DISSERTATION

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On some aspects of nonlinear water wave theory

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Abstract

This thesis is concerned with mathematical models for gravity water waves. I discuss applications of the full nonlinear governing equations for water waves with vorticity to tsunami-wave phenomena, as well as nonlinear dispersive equations which arise as approximations of the Euler equations for homogeneous, inviscid and incompressible fluids with a free boundary.

Regarding mathematical aspects of tsunami waves, I focus on the fluid dynamics in coastal regions and analyse the fluid motion prior to the arrival of a tsunami, which is due to the presence of currents beneath the surface. I present a model describing the state of the sea near the shore line in the absence of waves, the so-called background flow field, which is governed by the Euler equations. Rewriting the equations of motion in terms of a stream function and applying methods of dynamical systems, I prove existence of radially symmetric C^2-solutions of the governing equations with compact support in the fluid domain for a given family of vorticity distributions and a fixed sea bed. These solutions correspond to isolated regions of non-zero vorticity beneath the flat free surface outside of which the water is at rest in the absence of waves.

Furthermore, I study an equation for surface waves of moderate amplitude which has been recently derived as an approximation of the Euler equations in the shallow water regime. I focus on traveling wave solutions, i.e. unidirectional waves moving at constant speed without change of shape. I prove existence of solitary traveling waves, which, far out in the sea, decay to the undisturbed water level at zero. I present a qualitative description of the wave profile, showing that it has a unique maximum and is symmetric with respect to the crest. Moreover, I prove that faster moving waves are taller and that two wave profiles moving at different speeds intersect at precisely two points. A generalisation of this result is obtained when the assumption that solitary waves need to return to the zero water level is loosened, and one allows for a decay to an arbitrary constant far out. Rewriting the equation corresponding to traveling waves as a planar autonomous Hamiltonian system allows one to explicitly determine bounded orbits in the phase plane, which correspond to solitary and periodic traveling waves. In this way, I prove existence of traveling waves of elevation as well as depression for all wave speeds. Furthermore, I study in detail how the wave amplitude changes with the wave speed, and also prove existence of a family of solitary waves which have compact support. This approach is in fact applicable to a wider class of nonlinear dispersive evolution equations, which I exemplify by proving the existence of traveling waves of the well-known Camassa–Holm equation, including peaked continuous solitary waves.

Contents

Introduction

The study of water waves is an excitingly rich and complex area of research with a long history of contributions by mathematicians as well as physicists and engineers, incorporating both theoretical and experimental aspects of water wave phenomena. Despite the joint effort of generations of scientists, the picture their thorough investigations paint is far from complete. It is not just the profound fascination of the physical observation of water waves and the beauty of abstract mathematical theory arising from these observations, but also the multitude of unresolved issues and challenging questions that incite scientists to this day to further illuminate the subject.

This doctoral thesis is concerned with a variety of aspects of nonlinear water wave theory. It is composed of publications and preprints that I have written during my PhD studies at the Faculty of Mathematics at the University of Vienna, and may be divided into two parts. In the first part, presented in the research articles [29] and [28], I discuss some mathematical aspects of tsunami waves. More precisely, I describe the state of the sea in coastal regions prior to the arrival of tsunami waves and use dynamical systems methods to model isolated regions of vorticity in the background flow field governed by the Euler equations for water waves. In the second part of this thesis, laid out in the papers [30] and [26], I focus on an equation for surface waves of moderate amplitude which arises as an approximation of the governing equations in the shallow water regime. I prove existence of solitary and periodic traveling wave solutions as well as some qualitative features, including symmetry and decay properties. Furthermore, I study the dependence of the wave height on the speed and show the existence of a variety of solitary waves.

Before describing the results mentioned above in detail, let me give a brief introduction to the mathematical models that are commonly employed in the analysis of water waves. They are based on a number of simplifying assumptions regarding the fluid and the physical quantities that play a role in the equations of motion. In what follows, I assume that the water is inviscid, incompressible and that it has constant density. Furthermore I restrict our attention to gravity water waves, meaning that the only external force relevant for the propagation of waves is due to the gravitational acceleration g. I focus on a two dimensional fluid velocity field denoted by $\mathbf{u} = (u, v)$ which depends on space $\mathbf{x} = (x, y)$ and time t. The fluid domain Ω is bounded from below by the impermeable seabed $b(x)$ and bounded from above by the one dimensional free surface $\eta(x, t)$, which describes the elevation of the water waves above the bed. The equations governing the motion of the fluid are taken to be Euler's equations which arise from Newton's second law of motion. Based on the assumption that mass is neither generated nor destroyed anywhere in the fluid, I employ the equation of mass conservation. Furthermore, I impose kinematic and dynamic boundary conditions on the surface and the bottom of the fluid. They decouple the motion of the air from that of the water, and ensure that there is no flow through the bed as well as that fluid particles do not leave the fluid domain through the surface (see [11] and [42] for a justification of these assumptions in the context of inviscid homogeneous fluid flows for gravity water waves).

The first part of this thesis is concerned with mathematical models for tsunami waves. The Japanese word "tsunami" means"harbour wave" and denotes an extraordinary example of a gravity water wave, or rather, a succession of wave crests and troughs. Tsunami are typically generated by earthquakes, when a rapid vertical movement of the Earth's crust causes an abrupt disturbance of the column of water situated above the fault line which gives rise to the initial displacement of the wave, cf. [3]. The size and strength of a tsunami depends on the speed of the displacement, the magnitude of the earthquake, the size and shape of the rupture zone, as well as the water depth over the rupture zone. These fault regions, where oceanic plates collide with continental plates, are approximately elliptical with the major axis as long as $500km$. After the initial disturbance has been generated, the tsunami waves spread out from the source and may travel over enormous distances with little or no loss of energy and speed, cf. [31]. Their propagation is mostly unidirectional since the majority of the tsunami's energy is transmitted at right angles to the major axis of the fault region. Therefore, it is reasonable to use two-dimensional models for the description of tsunami wave propagation. Another extraordinary feature of tsunami is their wave length of hundreds of kilometers, which is very long compared to wind generated waves. Furthermore, their propagation speed may reach $800km/h$, while the wave height generally does not exceed $1m$, cf. [44]. Only when a tsunami approaches the shore do the waves pile up horizontally, due to the fact that the tsunami is slowed down as it enters regions of increasingly shallow water, while the back of the wave is still moving at very high speed out in the open ocean. The enormous amounts of water involved in this process and the sudden release of wave energy near the shore may cause local run-up and wave heights of tens of meters with devastating effects for coastal areas, as was the case in the 1960 Chilean tsunami, the Boxing Day tsunami in Thailand 2004 and the tsunami that hit the coast of Japan in March 2010, cf. [13, 14, 18]. Although it lies beyond the scope of science to avoid such catastrophes, a detailed analysis of the wave motion may help to obtain a better understanding of this phenomenon to eventually design accurate early warning systems.

Mathematical tsunami models face various exciting challenges, as they have to reconcile the dynamics of a tsunami's generation with the propagation in the far field and its arrival at the shore. In the following, we focus on the fluid dynamics near the coast and analyse the various states an ocean might exhibit prior to the arrival of a tsunami. Even in the absence of waves, there is considerable water motion due to the presence of currents beneath the flat surface, cf. [9] and [17]. Therefore, it seems essential in a reasonable tsunami model to include a background flow field describing the fluid motion in the absence of waves. In [15], Constantin and Johnson propose a tsunami model which includes a description of various states the ocean might exhibit prior to the arrival of tsunami waves. In the shallow water regime, they obtain a variety of vorticity distributions, confined to a region or extending over the entire fluid domain, which might enhance or supress the evolution of tsunami waves near the shore. However, none of these vorticity distributions allows for a flat surface, as this would invalidate even the most simple choice of constant non-zero vorticity throughout the flow field. To remedy this shortcoming, Constantin proves the existence of non trivial solutions for the exact equations governing the background flow with a flat surface [10]. For a particular choice of vorticity function, these solutions model isolated regions of vorticity in near shore regions prior to the arrival of waves. In [29], I discuss a generalization of the latter result by proving that there exist non-trivial solutions for a whole family of vorticity distributions (including the one presented in [10]). My approach is similar to the one pursued in [10] and, although not simple, is based only on the standard theory of ordinary differential equations and a dynamical systems approach. Since the evolution of tsunami waves can be reasonably modelled in a two-dimensional setting, one can write the equations of motion in terms of a stream function ψ for a given vorticity distribution $\gamma(\psi)$ and a fixed sea bed profile. The isolated regions of non-zero vorticity then correspond to radially symmetric C^2-solutions of the governing equations with compact support in the fluid domain. In [28], I was able to generalize the previous result by simplifying the existence and uniqueness proof therein, thus allowing for an even wider class of vorticity functions. Furthermore, an alternative proof of the fact that the solutions have compact support is presented under the additional but realistic assumption that the water is at rest close to the sea bed and far out in the open ocean.

The second part of this doctoral thesis is concerned with an equation for surface waves of moderate amplitude, which arises as an approximation to the Euler equations in the shallow water regime. Since the exact governing equations for water waves have proven to be nearly intractable (Gerstner waves being the only known non-trivial explicit solutions to the full equations, cf. [27] and [8]), the quest for suitable simplified model equations was initiated at the earliest stages of the development of hydrodynamics. To this end, the variables are nondimensionalised and scaled using appropriate reference quantities (cf. [32] for a detailed discussion). Depending on the ratio between the wavelength λ, the amplitude a and the water depth h, two parameters arise naturally in the problem:

$$\varepsilon = \frac{a}{h} \quad \text{and} \quad \delta = \frac{h}{\lambda},$$

called amplitude and shallowness parameter, respectively. These nondimensional quantities can be used to characterize various physical regimes according to their respective size. Furthermore, model equations can be derived by means of asymptotic expansions in terms of ε and δ, which serve as a basis to construct approximate solutions to the full governing equations, cf. [15, 16]. Many competing models are being proposed to this day, evolving from an analysis confined to linear theory which dominated most studies until the early twentieth century, cf. [20]. These models, based on linearizations of the governing equations, are applicable only for waves which are small perturbations of the flat surface. To gain insight into phenomena like wave breaking or solitary waves, a number of nonlinear models have been proposed, among them the prominent Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{KdV}$$

where $u(t, x)$ denotes the horizontal fluid velocity component, cf. [37]. It was derived in 1895 by means of an approximation procedure from the governing equations as a model for long waves of small amplitude in shallow water. In terms of the parameters defined above, the small-amplitude shallow water regime is characterized by the relation $\varepsilon = O(\delta^2)$ where $\delta \ll 1$. In accordance with experimental work on long waves by Scott Russel, Korteweg and de Vries put earlier considerations by Rayleigh and Boussinsq on a rigorous basis, cf. [5, 43]. Solitary wave solutions of the KdV equation capture soliton behaviour (cf. [25] and [38]) due to the inclusion of higher order nonlinear terms in the asymptotic expansion. The equation is completely integrable, as it exhibits a bi-Hamiltonian structure and therefore has an infinite number of conservation laws, cf. [39]. In the far field, there is a balance between dispersive and nonlinear terms which is responsible for the fact that the KdV equation is not suitable for modeling wave breaking. Furthermore, waves of higher amplitudes, whose nonlinear character is stronger than the dispersive effects, are not captured appropriately by this model. To account for these phenomena it is natural to investigate waves of moderate amplitude in shallow water. In this regime, characterized by the relation $\varepsilon = O(\delta)$ and $\delta \ll 1$, the Camassa–Holm (CH) equation

$$u_t + 2\kappa u_x + 3uu_x - u_{xxt} = 2u_x u_{xx} + uu_{xxx}, \tag{CH}$$

has attracted much attention in recent years. It was originally introduced by Fokas and Fuchssteiner in a paper on hereditary symmetries in 1981, cf. [24], and rederived by Camassa and Holm as a model for shallow water waves by means of asymptotic expansions in the Hamiltonian of the Euler equations, cf. [6] and [7]. It is a completely integrable equation whose solitary waves are smooth solitons when $\kappa > 0$, while in the limiting case where $\kappa = 0$ they are peaked solitons, cf. [34] and [6]. With the objective of showing the relevance of the CH equation (and also the related Degasperis-Procesi (DP) equation, cf. [21]) as models for the propagation of shallow water waves, Johnson demonstrates that in irrotational water of constant depth, the horizontal velocity component at a certain depth beneath the free surface is indeed described by a CH equation (cf. [33], and also [22] for a similar approach). Constantin and Lannes put earlier asymptotic procedures carried out in [33] on a mathematically rigorous footing, proving that in the Camassa-Holm scaling $\varepsilon = O(\delta)$ and $\delta \ll 1$, the Johnson equation

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \frac{\delta^2}{12}u_{xxx} - \frac{\delta^2}{12}u_{xxt} + \frac{7\varepsilon\delta^2}{24}u u_{xxx} + \frac{4\varepsilon\delta^2}{3}u_x u_{xx} = 0 \tag{J}$$

for the vertically averaged velocity component $u(x,t)$, coupled with an expression for the free surface η, provides a good approximation to the governing equations of water waves, cf. [16]. Furthermore, they show how this equation can be generalized to a one-parameter family of equations which approximate the governing equations in accuracy of order $O(\delta^4)$ (this is similar to the way the KdV is generalized by the Benjamin-Bona-Mahoney (BBM) equation, cf. [2]). However, none of these equations is integrable unless one considers the horizontal velocity at a specific depth in the fluid instead of the averaged horizontal velocity component. In this case, the resulting two-parameter family of equations can still be used to construct approximations of the governing equations of water waves. It turns out that among these equations there are precisely two which are integrable: CH at depth $\rho = 1/\sqrt{2}$ and DP at depth $\rho = \sqrt{23}/6$, cf. [16]. Unlike the KdV equation, which serves as a model for the fluid velocity as well as for the free surface of waves, the CH equation is valid for the velocity at a certain depth inside the fluid domain. In search of a corresponding equation for the free surface, the authors follow up on earlier considerations by Johnson (cf. [33]) and show that, using an expression for η from the asymptotic expansion in Johnson's equation (J), one obtains

$$\begin{aligned}\eta_t + \eta_x + &\frac{3}{2}\varepsilon\eta\eta_x - \frac{3}{8}\varepsilon^2\eta^2\eta_x + \frac{3}{16}\varepsilon^3\eta^3\eta_x \\ &+ \frac{\delta^2}{12}\eta_{xxx} - \frac{\delta^2}{12}\eta_{xxt} + \frac{7\varepsilon\delta^2}{24}\Big(\eta\eta_{xxx} + 2\eta_x\eta_{xx}\Big) = 0,\end{aligned} \tag{JS}$$

an equation for surface waves of moderate amplitude in shallow water, which approximates the governing equations, to the order $O(\delta^4)$, cf. [16].

The model equations described above form the basis and starting point of my investigations in the second part of this thesis, which are centered around traveling wave solutions of equation (JS). For the Korteweg-de Vries and Camassa-Holm equations, traveling wave solutions have been studied extensively in recent years, cf. [33, 41, 4, 46], and a detailed characterization of various types of traveling waves as weak solutions to the KdV and CH equations may be found in [40]. Regarding the equation for surface waves of moderate amplitude (JS), local well-posedness was first established by Constantin and Lannes in [16] and recently improved in [23], employing a semigroup approach for quasilinear hyperbolic equations due to Kato, cf. [35]. In addition Constantin and Lannes show that some of its classical solutions develop singularities in finite time in the form of wave breaking, e.g. the wave profile remains bounded but its slope becomes infinite, cf. [16, 12, 45]. Due to the intricate structure of (JS), a third order partial differential equation containing high order nonlinear terms, not a great deal is known so far about the global existence of solutions.

In the papers [30] and [26] I present some advances in this direction by proving existence of periodic and solitary traveling wave solutions of (JS). In the first article, I focus on the existence of solitary traveling waves, i.e. waves moving at constant speed c in one direction without changing their shape, and which decay to the flat surface far out. Using a traveling wave Ansatz the equation is rewritten as a planar autonomous system of ordinary differential equations. I employ methods of dynamical systems to determine the existence of a homoclinic orbit in the phase plane emerging from and returning to the origin, which corresponds to a solitary wave solution. It turns out that such an orbit exists for all wave speeds greater than one, which reflects the fact that solitary waves travel at supercritical speed (in accordance with the results in [36, 1, 19]). Because of the intricate structure of the equation, an explicit expression for the solution is out of reach. However, I describe the wave profile qualitatively and prove that it has a unique maximum and is symmetric with respect to the crest. Furthermore, I show that taller waves travel faster and that two wave profiles moving at different speeds intersect at precisely two points.

The follow up paper [26] is a joint work with Prof. Gasull from the Autonomous University of Barcelona (UAB). In this work, we loosen the assumption that solitary waves need to return to water level zero and allow for a decay to an arbitrary constant K. Essential in the existence proof is the observation that the corresponding equation for traveling waves of (JS) can be written as an autonomous planar Hamiltonian system. In fact, this observation holds in a more general context and may be used to analyse other nonlinear dispersive evolution equations in a similar way. As an example, we prove existence of traveling waves for the Camassa–Holm equation, including peaked continuous solitary waves. Owing to the fact that the Hamiltonian is constant along traveling wave solutions of (JS), we are able to explicitly determine closed orbits in the phase plane and find algebraic curves in terms of the parameters K and c, which provide bounds for the existence of periodic and homoclinic orbits corresponding to periodic and solitary waves, respectively. In this way, we find solitary waves traveling at all speeds $c > 0$, including waves of elevation and depression, and a family of solitary waves with compact support. Moreover, we extend the qualitative analysis of traveling wave solutions by analysing in detail how the wave amplitude changes with the wave speed. We determine algebraic curves and regions in terms of the aforementioned parameters, where the amplitude increases or decreases with the wave speed both for waves of depression and elevation.

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On some background flows for tsunami waves

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Abstract

With the aim to describe the state of the sea in a coastal region prior to the arrival of a tsunami, we show the existence of background flow fields with a flat free surface which model isolated regions of vorticity outside of which the water is at rest.

1 Introduction

Tsunami waves are generated by a sudden vertical displacement of a body of water on a massive scale, caused by landslides, volcanic eruptions or, most commonly, by undersea earthquakes [2]. Tectonic collisions in the form of thrust (or normal) faults sometimes make the ocean floor rise (or drop) by a few meters, causing the column of water directly above to rise (or fall) as well and thereby creating an initial wave profile of elevation (or of depression), as it was the case with the December 2004 tsunami cf. [16, 23, 9, 10, 4, 5]. Tsunami waves are a special type of gravity water waves, with typical wavelength of hundreds of kilometers, which can travel over thousands of kilometers at very high speed with little loss of energy, a spectacular example being the May 1960 tsunami that originated near the Chilean coast (due to the largest earthquake ever recorded) and propagated across the Pacific Ocean devastating coastal areas in Hawaii and Japan, $10000\,km$ respectively $17000\,km$ far from the Chilean coast [8, 24]. Away from the shore, where the ocean can be assumed to have uniform depth over large distances (e.g. the ocean floor of the Central Pacific Basin is relatively uniform, with a mean water depth of about $4300\,m$ cf. [8]), the evolution of the wave is governed essentially by linear water wave theory, the typical wave speed being $\sqrt{gh}$ with g the gravitational constant of acceleration and h the average depth of the sea [23, 11]. The amplitude of a tsunami wave out in the open sea is typically very small (roughly about $0.5\,m$ cf. [23]), but when it approaches a gently sloping beach the front of the wave slows down causing the water to pile up vertically, since the back of the wave is still hundreds of kilometers out in the sea, travelling at much higher speed. The enormous amounts of water involved in this process, account for much of the devastating effects tsunami waves have in coastal areas.

Before the arrival of the tsunami waves at the shore, the water in that region is unlikely to be still: even in the presence of surface waves of small amplitude or for a flat free surface, beneath the surface there could be considerable motion due to the presence of currents (already for irrotational flows with a free surface, an underlying uniform current complicates considerably the dynamics of the flow since without a current all particle paths describe a non-closed loop [3] whereas certain currents can produce closed particle paths [14]). Taking into account currents, it seems essential in a reasonable model for tsunami waves to allow for some kind of background flow field, which models the motion of water in the absence of waves. While most investigations are restricted to irrotational flows which model background states of still water, the possibility of incorporating pre-existing vorticity has only recently been studied in [10]. Various vorticity distributions were obtained in the shallow water regime and it was found that the requirement of a flat free surface is too restrictive, as it invalidates even the simple choice of constant non-zero vorticity throughout the flow field. As opposed to passing to the long wave limit and studying approximations for the shallow water regime, background flows that are governed by the full Euler equations and model isolated regions of vorticity outside of which the water is still have been only recently studied in [6], where a rigorous proof of the existence of a non-trivial solution to the equations governing such background flows which allow for a flat surface is given for a particular choice of vorticity distribution. The aim of the present work is to discuss a generalization of this result. While in [6] a special type of vorticity distribution was provided, we present a whole family of vorticity distributions (which includes that considered in [6]) admitting a vorticity region surrounded by still water.

2 Physical assumptions and the formulation of the problem

We can reasonably model the evolution of tsunami waves in a two dimensional setting, a simplifying assumption which is justified for the December 2004 tsunami off the coast of Indonesia [23] and the 1960 Chile tsunami [4]. The direction of propagation of tsunami waves was mainly perpendicular to the fault line, with the length of the rupture zone exceeding the wavelength, and the ocean depth over which the tsunami waves travelled was relatively uniform. Furthermore we assume the water to be inviscid and consider its density to be constant. As we are concerned with gravity water waves, we neglect surface tension. We want the model to admit a shoreline and assume that at the bottom we have a fixed impermeable bed. In Cartesian coordinates (x, y), let the origin be the intersection of the flat free surface and the seabed at the shoreline $x = 0$. Let the horizontal x-axis be in the direction of the incoming right-running waves and the vertical y-axis pointing upwards. We assume the fluid to extend to $-\infty$ in the negative horizontal direction and let the bed's topography for a gently sloping beach be given by the function $b(x)$ where $b(0) = 0$, $b(x) < 0$ for $x < 0$ and $b'(0) > 0$. In the open sea we assume uniform depth h_0 such that $b(x) = h_0$ for x far away from the shoreline $x = 0$. We will denote the fluid domain by $D = \{(x, y) \in \mathbb{R}^2 : x < 0,\, b(x) < y < 0\}$.

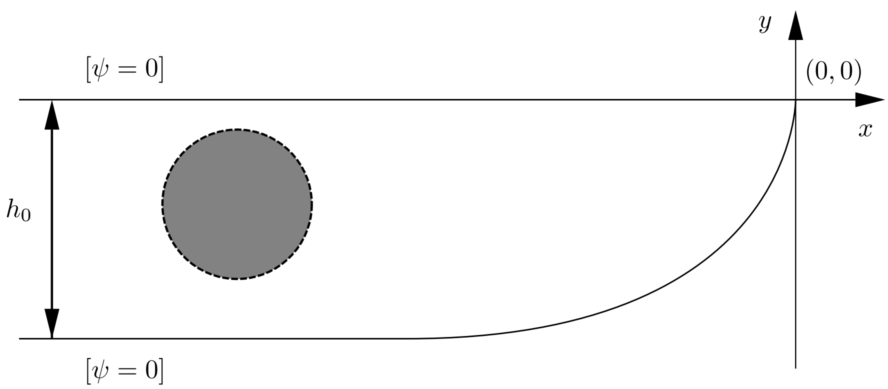

Figure 1: Fluid domain D with an isolated region of non-zero vorticity

In the two-dimensional setting we can introduce a stream function ψ, such that the fluid's velocity field is given by $(\psi_y, -\psi_x)$. We consider the vorticity ω to be a function of ψ, $\omega = \gamma(\psi)$, where γ is called vorticity function. Clearly $\omega = \gamma(\psi)$ specifies a vorticity distribution throughout the flow and notice that in the absence of stagnation points (that is, points where $\nabla\psi = (0,0)$), one can prove that the vorticity distribution is specified by means of a vorticity function cf. the discussion in [13, 7]. The equations governing a background state with flat free surface can be reformulated in terms of ψ as

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } D, \\ \psi = \psi_y = 0 & \text{on } y = 0, \\ \psi = 0 & \text{on } y = b(x), \end{cases} \tag{2.1}$$

given a vorticity distribution γ and the bottom profile b of the fluid domain D. For a detailed discussion of how these equations governing the fluid motion can be derived from the principle of mass conservation and the Euler equations we refer to [13] and [10].

Our aim is to show existence of an isolated region of non-zero vorticity in the fluid domain, outside of which the water is at rest (see Figure 1). That is, we have to find a suitable vorticity distribution γ and prove that (2.1) has a non-trivial radially symmetric solution with compact support in D.

Radial solutions are obtained via the Ansatz

$$\psi(x,z) = \psi(r) \text{ with } r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \text{ for } (x_0,y_0) \in D,$$

turning the system (2.1) into the semi-linear second order differential equation

$$\psi'' + \frac{1}{r}\,\psi' = -\gamma(\psi), \qquad r > 0, \tag{2.2}$$

where $'$ denotes the derivative with respect to r. Note that for solutions with compact support the boundary conditions in (2.1) will be trivially satisfied, as $\psi \equiv 0$ outside some compact region. To be able to uniquely determine a solution to (2.2), we have to specify initial values for ψ and ψ' at $r = 0$, say

$$(\psi(0), \psi'(0)) = (a, 0). \tag{2.3}$$

We require $\psi'(0) = 0$ to produce classical solutions.

The boundary value problem (2.1) is over-determined and it is expected that a non-trivial solution will only exist for certain classes of functions γ. The fact that our model admits a shoreline and we require the water to be still outside the region of vorticity imposes restrictions on the regularity of γ. For linear vorticity functions $\gamma(\psi) = a\psi + b$ it can be shown (see [10]) that system (2.1) admits only trivial solutions. The argument relies mainly upon maximum principles and the fact that the streamlines of the flat free surface and the seabed intersect at the shoreline and are equal to zero.

The argument uses essentially the restrictive boundary conditions on the flat free surface and the seabed and follows from maximum principles.

We can therefore not hope to find non-trivial solutions with compact support inside a circular boundary for a linear vorticity distribution as suggested for example in [1], since these arise in the context of an unbounded fluid which is at rest at infinity. As we are interested in classical solutions, γ has to be at least continuous. However, requiring $\gamma \in \mathcal{C}^1$ precludes radially symmetric solutions with compact support in the fluid domain, since we could find a value $T > 0$ sufficiently large, such that $\psi(T) = \psi'(T) = 0$. Then, by the backward uniqueness property [15] for (2.2) with $\gamma \in \mathcal{C}^1$, we would have $\psi(r) \equiv 0$ for all values of $r > 0$. The considerations made in [12] and [6] lead us to consider the vorticity function

$$\gamma(\psi) = \begin{cases} \psi - \psi|\psi|^{-\alpha} & \text{for } \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases} \qquad \alpha \in (0,1). \tag{2.4}$$

We now state the main result of this paper.

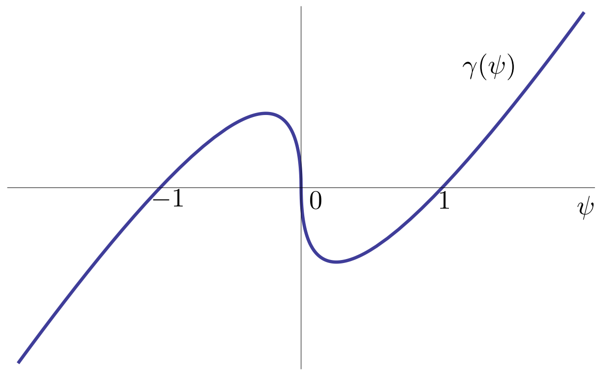

Figure 2: The vorticity function $\gamma(\psi)$ when $\alpha = \frac{1}{2}$.

Theorem 2.1. *For vorticity functions of type (2.4) there exists $a > 0$ such that* (2.2)-(2.3) *has a non-trivial, $\mathcal{C}^2$-solution ψ with compact support on $[0, \infty)$. This models a background state in the fluid domain D with an isolated region of non-zero vorticity outside of which the water is still.*

3 Proof of the main result

Instead of solving the second order initial value problem (2.2)-(2.3), consider the equivalent planar system of first order ordinary differential equations

$$\begin{cases} \psi' = \beta, \\ \beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}, \end{cases} \qquad r > 0, \tag{3.1}$$

with initial values

$$\psi(0) = a, \qquad \beta(0) = 0. \tag{3.2}$$

Define

$$a_\alpha := \left(\frac{2}{2-\alpha}\right)^{\frac{1}{\alpha}} > 1 \quad \text{and} \quad M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \le \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \le \alpha < 1. \end{cases} \tag{3.3}$$

The proof of Theorem 2.1, using a dynamical system approach which relies upon basic theory of ordinary differential equations, follows essentially from the results of the following two propositions, which we will prove in Sections 3.1 and 3.2, respectively.

Proposition 3.1. *For all $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to* (3.1)-(3.2) *which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$. Furthermore the solution satisfies $\psi > 1$ for $r \in [0, 1]$.*

Proposition 3.2. *There exists $a > M_\alpha > a_\alpha$ such that for the corresponding solution (ψ, β) of* (3.1)-(3.2) *there is a finite value $T > 0$ with $\psi(T) = \beta(T) = 0$.*

Proof of Theorem 2.1 By virtue of Propositions 3.1 and 3.2 there exists a value of $a > M_\alpha > a_\alpha$ such that for the corresponding uniquely defined $\mathcal{C}^2$-solution to (3.1)-(3.2), we can find $T > 0$ such that $\psi(T) = \beta(T) = 0$. Then by setting $\psi(r) = 0$ for $r \geq T$

we obtain a compactly supported solution of (2.2) defined for all $r \geq 0$. Furthermore, recall from Proposition 3.1 that $\psi(r) > 1$ for $r \in [0,1]$, which in view of (2.4) yields $\omega = \gamma(\psi) > 0$. Since ψ has compact support, we obtain an isolated region of non-zero vorticity ω which contains a ball of unit radius where $\omega > 0$, outside of which the water is at rest.

3.1 Proof of Proposition 3.1

We claim that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (3.1)-(3.2) which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$ and for which $\psi > 1$ for $r \in [0,1]$.

This is not immediately clear for two reasons:

- the right hand side of (3.1) displays a discontinuity at $r = 0$, so the system is not a classical initial value problem.
- since the vorticity function $\gamma(\psi)$ fails to be locally Lipschitz when $\psi = 0$ the right hand side of (3.1) is not locally Lipschitz and we cannot apriori expect uniqueness of solutions or continuous dependence on initial data from the standard theory of ordinary differential equations.

In the first part of the proof, summed up in Lemma 3.3, we consider the system in the vicinity of the discontinuity, for $r \in [0,1]$. By a simple change of variables (3.6) we overcome the problem of the discontinuity and solve the equivalent system (3.4) using an integral Ansatz and Banach's fixed point theorem. We ensure continuous dependence of solutions on the initial data $(a, 0)$ and find that the solutions of the integral equation (3.7) are always greater than one. In Lemma 3.4 we introduce an important functional (3.14) which decreases along solutions and will be helpful in deriving results throughout the proofs of both Proposition 3.1 and 3.2 as it ensures global existence of solutions. In Lemma 3.5 we tackle the second part of the proof by analyzing the system away from the discontinuity. The difficulty in this case lies in the fact that the right hand side of (3.1) fails to be locally Lipschitz continuous whenever $\psi = 0$. By rewriting the system in polar coordinates we obtain another equivalent formulation (3.16), for which existence and uniqueness of solutions as well as continuous dependence on initial data follows from standard results whenever the right hand side is $\mathcal{C}^1$. In the vicinity of points where $\psi = 0$ an application of the inverse function theorem yields yet another local reformulation (3.20), shifting the lack of Lipschitz continuity in the dependent variable for (3.16) to the independent variable for the new system and thereby gaining $\mathcal{C}^1$-regularity of the dependent variable for (3.20). We thus obtain local uniqueness and continuous dependence also at points where the right hand side of (3.16) fails to be Lipschitz.

Lemma 3.3. *For $r \in [0,1]$ system* (3.1) *can be equivalently written as*

$$v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \geq 0, \tag{3.4}$$

where the initial values (3.2) *are described by the limits*

$$v(s) \to a \quad \text{and} \quad v'(s)e^s \to 0 \quad \text{for } s \to \infty. \tag{3.5}$$

Equation (3.4) *has a unique $\mathcal{C}^2$-solution which depends continuously on the parameter a and with $v(s) > 1$ for $s \geq 0$. In particular, this means that for $r \leq 1$,* (3.1)-(3.2) *has a unique $\mathcal{C}^2$-solution (ψ, β) which depends continuously on a and is such that $\psi(r) > 1$ for $r \in [0,1]$.*

Proof. We perform the change of variables

$$s = -\ln r, \qquad \psi(r) = v(s), \tag{3.6}$$

and find that (3.1) is equivalent to

$$v'' + e^{-2s}(v - v|v|^{-\alpha}) = 0, \qquad s \in \mathbb{R}.$$

This follows from the fact that (3.1) is equivalent to (2.2) which in view of

$$\psi'' + \frac{1}{r}\psi' + \psi - \psi|\psi|^{-\alpha} = v''\frac{1}{r^2} + v'\frac{1}{r^2} - \frac{1}{r^2}v' + v - v|v|^{\alpha} = 0$$

yields

$$v'' + r^2(v - v|v|^{\alpha}) = 0.$$

The restriction $0 \leq r \leq 1$ is equivalent to $s \geq 0$ in the new variable.

Let an arbitrary $a > a_\alpha$ be fixed. We can deal with local existence and uniqueness issues of a solution to (3.1)-(3.2) by considering the integral equation

$$v(s) = a - \int_s^\infty (\tau - s)e^{-2\tau}\left(v(\tau) - v(\tau)|v(\tau)|^{-\alpha}\right) d\tau, \qquad s \geq 0. \tag{3.7}$$

The corresponding asymptotic behavior (3.5) is ensured by

$$v'(s) = \int_s^\infty e^{-2\tau}\gamma(v(\tau))\, d\tau, \qquad s \geq 0, \tag{3.8}$$

since

$$\lim_{s\to\infty} v(s) = \lim_{s\to\infty} a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau = a$$

and

$$\lim_{s\to\infty} v'(s)e^s = \lim_{s\to\infty} \frac{\int_s^\infty e^{-2\tau}\gamma(v(\tau))d\tau}{e^{-s}} = \lim_{s\to\infty} \frac{e^{-2s}\gamma(v(s))}{e^{-s}} = \lim_{s\to\infty} e^{-s}\,\gamma(v(s)) = 0,$$

where we used the rule of de l'Hospital in the second equality.

As long as $v(s) \geq 1$ we have that v is non-decreasing, since $\gamma(v) \geq 0$ which in view of (3.8) gives $v'(s) \geq 0$. We even have

$$v(s) > a^{1-\alpha} > 1 \quad \text{for } s \geq 0. \tag{3.9}$$

Indeed, if this were not so, define $s_1 := \sup\{s \geq 0 : v(s) = a^{1-\alpha}\}$. Then for all $s \geq s_1$ we have $1 < a^{1-\alpha} \leq v(s) \leq a$, which in view of (3.7) yields a contradiction as

$$\begin{aligned} 0 < a - a^{1-\alpha} = a - v(s_1) &= \int_{s_1}^\infty (\tau - s_1)e^{-2\tau}\Big(v(\tau) - v(\tau)^{1-\alpha}\Big)d\tau \\ &\leq (a - a^{1-\alpha})\int_{s_1}^\infty (\tau - s_1)e^{-2\tau}d\tau = (a - a^{1-\alpha})\frac{e^{-2s_1}}{4} \leq \frac{(a - a^{1-\alpha})}{4}, \end{aligned}$$

in view of the fact that $\gamma(v)$ is strictly increasing for $v \in [a^{1-\alpha}, a]$. So for $s \geq 0$ we have that $v(s) > a^{1-\alpha} > 1$ is non-decreasing.

These considerations allow us to view the solution of the integral equation (3.7) as the unique fixed point of the contraction T_a defined by

$$T_a(v)(s) := a - \int_s^\infty (\tau - s)e^{-2\tau}(v(\tau) - v(\tau)|v(\tau)|^{-\alpha})d\tau, \qquad s \geq 0, \tag{3.10}$$

on the closed subspace $X_a := \{v \in X : a^{1-\alpha} \le v(s) \le a,\ s \ge 0\}$ of the Banach space X of bounded continuous functions on $[0,\infty)$ endowed with the supremum norm $\|v\| = \sup_{s\ge 0}\{|v(s)|\}$. To be able to apply Banach's contraction principle (in the following form: *For $F \subset X$ a closed subspace of a Banach space X, any contraction $T : F \to F$ has a unique fixed point*) to (3.10) and subsequently to the integral equation (3.7), we have to check the hypotheses.

Notice that for $v \in X_a$ we have $v \ge 1$, since $a > a_\alpha$ and thus $a^{1-\alpha} > (\frac{2}{2-\alpha})^{\frac{1-\alpha}{\alpha}} > 1$ for $0 < \alpha < 1$. Let us check that $T_a(v) \in X_a$, i.e.

$$a^{1-\alpha} \le a - \int_s^\infty (\tau - s)e^{-2\tau}\gamma(v(\tau))d\tau \le a, \qquad s \ge 0.$$

The upper bound follows from the fact that the integral is positive, since for $v \in X_a$, $v \ge 1$ and thus $\gamma(v) \ge 0$. For the lower bound, we use the same reasoning as in the proof of (3.9).

Now we show that T_a as defined above is a contraction. Since the vorticity function γ defined in (2.4) is $\mathcal{C}^1$ on $[1,\infty)$, by the mean value theorem (cf. [17]) there exists $\xi \in (v, w)$ for $v, w \ge 1$ such that $\gamma(v) - \gamma(w) = \gamma'(\xi)(v - w)$. This yields

$$|\gamma(v) - \gamma(w)| \le |v - w| \qquad \text{for } v, w, \ge 1, \tag{3.11}$$

since $\gamma'(\xi) \le 1$ for $\xi \ge 1$. Then for $s \ge 0$ we have

$$\begin{aligned}\Big|\int_s^\infty (\tau-s)e^{-2\tau}[\gamma(v(\tau)) - \gamma(w(\tau))]d\tau\Big| &\le \int_s^\infty (\tau-s)e^{-2\tau}|v(\tau) - w(\tau)|d\tau \\ &\le \|v-w\| \int_s^\infty (\tau-s)e^{-2\tau}d\tau = \frac{1}{4}\,\|v-w\|\,,\end{aligned}$$

whenever $v, w \in X_a$. Thus

$$\begin{aligned}\|T_a(v) - T_a(w)\| &\le \Big|\int_s^\infty (\tau - s)e^{-2\tau}\Big(\gamma(v(\tau)) - \gamma(w(\tau))\Big)d\tau\Big| \\ &\le \frac{1}{4}\,\|v - w\| \qquad \text{for } v, w \in X_a, s \ge 0,\end{aligned}$$

which shows that T_a is a contraction on X_a with contraction constant $K \le \frac{1}{4}$.
Therefore, according to Banach's contraction principle, T_a has a unique fixed point, i.e. the integral equation (3.7) has a unique solution $v \in X_a$ which is of class $\mathcal{C}^2$ since $v''(s) = -e^{-2s}(v(s) - v(s)^{1-\alpha})$ is continuous for $s \ge 0$.

To show continuous dependence of the solution on the parameter a, let $v_1 \in X_{a_1}, v_2 \in X_{a_2}$. Then the integral equation (3.7) yields, in view of (3.11), that for $s \ge 0$

$$\begin{aligned}|v_1(s) - v_2(s)| &\le |a_1 - a_2| + \int_s^\infty (\tau - s)e^{-2\tau}|v_1(\tau) - v_2(\tau)|d\tau \\ &\le |a_1 - a_2| + \frac{1}{4}\,\|v_1 - v_2\|\,,\end{aligned} \tag{3.12}$$

and therefore

$$\|v_1 - v_2\| \le \frac{4}{3}|a_1 - a_2|, \tag{3.13}$$

so we actually even obtain that the solution is stable, cf. [15]. □

Before we proceed to the case where $r \ge 1$, we prove the following useful

Lemma 3.4. *The function*

$$E(r) = E(\psi, \beta) = \frac{1}{2}\beta^2 + \frac{1}{2}\psi^2 - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \tag{3.14}$$

satisfies

$$E'(r) = -\frac{1}{r}\beta^2, \qquad r > 0, \tag{3.15}$$

as long as solutions to (3.1)-(3.2) *exist and remains bounded for all* $r > 0$*. Furthermore,* $E(r)$ *is strictly decreasing whenever* $(\psi(r), \beta(r)) \notin \{(0,0), (\pm 1, 0)\}$*. We conclude that solutions to* (3.1)-(3.2) *are defined for all* $r \geq 0$ *and that* ψ *and* β *are bounded functions of* r*.*

Proof. As long as a solution to (3.1)-(3.2) exists, we have $E'(r) = -\frac{1}{r}\beta^2$, since the derivative with respect to r of the function $E(r)$ given by (3.14) in view of (3.1) can be computed as

$$E'(r) = \psi\beta - \beta|\psi|^{-\alpha}\psi + \beta(-\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha}) = -\frac{1}{r}\beta^2.$$

Notice that E attains its minimum $E_{min} = \frac{\alpha}{2(\alpha-2)} < 0$ at $(\psi, \beta) = (\pm 1, 0)$, so that (3.15) ensures that E remains bounded. Furthermore,

$$\inf_{\psi \in \mathbb{R}} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \right\} = \frac{\alpha}{2(\alpha-2)} \quad \text{and} \quad \lim_{|\psi| \to \infty} \left\{ \frac{\psi^2}{2} - \frac{1}{2-\alpha}|\psi|^{2-\alpha} \right\} = \infty.$$

Therefore ψ and β remain bounded as long as solutions exist, since otherwise E would become unbounded. We conclude that the solutions to (3.1)-(3.2) are defined for all $r \geq 0$.

Let us now prove that $E(r)$ is strictly decreasing whenever $(\psi(r), \beta(r)) \notin \{(0,0), (\pm 1, 0)\}$. Otherwise, for $r_2 > r_1 > 0$ with $E(r_2) = E(r_1)$, we would have

$$0 = E(r_2) - E(r_1) = \int_{r_1}^{r_2} E'(r)dr = -\int_{r_1}^{r_2} \frac{\beta^2(r)}{r} dr.$$

This implies $\beta(r) = 0$ on $[r_1, r_2]$ and consequently from (3.1) we have that $\psi'(r) = \beta'(r) = 0$ for all $r \in [r_1, r_2]$. Thus, $\psi(r) = \psi(r_1) = \psi(r_1)|\psi(r_1)|^{-\alpha}$ is constant in $[r_1, r_2]$, so that $\psi(r) \in \{0, \pm 1\}$ in view of (3.1), a contradiction. ☐

Now we consider the system away from the discontinuity at $r = 0$ and prove existence, uniqueness and continuous dependence of solutions on the parameter a as long as $\psi^2 + \beta^2 > 0$.

Lemma 3.5. *For* $r \geq 1$ *system* (3.1) *can be equivalently reformulated as*

$$\begin{cases} \theta'(r) = -\dfrac{1}{2r}\sin(2\theta) - 1 + R^{-\alpha}|\cos(\theta)|^{2-\alpha}, \\ R'(r) = -\dfrac{1}{r}R\sin^2(\theta) + R^{1-\alpha}\sin(\theta)\dfrac{\cos(\theta)}{|\cos(\theta)|^{\alpha}}, \end{cases} \qquad r \geq 1. \tag{3.16}$$

As long as $R > 0$ *this system of first order differential equations has a unique* $\mathcal{C}^2$*-solution which depends continuously on the initial data* $(\theta(1), R(1))$*, which in turn depends continuously on the parameter* a*.*

Proof. We introduce polar coordinates

$$\psi = R\cos(\theta), \qquad \beta = R\sin(\theta), \tag{3.17}$$

to show that (3.16) is yet another equivalent formulation of (3.1):

$$\begin{aligned}
\theta'(r) &= \frac{d}{dr}\arctan\left(\frac{\beta(r)}{\psi(r)}\right) = \frac{\beta'\psi - \beta\psi'}{\psi^2+\beta^2} \overset{(3.1)}{=} \frac{-\frac{1}{r}\beta\psi - \psi^2 + |\psi|^{2-\alpha} - \beta^2}{\psi^2+\beta^2} \\
&\overset{(3.17)}{=} \frac{-\frac{1}{r}R^2\sin(\theta)\cos(\theta) - R^2\cos^2(\theta) + R^{2-\alpha}|\cos(\theta)|^{2-\alpha} - R^2\sin^2(\theta)}{R^2} \\
&= -\frac{1}{2r}\sin(2\theta) + R^{-\alpha}|\cos(\theta)|^{2-\alpha} - 1,
\end{aligned}$$

and

$$\begin{aligned}
R'(r) &= \frac{\psi'\cos(\theta) + \psi\sin(\theta)\theta'}{\cos^2(\theta)} \\
&= \frac{R\sin(\theta)\cos(\theta) + R\cos(\theta)\sin(\theta)(-\frac{1}{r}\sin(\theta)\cos(\theta) + R^{-\alpha}|\cos(\theta)|^{2-\alpha} - 1)}{\cos^2(\theta)} \\
&= -\frac{1}{r}\sin^2(\theta)R + R^{1-\alpha}\sin(\theta)\cos(\theta)|\cos(\theta)|^{-\alpha}.
\end{aligned}$$

The initial data $(\theta(1), R(1))$ is specified after solving the integral equation (3.7) on $[0,\infty)$.
To show continuous dependence of $(\theta(1), R(1))$ on a, notice that (3.8) in view of (3.11) and (3.13) yields for $s \geq 0$

$$\begin{aligned}
|v_1'(s) - v_2'(s)| &\leq \int_s^\infty e^{-2\tau}|\gamma(v_1(\tau)) - \gamma(v_2(\tau))|d\tau \\
&\leq \int_s^\infty e^{-2\tau}|v_1(\tau) - v_2(\tau)|d\tau \\
&\leq \frac{1}{4}\|v_1 - v_2\| \leq \frac{1}{3}|a_1 - a_2|.
\end{aligned} \tag{3.18}$$

Evaluating inequalities (3.12) and (3.18) at $s = 0$ together with (3.13) yields

$$|v_1(0) - v_2(0)| + |v_1'(0) - v_2'(0)| \leq \frac{5}{3}|a_1 - a_2|. \tag{3.19}$$

In view of the formulation (3.16) of the initial value problem (3.1)-(3.2) this means that ψ and $\psi' = \beta$ vary little at $r = 1$. Thus, $\theta(1) = \arctan\left(\frac{\beta(1)}{\psi(1)}\right)$ and $R(1) = \sqrt{\psi^2(1) + \beta^2(1)}$ depend continuously on a. The considerations we made in Lemma 3.3 show that $\psi(1) > a^{1-\alpha} > 0$ and $R(1) > 0$, so $\cos(\theta(1)) = \frac{\psi(1)}{R(1)} > 0$. As long as $R > 0$ and $\cos(\theta) > 0$ the right hand side of (3.16) is $\mathcal{C}^1$. Thus we get local existence and uniqueness as well as continuous dependence on initial data $(\theta(1), R(1))$ for a solution to (3.16) by standard results.

We show that, as long as $R > 0$, this holds true even if $\cos(\theta(r)) = 0$, that is, at points where a solution intersects the vertical axis in the (ψ, β)-phase plane. At such points, the right hand side of (3.16) is still continuous and bounded, but fails to be locally Lipschitz. Thus, while we can still rely for the local existence of solutions on the Cauchy–Peano theorem [15], uniqueness and continuous dependence on initial data on the other hand are no longer guaranteed. We overcome this problem by transforming the system in a neighborhood of such values of r, taking advantage of its local structure.

Denote by r_0 the smallest value of $r > 1$ where $\cos(\theta(r_0)) = 0$, say $\theta(r_0) = -\frac{\pi}{2}$. Since for $r \in (1, r_0)$ the right hand side of (3.16) is $\mathcal{C}^1$, the solution is unique and depends continuously on the initial data $(\theta(1), R(1))$ up to r_0. We then select one of the possible continuations of the solution across $r = r_0$ and show that this selection is unique and depends continuously on $(\theta(1), R(1))$ close to $r = r_0$. Since $\cos(\theta(r_0)) = 0$ and $\theta'(r_0) = -1$, the inverse function theorem (cf. [17]) guarantees the existence of neighborhoods $(r_0 - \varepsilon, r_0 + \varepsilon)$ of r_0 and $(-\delta, \delta^+)$ of 0 for sufficiently small $\varepsilon > 0$ and $\delta, \delta^+ > 0$, as well as a uniquely determined $\mathcal{C}^1$-function

$$\varphi(\tau) = r$$

such that $\varphi(0) = r_0$, $\varphi(-\delta) = r_0 - \varepsilon$ and $\varphi(\delta^+) = r_0 + \varepsilon$ which allows us to locally set

$$\cos(\theta(r)) = -\tau.$$

Notice that this transformation preserves the monotonicity of the respective independent variables r and τ, since $\theta' < 0$ and $\cos(\theta)$ is increasing in a neighborhood of $-\frac{\pi}{2}$. Thus $r_0 - \varepsilon < r_0 < r_0 + \varepsilon$ implies $\cos(\theta(r_0 - \varepsilon)) > \cos(\theta(r_0)) > \cos(\theta(r_0 + \varepsilon))$, or, equivalently, $\cos(\varphi(-\delta)) > 0 > \cos(\varphi(\delta^+))$, which in view of $\cos(\theta(\varphi(\tau))) = -\tau$ implies $-\delta < 0 < \delta^+$. Differentiating the equation $\varphi(\tau) = \varphi(-\cos(\theta(r))) = r$ with respect to $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ yields

$$\varphi'(\tau) = \frac{1}{\theta'(r)\sin(\theta(r))}, \qquad \tau \in (-\delta, \delta^+).$$

Setting

$$\rho(\tau) = R(r)$$

yields

$$\rho'(\tau) = R'(r)\,\varphi'(\tau), \qquad \text{for } r \in (r_0 - \varepsilon, r_0 + \varepsilon),\ \tau \in (-\delta, \delta^+).$$

Now we transfer (3.16) for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and $\tau \in (-\delta, \delta^+)$ into the system

$$\begin{cases} \varphi'(\tau) = \dfrac{1}{\sqrt{1-\tau^2} + \frac{\tau(1-\tau^2)}{\varphi(\tau)} - \rho(\tau)^{-\alpha}|\tau|^{2-\alpha}\sqrt{1-\tau^2}}, \\[2ex] \rho'(\tau) = -\dfrac{\frac{1}{\varphi(\tau)}\rho(\tau)\sqrt{1-\tau^2} - \rho(\tau)^{1-\alpha}\frac{\tau}{|\tau|^\alpha}}{-1 - \frac{\tau\sqrt{1-\tau^2}}{\varphi(\tau)} + \rho^{-\alpha}(\tau)|\tau|^{2-\alpha}}. \end{cases} \tag{3.20}$$

A straightforward calculation and the fact that

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2(-\sqrt{1-\cos^2(\theta)})\cos(\theta) = 2\tau\sqrt{1-\tau^2}$$

for $\varepsilon > 0$ small enough shows that (3.20) and (3.16) are equivalent.

The advantage of the system (3.20) with respect to (3.16) is that the lack of $\mathcal{C}^1$-regularity in θ was shifted into a lack of $\mathcal{C}^1$-regularity in τ. Consequently, the new system is $\mathcal{C}^1$ in the unknown variables $(\varphi, \rho) \in (1, \infty) \times (0, \infty)$ and continuous in the independent variable τ. This is enough to ensure uniqueness and continuous dependence on initial data $(\varphi(-\delta), \rho(-\delta))$ of the solutions to (3.20). Furthermore, $(\varphi(-\delta), \rho(-\delta))$ depends continuously on $(\theta(r_0 - \varepsilon), R(r_0 - \varepsilon))$ via the $\mathcal{C}^1$-function $\varphi(\tau) = r$ and we already mentioned the continuous dependence of solutions $(\theta(r), R(r))$ on $(\theta(1), R(1))$ and thus on the parameter a for $r \in (1, r_0)$. We can therefore deduce that uniqueness and continuous dependence on a of the solution to (3.16) holds also in a neighborhood of r_0.

This procedure can be repeated in almost the same way for the next value of $r > r_0$ where $\cos(\theta(r)) = 0$, i.e. where the solution intersects the vertical axis in the upper half plane. Then again an application of the inverse function theorem (cf. [17]) guarantees the existence of a $\mathcal{C}^1$-function $\varphi(\tau) = r$ such that we can locally set

$$\cos(\theta(r)) = \tau.$$

Notice that this time we choose $\cos(\theta(r)) = \tau$ instead of $-\tau$ to preserves monotonicity of the respective independent variables. As before, we transfer (3.16) into a system which differs from (3.20) only by a change of sign in the second equation. Thus by the same reasoning as above we deduce that uniqueness and continuous dependence on a of the solution to (3.16) also holds in neighborhoods of points where the solution intersects the vertical axis in the upper half plane. This procedure can be repeated for all values of r where the right hand side of (3.16) fails to be locally Lipschitz, as long as $R > 0$.
Summing up, we can say that for values of r where $\cos(\theta(r)) = 0$, that is, where $\theta(r) = -\frac{\pi}{2} + 2k\pi$ or $\theta(r) = \frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$, the above local transformations guarantee uniqueness and continuous dependence on a of the solution to (3.16) also in neighborhoods of such values as long as $R > 0$. In between these values of r, the right hand side of (3.16) is $\mathcal{C}^1$ and everything follows from standard results. □

This concludes the proof of Proposition 3.1, as we have seen that for any $a > a_\alpha$ there exists a unique $\mathcal{C}^2$-solution (ψ, β) to (3.1)-(3.2) for which $\psi > 1$ on $[0, 1]$ by virtue of (3.9) and which depends continuously on the initial data $(a, 0)$ on any compact interval on which $\psi^2(r) + \beta^2(r) > 0$.

3.2 Proof of Proposition 3.2

We show that there exists a value of $a > a_\alpha$ such that for the corresponding solution to (3.1)-(3.2) we can find some $0 < T < \infty$ with $\psi(T) = \beta(T) = 0$.
The idea is to perform a detailed qualitative analysis for the system (3.1)-(3.2), similar to the phase-plane analysis of autonomous systems. We introduce two sets $\Omega_\pm$ defined by the solution sets of the equation $E(\psi, \beta) = 0$, where E is the functional defined in Section 3.1. In Lemma 3.4 we show that for initial data a large enough the solution can enter the region $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. We find that the value of r at which the solution can enter $\Omega_\pm$ tends to infinity as $a \to \infty$. After that, Lemma 3.5 ensures that there exists an initial value a_+ such that the corresponding solution stays outside $\Omega_- \cup \Omega_+$ for all $r \geq 0$. Finally, in Lemma 3.6, we prove that for solutions corresponding to such initial data a_+ there exists a finite value $T > 0$ such that $E(T) = 0$, and therefore also $\psi(T) = \beta(T) = 0$.

Let us start with defining the sets $\Omega_\pm$. From (3.14) in Lemma 3.4 we have that

$$E(\psi, \beta) = 0 \quad \text{if and only if} \quad \beta^2 = \frac{2}{2-\alpha}|\psi|^{2-\alpha} - \psi^2.$$

In the plane (ψ, β) the set where $E < 0$ consists of the interiors $\Omega_\pm$ of the closed curves representing the solution set of the above equation. These curves are symmetrical with respect to the vertical and the horizontal axis and are tangential to one another and to the vertical axis at the origin. Note from (3.3) that the curves reach their maximum $\beta_\alpha = (\frac{\alpha}{2-\alpha})^{\frac{1}{2}}$ at $\psi = \pm 1$ and they intersect the horizontal axis at the points $\psi = 0$ and $\psi = \pm(\frac{2}{2-\alpha})^{\frac{1}{\alpha}} = \pm a_\alpha$. To get a better understanding of the dynamics of the system (3.1), consider the right half plane, where $\psi > 0$. At $\beta = 0$ we have $\psi' = \beta = 0$ and $\beta' = -\frac{1}{r}\beta - \psi + \psi|\psi|^{-\alpha} > 0$ when $\psi|\psi|^{-\alpha} > \psi$ which is true for $0 < \psi < 1$, whereas $\beta' < 0$ for $\psi > 1$. In the left half plane, we have exactly the opposite situation.

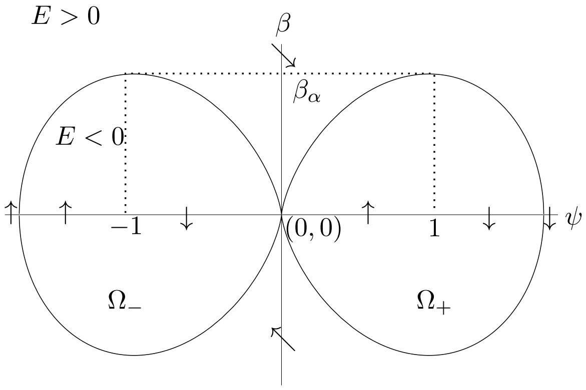

Figure 3: The solution set of $E = 0$ in the phase plane (ψ, β) with arrows indicating the dynamics of the system

Therefore, solutions intersect the horizontal axis perpendicularly from the upper to the lower half plane for $\psi > 1$ and for $-1 < \psi < 0$. On the complement of these sets, they intersect the axis in the opposite direction. For $\psi = 0$ and $\beta > 0$ we have that $\psi' > 0$ and $\beta' < 0$, which means that solutions intersect the vertical axis from left to right in the upper half plane. In the lower half plane, the opposite is true (see Figure 3).

By Lemma 3.4, E is strictly decreasing as long as $(\psi, \beta) \notin \{(0, 0), (\pm 1, 0)\}$. Therefore, once a solution reaches the boundary of $\Omega_\pm$ at a point other than $(0, 0)$ it will enter $\Omega_\pm$. Once inside, a solution will stay in either Ω_+ or Ω_- for all subsequent times, as E is strictly decreasing.

Recall from (3.3) that we defined

$$M_\alpha := \begin{cases} a_\alpha^{\frac{2}{\alpha^4}} & \text{for } 0 < \alpha \le \frac{1}{2}, \\ a_\alpha^{\frac{8}{(1-\alpha)\alpha^2}} & \text{for } \frac{1}{2} \le \alpha < 1. \end{cases}$$

For certain initial data, solutions stay outside of $\Omega_\pm$ for some time:

Lemma 3.4. *For $a > M_\alpha$ we have that $E(r) > 0$ as long as $r \in [0, \frac{2}{\alpha}]$. This means that a solution to* (3.1)-(3.2) *with $a > M_\alpha$ can enter $\Omega_\pm$ only for values of $r > \frac{2}{\alpha}$. Additionally, we find that the value of r such that a solution can enter $\Omega_\pm$ tends to infinity as $a \to \infty$.*

Proof. Let $a > a_\alpha^{\frac{2}{1-\alpha}}$. We know from the results in Lemma 3.3 that for $r \in [0, 1]$, $R(r) = \frac{\psi(r)}{\cos(\theta(r))} > a^{1-\alpha} > a_\alpha$. If a solution with initial data a enters the region $\Omega_+ \cup \Omega_-$, then for some value of $r^* > 1$ we will have $R(r^*) = a_\alpha < a^{\frac{1-\alpha}{2}}$. We can therefore define

$$\begin{aligned} r_0 &= \inf\{r > 0 : R(r) = a_\alpha\} \\ r_1 &= \sup\{0 < r < r_0 : R(r) = a^{1-\alpha}\} > 1 \\ r_2 &= \sup\{r_1 < r < r_0 : R(r) = a^{\frac{1-\alpha}{2}}\} \end{aligned} \tag{3.21}$$

so that

$$a^{1-\alpha} = R(r_1) \geq R(r) \geq R(r_2) = a^{\frac{1-\alpha}{2}} \qquad \text{for } r \in [r_1, r_2].$$

The argument leading to the desired result requires us to consider separately the case where $\alpha \in (0, \frac{1}{2}]$ and the case where $\alpha \in [\frac{1}{2}, 1)$. Some inequalities involving functions of α in the exponent will be denoted by (a)-(c) and will be shown at the end of the proof of this Lemma.

Let $\alpha \in (0, \frac{1}{2}]$. We claim that

$$r_2 \geq a^{\alpha^3}. \tag{3.22}$$

Indeed, assume to the contrary that $r_2 < a^{\alpha^3}$, then from (3.16) we infer that

$$R'(r) = -\frac{1}{r} R \sin^2(\theta) + R^{1-\alpha} \sin(\theta) \frac{\cos(\theta}{|\cos(\theta)|^\alpha} \geq -\frac{1}{r} R - R^{1-\alpha} > -\frac{2}{r} R.$$

In the last inequality we have used that $R(r) \geq a^{\frac{1-\alpha}{2}}$, thus for $\alpha \in (0, \frac{1}{2}]$ we have that $R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(a)}{\geq} a^{\alpha^3} > r_2 \geq r$ for $r \in [r_1, r_2]$, which yields $-R^{1-\alpha} > -\frac{1}{r} R$. Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2]$$

with respect to r on $[r_1, r_2]$ yields

$$\ln r_2 > \ln r_1 + \frac{1}{2} \ln\left(\frac{R(r_1)}{R(r_2)}\right) = \ln r_1 + \frac{1}{2} \ln\left(\frac{a^{1-\alpha}}{a^{\frac{1-\alpha}{2}}}\right) = \ln r_1 + \ln(a^{\frac{1-\alpha}{4}}),$$

which in turn gives

$$r_2 > r_1\, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}} \overset{(a)}{\geq} a^{\alpha^3}, \qquad \text{for } \alpha \in (0, \frac{1}{2}].$$

This last argument yields a contradiction and we are done proving the claim that $r_2 \geq a^{\alpha^3}$. Note that this also means that $r_0 > r_2 \geq a^{\alpha^3}$ and the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_\alpha^{\frac{2}{\alpha^4}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since $a > a_\alpha^{\frac{2}{\alpha^4}}$ implies

$$a^{\alpha^3} > a_\alpha^{\frac{2}{\alpha}} = (\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$$

and thus $r_0 > a^{\alpha^3} > \frac{2}{\alpha}$.

Now let $\alpha \in [\frac{1}{2}, 1)$. We claim that in this case

$$r_2 \geq a^{\frac{1-\alpha}{4}}. \tag{3.23}$$

If we assume to the contrary that $r_2 < a^{\frac{1-\alpha}{4}}$, then from (3.16) we infer again that

$$R'(r) \geq -\frac{1}{r} R - R^{1-\alpha} > -\frac{2}{r} R,$$

since $R(r) \geq a^{\frac{1-\alpha}{2}}$ and thus

$$R^\alpha \geq a^{\frac{(1-\alpha)\alpha}{2}} \overset{(b)}{\geq} a^{\frac{1-\alpha}{4}} > r_2 \geq r$$

for $r \in [r_1, r_2]$ and $\alpha \in [\frac{1}{2}, 1)$, which yields $-R^{1-\alpha} > -\frac{1}{r}R$. Integrating the differential inequality

$$\frac{R'(r)}{R(r)} > -\frac{2}{r}, \qquad r \in [r_1, r_2],$$

with respect to r on $[r_1, r_2]$ gives

$$r_2 > r_1\, a^{\frac{1-\alpha}{4}} > a^{\frac{1-\alpha}{4}}, \qquad \alpha \in [\frac{1}{2}, 1).$$

This last argument again yields a contradiction and we are done proving the claim that $r_2 \geq a^{\frac{1-\alpha}{4}}$. Note that this also means that $r_0 > r_2 \geq a^{\frac{1-\alpha}{4}}$ and as before the smallest value of r such that a solution can enter Ω_- or Ω_+ is r_0. We can therefore deduce that solutions corresponding to initial data $a > a_\alpha^{\overline{\frac{8}{(1-\alpha)\alpha^2}}}$ will stay outside $\Omega_- \cup \Omega_+$ at least for values of $r \in [0, \frac{2}{\alpha}]$, since

$$a^{\frac{(1-\alpha)\alpha}{4}} > a_\alpha^{\frac{2}{\alpha}} = (\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} \overset{(c)}{>} \frac{2}{\alpha}$$

and thus

$$r_0 > a^{\frac{1-\alpha}{4}} > a_\alpha^{\frac{2}{\alpha^2}} \geq a_\alpha^{\frac{2}{\alpha}} > \frac{2}{\alpha}.$$

Summing up, we can state that for $a > M_\alpha$ and $r \leq \frac{2}{\alpha}$ we have $R(r) > a_\alpha$ and it follows that

$$E(r) > 0 \quad \text{ for } r \in [0, \frac{2}{\alpha}], \quad \text{ if } a > M_\alpha, \tag{3.24}$$

since

$$E(r) = \frac{1}{2}(\beta^2 + \psi^2) - \frac{1}{2-\alpha}|\psi|^{2-\alpha} = \frac{1}{2}R^2 - \frac{1}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha} \tag{3.25}$$

$$\geq R^{2-\alpha}\left(\frac{1}{2}R^\alpha - \frac{1}{2-\alpha}\right) > a_\alpha^{2-\alpha}\left(\frac{1}{2}a_\alpha^\alpha - \frac{1}{2-\alpha}\right) = 0. \tag{3.26}$$

This means that a solution with $a > M_\alpha$ can enter Ω_- or Ω_+ only for a value of $r > \frac{2}{\alpha}$. Moreover, (3.22) and (3.23) show that as $a \to \infty$ the value of $r > 0$ at which a solution enters the region $\Omega_- \cup \Omega_+$ approaches infinity.

To finish the proof of this Lemma we show that the following inequalities hold:

(a) $a^{\frac{1-\alpha}{4}} > a^{\frac{(1-\alpha)\alpha}{2}} > a^{\alpha^3}$ for $\alpha \in (0, \frac{1}{2})$,

(b) $a^{\frac{(1-\alpha)\alpha}{2}} > a^{\frac{1-\alpha}{4}}$ for $\alpha \in (\frac{1}{2}, 1)$,

(c) $(\frac{2}{2-\alpha})^{\frac{2}{\alpha^2}} > \frac{2}{\alpha}$ for $\alpha \in (0, 1)$.

Since $a > 1$, in (a) and (b) it suffices to consider the exponent functions of α in and compare them in size (cf. Figure 4 (i)). Inequality (c) can be shown by noting (cf. Figure 4 (ii)) that the statement is equivalent to

$$\ln\frac{2}{2-\alpha} > \frac{\alpha^2}{2}\ln\frac{2}{\alpha} \qquad \text{for } \alpha \in (0, 1).$$

Then, define $f(\alpha) = \frac{\alpha}{2}$, $g(\alpha) = \ln\frac{2}{2-\alpha}$ for $\alpha \in (0, 1)$. Both functions are monotone increasing and equal to 0 at $\alpha = 0$. Since $g'(\alpha) = \frac{1}{2-\alpha} > \frac{1}{2} = f'(\alpha)$, $f(\alpha) < g(\alpha)$ for all $\alpha \in (0, 1)$. Furthermore, let $h(\alpha) = \alpha \ln\frac{2}{\alpha}$, then $h(\alpha) \to 0$ as $\alpha \to 0$ and $h(2) = 0$.

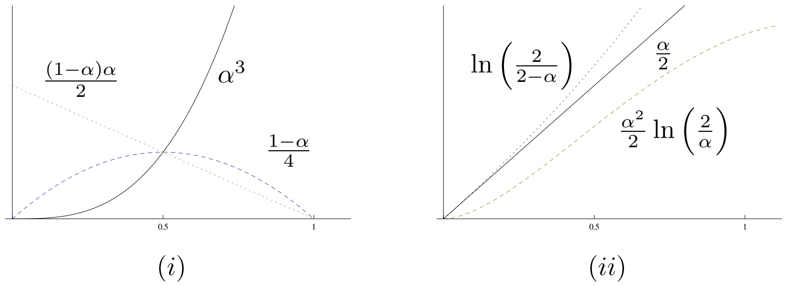

Figure 4:

It is easy to check that $h(\alpha)$ has its only maximum at $\alpha = \frac{2}{e}$ with $h(\frac{2}{e}) = \frac{2}{e} < 1$ which shows that $h(\alpha) = \alpha \ln \frac{2}{\alpha} < 1$. Thus

$$\frac{\alpha^2}{2} \ln \frac{2}{\alpha} = \frac{\alpha}{2} h(\alpha) < \frac{\alpha}{2} = f(\alpha) < g(\alpha) = \ln \frac{2}{2-\alpha}.$$

□

Now we know that the solution to certain initial conditions $a > M_\alpha$ can only enter the region $\Omega_\pm$ for a value of $r > \frac{2}{\alpha}$. Furthermore, this value of r increases if we let the starting point a tend to infinity. The question that arises is: will all solutions with initial data $a > M_\alpha$ enter $\Omega_\pm$? The answer is no, which represents our next result.

Lemma 3.5. *There exists initial data $a_+ > M_\alpha$ such that the corresponding solution to* (3.1)-(3.2) *stays outside of $\Omega_+ \cup \Omega_-$ for all $r \geq 0$.*

Proof. Outside of $\Omega_- \cup \Omega_+$ we have $E > 0$ so that $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$. Passing to polar coordinates (3.17) we find

$$-1 - \frac{1}{2r} \leq \theta'(r) \leq -\frac{\alpha}{2} + \frac{1}{2r}, \tag{3.27}$$

for the values of $r > 0$ where $E(r) > 0$. This is easy to see, as the derivative with respect to r of $\theta(r)$, in view of (3.1), is given by

$$\theta'(r) = \frac{\beta'\psi - \beta\psi'}{\psi^2 + \beta^2} \overset{(3.1)}{=} -1 - \frac{\beta\psi}{r(\psi^2 + \beta^2)} + \frac{|\psi|^{2-\alpha}}{\psi^2 + \beta^2} = -1 - A + B$$

where

$$A = \frac{\beta\psi}{r(\psi^2 + \beta^2)}, \quad B = \frac{|\psi|^{2-\alpha}}{\psi^2 + \beta^2}.$$

Since $r > 0$ we have $|A| = \frac{|\beta\psi|}{r(\psi^2+\beta^2)} \leq \frac{1}{2r}$, and since $E > 0$ we have $B < \frac{2-\alpha}{2}$. Thus

$$\theta'(r) = -1 - A + B \geq -1 - A \geq -1 - \frac{1}{2r},$$

and

$$\theta'(r) = -1 - A + B \leq -1 + \frac{1}{2r} + \frac{2-\alpha}{2} = -\frac{\alpha}{2} + \frac{1}{2r}.$$

These estimates on $\theta'(r)$ gives an upper and lower bound on the angular velocity of the solution for all values of $r > 0$ where $E(r) > 0$. In the previous lemma we showed that

$E(r) > 0$ at least for $r < \frac{2}{\alpha}$ if $a > M_\alpha$. Now, let us consider values of $r > \frac{2}{\alpha}$ for which $E(r)$ is still positive. For such values, (3.27) reads

$$-1 - \frac{\alpha}{4} < \theta'(r) < -\frac{\alpha}{4}. \tag{3.28}$$

Denote by D_+, D_- as the sets of points $\{(a, 0) : a > M_\alpha\}$ such that a solution (ψ, β) to (3.1)-(3.2) with initial data $(a, 0)$ will enter Ω_+, Ω_-, respectively, for some finite value of r. Both sets D_+ and D_- are open by continuous dependence of the solution on initial data, since for a point $(a^*, 0) \in D_+$, whose corresponding solution enters the region Ω_+ at time $r = r^*$, a solution whose starting point $(a, 0)$ lies sufficiently close to $(a^*, 0)$ will also enter Ω_+ at some time close to r^*. In the beginning of the discussion of the proof of Proposition 3.2 we analyzed the dynamics of the system and found that in the plane (ψ, β), outside of $\Omega_- \cup \Omega_+$ a solution to (3.1)-(3.2) intersects the horizontal axis from the upper to the lower half-plane on the right of the origin, and in the other direction on the left of the origin. Notice also that a solution intersects the horizontal axis a finite number of times as it winds around the region $\Omega_- \cup \Omega_+$ before entering it. Denote by $D_N \subseteq D_- \cup D_+$ the set of initial data such that corresponding solutions intersect the positive horizontal axis exactly N times prior to entering Ω_- or Ω_+. Since these intersections are transversal, they are stable under small perturbations (cf. [18]). Thus, again by continuous dependence on initial data, for any N these sets D_N are open. In view of the fact that solutions are unique once we specify the initial condition $(a, 0)$, they are disjoint. We can therefore write $D_- \cup D_+ = \dot{\bigcup}_N D_N$. Assume for a moment that all solutions will at one point enter Ω_+ or Ω_-. (3.22) and (3.23) in Lemma 3.4 show that as $a \to \infty$ the value of $r > 0$ at which a solution can enter the region where $E < 0$ approaches infinity. In view of the above inequality (3.28) and since by virtue of (3.15), E is strictly decreasing whenever $(\psi, \beta) \notin \{(0, 0), (\pm 1, 0)\}$, this means that for $r > \frac{2}{\alpha}$ a solution to (3.1)-(3.2) with $a > M_\alpha$ keeps winding around the region $\Omega_- \cup \Omega_+$ before entering it as $a \to \infty$. We deduce that there exist infinitely many open, non-empty sets D_N with $N \to \infty$ as a tends to infinity. By assumption, $D_- \cup D_+ = (a_\alpha, \infty)$. But this is an open interval in $\mathbb{R}$ which cannot be written as the disjoint union of open non-empty sets D_N. Hence, there exists $a_+ > M_\alpha$ such that $(a_+, 0) \notin D_+ \cup D_-$. Solutions to such initial data will therefore not enter the region $\Omega_- \cup \Omega_+$. $\square$

Lemma 3.6. *For solutions to* (3.1)-(3.2) *corresponding to initial data* $a > M_\alpha$ *such that they stay outside of* $\Omega_- \cup \Omega_+$ *for all* $r \geq 0$*, there exist* $0 < T < \infty$ *such that* $E(T) = 0$ *and* $\psi(T) = \beta(T) = 0$.

Proof. Let us assume that $E(r) > 0$ for all $r \geq 0$ and show that this leads us to a contradiction. Recall (3.28) from the previous lemma. Under the assumption that $E > 0$ for all $r \geq 0$, this bound on θ' holds in particular for all $r > \frac{2}{\alpha}$. Consequently, a solution to (3.1)-(3.2) with $a > M_\alpha$ and $r > \frac{2}{\alpha}$ would surround the region $\Omega_- \cup \Omega_+$ with angular velocity between $1 + \frac{\alpha}{4}$ and $\frac{\alpha}{4}$ in clockwise direction. Thus we can construct an increasing sequence $\{r_n\}_{n \geq 1}$ with $r_1 > \frac{2}{\alpha}$ such that $\theta(r_n) = \frac{\pi}{6} + 2(n-k)\pi$ where $k \in \mathbb{N}$ is fixed. We infer that

$$\frac{8\pi}{\alpha + 4} < r_{n+1} - r_n < \frac{8\pi}{\alpha} \qquad \text{for } r_1 > \frac{2}{\alpha},\ n \geq 1, \tag{3.29}$$

since in view of (3.28) we have $2\pi = \theta(r_{n+1}) - \theta(r_n) < (1 + \frac{\alpha}{4})(r_{n+1} - r_n)$ and $2\pi = \theta(r_{n+1}) - \theta(r_n) > \frac{\alpha}{4}(r_{n+1} - r_n)$. This shows that independent of the number of cycles n the solution has completed, the "time" it takes the solution to return to the ray $\theta(r_n) = \frac{\pi}{6}$ is bounded from above and below by constants. Now consider the region

$$A := \Big\{(\psi, \beta) : E > 0, \frac{\pi}{6}\Big(1 - \frac{1}{3}\alpha\Big) < \theta < \frac{\pi}{6}\Big\}. \tag{3.30}$$

From the dynamics of the system (3.1) we infer that the solutions enter the region A crossing the ray $\theta = \frac{\pi}{6}$ at some time $r = r_n$ and leave it crossing the ray $\theta = \frac{\pi}{6}(1 - \frac{1}{3}\alpha)$ at some bigger value of $r = r_n^+$. This value r_n^+ satisfies

$$r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha+4} < r_n^+ < r_n + \frac{2\pi}{9}, \tag{3.31}$$

since we infer from (3.28) that

$$(-1 - \frac{\alpha}{4})(r_n^+ - r_n) < \theta(r_n^+) - \theta(r_n) = \frac{\pi}{6}(1 - \frac{1}{3}\alpha) - \frac{\pi}{6} = -\alpha\frac{\pi}{18}$$
$$\text{and } -\frac{\alpha}{4}(r_n^+ - r_n) > \theta(r_n^+) - \theta(r_n) = -\alpha\frac{\pi}{18}.$$

Passing to polar coordinates (3.17), we find that

$$R > \left(\frac{\sqrt{3}}{2}\right)^{\frac{2-\alpha}{\alpha}}\left(\frac{2}{2-\alpha}\right)^{\frac{1}{\alpha}} \text{ in A.} \tag{3.32}$$

To see this, note that for $\theta < \frac{\pi}{6}$ we have $\cos\theta > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and thus $E > 0$ whenever $\psi^2 + \beta^2 > \frac{2}{2-\alpha}|\psi|^{2-\alpha}$ or, equivalently,

$$R^2 > \frac{2}{2-\alpha}R^{2-\alpha}|\cos\theta|^{2-\alpha} \geq \frac{2}{2-\alpha}R^{2-\alpha}\left(\frac{\sqrt{3}}{2}\right)^{2-\alpha}.$$

Consequently, in view of (3.17),

$$\beta^2(r) > \sin^2(\frac{\pi}{9})\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} =: K_\alpha \qquad \text{for } r \in (r_n, r_n^+), \tag{3.33}$$

since $\frac{\pi}{6}(1 - \frac{1}{3}\alpha) > \frac{\pi}{9}$ for $\alpha \in (0,1)$ and $\sin(\theta) > \sin\left(\frac{\pi}{6}(1 - \frac{1}{3}\alpha)\right)$ in A. From (3.32) we infer that

$$\beta^2(r) = R^2\sin^2(\theta) > \left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}}\left(\frac{2}{2-\alpha}\right)^{\frac{2}{\alpha}}\sin^2(\frac{\pi}{9})$$
$$> \sin^2(\frac{\pi}{9})\left(\frac{\sqrt{3}}{2}\right)^{\frac{2(2-\alpha)}{\alpha}} \qquad \text{in } A.$$

Furthermore, by virtue of (3.29) and (3.31),

$$r_n + \frac{2\pi}{9}\frac{\alpha}{\alpha+4} < r_n^+ < r_{n+1}, \tag{3.34}$$

since $r_n^+ < r_n + \frac{2\pi}{9} < r_n + \frac{8\pi}{\alpha+4} < r_{n+1}$ for all $\alpha \in (0,1)$. But now from (3.15), together with (3.33), (3.31) and (3.34), we get a contradiction:

$$E\left(\frac{2}{\alpha}\right) - E(\infty) = -\int_{2/\alpha}^{\infty} E'(r)dr = \int_{\frac{2}{\alpha}}^{\infty}\frac{\beta^2(r)}{r}dr$$
$$\geq K_\alpha\sum_{n\geq 1}\int_{r_n}^{r_n^+}\frac{1}{r}dr \geq K_\alpha\sum_{n\geq 1}\frac{1}{r_n^+}(r_n^+ - r_n)$$
$$> K_\alpha\frac{2\pi}{9}\frac{\alpha}{\alpha+4}\sum_{n\geq 1}\frac{1}{r_n + \frac{2\pi}{9}} = \infty.$$

The series is divergent since $r_n < \frac{8\pi}{\alpha}(n-1) + r_1$ in view of (3.29), and $E(\infty)$ is some finite number, as E is bounded by virtue of Lemma 3.4. Recall that we assumed $E(r)$ for all $r \geq 0$, which lead to the above contradiction. Thus there exists a finite value of $T > 0$ such that $E(T) = 0$. Notice that for such values of T we also have $\psi(T) = \beta(T) = 0$. If this were not the case, the dynamics of the system would force the solution to enter Ω_+ or Ω_- at this point, which is contradictory to the assumption of this Lemma. □

3.3 Limiting cases of the parameter α

In the case where $\alpha = 1$, the vorticity function γ simplifies to

$$\gamma(\psi) = \psi - \frac{\psi}{|\psi|} \tag{3.35}$$

which has a point of discontinuity at $\psi = 0$. As we are only interested in classical solutions, we will not consider this case.
In the case where $\alpha = 1$ we simply have

$$\gamma(\psi) \equiv 0. \tag{3.36}$$

Thus, system (2.3)-(2.2), for which we seek compactly supported $\mathcal{C}^2$-solutions, reads

$$\begin{cases} \psi'' + \frac{1}{r}\psi' = 0, & r > 0, \\ \psi(0) = a, \ \psi'(0) = 0, \end{cases} \tag{3.37}$$

which we can solve easily, obtaining

$$\psi(r) = C_1 \ln(r) + C_2,$$

for some constants $C_1, C_2 \in \mathbb{R}$. In view of the boundary conditions,

$$C_1 = 0 \text{ and } C_2 = a,$$

and we conclude that $\psi(r) \equiv a$ is constant for all $r \geq 0$. In the setting of ψ being the stream function on the fluid domain D, this means that $\psi \equiv a$ is constant throughout the flow field. The boundary conditions $\psi = \psi_y = 0$ on the flat free surface require this constant to be zero. So $\psi \equiv 0$ and the water is still throughout the fluid domain, which is why we do not consider this case.

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A note on uniqueness and compact support of solutions in a recent model for tsunami background flows

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Abstract

We present an elementary proof of uniqueness for solutions of an initial value problem which is not Lipschitz continuous, generalizing a technique employed in [20]. This approach can be applied for a wide class of vorticity functions in the context of [6], where, departing from a recent model for the evolution of tsunami waves developed in [10], the possibility of modelling background flows with isolated regions of vorticity is rigorously established.

1 Introduction

Tsunami waves are gravity water waves mostly generated by undersea earthquakes, cf. [2], which cause a vertical displacement of the entire column of water above the fault region, thus giving the tsunami its initial wave profile. The waves then propagate over large distances without essentially changing their shape, a characteristic feature observable for example in the May 1960 tsunami that set out off the coast of Chile and travelled almost $17000km$ across the Pacific Ocean until it hit Japan (cf. [8, 21, 22]). Tsunami waves travel at very high, almost constant speed and their wave length is typically hundreds of kilometers long whereas their amplitude is relatively small (about $0.5m$, cf. [21]). While out in the open ocean, where the water depth is relatively uniform (eg. the Central Pacific Basin is approximately $4.3km$ deep, cf. [8]), the evolution of a tsunami is essentially governed by linear theory, the typical wave speed being $\sqrt{gh}$, where g is the gravitational acceleration and h the average water depth, cf. [11]. When a tsunami approaches the shoreline, the front of the wave slows down as the depth decreases, causing the water to pile up vertically near the coast since the back of the wave is still out in the open ocean travelling at very high speed. The resulting damage by surging water and inundation is often far more devastating than the effects of the preceding earthquake itself. In the case of the tsunami that hit Japan on March 11, 2011, an undersea megathrust earthquake of magnitude 9 occurred in the region where the pacific plate is subducting under the plate beneath the Japanese island of Honshu, as reported by the U.S. Geological Survey. When the stresses that had been building up in this process were finally released, the break caused the sea floor to rise by several meters in a rupture zone $300km$ long and $150km$ wide, with the epicenter about $70km$ off the coast of the island of Honshu. The earthquake resulted in a major tsunami which devastated entire towns along the pacific coast of northern Japan, inundating a total area of approximately $500km^2$ and causing the loss of thousands of lives. The tsunami not only hit the coast of Japan but also travelled across the Pacific Ocean reaching as far as the coast of Chile, about $17,000km$ from the epicenter of the earthquake.

Although it is beyond the scope of mathematical analysis to predict such catastrophes, this recent tsunami has reminded the world once again of how dangerous the destructive forces of nature can be to human lives. To obtain a better understanding of such phenomena, several models have been developed which describe the time evolution of tsunamis after the wave has obtained its initial shape. The equations governing the behaviour of these very long waves can be derived from Euler's equations and the equation of mass conservation under some general assumptions on the water, the oceans bathymetry and the type of water wave under consideration. The resulting system of equation with suitable boundary conditions provides a general model for tsunami waves which applies not only in the situation of the recent Japan tsunami, but also for the 2004 boxing day tsunami and the Chilean tsunami of 1960, cf. [14, 9, 4, 8, 21]. Most investigations of tsunami waves do not take into account the various states the ocean might exhibit prior to the arrival of waves near the shore, that is, the models are restricted to irrotational flows, which model background states of still water, cf. [17, 18, 19]. However, underlying currents might have significant effects on the evolution of tsunami waves, cf. the discussion in [10]. The tsunami model analysed in this paper allows for slow bottom variations and includes the possibility of having a background flow with vorticity which might enhance or repress the evolution of tsunami waves. Departing from the analysis of [10], the possibility of incorporating a non-trivial background flow which models isolated regions of vorticity surrounded by still water near the shore is rigorously first established in [6] for the governing equations without passing to shallow-water approximations, and later generalized in [15] for a wider class of vorticity functions.

In the present paper, we improve the result obtained in [15] by simplifying the existence and uniqueness proof therein, thus allowing for even more general vorticity distributions in the model. Furthermore, we present an alternative proof of the fact that solutions have compact support under the additional assumption that far out in the ocean and close to the surface and bottom, the water is still.

2 Preliminaries

We can reasonably model the evolution of tsunami waves in a two-dimensional setting, since the direction of propagation is essentially perpendicular to the fault line. This simplifying assumption is justified for the prominent examples of tsunamis mentioned above, where the motion was almost uniform along the fault line, the length of the rupture zone exceeded the wave length and the ocean depth over which the tsunami travelled was relatively uniform, cf.[5, 21]. We assume the water to be inviscid, incompressible and to have constant density. Furthermore, we neglect the effect of surface tension which plays a minor role in the modelling of gravity water waves. We are interested in the motion of water near the coast beneath a water surface which is flat in the absence of waves. Hence, we want the model to admit a shoreline at the intersection of the flat surface and the sea bed. In Cartesian coordinates (x, y), let the shoreline be centered at the origin and assume that the water extends to $-\infty$ in the negative horizontal direction, with constant depth h_0 out in the open ocean. Denote the two-dimensional fluid domain by $D = \{(x, y) \in \mathbb{R}^2 : x < 0,\, b(x) < y < 0\}$, where $b(x)$ describes the fixed impermeable bed. In our two-dimensional setting we can introduce a stream function ψ such that the fluid's velocity field is given by $(\psi_y, -\psi_x)$. We assume that the vorticity ω can be written as a function $\gamma(\psi)$ called vorticity function and let $\omega = \gamma(\psi)$. This specifies a vorticity distribution throughout the entire fluid domain and it can be proven that in the absence of stagnation points a vorticity distribution may be given by means of a vorticity function (cf. the discussion in [12, 7]). The equations of motion, which can be derived from Euler's equation and the equation of mass conservation (cf. [10, 12]), and boundary conditions governing the background state of the water may be formulated in terms of the stream function ψ as

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } D, \\ \psi = \psi_y = 0 & \text{on } y = 0, \\ \psi = 0 & \text{on } y = b(x), \end{cases} \tag{2.1}$$

for a given seabed profile $b(x)$ and a vorticity function $\gamma(\psi)$. The goal is to model a background state of the ocean near the shore which contains isolated regions of vorticity surrounded by still water, assuming the surface is flat prior to the arrival of waves. That is, we look for radially symmetric solutions of (2.1) that have compact support in the fluid domain.

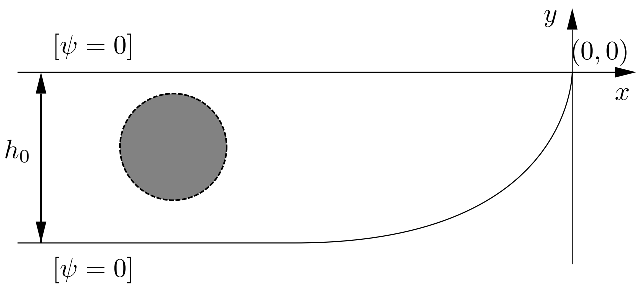

Figure 5: Fluid domain D with an isolated region of non-zero vorticity.

Using the Ansatz

$$\psi(x, y) = \psi(r), \quad \text{where} \quad r = \sqrt{(x-x_0)^2 + (y-y_0)^2},$$

for some $(x_0, y_0) \in D$, the problem is reduced to an initial value problem for the second order ordinary differential equation

$$\begin{cases} \psi''(r) + \dfrac{1}{r}\,\psi'(r) = -\gamma(\psi(r)), & r > 0, \\ \psi(0) = a, \quad \psi'(0) = 0, \end{cases} \tag{2.2}$$

for some initial value $a > 0$. Since (2.1) is an over determined boundary value problem, we expect non-trivial solutions to exist only for certain classes of functions γ. One can show using maximum principles, cf. [10], that for linear vorticity functions, the system (2.1) has only trivial solutions due to the fact that the model admits a shoreline, the free surface is flat in the absence of waves and the water is still outside the region of vorticity. It turns out that these requirements also impose restrictions on the regularity of γ: taking $\gamma \in \mathcal{C}^1$ precludes radially symmetric solutions with compact support in the fluid domain, as we could find $T > 0$ such that $\psi(T) = \psi'(T) = 0$ and it follows from the backward uniqueness property, cf. [13], that $\psi \equiv 0$. For the vorticity function

$$\gamma(\psi) = \begin{cases} \psi - \psi|\psi|^{-\alpha} & \text{for } \psi \neq 0, \\ 0 & \text{for } \psi = 0, \end{cases} \qquad \alpha \in (0, 1), \tag{2.3}$$

which is non-linear and continuous but not $\mathcal{C}^1$, non-trivial solutions of system (2.2) with compact support in $[0, \infty)$ were obtained for $\alpha = 1/2$ in [6], and arbitrary $\alpha \in (0, 1)$ in [15]. These solutions model background states of the ocean prior to the arrival of waves with isolated regions of vorticity under a flat free surface outside of which the

water is at rest, cf. Figure (5). The proof of this result is based on a dynamical systems approach, but relies only on basic tools from the theory of ordinary differential equations and consists essentially of two parts: In the first part, it is shown that for any initial value a greater than some constant $a_\alpha > 0$ there exists a unique $\mathcal{C}^2$-solution ψ which depends continuously on the initial data a. The second part shows that for certain initial data a big enough, the corresponding unique solution has compact support. There are essentially two difficulties to overcome: the first is due to the fact that (2.2) is not a classical initial value problem as the equation displays a discontinuity at $r = 0$. This can be remedied by performing a change of variables in the vicinity of the discontinuity and solving the resulting system using an integral Ansatz and a version of Banach's fixed point theorem. Furthermore, since γ is not locally Lipschitz in $\psi = 0$, one cannot merely rely on the classical theory to obtain existence and uniqueness of solutions. A complex chain of arguments involving the reparametrization of the system recast in polar coordinates and an application of the inverse function theorem is carried out to establish uniqueness of solutions also at points where $\psi = 0$. However, this approach might lead to serious difficulties if one wishes to work with more complicated vorticity functions, as pointed out in [20].
The aim of the present note is to avoid these problems by simplifying the uniqueness proof in [15], using elementary arguments to show that if a (vorticity) function γ satisfies the set of hypotheses given below, uniqueness of solutions of (2.2) is guaranteed in the neighborhood of points where the right hand side fails to be Lipschitz continuous, cf. Section (3). Furthermore, in Section (4) we present an alternative way of proving compact support of solutions of the initial value problem (2.2) under the additional assumption that for certain initial data a the corresponding solution ψ_a tends asymptotically to zero.

3 Uniqueness

Consider the initial value problem

$$\begin{cases} \psi'' + \frac{1}{r}\,\psi' = -\gamma(\psi), & r \geq r_0 \geq 1, \\ \psi(r_0) = 0, \quad \psi'(r_0) = \psi_1, \end{cases} \tag{3.4}$$

where $\psi_1 \neq 0$ and γ is given by (2.3). We are going to show that a unique solution to (3.4) exists and that it depends continuously on initial data. This is not straightforward, since the right hand side of the differential equation is not Lipschitz continuous for $\psi = 0$, hence we cannot apply classical existence and uniqueness theorems right away. To obtain the desired result we rely upon the following

Theorem 3.1. *Assume that a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses*

$$\begin{cases} (i) & \gamma(0) = 0, \\ (ii) & \psi \cdot \gamma(\psi) < 0, \\ (iii) & |\gamma(\psi_1) - \gamma(\psi_2)| \leq \frac{C}{(\min\{|\psi_1|,|\psi_2|\})^\alpha}|\psi_1 - \psi_2|, \;\; \alpha \in (0,1), \end{cases} \tag{3.5}$$

for any $\psi, \psi_1, \psi_2 \in [-\delta, 0) \cup (0, \delta]$ with $\psi_1 \cdot \psi_2 > 0$ and $C, \delta > 0$. Then, given $\psi_1 \neq 0$, the initial value problem (3.4) *has a unique solution to the right of r_0.*

This is a variation of Theorem 2.1 in [20] and can be proven in almost the same way. The vorticity function $\gamma(\psi)$ defined in (2.3) is continuous and satisfies the hypotheses (3.5) in Theorem (3.1) as long as $\delta < 1$. Indeed, by definition $\gamma(0) = 0$, whereas $\psi \cdot \gamma(\psi) < 0$ if and only if $\psi^2(1 - |\psi|^{-\alpha}) < 0$, which is true for $|\psi| \leq \delta < 1$. To show that the third hypothesis is fulfilled assume without loss of generality that $\psi_1 < \psi_2$. Under the assumption that $\psi_1 \cdot \psi_2 > 0$ it suffices to consider the case where $\psi_1, \psi_2 > 0$, since γ is an odd function. Then, since $-\psi_2(1-\psi_2^{-\alpha}) < -\psi_2(1-\psi_1^{-\alpha})$, we have

$$\begin{aligned}\left|\gamma(\psi_1) - \gamma(\psi_2)\right| &= \left|\psi_1 - \psi_1|\psi_1|^{-\alpha} - \psi_2 + \psi_2|\psi_2|^{-\alpha}\right| \\ &< \left|\psi_1(1-|\psi_1|^{-\alpha}) - \psi_2(1-|\psi_1|^{-\alpha})\right| \\ &< |\psi_1 - \psi_2|\left|1-|\psi_1|^{-\alpha}\right|,\end{aligned}$$

and there exists a constant $C > 0$ such that $|1-|\psi_1|^{-\alpha}| < C|\psi_1|^{-\alpha}$, since $|\psi_1| \leq \delta$. Therefore,

$$|\gamma(\psi_1) - \gamma(\psi_2)| < |\psi_1 - \psi_2|C|\psi_1|^{-\alpha} = \frac{C}{(\min\{|\psi_1|, |\psi_2|\})^\alpha}|\psi_1 - \psi_2|.$$

From continuity of the function γ we infer that a solution to (3.4) exists and that it is continuous for all $r \geq 1$. Hence there is a time interval centered at r_0 where $|\psi(r)| \leq \delta < 1$ and Theorem (3.1) applies. We conclude that solutions to (3.4) are uniquely determined by their initial values, at least in a small interval to the right of r_0. Away from the zeros of ψ, i.e. in any interval I where $|\psi(r)| > 0$ for $r \in I$, we can use the fact that γ is (locally) $\mathcal{C}^1$ to infer uniqueness from the theorem of Picard–Lindelöf. Once uniqueness is established, continuous dependence of the solution on initial conditions follows immediately (cf. Theorem 3.4 in [16]). Hence we can prove existence and uniqueness of solutions to the original initial value problem (2.2) by applying Theorem (3.1) in neighborhoods of values of r where $\psi(r) = 0$ and by employing standard results away from the zeros of ψ, where γ is locally $\mathcal{C}^1$.

4 Compact Support

To obtain isolated regions of vorticity for the background state in the model for tsunami waves presented above, one has to prove that the solutions of the initial value problem (2.2) have compact support. In [15] this is achieved by an involved argument using a coercive functional which decreases along solutions and performing a detailed analysis of the dynamics of the system in the (ψ, ψ')-plane. We present here a simpler approach which relies on the additional assumption that ψ tends asymptotically to zero, that is, close to the boundaries of the fluid domain, the water is at rest. More precisely, for a solution ψ_a of (2.2) corresponding to some initial value $a > 0$ and under the assumption that $\lim_{r\to\infty} \psi_a(r) = 0$, we give an elementary proof of the fact that $\psi_a(r)$ has compact support in $[0, \infty)$.

Consider the decreasing $\mathcal{C}^2$-function $\psi_+ : [0,\infty) \to [0,\infty)$ defined implicitly by

$$r = \int_{\psi_+(r)}^{\psi_+(0)} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}}, \qquad r \in [0, I], \tag{4.6}$$

where

$$I = \int_0^{(1-\alpha)^{\frac{1}{\alpha}}} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}}.$$

Let $\psi_+ \equiv 0$ for $r > I$. Then ψ_+ satisfies the second order ordinary differential equation

$$\psi_+'' + \psi_+ - \psi_+|\psi_+|^{-\alpha} = 0, \qquad \text{for } r > 0, \tag{4.7}$$

where the values at the boundary of $[0, I]$ are given by

$$\begin{aligned}&\psi_+(0) = (1-\alpha)^{\frac{1}{\alpha}}, \quad \psi_+'(0) = -(1-\alpha)^{\frac{1}{\alpha}}\sqrt{\frac{\alpha(3-\alpha)}{(2-\alpha)(1-\alpha)}}, \\ &\psi_+(I) = \psi_+'(I) = 0.\end{aligned}$$

This can be easily checked, as

$$r = \int_{\psi_+(r)}^{\psi_+(0)} \frac{ds}{\sqrt{\frac{2}{2-\alpha}|s|^{2-\alpha} - s^2}} = \int_r^0 \frac{\psi_+'(s)\, ds}{\sqrt{\frac{2}{2-\alpha}|\psi_+(s)|^{2-\alpha} - \psi_+^2(s)}}$$

is equivalent to

$$(\psi_+'(r))^2 = \frac{2}{2-\alpha}|\psi_+(r)|^{2-\alpha} - \psi_+^2(r).$$

Differentiating with respect to r yields

$$2\psi_+'\psi_+'' = 2|\psi_+|^{1-\alpha}\psi_+' - 2\psi_+\psi_+',$$

which in view of the fact that $\psi_+' \neq 0$ and $\psi_+ \geq 0$ gives (4.7). Furthermore,

$$\begin{aligned}\psi_+'(0) &= -\sqrt{\frac{2}{2-\alpha}|\psi_+(0)|^{2-\alpha} - \psi_+^2(0)} = -\sqrt{\frac{2}{2-\alpha}(1-\alpha)^{\frac{2-\alpha}{\alpha}} - (1-\alpha)^{\frac{2}{\alpha}}} \\ &= -\sqrt{(1-\alpha)^{\frac{2}{\alpha}}\left(\frac{2}{(2-\alpha)(1-\alpha)} - 1\right)} = -(1-\alpha)^{\frac{1}{\alpha}}\sqrt{\frac{\alpha(3-\alpha)}{(2-\alpha)(1-\alpha)}}.\end{aligned}$$

Notice that $(1-\alpha)^{\frac{1}{\alpha}} > 0$ is the minimum of the function $s \mapsto s - s|s|^{-\alpha}$ for $s > 0$, and since $\lim_{r\to\infty}\psi_a(r) = 0$ there exists $r_0 > 0$ such that $|\psi_a(r)| < (1-\alpha)^{\frac{1}{\alpha}}$ for all $r \geq r_0$. We claim that

$$|\psi_a(r)| \leq \psi_+(r - r_0) \quad \text{for } r \geq r_0. \tag{4.8}$$

If equation (4.8) holds then $\psi_a(r)$ vanishes for $r \geq r_0 + I$ since $\psi_+(r) = 0$ for $r > I$, and we have proved via another approach that a solution ψ_a to (2.2) has compact support in $[0, \infty)$.

To prove the claim, let us assume that the upper bound were false. By construction, $\lim_{r\to\infty}\psi_a(r) - \psi_+(r - r_0) = 0$ and $r \mapsto \psi_a(r) - \psi_+(r - r_0)$ is negative at $r = r_0$, since $\psi_a(r_0) < (1-\alpha)^{\frac{1}{\alpha}} = \psi_+(0)$. By assumption, there exists $R > r_0$ such that $\psi_a(R) > \psi_+(R - r_0)$. Therefore, the function $r \mapsto \psi_a(r) - \psi_+(r - r_0)$ has a positive maximum in $[r_0, \infty)$ at some point $r_1 > r_0$ with $\psi_a'(r_1) - \psi_+'(r_1 - r_0) = 0$ and $\psi_a''(r_1) - \psi_+''(r_1 - r_0) \leq 0$. Recalling that ψ_a is a solution of system (2.2) and that ψ_+ satisfies (4.7) leads to a contradiction, since

$$\begin{aligned}0 &\geq \psi_a''(r_1) - \psi_+''(r_1 - r_0) \\ &= -\frac{1}{r}\psi_a'(r_1) - [\psi_a(r_1) - \psi_a(r_1)|\psi_a(r_1)|^{-\alpha}] \\ &\quad + \psi_+(r_1 - r_0) - \psi_+(r_1 - r_0)|\psi_+(r_1 - r_0)|^{-\alpha} \\ &> -\frac{1}{r}\psi_+'(r_1 - r_0) \geq 0.\end{aligned}$$

The second to last inequality is due to the fact that $s \mapsto s - s|s|^{-\alpha}$ is strictly decreasing on $(0, (1-\alpha)^{\frac{1}{\alpha}})$ and $\psi_+(r_1 - r_0) < \psi_a(r_1) < (1-\alpha)^{\frac{1}{\alpha}}$. Analogously, we can show that the lower bound of inequality (4.8) holds. This proves the claim.

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Solitary traveling water waves of moderate amplitude

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Abstract

We prove the existence of solitary traveling wave solutions for an equation describing the evolution of the free surface for waves of moderate amplitude in the shallow water regime. This non-linear third order partial differential equation arises as an approximation of the Euler equations, modeling the unidirectional propagation of surface water waves. We give a description of the solitary wave profiles by performing a phase plane analysis and study some qualitative features of the solutions.

1 Introduction

Ever since Scott Russell's first recorded observation of "the great wave of translation" [33], there has been growing interest in the study of solitary wave solutions of the equations for water waves. The existence theory for irrotational waves of small amplitude dates back to works of Krasovski, Lavrentiev and Ter-Krikorov [26, 27, 34], and was later improved by Friedrichs and Hyers [17], Beale [6] and Amick and Toland [4]. Although at the time no existence results for waves with arbitrary amplitude were available, Keady and Pritchard [24] proved that symmetric and monotone solitary wave solutions are necessarily waves of elevation which propagate at supercritical Froude number. It was shown by Amick and Toland [3] that such waves of elevation actually exist for all amplitudes from zero up to the solitary wave of greatest height and that they decay exponentially at infinity, under the assumption that the wave profile is symmetric and monotone from crest to trough. Craig and Sternberg [16] proved that any supercritical solitary wave solution is symmetric and decays monotonically to a constant on either side of the crest. More recently, results on existence, symmetry and regularity were obtained in the rotational case, cf. [20, 29], and the flow beneath an irrotational solitary wave was investigated in [10] and [11]. In parallel with the aforementioned research on the exact water wave problem, the past fifty years have seen a resurgence in interest in approximate model equations. With the appearence of integrable equations like the KdV [25] or Camassa–Holm [15], whose solitary wave solutions are in fact solitons (cf. [31, 18, 1, 7, 22]), approximations to the full governing equations received renewed attention from the mathematical community. These same equations have long been a staple of the applied ocean sciences, with myriad applications in coastal engineering and tsunami modelling, cf. [12]. Many results have been obtained for waves of small amplitude, but it is also interesting and important to look at larger amplitude waves. Departing from an equation first derived by Johnson in [22], which at a certain depth below the fluid surface is a Camassa–Holm equation, one can derive a corresponding equation for the free surface valid for waves of moderate amplitude in the shallow water regime. Constantin and Lannes discuss large-time well-posedness of this equation in [13] and prove existence and uniqueness of solutions on some maximal time interval, also showing that singularites can develop only in the form of wave breaking. To our knowledge, not a great deal is known so far about global solutions. In the present paper we prove existence of solitary traveling wave solutions for this equation and provide some qualitative features of the wave profile including symmetry, exponential decay at infinity and the fact that the profile has a unique crest point.

2 Preliminaries

Our mathematical model is based on a number of simplifying assumptions regarding the fluid and the physical quantities that play a role in the equations of motion. We assume that the water is inviscid, incompressible and that it has constant density. Furthermore we restrict our attention to gravity water waves, meaning that the only external force relevant to the propagation of the waves is due to the gravitational acceleration g. Concerning the fluid domain Ω, our analysis is valid for fluid flows over a flat bed at depth $y = -h_0$ that extends to infinity in both horizontal directions. The fluid domain is bounded from above by the one dimensional free surface which describes the elevation of the wave above the bed by means of $\eta(x,t)$, a function of space x and time t. We denote the fluid velocity field by (u, v) and impose the additional assumption that the flow is irrotational. The equations governing the motion of the fluid are taken to be

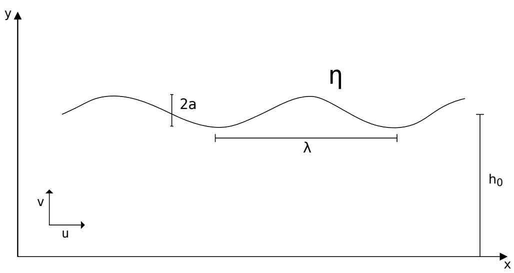

Figure 6: The fluid domain Ω for one dimensional surface waves.

Euler's equations (2.1), which arise from Newton's second law of motion. Furthermore, based on the assumption that mass is neither generated nor destroyed anywhere in the fluid, we employ the equation of mass conservation which reduces to (2.2) below, because the density is constant. Due to the fact that we consider irrotational flows the curl of the velocity field (i.e. the vorticity) must be zero, which leads to relation (2.3). In addition to these equations of motion we impose suitable conditions on the surface and the bottom of the fluid. The kinematic boundary conditions express the fact that the boundaries of the fluid domain behave like surfaces which move with the fluid. This ensures that there is no flow through the bottom and that fluid particles do not leave the fluid domain through the surface. The dynamic boundary condition decouples the motion of air from that of water by setting the atmospheric pressure equal to a constant, which is reasonable since the density of air is very small compared to that of water (see [8] and [28] for a justification of the assumptions on inviscid homogeneous fluid flows for gravity water waves). In what follows we will be interested mostly in approximations of the full governing equations. To this end, the variables are nondimensionalized and scaled using appropriate reference quantities (cf. [21] for a detailed discussion). The resulting system of equations valid in the fluid domain $\Omega = \{(x,y) | -h_0 < y < \eta(x,t)\}$ reads in nondimensionalized and scaled form

$$\left.\begin{aligned} u_t + \varepsilon(uu_x + vu_y) &= -P_x, \\ \delta^2\Big(v_t + \varepsilon(uv_x + vv_y)\Big) &= -P_y, \end{aligned}\right\} \quad \text{in } \Omega, \tag{2.1}$$

$$u_x + v_y = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$u_y - \delta^2 v_x = 0 \quad \text{in } \Omega, \tag{2.3}$$

with boundary conditions

$$P = \eta, \quad v = \eta_t + \varepsilon u \eta_x \quad \text{on } y = \varepsilon\eta,$$
$$v = 0 \quad \text{on } y = -1,$$

where P is the nondimensional pressure relative to the hydrostatic pressure distribution. The so called amplitude and shallowness parameters

$$\varepsilon = \frac{a}{h_0}, \qquad \delta = \frac{h_0}{\lambda},$$

appearing in this formulation arise naturally in the process of nondimensionalization and relate the average wave length λ, amplitude a and water depth h_0. They characterize various physical regimes in which simplified equations can be derived by means of asymptotic expansions in terms of ε and δ. The resulting model equations serve as a basis to construct approximate solutions to the full governing equations which, under certain circumstances, still model accurately the type of waves of interest. In the following we shall focus on the long-wave limit, or shallow water regime, where $\delta \ll 1$, and are concerned with waves of moderate amplitude, characterized by $\varepsilon = O(\delta)$.

3 An equation for waves of moderate amplitude in shallow water

In the shallow water regime, one can derive the Green–Naghdi equations (cf. [19]),

$$\begin{cases} \eta_t + [(1+\varepsilon\eta)u]_x = 0, \\ u_t + \eta_x + \varepsilon u u_x = \dfrac{\delta^2}{3}\dfrac{1}{1+\varepsilon\eta}\Big[(1+\varepsilon\eta)^3\left(u_{xt} + \varepsilon u u_{xx} - \varepsilon u_x^2\right)\Big]_x, \end{cases}$$

which couple the free surface η to the vertically averaged horizontal velocity component

$$u(x,t) = \frac{1}{1+\varepsilon\eta(x,t)}\int_{-1}^{\varepsilon\eta(x,t)} u(x,y,t)dy.$$

For one dimensional surface waves propagating over a flat bed, this set of equations provides an approximation to the Euler equations up to terms of order δ^2, cf. [2]. Under the additional assumption that $\varepsilon = O(\delta)$, one can study an equation for the velocity, the Johnson equation

$$u_t + u_x + \frac{3}{2}\varepsilon u u_x + \frac{\delta^2}{12}u_{xxx} - \frac{\delta^2}{12}u_{xxt} + \frac{7\varepsilon\delta^2}{24}u u_{xxx} + \frac{4\varepsilon\delta^2}{3}u_x u_{xx} = 0, \tag{3.4}$$

which was first derived in [22] by means of asymptotic expansions in terms of ε and δ. Constantin and Lannes showed in [13] that, defining η in terms of u by an expression which arises in the asymptotic derivation of (3.4),

$$\eta = u + \frac{\varepsilon}{4}u^2 - \frac{\delta^2}{6}u_{xx} - \frac{5\varepsilon\delta^2}{12}u u_{xx} - \frac{17\varepsilon\delta^2}{48}u_x^2, \tag{3.5}$$

a solution u of the Johnson equation satisfies the Green–Naghdi equations up to terms of order δ^4, providing thus a good approximation to the governing equations for water waves. The Johnson equation actually belongs to a wider family of equations of this type, none of which is integrable unless the averaged horizontal velocity component is replaced by the horizontal velocity evaluated at a specific depth in the fluid domain,

$$u_\rho(x,t) = \partial_x \Phi(x,y,t)\Big|_{y=(1+\varepsilon\eta)\rho-1},$$

where Φ is the velocity potential associated with the irrotational velocity field. Precisely at the level line $\rho = \frac{1}{\sqrt{2}}$ the Johnson equation turns out to be a Camassa–Holm equation and is therefore integrable, cf. [22, 13, 15]. However, all of these equations describe the evolution of the velocity at a certain depth below the water surface and, unlike the case of model equations like the Korteweg–de Vries equation [25], they are not identical to the equation for the free surface. Using the expression (3.5) in Johnson's equation (3.4), one can derive a corresponding evolution equation for the free surface of waves of moderate amplitude in the shallow water regime,

$$\begin{aligned}\eta_t + \eta_x + \frac{3}{2}\varepsilon\eta\eta_x - \frac{3}{8}\varepsilon^2\eta^2\eta_x + \frac{3}{16}\varepsilon^3\eta^3\eta_x + \frac{\delta^2}{12}\eta_{xxx} - \frac{\delta^2}{12}\eta_{xxt}& \\ + \frac{7\varepsilon\delta^2}{24}\Big(\eta\eta_{xxx} + 2\eta_x\eta_{xx}\Big) = 0.&\end{aligned} \tag{3.6}$$

Large-time well-posedness results were obtained for this equation in [13] using a semi-group approach due to Kato, cf. [23]. It is shown that for any initial data $\eta_0 \in H^3(\mathbb{R})$ and a maximal existence time $t_m > 0$ there exists a unique solution $\eta \in \mathcal{C}(H^3(\mathbb{R}); [0, t_m)) \cap \mathcal{C}^1(H^2(\mathbb{R}); [0, t_m))$ which depends continuously on initial data. Furthermore it is proved that if the maximal existence time is finite blow-up occurs in the form of wave breaking, e.g. the wave profile remains bounded but its slope becomes unbounded as t approaches t_m (cf. [9, 35] for discussions of this phenomenon). Proving results on global solutions seems to be quite an intricate task as the third order partial differential equation (3.6) contains non linear terms of high order. The aim of this paper is to prove existence of solitary traveling wave solutions of (3.6) by performing a phase plane analysis of the corresponding system of ordinary differential equations and providing a qualitative description of the wave profile.

4 Existence of solitary traveling wave solutions

Traveling wave solutions have the property that wave profiles propagate at constant speed $c > 0$ without changing their shape. Defining characteristic variables and scaling out the amplitude and shallowness parameters by means of the transformation

$$\tau = \frac{1}{\delta}\,(x - ct), \quad \eta(\tau) = \frac{\varepsilon}{2}\,\eta(x,t), \tag{4.7}$$

we transform (3.6) into the ordinary differential equation

$$\eta'(1-c) + 3\eta\eta' - \frac{3}{2}\eta^2\eta' + \frac{3}{2}\eta^3\eta' + \frac{1+c}{12}\eta''' + \frac{7}{12}\Big(\eta\eta''' + 2\eta'\eta''\Big) = 0,$$

which, upon integration with respect to τ, yields

$$12(1-c)\eta + 18\eta^2 - 6\eta^3 + \frac{9}{2}\eta^4 + (1+c)\eta'' + \frac{7}{2}\Big((\eta')^2 + 2\eta\eta''\Big) + C = 0, \tag{4.8}$$

where C is a constant of integration. Among all the traveling wave solutions of (3.6) we shall focus on solutions which have the additional property that the waves are localized and that η and its derivatives decay at infinity, that is,

$$\eta^{(n)}(\tau) \to 0 \quad \text{ as } |\tau| \to \infty, \ \text{ for } n \in \mathbb{N}. \tag{4.9}$$

Under this decay assumption the constant of integration in (4.8) vanishes and we can conveniently rewrite it as the planar autonomous system

$$\begin{cases} \eta' = \zeta, \\ \zeta' = \dfrac{12(c-1)\eta - 18\eta^2 + 6\eta^3 - 9/2\eta^4 - 7/2\zeta^2}{1 + c + 7\eta}. \end{cases} \tag{4.10}$$

Our goal is to determine a homoclinic orbit in the phase plane starting and ending in $(0,0)$ which corresponds to a solitary traveling wave solution of (3.6). The existence of such an orbit depends on the parameter c, the wave speed. In accordance with the results in [24, 3, 16, 30], it appears only for $c > 1$ which reflects the fact that solitary waves travel at supercritical speed with Froude number greater than one. We start our analysis by determining the critical points of (4.10), that is, points where $(\eta', \zeta') = (0,0)$. After linearizing the system in the vicinity of those points to determine the local behaviour, we prove existence of a homoclinic orbit by analyzing the phase plane.

System (4.10) has at most two critical points: one at the origin, $P_0 = (0,0)$, and one given by $P_c = (\eta_c, 0)$, where η_c is the unique real root of the polynomial

$$p(\eta) = 12(c-1) - 18\eta + 6\eta^2 - 9/2\eta^3. \tag{4.11}$$

Indeed, one can show that the discriminant of $p(\eta)$ is always negative, so there are no multiple roots. Furthermore, its derivative $p'(\eta) = -27/2\eta^2 + 12\eta - 18$ has no real roots, so $p(\eta)$ has no (local) extrema and therefore has exactly one real root. Since the highest coefficient is negative and the constant term is a multiple of $(c-1)$, the real root η_c is positive for $c > 1$ and negative for $c < 1$. When $c = 1$, the root is zero in which case the two fixed points coincide at the origin. Hence, only for $c > 1$ both fixed points lie in the right half-plane where $\eta > 0$ and we expect physically relevant solitary waves of elevation. To explicitly determine η_c (cf. also Figure 8 in Section 5) one can use Cardano's formula for third order polynomials to find that

$$\eta_c = \frac{1}{9}\Big(f(c)^{1/3} - 92 f(c)^{-1/3} + 4\Big), \tag{4.12}$$

where

$$f(c) = -1556 + 972c + 36\sqrt{2469 - 2334c + 729c^2}.$$

To linearize the system near its critical points we compute the Jacobian Matrix J of (4.10) and evaluate it at P_0 and P_c. All fixed points lie on the horizontal axis of the phase plane, so the Jacobian takes the form

$$J = \begin{pmatrix} 0 & 1 \\ J_c & 0 \end{pmatrix}, \text{ where } J_c = \partial_\eta \zeta'.$$

Since the trace of J is zero, all eigenvalues at the critical points are of the form

$$\lambda^\pm = \pm\sqrt{J_c},$$

and the behaviour of the system in the vicinity of the fixed points depends on the sign of J_c. At P_0 we find that

$$J_c\Big|_{(0,0)} = \frac{12(c-1)}{c+1} \begin{cases} < 0 & \text{if } c < 1, \\ = 0 & \text{if } c = 1, \\ > 0 & \text{if } c > 1, \end{cases}$$

so the eigenvalues of J at the origin are $\lambda_0^\pm = \pm\sqrt{\frac{12(c-1)}{c+1}}$. When $c > 1$, we get two distinct real eigenvalues of opposing sign and hence P_0 is a saddle point for the linearized system. For $c < 1$, the number J_c is negative, so the eigenvalues are purely imaginary and hence P_0 is a center for the linearized system. Evaluating J_c at the other critical point $P_c = (\eta_c, 0)$ we find that

$$J_c\Big|_{(\eta_c,0)} = \frac{p'(\eta_c)\,\eta_c}{1+c+7\eta_c} \begin{cases} > 0 & \text{if } c < 1, \\ = 0 & \text{if } c = 1, \\ < 0 & \text{if } c > 1, \end{cases}$$

where $p(\eta)$ was defined in (4.11) and has η_c as its unique real root. Indeed, since p' has no real roots and is negative in zero it is always negative, hence $J_c(\eta_c, 0) < 0$ if and only if $\eta_c > 0$ which holds true whenever $c > 1$. The other case follows by the same argument. Hence, the two fixed points P_0 and P_c exchange stability as c passes from less than 1 to greater than 1. Important for our analysis is the fact that only when $c > 1$, the fixed point P_c lies in the right half-plane where $\eta > 0$. In this case, we can hope to find a homoclinic orbit emerging and returning to the origin since P_c is a center whereas P_0 is a saddle point for the linearized system. Observe that, since J_c is non zero whenever $c \neq 1$, both fixed points are hyperbolic which means that a (topological) saddle point for the linearized system remains a saddle also for the non-linear system (cf. [32], p.140). Since (4.10) is symmetric with respect to the horizontal axis, i.e. invariant under the transformation $(t, \zeta) \mapsto (-t, -\zeta)$, a linear center remains a center for the non-linear system (cf. [32], p.144). When $c = 1$ the two fixed points coincide at the origin and the Jacobian evaluated at P_0 reduces to

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.13}$$

which is a nilpotent matrix, so $P_0 = P_c$ is a non-hyperbolic fixed point. In particular, J has two zero eigenvalues in which case one can show, using an approach described in [5], that the origin is a degenerate equilibrium state, i.e. a cusp[1].

To prove existence of a homoclinic orbit we look for a solution of (4.10) which leaves the saddle point P_0 in the direction of the unstable Eigenspace spanned by the eigenvector $(1, \lambda_0^+)$, encircles the center fixed point P_c and returns to the fixed point at the origin. Such a solution exists for all times, because the right hand side of (4.10) is analytic for $\eta > -(1+c)/7$. Since $\eta' > 0$ in the upper half-plane, a solution which leaves the origin in the direction $(1, \lambda_0^+)$ goes to the right and eventually has to cross the vertical line where $\eta = \eta_c$, because ζ' is bounded from above. Indeed,

$$\zeta' = \frac{\eta\, p(\eta) - 7/2\zeta^2}{1+c+7\eta} \leq \eta\, p(\eta) - 7/2\zeta^2 \leq P_m,$$

where P_m is the unique maximum of $\eta\, p(\eta)$, with $p(\eta)$ defined in (4.11). Then the solution goes down and to the right, since $\eta' = \zeta > 0$ and $\zeta' < 0$ whenever $\eta > \eta_c$.

[1] It is shown in Theorem 67 on p.362 of [5] that if a system can be put in the form

$$\begin{cases} \eta' = \zeta, \\ \zeta' = a_k \eta^k [1 + h(\eta)] + b_n \eta^n \zeta [1 + g(\eta)] + \zeta^2 f(\eta, \zeta). \end{cases}$$

where $h(\eta), g(\eta)$ and $f(\eta,\zeta)$ are analytic in a neighbourhood of the origin, $h(0) = g(0) = 0, k \geq 2, a_k \neq 0$ and $n \geq 1$, then, if $k = 2m$, $m \geq 1$ and $b_n = 0$, the equilibrium state is degenerate. In our case when $c = 1$, we can write the system (4.10) as

$$\begin{cases} \eta' = \zeta, \\ \zeta' = -9\eta^2[1 + h(\eta)] - \frac{7}{2(2+7\eta)}\zeta^2, \end{cases}$$

where $h(\eta) = -23/6\eta + O(\eta^2)$, which satisfies the assumptions of the theorem. Hence we infer that the origin is degenerate and that the neighbourhood of zero consists of the union of two hyperbolic sectors and two separatrices.

Consequently, it has to cross the horizontal axis since otherwise, assuming that ζ tends monotonically to a constant this would imply $\zeta' \to -\infty$ in view of (4.10), which is a contradiction. Once the solution has crossed the horizontal axis, it returns to the origin in the same way in the lower half-plane by symmetry. The solution cannot return to the origin without encircling the fixed point P_c since in this case, P_0 would have an elliptic sector but we already showed that it is a saddle point (i.e. there are four hyperbolic sectors plus separatrices in the neighbourhood of P_c, see [5]). This concludes the proof of existence of a homoclinic orbit starting and ending in the origin which corresponds to a solitary traveling wave solution of (3.6), cf. Figure 7.

Remark: For $c < 1$ and as long as $\eta_c > -(1+c)/7$, the fixed point at the origin is a center and $P_c = (\eta_c, 0)$ a saddle point which now lies in the left half-plane of the phase space, so we could still hope for the existence of a homoclinic orbit emerging from $(\eta_c, 0)$, which would correspond to a solitary wave solution of (3.6) with negative water level far out. This contradicts, however, the fact that we assumed η to decay to zero at infinity.

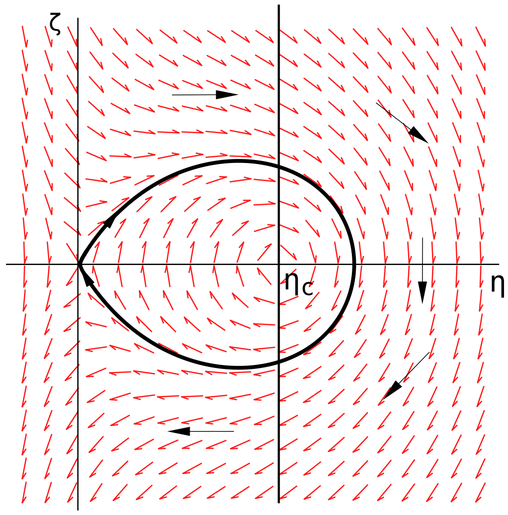

Figure 7: Phase portrait of system (4.10) for $c = 2$ with a homoclinic orbit emerging from the origin which corresponds to a solitary wave solution of the equation for surface waves of moderate amplitude in the shallow water regime.

5 Qualitative study of solitary traveling wave solutions

Despite the fact that we are not able to explicitly solve (4.10) we can work out certain features of its solitary traveling wave solutions for $c > 1$ along the lines of ideas in [14], to qualitatively describe the wave profile. Let η be a solitary traveling wave solution of (4.10). We claim that η has a single maximum. To this end, multiply equation (4.8) by η' and integrate, using the decay assumption (4.9). This gives

$$6(1-c)\eta^2 + 6\eta^3 - \frac{3}{2}\eta^4 + \frac{9}{10}\eta^5 + \frac{1+c}{2}\,[\eta']^2 + \frac{7}{2}\eta\,[\eta']^2 = 0,$$

which we rewrite as

$$[\eta']^2 = \eta^2 \frac{m(\eta)}{1+c+7\eta}, \tag{5.14}$$

where

$$m(\eta) = 12(c-1) - 12\eta + 3\eta^2 - 9/5\eta^3. \tag{5.15}$$

The discriminant of this third order polynomial is always negative, so there are no multiple roots. Furthermore, its derivative $m'(\eta)$ has no real roots, so $m(\eta)$ has no local extremum. Hence, $m(\eta)$ has only one real root which is positive since the highest coefficient is negative and the constant term is positive for $c > 1$. We conclude that η' vanishes precisely at the unique real root η_m of $m(\eta)$, so there exists a unique maximal wave height. Furthermore, this value η_m is attained at a single value of τ. To see this, assume to the contrary that there exists an interval (τ_1, τ_2) with $\eta(\tau) = \eta_m$ for all $\tau \in (\tau_1, \tau_2)$. Hence, η_m would have to satisfy equation (4.8) in that interval, which reduces to

$$12(1-c) + 18\eta_m - 6\eta_m^2 + 9/2\eta_m^3 = 0.$$

On the other hand, since η_m is a root of $m(\eta)$, it must also satisfy the equation

$$12(1-c) + 12\eta_m - 3\eta_m^2 + 9/5\eta_m^3 = 0,$$

but this is not possible. Also, assuming that there are two distinct and isolated values τ_1, τ_2 with $\eta(\tau_1) = \eta(\tau_2) = \eta_m$, there must be another maximum or minimum between these points since η decays for $|\tau| \to \infty$. Hence, there exists $\tau_0 \in (\tau_1, \tau_2)$ where $\eta'(\tau_0) = 0$ but $\eta(\tau_0) \neq \eta_m$ which contradicts the fact that $m(\eta)$ has a unique real root. To determine η_m, one can use again Cardano's formula to find that

$$\eta_m = \frac{5}{9} - \frac{155}{81} r(c)^{-1/3} + r(c)^{1/3}, \tag{5.16}$$

where

$$r(c) = 5/729(-731 + 486c + 18\sqrt{3(703 - 731c + 243c^2)}).$$

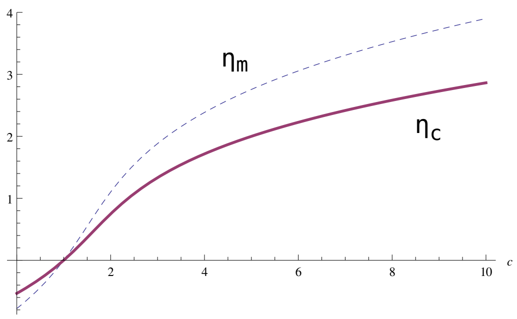

Figure 8: The x-coordinate η_c of the critical point $P_c = (\eta_c, 0)$ and the maximum height η_m of the wave profile as a function of the wave speed c.

It is important to notice that the maximal height of the wave η_m is an increasing function of the wave speed, cf. Figure 8, which means that higher waves travel faster. To see this, recall that η_m is the positive real root of the polynomial $m(\eta)$ which displays a dependence on c only in the constant term. Hence, since for $c_1 < c_2$ we have $m(\eta, c_1) - m(\eta, c_2) = 12(c_1 - c_2) < 0$, the graph of $m(\eta)$ is shifted upwards if we increase c and the zero of $m(\eta)$ is shifted to the right. We will use the fact that η_m grows with c in the comparison of solitary wave profiles of different speeds below.

We claim that the wave profile is symmetric with respect to the vertical axis, that is, we have to show that $\eta(\tau)$ is an even function of τ. To get a heuristic idea of this statement, recall (5.14) and regard η' as a function of η. This relation ensures that for every height of the profile η we get two values for the steepness of the wave at that point which only differ by sign. Hence the wave cannot be steeper on one side of the crest than on the other at the same height above the bed. To make this more precise, fix $c > 0$ and let η be a solution of (4.8) with crestpoint at $\tau = 0$. Define the function

$$\tilde{\eta}(\tau) = \begin{cases} \eta(\tau) & \text{for } \tau \in (-\infty, 0], \\ \eta(-\tau) & \text{for } \tau \in [0, \infty), \end{cases} \tag{5.17}$$

which is even by construction. On $[0, \infty)$, both η and $\tilde{\eta}$ satisfy (4.8) and since $\eta, \tilde{\eta} \in \mathcal{C}^1$ in $\tau = 0$, there exists a unique solution to the right of zero. Hence, $\eta = \tilde{\eta}$ on $[0, \infty)$ and in particular $\eta(\tau) = \tilde{\eta}(\tau) = \eta(-\tau)$ on $[0, \infty)$ which proves the claim.

The solitary wave profile decays exponentially at infinity, which can be seen by performing a Taylor expansion of the right hand side of (5.14) in η around zero. This yields $(\eta')^2 = 12\frac{c-1}{c+1}\eta^2 + O(\eta^3)$ for $|\tau| \to \infty$ and we conclude that

$$\eta(\tau) = O\left(\exp\left(-\sqrt{12\frac{c-1}{c+1}}|\tau|\right)\right) \text{ as } |\tau| \to \infty. \tag{5.18}$$

Note that the decay rate at infinity is given by the eigenvalue λ_0^- of the Jacobian matrix at the fixed point P_0, which determines the angle at which the homoclinic orbit corresponding to the solitary wave solution leaves the origin.

It is also interesting to investigate variations of the wave profile η upon changing the wave speed c. We show that two wave profiles with different speeds intersect precisely in two points. Indeed, let η be a solitary solution of (4.10) with crest point at $\tau = 0$. Since our system displays an analytic dependence on the parameter c, so does its solution and we can define the even function $f(\tau) = \partial_c \eta(\tau)$ for which we claim that it has precisely two zeros on $\mathbb{R}$. At $\tau = 0$, f is positive since the maximal height $\eta(0) = \eta_m$ is an increasing function of the wave speed c. Furthermore, notice that the decay rate of η at infinity is faster for larger c, since differentiating (5.18) with respect to c yields

$$\partial_c \eta \approx -\frac{12|\tau|}{\sqrt{12\frac{c-1}{c+1}}} \frac{1}{(1+c)^2} \eta < 0 \text{ for } |\tau| \to \infty,$$

so f is negative for large $|\tau|$. Moreover we show that the graph of f is decreasing whenever it crosses the horizontal axis, i.e. if there exists $\tau_0 > 0$ such that $f(\tau_0) = 0$, then $f'(\tau_0) < 0$. Indeed, differentiating (5.14) with respect to c gives

$$(1 + 7f)(\eta')^2 + (2 + 2c + 14\eta)\,\eta'\, f' = 2\eta\, f\, m(\eta) + \eta^2 f\,(-27/5\eta^2 + 6\eta - 12) + 12\eta^2,$$

where $'$ denotes the derivative with respect to τ and we used that $\partial_c \eta' = f'$. Evaluating this equation at τ_0, so that all the terms involving f disappear, we find that

$$(2 + 2c + 14\eta)\,\eta'\, f' = 12\eta^2 - (\eta')^2 > 0.$$

The fact that the right hand side is positive follows from (5.14), noting that $\frac{m(\eta)}{1+c+7\eta} < 12$[2]. Since the wave profile is decreasing to the right of the crest point, $\eta'(\tau_0) < 0$ and it

[2]This inequality follows by showing that $-9/5\eta^3 + 3\eta^2 - 96\eta - 24 < 0$, which amounts to proving that the polynomial has only one negative real root and then, since the highest order coefficient is negative, the left hand side is negative for all $\eta > 0$.

follows from the above inequality that also $f'(\tau_0) < 0$ which proves the claim. For the solitary wave solutions of (4.10) this means that, if we fix a wave speed c_1 and find for the corresponding wave profile $\eta(\tau, c_1)$ the unique value τ_0 at which f vanishes, then a wave profile $\eta(\tau, c_2)$ corresponding to a higher speed $c_2 > c_1$ lies above the original wave profile $\eta(\tau, c_1)$ to the left of τ_0 where f is positive, and below $\eta(\tau, c_1)$ to the right of τ_0 where f is negative. Since the wave profiles are symmetric with respect to the vertical axis, the same is true on the other side of the crest point, cf. Figure 9.

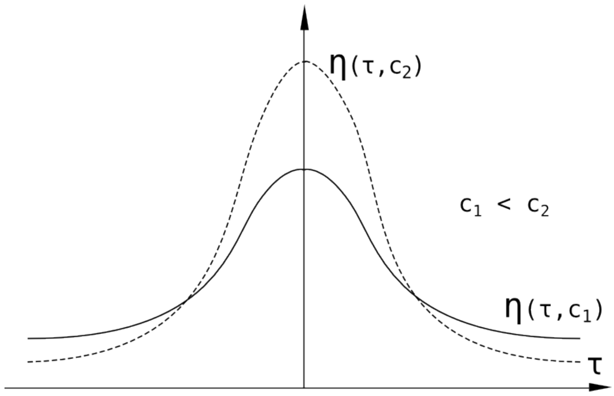

Figure 9: Solitary wave profiles with two different speeds.

Acknowledgments

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Traveling waves of moderate amplitude in shallow water

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Abstract

We prove existence of traveling wave solutions of an equation for surface waves of moderate amplitude arising as a shallow water approximation of the Euler equations for inviscid, incompressible and homogenous fluids. Our approach is based on techniques from dynamical systems and relies on a reformulation of the equation as an autonomous Hamiltonian system. We determine bounded orbits in the phase plane to establish existence of the corresponding periodic and solitary traveling wave solutions of elevation and depression, as well as of solitary waves with compact support. Furthermore, we provide a detailed analysis of the solitary waves' dependence on the wave speed.

1 Introduction

A number of competing nonlinear model equations for water waves have been proposed to this day to account for fascinating phenomena, such as wave breaking or solitary waves, which are not captured by the linear theory. The well-known Camassa–Holm equation [4] is one of the most prominent examples, due to its rich structural properties. It is an integrable infinite-dimensional Hamiltonian system [5, 8, 1] whose solitary waves are solitons [10, 12]. Some of its classical solutions develop singularities in finite time in the form of wave breaking [7], and recover in the sense of global weak solutions after blow up [3, 2]. The manifold of its fascinating features led Johnson to demonstrate the relevance of the Camassa–Holm equation as a model for the propagation of shallow water waves of moderate amplitude. He proved that the horizontal component of the fluid velocity field at a certain depth within the fluid is indeed described by a Camassa–Holm equation [15, 6]. Consequently, Constantin and Lannes followed up on the matter in search of a suitable corresponding equation for the free surface and derived an evolution equation for surface waves of moderate amplitude in the shallow water regime,

$$u_t + u_x + 6uu_x - 6u^2 u_x + 12u^3 u_x + u_{xxx} - u_{xxt} + 14uu_{xxx} + 28u_x u_{xx} = 0, \qquad (1.1)$$

which approximates the governing equations to the same order as the Camassa–Holm equation [9]. Local well-posedness for the initial value problem associated with (1.1) has been proven [9, 11] employing a semigroup approach due to Kato [16]. Owing to the involved structure of equation (1.1), not a great deal is known concerning global existence of solutions, unless one passes to a moving frame. The existence of solitary wave solutions traveling at constant speed $c > 1$ which decay to zero far out has been recently established [13], along with a detailed study of some qualitative features of the solutions. In the present paper we set out to improve this result by using techniques from the theory of dynamical systems and establish existence of traveling wave solutions of (1.1) for all waves speeds $c > 0$. We loosen the assumption that solitary waves need to return to zero water level and allow for a decay to an arbitrary constant K. This enables us to study the existence of traveling waves and their dependence on the wave speed in terms of two parameters c and K. A crucial point in our analysis is the observation that for traveling waves, equation (1.1) may be rewritten as an autonomous Hamiltonian system of nonlinear second order ordinary differential equations. This insight allows us to explicitly determine bounded orbits in the phase plane and regions in the parameter space (c, K) which correspond to solitary and periodic traveling waves of elevation as well as depression. Furthermore, we extend the qualitative analysis by studying in detail how the wave amplitude changes with the wave speed and also prove existence of a family of solitary waves which have compact support. Our approach is in fact applicable to a wider class of nonlinear dispersive evolution equations – as an example we show existence of traveling wave solutions of the aforementioned Camassa–Holm equation, including peaked continuous solitary waves.

2 Existence of traveling waves

For traveling waves $u(x,t) = u(x - c\,t)$ propagating at constant speed c to the right, the evolution equation for surface waves of moderate amplitude (1.1) takes the form

$$u'(1-c) + 3u\,u' - \frac{3}{2}u^2\,u' + \frac{3}{2}u^3\,u' + \frac{1+c}{12}u''' + \frac{7}{12}u\,u''' + \frac{7}{6}u'\,u'' = 0,$$

where $'$ denotes differentiation with respect to $\xi = x - c\,t$. Upon integration we obtain

$$\frac{12}{7}(1-c)u + \frac{18}{7}u^2 - \frac{6}{7}u^3 + \frac{9}{14}u^4 + \frac{1+c}{7}u'' + \frac{1}{2}(u')^2 + u\,u'' + K = 0,$$

for some constant K, which we may rewrite in a more compact form before stating the main existence result of this paper.

Definition 2.1. We say that $u \in \mathcal{C}^2$ is a traveling wave solution of (1.1) if it satisfies

$$u''\,(u - \bar{u}) + \frac{1}{2}(u')^2 + F'(u) = 0, \tag{2.2}$$

where

$$F(u) = K\,u - \frac{6}{7}(c-1)\,u^2 + \frac{6}{7}u^3 - \frac{3}{14}u^4 + \frac{9}{70}u^5, \tag{2.3}$$

and

$$\bar{u} = -\frac{1+c}{7}.$$

Theorem 2.2. *There exist solitary and periodic traveling wave solutions of* (1.1) *at every speed* $c > 0$. *More precisely:*

(*i*) *For each* $c > 0$ *and* $K_L(c) \le K < K_U(c)$ *there exist solitary and periodic solutions of* (2.2), *where* $K_L(c)$ *is given by* (2.10) *and* $K_U(c)$ *is on the curve* $\mathcal{D} = 0$ *given by* (2.11) *below. Periodic waves exist also when* $K < K_L(c)$. *These are all the bounded traveling wave solutions of* (1.1).

(*ii*) *For* $K = K_L(c)$ *the solitary wave solutions have compact support.*

2.1 Proof of Theorem 2.2

Essential in proving Theorem 2.2 is the observation that the scalar equation (2.2) can be rewritten as an autonomous Hamiltonian system of nonlinear ordinary differential equations. This fact can be used in a wider context for other similar evolution equations (cf. Section 4), which is why we state it in the following, slightely more general form.

Lemma 2.3. *An autonomous ordinary differential equation of the form*

$$u''\,(u-\bar{u})+\frac{a}{2}(u')^2+F'(u)=0, \tag{2.4}$$

where a and $\bar{u}$ are constants and $F(u)=a_1u+...+a_nu^n$ is a polynomial with constant coefficients, can be rewritten equivalently as the planar system

$$\begin{cases} u'=(u-\bar{u})\,v & =H_v, \\ v'=-F'(u)-\frac{a}{2}\,v^2 & =-H_u, \end{cases} \tag{2.5}$$

with Hamiltonian

$$H(u,v)=F(u)+\frac{a}{2}\,v^2\,(u-\bar{u}). \tag{2.6}$$

Along the solution curves of (2.5), $H(u,v)=h$ *is constant and*

$$v=\pm\sqrt{\frac{2}{a}\,\frac{h-F(u)}{u-\bar{u}}}. \tag{2.7}$$

Equation (2.2) *is of this form.*

Proof. Equation (2.4) can be rewritten as a planar system by introducing a new variable $v=u'$. Upon multiplication with the integrating factor[3] $u-\bar{u}$ we obtain (2.5). The system is Hamiltonian in view of the fact that there exists a function $H(u,v)$ given by (2.6), which satisfies Hamiltonian's equations $u'=H_v$, $v'=-H_u$. Furthermore, since $H(u,v)$ is autonomous, it is constant along each orbit in the (u,v) phase plane and one can express v in terms of u on each level set of H, cf. [14]. □

The Hamiltonian formulation enables us to prove existence of bounded solutions, as it provides complete and explicit knowledge of the critical points and (closed) orbits in the phase plane (u,v) of (2.5). More precisely:

Lemma 2.4. *Periodic orbits around center points of* (2.5) *correspond to periodic wave solutions of* (2.2)*, whereas homoclinic orbits leaving and returning to the same saddle point represent solitary wave solutions. All traveling wave solutions are symmetric with respect to the crest point with one maximum (per period).*

Proof. In order to determine the existence of bounded orbits, we shall study the critical points of the Hamiltonian system. These are closely related to the stationary points of H and in turn to the local extrema of the polynomial F, since $(u',v')=(-H_v,H_u)=(0,0)$ exactly when $v=0$ and $F'(u)=0$. Notice that all critical points lie on the horizontal axis of the phase plane and that $u=\bar{u}$ is an invariant line for (2.5). It is known that any non-degenerate critical point (i.e. the Jacobian of the vector field at that point has no zero eigenvalue) of an analytic Hamiltonian system is either a topological saddle or a center (cf. [17], p. 154). This fact simplifies the analysis considerably. Computing the Jacobian J of (2.5) and evaluating it at the critical points shows that $\det J=F''(u)\,(u-\bar{u})$. Recall that a non-degenerate critical point is a center whenever $\det J>0$ and a topological saddle when $\det J<0$, cf. [17]. Hence, all further analysis regarding the number, location and type of critical points in the phase plane is based on the specific structure of the polynomial F, depending on the parameters c and K. It is straightforward to

[3] Multiplication with an integrating factor means that we change the "time" of the system. To do this, we may introduce a new independent variable $\tau=\tau(\xi)$ such that $\frac{\mathrm{d}\tau}{\mathrm{d}\xi}=\frac{1}{u-\bar{u}}$. Then $\frac{\mathrm{d}u}{\mathrm{d}\tau}=\frac{\mathrm{d}u}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}\tau}=v(u-\bar{u})$ and similarly for the second equation. For convenience we maintain the original notation.

check that F has at most two local extrema[4] and we conclude that system (2.5) has at most two critical points. Next we show how to obtain the expressions for bounded orbits corresponding to bounded traveling wave solutions from relation (2.7), cf. Figure 10, and infer some basic properties of the waves. Homoclinic orbits are obtained by letting $h_s = F(s)$, where s solves

$$F'(y) = 0 \quad \text{and} \quad \begin{cases} F''(y) < 0 & \text{when } y > \bar{u}, \\ F''(y) > 0 & \text{when } y < \bar{u}. \end{cases} \tag{2.8}$$

Regarding v in (2.7) as a function of u and choosing $h = h_s$ allows one to plot the solution curve in the phase plane. The orbit leaves the critical saddle point $(s, 0)$ and crosses the horizontal axis once at $(m, 0)$ before returning to the saddle point symmetrically with respect to the horizontal axis. The value m is obtained at the unique intersection of the horizontal line h_s with the polynomial F where $F(m) = F(s)$. The corresponding solitary wave solution therefore has a unique maximum m, is symmetric with respect to the crest point and decays to the constant s on either side of the crest.

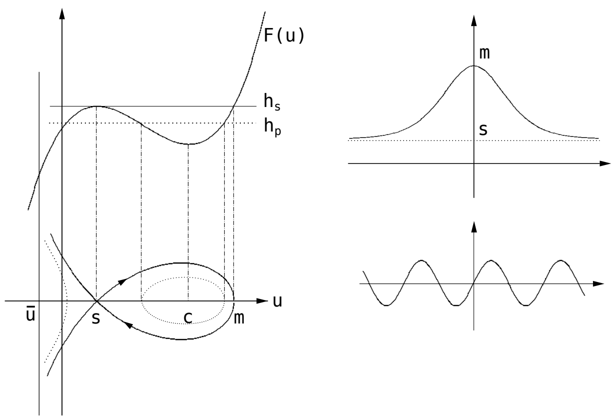

Figure 10: A sketch of how bounded orbits in the (u, v) phase plane are obtained using relation (2.7), where choosing $h = h_p$ and $h = h_s$ give rise to periodic and solitary traveling waves respectively.

Similarly, periodic waves are obtained by choosing $h \in (h_c, h_s)$ in (2.7) where $h_c = F(c)$ and c is a solution of

$$F'(y) = 0 \quad \text{and} \quad \begin{cases} F''(y) > 0 & \text{when } y > \bar{u}, \\ F''(y) < 0 & \text{when } y < \bar{u}. \end{cases} \tag{2.9}$$

□

The next Lemma provides conditions for the existence of homoclinic and periodic orbits corresponding to solitary and periodic wave solutions of (2.2). These are all the bounded traveling wave solutions of (1.1).

Lemma 2.5. *Homoclinic and periodic orbits in the phase plane of* (2.5) *exist if and only if* $K_L(c) < K < K_U(c)$ *for every* $c > 0$. *The number* $K_L(c)$ *is given by*

$$K_L(c) = \frac{6375}{33614} + -\frac{1908}{16807}\,c - \frac{5151}{16807}\,c^2 - \frac{60}{16807}\,c^3 - \frac{9}{33614}\,c^4 \tag{2.10}$$

[4]Indeed, $F(u)$ is a fifth order polynomial whose highest coefficient is a positive constant, i.e. $F(u) \to \infty$ as $u \to \infty$ and $F(u) \to -\infty$ as $u \to -\infty$. If we write $F(u) = u\,G(u)$, then $G''(u) > 0$ for all $u \in \mathbb{R}$ and so G' is strictly increasing with exactly one real root. Therefore, G has precisely one local minimum and hence $F(u)$ has at most two local extrema.

and $K_U(c)$ *is the solution of* $\mathcal{D} = 0$ *for every* c*, where*

$$\begin{aligned}\mathcal{D} = &-\frac{38397888}{117649} + \frac{21695040}{16807}K + \frac{13094144}{117649}c - \frac{122332032}{117649}c^2 \\ &- \frac{174960}{2401}K^2 + \frac{58973184}{117649}c^3 - \frac{11337408}{117649}c^4 + \frac{23328}{343}K^3 \\ &- \frac{22394880}{16807}K\,c + \frac{7418304}{16807}K\,c^2 - \frac{279936}{2401}K^2\,c. \end{aligned} \quad (2.11)$$

Periodic orbits exist also when $K < K_L(c)$*.*

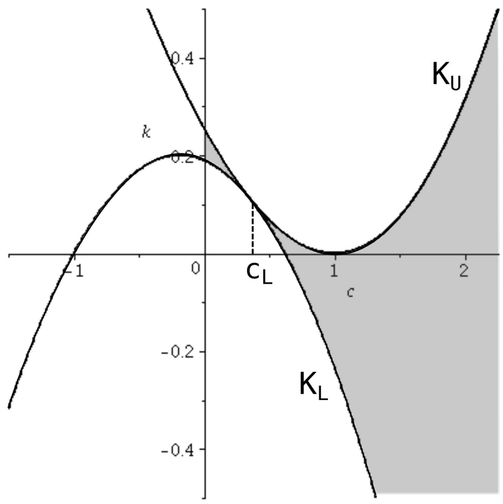

Figure 11: The shaded regions in the (c, K) parameter plane allow for homoclinic orbits which give rise to solitary traveling wave solutions of (2.2). For $0 < c < c_L$ we obtain solitary waves of depression, whereas for $c > c_L$ we obtain solitary waves of elevation.

Proof. From the previous discussion it is clear that the existence of bounded solutions relies on the fact that F has two local extrema. Indeed, bounded orbits in the phase plane arise only in the presence of critical points, which correspond to the extrema of F. To work out a condition for the appearence of bounded orbits we study the discriminant of F', which is given by the algebraic curve defined in (2.11). F has two distinct local extrema if and only if F' has two distinct roots. For each c we can compute a value $K_U(c)$ such that the discriminant of F', $\mathcal{D} = 0$, i.e. the roots of F' coincide. This guarantees existence of bounded orbits as long as $K < K_U(c)$ for each $c > 0$. To prove the lower bound on K, notice that homoclinic orbits exist only when there is one saddle point and one center point in the phase plane. This situation occurs when the extrema of F lie either to the left or to the right of the invariant line $u = \bar{u}$. Hence, we study the relation $F'(\bar{u}) = 0$ which yields relation (2.10) and conclude that homoclinic (and periodic) orbits exist for all $c > 0$ and $K_L(c) < K < K_U(c)$. □

Remark 2.6 (Solitary waves of depression and elevation). Notice that the homoclinic orbits obtained in Lemma 2.5 give rise to solitary waves of elevation and depression. More precisely, choosing $c < c_L < 1$ where c_L solves $F''(\bar{u}) = 0$,

$$c_L = \frac{1}{3}\Big(7(47 + 2\sqrt{2267})^{1/3} - 133(47 + 2\sqrt{2267})^{-1/3} - 10\Big) \approx 0.35, \quad (2.12)$$

we obtain solitary waves of depression. This is because both extrema of F lie to the left of $\bar{u}$ in this case. Hence, the orbit leaving the saddle point $(s, 0)$ corresponding to a local minimum of F has negative v-values before crossing the horizontal axis from below at $(m, 0)$, where $m < 0$, and returns to the critical point in the same way in the upper half plane. Thus, the solitary wave solution is decreasing from the constant value $s < 0$ to its minimum value m and then increasing to s again. Choosing $c > c_L$ on the other hand yields solitary waves of elevation by similar reasoning, cf. Figure 10.

Remark 2.7 (The algebraic curve $\mathcal{D}$). It is important to reflect for a moment on the algebraic curve given implicitely by $\mathcal{D}$. Denote $d_c(K) = \mathcal{D}$ and regard c as a parameter. We want to make sure that there exists a unique value K_U for each c such that $d_c(K_U) = 0$. For $c = 0$ this yields a polynomial of third order in K with one real root, so $d_0(K) = 0$ gives a unique value. Since the coefficient of the highest power in K is independent of c, no bifurcations can occur from infinity. Furthermore, we study the double roots of $d_c(K)$ which yield those values c where $\mathcal{D} = 0$ is satisfied for more than one K. To this end, consider the discriminant of $\mathcal{D}$ and K,

$$\operatorname{discrim}(d_c(K), K) = -a\,(-17 + 9c)^2\,(9c^2 - 34c + 46)^3,$$

where $a > 0$ is a constant. The unique real root of this expression is $\tilde{c} = 17/9$, and for this value

$$d_{\tilde{c}}(K) = b\,(126K - 31)\,(-128 + 63K)^2,$$

for another constant $b > 0$, which has a simple root in $K_1 = 31/126$ and a double root in $K_2 = 128/63$. Therefore, when $\tilde{c} = 17/9$ we obtain that $d_{\tilde{c}}(K_i) = 0$, so $\mathcal{D} = 0$ holds for two values K. However, since neither choice of K_i with $\tilde{c}$ in F leads to a bounded orbit as F does not have any local extrema in this case, this fact does not affect our analysis.

It turns out that homoclinic orbits exist also for $K = K_L(c)$ with the additional property that their existence time is finite. These orbits give rise to solitary wave solutions with compact support.

Lemma 2.8. *For any $c > 0$ and $K = K_L(c)$ defined in* (2.10)*, there exists a homoclinic orbit of* (2.5) *with finite exitence time $2T$, where*

$$T = \int_{\bar{u}}^{m} \frac{\mathrm{d}u}{\sqrt{(u-\bar{u})\,p(u)}} < \infty, \tag{2.13}$$

m is the non-trivial solution of $F(m) = F(\bar{u})$ and

$$p\,(u) := F''(\bar{u}) + \frac{2}{3!}F^{(3)}(\bar{u})(u-\bar{u}) + \cdots + \frac{2}{5!}F^{(5)}(\bar{u})(u-\bar{u})^3. \tag{2.14}$$

The corresponding solitary wave solution of (2.2) *has compact support on* $\mathbb{R}$.

Proof. When $K = K_L(c)$ then by construction $F'(\bar{u}) = 0$ and $h_s = F(\bar{u})$ in relation (2.7) (cf. the proof of the previous lemma and (2.8)) which yields

$$v = \pm\sqrt{2\,\frac{F(\bar{u}) - F(u)}{u - \bar{u}}} = \pm\sqrt{(u-\bar{u})\;p\,(u)}\;. \tag{2.15}$$

Notice that

$$T(u, u_0) := \int_{u_0}^{u} \frac{\mathrm{d}r}{\sqrt{(r-\bar{u})\,p(r)}}$$

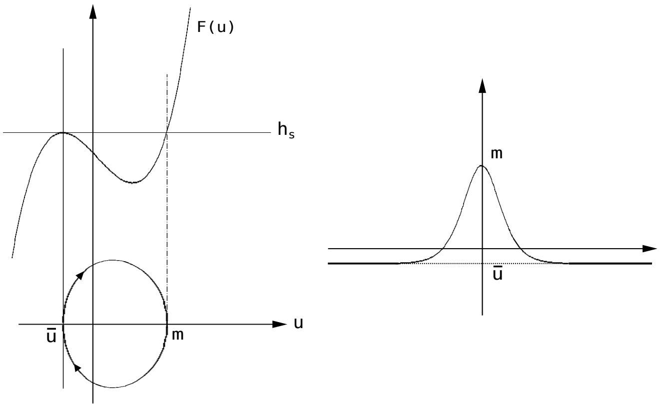

Figure 12: The choice of parameters c and $K = K_L(c)$ in F yields a homoclinic orbit with finite existence time, which gives rise to a solitary wave solution with compact support.

is an elliptic integral and therefore finite, since $p(r)$ is a third degree polynomial with no repeated roots and $r = \bar{u}$ is not a root of $p(r)$. Regarding v in (2.15) as a function of $u(\xi)$ this yields

$$T(u(\xi), \bar{u}) = \int_{u(\xi_0)}^{u(\xi)} \frac{\mathrm{d}r}{v(r)\,(r-\bar{u})} = \int_{\xi_0}^{\xi} \frac{v(u)\,(u-\bar{u})}{v(u)\,(u-\bar{u})}\,\mathrm{d}\xi = \xi - \xi_0$$

for a solution of $u'(\xi) = v(u)\,(u - \bar{u})$ with initial data $u(\xi_0) = \bar{u}$. Hence, the time it takes an orbit to get from $\bar{u}$ to m, where m is the non-trivial solution of $F(\bar{u}) = F(m)$, is given by

$$T = T(u(\xi), \bar{u}) - T(u(\xi), m) = \int_{\bar{u}}^{m} \frac{\mathrm{d}r}{\sqrt{(r-\bar{u})\,p(r)}} < \infty.$$

By symmetry it follows that the solitary traveling wave solution corresponding to the orbit with $h_s = F(\bar{u})$ is defined on the finite interval $(-T, T)$. We extend this solution to the real line by setting $u(\xi) = \bar{u}$ for $\xi \in \mathbb{R}\backslash(-T, T)$. This is possible because $u = \bar{u}$ is a constant solution of (2.2) for $K = K_L(c)$. Furthermore, $u(\xi) \to \bar{u}$ as $\xi \to \pm T$ and therefore $v \to 0$ and $v' = -F'(u) - \frac{a}{2}\,v^2 \to 0$ in view of (2.15) and (2.5). This proves that the extension to $\mathbb{R}$ is $\mathcal{C}^2$, which concludes the prove. □

Remark 2.9. The solitary wave solutions with compact support described in the previous Lemma decay exponentially to the constant $\bar{u}$. Indeed, a Taylor expansion of $(u')^2$ around $\bar{u}$ shows that $(u')^2 = -2F''(\bar{u})(u-\bar{u})^2 + O(u^3)$ in view of (2.15) and hence $u \approx \exp(\pm\sqrt{-2F''(\bar{u})}|\xi|)$ for $|\xi| \to T$. A similar result is true also in the non compact case.

3 Properties of traveling waves

We have already seen that the solutions of (2.2) are symmetric with respect to the crest point, that they have a single maximum and that solitary waves decay to a constant on either side of the crest. Next we pose the question of how the wave amplitude a, which is the difference between crest and trough, changes with respect to the wave speed c. In full generality, this task seems to be quite intricate as it involves studying complicated, high order algebraic curves. Nevertheless, it is possible to explicitely determine regions in the (c, K) plane where we can prove that $\partial_c\, a$ has a sign.

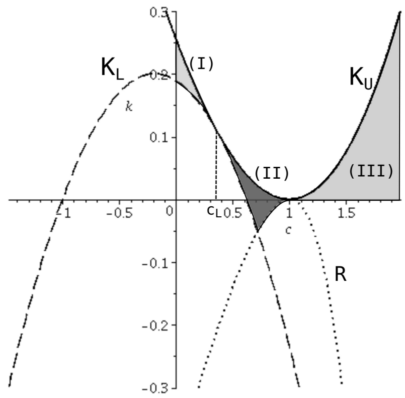

Figure 13: The choice of parameters in the regions *(I)* and *(III)* gives rise to solitary traveling wave solutions which increase with the wave speed, whereas solutions corresponding to parameters in region *(II)* are decreasing with c.

Proposition 3.1 (Amplitude as a function of the wave speed)**.**

(I) For $0 < c < c_L$ and $K_L \le K < K_U$, we find solitary waves of depression and periodic waves whose amplitude increases with the wave speed c.

(II) For $c_L < c < 1$, $K_L \le K < K_U$ and $K > R$, we find solitary waves of elevation and periodic waves whose amplitude decreases with c.

(III) For $c > 1$ and $K \ge 0$, we find solitary waves of elevation and periodic waves whose amplitude increases with the wave speed.

Before proving Proposition 3.1, we state the following preparatory Lemma (cf. Figures 10 and 12).

Lemma 3.2. *Let F be the polynomial defined in* (2.3) *and let* (s, m) *be a solution of*

$$\begin{cases} F'(s) = 0, \\ F(s) = F(m), \end{cases} \tag{3.16}$$

where $s \neq \bar{u}$, and denote the amplitude by $a = m - s$. Then

$$\partial_c\, a = \frac{-6/7}{F'(m)\, F''(s)} \Big((s^2 - m^2)\, F''(s) + 2s\, F'(m) \Big). \tag{3.17}$$

Proof of Lemma 3.2. Define

$$f(u, c) := F(u) = K\, u - \frac{6}{7}(c-1)\, u^2 + \frac{6}{7} u^3 - \frac{3}{14} u^4 + \frac{9}{70} u^5,$$

then the first equation in (3.16) rewrites as

$$f_u(s, c) = 0,$$

where subscripts denote partial differentiation. Implicit differentiation yields

$$f_{uu}(s, c)\, \dot{s} + f_{uc}(s, c) = F''(s)\, \dot{s} - \frac{12}{7} = 0,$$

where $\dot{}$ denotes differentiation with respect to c, and therefore

$$\dot{s} = \frac{12}{7}\frac{s}{F''(s)}.$$

The second equation in (3.16) reads

$$f(s,c) - f(m,c) = 0,$$

which upon implicit differentiation yields

$$f_u(s,c)\,\dot{s} + f_c(s,c) - f_u(m,c)\,\dot{m} - f_c(m,c) = 0$$

so

$$\dot{m} = \frac{f_c(s,c) - f_c(m,c)}{f_u(m,c)} = \frac{-6/7}{F'(m)}\,(s^2 - m^2).$$

Since $\dot{a} = \dot{m} - \dot{s}$, this proves (3.17). □

Proof of Proposition 3.1. Denote the wave amplitude by $a = m - s$. In case *(I)*, both critical points of F lie to the left of the invariant line $\bar{u}$, so $F'(m) > 0$ and $F''(s) > 0$. Furthermore, $s^2 < m^2$ and $s < 0$ which in view of (3.17) shows that $\dot{a} > 0$. In case *(III)*, both critical points of F are positive, so $F'(m) > 0$ and $F''(s) < 0$. Furthermore, $s^2 < m^2$ which in view of (3.17) shows that $\dot{a} > 0$. We conclude that F with parameter values in the regions determined in cases *(I)* and *(III)* yields solutions whose amplitude increases with wave speed. In case *(II)* both critical points lie between the invariant line $\bar{u}$ and 0, so $F'(m) > 0$ and $F''(s) < 0$ where $s < 0$. Hence, (3.17) has a sign only when $s^2 - m^2 > 0$. To determine the region in the (c, K)-plane where this is true, consider the system of equations

$$\begin{aligned}(s - m)\,(s + m) &= 0,\\ F'(s) &= 0,\\ F(s) - F(m) &= 0.\end{aligned}$$

Since $s = m$ is a trivial solution it suffices to study

$$\begin{aligned}&R_1 : F'(s) = 0,\\ &R_2 : F(s) - F(-s) = 0.\end{aligned}$$

This reduced system has a solution if the first and second polynomial have a common factor, i.e. if the resultant of R_1 and R_2 with respect to s is zero, which holds for $K = 0$ and along the curve given implicitely by

$$\begin{aligned}R(c,K) = {}&41220 - 1260c^2\,K + 17640\,c\,K^2 - 17280\,c^3 + 1620\,c^4 + 6174\,K^3 +\\ &70920\,c^2 - 43680\,c\,K - 96480\,c + 78540\,K - 43855\,K^2.\end{aligned} \tag{3.18}$$

Within the regions separated by these curves, the sign of $s^2 - m^2$ does not change. Hence, it is sufficient to pick one point in each region of interest and compute the absolute values of s and m and compare them in size. Let for example $c_1 = 0.6$, $K_1 = 0.03$ and $c_2 = 0.75$, $K_2 = -0.03$ be two points lying in the darker shaded region in the (c, K)-plane (cf. Figure 13). Plugging these values into F and solving system (3.16) it turns out that in both cases $|s| > |m|$. Therefore, F with parameter values in this region yields solutions with $\dot{a} < 0$, i.e. whose amplitude decreases with the wave speed. □

4 Traveling waves of the Camassa-Holm equation

We would like to point out that the method to prove existence of traveling wave solutions put forward in Section 2 is applicable to a wider class of nonlinear dispersive evolution equations. As an example, we apply this approach to the Camassa-Holm equation

$$u_t + u_{txx} + 3\,u\,u_x + 2\omega\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}, \tag{4.19}$$

for $x \in \mathbb{R}$, $t > 0$ and $\omega \in \mathbb{R}$, which is a parameter related to the critical shallow water speed. For traveling waves $u(x,t) = u(x - c\,t)$, equation (4.19) takes the form

$$u''(u-c) + \frac{(u')^2}{2} + K + (c - 2\omega)\,u - \frac{3}{2}u^2 = 0,$$

where K is a constant of integration. If instead of u we study the translate

$$w = u + c,$$

the previous equation reads

$$w''\,w + \frac{1}{2}(w')^2 + F'(w) = 0, \tag{4.20}$$

where

$$F(w) = A\,w + B\,w^2 - \frac{1}{2}w^3, \tag{4.21}$$

with constants $A = K + c\,(c - 2\omega) - \frac{3}{2}c^2$ and $B = -\omega - c$. Now, equation (4.20) is of a form appropriate to apply Lemma (2.3) to prove existence of traveling wave solutions. Indeed:

Theorem 4.1. *There exist solitary and periodic traveling wave solutions of the Camassa-Holm equation* (4.19) *for every c, K and ω satisfying*

$$-\frac{2}{3}\,B^2 < A < -\frac{1}{2}\,B^2, \tag{4.22}$$

with constants A and B defined as above. All traveling wave solutions are symmetric with respect ot the crest point and have a unique maximum (per period). Periodic orbits exist also for $A > -\frac{1}{2}B^2$.

Proof. Traveling wave solutions of the Camassa–Holm equation satisfy (4.20) which is of the form (2.4) in Lemma 2.3 with $a = 1$ and the invariant line $\bar{u} = 0$. Therefore, the corresponding planar system has a Hamiltonian structure which provides explicit knowledge of the critical points and closed orbits in the phase plane. More precisely, periodic orbits around center points of (2.5) correspond to period wave solutions of (4.20), whereas homoclinic orbits leaving and returning to a saddle point represent solitary wave solutions. These statements follow exactly as in the proof of Lemma 2.4 for F defined in (4.21). To work out conditions for the existence of homoclinic and periodic orbits we proceed as in the proof of Lemma 2.5. Bounded orbits exist as long as the local extrema of F are distinct, that is, when the discriminant of F', $discr\,F' = 6A + 4B^2$ is greater than zero. Furthermore, to guarantee homoclinic orbits there has to be one saddle point and one center point in the phase plane. This situation breaks down when $F(s) = F(\bar{u})$, where s solves $F'(s) = 0$ with $F''(s) < 0$ for $s > 0$ and $F''(s) > 0$ for $s < 0$. We find that $s = 1/3\,(2B \pm \sqrt{4B^2 + 6A})$ and $F(s) = 0$ if and only if $A = -\frac{1}{2}B^2$, which concludes the proof. □

Proposition 4.2. *There exist peaked continous solitary traveling wave solutions of* (4.19) *for* c, K *and* ω *satisfying* $A = -\frac{1}{2}B^2$.

Proof. For $A = -\frac{1}{2}B^2$ we obtain homoclinic orbits which give rise to continous solitary traveling wave solutions of (4.19) with a peaked crest. Indeed, for this choice of parameters $h = F(x) = 0$, so from (2.7) we get

$$v = \pm(B - u) \;\text{ and }\; v' = \pm 1.$$

Hence, $v \to \pm B$ when $u \to 0$, so there is a discontinuity of v at $(0, \pm B)$, the crestpoint of the solitary wave solution. However, it is straightforward to check that such a solution still satisfies the equation (4.20) in this point. □

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Deutsche Zusammenfassung

Diese Doktorarbeit befasst sich mit mathematischen Modellen, welche die Ausbreitung von Wasserwellen unter dem Einfluss von Gravitation beschreiben. Einerseits sollen Anwendungen der vollen, nicht linearen Euler Gleichungen unter Berücksichtigung von "vorticity"(Vortizität oder Wirbelströmung) auf Tsunami Wellen untersucht werden, und andererseits nichtlineare dispersive Gleichungen, die als Näherung der Euler Gleichungen für homogene, reibungsfreie und inkompressible Flüssigkeiten mit freiem Rand auftreten.

Bezüglich der mathematischen Aspekte von Tsunami Wellen präsentiere ich ein Modell, welches den Zustand des Meeres nahe der Küste in Abwesenheit von Wellen beschreibt und analysiere die Wasserbewegung vor der Ankunft eines Tsunami, welche auf Strömungen unter der Wasseroberfläche zurückzuführen ist. Die Dynamik dieser sogenannten Hintergrundströmung kann mit Hilfe der Euler Gleichungen beschrieben werden. Indem ich die Bewegungsgleichungen mittels einer geeigneten "stream function" (Strömungsfunktion) umforme und Methoden dynamischer Systeme anwende, kann ich die Existenz von radialsymmetrischen C^2-Lösungen mit kompaktem Träger für eine Familie von Wirbelverteilungen zeigen. Diese Lösungen modellieren isolierten Wirbelregionen unter der Wasseroberfläche außerhalb derer sich das Wasser in Ruhe befindet.

Weiters untersuche ich eine Gleichung für Oberflächenwellen von moderater Amplitude, welche erst kürzlich als Näherung für die Euler Gleichungen für Flachwasserwellen hergeleitet wurde. Ich konzentriere mich auf sogenannte "traveling wave solutions" dieser nichtlinearen dispersiven Gleichung. Das sind Wellen, die sich mit konstanter Geschwindigkeit in eine Richtung ausbreiten, ohne dabei ihre Form zu verändern. Ich zeige die Existenz von solitären Wellen, also traveling waves welche im Unendlichen zur flachen Wasseroberfläche zurückkehren. Weiters präsentiere ich eine qualitative Beschreibung des Wellenprofils, in dem ich zeige, dass jede Welle ein eindeutiges Maximum besitzt und symmetrisch bezüglich der Wellenscheitellinie ist. Weiters beweise ich dass Wellen, welche sich mit höherer Geschwindigkeit ausbreiten, auch eine höhere Amplitude besitzen, und dass sich die Profile zweier Wellen, welche sich mit verschiedenen Geschwindigkeiten ausbreiten, genau in zwei Punkten schneiden. Diese Resultate kann man verallgemeinern indem man die Bewegungsgleichung für traveling waves in ein planeres autonomes Hamiltonsches System transformiert. So kann man beschränkte Orbits im Phasenraum bestimmen und explizit berechnen, welche solitären und periodischen Oberflächenwellen entsprechen. Ich erhalte so die Existenz von Oberflächenwellen für beliebige Ausbreitungsgeschwindigkeit, und insbesondere solitäre Wellen mit kompaktem Träger. Weiters analysiere ich im Detail, wie die Wellenamplitude von der Wellenausbreitungsgeschwindigkeit abhängt. Die Tatsache, dass diese Methode auch auf eine größere Klasse von nichtlinearen dispersiven Gleichungen anwendbar ist, veranschauliche ich, indem ich die Existenz von traveling waves der bekannten Camassa–Holm Gleichung zeige.

Curriculum Vitae

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List of publications

1. A. Gasull, A. Geyer, *Traveling waves of moderate amplitude in shallow water*, in preparation.
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