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„A geometric construction of characteristic classes“

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Acknowledgments

I first encountered the notion of characteristic classes in an algebraic topology lecture given by Stefan Haller at the University of Vienna. A second appearance in a differential geometry seminar with Andreas Kriegl sparked my interest in the concept of characteristic classes from both an algebraic and a geometric point of view.

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Introduction

The beginnings of the study of characteristic classes can be traced back to works by Hassler Whitney [21], Eduard Stiefel [19], Lev Pontrjagin [17], and Shing-Shen Chern [4] in the first half of the twentieth century.

Historically, the main motivation in studying characteristic classes is to determine how many linearly independent sections a vector bundle can admit. A trivial vector bundle of dimension n admits n such sections. On an arbitrary vector bundle, characteristic classes are the first obstruction to the existence of such sections. To be more precise, if there are $n-i+1$ linearly independent sections, then the ith characteristic class vanishes.

The aim of this work is to give a geometric construction of the Stiefel-Whitney classes for real vector bundles and the Chern classes for complex vector bundles. We basically follow a paper by Marcelo Alberto Aguilar, José Luis Cisneros-Molina, and Martín Eduardo Frías-Armenta [1].

In order to carry out our construction we need some basic knowledge about vector bundles over Grassmann manifolds, which we will evolve in the first chapter (sections 1.1 and 1.2). Some well-known results on transversality will complete the preliminaries (section 1.3).

The main part (chapter 2) is the construction of the classes. We use methods from algebraic topology, in particular the Poincaré duality isomorphism will play a central role. Appendix A offers some basic definitions of (co-)homology and products on it as well as a proof of the Poincaré duality theorem. In the first section of chapter 2 we state the axiomatic definition of the Stiefel-Whitney classes and the Chern classes given by Friedrich Hirzebruch in [11].

Then we will construct the classes in the following way: Consider a smooth vector bundle ξ of rank n over a differentiable manifold M. A bundle morphism h from the product bundle ε of rank $n-i+1$ to ξ guarantees existence of $n-i+1$

linearly independent sections if and only if h is injective on each fibre. Let $\bar{Z}_1(h) \subset M$ be the set of points where h is not injective. These "singularity" subsets partition M. We are particularly interested in the subsets $Z_j(\tau)$ of tautological vector bundle morphisms τ and morphisms that are transverse to all these $Z_j(\tau)$. Such morphisms are said to be generic (section 2.2).

We consider a generic bundle morphism $h : \varepsilon^{n-i+1} \to \xi$ from the product bundle of rank $n-i+1$ to a (real or complex) vector bundle ξ of rank n. In that case, we can define a closed manifold $\tilde{Z}(h)$ and a map $\phi : \tilde{Z}(h) \to M$ whose image is $\bar{Z}_1(h)$. We will see that $\tilde{Z}(h)$ is compact and K_b-oriented. Therefore, the fundamental class $[\tilde{Z}(h)]$ exists and we consider its image under the morphism induced in homology by ϕ. Then we define the ith characteristic class $\mathbf{Cl}_i(\xi)$ of ξ as the Poincaré dual of this image (section 2.3).

The main theorem (2.4.1) will state that the so-constructed classes $\mathbf{Cl}_i(\xi)$ satisfy a set of axioms that is equivalent to the axiomatic definition of Stiefel-Whitney classes and Chern classes given by Friedrich Hirzebruch (section 2.4).

We will then generalise the result and prove that this construction actually works not only for smooth vector bundles, but for any numerable vector bundle (section 2.5).

Finally, in the last section we show the equivalence of the classes $\mathbf{Cl}_i(\xi)$ we worked with and the set of axioms defining the Stiefel-Whitney and Chern classes. In particular, this implies that the classes $\mathbf{Cl}_i(\xi)$ are well defined, which means they are independent of the choice of the generic vector bundle morphism that we used to define them (section 2.6).

1. Preliminaries

The first section of this chapter is about vector bundles, starting with the more general notion of fibre bundles. We collect some basic results and constructions such as the pullback bundle and the bundle of homomorphisms. The next section introduces Grassmann manifolds of both finite and infinite dimension and equips them with canonical vector bundles. In the last section we state and prove two transversality theorems by René Thom which reveal transversality as a generic property.

1.1. Vector bundles

1.1.1 Definition (Fibre bundle)**.**
Let F, P and M be smooth manifolds. A smooth map $p : P \to M$ is called a *fibre bundle*, if p is *locally trivial*, i.e. for every point x in M there exists an open neighborhood $U \subset M$ and a diffeomorphism $\phi_U : p^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{tikzcd} U \times F & p^{-1}(U) \arrow[l, "\phi_U"', "\cong"] \arrow[r, hook] \arrow[d, "p|_{p^{-1}(U)}"] & P \arrow[d, "p"] \\ & U \arrow[r, hook] \arrow[from=1-1, "\mathrm{pr}_1"'] & M. \end{tikzcd}$$

In this situation, P is called the *total space*, M the *base space*, p the *projection* and F the *typical fibre*. The submanifold $p^{-1}(x) = P_x \subset P$ is called the *fibre over* $x \in M$, the pair (U, ϕ_U) a *bundle chart*. We will also denote a fibre bundle by $(P, p, M; F)$.

1.1.2 Remark (Cover)**.**
A fibre bundle $p : P \to M$ is a cover if and only if the typical fibre F is discrete.

1.1.3 Definition (Sections).

(i) A continuous map $s : M \to P$ defined on (an open subset of) M is called a *(local) section* if it satisfies $p \circ s = \mathrm{id}_M$.

(ii) A *sheaf* of sets $\mathcal{F}$ on a topological space Y consists of the following data:

- a set $\mathcal{F}(U)$ of sections, also denoted $\Gamma(U, \mathcal{F})$ for any open subset $U \subset Y$,
- a (restriction) map $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for any pair $V \subset U$,
- for any open covering $U = \bigcup_{i \in I} U_i$ and for any family of sections $s_i \in \mathcal{F}(U_i)$ such that $r_{U_i, U_i \cap U_j}(s_i) = r_{U_j, U_i \cap U_j}(s_j)$ for all $i, j \in I$, there exists a unique section $s \in \mathcal{F}(U)$ such that $s_i = r_{U,U_i}(s)$ for all $i \in I$.

These data should satisfy the following conditions:

$$
\begin{aligned}
r_{UU} &= \mathrm{id}, \\
r_{VW} \circ r_{UV} &= r_{UW} \quad \text{for} \quad W \subset V \subset U.
\end{aligned}
$$

1.1.4 Remark. The local sections of a bundle $P \to M$ form a sheaf over M, called the *sheaf of sections* of P.

1.1.5 Definition (Isomorphic fibre bundles).
Two fibre bundles $(P, p, M; F)$ and $(\tilde{P}, \tilde{p}, M; \tilde{F})$ over the same base space M are called *isomorphic*, if there is a diffeomorphism $\varPhi : P \to \tilde{P}$ such that $\tilde{p} \circ \varPhi = p$ and such that the following diagram commutes:

$$
\begin{tikzcd}
P \arrow[rr, "\varPhi"] \arrow[dr, "p"'] & & \tilde{P} \arrow[dl, "\tilde{p}"] \\
& M. &
\end{tikzcd}
$$

1.1.6 Definition (Trivial fibre bundles).
Let M and F be manifolds and $\mathrm{pr}_1 : M \times F \to M$ be the projection onto the first component. Then $(M \times F, \mathrm{pr}_1, M; F)$ is a fibre bundle with an atlas consisting of one chart $\{(M \times F, \mathrm{id})\}$. Fibre bundles isomorphic to $(M \times F, \mathrm{pr}_1, M; F)$ are called *trivial*.

1.1.7 Definition (Subbundle)**.**
The set $P_V := p^{-1}(V)$ formed by the union of all fibres over an open neighbourhood $V \subset M$ is a bundle itself, the *subbundle over* V. For every point $x \in M$ the bundle chart (U, ϕ_U) induces a map given by

$$\phi_{(U,x)} := \mathrm{pr}_2 \circ \phi_U|_{P_x} : P_x \to F$$

which is a diffeomorphism between the fibre and the typical fibre. Let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be an open covering of M and let $\{(U_i, \phi_i)\}_{i \in \Lambda}$ be an atlas of bundle charts. The maps

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

are called *transition maps* between the bundle charts (U_i, ϕ_i) and (U_k, ϕ_k). They define maps from the intersections $U_i \cap U_k$ to the group of diffeomorphisms of F

$$\begin{aligned} \phi_{ik} : U_i \cap U_k &\to \mathrm{Diff}(F) \\ x &\mapsto \phi_{ix} \circ \phi_{kx}^{-1} : F \to F. \end{aligned}$$

These maps $\{\phi_{ik}\}_{i,k \in \Lambda}$ called the *cocycles of the bundle* satisfy the *cocycle conditions*

$$\begin{aligned} \phi_{ik}(x) \circ \phi_{kj}(x) &= \phi_{ij}(x), \\ \phi_{ii}(x) &= \mathrm{id}_F\,. \end{aligned}$$

Due to the smoothness of the bundle charts, the topology and the differential structure of the total space are uniquely defined by those of M and F respectively:

1.1.8 Proposition ([2])**.**

(i) Let M and F be manifolds and P be a set equipped with a map $p : P \to M$. Moreover, for each point $x \in M$ there shall exist a bundle chart (U, ϕ) with a bijective trivialisation $\phi : p^{-1}(U) \to U \times F$ and an atlas of bundle charts

$\{(U_i, \phi_i)\}_{i\in\Lambda}$ for which the transition maps

$$\phi_i \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times F$$

are smooth. Then one can define a topology and a manifold structure on P, such that $(P, p, M; F)$ is a fibre bundle.

(ii) Let $(E, \pi, M; F)$ and $(\tilde{E}, \tilde{\pi}, M; \tilde{F})$ be fibre bundles over M and $H : E \to \tilde{E}$ a bijective map satisfying $\tilde{\pi} \circ H = \pi$. Moreover, for atlases $\{(U_i, \phi_i)\}$ and $\{(U_i, \tilde{\phi}_i)\}$ of E and $\tilde{E}$ respectively, the corresponding charts

$$\tilde{\phi}_i \circ H \circ \phi_k^{-1} : (U_i \cap U_k) \times F \to (U_i \cap U_k) \times \tilde{F}$$

shall be diffeomorphisms. Then $H : E \to \tilde{E}$ is a diffeomorphism.

Proof. Without loss of generality, we may assume that the sets U_i of the atlas of E are domains of charts for M. We define a set $O \subset E$ to be open, if

$$\phi_i(O \cap \pi^{-1}(U_i)) \subset U_i \times F$$

is open in $U_i \times F$ for each bundle chart (U_i, ϕ_i), $i \in \Lambda$, (where $U_i \times F$ should carry the product topology inhabited by M and F). In this way we get a topology on E that is Hausdorff and possesses a countable basis (provided that M and F also have these properties).

Now consider atlases $\{(U_i, \varphi_i)\}_{i\in\Lambda}$ and $\{(V_j, \psi_j)\}_{j\in\Upsilon}$ of M and F respectively. To get a manifold structure on E, define charts (W_{ij}, θ_{ij}) of E by

$$\begin{aligned} W_{ij} &:= \phi_i^{-1}(U_i \times V_j) \subset E, \\ \theta_{ij} &:= (\varphi_i \times \psi_j) \circ \phi_i : W_{ij} \to \varphi_i(U_i) \times \psi_j(V_j) \subset \mathbb{R}^n \times \mathbb{R}^r \end{aligned}$$

where n and r denote the dimensions of M and F respectively. Then $\{(W_{ij}, \theta_{ij}) \mid i \in \Lambda,\ j \in \Upsilon\}$ is an atlas of E. This topology and also the manifold structure on E do not depend on the choice of atlases.

The second statement of the proposition immediately follows from the differential structures on E and $\tilde{E}$ respectively. □

1.1.9 Definition (Principal G-bundle)**.**
Let G be a Lie-group. A fibre bundle $p : P \to M$ is called a *principal G-bundle* if there is a free right-action by G on it that leaves the fibres invariant and acts transitively on them.
Two principal G-bundles $(P, p, M; G)$ and $(\tilde{P}, \tilde{p}, M; G)$ are called *isomorphic*, if there is a G-equivariant diffeomorphism $\Phi : P \to \tilde{P}$ such that $\tilde{p} \circ \Phi = p$.

1.1.10 Proposition ([2])**.** *A principal G-bundle $(P, p, M; G)$ is trivial if and only if it admits a global section.*

Proof. Let $s : M \to P$ be a global section. Then

$$\begin{aligned} \Phi : M \times G &\to P \\ (x, g) &\mapsto s(x) \cdot g \end{aligned}$$

is an isomorphism between P and the trivial principal G-bundle.
Conversely, if we are given an isomorphism $\Phi : M \times G \to P$, we can define a global section in P by $s(x) = \Phi(x, e)$. □

1.1.11 Remark. One can show ([12, Chapter 4, Corollary 8.3]) that for a principal G-bundle $\xi = (P, p, M; G)$ the following condition is equivalent to 1.1.10:
The bundle ξ is isomorphic to $f^*(\eta)$.
Here, η is the product bundle over a point, f is the unique constant map, and $f^*(\eta)$ denotes the pullback (see 1.1.19).

1.1.12 Definition (Structural group)**.**
We say a fibre bundle $p : P \to M$ with typical fibre F has *structural group G*, if there is a left-action by G on F, such that the transition maps factorize to smooth maps in G which respect the cocycle conditions (1.1.7).

1.1.13 Proposition (Associated bundle, [13])**.**
Let $p : P \to M$ be a principal G-bundle and let $\lambda : G \times S \to S$ be a smooth left-action. Then there is a right-action by G on $P \times S$ given by $(p, s) \cdot g := (p \cdot g, g^{-1} \cdot s)$ and the orbit space $P \times_G S := (P \times S)/G$ is a smooth manifold such that $P \times S \to P \times_G S$ is a principal G-bundle and $P \times_G S \to M$ is a fibre bundle with typical fibre S and structural group G.

Proof. Of course, $(p,s)\cdot g := (p\cdot g, g^{-1}\cdot s)$ is a right-action by G on $P\times S$. Let $\sigma : M \supseteq U \to P$ be a local section of the principal G-bundle $p : P \to M$. Consider the following diagram:

$$\begin{tikzcd}
 & P \arrow[dr, two heads, "\pi"] & \\
P\times S \arrow[ur, "\mathrm{pr}_1"] \arrow[r, two heads, "p"] & P\times_G S \arrow[r, dotted, two heads, "q"] & M \\
\pi^{-1}(U)\times S \arrow[u, hook] & U\times S \arrow[u, hook, "\tilde\sigma"'] \arrow[r, two heads, "\mathrm{pr}_1"] & U \arrow[u, hook] \\
\pi^{-1}(U)\times S \arrow[u, "\cong"] \arrow[ur, two heads, "\pi\times S"] \arrow[r, two heads, "\mathrm{pr}_1"'] & \pi^{-1}(U) \arrow[ur, "\pi"'] & U. \arrow[u, "\mathrm{id}"'] \arrow[l, "\sigma"]
\end{tikzcd}$$

There exists a surjective map q, because $\pi\circ\mathrm{pr}_1$ is constant on the G-orbits of $P\times S$:

$$\pi(\mathrm{pr}_1(yg, g^{-1}z)) = \pi(yg) = \pi(y) = \pi(\mathrm{pr}_1(y,z)).$$

The vertical isomorphism on the left hand side is given by

$$(y,z)\mapsto (y, \tau(\sigma(\pi(y)), y)^{-1}z),$$

in which the smooth map $\tau : P\times_M P \to G$ is given by $(y\cdot g, y)\mapsto g$.

The vertical arrow $\tilde\sigma$ is defined by $\tilde\sigma := p\circ(\sigma\times S)$ and therefore the pentagon on the left hand side commutes:

$$\begin{aligned}
(\tilde\sigma\circ(\pi\times S))(y,z) = (p\circ(\sigma\times S)\circ(\pi\times S))(y,z) &= p(\sigma\pi y, z) \\
&= p(y\cdot\tau(y,\sigma\pi y), z) \\
&= p(y, \tau(\sigma\pi y, y)^{-1}\cdot z).
\end{aligned}$$

Additionaly, $\tilde\sigma$ also ensures the commutativity of the square on the right hand side, because $\pi\times S$ is surjective and the boundary of the diagram obviously commutes. We then get a chart of the fibre bundle $P\times_G S$ with typical fibre S:

$$q^{-1}(U) = p(\pi^{-1}(U)\times S) = \tilde\sigma(\pi(\pi^{-1}(U))\times S) = \tilde\sigma(U\times S).$$

The transition maps of $\tilde{\sigma}$ are given by $(x,z) \mapsto (\tilde{\sigma}^{-1} \circ \tilde{\sigma}')(x,z) =: (x',z')$, i.e. $p(\sigma(x'),z') = \tilde{\sigma}(x',z') = \tilde{\sigma}'(x,z) = p(\tilde{\sigma}'(x),z)$. Therefore, there is $g \in G$ with $\sigma(x') = \sigma(x) \cdot g$ and $z' = g^{-1} \cdot z$ and consequently $x = x'$ and $g = \tau(\sigma'(x), \sigma(x)$. It follows that $z' = \tau(\sigma(x), \sigma'(x)) \cdot z$. This means that $P \times_G S \to M$ is a fibre bundle with structural group G. The computation also shows the injectivity of $\tilde{\sigma}$.

By construction the G-orbits of $P \times S$ are exactly the fibres of $p : P \times S \to P \times_G S$. Because of G acting freely on P, the same is true for the action on $P \times S$, i.e. $P \times S \to P \times_G S$ is a principal G-bundle. □

1.1.14 Remark (Real vector bundle)**.**
A fibre bundle $(P,p,M;F)$ with structural group G is a *real vector bundle* if and only if $F = \mathbb{R}^n$ and $G = GL(n)$ is acting in the standard way on F.

1.1.15 Definition (Vector bundle)**.**
A fibre bundle $(P,p,M;V)$ is called an $\mathbb{F}$-*vector bundle of rank* $n < \infty$ if it satisfies the following data:

- the typical fibre V is an n-dimensional vector space over $\mathbb{F}$,
- each fibre of the bundle is an $\mathbb{F}$-vector space,
- there is an atlas of bundle charts $\{U_i, \phi_i\}_{i \in \Lambda}$, such that the diffeomorphisms of fibres $\phi_{ix} : P_x \to V,\ i \in \Lambda$, are isomorphisms of vector spaces.

P is called a *real* vector bundle if $\mathbb{F} = \mathbb{R}$ and a *complex* vector bundle if $\mathbb{F} = \mathbb{C}$. If every fibre has dimension n we call it an *n-plane bundle*, the case $n = 1$ being the *line bundle*.

1.1.16 Definition (Vector bundle homomorphism)**.**
Let $(P,p,M;V)$ and $(\tilde{P},\tilde{p},M;\tilde{V})$ be two $\mathbb{K}$-vector bundles over the same base space M. A map $L : P \to \tilde{P}$ is called *vector bundle homomorphism* if it is smooth and preserves fibres and the restriction $L|_{P_x} : P_x \to \tilde{P}_x$ is linear for every $x \in M$. Two $\mathbb{K}$-vector bundles $(P,p,M;V)$ and $(\tilde{P},\tilde{p},M;\tilde{V})$ are called *isomorphic*, if there is a vector bundle isomorphism between them.

1.1.17 Remark. For every vector bundle the zero section is a global section.

1.1.18 Remark (Vector bundle versus frame bundle, [13])**.**
Let $p : P \to M$ be a principal G-bundle and let $\lambda : G \to GL(k)$ be a representation. Then the associated bundle $P \times_G \mathbb{R}^k$ is a fibre bundle with typical fibre $\mathbb{R}^k$ and structural group $GL(k)$, thus it is a vector bundle.
Conversely, we can construct every vector bundle $p : P \to M$ with fibres of dimension k if we consider the related *frame bundle* $GL(\mathbb{R}^k, E) \to M$. The set $GL(\mathbb{R}^k, E) := \{f \in L(M \times \mathbb{R}^k, E) : f \text{ invertible on each fibre}\}$ is an open subset of the vector bundle $L(M \times \mathbb{R}^k, E) \to M$. Its fibres are linear isomorphisms $\mathbb{R}^k \to E_x$ (i.e. the bases in E_x) and the right-action by $GL(k)$ is given by a reparametrization $f \mapsto f \circ g$. Now we check that $E \cong GL(\mathbb{R}^k, E) \times_{GL(k)} \mathbb{R}^k$:
The evaluation map $\mathrm{ev} : GL(\mathbb{R}^k, E) \times \mathbb{R}^k \to E$, $(f, v) \mapsto f(v)$ is surjective and invariant under $GL(k)$. Its local sections are given by local trivializations of $E \to M$. Thus ev induces a surjective fibrewise linear submersion $GL(\mathbb{R}^k, E) \times_{GL(k)} \mathbb{R}^k \to E$. For dimensional reasons this is a vector bundle isomorphism.

Essentially, the same constructions we know for vector spaces also hold for vector bundles. We point out some of these.

1.1.19 Definition (Pullback bundle)**.**
Let $f : M \to N$ be a smooth map between manifolds and $\xi = (E, p, M; F)$ a fibre bundle over M. The *pullback bundle* $f^*\xi := (f^*E, \hat{p}, N; F)$ is defined by

$$\begin{aligned} f^*\xi &:= \{(y, e) \in N \times E \mid f(y) = p(e)\} \subset N \times E, \\ \hat{p}(y, e) &:= y. \end{aligned}$$

Obviously the following diagram commutes

$$\begin{tikzcd} f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\hat{p}"'] & E \arrow[d, "p"] \\ N \arrow[r, "f"'] & M \end{tikzcd}.$$

1.1.20 Proposition ([2])**.** *$f^*\xi = (f^*E, \hat{p}, N; F)$ is a fiber bundle over N.*

Proof. Let $\{(U_i, \phi_i)\}_{i \in \Lambda}$ be an atlas of bundle charts for ξ. Now consider the open sets $V_i := f^{-1}(U_i) \subset N$ and the bijective maps

$$\begin{aligned}\psi_i : \hat{p}^{-1}(V_i) = (f^*E)_{V_i} &\to V_i \times F \\ (y, e) &\mapsto (y, \mathrm{pr}_2\, \phi_i(e)).\end{aligned}$$

Then the following maps are smooth:

$$\begin{aligned}\psi_i \circ \psi_k^{-1} : (V_i \cap V_k) \times F &\to (V_i \cap V_k) \times F \\ (y, v) &\mapsto (y, \mathrm{pr}_2\, \phi_i \phi_k^{-1}(f(y), v))\end{aligned}$$

According to 1.1.8 there exists a manifold structure on $f^*\xi$ turning $f^*\xi$ into a (smooth) fibre bundle with charts $\{(V_i, \psi_i)\}$. $\square$

1.1.21 Remark (Universal property)**.**
For every other fibre bundle $q : F \to N$ and bundle homomorphism $\bar{f} : F \to E$ there exists a unique bundle homomorphism $f^+ : F \to f^*E$ such that the following diagram commutes:

$$\begin{tikzcd}
F \arrow[drr, bend left, "\bar{f}"] \arrow[dr, dotted, "f^+"] \arrow[ddr, bend right, "q"'] & & \\
& f^*E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\hat{p}"'] & E \arrow[d, "p"] \\
& N \arrow[r, "f"'] & M.
\end{tikzcd}$$

1.1.22 Remark. If we take $N \subseteq M$ and $f := \mathrm{incl} : N \hookrightarrow M$ in the above construction, we see that $f^*\,\mathrm{incl}$ is a subbundle of ξ, called the *restriction of E to N*.

1.1.23 Remark. If $p : P \to M$ is a vector bundle or principal bundle, then so is the pullback bundle.

1.1.24 Definition (Whitney-sum)**.**
Let E and $\tilde{E}$ be two $\mathbb{F}$-vector bundles over M with typical fibres V and $\tilde{V}$ respectively, $E \oplus \tilde{E} := \bigcup_{x \in M} E_x \oplus \tilde{E}_x$ and $p_\oplus : (e_x, \tilde{e}_x) \in E \oplus \tilde{E} \mapsto x \in M$, the projection. The topology and differential structure on $E \oplus \tilde{E}$ is given by bundle

charts, as stated above. Let $(U, \phi_U = (p, \varphi_U))$ and $(U, \tilde{\phi}_U = (p, \tilde{\varphi}_U))$ be bundle charts over the open set $U \subset M$. Then

$$\begin{aligned} \phi_U : (E \oplus \tilde{E})_U &\to U \times (V \oplus \tilde{V}) \\ (e, \tilde{e}) &\mapsto (p(e), \varphi(e)_U \oplus \tilde{\varphi}_U(\tilde{e})) \end{aligned}$$

is a bundle chart of $E \oplus \tilde{E}$ over U. The vector bundle $(E \oplus \tilde{E}, p_\oplus, M; V \oplus \tilde{V})$ is called *Whitney-sum of E and $\tilde{E}$.*

1.1.25 Definition (Tensor product)**.**
Now we consider the set $E \otimes \tilde{E} := \bigcup_{x \in M} E_x \otimes_{\mathbb{F}} \tilde{E}_x$ and the projection $p_\otimes : (e_x, \tilde{e}_x) \in E \otimes \tilde{E} \mapsto x \in M$. Again the topology and the differential structure are given by the following bundle charts

$$\begin{aligned} \phi_U : (E \times \tilde{E})_U &\to U \times (V \otimes_{\mathbb{F}} \tilde{V}) \\ (e, \tilde{e}) &\mapsto (p(e), \varphi(e)_U \otimes \tilde{\varphi}_U(\tilde{e})). \end{aligned}$$

The vector bundle $(E \otimes \tilde{E}, p_\otimes, M; V \otimes \tilde{V})$ is called *tensor product of the bundles E and $\tilde{E}$.*

1.1.26 Remark. The universal property 1.1.21 generalises to tensor products.

1.1.27 Definition (Dual vector bundle)**.**
Let $(E, p, M; V)$ be a vector bundle and let V^* be the dual vector space of V. We consider the set $E^* := \bigsqcup_{x \in M} E_x^*$ and the projection $p^* : L_x \in E_x^* \mapsto x \in M$. The bundle charts

$$\begin{aligned} \phi_U^* : E_U^* &\to U \times V^* \\ L &\mapsto (p(L), \varphi_U^*(L)) \text{ with } \varphi_U^*(L)(v) = L(\phi_{U,x}^{-1}(v)) \end{aligned}$$

define the topology and differential structure of the vector bundle $(E^*, p^*, M; V^*)$, the *dual vector bundle of E.*

1.1.28 Definition (Bundle of homomorphisms)**.**
Let E and $\tilde{E}$ be two $\mathbb{F}$-vector bundles over M with typical fibres V and $\tilde{V}$ respectively. Let $\{(U, \phi)\}_{U \in \mathcal{U}}$ and $\{(U, \tilde{\phi})\}_{U \in \mathcal{U}}$ be charts for E and $\tilde{E}$. Now consider the

set

$$\mathrm{Hom}(E,\tilde{E}) := \bigsqcup_{x\in M} \mathrm{Hom}(E_x,\tilde{E}_x)$$

and the projection

$$\hat{p} : \mathrm{Hom}(E_x,\tilde{E}_x) \ni L_x \mapsto x \in M.$$

The bundle charts

$$\begin{aligned} \hat{\phi}_U : \mathrm{Hom}(E,\tilde{E})_U &\to U \times \mathrm{Hom}(V,\tilde{V}) \\ L_x &\mapsto (x,T) \text{ with } T(v) := (\tilde{\phi}_{U,x} \circ L_x \circ \phi_{U,x}^{-1})(v) \end{aligned}$$

define a new vector bundle $(\mathrm{Hom}(E,\tilde{E}),\hat{p},M;\mathrm{Hom}(V,\tilde{V}))$, *the bundle of homomorphisms from E to $\tilde{E}$.*

1.1.29 Remark. A vector bundle homomorphism $L : E \to \tilde{E}$ corresponds to a smooth section $M \ni x \mapsto L_x := L\,|_{E_x} \in \mathrm{Hom}(E_x,\tilde{E}_x)$ in the bundle of homomorphisms from E to $\tilde{E}$ and vice versa.

1.2. Grassmann manifolds

1.2.1 Notation. Let $\mathbb{F}(k,n)$ be the vector space of $k\times n$ matrices with entries in $\mathbb{F}$ and let $\mathbb{F}_r(k,n)$ be the subset of $k \times n$ matrices of rank r with $r \leqslant \min\{k,n\}$.

1.2.2 Lemma ([1, Lemma 2])**.** *$\mathbb{F}_r(k,n)$ is a submanifold of $\mathbb{F}(k,n)$ of $\mathbb{F}$-codimension $(k-r)(n-r)$.*

Proof. Consider $\alpha \in \mathbb{F}_r(k,n)$. By interchanging rows and columns of α, we can assume - without loss of generality - that the upper left submatrix of α has a non-zero determinant. A chart of $\mathbb{F}(k,n)$ around α is given by the set $U \subset \mathbb{F}(k,n)$ of matrices of the form

$$\begin{pmatrix} A & AB \\ C & CB+D \end{pmatrix}$$

with $A \in \mathbb{F}(r,r)$ and $\det(A) \neq 0$, $B \in \mathbb{F}(r,n-r)$, $C \in \mathbb{F}(k-r,r)$ and $D \in \mathbb{F}(k-r,n-r)$. Adding a multiple of the first column to the second one does not change the rank, so the previous matrix has the same rank as

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

which has rank r if and only if D is the zero matrix. Hence a chart for $\mathbb{F}_r(k,n)$ around α is given by the subset V of U of matrices with $D = 0$.
The chart $U \subset \mathbb{F}(k,n)$ has dimension kn and V has dimension $r^2 + (k-r)r + r(n-r)$. Since $kn - (r^2 + (k-r)r + r(n-r)) = (k-r)(n-r)$, this is the codimension of V in U. $\square$

1.2.3 Definition. Consider the space $\mathbb{F}_n(n, n+k)$. The set

$$V_n(\mathbb{F}^{n+k}) = \{A \in \mathbb{F}(n, n+k) \mid \ A^*A = \mathrm{id}\}$$

is called *Stiefel manifold*.

1.2.4 Remark.

(i) $V_n(\mathbb{F}^{n+k})$ is the set of all n-frames in $\mathbb{F}^{n+k}$ (i.e. n-tuples of linearly independent vectors in $\mathbb{F}^{n+k}$).

(ii) $V_n(\mathbb{F}^{n+k})$ carries the subspace topology inherited from the vector space $\mathbb{F}(n, n+k)$ making it a compact topological manifold.

1.2.5 Definition. The *Grassmann manifold* $G_n(\mathbb{F}^{n+k})$ is the set of all n-dimensional $\mathbb{F}$-subspaces in $\mathbb{F}^{n+k}$.

1.2.6 Remark (A topology for $G_n(\mathbb{F}^{n+k})$)**.**
Consider the canonical projection

$$q : V_n(\mathbb{F}^{n+k}) \to G_n(\mathbb{F}^{n+k})$$

which maps each n-frame to the n-plane it spans. Then $G_n(\mathbb{F}^{n+k})$ is given the quotient topology, i.e. a subset $U \subset G_n(\mathbb{F}^{n+k})$ is open if and only if its preimage

$q^{-1}(U) \subset V_n(\mathbb{F}^{n+k})$ is open.
Alternatively, we can also consider the commutative diagram

$$\begin{tikzcd} V_n^0(\mathbb{F}^{n+k}) \arrow[r, hook] \arrow[dr, "q_0"'] & V_n(\mathbb{F}^{n+k}) \arrow[r, "\text{Gram-Schmidt}", "\text{process}"'] \arrow[d, "q"] & V_n^0(\mathbb{F}^{n+k}) \arrow[dl, "q_0"] \\ & G_n(\mathbb{F}^{n+k}) & \end{tikzcd}$$

where $V_n^0(\mathbb{F}^{n+k})$ denotes the subset of $V_n(\mathbb{F}^{n+k})$ consisting of all orthonormal n-frames and q_0 is the restriction of q to $V_n^0(\mathbb{F}^{n+k})$.
Obviously, both constructions yield the same topolgy for $G_n(\mathbb{F}^{n+k})$.

1.2.7 Proposition ([16, 5.1])**.** *The Grassmann manifold $G_n(\mathbb{F}^{n+k})$ is a compact topological manifold of dimension nk. The correspondence $X \to X^{\perp}$ of assigning an orthogonal k-plane to each n-plane defines a homeomorphism between $G_n(\mathbb{F}^{n+k})$ and $G_k(\mathbb{F}^{n+k})$.*

Proof. We prove the proposition for the case $\mathbb{F} = \mathbb{R}$.

Fix $w \in \mathbb{R}^{n+k}$ and let $\rho_w(X)$ denote the square of the Euclidean distance from w to $X \in G_n(\mathbb{R}^{n+k})$. If $\{x_1, \ldots, x_n\}$ is an orthonormal basis for X, then the identity

$$\rho_w(X) = w \cdot w - (w \cdot x_1)^2 - \cdots - (w \cdot x_n)^2$$

shows that the composition

$$V_n^0(\mathbb{R}^{n+k}) \xrightarrow{q_0} G_n(\mathbb{R}^{n+k}) \xrightarrow{\rho_w} \mathbb{R}$$

is continuous and hence ρ_w is continuous. Now for distinct n-planes X and Y with the property $X \ni w \notin Y$, the continuous real valued function ρ_w gives $\rho_w(X) \neq \rho_w(Y)$, which proves that $G_n(\mathbb{R}^{n+k})$ is a Hausdorff space.

The set $V_n^0(\mathbb{R}^{n+k}) \subset \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$ is closed and bounded and thus compact by the Heine-Borel property. Then $G_n(\mathbb{F}^{n+k}) = q_0(V_n^0(\mathbb{F}^{n+k}))$ is also compact.

Consider $\mathbb{R}^{n+k}$ as the direct sum $X_0 \oplus X_0^{\perp}$, $X_0 \in G_n(\mathbb{R}^{n+k})$. Let U be the open subset of $G_n(\mathbb{R}^{n+k})$ consisting of all n-planes Y such that the orthogonal projection

$$p : X_0 \oplus X_0^{\perp} \to X_0$$

maps Y onto X_0 (i.e. all Y such that $Y \cap X_0^\perp = 0$). Then each $Y \in U$ can be considered as the graph of a linear transformation

$$T(Y) : X_0 \to X_0^\perp \ .$$

This defines a one-to-one correspondence

$$T : U \to \mathrm{Hom}(X_0, X_0^\perp) \cong \mathbb{R}^{nk} \ .$$

Let $\{x_1, \dots, x_n\}$ be a fixed orthonormal basis for X_0. Note that each n-plane $Y \in U$ has a unique basis $\{y_1, \dots, y_n\}$ such that

$$p(y_1) = x_1, \dots, \ p(y_n) = x_n \ .$$

The n-frame $(y_1, \dots, y_n)$ continuously depends on Y. Now note the identity

$$y_i = x_i + T(Y)x_i \ .$$

Since y_i depends continuously on Y, it follows that the image $T(Y)x_i \in X_0^\perp$ depends continuously on Y. Therefore the linear transformation $T(Y)$ depends continuously on Y.
On the other hand, this identity shows that the n-frame $(y_1, \dots, y_n)$ continuously depends on $T(Y)$, and hence Y depends continuously on $T(Y)$. Thus, the function T^{-1} is also continuous. This shows that $G_n(\mathbb{R}^{n+k})$ is a topological manifold, i.e. $G_n(\mathbb{R}^{n+k})$ is a Hausdorff space in which every point has a neighbourhood homeomorphic to $\mathbb{R}^{nk}$.

Now let $(\bar{x}_1, \dots, \bar{x}_k)$ be a fixed basis for $X_0^\perp$. Define a function

$$f : q^{-1}(U) \to V_k(\mathbb{R}^{n+k})$$

as follows: for each $(y_1, \dots, y_n) \in q^{-1}(U)$, apply the Gram-Schmidt process to the vectors $(y_1, \dots, y_n, \bar{x}_1, \dots, \bar{x}_k)$; thus obtaining an orthonormal $(n+k)$-frame $(y'_1, \dots, y'_{n+k})$ with $y'_{n+1}, \dots, y'_{n+k} \in Y^\perp$. Setting $f(y_1, \dots, y_n) = (y'_{n+1}, \dots, y'_{n+k})$, it follows that the diagram

$$\begin{tikzcd} q^{-1}(U) \arrow[r, "f"] \arrow[d, "q"'] & V_k(\mathbb{R}^{n+k}) \arrow[d, "q"] \\ U \arrow[r, "\perp"'] & G_k(\mathbb{R}^{n+k}) \end{tikzcd}$$

commutes. Now f is continuous, so $q \circ f$ is continuous, therefore the correspondence $Y \mapsto Y^{\perp}$ must also be continuous. This completes the proof. $\square$

1.2.8 Remark. The special case $k = 1$ gives equality of $G_1(\mathbb{F}^{n+1})$ and the projective space $\mathbb{F}\mathrm{P}^n$, hence $G_n(\mathbb{F}^{n+1}) \cong \mathbb{F}\mathrm{P}^n$.

1.2.9 Remark (A canonical vector bundle for $G_n(\mathbb{F}^{n+k})$)**.**
We will now construct a canonical vector bundle $\gamma^n(\mathbb{F}^{n+k})$ for the manifold $G_n(\mathbb{F}^{n+k})$. Let

$$E = E(\gamma^n(\mathbb{F}^{n+k}))$$

be the set of all pairs

$$(n\text{-plane in } \mathbb{F}^{n+k},\ \text{vector in that plane}).$$

This is to be topologized as a subset of $G_n(\mathbb{F}^{n+k}) \times \mathbb{F}^{n+k}$. The projection map $\pi : E \to G_n(\mathbb{F}^{n+k})$ is defined by $\pi(X, x) = X$, and the vector space structure in the fibre over X is defined by $t_1(X, x_1) + t_2(X, x_2) = (X, t_1x_1 + t_2x_2)$.

1.2.10 Lemma ([16, 5.2])**.** *The vector bundle $\gamma^n(\mathbb{F}^{n+k})$ constructed in 1.2.9 satisfies the local triviality condition.*

Proof. Let U be the neighborhood of X_0 as constructed in the proof of 1.2.7. Define the coordinate homeomorphism

$$h : U \times X_0 \to \pi^{-1}(U)$$

as follows. Let $h(Y, x) = (Y, y)$ where y denotes the unique vector in Y which is carried into x by the orthogonal projection

$$p : \mathbb{F}^{n+k} \to X_0.$$

The identities

$$h(Y, x) = (Y, x + T(Y)x) \quad \text{and}$$
$$h^{-1}(Y, y) = (Y, p(y))$$

show that h and h^{-1} are continuous, which completes the proof. $\square$

1.2.11 Lemma ([16, 5.3])**.** *For any n-plane bundle ξ over a compact base space B there exists a bundle map $\xi \to \gamma^n(\mathbb{F}^{n+k})$ provided k is sufficiently large.*

Proof. In order to construct a bundle map $f : \xi \to \gamma^n(\mathbb{F}^m)$ it is sufficient to construct a map

$$\hat{f} : E(\xi) \to \mathbb{F}^m$$

which is linear and injective on each fibre of ξ. Then the required function f can be defined by

$$f(e) = (\hat{f}(\text{fibre through e}),\ \hat{f}(e)).$$

Making use of the local triviality of ξ one concludes continuity of f.

Now choose open sets $U_1, \ldots, U_r$ covering B so that $\xi|_{U_i}$ is trivial. Since B is normal (i.e. B satisfies the separation axiom T_4), there exist open sets $V_1, \ldots, V_r$ covering B with $\overline{V_i} \subset U_i$, where $\overline{V_i}$ denotes the closure of V_i. Similarly construct $W_1, \ldots, W_r$ with $\overline{W_i} \subset V_i$. Let $\lambda : B \to \mathbb{F}$ denote a continuous function which takes the value 1 on $\overline{W_i}$ and the value 0 outside of V_i.
Since $\xi|_{U_i}$ is trivial there exists a map $h_i : \pi^{-1}(U_i) \to \mathbb{F}^n$ which maps each fibre of

$\xi|_{U_i}$ linearly onto $\mathbb{F}^n$. Define

$$\begin{aligned} h_i' &: E(\xi) \to \mathbb{F}^n, \\ &h_i'(e) = 0 \qquad \text{for } \pi(e) \notin V_i, \\ &h_i'(e) = \lambda_i(\pi(e))h_i(e) \qquad \text{for } \pi(e) \in U_i. \end{aligned}$$

h_i' is continuous and linear on each fibre. Now define

$$\begin{aligned} \hat{f} : E(\xi) &\to \underbrace{\mathbb{F}^n \oplus \cdots \oplus \mathbb{F}^n}_{r} \cong \mathbb{F}^{rn}, \\ \hat{f}(e) &= (h_1'(e), \ldots, h_r'(e)). \end{aligned}$$

Then $\hat{f}$ is continuous and maps each fibre injectively. $\square$

1.2.12 Remark. Because of 1.2.11 we call $\gamma^n(\mathbb{F}^{n+k})$ a *universal* bundle, also see 2.5.11 below.

1.2.13 Remark. For any $n \in \mathbb{N}$ set $\nu(n) = bn - 1$ and let $\mathbf{S}^{\nu(n)} = \{\mathbf{x} \in \mathbb{F}^n \mid \|\mathbf{x}\| = 1\}$ be the sphere or real dimension $\nu(n)$. (Set $b = 1$ if $\mathbb{F} = \mathbb{R}$ and $b = 2$ if $\mathbb{F} = \mathbb{C}$.)

Consider $\mathbb{F}\mathrm{P}^n$ as $\mathbf{S}^{\nu(n+1)}/\mathbf{S}^{\nu(1)}$ with $\mathbf{S}^{\nu(1)}$ acting on $\mathbf{S}^{\nu(n+1)}$ by

$$\lambda \cdot (x_1, \ldots, x_{n+1}) = (\lambda x_1, \ldots, \lambda x_{n+1}),$$

where $\lambda \in \mathbf{S}^{\nu(1)}$ and $(x_1, \ldots, x_{n+1}) \in \mathbf{S}^{\nu(n+1)}$. We denote the elements in $\mathbb{F}\mathrm{P}^n$ by $[\mathbf{x}] = [x_1, \ldots, x_{n+1}]$ with $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbf{S}^{\nu(n+1)}$.

Then we make the following definition.

1.2.14 Definition (Bundles over $\mathbb{F}\mathrm{P}^n$)**.**

(i) *(The vector bundle ζ^n over* $\mathbb{F}\mathrm{P}^n$*).* The total space of ζ^n is given by

$$E(\zeta^n) = (\mathbf{S}^{\nu(n+1)} \times \mathbb{F}^n)/\mathbf{S}^{\nu(1)}.$$

The action of $\mathbf{S}^{\nu(1)}$ on $\mathbf{S}^{\nu(n+1)} \times \mathbb{F}^n$ is given by $\lambda \cdot (\mathbf{x}, \mathbf{v}) = (\lambda\mathbf{x}, \lambda\mathbf{v})$, where $\lambda \in \mathbf{S}^{\nu(1)}$, $\mathbf{x} \in \mathbf{S}^{\nu(n+1)}$ and $\mathbf{v} \in \mathbb{F}^n$. Elements of $(\mathbf{S}^{\nu(n+1)} \times \mathbb{F}^n)/\mathbf{S}^{\nu(1)}$ are denoted by $[\mathbf{x}, \mathbf{v}]$, where $\mathbf{x} \in \mathbf{S}^{\nu(n+1)}$ and $\mathbf{v} \in \mathbb{F}^n$.

The projection map is given by $[\mathbf{x}, \mathbf{v}] \mapsto [\mathbf{x}]$.

(ii) *(The canonical line bundle γ_n^1 over $\mathbb{F}\mathrm{P}^n$).* Its total space is given by

$$E(\gamma_n^1) = (\mathbf{S}^{\nu(n+1)} \times \mathbb{F})/\mathbf{S}^{\nu(1)}.$$

The action of $\mathbf{S}^{\nu(1)}$ on $\mathbf{S}^{\nu(n+1)} \times \mathbb{F}$ is given by $\lambda \cdot (\mathbf{x}, v) = (\lambda\mathbf{x}, \lambda v)$, where $\lambda \in \mathbf{S}^{\nu(1)}$, $\mathbf{x} \in \mathbf{S}^{\nu(n+1)}$ and $v \in \mathbb{F}$.

The projection map is given by $[\mathbf{x}, v] \mapsto [\mathbf{x}]$.

1.2.15 Lemma ([1, Lemma 1])**.** *The bundle ζ^n is isomorphic to the Whitney sum of n copies of the canonical line bundle γ_n^1 over $\mathbb{F}\mathrm{P}^n$.*

Proof. There is a bundle map, i.e. a bundle morphism such that its restriction to each fibre is an isomorphism:

$$\begin{tikzcd}
(\mathbf{S}^{\nu(n+1)} \times \mathbb{F}^n)/\mathbf{S}^{\nu(1)} \arrow[r, "\tilde{\Delta}"] \arrow[d] & \gamma_n^1 \times \cdots \times \gamma_n^1 \arrow[d] \\
\mathbb{F}\mathrm{P}^n \arrow[r, "\Delta"'] & \underbrace{\mathbb{F}\mathrm{P}^n \times \cdots \times \mathbb{F}\mathrm{P}^n}_{n}
\end{tikzcd}$$

where Δ is the diagonal map and $\tilde{\Delta}[\mathbf{x}, v_1, \ldots, v_n] = ([\mathbf{x}, v_1], \ldots, [\mathbf{x}, v_n])$. Therefore

$$\Delta^*(\gamma_n^1 \times \cdots \times \gamma_n^1) = \gamma_n^1 \oplus \cdots \oplus \gamma_n^1 \cong \zeta^n.$$

□

1.2.16 Remark. For later use and to handle more exotic base spaces we want to let the dimension of $\mathbb{F}^{n+k}$ tend to infinity. Therefore we consider (infinite) sequences of the form

$$x = (x_1, x_2, x_3, \ldots)$$

for which all but a finite number of the x_i are zero. We denote the vector space consisting of these sequences by $\mathbb{F}^\infty$. For fixed k, the subspace consisting of sequences of the form

$$x = (x_1, x_2, \ldots, x_k, 0, 0, \ldots)$$

will be identified with $\mathbb{F}^k$.

We have inclusions $\mathbb{F}^1 \subset \mathbb{F}^2 \subset \mathbb{F}^3 \subset \dots$ and $\mathbb{F}^\infty := \bigcup_{k\geq 1} \mathbb{F}^k$.

1.2.17 Definition. The *infinite Grassmann manifold* $G_n = G_n(\mathbb{F}^\infty)$ is the set of all n-dimensional linear subspaces of $\mathbb{F}^\infty$, topologized as the direct limit of the sequence

$$G_n(\mathbb{F}^n) \subset G_n(\mathbb{F}^{n+1}) \subset G_n(\mathbb{F}^{n+2}) \subset \dots,$$

i.e. a subset of G_n is open (or closed) if and only if its intersection with $G_n(\mathbb{F}^{n+k})$ is open (or closed) as a subset of $G_n(\mathbb{F}^{n+k})$ for each k.

1.2.18 Remark. As a special case, the *infinite projective space* $\mathbb{F}\mathrm{P}^\infty = G_1(\mathbb{F}^\infty)$ is equal to the direct limit of the sequence $\mathbb{F}\mathrm{P}^1 \subset \mathbb{F}\mathrm{P}^2 \subset \mathbb{F}\mathrm{P}^3 \subset \dots$.

1.2.19 Remark (Canonical vector bundle for G_n)**.**
Similarly to the finite dimensional case (1.2.9) we construct a canonical vector bundle γ^n for G_n.
Let

$$E(\gamma^n) \subset G_n \times \mathbb{F}^\infty$$

be the set of all pairs

$$(n\text{-plane in } \mathbb{F}^\infty,\ \text{vector in that plane}),$$

topologized as a subspace of the Cartesian product. Define

$$\pi : E(\gamma^n) \to G_n,$$
$$\pi(X, x) = X.$$

The vector space structure on the fibres is defined as in the finite dimensional case 1.2.9.

1.2.20 Lemma ([16, 5.4])**.** *The vector bundle γ^n from 1.2.19 satisfies the local triviality condition.*

Proof. Let $X_0 \subset \mathbb{F}^\infty$ be a fixed n-plane and $U \subset G_n$ be the set of all n-planes Y which project onto X_0 under the orthogonal projection $p : \mathbb{F}^\infty \to X_0$. This set U is open since the intersection

$$U_k = U \cap G_n(\mathbb{F}^{n+k})$$

is an open set for each k. Defining

$$h : U \times X_0 \to \pi^{-1}(U)$$

as in 1.2.9, it follows from 1.2.9 that $h|_{U_k \times X_0}$ is continuous for each k. Now 1.2.21 below implies that h itself is continuous.
As in 1.2.9, the identity $h^{-1}(Y, y) = (Y, p(y))$ implies that h^{-1} is continuous, thus h is a homeomorphism, which completes the proof. □

1.2.21 Lemma ([16, 5.5]). *Let $A_1 \subset A_2 \subset A_3 \subset \ldots$ and $B_1 \subset B_2 \subset B_3 \subset \ldots$ be sequences of locally compact spaces with direct limits A and B respectively.*
Then the Cartesian product topology on $A \times B$ coincides with the direct limit topology which is associated with the sequence

$$A_1 \times B_1 \subset A_2 \times B_2 \subset A_3 \times B_3 \subset \ldots \quad .$$

Proof. Let W be open in the direct limit topology and let (a, b) be any point of W. Suppose that $(a, b) \in A_i \times B_i$. Choose a compact neighbourhood K_i of a in A_i and a compact neighbourhood L_i of b in B_i so that $K_i \times L_i \subset W$.
It is now possible to choose compact neighbourhoods K_{i+1} of K_i in A_{i+1} and L_{i+1} of L_i in B_{i+1} so that $K_{i+1} \times L_{i+1} \subset W$. Inductively we construct neighbourhoods $K_i \subset K_{i+1} \subset K_{i+2} \subset \ldots$ with union U and $L_i \subset L_{i+1} \subset L_{i+2} \subset \ldots$ with union V. Then U and V are open sets, and

$$(a, b) \in U \times V \subset W.$$

Thus W is open in the product topology. □

1.2.22 Theorem ([16, 5.7]). *Any two bundle maps from an $\mathbb{F}$-bundle of rank n to γ^n are bundle-homotopic.*

Proof. Any bundle map $f : \xi \to \gamma^n$ determines a map

$$\hat{f} : E(\xi) \to \mathbb{F}^\infty$$

whose restriction to each fibre is linear and injective. Conversely, $\hat{f}$ determines f by the identity

$$f(e) = (\hat{f}(\text{fiber through } e),\ \hat{f}(e)).$$

Let $f, g : \xi \to \gamma^n$ be any two bundle maps.

Case 1:

Suppose that the vector $\hat{f}(e) \in \mathbb{F}^\infty$ is never equal to a negative multiple of $\hat{g}(e)$ for $e \neq 0$, $e \in E(\xi)$. Then the formula

$$\hat{h}_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e), \qquad 0 \leq t \leq 1,$$

defines a homotopy between $\hat{f}$ and $\hat{g}$. To prove that $\hat{h}$ is continuous as a function of both variables, it is only necessary to prove that the vector space operations in $\mathbb{F}^\infty$ (i.e. addition and multiplication by scalars) are continuous, which follows from 1.2.21. Evidently $\hat{h}_t(e) \neq 0$ if e is a non-zero vector of $E(\xi)$. Hence we can define

$$\begin{aligned} h : E(\xi) \times [0,1] &\to E(\eta), \\ h_t(e) &= (\hat{h}_t(\text{fibre through } e,\ \hat{h}_t(e)). \end{aligned}$$

To prove that h is continuous, it is sufficient to prove that the corresponding function

$$\bar{h} : B(\xi) \times [0,1] \to G_n$$

on the base space is continuous.
Let U be an open subset of $B(\xi)$ with $\xi|_U$ trivial, and let $s_1, \dots, s_n$ be nowhere dependent cross-sections of $\xi|_U$. Then $\hat{h}|_{U\times[0,1]}$ can be considered as the composition

of

(i) a continuous function $b, t \mapsto (\hat{h}_t s_1(b), \ldots, \hat{h}_t s_n(b))$ from $U \times [0,1]$ to the 'infinite Stiefel manifold' $V_n(\mathbb{F}^\infty) \subset \underbrace{\mathbb{F}^\infty \times \cdots \times \mathbb{F}^\infty}_{n}$, and

(ii) the canonical projection $q : V_n(\mathbb{F}^\infty) \to G_n$.

By 1.2.21 q is continuous. Therefore, $\hat{h}$ is continuous, and hence, the bundle-homotopy h between f and g is continuous.

General case:

Let $f, g : \xi \to \gamma^n$ be arbitrary bundle maps. A bundle map

$$d_1 : \gamma^n \to \gamma^n$$

is induced by the linear transformation $\mathbb{F}^\infty \to \mathbb{F}^\infty$ which carries the i-th basis vector of $\mathbb{F}^\infty$ to the $(2i-1)$-th. Similarly $d_2 : \gamma^n \to \gamma^n$ is induced by the linear transformation which carries the i-th basis vector to the $2i$-th. Now we note that the three bundle-homotopies

$$f \sim d_1 \circ f \sim d_2 \circ g \sim g$$

are given by three applications of Case 1. Hence $f \sim g$. $\square$

1.2.23 Remark. We have shown that we can embed any bundle in the bundle of a suitable Grassmann manifold (1.2.11) and that this embedding is unique up to bundle-homotopy (1.2.22).

1.3. Transversality

In this section we collect some well-known results on transversality. The theorems we prove are due to René Thom. These and more can be found in [3] for example.

We start with the basic definition.

1.3.1 Definition (Transversality condition)**.**
Let M, N be manifolds, $L \subseteq N$ a submanifold and $p \in M$. A map $f : M \to N$ is called *transverse to* L (denoted $f \pitchfork L$) if the *transversality condition* holds:

$$T_p f(T_p M) + T_{f(p)} L = T_{f(p)} N \quad \text{for every} \quad p \in f^{-1}(L).$$

To state and prove the general theorems 1.3.7 and 1.3.8 below, we first have to consider the following special case.

1.3.2 Proposition ([3, 14.5])**.** *Let (E, π, M) be a differentiable vector bundle which is equipped with a Riemannian metric. Let $N \subset E$ be a differentiable submanifold and ε a continuous everywhere positive function on M. Then there exists a differentiable section $s : M \to E$, $|s(p)| < \varepsilon(p)$ for all $p \in M$, so that s is transverse to N. If $A \subset M$ is closed and the zero-section satisfies the transversality condition 1.3.1 with respect to N for all points of A, then one can choose the section s, so that $s|_A = 0$.*

To prove this proposition, we need *Sard's theorem.* A proof can be found in [3, Chapter 6] or [7, page 62].

1.3.3 Theorem (Sard)**.**
The set of critical values of a differentiable mapping of manifolds has Lebesgue measure zero.

Proof of 1.3.2. Choose first a complement (E', π', M) to the vector bundle (E, π, M), so that $E \oplus E'$ is the trivial bundle $M \times \mathbb{R}^k$. Let $f : E \oplus E' \to E$ be the projection on the first factor. Then the map $f : M \times \mathbb{R}^k \to E$ is a submersion, hence $f^{-1}(N) \subset M \times \mathbb{R}^k$ is a submanifold (for a submersion is certainly transverse), and the fibres of the normal bundle of $f^{-1}(N)$ in $M \times \mathbb{R}^k$ are mapped by Tf isomorphically onto the fibres of the normal bundle of N in E.
Hence a section s of the trivial bundle $M \times \mathbb{R}^k \to M$ is transverse to $f^{-1}(N)$ if and only if the section $f \circ s$ is transverse to N. That is, w.l.o.g., we may suppose that E is the trivial bundle $M \times \mathbb{R}^k \to M$. $f^{-1}(N)$ is the total space of the bundle $\pi^* E'|_N$ over N.
Therefore, suppose that $E = M \times \mathbb{R}^k$, $U = M \setminus A$ for the given closed set $A \subset M$

and $\delta = \varepsilon.\alpha$, where α is the function associated to A, $\alpha(x) := \exp(-\lambda(x)^{-2})$ where $\lambda = \sum_{i=1}^{\infty} \phi_i \lambda_i$ and $\{\phi_i | i \in \mathbb{N}\}$ a partition of unity. Then $0 \leq \alpha < 1$ and $\alpha^{-1}(0) = A$. All derivatives of α vanish on A, because $\exp(-t^2)$ vanishes if and only if $t = 0$.

Then $0 < \delta(p) < \varepsilon(p)$ for all $p \in U$, and both δ and all its derivatives vanish on A. We have a bundle map

$$\begin{aligned} g : E|_U = U \times \mathbb{R}^k &\to U \times \mathbb{R}^k, \\ (p, v) &\mapsto (p, (\delta(p))^{-1} v). \end{aligned}$$

Choose a regular value $w \in \mathbb{R}^k$, $|w| < 1$, of the composition

$$N \cap (E|_U) \xrightarrow{g} U \times \mathbb{R}^k \xrightarrow{\mathrm{pr}_2} \mathbb{R}^k,$$

and define the required section s by $s(p) = (p, \delta(p)w)$.
We are here using Sard's theorem, see 1.3.3. At the point $p \in A$, the value of the function s and its differential agree with those of the zero-section; by hypothesis, transversality is satisfied. If $p \in U$, one has only to convince oneself that at p the section $g \circ s|_U$ (which has the constant value w) is transverse to $g(N \cap E|_U)$. $\square$

1.3.4 Definition (Tubular neighbourhood)**.**
Let S be a submanifold of M and let N be the normal bundle of S in M. An embedding

$$i : N \to M$$

is called a *tubular neighbourhood* of S, if the image of the zero section of N is equal to S,

1.3.5 Theorem (Tubular neighbourhood theorem, [16, Theorem 11.1])**.**
Let A^n be a a smooth manifold which is smoothly and topologically embedded in a Riemannian manifold $M = M^{n+k}$.
Then there exists an open neighbourhood of A in M which is diffeomorphic to the total space of the normal bundle under a diffeomorphism which maps each point x of A to the zero normal vector of x.

1.3.6 Remark.

(i) In [16, Theorem 11.1] the above theorem 1.3.5 is proven under the additional assertion that A is compact. The general case is proven in [14] for example.

(ii) For a uniqueness theorem for tubular neighbourhoods of compact submanifolds see [3, 12.13] for example.

1.3.7 Theorem (Transversality theorem for sections, [3, 14.6])**.**
Let $f : E \to M$ be a differentiable map between differentiable manifolds, and let $s : M \to E$ be a differentiable section of f (i.e. $f \circ s = \mathrm{id}_M$). Let $N \subset E$ be a differentiable submanifold, then arbitrarily close to s there exists a section $t : M \to E$ transverse to N. If the transversality condition on s is already satisfied for all points of a closed subset $A \subset M$, then one can choose the section t, so that $t|_A = s|_A$.

Proof. We choose a well adapted tubular neighbourhood of $s(M)$ and apply proposition 1.3.2 in this tubular neighbourhood:
The section s is differentiable and an immersion ($Tf \circ Ts = \mathrm{id}$); also s is a homeomorphism with inverse map $f|_{s(M)}$ and therefore an embedding. Because $f|_{s(M)}$ has rank equal to the dimension of M, f is a submersion in some neighbourhood U of $s(M)$, and it is enough to prove the theorem for $f : U \to M$, $s : M \to U$ and $N \cap U \subset U$, i.e. we may assume that f is a submersion ($U = E$).
Consider the bundle $\ker(Tf)$ over E which is a subbundle of the tangent bundle TE. Then $\ker(Tf)|_{s(M)}$ is a complement of the tangent bundle of $s(M)$ in $TE|_{s(M)}$ and therefore may serve as a normal component:
The inclusion

$$\ker(Tf)|_{s(M)} \to TE|_{s(M)}$$

induces an ismorphism with the normal bundle of $s(M)$ in E. One can now define a spray

$$\xi : TE \to TTE,$$

so that $\xi(v) \in T(\ker(Tf))$ for vectors $v \in \ker(Tf)$ and such that the integral curves which begin in the direction of a vector out of $\ker(Tf)$, certainly preserve a direction from this subbundle. Otherwise put, the integral curves which at one point are tangential to the fibre $f^{-1}(p)$, $p \in M$, never leave $f^{-1}(p)$.
From this spray one obtains a tubular map

$$\tau : \ker(Tf)|_{s(M)} \to E,$$

such that the following diagram commutes:

$$\begin{tikzcd} \ker(Tf)|_{s(M)} \arrow[r, "\tau"] \arrow[d, "\pi"'] & E \arrow[d, "f"] \\ s(M) \arrow[r, "f", "\cong"'] & M. \end{tikzcd}$$

Since τ is an open embedding, one can directly apply proposition 1.3.2 to the left-hand side of the diagram. □

1.3.8 Theorem (Transversality theorem for maps, [3, 14.7])**.**
Let $f : M \to N$ be differentiable, and let $L \subset N$ be a differentiable submanifold. Then, arbitrarily close to f, there exists a map $g : M \to N$ transverse to L. If the transversality condition on f is already satisfied at the points of a closed subset $A \subset M$, then one can choose g, so that $f|_A = g|_A$.

Proof. Consider the composition

$$M \xrightarrow{s} M \times N \xrightarrow{\pi} N,$$

where $s = (\mathrm{id}, f)$ and π is the projection on the second factor. Then $f = \pi \circ s$ and π is a submersion, and hence transverse to L with preimage

$$\pi^{-1}(L) = M \times L \subset M \times N.$$

We may therefore by 1.3.7 approximate the section s of the projection $M \times N \to N$ by a section t transverse to $M \times L$. Hence the map $\pi \circ t : M \to N$ is transverse to $\pi(M \times L) = L$. □

1.3.9 Remark.

(i) If a smooth map $f : X \to Y$ is transversal to a regular submanifold $Z \subseteq Y$, then $f^{-1}(Z)$ is a regular submanifold of X.

(ii) Because of 1.3.7 and 1.3.8 transversality is said to be a *generic* property. In topology this means a property that holds on a dense open set, with the dual concept being a nowhere dense set. In measure theory a generic property holds almost everywhere, dual to a set of measure zero.

In that language, e.g. Sard's theorem 1.3.3 would read:
If $f : M \to N$ is a differentiable mapping between manifolds, then a generic point of N is not a critical value of f.

2. Characteristic classes and transversality

This chapter is the main part of this thesis. In the first section we state the axiomatic definition of characteristic classes given by Hirzebruch. Additionally, we state a slightly different set of axioms that turns out to be more convenient to work with.

Then we start the geometric construction of the classes $\mathbf{Cl}_i(\xi)$: The second section gives the definition of generic vector bundle morphisms. We then construct the manifold $\tilde{Z}(h)$ and prove some properties of it. In section four we state and prove the main theorem 2.4.1. The next section is dedicated to generalising the definition of the classes to numerable vector bundles. Finally, we prove uniqueness of the classes in the last section.

2.1. Axioms for characteristic classes

2.1.1 Notation (see Appendix A).

(i) We simultaneously work with both real and complex vector bundles and their (co)homology groups respectively. The ring of coefficients R is denoted by K_b; we set $b = 1$ for real vector bundles ($K_1 = \mathbb{Z}_2$ and $\mathbb{F} = \mathbb{R}$) and $b = 2$ for complex vector bundles ($K_2 = \mathbb{Z}$ and $\mathbb{F} = \mathbb{C}$).

(ii) We denote by g_n the canonical generator of $H^{bn}(\mathbb{F}\mathrm{P}^n; K_b)$, which is the dual of the fundamental class of $\mathbb{F}\mathrm{P}^n$ (see A.4.1).

2.1.2 Definition. A space is called *admissible* if it is locally compact, the union of a countable number of compact sets, and finite dimensional.

2.1.3 Remark.

(i) An admissible space is paracompact.

(ii) A space X is of dimension $\leq n$ if every open covering $\mathfrak{U}$ of X has a refinement $\mathfrak{S}$ such that each point of X lies in at most $n+1$ open sets of $\mathfrak{S}$.

(iii) In the sequel all spaces are assumed to be admissible.

2.1.4 Definition (First set of axioms)**.**
The *characteristic classes* $\mathbf{cl}_i(\xi) \in H^*(B; K_b)$ of an $\mathbb{F}$-vector bundle ξ over a base space B satisfy the following four axioms:

(A1) To each vector bundle ξ of rank n there corresponds a sequence of cohomology classes

$$\mathbf{cl}_i(\xi) \in H^{bi}(B; K_b), \; i = 0, 1, 2...$$

such that $\mathbf{cl}_0(\xi) = 1$ and $\mathbf{cl}_i(\xi) = 0$ if $i > n$.

(A2) (Naturality). If $f : B' \to B$ is a continuous map, then

$$\mathbf{cl}_i(f^*(\xi)) = f^*(\mathbf{cl}_i(\xi)).$$

(A3) (Whitney-sum). If ξ and η are vector bundles over B, then

$$\mathbf{cl}_i(\xi \oplus \eta) = \sum_{j=0}^{i} \mathbf{cl}_j(\xi) \cup \mathbf{cl}_{i-j}(\eta).$$

(A4) For the canonical line bundle γ_1^1 over $\mathbb{F}\mathrm{P}^1$,

$$\mathbf{cl}_1(\gamma_1^1) = -g_1 \in H^b(\mathbb{F}\mathrm{P}^1; K_b).$$

2.1.5 Definition (Characteristic classes)**.**

(i) The finite sum $\mathbf{cl}(\xi) = 1 + \mathbf{cl}_1(\xi) + \cdots + \mathbf{cl}_n(\xi) \in H^*(B; K_b)$ is called the *total characteristic class.*

(ii) For real vector bundles the classes $\mathbf{cl}_i(\xi)$ are called the *Stiefel-Whitney classes* $w_i(\xi)$.

(iii) For complex vector bundles the classes $\mathbf{cl}_i(\xi)$ are called the *Chern classes* $c_i(\xi)$.

2.1.6 Remark (on existence and uniqueness of the classes $\mathbf{cl}_i(\xi)$)**.**

(i) There exists at most one correspondence $\xi \mapsto \mathbf{cl}_i(\xi)$ which assigns to each vector bundle over a paracompact space a sequence of cohomology classes satisfying axioms (A1) to (A4). A proof of this uniqueness theorem can be found in [16, Theorem 7.3].

(ii) Existence of the classes $\mathbf{cl}_i(\xi)$ is shown in [16, Chapter 8] by giving a construction in terms of known operations.

The following set of axioms is slightly different from 2.1.4, but it will turn out to be more convenient to work with.

2.1.7 Definition (Second set of axioms)**.**
The classes satisfying the following axioms are denoted $\mathbf{Cl}_i(\xi)$.

(A1) as in 2.1.4.

(A2) as in 2.1.4.

(A3') Let ε^k be the trivial bundle of rank k. Then

$$\mathbf{Cl}_i(\xi \oplus \varepsilon^k) = \mathbf{Cl}_i(\xi).$$

(A4') Let ζ^n be the canonical n-bundle over $\mathbb{F}\mathrm{P}^n$. Then

$$\mathbf{Cl}_n(\zeta^n) = (-1)^n g_n \in H^{bn}(\mathbb{F}\mathrm{P}^n; K_b).$$

2.1.8 Remark. In the sequel we will work with the classes $\mathbf{Cl}_i(\xi)$ from 2.1.7 above. In 2.6.4 we will see that the two sets of axioms are equivalent.

2.1.9 Remark (Some immediate consequences of the axioms)**.**

(i) If the vector bundle ξ is isomorphic to another vector bundle η, then $\mathbf{cl}_i(\xi) = \mathbf{cl}_i(\eta)$ for all i.

(ii) If ε is a trivial vector bundle, then $\mathbf{cl}_i(\varepsilon) = 0$ for $i > 0$ by axiom (A2). (Since for a trivial vector bundle there exists a bundle map to a vector bundle over a point which has vanishing homology for $i > 0$.)

(iii) If ε is trivial, then $\mathbf{cl}_i(\varepsilon \oplus \eta) = \mathbf{cl}_i(\eta)$. In other words, (A3)+(A2) $\Rightarrow$ (A3').

(iv) (A4)+(A2) $\Rightarrow$ (A4').

(v) If ξ is an $\mathbb{F}^n$-bundle with a Euclidean metric which possesses a nowhere zero cross-section, then $\mathbf{cl}_n(\xi) = 0$.
If ξ possesses k cross-sections which are nowhere linearly dependent, then

$$\mathbf{cl}_{n-k+1}(\xi) = \mathbf{cl}_{n-k+2}(\xi) = \cdots = \mathbf{cl}_n(\xi) = 0.$$

This easily follows from [16, Theorem 3.3], since ξ splits as a Whitney sum $\varepsilon \oplus \varepsilon^{\perp}$ where ε is trivial and $\varepsilon^{\perp}$ has dimension $n - k$.

2.2. Generic vector bundle morphisms

2.2.1 Definition (Tautological bundle morphism and singularity subsets)**.**

(i) Let ζ and ξ be two smooth $\mathbb{F}$-vector bundles over a smooth manifold M and consider the bundle of morphisms $\pi : \mathrm{Hom}_{\mathbb{F}}(\zeta, \xi) \to M$.
A *tautological bundle morphism* over the total space of the bundle $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ is a morphism $\pi^*\zeta \to \pi^*\xi$ whose restriction to the fibre over $v \in \mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ is v itself considered as a linear transformation $\zeta_{\pi(v)} \cong (\pi^*\zeta)_v \to \xi_{\pi(v)} \cong (\pi^*\xi)_v$.

(ii) A bundle morphism $h : \zeta \to \xi$ (which corresponds to a cross section of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$, see 1.1.29) induces a partition of the manifold M given by *singularity subsets*

$$Z_j(h) = \{x \in M \mid \dim_{\mathbb{F}}(\ker h_x) = j\} \quad \text{where} \quad 0 \leq j \leq \mathrm{rank}\, \zeta.$$

2.2.2 Remark. Let $\tau : \pi^*\zeta \to \pi^*\xi$ be the tautological bundle morphism over the bundle of morphisms $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$. Its singularity subset $Z_j(\tau)$ is a subbundle of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ with fibre $Z_j(\tau)_x = \{v \in \mathrm{Hom}_{\mathbb{F}}(\zeta_x, \xi_x) \mid \dim_{\mathbb{F}}(\ker\, v) = j\}$.

2.2.3 Remark. Applying 1.2.2 to vector bundles ζ and ξ of rank k and n respectively, we see that $Z_j(\tau)_x$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\zeta_x, \xi_x)$ of $\mathbb{F}$-codimension

$$j(n-k+j) = kn - (k-j)(n+j).$$

Locally, we have isomorphisms $\zeta \cong M \times \mathbb{F}^k$, $\xi \cong M \times \mathbb{F}^n$, and $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi) \cong M \times L(\mathbb{F}^k, \mathbb{F}^n)$. It follows that $Z_j(\tau)$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ with

$$\mathrm{codim}_{\mathbb{R}}\, Z_j(\tau) = bj(n-k+j).$$

We have that $Z_{j+1}(\tau)$ belongs to the adherence of $Z_j(\tau)$, thus

$$\bar{Z}_1(\tau) = \bigcup_{l \geq 1} Z_l(\tau).$$

This can be seen as follows:

$\subseteq$. Consider a convergent sequence of linear maps such that every map has one-dimensional kernel. Then the kernel of the limit at least has dimension one. $\square$

$\supseteq$. Let $h : \zeta \to \xi$ be a bundle morphism with $\dim(\ker h) = j+1$ and consider a convergent sequence of morphisms h_n with $\dim(\ker h_n) \leq j$ and $h_n \to h$. Choose $0 \neq v \in \ker h$ and $w \in (\mathrm{Im}\ h)^\perp$. Define $h_n(v) = \frac{1}{n} w$ and $h_n|_{v^\perp} = h|_{v^\perp}$.
Then $v^\perp \supseteq \ker h_n \supseteq v^\perp \cap \ker h$. $\square$

2.2.4 Definition (Generic morphism)**.**
We call a vector bundle morphism $h : \zeta \to \xi$ *generic* if the corresponding section s_h of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ is transverse to all the submanifolds $Z_j(\tau)$.

2.2.5 Remark. Generic vector bundle morphisms form an open dense subset of the space of all vector bundle morphisms with the Whitney C^∞ topology. This follows (for instance) from 1.3.7.

2.2.6 Proposition ([1, Proposition 3]). *Let ζ and ξ be vector bundles over a manifold M of ranks k and n respectively. If $h : \zeta \to \xi$ is a generic bundle morphism over M, then $Z_j(h)$ is a submanifold of M of* real *codimension $bj(n-k+j)$.*

Proof. Let s_h be the section of $\mathrm{Hom}_{\mathbb{F}}(\zeta, \xi)$ corresponding to h. We have that $Z_j(h) = s_h^{-1}(Z_j(\tau))$, and since s_h is transverse to $Z_j(\tau)$, $Z_j(h)$ is a submanifold of M of *real* codimension $bj(n-k+j)$. □

2.3. The manifold $\tilde{Z}(h)$

2.3.1 Definition. Let ξ be a smooth $\mathbb{F}$-vector bundle of rank n over a smooth closed manifold M of dimension m and assume that the manifold M is K_b-oriented (see A.4.1 and 2.1.1). Let $h : \varepsilon^k \to \xi$ be a bundle morphism from the product bundle ε^k of rank k to ξ. We define:

$$\begin{aligned} \tilde{Z}(h) &= \{(x, L) \in M \times \mathbb{F}\mathrm{P}^{k-1} \mid L \subseteq \ker h_x\}, \\ \tilde{Z}^\circ(h) &= \{(x, L) \in \tilde{Z}(h) \mid L = \ker h_x\}. \end{aligned}$$

2.3.2 Remark.

(i) $\tilde{Z}^\circ(h)$ is an open subset of $\tilde{Z}(h)$. (Since the rank cannot decrease locally, the dimension of the kernel cannot increase locally, either.)

(ii) Let τ be the tautological bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$. We have that

$$\tilde{Z}(\tau) = \{(f, L) \in \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \mid L \subseteq \ker f_{\pi(f)}\}.$$

2.3.3 Proposition ([18, Proposition 1.1], [1, Proposition 4]). *Let $\hat{\phi} : \tilde{Z}(\tau) \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ be the projection onto the first factor. Then*

(i) $\hat{\phi}\big(\tilde{Z}(\tau)\big) = \bar{Z}_1(\tau)$.

(ii) $\tilde{Z}(\tau)$ *is a submanifold of* $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ *of* real *codimension bn.*

Proof. (i) If $(f, L) \in \tilde{Z}(\tau)$, then $\dim_{\mathbb{F}}(\ker f_{\pi(f)}) \geq 1$ and $f \in \bar{Z}_1(\tau)$.

On the other hand, if $f \in \bar{Z}_1(\tau)$, then the dimension of the kernel of $f_{\pi(f)}$ is at least 1 and it contains a line; hence, $(f, L) \in \tilde{Z}(\tau)$, and $\hat{\phi}(f, L) = f$.

(ii) Let $\pi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \to M$ be the bundle of morphisms and define $\varepsilon' = \hat{\phi}^* \pi^*(\varepsilon^k) = \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \times \mathbb{F}^k$ and $\xi' = \hat{\phi}^* \pi^*(\xi)$. Define the subbundle ε_1 of ε' by

$$\varepsilon_1 = \left\{ (f, L, v) \in \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \times \mathbb{F}^k \;\middle|\; v \in L \right\}.$$

Let $\pi' : \mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi') \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ be the bundle of morphisms from ε_1 to ξ'. Define the section $\Psi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \to \mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ by

$$\Psi(f, L) = f|_L \ ,$$

where $f|_L$ is the restriction of f to the line L.

We have that $\tilde{Z}(\tau)$ is the set of zeros of the section Ψ. The section Ψ is transverse to the zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$. Hence, $\tilde{Z}(\tau)$ is a submanifold of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. Clearly, the zero section has real codimension bn in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$, and therefore, $\tilde{Z}(\tau)$ has real codimension bn in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. $\square$

2.3.4 Proposition ([1, Proposition 5])**.** *Let $h : \varepsilon^k \to \xi$ be a generic bundle morphism. Then $\tilde{Z}(h)$ is a compact submanifold of $M \times \mathbb{F}\mathrm{P}^{k-1}$ of dimension $m + b(k - n - 1)$.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \arrow[r, "\hat{\phi}"] \arrow[d, "\pi \times \mathrm{id}"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \arrow[d, "\pi"'] \\
M \times \mathbb{F}\mathrm{P}^{k-1} \arrow[r, "\phi"'] \arrow[u, bend left, "\tilde{s}_h"] & M \arrow[u, bend right, "s_h"']
\end{tikzcd}$$

where $\pi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \to M$ is the bundle of morphisms, ϕ and $\hat{\phi}$ are the corresponding projections onto the first factors, s_h is the section of π corresponding to h, and the section $\tilde{s}_h$ is given by $\tilde{s}_h = s_h \times \mathrm{id}$.

Let τ be the tautological bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$. We have that

$$\tilde{Z}(h) = \tilde{s}_h^{-1}\big(\tilde{Z}(\tau)\big).$$

Therefore, by Proposition 2.3.3 it is enough to check that $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$.

Let $(x, L) \in \tilde{Z}(h)$, then $\tilde{s}_h(x, L) = (h_x, L) \in \tilde{Z}(\tau)$. By Proposition 2.3.3, $h_x \in Z_j(\tau)$ for some $j \geq 1$. Hence,

$$T_{h_x} Z_j(\tau) \subset d\hat{\phi}_{(h_x,L)} \left(T_{(h_x,L)} \tilde{Z}(\tau) \right).$$

On the other hand,

$$\begin{aligned} & d(\tilde{s}_h)_{(x,L)} \left(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \right) \oplus T_{(h_x,L)} \tilde{Z}(\tau) \\ \cong\ & d(s_h)_x (T_x M) \oplus d(\mathrm{id})_L \left(T_L \mathbb{F}\mathrm{P}^{k-1} \right) \oplus T_{(h_x,L)} \tilde{Z}(\tau) \\ \cong\ & d(s_h)_x (T_x M) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \oplus d\hat{\phi}_{(h_x,L)} \left(T_{(h_x,L)} \tilde{Z}(\tau) \right) \oplus d\hat{\varphi}_{(h_x,L)} \left(T_{(h_x,L)} \tilde{Z}(\tau) \right) \end{aligned}$$

where $\hat{\varphi} : \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1} \to \mathbb{F}\mathrm{P}^{k-1}$ is the projection onto the second factor. By the above and since

$$d\hat{\varphi}_{(h_x,L)} \big(T_{(h_x,L)} \tilde{Z}(\tau) \big) \subset T_L \mathbb{F}\mathrm{P}^{k-1},$$

we have

$$d(s_h)_x (T_x M) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \oplus T_{h_x} Z_j(\tau) \subset d(\tilde{s}_h)_{(x,L)} \left(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \right) \oplus T_{(h_x,L)} \tilde{Z}(\tau).$$

Since h is generic, s_h is transverse to $Z_j(\tau)$, thus

$$d(s_h)_x (T_x M) \oplus T_{h_x} Z_j(\tau) \cong T_{h_x} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$$

and

$$T_{h_x} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \subset d(\tilde{s}_h)_{(x,L)} \left(T_x M \oplus T_L \mathbb{F}\mathrm{P}^{k-1} \right) \oplus T_{(h_x,L)} \tilde{Z}(\tau).$$

Since the other inclusion is trivial, $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$. Hence,

$$\tilde{Z}(h) = \tilde{s}_h^{-1}\big(\tilde{Z}(\tau)\big)$$

is a submanifold of $M \times \mathbb{F}\mathrm{P}^{k-1}$. Since $\tilde{Z}(\tau)$ is closed in $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$, $\tilde{Z}(h)$ is closed in the compact space $M \times \mathbb{F}\mathrm{P}^{k-1}$ and therefore $\tilde{Z}(h)$ is compact.

By point (ii) in 2.3.3, $\tilde{Z}(h)$ has real codimension bn in $M \times \mathbb{F}\mathrm{P}^{k-1}$ which has real dimension $m + b(k-n-1)$. □

2.3.5 Proposition ([1, Proposition 6]). *The manifold $\tilde{Z}(h)$ is K_b-oriented (see 2.1.1). Therefore, it has a fundamental class $[\tilde{Z}(h)] \in H_{m+b(k-n-1)}(\tilde{Z}(h); K_b)$.*

Proof. We consider two cases.

First, let $b = 1$. In this case, every manifold is $\mathbb{Z}_2$-oriented (see A.4.10) and since $\tilde{Z}(h)$ is compact, it has a fundamental class, see [6, Corollary 22.28].

Second, let $b = 2$. If ε^k and ξ are complex vector bundles, then they have a canonical orientation. Hence, $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ has a canonical orientation. Since the complex manifold M is canonically $\mathbb{Z}$-oriented by its (holomorphic) atlas, $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi)$ is $\mathbb{Z}$-oriented as a manifold. Moreover, since $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ is a complex bundle (see the proof of 2.3.3), it also has a canonical orientation and since $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$ is a $\mathbb{Z}$-oriented manifold, $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ is also a $\mathbb{Z}$-oriented manifold. The zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ is a $\mathbb{Z}$-oriented manifold being diffeomorphic to $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^k, \xi) \times \mathbb{F}\mathrm{P}^{k-1}$. Since $\tilde{Z}(\tau)$ is the inverse image of the zero section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_1, \xi')$ under the section Ψ defined in the proof of 2.3.3(ii), it is also $\mathbb{Z}$-oriented.

On the other hand, if the bundle morphism $h : \varepsilon^k \to \xi$ is generic, by the proof of 2.3.4 the section $\tilde{s}_h$ is transverse to $\tilde{Z}(\tau)$.

Finally, since $\tilde{Z}(h) = \tilde{s}_h^{-1}\big(\tilde{Z}(\tau)\big)$, we have that $\tilde{Z}(h)$ is a $\mathbb{Z}$-oriented manifold. □

2.3.6 Proposition ([1, Proposition 7]). *Let $\phi : \tilde{Z}(h) \to M$ be the projection onto the first factor. Then ϕ is proper and maps $\tilde{Z}^{\circ}(h)$ diffeomorphically onto $Z_1(h)$.*

Proof. The image of ϕ is $\bar{Z}_1(h) = \bigcup_{l \geq 1} Z_l(h)$. Since $M \times \mathbb{F}\mathrm{P}^{k-1}$ and M are compact, $\phi : M \times \mathbb{F}\mathrm{P}^{k-1} \to M$ is proper, i.e. preimages of compact subsets are compact.

The subspaces $\tilde{Z}(h)$ and $\bar{Z}(h)$ are closed in $M \times \mathbb{F}\mathrm{P}^{k-1}$ and M respectively, hence the restriction of ϕ to $\tilde{Z}(h)$ is proper. We have that

$$\phi^{-1}\big(Z_1(h)\big) = \Big\{(x, L) \in \tilde{Z}(h) \,\Big|\, \ker h_x = L\Big\} = \tilde{Z}^\circ(h).$$

For every $x \in Z_1(h)$ we have that $\dim(\ker h_x) = 1$, thus x has only one preimage in $\tilde{Z}^\circ(h)$. Since $\tilde{Z}^\circ(h)$ is open in $\tilde{Z}(h)$, it is a submanifold, so ϕ restricted to $\tilde{Z}^\circ(h)$ is a diffeomorphism. □

To prove the following lemma 2.3.8, we need to put orientations on transversal preimages.

2.3.7 Remark (Preimage orientation [7, page 100])**.**
First, consider a direct sum $V = V_1 \oplus V_2$ and orientations on V_1 and V_2. These orientations induce the *direct sum orientation* as follows. Choose ordered bases β_1 and β_2 for V_1 and V_2 respectively and combine them to get an ordered basis $\beta = (\beta_1, \beta_2)$ for V. Then we simply demand that $\mathrm{sign}(\beta) = \mathrm{sign}(\beta_1) \cdot \mathrm{sign}(\beta_2)$, which uniquely assigns a sign to one ordered basis of the sum and thereby determines an orientation of the sum. Different choices of β_1 and β_2 would imply the same direct sum orientation, but note that the order of the summands V_1 and V_2 is crucial, for (β_1, β_2) may not be equivalent to (β_2, β_1) in orientation.

Let X, Y, Z be oriented manifolds, the last two without boundary, and let $f : X \to Y$ be a smooth map transversal to $Z \subset Y$. Now we define a *preimage orientation* on the manifold with boundary $S = f^{-1}(Z)$. If $f(x) = z \in Z$, then $T_x(S)$ is the preimage of $T_z(Z)$ under the derivative map $df_x : T_x(X) \to T_z(Y)$. Let $N_x(S; X)$ be the orthogonal complement to $T_x(S)$ in $T_x(X)$. Then

$$N_x(S; X) \oplus T_x(S) = T_x(X),$$

so that we only need to choose an orientation on $N_x(S; X)$ to obtain a direct sum orientation. Because

$$df_x T_x(X) + T_z(Z) = T_z(Y),$$

and $T_x(S)$ is the entire preimage of $T_z(Z)$, we get a direct sum

$$df_x N_x(S;X) \oplus T_z(Z) = T_Z(Y).$$

Thus the orientation on Z and Y induce a direct image orientation on $df_x N_x(S;X)$. But $T_x(S)$ contains the entire kernel of the linear map df_x, so df_x must map $N_x(S;X)$ isomorphically onto its image. Therefore the induced orientation on $df_x N_x(S;X)$ defines an orientation on $xN_x(S;X)$ via the isomorphism df_x.

2.3.8 Lemma ([1, Lemma 8])**.** *Let M and M' be two closed K_b-oriented differentiable manifolds and let Z be a closed K_b-oriented submanifold of M. Let $f : M' \to M$ be a differentiable map transverse to Z and set $Z' = f^{-1}(Z)$. Denote by i and j the inclusions of Z and Z' in M and M' respectively. If $[Z]$ and $[Z']$ are the corresponding fundamental classes of Z and Z', then*

$$j_*([Z']) = D_{M'} \circ f^* \circ D_M^{-1} \circ i_*([Z]),$$

where D_M and $D_{M'}$ denote the Poincaré duality isomorphisms in M and M' respectively.

Proof. Since f is transverse to Z, Z' is a K_b-oriented submanifold of M' with the preimage orientation, see 2.3.7.

Set dim $M = m$, dim $M' = m'$, dim $Z = r$, dim $Z' = r'$ and since Z and Z' have the same codimension let $q = m - r = m' - r'$.

Let ν be the normal bundle of Z in M; ν is oriented so that $\nu \oplus TZ$ is orientation preserving isomorphic to the restriction of TM to Z. Let ν' be the normal bundle of Z' in M', then we have that $\nu' = f^*\nu$ and in this way ν' gets its orientation from the orientation of ν.

Let $E(\nu)$ be the total space of ν and let $E(\nu)_0$ be the set of all non-zero elements of $E(\nu)$. Analogously define $E(\nu')$ and $E(\nu')_0$ for ν'. We have the following canonical isomorphisms of cohomology rings (see A.3.3)

$$H^*(E(\nu), E(\nu)_0; K_b) \cong H^*(M, M \setminus Z; K_b),$$
$$H^*(E(\nu'), E(\nu')_0; K_b) \cong H^*(M', M' \setminus Z'; K_b).$$

Under these isomorphisms, the Thom classes (see A.4.7) of ν and ν' correspond to canonical classes

$$u_\nu \in H^q(M, M \setminus Z; K_b),$$
$$u_{\nu'} \in H^q(M', M' \setminus Z'; K_b).$$

Consider the following commutative diagram

$$\begin{tikzcd} H^q(M, M \setminus Z; K_b) \arrow[r, "\iota"] \arrow[d, "f^*"'] & H^q(M; K_b) \arrow[r, "D_M"] \arrow[d, "f^*"'] & H_r(M; K_b) & H_r(Z; K_b) \arrow[l, "i_*"'] \\ H^q(M', M' \setminus Z'; K_b) \arrow[r, "\iota'"'] & H^q(M'; K_b) \arrow[r, "D_{M'}"'] & H_{r'}(M'; K_b) & H_{r'}(Z'; K_b). \arrow[l, "j_*"] \end{tikzcd}$$

We have that

$$\iota(u_\nu) = D_M^{-1} i_*([Z]),$$
$$\iota'(u_{\nu'}) = D_{M'}^{-1} j_*([Z']).$$

Now the lemma follows from the commutativity of the above diagram and the fact that $u_{\nu'} = f^*(u_\nu)$. □

2.3.9 Proposition ([1, Proposition 9]). *Let M, M' and P be K_b-oriented closed manifolds. Consider the commutative diagramm*

$$\begin{tikzcd} Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P \arrow[d, "g"] \\ M' \arrow[r, "f"'] & M \end{tikzcd}$$

where f and g are differentiable maps, Q is the fibred product given by

$$Q = \{(y, p) \in M' \times P \mid f(y) = g(p)\},$$

and π_1 and π_2 are the projections. Suppose that f and g are transverse and let $[P]$

and $[Q]$ *be the fundamental classes of* P *and* Q *respectively. Then*

$$(-1)^{(m-r)m}\pi_{1*}([Q]) = D_{M'} \circ f^* \circ D_M^{-1} \circ g_*([P]),$$

where D_M *and* $D_{M'}$ *denote the Poincaré duality isomorphisms in* M *and* M' *respectively and* $m = \dim M$ *and* $r = \dim P$.

Proof. Let $i : \Delta \to M \times M$ be the inclusion of the diagonal. We have that f is transverse to g if and only if $(f \times g)$ is transverse to Δ where $f \times g : M' \times P \to M \times M$. Therefore $Q = (f \times g)^{-1}(\Delta)$ is a K_b-oriented submanifold of $M' \times P$ of dimension $r' = r + m' - m$, with $m' = \dim M'$.

Let $[\Delta]$ be the fundamental class of Δ and set $u_\Delta = D_{M\times M}^{-1}(i_*([\Delta]))$. If $j : Q \to M' \times P$ is the inclusion, then by 2.3.8

$$\begin{aligned} j_*([Q]) &= D_{M'\times P} \circ (f \times g)^*(u_\Delta) \\ &= (f \times g)^*(u_\Delta) \cap [M' \times P] \end{aligned}$$

where $[M' \times P]$ is the fundamental class of $M' \times P$. Let $\bar{\pi}_1 : M' \times P \to M'$ be the projection onto the first factor. Since $\pi_1 = \bar{\pi}_1 \circ j$, we have (see A.2.17)

$$\begin{aligned} \bar{\pi}_{1*}\big(j_*([Q])\big) &= \pi_{1*}\big((f \times g)^*(u_\Delta) \cap [M' \times P]\big), \\ \pi_{1*}([Q]) &= \big((f \times g)^*(u_\Delta)/[P]\big) \cap [M'] \end{aligned}$$

where $[M']$ and $[P]$ are the fundamental classes of M' and P respectively. Another property of the product / (see A.2.17) gives the following commutative diagram

$$\begin{tikzcd} H_r(M;K_b) \arrow[r, "u_\Delta/"] & H^{m-r}(M;K_b) \arrow[d, "f^*"] & \\ H_r(P;K_b) \arrow[u, "g_*"] \arrow[r, "(g\times f)^*(u_\Delta)/"'] & H^{m-r}(M';K_b) \arrow[r, "\cap[M']"'] & H_{r'}(M';K_b). \end{tikzcd}$$

Applying the two different compositions to $(-1)^{(m-r)m}[P] \in H_r(P;K_b)$ and using the above we get

$$(-1)^{(m-r)m}\big((f\times g)^*(u_\Delta)/[P]\big)\cap[M'] = f^*\big((-1)^{(m-r)m}u_\Delta/g_*([P])\big)\cap[M'],$$
$$(-1)^{(m-r)m}\pi_{1*}([Q]) = D_{M'}\circ f^*\big((-1)^{(m-r)m}u_\Delta/g_*([P])\big).$$

This proves the proposition since the homomorphism $(-1)^{(m-r)m}u_\Delta/$ is the inverse of the Poincaré duality isomorphism D_M (A.4.5). $\square$

2.3.10 Remark. Given a smooth map $f : M' \to M$, the homomorphism $f_!$ defined by

$$f_! = D_{M'}\circ f^*\circ D_M^{-1}$$

is called the *Hopf-Umkehrhomomorphismus.*

In that language, the statement of 2.3.9 is

$$(-1)^{(m-r)m}\pi_{1*}\pi_{2!} = f_! g_*,$$

since $[Q] = \pi_{2!}([P])$.

2.4. Definition of the classes

Now we state the main result about the classes $\mathbf{Cl}_i(\xi)$.

2.4.1 Theorem ([1, Theorem 11])**.** *Let ξ be a smooth $\mathbb{F}$-vector bundle of rank n over a smooth closed K_b-oriented manifold M of dimension m. Let $h : \varepsilon^{n-i+1}\to\xi$ be a generic bundle morphism from the product bundle ε^{n-i+1} of rank $n-i+1$ to ξ. Then the classes*

$$\mathbf{Cl}_i(\xi) = \hat\phi\big([\tilde Z(h)]\big)\in H^{bi}(M;K_b)$$

satisfy the four axioms from 2.1.7, where $[\tilde Z(h)]$ is the fundamental class of $\tilde Z(h)$, and $\hat\phi$ is the composition of the Poincaré duality isomorphism with the homomor-

phism induced in homology by the projection onto the first factor $\phi : \tilde{Z}(h) \to M$.

$$\begin{tikzcd} H_{m-bi}(\tilde{Z}(h); K_b) \arrow[r, "\phi_*"] \arrow[dr, "\hat{\phi}"'] & H_{m-bi}(M; K_b) \arrow[d, leftrightarrow, "D"] \\ & H^{bi}(M; K_b) \end{tikzcd}$$

Proof. We have to show that the classes $\mathbf{Cl}_i(\xi)$ satisfy the four axioms from 2.1.7.

(A1). By 2.3.4, $\tilde{Z}(h)$ has dimension $m - bi$. Therefore $\mathbf{Cl}_i(\xi) \in H^{bi}(M; K_b)$.

If $i = 0$ then $\phi\big(\tilde{Z}(h)\big) = M$. Hence $\bar{Z}_1(h) = M$ and by 2.3.6 any $x \in Z_1(h)$ is a regular value of ϕ with only one preimage. Therefore ϕ has degree 1 and $\mathbf{Cl}_0(\xi) = 1 \in H^0(M; K_b)$.

The construction does not make sense for $i > n$ so we set $\mathbf{Cl}_i(\xi) = 0 \in H^{bi}(M; K_b)$ for $i > n$. □

(A2). Let $f : M' \to M$ be a differentiable map and consider the pullback diagram

$$\begin{tikzcd} f^*\xi \arrow[r, "\bar{f}"] \arrow[d, "p'"'] & \xi \arrow[d, "p"] \\ M' \arrow[r, "f"'] & M. \end{tikzcd}$$

Let $h : \varepsilon_M^{n-i+1} \to \xi$ be a generic bundle morphism. Recall that by 2.2.6 the singularity subsets $Z_j(h)$ of h are submanifolds of M. Without loss of generality we can assume that f is transverse to all the $Z_j(h)$ (if not, find a map homotopic to f which satisfies this, see 1.3.8).

Consider the bundle of morphisms $\pi : \mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1}, \xi) \to M$. Since

$$f^*\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1}, \xi) \cong \mathrm{Hom}_{\mathbb{F}}(f^*\varepsilon_M^{n-i+1}, f^*\xi)$$

and

$$f^*\varepsilon_M^{n-i+1} \cong \varepsilon_{M'}^{n-i+1},$$

we have the following pullback diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1}, f^*\xi) \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M}^{n-i+1}, \xi) \arrow[d, "\pi"'] \\ M' \arrow[r, "f"'] \arrow[u, bend left, "s_g"] & M \arrow[u, bend right, "s_h"'] \end{tikzcd}$$

where s_h is the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1}, \xi)$ corresponding to h.

Let τ and τ' be the tautological bundle morphisms (see 2.2.1) over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_M^{n-i+1}, \xi)$ and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1}, f^*\xi)$ respectively.

To simplify notation set $S_j = Z_j(\tau)$ and $S'_j = Z_j(\tau')$. We have that $S'_j = \tilde{f}^{-1}(S_j)$. Define a section s_g of $\pi' : \mathrm{Hom}_{\mathbb{F}}(\varepsilon_{M'}^{n-i+1}, f^*\xi) \to M'$ by

$$s_g(y) = \Big(y, \tilde{f}_y^{-1}\big(s_h\big(f(y)\big)\big)\Big)$$

which is well defined since $\tilde{f}$ is an isomorphism restricted to the fibres. We need to check that s_g is transverse to every S'_j.

Since h is generic, we have that s_h is transverse to S_j for every j. On the other hand, $Z_j(h) = s_h^{-1}(S_j)$, and we choose f such that f is transverse to $Z_j(h)$ for every j. This is equivalent to $s_h \circ f$ being transverse to S_j for every j. The above diagram commutes, implying that $\tilde{f} \circ s_g$ is transverse to S_j for every j, which is equivalent to s_g being transverse to S'_j for every j, since $S'_j = \tilde{f}^{-1}(S_j)$ and clearly $\tilde{f}$ is transverse to S_j. From the above diagram we also have that $Z_j(g) = f^{-1}\big(Z_j(h)\big)$ for every j, so $\bar{Z}_j(g) = f^{-1}\big(\bar{Z}_j(h)\big)$.

Let $\phi : \tilde{Z}(h) \to M$ and $\phi' : \tilde{Z}(g) \to M'$ respectively be the manifolds and maps corresponding to the generic bundle morphisms h and g given by 2.3.6. Since the image of ϕ is $\bar{Z}_1(h)$ and f is transverse to all the $Z_j(h)$, we have that f is transverse to ϕ. Moreover, the transverse intersection of f and ϕ is diffeomorphic to $\tilde{Z}(g)$, since

$$\begin{aligned}
M' \pitchfork \tilde{Z}(h) &= \{(y,x,L) \in M' \times \tilde{Z}(h) \mid f(y) = \phi(x,L),\ (x,L) \subset \ker_{h_x}\} \\
&= \{(y,f(y),L) \in M' \times \tilde{Z}(h) \mid \ (f(y),L) \subset \ker_{h_{f(y)}}\} \\
&\cong \{(y,L) \in M' \times \mathbb{F}\mathrm{P}^{n-i} \mid (y,L) \subset \ker_{g_y}\} \\
&= \tilde{Z}(g).
\end{aligned}$$

Hence, we have the following commutative diagram

$$\begin{tikzcd}
\tilde{Z}(g) \arrow[r] \arrow[d, "\phi'"'] & \tilde{Z}(h) \arrow[d, "\phi"] \\
M' \arrow[r, "f"'] & M
\end{tikzcd}$$

which satisfies the hypothesis of 2.3.9 and therefore

$$(-1)^{bim}\phi'_*\big([\tilde{Z}(g)]\big) = D_{M'} \circ f^* \circ D_M^{-1} \circ \phi_*\big([\tilde{Z}(h)]\big).$$

For the real case ($b = 1$), we use $\mathbb{Z}_2$-coefficients, so the sign is not important. In the complex case, the sign is always positive, since $b = 2$. Then one has that $f^* : H^*(M; K_b) \to H^*(M'; K_b)$ maps the Poincaré dual of the class $\phi_*\big([\tilde{Z}(h)]\big)$ to the Poincaré dual of the class $\phi'_*\big([\tilde{Z}(g)]\big)$. Thus $\mathbf{Cl}_i(f^*\xi) = f^*(\mathbf{Cl}_i(\xi))$. □

(A3'). Let $h : \varepsilon^{n-i+1} \to \xi$ be a generic vector bundle morphism. Consider the vector bundle morphism $h \oplus \mathrm{id}_{\varepsilon^k} : \varepsilon^{n-i+1} \oplus \varepsilon^k \to \xi \oplus \varepsilon^k$. Note that

$$Z_j(h \oplus \mathrm{id}_{\varepsilon^k}) = Z_j(h) \quad \text{for all } j.$$

We need to check that $h \oplus \mathrm{id}_{\varepsilon^k}$ is generic. Consider the bundle morphism

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi) \arrow[rr, "\hat{f}"] \arrow[dr] & & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k) \arrow[dl] \\
& M \arrow[ul, bend left, "s_h"] \arrow[ur, bend right, "s_{h\oplus \mathrm{id}_{\varepsilon^k}}"'] &
\end{tikzcd}$$

given by $\hat{f}(v) = v \oplus \mathrm{id}_{\varepsilon^k}$. Let τ and τ'' be the tautological bundle morphisms over

$\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k)$ respectively.

Again to simplify notation set $S_j = Z_j(\tau)$ and $S''_j = Z_j(\tau'')$. We have that $S_j = \hat{f}^{-1}(S''_j)$.

As before let s_h be the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ corresponding to h. Since h is generic, s_h is transverse to all the S_j. Let $s_h \oplus \mathrm{id}_{\varepsilon^k}$ be the section of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k)$ corresponding to $h \oplus \mathrm{id}_{\varepsilon^k}$. We have that $s_h \oplus \mathrm{id}_{\varepsilon^k} = \hat{f} \circ s_h$ and to prove that $h \oplus \mathrm{id}_{\varepsilon^k}$ is transverse to all the S''_j it suffices to show that $\hat{f}$ is transverse to all the S''_j. By the codimension formula in 2.2.3 we have the equalities

$$\begin{aligned}
&\dim\, \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi) = b^2(n-i+1)n + m, \\
&\dim\, S_j = b^2(n-i+1)n + m - bj(n-k+j), \\
&\dim\, \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k) = b^2(n-i+1+k)(n+k) + m, \\
&\dim\, S''_j = b^2(n-i+1+k)(n+k) + m - bj(n-k+j).
\end{aligned}$$

Let $v \in S_j$ and $w = \hat{f}(v)$. We need to prove that

$$\dim \left\{ d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)\big) \oplus T_w S''_j \right\} = \dim \left\{ T_w \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k) \right\}.$$

The vectors in $T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)$ which are sent to $T_w S''_j$ by $d\hat{f}_v$ are precisely the vectors in $T_v S_j$, and $d\hat{f}_v$, restricted to $T_v S_j$ is an isomorphism, so

$$d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)\big) \cap T_w S''_j \cong T_v S_j.$$

Then

$$\begin{aligned}
&\dim \left\{ d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)\big) \oplus T_w S''_j \right\} \\
= \;&\dim\, d\hat{f}_v\big(T_v \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1}, \xi)\big) + \dim\, T_w S''_j - \dim\, T_v S_j \\
= \;&b^2(n-i+1+k)(n+k) + m \\
= \;&\dim\, \mathrm{Hom}_{\mathbb{F}}(\varepsilon^{n-i+1} \oplus \varepsilon^k, \xi \oplus \varepsilon^k).
\end{aligned}$$

Hence $h \oplus \mathrm{id}_{\varepsilon^k}$ is generic. By Proposition 2.3.4, $\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})$ is a submanifold of $M \times \mathbb{F}\mathrm{P}^{n-i+k}$ of dimension $m - bi$. Let $\check{\phi} : \tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k}) \to M$ be the projection onto the first factor. Let $\Phi : M \times \mathbb{F}\mathrm{P}^{n-i} \to M \times \mathbb{F}\mathrm{P}^{n-i+k}$ be the inclusion given by

$$\Phi\big((x, [x_1, \ldots, x_{n-i+1}])\big) = \big(x, [x_1, \ldots, x_{n-i+1}, \underbrace{0, \ldots, 0}_{k}]\big).$$

where $[x_1, \ldots, x_{n-i}] \in \mathbb{F}\mathrm{P}^{n-i}$ is given in homogeneous coordinates. We have that Φ maps $\tilde{Z}(h)$ diffeomorphically onto $\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})$. Because of $Z_j(h \oplus \mathrm{id}_{\varepsilon^k}) = Z_j(h)$ for all j we have $\bar{Z}_1(h \oplus \mathrm{id}_{\varepsilon^k}) = \bar{Z}_1(h)$. Hence, the following diagram commutes:

$$\begin{tikzcd} \tilde{Z}(h) \arrow[rr, "\Phi"'] \arrow[dr, "\phi"'] & & \tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k}) \arrow[dl, "\check{\phi}"] \\ & M. & \end{tikzcd}$$

Thus $\phi_*\big([\tilde{Z}(h)]\big) = \check{\phi}_*\big([\tilde{Z}(h \oplus \mathrm{id}_{\varepsilon^k})]\big)$ and therefore $\mathbf{Cl}_i(\xi \oplus \varepsilon^k) = \mathbf{Cl}_i(\xi)$. $\square$

(A4'). We need to check that if ζ^n is the canonical n-bundle over $\mathbb{F}\mathrm{P}^n$, then we have that $\mathbf{Cl}_n(\zeta^n) = (-1)^n g_n \in H^{bn}(\mathbb{F}\mathrm{P}^n; K_b)$.

Define the vector bundle morphism $h : \varepsilon^1 \to \zeta^n$ by

$$h\big(([x_1, \ldots, x_{n+1}], t)\big) = [x_1, \ldots, x_{n+1}, tx_1, \ldots, tx_n].$$

We have that $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \cong \zeta^n$ and its section s_h corresponding to h is given by

$$s_h([x_1, \ldots, x_{n+1}]) = [x_1, \ldots, x_{n+1}, x_1, \ldots, x_n].$$

Let τ be the tautological vector bundle morphism over $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$. The only singularity subset is $S_1 = Z_1(\tau)$ and it is equal to the zero section. Let $\mathbf{x_0} = [0, \ldots, 0, 1] \in \mathbb{F}\mathrm{P}^n$ and let $R = s_h(\mathbb{F}\mathrm{P}^n) \subset \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$. We have that R intersects S_1 only in the point $[\mathbf{x_0}, \mathbf{0}]$.

We need to prove that R and S_1 are transverse. To simplify notation, let $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{F}^{n+1}$, $\tilde{\mathbf{x}} = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and write $\mathbf{x} = (\tilde{\mathbf{x}}, x_{n+1})$.

The tangent space of $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ at $[\mathbf{x}, \mathbf{v}]$ is given by

$$T_{[\mathbf{x},\mathbf{v}]}\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) = \{\langle \mathbf{y}, \mathbf{w}\rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{x} \cdot \mathbf{y} = 0\}$$

where $\langle \mathbf{y}, \mathbf{w} \rangle$ is the orbit of $(\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n$ under the action of $\mathbf{S}^{\nu(1)}$ given by $\lambda(\mathbf{y}, \mathbf{w}) = (\lambda \mathbf{y}, \lambda \mathbf{w})$. Hence,

$$T_{[\mathbf{x_0},\mathbf{0}]}\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) = \{\langle \mathbf{y}, \mathbf{w} \rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\tilde{\mathbf{y}}, 0)\},$$

since the condition $\mathbf{x_0} \cdot \mathbf{y} = 0$ is equivalent to $\mathbf{y}$ having the last coordinate equal to zero. We also have

$$\begin{aligned} T_{[\mathbf{x_0},\mathbf{0}]}S_1 &= \{\langle \mathbf{y}, \mathbf{0} \rangle \mid (\mathbf{y}, \mathbf{0}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\tilde{\mathbf{y}}, 0)\}, \\ T_{[\mathbf{x_0},\mathbf{0}]}R &= \{\langle \mathbf{y}, \mathbf{w} \rangle \mid (\mathbf{y}, \mathbf{w}) \in \mathbb{F}^{n+1} \times \mathbb{F}^n,\ \mathbf{y} = (\tilde{\mathbf{w}}, 0)\}. \end{aligned}$$

We can write $\langle (\tilde{\mathbf{y}}, 0), \mathbf{w} \rangle \in T_{[\mathbf{x_0},\mathbf{0}]}\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ as

$$\langle (\tilde{\mathbf{y}}, 0), \mathbf{w} \rangle = \langle (\mathbf{w}, 0), \mathbf{w} \rangle + \langle (\tilde{\mathbf{y}} - \mathbf{w}, 0), \mathbf{0} \rangle.$$

We have that $\langle (\mathbf{w}, 0), \mathbf{w} \rangle \in T_{[\mathbf{x_0},\mathbf{0}]}R$ and $\langle (\tilde{\mathbf{y}} - \mathbf{w}, 0), \mathbf{0} \rangle \in T_{[\mathbf{x_0},\mathbf{0}]}S_1$. Therefore, h is generic and $Z_1(h) = \{\mathbf{x_0}\}$ is a submanifold of dimension zero.

When $\mathbb{F} = \mathbb{C}$ we have to take into account orientations. Since $\mathbb{F}\mathrm{P}^0$ is a point we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \times \mathbb{F}\mathrm{P}^0 \arrow[r, "\hat{\phi}", "\cong"'] \arrow[d, "\pi\times\mathrm{id}"] & \mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n) \arrow[d, "\pi"'] \\ \mathbb{F}\mathrm{P}^n \times \mathbb{F}\mathrm{P}^0 \arrow[r, "\cong", "\phi"'] \arrow[u, bend left, "\tilde{s}_h"] & M. \arrow[u, bend right, "s_h"'] \end{tikzcd}$$

Hence we can identify $\tilde{Z}(\tau)$ with S_1 and $\tilde{Z}(h)$ with $Z_1(h)$. We have to take into account the orientations of $\mathbb{C}\mathrm{P}^n$, S_1 and $\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ in order to determine the orientation of $Z_1(h)$ and therefore the sign of $[\tilde{Z}(h)] = [Z_1(h)]$.

Let $\{e_1, \ldots, e_{n+1}\}$ and $\{f_1, \ldots, f_n\}$ be the canonical basis of $\mathbb{C}^{n+1}$ and of $\mathbb{C}^n$ respectively. The image of the canonical basis of $T_{\mathbf{x_0}}\mathbb{C}\mathrm{P}^n$ under $d(s_h)_{x_0}$ is the basis of $T_{[\mathbf{x_0},\mathbf{0}]}R$ given by $\{\langle e_1, f_1 \rangle, \ldots, \langle e_n, f_n \rangle\}$. On the other hand, the canonical basis for $T_{[\mathbf{x_0},\mathbf{0}]}S_1$ and for $T_{[\mathbf{x_0},\mathbf{0}]}\mathrm{Hom}_{\mathbb{F}}(\varepsilon^1, \zeta^n)$ are given by $\{\langle e_1, 0 \rangle, \ldots, \langle e_n, 0 \rangle\}$ and $\{\langle e_1, 0 \rangle, \ldots, \langle e_n, 0 \rangle, \langle 0, f_1 \rangle, \ldots, \langle 0, f_n \rangle\}$ respectively.

Since the section s_h is transverse to S_1, a basis for $T_{[\mathbf{x_0},\mathbf{0}]}R \oplus T_{[\mathbf{x_0},\mathbf{0}]}S_1$ is given by

$$\{\langle e_1, f_1\rangle, \dots, \langle e_n, f_n\rangle, \langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle\}.$$

Clearly,

$$\begin{aligned} &\operatorname{sgn}\{\langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle, \langle 0, f_1\rangle, \dots, \langle 0, f_n\rangle\} \\ = (-1)^{n^2} &\operatorname{sgn}\{\langle e_1, f_1\rangle, \dots, \langle e_n, f_n\rangle, \langle e_1, 0\rangle, \dots, \langle e_n, 0\rangle\}. \end{aligned}$$

Therefore, $[\tilde{Z}(h)] = (-1)^{n^2}\mathbf{x_0} \in H_0(\{\mathbf{x_0}\}; K_b)$ and (using the notation from 2.3.10)

$$\mathbf{Cl}_n(\zeta^n) = \phi_!\big([\tilde{Z}(h)]\big) = (-1)^{n^2} g_n \in H^{bn}(\mathbb{F}\mathrm{P}^n; K_b),$$

hence, axiom (A4') is satisfied. □

Indeed, the classes $\mathbf{Cl}_i(\xi)$ satisfy the four axioms from 2.1.7. □

2.5. Generalisation

We are going to extend the definition of the classes $\mathbf{Cl}_i(\xi)$ from smooth vector bundles to any numerable vector bundle.

2.5.1 Remark (Colimits, [5], [15]).
Let $\mathfrak{D}$ be a small category (i.e. $\mathrm{Ob}\mathfrak{D}$ is a set) and let $\mathfrak{C}$ be any category. A $\mathfrak{D}$-*shaped diagram* is a functor $F : \mathfrak{D} \to \mathfrak{C}$. A morphism $F \to F'$ of $\mathfrak{D}$-shaped diagrams is a natural transformation, and we have the category $\mathfrak{D}[\mathfrak{C}]$ of $\mathfrak{D}$-shaped diagrams in $\mathfrak{C}$.

The *colimit*, $\mathrm{colim}F$, of a $\mathfrak{D}$-shaped diagram F is an object of $\mathfrak{C}$ together with a morphism of diagrams $\iota : F \to \underline{\mathrm{colim}F}$ that is *initial* among all such morphisms, i.e. if $\eta : F \to \underline{A}$ is a morphism of diagrams, then there is a unique map $\tilde{\eta} : \mathrm{colim}F \to A$ in $\mathfrak{C}$ such that $\tilde{\eta} \circ \iota = \eta$. For each $d : D \to D'$ in $\mathfrak{D}$, this property is expressed by commutativity of the following diagram:

$$\begin{tikzcd}
F(D) \arrow[rr, "F(d)"] \arrow[dr, "\iota"] \arrow[ddr, "\eta"'] & & F(D') \arrow[dl, "\iota"'] \arrow[ddl, "\eta"] \\
& \operatorname{colim} F \arrow[d, "\tilde\eta"] & \\
& A. &
\end{tikzcd}$$

By reversing the arrows in the above diagram we dually define the *limit*, $\lim F$.

Colimits generalise constructions such as disjoint unions, direct sums, direct limits, coproducts and pushouts. Limits correspond to e.g. products, inverse limits and pullbacks.

2.5.2 Remark (G_n and γ^n)**.**
Recall from 1.2.17 that we have inclusions

$$\begin{aligned}
G_n(\mathbb{F}^{n+l}) \subset G_n(\mathbb{F}^{n+l+1}) \subset G_n(\mathbb{F}^{n+l+2}) \subset \dots, \\
\gamma^n(\mathbb{F}^{n+l}) \subset \gamma^n(\mathbb{F}^{n+l+1}) \subset \gamma^n(\mathbb{F}^{n+l+2}) \subset \dots \quad .
\end{aligned}$$

We set $G_n = G_n(\mathbb{F}^\infty) = \bigcup_l G_n(\mathbb{F}^{n+l})$ and $\gamma^n = \bigcup_l \gamma^n(\mathbb{F}^{n+l})$, both equipped with the direct limit topology.

2.5.3 Proposition. *Let $\iota_l : G_n(\mathbb{F}^{n+l}) \to G_n$ be the inclusions. Then there is an isomorphism*

$$\begin{aligned}
H^{bi}(G_n; K_b) &\xrightarrow{\lambda} \varprojlim H^{bi}(G_n(\mathbb{F}^{n+l}); K_b), \\
\omega &\longmapsto \big((\iota_0)_*(\omega), (\iota_1)_*(\omega), \dots, (\iota_k)_*(\omega), \dots\big).
\end{aligned}$$

Proof.

(i) The case $\mathbb{F} = \mathbb{R}$ follows from 2.5.4 below.

(ii) The case $\mathbb{F} = \mathbb{C}$ follows from [20, Proposition 7.66 and Theorem 7.75] since by [12, Theorem 20.3.2] and the pullback diagram in 2.5.5 below, the groups $H^{bi}(G_n(\mathbb{C}^{n+l}); \mathbb{Z})$ satisfy the *Mittag-Leffler condition* [20, Definition 7.74].

□

2.5.4 Proposition ([10, Proposition 3F.5])**.** *If the CW complex X is the union of an increasing sequence of subcomplexes X_i and G is one of the fields $\mathbb{Q}$ or $\mathbb{Z}_p$, then*

$$\lambda : H^n(X;G) \to \varprojlim H^n(X_i;G)$$

is an isomorphism for all n.

Proof. First we have an easy algebraic fact: Given a sequence of homomorphisms of abelian groups $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \longrightarrow \cdots$, we have that $\mathrm{Hom}(\varinjlim G_i, G) = \varprojlim \mathrm{Hom}(G_i, G)$ for any G. To be more precise, it follows from the definition of $\varinjlim G_i$ that a homomorphism $\phi : \varinjlim G_i \to G$ is the same thing as a sequence of homomorphisms $\phi_i : G_i \to G$ with $\phi_i = \phi_{i+1}\alpha_i$ for all i. Such a sequence (ϕ_i) is exactly an element of $\varprojlim \mathrm{Hom}(G_i, G)$.

Now if G is the field $\mathbb{Q}$ or $\mathbb{Z}_p$ we have

$$\begin{aligned} H^n(X;G) &= \mathrm{Hom}(H_n(X;G), G) \\ &= \mathrm{Hom}(\varinjlim H_n(X_i;G), G) \\ &= \varprojlim \mathrm{Hom}(H_n(X_i, G), G) \\ &= \varprojlim H^n(X_i;G) \end{aligned}$$

□

2.5.5 Remark. Consider the the pullback diagram

$$\begin{tikzcd} \gamma^n(\mathbb{F}^{n+l}) \arrow[r] \arrow[d] & \gamma^n(\mathbb{F}^{n+l+1}) \arrow[d] \\ G_n(\mathbb{F}^{n+l}) \arrow[r, hook, "\iota"] & G_n(\mathbb{F}^{n+l+1}) \end{tikzcd}$$

for the bundles $\gamma^n(\mathbb{F}^{n+l})$ and γ^n of $G_n(\mathbb{F}^{n+l})$ and G_n respectively.

Axiom (A2) implies

$$\iota^*\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l+1})\big)\Big) = \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big).$$

Hence, we have the element

$$\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^n)\big), \ldots, \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big), \ldots\Big) \in \varprojlim H^{bi}(G_n(\mathbb{F}^{n+l}); K_b)$$

and we can define

$$\mathbf{Cl}_i(\gamma^n) = \lambda^{-1}\bigg(\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^n)\big), \ldots, \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big), \ldots\Big)\bigg) \in H^{bi}(G_n; K_b)$$

where λ comes from the isomorphism in 2.5.3. Therefore, we have

$$\iota_l^*\big(\mathbf{Cl}_i(\gamma^n)\big) = \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big)$$

for all $l \leq 0$.

2.5.6 Remark. For $\mathbb{F} = \mathbb{C}$ the complex manifold $G_n(\mathbb{C}^{n+l})$ has a natural orientation and therefore the classes $\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big)$ are well-defined with $\mathbb{Z}$ coefficients.

2.5.7 Definition. A *classifying map* of a vector bundle ξ over a base space X is a map $\psi_\xi : X \to G_n$ such that ξ and $\psi_\xi^*(\gamma^n)$ are isomorphic.

2.5.8 Proposition ([1, Proposition 12])**.** *Let ξ be an $\mathbb{F}$-vector bundle of rank n over a compact K_b-oriented smooth manifold M. Let $\psi_\xi : M \to G_n$ be the classifying map of ξ. Then*

$$\mathbf{Cl}_i(\xi) = \psi_\xi^*\big(\mathbf{Cl}_i(\gamma^n)\big).$$

Proof. According to 1.2.11, for sufficiently large l there exists a map

$$\varrho_l : M \to G_n(\mathbb{F}^{n+l})$$

such that

$$\xi = \varrho_l^*\big(\gamma^n(\mathbb{F}^{n+l})\big).$$

By axiom (A2), see 2.1.7,

$$\mathbf{Cl}_i(\xi) = \varrho_l^*\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big)\Big).$$

On the other hand, since $\xi = (\iota_l \circ \varrho_l)^*(\gamma^n)$, where $\iota_l : G_n(\mathbb{F}^{n+l}) \to G_n$ is the inclusion, by 1.2.22 we have a homotopy

$$\psi_\xi \sim \iota_l \circ \varrho_l.$$

Hence, using the last line of 2.5.5 we can calculate

$$\begin{aligned}
\psi_\xi^*\big(\mathbf{Cl}_i(\gamma^n)\big) &= (\iota_l \circ \varrho_l)^*\big(\mathbf{Cl}_i(\gamma^n)\big)\\
&= \varrho_l^*\Big(\iota_l^*\big(\mathbf{Cl}_i(\gamma^n)\big)\Big)\\
&= \varrho_l^*\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big)\Big)\\
&= \mathbf{Cl}_i(\xi).
\end{aligned}$$

□

Using 2.5.8 we can now generalise the classes $\mathbf{Cl}_i(\xi)$ to numerable vector bundles over any (admissible) base space.

2.5.9 Definition (Numerable vector bundle)**.**

(i) An open covering $\{U_i\}_{i\in\Lambda}$ of a space B is called *numerable* provided there exists a partition of unity $\{\phi_i\}_{i\in\Lambda}$ such that the support of ϕ_i is contained in U_i for each $i \in \Lambda$.

(ii) An $\mathbb{F}$-vector bundle ξ over a base space B is called *numerable* provided that there is a numerable cover $\{U_i\}_{i\in\Lambda}$ of B such that $\xi|_{U_i}$ is trivial for each $i \in \Lambda$.

2.5.10 Remark. A Hausdorff space B is paracompact if and only if each open covering is numerable.

2.5.11 Definition (Universal bundle)**.**
A principal G-bundle $\omega = (E_0, p_0, B_0)$ is called *universal* if the following conditions hold:

- ω is numerable.

- For each numerable principal G-bundle ξ over X there exists a map $f : X \to B_0$ such that ξ and $f^*(\omega)$ are isomorphic over X.

- If $f, g : X \to B_0$ are two maps such that $f^*(\omega)$ and $g^*(\omega)$ are isomorphic over X, then f and g are homotopic.

To prove the main theorem 2.5.14 of this section, we need the following results:

2.5.12 Theorem ([12, Chapter 4, Theorems 12.2 & 12.4])**.**

(i) For each numerable principal G-bundle ξ over a base space B there exists a map $f : B \to B_G$ such that ξ and $f^*(\omega_G)$ are B-isomorphic principal G-bundles.

(ii) Let $f_0, f_1 : X \to B_G$ be two maps such that $f_0^*(\omega_G)$ and $f_1^*(\omega_G)$ are isomorphic. Then f_0 and f_1 are homotopic.

2.5.13 Remark (on 2.5.12)**.**

(i) G is a (topological) group, E_G is the infinite join of copies of G, $E_G = \operatorname{colim} G * G * \cdots * G$, and B_G is the quotient space $E_G \mod G$. This is the *Milnor construction.* The resulting bundle $\omega_G = (E_G, p, B_G)$ is a numerable principal G-bundle. For details see [12, Chapter 4, Section 11].

(ii) The proof of 2.5.12 shows that ω_G is universal.

(iii) 2.5.12 (i) and (ii) imply that every numerable $\mathbb{F}$-vector bundle ξ of rank n has a classifying map $f_\xi : B \to G_n$ which is unique up to homotopy, i.e.

$$\xi \cong f_\xi^*(\gamma^n).$$

2.5.14 Theorem ([1, Theorem 13])**.** *Let ξ be a numerable $\mathbb{F}$-vector bundle of rank n over a base space B. Let $\psi_\xi : B \to G_n$ be the classifying map of ξ. We define*

$$\mathbf{Cl}_i(\xi) = \psi_\xi^*\big(\mathbf{Cl}_i(\gamma^n)\big).$$

Then the classes $\mathbf{Cl}_i(\xi)$ satisfy the four axioms from 2.1.7.

Proof. Again, we verify the four axioms.

(A1). By definition 2.1.7,

$$\mathbf{Cl}_i(\xi) \in H^{bi}(B; K_b).$$

For $i = 0$, the proof of axiom (A1) in 2.4.1 implies $\mathbf{Cl}_0\big(\gamma^n(\mathbb{F}^{n+l})\big) = 1$ for all l. Then by definition of the classes $\mathbf{Cl}_i(\gamma^n)$ in 2.5.5 we have $\mathbf{Cl}_0(\gamma^n) = 1$, and therefore,

$$\mathbf{Cl}_0(\xi) = 1 \in H^0(B; K_b).$$

Analogously, if $i > n$, then by 2.4.1, axiom (A1), $\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big) = 0$ for all l, so again by 2.5.5 we have $\mathbf{Cl}_i(\gamma^n) = 0$ and therefore

$$\mathbf{Cl}_i(\xi) = 0 \in H^{bi}(B; K_b).$$

□

(A2). Let $f : B' \to B$ be a continuous map and ψ_ξ and $\psi_{f^*\xi}$ be the classifying maps of ξ and $f^*\xi$ respectively. Since

$$f^*\xi = \psi^*_{f^*\xi}(\gamma^n) = (\psi_\xi \circ f)^*(\gamma^n),$$

Theorem 2.5.12 says that $\psi_{f^*\xi}$ and $\psi_\xi \circ f$ are homotopic. Therefore

$$\psi^*_{f^*\xi}\big(\mathbf{Cl}_i(\gamma^n)\big) = (\psi_\xi \circ f)^*\big(\mathbf{Cl}_i(\gamma^n)\big)$$

and

$$\begin{aligned}\mathbf{Cl}_i(f^*\xi) &= f^*\Big(\psi^*_\xi\big(\mathbf{Cl}_i(\gamma^n)\big)\Big)\\ &= f^*\big(\mathbf{Cl}_i(\xi)\big).\end{aligned}$$

□

(A3'). Let ε^k_B be the trivial bundle of rank k over B. Let $\psi_\xi : B \to G_n$ and

$\psi_{\xi \oplus \varepsilon_B^k} : B \to G_{n+k}$ be the classifying maps of ξ and $\xi \oplus \varepsilon_B^k$ respectively. Let $l_n : G_n \to G_{n+k}$ be the classifying map of the bundle $\gamma^n \oplus \varepsilon_{G_n}^k$. We have that

$$\begin{aligned} \psi_\xi^*(\gamma^n \oplus \varepsilon_{G_n}^k) &= \psi_\xi^k(\gamma^n) \oplus \psi_\xi^k(\varepsilon_{G_n}^k) \\ &= \xi \oplus \varepsilon_B^k. \end{aligned}$$

Thus,

$$\xi \oplus \varepsilon_B^k = \psi_{\xi \oplus \varepsilon_B^k}^*(\gamma^{n+l}) = (l_n \circ \psi_\xi)^*(\gamma^{n+l})$$

and therefore $\psi_{\xi \oplus \varepsilon_B^k}$ and $l_n \circ \psi_\xi$ are homotopic by 2.5.12. It follows that

$$\mathbf{Cl}_i(\xi \oplus \varepsilon_B^k) = \psi_\xi^*\big(\mathbf{Cl}_i(\gamma^n \oplus \varepsilon_{G_n}^k)\big).$$

We have inclusions

$$\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k \subset \gamma^n(\mathbb{F}^{n+l+1}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l+1})}^k \subset \dots,$$

and therefore,

$$\gamma^n \oplus \varepsilon_{G_n}^k = \bigcup_k \big(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k\big).$$

The pullback diagram from 2.5.5 gives

$$\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k = \iota^*\big(\gamma^n(\mathbb{F}^{n+l+1}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l+1})}^k\big).$$

Since $\iota_l^*(\gamma^n) = \gamma^n(\mathbb{F}^{n+l})$, we also have

$$\iota_l^*(\gamma^n \oplus \varepsilon_{G_n}^k) = \gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k.$$

By 2.5.8 we have

$$\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon_{G_n(\mathbb{F}^{n+l})}^k\big) = \iota_l^*\Big(l_n^*\big(\mathbf{Cl}_i(\gamma^{n+k})\big)\Big) = \iota_l^*\big(\mathbf{Cl}_i(\gamma^n \oplus \varepsilon_{G_n}^k)\big),$$

and therefore,

$$\begin{aligned}&\mathbf{Cl}_i\big(\gamma^n \oplus \varepsilon^k_{G_n}\big) =\\ &= \lambda^{-1}\bigg(\Big(\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^n) \oplus \varepsilon^k_{G_n(\mathbb{F}^n)}\big), \dots, \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}\big), \dots\Big)\bigg).\end{aligned}$$

Now by axiom (A3') from 2.4.1,

$$\mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l}) \oplus \varepsilon^k_{G_n(\mathbb{F}^{n+l})}\big) = \mathbf{Cl}_i\big(\gamma^n(\mathbb{F}^{n+l})\big)$$

for all l. Hence, by definition 2.5.5 we have that

$$\mathbf{Cl}_i(\gamma^n \oplus \varepsilon^k_{G_n}) = \mathbf{Cl}_i(\gamma^n).$$

From the above it follows that $\mathbf{Cl}_i(\xi \oplus \varepsilon^k_B) = \mathbf{Cl}_i(\xi)$. □

(A4'). By 2.5.8, the verification of (A4') is the same as in 2.4.1. □

In summary we have shown that the classes $\mathbf{Cl}_i(\xi) = \psi^*_\xi\big(\mathbf{Cl}_i(\gamma^n)\big)$ indeed satisfy the four axioms. □

2.6. Uniqueness

In this section we show that the classes $\mathbf{Cl}_k(\xi)$ coincide with the classes $\mathbf{cl}_k(\xi)$. Therefore, the classes $\mathbf{Cl}_k(\xi)$ are unique.

2.6.1 Lemma ([1, Lemma 14]). *Let $p : \gamma^k \to G_k$ be the universal bundle over G_k. Let p_0 be the restriction of p to the subspace $E(\gamma^k)_0$ of non-zero vectors of the total space $E(\gamma^k)$. Then*

$$p_0^*(\gamma^k) = \eta^{k-1} \oplus \varepsilon^1 \quad \textit{and} \quad \mathbf{Cl}_k\big(p_0^*(\gamma^k)\big) = 0.$$

Proof. We have that

$$p_0^*(\gamma^k) = \{(l, v, w) \mid l \in G_k,\ v, w \in l \text{ and } v \neq 0\}.$$

The map

$$s : E_0 \to p_0^*(\gamma^k),$$
$$s(l, v) = (l, v, v)$$

is a non-vanishing global section. This section defines a trivial line bundle $\varepsilon^1 \subset p_0^*(\gamma^k)$. Since $\mathbb{F}^\infty$ has an Euclidean metric, there is a canonical Riemannian metric on γ^k, and we consider the pullback of this metric on $p_0^*(\gamma^k)$. We define η^{k-1} to be its orthogonal complement. Therefore,

$$p_0^*(\gamma^k) \cong \eta^{k-1} \oplus \varepsilon^1$$

and by 2.5.14 we have that $\mathbf{Cl}_k\big(p_0^*(\gamma^k)\big) = 0$. $\square$

To prove the main theorem 2.6.4 of this section, we need the following result by Werner Gysin [8] which relates the cohomology classes of the base space and the total space.

2.6.2 Theorem (The Gysin sequence of a vector bundle, [16, Theorem 12.2])**.** *To any oriented n-plane bundle ξ there is associated an exact sequence of the form*

$$\cdots \longrightarrow H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{p_0^*} H^{i+n}(E_0) \longrightarrow H^{i+1}(B) \xrightarrow{\cup e} \cdots$$

using integer coefficients.

2.6.3 Remark.

(i) As in 2.6.1, $p_0 : E_0 \to B$ denotes the restriction of $p : E \to B$ to non-zero vectors.

(ii) The symbol $\cup e$ stands for the homomorphism $a \mapsto a \cup e(\xi)$, where e denotes the Euler class.

(iii) There is a corresponding exact sequence for unoriented bundles using $\mathbb{Z}_2$-coefficients and the Stiefel-Whitney class $w_n(\xi)$ in place of the Euler class $e(\xi)$.

2.6.4 Theorem ([1, Theorem 15])**.** *Let ξ be a numerable $\mathbb{F}$-vector bundle of rank n. Then*

$$\mathbf{Cl}_k(\xi) = \mathbf{cl}_k(\xi)$$

for every k.

Proof. Let $\psi_\xi : B \to G_n$ be the classifying map of ξ, i.e. $\xi = \psi_\xi^*(\gamma^n)$. Let $k \leq n$ and let $l_k : G_k \to G_n$ be the classifying map of the bundle $\gamma^k \oplus \varepsilon^{n-k}$.
Due to 2.1.9, for $i = 0, \dots, n$ we have

$$\begin{aligned} l_k^*\big(\mathbf{cl}_i(\gamma^n)\big) &= \mathbf{cl}_i\big(l_k^*(\gamma^n)\big) \\ &= \mathbf{cl}_i(\gamma^k \oplus \varepsilon^{n-k}) \\ &= \mathbf{cl}_i(\gamma^k). \end{aligned}$$

On the other hand, again by axiom (A2) and (A3'), for $i = 0, \dots, n$ we have that

$$\begin{aligned} l_k^*\big(\mathbf{Cl}_i(\gamma^n)\big) &= \mathbf{Cl}_i\big(l_k^*(\gamma^n)\big) \\ &= \mathbf{Cl}_i(\gamma^k \oplus \varepsilon^{n-k}) \\ &= \mathbf{Cl}_i(\gamma^k). \end{aligned}$$

By 2.6.1, $p_0^*\big(\mathbf{Cl}_k(\gamma^k)\big) = 0$, and by the Gysin exact sequence (see 2.6.2) for the bundle $p_o^*(\gamma^k)$,

$$H^0(G_k; K_b) \xrightarrow{\cup \mathbf{cl}_k(\gamma^k)} H^{bk}(G_k; K_b) \xrightarrow{p_0^*} H^{bk}(E(\gamma^k)_0; K_b) \longrightarrow \cdots,$$

there exists $\alpha_k \in H^0(G_k; K_b)$ such that $\mathbf{Cl}_k(\gamma^k) = \alpha_k \cup \mathbf{cl}_k(\gamma^k)$.

Since G_n is path-connected for all n, l_k induces an isomorphism in 0-dimensional cohomology. Let ρ_k be the unique element in $H^0(G_n; K_b)$ such that $l_k^*(\rho_k) = \alpha_k$.

Then

$$
\begin{aligned}
l_k^*\bigl(\mathbf{Cl}_k(\gamma^n)\bigr) &= \mathbf{Cl}_k(\gamma^k)\\
&= \alpha_k \cup \mathbf{cl}_k(\gamma^k)\\
&= \alpha_k \cup l_k^*\bigl(\mathbf{cl}_k(\gamma^n)\bigr)\\
&= l_k^*\bigl(\rho_k \cup \mathbf{cl}_k(\gamma^n)\bigr).
\end{aligned}
$$

The cohomology ring $H^*(G_n; K_b)$ is the polynomial ring $K_b[\mathbf{cl}_1(\gamma^n), \ldots, \mathbf{cl}_n(\gamma^n)]$ on the Stiefel-Whitney classes of γ^n for $\mathbb{F} = \mathbb{R}$ [12, Theorem 20.5.2] or on the Chern classes of γ^n for $\mathbb{F} = \mathbb{C}$ [12, Theorem 20.3.2].

The first calculation in this proof shows that the homomorphisms

$$
H^{bi}(G_n; K_b) \xrightarrow{l_k^*} H^{bi}(G_k; K_b)
$$

are isomorphisms for $i \leq k$. Thus,

$$
\mathbf{Cl}_k(\gamma^n) = \rho_k \cup \mathbf{cl}_k(\gamma^n).
$$

Hence, we have that

$$
\begin{aligned}
\mathbf{Cl}_k(\xi) &= \mathbf{Cl}_k\bigl(\psi_\xi^*(\gamma^n)\bigr)\\
&= \psi_\xi^*\bigl(\mathbf{Cl}_k(\gamma^n)\bigr)\\
&= \psi_\xi^*\bigl(\rho_k \cup \mathbf{cl}_k(\gamma^n)\bigr)\\
&= \psi_\xi^*(\rho_k) \cup \psi_\xi^*\bigl(\mathbf{cl}_k(\gamma^n)\bigr)\\
&= \beta_k \cup \mathbf{cl}_k\bigl(\psi_\xi^*(\gamma^n)\bigr)\\
&= \beta_k \cup \mathbf{cl}_k(\xi)
\end{aligned}
$$

with $\beta_k = \psi_\xi^*(\rho_k)$. The element $\beta_k \in H^0(B; K_b)$ is independent of the bundle ξ since for any path-connected space B and any map $f : B \to G_n$ the induced homomorphism $f^* : H^0(G_n; K_b) \to H^0(B; K_b)$ is the same isomorphism.

Let ζ^k be the canonical k-bundle over $\mathbb{F}\mathrm{P}^k$. By axiom (A4') we have that

$$\mathbf{Cl}_k(\zeta^k) = (-1)^k g_k \in H^{bk}(\mathbb{F}\mathrm{P}^k; K_b).$$

Consider the class $\mathbf{cl}_1(\gamma_k^1) \in H^b(\mathbb{F}\mathrm{P}^k; K_b)$. Then by [16, page 170] we have that

$$g_k = (-1)^k \mathbf{cl}_1(\gamma_k^1)^k.$$

By 1.2.15, ζ^k is the Whitney sum of k copies of the canonical line bundle γ_k^1 over $\mathbb{F}\mathrm{P}^k$. Using axiom (A3) and the above this implies

$$\begin{aligned}
\mathbf{cl}_k(\zeta^k) &= \mathbf{cl}_k(\gamma_k^1 \oplus \cdots \oplus \gamma_k^1) \\
&= \mathbf{cl}_1(\gamma_k^1)^k \\
&= (-1)^k g_k.
\end{aligned}$$

Since $\mathbf{Cl}_k(\zeta^k) = \beta_k \cup \mathbf{cl}_k\big(\gamma(\zeta^k)\big)$, this implies that $\beta_k = 1$.

Therefore,

$$\mathbf{Cl}_k(\xi) = \mathbf{cl}_k(\xi)$$

for every bundle ξ and every k. □

2.6.5 Corollary (Uniqueness, [1, Corollary 16])**.**
The classes $\mathbf{Cl}_i(\xi)$ for a smooth $\mathbb{F}$-vector bundle are well defined and hence they are well defined for any numerable $\mathbb{F}$-vector bundle.

Proof. In 2.4.1, if we take a different generic vector bundle morphism $g : \varepsilon^{n-1+1} \to \xi$ and define the classes by

$$\mathbf{Cl}_i^g(\xi) = \phi_!\big([\tilde{Z}(g)]\big)$$

(see 2.3.10 for the definition of $\phi_!$), these classes also satisfy 2.6.4. Therefore, they coincide with the classes $\mathbf{cl}_i(\xi)$, which are unique (see 2.1.6). □

2.6.6 Corollary ([1, Corollary 17])**.** *The classes $\mathbf{Cl}_i(\xi)$ are well defined for any $\mathbb{F}$-vector bundle ξ over a paracompact base space.*

Proof. Any open cover of a paracompact space B admits a partition of unity subordinate to a locally finite refinement. Therefore, any $\mathbb{F}$-vector bundle over B is numerable; see 2.5.10. $\square$

A. Background from algebraic topology

A.1. Basic definitions

The definitions of this section can be found in most books on algebraic topology, for example in [5], [15], [9].

A.1.1 Definition.

(i) The *(standard) $n-$dimensional simplex* is the topological space

$$\Delta^n := \left\{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \right\} \subseteq \mathbb{R}^{n+1}.$$

The point e_i for which $x_i = 1$ is called the *i-th vertex of* Δ^n; the set of vertices is ordered: $e_0 < e_1 < \cdots < e_n$. More generally, with each subset $I \subset [n] := \{0, 1, \dots, n\}$ we associate the *I-th face of* Δ^n defined as the set of all points $(x_0, \dots, x_n) \in \Delta^n$ with $x_i = 0$ for $i \notin I$.
The map $\partial_n^i : [n-1] \to [n]$ is called *i-th face map* and is the only strictly increasing map not taking the value i.

(ii) Let X be a topological space. A *singular n-simplex* of X is a continuous map $x : \Delta^n \to X$. The free abelian group $C_n(X)$ generated by X_n (:= all singular n-simplices x of X) is called the n-th *singular chain group of* X. Elements of $C_n(X)$ are called *n-chains* and are finite formal linear combinations of the form $\sum_{x \in X_n} a(x)x$, where $a(x) \in \mathbb{Z}$, $a(x) \neq 0$ for a finite number of simplices x from X_n $(n = 0, 1, \dots)$.

(iii) The *boundary* of an n-chain $c \in C_n(X)$ is the $(n-1)$-chain $d_n c$ defined by

$$d_n \left(\sum_{x \in X_n} a(x)x \right) = \sum_{x \in X_n} a(x) \sum_{i=0}^{n} (-1)^i X(\partial_n^i)(x).$$

The *boundary operator* $d_n : C_n(X) \to C_{n-1}(X)$ defined by the above relation is a group homomorphism. For $n = 0$ we set $d_0 = 0$.

(iv) *Chains with coefficients in a unital commutative ring* R are formal linear combinations $\sum_{x \in X_n} a(x)x,\ a(x) \in R$. In other words,

$$C_n(X; R) = C_n(X) \otimes_{\mathbb{Z}} R,$$

so that $C_n(X) = C_n(X; \mathbb{Z})$. The *boundary operator* $d_n : C_n(X; R) \to C_{n-1}(X; R)$ is defined as above.

(v) Dually, we define *cochains with coefficients in* R.
$C^n(X; A)$ is the group of functions on X_n with values in R.
The *coboundary* $d^n : C^n(X; R) \to C^{n+1}(X; R)$ is given by the formula

$$(d^n f)(x) = \sum_{i=0}^{n+1} (-1)^i f\big(X(\partial_{n+1}^i)(x)\big).$$

A.1.2 Lemma ([5]). *Let d_n be the boundary operator and d^n the coboundary operator. Then*

(i) $d_{n-1} \circ d_n = 0$ *for* $n \geq 1$.

(ii) $d^{n+1} \circ d^n = 0$ *for* $n \geq 0$.

Proof. Note first that for any $0 \leq j < i \leq n-1$ we have

$$\partial_n^i \circ \partial_{n-1}^j = \partial_n^j \circ \partial_{n-1}^{i-1};$$

indeed, both sides of the equality give unique, increasing mapping of $[n-2]$ into $[n]$ not taking values i and j.

To prove part (i) of the lemma it suffices to check that $(d_{n-1} \circ d_n)(x) = 0$ for any $x \in X_n$. But

$$\begin{aligned} (d_{n-1} \circ d_n)(x) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} X\left(\partial_{n-1}^j\right) X\left(\partial_n^i\right)(x) \\ &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} X\left(\partial_n^i \circ \partial_{n-1}^j\right)(x). \end{aligned}$$

Compositions $\partial_n^i \circ \partial_{n-1}^j$ for different i, j all yield increasing maps of $[n-2]$ into $[n]$, and the map whose image does not contain i and j appears exactly twice: the first time as $\partial_n^i \circ \partial_{n-1}^j$ with the sign $(-1)^{i+j}$ and the second time as $\partial_n^j \circ \partial_{n-1}^{i-1}$ with the opposite sign $(-1)^{i+j-1}$. Hence $d_{n-1} \circ d_n(x) = 0$.

Similarly, one proves part (ii). $\square$

A.1.3 Definition.

(i) A *chain complex* is a sequence of abelian groups and homomorphisms

$$C_\bullet : \ldots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$$

with the property $d_n \circ d_{n+1} = 0$ for all n. Homomorphisms d_n are called *boundary maps* or *boundary operators*.

(ii) A *cochain complex* is a sequence

$$C^\bullet : \ldots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \ldots,$$

$d^n \circ d^{n-1} = 0$. Any chain complex can be transformed into a cochain complex by setting $D^n = C_{-n}$ and $d^n = d_{-n-1}$. Therefore one usually considers only cochain complexes.

(iii) *Homology groups* of a chain complex $C_\bullet$ are

$$H_n(C_\bullet) = \text{Ker } d_n \;/\; \text{Im } d_{n+1}.$$

Elements of the group $H_n(X; R) := H_n\big(C_\bullet(X; R)\big)$ are called *homology* *classes*. Each homology class is represented by an n-chain c called *cycle*, such that $d_n c = 0$. A cycle c in a given homology class is defined up to a summand of the form $b = d_{n+1}c'$; such chains are called *boundaries*. Two chains whose difference is a boundary are said to be *homological*.

(iv) *Cohomology groups* of a cochain complex $C^\bullet$ are

$$H^n(C^\bullet) = \text{Ker } d^n \ / \ \text{Im } d^{n-1}.$$

Elements of the group $H^n(X;R) := H^n\big(C^\bullet(X;R)\big)$ are called *cohomology classes*. Each cohomology class is represented by a cochain f such that $d^n f = 0$. Such cochains are called *cocycles*. Cochains of the form $d^{n-1}c'$ are called *coboundaries*.

(v) A *chain map* between to chain complexes $A_\bullet$ and $B_\bullet$ is a sequence $f_\bullet$ of homomorphisms $f_n : A_n \to B_n$ for each n that commutes with the boundary operators on the two chain complexes:

$$\begin{tikzcd} A_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "d'_n"'] & B_{n-1} \end{tikzcd}$$

In particular, a chain map sends cycles to cycles and boundaries to boundaries and thus descends to a map on homology $(f_\bullet)_* : H_\bullet(A) \to H_\bullet(B)$.

A.2. Products on homology

A.2.1 Definition (Eilenberg-Zilber equivalence)**.**
Let P and Q be chain maps

$$P : C(X) \otimes C(Y) \to C(X \times Y) \quad \text{and} \quad Q : C(X \times Y) \to C(X) \otimes C(Y),$$

such that $P(x \otimes y) = (x, y)$ and $Q(x, y) = x \otimes y$ for singular 0-simplices $x : \Delta^0 \to X$ and $y : \Delta^0 \to Y$. Additionally, P and Q are assumed to be natural, i.e. for

$f : X \to X'$ and $g : Y \to Y'$ continuous maps the diagrams

$$\begin{tikzcd} C(X) \otimes C(Y) \arrow[r, "P^{X,Y}"] \arrow[d, "f_\# \otimes g_\#"'] & C(X \times Y) \arrow[d, "(f \times g)_\#"] \\ C(X') \otimes C(Y') \arrow[r, "P^{X',Y'}"'] & C(X' \times Y') \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C(X \times Y) \arrow[r, "Q^{X,Y}"] \arrow[d, "(f \times g)_\#"'] & C(X) \otimes C(Y) \arrow[d, "f_\# \otimes g_\#"] \\ C(X' \times Y') \arrow[r, "Q^{X',Y'}"'] & C(X') \otimes C(Y') \end{tikzcd}$$

commute. Then we call P and Q *Eilenberg-Zilber equivalences.*

A.2.2 Definition (Augmentation)**.**
Let X be a topological space. We define the *augmentation* $\varepsilon^X : C_0(X) \to \mathbb{Z}$ on generators $x : \Delta^0 \to X$ to be $\varepsilon^X(x) := 1$.

A.2.3 Proposition ([9])**.** *There exist Eilenberg-Zilber equivalences P and Q as in A.2.1 and natural chain homotopies such that*

$$P^{X,Y} \circ Q^{X,Y} \simeq \mathrm{id}_{C(X \times Y)} \quad \textit{and} \quad Q^{X,Y} \circ P^{X,Y} \simeq \mathrm{id}_{C(X) \otimes C(Y)} .$$

In particular, every Eilenberg-Zilber equivalence is a chain homotopy equivalence. Moreover, up to natural chain homotopy, the following diagrams commute:

$$\begin{tikzcd} C(X) \otimes C(Y) \arrow[r, "P^{X,Y}"] \arrow[d, "\tau^{C(X),C(Y)}"'] & C(X \times Y) \arrow[r, "Q^{X,Y}"] \arrow[d, "T^{X,Y}"] & C(X) \otimes C(Y) \arrow[d, "\tau^{C(X),C(Y)}"] \\ C(Y) \otimes C(X) \arrow[r, "P^{Y,X}"'] & C(Y \times X) \arrow[r, "Q^{Y,X}"'] & C(Y) \otimes C(X) \end{tikzcd} \tag{A.1}$$

$$\begin{tikzcd} C(X) \otimes C(Y) \otimes C(Z) \arrow[r, "P^{X,Y} \otimes \mathrm{id}_{C(Z)}"] \arrow[d, "\mathrm{id}_{C(X)} \otimes P^{Y,Z}"'] & C(X \times Y) \otimes C(Z) \arrow[d, "P^{X \times Y, Z}"] \\ C(X) \otimes C(Y \times Z) \arrow[r, "P^{X, Y \times Z}"'] & C(X \times Y \times Z) \end{tikzcd} \tag{A.2}$$

$$\begin{tikzcd} C(X \times Y \times Z) \arrow[rr, "Q^{X\times Y,Z}"] \arrow[d, "Q^{X,Y\times Z}"'] & & C(X\times Y)\otimes C(Z) \arrow[d, "Q^{X,Y}\otimes \mathrm{id}_{C(Z)}"] \\ C(X)\otimes C(Y\times Z) \arrow[rr, "\mathrm{id}_{C(X)}\otimes Q^{Y,Z}"'] & & C(X)\otimes C(Y)\otimes C(Z) \end{tikzcd} \tag{A.3}$$

$$\begin{tikzcd} C(X)\otimes C(\{*\}) \arrow[r, "P^{X,\{*\}}"] \arrow[d, "\mathrm{id}_{C(X)}\otimes\varepsilon^{\{*\}}"'] & C(X\times\{*\}) \arrow[r, "Q^{X,\{*\}}"] \arrow[d, equal] & C(X)\otimes C(\{*\}) \arrow[d, "\mathrm{id}_{C(X)}\otimes\varepsilon^{\{*\}}"] \\ C(X)\otimes\mathbb{Z} \arrow[r, equal] & C(X) \arrow[r, equal] & C(X)\otimes\mathbb{Z} \end{tikzcd} \tag{A.4}$$

The map $T^{X,Y}$ is defined by $T^{X,Y} : X \times Y \to Y \times X, \quad T^{X,Y}(x, y) := (y, x)$.

A.2.4 Remark. The proof of A.2.3 can be found for example in [9] and [20]. It uses the method of *acyclic models*, see [20, 13.24] for the exact definition.

A.2.5 Definition (Homology cross product)**.**
The *(homology) cross product* $a \times b := (P_* \circ \lambda)(a \otimes b)$ is the composite

$$\begin{tikzcd} \times : \ H_*(X;R)\otimes_R H_*(Y;R) \arrow[r, "\lambda"] \arrow[dr, dotted] & H_*(C(X;R)\otimes_R C(Y;R)) \arrow[d, "P_*"] \\ & H_*(X\times Y;R) \end{tikzcd}$$

where λ is the (natural) homomorphism

$$\begin{aligned} \lambda^{C,D}_{p,q} : H_p(C) \times H_q(D) &\to H_{p+q}(C \otimes_R D) \\ ([c], [d]) &\mapsto [c \otimes d] \end{aligned} \tag{A.5}$$

and P_* is an isomorphism induced by an Eilenberg-Zilber equivalence. Equivalently, we may think of $\times$ as R-bilinear maps

$$H_p(X; R) \times H_q(Y; R) \xrightarrow{\times} H_{p+q}(X \times Y; R).$$

A.2.6 Proposition (Properties of the homology cross product, [9])**.**
Let R be a unital commutative ring, X, Y, Z topological spaces, $f : X \to X'$,

$g : Y \to Y'$ continuous maps, $a \in H_p(X;R)$, $b \in H_q(Y;R)$ and $c \in H_r(Z;R)$. Then:

(i) $(a \times b) \times c = a \times (b \times c)$ *(Associativity)*

(ii) $b \times a = (-1)^{pq}\, T_*(a \times b)$ *with* $T(x,y) := (y,x)$. *(Graded commutativity)*

(iii) $a \times 1_{\{*\}} = a$ *(Unital element)*

(iv) $(f \times g)_*(a \times b) = (f_*a) \times (g_*b)$. *(Naturality)*

Proof. (ii) In A.2.3 the left part of the diagram (A.1) commutes. Therefore we get the following commutative diagram on homology level

$$\begin{CD} H\left(C(X;R) \otimes_R C(Y;R)\right) @>{P_*^{X,Y;R}}>> H(X \times Y;R) \\ @V{\tau_*^{C(X;R),C(Y;R)}}VV @VV{T_*}V \\ H\left(C(Y;R) \otimes_R C(X;R)\right) @>>{P_*^{Y,X;R}}> H(Y \times X;R). \end{CD}$$

Now for the homomorphism λ, see (A.5), and chain complexes C and D over R, commutativity of

$$\begin{CD} H(C) \otimes_R H(D) @>{\lambda^{C,D}}>> H(C \otimes_R D) \\ @V{\tau^{H(C),H(D)}}V{\cong}V @V{\cong}V{\tau_*^{C,D}}V \\ H(D) \otimes_R H(C) @>>{\lambda^{D,C}}> H(D \otimes_R C) \end{CD}$$

implies commutativity of λ. So we have:

$$\begin{aligned} T_*(a \times b) &= \left(T_* \circ P_*^{X,Y;R} \circ \lambda^{C(X;R),C(Y;R)}\right)(a \otimes b) \\ &= \left(P_*^{Y,X;R} \circ \tau_*^{C(X;R),C(Y;R)} \circ \lambda^{C(X;R),C(Y;R)}\right)(a \otimes b) \\ &= \left(P_*^{Y,X;R} \circ \lambda^{C(Y;R),C(X;R)} \circ \tau^{H(X;R),H(Y;R)}\right)(a \otimes b) \\ &= \left(P_*^{Y,X;R} \circ \lambda^{C(Y;R),C(X;R)}\right)((-1)^{pq} b \otimes a) \\ &= (-1)^{pq} b \times a. \end{aligned}$$

(i) In A.2.3 the second diagram (A.2) commutes up to homotopy. Therefore, we get the following commutative diagram on homology level:

$$\begin{tikzcd}
H\left(C(X;R)\otimes_R C(Y;R)\otimes_R C(Z;R)\right) \arrow[r, "(P^{X,Y;R}\otimes\mathrm{id})_*"] \arrow[d, "\left(\mathrm{id}_{C(X;R)}\otimes P^{Y,Z;R}\right)_*"'] & H\left(C(X\times Y;R)\otimes_R C(Z;R)\right) \arrow[d, "P_*^{X\times Y,Z;R}"] \\
H\left(C(X;R)\otimes_R C(Y\times Z;R)\right) \arrow[r, "P_*^{X,Y\times Z;R}"'] & H(X\times Y\times Z;R)
\end{tikzcd}$$

Considering chain maps $\varphi : C \to C'$ and $\psi : D \to D'$ between chain complexes over R, naturality of λ from (A.5) is expressed by commutativity of

$$\begin{tikzcd}
H(C)\otimes_R H(D) \arrow[r, "\lambda^{C,D}"] \arrow[d, "\varphi_*\otimes\psi_*"'] & H(C\otimes_R D) \arrow[d, "(\varphi\otimes\psi)_*"] \\
H(C')\otimes_R H(D') \arrow[r, "\lambda^{C',D'}"'] & H(C'\otimes_E D')
\end{tikzcd}$$

and we therefore get the relations:

$$\begin{aligned}
\left(P^{X,Y;R}\otimes \mathrm{id}_{C(Z;R)}\right)_* \circ \lambda^{C(X\times Y;R),C(Z;R)} &= \lambda^{C(X\times Y;R),C(Z;R)} \circ \left(P_*^{X,Y;R}\otimes \mathrm{id}_{H(Z;R)}\right) \\
\left(\mathrm{id}_{C(X;R)}\otimes P^{Y,Z;R}\right)_* \circ \lambda^{C(X;R),C(Y\times Z;R)} &= \lambda^{C(X;R),C(Y\times Z;R)} \circ \left(\mathrm{id}_{H(X;R)}\otimes P_*^{Y,Z;R}\right).
\end{aligned}$$

Commutativity of

$$\begin{tikzcd}
H(C)\otimes_r H(D)\otimes_R H(E) \arrow[r, "\lambda^{C,D}\otimes\mathrm{id}_{H(E)}"] \arrow[d, "\mathrm{id}_{H(C)}\otimes\lambda^{D,E}"'] & H(C\otimes_R D)\otimes_R H(E) \arrow[d, "\lambda^{C\otimes_R D,E}"] \\
H(C)\otimes_R H(D\otimes_R E) \arrow[r, "\lambda^{C,D\otimes_R E}"'] & H(C\otimes_R D\otimes_R E)
\end{tikzcd}$$

yields associativity of λ and therefore

$$\begin{aligned}
&P_*^{X\times Y,Z;R}\circ\lambda^{C(X\times Y;R),C(Z;R)}\circ\left(P_*^{X,Y;R}\otimes\mathrm{id}_{H(Z;R)}\right)\circ\left(\lambda^{C(X;R),C(Y;R)}\otimes\mathrm{id}_{H(Z;R)}\right) \\
&= P_*^{X,Y\times Z;R}\circ\lambda^{C(X;R),C(Y\times Z;R)}\circ\left(\mathrm{id}_{H(X;R)}\otimes P_*^{Y,Z;R}\right)\circ\left(\mathrm{id}_{H(X)}\otimes\lambda^{C(Y;R),C(Z;R)}\right).
\end{aligned}$$

Evaluating this on $a\otimes b\otimes c \in H(X;R)\otimes_R H(Y;R)\otimes_R H(Z;R)$ we get $(a\times b)\times c = a\times(b\times c)$.

(*iii*) Since the left part of (A.4) commutes and λ is natural (see (*i*)), we have

$$\begin{tikzcd}
H(X;R)\otimes_R H(\{*\};R) \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}_{H(X;R)}\otimes\varepsilon_*^{\{*\};R}"'] & H(C(X;R)\otimes_R C(\{*\};R)) \arrow[r, "P_*"] \arrow[d, "\left(\mathrm{id}_{C(X;R)}\otimes\varepsilon_*^{\{*\};R}\right)_*"] & H(X\times\{*\};R) \arrow[d, equal] \\
H(X;R)\otimes_R H(R) \arrow[r, "\lambda"'] & H(C(X;R)\otimes_R R) \arrow[r, equal] & H(X;R)
\end{tikzcd}$$

where $\varepsilon^{X;R} : C(X;R) = C(X)\otimes C(R) \xrightarrow{\varepsilon^X\otimes\mathrm{id}_R} \mathbb{Z}\otimes R = R$ is the augmentation tensored with R. Now if we take R to be a chain complex concentrated in degree zero, then $H(R) = R$ and $C\otimes_R R = C$. For λ we have commutativity of

$$\begin{tikzcd}
H(C)\otimes_R H(R) \arrow[r, "\lambda^{C,R}"] \arrow[d, equal] & H(C\otimes_R R) \arrow[d, equal] \\
H(C)\otimes_R R \arrow[r, equal] & H(C)
\end{tikzcd}$$

, and therefore, we obtain

$$\begin{aligned}
a\times 1_{\{*\}} &= \left(P_*^{X,\{*\};R}\circ\lambda^{C(X;R),C(\{*\};R)}\right)(a\otimes 1_{\{*\}}) \\
&= \left(\lambda^{C(X;R),R}\circ(\mathrm{id}_{H(X;R)}\otimes\varepsilon_*^{\{*\};R})\right)(a\otimes 1_{\{*\}}) \\
&= \lambda^{C(X;R),R}(a\otimes 1_R) = a.
\end{aligned}$$

(*iv*) From the naturality of P and λ respectively, we obtain a commutative diagram

$$\begin{tikzcd}
H(X;R)\otimes_R H(Y;R) \arrow[r, "\lambda"] \arrow[d, "f_*\otimes g_*"'] & H(C(X;R)\otimes_R C(Y;R)) \arrow[r, "P_*"] \arrow[d, "(f_\#\otimes g_\#)_*"] & H(X\times Y;R) \arrow[d] \\
H(X';R)\otimes_R H(Y';R) \arrow[r, "\lambda"'] & H(C(X';R)\otimes_R C(Y';R)) \arrow[r, "P_*"'] & H(X'\times Y';R)
\end{tikzcd}$$

and thus the relation

$$(f\times g)_*\circ P_*^{X,Y;R}\circ\lambda^{C(X;R),C(Y;R)} = P_*^{X',Y';R}\circ\lambda^{C(X';R),C(Y';R)}\circ(f_*\otimes g_*).$$

Evaluation on $a\otimes b\in H(X;R)\otimes_R H(Y;R)$ gives $(f\times g)_*(a\times b) = f_*a\times g_*b$. $\square$

A.2.7 Definition (Pair of spaces, relative homology)**.**

(i) Let X be a topological space and $A \subseteq X$ a subspace. (X, A) is called a *pair of spaces*. A *map of pairs* $f : (X, A) \to (Y, B)$ is a continuous map with $f(A) \subseteq f(B)$.

(ii) We can think of the chain complex $C_\bullet(A)$ as a subchain complex of $C_\bullet(X)$ and define $C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A)$.

(iii) *Relative homology groups* of a pair of spaces (X, A) are defined by $H_n(X, A) := H_n(C_\bullet(X, A))$.

A.2.8 Remark.

(i) Consider the pair $(X, \emptyset)$. Then $C_\bullet(X)/C_\bullet(\emptyset) = C_\bullet(X)$ implies $H_n(X, \emptyset) = H_n(X)$.

(ii) If $P \in X$ is a point, the pair $(X, \{P\})$ yields the *reduced homology groups*

$$H_n(X, \{P\}) \cong \tilde{H}_n(X).$$

(iii) For $A \subseteq B \subseteq X$ one can extend definition A.2.7 to *triples of spaces* (X, A, B). The triple $(X, A, \emptyset)$ gives the same homology groups as the pair (X, A).

A.2.9 Lemma ([9])**.** *Let $U \subseteq X$ and $V \subseteq X$ and let one of the following maps be a homotopy equivalence. Then the other maps are homotopy equivalences, too.*

(i) $C(U, U \cap V) \to C(U \cup V, V)$

(ii) $C(V, U \cap V) \to C(U \cup V, U)$

(iii) $C(U, U \cap V) \oplus C(V, U \cap V) \to C(U \cup V, U \cap V)$

(iv) $C(U) + C(V) \subseteq C(U \cup V)$

(v) $\frac{C(U)+C(V)}{C(U\cap V} \to C(U \cup V, U \cap V)$

(vi) $\frac{C(X)}{C(U)+C(V)} \to C(X, U \cup V)$

Proof. The canonical inclusions induce short exact sequences of chain complexes:

$$
\begin{aligned}
0 \to \frac{C(U)}{C(U\cap V)} \to \frac{C(U\cup V)}{C(V)} \to \frac{C(U\cup V)}{C(U)+C(V)} \to 0 \\
0 \to C(U)+C(V) \to C(U\cup V) \to \frac{C(U\cup V}{C(U)+C(V)} \to 0 \\
0 \to \frac{C(U)+C(V)}{C(U\cap V)} \to \frac{C(U\cup V)}{C(U\cap V)} \to \frac{C(U\cup V)}{C(U)+C(V)} \to 0 \\
0 \to \frac{C(U\cup V)}{C(U)+C(V)} \to \frac{C(X)}{C(U)+C(V)} \to \frac{C(X)}{C(U\cup V)} \to 0
\end{aligned}
$$

Now we use the fact that a chain map is a homotopy equivalence if and only if it induces an isomorphism in homology. Thus, if we consider the long exact sequences on homology induced by the above short exact sequences, we see that $(i) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$ and by symmetry $(ii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$. By commutativity of the following diagram we get the equivalence $(iii) \Leftrightarrow (v)$.

$$
\begin{tikzcd}
\frac{C(U)}{C(U\cap V)} \oplus \frac{C(V)}{C(U\cap V)} \arrow[rr, equal] \arrow[dr] & & \frac{C(U)+C(V)}{C(U\cap V)} \arrow[dl] \\
& \frac{C(U\cup V)}{C(U\cap V)} &
\end{tikzcd}
$$

□

A.2.10 Definition (Excisive triad)**.**
Let $U \subseteq X$ and $V \subseteq X$. If the equivalent conditions from A.2.9 are fulfilled, we call $(X; U, V)$ an *excisive triad.*

A.2.11 Remark. In the sequel all spaces are assumed to be excisive.

A.2.12 Definition (Scalar product $\langle -, - \rangle$)**.**
We have R-bilinear maps

$$C^n(X, A; R) \times C_n(X, A; R) \to R, \quad \langle \alpha, c \otimes r\rangle := r\alpha(c)$$

called *scalar product* that satisfy $\langle \partial\alpha, a\rangle = \langle \alpha, \partial a\rangle$ and thus factorize to (surjective)

R-bilinear maps

$$H^n(X,A;R)\times H_n(X,A;R)\to R,\quad \langle[\alpha],[a]\rangle := \langle \alpha, a\rangle.$$

A.2.13 Remark. In the above definition, if $R=K$ is a field and the $H_n(X,A;R)$ are finite dimensional vector spaces, the adjoint

$$H^n(X,A;K)\xrightarrow{\cong}\mathrm{Hom}_K(H_n(X,A;K),K)$$

identifies $H^n(X,A;K)$ as vector space dual to $H_n(X,A;K)$.

A.2.14 Definition (Cohomology cross product)**.**
The *cohomology cross product* is the composite $\alpha\times\beta := Q^*\mu_*\lambda(\alpha\otimes\beta)$

$$\begin{tikzcd}
\times:\ H^*(X,A;R)\otimes_R H^*(Y,B;R) \arrow[r, "\lambda"] \arrow[dr, dotted] & H^*(C^*(X,A;R)\otimes_R C^*(Y,B;R)) \arrow[dl, "\mu_*"'] \\
H^*(\mathrm{Hom}(C(X,A)\otimes C(Y,B),R)) \arrow[r, "Q^*"'] & H^*(X\times Y, A\times Y\cup X\times B;R)
\end{tikzcd}$$

where μ_* is induced by the map $\mu(\alpha,\beta)(c\otimes d) := (-1)^{|\beta||c|}\alpha(c)\beta(d)$

$$\begin{aligned}\mu:\ C^*(X,A;R)\otimes_R C^*(Y,B;R) &= \mathrm{Hom}(C(X,A),R)\otimes_R \mathrm{Hom}(C(Y,B),R)\\ &\xrightarrow{\mu}\mathrm{Hom}(C(X,A)\otimes_R C(Y,B),R),\end{aligned}$$

Q^* comes from an Eilenberg-Zilber equivalence, and λ is induced by the natural homomorphism defined in $A.2.5$. We can think of $\times$ as R-bilinear maps

$$H^p(X,A;R)\times H^q(Y,B;R)\xrightarrow{\times}H^{p+q}(X\times Y, A\times Y\cup X\times B;R).$$

A.2.15 Proposition (Properties of the cohomology cross product, [9])**.**
Let R be a unital commutative ring, $\alpha\in H^*(X,A;R)$, $\beta\in H^*(X,B;R)$, $\gamma\in H^*(X,C;R)$, $a\in H_*(X,A;R)$, $b\in H_*(X,B;R)$, $\rho:R\to R'$ a homomorphism of rings and $f:X'\to X$, $g:Y'\to Y$ continuous. Then:

(i) $(\alpha\times\beta)\times\gamma = \alpha\times(\beta\times\gamma)$ *(Associativity)*

(ii) $\beta\times\alpha = (-1)^{|\alpha||\beta|}T^*(\alpha\times\beta)$ *(Graded commutativity)*

(iii) $\alpha \times 1_Y = \mathrm{pr}_X^* \alpha,\ 1_X \times \beta = \mathrm{pr}_Y^* \beta$ *(Unital element)*

(iv) $(f \times g)^*(\alpha \times \beta) = f^*\alpha \times f^*\beta$ *(Naturality)*

(v) $\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle$ *(Duality)*

(vi) $\rho_*(\alpha \times \beta) = \rho_*\alpha \times \rho_*\beta$ *(Naturality in the ring of coefficients)*

Proof. (i) For an Eilenberg-Zilber map Q we have $(Q \otimes \mathrm{id}) \circ Q \simeq (\mathrm{id} \otimes Q) \circ Q$ and therefore

$$\begin{aligned}
(\alpha \times \beta) \times \gamma &= Q^*\mu_*\lambda\,(Q^*\mu_*\lambda(\alpha \otimes \beta) \otimes \gamma) \\
&= Q^*\mu_*\lambda(Q^* \otimes \mathrm{id})(\mu_* \otimes \mathrm{id})(\alpha \otimes \beta \otimes \gamma) \\
&= Q^*(Q \otimes \mathrm{id})^*(\mu_*\lambda)(\mu_*\lambda \otimes \mathrm{id})(\alpha \otimes \beta \otimes \gamma) \\
&= Q^*(\mathrm{id} \otimes Q)^*(\mu_*\lambda)(\mathrm{id} \otimes \mu_*\lambda)(\alpha \otimes \beta \otimes \gamma) \\
&= Q^*\mu_*\lambda(\mathrm{id} \otimes Q^*)(\mathrm{id} \otimes \mu_*\lambda)(\alpha \otimes \beta \otimes \gamma) \\
&= Q^*\mu_*\lambda\,(\alpha \otimes Q^*\mu_*\lambda(\beta \otimes \gamma)) = \alpha \times (\beta \times \gamma).
\end{aligned}$$

(ii) From A.2.3 we have $\tau \circ Q \simeq Q \circ T_\#$ and therefore

$$\begin{aligned}
T^*(\alpha \times \beta) &= T^*Q^*\mu_*\lambda(\alpha \otimes \beta) = Q^*\tau^*\mu_*\lambda(\alpha \otimes \beta) = Q^*\mu_*\tau^*\lambda(\alpha \otimes \beta) \\
&= Q^*\mu_*\lambda\tau(\alpha \otimes \beta) = (-1)^{|\alpha||\beta|}Q^*\mu_*\lambda(\beta \otimes \alpha) = (-1)^{|\alpha||\beta|}\beta \times \alpha.
\end{aligned}$$

(iv) By naturality:

$$\begin{aligned}
(f \times g)^*(\alpha \times \beta) &= (f \times g)^*Q^*\mu_*\lambda(\alpha \otimes \beta) \\
&= Q^*(f_\# \otimes g_\#)^*\mu_*\lambda(\alpha \otimes \beta) \\
&= Q^*\mu_*(f_\# \otimes g_\#)_*\lambda(\alpha \otimes \beta) \\
&= Q^*\mu_*\lambda(f^* \otimes g^*)(\alpha \otimes \beta) \\
&= Q^*\mu_*\lambda(f^*\alpha \otimes g^*\beta) = f^*\alpha \times g^*\beta
\end{aligned}$$

(iii) Let $c : Y \to \{*\}$ be the constant map. Then by (iv):

$$\alpha \times 1_Y = \alpha \times c^*1_{\{*\}} = (\mathrm{id}_X \times c)^*(\alpha \times 1_{\{*\}}) = (\mathrm{id}_X \times c)^*\alpha = \mathrm{pr}_X^* \alpha$$

(v) Because of $Q \circ P \simeq \mathrm{id}$ we have:

$$\begin{aligned}\langle \alpha \times \beta, a \times b\rangle &= \langle Q^*\mu_*\lambda(\alpha \otimes \beta), P_*\lambda(a \otimes b)\rangle = \langle \mu_*\lambda(\alpha \otimes \beta), Q_*P_*\lambda(a \otimes b)\rangle \\ &= \langle \mu_*\lambda(\alpha \otimes \beta), \lambda(a \otimes b)\rangle = \langle \alpha, a\rangle\langle \beta, b\rangle\end{aligned}$$

(vi) Finally,

$$\begin{aligned}\rho_*(\alpha \times \beta) &= \rho_*Q^*\mu_*\lambda(\alpha \otimes \beta) = Q^*\rho_*\mu_*\lambda(\alpha \otimes \beta) = Q^*\mu_*\rho_*\lambda(\alpha \otimes \beta) \\ &= Q^*\mu_*\lambda(\rho_* \otimes \rho_*)(\alpha \otimes \beta) = Q^*\mu_*\lambda(\rho_*\alpha \otimes \rho_*\beta) = \rho_*\alpha \times \rho_*\beta.\end{aligned}$$

□

A.2.16 Definition (Slant product)**.**
Combining an Eilenberg-Zilber equivalence with the canonical chain map $\beta \otimes a \otimes b \mapsto (-1)^{|\beta||a|}\beta(b)a$,

$$\mathrm{Hom}(C_*(Y,B),R) \otimes_R C_*(X,A;R) \otimes_R C_*(Y,B;R) \to C_*(X,A;R),$$

we obtain an induced homomorphism called *slant product* $\beta \otimes c \mapsto \beta \setminus c$,

$$\setminus : \; H^q(Y,B;R) \otimes_R H_{p+q}(X \times Y, A \times Y \cup X \times B; R) \xrightarrow{\setminus} H_p(X,A;R).$$

A.2.17 Definition (Another slant product)**.**
The bilinear pairing

$$/ : H^{p+q}(X \times Y, A \times Y \cup X \times B; R) \otimes_R H_p(X,A;R) \xrightarrow{/} H^q(Y,B;R)$$

(referred to as *division by chains*) is also called *slant product*. We define $/$ in the following way. Consider chain complexes C and C'. Given a cocycle $\alpha \in \mathrm{Hom}(C \otimes C', R)$ and a cycle $z = \sum_i c'_i \otimes r_i \in C' \otimes R$, we form the cochain $\alpha/z \in \mathrm{Hom}(C,R)$ by taking

$$(\alpha/z)(c) = \sum_i \mu_*\big(\alpha(c \otimes c'_i) \otimes r_i\big)$$

for all $c \in C$, where μ comes from the multiplication in R.

Then α/z is a cocycle, thus we get an element of $H^*(C;R)$.

A.2.18 Remark. The slant products $\backslash$ and $/$ are referred to as *external products.*

A.2.19 Proposition (Properties of the slant product, [9])**.**
Let R be a unital commutative ring, $f : (X,A) \to (X',A')$ and $g : (Y,B) \to (Y',B')$ maps of pairs and $\rho : R \to R'$ a homomorphism of rings. Consider $\alpha \in H^*(X,A;R)$, $\beta \in H^*(Y,B;R)$, $\beta' \in H^*(Y',B';R)$, $\gamma \in H^*(Z,C;R)$, $a \in H_*(X,A;R)$, $b \in H_*(Y,B;R), c \in H_*(X \times Y, A \times Y \cup X \times B; R)$ and $e \in H_*(X \times Y \times Z, A \times Y \times Z \cup X \times B \times Z \cup X \times Y \times C; R)$.
Then:

(i) $\beta \backslash (\gamma \backslash\ e) = (\beta \times \gamma) \backslash e$ *(Associativity)*

(ii) $1_Y \backslash c = (\mathrm{pr}_X)_* c$ *(Unital element)*

(iii) $f_*(g^*\beta' \backslash c) = \beta' \backslash (f \times g)_* c$ *(Naturality)*

(iv) $\langle \alpha \times \beta, c\rangle = \langle \alpha, \beta \backslash c\rangle$ *(Duality)*

(v) $\beta \backslash (a \times b) = \langle \beta, b\rangle a$ *(Multiplicativtiy)*

(vi) $\rho_*(\beta \backslash c) = \rho_*\beta \backslash \rho_* c$ *(Naturality in the ring of coefficients)*

Proof. Analogously to that of A.2.15. □

A.2.20 Remark. The product $/$ has natural properties very similar to those of $\backslash$. Some of them are:

(i) For $a \in H_p(X)$, $b \in H_q(Y)$, $\gamma \in H^{p+q}(X \times Y)$ we have

$$\langle a \times b, \gamma\rangle = \langle b, \gamma/a\rangle.$$

(ii) Given maps $f : X \to X'$ and $g : Y \to Y'$ the product $/$ satisfies the formula

$$H^{p+q}(f \times g)(\gamma')/\alpha = H^q(f)(\gamma'/H_p(g)(\alpha)).$$

(iii) For $\alpha \in H^n(X \times Y)$, $b \in H_q(Y)$, $c \in H_p(X)$ we have

$$(\alpha/b) \cap c = \pi_{1*}(\alpha \cap (c \times b)) \in H_{p+q-n}(X)$$

where $\pi_1 : X \times Y \to X$ is the projection on the first factor.

For more details on / see [20] and [6].

A.2.21 Remark (Duality).
If R is a field, then the scalar product $\langle\ ,\ \rangle$ induces an isomorphism, cf. A.2.12. In that case, the slant product $\backslash$ is determined by property (*iv*) of A.2.19, $\langle \alpha \times \beta, c\rangle = \langle \alpha, \beta \backslash c\rangle$.

A.2.22 Definition (Cup product).
Let R be a unital commutative ring and (X, A) and (X, B) pairs of spaces such that $(X \times X, A \times X \cup X \times B)$ is an excisive triad. Then the *cup product* is defined by $\alpha \cup \beta := \Delta^*(\alpha \times \beta)$,

$$\begin{tikzcd} \cup:\ H^*(X,A;R) \otimes_R H^*(X,B;R) \arrow[r, "\times"] \arrow[dr, dotted] & H^*(X \times X, A \times X \cup X \times B; R) \arrow[d, "\Delta^*"] \\ & H^*(X, A \cup B; R). \end{tikzcd}$$

Here, $\times$ is the cross product on cohomology as defined in $A.2.14$ and Δ^* is induced by the diagonal map $\Delta : (X, A \cup B) \to (X \times X, A \times X \cup X \times B)$, $\quad \Delta(x) := (x, x)$. Equivalently, we may view $\cup$ as R-bilinear maps

$$H^p(X, A; R) \times H^q(X, B; R) \xrightarrow{\cup} H^{p+q}(X, A \cup B; R).$$

A.2.23 Proposition (Properties of the cup product, [9]).
Let R be a unital commutative ring, $\alpha \in H^*(X, A; R)$, $\beta \in H^*(X, B; R)$, $\gamma \in H^*(X, C; R)$, $a \in H_*(X, A; R)$, $b \in H_*(X, B; R)$, $\rho : R \to R'$ a homomorphism of rings and $f : X' \to X$ continuous.
Then:

(i) $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ *(Associativity)*

(ii) $\beta \cup \alpha = (-1)^{|\alpha||\beta|} \alpha \cup \beta$ *(Graded commutativity)*

(iii) $\alpha \cup 1_X = \alpha = 1_X \cup \alpha$ *(Unital element)*

(iv) $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$ *(Naturality)*

(v) $\alpha_1 \times \alpha_2 = \mathrm{pr}_{X_1}^* \alpha_1 \cup \mathrm{pr}_{X_2}^* \alpha_2$ *(Relation with cross product)*

(vi) $(\alpha_1 \times \alpha_2) \cup (\beta_1 \times \beta_2) = (-1)^{|\alpha_2||\beta_1|}(\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2)$ *(Distributivity)*

(vii) $\rho_*(\alpha \cup \beta) = \rho_*\alpha \cup \rho_*\beta$ *(Naturality in the ring of coefficients)*

Proof. Basically, the properties of the cup product $\cup$ follow from those of the cross product $\times$ on cohomolgy.

(i) Since $(\Delta \times \mathrm{id}_X) \circ \Delta = (\mathrm{id}_X \times \Delta) \circ \Delta$, we have

$$\begin{aligned}(\alpha \cup \beta) \cup \gamma &= \Delta^* \left(\Delta^*(\alpha \times \beta) \times \gamma\right) = \Delta^*(\Delta \times \mathrm{id}_X)^*(\alpha \times \beta \times \gamma) \\ &= \Delta^*(\mathrm{id}_X \times \Delta)^*(\alpha \times \beta \times \gamma) = \Delta^* \left(\alpha \times \Delta^*(\beta \times \gamma)\right) = \alpha \cup (\beta \cup \gamma).\end{aligned}$$

(ii) The commutator map $T(x,y) = (y,x)$ satisfies $T \circ \Delta = \Delta$ and so

$$\beta \cup \alpha = \Delta^*(\beta \times \alpha) = (T \circ \Delta)^*(\beta \times \alpha) = (-1)^{|\alpha||\beta|}\Delta^*(\alpha \times \beta) = (-1)^{|\alpha||\beta|}\alpha \cup \beta.$$

(iii) The relation $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$ gives

$$\alpha \cup 1_X = \Delta^*(\alpha \times 1_X) = \Delta^* \mathrm{pr}_1^* \alpha = (\mathrm{pr}_1 \circ \Delta)^*\alpha = \alpha.$$

(iv) We use $\Delta_X \circ f = (f \times f) \circ \Delta_{X'}$ to get

$$f^*(\alpha \cup \beta) = f^*\Delta_X^*(\alpha \times \beta) = \Delta_{X'}^*(f \times f)^*(\alpha \times \beta) = \Delta_{X'}^*(f^*\alpha \times f^*\beta) = f^*\alpha \cup f^*\beta.$$

(v) From the identity $(\mathrm{pr}_{X_1} \times \mathrm{pr}_{X_2}) \circ \Delta_{X_1 \times X_2} = \mathrm{id}_{X_1 \times X_2}$ it follows that

$$\begin{aligned}\mathrm{pr}_{X_1}^* \alpha_1 \cup \mathrm{pr}_{X_2}^* \alpha_2 &= \Delta_{X_1 \times X_2}^*(\mathrm{pr}_{X_1}^* \alpha_1 \times \mathrm{pr}_{X_2}^* \alpha_2) \\ &= \Delta_{X_1 \times X_2}^*(\mathrm{pr}_{X_1} \alpha_1 \times \mathrm{pr}_{X_2})^*(\alpha_1 \times \alpha_2) \\ &= \left((\mathrm{pr}_{X_1} \times \mathrm{pr}_{X_2}) \circ \Delta_{X_1 \times X_2}^*\right)^* (\alpha_1 \times \alpha_2) = \alpha_1 \times \alpha_2.\end{aligned}$$

(vi) From the above we get:

$$\begin{aligned}(\alpha_1 \times \alpha_2) \cup (\beta_1 \times \beta_2) &= (\mathrm{pr}^*_{X_1} \alpha_1 \cup \mathrm{pr}^*_{X_2} \alpha_2) \cup (\mathrm{pr}^*_{X_1} \beta_1 \cup \mathrm{pr}^*_{X_2} \beta_2) \\ &= (-1)^{|\alpha_2||\beta_1|} \mathrm{pr}^*_{X_1} \alpha_1 \cup \mathrm{pr}^*_{X_1} \beta_1 \cup \mathrm{pr}^*_{X_2} \alpha_2 \cup \mathrm{pr}^*_{X_2} \beta_2 \\ &= (-1)^{|\alpha_2||\beta_1|} \mathrm{pr}^*_{X_1}(\alpha_1 \cup \beta_1) \cup \mathrm{pr}^*_{X_2}(\alpha_2 \cup \beta_2) \\ &= (-1)^{|\alpha_2||\beta_1|}(\alpha_1 \cup \beta_1) \times (\alpha_2 \cup \beta_2).\end{aligned}$$

(vii) Finally,

$$\rho_*(\alpha \cup \beta) = \rho_*\Delta^*(\alpha \times \beta) = \Delta^*\rho_*(\alpha \times \beta) = \Delta^*(\rho_*\alpha \times \rho_*\beta) = \rho_*\alpha \cup \rho_*\beta.$$

□

A.2.24 Remark. Considering the cup product $\cup$ as multiplication, $H^*(X;R)$ becomes a graded, unital, associative, and graded commutative R-algebra, the *cohomology ring.*

A.2.25 Definition (Cap product)**.**
Combining the slant product $\backslash$ with the homomorphism induced by the diagonal map $\Delta(x) := (x,x)$,

$$\Delta_* : H_q(X, A \cup B; R) \to H_q(X \times X, A \times X \cup X \times B; R),$$

we obtain the *cap product* $\alpha \cap b := \alpha \backslash \Delta_* b$,

$$\cap : \ H^q(X,B;R) \otimes_R H_{p+q}(X, A \cup B; R) \xrightarrow{\cap} H_p(X,A;R).$$

A.2.26 Proposition (Properties of the cap product, [9])**.**
Let R be a unital commutative ring, $f : X \to X'$ continuous with $f(A) \subseteq A'$, $f(B) \subseteq B'$ and $\rho : R \to R'$ a homomorphism of rings. Consider elements $\alpha \in H^*(X,A;R)$, $\beta \in H^*(X,B;R)$, $\beta' \in H^*(X',B';R)$, $\gamma \in H^*(X,C;R)$, $a \in H_*(X,A;R)$, $c \in H_*(X,A \cup B;R)$, $e \in H_*(X, A \cup B \cup C; R)$ and $\xi \in H_*(X \times Y, A \times Y \cup X \times B; R)$.
Then:

(i) $\beta \cap (\gamma \cap e) = (\beta \cup \gamma) \cap e$ *(Associativity)*

(ii) $1_X \cap a = a$ *(Unital element)*

(iii) $f_*(f^*\beta' \cap c) = \beta' \cap f_*c$ *(Naturality)*

(iv) $\langle \alpha \cup \beta, c\rangle = \langle \alpha, \beta \cap c\rangle$ *(Duality)*

(v) $\langle \alpha, a\rangle = \varepsilon(\alpha \cap a)$ *(Relation with scalar product)*

(vi) $\beta \setminus \xi = (\mathrm{pr}_X)_*(\mathrm{pr}_Y^* \beta \cap \varepsilon)$ *(Relation with slant product)*

(vii) $\rho_*(\beta \cap c) = \rho_*\beta \cap \rho_* c$ *(Naturality in the ring of coefficients)*

Proof. (i) The definition of the cup product $\cup$ and the obvious relation $(\mathrm{id}_X \times \Delta) \circ \Delta = (\Delta \times \mathrm{id}_X) \circ \Delta$ give

$$\begin{aligned}
\beta \cap (\gamma \cap e) &= \beta \setminus \Delta_*(\gamma \setminus \Delta_* e) \\
&= \beta \setminus (\gamma \setminus ((\Delta \times \mathrm{id}_X)_* \Delta_* e)) \\
&= (\beta \times \gamma) \setminus ((\Delta \times \mathrm{id}_X)_* \Delta_* e) \\
&= (\beta \times \gamma) \setminus ((\mathrm{id}_X \times \Delta)_* \Delta_* e) \\
&= \Delta^*(\beta \times \gamma) \setminus \Delta_* e \\
&= (\beta \cup \gamma) \cap e.
\end{aligned}$$

(ii) Using $\mathrm{pr}_1 \circ \Delta = \mathrm{id}_X$ we get

$$1_X \cap a = 1_X \setminus \Delta_* a = (\mathrm{pr}_1)_* \Delta_* a = a.$$

(iii) Using the relation $(f \times f) \circ \Delta^X = \Delta^{X'} \circ f$ we get

$$f_*(f^*\beta' \cap c) = f_*(f^*\beta' \setminus \Delta_*^X c) = \beta' \setminus (f \times f)_* \Delta_*^X c = \beta' \setminus \Delta_*^{X'} f_* c = \beta' \cap f_* c.$$

(iv) This follows from the corresponding duality formula for the slant product:

$$\langle \alpha \cup \beta, c\rangle = \langle \Delta^*(\alpha \times \beta), c\rangle = \langle \alpha \times \beta, \Delta_* c\rangle = \langle \alpha, \beta \setminus \Delta_* c\rangle = \langle \alpha, \beta \cap c\rangle.$$

(*v*) From (*iv*) it follows that

$$\begin{aligned}\langle \alpha, a\rangle = \langle 1_X \cup \alpha, a\rangle &= \langle 1_X, \alpha \cap a\rangle \\ &= \langle c^* 1_{\{*\}}, \alpha \cap a\rangle = \langle 1_{\{*\}}, c_*(\alpha \cap a)\rangle = \varepsilon(\alpha \cap a).\end{aligned}$$

(*vi*) We have $(\mathrm{pr}_X \times \mathrm{pr}_Y) \circ \Delta^{X\times Y} = \mathrm{id}_{X\times Y}$, so

$$(\mathrm{pr}_X)_*(\mathrm{pr}_Y^* \beta \cap \xi) = (\mathrm{pr}_X)_*(\mathrm{pr}_Y^* \beta \setminus \Delta_*^{X\times Y}\xi) = \beta \setminus (\mathrm{pr}_X \times \mathrm{pr}_Y)_* \Delta_*^{X\times Y}\xi = \beta \setminus \xi.$$

(*vii*) Finally,

$$\rho_*(\beta \cap c) = \rho_*(\beta \setminus \Delta_* c) = \rho_*\beta \setminus \rho_*\Delta_* c = \rho_*\beta \setminus \Delta_*\rho_* c = \rho_*\beta \cap \rho_* c.$$

□

A.2.27 Remark. The products $\cup$ and $\cap$ are referred to as *internal products*.

A.2.28 Remark. The basic formula relating all the products is

$$\big((\alpha \times \beta) \cup \gamma\big)/a = (-1)^{r(s-q)} \beta \cup \big((\gamma/a) \cap \alpha\big),$$

see [6, Chapter 29].

A.3. Excision

There is an axiomatic approach to homology (of pairs of spaces) in which excision is included:

A.3.1 Theorem ([15, Ch. 13])**.** *For integers q there exist functors $H_q(X, A; R)$ from the homotopy category of pairs of spaces to the category of Abelian groups together with natural transformations $\partial : H_q(X, A; R) \to H_{q-1}(A; R)$, where $H_q(X; R)$ is defined to be $H_q(X, \emptyset; R)$. These functors and natural transformations satisfy and are characterized by the following axioms.*

- *(Dimension.) If X is a point, then $H_0(X; R) = R$ and $H_q(X; R) = 0$ for all $q \geq 1$.*

- *(Exactness.) The following sequence is exact, where the unlabeled arrows are induced by the inclusions $A \to X$ and $(X, \emptyset) \to (X, A)$*

$$\cdots \longrightarrow H_q(A; R) \longrightarrow H_q(X; R) \longrightarrow H_q(X, A; R) \xrightarrow{\partial} H_{q-1}(A; R) \longrightarrow \cdots$$

- *(Excision) If $(X; A, B)$ is an excisive triad, so that X is the union of the interiors of A and B, then the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism*

$$H_*(A, A \cap B; R) \to H_*(X, B; R).$$

- *(Additivity.) If (X, A) is the disjoint union of a set of pairs (X_i, A_i), then the inclusions $(X_i, A_i) \to (X, A)$ induce an isomorphism*

$$\sum_i H_*(X_i, A_i; R) \to H_*(X, A; R)$$

- *(Weak equivalence.) If $f : (X, A) \to (Y, B)$ is a weak equivalence (see A.3.2 below), then*

$$f_* : H_*(X, A; R) \to H_*(Y, B; R)$$

is an isomorphism.

A.3.2 Remark.

(i) (Weak equivalence.) Roughly speaking, a functor sends morphisms that are weak equivalent to the same isomorphism. For example, in the category of chain complexes the weak equivalent morphisms are by definition those morphisms $A \to B$ where $H_n(A) \to H_n(B)$ are isomorphisms for all $n \geq 0$. We say $f : (X, A) \to (Y, B)$ is a weak equivalence if its maps $A \to B$ and $X \to Y$ are.

(ii) The axiomatic description of cohomology is similar ([15, Ch. 18]); e.g. the excision isomorphism is given by

$$H^*(X, B; R) \to H^*(A, A \cap B; R).$$

Given a closed submanifold A in M, we can use 1.3.5 and *excision isomorphisms* to get the cohomology ring of the normal bundle of A in M.

A.3.3 Corollary ([16, Corollary 11.2]). *If A is closed in M, then the cohomology ring $H^*(E, E_0; R)$ associated with the normal bundle of A in M is canonically isomorphic to the cohomology ring $H^*(M, M \setminus A; R)$.*

Proof. Since the tubular neighbourhood N_ε and the complement $M \setminus A$ are open subsets with union M and intersection $N_\varepsilon \setminus A$, there is an excision isomorphism

$$H^*(M, M \setminus A) \to H^*(N_\varepsilon, N_\varepsilon \setminus A).$$

Therefore the embedding

$$\mathrm{Exp} : (E(\varepsilon), E(\varepsilon)_0) \to (N_\varepsilon, N_\varepsilon \setminus A) \subset (M, M \setminus A)$$

induces an isomorphism

$$\mathrm{Exp}^* : H^*(M, M \setminus A) \to H^*(E(\varepsilon), E(\varepsilon)_0).$$

Composing with the excision isomorphism

$$H^*(E(\varepsilon), E(\varepsilon)_0) \cong H^*(E, E_0)$$

we obtain an isomorphism which clearly does not depend on the particular choice of ε. $\square$

A.4. The Poincaré duality theorem

In this section we basically follow [15].

Let M be an n-manifold, $x \in M$ and U a coordinate chart $U \cong \mathbb{R}^n$ with $x \in U$. By excision, exactness, and homotopy invariance, we have isomorphisms

$$H_i(M, M \setminus x) \cong H_i(U, U \setminus x) \cong \tilde{H}_{i-1}(U \setminus x) \cong \tilde{H}_{i-1}(\mathbf{S}^{n-1}).$$

Thus $H_i(M, M \setminus x) = 0$ if $i \neq n$ and $H_n(M, M \setminus x) \cong R$ and we can choose a generator for the free R-module $H_n(M, M \setminus x)$ (corresponding to the unit of R).

A.4.1 Definition (Orientation, Fundamental class)**.**
An *R-fundamental class* of a manifold M at a subspace X is an element $z \in H_n(M, M \setminus X; R)$ such that, for each $x \in X$, the image of z under the map

$$H_n(M, M \setminus X) \to H_n(M, M \setminus x)$$

induced by the inclusion $(M, M \setminus X) \to (M, M \setminus x)$ is a generator. If $X = M$, we call $z \in H_n(M)$ a *fundamental class of M*. We define an *R-orientation of M* to be an open cover $\{U_i\}$ and R-fundamental classes z_i of M at U_i, such that if $U_i \cap U_j$ is non-empty, then u_i and u_j map to the same element of $H_n(M, M \setminus U_i \cap U_j)$.

A.4.2 Remark. An R-orientation of M corresponds to a consistent choice of generators. In particular, if $R = \mathbb{Z}$, then this coincides with the usual notion of orientability. An R-fundamental class determines an R-orientation.

The converse statement holds when M is compact. The proof uses the *vanishing theorem*, see [15, Chapter 20].

A.4.3 Remark (Constructing duality)**.**
For compact subspaces K of M we define

$$H_c^q(M) = \operatorname{colim}_K H^q(M, M \setminus K)$$

where the colimit (see 2.5.1) is taken with respect to the homomorphisms

$$H^q(M, M \setminus K) \to H^q(M, M \setminus L)$$

induced by the inclusions $(M, M \setminus L) \subset (M, M \setminus K)$ for $K \subset L$. This is the *cohomology of M with compact supports.* Intuitively, thinking in terms of singular cohomology, the elements are represented by cocyles that vanish off some compact subspace. For an open subspace U of M, we obtain a homomorphism $H_c^q(U) \to H_c^q(M)$ by passage to colimits from the excision isomorphisms

$$H^q(U, U \setminus K) \to H^q(M, M \setminus K)$$

for compact subspaces K of U. For these K, the R-orientation of M determines a fundamental class $[z_K] \in H_n(M, M \setminus K; R)$. Taking the cap product of an element $\eta \in H^p(M, M \setminus K)$ with $[z_K]$ we obtain a duality homomorphism

$$D_K : H^p(M, M \setminus K) \to H_{n-p}(M), \quad \eta \mapsto \eta \cap [z_K].$$

If $K \subseteq L$, the following diagram commutes

$$\begin{tikzcd} H^p(M, M \setminus K) \arrow[rr] \arrow[dr, "D_K"'] & & H^p(M, M \setminus L) \arrow[dl, "D_L"] \\ & H_{n-p}(M). & \end{tikzcd}$$

Passing to colimits we obtain a duality homomorphism

$$D : H_c^p(M) \to H_{n-p}(M).$$

If U is open in M and is given the induced R-orientation, then the following naturality diagram commutes:

$$\begin{tikzcd} H_c^p(U) \arrow[r, "D"] \arrow[d] & H_{n-p}(U) \arrow[d] \\ H_c^p(M) \arrow[r, "D"'] & H_{n-p}(M). \end{tikzcd}$$

A compact manifold M is cofinal[1] among its compact subspaces, therefore $H_c^p(M) = H_{n-p}(M)$.

A.4.4 Theorem (Poincaré duality theorem, [15, Chapter 20, Section 5])**.**

[1] A subset B of A is said to be *cofinal* if it satisfies the following condition: For every $a \in A$ there exists some $b \in B$ such that $a \leqslant b$ for a binary operation $\leqslant$ on A.

Let M be an R-oriented manifold of dimension n. Then the homomorphism

$$D : H^p_c(M) \to H_{n-p}(M)$$

from A.4.3 is an isomorphism.

Proof. We prove that $D : H^p_c(M) \to H_{n-p}(M)$ is an isomorphism for every open subspace U of M. The proof proceeds in five steps.

Step 1. The result holds for any coordinate chart U.
We may take $U = M = \mathbb{R}^n$. The compact cubes K are cofinal among the compact subspaces of $\mathbb{R}^n$. For such K and for $x \in K$,

$$H^p(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong H^p(\mathbb{R}^n, \mathbb{R}^n \setminus x) \cong \tilde{H}^{p-1}(\mathbf{S}^{n-1}) \cong \tilde{H}^p(\mathbf{S}^n).$$

The maps of the colimit system defining $H^p_c(\mathbb{R}^n)$ are clearly isomorphisms. By the definition of the cap product (A.2.25), we see that $D : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus x) \to H_0(\mathbb{R}^n)$ is an isomorphism. Therefore D_K is an isomorphism for every compact cube K and so $D : H^n_c(\mathbb{R}^n) \to H_0(\mathbb{R}^n)$ is an isomorphism. $\square$

Step 2. If the result holds for open subspaces U and V and their intersection, then it holds for their union.
Let $W = U \cap V$ and $Z = U \cup V$. The compact subspaces of Z that are unions of a compact subspace K of U and a compact subspace L of V are cofinal among all of the compact subspaces of Z. For such K and L, we have the following commutative diagram with exact rows. We let $J = K \cap L$ and $N = K \cup L$, and we write $U_K = (U, U \setminus K)$, and similarly for the other cases, to abbreviate notation.

$$\begin{tikzcd}
{} \arrow[r] & H^p(Z_J) \arrow[r] \arrow[d, "\cong"'] & H^p(Z_K) \oplus H^p(Z_L) \arrow[r] \arrow[d, "\cong"] & H^p(Z_N) \arrow[r] \arrow[d, equal] & H^{p+1}(Z_J) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H^p(W_J) \arrow[r] \arrow[d, "D"'] & H^p(U_K) \oplus H^p(V_L) \arrow[r] \arrow[d, "D\oplus D"] & H^p(Z_N) \arrow[r] \arrow[d, "D"] & H^{p+1}(W_J) \arrow[r] \arrow[d, "D"] & {} \\
{} \arrow[r] & H_{n-p}(W) \arrow[r] & H_{n-p}(U) \oplus H_{n-p}(V) \arrow[r] & H_{n-p}(Z) \arrow[r] & H_{n-p-1}(W) \arrow[r] & {}
\end{tikzcd}$$

The top row is the relative Mayer-Vietoris sequence of the triad $(Z; Z \setminus K, Z \setminus L)$. The middle row results from the top row by excision isomorphisms. The bottom row is the absolute Mayer-Vietoris sequence of the triad $(Z; U, V)$. The left two squares commute by naturality. The right square commutes by a diagram chase from the definition of the cap product. The entire diagram is natural with respect to pairs (K, L). We obtain a commutative diagram with exact rows on passage to colimits, and the conclusion follows by the five lemma. $\square$

Step 3. If the result holds for each U_i in a totally ordered set of open subspaces $\{U_i\}$, then it holds for the union U of the U_i.
Any compact subspace K of U is contained in a finite union of the U_i and therefore in one of the U_i. Since homology is compactly supported, it follows that colim $H_{n-p}(U_i) \cong H_{n-p}(U)$. On the cohomology side, we have

$$\begin{aligned} \operatorname{colim}_i\ H^p_c(U_i) &= \operatorname{colim}_i\ \operatorname{colim}_{\{K \mid K \subset U_i\}}\ H^p(U_i, U_i \setminus K) \\ &\cong \operatorname{colim}_{\{K \subset U\}}\ \operatorname{colim}_{\{i \mid K \subset U_i\}}\ H^p(U_i, U_i \setminus K) \\ &\cong \operatorname{colim}_{\{K \subset U\}}\ H^p(U, U \setminus K) = H^p_c(U). \end{aligned}$$

Here the first isomorphism is an (algebraic) interchange of colimits isomorphism: both composite colimits are isomorphic to colim $H^p_c(U_i, U_i \setminus K)$, where the colimit runs over the pairs (K, i) such that $K \subset U_i$. The second isomorphism holds since

$$\operatorname{colim}_{\{i \mid K \subset U_i\}}\ H^p(U_i, U_i \setminus K) \cong H^p(U, U \setminus K),$$

because the colimit is taken over a system of inverses of excision isomorphisms. The conclusion follows since a colimit of isomorphisms is an isomorphism. $\square$

Step 4. The result holds if U is an open subset of a coordinate neighbourhood.
We may take $M = \mathbb{R}^n$. If U is a convex subset of $\mathbb{R}^n$, then U is homeomorphic to $\mathbb{R}^n$ and Step 1 applies. Since the intersection of two convex sets is convex, it follows by induction from Step 2 that the conclusion holds for any finite union of convex open subsets of $\mathbb{R}^n$. Any open subset U of $\mathbb{R}^n$ is the union of countably many convex open subsets. By ordering them and letting U_i be the union of the first i, we see that the conclusion for U follows from Step 3. $\square$

Step 5. The result holds for any open subset U of M.
We may as well take $M = U$. By Step 3, we may apply *Zorn's lemma* to conclude that there is a maximal open subset V of M for which the conclusion holds. If V is not all of M, say $x \notin V$, we may choose a coordinate chart U such that $x \in U$. By Steps 2 and 4, the result holds for $U \cup V$, contradicting the maximality of V. □

This completes the proof of the Poincaré duality theorem. □

A.4.5 Theorem ([6, Theorem 30.6]). *Let M be a compact R-oriented manifold of dimension n with fundamental class $[M] \in H_n(M)$. Then for any $p \leq n$, the inverse to the Poincaré duality isomorphism $H^p(M) \to H_{n-p}(M)$ is given by*

$$a \mapsto (-1)^{pn}\mu'/a,$$

where $a \in H_{n-p}(M)$ and $/$ as in A.2.17.

Proof. If $\eta \in H^p(M)$, then $[M] \cap \eta$ is its image in $H_{n-p}(M)$, and

$$\begin{aligned}
\mu'/[M] \cap \eta &= 1 \cup (\mu'/[M] \cap \eta) \\
&= ((\eta \times 1) \cup \mu')/[M] \\
&= ((1 \times \eta) \cup \mu'/[M] \\
&= (-1)^{pn}(-1)^0 \eta \cup (\mu'/[M] \cap 1) \\
&= (-1)^{pn}\eta \cup (\mu'/[M]) \\
&= (-1)^{pn}\eta \cup 1 \\
&= (-1)^{pn}\eta
\end{aligned}$$

□

A.4.6 Remark. Let M be an R-oriented topological n-manifold and $A \subseteq M$ an R-oriented closed a-submanifold. By the above and by [10, Proposition 3.46], for every $b \in H_a(A; R)$ there is a unique element $\phi_A^M(b) \in H^{n-a}(M, M - A; R)$ with the following property:

For every open neighbourhood V of A the homomorphism

$$H^{n-a}(M, M \setminus A; R) = H^{n-a}(V, V \setminus A; R) \xrightarrow{D_A^V} H_a(V; R)$$

maps $\phi_a^M(b)$ to $(\iota_V)_*(b)$ where $\iota : A \to V$ is the inclusion, and we have an isomorphism

$$\phi_A^M : H_a(A; R) \xrightarrow{\cong} H^{n-a}(M, M \setminus A; R).$$

A.4.7 Definition (Thom class)**.**
Let $[A] \in H_a(A; R)$ be the fundamental class of A. The class

$$\phi_A^M([A]) \in H^{n-a}(M, M \setminus A; R)$$

from A.4.6 is called the *Thom class* of A in M.

A.4.8 Proposition (Orientation cover, [15, Chapter 20, Section 6])**.**
Let M be a connected manifold of dimension n. Then there is a 2-fold cover $p : \tilde{M} \to M$ such that $\tilde{M}$ is connected if and only if M is not orientable.

Proof. Define $\tilde{M}$ to be the set of pairs (x, α), where $x \in M$ and where $\alpha \in H_n(M, M \setminus x) \cong \mathbb{Z}$ is a generator. Define $p(x, \alpha) = x$. If $U \subset M$ is open and $\beta \in H_n(M, M \setminus U)$ is a fundamental class of M at U, define

$$\langle U, \beta \rangle = \{(x, \alpha) \mid x \in U \text{ and } \beta \text{ maps to } \alpha\}.$$

The sets $\langle U, \beta \rangle$ form a base for a topology on $\tilde{M}$. In fact, if $(x, \alpha) \in \langle U, \beta \rangle \cap \langle V, \gamma \rangle$, we can choose a coordinate neighbourhood $W \subset U \cap V$ such that $x \in W$. There is a unique class $\alpha' \in H_n(M, M \setminus W)$ that maps to α, and both β and γ map to α'. Therefore

$$\langle W, \alpha' \rangle \subset \langle U, \beta \rangle \cap \langle V, \gamma \rangle.$$

Clearly p maps $\langle U, \beta \rangle$ homeomorphically onto U and

$$p^{-1}(U) = \langle U, \beta \rangle \cup \langle U, -\beta \rangle.$$

Therefore $\tilde{M}$ is an n-manifold and p is a 2-fold cover. Moreover, $\tilde{M}$ is oriented. Indeed, if U is a coordinate chart and $(x, \alpha) \in \langle U, \beta \rangle$, then the following maps all induce isomorphisms on passage to homology:

$$\begin{tikzcd}
(\tilde{M}, \tilde{M} \setminus \langle U, \beta \rangle) \arrow[d] & (M, M \setminus U) \arrow[d] \\
(\tilde{M}, \tilde{M} \setminus (x, \alpha)) & (M, M \setminus x) \\
(\langle U, \beta \rangle, \langle U, \beta \rangle \setminus (x, \alpha)) \arrow[u] \arrow[r, "p", "\cong"'] & (U, U \setminus x). \arrow[u]
\end{tikzcd}$$

Via the diagram, $\beta \in H_n(M, M \setminus U)$ specifies an element $\tilde{\beta} \in H_n(\tilde{M}, \tilde{M} \setminus \langle U, \beta \rangle)$, and $\tilde{\beta}$ is independent of the choice of (x, α). These classes are easily seen to specify an orientation of $\tilde{M}$. Essentially by definition, an orientation of M is a cross-section $s : M \to \tilde{M}$: if $s(U) = \langle U, \beta \rangle$, then these β specify an orientation. Given one section s, changing the signs of the β gives a second section $-s$ such that $\tilde{M} = \mathrm{im}(s) \amalg \mathrm{im}(-s)$, showing that $\tilde{M}$ is not connected if M is oriented. □

The theory of covering spaces (see [15, Chapter 3] or [9] for example) gives the following consequence.

A.4.9 Corollary. *If M is simply connected, or if $\pi_1(M)$ contains no subgroup of index 2, then M is orientable. If M is orientable, then M admits exactly two orientations.*

Proof. If M is not orientable, then $p_*(\pi_1(\tilde{M}))$ is a subgroup of $\pi_1(M)$ of index 2. □

A.4.10 Remark. We can use homology with coefficients in a commutative unital ring R to construct an analogous R-orientation cover; it depends on the units of R. For example, if $R = \mathbb{Z}_2$, then the R-orientation cover is the identity map of M since there is a unique unit in R. This reproves the obvious fact that any manifold is $\mathbb{Z}_2$-oriented. The evident ring homomorphism $\mathbb{Z} \to R$ induces a natural homomorphism $H_*(X; \mathbb{Z}) \to H_*(X; R)$, and we see immediately that an orientation of M induces an R-orientation of M for any R.

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Abstract (English)

We give a geometric construction of characteristic classes, the Stiefel-Whitney classes for real vector bundles and the Chern classes for complex vector bundles. To achieve this, we consider the trivial bundle of rank $n - i + 1$, a smooth vector bundle ξ over a differentiable manifold M, and a generic vector bundle morphism $h : \varepsilon^{n-i+1} \to \xi$ between them. The singularity subsets of h are the image of a suitable projection map $\phi : \tilde{Z}(h) \to M$, where $\tilde{Z}(h)$ is a compact, K_b-oriented, differentiable manifold. We look at the image of the fundamental class of $\tilde{Z}(h)$ under the composite of the homological map induced by ϕ and the Poincaré duality isomorphism and define it as the ith characteristic class of ξ. We show equality of these classes with the axiomatic definition of the Stiefel-Whitney and Chern classes respectively given by Hirzebruch; hence, they are well-defined and unique. Finally, we generalise the definition to numerable vector bundles.

Abstract (German)

Wir geben eine geometrische Konstruktion von charakteristischen Klassen an, den Stiefel-Whitney Klassen für reelle Vektorbündel und den Chern Klassen für komplexe Vektorbündel. Dazu betrachten wir das triviale Bündel von Rang $n-i+1$, ein glattes Vektorbündel ξ über einer differenzierbaren Mannigfaltigkeit M und einen generischen Vektorbündelhomomorphismus $h : \varepsilon^{n-i+1} \to \xi$. Die Menge von Punkten, wo h nicht injektiv ist, ist das Bild einer bestimmten Projektion $\phi : \tilde{Z}(h) \to M$, wobei $\tilde{Z}(h)$ eine kompakte, K_b-orientierte, differenzierbare Mannigfaltigkeit ist. Wir kombinieren den von ϕ in der Homologie induzierten Morphismus mit dem Poincaré Dualitätsisomorphismus und definieren das Bild der Fundamentalklasse von $\tilde{Z}(h)$ unter dieser Zusammensetzung als die $i$te charakteristische Klasse von ξ. Dann zeigen wir, dass diese Klassen der axiomatischen Definition von Stiefel-Whitney Klassen und Chern Klassen, die Hirzebruch gegeben hat, genügen; sie sind daher wohldefiniert und eindeutig. Schließlich verallgemeinern wir die Definition auf abzählbare Vektorbündel.

Curriculum Vitae

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