# DISSERTATION

Titel der Dissertation

"On the special linear group over orders in quaternion division algebras - arithmetic and geometry"

Verfasser

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# Introduction

### Arithmetic subgroups of $SL_2(k)$, $k$ a number field

Let $k$ be an algebraic number field with ring of integers $\mathcal{O}_k$ and let $G$ be a linear algebraic group over $k$. So roughly speaking there exists a finite set of polynomial equations with coefficients in $k$, whose set of solutions in $k$ is a subgroup $G(k)$ of $GL_n(k)$. A subgroup $\Gamma$ of $G(k)$ is called arithmetic, if it is commensurable with $G(k) \cap GL_n(\mathcal{O}_k) =: G(\mathcal{O}_k)$, i.e. if $\Gamma \cap G(\mathcal{O}_k)$ has finite index in both groups. This notion includes classical groups such as $SL_2(\mathbb{Z})$ in the special linear group $SL_2(\mathbb{Q})$, or more generally, $SL_n(\mathcal{O}_k)$ in $SL_n(k)$. Further examples include congruence subgroups, which are subgroups of $G(\mathcal{O}_k)$ containing groups of the form $\Gamma(\mathfrak{q}) = \{\gamma \in G(\mathcal{O}_k) : \gamma \equiv 1 \mod \mathfrak{q}\}$ for some ideal $\mathfrak{q}$ in $\mathcal{O}_k$.

An arithmetic subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of the algebraic $\mathbb{Q}$-group $SL_2/\mathbb{Q}$ operates via right multiplication on $SL_2(\mathbb{R})$, and consequently on the symmetric space $X = SO_2(\mathbb{R})\backslash SL_2(\mathbb{R})$, the two dimensional hyperbolic space. It is well known that $X \cong \mathcal{H}_2$, the Poincaré upper half plane. Note that $SO_2(\mathbb{R})$ is a maximal compact subgroup in $SL_2(\mathbb{R})$. The space $X/\Gamma \cong \mathcal{H}_2/\Gamma$ is a Riemannian surface, which is noncompact but has finite volume. In fact $\mathcal{H}_2/SL_2(\mathbb{Z})$ can be identified with a moduli space of certain elliptic curves (see [BJ06] and [Leh64, Ch. 1A, 1B]).

Let $k$ be a number field and let $r_1$, respectively $r_2$, be the number of real, respectively complex, places of $k$. Then $G = SL_2/k$ is a linear algebraic group defined over $k$. The Weil restriction of scalars $\mathrm{Res}_{k/\mathbb{Q}}G$ of $G$ is an algebraic $\mathbb{Q}$-group, whose set of real points $\mathrm{Res}_{k/\mathbb{Q}}G(\mathbb{R}) =: G_\infty$ is a Lie group isomorphic to $SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2}$. An arithmetic subgroup $\Gamma$ of $SL_2(k)$ acts on $G_\infty$ and thus on the associated symmetric space $X = K\backslash G_\infty = \mathcal{H}_2^{r_1} \times \mathcal{H}_3^{r_2}$ where $K$ is a maximal compact subgroup of $G_\infty$ and $\mathcal{H}_3 = (SU_2\backslash SL_2(\mathbb{C}))$ is the three dimensional hyperbolic space. The latter action is properly discontinuous.

The arithmetic subgroups $\Gamma \subset SL_2(\mathcal{O}_k)$ and the corresponding quotient spaces $X/\Gamma$ are extensively studied. If $k$ is totally real, $SL_2(\mathcal{O}_k)$ is called the Hilbert modular group and the corresponding quotients $X/\Gamma$ are Hilbert modular varieties. If $\Gamma$ is torsion-free the manifolds $X/\Gamma$ can be viewed as the complex points of quasi-projective algebraic varieties, allowing one to use algebraic geometry in the investigation (compare [vdG88, Sec. II.7]).

**Compactification of $X/\Gamma$.** It is a classical result that $\mathcal{H}_2/SL_2(\mathbb{Z})$, respectively $\mathcal{H}_2/\Gamma$ for an arithmetic subgroup $\Gamma \subset SL_2(\mathbb{Z})$, can be compactified by adding finitely many points, the cusps. This might have been known as early as 1880 by Poincaré, though he did not mention it explicitly (see [BJ06] and [Leh64, Ch. 1]). More specifically, instead of adjoining the full boundary $\mathbb{R} \cup \{\infty\}$ of $\mathcal{H}_2$ one adds only the rational boundary points $\mathbb{Q} \cup \{\infty\}$. The $\Gamma$-action extends to a continuous action on $\mathcal{H}_2 \cup \mathbb{Q} \cup \{\infty\} =: \mathcal{H}_2^*$ and the quotient $\mathcal{H}_2^*/\Gamma = \mathcal{H}_2/\Gamma \cup \{\text{cusps}\}$ is compact.

This construction can be extended to arithmetic subgroups $\Gamma \subset SL_2(\mathcal{O}_k)$. If $k$ is totally real of degree $d$, then the points added to $X = \mathcal{H}_2^d$ are precisely $k \cup \{\infty\}$, where $k$ is mapped into the boundary $\mathbb{R}^d \cup \{\infty\}$ using the different embeddings $k \hookrightarrow \mathbb{R}$. The group $\Gamma$ acts on this extended space $(\mathcal{H}_2^d)^*$ and the quotient space $(\mathcal{H}_2^d)^*/\Gamma = \mathcal{H}_2^d/\Gamma \cup \{\text{cusps}\}$ is compact. It was first shown by Siegel in [Sie61, Prop. 20] that the number of cusps which are added to compactify $X/SL_2(\mathcal{O}_k)$ equals $h_k$, the class number of the number field $k$.

Since $X$ is contractible, $X/\Gamma$ is an Eilenberg-MacLane space $K(\Gamma, 1)$, i.e. the fundamental group of $X/\Gamma$ equals $\Gamma$ while all higher homotopy groups are trivial. Thus the cohomology of $\Gamma$ is isomorphic to the cohomology of $X/\Gamma$.

If the space $X/\Gamma$ is compact, it is homotopic to a finite CW-complex, which simplifies the study of the cohomology groups and implies certain finiteness properties of $\Gamma$, e.g. that $\Gamma$ is finitely presented. If $X/\Gamma$ is not compact, which is the case we are dealing with, a possible option to obtain a $K(\Gamma, 1)$ space, which is a finite CW-complex, is to compactify $X/\Gamma$ such that the inclusion $X/\Gamma \hookrightarrow \overline{X/\Gamma}$ is a homotopy equivalence. However the inclusion $X/\Gamma$ into the compact space $(\mathcal{H}_2^d)^*/\Gamma$ is not a homotopy equivalence and the cohomology groups of the two respective spaces are not equal. In some sense, this classical compactification is too small to serve this purpose (see [BJ06, Sec. .1.6]).

This problem was resolved in full generality when Borel and Serre ([BS73]) introduced a different compactification of $X/\Gamma$, the Borel-Serre compactification. This compactification is obtained as follows: Let $\Gamma$ be an arithmetic subgroup of a linear algebraic reductive $k$-group $G$. It acts on a homogeneous space $X$, which is defined by the Lie group $\operatorname{Res}_{k/\mathbb{Q}}(G)(\mathbb{R}) = G_\infty$. If the $k$-rank $\operatorname{rk}_k(G)$ is at least 1 one shows that $X/\Gamma$ is not compact. The Borel-Serre compactification $\overline{X}/\Gamma$ of $X/\Gamma$ is obtained by first adjoining infinitely many faces $e(P)$ to the space $X$, one for each proper parabolic $k$-subgroup $P$ of $G$. The action of $\Gamma$ on $X$ can be extended to $\overline{X}$, and $\Gamma$ permutes the $e(P)$. The image of $e(P)$ in $\overline{X}/\Gamma$ is denoted by $e'(P)$ and one shows that the $e'(P)$ correspond to the $\Gamma$-conjugacy classes of proper parabolic subgroups of $G$. We call the number of $e'(P)$ in $\overline{X}/\Gamma$ the cusp number $c(\Gamma)$ of $\Gamma$.

**The boundary - the case $SL_2(\mathcal{O}_k)$.** For $G = SL_2/\mathbb{Q}$ and $X = \mathcal{H}_2$, the space $\overline{X}$ is the union of $X$ and of countably many lines, one for each rational point on the boundary of $\mathcal{H}_2$. The quotient $\overline{X}/\Gamma$ is a compact surface, whose boundary consists of finitely many circles, one for each $\Gamma$-conjugacy class of minimal parabolic subgroups of $G$. The points, which are classically added, are here blown up to lines, and thus one obtains a space on which $\Gamma$ still acts properly (compare [BS73]).

The long exact sequence in cohomology attached to the pair $(\overline{X}/\Gamma, \partial\overline{X}/\Gamma)$ allows one to relate the cohomology of the boundary $\partial\overline{X}/\Gamma$ to the cohomology of $\overline{X}/\Gamma$, and thus to the cohomology of $\Gamma$. Therefore, a study of the structure of the boundary components is necessary. For $G = SL_2/k$ and $\Gamma \subset SL_2(\mathcal{O}_k)$ torsion-free, Harder showed in [Har75] that a connected component of the boundary of $\overline{X}/\Gamma$ has the structure of a torus bundle $\pi : e'(P) \to T^{r_1+r_2-1}$, whose fiber is a torus $T^d$ of dimension $d = [k : \mathbb{Q}] = r_1 + 2r_2$. The connected components of the boundary of the Borel-Serre compactification correspond to the cusps classically added, hence for $\Gamma = SL_2(\mathcal{O}_k)$, the number of boundary components equals $h_k$, the class number of $k$.

## Arithmetic subgroups of $SL_2(D)$, $D$ a quaternion division algebra

The main objective of this thesis is to extend some of these classical results to quaternion division algebras $D$ defined over a number field $k$. More precisely, let $G = SL_2(D)$ be the linear algebraic $k$-group of $2\times 2$ matrices with entries in $D$ of reduced norm $\mathrm{nrd}_{M_2(D)/k}$ equal to 1. Any arithmetic subgroup $\Gamma$ in $SL_2(D)$ acts on the homogeneous space $X$ associated to the ambient Lie group $G_\infty = (\mathrm{Res}_{k/\mathbb{Q}} SL_2(D))(\mathbb{R})$. We study the Borel-Serre compactification of $X/\Gamma$ in this case. In particular we are interested in the number of boundary components arising. This is called the cusp number of $\Gamma$ and equals the number of $\Gamma$-conjugacy classes of proper parabolic $k$-subgroups of $G$. Our objective is to see how this is related to the arithmetic of $D$.

Let $\Lambda$ be a maximal order in $D$. Then $\Lambda$ gives rise to an arithmetic subgroup $SL_2(\Lambda)$ in $SL_2(D)$, which is a natural generalization of $SL_2(\mathcal{O}_k)$ in $SL_2(k)$. Let $LF_1(\Lambda)$ denote the set of isomorphism classes of left $\Lambda$-ideals. Then $LF_1(\Lambda)$ is finite and its cardinality, denoted by $h_D$, is independent of the choice of maximal order $\Lambda$ in $D$. If $D = k$, then $LF_1(\mathcal{O}_k)$ equals the ideal class group of $\mathcal{O}_k$.

At this point we split our investigation in two major parts (for more details on central simple algebras see [Rei03, Ch. 8]):

**(td)** Our main case of interest is that $D$ is totally definite, i.e. $D_v \cong \mathbb{H}$, the Hamiltonian quaternions, for all infinite places $v$ of $k$. This implies that $k$ is totally real, i.e. $k$ has no complex places. For orders $\Lambda$ in totally definite quaternion algebras we obtain a rather intriguing description of the cusp number of $SL_2(\Lambda)$ in terms of various invariants attached to all maximal orders of $D$.

**(ntd)** If $D$ is not totally definite, $D_v \ncong \mathbb{H}$ for at least one finite place $v$ of $k$. The description of the cusp number of $SL_2(\Lambda)$ is rather short in this case, so we will start with it.

**The case (ntd):** Suppose $D$ is not totally definite. Then $LF_1(\Lambda)$ is a group, which is isomorphic to the ray class group $\mathrm{Cl}(\mathcal{O}_k)^D$. The latter is the quotient of the set of fractional ideals $I(k)$ of $\mathcal{O}_k$ by the principal ideals generated by $k^D = \mathrm{nrd}(D^*)$. Using the strong approximation property of $SL_1(D) = \ker(\mathrm{nrd}_{D/k})$ one shows that the cusp number of $SL_2(\Lambda)$ equals the class number of $D$, i.e.

$$c(SL_2(\Lambda)) = h_D$$

for any maximal order $\Lambda$ of $D$ (see Lemma 5.1). So in particular the cusp number $c(SL_2(\Lambda))$ is independent of the choice of maximal order $\Lambda$ of $D$.

More generally this calculation holds for any finite dimensional central division algebra $D$ over $k$, which is not totally definite. The constant $h_D$ can in fact be calculated using solely the class number $h_k$ of $k$, the unit group $\mathcal{O}_k^*$ and the behavior of $D$ at the infinite places $v$ of $k$.

These results can be generalized: An order $\Lambda$ in $D$ defines an arithmetic subgroup $SL_n(\Lambda)$ of $SL_n(D)$. In this thesis I have calculated the cusp number of $SL_n(\Lambda)$ in $SL_n(D)$, for $D$ not totally definite. More precisely (see Theorem 6.1):

**Theorem.** *Let $D$ be a central division algebra over $k$, such that $D$ is not totally definite. Let $\Lambda$ be a maximal order in $D$. Then the cusp number of $SL_n(\Lambda)$ equals*

$$c(SL_n(\Lambda)) = (1 + h_D)^{n-1} - 1.$$

In particular this holds for $D = k$.

The topological structure of a boundary component $e'(P)$ is another important aspect in the strudy of the boundary of the Bore-Serre compactification of $X/\Gamma$. Suppose that $k$ is totally real. Let $s$ denote the number of infinite places $v$ of $k$, where $D$ is split, i.e. $D_v \cong M_2(\mathbb{R})$, and let $r$ denote the number of ramified places of $k$, i.e. $D_v \cong \mathbb{H}$. So $s + r = d = [k : \mathbb{Q}]$.

Let $\Gamma_T \subset \Gamma \subset SL_2(\Lambda)$ be the group of diagonal matrices in $\Gamma$. Then we show (see Theorem 7.2):

**Theorem.** *Let $D$ be a quaternion division algebra over $k$ and let $\Lambda$ be an order in $D$. Let $\Gamma$ be a torsion-free subgroup of $SL_2(\Lambda)$ and let $\Lambda^1_\Gamma$ be the subgroup of reduced norm one elements which appear on the diagonal of $\Gamma_T$. Then the quotient map $P \to P/R_uP$ induces a fibration $\pi : e'(P) \to Y_T$ on a boundary component $e'(P)$. This is a fiber bundle whose base space is diffeomorphic to $T^{d-1} \times (\mathcal{H}_2^s/\Lambda^1_\Gamma)^2$ and the fiber is the torus $T^{4d}$.*

Here as usual, $\mathcal{H}_2$ denotes the upper half plane. It is well known that $\mathcal{H}_2^s/\Lambda^1$ is compact, hence so is $\mathcal{H}_2^s/\Lambda^1_\Gamma$. This theorem is also valid for $D$ totally definite. Then $s = 0$ and the base space is simply the torus $T^{d-1}$.

**The case (td):** Now suppose that $D$ is a totally definite quaternion algebra over $k$. Let $\Lambda$ be a maximal order in $D$. We see that the set of isomorphism classes of left $\Lambda$-ideals $LF_1(\Lambda)$ is not a group in general.

However we can remedy the situation by introducing a weaker concept, namely the notion of stable isomorphism: two left $\Lambda$-ideals $M$, $N$ are called stable isomorphic if $M \oplus \Lambda \cong N \oplus \Lambda$ as left $\Lambda$-modules. It is clear that isomorphic ideals are stable isomorphic, but the converse may not hold. The stable isomorphism class of a left $\Lambda$-ideal $M$ is denoted by $(M)$. The set of stable isomorphism classes $\mathrm{Cl}(\Lambda)$ is an abelian group, its unity element is given by $(\Lambda)$. One shows that $\mathrm{Cl}(\Lambda) \cong \mathrm{Cl}(\mathcal{O}_k)^+$ as groups, where $\mathrm{Cl}(\mathcal{O}_k)^+$ denotes the narrow class group of $\mathcal{O}_k$. More precisely $\mathrm{Cl}(\mathcal{O}_k)^+$ is the quotient of $I(k)$, the fractional ideals of $\mathcal{O}_k$, by the principal ideals generated by the totally positive elements $k^+$ in $k^*$.

We have a natural projection $LF_1(\Lambda) \to \mathrm{Cl}(\Lambda)$, mapping an ideal $M$ to its stable isomorphism class $(M)$. A left $\Lambda$-ideal $M$ is called stable locally free if it is in the kernel of this map, i.e. if $M \oplus \Lambda \cong \Lambda \oplus \Lambda$ as $\Lambda$-modules. The number of isomorphism classes of stable locally free left $\Lambda$-ideals is denoted by $s_\Lambda$ and depends on $\Lambda$.

We set $\mathcal{O}_k^+ = \mathcal{O}_k \cap k^+$. Then $\mathrm{nrd}(\Lambda^*)$ is a finite index subgroup of $(\mathcal{O}_k^+)^*$ and we set $o_\Lambda = [(\mathcal{O}_k^+)^* : \mathrm{nrd}(\Lambda^*)]$. For any two maximal orders $\Lambda_1, \Lambda_2$ we define

$o_{\Lambda_1,\Lambda_2} = [(\mathcal{O}_k^+)^* : (\mathrm{nrd}(\Lambda_1^*)\,\mathrm{nrd}(\Lambda_2^*))]$. We further define

$$r_\Lambda = \sum_\Gamma o_\Gamma$$

where the sum runs over all right orders of isomorphism classes of stable locally free left $\Lambda$-ideals, so there are $s_\Lambda$ many terms. To make this a little more transparent note that the invariant $r_\Lambda$ measures two things:

- how much $\Lambda$ fails to have the cancellation property, meaning that stable isomorphic $\Lambda$-ideals are in fact isomorphic,
- how $\mathrm{nrd} : \Delta^* \to (\mathcal{O}_k^+)^*$ fails to be surjective for the maximal orders $\Delta$ of $D$, which are right orders of the stable locally free left $\Lambda$-ideals.

Given a second maximal order $\Delta$, we define $r_{\Lambda,\Delta} = \sum\limits_\Gamma o_{\Gamma,\Delta}$. Let $H_\Lambda$ be cardinality of the set of isomorphism classes of two-sided $\Lambda$-ideals in $D$. Then we show (see Theorem 5.6):

**Theorem.** *Let $\Lambda$ be a maximal order in a totally definite quaternion algebra $D$ over a necessarily totally real number field $k$. Then the cusp number of $SL_2(\Lambda)$ equals*

$$\sum_\Gamma H_\Gamma \cdot r_{\Gamma,\Gamma},$$

*where the sum runs over all isomorphism classes of maximal orders $\Gamma$ of $D$. In particular the cusp number $c(SL_2(\Lambda))$ is independent of the choice of maximal order in $D$.*

More specifically, let $k$ be a number field such that $h_k^+ = h_k = 1$, for instance $k = \mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})\ldots$ and let $\Lambda$ be a maximal order in a totally definite quaternion algebra over $k$. It follows easily (see Corollary 5.2) that

$$c(SL_2(\Lambda)) = h_D^2$$

for all maximal orders $\Lambda$ of $D$. This has also been shown for $k = \mathbb{Q}$ in [KO90], using a completely different method.

In general it is clear that $c(SL_2(\Lambda)) \geq h_D$ holds. Above we have already seen examples where $c(SL_2(\Lambda)) = h_D$ (take $D$ not totally definite) or $c(SL_2(\Lambda)) = h_D^2$. It is interesting to see that other values can be taken as well. In this thesis we give various explicit examples where $h_D^2 > c(SL_2(\Lambda)) > h_D$, the most intriguing is the following:

**Example.** Let $k = \mathbb{Q}(\sqrt{219})$, then $\mathrm{Cl}(\mathcal{O}_k) \cong \mathbb{Z}/4\mathbb{Z}$, and the fundamental unit $e = 74 + 5\sqrt{219}$ of $\mathcal{O}_k^*$ is totally positive, hence $h_k^+ = 8$. Let $D$ be the quaternion algebra $Q(-1, -1 \,|\, \mathbb{Q}(\sqrt{219}))$. It is a totally definite quaternion algebra, having 84 isomorphism classes of maximal orders. Each maximal order $\Lambda$ has, up to isomorphism, 296 left $\Lambda$-ideals, hence $h_D = 296$. Depending on the order $\Lambda$ we obtain $s_\Lambda = 35$, 37 or 39 as actual values. The field $k$ is quadratic, hence $o_\Lambda = 1$ or 2, depending on whether $\mathrm{nrd} : \Lambda^* \to (\mathcal{O}_k^+)^*$ is surjective or not. The constants $r_{\Lambda,\Delta}$ range between 35 and 70 for maximal orders $\Lambda$, $\Delta$ in $D$. Altogether one obtains:

$$c(SL_2(\Lambda)) = 17\,712$$

for any maximal order $\Lambda$ in $D$.

## Description of the chapters

This thesis is organized as follows:

In the first chapter we state the basic properties of central simple algebras $D$ over an algebraic number field. We introduce the group of ideles $D_{\mathbb{A}}^*$ over $D$ and give an adelic description of the isomorphism classes of left-ideals of an order $\Lambda$ in $D$. Further we describe the class group and the unit group of an order.

In chapter 2 we give a short introduction to linear algebraic groups defined over number fields $k$. In particular we define the linear algebraic $k$-group $SL_n(D)$ and state its most important properties.

In chapter 3 we give an overview of the Borel-Serre compactification of a homogeneous space $X$ modulo an arithmetic subgroup $\Gamma$ of a reductive linear algebraic $k$-group $G$. In particular one will see that the number of faces $e'(P)$ adjoined to $X/\Gamma$ equals the number of $\Gamma$-conjugacy classes of proper parabolic $k$-subgroups of $G$, the cusp number of $\Gamma$.

In chapter 4 we show that the number of $SL_2(\Lambda)$ conjugacy classes of a proper parabolic subgroups $P$ of $SL_2(D)$ equals the class number of the Levi-component $T$ of $P$. This gives an adelic description of the cusp number of $SL_2(\Lambda)$, and more generally of $SL_n(\Lambda)$.

Chapter 5 is the main part of this thesis. We calculate the cusp number of $SL_2(\Lambda)$ and show how it corresponds to various invariants of $D$ and of all maximal orders of $D$. In particular we show that $c(SL_2(\Lambda))$ is independent of the choice of maximal order $\Lambda$ in $D$. Further we give a couple of explicit examples, among them the calculation of the cusp number of $SL_2(\Lambda)$ for a maximal order $\Lambda$ in the quaternion algebra $Q(-1,-1\,|\,\mathbb{Q}(\sqrt{219}))$.

In chapter 6 we describe the parabolic subgroups of $SL_n(D)$ and calculate the cusp number of $SL_n(\Lambda)$ in $SL_n(D)$, for $D$ not totally definite. We further give an adelic description of the cusp number of $GL_n(\Lambda)$ in $GL_n(D)$. We again calculate $c(GL_n(\Lambda))$ explicitly, for a maximal order $\Lambda$ in a not totally definite division algebra $D$. We also give $c(GL_2(\Lambda))$, for a maximal order $\Lambda$ in a totally definite quaternion algebra $D$.

In chapter 7 we describe the boundary components corresponding to parabolic subgroups in $SL_2(k)$, and later in $SL_2(D)$, for a quaternion division algebra $D$ over a totally real number field $k$. We also describe the cohomology $H^q(e'(P),\mathbb{C})$ of a boundary component.

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# Chapter 1

# Central simple algebras

In this chapter we give a short introduction on central simple algebras $D$ over algebraic number fields $k$. We will define orders and the ring of adeles $D_{\mathbb{A}}$ of an algebra. Then we give an adelic description of the isomorphism classes of ideals of an order $\Lambda$. More precisely these classes correspond to the finite set of double-cosets

$$U_\Lambda \backslash D_{\mathbb{A}}^* / D^*$$

where $U_\Lambda$ is a certain subgroup of idele group $D_{\mathbb{A}}^*$ depending on $\Lambda$. With this description at hand we introduce the class group $\mathrm{Cl}(\Lambda)$ of an order $\Lambda$, which is a generalization of the ideal class group $\mathrm{Cl}(\mathcal{O}_k)$ of the ring of integers $\mathcal{O}_k$ in $k$. Finally we describe the unit group of a maximal order in $D$, paying particular attention to the units of orders in totally definite quaternion algebras. More details can be found in [Rei03] and [Frf75].

## 1.1 Basic definitions

Let $k$ be an algebraic number field and let $D$ be a finite dimensional algebra over $k$. We assume that $D$ is simple, i.e. $D$ contains no non-trivial two-sided ideal, and that $D$ is central over $k$. So the center of $D$ equals the number field $k$. A division algebra is an algebra having the property that every nonzero element has a multiplicative inverse. We will mostly be working with division algebras $D$. Every division algebra over $\mathbb{Q}$ is automatically simple and has an algebraic number field as its center. Every algebra we consider will be finite dimensional, and is defined over a number field or over the completion of a number field.

For a central simple algebra $D$ over a field $k$ the reduced norm, respectively reduced trace, of $D$ over $k$ is denoted by $\mathrm{nrd}_{D/k}$, respectively $\mathrm{tr}_{D/k}$. If there is no risk of confusion we will omit the subscript $D/k$ and simply write nrd or tr. More precisely, let $L$ be a splitting field of $D$, i.e. let $L$ be a field containing $k$ such that

$$D \otimes L \cong M_n(L)$$

for some $n \in \mathbb{N}$. Let $\phi$ be an isomorphism $\phi : D \otimes L \tilde{\to} M_n(L)$ and let $i : D \hookrightarrow D \otimes L$ be the natural embedding. Then

$$\mathrm{nrd}(d) := \det(\phi(i(d))).$$

One shows that $\mathrm{nrd} : D \to k$ is independent of the choice of splitting field $L$ and isomorphism $\phi$. Analogously tr is defined by

$$\mathrm{tr}(d) = \mathrm{Tr}(\phi(i(d)))$$

and one proves similarly, that it does not depend on $L$ or $\phi$.

**Example 1. Quaternion Algebras:** A quaternion algebra $Q$ over a field $k$ is a central simple algebra over $k$ of dimension $[Q : k] = 4$ having a basis, denoted by $1$, $i$, $j$, $ij$ such that the multiplication rules are given by

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji$$

for some nonzero $a, b \in k$. We will denote this (uniquely determined) algebra by $Q(a, b \,|\, k)$. We define the conjugation $x \mapsto \overline{x}$ by the following: if $x = x_0 + x_1 i + x_2 j + x_3 ij$ for $x_0, \ldots, x_3 \in k$ then $\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij$. The reduced norm and trace are given by:

$$\begin{aligned} \mathrm{nrd}(x_0 + x_1 i + x_2 j + x_3 ij) &= x \cdot \overline{x} = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2 \\ \mathrm{tr}(x_0 + x_1 i + x_2 j + x_3 ij) &= x + \overline{x} = 2x_0 \end{aligned}$$

for $x_0, \ldots, x_3 \in k$. Exactly two cases can occur:

- $Q(a, b \,|\, k)$ is a division algebra over $k$.
- $Q(a, b \,|\, k)$ is isomorphic to the matrix algebra $M_2(k)$.

Whether $Q(a, b \,|\, k)$ is a division algebra or not depends on the parameters $a, b$. More precisely one shows that $Q(a, b \,|\, k)$ is not a division algebra if and only if there exists a nontrivial element $x \in Q(a, b \,|\, k)$ with $\mathrm{nrd}(x) = 0$.

One of the most important examples we consider is the Hamiltonian quaternion algebra $\mathbb{H} = Q(-1,-1\,|\,\mathbb{R})$. It is a division algebra, in fact the only finite dimensional non-commutative division algebra over $\mathbb{R}$.

Let us now fix some notation. For a number field $k$ let $\mathcal{O}_k$ denote the ring of integers in $k$. Let $V_k$, or simply $V$, be the set of places of $k$, and let $V_\infty$ (resp. $V_f$) be the set of infinite (resp. finite) places of $k$. The infinite places of $k$ are determined by the distinct embeddings of $k \hookrightarrow \mathbb{C}$. We will not distinguish between a place $v \in V_\infty$ and the corresponding embedding $k \hookrightarrow \mathbb{C}$. If an infinite place corresponds to a real embedding $k \hookrightarrow \mathbb{R}$ it is called real, otherwise we call it complex. For any $v \in V$ let $k_v$ be the completion of $k$ with respect to the valuation $|\,|_v$ corresponding to $v$. Every valuation corresponding to a place is assumed to be normalized.

For a finite place $v \in V_f$ let $\mathcal{O}_v$ be the completion of $\mathcal{O}_k$ with respect to $v$. So $\mathcal{O}_v = \{x \in k_v \,:\, |x|_v \leq 1\}$. Then $\mathcal{O}_v$ is a local ring, its unique maximal ideal is given by $\mathfrak{p}_v = \{x \in \mathcal{O}_v \,:\, |x|_v < 1\}$ and its unit group is $\mathcal{O}_v^* = \{x \in \mathcal{O}_v \,:\, |x|_v = 1\}$.

Let $\mathbb{A}_k$ be the ring of adeles over $k$. It is the restricted direct product of the (additive) groups $k_v$ with respect to $\mathcal{O}_v$. Then $\mathbb{A}_k$ is a locally compact ring. We will also give a different description using direct limits: for a finite set of places $S$ containing all infinite places of $k$ define

$$\mathbb{A}_k(S) := \prod_{v\in S} k_v \times \prod_{v \notin S} \mathcal{O}_v.$$

We define multiplication and addition component wise. With the product topology the ring $\mathbb{A}_k(S)$ is locally compact. Let $S'$ be another finite set containing $S$. Then $\mathbb{A}_k(S) \subset \mathbb{A}_k(S')$ and the topology of $\mathbb{A}_k(S)$ coincides with the topology induced by $\mathbb{A}_k(S')$. The ring of adeles $\mathbb{A}_k$ can then be defined to be the direct limit of the rings $\mathbb{A}_k(S)$, for $S$ as above. We define the topology on $\mathbb{A}_k$ by prescribing, that for each set $S$ the subring $\mathbb{A}_k(S)$ is open. We get a fundamental system of neighborhoods of 0 by taking a system of neighborhoods in any of the sets $\mathbb{A}_k(S)$. This definition induces the same topology as the restricted direct product.

We write $\mathbb{A}_k^*$ for the group of ideles, with its usual topology. So $\mathbb{A}_k^*$ is the restricted direct product of the groups $k_v^*$ with respect to $\mathcal{O}_v^*$. Note that this is not the topology induced by $\mathbb{A}_k$, for $\mathbb{A}_k^*$ is not a topological group with respect to the topology induced by $\mathbb{A}_k$. We will view $k$ as being diagonally embedded into $\mathbb{A}_k$ and similarly $k^* \hookrightarrow \mathbb{A}_k^*$. We will not distinguish between $k$, resp. $k^*$, and its image under the embedding.

Let $D$ be a central simple algebra over $k$. For any place $v$ of $k$ we define the completion of $D$ with respect to $v$ as

$$D_v := D \otimes_k k_v.$$

If $D$ is central simple over $k$, then $D_v$ is central and simple over $k_v$, and hence there exists a division algebra $S_v$, central over $k_v$, such that

$$D_v = M_{\kappa_v}(S_v)$$

for some integer $\kappa_v$. The algebra $D$ is called split at the place $v$ if $S_v = k_v$, or in other words if $D_v \cong M_{\kappa_v}(k_v)$. Otherwise $D$ is called ramified. One can show:

**Theorem 1.1** (Thm. 25.7 in [Rei03])**.** *Let $D$ be a central simple algebra over $k$. Then $D$ is split for almost all places of $k$.*

Now let $v$ be an infinite place of $k$. If $v$ is complex, we see that $D$ has to be split at the place $v$. If $v$ is real, then $D_v$ is of the form $M_{\kappa_v}(\mathbb{R})$ or $M_{\kappa_v}(\mathbb{H})$. This leads to the following definition.

**Definition 1.1.** *A central simple $k$-algebra $D$ is called totally definite quaternion algebra if every infinite place $v$ of $k$ is ramified in $D$ and if furthermore $D_v \cong \mathbb{H}$ for each such $v$.*

If $D$ is a totally definite quaternion algebra over $k$, then $k$ has to be totally real, i.e. $k$ has no complex places. Further the dimension of $D$ has to be equal to the dimension of $\mathbb{H}$ over $\mathbb{R}$, hence $[D : k] = 4$. So $D$ is totally definite if and only if

1. the dimension $[D : k]$ equals 4,

2. the field $k$ is totally real and

3. for every infinite place $v$ of $k$ we have $D_v \cong \mathbb{H}$.

We say that an algebra $D$ satisfies the Eichler condition relative $\mathcal{O}_k$, if $D$ is not totally definite.

Totally definite quaternion algebras will be rather important in this thesis. Usually they play the role of exceptions. A lot of theorems and properties, like cancellation property, strong approximation or properties of the class group, hold for not totally definite quaternion algebras only.

## 1.2 Orders

Let $D$ be a finite dimensional central simple algebra over a number field $k$.

**Definition 1.2.** *An $\mathcal{O}_k$-order in the $k$-algebra $D$ is a subring $\Lambda$ of $D$, having the same unity element as $D$, which is also a full $\mathcal{O}_k$-lattice in $D$, i.e. it is a finitely generated $\mathcal{O}_k$-submodule of $D$ such that $k.\Lambda = D$.*

If there is no risk of confusion we will write order instead of $\mathcal{O}_k$-order. A maximal order is an order not contained in any other order. Maximal orders exist. Using the Theorem of Skolem and Noether one sees that two orders are isomorphic if and only if they are conjugate by an element of $D^*$. The unique maximal order of the number field $k$ is the ring of integers $\mathcal{O}_k$. So orders, in particular maximal orders, are a natural generalization of $\mathcal{O}_k$. In this thesis we will mainly be interested in maximal orders, though most of the theorems in this section hold for arbitrary orders.

**Example 2.** Let $D = Q(-1,-1\,|\,\mathbb{Q})$. This is a central simple algebra over the rational numbers $\mathbb{Q}$. Define $\Lambda_1 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$. Then it is easily seen that $\Lambda_1$ is an order in $D$. It is not maximal though. It is contained in the maximal order $\Lambda_2 = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1+i+j+ij}{2}$. Any maximal order of $D$ is conjugate to $\Lambda_2$.

For an order $\Lambda$ in $D$ we define the completion of $\Lambda$ at a finite place $v$ of $k$ by

$$\Lambda_v = \Lambda \otimes_{\mathcal{O}_k} \mathcal{O}_v.$$

Note that $\Lambda_v$ is an order in $D_v$. Moreover one shows that $\Lambda$ is maximal if and only if $\Lambda_v$ is maximal for all finite places $v$. Let $\Lambda, \Gamma$ be two maximal orders. Then $\Lambda_v = \Gamma_v$ almost everywhere, i.e. for all but finitely many places $v$ of $k$. This allows us to define the ring of adeles of an algebra $D$ as the restricted direct product of the (additive) groups $D_v$ with respect to the orders $\Lambda_v$. More precisely

$$D_{\mathbb{A}} = \{(x_v)_v \in \prod_{v \in V} D_v \,:\, x_v \in \Lambda_v \text{ almost everywhere}\}.$$

This definition is independent of the choice of maximal order. The additive group $D_{\mathbb{A}}$ is a finite dimensional vector space over $\mathbb{A}_k$ and hence locally compact.

We can define the group of ideles of $D$ as the group of units of $D_{\mathbb{A}}$ with its usual topology. This is the topology for which the map $x \mapsto (x, x^{-1})$ is a homeomorphism of $D_{\mathbb{A}}^*$ onto its image in $D_{\mathbb{A}} \times D_{\mathbb{A}}$. Equivalently we could define it as the restricted direct product of the groups $D_v^*$ with respect to the unit groups

$\Lambda_v^*$. More precisely:

$$D_{\mathbb{A}}^* = \{(x_v)_v \in \prod_{v\in V} D_v^* \,:\, x_v \in \Lambda_v^* \text{ almost everywhere}\}.$$

Again we may embed $D$, resp. $\Lambda$, diagonally into $D_{\mathbb{A}}$ and similarly $D^*$, $\Lambda^*$ into $D_{\mathbb{A}}^*$. We will, as usual, not distinguish between $D$, resp. $D^*$, and their images under this embedding in $D_{\mathbb{A}}$, resp. $D_{\mathbb{A}}^*$.

### 1.2.1 Algebras over local fields

We will now look at the completions of $D$ in some more detail. Let $v$ be a finite place of $k$ and let $S_v$ be a central division algebra over $k_v$. One proves ([Rei03, Thm. 12.8]) that $S_v$ contains a unique maximal order, denoted by $\Delta_v$, which is given by

$$\Delta_v = \{x \in S_v \,:\, \mathrm{nrd}(x) \in \mathcal{O}_v\}.$$

Here nrd denotes the reduced norm of $S_v$ over $k_v$. Suppose $D_v = M_m(S_v)$ for some $m \in \mathbb{N}$. We set

$$\Lambda_v = M_m(\Delta_v).$$

Then $\Lambda_v$ is a maximal $\mathcal{O}_v$-order in $D_v$ and any other maximal order is conjugate to $\Lambda_v$ ([Rei03, Thm. 17.3]).

Any two orders are equal almost everywhere, so the definition of $D_{\mathbb{A}}$ does not depend on the choice of order $\Lambda$ in $D$. For a fixed order $\Lambda$ we define a subring of $D_{\mathbb{A}}$ via

$$\begin{aligned} D_\Lambda &= \{a = (a_v)_v \in D_{\mathbb{A}} \,:\, a_v \in \Lambda_v \text{ for all } v \in V_f\} \\ &= \prod_{v\in V_\infty} D_v \times \prod_{v\in V_f} \Lambda_v. \end{aligned}$$

We will see later (Theorem 1.3) that $D_\Lambda \cap D = \Lambda$. The unit group of $D_\Lambda$ is given by

$$\begin{aligned} U_\Lambda &= \{u = (u_v)_v \in D_{\mathbb{A}}^* \,:\, u_v \in \Lambda_v^* \text{ for all } v \in V_f\} \\ &= \prod_{v\in V_\infty} D_v^* \times \prod_{v\in V_f} \Lambda_v^*. \end{aligned}$$

It follows that $U_\Lambda \cap D^* = \Lambda^*$ and we may view $\Lambda^*$ as being embedded diagonally into $U_\Lambda \subset D^*_{\mathbb{A}}$.

### 1.2.2 The kernel of the reduced norm map

For every place $v$ of $k$, the algebra $D_v$ is central and simple over $k_v$, hence it admits a reduced norm map. These norms $\mathrm{nrd}_{D_v/k_v}$ induce a map, also denoted by nrd, from $D_{\mathbb{A}} \to \mathbb{A}_k$ via

$$\mathrm{nrd}((d_v)_v) := (\mathrm{nrd}_{D_v/k_v}(d_v))_v.$$

It is clear that nrd is well defined. Indeed let $\Lambda$ be any maximal order in $D$. Then $d_v \in \Lambda_v$ almost everywhere, and hence $\mathrm{nrd}_{D_v/k_v}(d_v) \in \mathrm{nrd}(\Lambda_v) = \mathcal{O}_v$ almost everywhere. Let

$$N^1(D_{\mathbb{A}}) = \{\alpha \in D_{\mathbb{A}} \,:\, \mathrm{nrd}(\alpha) = 1\}$$

denote the kernel of $\mathrm{nrd} : D_{\mathbb{A}} \to \mathbb{A}_k$. For any subset $R \subset D_{\mathbb{A}}$ we define $N^1(R) = N^1(D_{\mathbb{A}}) \cap R$. We see at once that $N^1(D_{\mathbb{A}}) = N^1(D^*_{\mathbb{A}})$ and $N^1(D_\Lambda) = N^1(U_\Lambda)$.

We say that $N^1(D^*_{\mathbb{A}})$ has the strong approximation property, if the following conditions are satisfied:

- Given a finite set $S$ of places of $k$, including $V_\infty$, and for each $v \in S$ an open subgroup $W_v \in N^1(D^*_v)$, then for each $\alpha \in N^1(D^*_{\mathbb{A}})$ there exists $a \in N^1(D)$ so that
  - $a^{-1}\alpha_v \in W_v$ for all $v \in S$,
  - $a \in N^1(\Lambda^*_v)$ for all $v \notin S$.

Note that the notion of strong approximation does not depend on the choice of order $\Lambda$. We can always replace $\Lambda$ by any order spanning $D$. Eichler's approximation theorem states, that if $D$ is not a totally definite quaternion algebra, then $N^1(D^*_{\mathbb{A}})$ has strong approximation.

For a fixed maximal order $\Lambda$ define $n_\Lambda$ to be the cardinality of the double quotient

$$N^1(U_\Lambda)\backslash N^1(D^*_{\mathbb{A}})/N^1(D).$$

We will see later that $n_\Lambda$ is necessarily finite. Now using strong approximation one shows:

**Theorem 1.2** (Cor. to Lemma 1 in chapter 3 in [Frf75])**.** *If $D$ is not a totally definite quaternion algebra, $n_\Lambda = 1$ for any maximal order $\Lambda$ of $D$.*

## 1.3 Ideals of orders

Let $D$ be a central simple algebra over an algebraic number field $k$ and let $\Lambda$ be an order in $D$. A left $\Lambda$-ideal $M$ is a left $\Lambda$-lattice in $D$ such that $kM = D$. As $\Lambda$ is finitely generated over $\mathcal{O}_k$, a left $\Lambda$-ideal is the same as a full $\mathcal{O}_k$-lattice $M$ in $D$, such that $\Lambda M \subset M$.

We define right $\Lambda$-ideals analogously. In the calculations following, we will work with left $\Lambda$-ideals only, however everything could be done using right $\Lambda$-ideals.

Let $M$ be a full $\mathcal{O}_k$-lattice in $D$, i.e. $M$ is finitely generated as an $\mathcal{O}_k$-module and $kM = D$. We define the left order of $M$ as

$$O_l(M) = \{d \in D \,:\, dM \subset M\}.$$

Then $O_l(M)$ is an order in $D$ and $M$ is a left $O_l(M)$-ideal. Likewise we define the right order $O_r(M) = \{d \in D \,:\, Md \subset M\}$. One can show ([Rei03, Thm. 21.2]) that $O_r(M)$ is maximal if and only if $O_l(M)$ is maximal.

Let $v$ be a finite place of $k$ and let $M$ be an $\mathcal{O}_k$-lattice in $D$. We define the completion of $M$ at a place $v$ as

$$M_v := M \otimes_{\mathcal{O}_k} \mathcal{O}_v.$$

One can show that the $\mathcal{O}_v$-lattices $M_v$, for $v \in V_f$, determine $M$ uniquely. More precisely:

**Theorem 1.3** (Thm 2, V in [Wei73])**.** *Let $M$ be an $\mathcal{O}_k$-lattice. For every finite place $v$ of $k$ let $N_v$ be an $\mathcal{O}_v$-lattice in $D_v$. Then there is a $\mathcal{O}_k$-lattice $N$ in $D$ whose completion equals $N_v$ for all finite $v$ if and only if $M_v = N_v$ for almost all $v$. When that is so there is only one such lattice and it is given by*

$$N = \bigcap_{v \in V_f} (D \cap N_v).$$

Let $M$ be a left $\Lambda$-ideal in $D$. One proves that for all finite places $v$ of $k$, the left $\Lambda_v$-ideal $M_v$ is principal, i.e. there exists an element $\alpha_v \in D_v$ such that $M_v = \Lambda_v \alpha_v$. Note that $\Lambda$ itself is also a left $\Lambda$-ideal, so by the theorem above it follows that $M_v = \Lambda_v$ for almost all places $v$. Consequently for each left $\Lambda$-ideal there exists an idele $\alpha \in D^*_{\mathbb{A}}$ such that $M_v = \Lambda_v \alpha_v$ for all finite places $v$. Contrary given $\alpha \in D^*_{\mathbb{A}}$ by the theorem above there exists a uniquely determined $\Lambda$-ideal $N$ in $D$ such that $N_v = \Lambda_v \alpha_v$ for all finite places $v$. We will write $\Lambda\alpha$ for this ideal.

Note that different ideles $\alpha, \beta$ may induce the same ideal $\Lambda\alpha = \Lambda\beta$. In particular if $\beta = u\alpha$ for some $u \in U_\Lambda$ then $\Lambda\alpha = \Lambda\beta$.

**Definition 1.3.** *A left $\Lambda$-module $M$ (not necessarily contained in $D$) is called locally free of rank $m$ if $M_v \cong (\Lambda_v)^m$ for one (all) finite places $v$ of $k$.*

The left $\Lambda$-ideals are precisely the locally free rank 1 modules. The set of isomorphism classes of locally free left $\Lambda$-modules of rank $n$ will be denoted by $LF_n(\Lambda)$. One can show ([Rei03, Thm. 26.4] or [Vig80, Thm. 5.4]) that for any maximal order $\Lambda$ the set $LF_1(\Lambda)$ is finite. We will prove later, that its cardinality is independent of the choice of maximal order in $D$.

The next theorem helps us to understand the isomorphism classes of left $\Lambda$-ideals using the adelic description given above:

**Theorem 1.4** (Thm. 1 in [Frf75])**.** *The locally free left $\Lambda$-modules of positive rank $m$ are precisely those isomorphic to a direct sum*

$$\Lambda\alpha_1 \oplus \cdots \oplus \Lambda\alpha_m, \qquad \alpha_i \in D^*_{\mathbb{A}}.$$

*Also*

$$\Lambda\alpha_1 \oplus \ldots \oplus \Lambda\alpha_m \cong \Lambda\beta_1 \oplus \ldots \oplus \Lambda\beta_n$$

*are isomorphic as left $\Lambda$-modules, if and only if firstly $n = m$, and secondly*

*(i) for $m = 1$, we have $\alpha_1 \in U_\Lambda\beta_1 D^*$*

*(ii) for $m > 1$, we have in the quotient group $U_\Lambda N^1(D^*_{\mathbb{A}})D^*\backslash D^*_{\mathbb{A}}$ the equation*

$$U_\Lambda N^1(D^*_{\mathbb{A}})D^*(\alpha_1 \cdots \alpha_m) = U_\Lambda N^1(D^*_{\mathbb{A}})D^*(\beta_1 \cdots \beta_m).$$

Using this theorem we see that a set of representatives $\{x_i\}_i$ of double-cosets $U_\Lambda\backslash D^*_{\mathbb{A}}/D^*$ corresponds to a set of representatives $\{\Lambda x_i\}_i$ of isomorphism classes of left $\Lambda$-ideals and vice versa.

An order $\Lambda$ is maximal, if and only if $\Lambda_v$ is maximal for all finite places $v$ of $k$. By the description of the local maximal orders and by Theorem 1.3, we see that for any two maximal orders $\Lambda$, $\Gamma$ there exists $\alpha \in D^*_{\mathbb{A}}$ such that

$$\Gamma = \alpha\Lambda\alpha^{-1}$$

where $\alpha\Lambda\alpha^{-1}$ is the unique maximal order such that

$$(\alpha\Lambda\alpha^{-1})_v = \alpha_v\Lambda_v\alpha_v^{-1}$$

for all finite places $v$ of $k$. Equivalently we have

$$\alpha\Lambda\alpha^{-1} = \bigcap_{v\in V_f} \alpha_v\Lambda_v\alpha_v^{-1} \cap D.$$

This leads us to the following definition:

**Definition 1.4.** *Let $\Lambda$ be an order in a central simple algebra $D$. An order $\Gamma$ of $D$ is in the genus of $\Lambda$ if there exists $\alpha \in D_{\mathbb{A}}^*$ such that $\Gamma = \alpha\Lambda\alpha^{-1}$.*

Evidently, any order in the genus of a maximal order is maximal and any two maximal orders are in the same genus. We can show:

**Theorem 1.5.** *For any two orders $\Lambda$, $\Gamma$ of $D$ in the same genus the number of left $\Lambda$-ideals equals the number of left $\Gamma$-ideals.*

*Proof.* Let $\alpha \in D_{\mathbb{A}}^*$ be such that $\Gamma = \alpha\Lambda\alpha^{-1}$. Let $D_{\mathbb{A}}^* = \bigcup_i U_\Lambda x_i D^*$ be a decomposition of $D_{\mathbb{A}}^*$ corresponding to the left $\Lambda$-ideals. The map

$$U_\Lambda\backslash D_{\mathbb{A}}^*/D^* \to U_\Gamma\backslash D_{\mathbb{A}}^*/D^*$$
$$U_\Lambda x_i D^* \mapsto U_\Gamma \alpha x_i D^*$$

is well defined. Indeed let $ux_id$ be another representative in the class $U_\Lambda x_i D^*$, with $u \in U_\Lambda$ and $d \in D^*$. Then we see that

$$U_\Gamma \alpha u x_i d D^* = \alpha U_\Lambda \alpha^{-1} \alpha u \alpha^{-1} \alpha x_i d D^* = U_\Gamma \alpha x_i D^*.$$

The inverse map is given by "multiplication" with $\alpha^{-1}$, which is well defined for the exact same reasons. This proves that the cardinality of the double quotient $U_\Lambda\backslash D_{\mathbb{A}}^*/D^*$ equals the cardinality of the double quotient $U_\Gamma\backslash D_{\mathbb{A}}^*/D^*$. □

Let $h_D$ denote the number of isomorphism classes of left $\Lambda$-ideals, where $\Lambda$ is any maximal order of $D$. The notation is justified by the previous theorem. It is usually called the class number of $D$.

**Example 3.** Consider the case $D = k$. Then the unique maximal order is $\mathcal{O}_k$ and $LF_1(\mathcal{O}_k)$ corresponds to

$$U_{\mathcal{O}_k}\backslash \mathbb{A}_k^*/k^*.$$

Let $I(k)$ denote the group of fractional ideals in $k$. Then we define a map $\psi$ : $U_{\mathcal{O}_k}\backslash \mathbb{A}_k^* \to I(k)$ via

$$(x_v) \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{a_\mathfrak{p}}$$

where, if $v$ corresponds to the prime ideal $\mathfrak{p}$, the integer $a_\mathfrak{p}$ is such that $|x_v|_v = [\mathcal{O}_k : \mathfrak{p}]^{a_\mathfrak{p}}$. Note that in this map we ignore the infinite places completely, as they coincide for $U_{\mathcal{O}_k}$ and $\mathbb{A}_k^*$ anyway. Under this map $k^*$ is identified with the set of principal ideals $P(k)$, which gives us a bijection

$$U_{\mathcal{O}_k} \backslash \mathbb{A}_k^* / k^* \leftrightarrow I(k)/P(k).$$

So $h_D$ equals the class number $h_k$ of $k$.

One sees that the number of isomorphism classes of maximal orders is finite. Indeed, for any two maximal orders $\Lambda$, $\Gamma$ in $D$, the set $\Lambda\Gamma$ is a left $\Lambda$-ideal and if $\Gamma$ and $\Gamma'$ are non-isomorphic maximal orders of $D$, $\Lambda\Gamma$ and $\Lambda\Gamma'$ are non-isomorphic ideals of $\Lambda$. The number of isomorphism classes of maximal orders in $D$ is called the type number of $D$ and will be denoted by $t_D$. We will also say $\Lambda$ and $\Gamma$ are of the same type when $\Lambda$ and $\Gamma$ are isomorphic.

The types of maximal orders of $D$ can also be described using a certain decomposition of $D_\mathbb{A}^*$. Let $N(D_\Lambda) = \{x = (x_v)_v \in D_\mathbb{A}^* : x_v D_\Lambda x_v^{-1} = D_\Lambda\}$ be the normalizer of $D_\Lambda$ in $D_\mathbb{A}^*$. This is not to be confused with $N^1(D_\Lambda)$, the subgroup of reduced norm one elements. Then the isomorphism classes of maximal orders of $D$ correspond to a decomposition

$$D_\mathbb{A}^* = \bigcup_{i=1}^{t_D} N(D_\Lambda) y_i D^*$$

with $y_i \in D_\mathbb{A}^*$. The corresponding representatives of the isomorphism classes of maximal orders are given by $y_i^{-1}\Lambda y_i$. Indeed suppose $x^{-1}\Lambda x \cong y^{-1}\Lambda y$ for two elements $x, y \in D_\mathbb{A}^*$. Then there exists $d \in D^*$ such that

$$d^{-1}x^{-1}\Lambda x d = y^{-1}\Lambda y$$

so by definition

$$d^{-1}x_v^{-1}\Lambda_v x_v d = y_v^{-1}\Lambda_v y_v$$

for all finite places $v$ of $k$. Hence $y \in N(D_\Lambda)xD^*$.

We will often write $N(\Lambda)$ instead of $N(D_\Lambda)$, as the normalizer contains exactly the elements $x \in D_\mathbb{A}^*$ such that $x\Lambda x^{-1} = \Lambda$. We will describe the normalizer of a fixed maximal order in some more detail. We are mainly interested in the case $[D : k] = 4$, so we will restrict our attention to that.

- If $v$ is an infinite place of $k$, then $(D_\Lambda)_v = D_v$, hence $N(\Lambda)_v = D_v^*$.

- If $v$ is finite and $D_v$ is a division algebra, then the unique maximal order of $D_v$ is
$$\Delta_v = \{x \in D_v \,:\, \mathrm{nrd}(x) \in \mathcal{O}_v\}.$$
It is evident, as the norm map is multiplicative, that the normalizer of $\Delta_v$ is $D_v^*$.

- If $v$ is finite and $D_v \cong M_2(k_v)$, then $\Lambda_v$ is conjugate to $M_2(\mathcal{O}_v)$. The normalizer of $M_2(\mathcal{O}_v)$ is $k_v^* M_2(\mathcal{O}_v)^*$. So $N(\Lambda)_v$ is conjugate to $k_v^* M_2(\mathcal{O}_v)^*$.

For a maximal order $\Lambda$ let $H_\Lambda$ be the number of two-sided $\Lambda$-ideals. Then $H_\Lambda$ is given as the cardinality of the double quotient

$$U_\Lambda \backslash N(\Lambda)/(D^* \cap N(\Lambda)).$$

**Theorem 1.6.** *One has*
$$h_D = \sum_{\Lambda} H_\Lambda$$
*where $\Lambda$ runs over all types of maximal orders.*

*Proof.* In this proof we are following [Vig80, Lemme 5.6]. Let $D_{\mathbb{A}}^* = \bigcup_{i=1}^{t_D} N(\Lambda) y_i D^*$ be a decomposition corresponding to the types of maximal orders. Let $\Lambda_i = y_i^{-1} \Lambda y_i$. Then $N(\Lambda_i) = y_i^{-1} N(\Lambda) y_i$ and $U_{\Lambda_i} = y_i^{-1} U_\Lambda y_i$. Hence $N(\Lambda) y_i D^* = y_i N(\Lambda_i) D^*$ and

$$|U_\Lambda \backslash N(\Lambda) y_i D^*/D^*| = |U_{\Lambda_i} \backslash N(\Lambda_i) D^*/D^*| = |U_{\Lambda_i} \backslash N(\Lambda_i)/(D^* \cap N(\Lambda_i))|$$

The first equality follows from the bijectivity of the map $U_\Lambda r y_i d D^* \mapsto U_{\Lambda_i} r d D^*$, which is proved similarly as Theorem 1.5. The second equality follows as $U_\Lambda \subset N(\Lambda)$ for any maximal order. But $U_{\Lambda_i} \backslash N(\Lambda_i)/(D^* \cap N(\Lambda_i)) = H_{\Lambda_i}$ which proves our claim. ☐

## 1.4 Class groups

Let $D$ be a central simple algebra over an algebraic number field $k$ and let $\Lambda$ be a maximal order in $D$. Let $LF_1(\Lambda)$ denote the set of isomorphism classes of left $\Lambda$-ideals. If $D = k$ is a field and $\Lambda = \mathcal{O}_k$ is its maximal order, then $LF_1(\Lambda)$ carries a group structure, it is the ideal class group $\mathrm{Cl}(\mathcal{O}_k)$. We would like to generalize this to orders in simple algebras.

Let $\alpha, \beta$ be representatives of double cosets in $U_\Lambda \backslash D_{\mathbb{A}}^*/D^*$ and let $\Lambda\alpha$, $\Lambda\beta$ be corresponding representatives of left $\Lambda$-ideals. Our first attempt could be to define a group structure on the isomorphism classes of left $\Lambda$-ideals via $\Lambda\alpha + \Lambda\beta = \Lambda\alpha\beta$. However for $u \in U_\Lambda$ we see that $\Lambda\beta = \Lambda u\beta$, but $\Lambda\alpha u\beta$ may not be isomorphic to $\Lambda\alpha\beta$. To avoid this problem we introduce stable isomorphism classes in $LF_1(\Lambda)$ and construct a group structure on these new classes.

**Definition 1.5.** *Two left $\Lambda$-ideals $\Lambda\alpha$ and $\Lambda\beta$ are called stable isomorphic if*

$$\Lambda\alpha \oplus \Lambda \cong \Lambda\beta \oplus \Lambda$$

*as $\Lambda$-ideals.*

It is clear that two isomorphic ideals are stable isomorphic, but the converse is false in general.

By Theorem 1.4 we see that $\Lambda\alpha$ and $\Lambda\beta$ are stable isomorphic if and only if $\alpha \in N^1(D^*_{\mathbb{A}})U_\Lambda D^*\beta$. We denote the set of stable isomorphism classes by $\mathrm{Cl}(\Lambda)$ and write $(\Lambda\alpha)$ for the set of all left $\Lambda$-ideals which are stable isomorphic to $\Lambda\alpha$.

**Theorem 1.7.** *For $(\Lambda\alpha)$, $(\Lambda\beta) \in \mathrm{Cl}(\Lambda)$ we define*

$$(\Lambda\alpha) + (\Lambda\beta) := (\Lambda\alpha\beta).$$

*With this addition $\mathrm{Cl}(\Lambda)$ is an abelian group, called the class group of $\Lambda$.*

*Proof.* A classical version (without the use of the adelic description) is given in [Rei03, Thm. 35.5]. An adelic version can be found in [Frf75, Section 2.II].

First we have to show that the addition defined above is well defined. Let $\alpha' = nud\alpha$ and $\beta' = n'u'd'\beta$ for $n$, $n' \in N^1(D^*_{\mathbb{A}})$, $u$, $u' \in U_\Lambda$ and $d$, $d' \in D^*$. We have to show that $\Lambda\alpha\beta$ is stable isomorphic to $\Lambda\alpha'\beta'$. We will show that $\alpha'\beta' \in N^1(D^*_{\mathbb{A}})U_\Lambda D^*\alpha\beta$ holds:

$$\begin{aligned}\alpha'\beta' &= nud\alpha n'u'd'\beta \\ &= \tilde{n}uu'dd'\alpha\beta\end{aligned}$$

where $\tilde{n} = nud\alpha n'u'd'\alpha^{-1}(d')^{-1}d^{-1}(u')^{-1}u^{-1} \in N^1(D^*_{\mathbb{A}})$. Hence $(\Lambda\alpha\beta) = (\Lambda\alpha'\beta')$. The commutativity follows from the fact that $\beta\alpha = (\beta\alpha\beta^{-1}\alpha^{-1})\alpha\beta$ and $(\beta\alpha\beta^{-1}\alpha^{-1}) \in N^1(D^*_{\mathbb{A}})$. The neutral element is given by $(\Lambda)$ and the inverse of $(\Lambda\alpha)$ is $(\Lambda\alpha^{-1})$. □

We have an obvious projection map from $LF_1(\Lambda)$ to $\mathrm{Cl}(\Lambda)$, mapping isomorphism classes of left $\Lambda$-ideals to stable isomorphism classes. Ideals in the kernel are called stable (locally) free. By definition a left $\Lambda$-ideal is stable free, if

$$\Lambda\alpha \oplus \Lambda \cong \Lambda \oplus \Lambda$$

and we see further by Theorem 1.4 that $\Lambda\alpha$ is stable free if and only if $\alpha \in U_\Lambda N^1(D^*_{\mathbb{A}})D^*$. So in terms of the double quotient $U_\Lambda \backslash D^*_{\mathbb{A}}/D^*$, the stable free ideals correspond to the double cosets which have a representative in $N^1(D^*_{\mathbb{A}})$. Hence the number of stable free left $\Lambda$-ideals is given by the cardinality of the double quotient

$$U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$$

and will be denoted by $s_\Lambda$.

It is obvious that $s_\Lambda$ and $|\operatorname{Cl}(\Lambda)|$ are finite, as they have to be less than or equal to $h_D$, which is finite. We have a canonical surjection

$$N^1(U_\Lambda)\backslash N^1(D^*_{\mathbb{A}})/N^1(D) \to U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$$

which shows that $s_\Lambda \leq n_\Lambda$. In particular if $D$ is not totally definite, $s_\Lambda = 1$ for any maximal order $\Lambda$. Note that $s_\Lambda = 1$ does not imply the injectivity of the projection map from $LF_1(\Lambda)$ to $\operatorname{Cl}(\Lambda)$.

**Definition 1.6.** *Let $\Lambda$ be a maximal order in $D$. Then $\Lambda$ satisfies the cancellation property, if the map $LF_1(\Lambda) \to \operatorname{Cl}(\Lambda)$ is injective. In other words $\Lambda$ satisfies the cancellation property, if for any two left $\Lambda$-ideals $M$ and $N$ the relation*

$$M \oplus \Lambda \cong N \oplus \Lambda \Rightarrow M \cong N$$

*holds. In this case isomorphism and stable isomorphism are equivalent.*

One can show

**Theorem 1.8** (Jacobinski, [Jac68])**.** *Let $D$ be a central simple algebra which is not totally definite. Then any order in $D$ has the cancellation property.*

*Proof.* If $D$ is not totally definite, then $D$ has the strong approximation property, so $N^1(D^*_{\mathbb{A}}) = N^1(U_\Lambda)N^1(D)$ for any maximal order $\Lambda$ in $D$. Let $\Lambda$ be a maximal order. We have to prove that for any $x \in D^*_{\mathbb{A}}$, the cardinality of

$$U_\Lambda \backslash U_\Lambda x N^1(D^*_{\mathbb{A}})D^*/D^*$$

equals one. Let $\Gamma = x^{-1}\Lambda x$. Define the map $\phi : U_\Lambda \backslash D^*_{\mathbb{A}}/D^* \to U_\Gamma \backslash D^*_{\mathbb{A}}/D^*$ via $U_\Lambda y D^* \mapsto U_\Gamma x^{-1} y D^*$. This is well defined and a bijection, as we have seen in the proof of Theorem 1.5. Then the classes $U_\Lambda \backslash U_\Lambda x N^1(D_{\mathbb{A}}^*)D^*/D^*$ are mapped to $U_\Gamma \backslash U_\Gamma N^1(D_{\mathbb{A}}^*)D^*/D^*$. As $\Gamma$ has the strong approximation property, $s_\Gamma = 1$ and the claim follows. □

If a maximal order in any central simple algebra $D$ has the cancellation property, then so does every other maximal order. Further if an order has the cancellation property, then so does any order containing it. One can show ([Vig76] and [HM06]), that there are only finitely many totally definite quaternion algebras $D$ containing maximal orders $\Lambda$ which have the cancellation property. In fact, if $D$ is a totally definite quaternion algebra over $k$ and if $\Lambda$ is a maximal order in $D$ and has the cancellation property, then $[k : \mathbb{Q}] \leq 6$. In [HM06] one finds a detailed list of all totally definite quaternion algebras containing orders which have the cancellation property.

We will now show that $\mathrm{Cl}(\Lambda)$ is independent of the choice of maximal order and write down the isomorphism explicitly.

**Theorem 1.9.** *Let $D$ be a central simple algebra and let $\Lambda, \Gamma$ be two maximal orders in $D$. Then their class groups $\mathrm{Cl}(\Lambda)$ and $\mathrm{Cl}(\Gamma)$ are isomorphic. Further if $\Gamma = \alpha\Lambda\alpha^{-1}$ for $\alpha \in D_{\mathbb{A}}^*$ the isomorphism is given by $(\Lambda x) \mapsto (\Gamma \alpha x \alpha^{-1}) = (\Gamma x)$.*

*Proof.* Suppose $\Lambda x'$ and $\Lambda x$ are stable isomorphic. We have to show that in that case $\Gamma \alpha x \alpha^{-1}$ and $\Gamma \alpha x' \alpha^{-1}$ are stable isomorphic. Let $u \in U_\Lambda$, $d \in D^*$ and $n \in N^1(D_{\mathbb{A}}^*)$ be such that

$$x' = nudx.$$

Then $\Gamma \alpha x' \alpha^{-1}$ and $\Gamma \alpha x \alpha^{-1}$ are in the same stable isomorphism class if and only if $\alpha x' \alpha^{-1} \in N^1(D_{\mathbb{A}}^*)\alpha U_\Lambda \alpha^{-1} D^* \alpha x \alpha^{-1}$. Now

$$\begin{aligned} \alpha x' \alpha^{-1} &= \alpha nudx\alpha^{-1} \\ &= \tilde{n}\alpha u \alpha^{-1} d \alpha x \alpha^{-1} \end{aligned}$$

where $\tilde{n} = \alpha nud\alpha^{-1}d^{-1}\alpha u^{-1}\alpha^{-1} \in N^1(D_{\mathbb{A}}^*)$. The inverse map is given by conjugation with $\alpha^{-1}$, which is, by the same calculations, well defined. It follows at once that $(\Lambda\alpha) + (\Lambda\beta) = (\Lambda\alpha\beta)$ is mapped to $(\Gamma\alpha\beta) = (\Gamma\alpha) + (\Gamma\beta)$, hence this is a homomorphism of groups. □

## 1.5 The reduced norm map - narrow class group

Let $D$ be a central simple algebra over $k$. We want to investigate the image of the reduced norm map $\mathrm{nrd} : D \to k$. It will depend mainly on the completions of $D$ at the real places of the algebraic number field $k$.

One proves:

**Theorem 1.10** (Thm. 33.4 in [Rei03]). *Let $v$ be a finite place of $k$ and let $D$ be a central simple $k$-algebra. Then*

$$\mathrm{nrd}(D_v) = k_v$$

*except for the special case where $k_v = \mathbb{R}$ and $D_v \cong M_{\kappa_v}(\mathbb{H})$. In that case*

$$\mathrm{nrd}(D_v) = \mathrm{nrd}(\mathbb{H}) = \mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}.$$

Let $x \in k$. Suppose there exists an element $d \in D$ such that $\mathrm{nrd}(d) = x$ then it follows at once that $x_v = \mathrm{nrd}_{D_v/k_v}(d_v)$ for all places $v$ of $k$. In particular $x_v \in \mathrm{nrd}(D_v)$ for all places $v$ of $k$. This condition is also sufficient. To make this more precise let $R(D)$ be the set of all infinite (necessarily real) places of $k$, such that $D_v \cong M_{\kappa_v}(\mathbb{H})$ for some $\kappa_v \in \mathbb{N}$. Then we define

$$k^D := \{x \in k \,:\, \sigma(x) > 0 \text{ for all } \sigma \in R(D)\}.$$

If $D$ is totally definite, we will write $k^+$ instead of $k^D$. One can show:

**Theorem 1.11** (Hasse-Schilling-Maass, Thm. 33.15 in [Rei03]). *Let $D$ be a central simple algebra over a global field $k$ and let $x \in k^*$. Then $x$ is the reduced norm of an element of $D^*$ if and only if $x \in k^D$.*

This theorem states, that an element $x \in k^*$ is the reduced norm of an element in $D^*$ if and only if $x$ is the reduced norm of an element in $D_v$ for every place $v$ of $k$.

### 1.5.1 Reduced norm of an order

Let $\Lambda$ be a maximal order in $D$. Then for each finite place $v$ the order $\Lambda_v$ is maximal in $D_v$. From the description of maximal orders in division algebras over local fields $k_v$ it follows that $\mathrm{nrd}_{D_v/k_v}(\Lambda_v) = \mathcal{O}_v$ for all finite places $v$ of $k$. We see at once that

$$\mathrm{nrd}(\Lambda) \subset \bigcap_{v \in V_f} \mathcal{O}_v \cap k^D =: \mathcal{O}_k^D.$$

If $[D : k] = m^2$ (note that the dimension of a central simple algebra has to be a square number) then it follows that for $x \in k \cap \Lambda$

$$\mathrm{nrd}(x) = x^m \in \mathcal{O}_k^*.$$

Hence by Dirichlet's Unit Theorem $\mathrm{nrd}(\Lambda^*)$ is a finite index subgroup of $\mathcal{O}_k^*$. We will prove later (Section 5.3) that, if $D$ is not totally definite, we have in fact $\mathrm{nrd}(\Lambda^*) = (\mathcal{O}_k^D)^*$.

### 1.5.2 Reduced norm - class group

Let $\mathrm{Cl}(\mathcal{O}_k)$ denote the ideal class group of $k$. By definition $\mathrm{Cl}(\mathcal{O}_k) = I(k)/P(k)$, where $I(k)$ is the set of fractional ideals and $P(k)$ is the set of principal ideals of $\mathcal{O}_k$.

Let $k$ be a number field of degree $[k : \mathbb{Q}] = r_1 + 2r_2$, where $r_1$ (resp. $r_2$) is the number of real (resp. complex) places of $k$. Let the embeddings be ordered as follows:

- Let $\sigma_1, \ldots, \sigma_s$, be the different embeddings of $k \hookrightarrow \mathbb{R}$ such that $\sigma_i \in R(D)$,
- let $\sigma_{s+1}, \ldots \sigma_{r_1}$ be the remaining real embeddings and
- let the complex embeddings be ordered as usual.

If $D$ is totally definite, $s = n = [k : \mathbb{Q}]$. An element $x \in k^*$ is in $\mathrm{nrd}(D^*)$, if and only if $\sigma_i(x) > 0$ for $i = 1, \ldots s$. If $D$ is totally definite, and $x \in k^D = \mathrm{nrd}(D^*)$ then $x$ is called totally positive. Let $P^D(k)$ be the group of principal ideals generated by the elements in $k^D$. Then we can define the ray class group with respect to $D$ as

$$\mathrm{Cl}^D(\mathcal{O}_k) = I(k)/P^D(k).$$

Note that we have a surjective group homomorphism $\mathrm{Cl}^D(\mathcal{O}_k) \to \mathrm{Cl}(\mathcal{O}_k)$, hence the cardinality $h_k^D$ of $\mathrm{Cl}^D(\mathcal{O}_k)$ is greater or equal to the class number $h_k$ of $k$.

If $D$ is totally definite we will write $P^+(k)$, $\mathrm{Cl}^+(\mathcal{O}_k)$, $h_k^+$ instead of $P^D(k)$, $\mathrm{Cl}^D(\mathcal{O}_k)$, $h_k^D$. We will call $h_k^+$ the narrow class number of $k$. We define a map $\psi : k^* \to (\mathbb{Z}/2\mathbb{Z})^s$ as follows:

$$x \mapsto (\mathrm{sg}(\sigma_1(x)), \ldots, \mathrm{sg}(\sigma_s(x)))$$

where

$$\mathrm{sg}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}.$$

This map is surjective and $\psi(xy) = \psi(x) + \psi(y)$. By construction, $k^D$ is the kernel of $\psi$ and one gets an induced map $\psi : k^*/k^D \to (\mathbb{Z}/2\mathbb{Z})^s$.

Now we have natural maps:

$$1 \to P(k)/P^D(k) \to I(k)/P^D(k) \to I(k)/P(k) \to 1$$

So $h_k^D/h_k = |P(k)/P^D(k)|$. It is easily seen that $P(k)/P^D(k) = k^*/k^D\mathcal{O}_k^*$ holds, hence we have an exact sequence:

$$1 \to \mathcal{O}_k^*/\mathcal{O}_k^D \to k^*/k^D \to P(k)/P^D(k) = k^*/k^D\mathcal{O}_k^* \to 1$$

Now $|k^*/k^D| = 2^s$ and $|\mathcal{O}_k^*/\mathcal{O}_k^D| = |\psi(\mathcal{O}_k^*)| = 2^m$ for some $m \leq s$. Hence $h_k^D = 2^{s-m}h_k$.

**Example 4.** Let $k = \mathbb{Q}(\sqrt{3})$. Then $\mathcal{O}_k^*$ isomorphic to $\mathbb{Z}$ by Dirichlet's Unit Theorem. It is generated by $(2+\sqrt{3})$. Now $(2+\sqrt{3}) > 0$ and $(2-\sqrt{3}) \approx 0.26 > 0$, so $\mathcal{O}_k^* = (\mathcal{O}_k^*)^+$, showing that $h_k^+ = 2h_k = 2$

**Example 5.** Let $k = \mathbb{Q}(\sqrt{10})$, then $\mathcal{O}_k^*$ is generated by $(3+\sqrt{10})$ and its inverse is $3-\sqrt{10} \approx -0.16$, so $h_k^+ = h_k = 2$

Let $\mathrm{Cl}(\Lambda)$ denote the class group of $\Lambda$, the group of stable isomorphism classes of left $\Lambda$-ideals. One can prove:

**Theorem 1.12** (Swan)**.** *Let $k$ be an algebraic number field. Then the reduced norm map induces an isomorphism*

$$\nu : \mathrm{Cl}(\Lambda) \tilde{\to} \mathrm{Cl}^D(\mathcal{O}_k)$$

*where $\mathrm{Cl}(\Lambda)$ is the additive group of stable isomorphism classes of left $\Lambda$-ideals in $D$ and $\mathrm{Cl}^D(\mathcal{O}_k)$ is the multiplicative ray class group (mod $R(D)$), where $R(D)$ is the set of all infinite places of $k$ which ramify in $D$.*

Here $\nu(M)$ is the fractional $\mathcal{O}_k$-ideal generated by $\mathrm{nrd}(M) = \{\mathrm{nrd}(m) : m \in M\}$, for a $\Lambda$-ideal $M$. Note that the set $\mathrm{nrd}(M)$ is not an ideal per se, as the reduced norm map does not respect the additive structure of $M$. We give a short proof here, using our adelic methods developed so far. A classical version can be found in [Rei03, Thm. 35.14].

*Proof.* Note that

$$\mathrm{Cl}^D(\mathcal{O}_k) = U_{\mathcal{O}_k}\backslash \mathbb{A}_k^*/k^D$$
$$\mathrm{Cl}(\Lambda) = U_\Lambda N^1(D_{\mathbb{A}}^*)D^*\backslash D_{\mathbb{A}}^*$$

Then $\nu$ is defined by

$$U_\Lambda N^1(D_{\mathbb{A}}^*)D^*x \mapsto U_{\mathcal{O}_k}\,\mathrm{nrd}(x)k^D.$$

This map is clearly well defined and surjective. The injectivity follows from Theorem 1.10 and the fact that $\mathrm{nrd}(U_\Lambda) = U_{\mathcal{O}_k}$ by definition. Further as $\mathrm{nrd}(xy) = \mathrm{nrd}(x)\,\mathrm{nrd}(y)$ for all $x, y \in D_{\mathbb{A}}^*$, we see that $\nu$ is an isomorphism of groups. □

If $D$ is not totally definite, $\nu$ gives a bijection from the isomorphism classes of left $\Lambda$-ideals to $\mathrm{Cl}^D(\mathcal{O}_k)$. For arbitrary central simple algebras $\nu$ gives us a surjective map from $LF_1(\Lambda)$ to $\mathrm{Cl}^D(\mathcal{O}_k)$, by mapping an isomorphism class of an ideal $M$ to the ideal $\mathrm{nrd}(M)$. The kernel consists of the stable free ideals of $\Lambda$.

## 1.6 The unit group of an order

In this section we describe the structure of the unit group of an order in a central simple algebra $D$ over an algebraic number field $k$. We will do a case by case analysis, first restricting our attention to orders in totally definite quaternion algebras and then study the general case.

### 1.6.1 Orders in totally definite quaternion algebras

Let $D$ be a totally definite quaternion over an algebraic number field $k$ of degree $n$ and let $\Lambda$ be an order in $D$. As $D$ is totally definite we see that

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \times \ldots \times \mathbb{H} = \mathbb{H}^n.$$

This gives a natural map $\Lambda \hookrightarrow \mathbb{H}^n$. Let $N^1(\mathbb{H})$ be the set of elements in the kernel of the reduced norm map $\mathrm{nrd} : \mathbb{H} \to \mathbb{R}^+$. Then

$$N^1(\mathbb{H}) \cong SU_2$$

where $SU_2$ is the special unitary group given by

$$SU_2 = \left\{ \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \in M_2(\mathbb{C}) \,:\, |x|^2 + |y|^2 = 1 \right\}.$$

Indeed we define a map from $\mathbb{H}^*$ to $M_2(\mathbb{C})$ as follows:

$$x_0 + x_1 i + x_2 j + x_3 ij \mapsto \begin{pmatrix} x_0 + x_1\sqrt{-1} & x_2 + x_3\sqrt{-1} \\ -x_2 + x_3\sqrt{-1} & x_0 - x_1\sqrt{-1} \end{pmatrix}.$$

It can be checked directly, that this map is indeed a morphism of groups. On $N^1(\mathbb{H})$ it restricts to an isomorphism to $SU_2$. Now we can show:

**Theorem 1.13.** *Let $D$ be a totally definite quaternion algebra over a number field $k$ and let $\Lambda$ be a maximal order in $D$. Then $\Lambda^1$, the subgroup of $\Lambda^*$ of elements of reduced norm equal to one, is finite.*

*Proof.* If $D$ is totally definite, then $k$ is totally real. Let $D_\infty = \prod_{v \in V_\infty} D_v$. Then we see that $N^1(D_\infty) = (N^1(\mathbb{H}))^n$ where $n$ is the degree of the number field $k$. We have an isomorphism of groups of $N^1(\mathbb{H}) \tilde{\to} SU_2$. Now $SU_2$ is compact, hence so is $N^1(D_\infty)$. We may embed $\Lambda^1 \hookrightarrow D_\infty$ diagonally. Then $\Lambda^1$ is a discrete subgroup of $N^1(D_\infty)$ and hence finite. $\square$

As $\Lambda^1$ is finite and every torsion element of $\Lambda^*$ has to be contained in $\Lambda^1$, it follows that

$$\Lambda^1 = \{x \in \Lambda \,:\, \mathrm{nrd}(x) = 1\} = \{x \in \Lambda \,:\, \exists n \in \mathbb{N}_{>0} \text{ such that } x^n = 1\}.$$

For any order $\Lambda$, the finite group $\Lambda^1$ is equal to a finite subgroup of $N^1(\mathbb{H})$. So it suffices to study the possible structures of finite subgroups of $\mathbb{H}^*$ to know the structure of the finite groups in $D$, and hence to know the structure of $\Lambda^1$.

One shows:

**Theorem 1.14** (Section 6 in [Cox40])**.** *Each finite subgroup of the real quaternions $\mathbb{H}^*$ containing $\{\pm 1\}$ is isomorphic to one of the following groups:*

| Group | Name | Presentation |
|---|---|---|
| $C_{2n}$ | Cyclic | $\langle a \mid a^{2n} = 1 \rangle$ |
| $H_{4n}$ | Binary dihedral | $\langle a, b \mid b^{2n} = 1, a^2 = b^n, aba^{-1} = b^{-1} \rangle$ |
| $E_{24}$ | Binary tetrahedral | $\langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle$ |
| $E_{48}$ | Binary octahedral | $\langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle$ |
| $E_{120}$ | Binary icosahedral | $\langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle$ |

These groups, in particular the exceptional groups $E_{24}$, $E_{48}$ and $E_{120}$ can be realized as finite subgroups of totally definite quaternion algebras (compare [HM06, Prop. 3] and [Vig76, Thm. 5 and Prop. 6]).

It is clear that $\mathcal{O}_k^*$ is contained in $\Lambda^*$, as the unit element 1 has to be contained in $\Lambda$ by definition. Further, as the reduced norm map is multiplicative, $\Lambda^1$ is a normal subgroup in $\Lambda^*$. It follows that $\Lambda^*/(\mathcal{O}_k^*\Lambda^1)$ is finite and depends on the image of $\mathrm{nrd}(\Lambda^*)$ in $(\mathcal{O}_k^+)^*$. More precisely

$$[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = [\mathrm{nrd}(\Lambda^*) : (\mathcal{O}_k^*)^2] = \frac{[(\mathcal{O}_k^+)^* : (\mathcal{O}_k^2)^*]}{[(\mathcal{O}_k^+)^* : \mathrm{nrd}(\Lambda^*)]}$$

One can show, that this index is less than or equal to 4. More precisely

**Theorem 1.15** (Theorem 6 in [Vig76])**.** *Notation as above. The index* $[\Lambda^* : \mathcal{O}_k^*\Lambda^1]$ *equals 1, 2 or 4. Depending on the structure of* $\Lambda^1$ *we have:*

- *Let* $\Lambda^1 = C_2 = \{\pm 1\}$*. Then*
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 4$ *if there exist units* $e_1,\, e_2 \in \Lambda^*$ *such that* $\mathrm{nrd}(e_i) \notin (k^*)^2$ *and* $e_1e_2 = -e_2e_1$*.*
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 2$ *if* $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] < 4$ *and* $\Lambda$ *contains one unit* $e$ *satisfying* $\mathrm{nrd}(e) \notin (k^*)^2$.
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 1$ *else.*
- *Let* $\Lambda^1 = C_{2n} = \langle \zeta_{2n} \rangle$ *with* $n \geq 2$*. Then*
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 4$ *if there exist two units* $e_1,\, e_2 \in \Lambda^*$ *with* $e_1 \in k(\zeta_{2n})$ *such that* $\mathrm{nrd}(e_i) \notin (k^*)^2$ *and* $e_2\zeta_{en}e_2^{-1} = \zeta_{2n}^{-1}$*.*
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 2$ *if* $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] < 4$ *and there exists* $e \in \Lambda^*$ *with* $\mathrm{nrd}(e) \notin (k^*)^2$ *satisfying either*
    - $e \in k(\zeta_{2n})^*$ *or*
    - $e\zeta_{2n}e^{-1} = \zeta_{2n}^{-1}$
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 1$ *else.*
- *Let* $\Lambda^1 = H_{4n} = \langle \sigma_4,\, \zeta_{2n} \rangle$*. Then*
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 2$ *if there exists* $e \in \Lambda^*$ *such that* $\mathrm{nrd}(e) \notin (k^*)^2$ *and* $e \in (1 + \sigma_4)k^*$.
  - $[\Lambda^* : \mathcal{O}_k^*\Lambda^1] = 1$ *else.*
- *Let* $\Lambda^1 = E_{24}$*. Then*

– $[\Lambda^* : \mathcal{O}_k^* \Lambda^1] = 2$ *if there exists* $e \in \Lambda^*$ *with* $\mathrm{nrd}(e) \notin (k^*)^2$ *and such that* $e \in (1+i)k^*$.

– $[\Lambda^* : \mathcal{O}_k^* \Lambda^1] = 1$ *else.*

- *Let* $\Lambda^1 = E_{48}$ *or* $E_{120}$*, then* $[\Lambda^* : \mathcal{O}_k^* \Lambda^1] = 1$.

This gives us a fairly well description of the unit group of an order in a totally definite quaternion algebra. In fact the finite subgroups of the reduced norm one elements $N^1(\mathbb{H})$ have a geometric interpretation. One can show that there is an exact sequence

$$1 \to \{\pm 1\} \to N^1(\mathbb{H}) \xrightarrow{\tau} SO_3(\mathbb{R}) \to 1$$

where $\tau$ sends an elements $u \in \mathbb{H}$ to the isometry $v \mapsto u^{-1}vu$ of $\mathbb{R}^3 \cong \mathbb{H}_0$. Here $\mathbb{H}_0$ denotes the pure quaternions, $\mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij$. Then $E_{24}$ is mapped to the group of rotations of $\mathbb{R}^3$ leaving a regular tetrahedron invariant. Likewise $E_{48}$ and $E_{120}$ correspond to the symmetry groups of a regular octahedron and icosahedron respectively.

### 1.6.2 General case

Let $\Lambda$ be an order in a central simple algebra $D$, which is not totally definite. We write $k^D$ for the image of the reduced norm map, so $k^D = \mathrm{nrd}(D^*)$. More precisely

$$k^D = \{x \in k^* : \sigma(x) > 0 \text{ for all } \sigma : k \hookrightarrow \mathbb{R} \text{ such that } D \text{ ramifies at } \sigma\}.$$

We will write $\mathcal{O}_k^D = k^D \cap \mathcal{O}_k$. Let $\Lambda^1$ denote the kernel of the reduced norm map $\Lambda \to \mathcal{O}_k^D$. We will show later, that $\mathrm{nrd}(\Lambda^*) = \mathcal{O}_k^D$. This follows from the strong approximation property. The index $[\Lambda^* : \mathcal{O}_k^* \Lambda^1]$ is finite, hence $\Lambda^*$ and $\mathcal{O}_k^* \Lambda^1$ are commensurable, as in the totally definite case. The difficulty of determining the structure of $\Lambda^*$ is concentrated in $\Lambda^1$. One shows that $\Lambda^1$ is infinite, unless $D$ is totally definite or a number field.

There is fairly little known on the structure of $\Lambda^1$ in general. Among the most extensively studied are Bianchi groups. These correspond to reduced norm one unit groups of orders in indefinite quaternion algebras isomorphic to $M_2(\mathbb{Q}(\sqrt{d}))$ with $d < 0$.

# Chapter 2

# Linear algebraic groups

In this chapter we will briefly review the most important definitions and properties of linear algebraic groups defined over algebraic number fields $k$. We will develop the necessary background material for the Borel-Serre compactification, which we explain in chapter 3.

Let $D$ be a central simple division algebra of finite dimension over an algebraic number field $k$. Then $M_n(D)$, the set of $n \times n$ matrices with entries in $D$, is a central simple algebra, hence it admits a reduced norm map. The kernel of this reduced norm will be denoted by $SL_n(D)$. It is a linear algebraic group defined over $k$. The most important properties of this group will be developed in the following sections.

## 2.1 Linear algebraic groups

In order to fix notation we will review some definitions and properties of linear algebraic groups defined over a field $k$. We will mainly follow [Bor91]. Every algebraic group we consider is assumed to be linear. The terms "affine" and "linear" will be used interchangeably.

A linear algebraic group is an affine algebraic variety $G$ together with

an identity element $e \in G$,

a morphism $\mu : G \times G \to G$, denoted by $\mu(x, y) = xy$,

a morphism $i : G \to G$, denoted by $x \mapsto x^{-1}$

with respect to which $G$ is a group. We call $G$ a $k$-group if $G$ is a $k$-variety and $\mu$ and $i$ are defined over $k$. We denote the connected component of an algebraic group

$G$ by $G^0$. One shows that $G$ is smooth and that $G^0$ is a normal subgroup of finite index. The affine ring of $G$ will be denoted by $k[G]$. Further $G$ is called connected, if $G = G^0$. Standard examples of algebraic groups are $GL_n$ and $SL_n$. It can be shown ([Bor91, Prop. 1.10]) that any linear algebraic $k$-group $G$ is $k$-isomorphic to a closed subgroup, defined over $k$, of $GL_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Let $D$ be a central simple algebra over $k$. Then the algebra

$$D \otimes_k M_n(k) \cong M_n(D)$$

is central and simple over $k$, hence it admits a reduced norm. Let $SL_n(D)$ denote the kernel of the reduced norm map [1], i.e.

$$SL_n(D) = \{X \in M_n(D) \,:\, \mathrm{nrd}_{M_n(D)/k}(X) = 1\}.$$

In the notation of the previous chapter, we have $SL_n(D) = N^1(M_n(D))$. One can show:

**Theorem 2.1.** *The group $SL_n(D)$ of reduced norm 1 elements of $M_n(D)$, $n \in \mathbb{N}$, is an affine algebraic $k$-group. The group of $k$-points is given by $SL_n(D)$.*

An explicit proof of this is given in [Lac05, Satz 3.1.1]. For any central simple $k$-algebra $L$, the set of $L$-points of $SL_n(D)$ is given by

$$SL_n(D)(L) = SL_n(D \otimes L).$$

We will write $SL_n(D)$ for the algebraic group and for its group of $k$-points, if there is no risk of confusion.

## 2.2 Reductive and semi-simple groups

Let $G$ be a linear algebraic group over $k$. The radical $RG$ of $G$ is the maximal connected solvable normal subgroup of $G$. Its unipotent part $R_uG$ is called the unipotent radical. It is the maximal connected unipotent normal subgroup of $G$. The group $G$ is called semi-simple (reductive) if $RG = \{e\}$ ($R_uG = \{e\}$). Evidently any semi-simple group is reductive.

[1] Note that this notation is ambiguous. In this thesis we write $SL_n(D)$ for the group of matrices in $M_n(D)$, having reduce norm $\mathrm{nrd}_{M_n(D)/k}$ equal to 1. However one could view $SL_n$ as an algebraic group over $k$ and $SL_n(D)$ as the group of $D$ points of $SL_n$, for the $k$-algebras $D$. As we do not assume that $D$ is commutative, this can become quite a problem and it may happen, that this is not well defined. We write $SL_n(D_v)$ for $SL_n(D)(k_v)$ in order to keep our notation simple. We do the same for subgroups of $SL_n(D)$, $GL_n(D)$ etc.

Any reductive $k$-subgroup of $G$ is contained in a maximal one, a so called Levi $k$-subgroup. One shows that any two Levi-subgroups of $G$ are conjugate under $R_uG(k)$ and further that $G = L \cdot R_uG$ for a Levi-subgroup $L$.

Most of the groups we consider are connected, reductive and defined over an algebraic number field $k$. In particular this holds for the group $SL_n(D)$, which is in fact semi-simple.

## 2.3 Characters and tori

Let $G$, $G'$ be two algebraic groups defined over $k$. Let $\mathrm{Mor}(G, G')$ be the set of algebraic group morphisms from $G$ to $G'$. The maps in $\mathrm{Mor}(G, G')$ are morphisms of algebraic varieties, which are also homomorphisms of groups. Further let $\mathrm{Mor}_k(G, G')$ denote the set of morphisms defined over $k$. We define the character group of $G$ to be

$$X(G) = \mathrm{Mor}(G, GL_1).$$

Analogously the group of $k$-characters is given by

$$X(G)_k = \mathrm{Mor}_k(G, GL_1).$$

For a connected algebraic $k$-group $G$ we define

$${}^0G := \bigcap_{\chi \in X_k(G)} \ker(\chi^2).$$

It is a normal $k$-subgroup of $G$ and its set of real points contains any compact subgroup of $G(\mathbb{R})$. Further ${}^0G = {}^0L \cdot R_uG$ for any Levi $k$-subgroup $L$ of $G$.

The characters $X(G)$ form a subgroup of $\overline{k}[G]$, where $\overline{k}$ is the algebraic closure of $k$. Likewise $X(G)_k$ is a subset of $k[G]$. We will call $G$ diagonalizable, if $X(G)$ spans $\overline{k}[G]$ as a $k$-module. If $X(G)_k$ spans $k[G]$ over $k$, then $G$ is called split over $k$. If $G$ is split over $k$, $G$ is diagonalizable, as $\overline{k}[G] \cong \overline{k} \otimes_k k[G]$.

**Theorem 2.2** (Prop. 8.4 in [Bor91]). *Let $G$ be a linear algebraic group. Then the following are equivalent:*

1. *$G$ is diagonalizable.*
2. *There exists $n \in \mathbb{N}_{>0}$ such that $G$ is isomorphic to a closed subgroup of the diagonal matrices $T_n$ in $GL_n$.*

An algebraic group $G$ isomorphic to $T_n$ is called an $n$-dimensional torus. If the isomorphism is defined over $k$, $G$ is said to be split over $k$. By definition, any torus is split over $\mathbb{C}$.

Let $G$ be a connected linear algebraic $k$-group. Then it can be shown that all maximal tori of $G$ are conjugate. One defines the rank, or $\mathbb{C}$-rank, of $G$ to be the dimension of a (any) maximal torus. Further one shows that all maximal $k$-split tori of $G$ are conjugate under $G(k)$. Therefore one defines the $k$-rank of $G$ to be the common dimension of these tori. It will be denoted by $\mathrm{rk}_k(G)$. If the $k$-rank of $G$ equals 0, $G$ is called anisotropic over $k$.

Let us return to our example $SL_n(D)$, where $D$ is a central simple algebra over $k$. If $D$ is a division algebra, a maximal $k$-split torus is given by the set of diagonal matrices in $SL_n(k) \subset SL_n(D)$. In particular

**Theorem 2.3.** *Let $D$ be a division algebra over $k$. Then* $\mathrm{rk}_k(SL_n(D)) = n - 1$.

## 2.4 Borel subgroups and parabolic subgroups

Let $G$ be a linear algebraic group. A Borel subgroup of $G$ is a maximal connected solvable (closed) subgroup of $G$. Borel subgroups exist and one can show that any two Borel subgroups of $G$ are conjugate.

A parabolic subgroup $P$ of $G$ is a closed subgroup such that the quotient $G/P$ is a projective variety. One shows that a closed subgroup $P$ is parabolic if and only if $P$ contains a Borel subgroup. A parabolic $k$-subgroup is a parabolic subgroup defined over $k$.

The parabolic $k$-subgroups of $G$ correspond to subsets $I$ of the set of simple $k$-roots $\Delta$ with respect to a maximal $k$-split torus. More precisely, any parabolic subgroup is conjugate under $G(k)$ to a unique standard rational parabolic subgroup $P_I$. With this notation, $G = P_\Delta$ and a minimal parabolic subgroup is given by $P_\emptyset$. As $|\Delta| = \mathrm{rk}_k(G)$ it follows that in a $k$-rank 1 group, there are, up to conjugacy, two parabolic $k$-subgroups, namely a minimal parabolic $k$-subgroup and $G$ itself.

Let $G = SL_2(D)$, where $D$ is a central simple division algebra over an algebraic number field $k$. The $k$-rank of $G$ equals 1, hence $G$ contains exactly one conjugacy class of non-trivial parabolic $k$-subgroups. A representative of this class is given by

$$P(D) = \left\{ \begin{pmatrix} d_1 & d_2 \\ & d_3 \end{pmatrix} \subset SL_2(D) \right\}.$$

The set of diagonal-matrices in $SL_2(D)$, which we denote by $T(D)$, is a Levi-subgroup of $P(D)$. The unipotent radical of $P$ is given by

$$R_u P(D) = \left\{ \begin{pmatrix} 1 & d \\ & 1 \end{pmatrix} \subset SL_2(D) \right\}.$$

# Chapter 3

# Borel - Serre compactification

Let $G$ be a connected reductive algebraic group over a number field $k$ and let $\Gamma$ be an arithmetic subgroup of $G$. If $\Gamma$ is torsion-free then $\Gamma$ operates properly and freely on a homogeneous space $X$ associated an ambient Lie group $G_\infty$. In general, the space $X/\Gamma$ is not compact. It has a natural compactification, the Borel-Serre compactification, which is obtained by adjoining finitely many so-called "corners" to the space. We will start this chapter by developing some necessary background material and continue by explaining this compactification in more detail. Most of the material can be found in [BJ06], [BS73] and [Sch10].

## 3.1 Arithmetic groups and quotients

Let $V$ be a finite dimensional $\mathbb{C}$-vector space endowed with a $\mathbb{Q}$-structure. So by definition there exists a $\mathbb{Q}$-vector space $V_\mathbb{Q}$, with $\dim_\mathbb{Q} V_\mathbb{Q} = \dim_\mathbb{C} V$ such that $V_\mathbb{Q} \otimes_\mathbb{C} \mathbb{C}$ is isomorphic to $V$. Let $G$ be a rational subgroup of $GL(V)$ and let $\Lambda$ be a lattice in $V_\mathbb{Q}$. We define the group of $\Lambda$-units of $G$ as

$$G_\Lambda = \{g \in G(\mathbb{Q}) \,:\, g\Lambda = \Lambda\}.$$

A subgroup $\Gamma \subset G(\mathbb{Q})$ is called arithmetic, or arithmetically defined, if there exists a lattice $\Lambda$ in $V_\mathbb{Q}$ such that $\Gamma$ and $G_\Lambda$ are commensurable, i.e. the intersection $\Gamma \cap G_\Lambda$ has finite index in both groups. Note that the $\Lambda$-units $G_\Lambda$ are precisely the elements in $G$, which have integral coefficients with respect to a basis induced by $\Lambda$.

The definition of arithmetic subgroups only depends on the structure of the group $G$. In particular it is independent of the choice of representation of $G$ in $GL(V)$.

We could replace our field of definition $\mathbb{Q}$ by an arbitrary algebraic number field $k$ of degree $n$ over $\mathbb{Q}$, and $\mathbb{Z}$ by the ring of integers $\mathcal{O}_k$ of $k$. So for an algebraic $k$-group $G \subset GL_n(\mathbb{C})$ for some $n \in \mathbb{N}$, we define a subgroup $\Gamma$ to be arithmetic if it is commensurable with $G(\mathcal{O}_k)$, the subgroup of $G(k)$ whose coefficients are in $\mathcal{O}_k$ and whose determinant is in $\mathcal{O}_k^*$. Doing so, one does not obtain any new groups. We can reduce groups defined over $k$ to groups defined over $\mathbb{Q}$ using the so-called Weil restriction of scalars. Details can be found in [Sch10, App. C] and [Oes84, App. 3].

### The homogeneous space $X$

Let $G$ be a linear algebraic group defined over an algebraic number field $k$. Let $V_\infty$ be the set of infinite places of $k$. We choose a fixed embedding $G \hookrightarrow GL_n(\mathbb{C})$ for some $n \in \mathbb{N}$. Let $G(\mathcal{O}_k)$ be as above. Further we assume that $G$ is reductive and connected. For every $v \in V_\infty$ we have a corresponding embedding $\sigma_v : k \hookrightarrow \mathbb{C}$. Then $k_v = \mathbb{R}$ or $\mathbb{C}$, depending on the embedding $\sigma_v$.

Suppose $G$ is defined by a set of polynomials $J$ with coefficients in $k$. Then $G^{\sigma_v}$ is the algebraic group defined by the set of polynomials obtained by applying $\sigma_v$ to the coefficients of all the polynomials in $J$. This is again an algebraic group. We write $G_v = G^{\sigma_v}(k_v)$ for all $v \in V_\infty$ and define

$$G_\infty := \prod_{v \in V_\infty} G_v.$$

Let $G' = \mathrm{Res}_{k/\mathbb{Q}} G$ be the Weil restriction of scalars obtained from $G$. Then $G_\infty = G'(\mathbb{R})$, the group of real points of $G'$. We can embed $G(k)$ in $G_\infty$ diagonally using the maps $\sigma_v$. We will not distinguish between $G(k)$ and its image in $G_\infty$. If $\Gamma$ is an arithmetic subgroup of $G$, then $\Gamma$ is discrete in $G_\infty$.

For $v \in V_\infty$, $G_v$ has finitely many connected components. In each $G_v$ we can find a maximal compact subgroup $K_v$ and any two maximal compact subgroups are conjugate. So for an infinite place $v$, the quotient $X_v := K_v \backslash G_v$ can be viewed as the space of maximal compact subgroups of $G_v$. Note that $X_v$ is diffeomorphic to $\mathbb{R}^{d(G_v)}$, where $d(G_v) = \dim(G_v) - \dim(K_v)$, hence $X_v$ is contractible. If $G$ is semi-simple, $X_v$ is the symmetric space associated to $G_v$. We define

$$X := \prod_{v \in V_\infty} X_v$$

and we set $d(G) = \sum_{v \in V_\infty} d(G_v)$.

Let $\Gamma$ be an arithmetic subgroup of $G$. Then $\Gamma$ acts properly discontinuously on the space $X$. So for any given compact subset $C$ in $X$ the set $\{\gamma \in \Gamma : \gamma C \cap C \neq$

$\emptyset\}$ is finite. In particular, the stabilizers of points are finite subgroups of $\Gamma$. Hence, if $\Gamma$ is torsion-free, the action of $\Gamma$ on $X$ is free. In this case the quotient $X/\Gamma$ has the structure of a smooth manifold. As $X$ is contractible and $\Gamma$ acts freely on $X$, the space $X/\Gamma$ is an Eilenberg-MacLane Space $K(\Gamma, 1)$. In other words the fundamental group of $X/\Gamma$ equals $\Gamma$ and all higher homotopy groups are trivial. In particular one can identify the cohomology of $X/\Gamma$ with the cohomology of $\Gamma$. This will be explained in more detail in section 3.3.

## 3.2 Borel-Serre compactification

Let $G$ be a linear algebraic group over $\mathbb{Q}$, let $\Gamma$ be an arithmetic subgroup of $G$ and let $X$ be the quotient space defined as above. For the sake of simplicity we will assume that $G$ is connected and reductive, and that the center of $G$ is anisotropic over $\mathbb{Q}$. This is always satisfied if $G$ is semi-simple. We will only study groups of this form here. However the construction can be done in greater generality.

If $\mathrm{rk}_k(G) > 0$, then $X/\Gamma$ is not compact. It admits a natural compactification $\overline{X/\Gamma}$, the so called Borel-Serre compactification (compare [BS73]). If $\Gamma$ is torsion-free, $\overline{X/\Gamma}$ is a compact manifold with boundary, which is homotopically equivalent to its interior $X/\Gamma$.

The compactification is obtained by first enlarging the space $X$ to a space $\overline{X}$ by adjoining infinitely many faces $e(P)$. This construction involves the $\mathbb{Q}$-structure of $X$, but is independent of the arithmetic group $\Gamma$. The group $G(\mathbb{Q})$, and hence $\Gamma$ acts on $\overline{X}$ and permutes the faces $e(P)$ of $\overline{X}$. One obtains the compact space $\overline{X}/\Gamma = \overline{X/\Gamma}$.

We will explain this construction in more detail.

### 3.2.1 The space $\overline{X}$

Let $P$ be a parabolic $\mathbb{Q}$-subgroup of $G$ and let $U_P$ denote its unipotent radical. Let $\kappa : P \to P/U_P = L_P$ be the canonical projection and let $S_P$ denote a maximal central $\mathbb{Q}$-split torus in $P/U_P$. Let $A_P$ denote the identity component of the group of real points of $S_P$, i.e. $A_P = S_P(\mathbb{R})^0$. It is often called the split component of $S_P$. Note that

$$L_P(\mathbb{R}) = A_P \cdot {}^0L_P(\mathbb{R})$$

as a semi-direct product ([BS73, Prop. 1.2]) , where ${}^0L_P = \bigcap_{\chi \in X_k(L_P)} \ker \chi^2$ is defined as in section 2.3.

Let $K$ be a maximal compact subgroup of $G(\mathbb{R})$ and let $\theta_K = \theta$ denote the Cartan involution corresponding to $K$. Then $P(\mathbb{R})$ contains one and only one Levi-subgroup $L_{P,K}$ of $P(\mathbb{R})$, which is stable under $\theta$ ([BS73, Cor. 1.9]). The projection $\kappa : L_{P,K} \to P(\mathbb{R})/U_P(\mathbb{R}) = L_P(\mathbb{R})$ yields an isomorphism of $\mathbb{R}$ groups. Using the inverse map, we obtain subgroups $A_{P,K}$ and ${}^0L_{P,K}$ of $L_{P,K}$. Note that they are not necessarily defined over $\mathbb{Q}$. However for each $P$ one can choose $K$ such that the groups $A_{P,K}$, ${}^0L_{P,K}$ and $L_{P,K}$ are defined over $\mathbb{Q}$ and such that $\kappa^{-1}$ is a rational map ([BJ06, Prop. III.1.11]). We obtain the so-called rational Langlands-decomposition of $P(\mathbb{R})$:

$$P(\mathbb{R}) = U_P(\mathbb{R}) \cdot A_{P,K} \cdot {}^0L_{P,K}.$$

Using the Iwasawa-decomposition we have $G(\mathbb{R}) = K \cdot P(\mathbb{R})$, hence we see that

$$X = K_P \backslash P(\mathbb{R})$$

where $K_P = K \cap P(\mathbb{R})$ is a maximal compact subgroup of $P(\mathbb{R})$. It is contained in the unique $\theta$-stable Levi-subgroup $L_{P,K}$ of $P(\mathbb{R})$, so the center of $L_{P,K}$ commutes with $K_P$ and acts (from the left) on the space $X$. By identifying $A_P$ with $A_{P,K}$ via $\kappa$ we obtain an action (from the left) of $A_P$ on $X$. This is the geodesic action of Borel and Serre, defined as in [BS73, Chapter 3]. It can be shown that this action is independent of the choice of maximal compact subgroup $K$ ([BS73, Lemma 3.2]), so we may drop the subscript $K$, and simply consider the action of $A_P$ on $X$. It may happen, that different parabolic $\mathbb{Q}$-groups $P$, $Q$ have the same split components $A_P$, $A_Q$. However the geodesic action depends crucially on the parabolic group. We will give a short example:

**Example 6.** Let $G = SL_2/\mathbb{Q}$, then $K = SO_2(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$. Let $P$, resp. $Q$, be the group of upper, resp. lower, triangular matrices in $G$. Then

$$A_P = A_Q = \left\{ \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} : x \in \mathbb{R}^+ \right\}.$$

Let $\mathcal{H}_2$ be the upper half plane. Then $K\backslash G(\mathbb{R})$ can be identified with $\mathcal{H}_2$. One can calculate explicitly that the orbits of $A_P$ are vertical lines, while the orbits of $A_Q$ are half-circles with one end-point in 0 and the other end on the boundary of $\mathcal{H}_2$.

We see that $A_P \cong (\mathbb{R}_+)^r$, where $r$ denotes the parabolic rank of $P$. The closure of $A_P$ in $\mathbb{R}^r$ equals the corner $\mathbb{R}^r_{\geq 0}$. We denote it by $\overline{A}_P$. Clearly the left multiplication of $A_P$ on itself extends to an action of $A_P$ on $\mathbb{R}^r_{\geq 0}$. Further $A_P$ is preserved under this action. Let $X_{P,K} = K_P \backslash {}^0L_{P,K}$ denote the boundary symmetric space associated with $P$. The space $X$ is a principal $A_P$-bundle over $U_P(\mathbb{R}) \cdot X_P$

under this geodesic action of $A_P$. We define the corner $X(P)$ associated to $P$ by

$$X(P) = \overline{A}_P \times_{A_P} X$$

which can be identified with $U_P(\mathbb{R}) \cdot X_P \cdot \overline{A}_P$. As $U_P(\mathbb{R})$ and $X_P$ are real analytic manifolds and $\overline{A}_P \cong \mathbb{R}^r_{\geq 0}$ is a corner, it follows that $X(P)$ is a real analytic manifold with corners. It contains $X$ as an open subset. One can show ([BJ06, Prop. III.5.5]), that the real analytic structure on $X(P)$ is canonical in the sense that it only depends on the geodesic action of $A_P$ on $X$. We set

$$e(P) = A_P \backslash X.$$

Then we see that $e(P)$ corresponds to the corner $\{o_P\} \times_{A_P} X$ in $X(P)$, where $o_P$ is the corner point in $\overline{A}_P = \mathbb{R}^r_{\geq 0}$. This is called the Borel-Serre boundary component. We will study it in the following chapters.

It can be shown, that $X(P)$ can be decomposed as

$$X(P) = X \cup \coprod_{P \subset Q} e(Q)$$

where $Q$ runs over all non-trivial parabolic $\mathbb{Q}$-subgroups of $G$ containing $P$. In particular, if the $\mathbb{Q}$-rank of $G$ equals one, we have $X(P) = X \cup e(P)$, as any two non-trivial parabolic $\mathbb{Q}$-groups are conjugate over $G(\mathbb{Q})$. In general $\overline{X}$ is the disjoint union of the sets $e(P)$, where $P$ runs over all parabolic $\mathbb{Q}$-subgroups of $G$, including $e(G) = X$. Given two parabolic $\mathbb{Q}$-subgroups $P$, $Q$, we have $X(P) \cap X(Q) = X(R)$, where $R$ is the smallest parabolic $\mathbb{Q}$ subgroup of $G$, containing both $P$ and $Q$. In particular if $\mathrm{rk}_{\mathbb{Q}}(G) = 1$ we see that $X(P) \cap X(Q) = X(G) = X$ for any two parabolic $\mathbb{Q}$-groups $P \neq Q$. There exists a uniquely determined structure of a manifold with corners on $\overline{X}$ such that for any parabolic $\mathbb{Q}$-subgroup $P$, $X(P)$ is an open submanifold with corners in $\overline{X}$.

### 3.2.2 The space $\overline{X}/\Gamma$

The natural action of $G(\mathbb{Q})$ (and hence of any arithmetic subgroup $\Gamma$) on $X$ can be extended to $\overline{X}$. This extended action preserves the structure of $\overline{X}$ as a manifold with corners. In particular it permutes the faces $e(P)$.

The quotient $\overline{X}/\Gamma$ is a compact manifold with corners (see [BS73, Thm. 9.3]). Let $e'(P)$ denote the image of $e(P)$ under the natural projection $\overline{X} \to \overline{X}/\Gamma$. One proves ([BS73, Prop. 9.4]) that $e'(P) = e(P)/(\Gamma \cap P)$ and further that for two parabolic subgroups $P$, $Q$, the corners $e'(P)$ and $e'(Q)$ have non-trivial intersection (and hence are in fact equal) if and only if $P$ and $Q$ are conjugate under $\Gamma$. It can be shown that the set of $\Gamma$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups is finite. The boundary of $\overline{X}/\Gamma$ is the disjoint union of finitely many faces $e'(P)$, one for each $\Gamma$-conjugacy class of proper parabolic $\mathbb{Q}$-subgroups. The number of $e'(P)$ in $\overline{X}/\Gamma$ is called the cusp number of $\Gamma$ and will be denoted by $c(\Gamma)$.

## 3.3 Cohomology of arithmetic groups

Let $G$ be a reductive connected algebraic group defined over $k$ and let $\Gamma$ be a torsion-free arithmetic subgroup of $G$. A topological space $Y$ is called a $K(\Gamma, 1)$-space or an Eilenberg-MacLane space of type $(\Gamma, 1)$ if $\pi_1(Y) = \Gamma$ and $\pi_i(Y) = \{1\}$ for $i \geq 2$. It can be proved that its cohomology (or homology) is isomorphic to the cohomology of $\Gamma$. More precisely let $E$ be a $\Gamma$-module and let $\tilde{E}$ denote the corresponding local system on $Y$. Then there is a canonical isomorphism

$$H^q(\Gamma, E) = H^q(Y, \tilde{E})$$

for any $q \in \mathbb{N}$.

Let $K$ be a maximal compact subgroup of $G_\infty$, defined as above. Then $X = K\backslash G_\infty$ is contractible, hence it is the universal cover of $K\backslash G_\infty/\Gamma$. So $K\backslash G_\infty/\Gamma$ is a $K(\Gamma, 1)$ space. One shows that the inclusion $X/\Gamma$ into the Borel-Serre compactification $\overline{X}/\Gamma$ is a homotopy equivalence. This allows us to use the long exact cohomology-sequence for the pair $(\overline{X}/\Gamma, \partial\overline{X}/\Gamma)$, where $\partial\overline{X}/\Gamma$ denotes the boundary of $\overline{X}/\Gamma$, to study the cohomology of $\Gamma$. We have the exact sequence

$$\ldots \to H^q(\overline{X}/\Gamma, \partial\overline{X}/\Gamma, \tilde{E}) \to H^q(\Gamma, E) \xrightarrow{r} H^q(\partial\overline{X}/\Gamma, \tilde{E}) \to H^{q+1}(\overline{X}/\Gamma, \partial\overline{X}/\Gamma, \tilde{E}) \to \ldots$$

for a $\Gamma$-module $E$.

This sequence is a helpful tool in the analysis of $H^q(\Gamma)$. It allows us to split the investigation of $H^q(\Gamma)$ into two parts:

- the interior cohomology, which is defined as the kernel of the restriction map $r : H^q(\overline{X}/\Gamma) \to H^q(\partial\overline{X}/\Gamma)$. It is usually denoted by $H^q_!(X/\Gamma)$.
- the cohomology at infinity, which is defined as the complement of the interior cohomology. The cohomology of infinity encodes all those phenomena in the cohomology that are due the to non-compactness of $X/\Gamma$.

# Chapter 4

# Cusp numbers

Let $D$ be a central simple division algebra of finite dimension over an algebraic number field $k$. Let $\Lambda$ be a maximal order in $D$. Then $SL_n(\Lambda)$ is an arithmetic subgroup of $SL_n(D)$, the group of $n \times n$ matrices of reduced norm $\mathrm{nrd}_{M_n(D)/k}$ equal to 1. In this chapter we give an adelic description of the number of faces $e'(P)$, which are added to $X/SL_n(\Lambda)$ to obtain the Borel-Serre compactification $\overline{X}/\Gamma$. This number equals the number of $SL_n(\Lambda)$-conjugacy classes of proper parabolic $k$-subgroups of $SL_n(D)$ and is called the cusp number $c(SL_n(\Lambda))$ of the arithmetic subgroup $SL_n(\Lambda)$.

We show that the number of $SL_n(\Lambda)$-conjugacy classes of parabolic subgroups of level $P$ equals the class number of $P$ with respect to a lattice induced by $\Lambda$. So we prove that there is a bijection

$$P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D) \longleftrightarrow SL_n(\Lambda)\backslash SL_n(D)/P(D).$$

Then we can calculate the class number of a parabolic subgroup using its Levi-component. In particular for the minimal parabolic subgroup $P$ of $SL_n(D)$ we prove that the class number of $P$ and the class number of its Levi-component $T$ coincide. The construction covers the case of the unique, up to conjugacy, proper parabolic subgroup of $SL_2(D)$. We use this description in the next chapter to determine the cusp number of $SL_2(\Lambda)$.

## 4.1 Class numbers of algebraic groups

In this section we introduce the class number of an algebraic group defined over an algebraic number field $k$ and state some basic properties. We are mainly following [Bor63] and chapter 5 and 8 in [PR94].

Let $k$ be an algebraic number field and let $\mathcal{O}_k$ be its ring of integers. We define

$$A_{\mathcal{O}_k} = \prod_{v \in V_\infty} k_v \times \prod_{v \in V_f} \mathcal{O}_v$$

and we denote its unit group by $U_{\mathcal{O}_k}$. The ring $A_{\mathcal{O}_k} \subset \mathbb{A}_k$ is sometimes called the group of integral adeles. Keep in mind that the class number of $k$ is equal to the cardinality of the double quotient

$$U_{\mathcal{O}_k} \backslash \mathbb{A}_k^* / k^*.$$

Now let $G$ be a linear algebraic group defined over $k$. Suppose that we have a fixed embedding $\varphi : G \hookrightarrow GL_r$ for some $r \in \mathbb{N}$. Let $L$ be an $\mathcal{O}_k$-lattice in $k^r$. Then the group of $L$-units is defined as

$$G_L = \{g \in G(k) \, : \, gL = L\}.$$

Using this group we can define a subgroup of $G(\mathbb{A}_k)$ as:

$$G(A_{\mathcal{O}_k})^L = \prod_{v \in V_\infty} G(k_v) \times \prod_{v \in V_f} G_L(k_v).$$

Note that by definition $G_L(k_v) = \{X \in G(k_v) \, : \, XL_v = L_v\}$. The elements in $G_L(k_v)$ are precisely those in $G(k_v) \subset GL_r(k_v)$, which are given by a matrix in $GL_r(\mathcal{O}_v)$ with respect to a base determined by the lattice $L_v$.

**Definition 4.1.** *The class number of an algebraic group $G$ with respect to the lattice $L$ is defined as*

$$cl(L, G) = |G(A_{\mathcal{O}_k})^L \backslash G(\mathbb{A}_k)/G(k)|.$$

This class number may depend on the lattice $L$ and on the embedding $\varphi{:}G \to GL_r$. If the reference to a lattice is clear we will simply write $cl(G)$.

The class number of an algebraic group is finite ([PR94, Thm. 5.1]). A unipotent group has class number equal to 1 for any embedding chosen.

Let $G = TU$ be a semi-direct product of two subgroups $T$ and $U$, with $U$ unipotent and normal in $G$. One shows ([Bor63, Prop. 2.7]) that $cl(G) \leq cl(T)$. More precisely for any disjoint union

$$T(\mathbb{A}_k) = \bigcup_{j=1}^{cl(T)} T(A_{\mathcal{O}_k}) t_j T(k)$$

with $t_j \in T(\mathbb{A}_k)$ for $1 \leq j \leq cl(T)$, one obtains

$$G(\mathbb{A}_k) = \bigcup_{j=1}^{cl(T)} G(A_{\mathcal{O}_k}) t_j G(k).$$

However this union may not be disjoint.

Let $D$ be a finite dimensional central division algebra over $k$. Let $G = SL_n(D)$, the algebraic group of reduced norm 1 matrices in $M_n(D)$ with its natural representation fixed. We write $SL_n(D_{\mathbb{A}})$ for $G(\mathbb{A}_k)$. Let $\Lambda$ be a maximal order in $D$. Then by definition

$$SL_n(D_{\mathbb{A}}) = \{X = (X_v)_v \in M_n(D_{\mathbb{A}}) : \mathrm{nrd}_{M_n(D_v)/k_v}(X_v) = 1 \text{ for all } v \in V \\ \text{and } X_v \in M_n(\Lambda_v) \text{ almost everywhere}\}.$$

Note that $SL_n(D_{\mathbb{A}})$ does not depend on the maximal order $\Lambda$ chosen. Let $P$ be a parabolic subgroup of $SL_n(D)$. We fix a maximal order $\Lambda$ in $D$ and write

$$cl(\Lambda, P) := |P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D)|.$$

To be precise, with the notation previously introduced, we should write $L(\Lambda)$ for the lattice induced by $\Lambda^n$ in $k^r$ for some $r \in \mathbb{N}$. However for the sake of simplicity we will simply write $cl(\Lambda, P)$.

## 4.2 From cusp numbers to class numbers

Let $k$ be an algebraic number field. Let $v$ be a finite place of $k$ and let $D_v$ be a central division algebra over $k_v$. Its unique maximal order $\Delta_v$ is given by $\{x \in D_v : \mathrm{nrd}(x) \in \mathcal{O}_v\}$. Let $P_G(D_v)$ denote the group of upper triangular matrices in $GL_n(D_v)$.

**Lemma 4.1.** *Let $\Delta_v$ be the unique maximal order in $D_v$. Then*

$$GL_n(D_v) = GL_n(\Delta_v) P_G(D_v).$$

*Proof.* We are following [Bum97, Prop. 4.5.2], where the lemma is proved for $D_v = k_v$. We prove this by induction on $n$. It is clear for $n = 1$. Now let $g \in GL_n(D_v)$. We have to find $k_0 \in GL_n(\Delta_v)$ such that $k_0 g$ is upper triangular. Our first step is to find $k_1$ in $GL_n(\Delta_v)$ such that $k_1 g$ has zeroes in the first column, except for the first entry on the main diagonal. Let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be the first column in $g$ and let us assume that $|\operatorname{nrd}(x_n)|_v$ is maximal. If not, we simply use a permutation matrix, for any permutation matrix is contained in $GL_n(\Delta_v)$. Then $x_i x_n^{-1} \in \Delta_v$ as $|\operatorname{nrd}(x_i x_n^{-1})|_v \leq 1$ by our assumption on $x_n$. We set

$$k_1 := \begin{pmatrix} & & & & 1 \\ & & & \iddots & \vdots \\ & & 1 & & -x_3 x_n^{-1} \\ & 1 & 0 & & -x_2 x_n^{-1} \\ 1 & 0 & 0 & \dots & -x_1 x_n^{-1} \end{pmatrix}$$

Then $k_1 g$ is of the form $\begin{pmatrix} * & \cdots \\ \mathbf{0} & g_1 \end{pmatrix}$ with $g_1 \in GL_{n-1}(D_v)$. By induction we find $k_2 \in GL_{n-1}(\Delta_v)$ such that $k_2 g_1$ is upper triangular. Now setting

$$k_0 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & k_2 \end{pmatrix} \cdot k_1$$

we obtain the required matrix. □

Now let $P_S(D_v)$ denote the group of upper triangular matrices in $SL_n(D_v)$. We show:

**Lemma 4.2.** *Let $\Delta_v$ be the unique maximal order in the division algebra $D_v$. Then*

$$SL_n(D_v) = SL_n(\Delta_v) P_S(D_v).$$

*Proof.* The map $\operatorname{nrd}_{M_n(D_v)/k_v} : P_G(D_v) \to k_v^*$ is surjective, hence $GL_n(D_v) = SL_n(\Delta_v) \cdot P_G(D_v)$. The natural map

$$SL_n(\Delta_v) \backslash SL_n(D_v) / P_S(D_v) \to SL_n(\Delta_v) \backslash GL_n(D_v) / P_G(D_v)$$

is well defined. We see at once that it is injective. This proves our claim. □

Let $D$ be a finite dimensional division algebra over $k$ with maximal order $\Lambda$. Any parabolic subgroup of $SL_n(D)$ is conjugate to one containing the minimal parabolic subgroup $P_S(D)$ of upper triangular matrices. We explain this in some more detail in Section 6.1.1.

Let $D_v$, resp. $\Lambda_v$, be the completion of $D$, resp. $\Lambda$, at the finite place $v$ of $k$. Then $D_v \cong M_r(S_v)$ for some division algebra $S_v$ over $k_v$ and $\Lambda_v$ is conjugate to

$M_r(\Delta_v)$, where $\Delta_v$ is the unique maximal order in $S_v$. Then it follows at once that:

$$GL_n(D_v) = GL_{nr}(S_v) = GL_n(M_r(\Delta_v))P_G(S_v) = GL_n(\Lambda_v)P_G(D_v)$$
$$SL_n(D_v) = SL_n(\Lambda_v)P_S(D_v).$$

**Theorem 4.1.** *Let $P$ be a parabolic subgroup in $SL_n(D)$, for $n \geq 2$, and let $\Lambda$ be a maximal order. Then $|SL_n(\Lambda)\backslash SL_n(D)/P(D)|$ is equal to $cl(\Lambda, P)$.*

*Proof.* We are following [Lac05, Lemma 4.4.6], where the equality is proved for the minimal parabolic $k$-subgroup $P$ in $SL_2(D)$. This theorem holds more generally, see [Bor63, Prop. 7.5]. Let $\psi_1$ be the natural map defined by

$$SL_n(\Lambda)\backslash SL_n(D)/P(D) \to SL_n(D_\Lambda)\backslash SL_n(D_\mathbb{A})/P(D)$$
$$SL_n(\Lambda)xP(D) \mapsto SL_n(D_\Lambda)xP(D).$$

It is well defined and injective. Indeed if $x, y \in SL_n(D)$ satisfy

$$x \in SL_n(D_\Lambda)yP(D)$$

it follows at once that $x \in SL_n(\Lambda)yP(D)$, as $D_\Lambda \cap D = \Lambda$. Further the central simple algebra $M_n(D)$ is not totally definite, hence it has the strong approximation property

$$SL_n(D_\mathbb{A}) = SL_n(D_\Lambda)SL_n(D).$$

So $\psi_1$ is a bijection.

For all finite places $v$ of $k$ we have shown that

$$SL_n(D_v) = SL_n(\Lambda_v)P(D_v).$$

Hence the representatives of $SL_n(D_\Lambda)\backslash SL_n(D_\mathbb{A})/P(D)$ can be chosen in $P(D_\mathbb{A})$. This shows that the map

$$P(D_\Lambda)\backslash P(D_\mathbb{A})/P(D) \to SL_n(D_\Lambda)\backslash SL_n(D_\mathbb{A})/P(D)$$
$$P(D_\Lambda)xP(D) \mapsto SL_n(D_\Lambda)xP(D)$$

is surjective. It is clear that it is injective as well, which proves our claim. □

Let $T(D)$ denote the group of diagonal matrices in the minimal parabolic subgroup $P_S(D)$ of $SL_n(D)$. We show:

**Theorem 4.2.** *Let $D$ be a division algebra with maximal order $\Lambda$. Then*

$$cl(\Lambda, T(D)) = cl(\Lambda, P_S(D)).$$

*Proof.* We already know that $cl(\Lambda, P_S(D)) \leq cl(\Lambda, T(D))$. It remains for us to show that if $t, t' \in T(D)$, the relation $t' \in P_S(D_\Lambda)tP_S(D)$ implies $t' \in T(D_\Lambda)tT(D)$. Suppose

$$\begin{pmatrix} t'_1 & & \\ & \ddots & \\ & & t'_n \end{pmatrix} = \begin{pmatrix} p_1 & * & * \\ & \ddots & * \\ & & p_n \end{pmatrix} \cdot \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \cdot \begin{pmatrix} d_1 & * & * \\ & \ddots & * \\ & & d_n \end{pmatrix}$$

Here $p_i \in U_\Lambda$ and $d_i \in D^*$ for $1 \leq i \leq n$ by definition. Then it follows that

$$\begin{pmatrix} t'_1 & & \\ & \ddots & \\ & & t'_n \end{pmatrix} = \begin{pmatrix} p_1 t_1 d_1 & & \\ & \ddots & \\ & & p_n t_n d_n \end{pmatrix}$$

Now as

$$\mathrm{nrd}_{M_n(D)/k}\left(\begin{pmatrix} x_1 & * & * \\ & \ddots & * \\ & & x_n \end{pmatrix}\right) = \prod_{i=1}^{n} \mathrm{nrd}_{D_v/k_v}(x_i)$$

we see that $\begin{pmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{pmatrix} \in T(D_\Lambda)$ and $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \in T(D)$. This finishes the proof. □

In particular we see that:

**Theorem 4.3.** *Let $T$ be a Levi-subgroup of a minimal parabolic subgroup of $SL_2(D)$. Then the cusp number $c(SL_2(\Lambda))$ of $SL_2(\Lambda)$ equals $cl(\Lambda, T) = |T(U_\Lambda)\backslash T(D^*_{\mathbb{A}})/T(D^*)|$.*

We will calculate this constant explicitly in the next chapter.

# Chapter 5

# The cusp number of $SL_2(\Lambda)$

Let $D$ be a finite dimensional central division algebra over an algebraic number field $k$. Let $\Lambda$ be a maximal order in $D$. We are now going to calculate the number of boundary components arising in the Borel-Serre compactification of the homogeneous space attached to $SL_2(D)$ under the action of the arithmetic group $SL_2(\Lambda)$. This is the cusp number of $SL_2(\Lambda)$, which we denote by $c(SL_2(\Lambda))$. It equals the number of $SL_2(\Lambda)$-conjugacy classes of proper parabolic $k$-subgroups of $SL_2(D)$. If $D$ is a field, i.e. $D = k$ it is well known ([Sie61, Prop. 20]) that the cusp number $c(SL_2(\mathcal{O}_k))$ of $SL_2(\mathcal{O}_k)$ equals the class number $h_k$ of $k$.

In the previous chapter we proved that there is a bijection between the set of $SL_2(\Lambda)$-conjugacy classes of parabolic $k$-subgroups of $SL_2(D)$ and the class number of the Levi-subgroup $T$ of a minimal parabolic subgroup $P$ in $SL_2(D)$. Using this approach we show that, if $D$ is not a totally definite quaternion algebra, $c(SL_2(\Lambda)) = h_D$, the class number of $D$.

However the calculation of the cusp number of $SL_2(\Lambda)$ for a maximal order $\Lambda$ in a totally definite quaternion algebra is much more difficult. It involves certain invariants of all maximal orders of $D$, which measure for instance how much the orders fail to have the cancellation property. A large part of this chapter is dedicated to this calculation. We start that part with some special cases, assuming for instance that the reduced norm map of the unit groups of all maximal orders to the group of totally positive units $(\mathcal{O}_k^*)^+$ is surjective. Afterward we give the final formula for totally definite quaternion algebras over arbitrary totally real number fields and show that the cusp number is independent of the choice of maximal order in $D$. At last we provide some explicit examples of calculations of cusp numbers $c(SL_2(\Lambda))$ for maximal orders $\Lambda$ in totally definite quaternion algebras.

## 5.1 Cusps

For now and for the rest of this chapter let $D$ be a finite dimensional central division algebra over an algebraic number field $k$. Let $\Lambda$ be a maximal order in $D$. It defines an arithmetic group $SL_2(\Lambda)$ in the algebraic $k$-group $SL_2(D)$. We are interested in the number of boundary components arising in the Borel-Serre-compactification of the homogeneous space attached to $SL_2(D)$ under the action of $SL_2(\Lambda)$. This number equals the number of $SL_2(\Lambda)$-conjugacy classes of (proper) parabolic $k$-subgroups of $SL_2(D)$ and will be denoted by $c(SL_2(\Lambda))$.

The group $SL_2(D)$ contains a minimal parabolic subgroup $P(D)$ of upper triangular matrices. It is the unique (up to conjugacy) proper parabolic $k$-subgroup of $SL_2(D)$. So

$$c(SL_2(\Lambda)) = |SL_2(\Lambda)\backslash SL_2(D)/P(D)|.$$

In the previous chapter (Theorem 4.3) we have further proved that $c(SL_2(\Lambda)) = cl(\Lambda, T)$, where $T$ denotes the Levi-subgroup of diagonal matrices in $P(D)$, i.e.

$$T(D^*) = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix} \in M_2(D) \ : \ \mathrm{nrd}(xy) = 1 \right\}.$$

So our aim is to calculate the cardinality of

$$T(U_\Lambda)\backslash T(D^*_{\mathbb{A}})/T(D^*)$$

for a maximal order $\Lambda$ in $D$.

## 5.2 Upper and lower bound

Let $D$ be a central division algebra over an algebraic number field $k$. Let $\Lambda$ be a maximal order in $D$. In this section we will give an upper and a lower bound for the cusp number $c(SL_2(\Lambda))$. We will use the adelic description of the previous section. We will further show that for a large class of division algebras those bounds are equal. However, in general the cusp number may be different from both of these bounds.

For $\alpha \in D^*_{\mathbb{A}}$ let $\Lambda\alpha$ be the unique left $\Lambda$-ideal in $D$, such that for all finite places $v$ of $k$,

$$(\Lambda\alpha)_v = \Lambda_v \alpha_v.$$

We have seen (Theorem 1.3), that the ideal $\Lambda\alpha$ is well defined. The isomorphism classes $\{\Lambda x_1, \ldots, \Lambda x_{h_D}\}$ of left $\Lambda$-ideals correspond to the decomposition of $D^*_{\mathbb{A}}$ given by

$$D^*_{\mathbb{A}} = \bigcup_{i=1}^{h_D} U_\Lambda x_i D^*$$

for some $x_i \in D^*_{\mathbb{A}}$, $1 \le i \le h_D$. For any maximal order $\Lambda$ in $D$ we define $s_\Lambda$ as

$$s_\Lambda = \left| U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}}) D^* / D^* \right|.$$

This is the cardinality of the kernel of the natural projection $LF_1(\Lambda) \to \mathrm{Cl}(\Lambda)$ mapping isomorphism classes of left $\Lambda$-ideals to stable isomorphism classes. Further let $n_\Lambda$ be defined by

$$n_\Lambda = \left| N^1(U_\Lambda) \backslash N^1(D^*_{\mathbb{A}}) / N^1(D) \right|.$$

Note that $n_\Lambda = cl(\Lambda, N^1(D))$ in the notation of the previous chapter. Hence both $s_\Lambda$ and $n_\Lambda$ are finite. One shows:

**Theorem 5.1.** *Let $D^*_{\mathbb{A}} = \bigcup_{i=1}^{h_D} U_\Lambda x_i D^*$ be a decomposition of $D^*_{\mathbb{A}}$ corresponding to a set of representatives of left $\Lambda$-ideal classes. Then*

$$\sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}} \le c(SL_2(\Lambda)) \le \sum_{i=1}^{h_D} n_{x_i \Lambda x_i^{-1}}.$$

*Proof.* This can also be found in [Lac05, Satz 5.3.1]. For the sake of simplicity we will write $[x]$ for the double coset $U_\Lambda x D^*$ in $D^*_{\mathbb{A}}$. Further we write $[x, y]$ for the double coset $T(U_\Lambda) \left(\begin{smallmatrix} x & \\ & y \end{smallmatrix}\right) T(D^*)$ in $T(D^*_{\mathbb{A}})$. Note that $\mathrm{nrd}(xy) = 1$ holds. So in particular $y \in x^{-1} N^1(D^*_{\mathbb{A}}) = N^1(D^*_{\mathbb{A}}) x^{-1}$.

Let $\phi$ denote the projection $T(U_\Lambda) \backslash T(D^*_{\mathbb{A}}) / T(D^*)$ to $U_\Lambda \backslash D^*_{\mathbb{A}} / D^*$ given by

$$T(U_\Lambda) \begin{pmatrix} x & \\ & y \end{pmatrix} T(D^*) \quad \mapsto \quad U_\Lambda x D^*.$$

This may also be written as $[x, y] \mapsto [x]$. The map $\phi$ is well defined and surjective. It is clear that

$$c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} \left| \phi^{-1}([x_i]) \right|,$$

hence we will investigate for each class $[x_i]$ the cardinality of the fiber $\phi^{-1}([x_i])$ to get a description of $c(SL_2(\Lambda))$.

Note that for each class $[x, y]$ in $\phi^{-1}([x_i])$ there exists an element $n \in N^1(D^*_{\mathbb{A}})$ such that $[x, y] = [x_i, x_i^{-1} n]$. This follows easily from the definitions. Further if

$[x_i, x_i^{-1}n] = [x_i, x_i^{-1}n']$ we see at once that $n \in x_i U_\Lambda x_i^{-1} n' D^*$. Now $x_i U_\Lambda x_i^{-1} = U_{x_i \Lambda x_i^{-1}}$, hence

$$s_{x_i \Lambda x_i^{-1}} \leq \left|\phi^{-1}([x_i])\right|.$$

On the other hand, if $n \in x_i N^1(U_\Lambda) x_i^{-1} n' N^1(D^*)$, then clearly $[x_i, x_i^{-1}n] = [x_i, x_i^{-1}n']$, so

$$\left|\phi^{-1}([x_i])\right| \leq n_{x_i \Lambda x_i^{-1}}.$$

□

Let us suppose that $D$ is not a totally definite quaternion algebra. Then Theorem 1.2 states that $n_\Lambda = 1$ for any maximal order $\Lambda$ of $D$. This shows:

**Lemma 5.1.** *If $D$ is not a totally definite quaternion algebra, then*

$$c(SL_2(\Lambda)) = h_D = h_k^D$$

*for any maximal order $\Lambda$ in $D$.*

Any number field is a division algebra over itself. Also note that if $D = k$, $h_D$ equals the class number $h_k$ of $k$. So as a special case we obtain:

**Corollary 5.1.** *Let $k$ be a number field. Then the cusp number of $SL_2(\mathcal{O}_k)$ is $h_k$.*

We have solved the cusp number problem for a large class of division algebras. What is still needed is a formula for totally definite quaternion algebras $D$. It has been shown ([KO90, Satz 2.1]), that for any maximal order $\Lambda$ in a totally definite quaternion algebra $D$ over $\mathbb{Q}$ the cusp number of $SL_2(\Lambda)$ equals the square of the class number $h_D$ of $D$, i.e.

$$c(SL_2(\Lambda)) = h_D^2.$$

In the following sections we will calculate the cusp number of a maximal order in a totally definite quaternion algebra over an arbitrary number field $k$. From now on we will assume that $D$ is totally definite.

### 5.2.1 The lower bound

Let $D$ be a totally definite quaternion algebra over an algebraic number field $k$ and let $\Lambda$ be a maximal order. Let

$$\bigcup_{i=1}^{h_D} U_\Lambda x_i D^*$$

be the decomposition of $D^*_{\mathbb{A}}$ corresponding to the isomorphism classes of left $\Lambda$-ideals with $x_i \in D^*_{\mathbb{A}}$ for $1 \leq i \leq h_D$. We denote the double coset $U_\Lambda x_i D^*$ by $[x_i]$.

Note that the isomorphism class of the $\Lambda$-ideal $\Lambda x_i$ is uniquely determined by $[x_i]$. Hence the isomorphism class of its right order $x_i^{-1}\Lambda x_i$ is uniquely determined by $[x_i]$. However this may not hold for the isomorphism class of $x_i\Lambda x_i^{-1}$. Due to the non-commutativity of $D^*_{\mathbb{A}}$, it is not clear whether $[x_i] \neq [x_j]$ implies $[x_i^{-1}] \neq [x_j^{-1}]$ or not. So in fact the sum $\sum_{i=1}^{h_D} s_{x_i\Lambda x_i^{-1}}$ might depend on the choice of representatives $x_i$. We prove that this is not the case.

So far we have interpreted $s$ as an invariant of a maximal order (more precisely of the isomorphism class of an order), i.e. as a map from the set of types of maximal orders to $\mathbb{N}$ defined in the obvious way by

$$s(\Lambda) = s_\Lambda = |U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*|.$$

An isomorphism class of left $\Lambda$-ideals determines, via its right order, a type of maximal order. So we may define a map, denoted by $s$ again, from $LF_1(\Lambda) \to \mathbb{N}$ via

$$s(\Lambda\alpha) = s_{\alpha^{-1}\Lambda\alpha}$$

for $\alpha \in D^*_{\mathbb{A}}$.

We have a projection from $LF_1$ to the class group $\mathrm{Cl}(\Lambda)$ of $\Lambda$, the set of stable isomorphism classes of left $\Lambda$-ideals. We will now show that $s$ factorizes through a map on $\mathrm{Cl}(\Lambda)$.

**Lemma 5.2.** *If $\Lambda\alpha$ and $\Lambda\beta$ are stable isomorphic left $\Lambda$-ideals in $D$, then $s_{\alpha^{-1}\Lambda\alpha} = s_{\beta^{-1}\Lambda\beta}$.*

*Proof.* We know that $\Lambda\alpha$ and $\Lambda\beta$ are stable isomorphic if and only if $\alpha \in U_\Lambda N^1(D^*_{\mathbb{A}})D^*\beta$ by Theorem 1.4, which holds if and only if $\Lambda\alpha\beta^{-1}$ is stable locally free. So we may restrict our attention to the case $\beta = 1$.

For a fixed $\alpha \in U_\Lambda N^1(D^*_{\mathbb{A}})D^*$ define a map

$$U_{\alpha^{-1}\Lambda\alpha} \backslash U_{\alpha^{-1}\Lambda\alpha} N^1(D^*_{\mathbb{A}})D^*/D^* \to U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$$
$$U_{\alpha^{-1}\Lambda\alpha} nD^* \mapsto U_\Lambda \alpha n D^*.$$

Now $\alpha = un'd$ for some $u \in U_\Lambda$, $n' \in N^1(D^*_{\mathbb{A}})$ and $d \in D^*$, hence

$$U_\Lambda \alpha n D^* = U_\Lambda u n' d n d^{-1} d D^* = U_\Lambda n' n^d D^*$$

where $n^d = dnd^{-1}$. So $\alpha n \in U_\Lambda N^1(D^*_{\mathbb{A}})D^*$, as required. It is well defined, for let

$\alpha^{-1}u_n\alpha n d_n$ be another representative of $U_{\alpha^{-1}\Lambda\alpha}nD^*$. Then

$$U_{\alpha^{-1}\Lambda\alpha}\alpha^{-1}u_n\alpha n d_n D^* \mapsto U_\Lambda \alpha\alpha^{-1}u_n\alpha n d_n D^* = U_\Lambda \alpha n D^*.$$

One shows analogously that the inverse given by "multiplication" with $\alpha^{-1}$ is well defined. This proves our claim. □

Let $(\Lambda\alpha)$ denote the stable isomorphism class of $\Lambda\alpha$. The set of all stable isomorphism classes of left $\Lambda$-ideals forms a commutative group $\mathrm{Cl}(\Lambda)$, with addition given by

$$(\Lambda\alpha) + (\Lambda\beta) = (\Lambda\alpha\beta).$$

The neutral element is $(\Lambda)$. Now for any $u \in U_\Lambda$ we have

$$(\Lambda\alpha) = (\Lambda\alpha) + (\Lambda) = (\Lambda\alpha) + (\Lambda u) = (\Lambda\alpha u).$$

**Theorem 5.2.** *Let $\{x_i\}_{i=1,\dots h_D}$ be a set of representatives in $D^*_{\mathbb{A}}$ of double cosets of $U_\Lambda\backslash D^*_{\mathbb{A}}/D^*$. Then the sum $S(\Lambda) = \sum_{i=1}^{h_D} s_{x_i\Lambda x_i^{-1}}$ is independent of the choice of representatives $x_i$.*

*Proof.* Elements in $D^*$ induce a (global) conjugation of the order, which is an isomorphism. This does not influence the invariant $s_{x_i\Lambda x_i^{-1}}$. The order $x_i\Lambda x_i^{-1}$ is defined as the right order of the ideal $\Lambda x_i^{-1}$. Replacing $x_i$ by $ux_i$, for some $u \in U_\Lambda$, gives the ideal $\Lambda x_i^{-1}u^{-1}$, which is, as we have seen above, stable isomorphic to $\Lambda x_i^{-1}$. Hence by Lemma 5.2, $s_{x_i\Lambda x_i^{-1}}$ is well defined. □

### 5.2.2 The choice of maximal order

Let $D$ be a totally definite quaternion algebra over $k$. In the above formula for the lower bound we see that we have to deal with a kind of inversion on the left $\Lambda$-ideals. More precisely for an ideal $\Lambda x$ we have to find the right order of $\Lambda x^{-1}$. We have seen above that it is in fact enough, to invert the class $(\Lambda x)$ and give one of the possible right orders. In fact all orders $nx\Lambda x^{-1}n^{-1}$, for $n \in N^1(D^*_{\mathbb{A}})$ have the same invariant $s_{x\Lambda x^{-1}}$.

**Definition 5.1.** *Two maximal orders $\Lambda$, $\Gamma$ are called stable isomorphic if there exists an element $n \in N^1(D^*_{\mathbb{A}})$ such that $\Gamma$ is isomorphic to $n\Lambda n^{-1}$.*

We can introduce stable isomorphism classes on the set of isomorphism classes of maximal orders, which we denote by $[\Lambda]$. The number of stable isomorphism classes of maximal orders is denoted by $s_D$. We have seen above (Lemma

5.2) that two stable isomorphic orders $\Lambda$, $\Gamma$ satisfy $s_\Lambda = s_\Gamma$. One shows analogously that $n_\Lambda = n_\Gamma$ holds as well.

We see at once that if two left $\Lambda$-ideals are stable isomorphic, then so are their right orders. The number of stable isomorphism classes of left $\Lambda$-ideals equals $h_k^+$, so in particular we see that $s_D \leq h_k^+$. However we may find ideals $I$, $J$, having stable isomorphic right orders, without being stable isomorphic, e.g. if $x \in D_{\mathbb{A}}^*$ is in the normalizer of $D_\Lambda$, then $x\Lambda x^{-1} = \Lambda$. However there is no reason why $x$ should be in $N^1(D_{\mathbb{A}}^*)$, respectively why $\Lambda x^{-1}$ and $\Lambda$ should be stable isomorphic as ideals.

For a maximal order $\Lambda$ we can describe its normalizer $N(\Lambda) := N(D_\Lambda) \subset D_{\mathbb{A}}^*$ quite accurately:

- If $v \in V_\infty$ we have $N(\Lambda)_v = \mathbb{H}^*$.
- If $v \in V_f$ and $D$ is ramified at $v$, then $\Lambda_v$ is equal to the unique maximal order

  $$\Delta = \{x \in D_v \,:\, \mathrm{nrd}(x_v) \in \mathcal{O}_v\}.$$

  Its normalizer is $D_v^*$.
- If $v \in V_f$ and $D_v$ is split at $v$, then $\Lambda_v$ is conjugate to $M_2(\mathcal{O}_v)$. Its normalizer equals $k^*\Lambda_v^*$.

We denote by $V_f^r$ the set of finite places $v$ of $k$, where $D_v$ is a division algebra, and further by $V_f^s$ the finite places $v$ of $k$, where $D_v \cong M_2(k_v)$. Then we see

$$\mathrm{nrd}(N(\Lambda)) = \prod_{v \in V_\infty} \mathbb{R}^+ \times \prod_{v \in V_f^r} k_v^* \times \prod_{v \in V_f^s} \mathcal{O}_v^* \cdot k_v^2$$

where $k_v^2$ is the set of squares in $k_v^*$. Note that $\mathrm{nrd}(N(\Lambda))$ is independent of the choice of maximal order $\Lambda$.

It follows that for any $x \in D_{\mathbb{A}}^*$,

$$x^2 \in N(\Lambda)N^1(D_{\mathbb{A}}^*).$$

So in particular note that $[x^2\Lambda x^{-2}] = [\Lambda]$ and hence

$$[x\Lambda x^{-1}] = [x^{-1}\Lambda x]$$

for any maximal order $\Lambda$ and $x \in D_{\mathbb{A}}^*$. This implies that $s_D = |\,\mathrm{Cl}(\mathcal{O}_k)^+/(\mathrm{Cl}(\mathcal{O}_k)^+)^2|$, where $(\mathrm{Cl}(\mathcal{O}_k)^+)^2$ is the group of squares in $\mathrm{Cl}(\mathcal{O}_k)^+$. We show:

**Lemma 5.3.** *The sum* $S(\Lambda) = \sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}}$ *is equal to*

$$\sum_{i=1}^{h_D} s_{x_i^{-1} \Lambda x_i} = \sum_{\Gamma} H_\Gamma s_\Gamma$$

*where the sum on the right hand side runs over all maximal orders of $D$. In particular $S(\Lambda)$ is independent of the choice of maximal order $\Lambda$ in $D$.*

*Proof.* We have already seen that $\sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}} = \sum_{i=1}^{h_D} s_{x_i^{-1} \Lambda x_i}$ holds. Note that $H_\Gamma$ is the number of two-sided $\Gamma$-ideals. In particular the number of left $\Lambda$-ideals, whose right order is $\Gamma$, equals $H_\Gamma$. So the equality follows. We see at once that $S(\Lambda) = S(\Gamma)$ for all maximal orders $\Lambda$, $\Gamma$ in $D$.

□

### 5.2.3 The case $h_k = h_k^+$

To motivate the general theorem at the end of this chapter, we will now give an explicit formula for $c(SL_2(\Lambda))$ for a maximal order $\Lambda$ in a totally definite quaternion algebra $D$ over a number fields $k$, satisfying $h_k = h_k^+$. This already solves the cusp number problem for a large class of totally definite quaternion algebras over algebraic number fields. As usual, we assume that $D$ is totally definite, so $k$ is totally real. First we need the following lemma:

**Lemma 5.4.** *If $h_k = h_k^+$ then $(\mathcal{O}_k^*)^+ = (\mathcal{O}_k^*)^2$.*

*Proof.* Denote by $\sigma_1, \dots \sigma_n$ the different embeddings $k \hookrightarrow \mathbb{R}$, where $n = [k : \mathbb{Q}]$. Let $\{e_1, \dots e_{n-1}\}$ be a set of fundamental units in $\mathcal{O}_k^*$, chosen such that $\sigma_n(e_i) > 0$ for $i = 1, \dots n-1$ (this is always possible, we can multiply with $(-1)$ if necessary). Define $\psi : \mathcal{O}_k^* \to (\mathbb{Z}/2\mathbb{Z})^{n-1}$ via

$$\psi(e) = (\mathrm{sg}(\sigma_1(e)), \dots, \mathrm{sg}(\sigma_{n-1}(e))) \subset (\mathbb{Z}/2\mathbb{Z})^{n-1}$$

for any element $e$ in $\mathcal{O}_k^*$ with $\sigma_n(e) > 0$.

Here $\mathrm{sg} : \mathbb{R}^* \to \{0, 1\}$ is defined by

$$\mathrm{sg}(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}.$$

Then $h_k = h_k^+$ holds, if and only if $\psi$ is surjective. We clearly have $\psi(xy) = \psi(x) + \psi(y)$ for all $x, y \in \mathcal{O}_k^*$.

Now suppose $\prod_{i=1}^{n-1} e_i^{r_i}$ is totally positive. This is the case if and only if

$$\psi\left(\prod_{i=1}^{n-1} e_i^{r_i}\right) = \sum_{i=1}^{n-1} r_i \psi(e_i) = 0.$$

Now $\psi$ is surjective if and only if the elements $\psi(e_1), \dots \psi(e_{n-1})$ form a basis of $(\mathbb{Z}/2\mathbb{Z})^{n-1}$. Any nontrivial relation of the above form forces a linear dependence of the $\psi(e_1), \dots, \psi(e_{n-1})$ in the vector space $(\mathbb{Z}/2\mathbb{Z})^{n-1}$, which is a contradiction. So all $r_i$ must be even and the result is established.

□

**Theorem 5.3.** *Let $D$ be a totally definite quaternion algebra over a number field $k$. If $h_k = h_k^+$ then $c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}} = S(\Lambda)$ for any maximal order $\Lambda$. In particular the cusp number is independent of the choice of maximal order $\Lambda$ in $D$.*

*Proof.* Let $D_{\mathbb{A}}^* = \bigcup_{i=1}^{h_D} U_\Lambda x_i D^*$ be a decomposition of $D_{\mathbb{A}}^*$ corresponding to the isomorphism classes of left $\Lambda$-ideals and let $\phi$ denote the projection map $[x, y] \mapsto [x]$, as in the proof of Theorem 5.1.

We will prove that for all $n, n' \in N^1(D_{\mathbb{A}}^*)$ if $x_i^{-1} n' \in U_\Lambda x_i^{-1} n D^*$, i.e. $x_i^{-1} n' = u x_i^{-1} n d$ for some $u \in U_\Lambda$, $d \in D^*$, there exist $n_u \in N^1(U_\Lambda)$ and $n_d \in N^1(D)$ such that

$$x_i = n_u u^{-1} x_i d^{-1} n_d.$$

In other words we will prove that the cardinality of the fiber $\phi^{-1}([x_i])$ equals $s_{x_i \Lambda x_i^{-1}}$.

Suppose $n' = x_i u x_i^{-1} n d$ for some $u \in U_\Lambda$, $d \in D^*$. Then $\mathrm{nrd}(ud) = 1$. On the other hand $\mathrm{nrd}(u_v) \in \mathcal{O}_v^*$ for all finite places $v$ and $\mathrm{nrd}(d) \in (k^*)^+$. Hence $\mathrm{nrd}(d) \in (\mathcal{O}_k^*)^+$. By the lemma above, $(\mathcal{O}_k^*)^+ = (\mathcal{O}_k^*)^2$, hence there exists a unit $e \in \mathcal{O}_k^*$ such that $\mathrm{nrd}(ed) = 1$. Then $\mathrm{nrd}(ue^{-1}) = 1$. Let $n_d \in N^1(D)$ and $n_u \in N^1(U_\Lambda)$ be such that $d^{-1} = e n_d$ and $u^{-1} = n_u e^{-1}$. Then

$$x_i^{-1} N^1(U_\Lambda) u^{-1} x_i d^{-1} N^1(D) = x_i^{-1} N^1(U_\Lambda) n_u e^{-1} x_i e n_d N^1(D).$$

As $e \in k^*$ commutes with every element in $D_{\mathbb{A}}^*$ the claim follows. □

**Corollary 5.2.** *Let $D$ be a totally definite quaternion algebra over an algebraic number field $k$ and let $\Lambda$ be a maximal order in $D$. If $h_k^+ = 1$, then $c(SL_2(\Lambda)) = h_D^2$.*

*Proof.* If $h_k^+ = 1$ then $h_k = 1 = h_k^+$ so we may apply the previous theorem. Then $h_k^+ = 1$ implies that the class group $\mathrm{Cl}(D)$ of $D$ is trivial, so $s_\Lambda = h_D$ for all maximal orders $\Lambda$. $\square$

Note that if $h_k = h_k^+$, the upper and lower bound of Theorem 5.1 coincide:

**Lemma 5.5.** *Let $D$ be a totally definite quaternion algebra over an algebraic number field $k$ satisfying $h_k = h_k^+$. Then for any maximal order $\Lambda$ in $D$ the equality*

$$s_\Lambda = n_\Lambda$$

*holds.*

*Proof.* Let $\tau : N^1(U_\Lambda)\backslash N^1(D_{\mathbb{A}}^*)/N^1(D) \to U_\Lambda \backslash U_\Lambda N^1(D_{\mathbb{A}}^*)D^*/D^*$ be defined as

$$N^1(U_\Lambda)nN^1(D) \mapsto U_\Lambda n D^*.$$

Then $\tau$ is well defined and surjective. We will show that $\tau$ is injective: Suppose $n' = und$ for $u \in U_\Lambda$, $d \in D^*$. Then $\mathrm{nrd}(ud) = 1$, hence $\mathrm{nrd}(d) \in (k^+)^* \cap U_{\mathcal{O}_k} = (\mathcal{O}_k^+)^* = (\mathcal{O}_k^*)^2$. There exists $e \in \mathcal{O}_k^*$ such that $e^2 = \mathrm{nrd}(d)$. In other words $d \in e \cdot N^1(D)$. A similar argument gives $u \in N^1(U_\Lambda)e^{-1}$. Now $\mathcal{O}_k^*$ commutes with $D_{\mathbb{A}}^*$, hence $n' = und = n_u e^{-1} n e n_d = n_u n n_d$ for some $n_u \in N^1(U_\Lambda)$, $n_d \in N^1(D)$. So $\tau$ is injective. $\square$

**Example 7.** Let $k$ be the real quadratic field $\mathbb{Q}(\sqrt{10})$. Then by Dirichlet's Unit Theorem $\mathcal{O}_k^*$ is a free $\mathbb{Z}$-module of rank 1. It can be generated by $e = 3 + \sqrt{10}$. Note that $e$ is not totally positive, hence $h_k^+ = h_k = 2$. Let $D = Q(-1, -1|k)$. Using the computer algebra system Magma we see that $D$ has exactly four types of maximal orders given by:

$$\begin{aligned}
\Lambda_1 =& \langle 1, 1+i, \frac{\sqrt{10}}{2}(1+i), 1+j, \frac{\sqrt{10}}{2}(1+j), \frac{1}{2}(1+i+j+ij)\rangle \\
\Lambda_2 =& \langle 1, i, \frac{1}{2}(\sqrt{10}+i+j), \frac{1}{2}(1+\sqrt{10}i+ij)\rangle \\
\Lambda_3 =& \langle 1, \frac{1}{2}(2+\sqrt{10}+(1+2\sqrt{10})i+j), \frac{8-\sqrt{10}}{4}(2+\sqrt{10}+(1+2\sqrt{10})i+j), \\
& \frac{1}{2}(1+(2+\sqrt{10})i+ij), (8+9\sqrt{10})i, \sqrt{10}i\rangle \\
\Lambda_4 =& \langle 1, 2i, \frac{1}{2}(2+\sqrt{10}+(2\sqrt{10}-3)i+j), \frac{10-\sqrt{10}}{4}(2+\sqrt{10}+(2\sqrt{10}-3)i+j), \\
& \frac{1}{2}(2+\sqrt{10}+\frac{1}{2}(8+3\sqrt{10})i+\frac{1}{2}\sqrt{10}j+ij), \\
& \frac{10+5\sqrt{10}}{4}(2+\sqrt{10}+\frac{1}{2}(8+3\sqrt{10})i+\frac{1}{2}\sqrt{10}j+ij)\rangle
\end{aligned}$$

Further, each maximal order is its unique two-sided ideal, hence $h_D = 4$. Some easy calculations show that $s_{\Lambda_i} = 2$ for $i = 1, \dots 4$. Hence by the above theorem we see that

$$c(SL_2(\Lambda)) = 8.$$

Note that we have now found an example with $c(SL_2(\Lambda)) \neq h_D, h_D^2$.

### 5.2.4 Surjective norms

Let $D$ be a totally definite quaternion algebra over an algebraic number field $k$. Let $\{x_i\}_{i=1,\dots,h_D}$ be a set of representatives of double cosets $U_\Lambda \backslash D_{\mathbb{A}}^*/D^*$ in $D_{\mathbb{A}}^*$. We will now prove a generalization of Theorem 5.3.

**Theorem 5.4.** *If* $\mathrm{nrd}(\Lambda^*) = (\mathcal{O}_k^*)^+$ *for all maximal orders* $\Lambda$ *in* $D$ *then* $c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}} = S(\Lambda)$. *The cusp number* $c(SL_2(\Lambda))$ *is independent of the choice of maximal order* $\Lambda$ *in* $D$.

*Proof.* We use the same notation as in the proof of Theorem 5.1. We want to show that $|\phi^{-1}([x_i])| = s_{x_i \Lambda x_i^{-1}}$ for $1 \leq i \leq h_D$.

Suppose $n = x_i u x_i^{-1} n' u$ for some $u \in U_\Lambda$, $d \in D^*$. We need to find $n_u \in N^1(U_\Lambda)$, $n_d \in N^1(D)$ such that

$$\begin{pmatrix} x_i & \\ & x_i^{-1} n \end{pmatrix} = \begin{pmatrix} u^{-1} n_u & \\ & u \end{pmatrix} \cdot \begin{pmatrix} x_i & \\ & x_i^{-1} n' \end{pmatrix} \cdot \begin{pmatrix} d^{-1} n_d & \\ & d \end{pmatrix}.$$

By construction we have $\mathrm{nrd}(ud) = 1$. On the other hand $\mathrm{nrd}(u_v) \in \mathcal{O}_v^*$ for all finite places $v$ and $\mathrm{nrd}(d) \in (k^*)^+$. Hence $\mathrm{nrd}(u), \mathrm{nrd}(d) \in (k^*)^+ \cap \bigcap_{v \in V_f} \mathcal{O}_v^* = (\mathcal{O}_k^*)^+$. We have assumed that $\mathrm{nrd}(\Gamma^*) = (\mathcal{O}_k^*)^+$ for any maximal order $\Gamma$ of $D$. Hence there exists an element $\lambda$ in the unit group of the maximal order $x_i^{-1} \Lambda x_i$ such that $\mathrm{nrd}(\lambda^{-1}) = \mathrm{nrd}(d)$. Therefore $d\lambda \in N^1(D)$. Say $d = \lambda^{-1} n_d$. Further $x_i \lambda x_i^{-1} \in U_\Lambda$ hence $u$ can be written as $u = n_u x_i \lambda x_i^{-1}$ for some $n_u \in N^1(U_\Lambda)$. Using the first line of the equation above, it remains to show that

$$x_i \in N^1(U_\Lambda) n_u x_i \lambda x_i^{-1} x_i \lambda^{-1} n_d N^1(D)$$

or equivalently that

$$1 \in N^1(U_{x_i^{-1} \Lambda x_i}) N^1(D)$$

which is trivial. This proves the claim. $\square$

**Remark** In this proof we have actually shown that $n_\Lambda = s_\Lambda$.

**Example 8.** Let $k = \mathbb{Q}(\sqrt{3})$, then $2+\sqrt{3}$ is the fundamental unit of $\mathcal{O}_k^*$ and so $h_k^+ = 2$ and $h_k = 1$. Let $D = Q(-1,-1|k)$. There are two types of maximal orders given by

$$\begin{aligned}\Lambda_1 =& \langle 1, i, \frac{1}{2}(\sqrt{3}i + j), \frac{1}{2}(\sqrt{3} + ij)\rangle \\ \Lambda_2 =& \langle 1, (1+\sqrt{3})i, \frac{1}{4}(1+\sqrt{3})(2-\sqrt{3}i + j), \frac{1}{2}(2+\sqrt{3}i + ij)\rangle\end{aligned}$$

Further we have $h_D = 2$ and we see that

$$\mathrm{nrd}(i + \frac{1}{2}(\sqrt{3}i + j)) = \left(1 + \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4} = 2+\sqrt{3}$$

$$\mathrm{nrd}\left(\frac{1}{4}(1+\sqrt{3})(2-\sqrt{3}i + j)\right) = \frac{2+\sqrt{3}}{8}(4+3+1) = 2+\sqrt{3}$$

Hence $\mathrm{nrd}(\Lambda_i^*) = (\mathcal{O}_k^*)^+$ for $i = 1, 2$. The fact that the reduced norm map maps to $\mathcal{O}_k^*$ shows that those elements are units. Using that the map $\mathrm{nrd}: LF_1(\Lambda) \to \mathrm{Cl}(\mathcal{O}_k)^+$ has to be surjective, we see that $s_{\Lambda_i} = 1$ for $i = 1, 2$. This gives

$$c(SL_2(\Lambda_1)) = c(SL_2(\Lambda_2)) = 2$$

and we also see that both orders have the cancellation property.

## 5.3 General result

Let $\Lambda$ be a maximal order in a totally definite quaternion algebra $D$ over a totally real number field $k$. Let $\{x_i\}_{i=1,\dots,h_D}$ be a set of representatives of double cosets in $D_\mathbb{A}^*$ such that $D_\mathbb{A}^* = \bigcup_{i=1}^{h_D} U_\Lambda x_i D^*$. For any maximal order $\Lambda$ we set

$$O_\Lambda = (\mathcal{O}_k^+)^* / \mathrm{nrd}(\Lambda^*).$$

This is a finite group, as $\mathcal{O}_k^* \subset \Lambda^*$. We denote its cardinality by $o_\Lambda$. Let $\Gamma$ be another maximal order in $D$. We set

$$o_{\Lambda,\Gamma} := [(\mathcal{O}_k^+)^* : (\mathrm{nrd}(\Lambda^*) \cdot \mathrm{nrd}(\Gamma^*))].$$

It is clear that $o_{\Lambda,\Gamma} \leq o_\Lambda$.

For any maximal order $\Lambda$ in $D$, let $n_{\Lambda,j}$ be a set of representatives of the double cosets $U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$. For the sake of simplicity we write

$$n_{i,j} := n_{x_i \Lambda x_i^{-1},j}.$$

Now we are able to show:

**Theorem 5.5.** *The cusp number of $SL_2(\Lambda)$ for a maximal order $\Lambda$ in a totally definite quaternion algebra $D$ equals*

$$c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} \sum_{j=1}^{s_{x_i\Lambda x_i^{-1}}} o_{x_i^{-1}\Lambda x_i, n_{i,j}^{-1} x_i \Lambda x_i^{-1} n_{i,j}}. \qquad (\circ)$$

*Proof.* We use the same notation as in the proof of Theorem 5.1. Let $n \in N^1(D^*_{\mathbb{A}})$ be fixed. Given $n'$ in the class $x_i U_\Lambda x_i^{-1} n D^*$ of $n$, we want to determine whether $[x_i, x_i^{-1}n'] = [x_i, x_i^{-1}n]$. Every $n'$ in $x_i U_\Lambda x_i^{-1} n D^*$ can be written in the form $n' = x_i u x_i^{-1} n d'$, with $u \in U_\Lambda$, $d \in D^*$, satisfying $\mathrm{nrd}(ud) = 1$. Now $n$ and $n'$ determine the same class if and only if

$$\begin{pmatrix} x_i & \\ & x_i^{-1}(x_i u x_i^{-1} n d) \end{pmatrix} \in \begin{pmatrix} u^{-1}N^1(U_\Lambda) & \\ & u \end{pmatrix} \begin{pmatrix} x_i & \\ & x_i^{-1} n \end{pmatrix} \begin{pmatrix} N^1(D^*)d^{-1} & \\ & d \end{pmatrix}.$$

This holds, if and only if

$$1 \in (x_i^{-1} u^{-1} x_i) x_i^{-1} N^1(U_\Lambda) x_i N^1(D^*) d^{-1},$$

which is true if and only if

$$x_i^{-1} u x_i d \in N^1(U_{x_i^{-1}\Lambda x_i}) N^1(D^*).$$

This implies that, up to an element in $N^1(D^*)$, $d$ is contained in $U_{x_i^{-1}\Lambda x_i} \cap D^*$, so, in particular, $\mathrm{nrd}(d) \in \mathrm{nrd}((x_i^{-1}\Lambda x_i)^*)$.

More generally one sees that for $u, u' \in U_\Lambda$, $d, d' \in D^*$ the classes $[x_i, x_i^{-1}(x_i u x_i^{-1} n d)]$ and $[x_i, x_i^{-1}(x_i u' x_i^{-1} n d')]$ are equal if and only if

$$\mathrm{nrd}(d^{-1}d') \in \mathrm{nrd}((x_i^{-1}\Lambda x_i)^*).$$

So the two classes coincide, if and only if $\mathrm{nrd}(d)$ and $\mathrm{nrd}(d')$ are in the same class in $O_\Lambda$.

However one has to be very careful here. Suppose $y \in \mathrm{nrd}(n^{-1}x_i\Lambda x_i^{-1}n)^*$, with $\lambda \in (n^{-1}x_i\Lambda x_i^{-1}n)^*$ such that $\mathrm{nrd}(\lambda) = x$. Then $u = n\lambda n^{-1} \in U_{x_i\Lambda x_i^{-1}}$ and so

$$u^{-1}n\lambda = n$$

even though there is no way to decide, whether or not $y \in \mathrm{nrd}(x_i^{-1}\Lambda x_i)^*$. It is also evident that if $n = und$, for some $u \in x_iU_\Lambda x_i^{-1}$, $d \in D^*$, it follows that $d \in (n^{-1}x_i\Lambda x_i^{-1}n)^*$, so only elements in $\mathrm{nrd}((n^{-1}x_i\Lambda x_i^{-1}n)^*)$ can appear additionally.

Let $n_{i,j}$ be a set of representatives of $U_{x_i\Lambda x_i^{-1}}\backslash U_{x_i\Lambda x_i^{-1}}N^1(D^*_{\mathbb{A}})D^*/D^*$ in $N^1(D^*_{\mathbb{A}})$. Then we have just seen that

$$\phi^{-1}([x_i]) = \sum_{j=1}^{s_{x_i\Lambda x_i^{-1}}} o_{x_i^{-1}\Lambda x_i, n_{i,j}^{-1}x_i\Lambda x_i^{-1}n_{i,j}}.$$

It is easy to see that $o_{x_i^{-1}\Lambda x_i, n_{i,j}^{-1}x_i\Lambda x_i^{-1}n_{i,j}}$ is independent of the choice of representatives $n_{i,j}$, but this is not clear for the choice of $x_i$. However in the course of the proof that $s_{x_i\Lambda x_i^{-1}}$ is independent of the choice of representatives, we gave an explicit bijection between the double cosets. So a different choice of $x_i$ shuffles the representatives $n_{i,j}$ and the above sum is independent of all choices made. This proves our claim.

□

We will now give a different description of $O_\Lambda$. It implies that $o_\Lambda \leq n_\Lambda$.

**Lemma 5.6.** *There is a bijection between $O_\Lambda$ and $N^1(U_\Lambda)\backslash N^1(U_\Lambda D^*)/N^1(D)$.*

*Proof.* We define a map $\psi$ from $\mathcal{O}_\Lambda$ to $N^1(U_\Lambda)\backslash N^1(U_\Lambda D^*)/N^1(D)$ as follows: let $x$ be a representative of a class of $\mathcal{O}_\Lambda$. As $x \in k^+ - \{0\}$ there exists an element $d \in D^*$ such that $\mathrm{nrd}(d) = x$ and further as $x^{-1} \in \mathcal{O}_v^*$ for all finite places $v$, there exists an element $u \in U_\Lambda$ such that $\mathrm{nrd}(u) = x^{-1}$. Set $\psi(x) = N^1(U_\Lambda)udN^1(D)$. We have to prove that $\psi$ is well defined.

First, $\psi(x)$ is independent of the choice of $u$, $d$: Let $d' \in D^*$ another element such that $\mathrm{nrd}(d) = \mathrm{nrd}(d') = x$. Then $d(d')^{-1} \in N^1(D)$ and there exists $n_d \in N^1(D)$ such that $d' = dn_d$. Similarly for $u, u' \in U_\Lambda$ with $\mathrm{nrd}(u) = \mathrm{nrd}(u') = x^{-1}$ we find $n_u \in N^1(U_\Lambda)$ such that $u' = n_uu$. Then

$$N^1(U_\Lambda)u'd'N^1(D) = N^1(U_\Lambda)n_uudn_dN^1(D) = N^1(U_\Lambda)udN^1(D)$$

and the claim follows.

Second, $\psi(x)$ is independent of the choice of representative $x$ in the class $x\,\mathrm{nrd}(\Lambda^*)$: Let $y \in \mathrm{nrd}(\Lambda^*)$ and let $\lambda \in \Lambda^*$ be such that $\mathrm{nrd}(\lambda) = y$. Let $d \in D^*$ be such that $\mathrm{nrd}(d) = x$ and $u \in U_\Lambda$ such that $\mathrm{nrd}(u) = x^{-1}$. Then $\lambda d \in D^*$ and $\mathrm{nrd}(d\lambda) = xy$, likewise $u\lambda^{-1} \in U_\Lambda$ and $\mathrm{nrd}(\lambda^{-1}u) = y^{-1}x^{-1}$. Hence

$$\psi(xy) = N^1(U_\Lambda)u\lambda^{-1}\lambda d N^1(D) = \psi(x).$$

We define an inverse map $\psi^{-1}$ to $\psi$ by setting $\psi^{-1}(ud) = \mathrm{nrd}(d)$. We have to show that $\psi^{-1}$ is well defined. Suppose we have another representative in $N^1(U_\Lambda)udN^1(D)$, so there is an equality $u'd' = ud$ for some $u' \in U_\Lambda$, $d' \in D^*$. Then

$$(u')^{-1}u = d'd^{-1} \in U_\Lambda \cap D^* = \Lambda^*$$

Therefore $\mathrm{nrd}(d') \in \mathrm{nrd}(d)\,\mathrm{nrd}(\Lambda^*)$ and the claim follows.

□

**Remark** If $D$ is not totally definite, then we know that $N^1(D^*_{\mathbb{A}}) = N^1(U_\Lambda)N^1(D)$ for any maximal order $\Lambda$. In particular by adjusting the lemma above one shows that for any maximal order $\Lambda$ of $D$ we have

$$\mathrm{nrd}(\Lambda^*) = (\mathcal{O}_k^*) \cap \mathrm{nrd}(D^*).$$

So for every unit $x \in (\mathcal{O}_k^*)^D = \mathcal{O}_k^* \cap k^D$ and every maximal order $\Lambda$ there exists $\lambda \in \Lambda^*$ such that $\mathrm{nrd}(\lambda) = x$.

**Remark** The above formula is also valid for central simple algebras $D$, which are not totally definite. Indeed by the remark above, $o_{\Lambda,\Gamma} = 1$ for any two maximal orders in $D$ and every order $\Lambda$ has the cancellation property, so $s_\Lambda = 1$. This gives again the formula $c(SL_2(\Lambda)) = h_D$.

This general formula ($\circ$) in Theorem 5.5 looks rather complicated. Further it is not clear, whether $c(SL_2(\Lambda))$ is independent of the choice of maximal order. In order to simplify the formula, we introduce some more notation. Let

$$r_\Lambda = \sum_\Gamma o_\Gamma$$

where the sum runs over all right orders of isomorphism classes of stable locally free left $\Lambda$-ideals, so there are $s_\Lambda$ many terms. More precisely if $n_1, \dots n_{s_\Lambda}$ is a set of representatives of $U_\Lambda \backslash U_\Lambda N^1(D_{\mathbb{A}}^*)D^*/D^*$ then

$$r_\Lambda = \sum_{i=1}^{s_\Lambda} o_{n_i^{-1}\Lambda n_i}.$$

It is clear that if $\Lambda$ and $\Gamma$ are stable isomorphic, then $r_\Lambda = r_\Gamma$. For any maximal order $\Delta$ , we set

$$r_{\Delta,\Lambda} = \sum_{\Gamma} o_{\Delta,\Gamma}.$$

We have just shown that:

**Theorem 5.6.** *Let $x_1, \dots x_{h_D} \in D_{\mathbb{A}}^*$ be a set of representatives of $U_\Lambda \backslash D_{\mathbb{A}}^*/D^*$. Let $\Lambda$ be a maximal order in $D$. The cusp number of $SL_2(\Lambda)$ for a maximal order $\Lambda$ is given by*

$$c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} r_{x_i^{-1}\Lambda x_i, x_i^{-1}\Lambda x_i} = \sum_{\Gamma} H_\Gamma r_{\Gamma,\Gamma}.$$

*Here the sum on the right hand side runs over all maximal orders $\Gamma$ of $D$. In particular for any two maximal orders $\Lambda_1$, $\Lambda_2$ in $D$ we have*

$$c(SL_2(\Lambda_1)) = c(SL_2(\Lambda_2)),$$

*so the cusp number is independent of the choice of maximal order in $D$.*

*Proof.* We have already shown that

$$c(SL_2(\Lambda)) = \sum_{i=1}^{h_D} r_{x_i\Lambda x_i^{-1}, x_i^{-1}\Lambda x_i}$$

and by the argumentation above, $x_i\Lambda x_i^{-1}$ and $x_i^{-1}\Lambda x_i$ are stable isomorphic. This proves the first equality. Each order $\Gamma$ is the right order of exactly $H_\Gamma$ left $\Lambda$-ideals, hence it appears exactly $H_\Gamma$-many times in the sum. This shows the second equality. The last assumption is now obvious. $\square$

Note that in general $c(SL_2(\Lambda))$ may be different from the upper and the lower bound given in Theorem 5.1. Indeed, one shows:

**Theorem 5.7.** *Let $\Lambda$ be a maximal order in a totally definite quaternion algebra $D$ over an algebraic number field $k$. Let $\{n_i\}_{i=1,\dots,s_\Lambda}$ be a set of representatives of the* *double quotient* $U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$ *in* $N^1(D^*_{\mathbb{A}})$. *Then*

$$n_\Lambda = \sum_{i=1}^{s_\Lambda} o_{n_i^{-1}\Lambda n_i} = r_\Lambda.$$

*Proof.* Let $\phi$ denote the natural map

$$N^1(U_\Lambda)\backslash N^1(D^*_{\mathbb{A}})/N^1(D^*) \to U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$$
$$N^1(U_\Lambda)nN^1(D^*) \mapsto U_\Lambda n D^*$$

Let, as usual, $[n]$ denote the class $U_\Lambda n D^*$ in $U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$. It is clear that $n_\Lambda = \sum_{i=1}^{s_\Lambda} |\phi^{-1}([n_i])|$, so we calculate the cardinality of each fiber $\phi^{-1}([n_i])$.

Let $n \in N^1(D^*_{\mathbb{A}})$ be a representative of a double coset $N^1(U_\Lambda)nN^1(D^*)$ in $N^1(U_\Lambda)\backslash N^1(D^*_{\mathbb{A}})/N^1(D^*)$. Then $n \in \phi^{-1}([n_i])$ if and only if there exist $u \in U_\Lambda, d \in D^*$ such that $n = un_id$. We see that $N^1(U_\Lambda)nN^1(D^*) = N^1(U_\Lambda)n_iN^1(D^*)$ if and only if

$$un_id \in N^1(U_\Lambda)n_iN^1(D^*)$$

which holds if and only if

$$n_i^{-1}un_id \in N^1(U_{n_i^{-1}\Lambda n_i})N^1(D).$$

This is true if and only if $\mathrm{nrd}(d) \in \mathrm{nrd}((n_i^{-1}\Lambda n_i)^*)$. If $u'n_id'$ is a different choice of elements, with $u' \in U_\Lambda, d' \in D^*$ satisfying $u'n_id' = n = un_id$, it follows that

$$(d(d')^{-1}) = n_i^{-1}u^{-1}u'n_i \in U_{n_i^{-1}\Lambda n_i} \cap D^* = (n_i^{-1}\Lambda n_i)^*.$$

This proves our claim. □

## 5.4 Examples

In this section we will give some examples of cusp numbers $c(SL_2(\Lambda))$ for a maximal order $\Lambda$ in a totally definite quaternion algebra. We have calculated them, using the computer algebra systems Magma, Sage and Mathematica. The first example can be done using Mathematica only, if the generators of the ideals are known.

Isomorphism classes of maximal orders, left ideal classes and norms of ideals are calculated using the computer algebra system Magma. To find the generator of a principal ideal, we have used Sage.

### 5.4.1 The algebra $Q(-1,-1|\mathbb{Q}(\sqrt{6}))$

Let $k = \mathbb{Q}(\sqrt{6})$. The ring of integers $\mathcal{O}_k$ is a principal ideal domain, hence $h_k = 1$. A fundamental unit of $\mathcal{O}_k^*$ is given by

$$e = 5 + 2\sqrt{6}$$

which is totally positive. Hence the narrow class number of $k$ equals $h_k^+ = 2$. Let $D = Q(-1,-1\,|\,k)$. It can be shown, using Magma, that $D$ has three types of maximal orders given by:

$$\begin{aligned}
\Lambda_1 &= \langle 1, \frac{1}{2}(2+\sqrt{6})(1+i), \frac{1}{2}(2+\sqrt{6})(1+j), \frac{1}{2}(1+i+j+ij)\rangle \\
\Lambda_2 &= \langle 1, i, \frac{1}{2}(\sqrt{6}+i+j), \frac{1}{2}(1+\sqrt{6}i+ij)\rangle \\
\Lambda_3 &= \langle 1, (2+\sqrt{6})i, \frac{1}{4}(2+\sqrt{6})(\sqrt{6}+i+j), \frac{1}{2}(1+\sqrt{6}i+ij)\rangle
\end{aligned}$$

We will first show that $5+2\sqrt{6}$ is contained in $\mathrm{nrd}(\Lambda_i^*)$ and thus $o_{\Lambda_i} = 1$ for $i = 1,2,3$. Indeed for $\Lambda_1$ we have

$$\mathrm{nrd}\left(\frac{1}{2}(2+\sqrt{6})(1+j)\right) = \frac{1}{2}(4+4\sqrt{6}+6) = 5+2\sqrt{6}.$$

For $\Lambda_2$ we see

$$\begin{aligned}
&\mathrm{nrd}\left((3+\sqrt{6})i + (-2-\sqrt{6})\frac{1}{2}(1+\sqrt{6}i+ij)\right) = \\
&= \left(\frac{-2-\sqrt{6}}{2}\right)^2 + \left(3+\sqrt{6}+\sqrt{6}\frac{-2-\sqrt{6}}{2}\right)^2 + \left(\frac{-2-\sqrt{6}}{2}\right)^2 \\
&= 2\left(\frac{4+4\sqrt{6}+6}{4}\right) + \left(\frac{6+2\sqrt{6}-2\sqrt{6}-6}{2}\right)^2 \\
&= 5+2\sqrt{6}.
\end{aligned}$$

For $\Lambda_3$ note that

$$\mathrm{nrd}\left(\frac{1}{4}(2+\sqrt{6})(\sqrt{6}+i+j)\right) = \frac{1}{16}(4+4\sqrt{6}+6)(6+1+1) = 5+2\sqrt{6}.$$

It remains to calculate $s_{\Lambda_i}$. We have $h_D = 3$ hence the left ideal classes of an order $\Lambda$ are given by $\Lambda\Lambda_1$, $\Lambda\Lambda_2$ and $\Lambda\Lambda_3$. It can be shown explicitly (using Magma or Mathematica) that one obtains the following table

| Order | ideal in $D$ | ideal in $k$ |
|---|---|---|
| $\Lambda_1$ | $\Lambda_1$ | $(1)$ |
| | $\Lambda_1\Lambda_2$ | $\left(1+\frac{\sqrt{6}}{2}\right)$ |
| | $\Lambda_1\Lambda_3$ | $\left(\frac{1}{2}\right)$ |
| $\Lambda_2$ | $\Lambda_2\Lambda_1$ | $\left(1+\frac{\sqrt{6}}{2}\right)$ |
| | $\Lambda_2$ | $(1)$ |
| | $\Lambda_2\Lambda_3$ | $\left(1+\frac{\sqrt{6}}{2}\right)$ |
| $\Lambda_3$ | $\Lambda_3\Lambda_1$ | $\left(\frac{1}{2}\right)$ |
| | $\Lambda_3\Lambda_2$ | $\left(1+\frac{\sqrt{6}}{2}\right)$ |
| | $\Lambda_3$ | $(1)$ |

Hence

$$s_{\Lambda_1} = 2$$
$$s_{\Lambda_2} = 1$$
$$s_{\Lambda_3} = 2$$

This shows that

$$c(SL_2(\Lambda_1)) = c(SL_2(\Lambda_2)) = c(SL_2(\Lambda_3)) = 5.$$

This is our first example of a cusp number $c(SL_2(\Lambda))$ which is not an integral multiple of the class number $h_D$.

### 5.4.2 The algebra $Q(-1,-1|\mathbb{Q}(\sqrt{219}))$

This example is calculated using a student version of the computer algebra system Magma. Some parts of the calculation are done with Sage. This subsection is organized as follows: We first give the general background information. Then we will explain in some detail how the calculations, resp. what kind of calculations, we have done and give the results. At the end we will write down the explicit source code for Magma.

#### Background

Let $k = \mathbb{Q}(\sqrt{219})$. The integer $219 = 3 \cdot 73$ is the smallest positive number having the following two properties:

1. The class group of $\mathcal{O}_k$ does not have exponent 2, so there is more than one stable isomorphism class of maximal orders.

2. The class number $h_k$ is not equal to the narrow class number $h_k^+$, so that the upper and lower bound do not necessarily coincide.

This integer 219 is found using Mathematica. Calculating

```
For [i = 1, i < 500, i++,
If[NumberFieldClassNumber[Sqrt[i]] >= 4, Print[i] Abort[],]]
```

and adapting the lower bound, we search for integers $z$ such that the class number of $\mathbb{Q}(\sqrt{z})$ is greater or equal to 4. Then we check using

```
NumberFieldFundamentalUnits[Sqrt[z]]
```

whether the fundamental unit is totally positive or not. After that we use Magma

```
P<x> := PolynomialRing(Rationals());
F<b> := NumberField(x^2-219);
O:=MaximalOrder(F);
ClassGroup(O);
```

to calculate the class group of the field.

The class group of $\mathbb{Q}(\sqrt{219})$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. The fundamental unit of $\mathcal{O}_k$ is given by

$$e = 74 + 5\sqrt{219}$$

and $74 - 5\sqrt{219} \approx 0.006757$, hence $e$ is totally positive. Therefore the narrow class group of $\mathcal{O}_k$ is isomorphic to $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and there are $4 = [G : G^2]$ stable isomorphism classes of maximal orders in $D$.

## Calculations

In this section we will use the same notation as in the code later on. Let $D = Q(-1, -1 \,|\, k)$. Then $D$ is a totally definite quaternion algebra. It has 84 types of maximal orders, which we denote by $L[1], \ldots L[84]$.

For the sake of simplicity we write $o_i := o_{L[i]}$ for the index $(\mathcal{O}_k^+)^*/(\mathrm{nrd}(\Lambda^*))$. Each $o_i$ is either equal to 1 or 2. Using the Magma-command Units and applying the norm map to these elements we obtain a list $os$ calculating the invariants. In this notation we have $os[i] = o_i$.

We will write $S = L[1]$, this is the first maximal order calculated. The remaining $L[i]$ are found using the left $S$-ideal classes. Any maximal order has 296 left ideal classes. The left ideal classes of $S$ are denoted $ideals[1], \dots ideals[296]$.

Two maximal orders $\Lambda, \Gamma$ are called stable isomorphic, if there exists an $n \in N^1(D^*_{\mathbb{A}})$ with $\Gamma = n\Lambda n^{-1}$. It is clear that stable isomorphism is an equivalence relation on the isomorphism classes of maximal orders. To calculate the stable isomorphism classes we use the function called $Norc$. It calculates for an order $L[i]$ (respectively the left ideal classes of an order) the right orders (with multiplicity) of the stable locally free ideal classes. We see that we have exactly $4 = h^+_k/2$ stable isomorphism classes of maximal orders which are

$$\begin{aligned} &\{L[1],\, L[2], \dots, L[21]\} \\ &\{L[22],\, L[23], \dots, L[42]\} \\ &\{L[43],\, L[44], \dots, L[63]\} \\ &\{L[64],\, L[65], \dots, L[84]\} \end{aligned}$$

This is done as follows: Using Magma we calculate the norm of a left ideal class and check, whether it is a principal ideal or not. We now have to find out, whether the generator of a given principal ideal is totally positive or not. Unfortunately we could not find an appropriate Magma-command, calculating a single generator of a given principal ideal. However only a short list of principal ideals appears as the norm of a left ideal of a maximal order. Using the program Sage, we calculated the generators of these few ideals (9 in total, not counting integral multiples) via the command

```
k.<a>=NumberField(x^2-219)
O=k.maximal_order()
I=O.ideal(32+3527936 *a, 12500000*a)
I.gens_reduced()
```

We define two lists of ideals, which we call *totpos* resp. *notpos*, accordingly whether the generators are totally positive or not. Then we use Magma and list the right order (respectively its index) of the ideals, having their norm in in the list *totpos*. If an ideal is principal and in neither list, it appears in the output as well and we can check it separately.

In adelic notation this is what happens: Let $n_i$ be a set of representatives of $U_\Lambda \backslash U_\Lambda N^1(D^*_{\mathbb{A}})D^*/D^*$. Then the function $Norc$ lists the orders (respectively their indexes) $n_i^{-1}\Lambda n_i$ and gives $s_\Lambda$. We see:

| $i$ | $s_{L[i]}$ |
|---|---|
| $1,\dots 21$ | 37 |
| $22,\dots 42$ | 37 |
| $43,\dots 63$ | 35 |
| $64,\dots 84$ | 39 |

We set

$$\begin{aligned} R_1 &:= Norc(ideals) \\ R_2 &:= Norc(LeftIdealClasses(L[22])) \\ R_3 &:= Norc(LeftIdealClasses(L[43])) \\ R_4 &:= Norc(LeftIdealClasses(L[64])) \end{aligned}$$

and define

$$r_{l,j} := \sum_{L[i]\in R_j} o_{l,i}$$

where the sum respects multiplicities and $j = 1,\dots 4$. Note that $o_{l,i} = o_{L[l],L[i]}$ for the sake of simplicity. We calculate:

$$r_{i,1} = \begin{cases} 37 & \text{for } o_i = 1 \\ 66 & \text{for } o_i = 2 \end{cases}$$

$$r_{i,2} = \begin{cases} 37 & \text{for } o_i = 1 \\ 66 & \text{for } o_i = 2 \end{cases}$$

$$r_{i,3} = \begin{cases} 35 & \text{for } o_i = 1 \\ 62 & \text{for } o_i = 2 \end{cases}$$

$$r_{i,4} = \begin{cases} 39 & \text{for } o_i = 1 \\ 70 & \text{for } o_i = 2 \end{cases}$$

The cusp number is then calculated using the function *cusps*. It equals

$$c(SL_2(L[i])) = 17\,712$$

for $i = 1,\dots 84$.

We have also calculated the upper and lower bound of Theorem 5.1. In adelic notation we have

$$\sum_{i=1}^{296} s_{x_i \Lambda x_i^{-1}} = 10\,968$$
$$\sum_{i=1}^{296} n_{x_i \Lambda x_i^{-1}} = 19\,568.$$

So the cusp number is different from the upper and the lower bound given in Theorem 5.1.

### Source code

We define the number field and $D$:

```
 P<x> := PolynomialRing(Rationals());
F<b> := NumberField(x^2-219);
O:=MaximalOrder(F);
A<x,y,z> := QuaternionAlgebra< F | -1, -1 >;
S:=MaximalOrder(A);
```

Then we calculate the left ideal classes of $S$ and the isomorphism classes of the maximal orders.

```
ideals:= LeftIdealClasses(S);
#ideals;
R:=AssociativeArray();
for x := 1 to #ideals by 1 do
R[x]:=RightOrder(ideals[x]);
end for;
 L:=[*S*];
for j:=1 to #ideals by 1 do
 T := function(n, j)
  if n le 0 then
       return true;
    else
       return $$(n-1, j) and not IsIsomorphic(L[n], R[j]);
    end if;
 end function;
```

```
if T(#L, j) then
    L:=L cat [*R[j]*];
else
end if;
end for;

#L;
```

These are the lists *totpos* and *notpos*, needed in the calculation of the stable isomorphism classes.

```
I0:=ideal<O| 1*O.1>;
I1:=ideal< O| 2*O.1+220496 *O.2, 781250*O.2>;
I2:=ideal < O| O.1+248*O.2, 625*O.2>;
I7:=ideal< O| 16*O.1+2295513968*O.2, 3906250000*O.2>;
I9:=ideal< O| 32*O.1+3527936 *O.2, 12500000*O.2>;

I8:=ideal< O| 8*O.1+16772756984*O.2, 97656250000*O.2>;
I3:=ideal<O| O.1+23*O.2, 50*O.2>;
I4:=ideal< O| 2*O.1+1746*O.2, 625000*O.2>;
I5:=ideal<O| 8*O.1+54006984*O.2, 156250000*O.2>;
I6:=ideal<O| 1*O.1+873*O.2, 31250*O.2>;
In1:=ideal<O| 32*O.1+27936 *O.2, 1000000*O.2>;


totpos:=[*I1, 2*I1, 4*I1, 8*I1, I2, 2*I2, 4*I2, 8*I2, 16*I2, I0,
2*I0, 4*I0, I7, 2*I7, 4*I7, I9, 2*I9 *];

notpos:=[* I3, 2*I3, 4*I3, 8*I3, 16*I3, I4, 2*I4, 4*I4, 8*I4, I5,
  2*I5, 4*I5, I6, 2*I6, 4*I6, In1, 2*In1, I8, 2*I8, 4*I8, 8*I8 *];
```

The ideals are equal to:

```
 J1:=ideal<O| -42*O.2+1396*O.1>;
J2:=ideal<O| 121*O.1+8*O.2>;
J7:=ideal<O| 762656*O.1+48688*O.2>;
J9:=ideal<O| 672*O.2- 22336*O.1>;
```

```
J8:=ideal<O| 64856*O.2 + 374072*O.1>;
J3:=ideal<O| 13*O.1-O.2>;
J4:=ideal<O| 34*O.2 + 358*O.1>;
J5:=ideal<O| -7768*O.2 + 109384*O.1>;
J6:=ideal<O|17*O.2 + 179*O.1>;
Jn1:=ideal<O| 544*O.2 + 5728*O.1>;
```

The stable isomorphism classes are calculated:

```
 findod:=function(T, i)
c:=1;
    repeat
    if  not IsIsomorphic(L[c], RightOrder(T[i])) then
            c:=c+1;
    else break;
    end if;
    until false;
return c ;
end function;

inside :=function (S,  i)
c:=1;
    repeat
    if c le #S and not S[c] eq i then
            c:=c+1;
    else break;
    end if;
    until false;
return c le #S ;
end function;


Norc:=function(S);
Y:=[* *];
 X:=[* *];
c:=0;
for i:=1 to #S by 1 do
    if IsPrincipal(Norm(S[i])) then
      if inside( totpos, Norm(S[i]))  then
```

```
      X := X cat [* findod(S,i) *];
        c:=c+1;
    elif not inside(notpos, Norm(S[i])) then
        Y := Y cat [*i, Norm(S[i])*];
    end if;
    end if;
end for;
return [* c, X, Y*];
end function;

R1:=Norc(ideals);
R1;
R2:=Norc(LeftIdealClasses(L[23]));
R2;
R3:=Norc(LeftIdealClasses(L[43]));
R3;
R4:=Norc(LeftIdealClasses(L[64]));
R4;
```

The output is:

```
 > R1:=Norc(ideals);
> R1;
[* 37, [* 2, 3, 4, 4, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10,
11, 10, 11, 12, 12, 13, 13, 2, 1, 14, 14, 15, 16, 15,
 16, 17, 18, 17, 19, 20, 21, 20, 21 *], [* *] *]
> R2:=Norc(LeftIdealClasses(L[23]));
> R2;
[* 37, [* 22, 23, 24, 24, 36, 36, 38, 34, 34, 33, 33,
 26, 38, 28, 32, 42, 42, 28, 29, 37, 37, 41, 41, 40,
40, 22, 30, 23, 35, 35, 27, 25, 25, 30, 39, 31, 39 *], [* *] *]
> R3:=Norc(LeftIdealClasses(L[43]));
> R3;
[* 35, [* 54, 59, 43, 50, 43, 55, 51, 51, 44, 44, 53,
 52, 53, 58, 58, 57, 57, 52, 61, 63, 63, 48, 60, 48,
56, 56, 54, 46, 46, 47, 47, 62, 45, 45, 49 *], [* *] *]
> R4:=Norc(LeftIdealClasses(L[64]));
> R4;
[* 39, [* 70, 84, 64, 71, 82, 82, 64, 76, 76, 81, 81,
```

```
79, 70, 79, 80, 75, 80, 75, 68, 68, 78, 78, 83, 83, 65,
 67, 65, 67, 72, 69, 69, 73, 73, 77, 77, 66, 66, 74, 71 *], [* *] *]
```

This gives the invariants $o_i$ as a list $os$.

```
 no:=function(M);
X:=[* *];
for i:=1 to #Units(M) do
    X := X cat [* Norm(Units(M)[i]) *];
end for;
return X;
end function;

os:=[* *];
for i:=1 to #L by 1 do
  if inside(no(L[i]), 74-5*b) or inside(no(L[i]), 74+5*b) then
    os:=os cat [* 1*];
   elif not inside(no(L[i]), 74-5*b) then
    os:=os cat [* 2*];
    end if;
end for;
```

We have as output:

```
 > os;
[* 2, 2, 1, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 2, 2,
2, 1, 2, 2, 1, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 1,
2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1,
2, 2, 2, 1, 2, 1, 2 *]
```

The calculation of $r_{i,j}$, if $o_i = 2$, note that $r_{i,j} = s_{L[j]}$ if $o_i = 1$:

```
  r1b:= 0;
for i:=1 to #R1[2] by 1 do
  r1b:=r1b+ os[R1[2][i]];
end for;
r1b;

 r2b:= 0;
for i:=1 to #R2[2] by 1 do
```

```
  r2b:=r2b+ os[R2[2][i]];
end for;
r2b;

 r3b:= 0;
for i:=1 to #R3[2] by 1 do
  r3b:=r3b+ os[R3[2][i]];
end for;
r3b;

 r4b:= 0;
for i:=1 to #R4[2] by 1 do
  r4b:=r4b+ os[R4[2][i]];
end for;
r4b;
```

And finally the function *cusps*:

```
cusps:=function(S);
cusp:=0;
for d:=1 to #S by 1 do
  i:=findod(S, d);
  if i le 42 then
    if os[i] eq 1 then
      cusp:=cusp + 37;
    elif os[i] eq 2 then
      cusp:= cusp + 66;
    end if;
  elif i le 63 and 43 le i then
    if os[i] eq 1 then
      cusp:= cusp +35;
    elif os[i] eq 2 then
      cusp:= cusp + 62;
    end if;
  elif 64 le i then
    if os[i] eq 1 then
      cusp:= cusp + 39;
    elif os[i] eq 2 then
      cusp:= cusp + 70;
```

```
    end if;
  end if;
end for;
return cusp;
end function;
```

# Chapter 6

# Further calculations of cusp numbers

Let $D$ be a finite dimensional central division algebra over an algebraic number field $k$. Let $\Lambda$ be a maximal order in $D$. In the previous chapter we have calculated the cusp number of $SL_2(\Lambda)$, which equals the number of boundary components of the Borel-Serre compactification of $X/SL_2(\Lambda)$, where $X$ is the homogeneous space associated to $SL_2(D)$. The cusp number of $SL_2(\Lambda)$ coincides with the number of $SL_2(\Lambda)$-conjugacy classes of proper parabolic $k$-subgroups of $SL_2(D)$ and is denoted by $c(SL_2(\Lambda))$. We will now extend these results to calculate the cusp number of $SL_n(\Lambda)$. More precisely we will calculate the number of $SL_n(\Lambda)$-conjugacy classes of all proper parabolic $k$-subgroups of $SL_n(D)$. We will handle the case of a totally definite quaternion algebras separately at the end of the section.

We will furthermore calculate the cusp number of $GL_n(\Lambda)$, by developing an adelic description of it first. We will also determine the cusp number of $GL_2(\Lambda)$, for a maximal order $\Lambda$ in a totally definite quaternion algebra.

## 6.1 The group $SL_n(D)$

Let $D$ be a finite dimensional central simple division algebra over an algebraic number field $k$ and assume that $D$ is not totally definite. Let $\Lambda$ be a maximal order in $D$. In this section we determine the parabolic $k$-subgroups of $SL_n(D)$ and their Levi-components. Then we calculate the cusp number of $SL_n(\Lambda)$ in $SL_n(D)$. First we need the following lemma:

**Lemma 6.1.** *Let $D$ be a central simple division algebra over $k$ which is not totally definite and let $n \in \mathbb{N}$. Then $cl(GL_n(D))$ equals the class number $h_D$ of $D$.*

*Proof.* In [Bor63, Prop. 2.2] this is proved for $D = k$ by different means. We see that $cl(SL_n(D)) = 1$, which follows from the strong approximation property of the central simple algebra $M_n(D)$.

Let $\Lambda$ be a maximal order in $D$ and let $f$ be a map defined by

$$\begin{aligned} GL_n(D_\Lambda)\backslash GL_n(D_{\mathbb{A}})/GL_n(D) &\to U_{\mathcal{O}_k}\backslash \mathbb{A}_k^*/k^D \\ GL_n(D_\Lambda)XGL_n(D) &\mapsto U_{\mathcal{O}_k}\operatorname{nrd}_{M_n(D)/k}(X)k^D \end{aligned}$$

where $k^D = \operatorname{nrd}(D)^*$. Further note that $|U_{\mathcal{O}_k}\backslash \mathbb{A}_k^*/k^D| = h_k^D = h_D$. Then $f$ is clearly well defined and surjective. We will show that it is injective. Indeed suppose $\operatorname{nrd}_{M_n(D)/k}(Y) = u_k \operatorname{nrd}_{M_n(D)/k}(X)t$ with $u_k \in U_{\mathcal{O}_k}$, $t \in k^D$. Then there exists $u \in U_\Lambda$, $d \in D^*$ such that $\operatorname{nrd}(u) = u_k$ and $\operatorname{nrd}(d) = t$. We see that $\begin{pmatrix} u & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in GL_n(D_\Lambda)$ and $\begin{pmatrix} d & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in GL_n(D)$, hence we may assume that $\operatorname{nrd}_{M_n(D)/k}(Y) = \operatorname{nrd}_{M_n(D)/k}(X) = x \in \mathbb{A}_k^*$.

Then $Y^{-1}M_n(\Lambda)Y$ is a maximal order in the central simple algebra $M_n(D)$ and $M_n(D)$ has the strong approximation property, hence

$$SL_n(D_{\mathbb{A}}) = Y^{-1}SL_n(D_\Lambda)YSL_n(D).$$

Now by construction $Y^{-1}X \in SL_n(D_{\mathbb{A}}) = Y^{-1}SL_n(D_\Lambda)YSL_n(D)$ and so $X \in SL_n(D_\Lambda)YSL_n(D)$. Hence $f$ is injective. □

### 6.1.1 The parabolic subgroups of $SL_n(D)$

The set of diagonal matrices with entries in $k$ is a maximal $k$-split torus $S$ in $SL_n(D)$, i.e. $S$ is given by

$$S = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in M_n(k) \ : \prod_{i=1}^n \operatorname{nrd}(x_i) = 1 \right\}.$$

It is of dimension $n-1$ (compare to [Bor91, Sec. 23.2]). A set of simple roots of $S$ is given by $\Delta = \{e_i - e_{i+1} \ : 1 \le i \le n-1\}$, where $e_i\left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\right) = x_i$. Let $\Phi$ be the set of all roots, $\Phi^+$ be the set of all positive roots of $S$. For any $\alpha \in \Phi$, the corresponding root space is denoted by $\mathfrak{g}_\alpha$. We set $\mathfrak{n} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ and let $N$ be the corresponding subgroup in $SL_n(D)$. It is the group of strictly (with 1 on the main diagonal) upper triangular matrices. Then a minimal parabolic $k$-subgroup of $SL_n(D)$ is given by $P(D) = Z(S)N$, the group of upper triangular matrices (with arbitrary entries on the diagonal) in $SL_n(D)$. Here $Z(S)$ denotes the centralizer of $S$ in $SL_n(D)$.

Let $I \subset \Delta$ be a proper subset. We set $S_I = \{X \in S : \alpha(X) = 1 \ \forall \alpha \in S\}$. Then $P_I = Z(S_I)N$ is a parabolic subgroup of $SL_n(D)$ containing $P(D) = Z(S)N$. It is called a standard parabolic subgroup of $SL_n(D)$. Any proper parabolic $k$-subgroup is conjugate to some $P_I$, and $P_I = P_J$ if and only if $I = J$.

### 6.1.2 The cusp number of $SL_n(\Lambda)$

Let $D$ be a central simple division algebra, which is not totally definite. Let $P_I$ be a standard parabolic subgroup. Its Levi-component $T_I$ consists of block-diagonal matrices, the size of the blocks depends on the roots in $I$. Now we show:

**Theorem 6.1.** *The cusp number of $SL_n(\Lambda)$, for $n \geq 2$ equals*

$$c(SL_n(\Lambda)) = \sum_{j=2}^{n} \binom{n-1}{j-1} h_D^{j-1} = (1 + h_D)^{n-1} - 1.$$

*Proof.* Theorem 4.1 shows that the number of $SL_n(\Lambda)$-conjugacy classes of $k$-parabolic subgroups of level $P_I$ (i.e. parabolic subgroups which are conjugate to $P_I$ over $SL_n(D)$) equals the class number $cl(\Lambda, P_I)$ of $P_I$. It follows easily from the description of $P_I$ that $cl(\Lambda, P_I) = cl(\Lambda, T_I)$. So let $T$ be a Levi-subgroup of a standard parabolic subgroup of $SL_n(D)$. Then $T(D_{\mathbb{A}})$ consists of block diagonal matrices of the form

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_j \end{pmatrix}$$

with $A_i \in GL_{n_i}(D_{\mathbb{A}})$ for $1 \leq i \leq j$ and such that $\sum_{i=1}^{j} n_i = n$ and $\prod_{i=1}^{j} \mathrm{nrd}(A_i) = 1$. We will first show that in this case the class number $cl(\Lambda, T)$ equals $h_D^{j-1}$. Let $\phi$ be defined as

$$T(D_\Lambda)\backslash T(D_{\mathbb{A}})/T(D) \to \prod_{i=1}^{j-1} GL_{n_i}(D_\Lambda)\backslash GL_{n_i}(D_{\mathbb{A}})/GL_{n_i}(D)$$

$$T(D_\Lambda)\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_j \end{pmatrix} T(D) \mapsto \prod_{i=1}^{j-1} GL_{n_i}(D_\Lambda)A_i GL_{n_i}(D)$$

It is well defined and easily seen to be surjective. Every class in the pre-image of $\phi$ of $\prod_{i=1}^{j-1} GL_{n_i}(D_\Lambda)A_i GL_{n_i}(D)$ has a representative of the form

$$\begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_{j-1} & \\ & & & B \end{pmatrix}$$

for some $B \in GL_{n_j}(D_{\mathbb{A}})$ satisfying $\mathrm{nrd}(B)^{-1} = \prod_{i=1}^{j-1} \mathrm{nrd}(A_i)$. Suppose $B'$ is another such matrix. The algebra $M_{n_j}(D)$ be the central and simple algebra and the conjugates of $M_{n_j}(\Lambda)$ are maximal orders in $M_{n_j}(D)$. In particular, the order $B^{-1}M_{n_j}(\Lambda)B$ is also maximal. By our assumption on $D$, $SL_{n_j}(D)$ has the strong approximation property and we obtain:

$$SL_{n_j}(D_{\mathbb{A}}) = B^{-1}SL_{n_j}(D_\Lambda)B\, SL_{n_j}(D).$$

Now $B^{-1}B' \in SL_{n_j}(D_{\mathbb{A}}) = B^{-1}SL_{n_j}(D_\Lambda)BSL_{n_j}(D)$ and so $B' \in SL_{n_j}(D_\Lambda)BSL_{n_j}(D)$ and $\phi$ is a bijection.

The number of parabolic subgroups $P$ (up to conjugation) with Levi-subgroups $T$ consisting of exactly $j$ blocks equals $\binom{n-1}{j-1}$. This gives the formula. □

Any number field is a central division algebra over itself, which is not totally definite. So in particular we obtain:

**Corollary 6.1.** *The cusp number of $SL_n(\mathcal{O}_k)$, for $n \geq 2$ equals*

$$c(SL_n(\mathcal{O}_k)) = \sum_{j=2}^{n} \binom{n-1}{j-1} h_k^{j-1} = (1+h_k)^{n-1} - 1.$$

**Remark** Some parts of the above construction hold for totally definite quaternion algebras as well. In particular the description of the parabolic subgroups and their Levi-components does not change. However, there are differences:

- If $D$ is totally definite, then $cl(GL_n(D)) = h_k^+$, for $n \geq 2$ while $cl(GL_1(D)) = h_D$.

- Let $P$ be the minimal parabolic subgroup of upper triangular matrices with Levi-subgroup $T_n$. For $n = 2$ the calculation of the class number of the Levi-component $T_2$ already proved to be quite involved. For greater $n$ the calculation of $cl(\Lambda, T_n)$ will involve not only orders of the form $x\Lambda x^{-1}$ but also the orders $(x_{i_1} \cdots x_{i_{n-1}})\Lambda(x_{i_1} \cdots x_{i_{n-1}})^{-1}$, for some $x, x_i \in D_{\mathbb{A}}^*$ representing left $\Lambda$-ideal classes.

- Let $P$ be a parabolic subgroup of $SL_n(D)$, which is not conjugate to the minimal parabolic subgroup. Let $T$ be its Levi-subgroup. Then the class number of $T$ equals $(h_k^+)^{j_1-1}(h_D)^{j_2}$, where $j_1$ is the number of blocks of size larger than 1 and $j_2$ is the number of blocks of size equal to 1. This makes writing down a nice formula for the cusp number considerably more difficult. For instance for $n = 4$ the cusp number of $SL_4(\Lambda)$, for a maximal order $\Lambda$ equals
$$2h_k^+ + h_D + 3(h_k^+)^2 h_D + cl(\Lambda, T_4).$$
Here $T_4$ denotes a Levi-component of the minimal parabolic subgroup of $SL_4(D)$.

## 6.2 The group $GL_n(D)$

Let $D$ be a finite dimensional central simple division algebra over $k$ and let $\Lambda$ be a maximal order in $D$. In this section we calculate the number of $GL_n(\Lambda)$-conjugacy classes of proper parabolic $k$-subgroups of $GL_n(D)$. This is the cusp number of $GL_n(\Lambda)$ and will be denoted by $c(GL_n(\Lambda))$.

### 6.2.1 The cusp number of $GL_n(\Lambda)$

A maximal $k$-split torus in $GL_n(D)$ is given by the set of diagonal matrices

$$S = \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in M_n(k) \,:\, x_i \in k^* \text{ for } 1 \leq i \leq n \right\}.$$

It is of dimension $n$. One sees (analogously to Section 6.1.1) that the group $P_n(D)$ of upper triangular matrices in $GL_n(D)$ is a minimal parabolic $k$-subgroup. It follows easily that the class number of a standard parabolic subgroup $P$ equals the class number of its Levi-subgroup $T$. The Levi-subgroups consist of block diagonal matrices, as above.

We first need to show:

**Theorem 6.2.** *Let $P(D)$ be a parabolic subgroup of $GL_n(D)$ for $n \geq 2$. Then there is a bijection*

$$GL_n(\Lambda)\backslash GL_n(D)/P(D) \to T(D_\Lambda)\backslash T(D_\Lambda)T_S(D_{\mathbb{A}})T(D)/T(D)$$

*where $T(D)$ is the Levi-component of $P(D)$ and $T_S(D) = T(D) \cap SL_n(D)$.*

*Proof.* A proof for the case $n = 2$, and $P$ the minimal parabolic subgroup of $GL_2(D)$ is given in [Lac05, Lemma 6.1.1]. Let $\phi_1$ be the natural projection

$$\begin{gathered} GL_n(\Lambda)\backslash GL_n(D)/P(D) \to GL_n(D_\Lambda)\backslash GL_n(D_{\mathbb{A}})/P(D) \\ GL_n(\Lambda)\, X\, P(D) \mapsto GL_n(D_\Lambda)\, X\, P(D). \end{gathered}$$

Then $\phi_1$ is well defined and easily seen to be injective. We will show that the image of $\phi_1$ consists of those classes having a representative in $SL_n(D_{\mathbb{A}})$. For the sake of simplicity we write

$$F := GL_n(D_\Lambda)\backslash GL_n(D_\Lambda)SL_n(D_{\mathbb{A}})P(D)/P(D).$$

As $\mathrm{nrd}(P(D)) = \mathrm{nrd}(GL_n(D))$, it follows that $\mathrm{im}(\phi_1) \subset F$. Now suppose that $X \in SL_n(D_{\mathbb{A}})$ is a representative of a class in $F$. As $SL_n(D)$ has the strong approximation property it follows that $X \in SL_n(D_\Lambda)SL_n(D)$. So the class $GL_n(D_\Lambda)\, X\, P(D)$ has a representative in $SL_n(D)$, hence it is in the image of $\phi_1$.

Now we define $\phi_2$ as the natural map

$$\begin{gathered} P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D) \to GL_n(D_\Lambda)\backslash GL_n(D_{\mathbb{A}})/P(D) \\ P(D_\Lambda)\, X\, P(D) \mapsto GL_n(D_\Lambda)\, X\, P(D). \end{gathered}$$

Then we see at once that $\phi_2$ is well defined and injective. We have proved earlier in Section 4.2 that $GL_n(D_v) = GL_n(\Lambda_v)P(D_v)$ holds for every finite place $v$ of $k$. This shows that $\phi_2$ is a bijection. By construction $\phi_2^{-1}(F)$ consists of the classes in

$P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D)$ having a representative in $SL_n(D_{\mathbb{A}}) \cap P(D_{\mathbb{A}})$.

We know that the representatives of the classes in $P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D)$ can be chosen in $T(D_{\mathbb{A}})$, where $T(D)$ is the Levi-component of $P(D)$. So we have a bijection $\phi_3$ :

$$T(D_\Lambda)\backslash T(D_{\mathbb{A}})/T(D) \to P(D_\Lambda)\backslash P(D_{\mathbb{A}})/P(D)$$
$$T(D_\Lambda)\, X\, T(D) \mapsto P(D_\Lambda)\, X\, P(D).$$

It is clear that $\phi_3^{-1}(\phi_2^{-1}(F))$ consists of those classes having a representative in $SL_n(D_{\mathbb{A}})$. This proves our claim. $\square$

We are now ready to show:

**Theorem 6.3.** *Let $D$ be a central simple division algebra over an algebraic number field $k$ which is not totally definite. Let $n \geq 2$. Then the cusp number of $GL_n(\Lambda)$ equals*

$$c(GL_n(\Lambda)) = \sum_{j=2}^{n} \binom{n-1}{j-1} h_D^{j-1} = (1 + h_D)^{n-1} - 1.$$

*Proof.* Let $P(D)$ be a standard parabolic $k$-subgroup of $GL_n(D)$. Let $T(D)$ be its Levi-component. We have shown above that the number of $GL_n(\Lambda)$-conjugacy classes of $P(D)$ equals the cardinality of the double quotient

$$T(D_\Lambda)\backslash T(D_\Lambda)T_S(D_{\mathbb{A}})T(D)/T(D)$$

where $T_S(D) = T(D) \cap SL_n(D)$. Then $T_S(D_{\mathbb{A}})$ consists of block diagonal matrices of the form

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_j \end{pmatrix}$$

with $A_i \in GL_{n_i}(D_{\mathbb{A}})$ for $i = 1, \dots j$. Furthermore $\sum_{i=1}^{j} n_i = n$ and $\prod_{i=1}^{n} \mathrm{nrd}(A_i) = 1$ holds. We define a map $\phi$ via

$$T(D_\Lambda)\backslash T(D_{\mathbb{A}})/T(D) \to \prod_{i=1}^{j-1} GL_{n_i}(D_\Lambda)\backslash GL_{n_i}(D_{\mathbb{A}})/GL_{n_i}(D)$$

$$T(D_\Lambda) \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_j \end{pmatrix} T(D) \mapsto \prod_{i=1}^{j-1} GL_{n_i}(D_\Lambda) A_i GL_{n_i}(D)$$

Then it is easily seen that $\phi$ is well defined and surjective. The restriction of $\phi$ to $T(D_\Lambda)\backslash T(D_\Lambda)T_S(D_\mathbb{A})T(D)/T(D)$ is injective, which follows from the strong approximation property of $SL_{n_j}(D)$ (compare to the proof of Theorem 6.1). Hence $|T(D_\Lambda)\backslash T(D_\Lambda)T_S(D_\mathbb{A})T(D)/T(D)| = h_D^{j-1}$. The number of parabolic subgroups with Levi-components consisting of exactly $j$ blocks equals $\binom{n-1}{j-1}$. This proves our claim. □

In the above proof we have assumed that $D$ is not a totally definite quaternion algebra. The main problem with $D$ being totally definite arises with the minimal parabolic $k$-subgroup of upper triangular matrices. This becomes apparent when studying the case $n = 2$:

**Theorem 6.4.** *Let $D$ be a totally definite quaternion algebra over a necessarily totally real algebraic number field $k$ and let $\Lambda$ be a maximal order in $D$. Then the cusp number of $GL_2(\Lambda)$ equals*

$$c(GL_2(\Lambda)) = \sum_\Gamma H_\Gamma s_\Gamma.$$

*Here the sum runs over all maximal orders $\Gamma$ of $D$. In particular $c(GL_2(\Lambda))$ is independent of the choice of maximal order $\Lambda$.*

*Proof.* Let $T(D)$, resp. $T_S(D)$, be the group of diagonal matrices in $GL_2(D)$, resp. $SL_2(D)$. By Theorem 6.2 we have to calculate the cardinality of the double quotient

$$T(U_\Lambda)\backslash T(U_\Lambda)T_S(D_\mathbb{A}^*)T(D^*)/T(D^*).$$

Let $\phi$ be the projection map defined by

$$T(U_\Lambda)\backslash T(U_\Lambda)T_S(D_\mathbb{A}^*)T(D^*)/T(D^*) \to U_\Lambda\backslash D_\mathbb{A}^*/D^*$$
$$T(U_\Lambda)\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} T(D^*) \mapsto U_\Lambda t_1 D^*.$$

It is well defined and surjective. Let $x_1, \dots, x_{h_D}$ be a set of representatives of the classes in $U_\Lambda\backslash D_\mathbb{A}^*/D^*$. We denote the class $U_\Lambda x_i D^*$ by $[x_i]$. Further we write $[t_1, t_2]$ for the class $T(U_\Lambda)\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} T(D^*)$. We have to study the cardinality of $\phi^{-1}([x_i])$. Each class in $\phi^{-1}([x_i])$ has a representative of the form $\begin{pmatrix} x_i & \\ & x_i^{-1} n \end{pmatrix}$ for some $n \in N^1(D_{\mathbb{A}}^*)$.

If $[x_i, x_i^{-1}n] = [x_i, x_i^{-1}n']$, then by definition $n' \in x_i U_\Lambda x_i^{-1} n D^*$. On the contrary, if $n' = x_i u x_i^{-1} n d$ for some $u \in U_\Lambda$ and $d \in D^*$, then

$$\begin{pmatrix} x_i & \\ & x_i^{-1}n \end{pmatrix} = \begin{pmatrix} 1 & \\ & u \end{pmatrix} \begin{pmatrix} x_i & \\ & x_i^{-1}n' \end{pmatrix} \begin{pmatrix} 1 & \\ & d \end{pmatrix}.$$

Hence $[x_i, x_i^{-1}n] = [x_i, x_i^{-1}n']$. This shows that $|\phi^{-1}([x_i])| = s_{x_i \Lambda x_i^{-1}}$.

We have seen already (Lemma 5.3) that

$$\sum_{i=1}^{h_D} s_{x_i \Lambda x_i^{-1}} = \sum_{i=1}^{h_D} s_{x_i^{-1} \Lambda x_i} = \sum_{\Gamma} H_\Gamma s_\Gamma.$$

Hence $c(GL_2(\Lambda))$ is independent of the choice of maximal order $\Lambda$ in $D$. This finishes the proof. □

# Chapter 7

# Boundary components

Let $k$ be an algebraic number field of degree $n = [k : \mathbb{Q}]$. Let $r_1$, resp. $r_2$, be the number of real, resp. complex, places of $k$, so that $n = r_1 + 2r_2$.

In the first section we will describe the structure of the boundary components of the Borel-Serre compactification of $X/\Gamma$, where $X$ is the homogeneous space associated to $SL_2(k)$ and $\Gamma$ is a torsion-free arithmetic subgroup in $SL_2(\mathcal{O}_k)$. We will see that the natural projection $P \to P/R_uP$ of a parabolic subgroup to its Levi-quotient induces a fibration of the boundary component $Y$ corresponding to $P$. More precisely $Y$ has the structure of a torus bundle, whose base space equals $T^{r_1+r_2-1}$ and whose fiber is $T^n$. We are closely following chapter 1 in [Har75].

Now suppose that $k$ is totally real and let $D$ be quaternion division algebra over $k$. The natural projection $P \to P/R_uP$ of a parabolic subgroup to its Levi-quotient induces again a fibration of the boundary component $Y$ corresponding to $P$. More precisely, we show that $Y$ is a torus bundle with fiber the torus $T^{4n}$. The base space is the product of a torus $T^{n-1}$ of dimension $n-1$ and a compact space. This compact space corresponds to the infinite places of $k$ with $D$ split. In particular, if $D$ is totally definite, the base space equals the torus $T^{n-1}$. This allows us to study the cohomology of $Y$ in some detail.

## 7.1 Boundary components for $D = k$

Let $k$ be an algebraic number field over $\mathbb{Q}$ of degree $n$ and let $\mathcal{O}_k$ be its ring of integers. Let $\sigma_i : k \hookrightarrow \mathbb{C}$, $1 \leq i \leq n$ be the distinct embeddings of $k$ into $\mathbb{C}$. Among these embeddings some factor through $k \to \mathbb{R}$. Let $\sigma_1, ..., \sigma_{r_1}$ denote the real embeddings $k \to \mathbb{R}$. Given one of the remaining embeddings $\sigma : k \to \mathbb{C}, \sigma(k) \not\subset \mathbb{R}$, to be called imaginary, there is the conjugate one $\bar{\sigma} : k \to \mathbb{C}$, defined by $x \mapsto \overline{\sigma(x)}$, where $\bar{z}$ denotes the usual complex conjugation of the complex number $z$. Then the number of imaginary embeddings is even and we denote it by $2r_2$. We number the $n = r_1 + 2r_2$ embeddings $\sigma_i : k \to \mathbb{C}$ , $i = 1, ..., n$ in such a way that, as above, $\sigma_i$ is real for $1 \leq i \leq r_1$, and $\overline{\sigma}_{r_1+i} = \sigma_{r_1+i+r_2}$ for $1 \leq i \leq r_2$.

Let $G = SL_2/k$ be the algebraic $k$-group of $2 \times 2$ matrices of determinant equal to 1. We need to construct the homogeneous space associated to $G$. Let $v$ be a finite place of $k$ corresponding to a real or complex embedding $\sigma$. Then $G_v := G^\sigma(k_v)$ is either equal to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$, depending on whether $\sigma$ is real or complex. The group $G_\infty$, defined as in section 3.1, is given by

$$G_\infty = SL_2(\mathbb{R})^{r_1} \times SL_2(\mathbb{C})^{r_2}.$$

Let $K$ be the maximal compact subgroup of $G_\infty$ given by

$$K = SO_2(\mathbb{R})^{r_1} \times SU_2^{r_2}$$

where $SO_2(\mathbb{R})$ denotes the special orthogonal group in $SL_2(\mathbb{R})$ and $SU_2$ is the special unitary group in $SL_2(\mathbb{C})$. Then the homogeneous space $X$ associated to $G$ equals

$$X = K \backslash G_\infty = \mathcal{H}_2^{r_1} \times \mathcal{H}_3^{r_2}$$

where $\mathcal{H}_2$ is the upper half-plane and $\mathcal{H}_3$ is the three-dimensional hyperbolic space. The dimension of $X$ equals

$$\dim X = 2r_1 + 3r_2.$$

We want to give a precise description of the structure of a connected component of the boundary. Let $P$ denote the set of upper triangular matrices in $SL_2(k)$. Then $P$ is a semi-direct product of the group of diagonal matrices $T$ and its unipotent radical $R_uP = U$, the group of upper triangular matrices with 1 on the main diagonal. Let $\alpha$ be the positive root, which we extend to $P$ via

$$\alpha\left(\begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix}\right) = x^2.$$

Then $\alpha$ induces a homomorphism

$$\alpha_\infty : P_\infty = P(k \otimes_\mathbb{Q} \mathbb{R}) \to (k \otimes_\mathbb{Q} \mathbb{R})^*$$

which we compose with the norm homomorphism

$$|\,.\,| : (k \otimes_{\mathbb{Q}} \mathbb{R})^* \to (\mathbb{R}^+)^*.$$

We can define a map $k \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ via $x \mapsto (\sigma_1(x), \ldots, \sigma_{r_1+r_2}(x))$. Then the norm map $|\,.\,|$ may be defined by

$$|x| = \prod_{i=1}^{r_1} x_i \cdot \prod_{i=r_1+1}^{r_2} |x_i|_{\mathbb{C}}^2$$

where $|\,.\,|_{\mathbb{C}}$ is the usual complex absolute value.

The composition induces a map

$$|\alpha| := |\,.\,| \circ \alpha : P_\infty \to (\mathbb{R}^+)^*.$$

Let $P_\infty(1)$ denote the kernel of $|\alpha|$. Then we see that

$$P_\infty(1) \cap K = \left( \pm \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right)^{r_1} \times \left\{ \begin{pmatrix} a & \\ & \overline{a} \end{pmatrix} \in SU_2 \,:\, |a|_{\mathbb{C}}^2 = 1 \right\}^{r_2}.$$

Hence $P_\infty(1) \cap K = P_\infty \cap K$. Let $\Gamma$ be a torsion-free finite index subgroup of $SL_2(\mathcal{O}_k)$. Every element in $\mathcal{O}_k^*$ has norm equal to $\pm 1$, hence $P_\infty(1) \cap \Gamma = P_\infty \cap \Gamma$ . A connected component $Y$ of the boundary is diffeomorphic to

$$Y = K \cap P_\infty \backslash P_\infty(1) / \Gamma \cap P_\infty.$$

The group $P$ is the semi-direct product of its unipotent radical $U$ and its Levi-component $T = P/U$. Then $T$ is a split torus of dimension 1. Let $\pi : P \to T$ denote the natural projection map. Let $K_T$, resp. $\Gamma_T$, denote the image of $K$, resp. $\Gamma$, under $\pi_\infty$ in $T_\infty$. Then the natural projection $\pi$ induces a fibration of the boundary component with base space

$$Y_T = K_T \backslash T_\infty(1) / \Gamma_T$$

and fiber $U_\infty / (U_\infty \cap \Gamma)$. Using Dirichlet's Unit Theorem and that $\Gamma_T$ is torsion-free we have $\Gamma_T \cong \mathbb{Z}^{r_1+r_2-1}$ and further

$$K_T \backslash T_\infty(1) / \Gamma_T \cong (S^1)^{r_1+r_2-1}.$$

The induced projection is a locally trivial fibration with fiber

$$U_\infty/(U_\infty \cap \Gamma) \cong (S^1)^n$$

as the structure of $U_\infty$ is additive. So we have just proved:

**Theorem 7.1** ([Har75])**.** *The fibration $\pi : Y \to Y_T$ is a torus bundle whose base space is diffeomorphic to the torus $T^r$ of dimension $r = r_1 + r_2 - 1$ and whose fiber is the torus $T^n$ of dimension $n = [k : \mathbb{Q}]$.*

## 7.2 Boundary components - general case

Let $D$ be a central simple quaternion division algebra over a totally real algebraic number field $k$. Let $[k : \mathbb{Q}] = r+s$, where $r$ denotes the number of infinite places $v$ of $k$ such that $D_v \cong \mathbb{H} = Q(-1,-1|\mathbb{R})$ and $s$ is the number of places with $D_v \cong M_2(\mathbb{R})$. Let $\Lambda$ be a maximal order in $D$ and let $\Gamma$ be a torsion-free subgroup of $SL_2(\Lambda)$. Let $G = SL_2(D)$ be the linear algebraic $k$-group of reduced norm one matrices in $M_2(D)$. Then we see

$$G_\infty = SL_2(\mathbb{H})^r \times SL_4(\mathbb{R})^s.$$

Let $1, i, j, ij$ be the standard basis of $\mathbb{H}$. For $x = x_0 + x_1 i + x_2 j + x_3 ij \in \mathbb{H}$, where $x_0, \ldots, x_3 \in \mathbb{R}$ let $\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij$ denote the conjugate of $x$. The reduced norm in $\mathbb{H}$ can be defined explicitly by $\mathrm{nrd}(x) = x\overline{x}$. For $A \in M_2(\mathbb{H})$ let

$$A^* = \overline{A}^T$$

the conjugate transpose of $A$ where $\overline{A}$ stands for conjugating each matrix entry of $A$. Let

$$Sp(2,\mathbb{H}) = \{A \in SL_2(\mathbb{H}) \,:\, AA^* = A^*A = 1\}.$$

This is the maximal compact subgroup in $SL_2(\mathbb{H})$ and is usually called the unitary group over the quaternions. In fact $\cdot^*$ is the Cartan involution in $SL_2(\mathbb{H})$. A maximal compact subgroup of $G_\infty$ is given by

$$K = Sp(2,\mathbb{H})^r \times SO_4(\mathbb{R})^s$$

and any other maximal compact subgroup is conjugate to $K$.

Let $P$ be the subgroup of upper triangular matrices in $SL_2(D)$. This is a parabolic $k$-subgroup of $G$ and any other non-trivial parabolic subgroup is conjugate to $P$ over $G(k)$. Let $U$ denote the unipotent radical and let $T = P/U$ be the Levi-quotient of $P$. Let $\alpha$ be a character of $P$ given by

$$\alpha\left(\begin{pmatrix} x & y \\ & z \end{pmatrix}\right) = \mathrm{nrd}(x).$$

Let $|\,.\,|$ be the norm homomorphism defined as in the previous section. The composition with $\alpha$ gives a map

$$|\alpha| = |\,.\,| \circ \alpha : P_\infty \to \mathbb{R}^*.$$

Let $P_\infty(1)$ be the kernel of $|\alpha|$. Then

$$P_\infty \cap K = \left(\begin{pmatrix} \mathbb{H}^1 & \\ & \mathbb{H}^1 \end{pmatrix}\right)^r \times \left(\begin{pmatrix} O_2(\mathbb{R}) & \\ & O_2(\mathbb{R}) \end{pmatrix} \cap SL_4(\mathbb{R})\right)^s$$

where $O_2(\mathbb{R})$ is the orthogonal group in $GL_2(\mathbb{R})$ and $\mathbb{H}^1$ is the group of Hamiltonian quaternions of reduced norm 1. Further it follows that $\Gamma \cap P_\infty(1) = \Gamma \cap P_\infty$, as $\mathrm{nrd}(\Lambda^*) \subset \mathcal{O}_k^*$ for any order $\Lambda$ of $D$. Hence a connected component $Y$ of the boundary is given by

$$Y = K \cap P_\infty \backslash P_\infty(1)/\Gamma \cap P_\infty.$$

Let $\pi$ be the natural projection $\pi : P \to P/U = T$ and let $K_T$ , resp. $\Gamma_T$, be the images of $K$, resp. $\Gamma$, under $\pi$. Then $\pi$ induces a fibration of $Y$ with fiber

$$U_\infty/\Gamma \cap U_\infty \cong ((\mathbb{H}^+)^r \times (M_4(\mathbb{R})^+)^s)/\mathbb{Z}^{4n} \cong (S^1)^{4n}$$

where $(\mathbb{H}^+)$ is the additive group of Hamiltonian quaternions, respectively $(M_4(\mathbb{R})^+)$ is $M_4(\mathbb{R})$, viewed as a vector space over $\mathbb{R}$. The base space $Y_T$ of the fibration is diffeomorphic to

$$Y_T = K_T \backslash T_\infty(1)/\Gamma_T.$$

The structure of $Y_T$ depends largely on the structure of $\Gamma_T$, and thus on $D$:

**Case 1:** If $s = 0$, then $D$ is totally definite. Let $\Lambda_\Gamma$ be the subgroup of units in $\Lambda^*$, appearing on the diagonal of $\Gamma_T$. Then $\Lambda^1 \not\subset \Lambda_\Gamma$, as $\Lambda^1$ finite and $\Gamma$ is by assumption torsion-free. Hence $\Lambda_\Gamma$ is commensurable with $(\mathcal{O}_k^*)^+$. So if $\Gamma$ satisfies some suitable conditions, we may assume that $\Gamma_T$ is in fact commutative and isomorphic to a finite index subgroup of $\mathcal{O}_k^*$.

**Case 2:** Now suppose $s > 0$. Then $\Lambda^1$ is not finite. However under some suitable conditions on $\Gamma$, we may still assume that set of diagonal entries in $\Gamma_T$ is a finite index subgroup of $\mathcal{O}_k^*\Lambda^1$. This allows us to consider the structure of $\Gamma_T$ and the reduced norms of its entries independent from each other. We see that $\Gamma_T$ is commensurable with $\mathcal{O}_k^* \times (\Lambda^1 \times \Lambda^1)$.

In summary we see that $\Gamma_T$ is a finite index subgroup of

$$\begin{cases} \mathcal{O}_k^* \times (\Lambda^1 \times \Lambda^1) & \text{if } s > 0 \\ \qquad \mathcal{O}_k^* & \text{if } s = 0 \end{cases}$$

Note that

$$\begin{pmatrix} \mathbb{H}^1 & \\ & \mathbb{H}^1 \end{pmatrix} \backslash \left( \begin{pmatrix} \mathbb{H} & \\ & \mathbb{H} \end{pmatrix} \cap SL_2(\mathbb{H}) \right) \cong \mathbb{R}^+$$

and

$$\left( \begin{pmatrix} O_2(\mathbb{R}) & \\ & O_2(\mathbb{R}) \end{pmatrix} \cap SL_4(\mathbb{R}) \right) \backslash \left( \begin{pmatrix} GL_2(\mathbb{R}) & \\ & GL_2(\mathbb{R}) \end{pmatrix} \cap SL_4(\mathbb{R}) \right) \cong (\mathcal{H}_2 \times \mathcal{H}_2) \times \mathbb{R}^+$$

where $\mathcal{H}_2$ denotes the upper half plane. If $s = 0$, it follows immediately, that $Y_T = (\mathbb{R}^+)^{r-1}/(\Gamma_T)$, which is, by Dirichlet's Unit Theorem, isomorphic to $(S^1)^{r-1}$.

Now suppose $s > 0$. Let $\mu_1, \ldots \mu_s$ denote the isomorphisms $D_v \cong M_2(\mathbb{R})$, corresponding to the places $v$ of $k$, where $D_v$ is split. Then $\mu = (\mu_1, \ldots, \mu_s)$ induces a map from $\Lambda^1 \to SL_2(\mathbb{R})^s$, hence we have an induced action, from the right, of $\Lambda^1$ on $\mathcal{H}_2^s$. The quotient space $\mathcal{H}_2^s/\Lambda^1$ is compact. This was first shown by Käte Hey in her doctoral thesis 1929. A proof can be found in [Kle00, Thm. 1.1]. Now we see that the group $\Gamma_T$ acts on the space

$$(\mathbb{R}^+)^{n-1} \times (\mathcal{H}_2^s \times \mathcal{H}_2^s)$$

as follows: an element $\gamma = \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \in \Gamma_T$, with $x \in \mathcal{O}_k^*$ and $\lambda_1, \lambda_2 \in \Lambda^1$, act coordinate-wise independently with $(\lambda_1, \lambda_2)$ on $\mathcal{H}_2^s \times \mathcal{H}_2^s$, while $x$, respectively the reduced norm of $x$, acts on $\mathbb{R}^{n-1}$. So it follows that

$$Y_T \cong (S^1)^{n-1} \times (\mathcal{H}_2^s/\Lambda^1)^2.$$

All in all we have proved:

**Theorem 7.2.** *Let $D$ be a quaternion division algebra over $k$ and let $\Lambda$ be a maximal order in $D$. Let $\Gamma$ be a torsion-free subgroup of $SL_2(\Lambda)$ and let $\Lambda_\Gamma^1$ be the subgroup* *of reduced norm one elements which appear on the diagonal of $\Gamma_T$. The fibration $\pi : Y \to Y_T$ is a fiber bundle whose base space is diffeomorphic to $T^{n-1} \times (\mathcal{H}_2^s/\Lambda_\Gamma^1)^2$, where $n = [k : \mathbb{Q}]$ and $s$ is the number of infinite places of $k$, where $D$ is split. The fiber is the torus $T^{4n}$.*

## 7.3 On the cohomology of the boundary components

In this section we study the cohomology of a boundary component of the Borel-Serre compactification of $X/\Gamma$, where $X$ is the homogeneous space associated to $SL_2(D)$ and $\Gamma$ is a torsion-free subgroup of $SL_2(\Lambda)$ for a maximal order $\Lambda$ of $D$. We will restrict our attention to the cases $D = k$ or $D$ a totally definite quaternion algebra over $k$. These allow a nice description of a boundary component as a torus bundle. We will again start with the case $D = k$, following [Har75].

### 7.3.1 The case $D = k$

We want to describe the cohomology of a boundary component $Y$, which we constructed in section 7.1. We are again following chapter 1 in [Har75]. The Levi-decomposition of a parabolic subgroup $P$ of $SL_2(k)$ induces a torus bundle structure $\pi : Y \to Y_T$ with base space denoted by $Y_T$. Let $U$ denote the unipotent radical of $P$, and let $U_\infty$ be defined as in section 3.1. Then the fiber of the fiber bundle $\pi : Y \to Y_T$ is equal to $U_\infty/(U_\infty \cap \Gamma) \cong T^n$, where $n = [k : \mathbb{Q}]$. The base space $Y_T$ is diffeomorphic to a torus $T^r$ of dimension $r = r_1 + r_2 - 1$, where $r_1$, resp. $r_2$ is the number of real, resp. complex, places of $k$.

The cohomology ring with coefficients in $\mathbb{C}$ of an $n$-dimensional torus $T^n$ is given by the exterior algebra $\bigwedge^* \mathbb{C}^n$, so we have:

$$H^j(U_\infty/(U_\infty \cap \Gamma); \mathbb{C}) \cong \bigwedge{}^j (\mathbb{C}^n).$$

One shows that the spectral sequence arising with the fibration $\pi : Y \to Y_T$ degenerates, hence

$$H^*(Y, \mathbb{C}) = H^*(Y_T; H^*(U_\infty/(U_\infty \cap \Gamma); \mathbb{C})),$$

so the cohomology of $Y$ equals the cohomology of the base space with coefficients in the local system induced by the cohomology of the fiber. Furthermore this local system is constant. A general proof of this is given in [Sch83, Thm. 2.7]. The endomorphisms induced by the elements of $\Gamma_T$ on $H^j(U_\infty/(U_\infty \cap \Gamma); \mathbb{C}) \cong \bigwedge^j (\mathbb{C}^n)$

are semi-simple and commute with one another. This shows:

**Theorem 7.3** (Prop. 1 in [Har75])**.** *Let $Y$ be a boundary component. Then*

$$H^*(Y;\mathbb{C}) = H^*((S^1)^r;\mathbb{C}) \otimes H^*(U_\infty/(U_\infty \cap \Gamma);\mathbb{C})^{\Gamma_T}.$$

In order to study the cohomology of $Y$ we have to determine the $\Gamma_T$-invariant elements of $H^*(U_\infty/U_\infty \cap \Gamma;\mathbb{C})$. We may view $\Gamma_T$ as a torsion-free finite index subgroup of $\mathcal{O}_k^*$. The action of $\Gamma_T$ is induced by the adjoint representation. Let

$$I_0 = \{\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \ldots, \sigma_{r_1+r_2}, \overline{\sigma_{r_1+r_2}}\}$$

be the set of embeddings $k \hookrightarrow \mathbb{C}$, where $\sigma_1, \ldots, \sigma_{r_1}$ are the real embeddings of $k \hookrightarrow \mathbb{R}$. For a subset $I \subset I_0$ we define $\varphi_I : k^* \to \mathbb{R}^*$ via

$$\varphi_I(x) := \prod_{\sigma \in I} \sigma(x).$$

For each subset $I \subset I_0$ we find a one-dimensional subspace $V_I$ generated by an element $v_I \in \bigwedge^{|I|} \mathbb{C}^n$ such that

$$e.v_I = \varphi_I(e) \cdot v_I$$

for all $e \in \Gamma_T$. Then $\bigwedge^* \mathbb{C}^n$ is a direct sum of these subspaces. We determine the subsets $I \subset I_0$ satisfying $\varphi_I(e) = 1$ for all $e \in \Gamma_T$.

**Case 1 :** Suppose that $r_1 > 0$. Note that if $\varphi_I(e) = 1$ it follows $\varphi_{I_0 - I}(e) = 1$ for all $e \in \Gamma_T$. Hence we may assume that $I$ contains at least one real embedding. Let us briefly recall the proof of Dirichlet's Unit Theorem: Define the map $\psi : \mathcal{O}_k^* \to \mathbb{R}^{r_1+r_2}$ by

$$x \mapsto (\log(|\sigma_1(x)|), \ldots \log(|\sigma_{r_1}(x)|), \log(|\sigma_{r_1+1}(x)|), \ldots, \log(|\sigma_{r_1+r_2}(x)|)).$$

One shows that the image $\psi(\mathcal{O}_k^*)$ generates a hyperplane $H$ of dimension $r_1 + r_2 - 1$. Hence so does the image of $\Gamma_T$. If $I$ is a proper subset of $I_0$ and $\varphi_I(e) = 1$ for each $e \in \Gamma_T$, then for each $a \in H \subset \mathbb{R}^{r_1+r_2}$ the relation $\sum_{j \in J} a_j = 0$ holds. Here $J$ denotes the subset of the coordinates of a vector in $\mathbb{R}^{r_1+r_2}$, corresponding to the embeddings in $I$. This contradicts the assumption that $H$ is a hyperplane, unless $I = \emptyset$ or $I = I_0$. Therefore we see that

$$H^*(U_\infty/U_\infty \cap \Gamma;\mathbb{C})^{\Gamma_T} = \mathbb{C} \oplus \mathbb{C}\xi$$

where $\xi$ is a generator of $H^n(U_\infty/U_\infty \cap \Gamma; \mathbb{C}) \cong \mathbb{C}$.

**Case 2:** Using the same arguments as above we see that if $r_1 = 0$, we find invariant elements in dimension $0, r_2$ and $n = 2r_2$. More precisely, a subset $I$ of $I_0 = \{\sigma_1, \overline{\sigma_1}, \ldots, \sigma_{r_2}, \overline{\sigma_{r_2}}\}$ is called a half system, if it contains of every pair of conjugate embeddings $\sigma_i, \overline{\sigma_i}$ exactly one. Now if $\Gamma_T$ satisfies some suitable conditions, it follows that $\varphi_I(e) = 1$, for all $e \in \Gamma_T$, and all half systems $I \subset I_0$. Then we see that

$$H^{r_2}(U_\infty/U_\infty \cap \Gamma; \mathbb{C})^{\Gamma_T} = \bigoplus_{I \text{ half system}} V_I.$$

### 7.3.2 Totally definite quaternion algebras

Let $D$ be a totally definite quaternion division algebra over a necessarily totally real algebraic number field $k$ of degree $n$ over $\mathbb{Q}$. Let $\Lambda$ be a maximal order in $D$ and let $\Gamma$ be a torsion-free finite index subgroup of $SL_2(\Lambda)$. Let $Y$ denote a boundary component arising in the Borel-Serre compactification of the space $X/\Gamma$, where $X$ is the homogeneous space corresponding to $SL_2(D)$.

We have seen in section 7.2 that the Levi-decomposition of the parabolic subgroup corresponding to $Y$ induces a fibration $\pi : Y \to Y_T$. The base space of this fibration equals $T^{n-1}$, a torus of dimension $n-1$, and the fiber is $U_\infty/(U_\infty \cap \Gamma) \cong T^{4n}$ a torus of dimension $4n$. Applying Theorem 7.3 shows that

$$H^*(Y; \mathbb{C}) = H^*((S^1)^{n-1}; \mathbb{C}) \otimes H^*(U_\infty/(U_\infty \cap \Gamma); \mathbb{C})^{\Gamma_T}.$$

We will determine the $\Gamma_T$ invariant subspace in $H^*(U_\infty/(U_\infty \cap \Gamma); \mathbb{C}) = \bigwedge^* \mathbb{C}^{4n}$. If $D$ is totally definite, $\Gamma_T$ is a finite index subgroup of $\mathcal{O}_k^*$, for we have assumed that $\Gamma$ is torsion-free. As above we see that $H^*(U_\infty/(U_\infty \cap \Gamma); \mathbb{C}) = \bigwedge^*(U_\infty \otimes \mathbb{C}) = \bigwedge^* \mathbb{C}^{4n}$.

Let

$$I_0 = \{\sigma_1, \ldots, \sigma_n\}$$

be the set of distinct embeddings $k \hookrightarrow \mathbb{R}$. We may choose a basis

$$\{u_1^1, u_1^2, u_1^3, u_1^4, u_2^1, \ldots, u_{n-1}^4, u_n^1, u_n^2, u_n^3, u_n^4\}$$

of $U_\infty$ such that the action of an element $e \in \Gamma_T$ on $u_i^j$ is given by multiplication with $\sigma_i(e)$. This gives rise to a basis of $\bigwedge^* U_\infty \otimes \mathbb{C} = \bigwedge^* \mathbb{C}^{4n}$ such that the action of $e \in \Gamma_T$ on an element of this basis is given by some product of the form $\prod_\sigma \sigma(e)$.

It may happen that some embeddings $\sigma$ appear more than once in this product. So the subsets $I \subset I_0$ do not describe the action of $\Gamma_T$ accurately.

However let $I$ be a subset of $I_0^4$, so $I$ may contain an embedding $\sigma \in I_0$ up to 4 times. Then for each $I \subset I_0^4$ there exists a subspace $V_I \subset \bigwedge^{|I|} \mathbb{C}^{4n}$ such that the action of $e \in \Gamma_T$ on $V_I$ is given by multiplication with

$$\varphi_I(e) := \prod_{\sigma \in I} \sigma(e).$$

We have to determine the subsets $I \subset I_0^4$, satisfying $\varphi_I(e) = 1$ for all $e \in \Gamma_T$. A similar argumentation as in the section above, where we considered number fields with at least one real place, shows that this can only happen for $I = \emptyset$, $I_0$, $I_0^2$, $I_0^3$ and $I_0^4$. The corresponding invariant subspaces $V_I$ are contained in $\bigwedge^0 \mathbb{C}^{4n}$, $\bigwedge^n \mathbb{C}^{4n}$, $\bigwedge^{2n} \mathbb{C}^{4n}$, $\bigwedge^{3n} \mathbb{C}^{4n}$ and $\bigwedge^{4n} \mathbb{C}^{4n}$. More precisely:

- If $I = \emptyset$, the subspace $V_\emptyset = \bigwedge^0 \mathbb{C}^{4n}$ is one dimensional.
- If $I = I_0$, then $V_I \subset \bigwedge^n \mathbb{C}^{4n}$ is of dimension $4^n$. It is generated by the elements

  $$u_1^{i_1} \wedge \ldots \wedge u_n^{i_n}$$

  for $1 \leq i_l \leq 4$ and $1 \leq l \leq n$.
- If $I = I_0^2$, then $V_I \subset \bigwedge^{2n} \mathbb{C}^{4n}$ is of dimension $\binom{4}{2}^n = 6^n$. It is generated by the elements

  $$u_1^{i_1} \wedge u_1^{j_1} \wedge \ldots \wedge u_n^{i_n} \wedge u_n^{j_n}$$

  where $1 \leq i_l < j_l \leq 4$ for all $1 \leq l \leq n$.
- If $I = I_0^3$, then $V_I \subset \bigwedge^{3n} \mathbb{C}^{4n}$ is of dimension $4^n$. It is generated by the elements

  $$u_1^{i_1} \wedge u_1^{j_1} \wedge u_1^{r_1} \wedge \ldots u_n^{i_n} \wedge u_n^{j_n} \wedge u_n^{r_n}$$

  where $1 \leq i_l < j_l < r_l \leq 4$ for all $1 \leq l \leq n$.
- If $I = I_0^4$, then $V_I = \bigwedge^{4n} \mathbb{C}^{4n}$ is one dimensional.

Hence we see that $H^*(U_\infty / U_\infty \cap \Gamma; \mathbb{C})^{\Gamma_T}$ is of dimension $2 + 2 \cdot 4^n + 6^n$.

If $k = \mathbb{Q}$, we see at once that $H^*(Y; \mathbb{C}) = H^*(U_\infty / U_\infty \cap \Gamma; \mathbb{C}) = \bigwedge^*(\mathbb{C}^4)$.

# Bibliography


[BJ06] A. Borel and L. Ji, *Compactifications of Symmetric and Locally Symmetric Spaces*, Birkhäuser, Boston, Mass., 2006.

[Bor63] A. Borel, *Some finiteness properties of adele groups over number fields*, Publ. Math. I.H.E.S **16** (1963), 5 – 30.

[Bor91] ______, *Linear Algebraic Groups*, 2nd ed., Springer, New York-Berlin-Heidelberg, 1991.

[BS73] A. Borel and J-P. Serre, *Corners and arithmetic groups*, Commentarii Mathematici Helvetici **48** (1973), 436 – 491.

[Bum97] D. Bump, *Automorphic Forms and Representations*, Cambridge studies in advanced mathematics: 55, Cambridge Univ. Press, Cambridge, 1997.

[Cox40] M. Coxeter, *The binary polyhedral groups, and other generalizations of the quaternion group*, Duke Math. Journal **7** (1940), 367 – 379.

[Frf75] A. Fröhlich, *Locally free modules over arithmetic orders*, Journal für die reine und angewandte Mathematik (Crelles Journal) **274 – 275** (1975), 112 – 124.

[Har75] G. Harder, *On the cohomology of* $SL(2, \mathfrak{O})$, Lie Groups and their Representations (I.M. Gelfand, ed.), Adam Hilger LTD, 1975, pp. 139 – 150.

[HM06] E. Hallouin and C. Maire, *Cancellation in totally definite quaternion algebras*, Journal für die reine und angewandte Mathematik (Crelles Journal) **595** (2006), 189 – 213.

[Jac68] H. Jacobinski, *Genera and decompositions of lattices over orders*, Acta Math. **121** (1968), 1 – 29.

[Kle00] E. Kleinert, *Units in Skew Fields*, Progress in Math. 186, Birkhäuser, Boston, 2000.

[KO90] V. Krafft and D. Osenberg, *Eisensteinreihen für einige arithmetisch definierte Untergruppen von $SL_2(H)$*, Mathematische Zeitschrift **204** (1990), 425 – 449.

[Lac05] C. Lacher, *Geometrie der affin-algebraischen reduzierten Norm-1 Gruppe einer zentral einfachen Algebra endlicher Dimension mit einem algebraischen Zahlkörper als Zentrum*, Ph.D. thesis, Univ. Wien, 2005.

[Leh64] J. Lehner, *Discontinuous Groups and Automorphic Forms*, Mathematical surveys, vol. 8, American Mathematical Soc., Providence, RI, 1964.

[Oes84] J. Oesterlé, *Nombres de Tamagawa et groupes unipotentes en charactéristique p*, Inventiones mathematicae **78** (1984), 13 – 88.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Acad. Press, Boston, 1994.

[Rei03] I. Reiner, *Maximal Orders*, Oxford Univ. Press, Oxford, 2003.

[Sch83] J. Schwermer, *Kohomologie arithmetisch definierter Gruppen und Eisensteinreihen*, Lecture Notes in Math. 988, Springer, Berlin-Heidelberg-New York, 1983.

[Sch10] ______, *Geometric cycles, arithmetic groups and their cohomology*, Bulletin AMS **47** (2010), 187 – 279.

[Sie61] C. L. Siegel, *Lectures on Advanced Analytic Number Theory*, Tata Inst. of Fundamental Research, Bombay, 1961.

[vdG88] G. van der Geer, *Hilbert Modular Surfaces*, Springer, Berlin-Heidelberg-New York, 1988.

[Vig76] M.-F. Vignéras, *Simplification pour les ordres des corps de quaternions totalement définis*, Journal für die reine und angewandte Mathematik (Crelles Journal) **286 – 287** (1976), 257 – 277.

[Vig80] ______, *Arithmétique des Algèbres de Quaternions*, Lecture Notes in Math. 800, Springer, Berlin-Heidelberg-New York, 1980.

[Wei73] A. Weil, *Basic Number Theory*, 2nd ed., Springer, Berlin-Heidelberg-New York, 1973.

# Abstract

Let $k$ be an algebraic number field and let $D$ be a finite dimensional central division algebra over $k$. The kernel of the reduced norm map $\mathrm{nrd}: M_2(D) \to k$ defines a linear algebraic $k$-group $G$, usually denoted by $SL_2(D)$. A maximal order $\Lambda$ of $D$ induces an arithmetic subgroup $\Gamma = SL_2(\Lambda)$ of $SL_2(D)$, which presents itself acting on a homogeneous space $X$, which is defined by the ambient Lie-group $G_\infty$. The quotient space $X/\Gamma$ is not compact, but has a natural compactification, due to Borel and Serre, $\overline{X}/\Gamma$, which is obtained by adjoining finitely many boundary components $e'(P)$, one for each $\Gamma$-conjugacy class of proper parabolic subgroups $P$ of $G$.

In the classical commutative case where $D = k$ and $\Gamma = SL_2(\mathcal{O}_k)$ the number of boundary components equals the class number $h_k$ of $k$. A boundary component has the structure of a torus bundle, whose base space is a torus $T^r$ and whose fiber is a torus $T^d$, where $d = [k : \mathbb{Q}] = s + 2t$ and $r = s + t - 1$, and $s$, resp. $t$, denotes the number of real, resp. complex, places of $k$.

In the general case of a maximal order $\Lambda$ in a division algebra $D$ we show that, if $D$ is not totally definite, the number of boundary components equals $h_D$, the class number of $D$. If $D$ is totally definite, this calculation is more involved. We show that this number $c(SL_2(\Lambda))$ is independent of the choice of maximal order $\Lambda$ in $D$. If $h_k^+ = 1$, we show that $c(SL_2(\Lambda)) = h_D^2$, but we also give examples such that $h_D < c(SL_2(\Lambda)) < h_D^2$. Let $D$ be a quaternion division algebra. The Levi-decomposition of a parabolic subgroup $P$ of $SL_2(D)$ induces the structure of a torus bundle on the boundary component $e'(P)$. Its base space is the product of a torus and, if $D$ is not totally definite, a compact space. Its fiber is a torus $T^{4d}$ where $d = [k : \mathbb{Q}]$.

We also calculate the cusp number of $SL_n(\Lambda)$ and $GL_n(\Lambda)$, for a maximal order $\Lambda$ in $D$, where $D$ is not totally definite and see that $c(SL_n(\Lambda)) = c(GL_n(\Lambda)) = (1 + h_D)^{n-1} - 1$. In particular this holds if $D$ is a number field.

# Zusammenfassung

Sei $k$ ein algebraischer Zahlkörper und sei $D$ eine zentrale Divisionsalgebra endlicher Dimension über $k$. Der Kern der reduzierten Norm-Abbildung $\mathrm{nrd}: M_2(D) \to k$ definiert eine lineare algebraische $k$-Gruppe $G$, die im Allgemeinen mit $SL_2(D)$ bezeichnet wird. Eine maximale Ordnung $\Lambda$ in $D$ induziert eine arithmetische Untergruppe $\Gamma = SL_2(\Lambda)$ in $SL_2(D)$. Diese operiert auf einem homogenen Raum $X$, der durch eine Lie-Gruppe, die $G$ zugeordnet ist, definiert wird. Der Quotientenraum $X/\Gamma$ ist nicht kompakt, kann aber in natürlicher Weise, nach Borel und Serre, kompaktifiziert werden. Diese Kompaktifizierung $\overline{X}/\Gamma$ erhält man, indem man endlich viele Randkomponenten $e'(P)$ hinzufügt, eine für jede eigentliche parabolische Untergruppe $P$ von $G$.

Im klassischen kommutativen Fall mit $D = k$ und $\Gamma = SL_2(\mathcal{O}_k)$ ist die Anzahl dieser Randkomponenten gleich der Klassenzahl $h_k$ von $k$. Eine Randkomponente hat die Struktur eines Torus-Bündels. Der Basisraum ist ein Torus $T^r$ und die Faser ist der Torus $T^d$, wobei $d = [k : \mathbb{Q}] = r + 2s$ und $r = r + s - 1$. Hier bezeichnen $r$, bzw. $s$, die Anzahl der reellen, bzw. komplexen, Stellen von $k$.

Wir zeigen in dieser Arbeit dass im allgemeinen Fall einer nicht total definiten Divisionsalgebra $D$ mit maximaler Ordnung $\Lambda$ die Anzahl dieser Randkomponenten, die wir mit $c(SL_2(\Lambda))$ bezeichnen, $h_D$ ist, wobei $h_D$ die Klassenzahl von $D$ bezeichnet. Ist $D$ total definit so ist die Spitzenzahl $c(SL_2(\Lambda))$ von verschiedenen Invarianten aller maximalen Ordnungen von $D$ abhängig. Allerdings ist $c(SL_2(\Lambda))$ unabhängig von der Wahl der maximalen Ordnung $\Lambda$ in $D$. Ist $h_k^+ = 1$ so kann man zeigen, das $c(SL_2(\Lambda)) = h_D^2$ gilt. Es existieren auch Beispiele so dass $h_D < c(SL_2(\Lambda)) < h_D^2$ gilt. Sei nun $D$ eine Quaternionen-Divisionsalgebra über einem total reellen algebraischen Zahlkörper $k$. Die Levi-Zerlegung einer parabolischen Untergruppe $P$ von $SL_2(D)$ induziert eine Torusbündel-Struktur auf der Randkomponente $e'(P)$. Der Basisraum ist das Produkt eines Torus und, wenn $D$ nicht total definit ist, eines kompakten Raumes. Die Faser ist ein Torus $T^{4d}$ wobei $d = [k : \mathbb{Q}]$.

Des weiteren bestimmen wir die Spitzenzahlen von $SL_n(\Lambda)$ und $GL_n(\Lambda)$ in $SL_n(D)$, bzw. $GL_n(D)$, wobei $\Lambda$ eine maximale Ordnung in einer nicht total definiten Divisionsalgebra $D$ ist. Wir erhalten $c(SL_n(\Lambda)) = c(GL_n(\Lambda)) = (1 + h_D)^{n-1} - 1$. Dies gilt insbesondere wenn $D$ ein Zahlkörper $k$ ist.

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