MASTERARBEIT

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„Coarse median structures on groups“

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Contents

Introduction

Median structures play a significant role in geometric group theory and arise in several different guises. For instance, they emerge in the theory of CAT(0) cubical complexes, where the 1-skeletons of such complexes are exactly the median graphs [Che00]. Median structures also appear on asymptotic cones of mapping class groups [BDS11]. Moreover, Behrstock–Minsky have proven the rapid decay property for the mapping class groups using a certain median-like construction [BM11]. Furthermore, Chatterji–Druţu–Haglund have characterized Gromov's a-T-menability, also known as the Haagerup property, and Kazhdan's property (T) for second-countable locally compact groups in terms of isometric actions on median metric spaces [CDH10].

The main topic of the present Master thesis are *coarse median spaces*, which have been introduced by Bowditch [Bow13a] in an attempt to put some of the concepts we have alluded to above into a more general framework. A *median metric space* is a metric space in which every triple of points possesses a unique *median point*, that is, an element which is simultaneously in between any two of the three given points. A point m in a metric space (X, d) is *between* two other points x, y if $d(x, y) = d(x, m) + d(m, y)$. For every median metric space (X, d), there is ternary operation $\mu\colon X \times X \times X \to X$, called the *median* of (X, d), which sends every triple to its median point. The structure of such ternary operations has been studied in the context of *median algebras* (for surveys see [BH83; Rol98]), and Bowditch' definition of a coarse median space is a generalization of this viewpoint. — Informally, a *coarse median* on a metric space (X, d) is a ternary operation $\mu\colon X \times X \times X \to X$ which is coarsely Lipschitz and satisfies the following approximation property: For every finite subset $A \subseteq X$, there exists a finite median algebra Λ, and a diagram,

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\pi"'] & X \\ \Lambda \arrow[ur, "\lambda"'] & \end{tikzcd}$$

which almost commutes and such that $\lambda\colon \Lambda \to X$ is almost a homomorphism of the respective ternary operations. Here, "almost" means up to bounded error with respect to the distance in (X, d), where the magnitude of the error only depends on the cardinality of the finite set A. If two ternary operations $\mu, \tilde{\mu}\colon X^3 \to X$ are close up to uniformly bounded error, that is $\sup_{x,y,z \in X} d(\mu(x, y, z), \tilde{\mu}(x, y, z)) < \infty$, then μ is a coarse median if and only if $\tilde{\mu}$ is. In this case, we say that μ and $\tilde{\mu}$ define the same *coarse median structure*.

A metric space which admits a coarse median is called a *coarse median space*. A finitely generated group is called *coarse median* if it is a coarse median space with respect to the word metric of some finite generating set. The coarse median property is a quasi-isometry invariant of the underlying metric space [Bow13a, Lemma 8.1]. In particular, a coarse median group is a coarse median space with respect to any finite generating set. The class of coarse median spaces contains all median metric spaces, finite dimensional CAT(0) cubical complexes, and the mapping class groups of orientable surfaces [Bow13a]. Furthermore, on any Gromov hyperbolic geodesic metric space, there is a *hyperbolic coarse median structure* which is defined in terms of geodesic triangles [Bow13a, Section 3]. Bowditch proves that every coarse median space satisfies a quadratic isoperimetric inequality [Bow13a, Proposition 8.2]. In particular, every finitely generated coarse median group is finitely presented with an at most quadratic Dehn function. In analogy to what has been established for the mapping class groups, asymptotic cones of coarse median spaces are metric median algebras [Bow13a, Section 9], and a group which admits a left-equivariant coarse median has rapid decay [Bow11, Section 9].

The main goal of this Master thesis is to study coarse median structures on many examples of metric spaces. We particularly focus on groups. The main new feature of our work is that not only we do care about the existence of a coarse median structure on a given group or a metric space, but we also aim to establish detailed descriptions and clarify the relation of coarse medians with other geometric structures on the underlying space.

As one source of examples, we consider classical $C'(1/6)$ small cancellation groups. On the one hand, these groups are Gromov hyperbolic, and on the other hand, by Wise's "cubulation" program [Wis04], they are known to act properly and cocompactly on a CAT(0) cubical complex. Hence there are two a priori distinct ways of defining a coarse median structure on such a group. — First from hyperbolicity, and second via the quasi-isometry to the CAT(0) cubical complex (by the Švarc–Milnor lemma). We compare these two approaches and show that they yield the same coarse median structure. The proof of this fact has lead us to establish several new more general results:

1. Below we introduce the *pushforward* (and *pullback*) of a coarse median structure via a quasi-isometry. In terms of this new terminology, we prove that for a quasi-isometry $X \to Y$ between Gromov hyperbolic geodesic spaces, the push-forward of the hyperbolic coarse median structure on X is the same as the hyperbolic coarse median structure on Y. This means that the hyperbolic coarse median structure is *stable under quasi-isometries.* In particular, the hyperbolic coarse median structure is fixed by the isometry group of the underlying metric space. This also implies that the coarse median on a hyperbolic group can be assumed to be left-equivariant. In contrast, the coarse median on the Euclidean plane, which is determined by the choice of a linear basis and the corresponding l^1-metric, is easily seen not to be equivariant with respect to Euclidean rotations. This makes it clear that — although in the hyperbolic case there is a natural coarse median structure determined by the quasi-isometry type of the underlying space — in general a coarse median is an additional structure imposed on a metric space.

2. We have mentioned above that a coarse median structure on a $C'(1/6)$ small cancellation group can be defined via the quasi-isometry to the CAT(0) cubical

complex on which the group acts properly and cocompactly. Note that Wise's construction of this CAT(0) cubical complex relies on an equivariant wall structure on the group [Wis04, Section 5]. In this Master thesis, we clarify the relation between the coarse median structure and the wall structure on a $C'(1/6)$ small cancellation group by introducing the corresponding *medianization*, which is a certain median metric space in which the wall space embeds isometrically with respect to the wall distance [CDH10, Section 3]. Indeed, using stability under quasi-isometries of the hyperbolic coarse median structure, we prove that the hyperbolic coarse median structure on a $C'(1/6)$ small cancellation group is the same as the pullback of the median on the medianization associated to Wise's wall structure.

3. The crucial observation which enables the previous result is that the image of the wall structure associated to a $C'(1/6)$ small cancellation group has finite Hausdorff distance to its entire medianization. This case also arises, for example, in the measured wall structure on the real hyperbolic space [CDH07, Corollary 3.25]. To further illuminate this setting, we study abstract spaces with measured walls which have finite Hausdorff distance to their medianizations. In such a case, the embedding into the medianization is trivially a quasi-isometry, hence there is a coarse median on the wall space obtained as a pullback from the medianization. Our contribution here is that we establish an intrinsic description of the so-defined coarse median structure only in terms of the walls, so that one can define the coarse median without referring to the medianization.

Moreover, we discuss the (non-)relation of the coarse median property on groups with a-T-menability and Kazhdan's property (T). There are two observations which would naively suggest a strong positive correlation between coarse median and a-T-menability, and a negative correlation between coarse median and property (T):

(a) The characterization of Chatterji–Druţu–Haglund [CDH10, Theorem 1.2] states that a second-countable locally compact group is a-T-menable iff it admits a proper isometric action on a median space, and it has property (T) iff every isometric action on a median space has bounded orbits. Note that a discrete group cannot have both properties unless it is finite.

(b) In Section 3.3 of this Master thesis, we compare known results about the permanence of these properties under standard group-theoretical constructions including (semi-)direct products, group extensions, quotients, and free products. As a result, we observe that the class of coarse median groups shares many stability properties with the class of a-T-menable groups, and these are very different from the stability results concerning Kazhdan's property (T).

However, despite these seemingly close relationships, there are no general implications between a-T-menability (or the negation of property (T), respectively) and existence of a *coarse* median on the group. Namely, we explain in detail the explicit combinatorial construction of infinite finitely generated hyperbolic groups with Kazhdan's property

(T) due to Ballmann–Świątkowski [BŚ97]. Such a group admits a left-equivariant coarse median, although it does not admit a proper isometric action on a median space. By considering direct products of such groups, we obtain many examples of non-hyperbolic groups with property (T) which admit a left-equivariant coarse median. Conversely, the Baumslag–Solitar groups are a-T-menable [GJ03], but most of them are not coarse median because they do not satisfy a quadratic isoperimetric inequality [Ger92, Theorem B].

In a slightly different direction, we generalize the definition of coarse median groups from finitely generated discrete groups to compactly generated locally compact groups. It follows that the isometry group of a locally compact coarse median length space is coarse median itself provided that the group acts cocompactly. In particular, the isometry group of Euclidean space is coarse median. This also holds for the isometry groups of the real, complex and quaternionic hyperbolic spaces. The isometry group of quaternionic hyperbolic space has Kazhdan's property (T) [BHV08, Section 3.3], hence this yields a non-discrete example of a coarse median group with property (T). Following Bowditch' proof of the finitely generated case, we prove that compactly generated locally compact coarse median groups are compactly presented.

Finally, we mention a few open questions on coarse median spaces.

We conclude the introduction with a more technical summary of the main original contributions of this Master thesis, together with references to the relevant parts of the text.

- In Section 2.4, we give formal proofs that a metric space with a coarse median satisfies the following approximation properties.
  - Every formal expression which holds in every median algebra holds up to bounded error with respect to the coarse median. (Theorem 2.4.3)
  - Every finite subset of a coarse median space is contained in the image of a quasi-isometric embedding of a finite CAT(0) cubical complex (with modified edge lengths), where the quasi-isometry constants only depend on the cardinality of the finite subset. (Theorem 2.4.6) This is an analogue of the result that in a Gromov hyperbolic geodesic space, every finite subset can be approximated by a finite tree up to an additive error which only depends on the cardinality of the finite subset [Gro87, §6.2].
- In Section 2.2, we formalize Bowditch' observation that the existence of a coarse median is a quasi-isometry invariant by defining the *pushforward* (and *pullback*, respectively) of a coarse median structure via a quasi-isometry. — If $f\colon X \to Y$ is a quasi isometry, and $\mu\colon X^3 \to X$ a coarse median on X, we choose a quasi-isometry $g\colon Y \to X$ which is quasi-inverse to f, and set

  $$f_*\mu := f \circ \mu \circ (g \times g \times g)\colon Y^3 \to Y.$$

  We show that this yields a coarse median on Y which up to uniformly bounded error does not depend on the choice of g. We prove various formal properties of this construction (see Lemma 2.2.3). If $\tilde{\mu}\colon Y^3 \to Y$ is a coarse median on Y, we proceed similarly to define a coarse median $f^*\tilde{\mu}$ on X.

- In Lemma 2.3.7, we state that the construction of the coarse median on hyperbolic geodesic spaces is stable under quasi-isometries. That is, for a quasi-isometry $f\colon X \to Y$ between hyperbolic spaces $(X, d_X), (Y, d_Y)$ with hyperbolic coarse medians $\mu_{\mathrm{hyp},X}, \mu_{\mathrm{hyp},Y}$, we have,

$$\sup_{y_1,y_2,y_3 \in Y} d_Y\left((f_*\mu_{\mathrm{hyp},X})(y_1,y_2,y_3), \mu_{\mathrm{hyp},Y}(y_1,y_2,y_3)\right) < \infty.$$

  In Proposition 3.1.12, we deduce that every hyperbolic group admits a left-equivariant coarse median.

- In Section 2.5, we study spaces with measured walls that are at finite Hausdorff distance to their associated median spaces. We give an intrinsic description (in terms of the measured wall structure) of the coarse median on such a space obtained by pulling back the median structure from the associated median space. To formulate this description, let $(X, \mathcal{W})$ be a space with (measured) walls, $\sigma\colon X \to \mathcal{M}(X)$ the embedding of X into its medianization such that $K := d_H(\sigma(X), \mathcal{M}(X)) < \infty$, and $\mu_{\mathcal{M}}\colon \mathcal{M}(X)^3 \to \mathcal{M}(X)$ the corresponding median. Then for all $x, y, z \in X$, the set

$$\mathfrak{M}(x, y, z; K) := \{m \in X \mid \#\left(\mathcal{W}(m|x, y) \cup \mathcal{W}(m|x, z) \cup \mathcal{W}(m|y, z)\right) \leqslant K\}$$

  is non-empty and has diameter bounded above by $2K$. (The notation $\#(S)$ refers to the cardinality of the set S, and $\mathcal{W}(A|B)$ is the set of walls separating two subsets $A, B \subseteq X$.) We prove that the coarse median $\sigma^*\mu_{\mathcal{M}}$ is uniformly close to a coarse median $\mu\colon X^3 \to X$ which satisfies $\mu(x, y, z) \in \mathfrak{M}(x, y, z; K)$ for all $x, y, z \in X$. For spaces with measured walls the same result holds with $\#$ replaced by the corresponding measure.

- In Section 3.1, we note the known facts (cf. [Cor10; Cap+12]) that the quasi-isometry type of the word metric of a compactly generated locally compact group does not depend on the particular choice of compact generating set, and that an appropriate version of the Švarc–Milnor lemma holds in this context. This allows us to define the concept of a coarse median locally compact group in the same way as in the finitely generated case. We conclude that locally compact groups which act continuously, properly and cocompactly by isometries on a coarse median length space are coarse median. In Section 3.2, we review Bowditch' proof that finitely generated coarse median groups are finitely presented [Bow13a, Proposition 8.2, Corollary 8.3], and observe that it still applies in this more general setting to show that locally compact coarse median groups are compactly presented.

- In Section 4.3, we prove that the hyperbolic coarse median structure on a $C'(1/6)$ small cancellation group agrees with the coarse median pulled back from the median space associated to the wall structure on such a group constructed by Wise. In the proof, we use the stability under quasi-isometries of the coarse median structure defined by a hyperbolic space as well as the formal properties of the pull-back/push-forward construction.

- In Section 4.4, we apply the results from Section 2.5 to the wall structure of a $C'(1/6)$ small cancellation group. For the special case of the fundamental group of an orientable closed surface of genus $\geqslant 2$, we get a concrete algorithm for computing a coarse median which only uses the wall structure and a geodesic between two of the three given points (see Remark 4.4.3 and Section 4.5).
- In Section 5.3, we take direct products to show that from the existence of hyperbolic groups with property (T), it follows that there are groups with Kazhdan's property (T) that admit equivariant coarse medians of arbitrary rank.

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Chapter 1

Median spaces

In this chapter, we discuss median metric spaces and related concepts. We also set up some notation to be used in later chapters.

1.1 Preliminaries on median metric spaces

We review basic definitions regarding (pseudo-)metric spaces, median metric spaces, median graphs and CAT(0) cubical complexes.

Metric spaces

A *pseudo-metric space* is a set X together with a function $d\colon X \times X \to \mathbb{R}_{\geqslant 0}$ which is symmetric and satisfies the triangle inequality as well as $d(x,x) = 0$ for all $x \in X$. If, in addition $d(x,y) = 0$ iff $x = y$, then (X,d) is a *metric space*.

For $x \in X$ and $r \in \mathbb{R}_{\geqslant 0}$, we define the *ball* $\mathrm{B}_r(x) := \{y \in X \mid d(x,y) \leqslant r\}$ of radius r centered at x. For a subset $S \subseteq X$, we define its *r-neighborhood* by $\mathcal{U}_r(S) := \{y \in X \mid \exists x \in S : d(x,y) \leqslant r\}$. A map $f\colon (X,d) \to (X',d')$ between (pseudo-)metric spaces is called an *isometric embedding* if $d'(f(x), f(y)) = d(x,y)$ for all $x, y \in X$. An isometric embedding is called an *isometry* if it is bijective.

A metric space (X,d) is a *length space* if for every $x, y \in X$, the distance $d(x,y)$ is equal to the infinum over the lengths of all rectifiable curves connecting x to y. Moreover, (X,d) is called a *geodesic space* if for every $x, y \in X$, there exists a curve of length $d(x,y)$ connecting x to y. In this case, there always exists an isometric embedding $c\colon [0, d(x,y)] \hookrightarrow X$ such that $c(0) = x$, $c(d(x,y)) = y$. Each such map (or sometimes its image) is called a *geodesic segment* connecting x to y. We use the notation $[x,y]$ to refer to a geodesic segment connecting x to y (note that such a geodesic segment is not unique in general). A *geodesic triangle* with vertices $x, y, z \in X$ in a geodesic space X is a triple of geodesic segments $([x,y],[x,z],[y,z])$. According to these definitions, every geodesic space is a length space, and every length space is connected.

Median metric spaces

Next we introduce the concept of a median (pseudo-)metric space following Chatterji–Druţu–Haglund [CDH07; CDH10] and state basic properties.

Definition 1.1.1. Let (X, d) be a pseudo-metric space and $x, y \in X$. A point $a \in X$ is said to be *between* x and y if $d(x, y) = d(x, a) + d(a, y)$. The set of all points between x and y is called the *interval* between x and y, and we denote it by $\mathrm{I}(x, y)$.

Definition 1.1.2. Let x_1, x_2, x_3 three points in a metric space X. We define[1]

$$M(x_1, x_2, x_3) := \bigcap_{i=1}^{3} \mathrm{I}(x_i, x_{i+1}) := \mathrm{I}(x_1, x_2) \cap \mathrm{I}(x_2, x_3) \cap \mathrm{I}(x_1, x_3)$$

and call any point in $M(x_1, x_2, x_3)$ a *median point* for x_1, x_2, x_3.

Definition 1.1.3. A pseudo-metric space X is called *median* if for any three points $x_1, x_2, x_3 \in X$, the set of median points $M(x_1, x_2, x_3)$ is non-empty and has diameter 0.

A pseudo-metric space is median iff its *metric quotient*, that is the space obtained by identifying points of distance zero, is median. We can always pass to the metric quotient without loss of generality. However, it is sometimes useful to work with general pseudo-metric spaces as an intermediate step.

If X is a median metric space, then we define a ternary operation $\mu\colon X^3 \to X$ by $\{\mu(x_1, x_2, x_3)\} = M(x_1, x_2, x_3)$.

Example 1.1.4 ([CDH10, Examples 2.8.])**.** We list some examples of median metric spaces.

(a) The real line $\mathbb{R}$ is a median metric space with the median function

$$\mu_{\mathbb{R}}(x_1, x_2, x_3) = x_1 + x_2 + x_3 - \max(x_1, x_2, x_3) - \min(x_1, x_2, x_3).$$

(b) Let $(X, \mathfrak{B}, \nu)$ be a measured space. Then $\mathrm{L}^1(X, \nu)$ is a median metric space, where the median is obtained by applying $\mu_{\mathbb{R}}$ point-wise, i.e.

$$\mu(f_1, f_2, f_3)(x) = \mu_{\mathbb{R}}(f_1(x), f_2(x), f_3(x)) \quad \text{for a.e. } x \in X.$$

(c) Let $\mathrm{S}(X, \nu) = \{\chi_A \mid A \in \mathfrak{B}, \nu(A) < \infty\} \subseteq \mathrm{L}^1(X, \nu)$, where $\chi_A\colon X \to \{0, 1\}$ denotes the characteristic function of A. The space $\mathrm{S}(X, \nu)$ is a median subspace of $\mathrm{L}^1(X, \nu)$ with distance $d(\chi_A, \chi_B) = \nu(A \vartriangle B)$ and

$$\mu_{\mathrm{S}}(\chi_{A_1}, \chi_{A_2}, \chi_{A_3}) = \chi_{\bigcup_{i=1}^{3}(A_i \cap A_{i+1})} = \chi_{\bigcap_{i=1}^{3}(A_i \cup A_{i+1})}.$$

Lemma 1.1.5 ([CDH10, Corollary 2.15])**.** *Let (X, d) be a median metric space. Then the associated median operation $\mu\colon X^3 \to X$ is 1-Lipschitz, i.e. for all $x_i, x_i' \in X$, $i \in \{1, 2, 3\}$, we have*

$$d(\mu(x_1, x_2, x_3), \mu(x_1', x_2', x_3')) \leqslant d(x_1, x_1') + d(x_2, x_2') + d(x_3, x_3').$$

[1] the indices in $\bigcap_{i=1}^{3} \mathrm{I}(x_i, x_{i+1})$ are taken $\bmod 3$. It will quite frequently occur that we have to iterate over the set of all unordered pairs of three elements, and then we will use a notational convention like this.

Median graphs

Unless specified otherwise, a *graph* will be a CW-complex of dimension at most 1. We call the 0-cells of a graph Γ *vertices*, and we denote the set of vertices by $\mathcal{V}(\Gamma)$. The closed 1-cells we call *edges*, and we denote the set of edges by $\mathrm{E}(\Gamma)$. A graph is *simplicial* if there are no multiple edges[2] and no loops[3]. A graph is a *tree* if it is simply connected.

Given a connected graph Γ, we may turn it into a geodesic metric space by declaring each 1-cell to be isometric to the real interval $[0,1]$.[4] The so-constructed metric on Γ we denote by d_Γ. If Γ is a tree, then (Γ, d_Γ) is a median metric space.

We call the metric induced on $\mathcal{V}(\Gamma) \subseteq \Gamma$ by d_Γ the *combinatorial metric*, and also denote it by d_Γ. A finite sequence $x_0, \dots, x_n \in \mathcal{V}(\Gamma)$ is called a *combinatorial geodesic* connecting x_0 to x_n if $d_\Gamma(x_i, x_{i+1}) = 1$ for all $i \in \{0, \dots, n-1\}$ and $d_\Gamma(x_0, x_n) = n$. We say a connected graph Γ is *median* if $(\mathcal{V}(\Gamma), d_\Gamma)$ is a median metric space. Every tree is a median graph.

CAT(0) **cubical complexes**

Many examples of (coarse) median spaces we consider in this thesis come from CAT(0) cubical complexes.

Definition 1.1.6 (Cubical complex [BH99, Chapter I, 7.32])**.** We consider the standard n-cube $I^n = [0,1]^n$ as a CW-complex, with closed i-cells the *i-faces* $\{(t_1, \dots, t_n) \mid t_{k_1} = \varepsilon_1, \dots, t_{k_{n-i}} = \varepsilon_{n-i}\}$, where $k_1, \dots, k_{n-i} \in \{1, \dots, n\}$ are distinct indices and $\varepsilon_1, \dots, \varepsilon_{n-i} \in \{0,1\}$. The *boundary* ∂I^n is the union of all $(n-1)$-faces. The *corner* $\operatorname{Cor} I^n$, $n \geqslant 3$, is the union of all 2-faces of I^n which contain the point $0 := (0, \dots, 0)$.

A *cubical complex* is a connected CW-complex C which is a quotient $X/{\sim}$ of a disjoint union of standard cubes $X = \bigsqcup_{j \in J} I^{n_j}$, where the gluing relation $\sim$ satisfies the following properties: Let $p_j \colon I^{n_j} \to C$ denote the restriction to I^{n_j} of the quotient map $X \to C$, then we require,

(i) for every $j \in J$, the map p_j is injective,

(ii) if $p_j(I^{n_j}) \cap p_{j'}(I^{n_{j'}}) \neq \emptyset$, then there exists an isometry $\Phi_{j',j} \colon I^{n_j} \supseteq F \xrightarrow{\cong} F' \subseteq I^{n_{j'}}$, where F, F' are faces of $I^{n_j}, I^{n_{j'}}$, respectively, such that $(I^{n_j} \ni x \sim x' \in I^{n_{j'}}) \Leftrightarrow x' = \Phi_{j',j}(x)$.

Each map of the form $p_j \circ \varphi \colon I^n \hookrightarrow C$ (or its image), where $\varphi \colon I^n \xrightarrow{\cong} F \subseteq I^{n_j}$ is an isometry from I^n to an n-face of I^{n_j}, is called an *n-cube* of C.

A continuous map $f \colon C \to D$ between cubical complexes is called *combinatorial* if for each n-cube $p \colon I^n \hookrightarrow C$ of C, the map $f \circ p$ is an n-cube of D.

[2] There exists a *multiple edge* if there are two different 1-cells such that the corresponding attaching maps have the same image.

[3] A *loop* is a 1-cell such that the image of its attaching consists of a single vertex.

[4] See Bridson–Haefliger [BH99, p. 7] for details on how to construct the metric.

Definition 1.1.7 (CAT(0) cubical complex)**.** A *corner* in C is a combinatorial map $\operatorname{Cor} I^n \to C$ with $n \geqslant 3$. A cubical complex C is called *non-positively curved* if for each corner $f\colon \operatorname{Cor} I^n \to C$, there exists an n-cube $p\colon I^n \to C$ and a combinatorial map $\iota\colon \operatorname{Cor} I^n \to I^n$ such that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Cor} I^n \arrow[d, "\iota"'] \arrow[dr, "f"] & \\ I^n \arrow[r, "p"'] & C \end{tikzcd}$$

A cubical complex is called CAT(0) if it is simply connected and non-positively curved.

Remark 1.1.8. The notation "CAT(0)" stands for a global non-positive curvature condition on geodesic metric spaces [BH99, Chapter II.1]. In this context, there is a *Cartan–Hadamard theorem* [BH99, Chapter II.4] which implies that a geodesic space is CAT(0) if and only if it is *locally* CAT(0) and simply connected. Moreover, a cubical complex can be turned into a geodesic metric space by declaring each of its n-cubes to be isometric to the standard Euclidean n-cube [BH99, Chapter I.7].

It is a theorem of Gromov that — in the case of finite dimensional cubical complexes — the local CAT(0) condition can be characterized in a purely combinatorial way [BH99, Theorem II.5.20]. This combinatorial characterization is exactly the definition of non-positive curvature we have given above.[5] Together with the Cartan–Hadamard theorem, this implies that the above definition of CAT(0) cubical complexes fits into the general picture of CAT(0) metric spaces (at least in the finite dimensional case).

Apart from the short digression above, we will not be interested in the metric induced by the Euclidean cubes on a CAT(0) cubical complex. Rather than that, we consider the combinatorial metric on the vertex set induced by the 1-skeleton. The following important facts explain our interest.

Proposition 1.1.9 ([Rol98, §10], [Che00, Theorem 6.1])**.** *The* 1*-skeleton of every* CAT(0) *cubical complex is a median graph.*

Theorem 1.1.10 ([Che00, Theorem 6.1])**.** *Every simplicial median graph is isomorphic to the* 1*-skeleton of some* CAT(0) *cubical complex.*

Given a CAT(0) cubical complex C, we write $\mu_C\colon C^{(0)} \times C^{(0)} \times C^{(0)} \to C^{(0)}$ for the ternary operation induced by the median metric space $(C^{(0)}, d_{C^{(1)}})$.

1.2 Median algebras

We now consider the abstract properties of the ternary operation $\mu\colon X^3 \to X$ defined by $\{\mu(x_1, x_2, x_3)\} = M(x_1, x_2, x_3)$, where (X, d) is a median metric space. This will be

[5]The condition in [BH99, Theorem II.5.20], that the link of every vertex is a simplicial flag complex, is immediately equivalent to the condition that every corner can be filled by a cube.

key to the definition of coarse median spaces in Chapter 2. The structure appearing in this context is that of a *median algebra*. For comprehensive surveys on median algebras we refer to [BH83; Rol98]; an overview from a different perspective can also be found in [Bow13a, Sections 4–7]. Here we summarize the properties relevant to our purposes.

Definition 1.2.1. Let Λ be a set. A ternary operation $\mu\colon \Lambda^3 \to \Lambda$ is called a *median* if for every $x, y, z, u, v \in \Lambda$, the following identities hold.

$$\mu(x,y,z) = \mu(x,z,y) = \mu(y,z,x), \tag{1.2.1}$$

$$\mu(x,x,y) = x, \tag{1.2.2}$$

$$\mu(\mu(x,y,z),u,v) = \mu(x,\mu(y,u,v),\mu(z,u,v)) \tag{1.2.3}$$

A *median algebra* (Λ, μ) is a set Λ together with a median $\mu\colon \Lambda^3 \to \Lambda$.

Given a median algebra (Λ, μ) and $x, y \in \Lambda$, we define the *interval* between x and y by

$$\mathrm{I}(x,y) = \{z \in \Lambda \mid \mu(x,y,z) = z\}\,.$$

It can be shown that for all $x_1, x_2, x_3 \in \Lambda$, we have,

$$\bigcap_{i=1}^{3} \mathrm{I}(x_i, x_{i+1}) = \{\mu(x_1,x_2,x_3)\}. \tag{1.2.4}$$

Thus, the intervals determine the median algebra completely. In fact, if Λ is a set and we are given an abstract interval function $\mathrm{I} : \Lambda \times \Lambda \to \mathcal{P}(\Lambda)$ with the properties,

(i) $\mathrm{I}(x,x) = \{x\}$ for all $x \in X$,

(ii) $\bigcap_{i=1}^{3} \mathrm{I}(x_i, x_{i+1})$ has cardinality 1 for all $x_1, x_2, x_3 \in X$,

(iii) $\mathrm{I}(x,y) \subseteq \mathrm{I}(x,z)$, whenever $y \in \mathrm{I}(x,z)$,

then the ternary operation $\mu\colon X^3 \to X$ determined by I via (1.2.4) is a median in the sense of Definition 1.2.1, cf. [BH83, Theorem 2.1]. These properties are clearly satisfied by the interval function of a median metric space, which shows that every median metric space is a median algebra.

Definition 1.2.2. Let $A \subseteq \Lambda$ be a subset of a median algebra (Λ, μ).

(i) The subset A is called a *subalgebra* if for any $a, b, c \in A$, their median $\mu(a,b,c)$ lies in A as well.

(ii) The smallest subalgebra of Λ containing A is called the *subalgebra generated by A*, and we denote it by $\langle A \rangle$.

(iii) A subset $A \subseteq \Lambda$ is called *convex*, if for all $a, b \in A$, we have $\mathrm{I}(a,b) \subseteq A$.

At several occasions we will make use of the following lemma.

Lemma 1.2.3 ([Bow13a, Lemma 4.2])**.** *For every finite subset $A \subseteq \Lambda$ of a median algebra, the subalgebra generated by A is finite and satisfies $|\langle A \rangle| \leqslant 2^{2^{|A|}}$.*

Remark 1.2.4 ([Bow13a, Section 5],[Rol98, §10])**.** Every finite median algebra is isomorphic to a median algebra obtained from the vertex set of a finite CAT(0) cubical complex (see Proposition 1.1.9). Moreover, the isomorphism class of a finite CAT(0) cubical complex is uniquely determined by its structure as a median algebra.

Lemma 1.2.3 together with Remark 1.2.4 gives the following characterization of median algebras.

Corollary 1.2.5. *Let Λ be a set and $\mu\colon \Lambda^3 \to \Lambda$ a ternary operation. Then (Λ, μ) is a median algebra iff it has the following property.*

For every finite subset $A' \subseteq \Lambda$, there is another finite subset $A \subseteq \Lambda$ containing A' such that A is closed under μ and (A, μ) is isomorphic to the median structure of some finite CAT(0) *cubical complex.*

Definition 1.2.6. The *rank* of a finite median algebra is the dimension of the corresponding cubical complex. The *rank* of a general median algebra is the (possibly infinite) supremum over the ranks of all finite subalgebras.

1.3 Spaces with measured walls

In this section, we introduce spaces with measured walls and discuss their relation to median spaces. We follow Chatterji–Druţu–Haglund [CDH10, Section 3]. We start with some preliminary definitions.

A *wall* on a set X is a partition $\{h, h^c\}$ of $X = h \sqcup h^c$ consisting of two disjoint subsets. We say a wall $\{h, h^c\}$ *separates* two subsets $A, B \subseteq X$, if $A \subseteq h$ and $B \subseteq h^c$ or vice-versa. Let $\mathcal{W}$ be a collection of walls on X, then we use the notation,

$$\mathcal{W}(A|B) = \{W \in \mathcal{W} \mid W \text{ separates } A{,}B\}\,,$$

and we write $\mathcal{W}(a_1, \dots, a_n|b_1, \dots, b_n)$ for $\mathcal{W}(\{a_1, \dots, a_n\}|\{b_1, \dots, b_n\})$.

Definition 1.3.1 (Space with measured walls)**.** Let X be a set together with a collection $\mathcal{W}$ of walls, $\mathfrak{B}$ a σ-algebra on $\mathcal{W}$ and $\nu\colon \mathfrak{B} \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ a measure. We say that the tuple $(X, \mathcal{W}, \mathfrak{B}, \nu)$ is a *space with measured walls*, if $\mathcal{W}(x|y) \in \mathfrak{B}$ and $\nu(\mathcal{W}(x|y)) < \infty$ for all $x, y \in X$.

If $\mathfrak{B} = \mathcal{P}(\mathcal{W})$, where $\mathcal{P}(\mathcal{W})$ denotes the power set of $\mathcal{W}$, and ν is the counting measure, then $(X, \mathcal{W})$ is called a *discrete space with walls* or just a *space with walls*.

Every space with measured walls has an associated wall-distance defined by

$$d_{\mathcal{W}}\colon X \times X \to \mathbb{R}_{\geqslant 0}, \quad d_{\mathcal{W}}(x, y) = \nu(\mathcal{W}(x|y)),$$

which is a pseudo-metric.

Definition 1.3.2. A *morphism* $\phi\colon (X, \mathcal{W}, \mathfrak{B}, \nu) \to (X', \mathcal{W}', \mathfrak{B}', \nu')$ of spaces with measured walls is a map $\phi\colon X \to X'$ such that

(i) for every $\{h', (h')^c\} \in \mathcal{W}'$, we have $\{\phi^{-1}(h'), \phi^{-1}(h')^c\} \in \mathcal{W}$, which means there is an induced map $\phi^{\mathcal{W}} \colon \mathcal{W}' \to \mathcal{W}$,

(ii) the map $\phi^{\mathcal{W}} \colon \mathcal{W}' \to \mathcal{W}$ is measurable, surjective and satisfies $\phi^{\mathcal{W}}(\nu') = \nu$, i.e. $\nu'((\phi^{\mathcal{W}})^{-1}(B)) = \nu(B)$ for all $B \in \mathfrak{B}$.

Note that a morphism of spaces with measured walls preserves the wall-distance. Furthermore, for any morphism $\phi \colon (X, \mathcal{W}) \to (X', \mathcal{W}')$ between discrete spaces with walls, the map $\phi^{\mathcal{W}} \colon \mathcal{W}' \to \mathcal{W}$ is a bijection.

Definition 1.3.3. A group G is said to *act on space with measured walls* $(X, \mathcal{W}, \mathfrak{B}, \nu)$, if it acts on the set X by morphisms of spaces with measured walls. Moreover, if G is a topological group, we say the action is continuous if it is point-wise continuous, that is $\mathrm{ev}_x \colon G \to X$, $g \mapsto g \cdot x$ is continuous with respect to the wall distance on X for every $x \in X$.

Example 1.3.4 ([CDH10, Theorem 5.1])**.** Every median metric space (X, d) admits a structure of a space with measured walls $(X, \mathcal{W}, \mathfrak{B}, \nu)$ such that $d = d_{\mathcal{W}}$ and every isometry $(X, d) \xrightarrow{\cong} (X, d)$ is an automorphism of $(X, \mathcal{W}, \mathfrak{B}, \nu)$.

Example 1.3.5 ([Sag95, Section 4.2], [NR98, Section 2.3], [Rol98, §10])**.** A CAT(0) cubical complex admits a structure of a discrete space with walls on its vertex set, determined by its "geometric hyperplanes". In this example, we give an overview of the construction of this wall structure by reviewing terminology and facts from the proof of [Rol98, Theorem 10.3] and from [NR98, Section 2.3].

Intuitively, a geometric hyperplane initially consists of a single "dual" hypercube of some cube in C, and is then successively constructed by continuation through appropriate hypercubes of adjacent cubes. More precisely, in the standard n-dimensional cube I^n, there are n *midplanes*, which are defined by $M_i = \left\{(t_1, \ldots, t_n) \in I^n \;\middle|\; t_i = \frac{1}{2}\right\}$ for fixed $i \in \{1, \ldots, n\}$. A *midplane in a cubical complex* C is the image $p(M_i)$, where $p \colon I^n \hookrightarrow C$ is some cube of C and $M_i \subseteq I^n$ some midplane. We write $M \square M'$ for two midplanes $M, M' \subseteq C$ if there exists a finite sequence of midplanes $M = M_0, \ldots, M_k = M' \subseteq C$ such that $M_i \cap M_{i+1}$ is a (non-empty) midplane in C for each $i \in \{0, \ldots, k-1\}$. Then $\square$ is an equivalence relation on the set of midplanes in C. A *geometric hyperplane* of C is an $\square$-equivalence class of midplanes in C. For each geometric hyperplane H of C, the space $C \setminus (\bigcup_{M \in H} M)$ has exactly two connected components. Similarly, the 1-skeleton of C splits into exactly two connected components if we delete all the edges which have non-empty intersection with a given geometric hyperplane.

We call two edges e, e' in the 1-skeleton of C *parallel*, and write $e \parallel e'$, if there exists a geometric hyperplane H such that both e and e' intersect non-trivially with H. For each n-dimensional cube $p(I^n) \subseteq C$ which has non-empty intersection with a hyperplane H, the edges of $p(I^n)$ intersecting H are a family of 2^{n-1} pairwise parallel edges. On the other hand, each edge (and thus all other edges parallel to it in some cube) intersects non-trivially with a unique geometric hyperplane.

For every geometric hyperplane H, we define a wall on $C^{(0)}$, the half spaces of which are the intersections of the components of $C \setminus (\bigcup_{M \in H} M)$ with $C^{(0)}$. We denote the set of walls thus obtained by $\mathcal{W}_C$. The half spaces of these walls are convex in the sense that a geodesic path in $C^{(1)}$ can cross any given geometric hyperplane at most once [Sag95, Theorem 4.13]. From this fact, we conclude that the half spaces of each $W \in \mathcal{W}_C$ are convex subsets (as in Definition 1.2.2 (iii)) of the median algebra $(C^{(0)}, \mu_C)$. It also follows that $d_{C^{(1)}}(x, y) = |\mathcal{W}_C(x|y)|$ for every $x, y \in C^{(0)}$, where $d_{C^{(1)}}$ is the combinatorial distance in the 1-skeleton. In particular, $(C^{(0)}, \mathcal{W}_C)$ is a discrete space with walls with $d_{\mathcal{W}_C} = d_{C^{(1)}}$.

A standard example of a non-discrete space with measured walls is real hyperbolic space. We also review the construction of this wall structure.

Example 1.3.6 ([CDH10, Example 3.7], [Rob98])**.** Let $\mathbb{H}^n$ be the hyperboloid model of n-dimensional real hyperbolic space, i.e. $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid g(x, x) = -1, x_0 > 0\}$, where g is the Lorentzian metric $-dx_0 \otimes dx_0 + \sum_{i=1}^n dx_i \otimes dx_i$ on $\mathbb{R}^{n+1}$. The metric g restricts to a positive definite bilinear form on each tangent space $T_x\mathbb{H}^n = x^\perp = \{\xi \in \mathbb{R}^{n+1} \mid g(\xi, x) = 0\}$, and this defines a Riemannian metric of constant sectional curvature -1 on $\mathbb{H}^n$. The group of orthochronous Lorentzian transformations $\mathrm{O}_+(1, n) = \{A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid A^*g = g, Ae_0 \in \mathbb{H}^n\}$, where $e_0 = (1, 0, \ldots, 0) \in \mathbb{H}^n$, acts by isometries on $\mathbb{H}^n$. For every $x, y \in \mathbb{H}^n$ and any linear isometry $\Phi : T_x\mathbb{H}^n \to T_y\mathbb{H}^n$, there exists a unique $A \in \mathrm{O}_+(1, n)$ such that $Ax = y$ and A restricts to Φ on $T_x\mathbb{H}^n$. In particular, $\mathrm{O}_+(1, n)$ acts transitively on $\mathbb{H}^n$ and coincides with the isometry group of $\mathbb{H}^n$ [BP92, Theorem A.2.4.].

Let Σ be the set of submanifolds of $\mathbb{H}^n$ of the form $\mathbb{H}^n \cap \operatorname{span}(E, x)$, where $x \in \mathbb{H}$ and E is a $(n-1)$-dimensional subspace of $T_x\mathbb{H}^n$. (Each $S \in \Sigma$ is a copy of $\mathbb{H}^{n-1}$ isometrically embedded in $\mathbb{H}^n$ as a totally geodesic submanifold.) For each $S \in \Sigma$, the space $\mathbb{H}^n \setminus S$ has two connected components, and we define a set of walls on $\mathbb{H}^n$ by

$$\mathcal{W}_{\mathbb{H}^n} = \{\{h, h^c\} \mid h \text{ is a component of } \mathbb{H}^n \setminus S \text{ for some } S \in \Sigma\}.$$

Since for two $(n-1)$-dimensional subspaces E and F of $T_x\mathbb{H}^n$ and $T_y\mathbb{H}^n$, respectively, there is a linear isometry $T_x\mathbb{H}^n \to T_y\mathbb{H}^n$ mapping E to F, we conclude that $\mathrm{O}_+(1, n)$ acts transitively on Σ. Moreover, if h is a connected component of $\mathbb{H}^n \setminus S$, then a reflection along S maps h to $(\overline{h})^c$. This implies that $\mathrm{O}_+(1, n)$ acts transitively on $\mathcal{W}_{\mathbb{H}^n}$ as well. Let $E_0 = \mathbb{R}^{n-1} \times \{0\}$ and $S_0 = \operatorname{span}(e_0, E_0) \cap \mathbb{H} \in \Sigma$, and let h_0 be the component of $\mathbb{H}^n \setminus S_0$ where the last coordinate is positive. The stabilizer of $W_0 = \{h_0, h_0^c\} \in \mathcal{W}_{\mathbb{H}^n}$ is $\mathrm{O}_+(1, n-1) \times \{1\} \subseteq \mathrm{O}_+(1, n)$. Thus we can identify $\mathcal{W}_{\mathbb{H}^n}$ with the homogeneous space $\mathrm{O}_+(1, n)/\mathrm{O}_+(1, n-1)$. Since the Lie groups $\mathrm{O}_+(1, k)$, $k \in \mathbb{N}$, are unimodular [BP92, Proposition C.4.11], the space $\mathcal{W}_{\mathbb{H}^n} = \mathrm{O}_+(1, n)/\mathrm{O}_+(1, n-1)$ admits a non-zero $\mathrm{O}_+(1, n)$-invariant regular Borel measure [BHV08, Corollary B.1.7], which we denote by $\nu_{\mathbb{H}^n}$. It can be shown [Rob98] that there exists a constant $K > 0$ such that for all $x, y \in \mathbb{H}^n$, we have,

$$\nu_{\mathbb{H}^n}(\mathcal{W}_{\mathbb{H}^n}(x|y)) = K d_{\mathbb{H}^n}(x, y),$$

where $d_{\mathbb{H}^n}$ denotes the geodesic distance in $\mathbb{H}^n$ (the essential ingredient for the proof of this fact is that both sides of the above equation are $\mathrm{O}_+(1, n)$-invariant). In particular, $(\mathbb{H}^n, \mathcal{W}_{\mathbb{H}^n}, \mathfrak{B}, \nu_{\mathbb{H}^n})$ is a space with measured walls, with $\mathfrak{B}$ the Borel σ-algebra on $\mathcal{W}_{\mathbb{H}^n} =$

$\mathrm{O}_+(1,n)/\mathrm{O}_+(1,n-1)$. Since $\nu_{\mathbb{H}^n}$ is $\mathrm{O}_+(1,n)$-invariant, we see that $\mathrm{O}_+(1,n)$ acts on this space with measured walls.

To every space with measured walls one can associate a median metric space. We discuss this in detail by closely following [CDH10, Section 3.3].

Let $(X,\mathcal{W},\mathfrak{B},\nu)$ be a space with measured walls and $\mathcal{H}=\{h\subseteq X \mid \{h,h^c\}\in\mathcal{W}\}$. There is a canonical surjection $\pi\colon \mathcal{H}\twoheadrightarrow\mathcal{W}$, $h\mapsto\{h,h^c\}$. Let $\mathfrak{B}^{\mathcal{H}}=\{\pi^{-1}(B)\mid B\in\mathfrak{B}\}$. Then $\mathfrak{B}^{\mathcal{H}}$ is a σ-algebra and there is a measure $\nu^{\mathcal{H}}\colon\mathfrak{B}^{\mathcal{H}}\to\mathbb{R}_{\geqslant 0}\cup\{\infty\}$ given by $\nu^{\mathcal{H}}(\pi^{-1}(B))=\nu(B)$.

A map $s\colon\mathcal{W}\to\mathcal{H}$ is called an *admissible section* if $\pi\circ s=\mathrm{id}_{\mathcal{W}}$ and for all $W,W'\in\mathcal{W}$ we have $s(W)\not\subseteq(s(W'))^c$. We denote the set of admissible sections by $\Gamma(\mathcal{W})$. Every point $x\in X$ gives rise to an admissible section s_x defined by the requirement that $x\in s_x(W)$ for all $W\in\mathcal{W}$. Since an admissible section is uniquely determined by its image, we will identify s and $s(\mathcal{W})$ for $s\in\Gamma(\mathcal{W})$ whenever convenient. Using this convention, it follows that $\pi(s_x\bigtriangleup s_y)=\mathcal{W}(x|y)$ for all $x,y\in X$.

Let $x_0\in X$ be a base-point and define

$$\mathfrak{B}^{\mathcal{H}}_X=\left\{A\in\mathcal{P}(\mathcal{H})\;\middle|\;A\bigtriangleup s_{x_0}\in\mathfrak{B}^{\mathcal{H}},\nu^{\mathcal{H}}(A\bigtriangleup s_{x_0})<\infty\right\}.$$

Observe that $A\bigtriangleup s_x=(A\bigtriangleup s_y)\bigtriangleup(s_x\bigtriangleup s_y)$ and $\nu^{\mathcal{H}}(s_x\bigtriangleup s_y)=\nu(\mathcal{W}(x|y))<\infty$ for all $x,y\in X$. Since the property of being measurable with finite measure is stable under perturbation by taking symmetric differences with finite measure sets, it follows that $\mathfrak{B}^{\mathcal{H}}_X$ does not depend on the choice of base-point. In particular, $s_x\in\mathfrak{B}^{\mathcal{H}}_X$ for all $x\in X$.

Definition 1.3.7 ([CDH10, Notation 3.13])**.** Let $(X,\mathcal{W},\mathfrak{B},\nu)$ be a space with measured walls. We define its *associated median space* (or *medianization*) by

$$\mathcal{M}(X)=\mathcal{M}(X,\mathcal{W},\mathfrak{B},\nu)=\Gamma(\mathcal{W})\cap\mathfrak{B}^{\mathcal{H}}_X,$$

i.e. $\mathcal{M}(X)$ consists of all those admissible sections which differ from every s_x only by a set of finite measure. We define a pseudo-metric on $\mathcal{M}(X)$ by

$$d_{\mathcal{M}}(s,t)=\nu^{\mathcal{H}}(s\bigtriangleup t),\qquad s,t\in\mathcal{M}(X).$$

Then the map $\sigma\colon(X,d_{\mathcal{W}})\to(\mathcal{M}(X),d_{\mathcal{M}})$, $x\mapsto s_x$ is an isometric embedding.

Proposition 1.3.8 ([CDH10, Proposition 3.14])**.** *Let $(X,\mathcal{W},\mathfrak{B},\nu)$ be a space with measured walls.*

(i) The associated median space $(\mathcal{M}(X),d_{\mathcal{M}})$ is a median pseudo-metric space with median operation

$$\mu_{\mathcal{M}}(s_1,s_2,s_3)=\bigcup_{i=1}^{3}(s_i\cap s_{i+1}).$$

(ii) Any morphism $\phi\colon(X,\mathcal{W},\mathfrak{B},\nu)\to(X',\mathcal{W}',\mathfrak{B}',\nu')$ of spaces with measured walls induces an isometric embedding $\mathcal{M}(\phi)\colon\mathcal{M}(X)\to\mathcal{M}(X')$. Moreover, if the induced map $\phi^{\mathcal{W}}\colon\mathcal{W}'\to\mathcal{W}$ is a bijection, then $\mathcal{M}(\phi)$ is also a bijection.

(iii) The construction is functorial, i.e. we have $\mathcal{M}(\mathrm{id}_X)=\mathrm{id}_{\mathcal{M}(X)}$ and $\mathcal{M}(\psi\circ\phi)=\mathcal{M}(\psi)\circ\mathcal{M}(\phi)$ for appropriate morphisms ϕ and ψ.

(iv) Any group acting by automorphisms of spaces with measured walls on $(X,\mathcal{W},\mathfrak{B},\nu)$ acts by isometries on $\mathcal{M}(X)$.

Proof. (i) Consider the map

$$\chi\colon \mathcal{M}(X) \to \mathrm{S}(\mathcal{H}, \nu^{\mathcal{H}}), \quad A \mapsto \chi_{A \triangle s_{x_0}},$$

compare Example 1.1.4. Note that χ is an isometric embedding, because for all $s, t \in \mathcal{M}(X)$, we have

$$\|\chi(s) - \chi(t)\|_1 = \nu^{\mathcal{H}}(s \triangle s_{x_0} \triangle t \triangle s_{x_0}) = \nu^{\mathcal{H}}(s \triangle t) = d_{\mathcal{M}}(s, t).$$

Thus, $(\mathcal{M}(X), d_{\mathcal{M}})$ can be considered as a subspace of $\mathrm{S}(\mathcal{H}, \nu^{\mathcal{H}})$. We have to show that it is a median subspace, and for that we need the following set-theoretic computation.

Lemma 1.3.9. *Let A_i, B be sets, $i \in \{1, 2, 3\}$. Then*

$$\left(\bigcup_{i=1}^{3} A_i \cap A_{i+1}\right) \triangle B = \bigcup_{i=1}^{3} (A_i \triangle B) \cap (A_{i+1} \triangle B),$$

where indices are taken $\operatorname{mod} 3$.

Proof. Let LHS and RHS be the left- and right hand side of the above identity, respectively. If $x \in LHS$, then there are two cases: First, there exists i such that $x \in A_i \cap A_{i+1}$ but $x \notin B$. Then we clearly have $x \in (A_i \triangle B) \cap (A_{i+1} \triangle B)$ and so $x \in RHS$. Second, $x \in B$ but x is not contained in two of the three sets A_1, A_2, A_3. In particular, there exists i such that $x \in (A_i \triangle B)$ and $x \in (A_{i+1} \triangle B)$, and thus $x \in RHS$.

Conversely, if $x \in RHS$, then there exists i such that $x \in (A_i \triangle B) \cap (A_{i+1} \triangle B)$. If, in addition, $x \notin B$ holds, then we must have $x \in A_i \cap A_{i+1}$ and hence $x \in LHS$. Similarly, if $x \in B$, then $x \notin A_i$ and $x \notin A_{i+1}$. This implies $x \notin \bigcup_{i=1}^{3} A_i \cap A_{i+1}$ and we again conclude $x \in LHS$. $\square$

By Example 1.1.4 and the above lemma it follows that

$$\mu_{\mathrm{S}}(\chi(s_1), \chi(s_2), \chi(s_3)) = \chi_{\bigcup_{i=1}^{3}\left((s_i \triangle s_{x_0}) \cap (s_{i+1} \triangle s_{x_0})\right)} = \chi_{\left(\bigcup_{i=1}^{3} s_i \cap s_{i+1}\right) \triangle s_{x_0}}$$

Observe that $\bigcup_{i=1}^{3} s_i \cap s_{i+1}$ represents an admissible section and thus

$$\mu_{\mathrm{S}}(\chi(s_1), \chi(s_2), \chi(s_3)) = \chi\left(\bigcup_{i=1}^{3} s_i \cap s_{i+1}\right) \in \chi(\mathcal{M}(X)).$$

This proves that $\mathcal{M}(X)$ is a median space and also the formula for the median.

(ii) The morphism ϕ induces a measurable surjective map $\phi^{\mathcal{H}}\colon \mathcal{H}' \to \mathcal{H}, h \mapsto \phi^{-1}(h)$ with the property that for all $B \in \mathfrak{B}^{\mathcal{H}}$, we have $(\nu')^{\mathcal{H}}((\phi^{\mathcal{H}})^{-1}(B)) = \nu^{\mathcal{H}}(B)$. (The definition of a morphism requires these properties for the map $\phi^{\mathcal{W}} : \mathcal{W}' \to \mathcal{W}$, but what we have stated here is equivalent.) Let $s \in \mathcal{M}(X)$ and define

$$\mathcal{M}(\phi)(s) = \left(\phi^{\mathcal{H}}\right)^{-1}(s) = \left\{h' \in \mathcal{H}' \;\middle|\; \phi^{\mathcal{H}}(h') \in s\right\}$$

We see that $\mathcal{M}(\phi)(s)$ represents a section, because for each $\{h,h'\} \in \mathcal{W}'$, we have either $\phi^{\mathcal{H}}(h') \in s$ or $\phi^{\mathcal{H}}(h'^c) = (\phi^{\mathcal{H}}(h'))^c \in s$, but not both, and this translates to the fact that we have either $h' \in \mathcal{M}(\phi)(s)$ or $h'^c \in \mathcal{M}(\phi)(s)$, but not both. It is admissible, because if we have $h' \in \mathcal{M}(\phi)(s)$ and $k' \in \mathcal{H}'$ with $h' \subseteq k'$, it follows that $s \ni \phi^{\mathcal{H}}(h') \subseteq \phi^{\mathcal{H}}(k')$ and, since s is admissible, $\phi^{\mathcal{H}}(k') \in s$ and thus $k' \in \mathcal{M}(\phi)(s)$. Observe that $\mathcal{M}(\phi)(s_x) = \{h' \in \mathcal{H}' \mid x \in \phi^{\mathcal{H}}(h')\} = \{h' \mid \phi(x) \in h'\} = s'_{\phi(x)}$ and

$$\mathcal{M}(\phi)(s) \triangle \mathcal{M}(\phi)(t) = (\phi^{\mathcal{H}})^{-1}(s) \triangle (\phi^{\mathcal{H}})^{-1}(t) = (\phi^{\mathcal{H}})^{-1}(s \triangle t),$$

which implies $\nu'^{\mathcal{H}'}(\mathcal{M}(\phi)(s) \triangle \mathcal{M}(\phi)(t)) = \nu^{\mathcal{H}}(s \triangle t)$. This shows that $\mathcal{M}(\phi)(s) \in \mathcal{M}(X')$ and that $\mathcal{M}(\phi)\colon \mathcal{M}(X) \to \mathcal{M}(X')$ is an isometric embedding. If $\phi^{\mathcal{W}}$ is bijective, then $\phi^{\mathcal{H}}$ is bijective and we can define a map

$$\Psi\colon \mathcal{M}(X') \to \mathcal{M}(X), \quad s' \mapsto \left\{\phi^{\mathcal{H}}(h') \;\middle|\; h' \in s'\right\},$$

and verify by direct calculation that it is inverse to $\mathcal{M}(\phi)$.

(iii) is clear by construction and (iv) follows immediately from functoriality. □

Remark 1.3.10 ([CDH10, Remark 3.16])**.** Let $(X, \mathcal{W}, \mathfrak{B}, \nu)$ be a space with measured walls and $\mathcal{M}(X)$ the associated median space. Then $\mathcal{M}(X)$ admits a unique structure of a space with measured walls $(\mathcal{M}(X), \mathcal{W}_{\mathcal{M}}, \mathfrak{B}_{\mathcal{M}}, \nu_{\mathcal{M}})$ such that $\sigma\colon X \to \mathcal{M}(X)$ is a morphism and the induced map $\sigma^{\mathcal{W}}\colon (\mathcal{W}_{\mathcal{M}}, \mathfrak{B}_{\mathcal{M}}, \nu_{\mathcal{M}}) \to (\mathcal{W}, \mathfrak{B}, \nu)$ is an isomorphism of measured spaces. In this case, the wall distance $d_{\mathcal{W}_{\mathcal{M}}}$ agrees with $d_{\mathcal{M}}$.

Indeed, for each half space $h \in \mathcal{H}$ on X, let $h_{\mathcal{M}}$ be the half space on $\mathcal{M}(X)$ determined by the condition that $s \in h_{\mathcal{M}}$ iff $h \in s$ for each $s \in \mathcal{M}(X)$. Let $\mathcal{W}_{\mathcal{M}}$ be defined by the collection of half spaces $\mathcal{H}_{\mathcal{M}} = \{h_{\mathcal{M}} \mid h \in \mathcal{H}\}$. By construction, there are bijections $\mathcal{H} \xrightarrow{\cong} \mathcal{H}_{\mathcal{M}}$ and $\mathcal{W} \xrightarrow{\cong} \mathcal{W}_{\mathcal{M}}$ given by $h \mapsto h_{\mathcal{M}}$. Using the latter, we transfer the measured space structure from $\mathcal{W}$ to $\mathcal{W}_{\mathcal{M}}$ and obtain $(\mathfrak{B}_{\mathcal{M}}, \nu_{\mathcal{M}})$. Moreover, we have $(\sigma^{\mathcal{W}})(\{h_{\mathcal{M}}, h_{\mathcal{M}}^c\}) = \{\sigma^{-1}(h_{\mathcal{M}}), \sigma^{-1}(h_{\mathcal{M}})^c\}$ and $\sigma^{-1}(h_{\mathcal{M}}) = \{x \in X \mid h \in s_x\} = h$. Thus, $\sigma^{\mathcal{W}}$ is the inverse to the bijection $\mathcal{W} \xrightarrow{\cong} \mathcal{W}_{\mathcal{M}}$. To see the claim about the wall distance, we observe that $\sigma^{\mathcal{W}}$ induces a bijection

$$\mathcal{W}_{\mathcal{M}}(s|t) = \{\{h_{\mathcal{M}}, h_{\mathcal{M}}^c\} \mid h \in \mathcal{H}, h \in s, h \notin t\} \xrightarrow{\cong} \pi(s \triangle t), \tag{1.3.1}$$

and consequently $d_{\mathcal{W}_{\mathcal{M}}}(s,t) = \nu_{\mathcal{M}}(\mathcal{W}_{\mathcal{M}}(s|t)) = \nu(\pi(s \triangle t)) = \nu^{\mathcal{H}}(s \triangle t) = d_{\mathcal{M}}(s,t)$ for every $t, s \in \mathcal{M}(X)$

Proposition 1.3.11. *Let C be a* CAT(0) *cubical complex and let $(C^{(0)}, \mathcal{W}_C)$ be the wall structure on its vertex set determined by geometric hyperplanes, see Example 1.3.5. Then $\mathcal{M}(C^{(0)})$ coincides with $C^{(0)}$.*

Proof. We need to prove that every element of $\mathcal{M}(C^{(0)})$ lies in the image of $\sigma\colon C^{(0)} \to \mathcal{M}(C^{(0)})$. For that, it suffices to show that for each $x \in C^{(0)}$, $s \in \mathcal{M}(C^{(0)})$ with $d_{\mathcal{M}}(s, \sigma(x)) = 1$, there exists $y \in C^{(0)}$ with $s = \sigma(y)$. Indeed, if $d_{\mathcal{M}}(s, \sigma(x)) = 1$, then there exists a half space $h_0 \in \mathcal{H}$ such that $s \triangle s_x = \{h_0, h_0^c\}$ and $x \in h_0$. Let $y \in h_0^c$

with the property that $d_{\mathcal{W}}(y,x)$ is minimal. To finish the proof, we will verify that $d_{\mathcal{W}}(y,x) = 1$, because then clearly $s_y = s$:

Assume by contradiction that $d_{\mathcal{W}}(y,x) \geqslant 2$. Then there exists $h \in \mathcal{H}$ such that $x \in h$, $y \in h^c$ and $\{h, h^c\} \neq \{h_0, h_0^c\}$. We see that $h \cap h_0^c = \emptyset$, because for any $z \in h \cap h_0^c$, we would have $z' = \mu_C(x,y,z) \in h_0^c$ and so

$$\mathcal{W}(z'|x) = \mathcal{W}(\{y,z\}|x) \subseteq \mathcal{W}(y|x) \setminus \{\{h, h^c\}\},$$

contradicting minimality of y. So it follows that $h \subseteq h_0$. However, this implies

$$s \ni h \subseteq h_0 \notin s,$$

which is impossible because s is admissible. □

Example 1.3.12 ([CN05])**.** Let $(X, \mathcal{W})$ be a discrete space with walls. In this case, $\mathcal{M}(X)$ has a concrete interpretation as the vertex set of a CAT(0) cubical complex C. We define the 1-skeleton of C and then sketch how to construct the higher dimensional cubes (for more details we refer to [CN05]).

Write $C^{(0)} := \mathcal{M}(X)$. We construct the 1-skeleton $C^{(1)}$ of the prospective cubical complex by adding an edge between $s, t \in C^{(0)}$ whenever $d_{\mathcal{M}}(s,t) = 1$, that is s and t differ on exactly one wall. The graph $C^{(1)}$ is connected and the induced combinatorial distance on $C^{(0)}$ agrees with $d_{\mathcal{M}}$. To see this, let $s, t \in C^{(0)}$. Clearly, if there is a path of length n from s to t, then $d_{\mathcal{M}}(s,t) \leqslant n$. Conversely, there are finitely many half spaces $h_1, \ldots, h_n \in \mathcal{H}$ such that $h_i \in s$ but $h_i \notin t$, where $n = d_{\mathcal{M}}(s,t)$. Assume that h_1 is minimal among those half spaces and define

$$\tilde{s} : \mathcal{W} \to \mathcal{H}, \quad \tilde{s}(W) = \begin{cases} h_1^c & \text{if } W = \{h_1, h_1^c\}, \\ s(W) & \text{otherwise.} \end{cases}$$

As in the proof of [CN05, Lemma 2.1], we can verify that $\tilde{s}$ is an admissible section. Indeed, if we assume the contrary, then there exist $W = \{h, h^c\}, W' = \{k, k^c\} \in \mathcal{W}$ such that $\tilde{s}(W) = h \subseteq k^c = (\tilde{s}(W'))^c$. Since $\tilde{s}$ agrees with s on walls other than $\{h_1, h_1^c\}$ and s is admissible, we conclude that $\{h_1, h_1^c\} \in \{W, W'\}$. If $\{h_1, h_1^c\} = W$, then $h_1^c = \tilde{s}(\{h_1, h_1^c\}) = \tilde{s}(W) = h \subseteq k^c$. Thus $k \subseteq h_1$. Note that $k = \tilde{s}(W') = s(W')$ and hence $s \in k$. Furthermore, since t is admissible, $h_1 \notin t$ implies that $k \notin t$. Due to minimality of h_1, we get $h_1 = k$. However, this is a contradiction, because it implies $W = \{h_1, h_1^c\} = \{k, k^c\} = W'$. A similar contradiction arises in the case $\{h_1, h_1^c\} = W'$. The section $\tilde{s}$ has been constructed in such a way that $d_{\mathcal{M}}(s, \tilde{s}) = 1$ and $d_{\mathcal{M}}(\tilde{s}, t) = n - 1$. Thus there is an edge between s and $\tilde{s}$ and, by proceeding inductively, we get a path of length n from s to t.

To construct a cubical complex out of $C^{(1)}$, we add a cube whenever its 1-skeleton appears. To describe this process more precisely, we consider the edges of $C^{(1)}$ labeled by elements of $\mathcal{W}$ such that any given edge is labeled by the wall on which the adjacent admissible sections differ. Then, we define an *n-corner* to be a vertex $s \in C^{(0)}$ together with n edges $e_1, \ldots e_n$ adjacent to s, such that the walls $W_1, \ldots, W_n$ labeling $e_1, \ldots, e_n$

pairwise cross. (Two walls $\{h, h^c\}$, $\{k, k^c\}$ are said to cross, if the sets $h \cap k$, $h \cap k^c$, $h^c \cap k$, $h^c \cap k^c$ are all non-empty.) It can be shown that each n-corner is contained in a unique subgraph of $C^{(1)}$ which is isomorphic to the 1-skeleton of a n-cube such that the walls $W_1, \ldots, W_n$ correspond to its midplanes. Finally, we glue higher dimensional cubes to $C^{(1)}$ such that each n-corner is contained in a unique n-dimensional cube in the resulting cubical complex C. It can be shown that C is CAT(0). Also, it follows from the construction that the induced wall structure $\mathcal{W}_{\mathcal{M}}$ on $\mathcal{M}(X)$ is in bijective correspondence with the geometric hyperplanes of C.

1.4 Kazhdan's property (T) and a-T-menability

In this section, we briefly recall Kazhdan's property (T) and a-T-menability (the Haagerup property) in terms of isometric actions on Hilbert spaces. Then we sketch the relation to group actions on median spaces and spaces with measured walls. For a thorough discussion of these properties we refer to the books [BHV08] and [Che+01], respectively, and for a complete treatment of the relation to median spaces to the article [CDH10].

Let G be a topological group. An *affine isometric action* $\alpha\colon G \to \mathrm{Aff}(H)$ of G on a Hilbert space H is a point-wise continuous G-action of the form

$$g \cdot \xi = \alpha_g(\xi) = \rho(g)\xi + b(g),$$

where $\rho\colon G \to \mathcal{U}(H)$ is a unitary representation and $b\colon G \to H$ a 1-cocycle with coefficients in ρ, i.e. b satisfies

$$b(gh) = b(g) + \rho(g)b(h) \quad \forall g, h \in G.$$

Let $\mathrm{Z}^1(G,\rho)$ be the abelian group of 1-cocycles with coefficients in ρ and let $\mathrm{B}^1(G,\rho)$ be the 1-coboundaries, that is the subgroup of $\mathrm{Z}^1(G,\rho)$ consisting of $b\colon G \to H$ of the form $b(g) = \rho(g)\xi_0 - \xi_0$, where $\xi_0 \in H$. Recall that the first cohomology of G with coefficients in ρ is the quotient group

$$\mathrm{H}^1(G,\rho) = \frac{\mathrm{Z}^1(G,\rho)}{\mathrm{B}^1(G,\rho)}.$$

In the following lemma, we collect some equivalent conditions for an affine isometric action to have a fixed point.

Lemma 1.4.1 ([BHV08, Proposition 2.2.9.])**.** *Let $\alpha\colon G \to \mathrm{Aff}(H)$ be an affine isometric action given by a unitary representation $\rho\colon G \to \mathcal{U}(H)$ and a 1-cocycle $b \in \mathrm{Z}^1(G,\rho)$. Then the following are equivalent.*

(i) The action α has a fixed point,

(ii) every α-orbit is bounded,

(iii) there exists a bounded α-orbit,

(iv) the cocycle b represents $0 \in \mathrm{H}^1(G,\rho)$,

(v) the cocycle b is bounded.

Proof. (i) $\Rightarrow$ (ii): Let $\xi_0 \in H$ such that $\alpha_g(\xi_0) = \xi_0$ for all $g \in G$. Then for every $\xi \in H$, we conclude

$$\|\alpha_g(\xi) - \xi_0\| = \|\alpha_g(\xi) - \alpha_g(\xi_0)\| = \|\rho(g)\xi - \rho(g)\xi_0\| = \|\xi - \xi_0\|,$$

which shows that $G \cdot \xi$ is bounded.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): The circumcentre of a bounded orbit, cf. [BH99, Proposition II.2.7], is a fixed point.

(i) $\Leftrightarrow$ (iv): A point $\xi_0 \in H$ is a fixed point of α, iff for all $g \in G$ we have $\xi_0 = \rho(g)\xi_0 + b(g)$, which is equivalent to $b(g) = \xi_0 - \rho(g)\xi_0$.

(iv) $\Rightarrow$ (v) is clear because $\|\rho(g)\xi_0 - \xi_0\| \leqslant 2\|\xi_0\|$.

(v) $\Rightarrow$ (iii): If $\|b(g)\| \leqslant R$ for some $R \in \mathbb{R}_{\geqslant 0}$ and all $g \in G$, then $G \cdot 0 = \{b(g) \mid g \in G\} \subseteq \mathrm{B}_R(0)$. □

Definition 1.4.2. Let G be a topological group acting pointwise continuously by isometries on a metric space X. We say that the action is *(metrically) proper* if for all bounded subsets $B \subseteq X$, the set $\{g \in G \mid g \cdot B \cap B \neq \emptyset\}$ is relatively compact in G.

If X is *proper metric space*, that is a metric space where all closed balls are compact, then the above is equivalent to the usual topological properness condition, i.e. that for all compact subsets $K \subseteq X$, the set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is compact.

Lemma 1.4.3. *For a continuous isometric action of a locally compact group G on a metric space X, the following are equivalent.*

(i) The action is proper,

(ii) for all $x \in X$ we have $d(g \cdot x, x) \to \infty$ as $g \to \infty$,

(iii) there exists $x_0 \in X$ such that $d(g \cdot x_0, x_0) \to \infty$ as $g \to \infty$.

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$ and assume by contraposition that there exists $R \in \mathbb{R}_{\geqslant 0}$ and a net $(g_i)_i$ in G with $g_i \to \infty$ such that $d(g_i \cdot x, x) \leqslant R$ for all i. Then $\{g_i\}_i \subseteq \{g \in G \mid g \cdot \mathrm{B}_R(x) \cap \mathrm{B}_R(x) \neq \emptyset\}$ is not relatively compact and the action is not proper.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Let $B \subseteq X$ be bounded and assume w.l.o.g. that $B = \mathrm{B}_R(x_0)$ for some $R \in \mathbb{R}_{\geqslant 0}$. We claim that

$$\{g \in G \mid g \cdot \mathrm{B}_R(x_0) \cap \mathrm{B}_R(x_0) \neq \emptyset\} \subseteq \{g \in G \mid d(g \cdot x_0, x_0) \leqslant 2R\},$$

which implies properness, as (iii) means that the set on the right hand side is relatively compact. Indeed, if $g \cdot \mathrm{B}_R(x_0) \cap \mathrm{B}_R(x_0) \neq \emptyset$, then there is $x \in X$ such that $d(x, x_0) \leqslant R$ and $d(g \cdot x, x_0) \leqslant R$ and therefore

$$d(g \cdot x_0, x_0) \leqslant d(g \cdot x_0, g \cdot x) + d(g \cdot x, x_0) = d(x_0, x) + d(g \cdot x, x_0) \leqslant 2R \qquad \square$$

Definition 1.4.4. Let G be a σ-compact[6], second-countable locally compact group. We say that

(i) G has *Kazhdan's property (T)* if every affine isometric action of G on a Hilbert space has a fixed point.

(ii) G is *a-T-menable* (or G has the *Haagerup property*) if it admits a proper affine isometric action on some Hilbert space.

Property (T) and a-T-menability are opposite to each other in the sense that a group has both properties if and only if it is compact.

We note two basic permanence properties of property (T) and a-T-menability.

Proposition 1.4.5. *The direct product of two a-T-menable groups is a-T-menable.*

Proof. Let G_i be a-T-menable groups with proper affine isometric actions $\alpha^i\colon G_i \to \mathrm{Aff}(H_i)$, $\alpha^i_{g_i}(\xi_i) = \rho_i(g_i)\xi_i + b_i(g_i)$, $i \in \{1,2\}$. Then $G_1 \times G_2$ acts affinely on $H_1 \oplus H_2$ via $\alpha_{(g_1,g_2)}(\xi_1 \oplus \xi_2) = \alpha^1_{g_1}(\xi_1) \oplus \alpha^2_{g_2}(\xi_2)$, $\xi_i \in H_i$, $g_i \in G_i$. Since α^1 and α^2 are proper, we have $\left\|\alpha_{(g_1,g_2)}(0)\right\|^2 = \left\|\alpha^1_{g_1}(0)\right\|^2 + \left\|\alpha^2_{g_2}(0)\right\|^2 \to \infty$ as $(g_1, g_2) \to \infty$, so α is proper and $G_1 \times G_2$ is a-T-menable. ☐

Proposition 1.4.6 ([BHV08, Proposition 2.5.4])**.** *Let $N \trianglelefteq G$ be a closed normal subgroup such that N and G/N have property (T). Then G has property (T). In particular, property (T) is preserved by (semi-)direct products.*

Proof. Let $\alpha\colon G \to \mathrm{Aff}(H)$, $\alpha_g(\xi) = \rho(g)\xi + b(g)$ be an affine isometric action. Since N has property (T), the set $H^N = \{\xi \in H \mid \alpha_g(\xi) = \xi \ \forall g \in N\}$ is non-empty. For $\xi \in H^N$, $g \in G$ and $n \in N$, we have $\alpha_n(\alpha_g(\xi)) = \alpha_{gg^{-1}ng}(\xi) = \alpha_g(\xi)$, as $g^{-1}ng \in N$. Therefore, the set H^N is G-invariant and there is an induced action of G/N on H^N, which is defined by $(gN) \cdot \xi = \alpha_g(\xi)$, where $g \in G$, $\xi \in H^N$. Moreover, note that for all $\xi \in H^N$ and $\eta \in H$, we have $\eta \in H^N$ if and only if $\rho(g)(\xi - \eta) = \xi - \eta$ for all $g \in N$. This means that H^N is an affine subspace of H modeled on the closed linear subspace $\bigcap_{g\in N} \ker(\rho(g) - \mathrm{id}_H)$. As a consequence, we can view the action of G/N on H^N as an affine isometric action which, due to Property (T) of G/N, has a fixed point $\xi_0 \in H^N$. Clearly, we have $\alpha_g(\xi_0) = (gN) \cdot \xi_0 = \xi_0$ for all $g \in G$, and thus ξ_0 is a fixed point of α, which proves that G has property (T). ☐

Now we indicate how to reformulate property (T) and a-T-menability in terms of actions on median spaces and spaces with measured walls. For simplicity, we restrict ourselves to countable discrete groups G in the following discussion. Let G act on a space with measured walls $(X, \mathcal{W}, \mathfrak{B}, \nu)$ and denote the induced measured space structure on

[6] A topological space is called *σ-compact* if it admits a countable cover by compact subsets. The way we define Kazhdan's property (T) here is not the original definition, and only in the case of σ-compact locally compact groups it is equivalent to the original one. This is the reason why we make this restriction here; see Remark 5.1.11 for a further discussion.

the set of half spaces by $(\mathcal{H}, \mathfrak{B}^{\mathcal{H}}, \nu^{\mathcal{H}})$. For every $g \in G$, the map $\phi_g^{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}, h \mapsto g^{-1}(h)$ is measure-preserving and thus

$$\rho(g) \colon \mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}}) \to \mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}}), \quad f \mapsto f \circ \phi_g^{\mathcal{H}}$$

is a linear isometry. These considerations show that to each action on a space with measured walls, there is an induced unitary representation $\rho \colon G \to \mathcal{U}\left(\mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}})\right)$.

Lemma 1.4.7. *Let G be a countable group acting on a space with measured walls $(X, \mathcal{W}, \mathfrak{B}, \nu)$ and let $\rho \colon G \to \mathcal{U}\left(\mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}})\right)$ be the induced unitary representation. Then for any base-point $x_0 \in X$, the map*

$$b \colon G \to \mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}}), \quad g \mapsto \chi_{s_{g \cdot x_0}} - \chi_{s_{x_0}}.$$

is a 1-cocycle with coefficients in ρ which satisfies $\|b(g)\|_2^2 = d_{\mathcal{W}}(g \cdot x_0, x_0)$.

Proof. To see that $b(g) \in \mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}})$, we just note that

$$\|\chi_{s_x} - \chi_{s_y}\|_2^2 = \nu^{\mathcal{H}}(s_x \triangle s_y) = \nu(\mathcal{W}(x|y)) < \infty$$

for all $x, y \in X$. Then, for all $g, h \in G$, we calculate

$$\begin{aligned} b(gh) &= \chi_{s_{g \cdot (h \cdot x_0)}} - \chi_{s_{g \cdot x_0}} + \chi_{s_{g \cdot x_0}} - \chi_{s_{x_0}} \\ &= \rho(g) \left(\chi_{s_{h \cdot x_0}} - \chi_{s_{x_0}}\right) + \chi_{s_{g \cdot x_0}} - \chi_{s_{x_0}} \\ &= \rho(g) b(h) + b(g), \end{aligned}$$

and thus we see that b is a 1-cocycle. □

Note that in the lemma above $b(g) = \chi_{s_{x_0}} \circ \phi_g^{\mathcal{H}} - \chi_{s_{x_0}}$, so b looks like the coboundary of $\chi_{s_{x_0}}$. However, usually we have $\nu^{\mathcal{H}}(s_{x_0}) = \nu(\mathcal{W}) = \infty$ which means $\chi_{s_{x_0}} \notin \mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}})$.

Corollary 1.4.8. *Let G be a countable group acting on X, where X is either a median space or a space with measured walls.*

(i) If G has Kazhdan's property (T), then every G-orbit in X is bounded.

(ii) If G acts properly on X, then G is a-T-menable.

In particular, an infinite group with property (T) cannot act properly on a median space.

Proof. Due to Example 1.3.4, we only need to consider the case that $X = (X, \mathcal{W}, \mathfrak{B}, \nu)$ is a space with measured walls. By Lemma 1.4.7, the action of G on X induces an affine isometric action $\alpha \colon G \to \mathrm{Aff}(\mathrm{L}^2(\mathcal{H}, \nu^{\mathcal{H}}))$, $\alpha_g(f) = \rho(g) f + b(g)$, where $\|b(g)\|_2^2 = d_{\mathcal{W}}(g \cdot x_0, x_0)$.

If G has property (T), α has a fixed point and by Lemma 1.4.1, the cocycle b is bounded. Thus $G \cdot x_0 \subseteq X$ is bounded. Since the base-point $x_0 \in X$ can be chosen arbitrarily, we conclude that every G-orbit in X is bounded.

On the other hand, if G acts properly on X, then $\|\alpha_g(0) - 0\|_2 = \|b(g)\|_2 = \sqrt{d_{\mathcal{W}}(g \cdot x_0, x_0)} \to \infty$ as $g \to \infty$. This shows that α is a proper action. (In this argument we have used both directions of Lemma 1.4.3.) □

We note that the above statements (and also the converse implications) are true for all second-countable locally compact groups [CDH10, Theorem 1.2, Theorem 1.3].

Chapter 2

Coarse median spaces

In this chapter, we define and study Bowditch' coarse median spaces which is the central theme of this Master thesis. In Sections 2.1, 2.2 and 2.4, we consider general properties of coarse median spaces, and in Sections 2.3 and 2.5, we discuss the relation of coarse median spaces to δ-hyperbolic spaces and certain spaces with measured walls, respectively.

2.1 Definition and basic properties

We follow Bowditch [Bow13a] to introduce the concept of a coarse median on a metric space. We start with some preliminary definitions.

Definition 2.1.1. Let S be a set, (X, d) a metric space, $f, f' \colon S \to X$ two functions, and $h \in \mathbb{R}_{\geqslant 0}$ some constant. We say that f and f' are *h-equivalent*, and write $f \sim_h f'$, if $\sup_{s \in S} d(f(s), f'(s)) \leqslant h$. We write $f \sim f'$ if there exists some $h \in \mathbb{R}_{\geqslant 0}$ such that $f \sim_h f'$.

Let $f \colon S \to \tilde{S}$ a map between two sets. For $n \in \mathbb{N}$, we write $f^{\sqcap n} \colon S^n \to \tilde{S}^n$ for the n-fold Cartesian product of f, that is $f^{\sqcap n}(s_1, \ldots, s_n) = (f(s_1), \ldots, f(s_n))$.

Definition 2.1.2. Let X be a metric space, S a set, and $\mu_Y \colon Y^3 \to Y$ some ternary operation for each $Y \in \{X, S\}$. A map $\phi \colon S \to X$ is called an *h-quasi-morphism* $(S, \mu_S) \to (X, \mu_X)$ if $\phi \circ \mu_S \sim_h \mu_X \circ \phi^{\sqcap 3}$.

Definition 2.1.3 (Coarse median space [Bow13a])**.** Let (X, d) be a metric space. A ternary operation $\mu \colon X^3 \to X$ is called a *coarse median* if there is a constant $k \in \mathbb{R}_{\geqslant 0}$ and a non-decreasing function $h \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that the following conditions are satisfied.

(i) The map μ is *coarsely Lipschitz* with multiplicative constant k and additive constant $h(0)$, that is, for every $x, y, z, x', y', z' \in X$, we have,

$$d\left(\mu(x, y, z), \mu(x', y', z')\right) \leqslant k\left(d(x, x') + d(y, y') + d(z, z')\right) + h(0). \tag{2.1.1}$$

(ii) For every finite non-empty subset $A \subseteq X$, there is a finite median algebra (Λ, μ_Λ) together with an $h(|A|)$-quasi-morphism $\lambda \colon \Lambda \to X$ and a map $\pi \colon A \to \Lambda$ such that

$$\iota_A \sim_{h(|A|)} \lambda \circ \pi,$$

where ι_A denotes the inclusion map $A \hookrightarrow X$.

We call k and h *parameters* of the coarse median μ. (Note that only their existence is required, we will not consider a particular choice of them to be part of the structure of a coarse median.)

A metric space that admits a coarse median is called a *coarse median space*.

Definition 2.1.4. A coarse median is said to have *rank at most n* if there exist[1] parameters such that we can choose all finite median algebras in Definition 2.1.3 (ii) to have rank at most n. It is said to have *rank n* if it has rank at most n but not rank at most $n-1$.

A coarse median space is said to have *rank at most n* if it admits a coarse median of rank at most n. It is said to have *rank n* if it has rank at most n but not rank at most $n-1$.

Remark 2.1.5. If we do not care about rank, we may always assume (after changing the parameters) that $\iota_A = \lambda \circ \pi$ in Definition 2.1.3 (ii).

To see this, let $A \subseteq X$ be a finite subset of a coarse median space, and $\lambda\colon \Lambda \to X$, $\pi\colon A \to \Lambda$ as in Definition 2.1.3 (ii). Let T be some finite tree with vertex set $\mathcal{V}(T) = A$, and $\mu_T\colon A^3 \to A$ the corresponding rank 1 median structure. Let

$$\tilde{\lambda}\colon \Lambda \times A \to X, \quad \tilde{\lambda}(y,a) = \begin{cases} a & \text{if } y = \pi(a), \\ \lambda(y) & \text{otherwise.} \end{cases}$$

Let $\mathrm{pr}_\Lambda\colon \Lambda \times A \to \Lambda$ be the projection on the first factor, and observe that by $\iota_A \sim_{h(|A|)} \lambda \circ \pi$, we have,

$$\tilde{\lambda} \sim_{h(|A|)} \lambda \circ \mathrm{pr}_\Lambda .$$

Moreover, define $\tilde{\pi}\colon A \to \Lambda \times A$ by $\tilde{\pi}(a) = (\pi(a), a)$ for all $a \in A$. Clearly, we have $(\tilde{\lambda} \circ \tilde{\pi})(a) = a$ for all $a \in A$. Since λ is an $h(|A|)$-quasi-morphism, it follows that

$$\tilde{\lambda} \circ \mu_{\Lambda \times A} \sim_{h(|A|)} \lambda \circ \mathrm{pr}_\Lambda \circ \mu_{\Lambda \times A} = \lambda \circ \mu_\Lambda \circ \mathrm{pr}_\Lambda^{\sqcap 3} \sim_{h(|A|)} \mu \circ \lambda^{\sqcap 3} \circ \mathrm{pr}_\Lambda^{\sqcap 3} = \mu \circ (\lambda \circ \mathrm{pr}_\Lambda)^{\sqcap 3},$$

and by Definition 2.1.3 (i), we have $\mu \circ (\lambda \circ \mathrm{pr}_\Lambda)^{\sqcap 3} \sim_{kh(|A|)+h(0)} \tilde{\lambda}$. Thus, $\tilde{\lambda}$ is an $((k+2)h(|A|) + h(0))$-quasi-morphism. This proves the claim, as we can replace λ, π by $\tilde{\lambda}, \tilde{\pi}$ if we also replace h by $n \mapsto (k+2)h(n) + h(0)$. However, by this construction we have increased the rank of the median algebra on which our new λ is defined (because $\mathrm{rank}(\Lambda \times A) = \mathrm{rank}(\Lambda) + 1$).

We give two interpretations of why Bowditch' concept of a coarse median space is a generalization of classical median structures.

First, note that by Lemma 1.2.3, a median algebra is characterized by the fact that every finite subset is contained in a finite subalgebra (compare also Corollary 1.2.5). In comparison, the second part of Definition 2.1.3 means that every finite subset is in some coarse sense contained in a finite median algebra. In view of this, we regard the notion of a coarse median space as a generalization of a median algebra. In fact, we could define a median algebra as a set X with a ternary operation $\mu\colon X^3 \to X$ such that μ is a coarse median with respect to the discrete metric $d(x,y) = \delta_{x,y}$ on X and parameters $k = 1$, $h \equiv 0$.

Second, every median metric space as in Definition 1.1.3 is a coarse median space. Indeed, Definition 2.1.3 (i) is true with $k = 1$, $h(0) = 0$ by Lemma 1.1.5. Property (ii)

[1] i.e. if we were to write this in logical formulas, then the $\exists$-quantifiers for h and k would come before the $\forall$-quantifier over the finite subsets of X.

is also immediately seen to be true for $h \equiv 0$, as a median metric space is a particular instance of a median algebra and a finite subset of such is always contained in a finite subalgebra. Thus we have the following proposition.

Proposition 2.1.6. *Let (X, d) be a median metric space with median $\mu\colon X^3 \to X$. Then μ is a coarse median with parameters $k = 1$ and $h \equiv 0$. Moreover, if the median on X is of rank n (in the sense of Definition 1.2.6), then its rank as a coarse median is* at most n.

Example 2.1.7. The median on the vertex set of a CAT(0) cubical complex is a coarse median (with respect to the combinatorial metric induced by the 1-skeleton), the rank of which is bounded by the dimension of the cubical complex.

Remark 2.1.8. By its definition, a coarse median is stable under bounded perturbations. — If $\mu\colon X^3 \to X$ is a coarse median and $\mu'\colon X^3 \to X$ some other ternary operation with $\mu \sim \mu'$, then μ' is a coarse median with the same rank as μ (but the corresponding parameters may differ accordingly). Thus, if we are not interested in fixing parameters, it is not useful to distinguish between coarse medians which agree up to uniformly bounded error.

Definition 2.1.9. A *coarse median structure* on a coarse median space X is a $\sim$-equivalence class of coarse medians $\mu\colon X^3 \to X$.

In later parts of this Master thesis, we will often consider the question of whether two coarse medians on a given metric space are equal. In fact, this is always to be understood as the question if the corresponding coarse median structures are the same. However, to keep the notation simple, in our discussions we will usually suppress the distinction between a coarse median and the coarse median structure it represents.

It is briefly mentioned in the introduction of [Bow13a] that the class of coarse median spaces is stable under Cartesian products. In the following, we give the straightforward formal proof of this fact.

Proposition 2.1.10. *Let $\mu_X\colon X^3 \to X$ and $\mu_Y\colon Y^3 \to Y$ be coarse median spaces. Then $\mu_X \times \mu_Y$ is a coarse median on $X \times Y$ (endowed with the l^1 product metric) with* $\operatorname{rank}(\mu_X \times \mu_Y) \leqslant \operatorname{rank}\mu_X + \operatorname{rank}\mu_Y$.

Proof. Denote the parameters of μ_X and μ_Y by k_X, h_X and k_Y, h_Y, respectively. Then (2.1.1) is true for $\mu = \mu_X \times \mu_Y$, $k = k_X + k_Y$ and $h = h_X + h_Y$, hence Definition 2.1.3 (i) is satisfied. To see (ii), let $A \subseteq X \times Y$ be a finite subset and let $A_X = \operatorname{pr}_X(A)$, $A_Y = \operatorname{pr}_Y(A)$, where $\operatorname{pr}_X\colon X \times Y \to X$, $\operatorname{pr}_Y\colon X \times Y \to Y$ are the respective coordinate projections. For each $Z \in \{X, Y\}$, there exists a finite median algebra $(\Lambda_Z, \mu_{\Lambda_Z})$, an $h_Z(|A_Z|)$-quasi-morphism $\lambda_Z\colon \Lambda_Z \to Z$ and a map $\pi_Z\colon A_Z \to \Lambda_Z$ such that $\iota_{A_Z} \sim_{h_Z(|A_Z|)} \lambda_Z \circ \pi_Z$, where ι_{A_Z} is the inclusion map $A_Z \hookrightarrow Z$. Let $\Lambda = \Lambda_X \times \Lambda_Y$, $\mu_\Lambda = \mu_{\Lambda_X} \times \mu_{\Lambda_Y}$, then (Λ, μ_Λ) is a finite median algebra, the rank of which is the sum $\operatorname{rank}(\lambda_X) + \operatorname{rank}(\lambda_Y)$. Furthermore, we define $\lambda = \lambda_X \times \lambda_Y\colon \Lambda \to X \times Y$ and $\pi\colon A \to \Lambda$, $\pi(x, y) = (\pi_X(x), \pi_Y(y))$. Then we have,

$$\begin{aligned}\lambda \circ (\mu_X \times \mu_Y) &= (\lambda_X \circ \mu_X) \times (\lambda_Y \circ \mu_Y)\\ &\sim_{h_X(|A_X|)+h_Y(|A_Y|)} (\mu_X \circ \lambda_X^{\sqcap 3}) \times (\mu_Y \circ \lambda_Y^{\sqcap 3})\\ &= (\mu_X \times \mu_Y) \circ (\lambda_X \times \lambda_Y)^{\sqcap 3}\\ &= (\mu_X \times \mu_Y) \circ \lambda^{\sqcap 3},\end{aligned}$$

which, since $h_X(|A_X|) + h_Y(|A_Y|) \leqslant h_X(|A|) + h_Y(|A|) = h(|A|)$, proves that λ is an $h(|A|)$-quasi-morphism. Similarly, we have,

$$\begin{aligned}\iota_A &= (\iota_{A_X} \times \iota_{A_Y}) \circ (\mathrm{pr}_X, \mathrm{pr}_Y)\\ &\sim_{h_X(|A_X|)+h_Y(|A_Y|)} ((\lambda_X \circ \pi_X) \times (\lambda_Y \times \pi_Y)) \circ (\mathrm{pr}_X, \mathrm{pr}_Y)\\ &= \lambda \circ (\pi_X \times \pi_Y) \circ (\mathrm{pr}_X, \mathrm{pr}_Y)\\ &= \lambda \circ \pi.\end{aligned}$$

□

2.2 Quasi-isometry invariance

It has been observed by Bowditch that the existence of a coarse median (of certain rank) is a quasi-isometry invariant of the underlying metric space [Bow13a, Lemma 8.1]. In this section, we formalize and extend this observation by introducing the “pushforward” and “pullback” of a coarse median via a quasi-isometry.

Definition 2.2.1. Let $(X, d), (X', d')$ be metric spaces and $f\colon X \to X'$ a map, $\alpha \geqslant 1$, $\varepsilon \geqslant 0$. The map f is called an *(α, ε)-quasi-isometric embedding*, if for all $x, y \in X$, we have

$$\frac{1}{\alpha} d(x, y) - \varepsilon \leqslant d'(f(x), f(y)) \leqslant \alpha d(x, y) + \varepsilon.$$

A map between metric spaces is called a *quasi-isometric embedding*, if there exist some α, ε such that it is an (α, ε)-quasi-isometric embedding. The map $f\colon X \to X'$ is called a *quasi-isometry*, if it is a quasi-isometric embedding and there exists another quasi-isometric embedding $f'\colon X' \to X$ such that $f \circ f' \sim \mathrm{id}_{X'}$ and $f' \circ f \sim \mathrm{id}_X$. In this case, we call f' a *quasi-inverse* to f.

Furthermore, the map f is called *K-quasi-surjective*, where $K \in \mathbb{R}_{\geqslant 0}$, if for all $x' \in X'$, there exists $x \in X$ such that $d'(f(x), x') \leqslant K$. It is *quasi-surjective*, if there exists K such that it is K-quasi-surjective.

Observe that a map between metric spaces is a quasi-isometry iff it is a quasi-surjective quasi-isometric embedding. Moreover, any two quasi-inverses to a fixed quasi-isometry are equivalent.

We now give the main definition of this section.

Definition 2.2.2. Let $f\colon X \to Y$ be a quasi-isometry between two metric spaces and μ_X, μ_Y coarse medians on X, respectively on Y. We choose a quasi-inverse $g\colon Y \to X$ to f, and define the *pushforward* of μ_X via f by

$$f_*\mu_X := f \circ \mu_X \circ g^{\sqcap 3}\colon Y^3 \to Y,$$

and similarly, the *pullback* of μ_Y via f by

$$f^*\mu_Y := g \circ \mu_Y \circ f^{\sqcap 3}\colon X^3 \to X.$$

Note that the above definitions depend on the choice of a quasi-inverse g, which is not included in the notation. This omission is justified by the first statement of the following lemma, which implies that the maps are well-defined on the level of coarse median *structures* (see Definition 2.1.9).

Lemma 2.2.3. *Under the notation of Definition 2.2.2, we have:*

(i) The ternary operations $f_*\mu_X$ and $f^*\mu_Y$ are both coarse medians and are independent of the choice of quasi-inverse g up to $\sim$.

(ii) We have $\operatorname{rank} f_*\mu_X = \operatorname{rank} \mu_X$ *and* $\operatorname{rank} f^*\mu_Y = \operatorname{rank} \mu_Y$.

(iii) If g is some quasi-inverse to f, then the relations $f_*\mu_X \sim g^*\mu_X$ and $f^*\mu_Y \sim g_*\mu_Y$ hold.

(iv) If $f_1\colon X \to Z$ and $f_2 : Z \to Y$ are quasi-isometries, then $(f_2 \circ f_1)_*\mu_X \sim f_{2*}f_{1*}\mu_X$ and $(f_2 \circ f_1)^*\mu_Y \sim f_1^* f_2^*\mu_Y$.

(v) If $f \sim f'$, where $f'\colon X \to Y$ is some other function, then $f_*\mu_X \sim f'_*\mu_X$ and $f^*\mu_Y \sim (f')^*\mu_Y$.

(vi) If $\mu_X \sim \mu'_X$, where μ'_X is some coarse median on X, then $f_*\mu_X \sim f_*\mu'_X$.

Proof. First we will establish (i); the other statements are then mostly formal consequences of the uniqueness part in (i). To avoid repetitions we will restrict our attention to the statements involving f_*, because the other case is analogous and, by (iii), can be formally reduced to the first.

(i) Assume that f is an (α, ε)-quasi-isometry, g an $(\tilde\alpha, \tilde\varepsilon)$-quasi-isometry and let k, h be the parameters of the coarse median μ_X. Then we may calculate

$$\begin{aligned}
&d_Y(f_*\mu_X(y,z,a), f_*\mu_X(y',z',a')) \\
&\quad\leqslant \alpha\, d_X\left(\mu(g(y),g(z),g(a))), \mu(g(y'),g(z'),g(a'))\right) + \varepsilon \\
&\quad\leqslant \alpha k\left(d_X(g(y),g(y')) + d_X(g(z),g(z')) + d_X(g(a),g(a')\right) + \alpha h(0) + \varepsilon \\
&\quad\leqslant \alpha\tilde\alpha k\left(d_Y(y,y') + d_Y(y,y') + d_Y(y,y')\right) + 3\alpha k\tilde\varepsilon + \alpha h(0) + \varepsilon,
\end{aligned}$$

which shows that Definition 2.1.3 (i) is satisfied for some parameters depending only on the parameters of μ_X and the quasi-isometry constants of f and g.

If B is a finite subset of Y, then there is a finite median algebra Λ, an $h(|g(B)|)$-quasi-morphism $\lambda\colon \Lambda \to X$ and a map $\pi\colon g(B) \to \Lambda$ satisfying 2.1.3 (ii) for $A = g(B)$. We set $\tilde\lambda = f \circ \lambda$ and $\tilde\pi = \pi \circ g|_B$. By definition of quasi-morphism we have,

$$\lambda \circ \mu_\Lambda \sim_{h(|g(B)|)} \mu_X \circ \lambda^{\sqcap 3}. \tag{2.2.1}$$

Moreover,

$$(f_*\mu_X) \circ \tilde\lambda^{\sqcap 3} = f \circ \mu_X \circ (g \circ f \circ \lambda)^{\sqcap 3},$$

and $g \circ f \sim_K \mathrm{id}_X$ for some $K \geqslant 0$. A direct computation using (2.1.1) for μ_X and the quasi-isometry property of f implies that

$$f \circ \mu_X \circ (g \circ f \circ \lambda)^{\sqcap 3} \sim_L f \circ \mu_X \circ \lambda^{\sqcap 3},$$

where L is some constant which only depends on the parameters of μ_X involved in (2.1.1), the quasi-isometry parameters of f and the constant K. Additionally, using (2.2.1), we get

$$f \circ \mu_X \circ \lambda^{\sqcap 3} \sim_{L'} f \circ \lambda \circ \mu_\Lambda = \tilde{\lambda} \circ \mu_\Lambda.$$

Here L' depends on $h(|g(B)|)$ and the quasi-isometry constants of f. Altogether we get $(f_*\mu_X) \circ \tilde{\lambda}^{\sqcap 3} \sim_{L+L'} \tilde{\lambda} \circ \mu_\Lambda$, so $\tilde{\lambda}$ is an $(L+L')$-quasi-morphism. For the last requirement we just note

$$\tilde{\lambda} \circ \tilde{\pi} = f \circ \underbrace{(\lambda \circ \pi)}_{\sim \iota_{g(B)}} \circ g|_B \sim f \circ \iota_{g(B)} \circ g|_B = f \circ g|_B \sim_K \iota_B,$$

where the first equivalence is implemented by a constant depending only on the quasi-isometry constants of f and monotonically on $h(g(B))$.

Thus we have shown that all the $\sim$-relations appearing in Definition 2.1.3 (ii) hold with respect to $\tilde{\lambda}$ and $\tilde{\pi}$, where the constant implementing them only depends on f, g and $h(|g(B)|)$. Since $|g(B)| \leqslant |B|$ this means that, by keeping track of the dependencies of all the involved constants, we can define an appropriate parameter function $\tilde{h}$ such that Definition 2.1.3 (ii) is satisfied.

To see that the choice of g does not matter, we note that if g and g' are both quasi-inverse to f, then $g \sim g'$. So if we employ the quasi-isometry property of f and (2.1.1) again, we can conclude that $f \circ \mu_X \circ g^{\sqcap 3} \sim f \circ \mu_X \circ (g')^{\sqcap 3}$.

(iii) If we choose our quasi-inverses appropriately, we actually have equality $f_*\mu_X = g^*\mu_X$. Thus, by (i), we always get at least $f_*\mu_X \sim g^*\mu_X$.

(iv) Clear by definition and (i), because if g_i is a quasi-inverse to f_i for $i = 1, 2$, then $g_1 \circ g_2$ is a quasi-inverse to $f_2 \circ f_1$.

(v) Fix a quasi-inverse g of f. Then f and f' are both a quasi-inverses to g and so (iii) implies that $f_*\mu_X \sim g^*\mu_X \sim f'_*\mu_X$.

(vi) We have $\mu_X \circ g^{\sqcap 3} \sim \mu'_X \circ g^{\sqcap 3}$ and since f is a quasi-isometry, this implies $f \circ \mu_X \circ g^{\sqcap 3} \sim f \circ \mu'_X \circ g^{\sqcap 3}$.

(ii) The proof of (i) shows that the rank of $f_*\mu_X$ is at most the rank of μ_X. But we also have $g_*f_*\mu_X \sim (g \circ f)_*\mu_X \sim (\mathrm{id}_X)_*\mu_X = \mu_X$ and so, by the same reasoning, we get $\operatorname{rank} \mu_X = \operatorname{rank} g_*f_*\mu_X \leqslant \operatorname{rank} f_*\mu_X$. ☐

Remark 2.2.4. Given a metric space (X, d), we denote the set of quasi-isometries $X \to X$ modulo the equivalence relation $\sim$ by $\mathrm{QIsom}(X, d)$. Composition of maps induces a well-defined binary operation on $\mathrm{QIsom}(X, d)$ which satisfies the group axioms [BH99, Exercise I.8.16 (3)]. The resulting group is called the *quasi-isometry group* of (X, d). By the lemma above, it follows that for a coarse median space (X, d), the group $\mathrm{QIsom}(X, d)$ acts on the set of coarse median structures on X. Indeed, if $[f] \in \mathrm{QIsom}(X, d)$ is represented by a quasi-isometry $f\colon X \to X$, and $[\mu]$ a coarse median structure on X represented by a coarse median $\mu\colon X^3 \to X$, we define the action of $[f]$ on $[\mu]$ by $[f] \cdot [\mu] := [f_*\mu]$.

Example 2.2.5. If C is CAT(0) cubical complex and μ_C denotes the median on its vertex set. As mentioned in Example 2.1.7, μ_C is a coarse median on $(C^{(0)}, d_{C^{(1)}})$. Since the inclusion $\iota\colon (C^{(0)}, d_{C^{(1)}}) \hookrightarrow (C^{(1)}, d_{C^{(1)}})$ is a quasi-isometry, this shows that $C^{(1)}$ has a coarse median given by $\iota_*\mu_C$.

A similar reasoning can be applied to see that a finite dimensional CAT(0) cubical complex, endowed with the metric induced by the Euclidean metric on each cube (cf. Remark 1.1.8), is coarse median with rank bounded above by its dimension.

By studying asymptotic cones of coarse median spaces [Bow13a, Section 9], Bowditch has obtained the following result.

Proposition 2.2.6 ([Bow13a, Corollary 2.4])**.** *Let X be a geodesic coarse median space of rank $\leqslant n$. Then there exists no quasi-isometric embedding $\mathbb{R}^{n+1} \to X$.*

Corollary 2.2.7. *Let $X_1, \dots, X_k$ be geodesic spaces that admit a quasi-isometric embedding $\mathbb{R} \to X_i$ and a coarse median $\mu_i\colon X_i^3 \to X_i$, for each $i \in \{1, \dots, k\}$. Then the product coarse median $\mu := \mu_1 \times \dots \times \mu_k$ on $X_1 \times \dots \times X_k$ satisfies*

$$k \leqslant \operatorname{rank}(\mu) \leqslant \sum_{i=1}^{k} \operatorname{rank}(\mu_i).$$

Proof. By assumption, there is a quasi-isometric embedding $\mathbb{R}^k \to X_1 \times \dots \times X_k$. Therefore, the previous proposition implies $\operatorname{rank}(\mu_1 \times \dots \times \mu_k) \geqslant k$. The upper bound on the rank follows by induction from Proposition 2.1.10. □

Example 2.2.8. Consider the median operation $\mu_{\mathbb{R}^n}$ on $\mathbb{R}^n$ coming from the l^1-norm and let $\mathbb{E}^n$ denote the metric space determined by $\mathbb{R}^n$ endowed with the *Euclidean norm*. Since all norms on a finite dimensional vector space are equivalent, it follows that $\mu_{\mathbb{R}^n}$ is a coarse median on $\mathbb{E}^n$. By Corollary 2.2.7, the rank of $\mu_{\mathbb{R}^n}$ is exactly n.

Let $A \in \operatorname{Isom}(\mathbb{E}^2)$ be the rotation by $\frac{\pi}{4}$ around the origin. For $n \in \mathbb{N}$, consider $x_n = (n, 0), y_n = (0, n) \in \mathbb{E}^2$. Then, by elementary Euclidean geometry, it follows that $Ax_n = (\sqrt{2}\,n, \sqrt{2}\,n)$ and $Ay_n = (-\sqrt{2}\,n, \sqrt{2}\,n)$. Thus

$$\begin{aligned} \mu_{\mathbb{R}^2}(x_n, y_n, 0) &= (0, 0), \\ \mu_{\mathbb{R}^2}(Ax_n, Ay_n, 0) &= (0, \sqrt{2}n), \end{aligned}$$

and subsequently $\|\mu_{\mathbb{R}^2}(Ax_n, Ay_n, A0) - A(\mu_{\mathbb{R}^2}(x_n, y_n, 0))\| \to \infty$ for $n \to \infty$, which means $A^*\mu_{\mathbb{R}^2} \not\sim \mu_{\mathbb{R}^2}$. In fact, if A_φ denotes the rotation by φ around the origin, then a similar computation as above shows that $A_\varphi^*\mu_{\mathbb{R}^2} \not\sim \mu_{\mathbb{R}^2}$, provided that φ is not an integer multiple of $\frac{\pi}{2}$. In particular, this shows that there are uncountably many pairwise non-equivalent coarse median structures on $\mathbb{E}^2$.

2.3 Hyperbolic spaces

In this section, we consider hyperbolic metric spaces and their relation to coarse median spaces. A geodesic metric space is called *δ-hyperbolic* if every one of its geodesic triangles is *δ-slim* [BH99, Chapter III.H], that is, each of its sides is contained in the δ-neighborhood of the union of the other two sides. We say that a space is *hyperbolic*, if it is δ-hyperbolic for some $\delta \geqslant 0$.

Bowditch proved that a geodesic metric space is hyperbolic if and only if it is *rank* 1 coarse median [Bow13a, Theorem 2.1]. Here we only present the construction to see that every hyperbolic metric space possesses a coarse median.

Definition 2.3.1. Let (X, d) be a geodesic metric space, three points $x, y, z \in X$ and $K \in \mathbb{R}_{\geqslant 0}$ some constant. We define a *K-center* of a geodesic triangle $([x, y], [x, z], [y, z])$ to be a point $m \in X$ such that

$$\max\{d([x, y], m), d([x, z], m), d([y, z], m)\} \leqslant K.$$

Requiring the existence of a K-center for each geodesic triangle (with $K \in \mathbb{R}_{\geqslant 0}$ fixed) is an equivalent way of defining hyperbolicity [Bow91, Definition 2]. We give a proof of the following statement which is sufficient for our purposes.

Lemma 2.3.2. *Let (X, d) be a δ-hyperbolic geodesic metric space.*

(i) Every geodesic triangle in X admits a δ-center.

(ii) For every constant $K \in \mathbb{R}_{\geqslant 0}$, there exists a constant $K' = K'(K, \delta)$ such that: If m_1 and m_2 are K-centers of geodesic triangles $\triangle_1$ and $\triangle_2$, respectively, where $\triangle_1$ and $\triangle_2$ have the same vertices, then $d(m_1, m_2) \leqslant K'$.

Proof. (i) Let $\triangle = ([x, y], [x, z], [y, z])$ be a geodesic triangle. Since $\triangle$ is δ-thin, we have $[x, y] \subseteq \mathcal{U}_\delta([x, z]) \cup \mathcal{U}_\delta([y, z])$ and $\mathcal{U}_\delta([x, z]) \cap [x, y] \neq \emptyset$, $\mathcal{U}_\delta([y, z]) \cap [x, y] \neq \emptyset$. It follows that $\mathcal{U}_\delta([x, z]) \cap \mathcal{U}_\delta([y, z]) \cap [x, y] \neq \emptyset$ because $[x, y]$ is connected, and any point in this set is a δ-center of $\triangle$.

(ii) We adapt the proof of [Bow91, Lemma 3.1.5]. By δ-hyperbolicity, two geodesics connecting the same points are within the δ-neighborhood of each other. Thus, it follows that m_2 is a $(K + \delta)$-center of $\triangle_1$. Let $\triangle_1 = ([x, y], [x, z], [y, z])$ and choose $\tilde{m}_i \in [x, y]$ as well as $p_i \in [x, z]$ such that $d(m_i, \tilde{m}_i) \leqslant K + \delta$ and $d(m_i, p_i) \leqslant K + \delta$ for $i \in \{1, 2\}$. Assume w.l.o.g. that $d(x, \tilde{m}_1) \leqslant d(x, \tilde{m}_2)$. It follows that $d(\tilde{m}_i, p_i) \leqslant 2(K + \delta)$ and hence $d(x, z) = d(x, p_i) + d(p_i, z) \geqslant d(x, \tilde{m}_i) + d(\tilde{m}_i, z) - 4(K + \delta)$. So, by the triangle inequality, we have

$$d(x, z) \sim_{4(K+\delta)} d(x, \tilde{m}_2) + d(\tilde{m}_2, z) = d(x, \tilde{m}_1) + d(\tilde{m}_1, \tilde{m}_2) + d(\tilde{m}_2, z),$$

and

$$d(x, z) \sim_{4(K+\delta)} d(x, \tilde{m}_1) + d(\tilde{m}_1, z).$$

This implies $d(\tilde{m}_1, z) \sim_{8(K+\delta)} d(\tilde{m}_1, \tilde{m}_2) + d(\tilde{m}_2, z)$. By the same argument, we also conclude $d(\tilde{m}_2, z) \sim_{8(K+\delta)} d(\tilde{m}_1, \tilde{m}_2) + d(\tilde{m}_1, z)$. Together this proves $2d(\tilde{m}_1, \tilde{m}_2) \leqslant 16(K + \delta)$, hence we can take $K' = 8(K + \delta)$. $\square$

Definition 2.3.3. Let X be a δ-hyperbolic metric space and fix $K \geqslant \delta$. For every three points $x, y, z \in X$, we choose a K-center $m \in X$ of some geodesic triangle with vertices x, y, z, and define $\mu_{\mathrm{hyp}}(x, y, z) = m$. We call $\mu_{\mathrm{hyp}} \colon X^3 \to X$ an *hyperbolic coarse median* on X.

If we assume that such a μ_{hyp} is a coarse median, then second part of Lemma 2.3.2 implies that the coarse median structure represented by μ_{hyp} does not depend on any of the choices. We refer to the resulting coarse median structure by *hyperbolic coarse median structure* on X.

To prove that μ_{hyp} is actually a coarse median, we need the following two fundamental results from the theory of δ-hyperbolic spaces.

An (α, ε)-*quasi-geodesic* in a metric space X is an (α, ε)-quasi-isometric embedding $c \colon J \to X$ with $J \subseteq \mathbb{R}$ some interval. The first result is called *stability of quasi-geodesics* and states that in hyperbolic spaces quasi-geodesics are always uniformly close to geodesics.

Lemma 2.3.4 ([BH99, Part III, Theorem 1.7])**.** *For all $\delta > 0$, $\alpha \geqslant 1$, $\varepsilon \geqslant 0$ there is a constant $R = R(\delta, \alpha, \varepsilon) \geqslant 0$ such that the following holds.*

If X is a δ-hyperbolic space, $c \colon [a, b] \to X$ is an (α, ε)-quasi-geodesic and $[p, q]$ a geodesic connecting $p = c(a)$ and $q = c(b)$, then the Hausdorff distance of $[p, q]$ and $c([a, b])$ is less than R.

The second results says that finite subsets of hyperbolic spaces can be approximated by trees, which is the essential observation for proving that hyperbolic spaces are coarse median (of rank 1).

Lemma 2.3.5 ([Gro87, §6.2],[Bow91, Section 3.3])**.** *Let $\delta \geqslant 0$. There is a function $h_0 \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ such that if (X, d) is a δ-hyperbolic space and $A \subseteq X$ a finite subset, then there exists a finite tree $T \subseteq X$, embedded in X with geodesic edges, such that*

(i) The prescribed set A is contained in the vertex set of T and the set of extremal points of T is contained in A.

(ii) Every two points $t, s \in T$ satisfy $d_T(t, s) \leqslant d(t, s) + \delta h_0(|A|)$, where $d_T(t, s)$ is the minimal length of a broken geodesic which stays in T and connects t to s.

Proposition 2.3.6 ([Bow13a, Section 3])**.** *Let X be a δ-hyperbolic geodesic metric space. Then the ternary operation $\mu_{\mathrm{hyp}} \colon X^3 \to X$ is a coarse median of rank 1.*

Proof. To verify Definition 2.1.3 (ii), let $T \subseteq X$ be an embedded tree associated to the finite subset A as in Lemma 2.3.5. Then we may take (Λ, μ_Λ) to be the vertex set of T with the usual median structure of a tree and $A \overset{\pi}{\hookrightarrow} \Lambda \overset{\lambda}{\hookrightarrow} X$ the inclusions. Then, by construction, $\lambda \circ \pi$ is the inclusion of A, so the last part of Definition 2.1.3 (ii) is trivial. To see that λ is a quasi-morphism, we use stability of quasi-geodesics. Let $a, b, c \in A$ and $\tilde{m} = \mu_\Lambda(a, b, c) \in T$. Then there is a tripod in X, the sides of which are broken geodesics contained in T, with vertices a, b, c and center $\tilde{m}$. This is then a $(1, \delta h_0(|A|))$-quasi-geodesic triangle in X. Thus, Lemmas 2.3.2 and 2.3.4 imply that there are constants $L, L' \in \mathbb{R}_{\geqslant 0}$, which depend only on δ and $h_0(|A|)$, such that $\tilde{m}$ and $\mu_{\mathrm{hyp}}(a, b, c)$ are both L-centers of a, b, c, and

$$d(\mu_\Lambda(a, b, c), \mu_{\mathrm{hyp}}(a, b, c)) \leqslant L'.$$

This means that λ is an L'-quasi-morphism and so we can choose an appropriate function $h\colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ to satisfy Definition 2.1.3 (ii).

To see that μ_{hyp} is coarsely Lipschitz, note that for any embedded tree $T \subseteq X$, by Lemma 1.1.5, the map $\mu_T\colon T^3 \to T$ is 1-Lipschitz with respect to d_T. Thus, to estimate the distance between $\mu_{\mathrm{hyp}}(x_1, x_2, x_3)$ and $\mu_{\mathrm{hyp}}(x'_1, x'_2, x'_3)$ for some $x_i, x'_i \in X$, we just have to apply Lemma 2.3.5 to the set $A = \{x_1, x_2, x_3, x'_1, x'_2, x'_3\}$ (and observe that the coarse Lipschitz constants we get do not depend on x_i, x'_i due to $|A| \leqslant 6$). ☐

Being hyperbolic is a quasi-isometry invariant of geodesic metric spaces [BH99, Chapter III.H, Theorem 1.9]. Moreover, the hyperbolic coarse median structure is *stable under quasi-isometries.*

Lemma 2.3.7. *Let $f\colon X \to Y$ be a quasi-isometry between hyperbolic spaces and $\mu_{\mathrm{hyp},X}$ and $\mu_{\mathrm{hyp},Y}$ corresponding hyperbolic coarse medians, respectively. Then $f_*\mu_{\mathrm{hyp},X} \sim \mu_{\mathrm{hyp},Y}$.*

Proof. Choose a quasi-inverse $g\colon Y \to X$ to f and fix constants K_Y and K_X as in Lemma 2.3.2 for Y and X, respectively. Say that f is an (α, ε)-quasi-isometry. Let $y_1, y_2, y_3 \in Y$ and choose $m = \mu_{\mathrm{hyp},X}(g(y_1), g(y_2), g(y_3))$ such that it is a K_X-center of $g(y_1), g(y_2), g(y_3)$ with respect to some geodesic triangle $([g(y_i), g(y_{i+1})])_{i=1,2,3}$ in X. Then $(f([g(y_i), g(y_{i+1})]))_{i=1,2,3}$ is an (α, ε)-quasi-geodesic triangle in Y with vertices $f(g(y_i))$ and we have that $f(m)$ has distance less than $\alpha K_X + \varepsilon$ to each $f([g(y_i), g(y_{i+1})])$ for $i \in \{1, 2, 3\}$. Thus, by stability of quasi-geodesics, we can assume that K_Y has a priori been chosen large enough such that $f(m)$ is a K_Y-center of $f(g(y_1)), f(g(y_2)), f(g(y_3))$.

Therefore we may assume that for all $y_1, y_2, y_3 \in Y$ we have

$$\begin{aligned} \mu_{\mathrm{hyp},Y}\left(f(g(y_1)), f(g(y_2)), f(g(y_3))\right) &= f\left(\mu_{\mathrm{hyp},X}\left(g(y_1), g(y_2), g(y_3)\right)\right) \\ &= f_*\mu_{\mathrm{hyp},X}(y_1, y_2, y_3). \end{aligned}$$

But since $f \circ g \sim \mathrm{id}_Y$, this implies that $f_*\mu_{\mathrm{hyp},X} \sim \mu_{\mathrm{hyp},Y}$, as required. ☐

Note that the construction of a rank 1 coarse median on a hyperbolic space we have presented here requires the space to be geodesic[2], whereas the general definition of a rank 1 coarse median does not require the underlying space to be geodesic. However, one can relax this restriction a little and consider metric spaces which are quasi-isometric to a hyperbolic geodesic spaces but are not necessarily geodesic themselves. Namely, quasi-isometry invariance of the existence of a coarse median shows that such a space

[2]There is a definition of hyperbolicity for general metric spaces which in the case of geodesic spaces is equivalent to the one given here [BH99, Chapter III.H, Definition 1.20].

admits a rank 1 coarse median. Moreover, the above lemma implies that the resulting coarse median structure does not depend on the choice of the quasi-isometry.

Furthermore, the lemma implies that the hyperbolic coarse median structure on a hyperbolic geodesic space is a fixed point with respect to the action of the quasi-isometry group (compare Remark 2.2.4). If we restrict ourselves to the isometry group, we get an even stronger corollary from the proof of the lemma.

Corollary 2.3.8. *Let X be a hyperbolic space with the hyperbolic coarse median μ_{hyp}. Then μ_{hyp} is equivariant up to uniformly bounded error with respect to the isometry group of X, that is, there exists a constant $h \in \mathbb{R}_{\geqslant 0}$ such that for any isometry $f\colon X \to X$, we have*

$$\mu_{\mathrm{hyp}} \circ f^{\sqcap 3} \sim_h f \circ \mu_{\mathrm{hyp}}.$$

Note that general coarse median structures are not necessarily fixed by isometries of the underlying space, see Example 2.2.8.

We conclude this section with an easy lemma which will be needed later to prove one of the main results of Chapter 4.

Lemma 2.3.9. *Let C be a* CAT(0) *cubical complex such that its* 1*-skeleton* $(C^{(1)}, d_{C^{(1)}})$ *is a hyperbolic metric space. Denote the median on $C^{(0)}$ by μ_C, the hyperbolic coarse median on $C^{(1)}$ by μ_{hyp} and let $\iota\colon C^{(0)} \hookrightarrow C^{(1)}$ be the inclusion map. Then $\iota_*\mu_C \sim \mu_{\mathrm{hyp}}$.*

Proof. Let $x_1, x_2, x_3 \in C^{(0)}$ and let $m = \mu_C(x_1, x_2, x_3)$. By definition of μ_C we can find a geodesic tripod in $C^{(1)}$ with vertices x_i and center m. This shows that m is a 0-center of x_1, x_2, x_3 with respect to this particular geodesic triangle.

We can choose a quasi-inverse $\pi\colon C^{(1)} \to C^{(0)}$ to ι such that $\pi \circ \iota = \mathrm{id}_{C^{(0)}}$. Thus, the previous paragraph implies that we can assume

$$\iota_*\mu_C(x_1, x_2, x_3) = \mu_C(x_1, x_2, x_3) = \mu_{\mathrm{hyp}}(x_1, x_2, x_3),$$

for all $x_1, x_2, x_3 \in C^{(0)}$. Since $C^{(0)}$ is at finite Hausdorff distance to $C^{(1)}$, this implies $\iota_*\mu_C \sim \mu_{\mathrm{hyp}}$. □

2.4 Approximation of finite subsets

In this section, we discuss two approximation properties of coarse median spaces. First, we show that every formal expression (see below for a precise definition) which holds in every median algebra holds up to bounded error with respect to a coarse median. Secondly, and more substantially, in Theorem 2.4.6 we generalize the fundamental result on approximating trees for hyperbolic geodesic spaces to “approximating CAT(0) cubical complexes” for arbitrary coarse median spaces. The reason for our interest in such a generalization is that approximating trees are a powerful tool for the understanding of hyperbolic geodesic spaces and hyperbolic groups; for formulations and applications of the classical result see [Gro87, §6.2], [GH90, Chapitre 2, §2], [Bow91, Section 3.3], [Coo93, Théorème 1.1], and Lemma 2.3.5 in this Master thesis.

Bowditch proves directly [Bow13a, p. 73] that one can assume without loss of generality that a coarse median is symmetric and satisfies $\mu(x, x, y) = x$ for all x, y. In Theorem 2.4.3 below, we extend his arguments to prove the first approximation result we have mentioned above. This sheds some light on how general formulas which hold in median algebras carry over to coarse medians. To state this theorem, we need to introduce a convention for referring to formal expressions built out of repeated application of some ternary operations. Let μ be a formal symbol which stands for a ternary operation. A *formal median expression* φ in the variables $x_1, \dots, x_k$ is recursively defined as follows. Either $\varphi(x_1, \dots, x_k) = x_i$ for some $i \in \{1, \dots, k\}$ or $\varphi(x_1, \dots, x_k) = \mu(\psi_1(x_1, \dots, x_k), \psi_2(x_1, \dots, x_k), \psi_3(x_1, \dots, x_k))$, where ψ_1, ψ_2, ψ_3 are formal median expressions in the variables $x_1, \dots, x_k$. We say that a formal equality $\varphi_1(x_1, \dots, x_k) = \varphi_2(x_1, \dots, x_k)$ of formal median expressions φ_1, φ_2 is a *formal median identity*. Given a set X together with a concrete ternary operation $\mu\colon X^3 \to X$, every formal median expression $\varphi(x_1, \dots, x_k)$ induces a map $\varphi_X = \varphi\colon X^k \to X$ in the obvious way. In this case, it makes sense to say that some formal median identity holds for μ.

Lemma 2.4.1. *Let (Λ, μ_Λ) be a median algebra and $\mu\colon X^3 \to X$ a coarse median. Consider a formal median expression $\varphi(x_1, \dots, x_k)$ and an L-quasi-morphism $f\colon (\Lambda, \mu_\Lambda) \to (X, \mu)$. Then there exists a constant $K \in \mathbb{R}_{\geqslant 0}$, which depends only on φ, L and the parameters of μ, such that*

$$f \circ \varphi_\Lambda \sim_K \varphi_X \circ f^{\sqcap k}. \tag{2.4.1}$$

Proof. We verify this claim by induction on the construction of φ. If φ consists of a single variable, there is nothing to prove. Otherwise, we can assume that $\varphi = \mu(\psi_1, \psi_2, \psi_3)$, where ψ_i are formal median expressions with constants K_i such that $f \circ (\psi_i)_\Lambda \sim_{K_i} (\psi_i)_X \circ f^{\sqcap k}$. The latter implies

$$\mu \circ ((f \circ \psi_1), (f \circ \psi_2), (f \circ \psi_3)) \sim_{\tilde{K}} \mu \circ \left((\psi_1 \circ f^{\sqcap k}), (\psi_1 \circ f^{\sqcap k}), (\psi_1 \circ f^{\sqcap k})\right),$$

where $\tilde{K} = k(K_1 + K_2 + K_3) + h(0)$ and k, h are the parameters of the coarse median μ. (For simplicity, we drop the subscripts indicating the induced maps of the formal expressions.) Since f is a quasi-morphism, we also have

$$f \circ \varphi = f \circ \mu_\Lambda \circ (\psi_1, \psi_2, \psi_3) \sim_L \mu \circ f^{\sqcap 3} \circ (\psi_1, \psi_2, \psi_3) = \mu \circ ((f \circ \psi_1), (f \circ \psi_2), (f \circ \psi_3)).$$

Putting these together, we get (2.4.1) with $K = L + \tilde{K}$. □

Lemma 2.4.2. *Let $\mu\colon X^3 \to X$ be a coarse median and $\varphi(x_1, \dots, x_k)$ a formal median expression. Then the induced map $\varphi_X\colon X^k \to X$ is coarsely Lipschitz with constants depending only on φ and the parameters of μ.*

Proof. This follows by an inductive proof similar to one in the previous lemma. □

Theorem 2.4.3. *Let $\mu\colon X^3 \to X$ be a coarse median and*

$$\varphi_1(x_1, \dots, x_k) = \varphi_2(x_1, \dots, x_k)$$

a formal median identity which holds in every median algebra. Then there exists a constant $R \in \mathbb{R}_{\geqslant 0}$, depending only on φ_1, φ_2 and the parameters of μ, such that

$$d((\varphi_1)_X(x_1, \dots, x_k), (\varphi_2)_X(x_1, \dots, x_k)) \leqslant R$$

for all $x_1, \dots, x_k \in X$.

Proof. Let $x_1, \dots, x_k \in X$ be fixed. Then, by Definition 2.1.3 (ii), there exists a finite median algebra (Λ, μ_Λ), an $h(k)$-quasi-morphism $\lambda\colon (\Lambda, \mu_\Lambda) \to (X, \mu)$, and points $\tilde{x}_1, \dots, \tilde{x}_k \in \Lambda$ such that $d(x_i, \lambda(\tilde{x}_i)) \leqslant h(k)$ for all $i \in \{1, \dots, k\}$. Let $K \in \mathbb{R}_{\geqslant 0}$ be the constant (depending on $h(k), \varphi_1, \varphi_2$ and the parameters of μ) we get from Lemma 2.4.1 so that (2.4.1) is true for $\varphi \in \{\varphi_1, \varphi_2\}$. Since the identity $\varphi_1 = \varphi_2$ holds in Λ, we see that

$$(\varphi_1)_X \circ \lambda^{\sqcap k} \sim_K f \circ (\varphi_1)_\Lambda = f \circ (\varphi_2)_\Lambda \sim_K (\varphi_2)_X \circ \lambda^{\sqcap k},$$

and so $d(\varphi_1(\lambda(\tilde{x}_1), \dots, \lambda(\tilde{x}_k)), \varphi_2(\lambda(\tilde{x}_1), \dots, \lambda(\tilde{x}_k)) \leqslant 2K$. By Lemma 2.4.2 and the fact that $d(x_i, \lambda(\tilde{x}_i)) \leqslant h(k)$, this concludes the proof. □

Corollary 2.4.4. *Let $\mu\colon X^3 \to X$ be a coarse median. Then there exists a coarse median μ' which is equivalent to μ and satisfies*

$$\mu'(x, y, z) = \mu'(x, z, y) = \mu'(y, z, x),$$
$$\mu'(x, x, y) = x,$$

for all $x, y, z \in X$.

Proof. Let $\mathcal{P}_{\leqslant 3} = \{\{x, y, z\} \mid x, y, z \in X\}$. Consider the obvious surjection $\pi\colon X^3 \to \mathcal{P}_{\leqslant 3}$, and choose a map $s\colon \mathcal{P}_{\leqslant 3} \to X^3$ that satisfies $\pi \circ s = \mathrm{id}_{\mathcal{P}_{\leqslant 3}}$. Define $\mu'\colon X^3 \to X$ by

$$\mu'(x, y, z) = \begin{cases} \mu(s(\pi(x, y, z))) & \text{if } x, y, z \text{ are pairwise distinct,} \\ u & \text{if } \pi(x, y, z) = \pi(u, u, v) \text{ for some } u, v \in X. \end{cases}$$

Clearly, the required identities are satisfied by μ', and since they hold in a general median algebra, Theorem 2.4.3 implies that $\mu' \sim \mu$. □

Next we discuss the approximation of finite subsets in a coarse median space by finite CAT(0) cubical complexes. We start with the following observations about quasi-morphisms into coarse median spaces.

Lemma 2.4.5. *Let $\mu\colon X^3 \to X$ be a coarse median, C a* CAT(0) *cubical complex and $f\colon (C^{(0)}, \mu_C) \to (X, \mu)$ an L-quasi-morphism. Then there exist constants $\beta, \gamma \in \mathbb{R}_{\geqslant 0}$, which depend on L and the parameters of μ, such that the following holds.*

(i) For any two parallel edges (x, y) and (x', y'), we have,

$$\frac{1}{\beta} d(f(x), f(y)) - \gamma \leqslant d(f(x'), f(y')) \leqslant \beta d(f(x), f(y)) + \gamma.$$

(ii) If $x_0, \dots, x_m \in C^{(0)}$ is a combinatorial geodesic in $C^{(1)}$, then we have,

$$d(f(x_i), f(x_{i+1})) \leqslant \beta\, d(f(x_0), f(x_n))) + \gamma$$

for all $i \in \{0, \dots, m-1\}$.

Proof. (i) By assumption, there exists a unique wall $W = \{h, h^c\} \in \mathcal{W}_C$ such that $x, x' \in h$ and $y, y' \in h^c$. Since the half spaces are convex, it follows that $x' = \mu_C(x', y', x)$ and $y' = \mu_C(x', y', y)$. Thus, since f is a quasi-morphism, we have $d(x', \mu(f(x'), f(y'), f(x)) \leqslant L$ and $d(y', \mu(f(x'), f(y'), f(y)) \leqslant L$. We conclude that

$$\begin{aligned} d(f(x'), f(y')) &\leqslant d\left(\mu(f(x'), f(y'), f(x)), \mu(f(x'), f(y'), f(y))\right) + 2L \\ &\leqslant k\, d(f(x), f(y)) + h(0) + 2L, \end{aligned}$$

where k, h are parameters of μ. By symmetry, this proves (i).

(ii) The unique wall which separates x_i from x_{i+1} also separates x_0 from x_m, and we have $x_i = \mu_C(x_i, x_{i+1}, x_0)$ as well as $x_{i+1} = \mu_C(x_i, x_{i+1}, x_m)$. Now the proof proceeds as in (i). □

To state the approximation theorem, we need to consider CAT(0) cubical complexes C together with assignments of positive lengths to each edge, with the property that parallel edges have the same length. We call such an assignment of edge lengths *compatible*. Every compatible length assignment induces a metric on the 1-skeleton of C. Although these metrics depend on the choice of compatible length assignment, the resulting geodesics are always the same (up to reparametrization) as the ones determined by the combinatorial metric (where each edge is deemed to have length 1).

Theorem 2.4.6. *Let (X, d) be a coarse median space. Then there exist functions $\alpha, \varepsilon \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ (which only depend on the parameters of a coarse median on X) such that the following holds.*

For each finite subset $A \subseteq X$, there exists a finite CAT(0) *cubical complex C with a compatible length assignment and an $(\alpha(|A|), \varepsilon(|A|))$-quasi-isometric embedding*

$$f \colon \left(C^{(0)}, d_{C^{(1)}}\right) \to (X, d),$$

where $d_{C^{(1)}}$ is the metric induced by the compatible length assignment, and such that $A \subseteq f(C^{(0)})$. Moreover, if (X, d) is coarse median of rank at most n, we can assume C to have dimension at most $n+1$.

Proof. Let h, k be parameters of a coarse median $\mu \colon X^3 \to X$. By Remark 2.1.5, we may assume that there exists a finite median algebra (Λ, μ_Λ) and an $h(|A|)$-quasi-morphism $\lambda \colon \Lambda \to X$ such that for each $a \in A$, there exists $\tilde{a} \in \Lambda$ with $\lambda(\tilde{a}) = a$. We can also assume that $|\Lambda| \leqslant 2^{2^{|A|}}$ by Lemma 1.2.3. By Remark 1.2.4, Λ is the vertex set of a finite CAT(0) cube complex C, and we set $f = \lambda \colon C^{(0)} \to X$. By assumption, we have $A \subseteq f(C^{(0)})$. If μ has rank at most n, by the construction in Remark 2.1.5, we can assume that C has dimension at most $n+1$.

In each class of parallel edges in C we choose a representative (x, y) and define the length of each edge in this class to be $d(f(x), f(y))$. By Lemma 2.4.5, this does not depend on the choice of representative up to fixed multiplicative and additive errors. We write $d_{C^{(1)}}$ for the metric on $C^{(1)}$ induced by this compatible length assignment.

Let $x_0, \dots, x_m \in C^{(0)}$ be a combinatorial geodesic in $C^{(1)}$. Then, by Lemma 2.4.5 (i), we can estimate

$$\begin{aligned} d(f(x_0), f(x_n)) &\leqslant \sum_{i=0}^{m-1} d(f(x_i), f(x_{i+1})) \\ &\leqslant \sum_{i=0}^{m-1} (\beta\, d_{C^{(1)}}(x_i, x_{i+1}) + \gamma) = \beta\, d_{C^{(1)}}(x_0, x_m) + m\gamma \\ &\leqslant \beta\, d_{C^{(1)}}(x_0, x_m) + |C^{(0)}|\gamma, \end{aligned}$$

where β, γ are the constants from Lemma 2.4.5. On the other hand,

$$\begin{aligned} d_{C^{(1)}}(x_0, x_m) &= \sum_{i=0}^{m-1} d_{C^{(1)}}(x_i, x_{i+1}) \\ &\leqslant \sum_{i=0}^{m-1} \beta\ (d(f(x_i), f(x_{i+1})) + \gamma) \\ &\leqslant \sum_{i=0}^{m-1} \beta\ ((\beta\, d(f(x_0), f(x_n)) + \gamma) + \gamma) \\ &= m\beta^2\, d(f(x_0), f(x_m)) + m(\beta\gamma + \gamma) \\ &\leqslant |C^{(0)}|\beta^2\, d(f(x_0), f(x_m)) + |C^{(0)}|(\beta\gamma + \gamma), \end{aligned}$$

where we have used both parts of Lemma 2.4.5. Since β, γ only depend on $L := h(|A|)$ and the parameters of μ, and $|C^{(0)}| = |\Lambda| \leqslant 2^{2^{|A|}}$, these two estimates show that $f\colon C^{(0)} \to X$ is an $(\alpha(|A|), \varepsilon(|A|))$-quasi-isometric embedding for functions α, ε that only have the dependencies allowed in the statement of the theorem. $\square$

The above result can be applied to hyperbolic geodesic spaces (since they are rank 1 coarse median). However, in this special case, Lemma 2.3.5 is much stronger than the result proved above. Namely, in Lemma 2.3.5 we only have additive errors and no multiplicative errors. Moreover, every finite subset is contained in a finite geodesic tree, that is, a CAT(0) cubical complex of dimension 1, whereas the theorem above would only give us approximations of finite subsets by vertex sets of 2-dimensional CAT(0) cubical complexes. This is due to the construction in Remark 2.1.5 and because we only use the definition of a coarse median space in the proof of Theorem 2.4.6 (in particular, we do not assume that the underlying space is geodesic).

2.5 Coarse median structure of measured wall spaces

Quasi-isometry invariance implies that every space with measured walls is a coarse median space provided it is at bounded distance to its associated median space. In this section, we describe the resulting coarse median in such a case intrinsically in terms of the measured wall structure.

Although we have defined coarse median spaces only in the context of metric spaces, the associated median space to a wall structure is — according to our definition — only a pseudo-metric space. To remedy this technical issue, we make the convention to implicitly pass to the metric quotient whenever necessary.

Definition 2.5.1. Let $K \in \mathbb{R}_{\geqslant 0}$. We say a space with measured walls X is K-*coarsely median*, if the embedding $\sigma\colon X \to \mathcal{M}(X)$ into the associated median space is K-quasi-surjective.

In a median algebra, the median of three points is characterized by the fact that there are no convex walls which separate it from two of the three given points.[3] The following theorem states that in a coarsely median space with measured walls, a coarse median can be defined by the requirement that the collection of walls with this property has globally bounded measure.

Theorem 2.5.2. *Let $(X, \mathcal{W}, \mathfrak{B}, \nu)$ be a space with measured walls which is L-coarsely median and let $K \geqslant L$. Then for every three points $x_1, x_2, x_3 \in X$, the set*

$$\mathfrak{M}(x_1, x_2, x_3; K) = \left\{ m \in X \;\middle|\; \nu\left(\bigcup_{i=1}^{3} \mathcal{W}(m|x_i, x_{i+1})\right) \leqslant K. \right\} \tag{2.5.1}$$

is non-empty and has diameter $\leqslant 2K$. Moreover, any ternary operation $\mu_{\mathcal{W}}\colon X^3 \to X$ which satisfies

$$\mu_{\mathcal{W}}(x_1, x_2, x_3) \in \mathfrak{M}(x_1, x_2, x_3; K) \qquad \forall x_1, x_2, x_3 \in X,$$

is a coarse median on X which defines the same coarse median structure as $\sigma^*\mu_{\mathcal{M}}$.

Proof. Let $t, s_i \in \mathcal{M}(X)$, $i \in \{1, 2, 3\}$ be arbitrary. Then for each i there is a measure-preserving bijection

$$\mathcal{W}_{\mathcal{M}}(t|s_i, s_{i+1}) \overset{\cong}{\to} \pi\left(\{h \in \mathcal{H} \mid h \in t, h \notin s_i, h \notin s_{i+1}\}\right) = \pi\left((s_i \vartriangle t) \cap (s_{i+1} \vartriangle t)\right),$$

compare Remark 1.3.10. We see that

$$\begin{aligned}
\nu_{\mathcal{M}}\left(\bigcup_{i=1}^{3} \mathcal{W}_{\mathcal{M}}(t|s_i, s_{i+1})\right) &= \nu\left(\bigcup_{i=1}^{3} \pi\left((s_i \vartriangle t) \cap (s_{i+1} \vartriangle t)\right)\right) \\
&= \nu^{\mathcal{H}}\left(\bigcup_{i=1}^{3} (s_i \vartriangle t) \cap (s_{i+1} \vartriangle t)\right) \\
&= \nu^{\mathcal{H}}\left(\left(\bigcup_{i=1}^{3} s_i \cap s_{i+1}\right) \vartriangle t\right) \\
&= d_{\mathcal{M}}(\mu_{\mathcal{M}}(s_1, s_2, s_3), t),
\end{aligned}$$

[3] A wall $\{h, h^c\}$ on the underlying set of a median algebra is called *convex* if both h and h^c are convex subsets in the sense of Definition 1.2.2 (iii). It can be shown [Rol98, Theorem 2.7] that any two disjoint convex subsets of a median algebra are separated from each other by some convex wall. The claim then follows from (1.2.4) and the fact that intervals are convex.

where the third equality follows from Lemma 1.3.9. Now we observe that

$$\nu\left(\bigcup_{i=1}^{3}\mathcal{W}(m|x_i,x_{i+1})\right)=\nu_{\mathcal{M}}\left(\bigcup_{i=1}^{3}\mathcal{W}_{\mathcal{M}}(\sigma(m)|\sigma(x_i),\sigma(x_{i+1}))\right),$$

which together with the above allows us to conclude

$$\mathfrak{M}(x_1,x_2,x_3;K)=\sigma^{-1}\left(\{t\in\mathcal{M}(X)\mid d_{\mathcal{M}}(\mu_{\mathcal{M}}(\sigma(x_1),\sigma(x_2),\sigma(x_3)),t)\leqslant K\}\right).$$

Since σ is a K-quasi-surjective isometric embedding, this proves the statements about the set $\mathfrak{M}(x_1,x_2,x_3;K)$. Finally, as any $m\in\mathfrak{M}(x_1,x_2,x_3;K)$ satisfies $d_{\mathcal{M}}(\mu_{\mathcal{M}}(\sigma(x_1),\sigma(x_2),\sigma(x_3)),\sigma(m))\leqslant K$, we conclude $\mu_{\mathcal{W}}\sim\sigma^*\mu_{\mathcal{M}}$. □

Example 2.5.3. The space with measured walls $(\mathbb{H}^n,\mathcal{W}_{\mathbb{H}^n},\mathfrak{B},\nu_{\mathbb{H}^n})$, as described in Example 1.3.6, is coarsely median [CDH07, Corollary 3.25]. Moreover, $(\mathbb{H}^n,d_{\mathcal{W}_{\mathbb{H}^n}})$ is a δ-hyperbolic geodesic space and thus this example is within the scope of the next proposition.

Proposition 2.5.4. *Let $(X,\mathcal{W},\mathfrak{B},\nu)$ be a coarsely median space with measured walls such that $(X,d_{\mathcal{W}})$ is a geodesic space which is hyperbolic.*

There is a constant $K\in\mathbb{R}_{\geqslant 0}$ such that for each $x_1,x_2,x_3\in X$ and every geodesic $[x_i,x_{i+1}]$ from x_i to x_{i+1}, we have $\mathfrak{M}(x_1,x_2,x_3;K)\cap[x_i,x_{i+1}]\neq\emptyset$, $i\in\{1,2,3\}$. As a consequence, we have $\mu_{\mathrm{hyp}}\sim\sigma^*\mu_{\mathcal{M}}$, where μ_{hyp} is a hyperbolic coarse median on X.

Proof. We use the notion of the *Gromov product* of $y,z\in X$ with respect to $x\in X$, which is defined by $(y,z)_x=\frac{1}{2}\left(d(y,x)+d(x,z)-d(y,z)\right)$. Since $(X,d_{\mathcal{W}})$ is hyperbolic, there is $\delta>0$ such that for each geodesic $[y,z]$ from y to z, we have $d_{\mathcal{W}}(x,[y,z])\leqslant(y,z)_x+\delta$, where δ is independent of x,y,z and $[y,z]$ (cf. [BH99, chapter III.H] for these properties of the Gromov product). Let $\tilde{K}\in\mathbb{R}_{\geqslant 0}$ such that $(X,\mathcal{W},\mathfrak{B},\nu)$ is $\tilde{K}$-coarsely median and let $K>2\tilde{K}+\delta$. Let $x_1,x_2,x_3\in X$ and $\tilde{m}=\mu_{\mathcal{M}}(\sigma(x_1),\sigma(x_2),\sigma(x_3))\in\mathcal{M}(X)$, and choose $m\in X$ such that $d_{\mathcal{M}}(\sigma(m),\tilde{m})\leqslant K$. Fix an arbitrary $i\in\{1,2,3\}$ and a geodesic $[x_i,x_{i+1}]$ from x_i to x_{i+1}. Clearly, in $\mathcal{M}(X)$ we have $(\sigma(x_i),\sigma(x_{i+1}))_{\tilde{m}}=0$. Thus,

$$\begin{aligned}d_{\mathcal{W}}(m,[x_i,x_{i+1}])&\leqslant(x_i,x_{i+1})_m+\delta\\&=(\sigma(x_i),\sigma(x_{i+1}))_{\sigma(m)}+\delta\\&\leqslant(\sigma(x_i),\sigma(x_{i+1}))_{\tilde{m}}+\tilde{K}+\delta\\&=\tilde{K}+\delta,\end{aligned}$$

and subsequently $d_{\mathcal{M}}(\tilde{m},\sigma([x_i,x_{i+1}]))\leqslant 2\tilde{K}+\delta<K$. By the proof of Theorem 2.5.2 we have $\mathfrak{M}(x_1,x_2,x_3;K)=\sigma^{-1}\left(\mathrm{B}_K(\tilde{m})\right)$ and so it follows that $\mathfrak{M}(x_1,x_2,x_3;K)\cap[x_i,x_{i+1}]\neq\emptyset$.

We have just proven that for a geodesic triangle $([x_i,x_{i+1}])_{i\in\{1,2,3\}}$, the uniformly bounded set $\mathfrak{M}(x_1,x_2,x_3;K)$ has non-empty intersection with each side $[x_i,x_{i+1}]$. Thus, according to Definition 2.3.3, any element of $\mathfrak{M}(x_1,x_2,x_3;K)$ will be a valid choice for $\mu_{\mathrm{hyp}}(x_1,x_2,x_3)$. Together with Theorem 2.5.2 this proves $\sigma^*\mu_{\mathcal{M}}\sim\mu_{\mathrm{hyp}}$. □

In Chapter 4, we encounter further examples of spaces with walls which are coarsely median. They arise from certain hyperbolic groups and there are analogous results as in the proposition above (see Proposition 4.3.1 and Corollary 4.4.1).

Chapter 3

Coarse median groups

3.1 Compactly generated groups

In this section, we apply the theory of coarse median spaces to groups. The concept of a *coarse median group* has already been defined for finitely generated groups in [Bow13a]. Here, we make a further step and note that the basic definition of a coarse median group works equally well for compactly generated locally compact groups.

Definition 3.1.1 (Word metric & Cayley graph)**.** Let G be a group and $S \subseteq G$ some generating set.

(i) For each $g \in G$, we define its *word length* with respect to S as follows.

$$|g|_S = \min\left\{n \in \mathbb{N} \;\middle|\; \exists s_1, \ldots, s_n \in S^{\pm 1} : g = s_1 \cdots s_n\right\}$$

For $g_1, g_2 \in G$, we define their *word distance* by $d_S(g_1, g_2) := |g_1^{-1} g_2|_S$.

(ii) The *Cayley graph* of G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the graph with vertex set G together with an oriented edge from g to gs labeled by s for each $g \in G, s \in S$. Usually, we will assume that $S = S^{-1}$, and then we consider the pair of oriented edges from g to gs and from gs to $gss^{-1} = g$ as a single unoriented edge. We may then view the Cayley graph $\mathrm{Cay}(G, S)$ as a graph as in Section 1.1 without multiple edges.

The word distance d_S defines a left-invariant metric on G, and it is the same as the combinatorial metric induced by the graph $\mathrm{Cay}(G, S)$.

Lemma 3.1.2 ([Cor10, Lemma 3.2])**.** *Let G be a locally compact group and S be a compact generating set. For each $n \in \mathbb{N}$, let $S^{\leqslant n} = \{s_1 \cdots s_k \mid s_i \in S^{\pm 1}, 0 \leqslant i \leqslant k \leqslant n\}$. Then for every compact subset $K \subseteq G$, there exists $n \in \mathbb{N}$ such that $K \subseteq S^{\leqslant n}$.*

Proof. We have $G = \bigcup_{n \in \mathbb{N}} S^{\leqslant n}$, where each $S^{\leqslant n}$ is compact (hence closed). The group G is locally compact and thus, by the Baire category theorem, there exists $n_0 \in \mathbb{N}$ such that $S^{\leqslant n_0}$ has non-empty interior $I_0 := \mathrm{Int}(S^{\leqslant n_0})$. For each $n \in \mathbb{N}$, the set $I_0\, S^{\leqslant n_0 + n}$ is open and has the property

$$S^{\leqslant n} \subseteq I_0\, S^{\leqslant n_0 + n} \subseteq S^{\leqslant 2n_0 + n}.$$

Thus $\left(I_0\, S^{\leqslant n_0 + n}\right)_{n \in \mathbb{N}}$ is an ascending sequence of open sets which covers G, hence it follows that there exists $n \in \mathbb{N}$ such that $K \subseteq I_0 S^{\leqslant n_0 + n} \subseteq S^{\leqslant 2n_0 + n}$. □

Remark 3.1.3. Let G be a locally compact group and S, S' two compact generating sets. Then the lemma above implies that the identity map $\mathrm{id}_G\colon (G, d_S) \to (G, d_{S'})$ is a bi-Lipschitz equivalence. Thus, if the metric spaces (G_1, d_{S_1}) and (G_2, d_{S_2}) are quasi-isometric, where G_i are locally compact groups with compact generating sets $S_i \subseteq G_i$, then this is also true for all other compact generating sets $S'_i \subseteq G_i$, $i \in \{1, 2\}$. In this case, we just say that G_1 and G_2 are *quasi-isometric*.

Definition 3.1.4. (i) A compactly generated locally compact group G is called *coarse median* (of rank n) if it admits a compact generating set $S \subseteq G$ such that the metric space (G, d_S) is coarse median (of rank n).

(ii) A compactly generated locally compact group G is called *hyperbolic* if it admits a compact generating set $S = S^{-1} \subseteq G$ such that the graph $\mathrm{Cay}(G, S)$ is hyperbolic.

If G is coarse median (of rank n), then (G, d_S) admits a coarse median (of rank n) for every compact generating set $S \subseteq G$. A similar remark applies to the notion of hyperbolicity. Moreover, by Section 2.3, a compactly generated locally compact hyperbolic group is coarse median of rank 1.

Definition 3.1.5. Let G be a compactly generated locally compact hyperbolic group and $S \subseteq G$ some compact generating set. We define $\mu_G\colon G \times G \times G \to G$ by $\mu_G = \iota^* \mu_{\mathrm{hyp}}$, where $\iota\colon G \hookrightarrow \mathrm{Cay}(G, S)$ is the inclusion map into the Cayley graph and μ_{hyp} a hyperbolic coarse median on $\mathrm{Cay}(G, S)$ as in Definition 2.3.3.

Observe that the coarse median structure determined by μ_G does not depend on S. Indeed, if $S, S' \subseteq G$ are two compact generating sets, then we can choose a quasi-isometry $\phi\colon\ \mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S')$ such that the following diagram commutes up to $\sim$.

$$\begin{tikzcd}
(G, d_S) \arrow[r, "\mathrm{id}_G"] \arrow[d, "\iota"'] & (G, d_{S'}) \arrow[d, "\iota'"] \\
\mathrm{Cay}(G, S) \arrow[r, "\phi"'] & \mathrm{Cay}(G, S')
\end{tikzcd}$$

Let μ_{hyp} and μ'_{hyp} be hyperbolic coarse medians on $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, S')$, respectively. Then, by Lemma 2.3.7, we have $\phi^* \mu'_{\mathrm{hyp}} \sim \mu_{\mathrm{hyp}}$ and we conclude

$$\iota^* \mu_{\mathrm{hyp}} \sim \iota^* \phi^* \mu'_{\mathrm{hyp}} \sim (\phi \circ \iota)^* \mu'_{\mathrm{hyp}} \sim (\iota' \circ \mathrm{id}_G)^* \mu'_{\mathrm{hyp}} = (\iota')^* \mu'_{\mathrm{hyp}},$$

where we have used the formal properties from Lemma 2.2.3.

Definition 3.1.6. Let G be a topological group acting on a metric space (X, d). The action is called *geometric*, if it is

(i) *by isometries*, i.e. for each $g \in G$, the map $X \to X$, $x \mapsto g \cdot x$ is an isometry,

(ii) *continuous*, i.e. for each $x \in X$, the map $G \to X$, $g \mapsto g \cdot x$ is continuous,

(iii) proper (see Definition 1.4.2),

(iv) and *cocompact*, i.e. there exists a compact subset $K \subseteq X$ such that $G \cdot K = X$.

We state a variant of the fundamental result known as the Švarc–Milnor[1] lemma.

Proposition 3.1.7 (Švarc–Milnor lemma)**.** *Let G be a topological group acting geometrically on a length space (X, d). Then for each $x_0 \in X$, there exists a compact generating set $S \subseteq G$ such that the map $g \mapsto g \cdot x_0$ is a quasi-isometry $(G, d_S) \to (X, d)$.*

Proof. We follow the proof in [BH99, Proposition I.8.19] and make the necessary changes to account for the fact that we are dealing with topological groups.

Let $K \subseteq X$ be a compact set such that $G \cdot K = X$, and choose $R > 0$ such that $K \subseteq \mathrm{B}_{\frac{R}{3}}(x_0)$. Let S denote the closure of $\{g \in G \mid g \cdot \mathrm{B}_R(x_0) \cap \mathrm{B}_R(x_0) \neq \emptyset\}$ in G, and we observe that this set is compact due to properness of the action. Also note that $S = S^{-1}$. We prove that S satisfies the required properties as follows.

(i) Let $g \in G$ and find a path $c\colon [0, 1] \to X$ such that $c(0) = x_0$, $c(1) = g \cdot x_0$, and the length of c is at most $d(x_0, g \cdot x_0) + 1$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of the interval $[0, 1]$ such that $d(c(t_i), c(t_{i+1})) \leqslant \frac{R}{3}$ for all $i \in \{0, \dots, n-1\}$. For each $i \in \{1, \dots, n-1\}$ choose an element $g_i \in G$ such that $g_i^{-1} \cdot c(t_i) \in \mathrm{B}_{\frac{R}{3}}(x_0)$, and set $g_0 = 1$, $g_n = g$. It follows that

$$d(g_i \cdot x_0, g_{i+1} \cdot x_0) \leqslant d(g_i \cdot x_0, c(t_i)) + d(c(t_i), c(t_{i+1})) + d(c(t_{i+1}), g_{i+1} \cdot x_0) \leqslant R,$$

and so $s_{i+1} := g_i^{-1} g_{i+1} \in S$. This implies that S generates G since $g = \prod_{i=0}^{n-1} g_i^{-1} g_{i+1} = \prod_{i=1}^{n} s_i$. Moreover, we can choose the partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$ so that $n\frac{R}{3} \leqslant d(x_0, g \cdot x_0) + 1 + \frac{R}{3}$, and thus

$$d_S(1, g) \leqslant n \leqslant \frac{3}{R} d(x_0, g \cdot x_0) + \frac{3}{R} + 1. \tag{3.1.1}$$

(ii) Let $M = \max\{d(x_0, s \cdot x_0) \mid s \in S\}$, where the maximum exists because the action is continuous and $S \subseteq G$ is compact. For each $g \in G$, there exist $s_1, \dots, s_n \in S$ with $g = s_1 \cdots s_n$ and $n = d_S(1, g)$. Let $g_i = s_1 \cdots s_i$. We then have

$$d(x_0, g \cdot x_0) \leqslant \sum_{i=0}^{n-1} d(g_i \cdot x_0, g_{i+1} \cdot x_0) = \sum_{i=1}^{n} d(x_0, s_i \cdot x_0) \leqslant Mn = M\, d_S(1, g). \tag{3.1.2}$$

Since the distance functions $(g, h) \mapsto d_S(g, h)$ and $(g, h) \mapsto d(g \cdot x_0, h \cdot x_0)$ are both left-invariant, we conclude from (3.1.1) and (3.1.2) that the map $(G, d_S) \to (X, d)$, $g \mapsto g \cdot x_0$ is a quasi-isometry. $\square$

Thus, to prove that a locally compact group is coarse median, it suffices to find a geometric action on a coarse median space.

We denote the isometry group of a metric space (X, d) by $\mathrm{Isom}(X, d)$ (or just $\mathrm{Isom}(X)$) and endow it with the topology of point-wise convergence. Then $\mathrm{Isom}(X)$ is a topological Hausdorff group and the natural action of $\mathrm{Isom}(X)$ on X is continuous. We note the following classical result.

[1] According to [BH99, p. 140], the theorem was originally discovered in the nineteen fifties by Russian mathematicians including A. S. Švarc, and later rediscovered by J. Milnor [Mil68].

Theorem 3.1.8. *Let (X, d) be a metric space which is connected and locally compact. Then $\operatorname{Isom}(X, d)$ is a locally compact group and the action on X is proper.*

Proof. For a proof that $\operatorname{Isom}(X, d)$ is locally compact, see [KN63, Chapter I, Theorem 4.7]. The fact that the action is proper also follows from the methods in the aforementioned reference, as described in [Ana12, Theorem 1.1]. □

Corollary 3.1.9. *Let X be locally compact length space. If $\operatorname{Isom}(X)$ acts cocompactly on X, then $\operatorname{Isom}(X)$ is compactly generated and quasi-isometric to X. In particular, if X is coarse median (of rank n), then $\operatorname{Isom}(X)$ is coarse median (of rank n).*

Example 3.1.10. (a) Let $\mathbb{E}^n$ be n-dimensional Euclidean space which, as in Example 2.2.8, admits a coarse median of rank n. The isometry group of $\mathbb{E}^n$ is isomorphic to the semi-direct product $\mathbb{R}^n \rtimes \mathrm{O}(n)$ and acts transitively on $\mathbb{E}^n$. Hence $\operatorname{Isom}(\mathbb{E}^n)$ is a coarse median group of rank n.

(b) The isometry group of real hyperbolic space acts transitively, so $\operatorname{Isom}(\mathbb{H}^n) \cong \mathrm{O}_+(1, n)$ (compare Example 1.3.6) is a hyperbolic group and thus coarse median of rank 1. The same is true for the isometry groups of complex and quaternionic hyperbolic space (for a discussion of these spaces and their isometry groups see e.g. [BH99, Chapter II.10]).

Note that $\mathrm{Sp}(1, n)$ — the isometry group of quaternionic hyperbolic space — has Kazhdan's property (T) [BHV08, Section 3.3]. Therefore, this yields an example of a hyperbolic (thus coarse median of rank 1) group with property (T). In Chapter 5, we present a different construction of groups with this combination of properties.

We finish this section by proving that on a hyperbolic group, we can always find a left-equivariant coarse median.

Lemma 3.1.11. *Let G be a group, $S \subseteq G$ a generating set and $\mu \colon G^3 \to G$ a coarse median on (G, d_S). Suppose that μ is equivariant up to uniformly bounded error, that is,*

$$\sup_{g, g_1, g_2, g_3 \in G} d_S\left(\mu(gg_1, gg_2, gg_3), g\mu(g_1, g_2, g_3)\right) < \infty.$$

Then there exists a coarse median $\tilde{\mu} \colon G^3 \to G$ such that $\mu \sim \tilde{\mu}$ (with respect to d_S) and $\tilde{\mu}(gg_1, gg_2, gg_3) = g\tilde{\mu}(g_1, g_2, g_3)$ for all $g, g_1, g_2, g_3 \in G$.

Proof. The diagonal action by left-multiplication of G on G^3 is *free*. Hence there exists a fundamental domain $\mathbf{T} \subseteq G^3$ such that for each triple $\mathbf{g} = (g_1, g_2, g_3) \in G^3$, there exists a unique $\mathbf{t}(\mathbf{g}) = (t_1(\mathbf{g}), t_2(\mathbf{g}), t_3(\mathbf{g})) \in \mathbf{T}$ and a *unique* $h(\mathbf{g}) \in G$ such that $\mathbf{g} = h(\mathbf{g}) \cdot \mathbf{t}(\mathbf{g}) := (h(\mathbf{g})t_1(\mathbf{g}), h(\mathbf{g})t_2(\mathbf{g}), h(\mathbf{g})t_3(\mathbf{g}))$. We define

$$\tilde{\mu} \colon G^3 \to G, \quad \mathbf{g} \mapsto h(\mathbf{g})\mu(\mathbf{t}(\mathbf{g})).$$

For $\mathbf{g} \in G^3$, $g \in G$, we have $\mathbf{t}(g \cdot \mathbf{g}) = \mathbf{t}(\mathbf{g})$ and $h(g \cdot \mathbf{g}) = gh(\mathbf{g})$, which implies that $\tilde{\mu}$ is left-equivariant. Moreover, by assumption we have $\sup_{\mathbf{g} \in G^3} d_S\left(\mu(\mathbf{g}), h(\mathbf{g})\mu(\mathbf{t}(\mathbf{g}))\right) < \infty$, whence $\mu \sim \tilde{\mu}$. □

Proposition 3.1.12. *Let G be a compactly generated locally compact hyperbolic group. Then there exists a left-equivariant coarse median $\mu_G\colon G^3 \to G$, i.e.*

$$\mu_G(gg_1, gg_2, gg_3) = g\mu_G(g_1, g_2, g_3) \qquad \forall g, g_1, g_2, g_3 \in G.$$

Proof. Let $S \subseteq G$ be a compact generating set, and $\mu\colon G^3 \to G$ a hyperbolic coarse median as in Definition 3.1.5. Left multiplication by an element of G defines an isometry $\mathrm{Cay}(G, S) \to \mathrm{Cay}(G, S)$. Thus, Corollary 2.3.8 implies that μ satisfies the assumption of Lemma 3.1.11, which proves the claim. □

3.2 Compact presentation and Dehn function

In this section, we show that every compactly generated locally compact coarse median group is compactly presented.

Definition 3.2.1 ([Cor10])**.** (i) Let G be a group and $S \subseteq G$ a (not necessarily finite) subset. We say that G is *boundedly presented* by S if it has a presentation with S as the set of generators, and relators of uniformly bounded length.

(ii) A locally compact group is called *compactly presented* if it is boundedly presented by a compact generating set.

Note that a group is finitely presented iff it is compactly presented with the discrete topology iff it is boundedly presented by some finite subset.

Definition 3.2.2. Let $\mathcal{P} = \langle S \mid \mathcal{R}\rangle$ be a group presentation. For a reduced word w in letters of S which represents the identity in the group defined by $\mathcal{P}$, we write $w =_{\mathcal{P}} 1$ and define its *area* by

$$\mathrm{A}_{\mathcal{P}}(w) = \min\left\{ n \in \mathbb{N} \;\middle|\; \exists r_1, \ldots, r_n \in \mathcal{R}^{\pm 1}, x_1, \ldots x_n \in \mathrm{F}(S) : w = \prod_{i=1}^{n} x_i r_i x_i^{-1} \right\}.$$

Moreover, the *Dehn function* of the presentation $\mathcal{P}$ is

$$\delta_{\mathcal{P}}\colon \mathbb{N} \to \mathbb{N} \cup \{\infty\}, \quad n \mapsto \sup\left\{\mathrm{A}_{\mathcal{P}}(w) \mid w \in \mathrm{F}(S), w =_{\mathcal{P}} 1, |w|_S \leqslant n\right\}.$$

Remark 3.2.3 (van Kampen diagrams [BRS07, Part III], [LS77, Chapter V.1])**.** Fix a group presentation $\mathcal{P} = \langle S \mid \mathcal{R}\rangle$, where we assume each element of $\mathcal{R}$ to be cyclically reduced, and the set $\mathcal{R}$ to be *symmetrized*, that is closed under inversions and cyclic permutations. Let D be a finite, connected, oriented, labeled, based, planar graph, together with a fixed embedding into $\mathbb{R}^2$ such that the base vertex lies on the boundary of the unbounded region of $\mathbb{R}^2 \setminus D$. We further assume that the edges are labeled by elements of S such that along the boundary of each bounded region, called a *face*, we read an element of $\mathcal{R}$. (Note that due to our assumptions on $\mathcal{R}$, this is independent of the starting point and the direction we choose to read.) The *boundary word* w is the word read on the boundary of the unbounded component of $\mathbb{R}^2 \setminus D$ starting from the base vertex in counter-clockwise direction. We then say that D is a *van Kampen diagram* for the word w over the presentation $\mathcal{P}$. The *area* of D is the number of its faces.

It is a classical result due to Ebert van Kampen that a word w in letters of S represents the identity element in the group determined by $\mathcal{P}$ if and only if there exists a van Kampen diagram for w over $\mathcal{P}$. Furthermore, in this case, the minimal area of a van Kampen diagram for w over $\mathcal{P}$ is exactly $\mathrm{A}_{\mathcal{P}}(w)$.

We also note that, in the definition of a van Kampen diagram, the assumption that each $r \in \mathcal{R}$ is cyclically reduced is not a restriction: Given a diagram as above, but possibly with faces that are not labeled in a cyclically reduced way and possibly with an unreduced boundary word w, then one can obtain a new diagram with cyclically reduced face labellings and with boundary word the freely reduced version of w [BRS07, Part III, Proposition 2.4, Proof of Theorem 2.2]. Moreover, this can be done without increasing the total number of faces or the maximum of their boundary lengths.

Usually we are only interested in the growth type of the Dehn function, and thus we introduce the following notation: For two non-decreasing functions $f_1, f_2 \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, we write $f_1 \preccurlyeq f_2$ if there exists $C \geqslant 0$ such that $f_1(n) \leqslant C f_2(Cn + C) + Cn + C$ for all $n \in \mathbb{N}$. We write $f_1 \approx f_2$ if $f_1 \preccurlyeq f_2$ and $f_2 \preccurlyeq f_1$. Then $\preccurlyeq$ is reflexive and transitive, and $\approx$ is an equivalence relation (on the set of non-decreasing functions $\mathbb{N} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$). We say that $f \colon \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is { linear, quadratic, polynomial of degree d, exponential } if $f \approx \{n, n^2, n^d, \exp\}$, respectively.

For *finite* presentations $\mathcal{P}$ and $\mathcal{Q}$ which define isomorphic groups, we have $\delta_{\mathcal{P}} \approx \delta_{\mathcal{Q}}$ [BRS07, Part III, Theorem 4.7]. In particular, for every finitely presented group G, we may define its *Dehn function* δ_G as the $\approx$-equivalence class represented by the Dehn function of some (hence, any) finite presentation of G.

Bowditch shows that every finitely generated group that admits a coarse median on its Cayley graph is necessarily finitely presented with an at most quadratic Dehn function [Bow13a, Corollary 8.3]. We follow Bowditch' proof and obtain an immediate extension to the case of compactly generated locally compact groups.

Theorem 3.2.4 (cf. [Bow13a, Proposition 8.2])**.** *Let G be a group, $S \subseteq G$ a generating set, and assume that there exists a coarse median $\mu \colon G \times G \times G \to G$ on (G, d_S). Then there exists a constant $K \geqslant 0$, depending only on the parameters of μ, and a set $\mathcal{R} \subseteq \mathrm{F}(S)$ of words, the length of each is bounded by K, such that $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ is a presentation of G with $\delta_{\mathcal{P}} \preccurlyeq n^2$. In particular, G is boundedly presented by S.*

Corollary 3.2.5. *Every compactly generated locally compact coarse median group is compactly presented. Every finitely generated coarse median group is finitely presented and has an at most quadratic Dehn function.*

Proof of Theorem 3.2.4. By Corollary 2.4.4, we can assume that the coarse median $\mu \colon G \times G \times G \to G$ is symmetric and satisfies $\mu(x, x, y) = x$ for all $x, y \in G$. Then, by Definition 2.1.3 (i), there exists a constant $L \geqslant 0$ such that $d_S(\mu(x, y, z), \mu(x, y, z')) \leqslant L$ whenever $d_S(z, z') \leqslant 1$, where $x, y, z, z' \in G$. To prove the theorem, we will verify that it is enough to choose $K = 6L$ and $\mathcal{R} = \{r \in \mathrm{F}(S) \mid r =_G 1, |r|_S \leqslant K\}$.

Indeed, let $w = s_1 \cdots s_l \in \mathrm{F}(S)$, where $s_i \in S^{\pm 1}$, with $w =_G 1$. To construct a van Kampen diagram for w, we define a simplicial graph D' with vertex set $\mathcal{V}(D') = \{\{0\}\} \cup \{\{i,j\} \mid 1 \leqslant i,j \leqslant l-1\}$, where $l = |w|_S$, and edges as follows.

- Connect $\{i,j\}$ to $\{i+1,j\}$, for $i \in \{1,\dots,l-2\}$, $j \in \{1,\dots,l-1\}$,
- and $\{i,j\}$ to $\{i+1,j+1\}$, for $i,j \in \{1,\dots,l-2\}$,
- and $\{0\}$ to $\{1,i\}$ as well as $\{0\}$ to $\{l-1,i\}$, for $1 \leqslant i \leqslant l-1$.

This graph admits an embedding into the plane as depicted in Figure 3.1, and it has $(l-2)^2+2(l-2) = l^2-2l$ faces. The word w determines a closed loop $1_G = g_0, g_1, \dots, g_l = 1_G$ in the Cayley graph of G, where $g_i =_G s_1 \cdots s_i$. We define a map

$$\psi' \colon \mathcal{V}(D') \to G, \quad \{i,j\} \mapsto \mu(1_G, g_i, g_j),$$

which is unambiguous due to our assumptions on μ, and it satisfies $\psi'(\{i\}) = g_i$ for all $i \in \{0,\dots,l-1\}$. If two vertices $\{i,j\}$, $\{i',j'\} \in \mathcal{V}(D')$ are adjacent we have $\max\{|i-j|, |i'-j'|\} \leqslant 1$, and hence we conclude $d_S(\psi'(\{i,j\}), \psi'(\{i',j'\})) \leqslant 2L$. We now construct a labeled graph D by appropriately subdividing each edge of D' and labeling the resulting edges so that ψ' extends to a label preserving graph morphism $\psi \colon D \to \mathrm{Cay}(G,S)$. Due to our considerations above, this can be done in a such a way that the exterior edges of D' are not subdivided at all, the word read along the external boundary is w, and the interior edges get replaced by a path of length at most $2L$. Since all faces in D' are triangles, the boundary length of each face in D is at most $6L$.

In summary, we have constructed a van Kampen diagram for w of area $\approx |w|_S^2$ over a presentation with generating set S and a set of (possibly unreduced) relators of length at most $6L = K$. Due to Remark 3.2.3 this finishes the proof. $\square$

Example 3.2.6. Consider the *Baumslag–Solitar groups*

$$\mathrm{BS}(m,n) = \left\langle a,t \;\middle|\; ta^m t^{-1} = a^n \right\rangle,$$

where $m,n \in \mathbb{Z} \setminus \{0\}$. We give a full description of which Baumslag–Solitar groups are coarse median.

- For $m \in \mathbb{Z} \setminus \{0\}$ arbitrary and $\epsilon \in \{\pm 1\}$, the group $\mathrm{BS}(m, \epsilon m)$ contains a finite index subgroup of the form $\mathbb{Z} \times \mathrm{F}_n$ [Why01, Theorem 0.1], hence it is coarse median of rank 2.
- For $|m| \neq |n|$, the Dehn function of $\mathrm{BS}(m,n)$ is not bounded by any polynomial [Ger92, Theorem B], so it is not coarse median by Corollary 3.2.5.

However, note that all $\mathrm{BS}(m,n)$ are a-T-menable [GJ03]. Thus, the second case yields a family of groups which are a-T-menable but not coarse median.

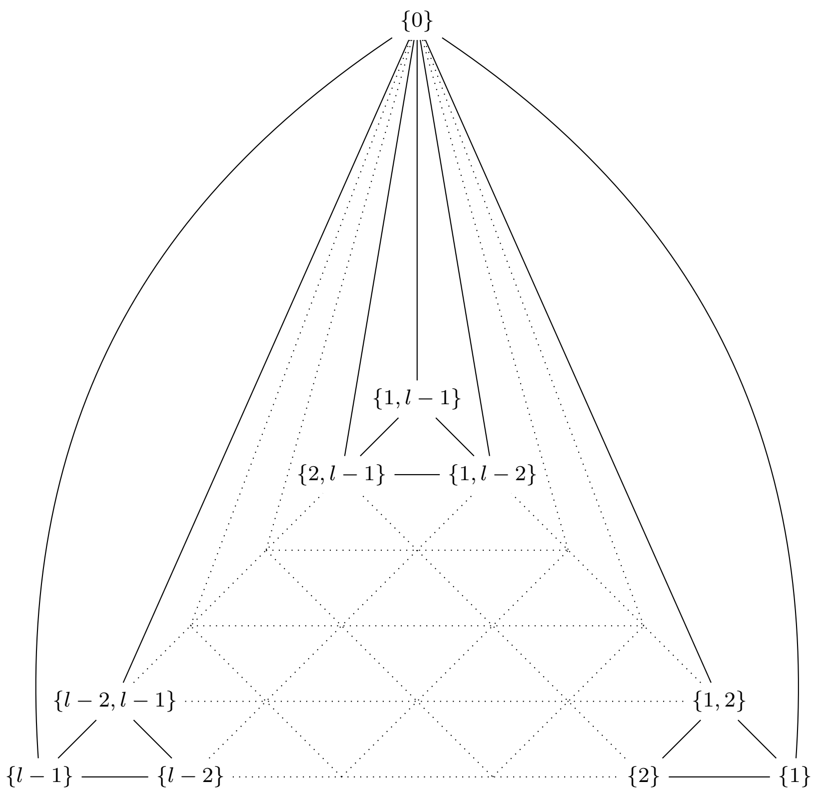

Figure 3.1: The graph D' embedded in the plane.

Remark 3.2.7. We call a finitely generated group CAT(0)-*cubical* if it acts properly and cocompactly by combinatorial automorphisms on some CAT(0) cube complex. Every CAT(0)-cubical group is a-T-menable by Corollary 1.4.8, and by Proposition 3.1.7, it is quasi-isometric to a CAT(0) cubical complex, hence coarse median. However, by the previous example, there are finitely generated a-T-menable groups which are not coarse median. Conversely, as discussed in detail in Chapter 5 (see also Example 3.1.10 (b)), there are coarse median groups which have Kazhdan's property (T). Thus, in general, no other implications between the properties CAT(0) cubical, coarse median and a-T-menable hold.

3.3 Permanence properties of finitely generated coarse median groups

Table 3.1: Stability of properties under group constructions

		Coarse median	**a-T-menability**	**(T)**
(a)	Direct Product	Yes	Yes	Yes
(b)	Semi-direct product	No	No	Yes
(c)	Group extension	No	No	Yes
(d)	(Normal) subgroup	No	Yes	No
(e)	Quotient	No	No	Yes
(f)	Amalgamated product over finite group	Yes	Yes	No
(g)	HNN-Extension with finite base groups	Yes	Yes	No

In this section, we focus on *finitely* generated groups and study how the coarse median property behaves under some of the standard group constructions. In Table 3.1, we give an overview of hereditary properties, and compare it with the situation of a-T-menability and Property (T). A list of references for the claims made in Table 3.1 can be found at the end of this section. We observe that — among the group constructions we consider — the property of being coarse median satisfies very similar permanence results as a-T-menability, and almost the opposite when compared to property (T). Nevertheless, there are no implications between a-T-menability and coarse median which hold in general (compare Remark 3.2.7).

An important permanence result due to Bowditch is the fact that being coarse median is preserved by relative hyperbolicity [Bow13b]. We state a variant of this result and then deduce the statement about coarse median in (f) and (g) of Table 3.1.

A connected graph Γ is said to be *fine*, if for each $n \in \mathbb{N}$, every edge of Γ is contained in only finitely many closed loops of length n.

Theorem 3.3.1 ([Bow13b, Theorem 2.1, Lemma 2.2])**.** *Let Γ be a fine connected graph which is hyperbolic with respect to the combinatorial metric. Assume that G is a finitely* *generated group which acts cocompactly by graph automorphisms on Γ such that,*

(i) each edge stabilizer is finite,

(ii) each vertex stabilizer is coarse median of rank $\leqslant n$.

Then G is coarse median of rank $\leqslant n$.

Corollary 3.3.2. *Let $\tilde{G}$ be either*

(i) an amalgamated product $\tilde{G} = G_1 *_H G_2$, where and G_i are finitely generated coarse median of rank $\leqslant n$, and $H \leq G_i$ is finite, $i \in \{1, 2\}$,

(ii) or an HNN-extension $\tilde{G} = G*_\alpha$, where G is finitely generated coarse median of rank $\leqslant n$ and $\alpha\colon H \to K$ an isomorphism between finite subgroups $H, K \leq G$.

Then $\tilde{G}$ is coarse median of rank $\leqslant n$.

Proof. In both cases, $\tilde{G}$ acts on the corresponding Bass–Serre tree (see [Ser80, p. 43] for the interpretation of amalgamated products and HNN-extensions as fundamental groups of certain graphs of groups; and [Ser80, §5.3] for the construction of the tree associated to a graph of groups). By assumption, the edge stabilizers are finite, and vertex stabilizers conjugate to G_1, G_2 and G, respectively. Thus, the result follows immediately by Theorem 3.3.1. □

We now mention two (counter-)examples to justify some of the negative statements in Table 3.1 about coarse medians.

Example 3.3.3 ([Gru78])**.** Let $\mathrm{C} = \langle t \rangle$ be the infinite cyclic group, and $\mathrm{F}_n = \langle s_1, \dots, s_n \rangle$ be some finitely generated non-abelian free group. We define $\phi\colon \mathrm{F}_n \to \mathrm{C}$ to be the group homomorphism which sends each s_i to t, $i \in \{1, \dots, n\}$, and observe that $\ker(\phi) \trianglelefteq \mathrm{F}_n$ is the normal closure of the finite set $\mathcal{R} := \{s_i {s_j}^{-1} \mid i \neq j \in \{1, \dots, n\}\}$.

Now we consider the fiber product,

$$\mathrm{F}_n \times_\phi \mathrm{F}_n := \{(x, y) \in \mathrm{F}_n \times \mathrm{F}_n \mid \phi(x) = \phi(y)\},$$

and claim that it is a finitely generated group. To see this, consider the following finite subset of $\mathrm{F}_n \times_\phi \mathrm{F}_n$,

$$T := \{(s_i, s_i) \mid i \in \{1, \dots, n\}\} \cup (\{1_{\mathrm{F}_n}\} \times \mathcal{R}),$$

and let $\langle T \rangle$ denote the subgroup generated by T. For every $(x, y) \in \mathrm{F}_n \times_\phi \mathrm{F}_n$, we can write $x^{-1}y \in \ker(\phi)$ as a product $x^{-1}y = \prod_k \gamma_k r_k \gamma_k^{-1}$ with $r_k \in \mathcal{R}$, $\gamma_k \in \mathrm{F}_n$. Since $(\gamma, \gamma) \in \langle T \rangle$ for all $\gamma \in \mathrm{F}_n$, we have,

$$(x, y) = (x, x)(1, x^{-1}y) = (x, x) \prod_k (\gamma_k, \gamma_k) \underbrace{(1, r_k)}_{\in T} (\gamma_k^{-1}, \gamma_k^{-1}) \in \langle T \rangle,$$

whence $\langle T \rangle = \mathrm{F}_n \times_\phi \mathrm{F}_n$.

However, because C is infinite, it follows from [Gru78, Proposition B] that $\mathrm{F}_n \times_\phi \mathrm{F}_n$ is not finitely presented, and so it cannot be coarse median by Corollary 3.2.5. Thus, $\mathrm{F}_n \times_\phi \mathrm{F}_n$ is a finitely generated group which does not admit a coarse median, although it is a subgroup of the rank 2 coarse median group $\mathrm{F}_n \times \mathrm{F}_n$. Also, note that $\mathrm{F}_n \times_\phi \mathrm{F}_n$ is even a normal subgroup of $\mathrm{F}_n \times \mathrm{F}_n$ because it is the kernel of the group homomorphism $\mathrm{F}_n \times \mathrm{F}_n \to \mathrm{C}$, $(x, y) \mapsto \phi(x)\phi(y)^{-1}$.

Example 3.3.4 ([Bri02, Theorem A])**.** For every $d \in \mathbb{N}$, there exists a semi-direct product $G = \mathrm{F}_n \rtimes \mathrm{F}_m$ of two finitely generated free groups such that $\delta_G \approx n^d$. For $d > 2$, this shows that being coarse median is not preserved by semi-direct products (again due to Corollary 3.2.5).

Proof of Table 3.1. (a) By Proposition 2.1.10, the product of two coarse median spaces is coarse median, hence the direct product of coarse median groups is coarse median. The corresponding statement for a-T-menability is proved in Proposition 1.4.5 and for property (T) it is a special case of (c).

(b) See Example 3.3.4 for the case of coarse median. For a-T-menability, note that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is not a-T-menable [BHV08, Theorem 4.2.2], whereas $\mathbb{Z}^2$ and $\mathrm{SL}_2(\mathbb{Z})$ are. The fact that property (T) is preserved by semi-direct products is again a special case of (c).

(c) The negative statements for coarse median and a-T-menability follow from (b). On the other hand, if $1 \to N \to G \to H \to 1$ is a short exact sequence of finitely generated groups such that N and H have property (T), then Proposition 1.4.6 implies that G has property (T).

(d) Example 3.3.3 shows that being coarse median is not inherited by finitely generated normal subgroups. On the other hand, by Definition 1.4.4, it is clear that a-T-menability passes to subgroups. Property (T) does not pass to normal subgroups, as $\mathbb{Z}^3 \rtimes \mathrm{SL}_3(\mathbb{Z})$ has property (T) [BHV08, Corollary 1.4.16] but $\mathbb{Z}^3$ has not.

(e) Finitely generated free groups are coarse median and a-T-menable, hence these properties cannot be inherited by quotient groups. However, it is clear from Definition 1.4.4 that Property (T) passes to quotients.

(f) For coarse median, this is the content of Corollary 3.3.2. For a-T-menability, the result follows from [Che+01, Theorem 6.2.8]. Since a non-trivial amalgamated product acts without fixed point on the corresponding Bass–Serre tree, it cannot have property (T).[2]

(g) Follows by the same arguments and references as (f). □

[2]To see this, note that a tree is a median space, and thus every (T)-group acting on it has bounded orbits, see Corollary 1.4.8. However, acting with bounded orbits on a tree implies the existence of a fixed point.

Chapter 4

Median structure of small cancellation groups

In this chapter, we consider a group G which is defined by a finite $C'(\frac{1}{6})$ small cancellation presentation. The reason of why these are of interest to us is two-fold.

1. The group G is hyperbolic and, by the considerations in Section 2.3, admits a rank 1 coarse median,

2. the group G acts properly and cocompactly on a CAT(0) cubical complex [Wis04], which comes with a natural discrete median structure (see Example 2.2.5).

The *main question* we address in this chapter is the following.

- *Are these two (coarse) median structures related, and if yes, in which way?*

In Section 4.3, we give an answer to this; we prove that — up to applying a certain (very natural) quasi-isometry — these two coarse median structures are the same.

In fact, the CAT(0) cubical complex on which G acts is constructed via an equivariant wall structure on G. In Section 4.4, we relate the hyperbolic coarse median on G to this wall structure (similarly to the general observations of Section 2.5).

4.1 Preliminaries on small cancellation theory

In this section, we give a very brief account of the relevant parts of classical small cancellation theory. A general introduction to this theory can be found in the book [LS77, Chapter V]. Here, we introduce a slightly different and more geometric language, which will be convenient in the next sections. See [Wis04, Section 2] and [MW02] for a more thorough development of these concepts.

Definition 4.1.1. Let X and Y be CW-complexes. A continuous map $f\colon Y \to X$ is called *combinatorial* if the restriction of f to each open cell of Y is a homeomorphism onto an open cell of X.

A 2-dimensional CW-complex X is called *combinatorial*, if the attaching map of each 2-cell is a combinatorial map $S^1 \to X$ for some subdivision of the circle.

A *combinatorial path* in a combinatorial 2-complex is a combinatorial map $P \to X$, where P is a subdivision of a compact real interval.

The examples of combinatorial 2-complexes we are interested in are defined by group presentations.

Example 4.1.2. Let $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ be a group presentation where each relator $r \in \mathcal{R}$ is cyclically reduced. The *presentation complex* of $\mathcal{P}$, denoted by $\mathrm{K}(\mathcal{P})$, is the combinatorial 2-complex consisting of one 0-cell v_0, 1-cells labeled bijectively by S, and 2-cells corresponding to $\mathcal{R}$. Namely, the edges are attached to v_0 in the only possible way and for each $r \in \mathcal{R}$, the attaching map of the corresponding 2-cell is the unique combinatorial path $S^1 \to \mathrm{K}^{(1)}(\mathcal{P})$ starting at v_0 such that we read r along it.

The *Cayley complex* of $\mathcal{P}$, denoted by $\mathrm{C}(\mathcal{P})$, is the universal covering of $\mathrm{K}(\mathcal{P})$. Since the CW-complex structure of $\mathrm{C}(\mathcal{P})$ is obtained by lifting the attaching maps of cells in $\mathrm{K}(\mathcal{P})$, it follows that $\mathrm{C}(\mathcal{P})$ is a combinatorial 2-complex and the covering $p\colon \mathrm{C}(\mathcal{P}) \twoheadrightarrow \mathrm{K}(\mathcal{P})$ is a combinatorial map. The group G determined by $\mathcal{P}$ is equal to $\pi_1(\mathrm{K}(\mathcal{P}))$, and therefore G acts on $\mathrm{C}(\mathcal{P})$ by combinatorial maps. Furthermore, this action is free and transitive on the fibers of p. In particular, G acts freely and transitively on the vertex set $\mathrm{C}^{(0)}(\mathcal{P}) = p^{-1}(v_0)$, and this, in turn, implies that the 1-skeleton $\mathrm{C}^{(1)}(\mathcal{P})$ is isomorphic to the Cayley graph of G with respect to S.

Definition 4.1.3. Let X be a combinatorial 2-complex. A *piece* in X is a non-trivial combinatorial path $P \to X$ such that there are 2-cells R_1, R_2 such that P factors as $P \to R_1 \to X$ and $P \to R_2 \to X$, but there exists *no* homeomorphism $\partial R_1 \overset{\cong}{\to} \partial R_2$ fitting into the commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \partial R_2 \arrow[d] \\ \partial R_1 \arrow[r] \arrow[ur, tail] & X \end{tikzcd}$$

The main idea of small cancellation theory is to impose certain conditions on combinatorial 2-complexes which ensure that pieces are relatively small, and to draw geometric conclusions (e.g. hyperbolicity) about the 2-complexes we consider. An example of such a condition is the metric $C'(\lambda)$ condition, which we define below.

Definition 4.1.4. Let X be a combinatorial 2-complex and $\lambda > 0$. We say that X is a $C'(\lambda)$ complex if for every 2-cell $R \to X$ and each piece $P \to R \to X$ which factors through R, we have

$$|P| < \lambda |\partial R|.$$

A group presentation $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ is called a $C'(\lambda)$ presentation provided that $\mathrm{C}(\mathcal{P})$ is a $C'(\lambda)$ complex.

Definition 4.1.5. A presentation $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ is called a *Dehn presentation* if $\mathcal{R}$ is symmetrized (cf. Remark 3.2.3) and the following holds.

For each reduced word w in letters from S which represents the identity in the group given by $\mathcal{P}$, there exists a subword u of w, which is contained as a subword of some $r \in \mathcal{R}$ and satisfies $|u| > \frac{1}{2}|r|$.

Proposition 4.1.6. *For a finitely generated group G, the following are equivalent.*

(i) G is hyperbolic.

(ii) G is finitely presented and its Dehn function δ_G is linear.

(iii) G admits a finite Dehn presentation.

Proof. To see (iii) $\Rightarrow$ (ii), let $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ be a finite Dehn presentation. We prove by an inductive procedure, called *Dehn's algorithm*, that $\mathrm{A}_{\mathcal{P}}(w) \leqslant |w|$ for all $w \in \mathrm{F}(S)$ with $w =_{\mathcal{P}} 1$. Let $w \in \mathrm{F}(S)$ with $|w| > 0$ and $w =_{\mathcal{P}} 1$. Since $\mathcal{R}$ is symmetrized, after possibly conjugating with some element in the free group, we can suppose that $w = uw'$ with $u\tilde{u} = r \in \mathcal{R}$ for some words $u, \tilde{u}$ that satisfy $|\tilde{u}| < |u|$. Hence, we have $w = r\tilde{w}$, where $\tilde{w} = \tilde{u}^{-1}w'$. Since $\tilde{w} =_{\mathcal{P}} w =_{\mathcal{P}} 1$ and $|\tilde{w}| \leqslant |w| - 1$, we have by the inductive hypothesis that $\mathrm{A}_{\mathcal{P}}(\tilde{w}) \leqslant |\tilde{w}|$, and so we see $\mathrm{A}_{\mathcal{P}}(w) \leqslant \mathrm{A}_{\mathcal{P}}(\tilde{w}) + 1 \leqslant |\tilde{w}| + 1 \leqslant |w|$.

A proof of the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) is contained in [BRS07, Part III, Proposition 4.11]. For (ii) $\Rightarrow$ (i), we refer to [BH99, III.H, Theorem 2.9]. $\square$

We state the following classical result, which says, in particular, that groups given by finite $C'(\frac{1}{6})$ presentations are hyperbolic.

Proposition 4.1.7 (Weak version of [LS77, Theorem V.4.5])**.** *Let $\mathcal{P}$ be a finite $C'(\lambda)$ presentation with $\lambda \leqslant \frac{1}{6}$. Then $\mathcal{P}$ is a Dehn presentation.*

4.2 Wall structure of $C'(1/6)$ complexes

In the following, we collect definitions and results from [Wis04] which are required in the next section.

Definition 4.2.1. Let X be a combinatorial 2-complex with the property that for each 2-cell R, the boundary length $|\partial R|$ is even. Let $\mathcal{G}_X$ be the graph with vertices corresponding to the 1-cells of X, where two vertices are connected by an edge if the corresponding 1-cells are antipodal[1] in a 2-cell of X. The connected components of $\mathcal{G}_X$ are called *hypergraphs*.

If Γ is a hypergraph, then we denote the graph obtained from $X^{(1)}$ by deleting all edges corresponding to vertices of Γ by $X^{(1)} \setminus \Gamma$.

Proposition 4.2.2 ([Wis04, Corollary 3.12, Lemma 3.13])**.** *Let X be a locally finite, simply connected $C'(\frac{1}{6})$ complex, where 2-cells have even boundary length. Then the following holds:*

(i) Every hypergraph is a tree.

(ii) For every hypergraph Γ, the graph $X^{(1)} \setminus \Gamma$ has exactly two connected components.

Therefore, every hypergraph Γ determines a wall W_Γ on $X^{(0)}$ where the half spaces of W_Γ are the vertex sets of the connected components of $X^{(1)} \setminus \Gamma$. This defines a wall structure $\mathcal{W}_X = \{W_\Gamma \mid \Gamma \text{ hypergraph}\}$ on the vertex set of X.

[1] This is then well defined because we have assumed our 2-cells to have even boundary length.

Let G be a group which acts by combinatorial homeomorphism on X. We claim that this induces an action on the space with walls $(X^{(0)}, \mathcal{W}_X)$. Indeed, each $g \in G$ gives rise to a bijection of the edges which preserves the relation of being antipodal, and thus induces an automorphism $\mathcal{G}_X \overset{\cong}{\to} \mathcal{G}_X$. Therefore, G acts on the set of hypergraphs. Moreover, for any hypergraph Γ we have $g \cdot (X^{(1)} \setminus \Gamma) = X^{(1)} \setminus (g \cdot \Gamma)$. It follows that $g \colon X^{(1)} \to X^{(1)}, x \mapsto g \cdot x$ maps the components of $X^{(1)} \setminus \Gamma$ to the components of $X^{(1)} \setminus (g \cdot \Gamma)$. This shows that $g \colon X^{(0)} \to X^{(0)}$ is a morphism of the space with walls $(X^{(0)}, \mathcal{W}_X)$.

Theorem 4.2.3 ([Wis04, Theorem 11.1])**.** *Let G be a group acting cocompactly and with finite vertex stabilizers on a 2-complex X which satisfies the assumptions of Proposition 4.2.2. Then the induced action on the associated median space $\mathcal{M}(X^{(0)})$ is proper and cocompact.*

Proof. This is a reformulation of (a particular case of) a theorem of Wise, and in this proof we just spell out the translation:

The theorem [Wis04, Theorem 11.1] says that the induced action of G on the CAT(0) cubical complex C constructed in [Wis04, Section 5] is proper and cocompact, provided that X is a $B(6)$-$C(7)$-complex. The notation $B(6)$-$C(7)$ refers to other small cancellation conditions which we do not discuss in this thesis. However, the $C'(\frac{1}{6})$ condition implies $B(6)$-$C(7)$, as mentioned in [Wis04, Section 2], and so we can apply Wise's theorem to our situation. Furthermore, we observe that C is exactly the cubical complex obtained from the space with walls $(X^{(0)}, \mathcal{W}_X)$ by an application of Example 1.3.12. Therefore, the vertex set of C coincides with $\mathcal{M}(X^{(0)})$. Putting all these results together, we see that Theorem 4.2.3 follows directly from [Wis04, Theorem 11.1]. $\square$

Corollary 4.2.4. *Under the assumptions of Theorem 4.2.3, the space with walls $(X^{(0)}, \mathcal{W}_X)$ is coarsely median in the sense of Section 2.5.*

Proof. Let $B \subseteq \mathcal{M}(X^{(0)})$ be a finite subset such that $G \cdot B = \mathcal{M}(X^{(0)})$, fix $x_0 \in X^{(0)}$, and let $R = \max_{t \in B} d_{\mathcal{M}}(\sigma(x_0), t)$. For every $s \in \mathcal{M}(X^{(0)})$, there exists $g \in G$ such that $s \in g \cdot B$. We have

$$d_{\mathcal{M}}(\sigma(g \cdot x_0), s) = d_{\mathcal{M}}(g \cdot \sigma(x_0), s) = d_{\mathcal{M}}(\sigma(x_0), g^{-1} \cdot s) \leqslant R,$$

and so $\sigma(X^{(0)}) \subseteq \mathcal{M}(X^{(0)})$ has bounded distance to $\mathcal{M}(X^{(0)})$. $\square$

4.3 Comparison of coarse median structures

We consider a $C'(\frac{1}{6})$ presentation $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ of a group G, where we assume for simplicity that each $r \in \mathcal{R}$ has even length. We let $X = \mathrm{C}(\mathcal{P})$ and, by the previous section, we get a wall structure $(X^{(0)}, \mathcal{W}_X)$. After fixing a base-point $x_0 \in X^{(0)}$, we can equivariantly identify $G = X^{(0)}$ via $g \mapsto g \cdot x_0$. This induces an equivariant wall structure on G, which we denote by $(G, \mathcal{W}_{\mathcal{P}})$. We fix this setup for the rest of this chapter.

The presentation complex $\mathrm{K}(\mathcal{P})$ is compact, as $\mathcal{P}$ is finite. Thus, the G-action on $\mathrm{C}(\mathcal{P})$ is cocompact. Since it is also free, Theorem 4.2.3 is applicable, and the induced G-action on $\mathcal{M}(G)$ is proper and cocompact. Recall that G is hyperbolic and thus possesses a rank 1 coarse median $\mu_G\colon G\times G\times G\to G$, see Definition 3.1.5. We can now answer the main question of this chapter. In the proposition below, we determine how μ_G is related to the median on $\mathcal{M}(G)$.

Proposition 4.3.1. *Let $\sigma\colon (G,d_{\mathcal{W}_\mathcal{P}})\to(\mathcal{M}(G),d_{\mathcal{M}})$ be the canonical isometric embedding. If we endow G with the combinatorial distance induced by the Cayley graph, then σ becomes a quasi-isometry and we have*

$$\mu_G\sim\sigma^*\mu_{\mathcal{M}},$$

where $\mu_{\mathcal{M}}$ is the median on $\mathcal{M}(G)$.

Proof. Recall that $\mathcal{M}(G)$ is the vertex set of a CAT(0) cubical complex C. From the description in Example 1.3.12 we conclude that the G-action extends to the 1-skeleton of C (in fact, to the whole cubical complex). The action on $C^{(1)}$ is proper and cocompact because it has these properties on $C^{(0)}=\mathcal{M}(G)$. Let $x_0=\sigma(1_G)\in\mathcal{M}(G)=C^{(0)}$. Since σ is G-equivariant, we see that σ coincides with the map $G\to C^{(0)}, g\mapsto g\cdot x_0$. By the Švarc–Milnor lemma (Proposition 3.1.7) we conclude that the map σ is a quasi-isometry, when considered as a map $G\to C^{(1)}$, where G is to be endowed with the combinatorial distance induced by the Cayley graph. Clearly, the same holds true when considered as a map $\sigma\colon G\to C^{(0)}=\mathcal{M}(G)$. Let $\iota\colon C^{(0)}\hookrightarrow C^{(1)}$, $j\colon G\hookrightarrow X^{(1)}$ be the inclusion maps, respectively (Note that we have identified G with $X^{(0)}$, so that $X^{(1)}$ is the Cayley graph of G). Choose a quasi-isometry $\phi\colon X^{(1)}\to C^{(1)}$ such that $\phi\circ j\sim\iota\circ\sigma$.

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & \mathcal{M}(G)\\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle\iota}\\ X^{(1)} & \xrightarrow[\phi]{} & C^{(1)}\end{array}$$

The 1-skeleton $C^{(1)}$ is hyperbolic, as $X^{(1)}$ is hyperbolic and ϕ is a quasi-isometry. Due to Lemma 2.3.7, the associated coarse median is $\phi_*j_*\mu_G$ (since $j_*\mu_G$ is the hyperbolic coarse median on $X^{(1)}$, see Definition 3.1.5). Thus Lemma 2.3.9 implies $\phi_*j_*\mu_G\sim\iota_*\mu_{\mathcal{M}}$ and we conclude

$$\sigma^*\mu_{\mathcal{M}}\sim\sigma^*\iota^*\iota_*\mu_{\mathcal{M}}\sim\sigma^*\iota^*\underbrace{\phi_*j_*\mu_G}_{\sim\iota_*\sigma_*\mu_G}\sim\mu_G,$$

where we have used the formal properties stated in Lemma 2.2.3. $\square$

By Corollary 4.2.4, the group G, endowed with the wall distance $d_{\mathcal{W}_\mathcal{P}}$, embeds isometrically and quasi-surjectively into $\mathcal{M}(G)$. Note that it is not immediate whether or not $d_{\mathcal{W}_\mathcal{P}}$ is equal to the combinatorial distance on the Cayley graph of G. However, from what we know, we obtain that there exist constants $\alpha>0$, $\varepsilon\geqslant 0$ such that for all $x,y\in G$, we have

$$d_{\mathcal{W}_\mathcal{P}}(x,y)\leqslant d_{X^{(1)}}(x,y)\leqslant\alpha\, d_{\mathcal{W}_\mathcal{P}}(x,y)+\varepsilon, \tag{4.3.1}$$

where $d_{X^{(1)}}$ is the combinatorial distance on the Cayley graph. Indeed, let $W_\Gamma \in \mathcal{W}_X(x|y)$, and observe that every edge path in $X^{(1)}$ which connects x and y, contains an edge that is a vertex in the hypergraph Γ. Since the sets of edges corresponding to different hypergraphs are disjoint, this immediately implies that $d_{\mathcal{W}_\mathcal{P}}(x, y) \leqslant d_{X^{(1)}}(x, y)$. To see the second inequality, note that by Proposition 4.3.1, the map $\sigma\colon (G, d_{X^{(1)}}) \to (\mathcal{M}(G), d_\mathcal{M})$ is a quasi-isometry, and so the existence of appropriate (α, ε) follows because $\sigma(G) \subseteq \mathcal{M}(G)$ is isometric to $(G, d_{\mathcal{W}_\mathcal{P}})$.

Remark 4.3.2. Using the argument above, we do not get any information on how large the constants (α, ε) have to be in order to satisfy (4.3.1). Although this is enough for our “coarse purposes”, it is not as good as one can do: In fact, it is possible to choose $\varepsilon = 0$ and $\alpha = \frac{1-6\lambda+4\lambda^2}{2-4\lambda}$ [AO12, Theorem 4.3], and this can be proven independently of a group action, i.e. the inequality

$$d_{\mathcal{W}_X}(x, y) \leqslant d_{X^{(1)}}(x, y) \leqslant \frac{1 - 6\lambda + 4\lambda^2}{2 - 4\lambda}\, d_{\mathcal{W}_X}(x, y) \quad \forall x, y \in X^{(0)}$$

is true for general $C'(\lambda)$ complexes X with $\lambda \leqslant \frac{1}{6}$.

4.4 Wall structure versus coarse median structure

We keep the setup and notation of the previous section. From Corollary 4.2.4, we know that $(G, \mathcal{W}_\mathcal{P})$ is coarsely median in the sense of Section 2.5. The coarse median $\sigma^*\mu_\mathcal{M}$ is thus described by Theorem 2.5.2. This means that to compute a coarse median of $x_1, x_2, x_3 \in G$, we have to find a point $m \in G$ such that the set $\bigcup_{i=1}^3 \mathcal{W}_\mathcal{P}(m|x_i, x_{i+1})$ is of uniformly bounded cardinality. From the previous section, we also know that this agrees with the coarse median induced by hyperbolicity of G. So, in the search for a coarse median, we can restrict our attention to points which lie on some geodesic between two of the three given points. More explicitly, we have the following corollary, which is a discrete analogue of Proposition 2.5.4.

Corollary 4.4.1. *There exists a constant $K \in \mathbb{R}_{\geqslant 0}$ such that:*

(i) For any geodesic triangle $([x_i, x_{i+1}])_{i\in\{1,2,3\}}$ in $X^{(1)}$, with vertices $x_1, x_2, x_3 \in G$, the bounded set

$$\mathfrak{M}(x_1, x_2, x_3; K) := \left\{ m \in G \;\middle|\; \left| \bigcup_{i=1}^{3} \mathcal{W}_\mathcal{P}(m|x_i, x_{i+1}) \right| \leqslant K. \right\} \tag{4.4.1}$$

has non-empty intersection with every $[x_i, x_{i+1}]$ for $i \in \{1, 2, 3\}$.

(ii) Every ternary operation $\mu\colon G^3 \to G$, with the property that $\mu(x_1, x_2, x_3) \in \mathfrak{M}(x_1, x_2, x_3; K)$ for all $x_1, x_2, x_3 \in G$, is a rank 1 coarse median defining the same coarse median structure as $\mu_G \sim \sigma^*\mu_\mathcal{M}$.

Proof. If $d_{\mathcal{W}_\mathcal{P}}$ and $d_{X^{(1)}}$ are equal, this is immediate from Theorem 2.5.2 and Proposition 4.3.1. Otherwise, we employ stability of quasi-geodesics to deal with the fact that geodesics in $X^{(1)}$ do not map to geodesics in $C^{(1)}$, where C is the cubical complex associated with $\mathcal{M}(G) = C^{(0)}$.

Let $\alpha \geqslant 1$ and $\varepsilon \geqslant 0$ be constants[2] satisfying (4.3.1). We then choose the constant K large enough such that $(G, \mathcal{W}_\mathcal{P})$ is K-coarsely median and $K \geqslant R(\delta, \alpha, \varepsilon) + 1$, where δ is the hyperbolicity constant of $C^{(1)}$ and R is from Lemma 2.3.4. As a preliminary remark to the ensuing argument, we recall that the distance $d_\mathcal{M}$ on $\mathcal{M}(G)$ agrees with the combinatorial distance induced by the graph $C^{(1)}$ (see Example 1.3.12).

(i) Let $\tilde{x}_i = \sigma(x_i) \in C^{(0)}$, $i \in \{1,2,3\}$ and $\tilde{m} = \mu_\mathcal{M}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in C^{(0)}$. Fix geodesics $[\tilde{x}_1, \tilde{m}], [\tilde{x}_2, \tilde{m}]$ connecting $\tilde{x}_1, \tilde{x}_2$ to $\tilde{m}$ in $C^{(1)}$, respectively. We denote the concatenation of $[\tilde{x}_1, \tilde{m}]$ and $[\tilde{x}_2, \tilde{m}]$ by $[\tilde{x}_1, \tilde{x}_2]$. Then $[\tilde{x}_1, \tilde{x}_2]$ is a geodesic connecting $\tilde{x}_1$ and $\tilde{x}_2$. Let c be a unit-speed parametrization of the geodesic $[x_1, x_2]$. Then $\sigma \circ c$ is an (α, ε)-quasi-geodesic with the same endpoints. (More precisely, we let $\sigma \circ c$ be some (α, ε)-quasi-geodesic $[0, d(x_1, x_2)] \to C^{(1)}$ which on integer points $t \in [0, d(x_1, x_2)] \cap \mathbb{Z}$ agrees with $\sigma(c(t))$.) Thus the images of $[\tilde{x}_1, \tilde{x}_2]$ and $\sigma \circ c$ are $R(\delta, \alpha, \varepsilon)$-close in the Hausdorff distance. In particular, there is $m \in \operatorname{im}(c) \cap X^{(0)}$ such that $d_\mathcal{M}(\sigma(m), \tilde{m}) \leqslant K$ and, as in the proof of Theorem 2.5.2, we can conclude

$$\left| \bigcup_{i=1}^{3} \mathcal{W}_\mathcal{P}(m | x_i, x_{i+1}) \right| \leqslant K.$$

(ii) Immediate by Theorem 2.5.2 and Proposition 4.3.1. □

If the wall distance and the combinatorial distance induced by the Cayley graph of G actually agree, then it gets even simpler:

Proposition 4.4.2. *Assume that $d_{\mathcal{W}_\mathcal{P}} = d_{X^{(1)}}$ and let K be the constant from Corollary 4.4.1. Then we have:*

For every $x_1, x_2, x_3 \in G$ and every $d_{X^{(1)}}$-geodesic c connecting x_1 and x_2, there is $m \in G$ contained in the image of c such that

$$\left| \bigsqcup_{i=1,2} \mathcal{W}_\mathcal{P}(m | x_i, x_3) \right| \leqslant K, \tag{4.4.2}$$

where $\bigsqcup$ indicates that the union is disjoint.

Moreover, for all $u \in G$ contained in the image of c, the following inclusion holds:

$$\bigsqcup_{i=1,2} \mathcal{W}_\mathcal{P}(u | x_i, x_3) \subseteq \mathcal{W}_\mathcal{P}(x_1 | x_2). \tag{4.4.3}$$

Proof. Since m lies on the geodesic c between x_1 and x_2, we necessarily have $\mathcal{W}_\mathcal{P}(m | x_1, x_2) = \emptyset$. (Any such wall would have to cross c at least twice, but then $d_{X^{(1)}}(x_1, x_2) > d_\mathcal{W}(x_1, x_2)$.) Since $\bigcap_{i=1,2} \mathcal{W}_\mathcal{P}(m | x_i, x_3) \subseteq \mathcal{W}_\mathcal{P}(m | x_1, x_2)$, this also shows that the union is disjoint.

[2] By Remark 4.3.2, we can even take $\varepsilon = 0$ so that $(G, d_{\mathcal{W}_\mathcal{P}})$ is a metric space, but the proof here does not rely on this.

To prove the additional inclusion, let $i \in \{1,2\}$ and assume that $W \in \mathcal{W}_\mathcal{P}(m|x_i,x_3)$ but $W \notin \mathcal{W}_\mathcal{P}(x_1|x_2)$. Then W would separate m from $\{x_1,x_2\}$, which we have proven to be impossible.

Taking these facts into account, the result follows from Corollary 4.2.4. □

Remark 4.4.3. We still consider the case that $d_{\mathcal{W}_\mathcal{P}} = d_{X^{(1)}}$. Equation (4.4.3) implies that, in order to compute the number in (4.4.2), it suffices to check those finitely many walls which correspond to edges contained in c. Thus, in view of Corollary 4.4.1, we can compute a coarse median of x_1,x_2,x_3 by the following algorithm:

- Determine a geodesic path between x_1 and x_2 and denote its consecutive vertices by $x_1 = u_0,\dots,u_n = x_2$. For each $i \in \{1,\dots n\}$ let W_i be the wall corresponding to the edge between u_{i-1} and u_i. Set $n_j = 0$ for $j \in \{0,\dots n\}$.
- For each $i \in \{1,\dots n\}$ determine "on which side" of W_i the point x_3 lies:
  - If $W_i \in \mathcal{W}_\mathcal{P}(x_1|x_3)$, then for each $j \in \{0,\dots i-1\}$ increase n_j by 1,
  - if $W_i \in \mathcal{W}_\mathcal{P}(x_2|x_3)$, then for each $j \in \{i,\dots n\}$ increase n_j by 1.
- Choose $i_0 \in \{0,\dots,n\}$ such that n_{i_0} is minimal and set $\mu(x_1,x_2,x_3) = u_{i_0}$.

This indeed works, because after completing the second step, each number n_i will be the cardinality of $\bigsqcup_{k=1,2}\mathcal{W}_\mathcal{P}(\{u_i\}|\{x_k,x_3\})$. To see this, we first note that (4.4.3) means $\bigsqcup_{k=1,2}\mathcal{W}_\mathcal{P}(\{u_i\}|\{x_k,x_3\}) \subseteq \{W_1,\dots,W_n\}$ and so we only need to check the walls W_i. Now let $i \in \{1,\dots n\}$ be fixed. Then u_j and x_1 are both in the same half space of W_i for all $j \in \{0,\dots,i-1\}$. Analogously for $j \in \{i,\dots n\}$, the points u_j and x_2 lie in a common half space of W_i. Thus for all $j \in \{0,\dots,n\}$ we have

$$W_i \in \mathcal{W}_\mathcal{P}(\{u_j\}|\{x_2,x_3\}) \quad \Leftrightarrow \quad j < i \text{ and } W_i \in \mathcal{W}_\mathcal{P}(x_1|x_3),$$

and similarly

$$W_i \in \mathcal{W}_\mathcal{P}(\{u_j\}|\{x_1,x_3\}) \quad \Leftrightarrow \quad j \geqslant i \text{ and } W_i \in \mathcal{W}_\mathcal{P}(x_2|x_3).$$

4.5 Surface groups

We consider the fundamental group of an orientable closed surface of genus $g \geqslant 2$ and verify that the standard presentation,

$$\mathcal{P} = \langle a_1,b_1,\dots,a_g,b_g \mid r := [a_1,b_1]\cdots[a_g,b_g] = 1\rangle\,,$$

satisfies all assumptions of the previous two sections, including $d_{X^{(1)}} = d_{\mathcal{W}_\mathcal{P}}$. In particular, one can apply Remark 4.4.3.

We give an explicit construction of the CAT(0) cubical complex associated to the space with walls $(G,\mathcal{W}_\mathcal{P})$. The presentation complex $\mathrm{K}(\mathcal{P})$ is homeomorphic to the closed orientable surface of genus g and the Cayley complex $X = \mathrm{C}(\mathcal{P})$ is the hyperbolic plane

$\mathbb{H}^2$. Thus, we may view the Cayley graph $X^{(1)}$ as being embedded in the hyperbolic plane, where we can think of each face as a $4g$-gon labeled by the relator r, and any two faces intersect in at most one edge. We see that every piece consists of a single edge. Thus, $\mathcal{P}$ is a $C'(\lambda)$ presentation for every $\lambda > \frac{1}{4g}$, where $\frac{1}{4g} \leqslant \frac{1}{8} < \frac{1}{6}$ (in particular, it is a $C'(\frac{1}{6})$ presentation).

In fact, X is the underlying 2-complex of the regular hyperbolic tessellation by $4g$-gons such that at every vertex $4g$ faces intersect. Since X is a 2-dimensional manifold, there are exactly two 2-cells meeting along any given edge, and so the degree of every vertex in a hypergraph is 2. Consequently, the hypergraphs of Definition 4.2.1 are not only trees but actually infinite lines.

There are $2g$ pairwise crossing walls at every face of X. Thus, in order to produce a $\mathrm{CAT}(0)$ cubical complex, we expect that we have to add at least a $2g$-dimensional cube for each face of X. It turns out that this is sufficient, and let us describe the construction of C right away: Choose an *isometric* embedding φ of a $4g$-cycle C_{4g} into the 1-skeleton of the $2g$-dimensional standard cube $I^{2g} = [0,1]^{2g}$. Note that for any $k \in \mathbb{N}$, we can define an isometric embedding $\varphi\colon C_{2k} \to I^k$ explicitly as follows. Denote the consecutive vertices of C_{2k} by $v_0, \dots, v_{k-1}, v_k = w_0, w_1, \dots, w_{k-1}, w_k = v_0$ and define $\{\varphi(v_i)\} = \{0\}^{k-i} \times \{1\}^i$ and $\{\varphi(w_i)\} = \{1\}^{k-i} \times \{0\}^i$. Now let C be the cubical complex obtained from $X^{(1)}$ by attaching a cube to every 2-cell of X, where we glue the isometrically embedded $4g$-cycle in the cube to the boundary of the 2-cell. The construction allows us to view the Cayley graph $X^{(1)}$ as being isometrically embedded in the 1-skeleton $C^{(1)}$. Note that the Hausdorff distance between a $2g$-dimensional cube and an isometrically embedded $4g$-cycle is g. Thus, the distance between $X^{(1)}$ and $C^{(1)}$ is also g.

Proposition 4.5.1. *The cubical complex C described above is* $\mathrm{CAT}(0)$*, and its vertex set (with the combinatorial metric of the* 1*-skeleton) is isometric to $\mathcal{M}(G)$. In particular, $d_{X^{(1)}} = d_{C^{(1)}} = d_{\mathcal{W}_{\mathcal{P}}}$.*

Proof. By construction (and the contractability of the added cubes), it is is clear that C is simply connected. We will verify that every corner in C is contained in some of cube of C, and thus show that it is non-positively curved in the sense of Definition 1.1.7.

Let $\mathrm{Cor}(I^n) \to C$ be a corner in C and let $x \in C^{(0)}$ be the image of $0 \in \mathrm{Cor}(I^n)$. If x is contained in $X^{(1)}$, then there are $4g$ cubes of dimension $2g$ which meet at x, where the intersection of any two of them is either empty or consists of a single edge. Thus, the corner must be completely contained in one of these $4g$ cubes. If, on the other hand, x is not in $X^{(1)}$, then all edges adjacent to x are contained in a unique $2g$-dimensional cube and thus the corner is again completely contained in one cube of C.

To see that $C^{(0)} \cong \mathcal{M}(G)$, we only need to verify that the inclusion

$$\iota\colon (X^{(0)}, d_{\mathcal{W}_{\mathcal{P}}}) \hookrightarrow (C^{(0)}, \mathcal{W}_C)$$

is a morphism of spaces with walls, because then Proposition 1.3.8 (ii) and Proposition 1.3.11 imply $\mathcal{M}(X^{(0)}) \cong \mathcal{M}(C^{(0)}) \cong C^{(0)}$. Consider an edge e in the 1-skeleton of I^{2g} which is contained in the embedded $4g$-cycle. Then there is exactly one other edge in the embedded $4g$-cycle which is parallel (in the cube) to e and it corresponds to the edge antipodal (in the $4g$-cycle) to e. Note that the unique midplane in I^{2g} which meets e, intersects exactly all edges parallel to e. These considerations show that the geometric hyperplanes of C are in 1 to 1 correspondence with the hypergraphs of X. In terms of the corresponding wall structures, this means that for every $\{h, h^c\} \in \mathcal{W}_C$, we have $\{X^{(0)} \cap h, X^{(0)} \cap h^c\} \in \mathcal{W}_X$ and every wall in $\mathcal{W}_X$ arises uniquely in this way. — In other words, the inclusion ι is a morphism of spaces with walls. $\square$

Chapter 5

Explicit hyperbolic groups with property (T)

In Chapter 4, we have considered hyperbolic groups, where the corresponding coarse median is determined by an equivariant and proper wall structure. The purpose of this chapter is to describe infinite hyperbolic groups, where the coarse median does not arise in this way. In fact, the examples we consider have Kazhdan's property (T) and thus cannot act properly on a space with measured walls or a median space, see Corollary 1.4.8.

5.1 A local criterion for property (T)

The goal of this section is to present a local condition on simplicial 2-complexes which implies Kazhdan's property (T) for groups acting geometrically on them. The criterion is due to Żuk [Żuk96] and Ballmann–Świątkowski [BŚ97]. Here we review a slightly more general version of the original result which is contained in the book of Bekka–de la Harpe–Valette [BHV08].

Let Y be a simplicial complex and $v \in Y^{(0)}$ some vertex. The *Link* of v in Y, denoted by Y_v, is the subcomplex consisting of all the simplices σ of X such that the vertices $\sigma^{(0)} \cup \{v\}$ span a simplex of Y. If Y is a 2-dimensional, then Y_v is a simplicial graph for each $v \in Y^{(0)}$. In that case, let $\lambda_1(Y_v)$ be the smallest non-zero eigenvalue of the Laplacian on the graph Y_v. (See Definition 5.1.9 below for the precise definition of the Laplacian we use.) We can now state the criterion.

Theorem 5.1.1 ([BHV08, Theorem 5.5.4])**.** *Let Y be a locally finite connected simplicial 2-complex such that every vertex belongs to some edge and every edge belongs to some triangle, and assume that for each $v \in Y^{(0)}$, the link at v is connected and satisfies $\lambda_1(Y_v) > \frac{1}{2}$.*

Then every unimodular locally compact group which acts continuously, properly and cocompactly by simplicial automorphisms on Y has Kazhdan's property (T).

The rest of this section is devoted to sketch the main ideas for the proof the theorem above. We closely follow [BHV08, Chapter 5] but omit some of the calculations.

First, we give an ad-hoc definition of the notion of a (discrete) random walk in a way that will be convenient to us.

Warning to avoid confusion: In this section, and only in this section, the letter μ will be used to denote a random walk (not a coarse median!).

Definition 5.1.2. A *random walk* on a set $X \neq \emptyset$ is a map $\mu\colon X \times X \to \mathbb{R}_{\geqslant 0}$ such that $\sum_{y \in X} \mu(x, y) = 1$ for all $x \in X$.

Such a random walk is called *irreducible* if for every two points $x \neq y \in X$, there exists a sequence $x = x_0, \dots, x_n = y \in X$ such that $\mu(x_{i-1}, x_i) > 0$ for all $i \in \{1, \dots, n\}$.

A random walk μ is said to have *finite range*, if for all $x \in X$, the set $\{y \in X \mid \mu(x,y) \neq 0\}$ is finite.

A *stationary measure* for a random walk μ is a function $\nu\colon X \to \mathbb{R}_{>0}$ such that $\nu(x)\mu(x,y) = \nu(y)\mu(y,x)$ for all $x, y \in X$. A random walk is called *reversible* if it admits a stationary measure.

Given a reversible random walk μ with stationary measure ν, we define

$$c\colon X \times X \to \mathbb{R}_{\geqslant 0}, \quad c(x,y) = \nu(x)\mu(x,y) = \nu(y)\mu(y,x).$$

Example 5.1.3. Let $\mathcal{V} \neq \emptyset$ be some set and E some subset of $\mathcal{V} \times \mathcal{V}$ with the property that $e = (x,y) \in \mathrm{E} \Leftrightarrow \bar{e} := (y,x) \in \mathrm{E}$. We consider the data $(\mathcal{V}, \mathrm{E})$ to be the same as a graph without multiple edges, where the set of vertices is $\mathcal{V}$ and $x, y \in \mathcal{V}$ are connected by an edge iff $(x,y) \in \mathrm{E}$. We further assume that this graph is locally finite, i.e. for all $x \in X$, we have that $\deg(x) = |\{y \in \mathcal{V} \mid (x,y) \in \mathrm{E}\}|$ is finite. Then we define

$$\mu\colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\geqslant 0}, \quad \mu(x,y) = \begin{cases} \frac{1}{\deg(x)} & (x,y) \in \mathrm{E}, \\ 0 & \text{otherwise.} \end{cases}$$

Then μ is a locally finite, reversible random walk on $\mathcal{V}$, and the map $x \mapsto \deg(x)$ is a stationary measure. Moreover, μ is irreducible iff the graph is connected.

Conversely, let μ be a random walk on some set X with the property that $\mu(x,y) \neq 0 \Leftrightarrow \mu(y,x) \neq 0$ for all $x, y \in X$. (Any reversible random walk has this property.) Then we can define a graph (X, E_μ) by $\mathrm{E}_\mu := \{(x,y) \in X \mid \mu(x,y) \neq 0\}$. This graph is locally finite iff μ has finite range.

Example 5.1.4. Let Y be a connected, locally finite simplicial 2-complex such that any vertex belongs to some edge and any edge belongs to some triangle. Let $X = Y^{(0)}$ be the set of vertices. For each $v, w \in X$, let $\tau(v,w)$ be the number of triangles containing v and w if $v \neq w$, and 0 if $v = w$. Define $\tau(v) = \sum_{w \in X} \tau(v,w)$, then $\tau(v)$ is exactly the number of flags[1] containing v. We define a random walk $\mu\colon X \times X \to \mathbb{R}_{\geqslant 0}$ by $\mu(v,w) = \frac{\tau(v,w)}{\tau(v)}$. Clearly, $v \mapsto \nu(v) := \tau(v)$ is a stationary measure.

Now we add a group action to the picture. Recall that every locally compact group G admits a non-zero left-invariant regular Borel measure, called the *Haar measure* of G, which is unique up to a positive constant. We choose such a measure m_G and we write $\int_G \alpha(g)\,\mathrm{d}g$ for the corresponding integral of a measurable function $\alpha\colon G \to \mathbb{R}_{\geqslant 0}$. Left-invariance means that $\int_G \alpha(x^{-1}g)\,\mathrm{d}g = \int_G \alpha(g)\,\mathrm{d}g$ for all $x \in G$. If $H \subseteq G$ is some open subgroup, then $m_G(H) \neq 0$ and the restriction of m_G to H is a Haar measure on H. If we have fixed a Haar measure on G, we will implicitly endow open subgroups of G with these induced measures.

[1] cf. Definition 5.2.1 below.

If G is discrete, the Haar measure is just (a multiple of) the counting measure and we have $\int_G \alpha(g)\, \mathrm{d}g = \sum_{g\in G} \alpha(g)$. A locally compact group is called *unimodular* if the left-invariant regular Borel measure is also right-invariant, i.e. $\int_G \alpha(gx^{-1})\, \mathrm{d}g = \int_G \alpha(g)\, \mathrm{d}g$ for all $x \in G$. In this case, we have $\int_G \alpha(g)\, \mathrm{d}g = \int_G \alpha(g^{-1})\, \mathrm{d}g$ for all measurable functions $\alpha\colon G \to \mathbb{R}_{\geqslant 0}$. Clearly, every discrete group is unimodular.

For the rest of this section, we let G be a unimodular locally compact group, and we fix a Haar measure m_G on it. We then consider a reversible random walk $\mu\colon X \times X \to \mathbb{R}_{\geqslant 0}$ with stationary measure $\nu\colon X \to \mathbb{R}_{>0}$, and assume that G acts on the set X such that the following conditions are satisfied.

(a) The random walk μ is irreducible and of finite range.

(b) The action is continuous with respect to the discrete topology on X. This is equivalent to the requirement that the stabilizers G_x are open subgroups for all $x \in X$.

(c) The action is proper, i.e. for every finite subset $F \subseteq X$, the set $\{g \in G \mid gF \cap F \neq \emptyset\}$ is compact. In particular, all stabilizers are compact open subgroups, and we write $|G_x| := m_G(G_x)$.

(d) The action is cofinite, i.e. the set of G-orbits in X is finite.

(e) The random walk and its stationary measure are G-invariant, i.e. $\mu(g\cdot x, g\cdot y) = \mu(x, y)$ and $\nu(g \cdot x) = \nu(x)$ for all $x, y \in X$, $g \in G$.

In the following, we let $T \subseteq X$ denote a minimal set of representatives for the orbits of G-action on X, i.e. $X = \bigsqcup_{t\in T} G \cdot t$.

Remark 5.1.5. These assumptions imply that G is compactly generated. Indeed, let $S = \{s \in G \mid \exists x, y \in T : \mu(x, s\cdot y) > 0\}$, and observe that S is compact because T is finite, μ has finite range and the action is proper. Then $S \cup G_{t_0}$ is a compact generating set of G for all $t_0 \in T$. To see that, let $g \in G$ and find a sequence $t_0 = x_0, x_1, \ldots, x_n = g \cdot t_0 \in X$ such that $\mu(x_0, x_1), \mu(x_1, x_2), \ldots, \mu(x_{n-1}, x_n)$ are all > 0. (This is possible because μ is irreducible.) Then there exist $t_1, \ldots, t_n \in T$, $g_1, \ldots g_n \in G$ such that $x_i = g_i t_i$. Let $s_i = g_{i-1}^{-1} g_i$ (with $g_0 = 1_G$) and calculate $\mu(t_{i-1}, s_i \cdot y_i) = \mu(t_{i-1}, g_{i-1}^{-1} g_i \cdot t_i) = \mu(g_{i-1} \cdot t_{i-1}, g_i \cdot t_i) = \mu(x_{i-1}, x_i) > 0$, which shows that $s_i \in S$ for all $i \in \{1, \ldots, n\}$. Clearly, we have $\left(\prod_{i\in\{1,\ldots n\}} s_i\right) \cdot t_n = g_n \cdot t_n = x_n = g \cdot t_0$, which implies $t_n = t_0$ and thus $g^{-1} \prod_{i\in\{1,\ldots,n\}} s_i \in G_{t_0}$.

Remark 5.1.6. The homogeneous space G/G_x is discrete because G_x is open for all $x \in X$, and for all measurable functions $\alpha\colon G \to \mathbb{R}_{\geqslant 0}$, we have,

$$\int_G \alpha(g)\, \mathrm{d}g = \sum_{[s]\in G/G_x} \int_{G_x} \alpha(sh)\, \mathrm{d}h,$$

where $\sum_{[s]\in G/G_x}$ denotes the sum over some set of representatives for the cosets G/G_x. In particular, if $\alpha\colon G \to \mathbb{R}_{\geqslant 0}$ is G_x-right-invariant, i.e. $\alpha(gh) = \alpha(g)$ for all $g \in G, h \in G_x$, then we have

$$\frac{1}{|G_x|}\int_G \alpha(g)\,\mathrm{d}g = \sum_{[s]\in G/G_x} \alpha(s). \tag{5.1.1}$$

Moreover, if $\beta\colon X \to \mathbb{R}_{\geqslant 0}$ is some function such that $\sum_{x\in X}\beta(x)$ is (absolutely) convergent, we have $\sum_{x\in X}\beta(x) = \sum_{r\in T}\sum_{[s]\in G/G_r}\beta(s\cdot r)$. Since $g\mapsto \beta(g\cdot r)$ is G_r-right-invariant, by (5.1.1) we get the formula,

$$\sum_{x\in X}\beta(x) = \sum_{r\in T}\frac{1}{|G_r|}\int_G \beta(g\cdot r)\,\mathrm{d}g. \tag{5.1.2}$$

Example 5.1.7. Let Y be a simplicial 2-complex as in Example 5.1.4 and let G be a unimodular locally compact group acting continuously, properly and cocompactly by simplicial automorphisms on Y. We observe that this situation gives rise to an action on $X = Y^{(0)}$ satisfying (a) to (e) with respect to the random walk μ defined in Example 5.1.4.

Indeed, since $X \subseteq Y$ is a closed, discrete and G-invariant subset, it follows that the induced action on X satisfies (b) to (d). It is clear from the construction in Example 5.1.4, that the random walk μ and its stationary measure ν are G-invariant, hence (e) is also satisfied. Condition (a) follows because Y is connected and locally finite.

We continue to work under the assumptions (a) to (e) and, in addition, let $\rho\colon G\to\mathcal{U}(H)$ be a unitary representation of G on a Hilbert space H. Let

$$\begin{aligned}
\mathcal{E}^0_\rho(X) &= \left\{ f\in H^X \;\middle|\; f(g\cdot x) = \rho(g)f(x) \quad \forall g\in X, x\in X \right\},\\
\mathcal{E}^1_\rho(X) &= \left\{ F\in H^{\mathrm{E}_\mu} \;\middle|\; F(e) = -F(\bar e), F(g\cdot e) = \rho(g)F(e) \quad \forall e\in \mathrm{E}_\mu, g\in G \right\},
\end{aligned}$$

and define inner products on $\mathcal{E}^0_\rho(X)$ and $\mathcal{E}^1_\rho(X)$, respectively by

$$\begin{aligned}
\langle f_1, f_2\rangle &= \sum_{t\in T}\langle f_1(t), f_2(t)\rangle\,\frac{\nu(t)}{|G_t|}, \quad f_1,f_2\in\mathcal{E}^0_\rho(X),\\
\langle F_1, F_2\rangle &= \sum_{[(x,y)]\in G\backslash \mathrm{E}_\mu}\frac{\langle F_1(x,y), F_2(x,y)\rangle}{c(x,y)|G_x\cap G_y|}, \quad F_1,F_2\in\mathcal{E}^1_\rho(X),
\end{aligned}$$

where $\sum_{[(x,y)]\in G\backslash \mathrm{E}_\mu}$ indicates a sum over some set of representatives for the quotient space $G\backslash \mathrm{E}_\mu$ (with respect to the diagonal G-action on $\mathrm{E}_\mu\subseteq X\times X$). Note that both sums are finite because $|G\backslash X| = |T|$ is finite and μ has finite range. It follows that $\mathcal{E}^i_\rho(X)$, $i\in\{1,2\}$, are both Hilbert spaces with respect to these inner products.

We define an exterior differential,

$$d\colon \mathcal{E}^0_\rho(X)\to\mathcal{E}^1_\rho(X), \quad (df)(x,y) = c(x,y)(f(y)-f(x)),$$

and the *Laplacian* $\Delta_{\mu,\rho}\colon \mathcal{E}^0_\rho(X)\to\mathcal{E}^0_\rho(X)$ as well as the *Markov operator* $M_{\mu,\rho}\colon \mathcal{E}^0_\rho(X)\to \mathcal{E}^0_\rho(X)$ by $\Delta_{\mu,\rho} = d^*d = \mathrm{id} - M_{\mu,\rho}$, where d^* is the adjoint operator of d (by the proposition below, d is bounded with respect to the inner products defined above).

Proposition 5.1.8 ([BHV08, Proposition 5.4.3]). *(i) The linear operator* $d\colon \mathcal{E}^0_\rho(X) \to \mathcal{E}^1_\rho(X)$ *is bounded with operator norm* $\|d\| \leqslant \sqrt{2}$.

(ii) The adjoint operator $d^*\colon \mathcal{E}^1_\rho(X) \to \mathcal{E}^0_\rho(X)$ *is given by the formula* $(d^*F)(y) = \frac{1}{\nu(y)}\sum_{x\in X} F(x,y)$.

(iii) The Laplacian is a positive operator, and the Markov operator has norm bounded above by 1.

(iv) For $f \in \mathcal{E}^0_\rho(X)$, *we have the formulas*

$$\begin{aligned} \langle \Delta_{\mu,\rho} f, f\rangle &= \frac{1}{2}\sum_{x\in X, t\in T} \|f(x)-f(t)\|^2 \,\frac{\nu(x)\mu(x,t)}{|G_t|}, \\ (M_{\mu,\rho} f)(x) &= \sum_{y\in X\colon (x,y)\in \mathrm{E}_\mu} f(y)\mu(x,y). \end{aligned}$$

(v) The kernel of the Laplacian $\ker(\Delta_{\mu,\rho}) \subseteq \mathcal{E}^0_\rho(X)$ *consists exactly of the constant functions with values in the* ρ*-invariant vectors of* H. *(Here we use the fact that* μ *is irreducible.)*

We briefly consider the special case that $G = 1$ is the trivial group, $\rho = 1_{\mathbb{C}}\colon 1 \to \mathcal{U}(\mathbb{C})$ is the trivial representation and X is a finite set of cardinality $n \in \mathbb{N}$. In that case, we must choose $T = X$. Then $\Delta_{\mu,1_{\mathbb{C}}}$ is a positive operator on the n-dimensional Hilbert space $\mathcal{E}^0_{1_{\mathbb{C}}}(X)$ and thus possesses n eigenvalues $0 = \lambda_0(\Delta_{\mu,1_{\mathbb{C}}}) < \lambda_1(\Delta_{\mu,1_{\mathbb{C}}}) \leqslant \cdots \leqslant \lambda_{n-1}(\Delta_{\mu,1_{\mathbb{C}}})$. Note that the eigenvalue 0 has multiplicity 1 by Proposition 5.1.8 (v).

Definition 5.1.9. Let μ be a random walk obtained as in Example 5.1.3 from a finite connected graph Γ with vertex set $\mathcal{V}(\Gamma)$ of cardinality n. The *Laplacian* and the *averaging operator* of Γ are, respectively,

$$\begin{aligned} \Delta_\Gamma &:= \Delta_{\mu,1_{\mathbb{C}}}\colon \mathbb{C}^{\mathcal{V}(\Gamma)} = \mathcal{E}^0_{1_{\mathbb{C}}}(X) \to \mathcal{E}^0_{1_{\mathbb{C}}}(X) = \mathbb{C}^{\mathcal{V}(\Gamma)}, \\ A_\Gamma &:= M_{\mu,1_{\mathbb{C}}}\colon \mathbb{C}^{\mathcal{V}(\Gamma)} = \mathcal{E}^0_{1_{\mathbb{C}}}(X) \to \mathcal{E}^0_{1_{\mathbb{C}}}(X) = \mathbb{C}^{\mathcal{V}(\Gamma)}. \end{aligned}$$

The *spectrum of* Γ is defined to be the spectrum of the Laplacian Δ_Γ, i.e. it consists of the values $\lambda_i(\Gamma) := \lambda_i(\Delta_{\mu,1_{\mathbb{C}}})$, $i \in \{0,\dots,n-1\}$. Moreover, we introduce the notation $\mu_i(\Gamma) := 1 - \lambda_i(\Gamma)$ for $i \in \{0,\dots,n-1\}$.

Clearly, $1 = \mu_0(\Gamma) > \mu_1(\Gamma) \geqslant \cdots \geqslant \mu_{n-1}(\Gamma)$, and $\{\mu_i(\Gamma)\}_{i\in\{0,\dots,n-1\}}$ is the spectrum of $A_\Gamma = \mathrm{id} - \Delta_\Gamma$. By Proposition 5.1.8 (iv), we have an explicit formula for A_Γ (and thus also for Δ_Γ).

$$(A_\Gamma f)(x) = \frac{1}{\deg(x)} \sum_{v\in\mathcal{V}(\Gamma)\colon (x,v)\in \mathrm{E}(\Gamma)} f(v)$$

In particular, if Γ is *k-regular*[2], then kA_Γ is the adjacency matrix of Γ.

[2] A graph is called *k-regular*, if the degree of every vertex is k.

Proposition 5.1.10 ([BHV08, Proposition 5.3.1])**.** *Let X be a finite set, μ a reversible, irreducible random walk on X with stationary measure ν. Let λ_1 be the smallest positive eigenvalue of $\Delta_{\mu,1_{\mathbb{C}}}$ on $\mathcal{E}^0_{1_{\mathbb{C}}}(X)$. For every Hilbert space H, we then have*

$$\lambda_1 = \inf\left\{ \frac{1}{2\|f\|^2} \sum_{x,y\in X} \|f(x)-f(y)\|^2\,\nu(x)\mu(x,y) \;\middle|\; f \in \ker(\Delta_{\mu,1_H})^\perp \setminus \{0\} \right\},$$

where $1_H\colon 1 \to \mathcal{U}(H)$ is the trivial representation. Moreover, for all $f \in \mathcal{E}^0_{1_H}(X)$ we have,

$$\sum_{x,y\in X} \|f(y)-f(x)\|^2\,\nu(x)\nu(y) \leqslant \frac{1}{\lambda_1} \sum_{z\in X} \nu(z) \sum_{x,y\in X} \|f(y)-f(x)\|^2\,\nu(x)\mu(x,y).$$

Proof. For every Hilbert space H, we have the formula[3],

$$\lambda_1 = \inf\left\{ \frac{\langle \Delta_{\mu,1_H} f, f\rangle}{\langle f,f\rangle} \;\middle|\; f \in \ker(\Delta_{\mu,1_H})^\perp \setminus \{0\} \right\}.$$

Therefore, the first formula in the statement of this proposition is a consequence of Proposition 5.1.8 (iv).

It suffices to prove the inequality for $f \in \ker(\Delta_{\mu,1_H})^\perp$, as $\ker(\Delta_{\mu,1_H})$ consists just of the constant functions. By the first statement, we have

$$2\lambda_1\|f\|^2 \leqslant \sum_{x,y\in X} \|f(y)-f(x)\|^2\,\nu(x)\mu(x,y). \tag{5.1.3}$$

Moreover, since f is orthogonal to the constant functions, we have $\sum_{x\in X} f(x)\nu(x) = 0$ and thus

$$\begin{aligned}
&\sum_{x,y\in X} \|f(y)-f(x)\|^2\,\nu(x)\nu(y)\\
&\quad= \sum_{x,y\in X} \Big(\|f(y)\|^2 + \|f(x)\|^2 - 2\operatorname{Re}\left(\langle f(y), f(x)\rangle\right)\Big)\,\nu(x)\nu(y)\\
&\quad= 2\underbrace{\sum_{x\in X}\|f(x)\|^2\nu(x)}_{=\|f\|^2}\sum_{y\in X}\nu(y) - 2\operatorname{Re}\left(\left\langle \sum_{y\in X} f(y)\nu(y), \sum_{x\in X} f(x)\nu(x)\right\rangle\right)\\
&\quad= 2\|f\|^2\sum_{z\in X}\nu(z),
\end{aligned}$$

which together with (5.1.3) proves the inequality. □

[3]Note that this formula is true for $H = \mathbb{C}$ because the self-adjoint operator $\Delta_{\mu,1_{\mathbb{C}}}$ (on the finite-dimensional space $\mathcal{E}^0_{1_{\mathbb{C}}}(X)$) restricts to a map $\ker(\Delta_{\mu,1_{\mathbb{C}}})^\perp \to \operatorname{im}(\Delta_{\mu,1_{\mathbb{C}}}) = \ker(\Delta_{\mu,1_{\mathbb{C}}})^\perp$, and — by definition — λ_1 is the smallest eigenvalue of this restriction. The general case is true because there are identifications $\mathcal{E}^0_{1_H}(X) = H^n = \mathbb{C}^n \otimes H = \mathcal{E}^0_{1_{\mathbb{C}}}(X) \otimes H$ and $\Delta_{\mu,1_H} = \Delta_{\mu,1_{\mathbb{C}}} \otimes \mathrm{id}_H$.

Remark 5.1.11. In Section 1.4, we have defined Kazhdan's property (T) for σ-compact locally compact groups in terms of isometric actions on Hilbert spaces. We now give a different definition (in fact, the original one) of Kazhdan's property (T) for general topological groups. To avoid confusion during the next paragraph, we will refer to the property defined in Definition 1.4.4 (i) as *property (FH)*, and reserve the name "Kazhdan's property (T)" for the concept we define in the following.

An *invariant vector* $\xi \in H$ of a unitary representation $\rho\colon G \to \mathcal{U}(H)$ is a vector such that $\rho(g)\xi = \xi$ for all $g \in G$. Let $K \subseteq G$ some subset and $\varepsilon > 0$. We say that $0 \neq \xi \in H$ is a (K, ε)-*almost-invariant vector* of ρ if $\sup_{g \in K} \|\rho(g)\xi - \xi\| < \varepsilon\|\xi\|$. A topological group G has *Kazhdan's property (T)* if there exists a compact subset $K \subseteq G$ and $\varepsilon > 0$ such that every unitary representation which has a (K, ε)-almost-invariant vector, has a non-zero invariant vector. The Delorme–Guichardet theorem [BHV08, Theorem 2.12.4] says that a topological group with Kazhdan's property (T) has property (FH), and conversely, if a σ-compact locally compact group has property (FH), then it has property (T). This means that the definition of property (T) in terms of almost-invariant vectors is an extension of Definition 1.4.4 (i).

Using the definition mentioned in the remark above, it is possible to characterize property (T) by the following *Poincaré inequality.*

Proposition 5.1.12 ([BHV08, Proposition 5.4.5])**.** *Let X be a set together with a G-action and μ a reversible random walk with stationary measure ν. Suppose that conditions (a) to (e) are satisfied.*

Then G has Kazhdan's property (T) if and only if for every unitary representation $\rho\colon G \to \mathcal{U}(H)$ without non-zero invariant vectors, there exists $\varepsilon > 0$ such that

$$\langle \Delta_{\mu,\rho} f, f\rangle \geqslant \varepsilon \|f\|^2, \quad \forall f \in \mathcal{E}^0_\rho(X). \tag{5.1.4}$$

In the proof of Theorem 5.1.1, we will use the (local) assumptions on the links of the simplicial complex to verify the (global) inequality (5.1.4) for the G-invariant random walk considered in Examples 5.1.4 and 5.1.7, and thus conclude that G has property (T). However, first we need a slight reformulation of (5.1.4), and for that we introduce the convolution of random walks.

Definition 5.1.13. Let μ, μ' be two random walks on X. We define their *convolution* by $(\mu * \mu')(x, z) = \sum_{y \in X} \mu(x, y)\mu'(y, z)$, which is again a random walk on X. The iterated convolution μ^{*k} is inductively defined by $\mu^{*k} = \mu^{*(k-1)} * \mu$, $\mu^{*0} = \delta$, where $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise.

By a direct calculation using the formula for $M_{\mu,\rho}$ from Proposition 5.1.8, one obtains the following lemma.

Lemma 5.1.14 ([BHV08, Proposition 5.2.3])**.** *Let $\mu\colon X \times X \to \mathbb{R}_{\geqslant 0}$ be a reversible random walk with stationary measure ν. Then ν is also a stationary measure for μ^{*k} for each $k \in \mathbb{N}$, and we have $M_{\mu^{*k},\rho} = (M_{\mu,\rho})^k$.*

Now we give a reformulation of the Poincaré inequality.

Lemma 5.1.15 ([BHV08, Proposition 5.4.8])**.** *Under the hypothesis of Proposition 5.1.12 let $k \in \mathbb{N}, k \geqslant 2$. Then* (5.1.4) *holds if and only if there exists a constant $0 \leqslant C < k$ such that*

$$\left\langle \Delta_{\mu^{*k},\rho} f, f \right\rangle \leqslant C \left\langle \Delta_{\mu,\rho} f, f \right\rangle, \quad \forall f \in \mathcal{E}^0_\rho(X). \tag{5.1.5}$$

Proof. The positive operator $\Delta_{\mu,\rho}$ possesses a unique positive square root $\Delta^{\frac{1}{2}}_{\mu,\rho} \colon \mathcal{E}^0_\rho(X) \to \mathcal{E}^0_\rho(X)$. Since ρ does not posses any non-zero invariant vectors, by Proposition 5.1.8 (v), we have $\ker(\Delta_{\mu,\rho}) = \{0\}$, and thus $\ker(\Delta^{\frac{1}{2}}_{\mu,\rho}) = \{0\}$. This shows that $\operatorname{im}(\Delta^{\frac{1}{2}}_{\mu})^\perp = \ker(\Delta^{\frac{1}{2}}_{\mu}) = 0$, and so $\operatorname{im}(\Delta^{\frac{1}{2}}_{\mu,\rho})$ is dense in H. Let

$$C = \sup_{f \in \mathcal{E}^0_\rho(X) \setminus \{0\}} \frac{\left\langle \Delta_{\mu^{*k},\rho} f, f \right\rangle}{\left\langle \Delta_{\mu,\rho} f, f \right\rangle}.$$

By Lemma 5.1.14 and the fact that $\Delta_{\mu,\rho}$ and $M_{\mu,\rho}$ commute, we have,

$$\begin{aligned} \Delta_{\mu^{*k},\rho} = \operatorname{id} - M_{\mu^{*k},\rho} = \operatorname{id} - M_{\mu^k,\rho} &= (\operatorname{id} - M_{\mu,\rho})(\operatorname{id} + M_{\mu,\rho} + \cdots + M^{k-1}_{\mu,\rho}) \\ &= \Delta^{\frac{1}{2}}_{\mu,\rho}(\operatorname{id} + M_{\mu,\rho} + \cdots + M^{k-1}_{\mu,\rho})\Delta^{\frac{1}{2}}_{\mu,\rho}, \end{aligned}$$

and thus, using the notation $P := \operatorname{id} + M_{\mu,\rho} + \cdots + M^{k-1}_{\mu,\rho}$, we conclude,

$$\frac{\left\langle \Delta_{\mu^{*k},\rho} f, f \right\rangle}{\left\langle \Delta_{\mu,\rho} f, f \right\rangle} = \frac{\left\langle P\Delta^{\frac{1}{2}}_{\mu,\rho} f, \Delta^{\frac{1}{2}}_{\mu,\rho} f \right\rangle}{\left\langle \Delta^{\frac{1}{2}}_{\mu,\rho} f, \Delta^{\frac{1}{2}}_{\mu,\rho} f \right\rangle}.$$

Since $\operatorname{im}(\Delta^{\frac{1}{2}}_{\mu,\rho})$ is dense, this shows that

$$C = \sup_{f \in \mathcal{E}^0_\rho(X), \|f\|=1} \langle Pf, f \rangle = \max_{\lambda \in \sigma(P)} \lambda = \max_{\lambda \in \sigma(M_{\mu,\rho})} 1 + \lambda + \cdots + \lambda^{k-1}.$$

Here we have used the spectral mapping theorem [Wer07, Satz VII.1.4 (d)] (for the polynomial function $z \mapsto 1 + z + \cdots + z^{k-1}$), and the fact that for a positive operator $A \colon H \to H$ on a Hilbert space H, the quantity $\sup_{f \in H, \|f\|=1} \langle Af, f \rangle$ is the same as its norm, and in turn, the same as its spectral radius [Wer07, Satz VI.1.7]. We see that

$$C < k \Leftrightarrow 1 \notin \sigma(M_{\mu,\rho}) \Leftrightarrow 0 \notin \sigma(\Delta_{\mu,\rho}),$$

and the latter is equivalent to the existence of $\varepsilon > 0$ such that (5.1.4) holds. □

To prove Theorem 5.1.1, we need to incorporate all the local data we have on the links, and to facilitate this, we introduce the following abstract framework (cf. [BHV08, Section 5.5]).

Suppose that for every $x \in X$, there exists a finite subset $X_x \subseteq X$ together with a reversible random walk μ_x on X_x with stationary measure ν_x, where we further require that $\sum_{y \in X_x} \nu_x(y) = 1$. Let $\tau : X \to \mathbb{R}_{>0}$ be a G-invariant function with the properties

(i) $\mu_{g\cdot x}(g\cdot y, g\cdot z) = \mu_x(y,z)$ for all $x,y,z \in X$.

(ii) $\sum_{x\in X} \tau(x)\nu_x(y)\mu_x(y,z) = \nu(y)\mu(y,z)$

(iii) $\sum_{x\in X} \tau(x)\nu_x(y)\nu_x(z) = \nu(y)(\mu * \mu)(y,z)$

Example 5.1.16. In this example, we continue to work with the setup of Examples 5.1.4 and 5.1.7. For each $v \in X$, let Γ_v be the graph with vertex set $X_v = \{w \in X \mid \tau(v,w) > 0\}$ and edge set $\mathrm{E}_v = \{(w_1,w_2) \in X_v^2 \mid (v,w_1,w_2)$ spans a triangle in $X\}$. Note that X_v is in 1–1 correspondence with the set of edges in X which are incident to v, and any two elements of X_v are connected by an edge in Γ_v iff the corresponding edges lie in a common triangle. In other words, the graph Γ_v is the same as the link Y_v.

The degree of $w \in X_v$ in the link Y_v is $\tau(v,w)$. By Example 5.1.3, a random walk on X_v is given by

$$\mu_v\colon X_v \times X_v \to \mathbb{R}_{\geqslant 0}, \quad \mu_v(w_1,w_2) = \begin{cases} \frac{1}{\tau(v,w_1)} & \text{if } (v,w_1,w_2) \text{ spans a triangle,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, μ_v is reversible, where $w \mapsto \nu_v(w) = \frac{\tau(v,w)}{\tau(v)}$ a stationary measure with the property $\sum_{w\in X_v} \nu_v(w) = 1$.

We observe that the expression $\tau(v)\nu_v(w_1)\mu_v(w_1,w_2)$ evaluates to 1 when (v,w_1,w_2) spans a triangle and 0 otherwise. Thus

$$\sum_{v\in X} \tau(v)\nu_v(w_1)\mu_v(w_1,w_2) = \tau(w_1,w_2) = \nu(v_1)\mu(v_1,v_2),$$

which proves that this data satisfies (ii). Furthermore, we have

$$\tau(v)\nu_v(w_1)\nu_v(w_2) = \nu(v)\mu(v,w_1)\mu(v,w_2) = \nu(w_1)\mu(w_1,v)\mu(v,w_2),$$

and so $\sum_{v\in X} \tau(v)\nu_v(w_1)\nu_v(w_2) = \nu(w_1)(\mu * \mu)(w_1,w_2)$, which is (iii).

Lemma 5.1.17 ([BHV08, Theorem 5.5.2])**.** *Assume (a) to (e) as well as (i) to (iii) and set $\lambda := \min_{x\in X} \lambda_1(\Delta_{\mu_x,1_{\mathbb{C}}})$. For every unitary representation $\rho\colon G \to \mathcal{U}(H)$ without non-zero invariant vectors, we have,*

$$\left\langle \Delta_{\mu^{*2},\rho} f, f \right\rangle \leqslant \frac{1}{\lambda} \left\langle \Delta_{\mu,\rho} f, f \right\rangle, \tag{5.1.6}$$

for all $f \in \mathcal{E}^0_\rho(X^{(0)})$.

Proof. By Proposition 5.1.8 (iv), inequality (5.1.6) is equivalent to

$$\sum_{t\in T}\sum_{x\in X} \|f(t)-f(x)\|^2 \mu^{*2}(t,x)\frac{\nu(t)}{|G_t|} \leqslant \frac{1}{\lambda}\sum_{t\in T}\sum_{x\in X} \|f(t)-f(x)\|^2 \mu(t,x)\frac{\nu(t)}{|G_t|}. \tag{5.1.7}$$

Due to Proposition 5.1.10, we have

$$\begin{aligned}
&\sum_{w,x\in X_r} \|f(w)-f(x)\|^2\nu_r(w)\nu_r(x)\\
&\leqslant \frac{1}{\lambda}\sum_{w,x\in X_r} \|f(w)-f(x)\|^2\mu_r(w,x)\nu_r(w),
\end{aligned} \tag{5.1.8}$$

for each $r\in X$. Now the strategy is to rewrite the quantities in (5.1.7) in such a way that we can use (5.1.8).

Recalling (5.1.2) from Remark 5.1.6, we compute for each $t\in T$ and $w\in X$,

$$\begin{aligned}
&\|f(t)-f(w)\|^2\mu^{*2}(t,w)\frac{\nu(t)}{|G_t|}\\
&\underset{(iii)}{=} \frac{1}{|G_t|}\sum_{x\in X}\|f(t)-f(w)\|^2\tau(x)\nu_x(t)\nu_x(w)\\
&= \frac{1}{|G_t|}\sum_{r\in T}\frac{1}{|G_r|}\int_G \|f(t)-f(w)\|^2\tau(g\cdot r)\nu_{g\cdot r}(t)\nu_{g\cdot r}(w)\,\mathrm{d}g\\
&\underset{(i)}{=} \frac{1}{|G_t|}\sum_{r\in T}\frac{1}{|G_r|}\int_G \|f(g^{-1}\cdot t)-f(g^{-1}\cdot w)\|^2\tau(r)\nu_r(g^{-1}\cdot t)\nu_r(g^{-1}\cdot w)\,\mathrm{d}g
\end{aligned}$$

Summing over $t\in T, w\in X$, we obtain for the left-hand side of (5.1.7),

$$\begin{aligned}
&\sum_{t\in T}\sum_{w\in X}\|f(t)-f(w)\|^2\mu^{*2}(t,w)\frac{\nu(t)}{|G_t|}\\
&=\sum_{t\in T}\sum_{w\in X}\frac{1}{|G_t|}\sum_{r\in T}\frac{1}{|G_r|}\int_G\|f(g\cdot t)-f(g\cdot w)\|^2\tau(r)\nu_r(g\cdot t)\nu_r(g\cdot w)\,\mathrm{d}g\\
&=\sum_{t\in T,r\in T}\frac{1}{|G_t||G_r|}\int_G\sum_{w\in X}\|f(g\cdot t)-f(g\cdot w)\|^2\tau(r)\nu_r(g\cdot t)\nu_r(g\cdot w)\,\mathrm{d}g\\
&=\sum_{t\in T,r\in T}\frac{1}{|G_t||G_r|}\int_G\sum_{w\in X}\|f(g\cdot t)-f(w)\|^2\tau(r)\nu_r(g\cdot t)\nu_r(w)\,\mathrm{d}g\\
&=\sum_{t\in T,r\in T}\sum_{w\in X}\frac{1}{|G_t||G_r|}\int_G\|f(g\cdot t)-f(w)\|^2\tau(r)\nu_r(g\cdot t)\nu_r(w)\,\mathrm{d}g\\
&=\sum_{r\in T}\sum_{w\in X}\frac{1}{|G_r|}\sum_{t\in T}\frac{1}{|G_t|}\int_G\|f(g\cdot t)-f(w)\|^2\tau(r)\nu_r(g\cdot t)\nu_r(w)\,\mathrm{d}g\\
&=\sum_{r\in T}\frac{1}{|G_r|}\sum_{w\in X}\sum_{x\in X}\|f(x)-f(w)\|^2\tau(r)\nu_r(x)\nu_r(w)\\
&=\sum_{r\in T}\frac{\tau(r)}{|G_r|}\underbrace{\sum_{x,w\in X_r}\|f(x)-f(w)\|^2\nu_r(x)\nu_r(w)}_{\text{left-hand side of (5.1.8)}},
\end{aligned}$$

where the second to last equality follows again by (5.1.2). Similarly, we have,

$$\begin{aligned}
&\|f(t)-f(w)\|^2\mu(t,w)\frac{\nu(t)}{|G_t|}\\
&\quad=\sum_{x\in X}\frac{1}{|G_t|}\|f(t)-f(w)\|^2\tau(x)\nu_x(t)\mu_x(t,w)\\
&\quad=\frac{1}{|G_t|}\sum_{r\in T}\frac{1}{|G_r|}\int_G\|f(t)-f(w)\|^2\tau(g\cdot r)\nu_{g\cdot r}(t)\mu_{g\cdot r}(t,w)\,\mathrm{d}g\\
&\quad=\frac{1}{|G_t|}\sum_{r\in T}\frac{1}{|G_r|}\int_G\|f(g^{-1}\cdot t)-f(g^{-1}\cdot w)\|^2\tau(r)\nu_r(g^{-1}\cdot t)\mu_r(g^{-1}\cdot t,g^{-1}\cdot w)\,\mathrm{d}g,
\end{aligned}$$

and thus for the right-hand side of (5.1.7),

$$\begin{aligned}
&\frac{1}{\lambda}\sum_{t\in T,w\in X}\|f(t)-f(w)\|^2\mu(t,w)\frac{\nu(t)}{|G_t|}\\
&\quad=\frac{1}{\lambda}\sum_{r,t\in T}\frac{1}{|G_t||G_r|}\int_G\sum_{w\in X}\|f(g\cdot t)-f(g\cdot w)\|^2\tau(r)\nu_r(g\cdot t)\mu_r(g\cdot t,g\cdot w)\,\mathrm{d}g\\
&\quad=\frac{1}{\lambda}\sum_{r,t\in T}\frac{1}{|G_t||G_r|}\int_G\sum_{w\in X}\|f(g\cdot t)-f(w)\|^2\tau(r)\nu_r(g\cdot t)\mu_r(g\cdot t,w)\,\mathrm{d}g\\
&\quad=\frac{1}{\lambda}\sum_{w\in X}\sum_{r\in T}\frac{1}{|G_r|}\sum_{x\in X}\|f(x)-f(w)\|^2\tau(r)\nu_r(x)\mu_r(x,w)\\
&\quad=\sum_{r\in T}\frac{\tau(r)}{|G_r|}\underbrace{\frac{1}{\lambda}\sum_{x,w\in X_r}\|f(x)-f(w)\|^2\nu_r(x)\mu_r(x,w)}_{\text{right-hand side of (5.1.8)}}.
\end{aligned}$$

These computations show that (5.1.7) is a consequence of (5.1.8) and the lemma is proved. □

Now we have everything for the proof of the property (T) criterion.

Proof of Theorem 5.1.1. Let Y be a simplicial 2-complex with properties as in the hypothesis of Theorem 5.1.1 and consider a unimodular locally compact group G acting continuously, properly and cocompactly by simplicial automorphisms on Y.

Let X, μ, ν, $(X_x)_{x\in X}$, $(\mu_x)_{x\in X}$, $(\nu_x)_{x\in X}$ and τ be as in Examples 5.1.4, 5.1.7 and 5.1.16. Then we are in the situation of (a) to (e) and (i) to (iii). By assumption, for every $v\in X$, we have $\lambda_1(\Delta_{\mu_v,1_{\mathbb{C}}})=\lambda_1(\Delta_{Y_v})=\lambda_1(Y_v)>\frac{1}{2}$. Thus, the constant λ from Lemma 5.1.17 is strictly greater than $\frac{1}{2}$. Let $\rho\colon G\to\mathcal{U}(H)$ be some unitary representation without non-zero invariant vectors. Lemma 5.1.17 implies that (5.1.5) holds with $k=2$ and $C=\frac{1}{\lambda}<2$. Thus, by Lemma 5.1.15 and Proposition 5.1.12, the group G has Kazhdan's property (T). □

5.2 Ballmann–Świątkowski complexes

We present an explicit construction due to Ballmann–Świątkowski of a simplicial 2-complex which is sufficiently symmetric (i.e. it admits a geometric action by some group), and such that the link at every vertex is isomorphic to some prescribed finite Cayley graph (provided that its girth is large enough). If we apply this to a finite Cayley graph with $\lambda_1 > \frac{1}{2}$, the construction will provide examples of groups with property (T) by Theorem 5.1.1. This entire section is based on the relevant parts of the original article [BŚ97, Section 4].

Definition 5.2.1. Let X be a simplicial 2-complex. A *flag* in X is a triple $(\sigma_0, \sigma_1, \sigma_2)$, such that each σ_i is an i-simplex and $\sigma_0 < \sigma_1 < \sigma_2$. We denote the set of all flags in X by $\mathcal{F}(X)$. Moreover, for a vertex $v \in X^{(0)}$, we denote by $\mathcal{F}_v(X)$ the set of all flags containing v.

Fix a vertex v and let $F = (v, e, t) \in \mathcal{F}_v(X)$. Then there exists exactly one $\bar{e} \in X^{(1)}$ such that $e \neq \bar{e}$ and $\bar{F} = (v, \bar{e}, t) \in \mathcal{F}_v(X)$. This gives rise to an involution $\mathcal{F}_v(X) \to \mathcal{F}_v(X), F \mapsto \bar{F}$.

Let $\mathrm{E}_v(X)$ be the set of all edges in X incident to v and consider the maps

$$\iota\colon \mathcal{F}_v(X) \to \mathrm{E}_v(X), \quad (v, e, t) \mapsto e,$$
$$\tau\colon \mathcal{F}_v(X) \to \mathrm{E}_v(X), \quad F \mapsto \iota(\bar{F}).$$

We give a slightly non-standard approach of the link at a vertex, which will be convenient in the ensuing constructions.

Definition 5.2.2. Let X be a simplicial 2-complex and $v \in X^{(0)}$. We define X_v to be the graph with vertex set $\mathrm{E}_v(X)$ and oriented edge set $\mathcal{F}_v(X)$ with initial and terminal vertex maps ι and τ, respectively.

Remark 5.2.3. For each oriented edge in X_v, there is exactly one other edge with the same endpoints and which is oriented in the opposite direction. Other than that, there are no double edges or loop edges. Thus we may realize X_v as a simplicial graph and then it is exactly the simplicial link of v in X as defined at the beginning of Section 5.1.

Let $f\colon X \to Y$ be a simplicial map between simplicial 2-complexes which is injective on each simplex of X. Then f induces maps $\mathcal{F}(f)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ and $\mathcal{F}_v(f)\colon \mathcal{F}_v(X) \to \mathcal{F}_{f(v)}(Y)$ by $(v, e, t) \mapsto (f(v), f(e), f(t))$. Similarly, there is an induced map $\mathrm{E}_v(X) \to \mathrm{E}_{f(v)}(Y)$. These are compatible with the respective initial and terminal vertex maps ι and τ. Thus there is an induced morphism of graphs $f_v\colon X_v \to Y_{f(v)}$. If f_v is an isomorphism for each $v \in X^{(0)}$, then we say that f is a *local isomorphism*. Note that a surjective local isomorphism is a covering map, and a local isomorphism $f\colon X \to Y$ is automatically surjective provided that Y is connected (since its image is clopen).

In the following we shall be concerned with a 2-complex X together with a labeling $l\colon \mathcal{F}(X) \to S$ on the flags. Such a labeling induces an edge labeling on X_v for each $v \in X^{(0)}$. Let X and Y both be endowed with labellings $l_Z\colon \mathcal{F}(Z) \to S$, for $Z \in \{X, Y\}$.

We say that $f\colon X \to Y$ is *label-preserving* if $l_Y \circ \mathcal{F}(f) = l_X$. This is equivalent to the requirement that $f_v\colon X_v \to Y_{f(v)}$ be label-preserving for each $v \in X^{(0)}$.

For the remainder of this section, we fix a finite group H together with a generating set $S \subseteq H$ satisfying $S = S^{-1}$. Let $\Gamma = \mathrm{Cay}(H, S)$ be the Cayley graph of H with respect to S. We consider Γ as a labeled graph, where each edge is labeled by the corresponding generator.

Remark 5.2.4. Let Λ be a finite graph with a reduced labeling on the edges by elements of S. (Reduced means no two oriented edges with the same initial vertex share a label.) Let x_0 and y_0 be vertices in Γ and Λ, respectively. Then there exists a unique label-preserving immersion $\phi\colon \Lambda \looparrowright \Gamma$ such that $\phi(y_0) = x_0$. In particular, for each $h \in H$, there is a unique label-preserving automorphism $\phi_h\colon \Gamma \to \Gamma$ such that $\phi(1_H) = h$, and ϕ is given by left translation via h. This shows that the group of label-preserving automorphisms of Γ is isomorphic to H. Moreover, for any two label-preserving immersions $\phi_1, \phi_2\colon \Lambda \looparrowright \Gamma$, there is a unique $h \in H$ such that $\phi_2 = \phi_h \circ \phi_1$.

Definition 5.2.5. A Γ-*complex* is a simplicial 2-complex X such that for each $v \in X^{(0)}$, there exists a graph isomorphism $\phi_v\colon X_v \xrightarrow{\cong} \Gamma$. A labeling $\mathcal{F}(X) \to S$ on a Γ-complex is said to be *modeled on* Γ, if each ϕ_v can be chosen to be label-preserving.

A *labeled* Γ-*complex* is a Γ-complex X together with a labeling $l\colon \mathcal{F}(X) \to S$ modeled on Γ such that, in addition, the following condition holds.

(L1) for each $F = (v, e, t), \tilde{F} = (\tilde{v}, e, t) \in \mathcal{F}(X)$ with $v \neq \tilde{v}$ we have $l(\tilde{F}) = (l(F))^{-1}$.

Remark 5.2.6. Due to the fact that Γ is a Cayley graph, the labeling of a labeled Γ-complex automatically satisfies:

(L2) For each $F = (v, e, t), \bar{F} = (v, \bar{e}, t)$ with $e \neq \bar{e}$ we have $l(\bar{F}) = (l(F))^{-1}$.

(L3) For every vertex v and every edge e, the labeling is injective on $\{(\tilde{v}, \tilde{e}, \tilde{t}) \in \mathcal{F}(X) | \tilde{v} = v, \tilde{e} = e\}$.

The condition (L2) means that edges in X_v which are inverse to each other have inverse labels and (L3) means that the labels of edges with a fixed initial vertex are pairwise different. Both statements are obviously true for a Cayley graph and thus for X_v.

Theorem 5.2.7. *There exists a unique*[4] *simply-connected labeled* Γ*-complex* (X, l)*, provided that* $\mathrm{girth}(\Gamma) \geqslant 6$*. Moreover,* X *is contractible and the group* $\mathrm{Aut}(X, l)$ *of label-preserving automorphisms acts transitively on vertices with stabilizer* $\left(\mathrm{Aut}(X, l)\right)_v \cong H$ *for each* $v \in X^{(0)}$*.*

Proof. We first prove the existence of a contractible labeled Γ-complex by an inductive construction:

Let $B_0 = \{v_0\}$ and $B_1 = \{v_0\} * \Gamma$ be the simplicial cone on Γ. Then $(B_1)_{v_0} = \Gamma$, which gives rise to a labeling on all the flags of B_1 containing v_0. This can be uniquely extended

[4] up to label preserving isomorphism

to a labeling $l_1 \colon \mathcal{F}(B_1) \to S$ such that (L1) is satisfied, and then l_1 also satisfies (L2) and (L3). By construction, B_1 deformation retracts onto B_0. Let $m = |S|$. For every $v \in B_1^{(0)} \setminus \{v_0\}$, the graph $(B_1)_v$ consists of a vertex x_0 corresponding to the edge e connecting v to v_0, together with m additional vertices, each of which corresponds to a triangle adjacent to e and is connected to x_0 by an edge. (Thus $(B_1)_v$ is a tree of diameter 2, where the degree of each vertex is either 1 or m.)

Assume inductively that we have constructed a chain of 2-complexes $B_1 \subseteq \cdots \subseteq B_n$ together with labellings $l_i \colon \mathcal{F}(B_i) \to S$, such that for every $i \in \{1, \ldots n\}$ the following properties hold:

(a) For every $v \in B_{i-1}^{(0)}$, there is a label-preserving isomorphism $\phi_v \colon (B_i)_v \xrightarrow{\cong} \Gamma$.

(b) The labeling l_i on B_i extends l_{i-1} and satisfies (L1)–(L3).

(c) Every closed edge of B_i which does not intersect B_{i-1} is adjacent to exactly one triangle in B_i.

(d) For every $v \in \left(B_i^{(0)} \setminus B_{i-1}^{(0)}\right)$, the graph $(B_i)_v$ is a finite tree of diameter 2 or 3 such that the valence of each vertex is either 1 or m.

(e) B_i deformation retracts onto B_{i-1}.

We call an edge as in (c) a *boundary edge*. To construct B_{n+1}, we first glue $m-1$ new triangles to each boundary edge of B_n and call the resulting 2-complex $\tilde{B}_n$. We can extend the labeling l_n to a labeling $\tilde{l}_n \colon \mathcal{F}(\tilde{B}_n) \to S$ such that (L1)–(L3) is satisfied and this extension is unique up to permutations of the newly added triangles. Let $v \in B_n^{(0)} \setminus B_{n-1}^{(0)}$. Leaf vertices in $(B_n)_v$ correspond exactly to boundary edges in B_n adjacent to v. This implies that $(\tilde{B}_n)_v$ is obtained from $(B_n)_v$ by gluing $(m-1)$ edges to each leaf vertex. So $(\tilde{B}_n)_v$ is a finite tree of diameter 4 or 5 with valences either 1 or m. Since $\tilde{l}_n$ satisfies (L3), the induced labeling on $(\tilde{B}_n)_v$ is reduced. Together with girth$(\Gamma) \geqslant 6$, this implies that there is a label-preserving embedding $\tilde{\phi}_v \colon (\tilde{B}_n)_v \to \Gamma$, and any two such embeddings only differ by a label-preserving automorphism $\Gamma \to \Gamma$, see Remark 5.2.4. The map $\tilde{\phi}_v$ determines a label-preserving embedding $\Phi_v \colon \bar{\mathrm{st}}_v(\tilde{B}_n) \to \{v\} * \Gamma$ mapping v to v, where $\bar{\mathrm{st}}_v(\tilde{B}_n)$ denotes the closed star of v in $\tilde{B}_n$. We define B_{n+1} as the following quotient

$$B_{n+1} = \left(\tilde{B}_n \sqcup \bigsqcup_v (\{v\} * \Gamma)\right) / \sim,$$

where v runs over all $B_n^{(0)} \setminus B_{n-1}^{(0)}$ and we identify $\Phi_v(\sigma) \sim \sigma$ for all such v and $\sigma \in \bar{\mathrm{st}}_v(\tilde{B}_n)$. This construction amounts to (independently) completing each vertex link in $\tilde{B}_n$ to be isomorphic to Γ. Note that $\tilde{B}_n$ naturally embeds into B_{n+1} and, since each Φ_v is label-preserving, there is an induced labeling $l_{n+1} \colon \mathcal{F}(B_{n+1}) \to S$ which extends $\tilde{l}_n$ and satisfies (L1)–(L3). Moreover, l_{n+1} is the unique labeling with these properties.

We now verify the above conditions for B_{n+1}, where (a), (b) and (c) are immediate by construction. To prove (d), let $v \in B_{n+1}^{(0)} \setminus B_n^{(0)}$ and distinguish two cases: If $v \notin \tilde{B}_n$, then $(B_{n+1})_v$ is isomorphic to the link of a boundary vertex in the cone on Γ. Thus it is a tree of diameter two with one non-leaf node of degree m. On the other hand, if $v \in \tilde{B}_n$, then it is contained in a triangle $t \in \tilde{B}_n$ such that the edge opposite to v is attached to a boundary edge of B_n. Let v, v_1, v_2 be the vertices of t and denote by e_i the edge between v and v_i, $i = 1, 2$. Then e_1 and e_2 represent vertices in $(B_{n+1})_v$ which are connected by an edge. Since $v_i \in B_n^{(0)}$, by (a), we have $(B_{n+1})_{v_i} \cong \Gamma$ and thus there are m triangles $t, t_1^i, \ldots, t_{m-1}^i$ adjacent to e_i, for each $i = 1, 2$. The triangles t_j^i correspond to $(m-1)$ additional edges in $(B_{n+1})_v$ which are attached to e_i. This is all we need to construct $(B_{n+1})_v$, because by construction of B_{n+1} every edge adjacent to v is contained in one of the triangles t_j^i for $i \in \{1, 2\}, j \in \{1, \ldots, m-1\}$. Thus we see that $(B_{n+1})_v$ is tree of diameter 3, where the two non-leaf vertices have degree m.

For (e), observe that every subcomplex of $\{v\} * \Gamma$, which is the union of a set of closed triangles, is a deformation retract of the whole cone $\{v\} * \Gamma$. Thus it follows that B_{n+1} deformation retracts onto $\tilde{B}_n$ and, clearly, $\tilde{B}_n$ deformation retracts onto B_n. This concludes the inductive step of the construction.

The desired simplicial 2-complex is the union $X = \bigcup_{n \geqslant 1} B_n$. We define a labeling $l\colon \mathcal{F}(X) \to S$ by the requirement that l restricts to l_n on each X_n. By (a) and (b), this is well defined and turns (X, l) into a labeled Γ-complex. Using (e), we see inductively that each B_n is contractible. Since every finite subcomplex of X is contained in some B_n, this implies that X is weakly contractible. Hence, by Whitehead's theorem, it is contractible.

To prove uniqueness, we can observe that every labeled Γ-complex arises from the above construction and that there was essentially no choice involved in it. More precisely, the following lemma implies that X is the universal covering of any connected labeled Γ-complex, thus proving uniqueness of simply connected ones.

Lemma 5.2.8. *Let Y be any connected labeled Γ-complex and $w_0 \in Y^{(0)}$. Choose a label-preserving isomorphism $\phi\colon X_{v_0} \to Y_{w_0}$. Then there exists a unique label-preserving local isomorphism $p\colon X \to Y$ with $p(v_0) = w_0$ such that $p_{v_0} = \phi$.*

Proof. Let $p_1\colon B_1 \to \bar{\mathrm{st}}(w_0, Y)$ be the unique label-preserving isomorphism satisfying $(p_1)_{v_0} = \phi$. Assume inductively that there is a unique label-preserving map $p_n\colon B_n \to Y$ extending p_1 such that the induced map $(p_n)_v\colon (B_n)_v \to (Y)_{p(v)}$ is an isomorphism for all $v \in B_{n-1}^{(0)}$. Since Y is a labeled Γ-complex, the map p_n admits a unique label-preserving extension $\tilde{p}_n\colon \tilde{B}_n \to Y$. Moreover, observe that for each $v \in B_n^{(0)} \setminus B_{n-1}^{(0)}$ there is a unique label-preserving isomorphism $(p_{n+1})_v\colon (B_{n+1})_v \xrightarrow{\cong} Y_{p(v)}$ such that

$$
\begin{tikzcd}
(\tilde{B}_n)_v \arrow[r, hook] \arrow[d, hook, "(\tilde{p}_n)_v"'] & (B_{n+1})_v \arrow[dl, "(p_{n+1})_v"] \\
Y_{p_n(v)} &
\end{tikzcd}
$$

commutes, compare Remark 5.2.4. This translates to the fact that there is a unique label-preserving extension $p_{n+1}\colon B_{n+1} \to Y$. This finishes the inductive step and thus the proof of the lemma, since $p = \bigcup_n p_n$ is the desired map. $\square$

The claim about transitivity and stabilizers follows immediately from the lemma above: Taking $Y = X$ implies that $\mathrm{Aut}(X, l)$ acts transitively on vertices. In addition, note that for every $v \in X^{(0)}$, there is a natural homomorphism

$$(\mathrm{Aut}(X, l))_v \to \mathrm{Aut}(X_v, l), \quad f \mapsto f_v,$$

where $\mathrm{Aut}(X_v, l)$ denotes the group of label-preserving graph automorphisms of X_v. By the lemma, this homomorphism is an isomorphism and, by Remark 5.2.4, $\mathrm{Aut}(X_v, l) \cong H$. Thus the theorem is proved. $\square$

Remark 5.2.9. Lemma 5.2.8 implies that the action of $\mathrm{Aut}(X, l)$ is simply transitive on oriented edges of X, i.e. for vertices $v_i, w_i \in X^{(0)}$, $i \in \{1, 2\}$, such that v_i and w_i are adjacent in $X^{(1)}$, there exists a unique $f \in \mathrm{Aut}(X, l)$ such that $f(v_1) = f(v_2)$ and $f(w_1) = f(w_2)$. In particular, for every edge e in X, there is a unique $\tau_e \in \mathrm{Aut}(X, l)$ which reverses e. Note that by uniqueness each τ_e is an involution, i.e. $\tau_e^2 = \mathrm{id}$.

Similarly, for each oriented triangle t in X, there is a unique label-preserving automorphism r which acts by rotation on t. (In particular, $r^3 = \mathrm{id}$.) Indeed, if t is spanned by vertices v_0, v_1, v_2, then by simple transitivity on oriented edges there is a unique r which satisfies $r(v_i) = v_{i+1}$, for $i = 0, 1$. It follows that $r(v_2) = v_0$ and thus r is the desired rotation, because otherwise $v_1, v_2, r(v_2)$ would span a different triangle adjacent to t which would lead to a contradiction with the conditions on the labeling.

Proposition 5.2.10. *Let (X, l) be the contractible labeled Γ-complex. Then $\mathrm{Aut}(X, l)$ has a presentation*

$$\left\langle S \cup \{\tau\} \;\middle|\; R \cup \{\tau^2\} \cup \{(\tau s)^3 | s \in S\} \right\rangle, \tag{5.2.1}$$

where $\langle S|R \rangle$ is a presentation of H and τ corresponds to an automorphism reversing an edge.

Proof. Fix a vertex $v_0 \in X^{(0)}$ and an identification $\Gamma = X_{v_0}$ and correspondingly $H = (\mathrm{Aut}(X, l))_{v_0} \subseteq \mathrm{Aut}(X, l)$. Let e_0 be the edge adjacent to v_0 corresponding to $1_H \in \Gamma$. Let $\tau = \tau_{e_0} \in \mathrm{Aut}(X, l)$ as in Remark 5.2.9. We first prove that $H \cup \{\tau\}$ generates $\mathrm{Aut}(X, l)$. Let $G \subseteq \mathrm{Aut}(X, l)$ be the group generated by $H \cup \{\tau\}$. Note that for each $h \in H$ we have $\tau_{he_0} = h\tau h^{-1}$ and, since H acts transitively on the edges adjacent to v_0, it follows that $\tau_e \in G$ for all $e \in \mathrm{E}_{v_0}(X)$. We claim that for each $v \in X^{(0)}$ which is connected to v_0 by an edge, we have

$$(\mathrm{Aut}(X, l))_v \subseteq G \quad \text{and} \quad \{\tau_e \mid e \in \mathrm{E}_v(X)\} \subseteq G \tag{5.2.2}$$

Let $\tilde{e} \in \mathrm{E}_{v_0}(X) \cap \mathrm{E}_v(X)$ be the edge connecting v and v_0. Then, because $v = \tau_{\tilde{e}}(v_0)$, it follows that $(\mathrm{Aut}(X, l))_v = \tau_{\tilde{e}} H \tau_{\tilde{e}}$, and consequently $(\mathrm{Aut}(X, l))_v \subseteq G$. By the same argument as before, we conclude $\tau_e \in G$ for all $e \in \mathrm{E}_v(X)$. We can now proceed inductively to see that (5.2.2) holds for all $v \in X^{(0)}$. In particular, all edge reversals are contained in G. This shows that G acts transitively on $X^{(0)}$, because if $e_1, \ldots, e_n$ is any edge path from v_0 to some $v \in X^{(0)}$, then $(\tau_{e_n} \tau_{e_{n-1}} \cdots \tau_{e_1})(v_0) = v$. Finally, let

$f \in \mathrm{Aut}(X,l)$ be arbitrary and choose $g \in G$ such that $g(v_0) = f(v_0)$. Then $g^{-1}f \in H$ and, due to $H \subseteq G$, this means that $f \in G$. Thus we have shown $G = \mathrm{Aut}(X,l)$.

Let $s \in S \subseteq H$ and t be the triangle in X spanned by the vertices v_0, v_1, v_2, where $v_1 = \tau(v_0)$ and $v_2 = (\tau s)(v_1) = s^{-1}(v_1)$. (This indeed defines a triangle in X, because $e_0 = 1_H$ and $s^{-1}e_0 = s^{-1}$ are adjacent in $X_{v_0} = \Gamma$.) We see that $(\tau s)(v_0) = \tau(v_0) = v_1$ and $(\tau s)(v_1) = v_2$. Therefore τs coincides with a rotation of t, see Remark 5.2.9, and the relation $(\tau s)^3 = \mathrm{id}$ follows. Thus all relations of the presentation (5.2.1) hold in G. As explained in [BŚ97, Section 4], we can use [BB94, Proposition 2.1] to conclude that these relations are sufficient to present G. □

5.3 Coarse median groups with property (T)

As we have already mentioned, applying the construction of Section 5.2 to a finite Cayley graph Γ with $\lambda_1(\Gamma) > \frac{1}{2}$ and $\mathrm{girth}(\Gamma) \geqslant 6$ yields a finitely generated group with property (T). In this section, we review an additional condition on Γ such that the resulting group is hyperbolic (and thus coarse median of rank 1). In fact, it suffices to require $\mathrm{girth}(\Gamma) \geqslant 7$. Subsequently, we explain how to construct property (T) groups which are coarse median of arbitrary prescribed rank.

Proposition 5.3.1. *Let Y be a simply connected simplicial 2-complex with $\mathrm{girth}(Y_v) \geqslant 7$ for all $v \in Y^{(0)}$. Then every locally compact group G acting continuously, properly and cocompactly by simplicial automorphisms on Y is compactly generated and hyperbolic.*

Proof. Let $g = \min_{v \in Y^{(0)}} \mathrm{girth}(Y_v) > 6$ and choose $\alpha \in [\frac{2\pi}{g}, \frac{\pi}{3})$. Let $\triangle$ be the equilateral triangle in the hyperbolic plane $\mathbb{H}^2$ with all inner angles equal to α. We turn the simplicial complex into a metric space by declaring each 2-simplex of Y to be isometric to $\triangle$;[5] the resulting metric space we denote by (Y,d). We claim that (Y,d) is a $\mathrm{CAT}(-1)$ metric space. By assumption Y is simply connected, so to show the claim, it suffices to prove that for every $v \in Y^{(0)}$, the length of every simply closed loop in Y_v is at least 2π (with respect to the angular metric) [BH99, Theorem II.5.2, Lemma II.5.6]. This condition is immediate by construction because the length of a simply closed loop in Y_v is $\geqslant \alpha\,\mathrm{girth}(Y_v) \geqslant \frac{2\pi}{g} g = 2\pi$. Since a $\mathrm{CAT}(-1)$ space is hyperbolic [BH99, Proposition III.H.1.2], it follows that (Y,d) is hyperbolic.

Since we have used the same equilateral triangle for each 2-simplex, any simplicial automorphism of Y gives rise to an isometry of (Y,d). In particular, G acts by isometries on the metric space Y, and the action is still continuous, metrically proper and cocompact. By Proposition 3.1.7, it follows that G is generated by some compact set $S \subseteq G$ and (G, d_S) is quasi-isometric to (Y,d). In particular, $\mathrm{Cay}(G,S)$ is quasi-isometric to (Y,d), which proves the claim due to quasi-isometry invariance of hyperbolicity on geodesic spaces. □

In Section 5.4, we mention an explicit construction of finite Cayley graphs Γ such that $\mathrm{girth}(\Gamma) > 6$ and $\lambda_1(\Gamma) > \frac{1}{2}$. Applying Theorem 5.2.7 and Proposition 5.3.1, we immediately get the next corollary.

[5] See Bridson–Haefliger [BH99, Chapter I.7] for details on how to construct the metric.

Corollary 5.3.2. *There exist finitely generated hyperbolic groups with Kazhdan's property (T).*

In fact, we even get an explicitly constructed simplicial 2-complex endowed with a CAT(-1) metric (see the proof of Proposition 5.3.1) on which G acts geometrically, and an explicit description of a presentation (see Proposition 5.2.10).

Theorem 5.3.3. *For each $n \in \mathbb{N}$, there exists a finitely generated group G with Kazhdan's property (T) which is coarse median of rank n. Moreover, one can choose the coarse median to be equivariant with respect to the left action of G on itself.*

Proof. Let $G_1, \dots, G_n$ be infinite hyperbolic groups with property (T). We claim that it is enough to choose $G = G_1 \times \cdots \times G_n$.

Every G_i is coarse median of rank 1 and, since it is infinite, there exists an isometric embedding of $\mathbb{R}$ into the Cayley graph of G_i. Therefore, by Corollary 2.2.7 we conclude that G is coarse median of rank n. By Proposition 1.4.6, G has property (T).

By Proposition 3.1.12, we can assume that the coarse median $\mu_{G_i} \colon G_i^3 \to G_i$ is G_i-left-equivariant for all $i \in \{1, \dots, n\}$. This implies that $\mu_G := \mu_{G_1} \times \cdots \times \mu_{G_n} \colon G^3 \to G$ is G-left-equivariant. □

5.4 Ramanujan Graphs

In view of the previous section, it remains to find a finite Cayley graph Γ with $\lambda_1(\Gamma) > \frac{1}{2}$ (equivalently, $\mu_1(\Gamma) < \frac{1}{2}$) and $\mathrm{girth}(\Gamma) > 6$. The construction we describe is due to Lubotzky–Phillips–Sarnak [LPS88]; it produces explicit generators for the group $\mathrm{PSL}_2(\mathbb{F}_q)$ (with q certain prime numbers and $\mathbb{F}_q$ the finite field with q elements) such that the corresponding Cayley graph satisfies a certain optimal lower bound on λ_1 (and there also exists a lower bound on its girth). The proof of the bound on λ_1 for these Cayley graphs uses the solution of the Ramanujan conjecture, and for that reason, such graphs have been named *Ramanujan graphs*. In this section, we follow the exposition [Sar90, Chapter 3] and describe the construction of such Ramanujan graphs (but we omit almost all of the proofs since the required number-theoretic tools are beyond the scope of this Master thesis).

Definition 5.4.1. Let Γ be a finite k-regular graph. The graph Γ is called a *Ramanujan graph* if $\mu_1(\Gamma) \leqslant 2\frac{\sqrt{k-1}}{k}$.

Asymptotically — as the size of the graphs we consider grows — Ramanujan graphs satisfy an optimal upper bound on μ_1, cf. [Sar90, Proposition 3.2.7][6].

Note that $\lambda_1(\Gamma) = 1 - \mu_1(\Gamma)$; so a Ramanujan graph Γ satisfies $\lambda_1(\Gamma) > \frac{1}{2}$ if $k \geqslant 15$. Thus, we need to find a finite Cayley graph Γ with respect to a symmetric generating set of cardinality > 15 which is Ramanujan and such that $\mathrm{girth}(\Gamma) > 6$.

[6] We warn the reader that Sarnak's notation is different than ours. – In fact, for a k-regular graph $X_{n,k}$, the quantity $\lambda_1(X_{n,k})$ in [Sar90] is $k \cdot \mu_1(X_{n,k})$ in our notation.

We now review the construction [Sar90, Section 3.3]; the proofs of the claims made in the paragraph below can also be found in the same book [Sar90, Section 3.4].

We start with some remarks on the matrix groups used in the construction. For a field F, let $\mathrm{GL}_n(F)$ denote the group of invertible $n \times n$-matrices and $\mathrm{SL}_n(F)$ the subgroup of $\mathrm{GL}_n(F)$ consisting of the matrices with determinant 1. Let $\mathrm{PGL}_n(F)$ be the quotient of $\mathrm{GL}_n(F)$ modulo scalar multiples of the identity. For a matrix $M \in \mathrm{GL}_n(F)$, we write $[M]$ for its image in $\mathrm{PGL}_n(F)$. The image of $\mathrm{SL}_n(F)$ in $\mathrm{PGL}_n(F)$ is denoted by $\mathrm{PSL}_n(F)$. For every $M \in \mathrm{GL}_n(F)$, we have $[M] \in \mathrm{PSL}_n(F)$ if and only if $\det(M) \in \{a^n \mid a \in F \setminus \{0\}\}$.

Now let $p \neq q$ be prime numbers which satisfy $p, q \equiv 1 \mod 4$ and $\left(\frac{p}{q}\right) = 1$, where $\left(\frac{p}{q}\right)$ is the *Legendre symbol*[7]. Let

$$\tilde{S}_p = \left\{ (a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \;\middle|\; \sum_{i=0}^{3} a_i^2 = p, \text{with } a_0 > 0 \text{ odd and } a_i \text{ even for } i \in \{1,2,3\}. \right\}$$

Then the cardinality of $\tilde{S}_p$ is $p+1$. Let $\mathbb{F}_q$ be the finite field with q elements and choose $\mathrm{i} \in \mathbb{F}_q$ such that $\mathrm{i}^2 = -1$. (This is possible due to $q \equiv 1 \mod 4$.) For each $\alpha = (a_0, a_1, a_2, a_3) \in \tilde{S}_p$, we define $M_\alpha \in \mathrm{GL}_2(\mathbb{F}_q)$ by

$$M_\alpha = \begin{pmatrix} a_0 + \mathrm{i}a_1 & a_2 + \mathrm{i}a_3 \\ -a_2 + \mathrm{i}a_3 & a_0 - \mathrm{i}a_1 \end{pmatrix}, \tag{5.4.1}$$

and we set $S_{p,q} = \{[M_\alpha] \mid \alpha \in \tilde{S}_p\} \subseteq \mathrm{PGL}_2(\mathbb{F}_q)$. Note that $\det(M_\alpha) = a_0^2 - (\mathrm{i}a_1)^2 - ((\mathrm{i}a_3)^2 - a_2^2) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$. Since $\left(\frac{p}{q}\right) = 1$, there exists $x \in \mathbb{F}_q$ with $x^2 = p$, so $[M_\alpha] \in \mathrm{PSL}_2(\mathbb{F}_q)$. Moreover, observe that for each $\alpha = (a_0, a_1, a_2, a_3) \in \tilde{S}_p$ we have

$$(M_\alpha)^{-1} = \frac{1}{p} \begin{pmatrix} a_0 + \mathrm{i}(-a_1) & -a_2 + \mathrm{i}(-a_3) \\ -(-a_2) + \mathrm{i}(-a_3) & a_0 - \mathrm{i}(-a_1) \end{pmatrix} = \frac{1}{p} M_{\bar{\alpha}},$$

where $\bar{\alpha} = (a_0, -a_1, -a_2, -a_3) \in \tilde{S}_p$. In particular, in $\mathrm{PSL}_2(\mathbb{F}_q)$ we have $[M_\alpha]^{-1} = [M_{\bar{\alpha}}]$, hence the set $S_{p,q} \subseteq \mathrm{PSL}_2(\mathbb{F}_q)$ is symmetric.

Theorem 5.4.2 ([Sar90, Theorem 3.3.1, Case (ii)])**.** *Suppose that p and q have been chosen as above. Then $S_{p,q}$ generates the group* $\mathrm{PSL}_2(\mathbb{F}_q)$*, and the Cayley graph* $X^{p,q} :=$ $\mathrm{Cay}(\mathrm{PSL}_2(\mathbb{F}_q), S_{p,q})$ *is a $(p+1)$-regular Ramanujan graph which satisfies*

$$\operatorname{girth}(X^{p,q}) \geqslant 2\log_p(q). \tag{5.4.2}$$

In view of (5.4.2) and the Ramanujan bound $\lambda_1(X^{p,q}) \geqslant 1 - 2\frac{\sqrt{p}}{p+1}$, it is to be expected that by choosing p, q appropriately, we can ensure both $\lambda_1(X^{p,q}) > \frac{1}{2}$ and $\operatorname{girth}(X^{p,q}) > 6$. In the following, we explicitly work out the minimal such example which can be obtained by using these bounds.

[7] i.e. $\left(\frac{p}{q}\right) = 1$ if there exists $x \in \mathbb{N}$ such that $x^2 \equiv p \mod q$, and $\left(\frac{p}{q}\right) = -1$ otherwise.

Example 5.4.3. We note all conditions on the prime numbers p and q which need to be satisfied.

$$p+1 \geqslant 15, \qquad p \equiv 1 \mod 4, \tag{5.4.3}$$

$$\log_p(q) > 3, \qquad q \equiv 1 \mod 4, \qquad \left(\frac{p}{q}\right) = 1. \tag{5.4.4}$$

The two smallest prime numbers p which satisfy (5.4.3) are 17 and 29. If we fix $p = 17$, then $q = 4973$ is the smallest possible choice such that (5.4.4) is satisfied.[8] Conversely, if $17 \neq p \neq q$ are two prime numbers satisfying (5.4.3) and (5.4.4), then $p \geqslant 29$ and $q > p^3 \geqslant 29^3 > 4973$. This shows that $p = 17$ and $q = 4973$ is the minimal choice (for both p and q) such that (5.4.3) and (5.4.4) are satisfied.

From now on, we fix $p = 17$ and $q = 4973$. We have $\tilde{S}_p = \tilde{S}_p^+ \sqcup \tilde{S}_p^-$, where we define $\tilde{S}_p^+$ to consist of the following nine tuples,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix},$$

and $\tilde{S}_p^- = \{\bar{\alpha} \mid \alpha \in \tilde{S}_p^+\}$. To construct $S_{p,q}$, we need to choose $\mathrm{i} \in \mathbb{Z}$ such that $\mathrm{i}^2 \equiv -1 \mod q$; the smallest positive integer which satisfies this condition is $\mathrm{i} = 223$. Using this choice, we plug the list of tuples $\tilde{S}_p^+$ into (5.4.1) and obtain matrices as follows.

$$\begin{pmatrix} 1 & 892 \\ 892 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}, \begin{pmatrix} 893 & 0 \\ 0 & -891 \end{pmatrix}, \begin{pmatrix} 3 & 448 \\ 444 & 3 \end{pmatrix}, \begin{pmatrix} 449 & 446 \\ 446 & -443 \end{pmatrix},$$
$$\begin{pmatrix} 449 & 2 \\ -2 & -443 \end{pmatrix}, \begin{pmatrix} 3 & 444 \\ 448 & 3 \end{pmatrix}, \begin{pmatrix} -443 & 446 \\ 446 & 449 \end{pmatrix}, \begin{pmatrix} -443 & 2 \\ -2 & 449 \end{pmatrix}.$$

The set of elements of $\mathrm{PSL}_2(\mathbb{F}_q)$ represented by these matrices we denote by $S_{p,q}^+$, and it follows that $S_{p,q} = S_{p,q}^+ \sqcup S_{p,q}^-$, where $S_{p,q}^- = (S_{p,q}^+)^{-1}$. (Alternatively, we could directly calculate representatives for the elements of $S_{p,q}^-$ by plugging $\tilde{S}_p^-$ into (5.4.1).)

In summary, we have computed nine explicit generators (or 18, if we also count inverses) for the group $\mathrm{PSL}_2(\mathbb{F}_{4973})$ such that the corresponding Cayley graph has girth > 6 and $\lambda_1 > \frac{1}{2}$.

[8] We have verified this claim using the software package Wolfram *Mathematica* 8.

Open questions

In the following, we collect a few interesting and open[9] questions about coarse median spaces and groups.

1. *Let (X, d) and (X', d') be coarse median spaces with coarse medians denoted by μ and μ', respectively. Is it always true that*

$$\operatorname{rank}(\mu \times \mu') = \operatorname{rank}(\mu) + \operatorname{rank}(\mu') \quad ?$$

   We have noted in Proposition 2.1.10 that $\operatorname{rank}(\mu \times \mu') \leqslant \operatorname{rank}(\mu) + \operatorname{rank}(\mu')$. In addition, we know from the characterization of hyperbolicity in terms of rank 1 coarse median spaces [Bow13a, Theorem 2.1] that $\operatorname{rank}(\mu \times \mu') \geqslant 2$ provided X and X' are both infinite geodesic spaces (because then $X \times X'$ is not hyperbolic). More generally, Bowditch shows that asymptotic cones of rank n coarse median spaces do not admit bi-Lipschitz embeddings of $\mathbb{R}^{n+1}$ (cf. [Bow13a, Theorem 2.2]). But conversely, it is not clear whether this estimate is sharp, i.e. if for a coarse median space of rank $n \geqslant 2$, there always exists an asymptotic cone with a bi-Lipschitz embedding of $\mathbb{R}^n$. Hence, these techniques don't seem to suffice for proving a positive answer to this question (if it is true at all).

2. *Are there coarse median groups with Property (T) that have an infinite outer automorphism group?* This question is motivated by a result of Paulin [Pau91] which states that a hyperbolic group with property (T) has finite outer automorphism group. The proof of Paulin's theorem works by constructing an action without fixed points on an asymptotic cone of the group under the indirect assumption that the outer automorphism group is infinite. By hyperbolicity, such an asymptotic cone is an $\mathbb{R}$-tree, whence the existence of an action without fixed points contradicts property (T). On first sight, it seems that one could proceed similarly for general coarse median groups using Bowditch' observation that asymptotic cones of coarse median spaces are metric median algebras [Bow13a, Theorem 2.3] and the characterization of property (T) in terms of isometric actions on median spaces [CDH10]. However, among other problems with this approach, to conclude that a group does not have property (T), one needs an action on a median space with unbounded orbits; not just an action without fixed points. In this context, note that Behrstock–Druţu–Sapir [BDS11] prove a related result about homomorphisms from (T)-groups into the mapping class group (but with an additional effort).

3. The following questions (ordered by descending level of generality) are a restatement of a conjecture made by Bowditch [Bow13a, Introduction].

[9] to the author's knowledge as of May 2013.

- *Does every (finite dimensional)* $\mathrm{CAT}(0)$ *geodesic metric space admit a coarse median?*
- *In particular, does every simply connected non-positively curved Riemannian manifold admit a coarse median?*
- *More specifically, do (simply connected, irreducible) symmetric spaces of non-compact type with real rank* $\geqslant 2$ *admit a coarse median?*

From our point of view, the last question is particularly interesting because higher rank simple Lie groups have property (T);[10] hence a positive advancement in this direction would provide new non-hyperbolic examples of coarse median groups with property (T).

[10] A connected real simple Lie group has Kazhdan's property (T) iff its Lie algebra is not isomorphic to $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$, see e.g. [BHV08, Theorem 3.5.4].

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Summary

In the present Master thesis, we study Bowditch' concept of a coarse median on many examples of metric spaces and groups. We develop formal tools to compare different coarse medians on metric spaces. We also illuminate the interrelations between the existence of a coarse median on a group, a-T-menability, and Kazhdan's property (T).

A coarse median on a metric space (X, d) is a ternary operation $\mu\colon X^3 \to X$ which satisfies the axioms of a median algebra up to bounded error (in a certain precise sense). In order to meaningfully compare different coarse medians, we introduce the concept of a *coarse median structure* as an equivalence class of coarse medians up to uniformly bounded error. We formalize Bowditch' observation of quasi-isometry invariance of the existence of a coarse median on a metric space by defining the *pushforward* and the *pullback* of a coarse median structure via a quasi-isometry. Bowditch has shown that on any Gromov hyperbolic geodesic metric space, there is a coarse median structure defined in terms of geodesic triangles. In terms of the pushforward/pullback notation, we note that the hyperbolic coarse median structure is *stable under quasi-isometries*. We also prove that any hyperbolic group admits a *left-equivariant* coarse median.

Furthermore, using the pullback construction, we define a coarse median structure on spaces with measured walls which are at finite Hausdorff distance to their associated median spaces. For these examples, we establish a description of the coarse median structure only in terms of the wall structure. We apply these ideas to the measured wall structure on the real hyperbolic space and to the discrete wall structures obtained from Wise's cubulation of classical small cancellation groups. In both these cases, we show that the coarse median structure obtained from the wall structure is the same as the one determined by hyperbolicity of the underlying metric space.

We prove the existence of "*approximating* CAT(0) *cubical complexes*" in coarse median spaces and thereby generalize the fundamental result on approximating trees in Gromov hyperbolic geodesic spaces.

Moreover, we extend Bowditch' theory of coarse median groups from finitely generated discrete groups to compactly generated locally compact groups. We show that a compactly generated locally compact coarse median group is *compactly presented*.

Among finitely generated groups, we compare known permanence results concerning the classes of coarse median groups, a-T-menable groups, and groups with Kazhdan's property (T). In addition, we carefully present the explicit construction due to Ballmann–Świątkowski of finitely generated hyperbolic groups with Kazhdan's property (T). By considering direct products of such groups, we show that there exist many examples of non-hyperbolic groups with Kazhdan's property (T) which admit a left-equivariant coarse median.

Zusammenfassung

In der vorliegenden Masterarbeit behandeln wir das von Bowditch eingeführte Konzept eines „coarse median". Wir studieren „coarse medians" auf vielen Beispielen von metrischen Räumen und insbesondere auf Gruppen. Wir entwickeln formale Werkzeuge um verschiedene „coarse medians" auf einem metrischen Raum zu vergleichen. Weiters beleuchten wir die Zusammenhänge zwischen der Existenz eines „coarse median" auf einer Gruppe, Gromovs „a-T-menability", und Kazhdans Eigenschaft (T).

Ein „coarse median" auf einem metrischen Raum (X, d) ist eine ternäre Operation $\mu\colon X^3 \to X$, die — in einem gewissen präzisen Sinn — die Axiome einer „median algebra" bis auf beschränkten Fehler erfüllt. Um verschiedene „coarse medians" vergleichen zu können, führen wir das Konzept einer „*coarse median structure*" als Äquivalenzklasse von „coarse medians" bezüglich der Relation „bis auf beschränkten Fehler übereinstimmen" ein. Bowditch bemerkte, dass die Existenz eines „coarse median" eine Invariante des Quasi-Isometrietyps des zugrunde liegenden metrischen Raums ist. Wir formalisieren dies und definieren den *Pushforward* beziehungsweise den *Pullback* einer „coarse median structure" durch eine Quasi-Isometrie. Bowditch bewies, dass jeder Gromov hyperbolische geodätische metrische Raum einen „coarse median" besitzt. Im Sinne der Pushforward/Pullback Notation stellen wir fest, dass diese hyperbolische „coarse median structure" *stabil unter Quasi-Isometrien* ist. Wir beweisen weiters, dass jede hyperbolische Gruppe einen *links-äquivarianten* „coarse median" besitzt.

Auf „spaces with measured walls", die endliche Hausdorffdistanz zu ihrer „medianization" haben, definieren wir einen „coarse median" unter Verwendung der Pullback Konstruktion. Wir erhalten eine nur auf die „walls" zurückgreifende Beschreibung der so definierten „coarse median structure". Diese Ideen werden auf den reell hyperbolischen Raum sowie auf klassische „small cancellation" Gruppen angewandt.

Wir verallgemeinern das fundamentale Resultat über approximierende Bäume in Gromov hyperbolischen geodätischen Räumen und erhalten einen Satz über „*approximierende* CAT(0) *kubische Komplexe*" in Räumen mit einem „coarse median".

Außerdem erweitern wir Bowditch' Definition von „coarse median groups" von der Klasse der endlich erzeugten diskreten Gruppen in die Klasse der kompakt erzeugten lokal-kompakten Gruppen. Wir beweisen, dass kompakt erzeugte lokal-kompakte „coarse median" Gruppen *kompakt präsentiert* sind.

Wir vergleichen bekannte Stabilitätsresultate, die die Klassen der endlich erzeugten „coarse median groups", a-T-menable Gruppen, und Gruppen mit Kazhdans Eigenschaft (T) betreffen. Zusätzlich präsentieren wir die auf Ballmann–Świątkowski zurückgehende Konstruktion von endlich erzeugten hyperbolischen Gruppen mit Kazhdans Eigenschaft (T). Indem wir direkte Produkte von solchen Gruppen betrachten, zeigen wir, dass es viele Beispiele von nicht hyperbolischen Gruppen mit Kazhdans Eigenschaft (T) gibt, die einen links-äquivarianten „coarse median" besitzen.

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