma \in Y$. Let $[\sigma]_\sim$ be a triangle in $\bar{S}$ with $i([\sigma]_\sim) = \sigma$ , $\partial[\sigma]_\sim = [\tau_1]_\sim + [\tau_2]_\sim + [\tau_3]_\sim$ and $i([\tau_i]_\sim) = \tau_i$ for $i \in \{1, 2, 3\}$. Then we have,

$$\delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(\tau_1 + \tau_2 + \tau_3) = \varphi(\tau_1) + \varphi(\tau_2) + \varphi(\tau_3) =$$
$$\bar{\varphi}([\tau_1]_\sim) + \bar{\varphi}([\tau_2]_\sim) + \bar{\varphi}([\tau_3]_\sim) = \bar{\varphi}([\tau_1]_\sim + [\tau_2]_\sim + [\tau_3]_\sim) = \bar{\varphi}(\partial[\sigma]_\sim) = \partial\bar{\varphi}([\sigma]_\sim) = 0.$$

Finally we show that $[\varphi] \in H^1(Y;\mathbb{Z})$ is nontrivial. Suppose that $\varphi = \delta\alpha$ for some $\alpha \in C^0(Y;\mathbb{Z})$. We define an $\bar{\alpha} \in C^0(\bar{S};\mathbb{Z})$ as $\bar{\alpha}([v]_\sim) := \alpha(i([v]_\sim))$. Then $\bar{\varphi}([\tau]_\sim) = \varphi(i([\tau]_\sim)) = \delta\alpha(i([\tau]_\sim)) = \alpha(\partial i([\tau]_\sim)) = \alpha(i([v_1]_\sim)) - \alpha(i([v_2]_\sim)) = \bar{\alpha}([v_1]_\sim) - \bar{\alpha}([v_2]_\sim)$, where $[\tau]_\sim = [v_1]_\sim[v_2]_\sim$ and $i([v_i]_\sim) = v_i$. This is a contradiction since $\bar{\varphi}$ is not a coboundary.

Since $\bar{S}$ is 6–large, by Lemma 2.4.3 it can only be a connected sum of tori (single torus included) or a connected sum of $\mathbb{RP}^2$ (minimum two projective planes). Then by Lemmas 3.1.3 and 3.1.4, we know that $\bar{S}$ has nontrivial first cohomology group. So there exists $[\bar{\varphi}] \neq 0$ in $H^1(\bar{S};\mathbb{Z})$, thus $[\varphi] \neq 0$ in $H^1(Y;\mathbb{Z})$.

□

Finally, we have all the necessary tools to prove the main theorem of this paper.

**Theorem 4.3.2.** Let $X$ be a systolic 3–pseudomanifold. Then $H^2_c(X;\mathbb{Z}) \neq \{0\}$ .

*Proof.* To simplify the notation, we denote the ball and sphere of radius $k$ around a fixed simplex $\sigma \in X$ by $B_k$ and $S_k$ respectively. To prove this theorem, we proceed as in examples in Sections 4.1 and 4.2.

Given the long exact sequence of cohomology groups (see Section 3.2)

$$\cdots \longrightarrow H^1(X;\mathbb{Z}) \xrightarrow{i^*} H^1(X - B_k;\mathbb{Z}) \xrightarrow{\delta} H^2(X, X - B_k;\mathbb{Z}) \xrightarrow{j^*} H^2(X;\mathbb{Z}) \longrightarrow \cdots$$

we know that $H^1(X;\mathbb{Z}) = \{0\}$ and $H^2(X;\mathbb{Z}) = \{0\}$ since $X$ is contractible (by Theorem 2.3.6). Therefore we have the sequence

$$\cdots \longrightarrow 0 \xrightarrow{i^*} H^1(X - B_k;\mathbb{Z}) \xrightarrow{\delta} H^2(X, X - B_k;\mathbb{Z}) \xrightarrow{j^*} 0 \longrightarrow \cdots$$

In this situation, $\delta$ is an isomorphism, which means that $H^1(X - B_k;\mathbb{Z}) \simeq H^2(X, X - B_k;\mathbb{Z})$.

Now, by Projection Lemma (Lemma 2.3.4), $(X - B_k)$ deformation retracts into $S_k$, thus $H^2(X, X - B_k;\mathbb{Z}) \simeq H^1(S_k;\mathbb{Z})$.

Thus we have that

$$H^2_c(X;\mathbb{Z}) = \varinjlim H^2(X, X - B_k;\mathbb{Z}) = \varinjlim H^1(S_k;\mathbb{Z})$$

So our problem reduces to work on the cohomology group of pseudosurfaces $S_k$, which we know is nontrivial by Lemma 4.3.1. Thus all we need to prove is that given a cocycle different from 0 in $S_k$ it can be mapped to $S_{k+1}$ by the induced (by projection) homomorphism of cohomology groups so that its image is nontrivial too. For that we use the contraction defined in Section 2.3

$$\pi_{S_k} : S'_{k+1} \to S'_k$$

between barycentric subdivisions of spheres. For the induced map

$$\pi^*_{S_k} : H^1(S'_k) \to H^1(S'_{k+1})$$

we want to see that a cocycle $\tilde{\varphi} \in Z^1(S'_{k+1};\mathbb{Z})$ defined as $\tilde{\varphi} := \pi^*_{S_k}(\varphi)$ is nontrivial, if the cocycle $\varphi \in Z^1(S'_k;\mathbb{Z})$ is nontrivial.

Suppose that $\tilde{\varphi} = \delta\tilde{\alpha}$ for some $\tilde{\alpha} \in C^0(S'_{k+1};\mathbb{Z})$. We show that in this case we can construct an $\alpha \in C^0(S'_k;\mathbb{Z})$ such that $\varphi = \delta\alpha$, reaching then a contradiction. To define such $\alpha$ in $S'_k$ we need to do three steps:

1. Define $\alpha$ on all the vertices of $S'_k$ corresponding to barycenters of triangles in $S_k$. Consider a triangle $\tau$ in $S_k$. Since $X$ is a 3–pseudomanifold its link $X_\tau$ consists of two vertices $\tilde{v}_1$ and $\tilde{v}_2$. By Projection Lemma (Lemma 2.3.4), $S_{k-1} \cap X_\tau$ is a single (nonempty) simplex. Thus one of the vertices of the link, say $\tilde{v}_2$, belongs to $S_{k-1}$. Therefore the other vertex $\tilde{v}_1$ must belong to $S_{k+1}$. We can see this situation in Figure 10, where we name the barycenter of the triangle as $v$. We define $\alpha(v) := \tilde{\alpha}(\tilde{v}_1)$.

2. Define $\alpha$ on all the vertices of $S'_k$ corresponding to barycenters of edges in $S_k$. Consider an edge $e$. Since $S_k$ is a pseudosurface, $e$ belongs to two triangles $\tau_1$ and $\tau_2$. Let $w$

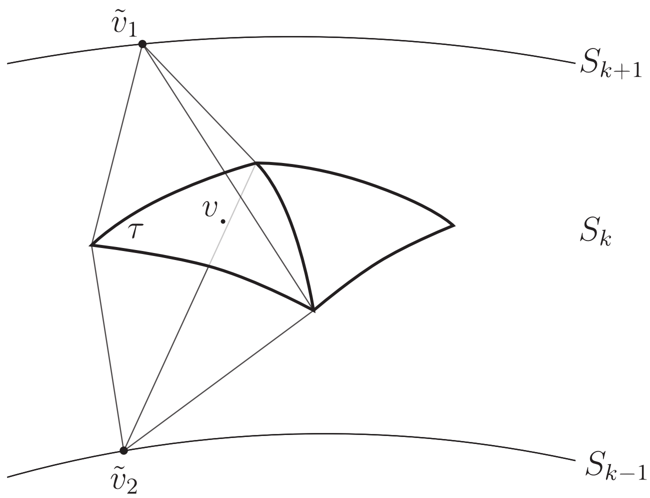



Figure 10: *Link of a triangle* $\tau \in S_k$.

be the barycenter of $e$. By Lemma 2.5.2 and Example 2.5.1, the link of this edge is a finite 1–pseudomanifold, thus a disjoint union of circles. By Projection Lemma (Lemma 2.3.4) only one of this circles intersects with $B_k$ resulting in an interval. The rest of the interval that complements the circle belongs to $S_{k+1}$. We denote it by $L = (\tilde{v}_1, \ldots, \tilde{v}_m)$, where $\tilde{v}_1$ and $\tilde{v}_m$ span a simplex with $\tau_1$ and $\tau_2$ respectively. Let $L' = (\tilde{v}_1, \tilde{w}_1, \tilde{v}_2, \tilde{w}_2, \ldots, \tilde{w}_{m-1}, \tilde{v}_m) \subseteq S'_{k+1}$ be the barycentric subdivision of $L$ (see Figure 11). Now we use that $\pi_{S_k}$ projects the edges $\tilde{v}_1\tilde{w}_1$ and $\tilde{w}_{m-1}\tilde{v}_m$ into the edges $v_1 w$ and $wv_m$ respectively and we obtain

$$\varphi(v_1 w) = \delta\tilde{\alpha}(\tilde{v}_1\tilde{w}_1) = \tilde{\varphi}(\tilde{v}_1\tilde{w}_1) = \tilde{\alpha}(\tilde{w}_1) - \tilde{\alpha}(\tilde{v}_1) = \tilde{\alpha}(\tilde{w}_1) - \alpha(v_1);$$

but since we want that $\varphi = \delta\alpha$, we have to have

$$\varphi(v_1 w) = \delta\alpha(v_1 w) = \alpha(w) - \alpha(v_1).$$

Thus we have to set $\alpha(w) = \tilde{\alpha}(\tilde{w}_1)$. Now we can see that if we do the same argument as above for the edge $wv_m$ we obtain $\alpha(w) = \tilde{\alpha}(\tilde{w}_{m-1})$. For $\alpha$ to be well defined on all the barycenters of the edges, we have to check that this two values are in fact the same. However, $\pi_{S_k}$ projects all the edges of $L'$ different from $\tilde{v}_1\tilde{w}_1$ and $\tilde{w}_{m-1}\tilde{v}_m$ into $w$. That gives $\tilde{\varphi}(\tilde{w}_1\tilde{v}_2) = \ldots = \tilde{\varphi}(\tilde{v}_{m-1}\tilde{w}_{m-1}) = 0$. This means that $0 = \tilde{\varphi}(\tilde{w}_1\tilde{v}_2) = \delta\tilde{\alpha}(\tilde{w}_1\tilde{v}_2) = \tilde{\alpha}(\tilde{v}_2) - \tilde{\alpha}(\tilde{w}_1)$, thus $\tilde{\alpha}(\tilde{v}_2) = \tilde{\alpha}(\tilde{w}_1)$. The same argument with the rest of the edges results in $\tilde{\alpha}(\tilde{w}_1) = \tilde{\alpha}(\tilde{v}_2) = \ldots = \tilde{\alpha}(\tilde{v}_{m-1}) = \tilde{\alpha}(\tilde{w}_{m-1})$. Therefore it follows that $\alpha$ is well defined.

3. Define $\alpha$ on all the vertices $u$ of $S_k$. Consider a triangle $\triangle uu'u''$, where $w$ is the barycenter of $uu'$ and $v$ the barycenter of the triangle (see Figure 12). We want that

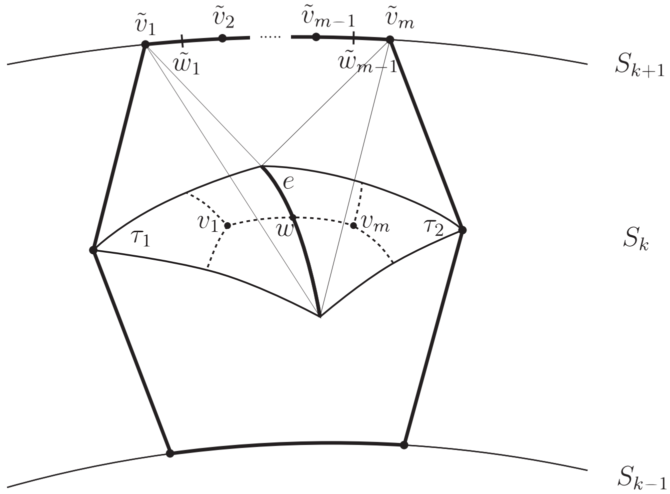



Figure 11: *Link of an edge* $e \in S_k$.

$\varphi(wu) = \alpha(u) - \alpha(w) \Longrightarrow \alpha(u) = \varphi(wu) + \alpha(w)$. That is the value we set for $\alpha(u)$. Notice that we want to have $\varphi(uv) = \alpha(v) - \alpha(u) \Longrightarrow \alpha(u) = \alpha(v) - \varphi(uv)$. Thus we have to check that this two values are the same. Joining both equations we get

$$\varphi(wu) + \alpha(w) - \alpha(v) + \varphi(uv) = 0 \iff \varphi(wu) + \varphi(vw) + \varphi(uv) = 0,$$

but since $\varphi$ is a cocycle, we have $\delta\varphi = 0$. Therefore $\varphi(wu) + \varphi(vw) + \varphi(uv) = \delta\varphi(uvw) = 0$, so $\alpha(u)$ is well defined.

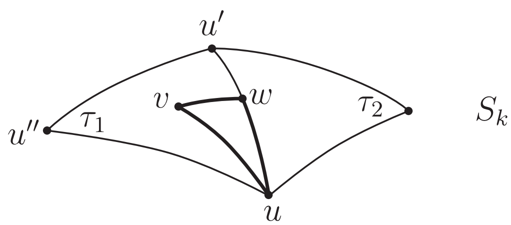



Figure 12: *Triangle* $\triangle uvw \in S'_k$

In all cases 1.,2. and 3. $\alpha$ was defined so that $\delta\alpha = \varphi$. Thus the proof is completed.

$\square$

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