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Caroline Moosmüller, BSc

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Betreuer:	Univ.-Prof. Mag. Dr. Andreas Čap

Abstract

This thesis studies projective structures on smooth manifolds M, i.e. equivalence classes of torsion-free affine connections on M, which induce the same unparametrised geodesics. Questions, that arise naturally in this setting, are concerned with the existence of representatives in the projective class, which have special properties. For example, one may ask if a given projective structure can be represented by a connection coming from a metric or an Einstein metric. Furthermore, the existence of invariants of the geometric structure can be studied. In this setting, differential operators, which are intrinsic to the projective structure, are of particular interest.

We treat these problems in the framework of tractor calculus, which is an invariant calculus for projective structures. On a manifold endowed with a projective structure, there is no distinguished connection on the tangent bundle. Nevertheless, there exist natural vector bundles, called tractor bundles, which can be associated to any projective structure and they carry an invariant connection, called tractor connection. Therefore, it is natural to calculate with these bundles.

Furthermore, ideas from the general theory of BGG-sequences are used to approach questions associated to projective structures. The BGG-sequence provides a sequence of projectively invariant differential operators, where we are especially interested in the first operators of sequences coming from certain tractor bundles. If D is a first BGG-operator, then the equation $D\sigma = 0$ is called first BGG-equation. For the tractor bundles we consider, the existence of appropriate solutions of the first BGG-equation is equivalent to the existence of special representatives in the projective class. In particular, we provide conditions for the existence of (pseudo-) Riemannian metrics and Einstein metrics and for Ricci flat connections in the projective class.

As an application, tractor bundles, their connections and solutions of the first BGG-equation are discussed on the homogeneous model for oriented projective structures, which is the sphere viewed as a homogeneous space of the special linear group.

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Contents

Chapter 1
Introduction

Originally, projective geometry goes back to E. Cartan and T. Thomas, but it has recently attracted much interest. The defining data is a smooth manifold M equipped with a projective structure, i.e. an equivalence class of torsion-free connections, whose geodesics, viewed as unparametrised curves, coincide. Traditionally, the connection between projective geometry and Riemannian geometry is of great interest, but projective geometry also plays a crucial role in the study of overdetermined systems of partial differential equations. Furthermore, it provides an important example of a parabolic geometry as well as for the BGG machinery.

Given a projective structure on a manifold, one may ask for representatives in the equivalence class that have special properties. There is, for example, the classical problem of metrisability, which has been addressed by many authors. It is concerned with the question of whether a given projective structure can be represented by the Levi-Civita connection of a (pseudo-) Riemannian metric. Due to Mikeš and Sinjukov [15, 16], the existence of a Levi-Civita connection in the projective class is equivalent to the existence of a positive definite solution of a certain overdetermined linear partial differential equation. Similarly, one may ask for Einstein metrics in a given projective class, compare [12, 7].

Another question that naturally arises is concerned with the existence of invariant differential operators, i.e. operators which are intrinsic to the projective structure. One can try to construct first order operators directly, that is, without using any further theory, and may succeed. But by raising the order, this becomes a rather difficult task. Nevertheless, the BGG machinery provides us with a general construction for a sequence (called BGG-sequence) of invariant differential operators, compare [10].

This is actually closely related to the search of special representatives in the projective class. Indeed, the overdetermined partial differential equation, which we addressed above, is a special case of a first BGG-equation. This is an equation of the form $D\sigma = 0$, where D is an invariant differential operator arising as the first operator in a BGG-sequence. Hence, the existence of an appropriate solution of a first BGG-equation $D\sigma = 0$ is equivalent to the existence of a Levi-Civita connection in the projective class.

The general theory of BGG-sequences provides a natural subclass of solutions, called normal solutions. In [7] it is shown, that the existence of a positive definite normal solution of a first BGG-equation $D\sigma = 0$ (which is actually the same equation as above) is equivalent to the existence of an Einstein metric in the projective class.

Most of the results we commented on here, strongly rely on an invariant calculus for projective geometry, called tractor calculus. Given a projective structure, there is no distinguished connection on the tangent bundle. Nevertheless, we can associate natural vector bundles to a projective structure, which are equipped with invariant connections. These are called tractor bundles and the resulting connections are called tractor connections.

As a matter of fact, tractor calculus was uniformly developed for all parabolic geometries, see [4]. It can be constructed directly, that is, without using the Cartan connection (which is a basic ingredient for a parabolic geometry). The Cartan bundle and connection can however be regained from the tractor bundle and its connection.

This thesis is concerned with explicity calculating certain tractor bundles and their connections for a given projective geometry. Furthermore, we give formulae for the invariant differential operators which arise from the BGG-sequence applied to these tractor bundles. We provide conditions for the existence of Riemannian metrics, (pseudo-) Riemannian Einstein metrics and Ricci-flat connections in a given projective class. As a final part, these tractor bundles, their connections and (normal) solutions of the first BGG-equation are interpreted on the sphere, which is the homogeneous model for oriented projective structures.

Overview of the text

Chapter 2 is mainly concerned with fixing conventions and notations as well as recalling basic facts on linear connections. A short introduction to abstract index notation is given, which will be used extensively throughout the whole text. Besides the general theory of linear connections on vector bundles, it is discussed how principal connection forms on the linear frame bundle induce linear connections on the tangent bundle (which are called affine connections). Furthermore, the homogeneous model of the Cartan geometry describing affine connections is considered.

In chapter 3 projective structures on smooth manifolds are introduced as equivalence classes of torsion-free connections that define the same geodesics (as unparametrised curves). An equivalent definition is presented, which is less geometric, but more useful in practice. Similarities with terms and results of chapter 2 are pointed out throughout this chapter, especially when discussing the model spaces for projective structures.

Tractor calculus is developed in chapter 4. We start by defining the cotractor bundle and its tractor connection. The other tractor bundles are obtained by natural constructions on the cotractor bundle: Dualising gives the tractor bundle with canonically the dual connection, symmetrizing gives the symmetric power of the (co-) tractor bundle with canonical induced connection. Furthermore, the curvatures of the obtained tractor connections are computed. Also, we remark on how the Cartan bundle and connection can be obtained from the tractor bundle and its connection.

In chapter 5 the main results are presented. Each tractor bundle considered in chapter 4 gives rise to a first, second or third order invariant differential operator, which is the first operator in a BGG-sequence. These operators are constructed directly, i.e. without using the theory of BGG-sequences. The resulting first BGG-equations provide conditions for the existence of Ricci-flat connections (Theorem 5.3.1), Riemannian (Einstein) metrics (Remark 5.3.1) and pseudo-Riemannian Einstein metrics (Theorem 5.3.2) in the projective class.

Chapter 6 is dedicated to interpreting results from the last chapters on the homogeneous model for oriented projective structures, which is the sphere. We will show that each tractor bundle is trivial and that it carries the trivial connection, i.e. all tractor connections are trivial. Furthermore, in the case of the homogeneous model, all solution of the first BGG-equations are normal. Normal solutions, in turn, correspond to elements in the standard fiber of the tractor bundle considered.

Chapter 2
Preliminaries

2.1 Conventions and Abstract Index Notation

This thesis is founded on general concepts from differential geometry and Lie theory. Therefore, the notions of smooth manifolds, diffeomorphisms, vector fields, tensor fields, Lie groups, Lie algebras, etc. are used without further explanation. Moreover, the theory of vector bundles, principal bundles and associated bundles is presumed. All of these prerequisites are briefly reviewed in [9, Section 1.2.]. A more detailed presentation can be found in [14, Chapter 1-4].

The notation is also adopted from [9, Section 1.2.]. Thus, throughout this thesis, M denotes a smooth manifold of dimension $n \geq 2$. Its tangent respectively cotangent bundle is denoted by TM respectively T^*M. We write $\mathfrak{X}(M)$ for the space of vector fields on M and $\Omega^k(M)$ for the space of $k-$forms. If V is a vector space, then by $\Omega^k(M, V)$ we denote the space of $k-$forms with values in V, by S^kV the $k-$th symmetric power and by Λ^kV the $k-$th exterior power of V.

A fiber bundle is denoted by $p : Y \to M$, its fiber over $x \in M$ by Y_x and the space of smooth sections of Y by $\Gamma(Y)$. To indicate the type of a given bundle, we write $p : \mathcal{V} \to M$ for vector bundles and $p : \mathcal{P} \to M$ for principal bundles. Given a principal bundle $p : \mathcal{P} \to M$ with structure group H and a smooth manifold S, which is endowed with a left action $H \times S \to S$, the associated bundle to the principal bundle $\mathcal{P}$ with standard fiber S is denoted by $\mathcal{P} \times_H S$.

We will also make use of abstract index notation. In this notation indices act as labels to specify the kind of tensor we are working with. In particular, it gives rise to a coordinate-free way of manipulating tensors. We give a brief introduction here.

A lower index indicates covariance of a given tensor with respect to that index, hence ω_j is a $1-$form. Similarly an upper index indicates contravariance and thus ξ^i is a vector field. Combining upper and lower indices, we can form any kind of tensor. For example, a $\binom{1}{1}-$tensor field A is written as $A_j{}^i$. Given a vector field ξ and a $1-$form ω, their tensor product $\xi \otimes \omega$ is a $\binom{1}{1}-$tensor field. We write $\xi^i\omega_j$ for $(\xi \otimes \omega)_j{}^i$, hence we omit the product sign. A $\binom{1}{1}-$tensor field A can be viewed as a map associating to each $x \in M$ an endomorphism of T_xM. The trace or contraction of A is then given by the trace of this endomorphism in each point $x \in M$. In abstract index notation the trace of $A_j{}^i$ is indicated by $A_k{}^k$. There exists a canonical $\binom{1}{1}-$tensor field $\delta_j{}^i$, which, viewed as an endomorphism, is given by the identity map. Thus $\delta_k{}^k = n$.

Indicating a contraction in abstract indices is especially useful for tensors that allow more than one trace. For example, a tensor of the form $B_{ij}{}^k$ allows two different traces, which are written as $B_{kj}{}^k$ and $B_{ik}{}^k$. Note that taking a trace turns $B_{ij}{}^k$ into a $1-$form. Furthermore, viewing a $\binom{1}{1}-$tensor field A as a map $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ and evaluating at a point $\xi \in \mathfrak{X}(M)$ gives a vector field $A(\xi)^i$, which in abstract index notation is written as $A_j{}^i \xi^j$. Hence, the contraction in this case indicates the evaluation of A at ξ.

Now as an example, consider the curvature R of a linear connection (details on linear connections and curvature can be found in the next section). The curvature is a map $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ and we will see in the next section that it is a $\binom{1}{3}-$tensor field. Therefore, in abstract index notation it can be written as $R_{ij}{}^k{}_\ell$ and using the paragraph above, we obtain $(R(\xi, \eta)(\zeta))^k = R_{ij}{}^k{}_\ell \xi^i \eta^j \zeta^\ell$ for vector fields $\xi, \eta, \zeta \in \mathfrak{X}(M)$. Also the curvature $R_{ij}{}^k{}_\ell$ of a linear connection allows two different traces (note that it is antisymmetric in i and j) and $R_{j\ell} = R_{kj}{}^k{}_\ell$ is exactly the Ricci curvature.

We introduce notation for the symmetric respectively the antisymmetric part of a tensor. Parentheses around indices indicate the symmetrization of a given tensor. Thus

$$\omega_{(i_1 \cdots i_n)} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \omega_{i_{\sigma(1)} \cdots i_{\sigma(n)}}.$$

Similarly we use square brackets for the antisymmetric part:

$$\omega_{[i_1 \cdots i_n]} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \omega_{i_{\sigma(1)} \cdots i_{\sigma(n)}}.$$

In this notation, an $n-$form is a tensor $\omega_{i_1 \cdots i_n}$ satisfying $\omega_{i_1 \cdots i_n} = \omega_{[i_1 \cdots i_n]}$. Note however, that indices, which are not inside the brackets or appear on different levels, are not part of the symmetration respectively antisymmetration. Hence

$$\omega_{(i}{}^\ell \xi_{j)k} = \frac{1}{2}(\omega_i{}^\ell \xi_{jk} + \omega_j{}^\ell \xi_{ik}).$$

In the case of bundles, there is a similar notation with abstract indices. Denote by $\mathcal{E}$ the trivial bundle over M, by $\mathcal{E}^a$ the tangent bundle TM and by $\mathcal{E}_a$ the cotangent bundle T^*M. Tensor products of these bundles are indicated by iterating this notation. For example, $\otimes^2 TM$ is written as $\mathcal{E}^{ab}$. Similarly we write $\mathcal{E}^{(ab)}$ respectively $\mathcal{E}^{[ab]}$ for $S^2(TM)$ respectively $\Lambda^2(TM)$.

In the case a chart is chosen and we are working with explicit indices, we make use of Einstein summation convention: An index appearing twice as a lower and an upper index indicates summation over that index. Thus for example coordinate expressions of the form $\xi|_U = \sum_i \xi^i \partial_i$ are written as $\xi|_U = \xi^i \partial_i$.

2.2 Linear Connections

In this section, we recall basic definitions of linear connections on vector bundles. We find a local representation of a linear connection and define its curvature. When restricting to the case of linear connections on the tangent bundle, we obtain a decomposition of the curvature containing a trace-free component. In the final part of this section, we remark on a construction, which induces linear connections on the tangent bundle from connection forms on the linear frame bundle. The theory presented in this chapter can be found in [9, Section 1.3], [1, Section 3.1] and [2, Chapter 3.2].

General Theory

A linear connection on a vector bundle $\mathcal{V} \to M$ provides a concept of directional derivative of sections of this bundle in the direction of vector fields.

Definition 2.2.1. A linear connection on a vector bundle $\mathcal{V} \to M$ is a bilinear operator $\nabla : \mathfrak{X}(M) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$, given by $(\xi, s) \mapsto \nabla_\xi s$, which additionally satisfies

$$\nabla_{f\xi} s = f \nabla_\xi s$$

and

$$\nabla_\xi f s = (\xi \cdot f) s + f \nabla_\xi s,$$

for $f \in C^\infty(M, \mathbb{R})$.

Thus a linear connection is a bilinear operator, which is tensorial in the first argument and satisfies a Leibniz rule in the second argument.

On the trivial vector bundle $\mathcal{V} = M \times V$ for V a vector space, a linear connection is given by $\nabla_\xi s = \xi \cdot s$, for $s \in \Gamma(\mathcal{V}) = \mathcal{C}^\infty(M, V)$ and $\xi \in \mathfrak{X}(M)$. This linear connection is called the trivial connection on $\mathcal{V}$.

Remark 2.2.1.
1. A linear connection ∇ on a vector bundle $\mathcal{V} \to M$ induces linear connections on its dual bundle as well as on tensor products. We will denote these induced linear connections by the same symbol.

   Let $\mathcal{V}^* \to M$ be the dual bundle of $\mathcal{V}$. Then for $\sigma \in \Gamma(\mathcal{V}^*), \xi \in \mathfrak{X}(M)$ and $s \in \Gamma(\mathcal{V})$

   $$(\nabla_\xi \sigma)(s) = \xi \cdot (\sigma(s)) - \sigma(\nabla_\xi s),$$

   defines a linear connection on $\mathcal{V}^*$.

Similarly for $\mathcal{V} \to M$ and $\mathcal{W} \to M$ two vector bundles, there is an induced connection on $\mathcal{V} \otimes \mathcal{W} \to M$ given by

$$\nabla_\xi(s \otimes t) = s \otimes \nabla_\xi t + (\nabla_\xi s) \otimes t,$$

for $s \in \Gamma(\mathcal{V}), t \in \Gamma(\mathcal{W})$ and $\xi \in \mathfrak{X}(M)$. Of course this can be iterated to tensor products of finite size. Note that we used the same symbol for the linear connections on $\mathcal{V}$ and $\mathcal{W}$.

2. On every vector bundle $\mathcal{V} \to M$ there exists a linear connection. In a vector bundle chart, we can locally pull back the trivial connection and piece these locally defined connections together by using a partition of unity to obtain a globally defined linear connection on $\mathcal{V}$.

Now we want to find a local description of a linear connection ∇. Let $p : \mathcal{V} \to M$ be a vector bundle of rank k, hence given a vector bundle atlas (U_α, ϕ_α), we have that $p^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{R}^k$. Consider a local frame $\{e_1, \ldots, e_k\}$ for U_α and write $\sigma = \sigma^i e_i$ for $\sigma \in \Gamma(\mathcal{V})$ and $\sigma^i : U_\alpha \to \mathbb{R}$ smooth. Then for a vector field $\xi \in \mathfrak{X}(M)$, on U_α we have:

$$\nabla_\xi \sigma = \nabla_\xi \sigma^i e_i = (\xi \cdot \sigma^i) e_i + \sigma^i \nabla_\xi e_i.$$

Definition 2.2.2. The curvature of a linear connection ∇ on $\mathcal{V} \to M$ is defined as

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$$

$$R(\xi, \eta)s = \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s$$

By definition the curvature is skew-symmetric in the first two arguments.

Lemma 2.2.1. The curvature is $\mathcal{C}^\infty(M, \mathbb{R})-$linear in all three arguments. Thus R is a section of $\Lambda^2 T^* M \otimes L(\mathcal{V}, \mathcal{V})$.

Proof. We show that it is $\mathcal{C}^\infty(M, \mathbb{R})-$linear in the first and third argument. By skew symmetry, the linearity in the second argument follows. Let $f : M \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}
R(f\xi, \eta)s &= \nabla_{f\xi} \nabla_\eta s - \nabla_\eta \nabla_{f\xi} s - \nabla_{[f\xi,\eta]} s \\
&= f\nabla_\xi \nabla_\eta s - \nabla_\eta f \nabla_\xi s - \nabla_{f[\xi,\eta]-(\eta \cdot f)\xi} s \\
&= f\nabla_\xi \nabla_\eta s - ((\eta \cdot f)\nabla_\xi s + f\nabla_\eta \nabla_\xi s) - (f\nabla_{[\xi,\eta]} s - (\eta \cdot f)\nabla_\xi s) \\
&= fR(\xi, \eta)s
\end{aligned}$$

and

$$\begin{aligned}
R(\xi,\eta)fs &= \nabla_\xi\nabla_\eta fs - \nabla_\eta\nabla_\xi fs - \nabla_{[\xi,\eta]}fs \\
&= \nabla_\xi((\eta\cdot f)s + f\nabla_\eta s) - \nabla_\eta((\xi\cdot f)s + f\nabla_\xi s) \\
&\quad -([\xi,\eta]\cdot f + f\nabla_{[\xi,\eta]}s) \\
&= \xi\cdot(\eta\cdot f)s + (\eta\cdot f)\nabla_\xi s + (\xi\cdot f)\nabla_\eta s + f\nabla_\xi\nabla_\eta s \\
&\quad -(\eta\cdot(\xi\cdot f)s + (\xi\cdot f)\nabla_\eta s + (\eta\cdot f)\nabla_\xi s + f\nabla_\eta\nabla_\xi s) \\
&\quad -([\xi,\eta]\cdot f + f\nabla_{[\xi,\eta]})s \\
&= fR(\xi,\eta)s.
\end{aligned}$$

Note that we used $\xi\cdot(\eta\cdot f) - \eta\cdot(\xi\cdot f) = [\xi,\eta]\cdot f$. □

Affine Connections

Now consider a linear connection on the tangent bundle $TM \to M$. Sections of this bundle are simply given by vector fields and thus Definition 2.2.1 yields a bilinear operator $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying the defining equations. Linear connections on the tangent bundle are called affine connections on M.

The induced connection on T^*M as in Remark 2.2.1 is then given by

$$(\nabla_\xi\omega)(\eta) = \xi\cdot(\omega(\eta)) - \omega(\nabla_\xi\eta),$$

for $\omega \in \Gamma(T^*M) = \Omega^1(M)$ and $\xi,\eta \in \mathfrak{X}(M)$.

In the case of affine connections, the local description can be expressed in terms of the Christoffel symbols. Consider a chart $(U,\{x_1,\dots,x_n\})$ for M and vector fields $\xi,\eta \in \mathfrak{X}(M)$. Then locally on U we have $\xi = \xi^i\partial_i$ and $\eta = \eta^j\partial_j$. Therefore we obtain

$$\nabla_\xi\eta|_U = \xi^i\frac{\partial\eta^j}{\partial x^i}\partial_j + \eta^j\xi^i\Gamma^k_{ij}\partial_k,$$

for Γ^k_{ij} the Christoffel symbols, which are defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$.

Definition 2.2.3. The torsion of a linear connection ∇ on TM is defined as

$$\mathrm{Tor}^\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$\mathrm{Tor}^\nabla(\xi,\eta) = \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta].$$

The torsion is obviously skew-symmetric and it is $\mathcal{C}^\infty(M,\mathbb{R})-$linear in both arguments:

Let f be a smooth function $M \to \mathbb{R}$, then

$$\begin{aligned}\mathrm{Tor}^\nabla(f\xi,\eta) &= \nabla_{f\xi}\eta - \nabla_\eta f\xi - [f\xi,\eta] \\ &= f\nabla_\xi\eta - (\eta\cdot f)\xi - f\nabla_\eta\xi - f[\xi,\eta] + (\eta\cdot f)\xi \\ &= f\mathrm{Tor}^\nabla(\xi,\eta)\end{aligned}$$

and by using the skew-symmetry of Tor^∇, we obtain the result for the second argument. Thus Tor^∇ is a section of $\Lambda^2 T^*M \otimes TM$.

Lemma 2.2.2. Two linear connections ∇ and $\hat{\nabla}$ on TM have the same torsion if and only if their difference tensor $A(\xi,\eta) := \hat{\nabla}_\xi\eta - \nabla_\xi\eta$ is symmetric.

Proof. First of all note that A is indeed a tensor field: It is obviously $\mathcal{C}^\infty(M,\mathbb{R})-$linear in the first argument and since

$$\begin{aligned}A(\xi,f\eta) &= \hat{\nabla}_\xi f\eta - \nabla_\xi f\eta \\ &= (\xi\cdot f)\eta + \hat{\nabla}_\xi\eta - ((\xi\cdot f)\eta + \nabla_\xi\eta) \\ &= fA(\xi,\eta),\end{aligned}$$

it is also $\mathcal{C}^\infty(M,\mathbb{R})-$linear in the second argument.

Now the following calculation proves the statement:

$$\begin{aligned}\mathrm{Tor}^\nabla(\xi,\eta) &= \mathrm{Tor}^{\hat{\nabla}}(\xi,\eta) \\ \nabla_\xi\eta - \nabla_\eta\xi - [\xi,\eta] &= \hat{\nabla}_\xi\eta - \hat{\nabla}_\eta\xi - [\xi,\eta] \\ \hat{\nabla}_\eta\xi - \nabla_\eta\xi &= \hat{\nabla}_\xi\eta - \nabla_\xi\eta \\ A(\eta,\xi) &= A(\xi,\eta).\end{aligned}$$

□

Definition 2.2.4. A linear connection ∇ on TM is called torsion-free, if $\mathrm{Tor}^\nabla(\xi,\eta) = 0$ for all vector fields $\xi,\eta \in \mathfrak{X}(M)$.

Given any linear connection ∇ on TM, we can always construct a torsion-free connection by defining $\hat{\nabla}_\xi\eta := \nabla_\xi\eta - \frac{1}{2}\mathrm{Tor}^\nabla(\xi,\eta)$. Then

$$\begin{aligned}\mathrm{Tor}^{\hat{\nabla}}(\xi,\eta) &= \hat{\nabla}_\xi\eta - \hat{\nabla}_\eta\xi - [\xi,\eta] \\ &= \nabla_\xi\eta - \frac{1}{2}\mathrm{Tor}^\nabla(\xi,\eta) - (\nabla_\eta\xi - \frac{1}{2}\mathrm{Tor}^\nabla(\eta,\xi)) - [\xi,\eta] \\ &= \mathrm{Tor}^\nabla(\xi,\eta) - \frac{1}{2}\mathrm{Tor}^\nabla(\xi,\eta) + \frac{1}{2}\mathrm{Tor}^\nabla(\eta,\xi) \\ &= 0,\end{aligned}$$

where we used that Tor^∇ is skew-symmetric in the last line.

We can view an affine connection as an operator acting on tensor fields. Given a tensor field t, then ∇t is a tensor field with covariance raised by one. For example, consider a vector field ξ, which is a $\binom{1}{0}-$ tensor field. Then $\nabla\xi$ is the $\binom{1}{1}-$ tensor field given by $(\nabla\xi)(\eta) := \nabla_\eta \xi$, for a vector field η. Moreover, we can consider $\nabla^2 t = \nabla(\nabla t)$, which is a tensor field with convariance raised by two. Inductively, we obtain that $\nabla^r t$ is a tensor field with covariance raised by r.

Lemma 2.2.3. Let ∇ be a torison-free linear connection on TM, then the curvature has the form

$$R(\xi,\eta)\zeta = (\nabla^2\zeta)(\xi,\eta) - (\nabla^2\zeta)(\eta,\xi),$$

for $\xi,\eta,\zeta \in \mathfrak{X}(M)$.

Proof. Since we are working with a torison-free connection, we have $[\xi,\eta] = \nabla_\xi\eta - \nabla_\eta\xi$ and thus

$$\begin{aligned}(\nabla^2\zeta)(\xi,\eta) - (\nabla^2\zeta)(\eta,\xi) &= (\nabla_\xi\nabla_\eta\zeta - \nabla_{\nabla_\xi\eta}\zeta) - (\nabla_\eta\nabla_\xi\zeta - \nabla_{\nabla_\eta\xi}\zeta)\\ &= \nabla_\xi\nabla_\eta\zeta - \nabla_\eta\nabla_\xi\zeta - \nabla_{[\xi,\eta]}\zeta\\ &= R(\xi,\eta)\zeta\end{aligned}$$

□

From now on all connections ∇ on the tangent bundle are assumed to be torsion-free and thus Lemma 2.2.3 can be used as the definition of the curvature of ∇. In order to decompose R it is easier to working with abstract indices. Therefore, we have to introduce abstract index notation for linear connections and fix conventions for calculating. We illustrate this on the example of a $\binom{1}{1}-$ tensor field $t_j{}^i$; it works analogously for any kind of tensor field. Starting with $t_j{}^i$, then we have that ∇t is a $\binom{1}{2}-$ tensor field and thus has indices $(\nabla t)_{kj}{}^i$. This is written as $\nabla_k t_j{}^i$. Iterating this notation, we obtain $(\nabla(\nabla t))_{k\ell j}{}^i = \nabla_k\nabla_\ell t_j{}^i$ and more generally $(\nabla^r t)_{j_1\cdots j_r k}{}^i = \nabla_{j_1}\cdots\nabla_{j_r} t_k{}^i$.

When working with abstract indices, the order of the indices is not important, hence $\xi^i\omega_j = \omega_j\xi^i$. However, this is not the case for terms containing linear connections. A linear connection acts on everthing to its right, unless there are brackets. For example,

$$\nabla_i\xi^j\omega_k = \xi^j\nabla_i\omega_k + \omega_k\nabla_i\xi^j = (\nabla_i\omega_k)\xi^j + (\nabla_i\xi^j)\omega_k.$$

Now we use abstract index notation to refomulate Lemma 2.2.3. Given vector fields $\xi^i,\eta^j,\zeta^\ell \in \mathfrak{X}(M)$, we obtain

$$R_{ij}{}^k{}_\ell\xi^i\eta^j\zeta^\ell = \xi^i\eta^j\nabla_i\nabla_j\zeta^k - \xi^i\eta^j\nabla_j\nabla_i\zeta^k.$$

We can also omit ξ^i and η^j and simply write $R_{ij}{}^k{}_\ell\zeta^\ell = (\nabla_i\nabla_j - \nabla_j\nabla_i)\zeta^k$.

Lemma 2.2.4. Given an affine connection ∇ with curvature $R_{ij}{}^{k}{}_{\ell}$. Then the curvature of the induced connection on the cotangent bundle T^*M is given by

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)\omega_\ell = -R_{ij}{}^{k}{}_{\ell}\omega_k.$$

Proof. We use the following fact: $(\nabla_\xi\omega)(\zeta) = \xi\cdot\omega(\zeta) - \omega(\nabla_\xi\zeta)$. Thus

$$\begin{aligned}(\nabla_\xi\nabla_\eta\omega)(\zeta) &= \xi\cdot(\nabla_\eta\omega(\zeta)) - (\nabla_\eta\omega)(\nabla_\xi\zeta)\\ &= \xi\cdot(\eta\cdot\omega(\zeta)) - \xi\cdot\omega(\nabla_\eta\zeta) - (\eta\cdot\omega(\nabla_\xi\zeta) - \omega(\nabla_\eta\nabla_\xi\zeta))\end{aligned}$$

Using this calculation as well as $\xi\cdot(\eta\cdot\omega(\zeta)) - \eta\cdot(\xi\cdot\omega(\zeta)) = [\xi,\eta]\cdot\omega(\zeta)$ we obtain

$$(\nabla_\xi\nabla_\eta\omega)(\zeta) - (\nabla_\eta\nabla_\xi\omega)(\zeta) - (\nabla_{[\xi,\eta]}\omega)(\zeta) = -\omega(R(\xi,\eta)(\zeta)),$$

which in abstract index notation is given by $-R_{ij}{}^{k}{}_{\ell}\omega_k\zeta^\ell$. □

Definition 2.2.5. Let ∇ be an affine connection with curvature $R_{ij}{}^{k}{}_{\ell}$. Then we define β_{ij} and the projective Schouten tensor P_{ij} by

$$(n+1)\beta_{ij} = -2R_{[ij]} \text{ and } (n-1)\mathrm{P}_{ij} = R_{ij} + \beta_{ij},$$

where $R_{ij} = R_{ki}{}^{k}{}_{j}$ is the Ricci curvature.

From the definition we immediately see that β_{ij} is skew symmetric and that $2\mathrm{P}_{[ij]} = -\beta_{ij}$.

Definition 2.2.6. Let ∇ be an affine connection with curvature $R_{ij}{}^{k}{}_{\ell}$. Then Weyl tensor $C_{ij}{}^{k}{}_{\ell}$ is defined by the equation

$$R_{ij}{}^{k}{}_{\ell} = C_{ij}{}^{k}{}_{\ell} + 2\delta_{[i}{}^{k}\mathrm{P}_{j]\ell} + \beta_{ij}\delta_\ell{}^k.$$

Lemma 2.2.5. The Weyl tensor is totally trace-free, i.e. $C_{kj}{}^{k}{}_{\ell} = C_{ik}{}^{k}{}_{\ell} = C_{ij}{}^{k}{}_{k} = 0$.

Proof. There are three possible traces we can form. As $R_{ij}{}^{k}{}_{\ell}$ and β_{ij} are skew symmetric in i and j and the defining equation for $C_{ij}{}^{k}{}_{\ell}$ is antisymmetrized in i and j in the part containing $\mathrm{P}_{j\ell}$, it suffices to compute the trace in j and ℓ.

$$\begin{aligned}C_{ij}{}^{j}{}_{\ell} &= R_{ij}{}^{j}{}_{\ell} - 2\delta_{[i}{}^{j}\mathrm{P}_{j]\ell} - \beta_{ij}\delta_\ell{}^j\\ &= -R_{i\ell} - (\mathrm{P}_{i\ell} - n\mathrm{P}_{i\ell}) - \beta_{i\ell}\\ &= -R_{i\ell} + (n-1)\mathrm{P}_{i\ell} - \beta_{i\ell}\\ &= 0\end{aligned}$$

The Bianchi symmetry $R_{[ij}{}^{k}{}_{\ell]} = 0$ implies

$$R_{ij}{}^{k}{}_{k} = -R_{jk}{}^{k}{}_{i} - R_{ki}{}^{k}{}_{j} = R_{ji} - R_{ij} = (n+1)\beta_{ij}. \tag{2.1}$$

We use identity (2.1) to compute the trace of the Weyl tensor in ℓ:

$$\begin{aligned} C_{ij}{}^{k}{}_{k} &= R_{ij}{}^{k}{}_{k} - \delta_i{}^{k}\mathrm{P}_{jk} - \delta_j{}^{k}\mathrm{P}_{ik} - \beta_{ij}\delta_k{}^{k} \\ &= (n+1)\beta_{ij} - \mathrm{P}_{ji} + \mathrm{P}_{ij} - n\beta_{ij} \\ &= (n+1)\beta_{ij} - \beta_{ij} - n\beta_{ij} \\ &= 0 \end{aligned}$$

□

Thus Definition 2.2.6 gives a decomposition of the curvature into trace-free and trace parts. For later purpose, we prove the following identities, which are based on the Bianchi symmetry $R_{[ij}{}^{k}{}_{\ell]} = 0$ and the Bianchi identity $\nabla_{[i}R_{jk]}{}^{\ell}{}_{m} = 0$. Details on these identities can be found in [13, Chapter 7].

Lemma 2.2.6. Let ∇ be an affine connection with curvature $R_{ij}{}^{k}{}_{\ell}$ and consider the decomposition of $R_{ij}{}^{k}{}_{\ell}$ as in Definition 2.2.6. Then we have the following identities:

$$\nabla_{[i}\beta_{jk]} = 0 \quad \text{and} \quad \nabla_k C_{ij}{}^{k}{}_{\ell} = 2(n-2)\nabla_{[i}\mathrm{P}_{j]\ell}.$$

Proof. Starting with the Bianchi identity $\nabla_{[i}R_{jk]}{}^{\ell}{}_{m} = 0$ and taking the trace over j and ℓ, we obtain:

$$\begin{aligned} 0 &= \nabla_i R_{\ell k}{}^{\ell}{}_{m} - \nabla_\ell R_{ik}{}^{\ell}{}_{m} - \nabla_k R_{\ell i}{}^{\ell}{}_{m} \\ &= \nabla_i R_{km} - \nabla_\ell R_{ik}{}^{\ell}{}_{m} - \nabla_k R_{im}, \end{aligned}$$

where $R_{ij} = R_{ki}{}^{k}{}_{j}$ denotes the Ricci curvature. Therefore, we obtain the following identity for the Ricci curvature:

$$\nabla_i R_{km} - \nabla_k R_{im} = \nabla_\ell R_{ik}{}^{\ell}{}_{m}. \tag{2.2}$$

Now by Definition 2.2.5, we have that $(n+1)\beta_{ij} = -R_{[ij]}$ and thus

$$\begin{aligned} (n+1)\nabla_{[i}\beta_{jk]} &= -2\nabla_{[i}R_{jk]} \\ &= 2((\nabla_j R_{ik} - \nabla_i R_{jk}) + (\nabla_i R_{kj} - \nabla_k R_{ij}) + (\nabla_k R_{ji} - \nabla_j R_{ki})) \\ &= 2(\nabla_\ell R_{ji}{}^{\ell}{}_{k} + \nabla_\ell R_{ik}{}^{\ell}{}_{j} + \nabla_\ell R_{kj}{}^{\ell}{}_{i}) \\ &= 0, \end{aligned}$$

where we used (2.2) in the second line and the Bianchi symmetry in the third line.

In order to prove the second statement, consider the decomposition of the curvature of Definition 2.2.6, apply the connection ∇ and antisymmetrize in the first three indices to obtain:

$$0 = \nabla_{[i}C_{jk]}{}^{\ell}{}_{m} + \nabla_{[i}\delta_j{}^{\ell}\mathrm{P}_{k]m} - \nabla_{[i}\delta_k{}^{\ell}\mathrm{P}_{j]m}.$$

Here we already used the Bianchi identity and $\nabla_{[i}\beta_{jk]} = 0$. Now taking the trace over i and ℓ and using that the Weyl tensor is totally trace free, we get:

$$\nabla_i C_{jk\ m}^{\ \ i} - \nabla_i C_{kj\ m}^{\ \ i} = -2(\nabla_j \mathrm{P}_{km} - n\nabla_j \mathrm{P}_{km} + \nabla_j \mathrm{P}_{km} - \nabla_k \mathrm{P}_{jm} - \nabla_k \mathrm{P}_{jm} + n\nabla_k \mathrm{P}_{jm})$$

and since the Weyl tensor is antisymmetric in the first two indices, this gives

$$\begin{aligned} \nabla_i C_{jk\ m}^{\ \ i} &= (n-2)(\nabla_j \mathrm{P}_{km} - \nabla_k \mathrm{P}_{jm}) \\ &= 2(n-2)\nabla_{[j}\mathrm{P}_{k]m}. \end{aligned}$$

□

Induced Linear Connections

A principal connection form on the linear frame bundle induces a linear connection on the tangent bundle as well as on any associated bundle. The case of the tangent bundle is very similar to what is presented in [2, Chapter 2.3], replacing the orthonormal frame bundle by the linear frame bundle. Induced linear connections on associated bundles from arbitrary principal bundles are discussed in [9, Section 1.3.4]. We give a short description of these constructions.

The linear frame bundle $p : P^1M \to M$ consists of the fibers P^1_xM for $x \in M$, where P^1_xM is the set of all linear isomorphisms $\mathbb{R}^n \to T_xM$. There exists a smooth structure on P^1_xM, which makes $p : P^1M \to M$ into a $GL(n,\mathbb{R})-$principal bundle. The principal right action is given by composition from the right, hence

$$r^A : P^1M \to P^1M$$

$$r^A(\phi) = \phi \circ A,$$

for $A \in GL(n,\mathbb{R})$ and $\phi \in P^1M$. Furthermore, the fact that we can put a smooth structure on P^1M exactly means that local sections correspond to local frames.

Recall that on the linear frame bundle we have the fundamental vector fields $\zeta_X \in \mathfrak{X}(P^1M)$, which are given by $\zeta_X(u) = \frac{d}{dt}|_{t=0} r^{\exp tX}(u)$ for $X \in \mathfrak{gl}(n,\mathbb{R}), u \in P^1M$. A principal connection form on P^1M is a one-form $\gamma \in \Omega^1(P^1M, \mathfrak{gl}(n,\mathbb{R}))$ satisfying

$$\begin{aligned} \gamma(\zeta_X) = X \quad & \text{for } X \in \mathfrak{gl}(n,\mathbb{R}) \\ (r^A)^*\gamma = \mathrm{conj}_A \circ \gamma \quad & \text{for } A \in GL(n,\mathbb{R}) \end{aligned} \tag{2.3}$$

The second part of (2.3) should be understood as $\gamma(u \circ A)(T_u r^A \cdot \xi) = A^{-1} \cdot \gamma(u)(\xi) \cdot A$, for $u \in P^1M, \xi \in T_uP^1M$ and $A \in GL(n,\mathbb{R})$. Now any principal connection form on P^1M induces a linear connection on TM via the notion of horizontal lifts.

If $\xi \in \mathfrak{X}(M)$ is a vector field, then the horizontal lift is the unique vector field $\xi^{\mathrm{hor}} \in \mathfrak{X}(P^1M)$, satisfying $Tp \circ \xi^{\mathrm{hor}} = \xi \circ p$ and $\gamma(\xi^{\mathrm{hor}}) = 0$. The linear connection on TM

can be obtained by identifying vector fields on M with $GL(n,\mathbb{R})-$equivariant functions $F : P^1M \to \mathbb{R}^n$, where the equivariancy condition is given by

$$F(\varphi \circ A) = A^{-1}F(\varphi) \qquad \forall A \in GL(n,\mathbb{R}).$$

Denote the set of all equivariant functions by $\mathcal{C}^\infty(P^1M,\mathbb{R}^n)^{GL(n,\mathbb{R})}$. If $\eta \in \mathfrak{X}(M)$ corresponds to $F \in \mathcal{C}^\infty(P^1M,\mathbb{R}^n)^{GL(n,\mathbb{R})}$, then the vector field corresponding to the equivariant function $\xi^{\text{hor}} \cdot F : P^1M \to \mathbb{R}^n$ is $\nabla_\xi \eta$. Here the dot denotes the directional derivative.

Thus any principal connection on P^1M induces a linear connection on TM. Actually there is a one-to-one correspondence between the set of linear connections on TM and the set of principal connection forms on P^1M, see [9, Section 1.3.5].

For the case of vector bundles, which are associated bundles to P^1M, there is a similar construction which induces linear connections on these bundles. Consider $\mathcal{V} = P^1M \times_{GL(n,\mathbb{R})} V$, where V is a vector space. Sections σ of this bundle are in bijective correspondence to smooth maps $f : P^1M \to V$, which are $GL(n,\mathbb{R})-$equivariant, i.e. which satisfy $f(\varphi \circ A) = A^{-1} \cdot f(\varphi)$ for $A \in GL(n,\mathbb{R})$. Here the dot on the right side of the equation indicates the left action of $GL(n,\mathbb{R})$ on V. For a proof of this fact see [9, Proposition 1.2.7.].

If a section σ corresponds to $f \in \mathcal{C}^\infty(P^1M,V)^{GL(n,\mathbb{R})}$, then $\nabla_\xi \sigma$ is the section corresponding to $\xi^{\text{hor}} \cdot f : P^1M \to V$, see [9, Proposition 1.3.4.].

Note that the construction of the linear connection on the tangent bundle via a principal connection on P^1M is actually a special case of the one of associated bundles to P^1M. The associated bundle $P^1M \times_{GL(n,\mathbb{R})} \mathbb{R}^n$, where the left action of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$ is given by multiplication from the left, can be identified with the tangent bundle.

2.3 Geodesics

Geodesics of a linear connection on a manifold M provide an analog concept to straight lines in $\mathbb{R}^n$. The defining property of a straight line is the vanishing of its acceleration. In order to define geodesics we need an interpretation of the second derivative of curves on a manifold. This is done by the notion of linear connections along curves. In the following we will only recall basic properties of linear connections along curves and geodesics, referencing [13, Chapter 4] for more detail.

Definition 2.3.1. Let $c : I \to M$ be a smooth curve. A vector field along c is a smooth map $\xi : I \to TM$ satisfying $p \circ \xi = c$, i.e. such that $\xi(t) \in T_{c(t)}M$. Denote the set of all vector fields along c by $\mathfrak{X}_c(M)$.

In this sense $c' : I \to TM$ is a vector field along c.

Lemma 2.3.1. Let ∇ be a linear connection on TM and $c : I \to M$ a smooth curve. Then there exists a unique linear operator $\nabla_{c'} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$, such that for every vector field ξ along c and for every smooth function $f : I \to \mathbb{R}$ we have

$$\nabla_{c'} f\xi = f'\xi + f\nabla_{c'}\xi. \tag{2.4}$$

If furthermore $\phi : J \to I$ is a reparametrisation, then

$$\nabla_{(c\circ\phi)'}(c\circ\phi)' = \phi''(c'\circ\phi) + (\phi')^2(\nabla_{c'}c')\circ\phi. \tag{2.5}$$

The linear operator $\nabla_{c'}$ is called the linear connection along c induced by ∇.

Proof. Let $t_0 \in \mathbb{R}$ and choose a chart $(U, \{x_1, \dots, x_n\})$ of M around $c(t_0)$. Write ∂_i for the vector fields $\frac{\partial}{\partial x_i}$. Then we have $\xi = \xi^i\partial_i$ and $\xi' = (\xi^i)'\partial_i$ on $c^{-1}(U)$. Similarly $c' = (c^i)'\partial_i$, where c^i denote the components of c. Define the linear connection on $c^{-1}(U)$ by

$$(\nabla_{c'}\xi)|_{c^{-1}(U)} = ((\xi^k)' + \xi^j(c^i)'\Gamma^k_{ij}\circ c)\partial_k. \tag{2.6}$$

From the definition, it is obvious that locally $\nabla_{c'}$ is uniquely determined. In order to show existence, we cover $c(I)$ with charts and on each chart we define $\nabla_{c'}$ by formula (2.6). By uniqueness, these linear connections have to agree on intersections of different charts and thus we obtain a well-defined operator. Equation (2.6) obviously defines a linear operator and satisfies (2.4):

$$\begin{aligned}
\nabla_{c'} f\xi &= (f'\xi^k + f(\xi^k)' + f\xi^j(c^i)'\Gamma^k_{ij}\circ c)\partial_k \\
&= f'\xi^k\partial_k + f((\xi^k)' + \xi^j(c^i)'\Gamma^k_{ij}\circ c)\partial_k \\
&= f'\xi + f\nabla_{c'}\xi.
\end{aligned}$$

Now given a reparametrisation ϕ, then by the chain rule we have the following expressions in coordinates:

$$(c\circ\phi)' = (\phi'((c^i)'\circ\phi))\partial_i \text{ and } (c\circ\phi)'' = ((\phi')^2((c^i)''\circ\phi) + \phi''((c^i)'\circ\phi))\partial_i.$$

The coordinate representation (2.6) of a linear connection along c gives (2.5):

$$\begin{aligned}
\nabla_{(c\circ\phi)'}(c\circ\phi)' =& \phi''((c^k)'\circ\phi)\partial_k \\
&+ ((\phi')^2((c^k)''\circ\phi) + ((c^j)'\circ\phi)\phi'((c^i)'\circ\phi)\phi'\Gamma^k_{ij}(c\circ\phi))\partial_k \\
=& \phi''((c^k)'\circ\phi)\partial_k \\
&+ (\phi')^2((((c^k)''\circ\phi) + ((c^j)'\circ\phi)((c^i)'\circ\phi)\Gamma^k_{ij}(c\circ\phi))\partial_k) \\
=& \phi''(c'\circ\phi) + (\phi')^2(\nabla_{c'}c')\circ\phi.
\end{aligned}$$

□

Definition 2.3.2. Let ∇ be an affine connection on M. A smooth curve $c : I \to M$ is called geodesic for ∇ if $\nabla_{c'}c' = 0$.

Working in a chart $(U, \{x_1, \dots, x_n\})$, the local representation of $\nabla_{c'}c'$ in Lemma 2.3.1 shows that a smooth curve $c : I \to M$ is a geodesic if and only if it satisfies the following system of second-order ordinary differential equations on its components:

$$(c^k)''(t) + (c^i)'(t)(c^j)'(t)\Gamma^k_{ij}(c(t)) = 0. \tag{2.7}$$

Remark 2.3.1. For every $x \in M$ and every $\xi_x \in T_xM$ there exists a unique open maximal interval I containing 0 and a geodesic $c : I \to M$ such that $c(0) = x$ and $c'(0) = \xi_x$. This can be deduced from the existence and uniqueness theorem of ODEs, using the local equation (2.7). For details see [13, Theorem 4.10].

Lemma 2.3.2. Two linear connections $\hat{\nabla}$ and ∇ on TM have the same geodesics if and only if their difference tensor $A(\xi, \eta) = \hat{\nabla}_\xi \eta - \nabla_\xi \eta$ (see Lemma 2.2.2) is skew-symmetric.

Proof. First of all note that A is skew-symmetric if and only if $A(\xi, \xi) = 0$ for all $\xi \in \mathfrak{X}(M)$. If A is skew-symmetric, then obviously $A(\xi, \xi) = -A(\xi, \xi)$ and thus $A(\xi, \xi) = 0$. On the other hand if $A(\xi, \xi) = 0$, then

$$0 = A(\xi + \eta, \xi + \eta) = A(\xi, \xi) + A(\xi, \eta) + A(\eta, \xi) + A(\eta, \eta) = A(\xi, \eta) + A(\eta, \xi),$$

for all $\xi, \eta \in \mathfrak{X}(M)$ and thus A is skew-symmetric.

Now let $c : I \to M$ be a geodesic for ∇ and $\hat{\nabla}$. Then $A(c', c') = 0$ and since every tangent vector occurs as the derivative of some geodesic, we obtain that A is skew-symmetric. Conversely let A be skew-symmetric and $c : I \to M$ a geodesic for ∇. Then $\hat{\nabla}_{c'}c' = A(c', c') + \nabla_{c'}c' = 0$ and thus c is a geodesic for $\hat{\nabla}$. We can argue in exactly the same way to show that every geodesic for $\hat{\nabla}$ is a geodesic for ∇. □

Note that the change from a linear connection ∇ to a torsion-free connection $\hat{\nabla}$ as stated after Definition 2.2.4 doesn't change the geodesics: Their difference tensor is given by $A(\xi, \eta) = -\frac{1}{2}\mathrm{Tor}^\nabla(\xi, \eta)$, which is skew-symmetric.

Lemma 2.3.3. Consider two linear connection ∇ and $\hat{\nabla}$ on TM. Then we have the following equivalence:

$$\hat{\nabla} = \nabla \Leftrightarrow \mathrm{Tor}^\nabla = \mathrm{Tor}^{\hat{\nabla}} \text{ and they have the same geodesics.}$$

Proof. ($\Rightarrow$) is trivial. Conversely, by Lemma 2.2.2, the difference tensor $A = \hat{\nabla} - \nabla$ is symmetric and by Lemma 2.3.2 it is skew-symmetric. Thus $A = 0$ and $\hat{\nabla} = \nabla$. □

2.4 The Homogeneous Model for Affine Connections

In this section we consider the homogeneous model for linear connections on the tangent bundle. Recall from section 2.2, that a linear connection on the tangent bundle is equivalent to a principal connection form on the linear frame bundle. We will understand in the end of this section, why the space we consider is exactly the homogeneous model. Furthermore, we will find an invariant geometric structure on the considered homogeneous space. The theory presented in this section can be found in [9, Section 1.3.5].

Consider affine $n-$space A^n, which as a set is given by $A^n = \mathbb{R}^n$. An affine motion is a map of the form $v \mapsto Av + b$ for $A \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The group of all affine motions is denoted by $\mathrm{Aff}(n, \mathbb{R})$. If we view A^n as the affine hyperplane $x_1 = 1$ in $\mathbb{R}^{n+1}$, then $\mathrm{Aff}(n, \mathbb{R})$ is the subgroup of $GL(n+1, \mathbb{R})$, which maps this affine hyperplane to itself. Thus

$$\mathrm{Aff}(n, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & A \end{pmatrix}, A \in GL(n, \mathbb{R}), b \in \mathbb{R}^n \right\}.$$

The group of affine motions acts on A^n and this action is transitive. Let $v \in \mathbb{R}^n$ with $v \neq 0$. Then complete $\{v\}$ to a basis $B = \{v, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$. Let $A \in GL(n, \mathbb{R})$ be the matrix whose columns are the elements of B. Then $Ae_1 = v$ and thus

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 1 \\ e_1 \end{pmatrix} = \begin{pmatrix} 1 \\ v \end{pmatrix}.$$

The isotropy subgroup of the first unit vector of $\mathbb{R}^{n+1}$ is given by those elements in $\mathrm{Aff}(n, \mathbb{R})$ with $b = 0$ and is therefore isomorphic to $GL(n, \mathbb{R})$. Thus we obtain a description of A^n as a homogeneous space: $A^n = \mathrm{Aff}(n, \mathbb{R})/GL(n, \mathbb{R})$.

The natural projection $p : \mathrm{Aff}(n, \mathbb{R}) \to A^n$ is a principal bundle with structure group $GL(n, \mathbb{R})$. This follows from a general fact of the quotient projection $p : G \to G/H$: Since p is a surjective submersion, there exist smooth local sections and any such local section gives rise to a principal bundle chart, see [9, Section 1.2.6].

Now consider the Lie algebra of $\mathrm{Aff}(n, \mathbb{R})$. It is given by

$$\mathfrak{a}(n, \mathbb{R}) = \left\{ \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix}, B \in \mathfrak{gl}(n, \mathbb{R}), X \in \mathbb{R}^n \right\}$$

and therefore as vector spaces we have $\mathfrak{a}(n, \mathbb{R}) \cong \mathbb{R}^n \oplus \mathfrak{gl}(n, \mathbb{R})$. The restriction of the adjoint action $\mathrm{Ad} : \mathrm{Aff}(n, \mathbb{R}) \to GL(\mathfrak{a}(n, \mathbb{R}))$ to $GL(n, \mathbb{R})$ is given by the direct sum of the standard action on $\mathbb{R}^n$ and the adjoint action on $\mathfrak{gl}(n, \mathbb{R})$. Indeed, the adjoint action of $\mathrm{Aff}(n, \mathbb{R})$ is given by matrix conjugation and thus

$$\mathrm{Ad}(\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}) \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ AX & ABA^{-1} \end{pmatrix}.$$

Furthermore, the splitting of $\mathfrak{a}(n, \mathbb{R})$ is invariant under the action of $GL(n, \mathbb{R})$.

Recall that on any Lie group G with Lie algebra $\mathfrak{g}$, we have the Maurer-Cartan form, which gives a left-trivialisation of the tangent bundle. The Maurer-Cartan form is the one-form $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$ defined by $\omega^{MC}(g)(\xi) = T_g\lambda_{g^{-1}}(\xi)$, for $g \in G, \xi \in T_gG$ and $\lambda_g : G \to G$ the left multiplication with g. It has the following properties:

1. $\omega^{MC}(L_X) = X$, for L_X the left-invariant vector field generated by $X \in \mathfrak{g}$,
2. $(\lambda_g)^*\omega^{MC} = \omega^{MC}$,
3. $(\rho^g)^*\omega^{MC} = \mathrm{Ad}(g^{-1}) \circ \omega^{MC}$, for $\rho^g : G \to G$ the multiplication with g from the right,
4. $\omega^{MC}(g) : T_gG \to \mathfrak{g}$ is a linear isomorphism for all $g \in G$ and
5. the Maurer-Cartan equation
$$d\omega^{MC}(\xi, \eta) + [\omega^{MC}(\xi), \omega^{MC}(\eta)] = 0,$$
for all $\xi, \eta \in \mathfrak{X}(M)$.

See [9, Section 1.2.4] for more details and the proof of these properties.

Furthermore, given a Lie group G and a Lie subgroup $H \subset G$, there is the notion of a Cartan geometry of type (G, H). It consists of an $H-$principal bundle $p : \mathcal{P} \to M$ and a $\mathfrak{g}-$valued one-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called Cartan connection. The Cartan connection is required to satisfy the following properties:

$$\begin{aligned} (\rho^h)^*\omega &= \mathrm{Ad}(h^{-1}) \circ \omega \text{ for all } h \in H \\ \omega(\zeta_X(u)) &= X \quad \forall X \in \mathfrak{h} \\ \omega(u) : T_u\mathcal{P} &\to \mathfrak{g} \text{ is a linear isomorphism for all } u \in \mathcal{P}. \end{aligned} \tag{2.8}$$

These are the properties of the Maurer-Cartan form, which can be generalized to a curved setting. The homogeneous model for a Cartan geometry of type (G, H) is the homogeneous bundle $p : G \to G/H$ endowed with the Maurer-Cartan form $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$. See [9, Chapter 1] for the general theory on Cartan geometries.

Now for the group of affine motions, we may split the Maurer-Cartan form $\omega^{MC} \in \Omega^1(\mathrm{Aff}(n, \mathbb{R}), \mathfrak{a}(n, \mathbb{R}))$ with respect to the decomposition of $\mathfrak{a}(n, \mathbb{R})$, hence $\omega^{MC} = \theta + \gamma$, for θ a one-form with values in $\mathbb{R}^n$ and γ a one-form with values in $\mathfrak{gl}(n, \mathbb{R})$. Since ω^{MC} is equivariant with respect to the adjoint action of $GL(n, \mathbb{R})$ (Property 3) and the splitting is invariant under this action, we see that θ and γ are equivariant as well.

We will now identify $p : \mathrm{Aff}(n, \mathbb{R}) \to A^n$ with the linear frame bundle $\pi : P^1A^n \to A^n$. Therefore we need a smooth fiber bundle isomorphism, which is compatible with the principal right action, hence a commutative diagram of the form:

$$\begin{tikzcd} \mathrm{Aff}(n, \mathbb{R}) \arrow[r, "\phi"] \arrow[d, "p"'] \arrow[dr, phantom, "\#"] & P^1A^n \arrow[d, "\pi"] \\ A^n \arrow[r, "\mathrm{id}"'] & A^n \end{tikzcd},$$

such that $\phi(A \circ g) = \phi(A) \circ g$ for $A \in \mathrm{Aff}(n, \mathbb{R})$ and $g \in GL(n, \mathbb{R})$. Denote by $V\mathrm{Aff}(n, \mathbb{R})$ the vertical bundle of the principal bundle $p : \mathrm{Aff}(n, \mathbb{R}) \to A^n$. Hence $V_A\mathrm{Aff}(n, \mathbb{R}) :=$

$\ker(T_A p)$. The vertical bundle is trivialized by the fundamental vector fields, hence for every $\xi \in V_A \mathrm{Aff}(n, \mathbb{R})$, there exists a unique $X \in \mathfrak{gl}(n, \mathbb{R})$, such that $\xi = \zeta_X(A)$. Now the fundamental vector fields for the right multiplication are exactly the left-invariant vector fields, hence $L_X = \zeta_X$.

Claim 1. The vertical subspace $V_A \mathrm{Aff}(n, \mathbb{R})$ equals the kernel of $\theta(A)$ for every $A \in \mathrm{Aff}(n, \mathbb{R})$.

Proof. Let $\xi \in V_A \mathrm{Aff}(n, \mathbb{R})$, then $\xi = \zeta_X(A)$ for some $X \in \mathfrak{gl}(n, \mathbb{R})$. By property 1 of the Maurer-Cartan form, we obtain that

$$\omega^{MC}(A)(\xi) = \omega^{MC}(A)(\zeta_X(A)) = \omega^{MC}(A)(L_X(A)) = X \in \mathfrak{gl}(n, \mathbb{R}).$$

Therefore $\theta(A)(\xi) = 0$ and ξ lies in the kernel of $\theta(A)$. By property 4, the map $\theta(A)$ has to be surjective and thus $\ker(\theta(A))$ has dimension n^2. Since p is a surjective submersion, we have that $T_A \mathrm{Aff}(n, \mathbb{R})/V_A \mathrm{Aff}(n, \mathbb{R}) \cong T_{p(A)} A^n$ and therefore $V_A \mathrm{Aff}(n, \mathbb{R})$ has dimension n^2 as well. Thus we see that these spaces are isomorphic. $\square$

This also shows, that $\theta(A)$ induces an injective linear map

$$T_A \mathrm{Aff}(n, \mathbb{R})/V_A \mathrm{Aff}(n, \mathbb{R}) \cong T_{p(A)} A^n \to \mathbb{R}^n,$$

which has to be an isomorphism, since both spaces have the same dimension. Thus we can now define the fiber bundle map which induces the identification of $\mathrm{Aff}(n, \mathbb{R})$ with $P^1 A^n$:

$$\phi(A) = \theta(A)^{-1} : \mathbb{R}^n \to T_{p(A)} A^n.$$

Obviously this covers the identity map and by property 3 of the Maurer-Cartan form it is compatible with the principal right action. Thus we obtain an isomorphism of principal bundles.

On the linear frame bundle there exists a canonical form $\theta^s \in \Omega^1(P^1 M, \mathbb{R}^n)$, called the soldering form. It is defined by $\theta^s(u)(\xi) := u^{-1}(T_u \pi \cdot \xi)$, for $u \in P^1 M$ and $\xi \in T_u P^1 M$ and satisfies $(r^g)^* \theta^s = g^{-1} \circ \theta^s$ and $\theta^s(\eta) = 0$ for every η a vertical vector field, i.e. a vector field with values in the vertical bundle. Now via the morphism ϕ of principal bundles, the soldering form is pulled back to the $\mathbb{R}^n-$ component of the Maurer-Cartan form. Indeed, we have

$$\begin{aligned} \phi^* \theta^s(A)(\xi) &= \theta^s(\phi(A))(T_A \phi \cdot \xi) \\ &= \phi(A)^{-1}(T_{\phi(A)} \pi \cdot T_A \phi \cdot \xi) \\ &= \theta(A)(T_A \pi \circ \phi \cdot \xi) \\ &= \theta(A)(T_A \pi \cdot \xi) \end{aligned} \tag{2.9}$$

The one-form γ gives rise to a principal connection form on $P^1 A^n$. By properties 1 and 2 of the Maurer-Cartan form, it satisfies the defining properties of a principal connection form (see Section 2.2). This connection form induces the canonical linear connection on A^n.

We have seen, that the data defining a principal connection form on P^1A^n (and hence a linear connection on TA^n) is given by the principal bundle $\mathrm{Aff}(n,\mathbb{R}) \to A^n$ with the Maurer-Cartan form. Arguing similarly as before, one can show, that for a smooth manifold M, any $GL(n,\mathbb{R})-$principal bundle $p : \mathcal{P} \to M$ together with a one-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{a}(n,\mathbb{R}))$ satisfying the properties 1, 3 and 4 of the Maurer-Cartan form, is isomorphic to the linear frame bundle $\pi : P^1M \to M$ and induces a principal connection form via the splitting of $\mathfrak{a}(n,\mathbb{R})$. Such a principal bundle is refered to as an affine structure. Thus an affine structure induces a principal connection form on P^1M and conversely any principal connection form γ on P^1M induces an affine structure, namely the $GL(n,\mathbb{R})-$principal bundle $p : P^1M \to M$ with $\omega := \theta^s + \gamma$, where θ^s is the soldering form.

Now we can understand, why $\mathrm{Aff}(n,\mathbb{R}) \to A^n$ is considered the homogeneous model for linear connections. The Cartan geometry of type $(\mathrm{Aff}(n,\mathbb{R}), GL(n,\mathbb{R}))$ on an $n-$ dimensional manifold M is an affine structure and therefore equivalent to a principal connection form on P^1M (and hence to a linear connection on the tangent bundle). The homogeneous model for the Cartan geometry is exactly $p : \mathrm{Aff}(n,\mathbb{R}) \to \mathrm{Aff}(n,\mathbb{R})/GL(n,\mathbb{R})$.

Affine motions are maps that preserve the geometric structure of A^n. We will see in the next Lemma, that in order to preserve this structure, for a diffeomorphism it suffices to preserve the flat connection.

Lemma 2.4.1. The affine motions on $\mathbb{R}^n$ are exactly those diffeomorphisms of $\mathbb{R}^n$, which preserve the flat connection.

Proof. Consider an affine transformation $f(x) = Ax + b$ for $A \in GL(n,\mathbb{R})$ and $b \in \mathbb{R}^n$. Then we have to show that $f^*\nabla = \nabla$ hence $f^*\nabla_{f^*\xi} f^*\eta = f^*(\nabla_\xi \eta) = \nabla_{f^*\xi} f^*\eta$ for smooth functions $\xi, \eta : \mathbb{R}^n \to \mathbb{R}^n$. Note that $(f^*\xi)(x) = A^{-1}\xi(f(x))$ and $\nabla_\xi \eta = D\xi \circ \eta$ thus

$$\begin{aligned} \nabla_{f^*\xi} f^*\eta &= \nabla_{A^{-1}\circ\xi\circ f} A^{-1} \circ \eta \circ f = D(A^{-1} \circ \xi \circ f) \circ A^{-1} \circ \eta \circ f \\ &= A^{-1} \circ (D\xi \circ f) \circ (\eta \circ f) = A^{-1}(\nabla_\xi \eta) \circ f = f^*(\nabla_\xi \eta) \end{aligned}$$

Conversely given a diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$, there is a unique lift $\tilde{f} : \mathrm{Aff}(n,\mathbb{R}) \to \mathrm{Aff}(n,\mathbb{R})$ satisfying $f^*\theta = \theta$, for θ the $\mathbb{R}^n-$component of the Maurer-Cartan form. This is due to the fact, that any diffeomorphism on an arbitrary manifold M can be uniquely lifted to the linear frame bundle such that this lift preserves the soldering form. Indeed, the lift is given by $\tilde{f}(u) = T_{\pi(u)} f \circ u \in T_{f(\pi(u))} P^1M$, for $u \in P^1M$. Via the identification $\phi : \mathrm{Aff}(n,\mathbb{R}) \to P^1A^n$, we obtain a unique lift $\tilde{f} : \mathrm{Aff}(n,\mathbb{R}) \to \mathrm{Aff}(n,\mathbb{R})$ and since ϕ pulls back the soldering form to θ (see (2.9)), $\tilde{f}$ preserves θ.

Since the flat connection ∇ on $\mathbb{R}^n$ is induced by the $\mathfrak{gl}(n,\mathbb{R})-$component γ of the Maurer-Cartan form, we see that f preserves ∇ if and only if $\tilde{f}$ preserves γ. Thus starting with a diffeomorphism f on $\mathbb{R}^n$ that preserves ∇, the unique lift $\tilde{f}$ preserves the

Maurer-Cartan form, i.e. $\tilde{f}^*\omega = \omega$. By [9, Theorem 1.2.4], any two smooth maps $f_1, f_2 : \mathrm{Aff}(n,\mathbb{R}) \to \mathrm{Aff}(n,\mathbb{R})$ with $f_1^*\omega = f_2^*\omega$ satisfy $f_2 = G \circ f_1$ for a unique $G \in \mathrm{Aff}(n,\mathbb{R})$. Using this on $\tilde{f}$ and id, we obtain that $\tilde{f}$ is given by left multiplication with an element $G \in \mathrm{Aff}(n,\mathbb{R})$. If $G = \left(\begin{smallmatrix} 1 & 0 \\ b & \tilde{G} \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} 1 & 0 \\ X & \tilde{A} \end{smallmatrix}\right)$, then

$$f(X) = p(\tilde{f}(A)) = p(GA) = \tilde{G}X + b$$

and thus f is an affine motion. □

We will now show, that for a diffeomorphism on an arbitrary manifold M, preserving a torsion-free linear connection is equivalent to preserving the geodesics, viewed as parametrised curves. Thus rephrasing Lemma 2.4.1, we obtain that a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the geometric structure of A^n if and only if it preserves the straight lines (hence the geodesics of the flat connection).

Lemma 2.4.2. A diffeomorphism $f : M \to M$ preserves a torsion-free connection if and only if it preserves the geodesics, i.e. $f^*\nabla = \nabla$ for ∇ torsion-free $\Leftrightarrow$ c is a geodesic for ∇ if and only if $f \circ c$ is a geodesic for ∇.

Proof. First of all note that $f \circ c$ is a geodesic for ∇ if and only if c is a geodesic for $f^*\nabla$. This is due to the following equation:

$$f^*(\nabla_{(f\circ c)'}(f \circ c)') = f^*\nabla_{c'}c'.$$

Thus we can rephrase the statement of the Lemma: $f^*\nabla = \nabla$ for ∇ torsion-free $\Leftrightarrow$ ∇ and $f^*\nabla$ have the same geodesics. Furthermore since ∇ is torsion-free, so is $f^*\nabla$. This follows from the following equation:

$$\mathrm{Tor}^{f^*\nabla}(\xi, \eta) = f^*\mathrm{Tor}^{\nabla}((f^{-1})^*\xi, (f^{-1})^*\eta).$$

Thus by Lemma 2.3.3, we obtain the result. □

On the homogeneous space $A^n = \mathrm{Aff}(n,\mathbb{R})/GL(n,\mathbb{R})$, we have found a geometric structure, namely the family of straight lines, which is invariant under the action of $\mathrm{Aff}(n,\mathbb{R})$ and the elements of $\mathrm{Aff}(n,\mathbb{R})$ are exactly those diffeomorphisms, which preserve this structure.

Chapter 3

Projective Structures

3.1 Projective Equivalence

The concept of projective equivalence is motivated by weakening Lemma 2.3.3. This Lemma states that two affine connections are equal if and only if they have the same torsion and induce the same geodesics. Geodesics are parametrised curves, thus changing the parametrisation of a geodesic will in general not lead to another geodesic. Now, instead of considering parametrised geodesics, we look at the family of unparametrised curves, which are defined by geodesics. This means that we have to study all affine connections, which induce the same geodesics, regardless of the parametrisation. Any two such affine connections are called projectively equivalent. Thus, on the basis of Lemma 2.3.3, two affine connections are projectively equivalent if and only if they have the same torsion and induce the same geodesics up to reparametrisation. Furthermore, we get a similar result to Lemma 2.4.2: a diffeomorphism preserves projective equivalence if and only if it preserves the geodesics, up to reparametrisation.

In this chapter we give the definition of a projective structure on an arbitrary manifold M as well as basic properties. This can be found in [9, Section 4.1.5-4.1.6].

As stated after Lemma 2.3.2, changing an affine connection ∇ to a torsion-free connection $\hat{\nabla}$ doesn't change the geodesics. When working with projective structures we thus restrict to torsion-free affine connections.

Definition 3.1.1. Two torsion-free affine connections ∇ and $\hat{\nabla}$ on M are called projectively equivalent if they have the same geodesics up to reparametrisation.

This defines an equivalence relation on the set of affine connections on M and a choice of an equivalence class is called a projective structure on M. We indicate a projective structure on M by $(M, [\nabla])$. The following Lemma gives an easier approach to Definition 3.1.1:

Lemma 3.1.1. A curve $c : I \to M$ is a geodesic of ∇ up to reparametrisation if and only if $\nabla_{c'} c' = ac'$, for some smooth function $a : I \to \mathbb{R}$.

Proof. A curve c is a geodesic of ∇ up to reparametrisation if there exists a reparametrisation ϕ such that $c \circ \phi$ is a geodesic of ∇. By (2.5), we have the following equation:

$$\nabla_{(c\circ\phi)'}(c \circ \phi)' = \phi''(c' \circ \phi) + (\phi')^2(\nabla_{c'}c') \circ \phi.$$

Therefore, if $c \circ \phi$ is a geodesic for ∇, we obtain:

$$(\nabla_{c'}c') \circ \phi = -\frac{\phi''}{(\phi')^2}(c' \circ \phi)$$

and hence $\nabla_{c'}c'$ is proportional to c'.

Conversely, suppose that a smooth curve c satisfies $\nabla_{c'}c' = ac'$ for a smooth function $a : I \to \mathbb{R}$. We have to show that there exists a reparametrisation ϕ such that $c \circ \phi$ is a geodesic for ∇. Using formula (2.5), we are looking for solutions to

$$(\phi')^2(a \circ \phi)(c' \circ \phi) + \phi''(c' \circ \phi) = 0,$$

for given c and a. This is a second-order ordinary differential equation and thus by the existence and uniqueness theorem of ODEs, there exists a unique solution. □

Lemma 3.1.2. [9, Corollary 4.1.6] Consider two manifolds M and $\tilde{M}$ with projective structures $(M, [\nabla])$ respectively $(\tilde{M}, [\tilde{\nabla}])$. Let $f : M \to \tilde{M}$ be a local diffeomorphism. Then $f^*\tilde{\nabla}$ is projectively equivalent to ∇ if and only if for every geodesic $c : I \to M$ of ∇ the curve $f \circ c$ is a geodesic up to reparametrisation for $\tilde{\nabla}$.

A local diffeomorphism satisfying this property is called a morphism of projective structures or a projective (local) diffeomorphism.

Proof. We first show that c is a geodesic up to reparametrisation for $f^*\tilde{\nabla}$ if and only if $f \circ c$ is a geodesic up to reparametrisation for $\tilde{\nabla}$. This is established via the equation

$$f^*(\tilde{\nabla}_{(f\circ c)'}(f \circ c)') = f^*\tilde{\nabla}_{c'}c'.$$

Suppose $f^*\tilde{\nabla}$ is projectively equivalent to ∇ and let $c : I \to M$ be a geodesic for ∇. Then c is a geodesic for $f^*\tilde{\nabla}$ up to reparametrisation, hence $f \circ c$ is a geodesic up to reparametrisation for $\tilde{\nabla}$. Conversely if c is a geodesic for ∇, then $f \circ c$ is a geodesic for $\tilde{\nabla}$ up to reparametrisation and thus c is a geodesic up to reparametrisation for $f^*\tilde{\nabla}$. This means that $f^*\tilde{\nabla}$ is projectively equivalent to ∇. □

The following theorem gives an algebraic criterion for projective equivalence, which will be used as the definition throughout the upcoming chapters.

Theorem 3.1.1. [9, Prop. 4.1.6.] Let ∇ and $\hat{\nabla}$ be two torsion-free affine connections on M. Then ∇ and $\hat{\nabla}$ are projectively equivalent if and only if there exists a one-form $\Upsilon \in \Omega^1(M)$ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\eta)\xi + \Upsilon(\xi)\eta, \tag{3.1}$$

for all vector fields $\xi, \eta \in \mathfrak{X}(M)$.

To abbreviate notation, we write $\hat{\nabla} = \nabla + \Upsilon$ to express property (3.1).

Proof. We prove the statement of the Theorem using Lemma 3.1.1. Suppose that ∇ and $\hat{\nabla}$ satisfy property (3.1). Let $c : I \to M$ be a geodesic for ∇, then along c we get:

$$\hat{\nabla}_{c'}c' = \nabla_{c'}c' + \Upsilon(c')c' + c'\Upsilon(c') = 2\Upsilon(c')c'$$

and thus c is a geodesic up to reparametrisation for $\hat{\nabla}$. This shows that $\hat{\nabla}$ and ∇ have the same geodesics up to reparametrisation.

Conversely, suppose that $\hat{\nabla}$ and ∇ have the same geodesics up to reparametrisation. We claim that there exists a smooth function $b(\xi) : M \to \mathbb{R}$ such that

$$A(\xi, \xi) = b(\xi)\xi, \tag{3.2}$$

for every vector field $\xi \in \mathfrak{X}(M)$. Consider a geodesic c for ∇ satisfying $c(0) = x$ and $c'(0) = \xi(x)$. Then $\hat{\nabla}_{c'}c' = ac'$, since c is a geodesic up to reparametrisation of $\hat{\nabla}$ and we obtain $A_x(\xi(x), \xi(x)) = a(0)c'(0) = a(0)\xi(x)$, where a depends smoothly on $\xi(x)$. Define $b(\xi(x)) := a(0)$, then $A(\xi, \xi) = b(\xi)\xi$ for $b(\xi) : M \to \mathbb{R}$. In the following we will show that b is a $1-$form.

Claim 2. The function $b(\xi) : M \to \mathbb{R}$ as defined in (3.2), satisfies

$$b(t\xi) = tb(\xi) \text{ and } b(\xi + \eta) = b(\xi) + b(\eta),$$

for $t \in \mathbb{R}$ and $\xi, \eta \in \mathfrak{X}(M)$. Furthermore, considered as a function $b : \mathfrak{X}(M) \to \mathcal{C}^\infty(M, \mathbb{R})$, it is $\mathcal{C}^\infty(M, \mathbb{R})-$ linear and hence defines a $1-$form.

For $t \in \mathbb{R}$ we have

$$b(t\xi)t\xi = A(t\xi, t\xi) = t^2 A(\xi, \xi) = tb(\xi)t\xi$$

and thus $b(t\xi) = tb(\xi)$. Similarly it follows, that b is $\mathcal{C}^\infty(M, \mathbb{R})-$ linear.

Since ∇ and $\hat{\nabla}$ have the same torsion, their difference tensor $A(\xi, \eta) = \hat{\nabla}_\xi \eta - \nabla_\xi \eta$ is symmetric, compare Lemma 2.2.2. We use symmetry and bilinearity of A as well as equation (3.2) to show that b is additive:

$$\begin{aligned} b(\xi + \eta)(\xi + \eta) &= A(\xi + \eta, \xi + \eta) \\ &= A(\xi, \xi) + 2A(\xi, \eta) + A(\eta, \eta) \\ &= b(\xi)\xi + 2A(\xi, \eta) + b(\eta)\eta. \end{aligned} \tag{3.3}$$

Now manipulating equation (3.3), we obtain:

$$2A(\xi, \eta) = (b(\xi + \eta) - b(\xi))\xi + (b(\xi + \eta) - b(\eta))\eta. \tag{3.4}$$

Since A is linear, we can use $A(\xi, t\eta) = tA(\xi, \eta)$ to rewrite equation (3.4) and by assuming that ξ and η are linearly independent as well as that $t \neq 0$, we get:

$$b(\xi + t\eta) - tb(\eta) = b(\xi + \eta) - b(\eta).$$

Thus letting $t \to 0$ shows that b is additive. This proves the claim.

Continuing with the proof of the Theorem, we use additivity of b in equation (3.4) to obtain:

$$2A(\xi, \eta) = b(\eta)\xi + b(\xi)\eta.$$

Thus there exists a $1-$form Υ, namely $\Upsilon = \frac{1}{2}b$, such that $\hat{\nabla} = \nabla + \Upsilon$. □

3.2 Model Spaces for Projective Structures

In this section we discuss model spaces for projective structures. These spaces also act as basic examples for the theory developed in the upcoming chapters, which is discussed in chapter 6.

As a model space for projective structures we consider $n-$dimensional real projective space. We find two projectively equivalent linear connections on real projective space. It is however easier to approach the oriented model, which is given by the $n-$dimensional sphere S^n. The sphere can be viewed as a homogeneous space of $G = SL(n+1, \mathbb{R})$ and we find a geometric structure, which is invariant under the action of G. Similarly to the case of affine structures (see section 2.4), we will see in the following chapters, that elements of G are exactly the diffeomorphisms which preserve this structure. Indeed, S^n is the homogeneous model for the Cartan geometry, which describes oriented projective structures. This is why S^n is referred to as the homogeneous model for oriented projective structures.

The theory of this chapter can be found in [9, Section 1.1.3].

Definition 3.2.1. Real projective space $\mathbb{R}P^n$ is defined as the space of all lines through 0 in $\mathbb{R}^{n+1}$.

Consider the following equivalence relation on $C := \mathbb{R}^{n+1} \backslash 0$: $x \sim y$ if and only if there exists a $t \in \mathbb{R}$ such that $y = tx$. Then real projective space can be equivalently defined by $\mathbb{R}P^n = C/\sim$. The equivalence class of $x \in \mathbb{R}^{n+1}$ is denoted by $[x]$ or $[x^1 : \cdots : x^{n+1}]$, if we work with the components of x.

The action of $\mathbb{R}^* = \mathbb{R} \backslash 0$ on $\mathbb{R}^{n+1}$ given by multiplication, restricts to a free action on C. Furthermore, the natural projection $\pi : C \to \mathbb{R}P^n$ is a surjective submersion and the orbits of the action by $\mathbb{R}^*$ are exactly the fibers of π. Thus by [14, Lemma 18.3], $\pi : C \to \mathbb{R}P^n$ is a principal bundle with structure group $\mathbb{R}^*$.

As a model space for projective structures, we consider $\mathbb{R}P^n$ with the equivalence class of the connection induced from the flat connection on $\mathbb{R}^n$. We will show, that the geodesics of $\mathbb{R}P^n$ induced by this connection are exactly the projective lines.

Consider the following chart for real projective space: $U_i = \{[x^1 : \dots : x^{n+1}] : x_i \neq 0\}$ with diffeomorphisms $u_i : U_i \to F_i = \{x \in \mathbb{R}^{n+1} : x_i = 1\}$,

$$u_i([x^1 : \dots : x^{n+1}]) = (\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}).$$

Note that F_i is diffeomorphic to $\mathbb{R}^n$ and that the inverse map of u_i is given by

$$u_i^{-1}(x_1, \dots, 1, \dots, x_{n+1}) = [x_1 : \dots : x_{n+1}].$$

Now on F_i we have the flat connection induced by the diffeomorphism $F_i \cong \mathbb{R}^n$. Via u_i we can locally pull back this connection to a flat linear connection on $\mathbb{R}P^n$. Geodesics on F_i are given by affine lines. Every affine line in F_i spans a 2−dimensional linear subspace of $\mathbb{R}^{n+1}$. Conversely, every 2−dimensional linear subspace defines an affine line by intersecting with F_i. By projecting the 2−dimensional subspace defined by an affine line to real projective space, we obtain a projective line. Thus the flat connection on $\mathbb{R}^n$ induces a linear connection on $\mathbb{R}P^n$, whose geodesics are exactly the projective lines.

Another linear connection on $\mathbb{R}P^n$, which has the projective lines as geodesics, is induced by the Levi-Civita connection on S^n. Identifying antipodal points on S^n generates a free group action of $\mathbb{Z}_2$ on S^n by isometries. The orbit space of this action is then given by $\mathbb{R}P^n$ and thus $p : S^n \to \mathbb{R}P^n$ is a two-fold covering. By projecting the great circles, we obtain the geodesics of the induced linear connection on $\mathbb{R}P^n$. A great circle is given by the intersection of a 2−dimensional linear subspace with the sphere. Identifying a pair of antipodal points of a great circle with the line through these points, this line lies in a 2−dimensional linear subspace of $\mathbb{R}^{n+1}$. Thus the projection of a great circle gives the set of all lines in a 2−dimensional subspace and hence a projective line. Thus the Levi-Civita connection of S^n induces a linear connection on $\mathbb{R}P^n$, whose geodesics are exactly the projective lines.

We have seen, that the flat connection on $\mathbb{R}^n$ and the (non-flat) Levi-Civita connection on S^n, induce projectively equivalent connections on $\mathbb{R}P^n$. This is an instance of the Beltrami-theorem, which states that the Levi-Civita connection of a Riemannian metric is projectively flat, i.e. projectively equivalent to a flat connection, if and only if the metric has constant sectional curvature.

Consider the space of rays in $\mathbb{R}^{n+1}$. This can be defined by the equivalence relation $x \sim y$ if and only if there exists a $t > 0$ such that $y = tx$ on $C = \mathbb{R}^{n+1} \backslash 0$. We identify the space of rays with S^n: Every ray intersects S^n in exactly one point, namely the point of the ray with length equal to 1. Conversely, every element of S^n certainly defines a ray in $\mathbb{R}^{n+1}$. As a projective structure we can consider $(S^n, [\nabla])$, where ∇ is the Levi-Civita connection.

The natural projection $\pi : C \to S^n$ is a principal bundle with structure group $\mathbb{R}^+$: Arguing as in the case of real projective space, we see that π is a surjective submersion and the action of $\mathbb{R}^+$ restricts to a free action on C, such that the orbits of this action coincide with the fibers of π.

The standard action of $G = SL(n, \mathbb{R})$ on $\mathbb{R}^{n+1}$ induces a well-defined action on S^n: If $x \sim y$, then there exists a $t > 0$ such that $y = tx$. Hence $Ay = tAx$ for $A \in G$ and thus $Ay \sim Ax$.

The induced action on S^n is transitive. Let ℓ be a ray in $\mathbb{R}^{n+1}$ through 0 and choose $v \in \mathbb{R}^{n+1}$ with $[v] = \ell$. Complete v to a basis $B = \{v, v_1, \ldots, v_n\}$ of $\mathbb{R}^{n+1}$ and let A be the matrix whose columns are exactly the vectors of B. If necessary rescale v_n to obtain that $\det(A) = 1$. Now $A \in G$ and $Ae_1 = v$. Thus the action of G is transitive.

The stabilizer of the ray through the first basis vector is given by

$$P = \left\{ \begin{pmatrix} \det(A)^{-1} & \phi \\ 0 & A \end{pmatrix} : A \in GL(n, \mathbb{R}) \text{ and } \det(A) > 0, \phi \in \mathbb{R}^{n*} \right\}.$$

If a matrix C stabilizes the ray through the first basis vector, it has to have the form

$$C = \begin{pmatrix} a & \phi \\ 0 & A \end{pmatrix},$$

for $a \in \mathbb{R}$, $\phi \in \mathbb{R}^{n*}, A \in M_n(\mathbb{R})$ and $a > 0$. Since $C \in G$ and $\det(C) = a\det(A)$, we have that $a\det(A) = 1$. Thus $A \in GL(n, \mathbb{R}), a = \det(A)^{-1}$ and $a > 0$ if and only if $\det(A) > 0$. Therefore we obtain $S^n = G/P$.

To find the geometric structure which is invariant under the action of G, we consider the family of great circles on S^n. They are given by intersections of S^n with $2-$dimensional linear subspaces of $\mathbb{R}^{n+1}$. Obviously, G maps $2-$dimensional subspaces to $2-$dimensional subspaces and thus restricting to S^n maps great circles to great circles. As remarked above, we will see in section 4.3, that the converse is also true: every diffeomorphism of S^n, which maps great circles to great circles is given by the action of an element of G.

3.3 Formulae

We can now make use of abstract index notation, as introduced in section 2.1 and reformulate the transformation law connected to a projective structure on a manifold. A projective structure is given by an equivalence class of torsion-free affine connections on M, where two affine connections ∇ and $\hat{\nabla}$ are equivalent if there exists a $1-$form Υ such that

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi,$$

for $\eta, \xi \in \mathfrak{X}(M)$, compare Theorem 3.1.1.

Recall from section 2.1, that $\delta_i{}^j$ is the $\binom{1}{1}-$tensor field, which, viewed as an endomorphism, is the identity map. Thus in abstract index notation we have

$$\hat{\nabla}_i \eta^j = \nabla_i \eta^j + \Upsilon_i \eta^j + \Upsilon_k \eta^k \delta_i{}^j.$$

We can compute the transformation for 1−forms ω as well by using $(\nabla_\xi \omega)(\eta) = \xi \cdot \omega(\eta) - \omega(\nabla_\xi \eta)$, which expresses the connection on T^*M in terms of the connection on TM.

$$\begin{aligned}(\hat{\nabla}_\xi \omega)(\eta) &= \xi \cdot \omega(\eta) - \omega(\hat{\nabla}_\xi \eta) \\ &= \xi \cdot \omega(\eta) - \omega(\nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi) \\ &= (\nabla_\xi \omega)(\eta) - \Upsilon(\xi)\omega(\eta) - \Upsilon(\eta)\omega(\xi).\end{aligned}$$

Thus in abstract index notation the transformation is given by:

$$\hat{\nabla}_i \omega_j = \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i.$$

Now we have established the transformations for sections of the tangent respectively cotangent bundle. Of course this can be iterated to powers of these bundles which is stated in the next Lemma.

Lemma 3.3.1. By changing the connection in the projective class from ∇ to $\hat{\nabla} = \nabla + \Upsilon$, the induced connections on $\otimes^k TM$ respectively on $\otimes^k T^*M$ transform as

$$\hat{\nabla}_a \xi^{b_1 \cdots b_k} = \nabla_a \xi^{b_1 \cdots b_k} + k\Upsilon_a \xi^{b_1 \cdots b_k} + \sum_{i=1}^{k} \Upsilon_c \xi^{b_1 \cdots b_{i-1} c b_{i+1} \cdots b_k} \delta_a{}^{b_i},$$

respectively

$$\hat{\nabla}_a \eta_{b_1 \cdots b_k} = \nabla_a \eta_{b_1 \cdots b_k} - k\Upsilon_a \eta_{b_1 \cdots b_k} - \sum_{i=1}^{k} \Upsilon_{b_i} \eta_{b_1 \cdots b_{i-1} a b_{i+1} \cdots b_k}.$$

Combining these, we obtain the following transformation for the induced connection on $(\otimes^k T^*M) \otimes (\otimes^\ell TM)$:

$$\begin{aligned}\hat{\nabla}_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} &= \nabla_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} + (\ell - k)\Upsilon_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} \\ &\quad + \sum_{i=1}^{\ell} \Upsilon_d \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_{i-1} d c_{i+1} \cdots c_\ell} \delta_a{}^{c_i} - \sum_{j=1}^{k} \Upsilon_{b_i} \zeta_{b_1 \cdots b_{i-1} a b_{i+1} \cdots b_k}{}^{c_1 \cdots c_\ell}.\end{aligned}$$

Proof. We prove the first part of this Lemma using induction on k. The second part is proved analogously. The third part is obtained by combining these formulae.

Note that the induced connection on $\otimes^k TM, \otimes^k T^*M$ and $(\otimes^k T^*M) \otimes (\otimes^\ell TM)$ is given by iterating the definitions given in Remark 2.2.1.

The case $k = 1$ is just the definition of projective equivalence. Suppose that the formula is true for $k - 1$ and consider the tensor $\xi^{b_1 \cdots b_{k-1}} \otimes \eta^{b_k}$ on $\otimes^k TM$. Then we get:

$$\hat{\nabla}_a \xi^{b_1 \cdots b_{k-1}} \otimes \eta^{b_k} = (\hat{\nabla}_a \xi^{b_1 \cdots b_{k-1}})\eta^{b_k} + (\xi^{b_1 \cdots b_{k-1}})\hat{\nabla}_a \eta^{b_k}.$$

Now using the induction hypothesis we obtain:

$$
\begin{aligned}
&\hat{\nabla}_a \xi^{b_1\cdots b_{k-1}} \otimes \eta^{b_k} = \\
=&(\nabla_a \xi^{b_1\cdots b_{k-1}} + (k-1)\Upsilon_a \xi^{b_1\cdots b_{k-1}} + \sum_{i=1}^{k-1} \Upsilon_c \xi^{b_1\cdots b_{i-1} c b_{i+1}\cdots b_{k-1}} \delta_a{}^{b_i})\eta^{b_k} \\
&+ (\xi^{b_1\cdots b_{k-1}})(\nabla_a \eta^{b_k} + \Upsilon_a \eta^{b_k} + \Upsilon_{c_k} \eta^{c_k} \delta_a{}^{b_k}) \\
=&\nabla_a \xi^{b_1\cdots b_{k-1}} \otimes \eta^{b_k} + k\Upsilon_a \xi^{b_1\cdots b_{k-1}} \otimes \eta^{b_k} + \sum_{i=1}^{k} \Upsilon_c \xi^{b_1\cdots b_{i-1} c b_{i+1}\cdots b_{k-1}} \otimes \eta^{b_k} \delta_a{}^{b_i}.
\end{aligned}
$$

□

In the case of $k-$ forms respectively $n-$forms, due to skew symmetry, there are simpler formulae available for the second part of Lemma 3.3.1:

Lemma 3.3.2. Let $\eta_{b_1\cdots b_k} \in \Omega^k(M)$ be a $k-$form and $\nu_{b_1\cdots b_n} \in \Omega^n(M)$ an $n-$form. If we change the connection within the projective class from ∇ to $\hat{\nabla} = \nabla + \Upsilon$, then we obtain the following formulae:

$$\hat{\nabla}_a \eta_{b_1\cdots b_k} = \nabla_a \eta_{b_1\cdots b_k} - (k+1)\Upsilon_a \eta_{b_1\cdots b_k} + (k+1)\Upsilon_{[a} \eta_{b_1\ldots b_k]}$$

and

$$\hat{\nabla}_a \nu_{b_1\cdots b_n} = \nabla_a \nu_{b_1\cdots b_n} - (n+1)\Upsilon_a \nu_{b_1\cdots b_n}.$$

Proof. To prove the first statement, we are going to work with the definition of the antisymmetric part of a tensor, see section 2.1. This involves permutations and in order to make notation more convenient, we replace the index a by b_1. Thus we have to show that

$$(k+1)\Upsilon_{[b_1} \eta_{b_2\cdots b_{k+1}]} = \Upsilon_{b_1} \eta_{b_2\cdots b_{k+1}} - \sum_{i=2}^{k+1} \Upsilon_{b_i} \eta_{b_2\cdots b_{i-1} b_1 b_{i+1}\cdots b_{k+1}}.$$

Now starting with the left hand side and using the definition, we get:

$$
\begin{aligned}
(k+1)\Upsilon_{[b_1} \eta_{b_2\cdots b_{k+1}]} &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_{k+1}} \operatorname{sgn}(\sigma) \Upsilon_{b_{\sigma(1)}} \eta_{b_{\sigma(2)}\cdots b_{\sigma(k+1)}} \qquad (3.4) \\
&= \frac{1}{k!} \sum_{j=1}^{k+1} \sum_{\substack{\sigma \in \mathfrak{S}_{k+1}, \\ \sigma(1)=j}} \operatorname{sgn}(\sigma) \Upsilon_{b_j} \eta_{b_{\sigma(2)}\cdots b_{\sigma(k+1)}}
\end{aligned}
$$

In the last equation, we split a permutation $\sigma \in \mathfrak{S}_{k+1}$ into two parts, one containing Υ and the other containing η, by fixing the value of $\sigma(1) = j \in \{1, \ldots, k+1\}$. The part containing η is left with a bijective map $\tilde{\sigma} : \{2, \ldots, j, \ldots, k+1\} \to \{1, \ldots, j-1, j+1, \ldots, k+1\}$. Denote the set of all bijective maps $\{2, \ldots, j, \ldots, k+1\} \to \{1, \ldots, j-1, j+1, \ldots, k+1\}$ by $\mathcal{B}$.

Recall that given a permutation σ, an inversion is a pair $(\sigma(i), \sigma(j))$, satisfying $j < i$ and $\sigma(j) > \sigma(i)$. The number of inversions in a permutation is denoted by $\operatorname{inv}(\sigma)$ and we have that $\operatorname{sgn}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}$. The notion of inversion also makes sense for elements of $\mathcal{B}$ and therefore we get $\operatorname{sgn}(\sigma) = (-1)^{j-1}(-1)^{\operatorname{inv}(\tilde{\sigma})}$, since $\sigma(1) = j$ adds $(j-1)$ inversions in σ. Thus (3.4) can be rewritten as

$$\sum_{j=1}^{k+1}(-1)^{j-1}\Upsilon_{b_j}\frac{1}{k!}\sum_{\tilde{\sigma}\in\mathcal{B}}(-1)^{\operatorname{inv}(\tilde{\sigma})}\eta_{b_{\tilde{\sigma}(2)}\cdots b_{\tilde{\sigma}(k+1)}} \tag{3.5}$$

If $\sigma(1) = 1$, then in fact $\tilde{\sigma}$ is a permutation of $\{2, \ldots, k+1\}$. If $\sigma(1) \neq 1$, this is not true. In this case we relabel the elements of $\{2, \ldots, j, \ldots, k+1\}$ to obtain the set $\{1, \ldots, j-1, j+1, \ldots, k+1\}$, hence all elements $\leq j$ are lowered by 1. Note that this relabeling does not change the number of inversions of an element of $\mathcal{B}$. Thus, an element of $\mathcal{B}$ can be viewed as a bijection $\{1, \ldots, j-1, j+1, \ldots, k+1\} \to \{1, \ldots, j-1, j+1, \ldots, k+1\}$. Furthermore, such a bijection is a permutation of k elements and therefore we can continue with (3.5):

$$\begin{aligned}
&\sum_{j=1}^{k+1}(-1)^{j-1}\Upsilon_{b_j}\frac{1}{k!}\sum_{\tau\in\mathfrak{S}_k}(-1)^{\operatorname{inv}(\tau)}\eta_{b_{\tau(1)}\cdots b_{\tau(j-1)}b_{\tau(j+1)}\cdots b_{\tau(k+1)}} && (3.6)\\
=\;& \sum_{j=1}^{k+1}(-1)^{j-1}\Upsilon_{b_j}\eta_{b_1\cdots b_{j-1}b_{j+1}\cdots b_{k+1}}\\
=\;& \Upsilon_{b_1}\eta_{b_2\cdots b_{k+1}} - \sum_{j=2}^{k+1}\Upsilon_{b_j}\eta_{b_2\cdots b_{j-1}b_1 b_{j+1}\cdots b_{k+1}},
\end{aligned}$$

where we used skew-symmetry of η in both lines.

Applying this formula to $n-$forms, we obtain

$$\hat{\nabla}_a\nu_{b_1\cdots b_n} = \nabla_a\nu_{b_1\cdots b_n} - (n+1)\Upsilon_a\nu_{b_1\cdots b_n} + (n+1)\Upsilon_{[a}\nu_{b_1\cdots b_n]}.$$

The last part of this equation is antisymmetrized in $n+1$ indices and thus has to vanish on an $n-$dimensional manifold. □

We now compute how the curvatures of projectively equivalent connections are related.

Lemma 3.3.3. If we change the connection in the equivalence class from ∇ to $\hat{\nabla} = \nabla + \Upsilon$, the curvature transforms as

$$\widehat{R_{ij}{}^k{}_\ell} = R_{ij}{}^k{}_\ell + 2\Upsilon_\ell\Upsilon_{[j}\delta_{i]}{}^k + 2\delta_{[j}{}^k\nabla_{i]}\Upsilon_\ell + 2\delta_\ell{}^k\nabla_{[i}\Upsilon_{j]}.$$

Proof. By Definition 2.2.2 we have $\hat{R}(\xi,\eta)\zeta = \hat{\nabla}_\xi\hat{\nabla}_\eta\zeta - \hat{\nabla}_\eta\hat{\nabla}_\xi\zeta - \hat{\nabla}_{[\xi,\eta]}\zeta$. We now compute the first part of this expression:

$$\begin{aligned}
\hat{\nabla}_\xi\hat{\nabla}_\eta\zeta &= \hat{\nabla}_\xi(\nabla_\eta\zeta + \Upsilon(\eta)\zeta + \Upsilon(\zeta)\eta) \\
&= \hat{\nabla}_\xi\nabla_\eta\zeta + \Upsilon(\eta)\hat{\nabla}_\xi\zeta + (\xi\cdot\Upsilon(\eta))\zeta + \Upsilon(\zeta)\hat{\nabla}_\xi\eta + (\xi\cdot\Upsilon(\zeta))\eta \\
&= \nabla_\xi\nabla_\eta\zeta + \Upsilon(\nabla_\eta\zeta)\xi + \Upsilon(\xi)\nabla_\eta\zeta + \Upsilon(\eta)(\nabla_\xi\zeta + \Upsilon(\xi)\zeta + \Upsilon(\zeta)\xi) \\
&\quad + (\xi\cdot\Upsilon(\eta))\zeta + \Upsilon(\zeta)(\nabla_\xi\eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi) + (\xi\cdot\Upsilon(\zeta))\eta.
\end{aligned}$$

By exchanging ξ and η in the above equation, we obtain the second part in the expression for $\hat{R}$. The third part is given by $\hat{\nabla}_{[\xi,\eta]}\zeta = \nabla_{[\xi,\eta]}\zeta + \Upsilon([\xi,\eta])\zeta + \Upsilon(\zeta)[\xi,\eta]$. Now using $(\nabla_\eta\Upsilon)(\zeta) = (\eta\cdot\Upsilon(\zeta)) - \Upsilon(\nabla_\eta\zeta)$ as well as that ∇ is torsion-free, we obtain

$$\begin{aligned}
\hat{R}(\xi,\eta)\zeta = R(\xi,\eta)\zeta &+ \Upsilon(\eta)\Upsilon(\zeta)\xi - \Upsilon(\xi)\Upsilon(\zeta)\eta - (\nabla_\eta\Upsilon)(\zeta)\xi + (\nabla_\xi\Upsilon)(\zeta)\eta \\
&+ (\xi\cdot\Upsilon(\eta) - \eta\cdot\Upsilon(\xi) - \Upsilon([\xi,\eta]))\zeta
\end{aligned}$$

Again using that ∇ is torsion-free, we rewrite $\Upsilon([\xi,\eta]) = \Upsilon(\nabla_\xi\eta) - \Upsilon(\nabla_\eta\xi)$ and obtain

$$((\nabla_\xi\Upsilon)(\eta) - (\nabla_\eta\Upsilon)(\xi))\zeta$$

for the last line of the equation. Rewriting this in index notation we get:

$$\begin{aligned}
\widehat{R_{ij}{}^k{}_\ell}\zeta^\ell &= R_{ij}{}^k{}_\ell\zeta^\ell + \Upsilon_j\Upsilon_\ell\zeta^\ell\delta_i{}^k - \Upsilon_i\Upsilon_\ell\zeta^\ell\delta_j{}^k \\
&\quad -\zeta^\ell\delta_i{}^k\nabla_j\Upsilon_\ell + \zeta^\ell\delta_j{}^k\nabla_i\Upsilon_\ell + \zeta^k\nabla_i\Upsilon_j - \zeta^k\nabla_j\Upsilon_i.
\end{aligned}$$

By removing ζ and combining each antisymmetrized pair, we obtain the result. □

Recall from section 2.2 that there is a decomposition of the curvature $R_{ij}{}^k{}_\ell$ containing the trace-free Weyl tensor $C_{ij}{}^k{}_\ell$. This decomposition is given by

$$R_{ij}{}^k{}_\ell = C_{ij}{}^k{}_\ell + 2\delta_{[i}{}^k\mathrm{P}_{j]\ell} + \beta_{ij}\delta_\ell{}^k,$$

with P_{ij} and β_{ij} as in Definition 2.2.5. Under change of connection the Weyl tensor is invariant and the transformations of β_{ij} and P_{ij} are stated in the next Lemma.

Lemma 3.3.4. If we change the connection in the equivalence class from ∇ to $\hat{\nabla}$, then $C_{ij}{}^k{}_\ell$ is invariant and β_{ij} and P_{ij} transform as

$$\hat{\beta}_{ij} = \beta_{ij} + 2\nabla_{[i}\Upsilon_{j]} \quad \text{and} \quad \hat{\mathrm{P}}_{ij} = \mathrm{P}_{ij} - \nabla_i\Upsilon_j + \Upsilon_i\Upsilon_j.$$

Proof. As both β_{ij} and P_{ij} are defined via the Ricci tensor, we first compute the transformation of R_{ij} by using Lemma 3.3.3.

$$\begin{aligned}
\hat{R}_{ij} &= \widehat{R_{ki}{}^k{}_j} \\
&= R_{ij} + \Upsilon_i \Upsilon_j \delta_k{}^k - \Upsilon_k \Upsilon_j \delta_i{}^k - \nabla_i \Upsilon_j \delta_k{}^k + \nabla_k \Upsilon_j \delta_i{}^k + \nabla_k \Upsilon_i \delta_j{}^k - \nabla_i \Upsilon_k \delta_j{}^k \\
&= R_{ij} + n \Upsilon_i \Upsilon_j - \Upsilon_i \Upsilon_j - n \nabla_i \Upsilon_j + \nabla_i \Upsilon_j + \nabla_j \Upsilon_i - \nabla_i \Upsilon_j \\
&= R_{ij} + (n-1) \Upsilon_i \Upsilon_j - n \nabla_i \Upsilon_j + \nabla_j \Upsilon_i .
\end{aligned}$$

Thus

$$\begin{aligned}
-(n+1)\hat{\beta}_{ij} &= \hat{R}_{ij} - \hat{R}_{ji} \\
&= R_{ij} - R_{ji} - (n+1)\nabla_i \Upsilon_j + (n+1)\nabla_j \Upsilon_i \\
&= -(n+1)(\beta_{ij} + \nabla_i \Upsilon_j - \nabla_j \Upsilon_i)
\end{aligned}$$

and

$$\begin{aligned}
(n-1)\hat{\mathrm{P}}_{ij} &= \hat{R}_{ij} + \hat{\beta}_{ij} \\
&= R_{ij} + \beta_{ij} + (n-1)\Upsilon_i \Upsilon_j - (n-1)\nabla_i \Upsilon_j \\
&= (n-1)(\mathrm{P}_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j),
\end{aligned}$$

which gives the result for β_{ij} and P_{ij}. From Lemma 3.3.3, we see that the change of the curvature contains only trace components. Therefore, the trace-free part has to be invariant. To be on the safe side, we compute:

$$\begin{aligned}
\widehat{C_{ij}{}^k{}_\ell} &= \widehat{R_{ij}{}^k{}_\ell} - 2\delta_{[i}{}^k \hat{\mathrm{P}}_{j]\ell} - \hat{\beta}_{ij}\delta_\ell{}^k \\
&= R_{ij}{}^k{}_\ell + \Upsilon_j \Upsilon_\ell \delta_i{}^k - \Upsilon_i \Upsilon_\ell \delta_j{}^k - \nabla_j \Upsilon_\ell \delta_i{}^k \\
&\quad + \nabla_i \Upsilon_\ell \delta_j{}^k + \nabla_i \Upsilon_j \delta_\ell{}^k - \nabla_j \Upsilon_i \delta_\ell{}^k \\
&\quad - \delta_i{}^k(\mathrm{P}_{j\ell} - \nabla_j \Upsilon_\ell + \Upsilon_j \Upsilon_\ell) + \delta_j{}^k(\mathrm{P}_{i\ell} - \nabla_i \Upsilon_\ell + \Upsilon_i \Upsilon_\ell) \\
&\quad - \delta_\ell{}^k(\beta_{ij} + \nabla_i \Upsilon_j - \nabla_j \Upsilon_i) \\
&= R_{ij}{}^k{}_\ell - \delta_i{}^k \mathrm{P}_{j\ell} + \delta_j{}^k \mathrm{P}_{i\ell} - \delta_\ell{}^k \beta_{ij} \\
&= C_{ij}{}^k{}_\ell .
\end{aligned}$$

□

Chapter 4

Tractor Bundles and Calculus

Tractor calculus is an invariant calculus for projective structures, or more generally, for any parabolic geometry. It is based on natural vector bundles, called tractor bundles, which admit a canonical linear connection, called tractor connection. These bundles and connections can be associated to any parabolic geometry, compare [4].

In this chapter, we explicitely construct certain tractor bundles and their connections for projective structures on manifolds. Furthermore, we show how the Cartan bundle and Cartan connection can be regained from the tractor bundle and its connection. In chapter 5, we will then make use of tractor calculus to construct projectively invariant differential operators, i.e. differential operators which are intrinsic to the projective structure.

4.1 Densities

In the following we want to define the bundle $\mathcal{E}(\omega)$ of densities of projective weight $\omega \in \mathbb{R}$. This can be constructed by taking a certain power of the volume bundle. Details on the volume bundle can be found in [14, Section 10], we give a brief introduction here.

On any (not necessarily oriented) manifold M, there exists a line bundle, called volume bundle, whose sections can be integrated. It is defined as the associated bundle to the linear frame bundle $P^1M \to M$ with standard fiber $\mathbb{R}$, where the left action of $GL(n, \mathbb{R})$ on $\mathbb{R}$ is given by

$$A \cdot t = |\det A|^{-1} t.$$

We denote this bundle by $\mathrm{Vol}(M)$ and its sections are called densities. It follows that this bundle is trivial, but there is no canonical trivialisation, see [14, Lemma 10.2.]. In the special case of an oriented manifold, $\mathrm{Vol}(M)$ can be identified with $\Lambda^n T^*M$. Similarly, for $p \in \mathbb{R}$ we can define the bundle $\mathrm{Vol}^p(M)$ by changing the action of $GL(n, \mathbb{R})$ on $\mathbb{R}$ to

$$A \cdot t = |\det A|^{-p} t.$$

Note that the dual bundle to $\mathrm{Vol}^p(M)$ is $\mathrm{Vol}^{-p}(M)$.

Definition 4.1.1. The bundle $\mathcal{E}(\omega)$ is defined as $\mathcal{E}(\omega) = \mathrm{Vol}^{-\frac{\omega}{n+1}}(M)$. Sections of this bundle are called densities of projective weight ω.

Note that $\mathcal{E}(0)$ is the trivial bundle $M \times \mathbb{R} \to M$. Given an arbitrary bundle $\mathcal{V}$ we write $\mathcal{V}(\omega)$ for $\mathcal{V} \otimes \mathcal{E}(\omega)$. Thus using the notation introduced in chapter 2.1, we write $\mathcal{E}_a(\omega)$ for $T^*M \otimes \mathcal{E}(\omega)$ and $\mathcal{E}^a(\omega)$ for $TM \otimes \mathcal{E}(\omega)$.

Since $\mathcal{E}(\omega)$ is an associated bundle to the linear frame bundle, by choosing an affine connection on the manifold M, we obtain an induced connection on $\mathcal{E}(\omega)$, see section 2.2. We will denote the induced connection by the same symbol.

Recall from section 2.2, that any affine connection ∇ on M defines a curvature $R_{ij}{}^k{}_\ell$, which can be decomposed in trace and trace-free components:

$$R_{ij}{}^k{}_\ell = C_{ij}{}^k{}_\ell + 2\delta_{[i}{}^k \mathrm{P}_{j]\ell} + \beta_{ij}\delta_\ell{}^k. \tag{4.1}$$

Given an associated bundle to a principal frame bundle, its curvature can be calculated using the last part of [9, Proposition 1.3.4.]. We now calculate the curvature of the induced connection on $\mathcal{E}(\omega)$.

Lemma 4.1.1. Let ∇ be an affine connection on M with curvature $R_{ij}{}^k{}_\ell$ and decomposition (4.1). Then the curvature of the induced connection on $\mathcal{E}(\omega)$ is given by $\omega\beta_{ab}$.

Proof. The defining property of the curvature κ of the induced connection is

$$\nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma = \kappa(\xi,\eta)\sigma,$$

for σ a section of $\mathcal{E}(\omega)$. By [9, Proposition 1.3.4.] this section corresponds to the equivariant function $\lambda'(R(\xi^{\mathrm{hor}}, \eta^{\mathrm{hor}})) \circ f$, where f is the equivariant function corresponding to σ and λ' is the Lie algebra representation of the associated bundle $\mathcal{E}(\omega)$. By Definition 4.1.1, the representation λ of $GL(n,\mathbb{R})$ on $\mathbb{R}$ is given by

$$\lambda(A) = (t \mapsto |\det(A)|^{\frac{\omega}{n+1}} t).$$

Thus the Lie algebra representation corresponding to λ is given by its derivative. Since $T_{\mathbb{I}}\det(B) = \mathrm{tr}(B)$, we obtain that

$$\lambda'(X) = (t \mapsto \frac{\omega}{n+1}\mathrm{tr}(X)t),$$

for $X \in \mathfrak{gl}(n,\mathbb{R})$ and $t \in \mathbb{R}$. Thus we have to take the trace in k and ℓ of $R_{ij}{}^k{}_\ell$. Using Definition 2.2.5, we obtain

$$R_{ij}{}^k{}_k = (n+1)\beta_{ij}.$$

Thus the curvature κ_{ab} on $\mathcal{E}(\omega)$ is given by $\omega\beta_{ab}$.

$\square$

Induced Connections on Densities

In this section we compute the change of the induced connection on $\mathcal{E}(\omega)$ within a projective class. For later purpose we also calculate the transformation for products of the form $\otimes^k T^*M \otimes \mathcal{E}(\omega)$, $\otimes^\ell TM \otimes \mathcal{E}(\omega)$ and $(\otimes^k T^*M) \otimes (\otimes^\ell TM) \otimes \mathcal{E}(\omega)$.

Theorem 4.1.1. If we change the connection ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ on M, the induced connection on $\mathcal{E}(\omega)$ transforms via

$$\hat{\nabla}_a \sigma = \nabla_a \sigma + \omega \Upsilon_a \sigma.$$

Proof. This proof uses basic facts on linear connections on the linear frame bundle $P^1M \to M$, which we have partly discussed in section 2.2. We sketch the constructions needed here, again referencing [2, Chapter 2.3.] for more detail.

The linear frame bundle $p : P^1M \to M$ is a $GL(n, \mathbb{R})-$principal bundle. A principal connection form on P^1M is given by $\gamma \in \Omega^1(P^1M, \mathfrak{gl}(n, \mathbb{R}))$ satisfying certain properties, see Remark 2.2. Any two principal connection forms are related via

$$\hat{\gamma}_i^j = \gamma_i^j + \sum_{k=1}^{n} \Psi_{ki}{}^j \theta^k,$$

where θ^k denotes the soldering form. Here $\Psi_{ki}{}^j$ may be interpreted as a function $\Psi : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, where the index k represents the first argument and i and j the endomorphism.

Via the principal connection form we can lift vector fields of M horizontally to P^1M. Two horizontal lifts coming from different connection forms are related via

$$\hat{\xi}^{\mathrm{hor}} = \xi^{\mathrm{hor}} - \zeta_{\Psi(\xi,-)},$$

where ζ_X denotes the fundamental vector field generated by X.

The linear connection on any associated bundle to P^1M can be obtained by identifying section of this bundle with $GL(n, \mathbb{R})-$equivariant functions. We are interested in the case of $\mathcal{E}(\omega)$, which is an associated bundle to P^1M, see Definition 4.1.1. Sections σ of this bundle are in bijective correspondence to smooth maps $f : P^1M \to \mathbb{R}$, which are $GL(n, \mathbb{R})-$equivariant, i.e. which satisfy $f(\varphi \circ A) = |\det(A)|^{-\frac{\omega}{n+1}} f(\varphi)$ for $A \in GL(n, \mathbb{R})$. If a section σ corresponds to $f \in \mathcal{C}^\infty(P^1M, \mathbb{R})^{GL(n,\mathbb{R})}$, then $\nabla_\xi \sigma$ is the section corresponding to the $GL(n, \mathbb{R})-$equivariant function $\xi^{\mathrm{hor}} \cdot f : P^1M \to \mathbb{R}$. Now we have the basic information to prove the Theorem.

Let σ be a section of $\mathcal{E}(\omega)$, we want to compute $\hat{\nabla}_\xi \sigma - \nabla_\xi \sigma$. In the projective class we have the transformation

$$\hat{\nabla}_\xi \eta = \nabla_\xi \eta + \Upsilon(\xi)\eta + \Upsilon(\eta)\xi,$$

so any two connections differ by $\Psi(\xi,\eta) = \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$. Thus, as stated above, the horizontal lifts of any vector field ξ differ by $-\zeta_{\Psi(\xi,-)}$. Let $f : P^1M \to \mathbb{R}$ be the $GL(n,\mathbb{R})-$equivariant function corresponding to σ. Then $\hat{\nabla}_\xi\sigma - \nabla_\xi\sigma$ is the section corresponding to

$$\hat{\xi}^{\text{hor}} \cdot f - \xi^{\text{hor}} \cdot f,$$

which is given by $-\zeta_{\Psi(\xi,-)} \cdot f$. So first we have to compute the directional derivative of the fundamental vector field on equivariant functions.

Claim 3. The directional derivative of the fundamental vector field of a $GL(n,\mathbb{R})-$equivariant function f is given by

$$\zeta_A \cdot f = -A \cdot f,$$

where $A \in \mathfrak{gl}(n,\mathbb{R})$ and the second dot denots the Lie algebra action.

This is the infitesimal equivariancy. The fundamental vector field is defined by $\zeta_A(\varphi) = \frac{d}{dt}|_{t=0}\varphi \cdot \exp(tA)$, for $\varphi \in P^1M$. Then

$$\begin{aligned}
(\zeta_A \cdot f)(\varphi) &= T_\varphi f \cdot \zeta_A \\
&= \frac{d}{dt}|_{t=0} f(\varphi \cdot \exp(tA)) \\
&= \frac{d}{dt}|_{t=0} \exp(tA)^{-1} \cdot f(\varphi) = \frac{d}{dt}|_{t=0} \exp(-tA) \cdot f(\varphi) \\
&= -A \cdot f(\varphi),
\end{aligned}$$

where we used the equivariancy of f in the third line. This proves the claim.

We now continue with the proof of the Theorem. In Lemma 4.1.1 we already computed the Lie algebra action on $\mathbb{R}$; it is given by

$$X \cdot t = \frac{\omega}{n+1}\text{tr}(X)t,$$

for $X \in \mathfrak{gl}(n,\mathbb{R})$. So until now we have found that $\hat{\nabla}_\xi\sigma - \nabla_\xi\sigma$ is given by the section, which corresponds to the equivariant function

$$\frac{\omega}{n+1}\text{tr}(\Psi(\xi,-))f.$$

At this point it is easier to compute with abstract indices. Since $\Psi(\xi,\eta) = \Upsilon(\xi)\eta + \Upsilon(\eta)\xi$, we find that

$$\Psi_{ki}{}^j = \delta_i{}^j\Upsilon_k + \delta_k{}^j\Upsilon_i.$$

Now the index k corresponds to ξ and thus $\text{tr}(\Psi(\xi,-))$ is the trace over i and j. This is given by $\delta_a{}^a\Upsilon_k + \delta_k{}^a\Upsilon_a = (n+1)\Upsilon_k$. In abstract index notation, we have that $\hat{\nabla}_k\sigma - \nabla_k\sigma$ corresponds to the section given by $\omega\Upsilon_k f$, which simply is $\omega\Upsilon_k\sigma$, again using [9, Proposition 1.2.7.]. Thus we obtain

$$\hat{\nabla}_k \sigma = \nabla_k \sigma + \omega \Upsilon_k \sigma.$$

□

In order to compute transformations on tensor bundles as stated at the beginning of the section, we use the transformations proved in Lemma 3.3.1.

Lemma 4.1.2. By changing the connection in the projective class from ∇ to $\hat{\nabla} = \nabla + \Upsilon$, the induced connections on $\otimes^k TM \otimes \mathcal{E}(\omega)$ respectively on $\otimes^k T^*M \otimes \mathcal{E}(\omega)$ transform via

$$\hat{\nabla}_a \xi^{b_1 \cdots b_k} = \nabla_a \xi^{b_1 \cdots b_k} + (\omega + k)\Upsilon_a \xi^{b_1 \cdots b_k} + \sum_{i=1}^{k} \Upsilon_c \xi^{b_1 \cdots b_{i-1} c b_{i+1} \cdots b_k} \delta_a{}^{b_i},$$

respectively

$$\hat{\nabla}_a \eta_{b_1 \cdots b_k} = \nabla_a \eta_{b_1 \cdots b_k} + (\omega - k)\Upsilon_a \eta_{b_1 \cdots b_k} - \sum_{i=1}^{k} \Upsilon_{b_i} \eta_{b_1 \cdots b_{i-1} a b_{i+1} \cdots b_k}.$$

Combining these, we obtain the following transformation for the induced connection on $(\otimes^k T^*M) \otimes (\otimes^\ell TM) \otimes \mathcal{E}(\omega)$:

$$\begin{aligned} \hat{\nabla}_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} &= \nabla_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} + (\ell - k + \omega)\Upsilon_a \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_\ell} \\ &\quad + \sum_{i=1}^{\ell} \Upsilon_d \zeta_{b_1 \cdots b_k}{}^{c_1 \cdots c_{i-1} d c_{i+1} \cdots c_\ell} \delta_a{}^{c_i} - \sum_{j=1}^{k} \Upsilon_{b_i} \zeta_{b_1 \cdots b_{i-1} a b_{i+1} \cdots b_k}{}^{c_1 \cdots c_\ell}. \end{aligned}$$

Proof. We use the induced connection on $\otimes^k T^*M \otimes \mathcal{E}(\omega)$ on tensors of the form $\xi^{b_1 \cdots b_k} \otimes \sigma$, hence $\nabla_a(\xi^{b_1 \cdots b_k} \otimes \sigma) = \xi^{b_1 \cdots b_k} \otimes \nabla_a \sigma + (\nabla_a \xi^{b_1 \cdots b_k}) \otimes \sigma$. Then by using the transformations established in Lemma 3.3.1 and Theorem 4.1.1, we obtain the result. An analogous argument proves the other statements. □

Lemma 4.1.3. Given a nowhere vanishing section of $\mathcal{E}(\omega)$ with $\omega \neq 0$, then there is a unique connection in the projective class, for which this section is parallel.

Proof. In order to show existence, let σ be a nowhere vanishing section of $\mathcal{E}(\omega)$ and ∇ a connection on M. Define $\hat{\nabla} = \nabla + \Upsilon$ with $\Upsilon = -\frac{1}{\omega}\sigma^{-1}\nabla\sigma$. Then

$$\begin{aligned} \hat{\nabla}_a \sigma &= \nabla_a \sigma + \omega \Upsilon_a \sigma \\ &= \nabla_a \sigma - (\sigma^{-1} \nabla_a \sigma)\sigma \\ &= 0 \end{aligned} \tag{4.2}$$

Now suppose ∇ and $\hat{\nabla}$ are two equivalent connections, for which σ is parallel. Then $\hat{\nabla}_a \sigma = \nabla_a \sigma + \omega \Upsilon_a \sigma$ and thus $0 = \omega \Upsilon_a \sigma$. Since $\sigma \neq 0$ and $\omega \neq 0$ by assumption, we obtain $\Upsilon_a = 0$ and thus $\hat{\nabla} = \nabla$. □

Remark 4.1.1. Since $\mathcal{E}(\omega)$ is a trivial bundle, there always exists a nowhere vanishing section. A nowhere vanishing section is called a scale and the resulting connection is called the connection determined by that scale. If ∇ is a connection determined by a scale σ and $\omega \neq 0$, then β_{ab} has to vanish. Let $\xi, \eta \in \mathfrak{X}(M)$ be two vector fields, then by Lemma 4.1.1 we have

$$(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]})\sigma = \omega \beta(\xi, \eta)\sigma$$

and since σ is parallel, the left hand side vanishes and therefore $\beta_{ab} = 0$. Thus we always find a connection in the projective class, such that β_{ab} vanishes.

Lemma 4.1.4. The following statements for an affine connection ∇ on M are equivalent:

1. The Ricci curvature R_{ab} is symmetric
2. P_{ab} is symmetric
3. $\beta_{ab} = 0$
4. ∇ induces a flat connection on $\mathcal{E}(\omega)$

If a connection is determined by a scale, then it satisfies (1-4) and locally, the converse is also true.

Proof. Certainly R_{ab} is symmetric if and only if $\beta_{ab} = 0$, by Definition 2.2.5. Also P_{ab} is symmetric if and only if $\beta_{ab} = 0$. The curvature on $\mathcal{E}(\omega)$ is given by $\omega\beta_{ab}$ and thus ∇ on $\mathcal{E}(\omega)$ is flat if and only if $\beta_{ab} = 0$. This established the equivalence of (1)-(4).

Given a connection, which is determined by a scale, then we have seen in Remark 4.1.1, that $\beta_{ab} = 0$. Conversely, if the induced connection on $\mathcal{E}(\omega)$ is flat, then locally it admits a nowhere vanishing parallel section. □

4.2 The Cotractor Bundle

In this section we want to define the cotractor bundle as the first jet prolongation of the density bundle $\mathcal{E}(1)$. Therefore, we briefly introduce jet bundles, referencing [14, Section 21] for more details.

The aim of jets is to provide a coordinate-free version of Taylor expansions of smooth functions defined on manifolds and smooth sections of fiber bundles.

Consider two smooth manifolds M and N as well as a point $x \in M$. Then two smooth maps $f, g : M \to N$ have the same $k-$jet at x, if $f(x) = g(x)$ and their partial derivatives at x up to order k in some local chart (or equivalently all local charts) around x and

$f(x)$ coincide. In this case we write $j_x^k f = j_x^k g$. Having the same $k-$jet at x defines an equivalence relation and the set of all such equivalence classes is denoted by $J_x^k(M,N)$. We then define $J^k(M,N) := \bigsqcup_{x\in M} J_x^k(M,N)$. This set is also called the space of $k-$jets of smooth maps from M to N.

There are some natural projections on $J^k(M,N)$. Consider $\pi_M : J^k(M,N) \to M$ given by $j_x^k f \mapsto x$ and $\pi_N : J^k(M,N) \to N$ given by $j_x^k f \mapsto f(x)$. Combining these projections, we also obtain $\pi_M \times \pi_N : J^k(M,N) \to M\times N$, $j_x^k f \mapsto (x, f(x))$. Charts on M and N give rise to charts on $J^k(M,N)$, which make $J^k(M,N)$ into a smooth manifold, such that the considered projections are also smooth. Furthermore, $\pi_M \times \pi_N : J^k(M,N) \to M \times N$ is a fiber bundle, see [14, Theorem 21.5.].

Given a fiber bundle $p : Y \to M$, we write $J^k(Y \to M)$ or $J^k(Y)$ to indicate the subset of $J^k(M,Y)$, given by all jets of local sections of p. A smooth section $\sigma \in \Gamma(Y)$ induces a smooth section of $J^k(Y)$ by $j^k\sigma : M \to J^k(Y), x \mapsto j_x^k\sigma$.

Definition 4.2.1. The cotractor bundle $\mathcal{T}^*$ is defined as the bundle of $1-$jets of sections of $\mathcal{E}(1)$, hence $\mathcal{T}^* = J^1\mathcal{E}(1)$.

Theorem 4.2.1. There is a short exact sequence of vector bundles given by

$$0 \to T^*M \otimes \mathcal{E}(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0 \tag{4.3}$$

By choosing a connection ∇ on M, we obtain an isomorphism $\mathcal{T}^* \cong_\nabla (T^*M \otimes \mathcal{E}(1)) \oplus \mathcal{E}(1)$.

Proof. We have the natural projection $p : J^1\mathcal{E}(1) \to \mathcal{E}(1)$ given by $j_x^1\sigma \mapsto \sigma(x)$. The fiberwise kernel of this projection may be identified with $T_x^*M \otimes \mathcal{E}_x(1)$, by choosing a connection ∇ and mapping $j_x^1\sigma$ in the kernel to $\nabla\sigma(x)$. We can use a local description of $\nabla\sigma(x)$, to see that this map is well-defined and independent of the choice of connection ∇ on M.

Recall from section 2.2, that we have a local description of ∇ using a vector bundle atlas (U_α, ϕ_α). Locally on U_α we can write $\sigma = \sigma^i e_i$ for $\{e_i\}$ a local frame and $\sigma^i : U_\alpha \to \mathbb{R}$ smooth. Then for a vector field $\xi \in \mathfrak{X}(M)$, on U_α we obtain:

$$\nabla_\xi \sigma = \nabla_\xi \sigma^i e_i = (\xi \cdot \sigma^i) e_i + \sigma^i \nabla_\xi e_i.$$

So we conclude that $\nabla\sigma(x)$ only depends on $\sigma(x)$ and its derivatives up to first order, hence only depends on $j_x^1\sigma$. Also two connections only differ in the part $\sigma^i\nabla_\xi e_i$. If $j_x^1\sigma$ is in the kernel of p, then $\sigma(x) = 0$, hence this part vanishes. Thus this map is independent of the choice of connection on M.

This gives a linear isomorphism on the fibers and thus a vector bundle isomorphism $\ker(p) \cong T^*M \otimes \mathcal{E}(1)$.

To prove the second statement, choose a connection ∇ on M and define a map $l : J_x^1\mathcal{E}(1) \to T_x^*M \otimes \mathcal{E}_x(1)$ by $j_x^1\sigma \mapsto \nabla\sigma(x)$. This map of course depends on the choice of connection, but defines a left inverse map to the inclusion of $\ker_x(p)$ to $J_x^1\mathcal{E}(1)$. Thus the sequence splits and we obtain an isomorphism $J^1_x\mathcal{E}(1) \cong_\nabla (T^*_xM \otimes \mathcal{E}_x(1)) \oplus \mathcal{E}_x(1)$ on the fibers, which again gives rise to an isomorphism on the vector bundles. $\square$

Remark 4.2.1. To summerize the composition structure (4.3), we write $\mathcal{T}^* = \mathcal{E}_a(1) \mathbin{\lefteqn{\subset}+} \mathcal{E}(1)$, where $\mathcal{E}_a(1)$ denotes $T^*M \otimes \mathcal{E}(1)$ as defined in section 4.1.1. This, of course, is valid without the choice of a connection.

We now fix a projective structure $(M, [\nabla])$ and via the isomorphism $\mathcal{T}^* \cong_\nabla \mathcal{E}_a(1) \oplus \mathcal{E}(1)$ we identify elements of $\mathcal{T}^*$ with pairs (μ_a, σ).

Corollary 4.2.1. If an element of $\mathcal{T}^*$ splits as (μ_a, σ) with respect to ∇ on M, then by changing the connection within the projective class to $\hat{\nabla} = \nabla + \Upsilon$, the splitting changes according to

$$\widehat{(\mu_a, \sigma)} = (\mu_a + \Upsilon_a \sigma, \sigma).$$

Proof. Under change of connection the isomorphism $J^1\mathcal{E}(1) \cong_{\hat{\nabla}} (T^*M \otimes \mathcal{E}(1)) \oplus \mathcal{E}(1)$ is given by $j^1_x\sigma \mapsto (\hat{\nabla}\sigma(x), \sigma(x))$. Since the connections on $\mathcal{E}(1)$ transform via $\hat{\nabla}\sigma = \nabla\sigma + \Upsilon_a\sigma$, we obtain that $(\hat{\nabla}\sigma(x), \sigma(x)) = (\nabla\sigma(x) + \Upsilon_a\sigma(x), \sigma(x))$ and thus the result. $\square$

We can associate a connection on the cotractor bundle to the given projective class. Note that the definition of the cotractor bundle was independent of the choice of the projective class, whereas we will see that projective class is encoded in the tractor connection. In order to give a definition, we have to introduce some notation. Choose a connection ∇ in the projective class and split $\mathcal{T}^*$ with respect to ∇, as in Theorem 4.2.1. Then this splitting induces a splitting of $\mathcal{E}_a \otimes \mathcal{T}^*$. Therefore, if $\mathcal{T}^* \cong_\nabla (\mathcal{E}_b \otimes \mathcal{E}(1)) \oplus \mathcal{E}(1)$, then $\mathcal{E}_a \otimes \mathcal{T}^* \cong_\nabla (\mathcal{E}_{ab} \otimes \mathcal{E}(1)) \oplus (\mathcal{E}_a \otimes \mathcal{E}(1))$, via the map $\tau_a \otimes (\mu_b, \sigma) \mapsto (\tau_a\mu_b, \tau_a\sigma)$. If an element of $\mathcal{E}_a \otimes \mathcal{T}^*$ splits as (τ_{ab}, ρ_a) with respect to ∇, then by changing the connection in the projective class of M to $\hat{\nabla} = \nabla + \Upsilon$, the splitting changes according to $(\tau_{ab} + \Upsilon_b\rho_a, \rho_a)$. We can illustrate this fact by the following diagram:

$$\begin{tikzcd}
\mathcal{T}^* \cong_\nabla \mathcal{E}_b(1) \oplus \mathcal{E}(1) \arrow[r, squiggly, "\text{induces}"] \arrow[d, "(\mu_b,\sigma)\mapsto(\mu_b+\Upsilon_b\sigma,\sigma)"'] \arrow[dr, phantom, "\#"] & \mathcal{E}_a \otimes \mathcal{T}^* \cong_\nabla \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1) \arrow[d, "(\tau_{ab},\rho_a)\mapsto(\tau_{ab}+\Upsilon_b\rho_a,\rho_a)"] \\
\mathcal{T}^* \cong_{\hat{\nabla}} \mathcal{E}_b(1) \oplus \mathcal{E}(1) \arrow[r, squiggly, "\text{induces}"'] & \mathcal{E}_a \otimes \mathcal{T}^* \cong_{\hat{\nabla}} \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1)
\end{tikzcd}$$

Definition 4.2.2. Choose a connection ∇ on M in the projective class. Split $\mathcal{T}^*$ with respect to this connection and via this splitting let (μ_b, σ) be section of $\mathcal{T}^*$. Define the tractor connection on $\mathcal{T}^*$ by

$$\nabla_a^{\mathcal{T}^*}(\mu_b, \sigma) := (\nabla_a\mu_b + \mathrm{P}_{ab}\sigma, \nabla_a\sigma - \mu_a),$$

where the splitting on the right hand side is the splitting of $\mathcal{E}_a \otimes \mathcal{T}^*$ induced by the one of $\mathcal{T}^*$.

Obviously, this definition depends on the choice of connection. However, the following Theorem shows, that it is independent of the choice of connection in the projective class. Therefore, the tractor connection is canonical, associated to the projective structure on M.

Theorem 4.2.2. The definition of the tractor connection is independent of the choice of connection in the projective class of M, i.e. if we change ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ on M, we obtain the following commutative diagram:

$$\begin{tikzcd}
\Gamma(\mathcal{E}_b(1) \oplus \mathcal{E}(1)) \arrow[r, "\nabla_a^{\mathcal{T}^*}"] \arrow[d, "(\mu_b,\sigma)\mapsto(\mu_b+\Upsilon_b\sigma,\sigma)"'] \arrow[dr, phantom, "\#"] & \Gamma(\mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1)) \arrow[d, "(\tau_{ab},\rho_a)\mapsto(\tau_{ab}+\Upsilon_b\rho_a,\rho_a)"] \\
\Gamma(\mathcal{E}_b(1) \oplus \mathcal{E}(1)) \arrow[r, "\hat{\nabla}_a^{\mathcal{T}^*}"'] & \Gamma(\mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1))
\end{tikzcd}$$

Proof. For the following calculation we will use Corollary 4.2.1, Lemma 3.3.4, Theorem 4.1.1 and the first part of Lemma 4.1.2 with $k = \omega = 1$ as well as the transformation laws for $1-$forms.

Split $\mathcal{T}^*$ with respect to ∇ and via this splitting let (μ_b, σ) be an element of $\mathcal{T}^*$. Then apply the connection $\nabla_a^{\mathcal{T}^*}$ and the map $(\tau_{ab}, \rho_a) \mapsto (\tau_{ab} + \Upsilon_b\rho_a, \rho_a)$ to obtain

$$(\nabla_a\mu_b + \mathrm{P}_{ab}\sigma + \Upsilon_b(\nabla_a\sigma - \mu_a), \nabla_a\sigma - \mu_a). \tag{4.4}$$

On the other hand applying $(\mu_b, \sigma) \mapsto (\mu_b + \Upsilon_b\sigma, \sigma)$ first and then the connection $\hat{\nabla}_a^{\mathcal{T}^*}$, we obtain $(\hat{\nabla}_a(\mu_b + \Upsilon_b\sigma) + \hat{\mathrm{P}}_{ab}\sigma, \hat{\nabla}_a\sigma - (\mu_a + \Upsilon_a\sigma))$. Now we calculate:

$$\begin{aligned}
& (\hat{\nabla}_a(\mu_b + \Upsilon_b\sigma) + \hat{\mathrm{P}}_{ab}\sigma, \hat{\nabla}_a\sigma - (\mu_a + \Upsilon_a\sigma)) \\
= \; & (\hat{\nabla}_a\mu_b - \Upsilon_b\hat{\nabla}_a\sigma + (\hat{\nabla}_a\Upsilon_b)\sigma + \hat{\mathrm{P}}_{ab}\sigma, \hat{\nabla}_a\sigma - \mu_a - \Upsilon_a\sigma) \\
= \; & (\nabla_a\mu_b - \Upsilon_b\mu_a + \Upsilon_b(\nabla_a\sigma + \Upsilon_a\sigma) + (\nabla_a\Upsilon_b - 2\Upsilon_a\Upsilon_b)\sigma \\
& + \mathrm{P}_{ab}\sigma - (\nabla_a\Upsilon_b)\sigma + \Upsilon_a\Upsilon_b\sigma, \nabla_a\sigma + \Upsilon_a\sigma - \mu_a - \Upsilon_a\sigma) \\
= \; & (\nabla_a\mu_b + \mathrm{P}_{ab}\sigma + \Upsilon_b(\nabla_a\sigma - \mu_a), \nabla_a\sigma - \mu_a).
\end{aligned}$$

This coincides with (4.4) and therefore the diagram commutes. □

Until now we have defined the cotractor bundle and via the choice of an affine connection on M, we obtained a splitting. Furthermore, to a given projective structure, we can assign a canonical tractor connection. In this chapter we will define further bundles by natural constructions on the cotractor bundle, such as for example the endomorphism bundle. A splitting of the cotractor bundle induces a splitting of a bundle coming from a natural construction, at least abstractly. By fixing a concrete representation of this construction, we can make the splitting explicit.

As discussed in chapter 2.2, every connection on a vector bundle possesses a curvature.

When thinking about the curvature of the tractor connection, we realize, that we have to understand the endomorphism bundle $L(\mathcal{T}^*, \mathcal{T}^*)$ first, since the curvature is a section of $\mathcal{E}_{[ab]} \otimes L(\mathcal{T}^*, \mathcal{T}^*)$.

Consider a splitting $\mathcal{T}^* \cong_\nabla \mathcal{E}_a(1) \oplus \mathcal{E}(1)$ and let (μ_a, σ) be an element of $\mathcal{T}^*$ via the given splitting. Viewing elements of $L(\mathcal{T}^*, \mathcal{T}^*)$ as matrices, by

$$(\mu_a, \sigma)\begin{pmatrix} A_c{}^b & \xi^d \\ \phi_e & \psi \end{pmatrix} := (A_c{}^i\mu_i + \sigma\phi_c, \psi\sigma + \mu_i\xi^i),$$

we define the splitting $L(\mathcal{T}^*, \mathcal{T}^*) = \mathcal{E}_a^b \oplus \mathcal{E}_c \oplus \mathcal{E}^d \oplus \mathcal{E}$. Therefore, the curvature of the cotractor connection is of the form: $\left(\begin{smallmatrix} A_{ab}{}^c{}_d & \xi_{ab}{}^e \\ \phi_{abf} & \psi_{ab} \end{smallmatrix}\right)$ with antisymmetric indices a and b.

If we change ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ on M, we claim that this splitting changes according to

$$\widehat{\begin{pmatrix} A_c{}^b & \xi^d \\ \phi_e & \psi \end{pmatrix}} = \begin{pmatrix} A_c{}^b + \Upsilon_c\xi^b & \xi^d \\ \phi_e - A_e{}^i\Upsilon_i - \xi^j\Upsilon_j\Upsilon_e + \Upsilon_e\psi & \psi - \Upsilon_i\xi^i \end{pmatrix}. \tag{4.5}$$

By the definition of the splittings on $L(\mathcal{T}^*, \mathcal{T}^*)$, we have that

$$(A_c{}^i\mu_i + \widehat{\sigma\phi_c, \psi}\sigma + \mu_i\xi^i) = \widehat{(\mu_a, \sigma)}\begin{pmatrix} \widehat{A_c{}^b} & \hat{\xi}^d \\ \hat{\phi}_e & \hat{\psi} \end{pmatrix}. \tag{4.6}$$

The left side of equation (4.6) yields:

$$(A_c{}^i\mu_i + \widehat{\sigma\phi_c, \psi}\sigma + \mu_i\xi^i) = ((A_c{}^i + \Upsilon_c\xi^i)\mu_i + (\phi_c + \Upsilon_c\psi)\sigma, \mu_i\xi^i + \psi\sigma). \tag{4.7}$$

On the other hand the right side of equation (4.6) gives:

$$\begin{aligned} \widehat{(\mu_a, \sigma)}\begin{pmatrix} \widehat{A_c{}^b} & \hat{\xi}^d \\ \hat{\phi}_e & \hat{\psi} \end{pmatrix} &= (\widehat{A_c{}^i}\mu_i + \widehat{A_c{}^i}\Upsilon_i\sigma + \hat{\phi}_c\sigma, \hat{\xi}^i\mu_i + \Upsilon_i\hat{\xi}^i\sigma + \sigma\hat{\psi}) \\ &= (\widehat{A_c{}^i}\mu_i + (\widehat{A_c{}^i}\Upsilon_i + \hat{\phi}_c)\sigma, \hat{\xi}^i\mu_i + (\Upsilon_i\hat{\xi}^i + \hat{\psi})\sigma) \end{aligned} \tag{4.8}$$

and therefore comparing (4.7) and (4.8) immediately gives the result for $\widehat{A_c{}^b}, \hat{\xi}^d$ and $\hat{\psi}$. Moreover, for $\hat{\phi}_c$ we obtain:

$$\begin{aligned} \hat{\phi}_c &= -\widehat{A_c{}^i}\Upsilon_i + \phi_c + \Upsilon_c\psi \\ &= \phi_c - A_c{}^i\Upsilon_i - \xi^i\Upsilon_i\Upsilon_c + \Upsilon_c\psi. \end{aligned}$$

Another ingredient we need in order to calculate the curvature of the tractor connection, is the notion of coupled connections. Given the tractor connection and a splitting, we already remarked before, that this induces a splitting of $\mathcal{E}_a \otimes \mathcal{T}^*$ and that the tractor connection $\nabla_a^{\mathcal{T}^*} : \Gamma(\mathcal{T}^*) \to \Gamma(\mathcal{E}_a \otimes \mathcal{T}^*)$ is well-defined within the projective class. Now if we want to apply the connection for a second time, we certainly need to couple the tractor connection with a connection on $\mathcal{E}_a$. This is approached as follows: Choose a connection ∇ in the projective class, split the cotractor bundle and $\mathcal{E}_a \otimes \mathcal{T}^*$ with this connection. Choose the tractor connection on $\mathcal{T}^*$ and the connection induced by ∇ on $\mathcal{E}_a$, see Remark 2.2.1. Now again by Remark 2.2.1, we have an induced connection on $\mathcal{E}_a \otimes \mathcal{T}^*$. This connection is a map $\nabla_c^{\mathcal{E}_a \otimes \mathcal{T}^*} : \Gamma(\mathcal{E}_a \otimes \mathcal{T}^*) \to \Gamma(\mathcal{E}_c \otimes \mathcal{E}_a \otimes \mathcal{T}^*)$. Eventhough the connection $\nabla_c^{\mathcal{E}_a \otimes \mathcal{T}^*}$ depends on the choice of connection in the projective class of M, the following expression is independent of the choice of connection:

$$(\nabla_a^{\mathcal{E}_a \otimes \mathcal{T}^*} \nabla_b^{\mathcal{T}^*} - \nabla_b^{\mathcal{E}_b \otimes \mathcal{T}^*} \nabla_a^{\mathcal{T}^*})(\mu_d, \sigma) = (\mu_c, \sigma) \begin{pmatrix} A_{ab}{}^c{}_d & \xi_{ab}{}^f \\ \phi_{abd} & \psi_{ab} \end{pmatrix}$$

and this is exactly the definition of the curvature. Hence we can choose any connection in the projective class of M to compute the curvature on $\mathcal{T}^*$.

Lemma 4.2.1. Given an affine connection ∇ on M with curvature $R_{ab}{}^c{}_d$ and decomposition of the curvature (4.1). Then the curvature of the tractor connection on $\mathcal{T}^*$ is given by

$$\begin{pmatrix} -C_{ab}{}^e{}_d & 0 \\ 2\nabla_{[a} \mathrm{P}_{b]d} & 0 \end{pmatrix}.$$

Proof. For this calculation we need the curvature of the induced connection on $T^*M \otimes \mathcal{E}(1)$. Recall from Lemma 2.2.4, that the curvature of the induced connection on T^*M is given by $-R_{ab}{}^d{}_c$, i.e. for $\mu_c \in \Gamma(T^*M)$ we have

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\mu_c = -R_{ab}{}^d{}_c \mu_d.$$

Furthermore, by Lemma 4.1.1, the curvature of the induced connection on $\mathcal{E}(1)$ is given by β_{ab} and thus for $\sigma \in \Gamma(\mathcal{E}(1))$:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma = \beta_{ab}\sigma.$$

Now consider the induced connection on $T^*M \otimes \mathcal{E}(1)$, which we again denote by ∇. Then we have

$$\begin{aligned} \nabla_a \nabla_b \mu_c \sigma &= \nabla_a(\mu_c \nabla_b \sigma + (\nabla_b \mu_c)\sigma) \\ &= (\nabla_a \mu_c)(\nabla_b \sigma) + \mu_c \nabla_a \nabla_b \sigma + (\nabla_a \nabla_b \mu_c)\sigma + (\nabla_b \mu_c)(\nabla_a \sigma) \end{aligned}$$

and therefore

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a)\mu_c \sigma &= (\nabla_a \nabla_b \mu_c - \nabla_b \nabla_a \mu_c)\sigma + \mu_c(\nabla_a \nabla_b \sigma - \nabla_b \nabla_a \sigma) \\ &= (-R_{ab}{}^d{}_c + \beta_{ab}\delta_c{}^d)\mu_d \sigma. \end{aligned} \tag{4.9}$$

This proves, that the curvature of the induced connection on $T^*M \otimes \mathcal{E}(1)$ is given by $(-R_{ab}{}^d{}_c + \beta_{ab}\delta_c{}^d)$.

As remarked before, in order to prove the Lemma, we have to consider coupled connections of the form $\nabla_a^{\mathcal{E}_a\otimes\mathcal{T}^*}\nabla_b^{\mathcal{T}^*}$. To abbreviate notation, we denote both connections by ∇. Given a pair (μ_d,σ), induced by a splitting, with μ_d a section of $T^*M\otimes\mathcal{E}(1)$, we calculate:

$$\begin{aligned}
\nabla_a\nabla_b(\mu_d,\sigma) &= \nabla_a(\nabla_b\mu_d+\mathrm{P}_{bd}\sigma,\nabla_b\sigma-\mu_b)\\
&= (\nabla_a(\nabla_b\mu_d+\mathrm{P}_{bd}\sigma)+\mathrm{P}_{ad}(\nabla_b\sigma-\mu_b),\nabla_a(\nabla_b\sigma-\mu_b)-\nabla_b\mu_a-\mathrm{P}_{ba}\sigma)\\
&= (\nabla_a\nabla_b\mu_d+(\nabla_a\mathrm{P}_{bd})\sigma+\mathrm{P}_{bd}\nabla_a\sigma+\mathrm{P}_{ad}\nabla_b\sigma-\mathrm{P}_{ad}\mu_b,\\
&\qquad \nabla_a\nabla_b\sigma-\nabla_a\mu_b-\nabla_b\mu_a-\mathrm{P}_{ba}\sigma)
\end{aligned}$$

Therefore, when calculating the curvature, terms, which are symmetric in a and b, can be ignored. Thus we obtain:

$$\begin{aligned}
(\nabla_a\nabla_b-\nabla_b\nabla_a)(\mu_d,\sigma) &= ((\nabla_a\nabla_b-\nabla_b\nabla_a)\mu_d+2\nabla_{[a}\mathrm{P}_{b]}\sigma-\mathrm{P}_{ad}\mu_b+\mathrm{P}_{bd}\mu_a,\\
&\qquad (\nabla_a\nabla_b-\nabla_b\nabla_a)\sigma+2\mathrm{P}_{[ab]}\sigma)\\
&= ((\beta_{ab}\delta_d{}^e-R_{ab}{}^e{}_d+2\delta_{[a}{}^e\mathrm{P}_{b]d})\mu_e+2(\nabla_{[a}\mathrm{P}_{b]d})\sigma,\\
&\qquad \beta_{ab}\sigma+2\mathrm{P}_{[ab]}\sigma)\\
&= (-C_{ab}{}^e{}_d\mu_e+2(\nabla_{[a}\mathrm{P}_{b]d})\sigma,0),
\end{aligned}\tag{4.10}$$

where we used the decomposition (4.1) and $\beta_{ab}\sigma=-2\mathrm{P}_{[ab]}\sigma$ (see Remark after Definition 2.2.5) in (4.10). If the curvature has the form $\begin{pmatrix} A_{ab}{}^e{}_d & \xi_{ab}{}^f\\ \phi_{abd} & \psi_{ab}\end{pmatrix}$, then we see that $A_{ab}{}^e{}_d=-C_{ab}{}^e{}_d$, $\phi_{abd}=2(\nabla_{[a}\mathrm{P}_{b]d})$ and $\xi_{ab}{}^f=\psi_{ab}=0$. □

Note, that by naturality of the construction, the curvature has to transform in the right way, if we change the connection from ∇ to $\hat\nabla=\nabla+\Upsilon$ in the projective class of M. Nevertheless, checking this provides a good possibility to verify our calculation.

If we change the connection from ∇ to $\hat\nabla=\nabla+\Upsilon$, then by (4.5), the curvature on the cotractor bundle has to transform as

$$\begin{pmatrix} -C_{ab}{}^e{}_d & 0\\ 2\nabla_{[a}\mathrm{P}_{b]d}+C_{ab}{}^e{}_d\Upsilon_e & 0\end{pmatrix}.$$

On the other hand, under change of connection, the curvature is given by

$$\begin{pmatrix} -\widehat{C_{ab}{}^e{}_d} & 0\\ 2\hat\nabla_{[a}\hat{\mathrm{P}}_{b]d} & 0\end{pmatrix}$$

and we have to show, that these two matrices coincide. First of all, note that by Lemma 3.3.4, the Weyl tensor is invariant under change of connection. Thus we only have to compute the second components of the matrix.

$$\begin{aligned}\hat{\nabla}_a \hat{\mathrm{P}}_{bd} &= \hat{\nabla}_a(\mathrm{P}_{bd} - \nabla_b \Upsilon_d + \Upsilon_b \Upsilon_d) \\ &= (\nabla_a \mathrm{P}_{bd} - 2\Upsilon_a \mathrm{P}_{bd} - \Upsilon_b \mathrm{P}_{ad} - \Upsilon_d \mathrm{P}_{ba}) \\ &\quad -(\nabla_a \nabla_b \Upsilon_d + 2\Upsilon_a \nabla_b \Upsilon_d + \Upsilon_b \nabla_a \Upsilon_d + \Upsilon_d \nabla_b \Upsilon_a) \\ &\quad +(\nabla_a \Upsilon_b \Upsilon_d - 4\Upsilon_a \Upsilon_b \Upsilon_d),\end{aligned} \tag{4.11}$$

where we used Lemma 3.3.1 in the case $k = 2$. Therefore, by removing all elements, which are symmetric in a and b, we obtain:

$$\begin{aligned}2\hat{\nabla}_{[a} \hat{\mathrm{P}}_{b]d} &= 2\nabla_{[a} \mathrm{P}_{b]d} - \Upsilon_a \mathrm{P}_{bd} + \Upsilon_b \mathrm{P}_{ad} + 2\Upsilon_d \mathrm{P}_{[ab]} - (\nabla_a \nabla_b - \nabla_b \nabla_a)\Upsilon_d \\ &= 2\nabla_{[a} \mathrm{P}_{b]d} + (R_{ab}{}^c{}_d - \beta_{ab}\delta_d{}^c - \delta_{[a}{}^c \mathrm{P}_{b]d})\Upsilon_c \\ &= 2\nabla_{[a} \mathrm{P}_{b]d} + C_{ab}{}^c{}_d \Upsilon_c,\end{aligned} \tag{4.12}$$

where we again used Lemma 2.2.4, the decomposition (4.1) of the curvature and $\beta_{ab} = -2\mathrm{P}_{[ab]}$.

Corollary 4.2.2. The tractor connection is flat if and only if $C_{ab}{}^c{}_d = 0$ for $n \geq 3$ or $\nabla_{[a} \mathrm{P}_{b]d} = 0$ for $n = 2$.

Proof. If $n = 2$, then by symmetry considerations, the Weyl tensor vanishes. Therefore, this case is obvious. If $n \geq 3$, then the vanishing of the curvature implies $C_{ab}{}^c{}_d = 0$. Conversely, if the Weyl tensor vanishes, then by Lemma 2.2.6, also $\nabla_{[a} \mathrm{P}_{b]d} = 0$. □

Remark 4.2.2. Any diffeomorphism $f : M \to N$, with M and N two smooth manifolds lifts to a diffeomorphism on the cotractor bundle. Indeed, we have seen in section 2.4, that any diffeomorphism lifts to the linear frame bundle and thus also lifts to the associated bundle $\mathcal{E}(1)$. Furthermore, any diffeomorphism on $\mathcal{E}(1)$ lifts to a diffeomorphism on $J^1\mathcal{E}(1)$ and hence to the cotractor bundle. If in addition, the diffeomorphism $f : M \to N$ is a morphism of projective structures, i.e. a diffeomorphism $f : (M, [\nabla]) \to (N, [\tilde{\nabla}])$, satisfying $f^*\tilde{\nabla}$ is projectively equivalent to ∇, then it is compatible with the tractor connection. Therefore, the construction of tractor bundles is functorial.

4.3 The Tractor Bundle

We now continue with constructing natural bundles from the cotractor bundle. In this section, we consider the dual bundle of the cotractor bundle, called the tractor bundle. It carries the dual connection to the tractor connection, which we will denote by the same name. Also, we calculate its curvature. In the second part of this section, we will see how the tractor bundle and its connection give rise to a Cartan geometry of type $(SL(n, \mathbb{R}), P)$, compare section 3.2.

Definition 4.3.1. The tractor bundle $\mathcal{T}$ is defined as the dual bundle of the cotractor bundle.

Recall from Theorem 4.2.1, that we have a short exact sequence of vector bundles

$$0 \to \mathcal{E}_a(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0,$$

which induces a short exact sequence for the dual bundle:

$$0 \to \mathcal{E}(-1) \to \mathcal{T} \to \mathcal{E}^a(-1) \to 0, \tag{4.13}$$

with notation as is section 4.1. To summerize this composition structure, we write $\mathcal{T} = \mathcal{E}(-1) \mathbin{+\!\!\!\!\supset} \mathcal{E}^a(-1)$. Fixing a projective structure $(M, [\nabla])$ splits the cotractor bundle $\mathcal{T}^* = \mathcal{E}_a(1) \oplus \mathcal{E}(1)$ and we again denote elements by (μ_c, σ) via this splitting. By

$$\langle (\mu_c, \sigma), \left(\begin{smallmatrix} \nu^b \\ \tau \end{smallmatrix}\right) \rangle := \mu_a \nu^a + \tau\sigma$$

we define the splitting $\mathcal{T} = \mathcal{E}(-1) \oplus \mathcal{E}^a(-1)$.

Lemma 4.3.1. If we change the connection ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M, then the splitting changes according to

$$\widehat{\begin{pmatrix} \nu^b \\ \tau \end{pmatrix}} = \begin{pmatrix} \nu^b \\ \tau - \Upsilon_b \nu^b \end{pmatrix}.$$

Proof. By the definition of the splittings on $\mathcal{T}$, we have that $\langle (\mu_a, \sigma), \left(\begin{smallmatrix} \nu^b \\ \tau \end{smallmatrix}\right) \rangle = \langle \widehat{(\mu_a, \sigma)}, \widehat{\left(\begin{smallmatrix} \nu^b \\ \tau \end{smallmatrix}\right)} \rangle$. Working this out we obtain

$$\mu_b \nu^b + \tau\sigma = \mu_b \hat{\nu}^b + (\Upsilon_b \hat{\nu}^b + \hat{\tau})\sigma,$$

and thus the result. $\square$

The tractor connection on $\mathcal{T}$ is canonically the dual connection to the tractor connection on $\mathcal{T}^*$. The following theorem computes this connection in terms of splittings. Note, that as in the case of the cotractor bundle, a splitting of $\mathcal{T}$ induces a splitting of $\mathcal{E}_a \otimes \mathcal{T}$.

Theorem 4.3.1. The connection $\nabla^{\mathcal{T}}$ on the tractor bundle dual to $\nabla^{\mathcal{T}^*}$ is given by

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \nu^b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \nu^b + \tau \delta_a{}^b \\ \nabla_a \tau - \mathrm{P}_{ab} \nu^b \end{pmatrix},$$

where the splitting on the right side is the induced splitting on $\mathcal{E}_a \otimes \mathcal{T}$.

Proof. Given the connection on $\mathcal{T}^*$ the connection on its dual bundle satisfies the following relation:

$$\langle (\mu_b, \sigma), \nabla_a \left(\begin{smallmatrix} \nu^c \\ \tau \end{smallmatrix}\right) \rangle + \langle \nabla_a (\mu_b, \sigma), \left(\begin{smallmatrix} \nu^c \\ \tau \end{smallmatrix}\right) \rangle = \nabla_a \langle (\mu_b, \sigma), \left(\begin{smallmatrix} \nu^c \\ \tau \end{smallmatrix}\right) \rangle,$$

where we denote all connections by the same symbol. Therefore, we obtain:

$$\begin{aligned}
\langle(\mu_b,\sigma),\nabla_a\left(\begin{smallmatrix}\nu^c\\ \tau\end{smallmatrix}\right)\rangle &= \nu^b\nabla_a\mu_b+(\nabla_a\nu^b)\mu_b+\tau\nabla_a\sigma+(\nabla_a\tau)\sigma\\
&\quad -(\nu^b(\nabla_a\mu_b)+\mathrm{P}_{ab}\nu^b\sigma+\tau\nabla_a\sigma-\tau\mu_a)\\
&= (\nabla_a\nu^b+\tau\delta_a{}^b)\mu_b+(\nabla_a\tau-\mathrm{P}_{ab}\nu^b)\sigma\\
&= \langle(\mu_b,\sigma),\left(\begin{smallmatrix}\nabla_a\nu^c+\tau\delta_a{}^c\\ \nabla_a\tau-\mathrm{P}_{ac}\nu^c\end{smallmatrix}\right)\rangle.
\end{aligned}$$

□

It is again clear by construction, that the tractor connection transforms correctly under change of connection in the projective class of M. However, it can be checked analogously to Theorem 4.2.2, using Lemma 4.3.1, Theorem 4.1.1 with $\omega=1$, the second part of Lemma 4.1.2 with $k=\omega=1$ and Theorem 4.3.1.

Similarly to the case of the cotractor bundle, we calculate the curvature of the tractor connection on $\mathcal{T}$. Note, that as in the case of the cotractor bundle, we have to consider coupled connections and we denote all types of connections by the same symbol. Given a splitting, the curvature satisfies

$$(\nabla_a\nabla_b-\nabla_b\nabla_a)\left(\begin{smallmatrix}\nu^d\\ \tau\end{smallmatrix}\right)=\left(\begin{smallmatrix}A_{ab}{}^d{}_e & \xi_{ab}{}^d\\ \phi_{abf} & \psi_{ab}\end{smallmatrix}\right)\left(\begin{smallmatrix}\nu^c\\ \tau\end{smallmatrix}\right).$$

Lemma 4.3.2. Let ∇ be an affine connection on M with curvature $R_{ab}{}^c{}_d$ and decomposition of curvature (4.1). The curvature of the tractor connection on the tractor bundle is given by

$$\begin{pmatrix} C_{ab}{}^c{}_d & 0\\ -2\nabla_{[a}\mathrm{P}_{b]d} & 0\end{pmatrix}.$$

Proof. Consider splittings of $\mathcal{T}$ and $\mathcal{T}^*$ with elements (μ_d,σ) and $\left(\begin{smallmatrix}\nu^c\\ \tau\end{smallmatrix}\right)$. Then the curvature on $\mathcal{T}$ is of the form $\left(\begin{smallmatrix}A_{ab}{}^d{}_e & \xi_{ab}{}^d\\ \phi_{abf} & \psi_{ab}\end{smallmatrix}\right)$ and satisfies:

$$\langle(\mu_h,\sigma),\left(\begin{smallmatrix}A_{ab}{}^d{}_e & \xi_{ab}{}^d\\ \phi_{abf} & \psi_{ab}\end{smallmatrix}\right)\left(\begin{smallmatrix}\nu^g\\ \tau\end{smallmatrix}\right)\rangle+\langle(\mu_h,\sigma)\left(\begin{smallmatrix}-C_{ab}{}^c{}_d & 0\\ 2\nabla_{[a}\mathrm{P}_{b]d} & 0\end{smallmatrix}\right),\left(\begin{smallmatrix}\nu^g\\ \tau\end{smallmatrix}\right)\rangle=0.$$

Therefore, we obain:

$$\langle(\mu_h,\sigma),\left(\begin{smallmatrix}A_{ab}{}^d{}_e\nu^e+\xi_{ab}{}^d\tau\\ \phi_{abf}\nu^f+\psi_{ab}\tau\end{smallmatrix}\right)\rangle=-\langle(-C_{ab}{}^c{}_d\mu_c+2\nabla_{[a}\mathrm{P}_{b]d}\sigma,0),\left(\begin{smallmatrix}\nu^g\\ \tau\end{smallmatrix}\right)\rangle$$

and thus

$$A_{ab}{}^d{}_e(\nu^e\mu_d)+\xi_{ab}{}^d(\tau\mu_d)+\phi_{abf}(\nu^f\sigma)+\psi_{ab}(\tau\sigma)=C_{ab}{}^d{}_e(\nu^e\mu_d)-2\nabla_{[a}\mathrm{P}_{b]f}(\nu^f\sigma),$$

which gives the result. □

Since the curvature on $\mathcal{T}$ only differs by sign from the curvature on $\mathcal{T}^*$, it also transforms correctly under the change of connection in the projective class of M.

Relation to Cartan Geometries

For oriented manifolds M, the tractor bundle gives rise to a Cartan geometry of type (G, P), with $G = SL(n, \mathbb{R})$ and P the stabilizer of the ray through the first basis vector in $\mathbb{R}^{n+1}$, compare chapter 3.2.

Recall that we have the short exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{T} \to \mathcal{E}^a(-1) \to 0. \tag{4.14}$$

Thus $\mathcal{T}$ contains a distinguished line bundle, which is $\mathcal{E}(-1)$. Furthermore, the orientation on M induces an orientation on $\Lambda^n T^*M$ and hence on all density bundles. In particular, the distinguished line bundle $\mathcal{E}(-1)$ is oriented. Furthermore, there exists a distinguished volume form on $\mathcal{T}$. The following calculation shows, that the highest exterior power of $\mathcal{T}$ is trivial:

$$\begin{aligned} \Lambda^{n+1}\mathcal{T} &\cong \mathcal{E}(-1) \otimes \Lambda^n \mathcal{E}^a(-1) \cong (\Lambda^n \mathcal{E}^a) \otimes \mathcal{E}(-n-1) \\ &\cong (\Lambda^n \mathcal{E}^a) \otimes (\Lambda^n \mathcal{E}_a) \cong \mathbb{R} \end{aligned} \tag{4.15}$$

The first isomorphism is due to the fact that highest exterior power of a decomposition given by an exact sequence is isomorphic to the tensor product of the highest exterior power of the two components. Furthermore, in section 4.1 we mentioned that the volume bundle is isomorphic to $\Lambda^n \mathcal{E}_a$ in the case of oriented manifolds. Hence $\mathcal{E}(-n-1) \cong \Lambda^n \mathcal{E}_a$, which explains the third isomorphism. This construction fits together with the induced connection, hence $\Lambda^{n+1}\mathcal{T}$ is a trivial bundle with an induced flat connection. Thus there exists a parallel section, which is unique up to multiplication with a constant. We choose one of these parallel sections and fix the volume form on $\mathcal{T}$, which is induced by the chosen section.

On $\mathbb{R}^{n+1}$ we always work with the standard orientation and volume form. An orientation on an $1-$dimensional linear subspace of $\mathbb{R}^{n+1}$ is determined by the choice of a positive and negative oriented ray. We define an orientation on the $1-$dimensional linear subspace, which is generated by the first vector e_1 of the standard basis, by fixing the ray through e_1 as the positive ray.

Consider the adapted frame bundle $\mathcal{G}$ for $\mathcal{T}$, i.e. the fiber $\mathcal{G}_x$ over $x \in M$ is given by the set of all linear isomorphisms $\varphi : \mathbb{R}^{n+1} \to \mathcal{T}_x$, which preserve the volume form and the line through e_1 with its orientation. This defines a principal bundle $\mathcal{G}$ with structure group $P \subset SL(n+1, \mathbb{R})$. Since the volume forms are preserved, we obtain a subgroup of $SL(n+1, \mathbb{R})$ and stabilizing an oriented line is the same as stabilizing a ray. The principal right action is given by composition from the right.

Now consider the following map:

$$\phi : \mathcal{G} \times \mathbb{R}^{n+1} \to \mathcal{T}$$

$$(\psi, v) \mapsto \psi(v).$$

Note that for $A \in P$ we have $\phi(\psi \circ A, A^{-1}v) = (\psi \circ A)(A^{-1}v) = \phi(\psi, v)$ and thus ϕ factors to a well-defined map on the associated bundle $\mathcal{G} \times_P \mathbb{R}^{n+1} \to \mathcal{T}$. This map covers the identity on M and is a bijection on each fiber and thus an isomorphism of vector bundles.

Following [4], we will now prove that the tractor connection induces a Cartan connection on $\mathcal{G}$. Since $\mathcal{T} = \mathcal{G} \times_P \mathbb{R}^{n+1}$, we can identify sections of $\mathcal{T}$ with $P-$equivariant functions $\mathcal{G} \to \mathbb{R}^{n+1}$, see [9, Prop. 1.2.7.]. If $s \in \Gamma(\mathcal{T})$, then the corresponding equivariant function is given by $f : \mathcal{G} \to \mathbb{R}^{n+1}$, $f(u) = u^{-1}(s(p(u)))$, for $u \in \mathcal{G}$. Furthermore, the equivariancy condition reads as $f(u \cdot g) = g^{-1} \cdot f(u)$, for $u \in \mathcal{G}$ and $g \in G$.

Starting with a point $u \in \mathcal{G}$, satisfying $p(u) = x$, and a tangent vector $\xi \in T_u\mathcal{G}$, we have that $\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x) \in \mathcal{T}_x$. Thus we can consider the element $u^{-1}(\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x)) - \xi \cdot f(u) \in \mathbb{R}^{n+1}$. Given a smooth function $g : M \to \mathbb{R}$, we obtain

$$\nabla^{\mathcal{T}}_{T_u p \cdot \xi} g s(x) = s(x)(T_u p \cdot \xi) \cdot g(x) + g(x) \nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x).$$

Furthermore, the equivariant function corresponding to gs is given by $(p^*g)f$ and thus $\xi \cdot ((p^*g)f(u)) = \xi \cdot (p^*g)(u)f(u) + (p^*g)(u)\xi \cdot f(u)$. This shows, that the element $u^{-1}(\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x)) - \xi \cdot f(u)$ only depends on $s(x)$ and therefore ξ gives a linear map $\omega(\xi) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, satisfying the following equation:

$$u^{-1}(\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x)) = \xi \cdot f(u) + \omega(\xi)(f(u)). \tag{4.16}$$

It can be shown that $\omega(\xi)$ is actually an element of $\mathfrak{sl}(n+1, \mathbb{R})$, by using the fact that the volume form on $\mathcal{T}$ is preserved by the tractor connection. In addition, we notice that $\omega(\xi)$ is uniquely determined by ξ and thus gives a well-defined map $\omega : T_u\mathcal{G} \to \mathfrak{g}$. Furthermore, from the equation it can be seen, that ω considered as a map $\omega : T\mathcal{G} \to \mathfrak{g}$ is smooth. In order to show that ω is a Cartan connection on $\mathcal{G}$, we have to prove that properties (2.8) are satisfied.

We start by showing that ω reproduces fundamental vector fields, hence we have to show that $\omega(\zeta_A) = A$ for all $A \in \mathfrak{g}$. Recall that the fundamental vector field is defined by $\zeta_A(u) = \frac{d}{dt}|_{t=0} u \cdot \exp(tA)$ for $u \in \mathcal{G}$ and $A \in \mathfrak{g}$. Furthermore, fundamental vector fields are vertical, hence satisfy $Tp \cdot \zeta_A = 0$. In Claim 3 we have seen, that $\zeta_A \cdot f = -A \cdot f$ for equivariant functions f. Therefore

$$\begin{aligned} \omega(\zeta_A)(f(u)) &= u^{-1}(\nabla^{\mathcal{T}}_{Tp \cdot \zeta_A} s(x)) - \zeta_A \cdot f(u) \\ &= -\zeta_A \cdot f(u) \\ &= A \cdot f(u) \end{aligned}$$

and since the action of $\mathfrak{g}$ is effective, we obtain $\omega(\zeta_A) = A$. Note, that this also shows that ω is injective on each vertical subspace.

Secondly, we have to show that ω is equivariant with respect to the principal right action r on $\mathcal{G}$, i.e. $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for all $g \in G$. Since the principal right action preserves the fibers, hence $p \circ r^g = p$, we have that $Tp \circ Tr^g = Tp$. For $u \in \mathcal{G}$ and $\xi \in T_u\mathcal{G}$ we obtain:

$$\begin{aligned} ((r^g)^*\omega)(u)(\xi)(f(u)) &= \omega(u \cdot g)(T_u r^g \cdot \xi)(f(u \cdot g)) \\ &= (u \cdot g)^{-1}(\nabla^{\mathcal{T}}_{Tp \cdot Tr^g \cdot \xi} s(x)) - ((T_u r^g \cdot \xi) \cdot f)(u \cdot g) \\ &= g^{-1} \cdot u^{-1}(\nabla^{\mathcal{T}}_{Tp \cdot \xi} s(x)) - (T_u(f \circ r^g) \cdot \xi) \\ &= g^{-1} \cdot u^{-1}(\nabla^{\mathcal{T}}_{Tp \cdot \xi} s(x)) - \xi \cdot (f \circ r^g)(u). \end{aligned} \tag{4.17}$$

By the equivariancy of f and linearity of the action of G, we have that $\xi \cdot (f \circ r^g) = \xi \cdot (g^{-1} \cdot f) = g^{-1} \cdot (\xi \cdot f)$ and therefore we can continue with equation (4.17):

$$\begin{aligned} \omega(u \cdot g)(T_u r^g \cdot \xi)(f(u \cdot g)) &= g^{-1} \cdot (u^{-1}(\nabla^{\mathcal{T}}_{Tp \cdot \xi} s(x)) - (\xi \cdot f)(u)) \\ &= g^{-1} \cdot (\omega(u)(\xi)(f(u))) \\ &= \mathrm{Ad}(g^{-1})(\omega(u)(\xi))(g^{-1} \cdot f(u)) \\ &= \mathrm{Ad}(g^{-1})(\omega(u)(\xi))(f(u \cdot g)). \end{aligned}$$

Since the action is effective, we obtain $\omega(u \cdot g)(T_u r^g \cdot \xi) = \mathrm{Ad}(g^{-1})(\omega(u)(\xi))$ and therefore $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$.

Finally, we have to show that $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism. Since $\mathcal{G}$ and $\mathfrak{g}$ have the same dimension, it suffices to show that $\omega(u)$ is injective. Consider a tangent vector $\xi \in T_u\mathcal{G}, \xi \neq 0$, such that $T_u p \cdot \xi \neq 0$. Furthermore, consider the subbundle $\mathcal{S} \subset \mathcal{T}$, consisting of all pairs with vanishing first slot, in a given splitting and a section $s \in \Gamma(\mathcal{S})$. Then we have $s(x) \in \mathcal{S}_x$, but $\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x) \notin \mathcal{S}_x$. Given a splitting of $\mathcal{T}$, let s have the form $\binom{0}{\tau}$ in this splitting. Then by Theorem 4.3.1, we obtain

$$\nabla^{\mathcal{T}}_{T_u p \cdot \xi} \binom{0}{\tau} = \binom{T_u p \cdot \xi \tau}{\nabla_{T_u p \cdot \xi} \tau},$$

which is not an element of $\mathcal{S}$. On the other hand, if s corresponds to the equivariant function $f : \mathcal{G} \to \mathbb{R}^{n+1}, f(u) = u^{-1}(s(x))$, then certainly $u(f(u)) \in \mathcal{S}_x$. Since $u : \mathbb{R}^{n+1} \to \mathcal{T}$ is an isomorphism, we also have $u(\xi \cdot f(u)) \in \mathcal{S}_x$. Thus by equation (4.16), $\omega(u)(\xi) \neq 0$, because otherwise, $\nabla^{\mathcal{T}}_{T_u p \cdot \xi} s(x)$ would be an element of $\mathcal{S}_x$. We already mentioned, that $\omega(u)$ is injective on each vertical subspace, which covers the case of $T_u p \cdot \xi = 0$.

In fact the converse is also true. By construction, the standard tractor bundle is the associated bundle to $\mathcal{G}$ with respect to the standard action and equation (4.16) gives rise to a tractor connection, when starting with a Cartan connection.

Any morphism $f : (M, [\nabla]) \to (N, [\tilde{\nabla}])$ of projective structures, lifts to an automorphism of the associated Cartan geometries. Denoting the Cartan geometry associated to M by $(\mathcal{G}_M, \omega_M)$ and the one associated to N by $(\mathcal{G}_N, \omega_N)$, then there exists a principal bundle morphism $\tilde{f} : \mathcal{G}_M \to \mathcal{G}_N$, satisfying $\tilde{f}^*\omega_N = \omega_M$. We already mentioned in Remark 4.2.2, that any morphism of projective structures lifts to a diffeomorphism on the cotractor bundle, which is compatible with the cotractor connection. Since the tractor bundle is obtain from the cotractor bundle by dualizing, the same is true for the tractor bundle. Any diffeomorphism of the tractor bundles certainly restricts to a principal bundle morphism $\mathcal{G}_M \to \mathcal{G}_N$ and the compatibility with the tractor connections gives $\tilde{f}^*\omega_N = \omega_M$ by equation (4.16).

This also shows, that the transformations of the homogeneous model are exactly the automorphisms of the projective structures. Recall from section 3.2, that the action of G on S^n is an automorphism of projective structures. Now conversely, consider an automorphism of projective structures $f : (S^n, [\nabla]) \to (S^n, [\nabla])$ with ∇ the Levi-Civita connection. The Cartan geometry of S^n is given by $G \to G/P \cong S^n$ with the Maurer-Cartan form. Thus by the above paragraph the automorphism f lifts to an automorphism of the Cartan geometry $(G \to G/P, \omega^{MC})$ which, by [9, Proposition 1.5.2], is given by the left multiplication with an element of G.

4.4 Symmetric Powers of Tractor Bundles

In this section we discuss symmetric powers of the tractor bundle and the cotractor bundle, which are natural constructions on these bundles. Again, the symmetric powers carry natural connections and we calculate their curvatures.

Symmetric Power of the Cotractor Bundle

We start by inducing a splitting on the symmetric tensor power of the cotractor bundle from a splitting of the cotractor bundle.

The short exact sequence of Theorem 4.2.1

$$0 \to \mathcal{E}_a(1) \to \mathcal{T}^* \to \mathcal{E}(1) \to 0,$$

which we summerize by $\mathcal{T}^* = \mathcal{E}_a(1) \mathbin{+\!\!\!\!\supset} \mathcal{E}(1)$, induces a composition of the symmetric power:

$$S^2\mathcal{T}^* = \mathcal{E}_{(ab)}(2) \mathbin{+\!\!\!\!\supset} \mathcal{E}_a(2) \mathbin{+\!\!\!\!\supset} \mathcal{E}(2).$$

This is due to the following: $\mathcal{E}_{(ab)}(2) = S^2\mathcal{E}_a(1) \hookrightarrow S^2\mathcal{T}^*$ and $\mathcal{E}_a(2) = \mathcal{E}_a \otimes \mathcal{E}(1)$ includes into the quotient $S^2\mathcal{T}^*/\mathcal{E}_{(ab)}(2)$. Furthermore, the quotient of the last map is given by $S^2\mathcal{E}(1) \cong \mathcal{E}(2)$.

View elements of $S^2\mathcal{T}^*$ as bilinear maps on $\mathcal{T}$ and choose a connection in the projective class of M giving rise to splitting of $\mathcal{T}$ with elements $\binom{\nu^d}{\tau}$. Then by

$$\begin{pmatrix}\mu_{ab} & \eta_a & \sigma\end{pmatrix}\left(\binom{\nu^d}{\tau}, \binom{\tilde{\nu}^e}{\tilde{\tau}}\right) := \mu_{ab}\nu^a\tilde{\nu}^b + \eta_a\nu^a\tilde{\tau} + \eta_a\tilde{\nu}^a\tau + \sigma\tau\tilde{\tau},$$

we define the splitting of $S^2\mathcal{T}^*$. Note that μ_{ab} is symmetric.

Lemma 4.4.1. If we change the connection ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M, then the splitting of $S^2\mathcal{T}^*$ changes according to

$$\widehat{\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}} = \begin{pmatrix}\mu_{bc} + 2\Upsilon_{(b}\eta_{c)} + \Upsilon_b\Upsilon_c\sigma & \eta_b + \Upsilon_b\sigma & \sigma\end{pmatrix}.$$

Proof. This calculation uses Lemma 4.3.1. Consider elements $\binom{\nu^d}{\tau}$ and $\binom{\tilde{\nu}^d}{\tilde{\tau}}$ in $\mathcal{T}$. Then we have:

$$\begin{aligned}
\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\left(\binom{\nu^d}{\tau}, \binom{\tilde{\nu}^d}{\tilde{\tau}}\right) &= \widehat{\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}}\left(\widehat{\binom{\nu^d}{\tau}}, \widehat{\binom{\tilde{\nu}^d}{\tilde{\tau}}}\right)\\
&= \hat{\mu_{bc}}\nu^b\tilde{\nu}^c + \hat{\eta}_b\nu^b(\tilde{\tau} - \Upsilon_c\tilde{\nu}^c) + \hat{\eta}_b\tilde{\nu}^b(\tau - \Upsilon_c\nu^c)\\
&\quad + \hat{\sigma}(\tau - \Upsilon_c\nu^c)(\tilde{\tau} - \Upsilon_c\tilde{\nu}^c)\\
&= (\hat{\mu}_{bc} - 2\hat{\eta}_{(b}\Upsilon_{c)} + \hat{\sigma}\Upsilon_b\Upsilon_c)(\nu^b\tilde{\nu}^c)\\
&\quad + (\hat{\eta}_b - \hat{\sigma}\Upsilon_b)(\nu^b\tilde{\tau}) + (\hat{\eta}_b - \hat{\sigma}\Upsilon_b)(\tilde{\nu}^b\tau) + \hat{\sigma}\tau\tilde{\tau}.
\end{aligned}$$

Comparing both sides of the equation yields:

$$\hat{\sigma} = \sigma, \quad \hat{\eta}_b = \eta_b + \Upsilon_b\sigma \quad \text{and} \quad \hat{\mu}_{bc} = \mu_{bc} + 2\Upsilon_{(b}\eta_{c)} + \Upsilon_b\Upsilon_c\sigma. \qquad \square$$

The following theorem computes the connection on $S^2\mathcal{T}^*$ induced by the tractor connection in terms of splittings. Note, that again a splitting of $S^2\mathcal{T}^*$ induces a splitting of $\mathcal{E}_a \otimes S^2\mathcal{T}^*$.

Theorem 4.4.1. The induced connection on $S^2\mathcal{T}^*$ is given by

$$\nabla_a^{S^2\mathcal{T}^*}\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix} = \begin{pmatrix}\nabla_a\mu_{bc} + 2\mathrm{P}_{a(b}\eta_{c)} & \nabla_a\eta_b + \mathrm{P}_{ab}\sigma - \mu_{ab} & \nabla_a\sigma - 2\eta_a\end{pmatrix},$$

where the splitting on the right hand side is the induced splitting on $\mathcal{E}_a \otimes S^2\mathcal{T}^*$.

Proof. Given the tractor connection on $\mathcal{T}$ and a bilinear form Φ, the induced connection is given by

$$(\nabla_a\Phi)(X, Y) = \nabla_a(\Phi(X, Y)) - \Phi(\nabla_a X, Y) - \Phi(X, \nabla_a Y), \tag{4.18}$$

where we denote all connections by the same symbol. Working out each part on the right side seperately, we obtain

$$\begin{aligned}\nabla_a \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix} \left(\begin{pmatrix}\nu^d \\ \tau\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^d \\ \tilde{\tau}\end{pmatrix}\right) &= \nabla_a(\mu_{bc}\nu^b\tilde{\nu}^c + \eta_b\nu^b\tilde{\tau} + \eta_a\tilde{\nu}^b\tau + \sigma\tau\tilde{\tau}) \\ &= \nabla_a\mu_{bc}(\nu^b\tilde{\nu}^c) + \mu_{bc}\nabla_a\nu^b\tilde{\nu}^c + \nabla_a\eta_b(\nu^b\tilde{\tau}) + \eta_b\nabla_a\nu^b\tilde{\tau} \\ &\quad + \nabla_a\eta_b(\tilde{\nu^b}\tau) + \eta_b\nabla_a\tilde{\nu}^b\tau + \nabla_a\sigma(\tau\tilde{\tau}) + \sigma\nabla_a\tau\tilde{\tau}.\end{aligned}$$

For the second part we use the tractor connection from Theorem 4.3.1:

$$\begin{aligned}\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix} \left(\nabla_a\begin{pmatrix}\nu^d \\ \tau\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^d \\ \tilde{\tau}\end{pmatrix}\right) &= \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix} \left(\begin{pmatrix}\nabla_a\nu^d + \tau\delta_a{}^d \\ \nabla_a\tau - \mathrm{P}_{ac}\nu^c\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^d \\ \tilde{\tau}\end{pmatrix}\right) \\ &= \mu_{bc}(\nabla_a\nu^b)\tilde{\nu}^c + \mu_{ac}\tau\tilde{\nu}^c + \eta_b(\nabla_a\nu^b)\tilde{\tau} + \eta_a\tau\tilde{\tau} \\ &\quad + \eta_b\tilde{\nu}^b\nabla_a\tau - \eta_b\tilde{\nu}^b\mathrm{P}_{ac}\nu^c + \sigma\tilde{\tau}\nabla_a\tau - \sigma\tilde{\tau}\mathrm{P}_{ac}\nu^c\end{aligned}$$

By exchanging $\begin{pmatrix}\nu^d \\ \tau\end{pmatrix}$ and $\begin{pmatrix}\tilde{\nu}^d \\ \tilde{\tau}\end{pmatrix}$ in the above expression, we get the third part of the right side of equation (4.18). Combining all three parts, we obtain:

$$\begin{aligned}(\nabla_a \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix})\left(\begin{pmatrix}\nu^d \\ \tau\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^d \\ \tilde{\tau}\end{pmatrix}\right) &= (\nabla_a\mu_{bc} + 2\mathrm{P}_{a(b}\eta_{c)})\nu^b\tilde{\nu}^c \\ &\quad + (\nabla_a\eta_b + \mathrm{P}_{ab}\sigma - \mu_{ab})\nu^b\tilde{\tau} \\ &\quad + (\nabla_a\eta_b + \mathrm{P}_{ab}\sigma - \mu_{ab})\tilde{\nu}^b\tau + (\nabla_a\sigma - 2\eta_a)\tau\tilde{\tau},\end{aligned}$$

which gives the result. □

Again by construction, the tractor connection on $S^2\mathcal{T}^*$ transforms correctly under change of connection in the projective class of M. Nevertheless, it can be checked directly by using Theorem 4.1.1 and Lemma 4.1.2 in the cases $k = 2, \omega = 2$ and $k = 1, \omega = 2$.

We now want to calculate the curvature of the induced connection on $S^2\mathcal{T}^*$. Similarly to the cases of the cotractor and tractor bundle, we have to consider $L(S^2\mathcal{T}^*, S^2\mathcal{T}^*)$, as the curvature is a section of $\mathcal{E}_{[ab]} \otimes L(S^2\mathcal{T}^*, S^2\mathcal{T}^*)$. Again, we view elements of $L(S^2\mathcal{T}^*, S^2\mathcal{T}^*)$ as matrices and for a splitting of $S^2\mathcal{T}^*$, we define the induced splitting by

$$\begin{aligned}&\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\begin{pmatrix}A_{ef}{}^{bc} & B_e{}^{bc} & C^{bc} \\ \psi_{ef}{}^b & \xi_e{}^b & \zeta^b \\ \phi_{ef} & \rho_e & \pi\end{pmatrix} \qquad (4.19) \\ :=\; &\begin{pmatrix}A_{ef}{}^{bc}\mu_{bc} + \psi_{ef}{}^b\eta_b + \phi_{ef}\sigma & B_e{}^{bc}\mu_{bc} + \xi_e{}^b\eta_b + \rho_e\sigma & C^{bc}\mu_{bc} + \zeta^b\eta_b + \pi\sigma\end{pmatrix}.\end{aligned}$$

If we change the connection ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ on M, then this splitting changes to

$$\begin{pmatrix}\widehat{A_{ef}{}^{bc}} & \widehat{B_e{}^{bc}} & \widehat{C^{bc}} \\ \widehat{\psi_{ef}{}^b} & \widehat{\xi_e{}^b} & \widehat{\zeta^b} \\ \widehat{\phi_{ef}} & \widehat{\rho_e} & \widehat{\pi}\end{pmatrix},$$

with first column given by

$$
\begin{aligned}
\widehat{A_{ef}{}^{bc}} &= A_{ef}{}^{bc} + 2\Upsilon_{(e}B_{f)}{}^{bc} + \Upsilon_e\Upsilon_f C^{bc},\\
\widehat{\psi_{ef}{}^{b}} &= \psi_{ef}{}^{b} - 2A_{ef}{}^{(bc)}\Upsilon_c + 2\Upsilon_{(e}\xi_{f)}{}^{b} + \Upsilon_e\Upsilon_f\zeta^b - 4\Upsilon_{(e}B_{f)}{}^{(bc)}\Upsilon_c - 2\Upsilon_e\Upsilon_f\Upsilon_c C^{(bc)},\\
\widehat{\phi_{ef}} &= \phi_{ef} - \psi_{ef}{}^{b}\Upsilon_b + A_{ef}{}^{cb}\Upsilon_b\Upsilon_c + 2\Upsilon_{(e}\rho_{f)} + \Upsilon_e\Upsilon_f\pi - 2\Upsilon_{(e}\xi_{f)}{}^{b}\Upsilon_b\\
&\quad - \Upsilon_e\Upsilon_f\Upsilon_b\zeta^b + 2\Upsilon_{(e}B_{f)}{}^{cb}\Upsilon_c\Upsilon_b + \Upsilon_e\Upsilon_f\Upsilon_c\Upsilon_b C^{cb},
\end{aligned}
$$

second column by

$$
\begin{aligned}
\widehat{B_e{}^{bc}} &= B_e{}^{bc} + C^{bc}\Upsilon_e,\\
\widehat{\xi_e{}^{b}} &= \xi_e{}^{b} - 2\Upsilon_c B_e{}^{(bc)} + \Upsilon_e\zeta^b - 2C^{(bc)}\Upsilon_e\Upsilon_c,\\
\widehat{\rho_e} &= \rho_e + \Upsilon_e\pi - \Upsilon_b\xi_e{}^{b} - \zeta^b\Upsilon_b\Upsilon_e + B_e{}^{cb}\Upsilon_b\Upsilon_c + \Upsilon_c\Upsilon_e\Upsilon_b C^{cb}
\end{aligned}
$$

and the third column by

$$
\begin{aligned}
\widehat{C^{bc}} &= C^{bc},\\
\widehat{\zeta^{b}} &= \zeta^b - 2\Upsilon_c C^{(bc)},\\
\widehat{\pi} &= \pi - \zeta^b\Upsilon_b + C^{cb}\Upsilon_b\Upsilon_c.
\end{aligned}
$$

This can be shown by working out the following equation:

$$
\begin{aligned}
&\widehat{\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\begin{pmatrix} A_{ef}{}^{bc} & B_e{}^{bc} & C^{bc}\\ \psi_{ef}{}^{b} & \xi_e{}^{b} & \zeta^b\\ \phi_{ef} & \rho_e & \pi\end{pmatrix}}\\
= \quad &\widehat{\begin{pmatrix} A_{ef}{}^{bc}\mu_{bc} + \psi_{ef}{}^{b}\eta_b + \phi_{ef}\sigma & B_e{}^{bc}\mu_{bc} + \xi_e{}^{b}\eta_b + \rho_e\sigma & C^{bc}\mu_{bc} + \zeta^b\eta_b + \pi\sigma\end{pmatrix}}.
\end{aligned}
$$

Since the calculation is analogous to the case of the cotractor bundle (see (4.6)), it is ommited here.

As in the cases of the cotractor and tractor bundle, in order to write down the defining equation for the curvature, we have to consider coupled connections. We will again denote all connections by the same symbol. Therefore, the curvature on $S^2\mathcal{T}^*$ satisfies:

$$
\left(\nabla_a\nabla_b - \nabla_b\nabla_a\right)\begin{pmatrix}\mu_{ef} & \eta_e & \sigma\end{pmatrix} = \begin{pmatrix}\mu_{cd} & \eta_c & \sigma\end{pmatrix}\begin{pmatrix} A_{abef}{}^{cd} & B_{abe}{}^{cd} & C_{ab}{}^{cd}\\ \psi_{abef}{}^{c} & \xi_{abe}{}^{c} & \zeta_{ab}{}^{c}\\ \phi_{abef} & \rho_{abe} & \pi_{ab}\end{pmatrix},
$$

with antisymmetric indices a and b.

Lemma 4.4.2. Let ∇ be an affine connection on M with curvature $R_{ab}{}^{c}{}_{d}$ and decomposition of the curvature (4.1). The curvature of the induced connection on $S^2\mathcal{T}^*$ is then given by

$$\begin{pmatrix} -C_{ab}{}^{c}{}_{e}\delta_f{}^d - C_{ab}{}^{d}{}_{f}\delta_e{}^c & 0 & 0 \\ 2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^c + 2(\nabla_{[a}\mathrm{P}_{b]f})\delta_e{}^c & -C_{ab}{}^{c}{}_{e} & 0 \\ 0 & 2\nabla_{[a}\mathrm{P}_{b]e} & 0 \end{pmatrix}.$$

Proof. The induced curvature R of a bilinear form Φ in $S^2\mathcal{T}^*$ on elements X, Y in $\mathcal{T}$ satisfies:

$$(R\cdot\Phi)(X,Y) = -\Phi(\hat{R}\cdot X, Y) - \Phi(X, \hat{R}\cdot Y), \tag{4.20}$$

where $\hat{R}$ denotes the curvature on $\mathcal{T}$, see Lemma 4.3.2. Now consider a splitting and let $\Phi = \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}$, $X = \begin{pmatrix}\nu^e \\ \tau\end{pmatrix}, Y = \begin{pmatrix}\tilde{\nu}^e \\ \tilde{\tau}\end{pmatrix}$ and the curvature on $S^2\mathcal{T}^*$ have the form

$$R = \begin{pmatrix} A_{abef}{}^{cd} & B_{abe}{}^{cd} & D_{ab}{}^{cd} \\ \psi_{abef}{}^{c} & \xi_{abe}{}^{c} & \zeta_{ab}{}^{c} \\ \phi_{abef} & \rho_{abe} & \pi_{ab} \end{pmatrix}.$$

Now working out the right side of equation (4.20), the first part yields:

$$\begin{aligned} \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\left(\hat{R}\cdot\begin{pmatrix}\nu^e \\ \tau\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^e \\ \tilde{\tau}\end{pmatrix}\right) &= \begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\left(\begin{pmatrix} C_{ab}{}^{e}{}_{f}\nu^f \\ -2\nabla_{[a}\mathrm{P}_{b]f}\nu^f\end{pmatrix}, \begin{pmatrix}\tilde{\nu}^e \\ \tilde{\tau}\end{pmatrix}\right) \\ &= \mu_{cd}C_{ab}{}^{c}{}_{f}\nu^f\tilde{\nu}^d + \eta_c C_{ab}{}^{c}{}_{f}\nu^f\tilde{\tau} \\ &\quad -2\eta_c\tilde{\nu}^c\nabla_{[a}\mathrm{P}_{b]f}\nu^f - 2\nabla_{[a}\mathrm{P}_{b]f}\nu^f\sigma\tilde{\tau}. \end{aligned}$$

By exchanging $\begin{pmatrix}\nu^e \\ \tau\end{pmatrix}$ with $\begin{pmatrix}\tilde{\nu}^e \\ \tilde{\tau}\end{pmatrix}$, we obtain the result for the second part of the right side of equation (4.20). Combining these two parts, we get:

$$\begin{aligned} &\Phi(\hat{R}\cdot X, Y) + \Phi(X, \hat{R}\cdot Y) \\ = &\ (C_{ab}{}^{c}{}_{e}\delta_f{}^d\mu_{cd} + C_{ab}{}^{d}{}_{f}\delta_e{}^c\mu_{cd} - 2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^c\eta_c - 2(\nabla_{[a}\mathrm{P}_{b]f})\delta_e{}^c\eta_c)\nu^e\tilde{\nu}^f \\ &+(\eta_c C_{ab}{}^{c}{}_{e} - 2(\nabla_{[a}\mathrm{P}_{b]e})\sigma)\nu^e\tilde{\tau} + (\eta_c C_{ab}{}^{c}{}_{e} - 2(\nabla_{[a}\mathrm{P}_{b]e})\sigma)\tilde{\nu}^e\tau. \end{aligned} \tag{4.21}$$

On the other hand, the left side of equation (4.20) yields:

$$R\cdot\Phi = \begin{pmatrix}\mu_{cd} & \eta_c & \sigma\end{pmatrix}\begin{pmatrix} A_{abef}{}^{cd} & B_{abe}{}^{cd} & D_{ab}{}^{cd} \\ \psi_{abef}{}^{c} & \xi_{abe}{}^{c} & \zeta_{ab}{}^{c} \\ \phi_{abef} & \rho_{abe} & \pi_{ab} \end{pmatrix} =$$

$$\begin{pmatrix} A_{abef}{}^{cd}\mu_{cd} + \psi_{abef}{}^{c}\eta_c + \phi_{abef}\sigma & B_{abe}{}^{cd}\mu_{cd} + \xi_{abe}{}^{c}\eta_c + \rho_{abe}\sigma & D_{ab}{}^{cd}\mu_{cd} + \zeta_{ab}{}^{c}\eta_c + \pi_{ab}\sigma \end{pmatrix}.$$

Applying this to $\begin{pmatrix}\nu^e \\ \tau\end{pmatrix}$ with $\begin{pmatrix}\tilde{\nu}^e \\ \tilde{\tau}\end{pmatrix}$, we obtain:

$$\begin{aligned} (R\cdot\Phi)(X,Y) =& (A_{abef}{}^{cd}\mu_{cd} + \psi_{abef}{}^{c}\eta_c + \phi_{abef}\sigma)\nu^e\tilde{\nu}^f \\ &+ (B_{abe}{}^{cd}\mu_{cd} + \xi_{abe}{}^{c}\eta_c + \rho_{abe}\sigma)\nu^e\tilde{\tau} + (B_{abe}{}^{cd}\mu_{cd} + \xi_{abe}{}^{c}\eta_c + \rho_{abe}\sigma)\tilde{\nu}^e\tau \\ &+ (D_{ab}{}^{cd}\mu_{cd} + \zeta_{ab}{}^{c}\eta_c + \pi_{ab}\sigma)\tau\tilde{\tau} \end{aligned}$$

Now comparing this with equation (4.21) immediately yields that all coefficients of $\tau\tilde{\tau}$ have to vanish and hence $D_{ab}{}^{cd} = \zeta_{ab}{}^{c} = \pi_{ab} = 0$. Continuing with the coefficients of $\nu^e\tilde{\tau}$, we obtain $B_{abe}{}^{cd} = 0, \xi_{abe}{}^{c} = C_{ab}{}^{c}{}_{e}$ and $\rho_{abe} = -2\nabla_{[a}\mathrm{P}_{b]e}$. Finally, combining the coefficients of $\nu^e\tilde{\nu}^f$ gives $A_{abef}{}^{cd} = C_{ab}{}^{c}{}_{e}\delta_f{}^d + C_{ab}{}^{d}{}_{f}\delta_e{}^c, \psi_{abef}{}^{c} = -2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^c - 2(\nabla_{[a}\mathrm{P}_{b]f})\delta_e{}^c$ and $\phi_{abef} = 0$. □

It is again clear by construction, that the curvature transforms correctly under the change of connection from ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M. It can however be checked analogously to the case of the cotractor bundle, by using equation (4.12) and the fact that the Weyl tensor is invariant.

Symmetric Power of the Tractor Bundle

The symmetric tensor power of the tractor bundle is approached analogously to the case of the cotractor bundle, compare the last section. Thus we start by inducing a composition of $S^2\mathcal{T}$ from the composition of the tractor bundle.

Using the short exact sequence

$$0 \to \mathcal{E}(-1) \to \mathcal{T} \to \mathcal{E}^a(-1) \to 0,$$

with composition $\mathcal{T} = \mathcal{E}(-1) \niplus \mathcal{E}^a(-1)$, we obtain a composition of $S^2\mathcal{T}$ by

$$S^2\mathcal{T} = \mathcal{E}(-2) \niplus \mathcal{E}^a(-2) \niplus \mathcal{E}^{(ab)}(-2).$$

As in the case of the cotractor bundle, we have inclusions $S^2\mathcal{E}(-1) \cong \mathcal{E}(-2) \hookrightarrow S^2\mathcal{T}$ and $\mathcal{E}(-1) \otimes \mathcal{E}^a(-1) = \mathcal{E}^a(-2) \hookrightarrow S^2\mathcal{T}/\mathcal{E}(-2)$, where the quotient of the last map is given by $S^2(\mathcal{E}^a(-1)) = \mathcal{E}^{(ab)}(-2)$.

View elements of $S^2\mathcal{T}$ as symmetric bilinear forms on $\mathcal{T}^*$ and choose a splitting of $\mathcal{T}^*$ with elements (μ_a, σ). Then by

$$\left(\begin{smallmatrix} \tau^{ab} \\ \rho^a \\ \nu \end{smallmatrix}\right)((\mu_a, \sigma), (\tilde{\mu}_a, \tilde{\sigma})) := \tau^{ab}\mu_a\tilde{\mu}_b + \rho^a\mu_a\tilde{\sigma} + \rho^a\tilde{\mu}_a\sigma + \nu\sigma\tilde{\sigma}, \tag{4.22}$$

we define the splitting of $S^2\mathcal{T}$. Note that τ^{ab} is symmetric.

Lemma 4.4.3. If we change the connection ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M, then the splitting of $S^2\mathcal{T}$ changes according to

$$\widehat{\begin{pmatrix} \tau^{ab} \\ \rho^a \\ \nu \end{pmatrix}} = \begin{pmatrix} \tau^{ab} \\ \rho^a - \Upsilon_b\tau^{ab} \\ \nu - 2\Upsilon_a\rho^a + \Upsilon_a\Upsilon_b\tau^{ab} \end{pmatrix}.$$

Proof. We will use Corollary 4.2.1 in the following calculation, which gives the transformation for elements of $\mathcal{T}^*$ under the change of connection in the projective class.

Consider a splitting of $\mathcal{T}^*$ with elements (μ_a, σ) and $(\tilde{\mu}_a, \tilde{\sigma})$. Then the induced splittings on $S^2\mathcal{T}$ satisfy:

$$\left(\begin{smallmatrix} \tau^{ab} \\ \rho^a \\ \nu \end{smallmatrix}\right)((\mu_a, \sigma), (\tilde{\mu}_a, \tilde{\sigma})) = \left(\begin{smallmatrix} \hat{\tau}^{ab} \\ \hat{\rho}^a \\ \hat{\nu} \end{smallmatrix}\right)(\widehat{(\mu_a, \sigma)}, \widehat{(\tilde{\mu}_a, \tilde{\sigma})}).$$

Now calculating the right side of the equation gives:

$$\begin{aligned} \left(\begin{smallmatrix} \hat{\tau}^{ab} \\ \hat{\rho}^a \\ \hat{\nu} \end{smallmatrix}\right)(\widehat{(\mu_a, \sigma)}, \widehat{(\tilde{\mu}_a, \tilde{\sigma})}) &= \hat{\tau}^{ab}(\mu_a + \Upsilon_a\sigma)(\tilde{\mu}_b + \Upsilon_b\tilde{\sigma}) + \hat{\rho}^a(\mu_a + \Upsilon_a\sigma)\tilde{\sigma} \\ &\quad + \hat{\rho}^b(\tilde{\mu}_b + \Upsilon_b\tilde{\sigma})\sigma + \hat{\nu}\sigma\tilde{\sigma} \\ &= \hat{\tau}^{ab}\mu_a\tilde{\mu}_b + (\hat{\tau}^{ab}\Upsilon_b + \hat{\rho}^a)\mu_a\tilde{\sigma} + (\hat{\tau}^{ab}\Upsilon_a + \hat{\rho}^b)\tilde{\mu}_b\sigma \\ &\quad + (\hat{\tau}^{ab}\Upsilon_a\Upsilon_b + 2\hat{\rho}^a\Upsilon_a + \hat{\nu})\sigma\tilde{\sigma}. \end{aligned}$$

Comparing this with (4.22), we immediately obtain the result for $\hat{\tau}^{ab}$ and $\hat{\rho}^a$. Furthermore, we get:

$$\begin{aligned} \hat{\nu} &= \nu - \tau^{ab}\Upsilon_a\Upsilon_b - 2(\rho^a - \tau^{ab}\Upsilon_b)\Upsilon_a \\ &= \nu - 2\rho^a\Upsilon_a + \tau^{ab}\Upsilon_a\Upsilon_b. \end{aligned}$$

□

Again we compute the induced connection on $S^2\mathcal{T}$ in terms of splittings. Recall that given a splitting on $\mathcal{T}$, by (4.22) we defined the splitting of $S^2\mathcal{T}$. Furthermore, this induces a splitting of $\mathcal{E}_a \otimes S^2\mathcal{T}$.

Theorem 4.4.2. The induced connection on $S^2\mathcal{T}$ is given by

$$\nabla_a^{S^2\mathcal{T}}\left(\begin{smallmatrix} \tau^{bc} \\ \rho^b \\ \nu \end{smallmatrix}\right) = \left(\begin{smallmatrix} \nabla_a\tau^{bc} + \rho^c\delta_a{}^b + \rho^b\delta_a{}^c \\ \nabla_a\rho^b + \nu\delta_a{}^b - \tau^{bc}\mathrm{P}_{ac} \\ \nabla_a\nu - 2\rho^b\mathrm{P}_{ab} \end{smallmatrix}\right),$$

where the splitting on the right side is the induced splitting on $\mathcal{E}_a \otimes S^2\mathcal{T}$.

Proof. Similar to Theorem 4.4.1, given the tractor connection on $\mathcal{T}^*$ and a bilinear form on $\mathcal{T}^*$, the induced connection is given by

$$(\nabla_a\Phi)(X, Y) = \nabla_a(\Phi(X, Y)) - \Phi(\nabla_a X, Y) - \Phi(X, \nabla_a Y) \tag{4.23}$$

and we again denote all connections by the same symbol. Working out the right side of the equation by using Definition 4.2.2, we obtain:

$$\begin{aligned} \nabla_a\left(\begin{smallmatrix} \tau^{bc} \\ \rho^b \\ \nu \end{smallmatrix}\right)\Big((\mu_d, \sigma), (\tilde{\mu}_d, \tilde{\sigma})\Big) &= \nabla_a(\tau^{ab}\mu_a\tilde{\mu}_b + \rho^a\mu_a\tilde{\sigma} + \rho^a\tilde{\mu}_a\sigma + \nu\sigma\tilde{\sigma}) \\ &= (\nabla_a\tau^{bc})\mu_b\tilde{\mu}_c + \tau^{bc}\nabla_a\mu_b\tilde{\mu}_c + (\nabla_a\rho^b)\mu_b\tilde{\sigma} + \rho^b\nabla_a\mu_b\tilde{\sigma} \\ &\quad + (\nabla_a\rho^c)\tilde{\mu}_c\sigma + \rho^c\nabla_a\tilde{\mu}_c\sigma + (\nabla_a\nu)\sigma\tilde{\sigma} + \nu\nabla_a\sigma\tilde{\sigma} \end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix}\tau^{bc}\\ \rho^b\\ \nu\end{pmatrix}\Big(\nabla_a(\mu_d,\sigma),(\tilde\mu_d,\tilde\sigma)\Big) &= \begin{pmatrix}\tau^{bc}\\ \rho^b\\ \nu\end{pmatrix}\Big((\nabla_a\mu_d+\mathrm{P}_{ad}\sigma,\nabla_a\sigma-\mu_a),(\tilde\mu_d,\tilde\sigma)\Big)\\ &= \tau^{bc}(\nabla_a\mu_b+\mathrm{P}_{ab}\sigma)\tilde\mu_c+\rho^b(\nabla_a\mu_b+\mathrm{P}_{ab}\sigma)\tilde\sigma\\ &\quad \rho^c\tilde\mu_c(\nabla_a\sigma-\mu_a)+\nu(\nabla_a\sigma-\mu_a)\tilde\sigma.\end{aligned}$$

If we exchange (μ_d,σ) and $(\tilde\mu_d,\tilde\sigma)$ in the last expression, we obtain the third part of right side of equation (4.23). Thus by combining all three parts, we obtain:

$$\begin{aligned}\nabla_a\begin{pmatrix}\tau^{bc}\\ \rho^b\\ \nu\end{pmatrix}\Big((\mu_d,\sigma),(\tilde\mu_d,\tilde\sigma)\Big) &= (\nabla_a\tau^{bc}+\rho^c\delta_a{}^b+\rho^b\delta_a{}^c)\mu_b\tilde\mu_c\\ &\quad +(\nabla_a\rho^b+\nu\delta_a{}^b-\tau^{bc}\mathrm{P}_{ac})\mu_b\tilde\sigma\\ &\quad +(\nabla_a\rho^c+\nu\delta_a{}^c-\tau^{cd}\mathrm{P}_{ad})\tilde\mu_c\sigma+(\nabla_a\nu-2\rho^b\mathrm{P}_{ab})\sigma\tilde\sigma,\end{aligned}$$

which gives the result. □

By construction, the induced connection on $S^2\mathcal{T}$ transforms correctly under change of connection in the projective class of M. This can be checked using Lemma 4.1.2 in the cases $k=2,\omega=-2$ and $k=1,\omega=-2$ as well as Theorem 4.1 in the case $\omega=-2$.

We again calculate the curvature of the induced connection on $S^2\mathcal{T}$, which is a section of $\mathcal{E}_{[ab]}\otimes L(S^2\mathcal{T},S^2\mathcal{T})$. Considering $L(S^2\mathcal{T},S^2\mathcal{T})$, we induce a splitting from a splitting of $S^2\mathcal{T}$ by

$$\begin{pmatrix}A_{ef}{}^{cd} & B_e{}^{cd} & D^{cd}\\ \psi_{ef}{}^c & \xi_e{}^c & \zeta^c\\ \phi_{ef} & \eta_e & \pi\end{pmatrix}\begin{pmatrix}\tau^{ef}\\ \rho^e\\ \nu\end{pmatrix}:=\begin{pmatrix}A_{ef}{}^{cd}\tau^{ef}+B_e{}^{cd}\rho^e+D^{cd}\nu\\ \psi_{ef}{}^c\tau^{ef}+\xi_e{}^c\rho^e+\zeta^c\nu\\ \phi_{ef}\tau^{ef}+\eta_e\rho^e+\pi\nu\end{pmatrix}. \tag{4.24}$$

If we change the connection ∇ to an equivalent connection $\hat\nabla=\nabla+\Upsilon$ on M, then this splitting changes to

$$\begin{pmatrix}\widehat{A_{ef}{}^{cd}} & \widehat{B_e{}^{cd}} & \widehat{D^{cd}}\\ \widehat{\psi_{ef}{}^c} & \widehat{\xi_e{}^c} & \widehat{\zeta^c}\\ \widehat{\phi_{ef}} & \widehat{\eta_e} & \widehat{\pi}\end{pmatrix}, \tag{4.25}$$

where the first column is given by

$$\begin{aligned}\widehat{A_{ef}{}^{cd}} &= A_{ef}{}^{cd}+B_e{}^{cd}\Upsilon_f-D^{cd}\Upsilon_e\Upsilon_f,\\ \widehat{\psi_{ef}{}^c} &= \psi_{ef}{}^c+\xi_e{}^c\Upsilon_f+\Upsilon_e\Upsilon_f\zeta^c-\Upsilon_e\Upsilon_f\Upsilon_bD^{cb}-\Upsilon_f\Upsilon_dB_e{}^{cd}-\Upsilon_dA_{ef}{}^{cd},\\ \widehat{\phi_{ef}} &= \phi_{ef}+\Upsilon_f\eta_e+\Upsilon_e\Upsilon_f\pi-2\Upsilon_c\Upsilon_e\Upsilon_f\zeta^c+D^{bc}\Upsilon_b\Upsilon_c\Upsilon_e\Upsilon_f,\\ &\quad -2\Upsilon_c\Upsilon_f\xi_e{}^c+B_e{}^{cd}\Upsilon_c\Upsilon_d\Upsilon_f-2\Upsilon_c\psi_{ef}{}^c+\Upsilon_d\Upsilon_cA_{ef}{}^{cd},\end{aligned}$$

the second column by

$$\begin{aligned}
\widehat{B_e{}^{cd}} &= B_e{}^{cd} + 2\Upsilon_e D^{cd},\\
\widehat{\xi_e{}^c} &= \xi_e{}^c + 2\Upsilon_e\zeta^c - 2\Upsilon_b\Upsilon_e D^{cb} - \Upsilon_d B_e{}^{cd},\\
\widehat{\eta_e} &= \eta_e + 2\Upsilon_e\pi - 4\Upsilon_b\Upsilon_e\zeta^b + 2D^{bc}\Upsilon_b\Upsilon_c\Upsilon_e - 2\Upsilon_c\xi_e{}^c + B_e{}^{cd}\Upsilon_c\Upsilon_d
\end{aligned}$$

and the third column by

$$\begin{aligned}
\widehat{D^{cd}} &= D^{cd},\\
\widehat{\zeta^c} &= \zeta^c - \Upsilon_d D^{cd},\\
\widehat{\pi} &= \pi - 2\Upsilon_b\zeta^b + D^{bc}\Upsilon_b\Upsilon_c.
\end{aligned}$$

This can be shown using a similar calculation as in (4.6).

As in the case of $S^2\mathcal{T}^*$, we have to consider coupled connections to write down the defining equation for the curvature. We will denote all connections by the same symbol. Thus in a splitting the curvature is given by:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{pmatrix}\tau^{cd}\\ \rho^c\\ \nu\end{pmatrix} = \begin{pmatrix} A_{abef}{}^{cd} & B_{abe}{}^{cd} & D_{ab}{}^{cd}\\ \psi_{abef}{}^c & \xi_{abe}{}^c & \zeta_{ab}{}^c\\ \phi_{abef} & \eta_{abe} & \pi_{ab}\end{pmatrix}\begin{pmatrix}\tau^{ef}\\ \rho^e\\ \nu\end{pmatrix}.$$

However, as in the case of $S^2\mathcal{T}^*$, it can be calculated using the curvature of the tractor connection on $\mathcal{T}^*$.

Lemma 4.4.4. Let ∇ be an affine connection on M with curvature $R_{ab}{}^c{}_d$ and decomposition of the curvature (4.1). The curvature of the induced connection on $S^2\mathcal{T}$ is then given by

$$\begin{pmatrix} C_{ab}{}^c{}_e\delta_f{}^d + C_{ab}{}^d{}_e\delta_f{}^c & 0 & 0\\ -2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^c & C_{ab}{}^c{}_e & 0\\ 0 & -4\nabla_{[a}\mathrm{P}_{b]e} & 0\end{pmatrix}.$$

Proof. This is proved analogously to Lemma 4.4.2. Given a bilinear form Φ in $S^2\mathcal{T}$ and elements X, Y in $\mathcal{T}^*$, then the curvature R of the induced connection on $S^2\mathcal{T}$ satisfies:

$$(R\cdot\Phi)(X,Y) = -\Phi(\hat{R}\cdot X, Y) - \Phi(X, \hat{R}\cdot Y), \tag{4.26}$$

where $\hat{R}$ denotes the curvature on $\mathcal{T}^*$, compare Lemma 4.2.1. Consider a splitting with $\Phi = \begin{pmatrix}\tau^{ef}\\ \rho^e\\ \nu\end{pmatrix}$, $X = (\mu_d, \sigma), Y = (\tilde{\mu}_d, \tilde{\sigma})$ and

$$R = \begin{pmatrix} A_{abef}{}^{cd} & B_{abe}{}^{cd} & D_{ab}{}^{cd}\\ \psi_{abef}{}^c & \xi_{abe}{}^c & \zeta_{ab}{}^c\\ \phi_{abef} & \eta_{abe} & \pi_{ab}\end{pmatrix}.$$

We now calculate the right side of equation (4.26):

$$\begin{aligned}\begin{pmatrix}\tau^{ef}\\ \rho^e\\ \nu\end{pmatrix}\Big(\hat{R}\cdot(\mu_d,\sigma),(\tilde{\mu}_d,\tilde{\sigma})\Big) &= \begin{pmatrix}\tau^{ef}\\ \rho^e\\ \nu\end{pmatrix}\Big((-C_{ab}{}^{e}{}_{d}\mu_e+2\nabla_{[a}\mathrm{P}_{b]d}\sigma,0),(\tilde{\mu}_d,\tilde{\sigma})\Big)\\ &= (-C_{ab}{}^{c}{}_{e}\tau^{ed})\mu_c\tilde{\mu}_d+(-C_{ab}{}^{c}{}_{e}\rho^e)\mu_c\tilde{\sigma}\\ &\quad +(2(\nabla_{[a}\mathrm{P}_{b]e})\tau^{ed})\tilde{\mu}_d\sigma+(2(\nabla_{[a}\mathrm{P}_{b]e}\rho^e)\sigma\tilde{\sigma}.\end{aligned}$$

By changing (μ_d,σ) to $(\tilde{\mu}_d,\tilde{\sigma})$, we obtain $\Phi(X,\hat{R}\cdot Y)$. Combining these gives:

$$\begin{aligned}&\Phi(\hat{R}\cdot X,Y)+\Phi(X,\hat{R}\cdot Y) \qquad (4.27)\\ =\ &((-C_{ab}{}^{c}{}_{e}\delta_f{}^d-C_{ab}{}^{d}{}_{e}\delta_f{}^c)\tau^{ef})\mu_c\tilde{\mu}_d+(2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^c\tau^{ef}-C_{ab}{}^{c}{}_{e}\rho^e)\mu_c\tilde{\sigma}\\ &+(2(\nabla_{[a}\mathrm{P}_{b]e})\delta_f{}^d\tau^{ef}-C_{ab}{}^{d}{}_{e}\rho^e)\tilde{\mu}_d\sigma+(4(\nabla_{[a}\mathrm{P}_{b]e})\rho^e)\sigma\tilde{\sigma}.\end{aligned}$$

On the other hand we have:

$$\begin{aligned}(R\cdot\Phi)(X,Y) &= (A_{abef}{}^{cd}\tau^{ef}+B_{abe}{}^{cd}\rho^e+D_{ab}{}^{cd}\nu)\mu_c\tilde{\mu}_d \qquad (4.28)\\ &\quad+(\psi_{abef}{}^{c}\tau^{ef}+\xi_{abe}{}^{c}\rho^e+\zeta_{ab}{}^{c}\nu)\mu_c\tilde{\sigma}\\ &\quad+(\psi_{abef}{}^{c}\tau^{ef}+\xi_{abe}{}^{c}\rho^e+\zeta_{ab}{}^{c}\nu)\tilde{\mu}_c\sigma\\ &\quad+(\phi_{abef}\tau^{ef}+\eta_{abe}\rho^e+\pi_{ab}\nu)\sigma\tilde{\sigma}.\end{aligned}$$

Now comparing (4.27) and (4.28) as well as changing the sign, we obtain the result. □

By construction, the curvature transforms correctly under change of connection in the projective class of M. Nevertheless, it can be checked using (4.25) as well as (4.12) and the fact that the Weyl tensor is invariant.

Remark 4.4.1. An interesting question we want to address here regards the duality between $S^2\mathcal{T}^*$ and $S^2\mathcal{T}$. We have constructed $S^2\mathcal{T}^*$ and $S^2\mathcal{T}$ independently of one another, that is we (seperately) considered these bundles as bilinear forms on $\mathcal{T}$ respectively on $\mathcal{T}^*$. On the other hand, we we could have worked with $S^2\mathcal{T}$ as the dual bundle of $S^2\mathcal{T}^*$, just as in the case of the tractor bundle, compare section 4.3. This, of course, involves the choice of a duality, which in turn determines a splitting. Now in the case of $S^2\mathcal{T}^*$ and $S^2\mathcal{T}$, there are different choices available, as for all constants $A,B,C\in\mathbb{R}$ we can define this duality by

$$\langle\begin{pmatrix}\tau^{bc}\\ \rho^b\\ \nu\end{pmatrix},\begin{pmatrix}\mu_{bc} & \eta_b & \sigma\end{pmatrix}\rangle := A\mu_{bc}\tau^{bc}+B\eta_b\rho^b+C\sigma\nu. \qquad (4.29)$$

It is not obvious, which of the dualities in (4.29) induces exactly the splitting of $S^2\mathcal{T}$, which we obtained by regarding $S^2\mathcal{T}$ as the bundle of bilinear forms on $\mathcal{T}^*$. A calculation, however, shows that that these splittings coincide, if we set $A=C=1$ and $B=2$. Note, that this also means, that the dual connection to the tractor connection on $S^2\mathcal{T}^*$ is exactly the connection we obtained in Theorem 4.4.2.

Remark 4.4.2. Finally, we would like to remark on the conventions used in this chapter.

In section 4.4 we viewed elements of $S^2\mathcal{T}^*$ respectively $S^2\mathcal{T}$ as bilinear forms on $\mathcal{T}$ respectively on $\mathcal{T}^*$ and induced a splitting into tripels. This led to conventions for calculating with these bundles. It is also common, however, to represent symmetric bilinear forms by symmetric matrices. We briefly introduce this approach.

For a given splitting of $\mathcal{T}^*$ and induced splitting of $\mathcal{T}$, we define the splitting of $S^2\mathcal{T}^*$ by

$$\begin{pmatrix}\tilde{\nu}^a & \tilde{\tau}\end{pmatrix}\begin{pmatrix}\mu_{ab} & \rho_a \\ \rho_a & \sigma\end{pmatrix}\begin{pmatrix}\nu^b \\ \tau\end{pmatrix} := \mu_{ab}\tilde{\nu}^a\nu^b + \rho_a\nu^a\tilde{\tau} + \rho_a\tilde{\nu}^a\tau + \tau\tilde{\tau}\sigma. \tag{4.30}$$

Note that even though $\begin{pmatrix}\tilde{\nu}^a & \tilde{\tau}\end{pmatrix}$ denotes an element of $\mathcal{T}$, we use its transpose to relate to usual matrix multiplication. Therefore, instead of working with triples, we can equivalently work with symmetric matrices and either convention certainly leads to the same results. Similary, we can discuss $S^2\mathcal{T}$.

Chapter 5
Invariant Differential Operators

5.1 Motiviation

Given a manifold M and two tensor bundles E and F over M, we can use an affine connection ∇ and its curvature to write down differential operators $\Gamma(E) \to \Gamma(F)$. Starting with a projective structure on M, such a differential operator can be expressed in terms of any representative in the projective class. If every representative gives rise to the same operator, i.e. the operator is independent of the chosen representative in the projective class, then this operator is called projectively invariant. Hence a given differential operator is projectively invariant, if replacing all derivative and curvature terms by the ones of a projectively equivalent connection, does not change the operator.

Let us consider examples of a first order differential operator on $\mathcal{E}_b(\omega)$. Recall from Lemma 4.1.2, that changing ∇ to an equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ gives the following transformation for $\eta_b \in \Gamma(\mathcal{E}_b(\omega))$:

$$\hat{\nabla}_a \eta_b = \nabla_a \eta_b + (\omega - 1)\Upsilon_a \eta_b - \Upsilon_b \eta_a. \tag{5.1}$$

Now considering its antisymmetrization gives

$$\hat{\nabla}_{[a} \eta_{b]} = \nabla_{[a} \eta_{b]} + \omega \Upsilon_{[a} \eta_{b]} \tag{5.2}$$

and therefore, in the case of $\omega = 0$, the first order differential operator $\Gamma(\mathcal{E}_b(\omega)) \to \Gamma(\mathcal{E}_{[ab]}(\omega)), \eta_b \mapsto \nabla_{[a}\eta_{b]}$, is projectively invariant. This result is not too surprising, since $\nabla_{[a}\eta_{b]}$ is the exterior derivative of the $1-$form η_b. A more interesting outcome is obtained by symmetrizing equation (5.1):

$$\hat{\nabla}_{(a} \eta_{b)} = \nabla_{(a} \eta_{b)} + (\omega - 2)\Upsilon_{(a} \eta_{b)}. \tag{5.3}$$

Hence if $\omega = 2$, the first order differential operator $\eta_b \mapsto \nabla_{(a}\eta_{b)}$ is projectively invariant. This also shows the importance of densities, as without the additional weighting, the operator would not be projectively invariant. Thus the notion of densities is essential to obtain invariant differential operators.

The general theory of projectively invariant differential operators of first order looks very similar to examples (5.2) and (5.3). The explicit description of all such operators (for any parabolic geometry) is due to [17].

The case of second order projectively invariant differential operators is more complicated. For $\sigma \in \Gamma(\mathcal{E}(1))$ consider the following example:

$$\nabla_{(a}\nabla_{b)}\sigma + \mathrm{P}_{(ab)}\sigma. \tag{5.4}$$

A calculation shows, that this defines a projectively invariant differential operator. The additional curvature term of course makes the expression more complicated compared to (5.3), but is necessary to guarantee the projective invariance. We will come across this differential operator in section 5.3, where we will also prove projective invariance.

If we want to consider invariant differential operators of higher order, we could try to generalize (5.4). In particular, this example shows that we need to find curvature terms to compensate the change of the terms involving derivatives when replacing ∇ by $\hat{\nabla}$. This gets more complicated when raising the order, as in this case more curvature terms have to be involved. For example, the third order operator

$$\tfrac{1}{2}\nabla_{(a}\nabla_b\nabla_{c)}\sigma + (\nabla_{(a}\mathrm{P}_{bc)})\sigma + 2\mathrm{P}_{(ab}\nabla_{c)}\sigma, \tag{5.5}$$

for $\sigma \in \Gamma(\mathcal{E}(2))$, can be understood as a generalisation of (5.4). Here we already need two curvature terms to compensate the change of $\nabla_{(a}\nabla_b\nabla_{c)}\sigma$. In section 5.3, we will prove projective invariance of this operator.

For these reasons, in order to find projectively invariant differential operators, a more systematic approach is needed. We will present this approach in the following chapters.

5.2 The Kostant Codifferential

In this section we consider the Kostant codifferential on the tractor bundles defined in chapter 4. We can construct this codifferential by considering a certain action of an abelian Lie algebra on the tractor bundle in question and extending it to a Lie algebra homology differential. Even though we do not need this, it should be mentioned, that the Kostant codifferential arises from a general theory, compare [8, Section 2.5, 2.6, 2.13].

We will briefly discuss how an action of an abelian Lie algebra extends to a Lie algebra homology differential. This is then applied to the tractor bundles and the first two maps, which arise from this construction, are calculated explicitely (in a splitting). They will be of interest in the next section.

Consider an abelian Lie algebra $\mathfrak{g}$, a vector space V and an action of $\mathfrak{g}$ on V:

$$\delta_1 : \mathfrak{g} \otimes V \to V,$$

$$g \otimes v \mapsto g \cdot v.$$

The action δ_1 can be extended to a Lie algebra homology differential, i.e. a sequence of maps

$$0 \xleftarrow{\delta_0} V \xleftarrow{\delta_1} \mathfrak{g} \otimes V \xleftarrow{\delta_2} \Lambda^2\mathfrak{g} \otimes V \xleftarrow{\delta_3} \cdots,$$

satisfying $\delta_k \circ \delta_{k+1} = 0$, for $k \geq 0$. This extension is given by

$$\delta_k : \Lambda^k\mathfrak{g} \otimes V \to \Lambda^{k-1}\mathfrak{g} \otimes V,$$

$$g_1 \wedge \ldots \wedge g_k \otimes v \mapsto \sum_{i=1}^{k} (-1)^i g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_k \otimes (g_i \cdot v), \tag{5.6}$$

where the hat denotes omission. We prove that $\delta_k \circ \delta_{k+1} = 0$ for $k \geq 0$ by using the fact that the action δ_1 satisfies $g_1 \cdot (g_2 \cdot v) = g_2 \cdot (g_1 \cdot v)$.

$$(\delta_k \circ \delta_{k+1})(g_1 \wedge \ldots \wedge g_{k+1} \otimes v) = \delta_k(\sum_{i=1}^{k} (-1)^i g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_k \otimes (g_i \cdot v))$$

$$= \sum_{1 \leq j < i \leq k+1} (-1)^{i+j} g_1 \wedge \ldots \wedge \hat{g}_j \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge g_{k+1} \otimes (g_j \cdot (g_i \cdot v))$$
$$+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j-1} g_1 \wedge \ldots \wedge \hat{g}_i \wedge \ldots \wedge \hat{g}_j \wedge \ldots \wedge g_{k+1} \otimes (g_j \cdot (g_i \cdot v)). \tag{5.7}$$

Note that in line (5.7), we have the sign $(-1)^{i+j-1}$. In the case $j > i$, the omission of g_i implies that the element g_j is located at the $(j-1)$th spot, which gives the sign $(-1)^{i+j-1}$. Now, relabeling line (5.7) and using $g_i \cdot (g_j \cdot v) = g_j \cdot (g_i \cdot v)$, we obtain that $\delta_k \circ \delta_{k+1} = 0$. This gives rise to homologies $H_k(M, V) = \ker(\delta_k)/\mathrm{im}(\delta_{k+1})$ for $k \geq 0$.

Now consider the cotractor bundle $\mathcal{T}^*$, with short exact sequence

$$0 \longrightarrow T^*M \otimes \mathcal{E}(1) \xrightarrow{i} \mathcal{T}^* \xrightarrow{p} \mathcal{E}(1) \longrightarrow 0.$$

This induces the following bundle maps:

$$T^*M \otimes \mathcal{T}^* \xrightarrow{\mathrm{id} \otimes p} T^*M \otimes \mathcal{E}(1) \xrightarrow{i} \mathcal{T}^*.$$

The composition of these maps gives

$$\partial_1^* := i \circ (\mathrm{id} \otimes p) : \quad T^*M \otimes \mathcal{T}^* \to \mathcal{T}^*,$$

which is called the Kostant codifferential. We prove that fiberwise this is an action of the abelian Lie algebra T_x^*M on the vector space $\mathcal{T}_x^*$, for $x \in M$. Starting with $\tau \in \mathcal{T}_x^*$ and $g_1, g_2 \in T_x^*M$, we have that $g_1 \cdot \tau \in T_x^*M \otimes \mathcal{E}_x(1)$ and thus its projection onto $\mathcal{E}_x(1)$ vanishes. Therefore, we obtain $g_1 \cdot (g_2 \cdot \tau) = 0$, which implies that ∂_1^* defines a fiberwise action. Thus we obtain a fiberwise extention to a Lie algebra homology differential by

(5.6), which can be put together to obtain bundle maps

$$\partial_k^* : \Lambda^k T^*M \otimes \mathcal{T}^* \to \Lambda^{k-1} T^*M \otimes \mathcal{T}^*. \tag{5.8}$$

Note, that these bundle maps induce operators on the space of sections $\Gamma(\Lambda^k T^*M \otimes \mathcal{T}^*) \to \Gamma(\Lambda^{k-1} T^*M \otimes \mathcal{T}^*)$, which we denote by the same symbol.

We now want to understand what the maps of (5.8) look like in a given splitting. Starting with a splitting of $\mathcal{T}^* \cong_\nabla \mathcal{E}_b(1) \oplus \mathcal{E}(1)$ induced by an affine connection ∇ on M, we already remarked in section 4.2, that this induces a splitting $T^*M \otimes \mathcal{T}^* \cong_\nabla \mathcal{E}_{ab}(1) \oplus \mathcal{E}(1)$ and showed how this splitting transforms under change of connection in the projective class. Similarly, a given splitting of $\mathcal{T}^*$ induces a splitting $\Lambda^2 T^*M \otimes \mathcal{T}^* \cong_\nabla \mathcal{E}_{[ab]c}(1) \oplus \mathcal{E}_{[ab]}(1)$ and thus also on $\Lambda^k T^*M \otimes \mathcal{T}^*$, for $k \geq 3$. As the first two bundle maps of (5.8) are especially important in the next section, we compute these in the following Lemma.

Lemma 5.2.1. Let $\mathcal{T}^* \cong_\nabla \mathcal{E}_b(1) \oplus \mathcal{E}(1)$ be a splitting of $\mathcal{T}^*$ induced by an affine connection ∇ on M. Then in this splitting the first two bundle maps of (5.8) are given by

$$\partial_1^* : \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1) \to \mathcal{E}_a(1) \oplus \mathcal{E}(1)$$
$$(\mu_{ab}, \sigma_a) \mapsto (\sigma_a, 0)$$

and

$$\partial_2^* : \mathcal{E}_{[ab]c}(1) \oplus \mathcal{E}_{[ab]}(1) \to \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1)$$
$$(\mu_{abc}, \sigma_{ab}) \mapsto (2\sigma_{ab}, 0).$$

The first bundle map ∂_1^* transforms correctly under the change of connection in the projective class, i.e. changing the connection ∇ to $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M and splitting $\mathcal{T}^*$ with respect to $\hat{\nabla}$, gives the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}_a \otimes \mathcal{T}^* \cong_\nabla \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1) \arrow[r, "\partial_1^*"] \arrow[d, "(\mu_{ab},\sigma_a)\mapsto(\mu_{ab}+\Upsilon_b\sigma_a,\sigma_a)"'] \arrow[dr, phantom, "\#"] & \mathcal{T}^* \cong_\nabla \mathcal{E}_b(1) \oplus \mathcal{E}(1) \arrow[d, "(\tau_b,\rho)\mapsto(\tau_b+\Upsilon_b\rho,\rho)"] \\
\mathcal{E}_a \otimes \mathcal{T}^* \cong_{\hat{\nabla}} \mathcal{E}_{ab}(1) \oplus \mathcal{E}_a(1) \arrow[r, "\partial_1^*"'] & \mathcal{T}^* \cong_{\hat{\nabla}} \mathcal{E}_b(1) \oplus \mathcal{E}(1)
\end{tikzcd}$$

By construction, we have a similar diagram in the case of ∂_2^* (and also for ∂_k^*, for $k \geq 3$). Furthermore, the first two homologies are given by $H_0(M, \mathcal{T}^*) = \mathcal{E}(1)$ and $H_1(M, \mathcal{T}^*) = \mathcal{E}_{(ab)}(1)$.

Proof. We show the first statement on elements of the form $\rho_a \otimes \tau$, with $\rho_a \in \mathcal{E}_a$ and $\tau \in \mathcal{T}^*$. Given a splitting of $\mathcal{T}^*$ with $\tau = (\mu_b, \sigma)$, we obtain $\rho_a \otimes \tau = (\rho_a \mu_b, \rho_a \sigma)$ for the induced splitting on $\mathcal{E}_a \otimes \mathcal{T}^*$. Therefore,

$$\partial_1^*(\rho_a \otimes \tau) = (i \circ (\mathrm{id} \otimes p))(\rho_a \mu_b, \rho_a \sigma) = i(\rho_a \sigma) = (\rho_a \sigma, 0)$$

and we obtain the result. The commutativity of the diagram is obvious.

Analogously, we prove the second statement on elements of the form $\eta_a\omega_b \otimes \tau$, with $\eta_a\omega_b \in \mathcal{E}_{[ab]}$, i.e. satisfying $\eta_a\omega_b = -\omega_a\eta_b$ and $\tau \in \mathcal{T}^*$. Considering a splitting of $\mathcal{T}^*$ with $\tau = (\mu_c, \sigma)$, we obtain:

$$\partial_2^*(\eta_a\omega_b \otimes \tau) = \eta_a \otimes (\omega_b \cdot \tau) - \omega_a \otimes (\eta_b \cdot \tau) = (\eta_a\omega_b\sigma - \omega_a\eta_b\sigma, 0) = (2\eta_a\omega_b\sigma, 0),$$

where we used the first part of this Lemma, equation (5.6) and the induced splitting on $\Lambda^2 T^*M \otimes \mathcal{T}^*$.

Furthermore, we immediately obtain $\ker(\partial_0^*) = \mathcal{T}^*$, since $\partial_0^* = 0$, $\ker(\partial_1^*) = \mathcal{E}_{ab}(1)$, $\mathrm{im}(\partial_1^*) = \mathcal{E}_a(1)$ and $\mathrm{im}(\partial_2^*) = \mathcal{E}_{[ab]}(1)$. Thus $H_0(M, \mathcal{T}^*) = \mathcal{T}^*/\mathcal{E}_a(1) \cong \mathcal{E}(1)$ and $H_1(M, \mathcal{T}^*) = \mathcal{E}_{ab}(1)/\mathcal{E}_{[ab]}(1) \cong \mathcal{E}_{(ab)}(1)$.

$\square$

We now want to define the Kostant codifferential on the tractor bundle $\mathcal{T}$. Similar to the case of the cotractor bundle, we first define an action of the abelian Lie algebra T^*M and then extend this action to a differential.

Since we are working with the dual bundle of the cotractor bundle, we can use the dual action on $\mathcal{T}$. Thus, the action $\partial_1^* : T^*M \otimes \mathcal{T} \to \mathcal{T}$ is given by the equation

$$\langle \mu, g_1 \cdot \nu \rangle + \langle g_1 \cdot \mu, \nu \rangle = 0,$$

for $\mu \in \mathcal{T}^*, \nu \in \mathcal{T}$ and $g_1 \in T^*M$. Again this action can be extended to a differential via equation (5.6):

$$\partial_k^* : \Lambda^k T^*M \otimes \mathcal{T} \to \Lambda^{k-1} T^*M \otimes \mathcal{T}. \tag{5.9}$$

Note, that these bundle maps induce operators on the space of sections

$$\Gamma(\Lambda^k T^*M \otimes \mathcal{T}) \to \Gamma(\Lambda^{k-1} T^*M \otimes \mathcal{T}),$$

which we denote by the same symbol.

The maps of (5.9) can be computed in terms of splittings, which is done in the next Lemma. Note, that given a splitting of $\mathcal{T}$, we again use induced splittings of $T^*M \otimes \mathcal{T}$ and $\Lambda^2 T^*M \otimes \mathcal{T}$. By construction, ∂_k^* transforms correctly under change of connection in the projective class.

Lemma 5.2.2. Given a splitting of $\mathcal{T} \cong_\nabla \mathcal{E}(-1) \oplus \mathcal{E}^b(-1)$ the first two bundle maps of (5.9) are given by

$$\partial_1^* : \mathcal{E}_a(-1) \oplus \mathcal{E}_a^b(-1) \to \mathcal{E}(-1) \oplus \mathcal{E}^b(-1)$$
$$\begin{pmatrix} \nu_a{}^b \\ \rho_a \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -\nu_a{}^a \end{pmatrix}$$

and

$$\partial_2^* : \mathcal{E}_{[ab]}(-1) \oplus \mathcal{E}_{[ab]}{}^c(-1) \to \mathcal{E}_a(-1) \oplus \mathcal{E}_a^b(-1)$$

$$\begin{pmatrix} \nu_{ab}{}^c \\ \rho_{ab} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -2\nu_{ac}{}^c \end{pmatrix}.$$

The first two homologies are given by $H_0(M,\mathcal{T}) = \mathcal{E}^a(-1)$ and $H_1(M,\mathcal{T}) = (\mathcal{E}_a^b)_0(-1)$, where $(\mathcal{E}_a^b)_0(-1)$ denotes the trace-free part of $\mathcal{E}_a^b(-1)$.

Proof. A given splitting of $\mathcal{T}^* \cong_\nabla \mathcal{E}_c(1) \oplus \mathcal{E}(1)$ induces $\mathcal{T} \cong_\nabla \mathcal{E}(-1) \oplus \mathcal{E}^b(-1)$ and in this splitting consider $\tau = (\mu_c, \sigma)$ and $\psi = \binom{\nu^b}{\rho}$. Furthermore, for $g_a \in \mathcal{E}_a$ we have $g_a \otimes \tau = (g_a\mu_c, g_a\sigma)$ and $g_a \otimes \psi = \binom{g_a\nu^b}{g_a\rho}$ in the induced splittings on $\mathcal{E}_a \otimes \mathcal{T}^*$ respectively $\mathcal{E}_a \otimes \mathcal{T}$. The Lemma is proved for elements of the form $g_a \otimes \psi$. Then by definition we obtain

$$\begin{aligned} \langle(\mu_b,\sigma), \partial_1^*(\tbinom{g_a\nu^b}{g_a\rho})\rangle &= -\langle \partial_1^*(g_a\mu_b, g_a\sigma), \tbinom{\nu^b}{\rho}\rangle \\ &= -\langle (g_a\sigma, 0), \tbinom{\nu^b}{\rho}\rangle \\ &= -g_a\nu^a\sigma, \end{aligned} \tag{5.10}$$

which gives the result.

For the second statement consider $\eta_a\omega_b \in \mathcal{E}_{[ab]}$, i.e. $\eta_a\omega_b = -\eta_b\omega_a$ and in the splitting already considered, we have $\eta_a\omega_b \otimes \psi = \binom{\eta_a\omega_b\nu^c}{\eta_a\omega_b\rho}$. Again we prove the statement on elements of the form $\eta_a\omega_b \otimes \psi$. Thus we have

$$\begin{aligned} \partial_2^*(\eta_a\omega_b \otimes \psi) &= \eta_a \otimes (\omega_b \cdot \psi) - \omega_a \otimes (\eta_b \cdot \psi) \\ &= \tbinom{0}{-\eta_a\omega_c\nu^c + \omega_a\eta_c\nu^c} \\ &= \tbinom{0}{-2\eta_a\omega_c\nu^c}. \end{aligned}$$

Furthermore, we obtain $\mathrm{im}(\partial_1^*) = \mathcal{E}(-1)$ and thus $H_0(M,\mathcal{T}) = \mathcal{T}/\mathcal{E}(-1) \cong \mathcal{E}^a(-1)$. Similarly, $\ker(\partial_1^*) = \mathcal{E}_a(-1) \oplus (\mathcal{E}_a^b)_0(-1)$ and $\mathrm{im}(\partial_2^*) = \mathcal{E}_a(-1)$, which gives $H_1(M,\mathcal{T}) \cong (\mathcal{E}_a^b)_0(-1)$. □

We continue by defining the Kostant codifferential for $S^2\mathcal{T}^*$. Given a bilinear form Φ on $\mathcal{T}$, then for $g \in T^*M$, we use the induced action of T^*M on $S^2\mathcal{T}^*$ given by the equation

$$(g\cdot\Phi)(v,w) + \Phi(g\cdot v, w) + \Phi(v, g\cdot w) = 0. \tag{5.11}$$

This action can again be extended to a differential via equation (5.6):

$$\partial_k^* : \Lambda^k T^*M \otimes S^2\mathcal{T}^* \to \Lambda^{k-1}T^*M \otimes S^2\mathcal{T}^*. \tag{5.12}$$

Note, that these bundle maps induce operators on the space of sections

$$\Gamma(\Lambda^k T^*M \otimes S^2\mathcal{T}^*) \to \Gamma(\Lambda^{k-1}T^*M \otimes S^2\mathcal{T}^*),$$

which we denote by the same symbol.

Furthermore, we calculate the first two bundle maps of (5.12) in terms of splittings. Starting with a splitting of $\mathcal{T}^* \cong_\nabla \mathcal{E}_a(1) \oplus \mathcal{E}(1)$, induced by an affine connection ∇ on M, we have the induced splitting $S^2\mathcal{T}^* \cong_\nabla \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}(2)$, see section 4.4. This splitting of course induces splittings of $T^*M \otimes S^2\mathcal{T}^*$ and $\Lambda^2 T^*M \otimes S^2\mathcal{T}^*$, which are used in the next Lemma. Of course, all the maps ∂_k^* transform correctly under change of connection in the projective class of M.

Lemma 5.2.3. Given a splitting $S^2\mathcal{T}^* \cong_\nabla \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}(2)$, then the first two bundle maps of (5.12), in terms of this splitting, are given by

$$\partial_1^* : \mathcal{E}_{a(bc)}(2) \oplus \mathcal{E}_{ab}(2) \oplus \mathcal{E}_a(2) \to \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2) \oplus \mathcal{E}(2)$$

$$\begin{pmatrix}\mu_{abc} & \rho_{ab} & \sigma_a\end{pmatrix} \mapsto \begin{pmatrix}2\rho_{(ab)} & \sigma_a & 0\end{pmatrix}$$

and

$$\partial_2^* : \mathcal{E}_{[ab](cd)}(2) \oplus \mathcal{E}_{[ab]c}(2) \oplus \mathcal{E}_{[ab]}(2) \to \mathcal{E}_{a(bc)}(2) \oplus \mathcal{E}_{ab}(2) \oplus \mathcal{E}_a(2)$$

$$\begin{pmatrix}\mu_{abcd} & \rho_{abc} & \sigma_{ab}\end{pmatrix} \mapsto \begin{pmatrix}4\rho_{a(bc)} & 2\sigma_{ab} & 0\end{pmatrix}.$$

The first two homologies are given by $H_0(M, S^2\mathcal{T}^*) = \mathcal{E}(2)$ and $H_1(M, S^2\mathcal{T}^*) = \mathcal{E}_{(abc)}(2)$.

Proof. Using Definition (5.11) and the given splitting, set $\Phi = \begin{pmatrix}\mu_{bc} & \rho_b & \sigma\end{pmatrix}$, $v = \left(\begin{smallmatrix}\nu^b \\ \tau\end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix}\eta^c \\ \pi\end{smallmatrix}\right)$. We prove the first statement on elements of the form $g_a \otimes \Phi$:

$$\begin{aligned}
\partial_1^*(g_a \otimes \Phi)\left(\left(\begin{smallmatrix}\nu^b \\ \tau\end{smallmatrix}\right), \left(\begin{smallmatrix}\eta^c \\ \pi\end{smallmatrix}\right)\right) &= -\begin{pmatrix}\mu_{bc} & \rho_b & \sigma\end{pmatrix}\left(\left(\begin{smallmatrix}0 \\ -g_b\nu^b\end{smallmatrix}\right), \left(\begin{smallmatrix}\eta^c \\ \pi\end{smallmatrix}\right)\right) \\
&\quad -\begin{pmatrix}\mu_{bc} & \rho_b & \sigma\end{pmatrix}\left(\left(\begin{smallmatrix}\nu^b \\ \tau\end{smallmatrix}\right), \left(\begin{smallmatrix}0 \\ -g_c\eta^c\end{smallmatrix}\right)\right) \\
&= (g_b\rho_c + g_c\rho_b)\nu^b\eta^c + (g_b\sigma)\nu^b\pi + (g_c\sigma)\eta^c\tau,
\end{aligned}$$

which gives the result.

The prove the second statement, we consider $\eta_a\omega_b \in \mathcal{E}_{[ab]}$, thus satisfying $\eta_a\omega_b = -\omega_a\eta_b$. In the given splitting consider $\Phi = \begin{pmatrix}\mu_{cd} & \rho_c & \sigma\end{pmatrix}$ and we show the result for elements of the form $\eta_a\omega_b \otimes \Phi$:

$$\begin{aligned}
\partial_2^*(\eta_a\omega_b \otimes \Phi) &= \eta_a \otimes (\omega_b \cdot \Phi) - \omega_a \otimes (\eta_b \cdot \Phi) \\
&= \begin{pmatrix}2\eta_a\omega_{(b}\rho_{c)} & \eta_a\omega_b\sigma & 0\end{pmatrix} - \begin{pmatrix}2\omega_a\eta_{(b}\rho_{c)} & \omega_a\eta_b\sigma & 0\end{pmatrix} \\
&= \begin{pmatrix}4\eta_a\omega_{(b}\rho_{c)} & 2\eta_a\omega_b\sigma & 0\end{pmatrix}.
\end{aligned}$$

We have $\operatorname{im}(\partial_1^*) = \mathcal{E}_{(ab)}(2) \oplus \mathcal{E}_a(2)$ and thus $H_0(M, S^2\mathcal{T}^*) \cong \mathcal{E}(2)$. Furthermore, $\ker(\partial_1^*) = \mathcal{E}_{a(bc)}(2) \oplus \mathcal{E}_{[ab]}(2)$ and $\operatorname{im}(\partial_2^*) = \operatorname{im}(T) \oplus \mathcal{E}_{[ab]}(2)$, where $T : \mathcal{E}_{[ab]c}(2) \to \mathcal{E}_{a(bc)}(2)$ is the map given by $\rho_{abc} \mapsto 4\rho_{a(bc)}$. In order to compute $\operatorname{im}(T)$, we first note that $\ker(T) = \mathcal{E}_{[abc]}(2)$. This follows from the fact that a map which is antisymmetric in a, b and b, c is antisymmetric in all three indices. In particular, $\ker(T) \subsetneq \mathcal{E}_{[ab]c}(2)$ implies that $T \neq 0$. Now consider the symmetrization $\mathrm{Symm} : \mathcal{E}_{a(bc)}(2) \to \mathcal{E}_{(abc)}(2)$, which is given by $\omega_{abc} \mapsto \omega_{(abc)}$. Then Symm $\circ\, T = 0$ and therefore, $\mathrm{im}(T) \subseteq \ker(\mathrm{Symm})$. On the other hand, since $T \neq 0$ and $\ker(\mathrm{Symm})$ is an irreducible representation, we actually have $\mathrm{im}(T) = \ker(\mathrm{Symm})$. This implies $\mathcal{E}_{a(bc)}(2)/\mathrm{im}(T) = \mathcal{E}_{(abc)}(2)$, which gives $H_1(M, S^2\mathcal{T}^*) \cong \mathcal{E}_{(abc)}(2)$. $\square$

In the last part of this section, we compute the Kostant codifferential on $S^2\mathcal{T}$. This is approached exactly as in the case of $S^2\mathcal{T}^*$. Thus, the action of T^*M is given by (5.11), where Φ now denotes a bilinear map on $\mathcal{T}^*$ and v, w elements of $\mathcal{T}^*$. This, in turn, gives rise to a differential, which we will denote by

$$\partial_k^* : \Lambda^k T^*M \otimes S^2\mathcal{T} \to \Lambda^{k-1}T^*M \otimes S^2\mathcal{T}. \tag{5.13}$$

Note, that these bundle maps induce operators on the space of sections

$$\Gamma(\Lambda^k T^*M \otimes S^2\mathcal{T}) \to \Gamma(\Lambda^{k-1}T^*M \otimes S^2\mathcal{T}),$$

which we denote by the same symbol.

The first two maps of (5.13) are computed in the next Lemma, using splittings of $S^2\mathcal{T}$, $T^*M \otimes S^2\mathcal{T}$ and $\Lambda^2 T^*M \otimes S^2\mathcal{T}$.

Lemma 5.2.4. Given a splitting $S^2\mathcal{T} \cong_\nabla \mathcal{E}(-2) \oplus \mathcal{E}^b(-2) \oplus \mathcal{E}^{(bc)}(-2)$, then in this splitting, the first two maps of (5.13) are given by

$$\partial_1^* : \mathcal{E}_a(-2) \oplus \mathcal{E}_a^b(-2) \oplus \mathcal{E}_a{}^{(bc)}(-2) \to \mathcal{E}(-2) \oplus \mathcal{E}^b(-2) \oplus \mathcal{E}^{(bc)}(-2)$$

$$\begin{pmatrix} \tau_a{}^{bc} \\ \rho_a{}^b \\ \nu_a \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -\tau_d{}^{bd} \\ -2\rho_a{}^a \end{pmatrix}$$

and

$$\partial_2^* : \mathcal{E}_{[ab]}(-2) \oplus \mathcal{E}_{[ab]}{}^c(-2) \oplus \mathcal{E}_{[ab]}{}^{(cd)}(-2) \to \mathcal{E}_a(-2) \oplus \mathcal{E}_a^c(-2) \oplus \mathcal{E}_a{}^{(cd)}(-2)$$

$$\begin{pmatrix} \tau_{ab}{}^{cd} \\ \rho_{ab}{}^c \\ \nu_{ab} \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -2\tau_{ad}{}^{cd} \\ -4\rho_{ad}{}^d \end{pmatrix}.$$

The first two homologies are given by $H_0(M, S^2\mathcal{T}) = \mathcal{E}^{(ab)}(-2)$ and $H_1(M, S^2\mathcal{T}) = (\mathcal{E}_a^{(bc)})_0(-2)$, where $(\mathcal{E}_a^{(bc)})_0(-2)$ denotes the trace-free part of $\mathcal{E}_a^{(bc)}(-2)$.

Proof. In the given splitting consider $\Phi = \begin{pmatrix} \tau^{bc} \\ \rho^b \\ \nu \end{pmatrix}$, $v = (\mu_b, \sigma)$ and $w = (\eta_c, \pi)$. We prove the first statement for elements of the form $g_a \otimes \Phi$, for $g_a \in \mathcal{E}_a$. Then by (5.11), we have

$$\begin{aligned} \partial_1^*(g_a \otimes \Phi)\Big((\mu_b, \sigma), (\eta_c, \pi)\Big) &= -\begin{pmatrix} \tau^{bc} \\ \rho^b \\ \nu \end{pmatrix}\Big((g_a\sigma, 0), (\eta_c, \pi)\Big) - \begin{pmatrix} \tau^{bc} \\ \rho^b \\ \nu \end{pmatrix}\Big((\mu_b, \sigma), (g_a\pi, 0)\Big) \\ &= (-g_c\tau^{bc})\mu_b\pi + (-g_b\tau^{bc})\eta_c\sigma + (-2g_b\rho^b)\pi\sigma, \end{aligned}$$

which gives the result.

The second statement is proved for elements of the form $\eta_a\omega_b \otimes \Phi$, for $\eta_a\omega_b = -\eta_b\omega_a$.

$$\begin{aligned} \partial_2^*(\eta_a\omega_b \otimes \Phi) &= \eta_a \otimes (\omega_b \cdot \Phi) - \omega_a \otimes (\eta_b \cdot \Phi) \\ &= \begin{pmatrix} 0 \\ -\eta_a\omega_b\tau^{bc} \\ -2\eta_a\omega_b\rho^b \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_a\eta_b\tau^{bc} \\ 2\omega_a\eta_b\rho^b \end{pmatrix} = \begin{pmatrix} 0 \\ -2\eta_a\omega_b\tau^{bc} \\ -4\eta_a\omega_b\rho^b \end{pmatrix}. \end{aligned} \tag{5.14}$$

Continuing with computing the homologies, we obtain $\mathrm{im}(\partial_1^*) = \mathcal{E}(-2)\oplus\mathcal{E}^b(-2)$ and thus $H_0(M, S^2\mathcal{T}) = \mathcal{E}^{(ab)}(-2)$. Furthermore, $\ker(\partial_1^*) = \mathcal{E}_a(-2)\oplus(\mathcal{E}_a^b)_0(-2)\oplus(\mathcal{E}_a{}^{(bc)})_0(-2)$ and $\mathrm{im}(\partial_2^*) = \mathcal{E}_a(-2) \oplus (\mathcal{E}_a^c)_0(-2)$. Note, that we obtain $(\mathcal{E}_a^c)_0(-2)$ in the second component of $\mathrm{im}(\partial_2^*)$, since $\tau_{ad}{}^{cd}$ with skew symmetric lower indices and symmetric upper indices, is trace free. Therefore, $H_1(M, S^2\mathcal{T}) = (\mathcal{E}_a^{(bc)})_0(2)$. □

5.3 Some First BGG-Operators

In this section we construct projectively invariant differential operators using the tractor bundles considered in chapter 4. Eventhough these operators are constructed directly, i.e. without using any additional theory, it should be mentioned, that they actually result from a general construction. Indeed, they appear as the first operators in a Bernstein-Gelfand-Gelfand (BGG) sequence. The definition as well as the construction of such sequences can be found in [10]. For a sketch see [3].

We outline the construction, which will be then discussed in case of each tractor bundle seperately in the following subsections. Working on any tractor bundle, consider the projections $\pi_k : \ker(\partial_k^*) \to H_k(M)$, for $k \geq 0$. These bundle maps certainly induce tensorial operators on the space of sections, i.e. maps $\Gamma(\ker(\partial_k^*)) \to \Gamma(H_k(M))$, which are denoted by the same symbol. Note, that these induced maps are projectively invariant. In the case of $k = 0$, we will obtain a projectively invariant differential operator $L : \Gamma(H_0(M)) \to \Gamma(\ker(\partial_0^*))$, which is a differential splitting of π_0 and satisfies $\partial_1^*(\nabla L(\sigma)) = 0$ for all $\sigma \in \Gamma(H_0(M))$. Furthermore, via the following diagram, the differential operator L gives rise to a projectively invariant differential operator $D : \Gamma(H_0(M)) \to \Gamma(H_1(M))$:

$$\begin{tikzcd} \Gamma(\ker(\partial_0^*)) \arrow[d, shift left, "\pi_0"] & \Gamma(\ker(\partial_1^*)) \arrow[d, "\pi_1"] \\ \Gamma(H_0(M)) \arrow[u, shift left, "L"] \arrow[ur, "\nabla L"] \arrow[r, dotted, "D"'] & \Gamma(H_1(M)) \end{tikzcd} \tag{5.15}$$

Diagram (5.15) can be extended to higher degrees, as its form already suggests. This is however not necessary for our purposes.

Now a solution of the first BGG-equation is a section σ of $H_0(M)$, satisfying $D(\sigma) = 0$.

There is an obvious subclass of solutions, namely those, for which ∇L is actually parallel. Such solutions are called normal.

Although we can define operators which satisfy diagram (5.15) on every tractor bundle, the results connected with these operators are different. This is why we treat each case seperately in the sequel.

Most of the results, we will discuss, are due to [5, 7]. We give the corresponding reference within the text.

The First BGG-Operator on $\mathcal{T}^*$

From Lemma 5.2.1, we know that $\ker(\partial_0^*) = \mathcal{T}^*$ and $H_0(M, \mathcal{T}^*) = \mathcal{E}(1)$. Therefore, in a splitting, the projection $\pi_0 : \Gamma(\mathcal{T}^*) \to \Gamma(\mathcal{E}(1))$ is given by projecting onto the right slot. The next Proposition shows the existence of an invariant differential operator, which splits this projection.

Proposition 5.3.1. Given a smooth section σ of $\mathcal{E}(1)$, then there exists a unique section s of $\mathcal{T}^*$, satisfying $\pi_0(s) = \sigma$ and $\partial_1^*(\nabla s) = 0$. Mapping σ to s defines a projectively invariant differential operator L.

Proof. In a given splitting of $\mathcal{T}^*$, let $s = (\mu_b, \tau)$. Since s satisfies $\Pi(s) = \sigma$, we have $\tau = \sigma$. Moreover, using Definition 4.2.2 and Lemma 5.2.1, gives

$$\nabla_a(\mu_b, \sigma) = (\nabla_a \mu_b + \mathrm{P}_{ab}\sigma, \nabla_a\sigma - \mu_a) \text{ and } \partial_1^*(\nabla_a(\mu_b, \sigma)) = (\nabla_a\sigma - \mu_a, 0).$$

Thus, $\partial_1^*(\nabla s) = 0$ implies $\mu_b = \nabla_b\sigma$. This proves that $s = (\nabla_a\sigma, \sigma)$ is unique and certainly $\sigma \mapsto (\nabla_a\sigma, \sigma)$ defines a smooth differential operator.

We use Corollary 4.2.1 and Theorem 4.1.1, to prove that L is projectively invariant. Change ∇ to a projectively equivalent connection $\hat{\nabla} = \nabla + \Upsilon$ and split $\mathcal{T}^*$ with respect to $\hat{\nabla}$. Then in this splitting we have

$$L(\sigma) \overset{\hat{\nabla}}{=} (\hat{\nabla}\sigma, \sigma) = (\nabla_a\sigma + \Upsilon_a\sigma, \sigma).$$

On the other hand, the change of splitting transforms $(\nabla_a\sigma, \sigma)$ to $(\nabla_a\sigma + \Upsilon_a\sigma, \sigma)$, which gives the invariance of L. $\square$

Note, that since $\partial_1^*(\nabla L(\sigma)) = 0$ by Proposition 5.3.1, $\nabla L(\sigma)$ is in fact an element of $\ker(\partial_1^*)$ and can therefore be projected to the quotient.

Definition 5.3.1. Given a section σ of $\mathcal{E}(1)$, then we define $D(\sigma)$ by projecting $\nabla L(\sigma)$ to the quotient bundle $H_1(M, \mathcal{T}^*) = \mathcal{E}_{(ab)}(1)$ (compare Lemma 5.2.1).

A section σ satisfying $D(\sigma) = 0$, is called a solution of the first BGG-equation, whereas it is called a normal solution, if it in addition satisfies $\nabla L(\sigma) = 0$.

Therefore, in the case of the cotractor bundle, diagram (5.15) has the following form:

$$\begin{tikzcd}
\Gamma(\mathcal{T}^*) \arrow[d, shift left, "\pi_0"] & & \Gamma(\mathcal{E}_{ab}(1)) \arrow[d, "\pi_1"] \\
\Gamma(\mathcal{E}(1)) \arrow[u, shift left, "L"] \arrow[urr, "\nabla L"] \arrow[rr, dotted, "D"'] & & \Gamma(\mathcal{E}_{(ab)}(1))
\end{tikzcd} \tag{5.16}$$

We can give a description of D in terms of a splitting. Choose a splitting of $\mathcal{T}^*$, then in this splitting $L(\sigma) = (\nabla_a \sigma, \sigma)$ and $\nabla_a L(\sigma) = (\nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma, 0)$. Projecting this to the symmetric part of $\mathcal{E}_{ab}(1)$, gives

$$D(\sigma) = \nabla_{(a}\nabla_{b)}\sigma + \mathrm{P}_{(ab)}\sigma,$$

which is exactly the example we considered in section 5.1.

Lemma 5.3.1. A smooth section σ of $\mathcal{E}(1)$ is a solution of the first BGG-equation if and only if it is a normal solution, hence

$$D(\sigma) = 0 \Leftrightarrow \nabla L(\sigma) = 0.$$

Proof. By Definition, a normal solution is also a solution of the first BGG-equation. On the other hand, for a smooth section σ, we remarked before, that D is given by

$$D(\sigma) = \nabla_{(a}\nabla_{b)}\sigma + \mathrm{P}_{(ab)}\sigma.$$

Moreover, we showed that $\nabla_a L(\sigma) = (\nabla_a \nabla_b \sigma + \mathrm{P}_{ab}\sigma, 0)$. Thus we have to show that the first slot of $\nabla_a L(\sigma)$ vanishes, if its symmetric part vanishes. Choose a connection ∇ in the projective class for which $\beta_{ab} = 0$ (compare Lemma 4.1.3 and Remark 4.1.1). Then P_{ab} is symmetric and we have that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma = 0,$$

since β_{ab} is the curvature of ∇. Thus $\nabla_a \nabla_b \sigma$ is symmetric and we obtain the result. □

Theorem 5.3.1. [5, Theorem 9] Let $\sigma \not\equiv 0$ be a section of $\mathcal{E}(1)$ and consider the first BGG-equation $D\sigma = 0$. Then we have the following statements:

1. A solution σ of the first BGG-equation induces a Ricci flat connection in the projective class on the open subset $U = \{x \in M : \sigma(x) \neq 0\}$.
2. Given a Ricci flat connection in the projective class, we obtain a local solution of the first BGG-equation.
3. If σ is a solution of the first BGG-equation, then U is dense in M.

Proof. 1. Suppose that $D(\sigma) = 0$, then $\nabla_{(a}\nabla_{b)}\sigma + \mathrm{P}_{(ab)}\sigma = 0$. As D is projectively invariant, we may change the connection in the equivalence class of ∇ without changing D. Locally on U there exists a unique connection $\hat{\nabla}$ in the projective class, such that $\hat{\nabla}\sigma = 0$, compare Lemma 4.1.3. Now for this connection, the equation $D(\sigma) = 0$ reduces to $\hat{\mathrm{P}}_{(ab)}\sigma = 0$, which implies that $\hat{\mathrm{P}}_{(ab)} = 0$. Hence $\hat{\mathrm{P}}_{ab}$ is antisymmetric. Furthermore, by Lemma 4.1.4, $\hat{\beta}_{ab} = 0$ and $\hat{\mathrm{P}}_{ab}$ is symmetric, thus $\hat{\mathrm{P}}_{ab} = 0$. The second part of Definition 2.2.5 states that $(n-1)\hat{\mathrm{P}}_{ab} = \hat{R}_{ab} + \hat{\beta}_{ab}$ and therefore also $\hat{R}_{ab} = 0$.

2. A Ricci flat connection ∇ implies $\mathrm{P}_{ab} = 0$, by Definition 2.2.5. Therefore, the first BGG-equation reduces to $\nabla_{(a}\nabla_{b)}\sigma = 0$. Also the induced connection on $\mathcal{E}(1)$ is flat and thus admits a local section, which is parallel. This section is a local solution of the first BGG-equation.

3. Let σ be a solution of the first BGG-equation. Then by Lemma 5.3.1, this solution is normal, hence $\nabla L(\sigma) = 0$. Furthermore, by Lemma 5.3.1, we have $L(\sigma) = (\nabla_a\sigma, \sigma)$. We claim that $L(\sigma)(x) \neq 0$ for all $x \in M$. The equation $\nabla L(\sigma) = 0$ locally defines an ordinary differential equation and by the uniqueness of solutions for a given initial value, assuming $L(\sigma)(x) = 0$ for some $x \in M$ gives $L(\sigma) \equiv 0$. This is a contradiction, as $L(\sigma) \equiv 0$ implies $\sigma \equiv 0$. Now if $\sigma(x) = 0$ for some $x \in M$, then $\nabla\sigma(x) \neq 0$ (otherwise $L(\sigma)(x) = 0$). This shows that the zero set of σ is an embedded hypersurface. Taking the complement of the zero set of σ, we obtain an open and dense subset and this is exactly U.

□

The First BGG-Operator on $\mathcal{T}$

On $\mathcal{T}$ we have that $\ker(\partial_0^*) = \mathcal{T}$ and $H_0(M, \mathcal{T}) = \mathcal{E}^a(-1)$, compare Lemma 5.2.2. Therefore, the projection is a map $\pi_0 : \Gamma(\mathcal{T}) \to \Gamma(\mathcal{E}^a(-1))$ and in a splitting, it is given by projecting onto the top slot. Furthermore, we have an analog to Proposition 5.3.1:

Proposition 5.3.2. Given a smooth section ν^a of $\mathcal{E}^a(-1)$, then there exists a unique section s of $\mathcal{T}$, satisfying $\pi_0(s) = \nu^a$ and $\partial_1^*(\nabla s) = 0$. Mapping ν^a to s defines a projectively invariant differential operator L.

Proof. Given a splitting of $\mathcal{T}$, let the section s be given by $s = \binom{\eta^a}{\tau}$. Then from $\Pi(s) = \nu^a$, we immediately obtain $\eta^a = \nu^a$. Furthermore, by Theorem 4.3.1 and Lemma 5.2.2 we have

$$\nabla_a\binom{\eta^b}{\tau} = \binom{\nabla_a\eta^b + \tau\delta_a{}^b}{\nabla_a\tau - \mathrm{P}_{ab}\nu^b} \quad \text{and} \quad \partial_1^*\Big(\nabla_a\binom{\eta^b}{\tau}\Big) = \binom{0}{-\nabla_a\nu^a - n\tau}.$$

Now $\partial_1^*(\nabla s) = 0$ implies $\tau = -\frac{1}{n}\nabla_a\nu^a$, which proves the uniqueness of $s = \binom{\nu^a}{-\frac{1}{n}\nabla_a\nu^a}$. Certainly,

$$L : \nu^a \mapsto \binom{\nu^a}{-\frac{1}{n}\nabla_a\nu^a}$$

is a smooth differential operator.

We prove projective invariance using Lemma 4.3.1 and Lemma 4.1.2 in the case $k = 1, \omega = -1$. Consider a change of connection from ∇ to $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M. In the splitting induced by $\hat{\nabla}$, the the map L is given by

$$L(\nu^a) \overset{\hat{\nabla}}{=} \begin{pmatrix} \nu^a \\ -\frac{1}{n}\hat{\nabla}_a\nu^a \end{pmatrix} = \begin{pmatrix} \nu^a \\ -\frac{1}{n}(\nabla_a\nu^a + n\Upsilon_a\nu^a) \end{pmatrix}.$$

Furthermore, the change of splitting transforms $\begin{pmatrix} \nu^a \\ -\frac{1}{n}\nabla_a\nu^a \end{pmatrix}$ to $\begin{pmatrix} \nu^a \\ -\frac{1}{n}\nabla_a\nu^a - \Upsilon_a\nu^a \end{pmatrix}$, which gives the result. $\square$

Since $\partial_1^*(\nabla L(\nu^a)) = 0$ by Proposition 5.3.2, it can be projected to the homology $H_1(M, \mathcal{T})$.

Definition 5.3.2. Given a section ν^a of $\mathcal{E}^a(-1)$, then we define $D(\nu^a)$ by projecting $\nabla L(\nu^a)$ to the quotient bundle $H_1(M, \mathcal{T}) = (\mathcal{E}_a^b)_0(-1)$, compare Lemma 5.2.2.

A section ν^a satisfying $D(\nu^a) = 0$, is called a solution of the first BGG-equation, whereas it is called a normal solution, if it additionally satisfies $\nabla L(\nu^a) = 0$.

Therefore, in the case of the tractor bundle (and in a given splitting), diagram (5.15) yields:

$$\begin{tikzcd} \Gamma(\mathcal{T}) \arrow[d, shift left, "\pi_0"] & & \Gamma(\mathcal{E}_a(-1) \oplus (\mathcal{E}_a^b)_0(-1)) \arrow[d, "\pi_1"] \\ \Gamma(\mathcal{E}^a(-1)) \arrow[u, shift left, "L"] \arrow[urr, "\nabla L"] \arrow[rr, dotted, "D"'] & & \Gamma((\mathcal{E}_a^b)_0(-1)) \end{tikzcd} \tag{5.17}$$

We again describe D in terms of a splitting. Theorem 4.3.1 gives

$$\nabla_a L(\nu^b) = \nabla_a \begin{pmatrix} \nu^b \\ -\frac{1}{n}\nabla_c\nu^c \end{pmatrix} = \begin{pmatrix} \nabla_a\nu^b - \frac{1}{n}(\nabla_c\nu^c)\delta_a{}^b \\ -\frac{1}{n}\nabla_a\nabla_c\nu^c - \mathrm{P}_{ab}\nu^b \end{pmatrix} \tag{5.18}$$

and projecting onto the trace-free part of $\mathcal{E}_a^b$ implies:

$$D(\nu^b) = \nabla_a\nu^b - \frac{1}{n}(\nabla_c\nu^c)\delta_a{}^b, \tag{5.19}$$

which is exactly the trace-free part of $\nabla_a\nu^b$. Furthermore, we have an equivalent statement to Lemma 5.3.1:

Lemma 5.3.2. A smooth section ν^a of $\mathcal{E}^a(1)$ is a solution of the first BGG-equation if and only if it is a normal solution, hence

$$D(\nu^a) = 0 \Leftrightarrow \nabla L(\nu^a) = 0.$$

Proof. Certainly, a normal solution is a solution of the first BGG-equation. Conversely, using equation (5.19), the vanishing of D implies

$$\nabla_a \nu^b = \frac{1}{n}(\nabla_c \nu^c)\delta_a{}^b. \tag{5.20}$$

We have to show, that (5.20) forces $\nabla L(\nu^a) = 0$. First of all, by (5.18), we immediately obtain that the top slot of $\nabla L(\nu^a)$ vanishes. Thus, we have to show that (5.20) implies $\mathrm{P}_{ab}\nu^b = -\frac{1}{n}\nabla_a \nabla_b \nu^b$. By Definition 2.2.5, we have that

$$(n+1)\beta_{ab} = -2R_{[ab]} \quad \text{and} \quad (n-1)\mathrm{P}_{ab} = R_{ab} + \beta_{ab}.$$

Therefore, we first have to calculate the Ricci curvature. Set $\rho = \frac{1}{n}\nabla_c \nu^c$, then we obtain:

$$\begin{aligned} R_{ab}{}^c{}_d \nu^d &= \nabla_a \nabla_b \nu^c - \nabla_b \nabla_a \nu^c \\ &= \nabla_a(\rho \delta_b{}^c) - \nabla_b(\rho \delta_a{}^c) \\ &= \delta_b{}^c \nabla_a \rho - \delta_a{}^c \nabla_b \rho. \end{aligned}$$

Taking the trace implies $R_{ab}\nu^b = -(n+1)\nabla_a \rho$. This, in turn, indicates $\beta_{ab}\nu^b = 2\nabla_a \rho$ and $(n-1)\mathrm{P}_{ab}\nu^b = -(n+1)\nabla_a \rho + 2\nabla_a \rho = -(n-1)\nabla_a \rho$. Dividing the last equation by $(n-1)$ gives the result:

$$\mathrm{P}_{ab}\nu^b = -\frac{1}{n}\nabla_a \nabla_b \nu^b.$$

□

The First BGG-Operator on $S^2\mathcal{T}^*$

By Lemma 5.2.3, we have $\ker(\partial_0^*) = S^2\mathcal{T}^*$ and $H_0(M, S^2\mathcal{T}^*) = \mathcal{E}(2)$, which gives $\pi_0 : \Gamma(S^2\mathcal{T}^*) \to \Gamma(\mathcal{E}(2))$. In a splitting, this map is given by projecting onto the last slot. Then analogously to the cases of the cotractor and tractor bundle, there exists a projectively invariant differential operator, which splits the projection.

Proposition 5.3.3. Given a smooth section σ of $\mathcal{E}(2)$, then there exists a unique section s of $S^2\mathcal{T}^*$, satisfying $\pi_0(s) = \sigma$ and $\partial_1^*(\nabla s) = 0$. Mapping σ to s defines a projectively invariant differential operator L.

The proof of this Proposition requires the following Lemma, which gives the transformation of ∇^2 under change of connection in the projective class.

Lemma 5.3.3. Changing the connection ∇ to $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M, gives the following transformation for sections σ of $\mathcal{E}(\omega)$:

$$\hat{\nabla}_a \hat{\nabla}_b \sigma = \nabla_a \nabla_b \sigma + 2(\omega - 1)\Upsilon_{(a}\nabla_{b)}\sigma + \omega(\nabla_a \Upsilon_b)\sigma + (\omega^2 - 2\omega)\Upsilon_a \Upsilon_b \sigma.$$

Proof. This can be shown using $(\hat{\nabla}^2\sigma)(\xi, \eta) = \hat{\nabla}_\xi \hat{\nabla}_\eta \sigma - \hat{\nabla}_{\hat{\nabla}_\xi \eta}\sigma$, for $\xi, \eta \in \mathfrak{X}(M)$. Working this out by using Theorem 4.1.1 as well as Lemma 4.1.2 , we obtain:

$$\begin{aligned}(\hat{\nabla}^2\sigma)(\xi,\eta) &= \nabla_\xi(\nabla_\eta\sigma+\omega\Upsilon(\eta)\sigma)+\omega\Upsilon(\xi)(\nabla_\eta\sigma+\omega\Upsilon(\eta)\sigma)\\ &\quad -(\nabla_{\nabla_\xi\eta}\sigma+\omega\Upsilon(\nabla_\xi\eta)\sigma+\Upsilon(\xi)\nabla_\eta\sigma+2\omega\Upsilon(\xi)\Upsilon(\eta)\sigma+\Upsilon(\eta)\nabla_\xi\sigma)\\ &= (\nabla_\xi\nabla_\eta\sigma-\nabla_{\nabla_\xi\eta}\sigma)+\omega(\nabla_\xi\Upsilon(\eta)-\Upsilon(\nabla_\xi\eta))\sigma+(\omega-1)\Upsilon(\eta)\nabla_\xi\sigma\\ &\quad +(\omega-1)\Upsilon(\xi)\nabla_\eta\sigma+(\omega^2-2\omega)\Upsilon(\xi)\Upsilon(\eta)\sigma\\ &= (\nabla^2\sigma)(\xi,\eta)+\omega(\nabla_\xi\Upsilon)(\eta)\sigma+(\omega-1)(\Upsilon(\xi)\nabla_\eta\sigma+\Upsilon(\eta)\nabla_\xi\sigma)\\ &\quad +(\omega^2-2\omega)\Upsilon(\xi)\Upsilon(\eta)\sigma.\end{aligned}$$

Rewriting this in abstract index notation gives the result. □

Proof of Proposition 5.3.3. In a given splitting, let $s=\begin{pmatrix}\mu_{bc} & \eta_b & \nu\end{pmatrix}$. Then by $\Pi(s)=\sigma$, we immediately obtain $\nu=\sigma$. Furthermore, Theorem 4.4.1 and Lemma 5.2.3 imply

$$\nabla_a s=\begin{pmatrix}\nabla_a\mu_{bc}+2\mathrm{P}_{a(b}\eta_{c)} & \nabla_a\eta_b+\mathrm{P}_{ab}\sigma-\mu_{ab} & \nabla_a\sigma-2\eta_a\end{pmatrix}$$

and

$$\partial_1^*(\nabla s)=\begin{pmatrix}2(\nabla_{(a}\eta_{b)}+\mathrm{P}_{(ab)}\sigma-\mu_{(ab)}) & \nabla_a\sigma-2\eta_a & 0\end{pmatrix}.$$

Therefore, from $\partial_1^*(\nabla s)=0$ we obtain $\eta_a=\frac{1}{2}\nabla_a\sigma$ and

$$\mu_{ab}=\nabla_{(a}\eta_{b)}+\mathrm{P}_{(ab)}\sigma=\tfrac{1}{2}\nabla_{(a}\nabla_{b)}\sigma+\mathrm{P}_{(ab)}\sigma.$$

This shows, that $s=\begin{pmatrix}\frac{1}{2}\nabla_{(a}\nabla_{b)}\sigma+\mathrm{P}_{(ab)}\sigma & \frac{1}{2}\nabla_a\sigma & \sigma\end{pmatrix}$ is unique and that mapping σ to s defines a smooth differential operator.

In order to prove projective invariance of L, we use Lemma 5.3.3. Thus, consider a change from ∇ to $\hat{\nabla}=\nabla+\Upsilon$ in the projective class of M. Then in the splitting induced by $\hat{\nabla}$, the differential operator L is given by:

$$L(\sigma)\overset{\hat{\nabla}}{=}\begin{pmatrix}\frac{1}{2}\hat{\nabla}_{(a}\hat{\nabla}_{b)}\sigma+\hat{\mathrm{P}}_{(ab)}\sigma & \frac{1}{2}\hat{\nabla}_a\sigma & \sigma\end{pmatrix}.$$

Now using Lemma 5.3.3 in the case of $\omega=2$, Lemma 3.3.4 and Lemma 4.1.1, we obtain:

$$L(\sigma)\overset{\hat{\nabla}}{=}\begin{pmatrix}\frac{1}{2}\nabla_{(a}\nabla_{b)}\sigma+\mathrm{P}_{(ab)}\sigma+\Upsilon_{(a}\nabla_{b)}\sigma+\Upsilon_a\Upsilon_b\sigma & \frac{1}{2}\nabla_a\sigma+\Upsilon_a\sigma & \sigma\end{pmatrix}. \tag{5.21}$$

On the other hand, by Lemma 4.4.1, the element $\begin{pmatrix}\frac{1}{2}\nabla_{(a}\nabla_{b)}\sigma+\mathrm{P}_{(ab)}\sigma & \frac{1}{2}\nabla_a\sigma & \sigma\end{pmatrix}$ exactly transforms to (5.21), showing the projective invariance. □

By Proposition 5.3.3, $\partial_1^*(\nabla L(\sigma))=0$ and therefore, $\nabla L(\sigma)$ can be projected to the homology $H_1(M,S^2\mathcal{T}^*)$.

Definition 5.3.3. Given a section σ of $\mathcal{E}(2)$, then we define $D(\sigma)$ by projecting $\nabla L(\sigma)$ to the quotient bundle $H_1(M,S^2\mathcal{T}^*)=\mathcal{E}_{(abc)}(2)$, compare Lemma 5.2.3.

A section σ satisfying $D(\sigma) = 0$, is called a solution of the first BGG-equation, whereas it is called a normal solution, if it in addition satisfies $\nabla L(\sigma) = 0$.

This yields the following diagram in the case of $S^2\mathcal{T}^*$ in a given splitting:

$$\begin{tikzcd}
\Gamma(S^2\mathcal{T}^*) \arrow[d, shift left, "\pi_0"] & & \Gamma(\mathcal{E}_{a(bc)}(2) \oplus \mathcal{E}_{[ab]}(2)) \arrow[d, "\pi_1"] \\
\Gamma(\mathcal{E}(2)) \arrow[u, shift left, "L"] \arrow[urr, "\nabla L"] \arrow[rr, dotted, "D"'] & & \Gamma(\mathcal{E}_{(abc)}(2))
\end{tikzcd} \tag{5.22}$$

Moreover, we can describe D in a given splitting. Choose a splitting of $S^2\mathcal{T}^*$, then by Theorem 4.4.1 we have:

$$\begin{aligned}
&\nabla_a L(\sigma) = \\
&= \big(\nabla_a(\tfrac{1}{2}\nabla_{(b}\nabla_{c)}\sigma + \mathrm{P}_{(bc)}\sigma) + \mathrm{P}_{a(b}\nabla_{c)}\sigma \quad \tfrac{1}{2}\nabla_a\nabla_b\sigma + \mathrm{P}_{ab}\sigma - \tfrac{1}{2}\nabla_{(a}\nabla_{b)}\sigma - \mathrm{P}_{(ab)}\sigma \quad 0\big) \\
&= \big(\tfrac{1}{2}\nabla_a\nabla_{(b}\nabla_{c)}\sigma + \nabla_a\mathrm{P}_{(bc)}\sigma + \mathrm{P}_{a(b}\nabla_{c)}\sigma \quad \tfrac{1}{2}\nabla_{[a}\nabla_{b]}\sigma + \mathrm{P}_{[ab]}\sigma \quad 0\big)\,.
\end{aligned} \tag{5.23}$$

Therefore, projecting onto $\mathcal{E}_{(abc)}(2)$ implies

$$\begin{aligned}
D(\sigma) &= \tfrac{1}{2}\nabla_{(a}\nabla_b\nabla_{c)}\sigma + \nabla_{(a}\mathrm{P}_{bc)}\sigma + \mathrm{P}_{(ab}\nabla_{c)}\sigma \\
&= \tfrac{1}{2}\nabla_{(a}\nabla_b\nabla_{c)}\sigma + (\nabla_{(a}\mathrm{P}_{bc)})\sigma + 2\mathrm{P}_{(ab}\nabla_{c)}\sigma.
\end{aligned} \tag{5.24}$$

This is a multiple of the operator considered in [6, Section 3.3].

Theorem 5.3.2. [5, Theorem 11] Let σ be a section of $\mathcal{E}(2)$. Consider the set $U = \{x \in M : \sigma(x) \neq 0\}$ and the unique connection ∇ in the projective class of M satisfying $\nabla\sigma = 0$ on U, compare Lemma 4.1.3. Then on U we have the following statements:

1. σ is a solution of the first BGG-equation $D(\sigma) = 0$, if and only if $\nabla_{(a}\mathrm{P}_{bc)} = 0$.
2. σ is a normal solution, i.e. satisfies $\nabla L(\sigma) = 0$, if and only if $\nabla_a\mathrm{P}_{bc} = 0$. If, in addition, P_{ab} is non-degenerate, then it defines a pseudo-Riemannian Einstein metric on M and its Levi-Civita connection is exactly ∇.

Proof. 1. This follows immediately from equation (5.24).

2. Assuming that $\nabla L(\sigma) = 0$, then in the splitting induced by ∇, we have

$$\big(\sigma\nabla_a\mathrm{P}_{(bc)} \quad \mathrm{P}_{[ab]}\sigma \quad 0\big) = 0. \tag{5.25}$$

Lemma 4.1.4 implies that P_{ab} is symmetric, which reduces equation (5.25) to $\big(\sigma\nabla_a\mathrm{P}_{bc} \quad 0 \quad 0\big) = 0$. Since σ is non-zero, $\nabla_a\mathrm{P}_{bc}$ has to vanish.

Conversely, $\nabla_a\mathrm{P}_{bc} = 0$ and the fact that P_{ab} is symmetric, immediately imply that $\nabla L(\sigma) = 0$. This proves the first part of (2).

Now if P_{ab} is non-degenerate, it can be understood as a pseudo-Riemannian metric, as it satisfies all defining properties of a metric, except for positive definitness. Furthermore, since $\nabla_a \mathrm{P}_{bc} = 0$, the connection ∇ has to be its Levi-Civita connection, as there exists only one torsion-free connection, which preserves a pseudo-Riemannian metric. By Definition 2.2.5, we have $R_{ab} = (n-1)\mathrm{P}_{ab}$, hence the Ricci curvature is a multiple of the pseudo-Riemannian metric. Thus, we obtain a pseudo-Riemannian Einstein metric on M.

□

The First BGG-Operator on $S^2\mathcal{T}$

On $S^2\mathcal{T}$ we have $\ker(\partial_0^*) = S^2\mathcal{T}$ and $H_0(M, S^2\mathcal{T}) = \mathcal{E}^{(bc)}(-2)$, compare Lemma 5.2.4. Therefore, $\pi_0 : \Gamma(S^2\mathcal{T}) \to \Gamma(\mathcal{E}^{(bc)}(-2))$ is given by projecting onto the first slot in a splitting. We have the following result:

Proposition 5.3.4. [7, Proposition 3.1] Given a smooth section τ^{bc} of $\mathcal{E}^{(bc)}(-2)$, then there exists a unique section s of $S^2\mathcal{T}$, satisfying $\pi_0(s) = \tau^{bc}$ and $\partial_1^*(\nabla s) = 0$. Mapping τ^{bc} to s defines a projectively invariant differential operator L.

In order to prove this Proposition, we need the following Lemma:

Lemma 5.3.4. Let ∇ be an affine connection on M. If we change ∇ to $\hat{\nabla} + \Upsilon$ in the projective class of M, then we have the following transformations for $\tau^{bc} \in \Gamma(\mathcal{E}^{(bc)}(-2))$:

$$\begin{aligned} \hat{\nabla}_a \tau^{bc} &= \nabla_a \tau^{ab} + (n+1)\Upsilon_a \tau^{ab}, && (5.26)\\ \hat{\nabla}_a \hat{\nabla}_b \tau^{ab} &= \nabla_a \nabla_b + 2n\Upsilon_a \nabla_b \tau^{ab} + (n^2-1)\Upsilon_a \Upsilon_b \tau^{ab} && (5.27)\\ &\quad +(n+1)(\nabla_a \Upsilon_b)\tau^{ab}. \end{aligned}$$

Proof. Equation (5.26) can be shown using Lemma 4.1.2 in the case $k = 2$, $\omega = -2$. Similarly, equation (5.27) is an application of this Lemma. Note, however, that calculating transformations of the form $\hat{\nabla}_a \nabla_b \tau^{cd}$, uses its third part in the case $k = 1$, $l = 2$ and $\omega = -2$. □

Proof of Proposition 5.3.4. Suppose that in a splitting of $S^2\mathcal{T}$, the smooth section s is given by $s = \begin{pmatrix} \mu^{bc} \\ \rho^c \\ \nu \end{pmatrix}$. Again, the first slot is already determined by $\Pi(s) = \tau^{bc}$, hence $\mu^{bc} = \tau^{bc}$. Using Theorem 4.4.2 and Lemma 5.2.4, we obtain

$$\nabla_a s = \begin{pmatrix} \nabla_a \tau^{bc} + \rho^c \delta_a{}^b + \rho^b \delta_a{}^c \\ \nabla_a \rho^b + \nu \delta_a{}^b - \tau^{bc}\mathrm{P}_{ac} \\ \nabla_a \nu - 2\rho^b \mathrm{P}_{ab} \end{pmatrix} \quad \text{and} \quad \partial_1^*(\nabla_a s) = \begin{pmatrix} 0 \\ -\nabla_a \tau^{ba} - (n+1)\rho^b \\ -2(\nabla_a \rho^a + n\nu - \tau^{ab}\mathrm{P}_{ab}) \end{pmatrix}.$$

Therefore, $\partial_1^*(\nabla_a s)$ implies $\rho^b = -\frac{1}{n+1}\nabla_a \tau^{ba}$ and $\nu = \frac{1}{n}(\frac{1}{n+1}\nabla_a \nabla_b + \mathrm{P}_{ab})\tau^{ab}$. This shows, that s is unique and mapping τ^{bc} to s defines a smooth differential operator.

In order to show that L is projectively invariant, we need Lemma 4.4.3 and Lemma 5.3.4. Consider a change of connection from ∇ to $\hat{\nabla} = \nabla + \Upsilon$ in the projective class of M. Then in the splitting induced by $\hat{\nabla}$ the differential operator L is given by:

$$L(\tau^{bc}) \overset{\hat{\nabla}}{=} \begin{pmatrix} \tau^{bc} \\ -\frac{1}{n+1}\hat{\nabla}_a\tau^{ba} \\ \frac{1}{n}(\frac{1}{n+1}\hat{\nabla}_a\hat{\nabla}_b + \hat{\mathrm{P}}_{ab})\tau^{ab} \end{pmatrix}.$$

Using Lemma 5.3.4 as well as Lemma 3.3.4, we obtain

$$L(\tau^{bc}) \overset{\hat{\nabla}}{=} \begin{pmatrix} \tau^{bc} \\ -\frac{1}{n+1}\nabla_a\tau^{ba} - \Upsilon_a\tau^{ab} \\ \frac{1}{n}(\frac{1}{n+1}\nabla_a\nabla_b + \mathrm{P}_{ab})\tau^{ab} + \frac{2}{n+1}\Upsilon_a\nabla_b\tau^{ab} + \Upsilon_a\Upsilon_b\tau^{ab} \end{pmatrix}. \tag{5.28}$$

On the other hand, by Lemma 4.4.3, changing the splitting transforms $L(\tau^{bc})$ to (5.28), which proves projective invariance of L. □

Since $\partial_1^*(\nabla L(\tau^{bc})) = 0$ by Proposition 5.3.4, $\nabla L(\tau^{bc})$ can be projected to the homology $H_1(M, S^2\mathcal{T})$.

Definition 5.3.4. Given a section τ^{bc} of $\mathcal{E}^{(bc)}(-2)$, then we define $D(\tau^{bc})$ by projecting $\nabla L(\tau^{bc})$ to the quotient bundle $H_1(M, S^2\mathcal{T}) = (\mathcal{E}_a^{(bc)})_0(-2)$, compare Lemma 5.2.4.

A section τ^{bc} satisfying $D(\tau^{bc}) = 0$, is called a solution of the first BGG-equation, whereas it is called a normal solution, if it additionally satisfies $\nabla L(\tau^{bc}) = 0$.

In a given splitting, diagram (5.15) is given by

$$\begin{tikzcd}
\Gamma(S^2\mathcal{T}) \arrow[d, shift left, "\pi_0"] & \Gamma(\mathcal{E}(2) \oplus (\mathcal{E}_a^b)_0(-2) \oplus ({\mathcal{E}_a}^{(bc)})_0(-2)) \arrow[d, "\pi_1"] \\
\Gamma(\mathcal{E}^{(bc)}(-2)) \arrow[u, shift left, "L"] \arrow[ur, "\nabla L"] \arrow[r, dotted, "D"'] & \Gamma((\mathcal{E}_a^{(bc)})_0(-2))
\end{tikzcd} \tag{5.29}$$

In analog to the previous sections, we compute D in terms of a splitting. Thus, choose a splitting of $S^2\mathcal{T}$ and calculate $\nabla_a L(\tau^{bc})$ using Theorem 4.4.2. We see that the first slot of $\nabla_a L(\tau^{bc})$ is given by the trace-free part of $\nabla_a\tau^{bc}$. Therefore, we obtain:

$$D(\tau^{bc}) = \nabla_a\tau^{bc} - \tfrac{1}{n+1}(\nabla_d\tau^{dc}){\delta_a}^b - \tfrac{1}{n+1}(\nabla_d\tau^{db}){\delta_a}^c. \tag{5.30}$$

Remark 5.3.1. For $S^2\mathcal{T}$ there exists an analogous result to Theorem 5.3.1 respectively Theorem 5.3.2. Non-degenerate solutions of the first BGG-equation (5.30) give rise to Levi-Civita connections in the projective class, whereas non-degenerate normal solutions yield Einstein metrics. Compare [7, 11].

Chapter 6
Interpretation on the Homogeneous Model

In this chapter we discuss the theory developed so far on the model for oriented projective structures. Recall from chapter 3 that this is given by S^n viewed as the ray projectivizations of $C = \mathbb{R}^{n+1}/\{0\}$. Hence $S^n = C/\mathbb{R}_+$ and the projection $\pi : C \to S^n$ is a surjective submersion.

6.1 The Cotractor Bundle of the Homogeneous Model

Consider the tangent map $T_v\pi : T_vC \to T_{\pi(v)}S^n$ for $v \in \mathbb{R}^{n+1}$. Then the kernel of this map is given by $\mathbb{R}v$ and we obtain a short exact sequence

$$0 \longrightarrow \mathbb{R}v \longrightarrow T_vC \xrightarrow{T_v\pi} T_{\pi(v)}S^n \longrightarrow 0. \tag{6.1}$$

Furthermore, since $T_vC = \mathbb{R}^{n+1}$, we have an isomorphism $\mathbb{R}^{n+1}/\mathbb{R}v \cong T_{\pi(v)}S^n$, which we denote by φ_v.

Lemma 6.1.1. We have the following transformation:

$$\varphi_{\lambda v} = \varphi_v \circ \lambda^{-1}\mathrm{id},$$

for $\lambda > 0$.

Proof. Consider the map $\lambda\mathrm{id} : C \to C$. Then we have $\pi \circ \lambda\mathrm{id} = \pi$ and this induces $T_v\pi = T_v(\pi \circ \lambda\mathrm{id}) = T_{\lambda v}\pi \circ \lambda\mathrm{id}$ on the tangent bundles, since $\lambda\mathrm{id}$ is a linear function. Also $\lambda\mathrm{id}$ factors to a well-defined function $\mathbb{R}^{n+1}/\mathbb{R}v \to \mathbb{R}^{n+1}/\mathbb{R}v$ and hence we obtain a commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^{n+1}/\mathbb{R}v \arrow[r, "\lambda\mathrm{id}"] \arrow[d, "\varphi_v"] & \mathbb{R}^{n+1}/\mathbb{R}v \arrow[dl, "\varphi_{\lambda v}"] \\
T_{\pi(v)}S^n &
\end{tikzcd}$$

and thus the claim. □

Now for $v \in C$ define $0 \neq \mu_v \in \Lambda^n T^*_{\pi(v)} S^n$ by

$$\mu_v(\xi_1, \ldots, \xi_n) = \det(v, v_1, \ldots, v_n),$$

where $\xi_i \in T_{\pi(v)}S^n$ and $v_i \in \mathbb{R}^{n+1}$ with $T_v\pi \cdot v_i = \xi_i$ for all i. First of all note that there is always a lift v_i for a given ξ_i, since π is a submersion. Furthermore, the definition of μ_v is independet of the choice of lifts:

Assume that $v_i, \tilde{v}_i \in \mathbb{R}^{n+1}$ are two lifts of ξ_i. Then $T_v\pi(\tilde{v}_i - v_i) = 0$ and thus $\tilde{v}_i - v_i \in \ker(T_v\pi) = \mathbb{R}v$. Thus there exists a $t \in \mathbb{R}$ such that $\tilde{v}_i = v_i + tv$, which gives

$$\det(v, v_1, \ldots, v_n) = \det(v, \tilde{v}_1, \ldots, \tilde{v}_n),$$

by linearity and the fact that the determinant vanishes if there occur two linearly dependent vectors in the argument.

Lemma 6.1.2. For $\lambda > 0$ we have the following transformation:

$$\mu_{\lambda v} = \lambda^{n+1} \mu_v.$$

Proof. Let $\xi_i \in T_{\pi(v)}S^n$ and $v_i \in \mathbb{R}^{n+1}$ such that $T_{\lambda v}\pi \cdot v_i = \xi_i$. Then by the definition we have $\mu_{\lambda v}(\xi_1, \ldots, \xi_n) = \det(\lambda v, v_1, \ldots, v_n)$. From above we know that $T_{\lambda v}\pi = T_v\pi \circ \lambda^{-1}\mathrm{id}$ and thus $\lambda^{-1}v_i$ is a lift of ξ_i for the map $T_v\pi$. Thus

$$\begin{aligned}
\mu_{\lambda v}(\xi_1, \ldots, \xi_n) &= \lambda \det(v, v_1, \ldots, v_n) \qquad (6.2)\\
&= \lambda^{n+1} \det(v, \lambda^{-1}v_1, \ldots, \lambda^{-1}v_n)\\
&= \lambda^{n+1} \mu_v(\xi_1, \ldots, \xi_n).
\end{aligned}$$

□

Theorem 6.1.1. Sections of the bundle $\Lambda^n T^* S^n$ can be identified with smooth functions $C \to \mathbb{R}$ which are homogeneous of degree $(-n-1)$.

Proof. Let ν be a section of this bundle. Then there exists a unique function $f : C \to \mathbb{R}$ such that $\nu(v) = f(v)\mu_v$. Since v is an element of S^n, we have that $\nu(v) = \nu(\lambda v)$ for

$\lambda > 0$. Thus

$$\begin{aligned} f(v)\mu_v &= f(\lambda v)\mu_{\lambda v} \\ &= f(\lambda v)\lambda^{n+1}\mu_v, \end{aligned} \tag{6.3}$$

which gives $f(\lambda v) = \lambda^{-n-1} f(v)$. □

Recall from chapter 4.1, that since S^n is oriented, we can identify $n-$forms on S^n with sections of the volume bundle. We can use Theorem 6.1.1 to find a similar identification of sections of the bundle $\mathcal{E}(\omega)$.

Lemma 6.1.3. Sections of the bundle $\mathcal{E}(\omega)$ can be identified with smooth functions $C \to \mathbb{R}$ which are homogeneous of degree ω.

Proof. The volume bundle is defined as an associated bundle to the linear frame bundle. Sections of this bundle can be identified with $GL(n,\mathbb{R})-$equivariant functions $h : PS^n \to \mathbb{R}$, with equivariancy given by $h(\varphi \circ A) = |\det(A)| h(\varphi)$, see chapter 4.1 and [9, Proposition 1.2.7.]. On the level of equivariant functions, the equation which provides the identification in Theorem 6.1.1, is given by $h = (f \circ \pi)g$, where h corresponds to ν, g to μ and π is the projection $PS^n \to S^n$.

Now if σ is a section of the bundle $\mathcal{E}(\omega)$, then it corresponds to an $GL(n,\mathbb{R})-$equivariant function $\tilde{h} : PS^n \to \mathbb{R}$, where in this case the equivariancy condition is $\tilde{h}(\varphi \circ A) = |\det(A)|^{-\frac{\omega}{n+1}} \tilde{h}(\varphi)$. Define $h := \tilde{h}^{-\frac{n+1}{\omega}}$, then h is an equivariant function of the type above and thus there exists a unique smooth function $f : C \to \mathbb{R}$ which is homogeneous of degree $-n-1$, such that $h = (f \circ \pi)g$. Then the equation $\tilde{h} = (f^{-\frac{\omega}{n+1}} \circ \pi) g^{-\frac{\omega}{n+1}}$ gives an identification of $\tilde{h}$ with a homogeneous function of degree ω. □

We denote the set of smooth homogeneous functions $C \to \mathbb{R}$ of degree ω by $\mathcal{C}^\infty(C,\mathbb{R})_\omega$. In particular Lemma 6.1.3 shows that sections of the density bundle $\mathcal{E}(1)$ correspond to smooth homogeneous functions of degree 1. We now want to show, that the cotractor bundle of S^n is a trivial bundle. This can be done by computing the bundle of $1-$jets of $\mathcal{E}(1)$. Before we do that, we need some basic facts about homogeneous functions.

Lemma 6.1.4. Let $f : C \to \mathbb{R}$ be a homogeneous function of degree ω. Then

1. $Df : C \to \mathbb{R}^{n+1*}$ is homogeneous of degree $\omega - 1$, hence $Df(\lambda x) = \lambda^{\omega-1} Df(x)$ for all $\lambda > 0$.
2. $Df(x)x = \omega f(x)$.

Proof. 1. $D(f(\lambda x)) = Df(\lambda x)\lambda$, on the other hand $D(f(\lambda x)) = \lambda^\omega Df(x)$ and thus the claim.

2. Differentiating the equation $f(\lambda x) = \lambda^\omega f(x)$ with respect to λ, we obtain $Df(\lambda x)x = \omega\lambda^{\omega-1}f(x)$. Using the first part of this Lemma, we get $\lambda^{\omega-1}Df(x)x = \omega\lambda^{\omega-1}f(x)$, which gives the result.

□

Theorem 6.1.2. The cotractor bundle $\mathcal{T}^*$ on S^n can be identified with the trivial bundle $S^n \times \mathbb{R}^{n+1*}$.

Proof. We want to show that $J^1(\mathcal{E}(1) \to S^n)$ is isomorphic to $S^n \times \mathbb{R}^{n+1*}$. A section σ of $\mathcal{E}(1)$ can be identified with a homogeneous functions f of degree one and its derivative Df is homogeneous of degree 0. This means that $Df(\lambda x) = Df(x)$ and thus $Df : C \to \mathbb{R}^{n+1*}$ factors to a well-defined function on S^n. By Lemma 6.1.4 the value of f at a point x can be recovered from its derivative. We get a well-defined map $J^1_x(\mathcal{E}(1) \to S^n) \to \{x\} \times \mathbb{R}^{n+1*}$ on the fibers by $j^1_x\sigma \mapsto Df(x)$. This map is injective and since both fibers have the same dimension, we obtain an isomorphism of vector bundles.

□

Of course Theorem 6.1.2 also identifies sections of the cotractor bundle with $\mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*})$. Via this identification the jet projection corresponds to the map

$$\tilde{p} : \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*}) \to \mathcal{C}^\infty(C, \mathbb{R})_1$$

$$F \mapsto (v \mapsto F(\mathbb{R}_+ v)(v)).$$

This map is homogeneous of degree 1, since F is invariant under multiplication with a scalar in the first component and linear in the second component. Thus we have the following diagram for the jet projection:

$$\begin{tikzcd} \Gamma(\mathcal{T}^*) \arrow[r, leftrightarrow] \arrow[d, "p"'] & \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*}) \arrow[d, "\tilde{p}"] \\ \Gamma(\mathcal{E}(1)) \arrow[r, leftrightarrow] & \mathcal{C}^\infty(C, \mathbb{R})_1 \end{tikzcd} \tag{6.4}$$

Note that until now we have been working with S^n as ray projectivizations in C. Thus we considered S^n as a set of equivalence classes, which has no preferred embedding in $\mathbb{R}^{n+1}$. The choice of a section $\sigma : S^n \to C$ defines such an embedding.

We prove that a section $\sigma : S^n \to C$ splits the short exact sequence (6.1), for all $v \in C$ in the image of σ. The map $T_x\sigma : T_xS^n \to T_{\sigma(x)}C$ gives the following diagramm

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}\sigma(x) \arrow[r] & T_{\sigma(x)}C \arrow[r, shift left, "T_{\sigma(x)}\pi"] & T_xS^n \arrow[l, shift left, "T_x\sigma"] \arrow[r] & 0\,. \end{tikzcd}$$

Since $\mathrm{id} = \pi \circ \sigma$, we have $\mathrm{id} = T_x(\pi \circ \sigma) = T_{\sigma(x)}\pi \cdot T_x\sigma$ for $x \in S^n$. This shows, that $T_x\sigma$ is right inverse to $T_{\sigma(x)}\pi$ and thus induces a splitting

$$T_{\sigma(x)}C = \mathbb{R}\sigma(x) \oplus T_xS^n, \tag{6.5}$$

for all $x \in S^n$.

Furthermore, the choice of a section σ gives rise to an affine connection on S^n. Consider a vector field $\xi \in \mathfrak{X}(S^n)$. This defines a vector field along $\sigma(S^n) \subseteq C$ by setting $\tilde{\xi}(\sigma(x)) := T_x\sigma \cdot \xi(x)$. Then for $\xi, \eta \in \mathfrak{X}(S^n)$, we define:

$$(\nabla^{S^n}_\xi \eta)(x) := T_{\sigma(x)}\pi((\nabla_{\hat{\xi}}\hat{\eta})(\sigma(x))), \tag{6.6}$$

where $\hat{\xi}$ respectively $\hat{\eta}$ are extensions of $\tilde{\xi}$ respectively $\tilde{\eta}$ to C, i.e. $\hat{\xi}, \hat{\eta} \in \mathfrak{X}(C)$ satisfying $\hat{\xi}(\sigma(x)) = \tilde{\xi}(\sigma(x))$ respectively $\hat{\eta}(\sigma(x)) = \tilde{\eta}(\sigma(x))$ for all $x \in S^n$. We briefly explain why the restriction of $\nabla_{\hat{\xi}}\hat{\eta}$ to $\sigma(S^n)$ is independet of the choice of extensions: Certainly, the value of $\nabla_{\hat{\xi}}\hat{\eta}$ along $\sigma(S^n)$ only depends on the value of $\hat{\xi}$ on $\sigma(S^n)$ and since $\hat{\xi}$ is tangential to $\sigma(S^n)$, it is also independet of the choice of $\hat{\eta}$. Therefore, $\nabla_{\hat{\xi}}\hat{\eta}$ is a well-defined vector field along $\sigma(S^n)$ and projecting to TS^n gives a vector field on S^n. Furthermore, it can be verified that (6.6) satisfies the defining properties of a linear connection.

Until now we have seen, that the choice of a section $\sigma : S^n \to C$ gives rise to an affine connection on S^n. The section σ also defines a section of $\mathcal{E}(1)$, which corresponds to the function $f : C \to \mathbb{R}$ characterized by $v = f(v)\sigma(\pi(v))$, i.e. on $\sigma(S^n)$ it is identically 1 and on C it is homogeneous of degree 1. We will also denote this function by σ. One can verify directly that the connection on $\mathcal{E}(1)$ induced by the affine connection on S^n from (6.6) has the property that $\nabla\sigma = 0$. (Of course this completely determines the connection on $\mathcal{E}(1)$ and the affine one within its projective class).

Furthermore, one can verify that the splitting of $\mathcal{T}^*$ determined by the affine connection coming from (6.6) is induced by the splitting (6.5).

Recall that in section 3.2, we decided on working with the projective structure induced by the Levi-Civita connection on S^n. Therefore, it is natural to consider the splitting σ, which embeds S^n as the unit sphere in C. In this case, the tangent space T_xS^n is identified with $\{x\}^\perp$ and the splitting (6.5) is orthogonal. Thus, $T\pi$ coincides with the orthogonal projection and the linear connection (6.6) induced by σ is exactly the Levi-Civita connection. We now induce a connection on S^n from the flat connection on $\mathbb{R}^n$, which is projectively equivalent to the Levi-Civita connection. This will be useful later, as it is easier to calculate with the flat connection on $\mathbb{R}^n$.

Consider coordinates $x_0, x_1, \dots, x_n$ for $\mathbb{R}^{n+1}$ and the affine hyperplane A^n, given by $x_0 = 1$. Via central projection, we identify the northern hemnisphere of S^n with A^n. Hence we define the function

$$f : \{x \in S^n : x_0 > 0\} \to A^n,$$

$$(x_0, \dots, x_n) \mapsto (1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

with inverse

$$f^{-1} : A^n \to \{x \in S^n : x_0 > 0\},$$
$$y \mapsto \tfrac{y}{||y||}.$$

The flat connection on A^n gives a connection on the northern hemnisphere, by pulling back along f. Furthermore, this connection is projectively equivalent to the Levi-Civita connection on S^n. Geodesics of the Levi-Civita connection on S^n are given by great circles, hence intersections of S^n with $2-$dimensional linear subspaces in $\mathbb{R}^{n+1}$. Via f, such an intersection is mapped to the intersection of A^n with the same $2-$dimensional linear subspace, hence a straight line. This is a geodesic for the flat connection on A^n. Thus we have seen that the Levi-Civita connection and the pullback of the flat connection on S^n have the same geodesics and thus by Definition 3.1.1, they are projectively equivalent.

We will now prove that the cotractor connection induces the trivial connection via the identification $\mathcal{T}^* \cong S^n \times \mathbb{R}^{n+1*}$.

Theorem 6.1.3. The cotractor connection on $\mathcal{T}^*$ gives the trivial connection on $S^n \times \mathbb{R}^{n+1*}$.

Proof. In order to compute the connection induced on $S^n \times \mathbb{R}^{n+1*}$ via the isomorphism of Theorem 6.1.2, we consider the following diagram:

$$\begin{tikzcd}
\Gamma(\mathcal{T}^*) \arrow[r, leftrightarrow] \arrow[d, "\nabla^{\mathcal{T}^*}"'] & \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*}) \arrow[d, dotted] \\
\Gamma(\mathcal{T}^*) \arrow[r, leftrightarrow] & \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*})
\end{tikzcd}$$

We want to show that the dashed arrow is given by $F \mapsto DF$, for $F \in \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*})$. As already remarked above, we will work with the flat connection on S^n induced via the central projection. Recall from Definition 4.2.2 that the cotractor connection (in abstract indices) is given by

$$\nabla_a^{\mathcal{T}^*}(\mu_b, \sigma) = (\nabla_a \mu_b + \mathrm{P}_{ab}\sigma, \nabla_a \sigma - \mu_a),$$

where $\mathrm{P}_{ab} = 0$, since we are working with a flat connection. Starting with an element $F \in \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*})$ we have a representation in coordinates $F = \sum_{i=0}^n F_i dx^i$. Recall from diagram (6.4) that σ is given by $\tilde{p}(F)$, where $\tilde{p}$ is the projection

$$\tilde{p} : \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*}) \to \mathcal{C}^\infty(S^n, \mathbb{R})$$

$$F \mapsto (x \mapsto F(x)(x)).$$

Restricting to A^n in coordinates gives:

$$\sigma(x) = F_0(x) + \sum_{i=1}^{n} x_i F_i(x). \tag{6.7}$$

Therefore $\nabla\sigma = DF_0 + \sum_{i=1}^{n} x_i DF_i + F_i dx^i$. Similary by restricting to A^n, we obtain

$$\mu(x) = \sum_{i=1}^{n} F_i(x) dx^i \tag{6.8}$$

and $\nabla\mu = \sum_{i=1}^{n} DF_i(x)dx^i$. Now this is the first slot of the cotractor connection and the second slot is given by $DF_0 + \sum_{i=1}^{n} x_i DF_i$. We see that this pair corresponds to DF via the identification of Theorem 6.1.2, since replacing F_i by DF_i in (6.7) and (6.8) exactly gives the first and second slot of the cotractor connection. □

6.2 Solutions of first BGG-Equations

We continue by interpreting solutions of the first BGG-equation $D(\sigma) = 0$, for σ a smooth section of $\mathcal{E}(1)$. Recall from Lemma 5.3.1, that every solution is a normal solution. Hence

$$D(\sigma) = 0 \Leftrightarrow \nabla L(\sigma) = 0.$$

This means, that a solution of the first BGG-equation corresponds to a parallel cotractor. If $L(\sigma) \in \Gamma(\mathcal{T}^*)$ corresponds to $F \in \mathcal{C}^\infty(S^n, \mathbb{R}^{n+1*})$ via Theorem 6.1.2, then $\nabla L(\sigma) = 0$ is equivalent to $DF = 0$ by Theorem 6.1.3. This, in turn, implies that F is constant, hence $F(x) = \lambda$ for all $x \in S^n$ and some $\lambda \in \mathbb{R}^{n+1*}$. Thus, we have seen that (normal) solutions correspond to elements in the standard fiber of the cotractor bundle.

Furthermore, we can consider the zero set $\mathcal{Z}$ of a normal solution $\sigma \in \Gamma(\mathcal{E}(1))$ of the first BGG-equation. A normal solution σ gives rise to a parallel section $s \in \Gamma(\mathcal{T}^*)$ satisfying $\pi_0(s) = \sigma$, where π_0 is the projection as considered in section 5.3. The parallel cotractor s corresponds to an element $\lambda \in \mathbb{R}^{n+1*}$. Considering diagram (6.4), the homogeneous function of degree 1, which corresponds to the section σ, is λ, now interpreted as a homogeneous function on the space of rays in $\mathbb{R}^{n+1}$. The zero set of this homogeneous function is exactly the projectivization of the kernel of λ (viewed as linear functional). The kernel of λ is an $n-$dimensional subspace in $\mathbb{R}^{n+1}$ (as long as $\lambda \neq 0$). Under the projectivization $\pi : C \to \mathbb{R}^{n+1}$, the kernel of λ is mapped to an $(n-1)-$dimensional sphere, which is the equator of S^n. This especially shows that $\mathcal{Z}$ is a hypersurface of S^n. In fact, it is a totally geodesic hypersurface, that is, for any vector v which is tangential to $\mathcal{Z}$, the great circle γ in S^n with $\gamma'(0) = v$, stays in $\mathcal{Z}$. This is certainly a projectively invariant property of a hypersurface. Furthermore, it is a very nice feature, as knowing the tangent space in a point of a totally geodesic hypersurface, already means that we know the hypersurface locally around this point.

Similar results on the other tractor bundles follow immediately. The tractor bundle is defined as the dual bundle of the cotractor bundle and carries the dual connection, hence $\mathcal{T} = S^n \times \mathbb{R}^{n+1}$ and the tractor connection is the trivial connection. By Lemma 5.3.2, every solution of the first BGG-equation is normal and therefore corresponds to an element of $\mathbb{R}^{n+1}$.

In order to compute the zero set $\mathcal{Z}$ of a normal solution $\nu^a \in \Gamma(\mathcal{E}^a(-1))$, we have to compute the projection $\pi_0 : \mathcal{T} \to \mathcal{E}^a(-1)$ first. Via the identification $\mathcal{T} = S^n \times \mathbb{R}^{n+1}$, we see that the bundle $\mathcal{E}(-1) \hookrightarrow \mathcal{T}$ is given by the tautological line bundle. By the short exact sequence (4.13), the weighted tangent bundle $\mathcal{E}^a(-1)$ is given by the quotient of $\mathcal{T}$ by $\mathcal{E}(-1)$. Therefore, given $v \in \mathbb{R}^{n+1}$ in the fiber over $x \in S^n$, the projection is given by $\pi_0(v) = v + \mathbb{R}x$. Consider a normal solution ν^a and its corresponding parallel tractor s, which, in turn, is equivalent to an element $v \in \mathbb{R}^{n+1}$. Via the projection π_0, the section ν^a corresponds to the function $g(x) = v + \mathbb{R}x$, for $x \in S^n$. This function is zero if and only if $x \in \mathbb{R}v$, hence the zero set of g is given by $\mathbb{R}v$. Under the projectivization this is mapped to a pair of antipodal points on the sphere S^n.

For the symmetric power of the cotractor bundle, we obtain $S^2\mathcal{T}^* = S^n \times S^2\mathbb{R}^{n+1*}$ and the tractor connection is trivial. Furthermore, we can see directly from the formulae for $D(\sigma)$ and $\nabla L(\sigma)$, that every solution is normal: Recall from (5.23) and (5.24) that these are given by

$$\nabla_a L(\sigma) = \begin{pmatrix} \frac{1}{2}\nabla_a \nabla_{(b}\nabla_{c)}\sigma + \nabla_a \mathrm{P}_{(bc)}\sigma + \mathrm{P}_{a(b}\nabla_{c)}\sigma & \frac{1}{2}\nabla_{[a}\nabla_{b]}\sigma + \mathrm{P}_{[ab]}\sigma & 0 \end{pmatrix}$$

and

$$D(\sigma) = \tfrac{1}{2}\nabla_{(a}\nabla_b\nabla_{c)}\sigma + \nabla_{(a}\mathrm{P}_{bc)}\sigma + \mathrm{P}_{(ab}\nabla_{c)}\sigma.$$

Since we are working with a flat connection, we obtain that $\mathrm{P}_{ab} = 0$ and $\beta_{ab} = 0$. Therefore, $\nabla_a\nabla_b$ is symmetric, which implies the vanishing of the second component of $\nabla L(\sigma)$. Also $\nabla_a\nabla_b\nabla_c$ is symmetric in all three indices, which shows that $D(\sigma) = 0$ implies $\nabla L(\sigma) = 0$. Therefore, we obtain that solutions of the first BGG-equation correspond to elements of $S^2\mathbb{R}^{n+1*}$, hence bilinear forms on $\mathbb{R}^{n+1}$.

We can again consider the zero set $\mathcal{Z}$ of a normal solution $\sigma \in \Gamma(\mathcal{E}(2))$ of the first BGG-equation. A normal solution σ gives rise to a parallel section $s \in \Gamma(S^2\mathcal{T}^*)$, which, in turn, corresponds to a bilinear form b on $\mathbb{R}^{n+1}$. A diagram similar to (6.4) shows that the homogeneous function of degree 2, to which σ corresponds, is given by the quadratic form $q(v) = b(v, v)$. The zero set $\mathcal{Z}$ of σ is then given by the projectivization of the zero set of q, which is a quadric. There are a lot of possibilies for the form of $\mathcal{Z}$, but if q is non-degenerate, it is an embedded hypersurface. We consider some special cases. The case of a positive definite bilinear form can be eliminated, as in this case, the corresponding quadratic form does not have a zero set. Considering a quadratic form with Lorentzian signature, its zero set defines a cone and the projectivization of this cone gives rise to the union of two $(n-1)$-dimensional spheres.

The symmetric power of the tractor bundle satisfies $S^2\mathcal{T} = S^n \times S^2\mathbb{R}^{n+1}$ and the tractor connection is again trivial. As before, we can see from the defining equations, that a solution of the first BGG-equation is normal. Recall from (5.30), that

$$D(\tau^{bc}) = \nabla_a \tau^{bc} - \tfrac{1}{n+1}(\nabla_d \tau^{dc})\delta_a{}^b - \tfrac{1}{n+1}(\nabla_d \tau^{db})\delta_a{}^c. \tag{6.9}$$

Furthermore, a calculation shows that

$$\nabla_a L(\tau^{bc}) = \begin{pmatrix} \nabla_a \tau^{bc} - \frac{1}{n+1}(\nabla_e \tau^{ce})\delta_a{}^b - \frac{1}{n+1}(\nabla_e \tau^{be})\delta_a{}^c \\ -\frac{1}{n+1}\nabla_a \nabla_c \tau^{bc} + \frac{1}{n(n+1)}(\nabla_e \nabla_f \tau^{ef})\delta_a{}^b \\ \frac{1}{n(n+1)}\nabla_a \nabla_b \nabla_c \tau^{bc} \end{pmatrix},$$

where we already omitted parts containing P_{ab}. Now if $D(\tau^{bc}) = 0$, then the first component of $\nabla_a L(\tau^{bc})$ immediately vanishes. We now show, how the vanishing of the first component implies the vanishes of the other components. We first want to show $\nabla_a \nabla_c \tau^{bc} = \frac{1}{n}(\nabla_e \nabla_f \tau^{ef})\delta_a{}^b$. Set $\rho^b = \frac{1}{n+1}\nabla_e \tau^{be}$, then the vanishing of the curvature $R_{ab}{}^c{}_d$ (and the resulting vanishing of β_{ab}) imply:

$$\begin{aligned} 0 &= \nabla_a \nabla_b \tau^{cd} - \nabla_b \nabla_a \tau^{cd} \\ &= (\nabla_a \rho^d)\delta_b{}^c + (\nabla_a \rho^c)\delta_b{}^d - (\nabla_b \rho^d)\delta_a{}^c - (\nabla_b \rho^c)\delta_a{}^d. \end{aligned} \tag{6.10}$$

Taking the trace over a and c, we obtain $\nabla_b \rho^d = \frac{1}{n}(\nabla_e \rho^e)\delta_b{}^d$ and thus

$$\nabla_a \nabla_e \tau^{be} = (n+1)\nabla_a \rho^b = \tfrac{n+1}{n}(\nabla_f \rho^f)\delta_a{}^b = \tfrac{1}{n}(\nabla_e \nabla_f \tau^{ef})\delta_a{}^b.$$

This implies the vanishing of the second component. We are left with showing that the vanishing of (6.9) implies $\nabla_a \nabla_b \nabla_c \tau^{bc} = 0$. Set $\rho = \frac{1}{n}\nabla_e \nabla_f \tau^{ef}$, then again we have:

$$\begin{aligned} 0 &= \nabla_a \nabla_b \nabla_e \tau^{ce} - \nabla_b \nabla_a \nabla_e \tau^{ce} \\ &= (\nabla_a \rho)\delta_b{}^c - (\nabla_b \rho)\delta_a{}^c. \end{aligned}$$

Taking the trace over b and c implies $0 = \nabla_a \rho = \frac{1}{n}\nabla_a \nabla_b \nabla_c \tau^{bc}$.

Therefore, we obtain that solutions correspond to elements of $S^2\mathbb{R}^{n+1}$, hence bilinear forms on $\mathbb{R}^{n+1*}$.

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Abstract (German)

Die vorliegende Arbeit beschäftigt sich mit projektiven Strukturen auf glatten Mannigfaltigkeiten, d.h. mit Äquivalenzklassen torsionsfreier affiner Konnexionen, die die selben unparametrisierten Geodäten definieren. In diesem Zusammenhang stellt sich zunächst die Frage, ob es in einer gegebenen Äquivalenzklasse Konnexionen gibt, die spezielle Eigenschaften haben. Von besonderem Interesse sind z.B. Konnexionen, die von Metriken oder Einstein Metriken induziert werden. Des Weiteren interessiert man sich für Invarianten der geometrischen Struktur, insbesondere für invariante Differentialoperatoren, d.h. Differentialoperatoren, die für die projektive Struktur intrinsisch sind.

Wir behandeln diese Fragestellungen mit Hilfe des Traktorkalküls, einem invarianten Kalkül für projektive Geometrien. Einer projektiven Struktur können natürliche Vektorbündel zugeordnet werden, die eine invariante Konnexion besitzen. Diese Bündel heißen Traktorbündel und die zugehörigen Konnexionen Traktorkonnexion. Da es auf einer projektiven Mannigfaltigkeit keine ausgezeichnete Konnexion auf dem Tangentialbündel gibt, ist es natürlich mit diesen Bündeln zu arbeiten.

Des Weiteren verwenden wir die Theorie der BGG-Sequenzen. Jede BGG-Sequenz definiert eine Folge von invarianten Differentialoperatoren, wobei insbesondere die ersten Operatoren dieser Folgen von Interesse sind. Bezeichnet D einen ersten BGG-Operator, dann heißt die Gleichung $D\sigma = 0$ erste BGG-Gleichung. Werden die BGG-Sequenzen auf passende Traktorbündel angewendet, so ist die Existenz von bestimmten Lösungen der ersten BGG-Gleichung äquivalent zur Existenz von Konnenxionen in der projektiven Klasse, die spezielle Eigenschaften haben. Insbesondere werden wir daraus Bedingungen für die Existenz von (pseudo-) Riemann'schen Metriken bzw. Riemann'schen Einstein Metriken und Ricci-flachen Konnenxionen in der projektiven Klasse erhalten.

Als Anwendung werden Traktorbündel, ihre Konnexionen und Lösungen der ersten BGG-Gleichung auf dem homogenen Modell für orientierte projektive Strukturen diskutiert.

Curriculum Vitae

Personal Details

Name	Caroline Moosmüller
Date of birth	08.02.1989 in Vienna
Nationality	Austria
E-Mail	caroline.moosmueller@univie.ac.at

Education

Since 03/2011	Master's studies in Mathematics at the University of Vienna
03/2011-06/2011	Erasmus Exchange at the University of Leeds (GB)
12/2010	BSc in Mathematics, with distinction
2007-2010	Bachelor's studies in Mathematics at the University of Vienna
1999-2007	Bundesgymnasium und Bundesrealgynmasium Rahlgasse (Grammar school), graduated with distinction
1995-1999	Volksschule Schulkollektiv-Wien WUK (Elementary School)

Scholarships

2013	Merit scholarship from the University of Vienna
2011	Merit scholarship from the University of Vienna

Teaching

03/2010-02/2013	Tutor at the Faculty of Mathematics, University of Vienna, for the problem session "Introduction to Computer Infrastructure"
10/2011-07/2012	Tutor at the Institute of Discrete Mathematics and Geometry, Vienna University of Technology, for the problem sessions "Mathematics for Civil Engineers" (part 1 and 2)