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DISSERTATION

Exponential laws for classes of Denjoy-Carleman differentiable mappings

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1 Introduction

1.1 Abstract

Spaces of so-called "ultra-differentiable functions" are special sub-classes of all smooth functions $\mathcal{E}$ with certain growing conditions on all their derivatives. Such classes are usually defined by using either weight sequences $M = (M_p)_p$ or a weight functions ω and one can distinguish between classes of *Roumieu-type* $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{\{\omega\}}$ and of *Beurling-type* $\mathcal{E}_{(M)}$ resp. $\mathcal{E}_{(\omega)}$.
In [21], [23] and finally in [22] my PhD-Advisor *A. Kriegl*, *A. Rainer* and *P. W. Michor* were able to introduce a "convenient setting" for ultradifferentiable classes defined by certain weight sequences M: More precisely they succeeded to transfer the results from [20] for the smooth case first to non-quasi-analytic classes of Roumieu-type, in the second paper to certain quasi-analytic classes of Roumieu-type (which can be written as intersection of non-quasi-analytic classes), and finally in the third paper to the general case, i.e. to both non-quasi- and quasi-analytic classes of Roumieu- and Beurling-type.

The main goal of this PhD-Thesis is to extend and generalize these results and notations to other classes of ultradifferentiable functions. More precisely we will extend them to classes $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ defined by a weight matrix $\mathcal{M}$, and as particular cases one can describe the classes defined by ω or by a (single) sequence M simultaneously.

We give now a brief summary of the structure of this PhD-Thesis:

First, after introducing and recalling important definitions and conditions, we will study systematically several properties for a weight function ω in detail. After that, we are going to study the most important technique and new idea in this work: To each ω we can associate a weight matrix $\mathcal{M} = \{M^l : l > 0\}$, i.e. a family of weight sequences with parameter $l > 0$. We will prove new versions of a comparison theorem resp. new representations for $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ by using these sequences M^l with the following consequence: We can define now classes of ultra-diff. functions by using a weight matrix $\mathcal{M}$ abstractly, and then this definition is a common generalization of defining them by using ω or a (single) sequence M. But we will also show that by using $\mathcal{M}$ one can describe in fact really new classes which can neither be described by a sequence M nor by a function ω.

In the next step we introduce several new conditions on $\mathcal{M}$ and study the behavior and properties of these new classes $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ in detail. Moreover we will characterize closedness under composition and some further stability properties of $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ in terms of conditions on the weight matrix $\mathcal{M}$.

Using these tools together with *diagonal techniques* we are able to extend the results of [21], [23] and [22] to this much more general situation, where the classes are defined by a weight matrix $\mathcal{M}$.

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1.2 Zusammenfassung

Räume sogenannter "ultra-differenzierbarer Funktionen" sind spezielle Teilklassen aller glatten Funktionen $\mathcal{E}$, deren Ableitungen alle gewisse Wachstumseigenschaften erfüllen. Solche Klassen werden in der Literatur entweder durch eine Gewichtsfolge $M=(M_p)_p$ oder durch eine Gewichtsfunktion ω definiert und man kann immer zwischen dem *Roumieu-Typ* $\mathcal{E}_{\{M\}}$ bzw. $\mathcal{E}_{\{\omega\}}$ und dem *Beurling-Typ* $\mathcal{E}_{(M)}$ bzw. $\mathcal{E}_{(\omega)}$ unterscheiden.

In [21], [23] und schließlich in [22] ist es meinem Betreuer *A. Kriegl*, *A. Rainer* und *P. W. Michor* gemeinsam gelungen, ein sogenanntes "convenient setting" für ultra-differenzierbare Funktionenklassen zu entwickeln, welche mittels bestimmter Gewichtsfolgen M definiert sind. Zuerst haben sie die wichtigen Resultate der Klasse aller glatten Funktionen in [20] auf nicht-quasi-analytische Klassen vom Roumieu-Typ übertragen, im zweiten Paper auf bestimmte quasi-analytische Klassen vom Roumieu-Typ (welche eine Darstellung als Durchschnitt nicht-quasi-analytischer Klassen besitzen). Schließlich ist es ihnen im dritten Paper gelungen, die Resultate auf den allgemeinen Fall zu übertragen: Auf Klassen ultra-differenzierbarer Funktionen vom Roumieu- und vom Beurling-Typ, welche sowohl quasi- als auch nicht-quasi-analytisch sind.

Das Hauptziel dieser Arbeit ist es gewesen, all diese Ergebnisse und Beweise auch auf andere Klassen ultra-differenzierbarer Funktionen zu übertragen bzw. zu verallgemeinern. Wir werden das "convenient setting" auf Klassen $\mathcal{E}_{\{\mathcal{M}\}}$ bzw. $\mathcal{E}_{(\mathcal{M})}$ übertragen, welche mittels einer sogennanten Gewichts-Matrix $\mathcal{M}$ definiert werden. Verwendet man nämlich diese Notation so kann man sowohl Klassen welche über eine Folge M als auch über eine Funktion ω definiert werden gleichzeitig beschreiben.

Wir geben nun einen kurzen strukturellen Überblick über die vorliegende Arbeit:

Zu Beginn, nachdem wir die wichtigsten (bekannten) Definitionen und Eigenschaften wiederholen, werden wir systematisch die wichtigen Bedingungen für eine Gewichtsfunktion ω untersuchen. Danach werden wir eine neue Methodik und die zentrale neue Idee in dieser Arbeit studieren: Zu jeder Gewichtsfunktion ω können wir eine Gewichtsmatrix $\mathcal{M}=\{M^l : l>0\}$ konstruieren, also eine Familie von Gewichtsfolgen mit einem Parameter $l>0$. Mit Hilfe dieses neuen Konzepts und der Folgen M^l werden wir neue Versionen von Vergleichsresultaten bzw. neue Darstellungen für die Klassen $\mathcal{E}_{\{\omega\}}$ und $\mathcal{E}_{(\omega)}$ beweisen, mit der folgenden Konsequenz: Man kann nun Klassen ultra-differenzierbarer Funktionen abstrakt mittels einer Gewichtsmatrix $\mathcal{M}$ definieren, und dann ist diese neue Definition eine gemeinsame Verallgemeinerung der Definition durch eine Gewichtsfunktion ω und durch eine (einzelne) Gewichtsfolge M. Aber wir werden zeigen, dass mittels dieser neuen Methode auch neue Klassen beschrieben werden können, welche weder durch eine einzelne Folge M noch durch eine Funktion ω beschrieben werden können.

Im nächsten Schritt werden wir Eigenschaften von Gewichts-Matrizen $\mathcal{M}$ einführen und diese sowie die Eigenschaften und neue Phänomene der neuen Klassen $\mathcal{E}_{\{\mathcal{M}\}}$ bzw. $\mathcal{E}_{(\mathcal{M})}$ genau studieren. Weiters werden wir die Abgeschlossenheit unter Komposition und einige weitere wichtige Stabilitätseigenschaften der Klassen $\mathcal{E}_{\{\mathcal{M}\}}$ bzw. $\mathcal{E}_{(\mathcal{M})}$ durch

Eigenschaften der Gewichts-Matrix $\mathcal{M}$ charakterisieren.

Mit dieser entwickelten Theorie und mit Hilfe von *Diagonalargumenten* können wir nun die Resultate von [21], [23] und [22] auf diese neue Situation verallgemeinern, wo die Klassen ultra-differenzierbarer Funktionen durch eine Gewichts-Matrix $\mathcal{M}$ definiert sind.

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2 Basic definitions

2.1 General notation

Throughout the whole PhD-thesis we will use the following notation: We denote by $\mathcal{E}$ the class of smooth functions, $\mathcal{C}^\omega$ the class of all real analytic functions and $\mathcal{H}$ the class of all holomorphic functions. We will write $\mathbb{N}_{>0} = \{1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$. Moreover we put $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$, i.e. the set of all positive real numbers. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we use the usual multi-index notation, write $\alpha! := \alpha_1! \dots \alpha_n!$, $|\alpha| := \alpha_1 + \dots + \alpha_n$ and for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also put $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and we denote by $f^{(k)}$ the k-th order *Fréchet derivative* of f. Let $E_1, \dots, E_k$ and F be topological vector spaces, then $L(E_1 \times \dots \times E_k, F)$ is the space of all bounded k-linear mappings $E_1 \times \dots \times E_k \to F$. If $E = E_i$ for $i = 1, \dots, k$, then we write $L^k(E, F)$. The natural logarithm will always be denoted by log.

2.2 Weight functions

A function $\omega : [0, \infty) \to [0, \infty)$ (sometimes ω is extended to an even function on $\mathbb{R}$, by $\omega(-x) := \omega(x)$ for $x \in [0, \infty)$) is called a *weight function* if we assume that

(ω_0) ω is continuous, on $[0, \infty)$ increasing, $\omega(x) = 0$ for $x \in [0, 1]$ (w.l.o.g.) and $\lim_{x\to\infty} \omega(x) = +\infty$.

Moreover we consider the following conditions:

(ω_1) $\omega(2t) = O(\omega(t))$ as $t \to +\infty$, i.e. $\limsup_{t\to\infty} \frac{\omega(2t)}{\omega(t)} < +\infty$.

(ω_2) $\omega(t) = O(t)$ as $t \to \infty$, i.e. $\limsup_{t\to\infty} \frac{\omega(t)}{t} < +\infty$.

(ω_3) $\log(t) = o(\omega(t))$ as $t \to +\infty$, i.e. $\lim_{t\to+\infty} \frac{\log(t)}{\omega(t)} = 0$ ($\Leftrightarrow \lim_{t\to+\infty} \frac{t}{\varphi_\omega(t)} = 0$).

(ω_4) $\varphi_\omega : t \mapsto \omega(e^t)$ is a convex function on $\mathbb{R}$.

(ω_5) $\omega(t) = o(t)$ as $t \to +\infty$, i.e. $\lim_{t\to+\infty} \frac{\omega(t)}{t} = 0$.

(ω_6) $\exists\, H \geq 1\, \forall\, t \geq 0 :\ 2 \cdot \omega(t) \leq \omega(H \cdot t) + H$, i.e. $\limsup_{t\to\infty} \frac{\omega(t)}{\omega(H\cdot t)} \leq \frac{1}{2}$.

(ω_7) $\exists\, H > 0\, \exists\, C > 0\, \forall\, t \geq 0 : \omega(t^2) \leq C \cdot \omega(H \cdot t) + C$.

(ω_{nq}) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty \Leftrightarrow \int_1^\infty \frac{\omega(t)}{1+t^2} dt < +\infty$.

We define the set

$$\mathcal{W} := \{\omega : [0, \infty) \to [0, \infty) : \omega \text{ has } (\omega_0), (\omega_3), (\omega_4)\}.$$

The function φ_ω in condition (ω_4) is increasing by definition and by (ω_4) convex on $[0,\infty)$. By (ω_3) we have $0 = \lim_{t\to\infty}\frac{\log(t)}{\omega(t)} = \lim_{x\to\infty}\frac{x}{\varphi_\omega(x)}$, finally by (ω_0) also $\varphi_\omega(0) = 0$ holds. Hence, by [5, 1.3. Remark], we can define the *Legendre-Fenchel-Young-conjugate* φ^*_ω via

$$\varphi^*_\omega(x) := \sup_{y\geq 0}(x\cdot y - \varphi_\omega(y)),$$

which is a convex increasing function, $\varphi^*_\omega(0) = 0$, $\varphi^{**}_\omega = \varphi_\omega$ and furthermore $\lim_{x\to\infty}\frac{x}{\varphi^*_\omega(x)} = 0$. Finally, by [5, 1.5. Lemma], the functions $x \mapsto \frac{\varphi_\omega(x)}{x}$ resp. $x \mapsto \frac{\varphi^*_\omega(x)}{x}$ are increasing on $[0,+\infty)$.
We summarize: Conditions (ω_0), (ω_3) and (ω_4) together guarantee nice properties for the function φ_ω and its Legendre-Fenchel-Young conjugate φ^*_ω. Note: Because ω vanishes on $[0,1]$ we have that φ_ω vanishes on $(-\infty,0]$ and so in the definition of the conjugate we get $\sup_{y\geq 0} = \sup_{y\in\mathbb{R}}$. For more details about these conditions 3.1.1 and 3.1.2.

All other properties will be studied in detail below, a short summary: (ω_1) and (ω_6) are certain growth conditions and they imply strong inequalities, which will be used several times. (ω_2) and (ω_5) guarantee, that the real-analytic functions are contained in the considered Roumieu- resp. Beurling-space. (ω_{nq}) is called the non-quasi-analyticity-condition. For $\omega \in \mathcal{W}$ with additionally (ω_1) and (ω_{nq}) it was shown in [5], that the classes of ultradifferentiable functions defined by ω (for the definition see in the next section) contain functions with compact support! In [5, 4.4. Proposition], the convexity of φ^*_ω together with property (ω_1) was used to show that such classes defined by ω are closed under pointwise multiplication.
All these conditions have already been introduced in the literature, but condition (ω_7) seems to be new.

For two weight functions $\omega_1, \omega_2 \in \mathcal{W}$ we write

$$\omega_1 \preceq \omega_2 :\Leftrightarrow \omega_2(t) = O(\omega_1(t)), t \to +\infty$$

$$\omega_1 \sim \omega_2 :\Leftrightarrow \omega_1 \preceq \omega_2, \omega_2 \preceq \omega_1$$

$$\omega_1 \vartriangleleft \omega_2 :\Leftrightarrow \omega_2(t) = o(\omega_1(t)), t \to +\infty$$

2.3 Weight sequences

A weight sequence $M = (M_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0}$ is a sequence of positive real numbers. We introduce also the sequences $m = (m_k)_k$ and $\mu = (\mu_k)_k$ defined by $m_k := \frac{M_k}{k!}$ and $\mu_k := \frac{M_k}{M_{k-1}}$, $\mu_0 := 1$. So alternatively we have $M_k = \prod_{i=0}^k \mu_i$. The sequence M is called *normalized* if $1 = M_0 \leq M_1$ holds.

(1) M is called *weakly log. convex* if

$$(\text{lc}) :\Leftrightarrow\ \forall\, j \in \mathbb{N}:\ M_j^2 \leq M_{j-1}\cdot M_{j+1}$$

which means that the set above the graph $\{(j,\log(M_j)) : j \in \mathbb{N}\}$ is a convex set in $\mathbb{R}^2$ and this condition is equivalent to the fact that the sequence μ is increasing. M is called *strongly log. convex*, if

$$(\text{slc}) :\Leftrightarrow\ \forall\, j \in \mathbb{N}:\ m_j^2 \leq m_{j-1}\cdot m_{j+1}.$$

We recall the following well-known facts: If M is weakly log. convex and $M_0 = 1$, then

$$\text{(alg)} \;:\Leftrightarrow\; \exists\, C \geq 1\, \forall\, j,k \in \mathbb{N}:\; M_j \cdot M_k \leq C^{j+k} \cdot M_{j+k}$$

holds with $C = 1$ and moreover the mapping $j \mapsto (M_j)^{1/j}$ is increasing (see e.g. [38, Lemma 2.0.4, Lemma 2.0.6]). More general, if M is weakly log. convex and $M_0 \neq 1$, then the mentioned proofs show $M_j \cdot M_k \leq M_0 \cdot M_{j+k}$ for all $j,k \in \mathbb{N}$ and $(M_k)^{1/k} \leq (M_0 \cdot M_{k+1})^{1/(k+1)}$ for $k \geq 1$. But since in this case $(\mu_k)_k$ is increasing and $(M_k)^{1/k} \leq (M_{k+1})^{1/(k+1)} \Leftrightarrow M_k \cdot (M_k)^{1/k} = (M_k)^{(k+1)/k} \leq M_{k+1} \Leftrightarrow \mu_1 \cdots \mu_k = M_k \leq (\mu_{k+1})^k$ also for $M_0 \neq 1$ the mapping $j \mapsto (M_j)^{1/j}$ is increasing.
(slc) implies (lc) and if M is weakly log. convex and normalized, then $\mu_k \geq 1$ for all $k \in \mathbb{N}$, hence M is automatically increasing.

In this context sometimes regularizations of M are used, see e.g. [15, Theorem 1.3.8.] for their importance: First we introduce

$$M^{\mathrm{i}} := (M^{\mathrm{i}}_k)_k, \qquad M^{\mathrm{i}}_k := \left(\inf\{(M_j)^{1/j} : j \geq k\}\right)^k \text{ for } k \geq 1, \qquad M^{\mathrm{i}}_0 := 1,$$

so the sequence $\left((M^{\mathrm{i}}_k)^{1/k}\right)_k$ is the increasing minorant of the sequence $((M_k)^{1/k})_k$. If $k \mapsto (M_k)^{1/k}$ is increasing and $M_0 = 1$ (w.l.o.g.), then $M_k = M^{\mathrm{i}}_k$ follows for all $k \in \mathbb{N}$. Second we define

$$M^{\mathrm{lc}} := (M^{\mathrm{lc}}_k)_k, \quad M^{\mathrm{lc}}_k := \inf\left\{(M_j)^{\frac{l-k}{l-j}} \cdot (M_l)^{\frac{k-j}{l-j}} : j \leq k \leq l, j < l\right\},$$

the (weakly) log. convex minorant of M. If M is log. convex, then $M = M^{\mathrm{lc}}$ holds.
Furthermore we point out: If one starts with M and obtains the sequence $m = (m_k)_k$ by $m_k := \frac{M_k}{k!}$, then we can derive m^{lc}, the log. convex minorant of the sequence m. Since m^{lc} is log. convex, also the sequence $(k! \cdot m^{\mathrm{lc}}_k)_k$ is log. convex with $k! \cdot m^{\mathrm{lc}} \leq k! \cdot m_k = M_k$ for each k. This shows $M^{\mathrm{lc}}_k \geq k! \cdot m^{\mathrm{lc}}_k$ and analogously $M^{\mathrm{i}}_k \geq (k!)^{1/k} \cdot m^{\mathrm{i}}_k$ (note that $k!^{1/k} \leq (k+1)!^{1/(k+1)} \Leftrightarrow k! \cdot k!^{1/k} \leq (k+1)! \Leftrightarrow k!^{1/k} \leq k+1 \Leftrightarrow k! \leq (k+1)^k$). If m is log. convex, then M too and we obtain equality. But in general we cannot have equality: If e.g. M is log. convex but m not, then we would get $m_k = \frac{M_k}{k!} = \frac{M^{\mathrm{lc}}_k}{k!} = m^{\mathrm{lc}}_k$, a contradiction (and analogously for M^{i}, m^{i}).

Except these both cases in the literature there are mentioned several more regularizations, which we don't need necessarily. For their importance (characterization of equality of ultradiff. function classes in terms of properties of weight sequences), the precise technique of regularizing a weight sequence and more details we refer to [7], [8] and [25], see also [39] and [35].

(2) M satisfies *moderate growth* if

$$\text{(mg)} :\Leftrightarrow \sup_{j,k \geq 1} \left(\frac{M_{j+k}}{M_j \cdot M_k}\right)^{1/(j+k)} < +\infty,$$

which means that

$$\exists\, C \geq 1\, \forall\, j,k \in \mathbb{N}:\; M_{j+k} \leq C^{j+k} \cdot M_j \cdot M_k.$$

Since $1 \leq \binom{j+k}{j} \leq 2^{j+k}$ we can replace in the condition above M also by m. Sometimes instead of moderate growth the following equivalent condition is used

$$\text{(udo)} \;:\Leftrightarrow \exists\, C_1, C_2 \geq 1\, \forall\, p \in \mathbb{N}:\; M_p \leq C_1 \cdot C_2^p \cdot \min_{0 \leq q \leq p} M_q \cdot M_{p-q},$$

which is also known as *stability under applying ultradifferential operators.* A weaker condition than moderate growth is *derivation closedness:*

$$(\mathrm{dc}) \;:\Leftrightarrow \exists\, C \geq 1\, \forall\, j \in \mathbb{N}:\; M_{j+1} \leq C^{j+1} \cdot M_j.$$

(3) M satisfies condition (β_3), if

$$(\beta_3) :\Leftrightarrow\; \exists\, Q \in \mathbb{N}:\; \liminf_{p\to\infty} \frac{\mu_{Qp}}{\mu_p} > 1.$$

A stronger condition than (β_3) is (β_1) (introduced in [33]), which means

$$(\beta_1) :\Leftrightarrow\; \exists\, Q \in \mathbb{N}:\; \liminf_{p\to\infty} \frac{\mu_{Qp}}{\mu_p} > Q.$$

(4) M is called *non-quasi-analytic,* if

$$(\mathrm{nq}) :\Leftrightarrow\; \sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} = \sum_{p=1}^{\infty} \frac{1}{\mu_p} < +\infty$$

otherwise M is called *quasi-analytic.* A stronger condition is *strong non-quasi-analyticity,* which means

$$(\mathrm{snq}) :\Leftrightarrow\; \exists\, C \geq 1\, \forall\, j \in \mathbb{N}:\; \sum_{k=j}^{\infty} \frac{M_k}{M_{k+1}} \leq C \cdot \frac{(j+1)\cdot M_j}{M_{j+1}}.$$

If M is weakly log. convex, then one can show directly by using *Carleman's inequality* (see e.g. [38, Proposition 4.1.7]):

$$\sum_{p=1}^{\infty} \frac{1}{\mu_p} < +\infty \Leftrightarrow \sum_{p=1}^{\infty} \frac{1}{(M_p)^{1/p}} < +\infty.$$

(5) For two arbitrary weight sequences $M = (M_p)_p$ and $N = (N_p)_p$ we write $M \leq N$ if and only if $M_p \leq N_p$ holds for all $p \in \mathbb{N}$. Moreover we define

$$M \preceq N :\Leftrightarrow\; \exists\, C_1, C_2 \geq 1\, \forall\, j \in \mathbb{N}:\; M_j \leq C_2 \cdot C_1^j \cdot N_j \Longleftrightarrow \sup_{p\in\mathbb{N}_{>0}} \left(\frac{M_p}{N_p}\right)^{1/p} < +\infty$$

and we call the sequences equivalent and write

$$M \approx N :\Leftrightarrow\; M \preceq N, N \preceq M.$$

Furthermore we will write

$$M \vartriangleleft N :\Leftrightarrow\; \forall\, h > 0\, \exists\, C_h \geq 1\, \forall\, j \in \mathbb{N}:\; M_j \leq C_h \cdot h^j \cdot N_j \Longleftrightarrow \lim_{p\to\infty} \left(\frac{M_p}{N_p}\right)^{1/p} = 0.$$

In the literature sometimes one can find the following numbers for these conditions (introduced by [16]): Weakly log. convexity is called $(M1)$, closedness under taking derivatives $(M2)'$, moderate growth $(M2)$, non-quasi-analyticity $(M3)'$ and strong non-quasi-analyticity $(M3)$.

Some properties for weight sequences are very basic and so we introduce for convenience the following set:

$$\mathcal{LC} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} :\; M \text{ is normalized, weakly log. convex, } \lim_{k\to\infty} (M_k)^{1/k} = +\infty\}.$$

2.4 Spaces of ultradiff. functions defined by a weight sequence M resp. a weight function ω

Let $n, m \in \mathbb{N}_{>0}$, let $U \subseteq \mathbb{R}^n$ be a non-empty open set and we denote by $\mathcal{E}(U, \mathbb{R}^m)$ the space of all smooth functions $f : U \to \mathbb{R}^m$ and write $\mathcal{E}(U)$ instead of $\mathcal{E}(U, \mathbb{R})$. There are several ways to define spaces of *ultradifferentiable functions* and basically one can distinguish in each way between the *Roumieu-* and the *Beurling-type*.
First we can introduce them by using a weight function $\omega \in \mathcal{W}$. The Roumieu-type space is defined by

$$\mathcal{E}_{\{\omega\}}(U, \mathbb{R}^m) := \{f \in \mathcal{E}(U, \mathbb{R}^m) : \ \forall\, K \subseteq U \text{ compact } \exists\, l > 0 : \ \|f\|_{\omega,K,l} < +\infty\}$$

and the Beurling-type space by

$$\mathcal{E}_{(\omega)}(U, \mathbb{R}^m) := \{f \in \mathcal{E}(U, \mathbb{R}^m) : \ \forall\, K \subseteq U \text{ compact } \forall\, l > 0 : \ \|f\|_{\omega,K,l} < +\infty\},$$

where we have put

$$\|f\|_{\omega,K,l} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R}^m)}}{\exp(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot k))} \tag{2.4.1}$$

and $f^{(k)}(x)$ denotes the k-th order Fréchet derivative at x. For compact sets K with smooth boundary the space

$$\mathcal{E}_{\omega,l}(K, \mathbb{R}^m) := \{f \in \mathcal{E}(K, \mathbb{R}^m) : \|f\|_{\omega,K,l} < +\infty\}$$

is Banach. For $l_1 \le l_2$ we obtain the bounded canonical inclusion mapping $i_{l_1,l_2} : \mathcal{E}_{\omega,l_1}(K, \mathbb{R}^m) \to \mathcal{E}_{\omega,l_2}(K, \mathbb{R}^m)$ and we have the topological vector space representations

$$\mathcal{E}_{\{\omega\}}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varinjlim_{l > 0} \mathcal{E}_{\omega,l}(K, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \mathcal{E}_{\{\omega\}}(K, \mathbb{R}^m) \tag{2.4.2}$$

resp.

$$\mathcal{E}_{(\omega)}(U, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \varprojlim_{l > 0} \mathcal{E}_{\omega,l}(K, \mathbb{R}^m) = \varprojlim_{K \subseteq U} \mathcal{E}_{(\omega)}(K, \mathbb{R}^m), \tag{2.4.3}$$

where the limits run over all compact sets $K \subseteq U$ and $l > 0$.
Similarly we introduce such classes by using an arbitrary *weight sequence* $M = (M_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0}$ as follows: First again the Roumieu-type

$$\mathcal{E}_{\{M\}}(U, \mathbb{R}^m) := \{f \in \mathcal{E}(U, \mathbb{R}^m) : \ \forall\, K \subseteq U \text{ compact } \exists\, h > 0 : \ \|f\|_{M,K,h} < +\infty\}$$

and the Beurling-type-space by

$$\mathcal{E}_{(M)}(U, \mathbb{R}^m) := \{f \in \mathcal{E}(U, \mathbb{R}^m) : \ \forall\, K \subseteq U \text{ compact } \forall\, h > 0 : \ \|f\|_{M,K,h} < +\infty\},$$

where we have put

$$\|f\|_{M,K,h} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R}^m)}}{h^k \cdot M_k}. \tag{2.4.4}$$

For compact sets K with smooth boundary the space

$$\mathcal{E}_{M,h}(K, \mathbb{R}^m) := \{f \in \mathcal{E}(K, \mathbb{R}^m) : \|f\|_{M,K,h} < +\infty\}$$

is Banach. For $h_1 \leq h_2$ we obtain the bounded canonical inclusion mapping ι_{h_1,h_2} : $\mathcal{E}_{M,h_1}(K,\mathbb{R}^m) \to \mathcal{E}_{M,h_2}(K,\mathbb{R}^m)$ and we have the topological vector space representations

$$\mathcal{E}_{\{M\}}(U,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \varinjlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{M\}}(K,\mathbb{R}^m) \tag{2.4.5}$$

resp.

$$\mathcal{E}_{(M)}(U,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \varprojlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \mathcal{E}_{(M)}(K,\mathbb{R}^m), \tag{2.4.6}$$

where the limits run over all compact sets $K \subseteq U$ and $h > 0$.

Convention: Let $\star \in \{M,\omega\}$, then for convenience we will write $\mathcal{E}_{[\star]}$ if either $\mathcal{E}_{\{\star\}}$ or $\mathcal{E}_{(\star)}$ is considered, but not mixing the cases if statements involve more than one $\mathcal{E}_{[\star]}$ symbol. We will write $\mathcal{E}_{[\star]}(U)$ instead of $\mathcal{E}_{[\star]}(U,\mathbb{R})$.

Observations:

(1) It's no restriction that all occurring limits are countable: Consider a countable exhaustion $(K_j)_{j\in\mathbb{N}}$ of U and in the Roumieu-cases one can restrict to $l, h \in \mathbb{N}$, in the Beurling-cases to $l', h' \in \mathbb{N}$ with $l' := \frac{1}{l}$ resp. $h' := \frac{1}{h}$.

(2) By definition $M \preceq N$ implies $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{N\}}$ resp. for the Beurling-case, $M \lhd N$ implies $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{(N)}$.

(3) The inclusion $\mathcal{E}_{(\star)} \subseteq \mathcal{E}_{\{\star\}}$ is clearly satisfied for $\star \in \{M,\omega\}$.

(4) We can replace a sequence $M = (M_k)_k$ by $(C^k \cdot M_k)_k$ for any $C > 0$ without changing the associated function space $\mathcal{E}_{[M]}$. Moreover we can assume w.l.o.g. $1 = M_0 \leq M_1$ (*normalization*) for the defining weight sequence: We can replace a sequence $M = (M_k)_k$ by $\left(\frac{M_k}{M_0}\right)_k$ and assume that $C \cdot h > \frac{1}{M_1}$ without changing the associated function space $\mathcal{E}_{[M]}$.

(5) See also the remark in [21, 2.8.]: In the above description of the steps $\mathcal{E}_{M,h}(K,\mathbb{R}^m)$ resp. $\mathcal{E}_{\omega,l}(K,\mathbb{R}^m)$ instead of compact sets K with smooth boundary one can also consider open $K \subseteq U$ with $\overline{K}$ compact in U resp. one can work with *Whitney jets* on compact K (e.g. in [4]).

For the *Beurling cases* in both definitions we obtain a *Fréchet-space,* because here we take the seminorms $\|\cdot\|_{\omega,K_j,1/l}$ resp. $\|\cdot\|_{M,K_j,1/h}$ for $l, h \in \mathbb{N}_{>0}$ and an exhaustion of U with countably many compact sets K_j. In [5, Proposition 4.9.] it is shown for $\omega \in \mathcal{W}$ with (ω_1) and (ω_{nq}), that it is in fact a nuclear Fréchet-space.

For the *Roumieu-cases* we take again a countable exhaustion $(K_j)_{j\in\mathbb{N}}$ of U with compact sets and for the inductive limit we can also reduce to $l, h \in \mathbb{N}_{>0}$, so one has a countable inductive limit with continuous (bounded) inclusions.
$\varinjlim_{l>0} \mathcal{E}_{\omega,l}(K,\mathbb{R}^m)$ resp. $\varinjlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^m)$ are clearly (LF)-spaces (more precisely (LB)-spaces), hence webbed because a (LF)-space is a quotient of a countable direct sum of Fréchet-spaces. Furthermore they are ultrabornological, hence bornological and barreled.
For an arbitrary weight sequence $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ the space $\varinjlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^m)$ is a *Silva-sapce,* i.e. a countable inductive limit of Banach-spaces with compact connecting mappings, see e.g. [16, Proposition 2.2], [38, Lemma 3.1.8].

We show: For $\omega \in \mathcal{W}$ with (ω_1) the space $\varinjlim_{l>0} \mathcal{E}_{\omega,l}(K,\mathbb{R}^m)$ is a Silva-space: It suffices to prove that for all $l_1 > 0$ there exists $l_2 > 0$, $l_2 > l_1$, such that the mapping

$$\iota : \mathcal{E}_{M^{l_1},1}(K,\mathbb{R}^m) \to \mathcal{E}_{M^{l_2},1}(K,\mathbb{R}^m)$$

is compact, where we have put $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$ for $l > 0$ and $j \in \mathbb{N}$. (For more details concerning this definition we refer to the fifth chapter below!) For this consider

$$\mathcal{E}_{M^{l_1},1}(K,\mathbb{R}^m) \underset{\iota_1}{\to} \mathcal{E}_{M^{l_1},h_1}(K,\mathbb{R}^m) \underset{\iota_2}{\to} \mathcal{E}_{M^{l_1},h_2}(K,\mathbb{R}^m) \underset{\iota_3}{\to} \mathcal{E}_{M^{l_2},1}(K,\mathbb{R}^m).$$

The first inclusion holds for all $h_1 \geq 1$, the second for $h_2 \geq h_1$ and finally the third by (3.3.2) below (see also 5.1.4 below): $h_2^k \cdot M^{l_1}_k \leq C \cdot M^{l_2}_k$ is satisfied for all $k \in \mathbb{N}$ if we put $l_2 = L^s \cdot l_1$ and $s \in \mathbb{N}$ chosen minimal with $\exp(s) \geq h_2$.
We put now $\iota := \iota_3 \circ \iota_2 \circ \iota_1$, the mappings ι_1 and ι_3 are clearly bounded mappings (between Banach spaces), for $h_2 > h_1$ the mapping ι_2 is compact (again by [16, Proposition 2.2], [38, Lemma 3.1.8]). Hence by the ideal property of compact mappings also ι is compact.

Finally for $U \subseteq \mathbb{R}^n$ non-empty open we can define ultradiff. functions classes defined by global estimates (no compact set $K \subseteq U$ is involved) as follows:

$$\mathcal{E}^{\text{global}}_{\{\omega\}}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \exists\, l > 0\ \|f\|_{\omega,U,l} < +\infty\}$$

resp.

$$\mathcal{E}^{\text{global}}_{(\omega)}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \forall\, l > 0\ \|f\|_{\omega,U,l} < +\infty\}$$

and

$$\mathcal{E}^{\text{global}}_{\{M\}}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \exists\, h > 0\ \|f\|_{M,U,h} < +\infty\}$$

resp.

$$\mathcal{E}^{\text{global}}_{(M)}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \forall\, h > 0\ \|f\|_{M,U,h} < +\infty\},$$

where $\|f\|_{\star,U,*}$ denotes the semi-norms from (2.4.1) resp. (2.4.4) where "$\sup_{k\in\mathbb{N},x\in K}$" is replaced by "$\sup_{k\in\mathbb{N},x\in U}$".

Remark 2.4.1. *During this PhD-Thesis we will deal with classes of ultradiff. functions which are defined locally. But it turns out that many important theorems are also valid for globally defined classes. More precisely we point out:*

(i) *In many proofs we will deal with certain inequalities concerning only the denominator of the quotient in the defining semi-norms* (2.4.1) *resp.* (2.4.4). *So in the numerator there we can use instead of* $f_k := \sup_{x\in K,k\in\mathbb{N}}$ *also the sequence* $f_k := \sup_{x\in U,k\in\mathbb{N}}$.

(ii) *In globally defined Roumieu-type classes introduced by (at least) weakly log. convex weight sequences M we still can find the important functions* $\theta_M \in \mathcal{E}^{\text{global}}_{\{M\}}(\mathbb{R},\mathbb{R})$ *resp.* $\tilde{\theta}_M \in \mathcal{E}^{\text{global}}_{\{M\}}(\mathbb{R},\mathbb{C})$ *from* (chf) *resp.* (2.4.7) *below.*

(iii) *Note that for globally defined classes the above top. vector space representations* (2.4.2), (2.4.3), (2.4.5) *and* (2.4.6) *become more easier since no projective limit over all compact sets $K \subseteq U$ is involved.*

More general for formal calculations also the following abstract spaces of sequences of positive real numbers can be considered:

$$\mathcal{F}_{M,h} := \left\{(f_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \exists\, C > 0 : \forall\, k \in \mathbb{N} : |f_k| \le C \cdot h^k \cdot M_k\right\}$$

with

$$\mathcal{F}_{\{M\}} := \bigcup_{h>0} \mathcal{F}_{M,h} \qquad \mathcal{F}_{(M)} := \bigcap_{h>0} \mathcal{F}_{M,h}$$

resp.

$$\mathcal{F}_{\omega,l} := \left\{(f_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \exists\, C > 0 : \forall\, k \in \mathbb{N} : |f_k| \le C \cdot \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot k)\right)\right\}$$

with

$$\mathcal{F}_{\{\omega\}} := \bigcup_{l>0} \mathcal{F}_{\omega,l} \qquad \mathcal{F}_{(\omega)} := \bigcap_{l\in\Lambda} \mathcal{F}_{\omega,l}.$$

We recall now some important facts and consequences if the weight sequence M is weakly log. convex, i.e. satisfies (lc) is satisfied:

(i) We can find in such classes special functions:

$$\text{(chf)} \;:\Leftrightarrow \exists\, \theta_M \in \mathcal{E}^{\text{global}}_{\{M\}}(\mathbb{R},\mathbb{R}) : \forall\, j \in \mathbb{N} : \left|\theta_M^{(j)}(0)\right| \ge M_j.$$

Sometimes in the literature such a function is called a *characteristic function*, see e.g. [39]. We recall the explicit construction: First consider the complex-valued function $\tilde{\theta}_M(t) := \sum_{k=0}^{\infty} \frac{M_k}{(2\mu_k)^k} \cdot \exp(2\sqrt{-1}\mu_k t)$ with $\mu_k = \frac{M_k}{M_{k-1}}$, see [40, Theorem 1]. We obtain $\tilde{\theta}_M \in \mathcal{E}^{\text{global}}_{\{M\}}(\mathbb{R},\mathbb{C})$ and moreover

$$\forall\, j \in \mathbb{N}:\ \tilde{\theta}_M^{(j)}(0) = (\sqrt{-1})^j \cdot s_j, \qquad s_j := \sum_{k=0}^{\infty} M_k \cdot (2\mu_k)^{j-k} \underbrace{\ge}_{j=k} M_j, \tag{2.4.7}$$

hence $\left|\tilde{\theta}_M^{(j)}(0)\right| \ge M_j$ for all $j \in \mathbb{N}$. We also get by definition $\left|\tilde{\theta}_M^{(j)}(t)\right| \le s_j = \left|\tilde{\theta}_M^{(j)}(0)\right|$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$, hence $\|\tilde{\theta}_M\|_\infty = \left|\tilde{\theta}_M(0)\right|$. We can put now $\theta_M := \Re(\tilde{\theta}_M) + \Im(\tilde{\theta}_M)$, see also [38, 3.1.2. Proposition]. More precisely we get $|\Re(\tilde{\theta}_M)^{(j)}(0)| = s_j$ for even j and $|\Re(\tilde{\theta}_M)^{(j)}(0)| = 0$ for odd j, analogously $|\Im(\tilde{\theta}_M)^{(j)}(0)| = s_j$ for odd j and $|\Im(\tilde{\theta}_M)^{(j)}(0)| = 0$ for even j. Finally in [38, 3.1.2. Proposition] it is also pointed out that the Beurling-class $\mathcal{E}^{\text{global}}_{(M)}(\mathbb{R},\mathbb{R})$ cannot contain such a characteristic function θ_M.

(ii) Using such characteristic functions one is able to prove: Let M be a weakly log. convex sequence, then $M \preceq N \Longleftrightarrow \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{N\}}$, for a proof see e.g. [38, Theorem 3.1.3].

To obtain this equivalence also for the Beurling-case one has to assume that $M \in \mathcal{LC}$ and then one can apply the second chapter in [6].

Precisely the same argument as in [38, Theorem 3.1.3] can be used to prove the following: Let M be a weakly log. convex sequence, then $M \lhd N \Longleftrightarrow \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{(N)}$.

We point out: In all equivalences above to prove $\Longleftarrow$ we need only the assumption $\mathcal{E}_{[M]}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{[N]}(\mathbb{R},\mathbb{R})$ resp. $\mathcal{E}_{\{M\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(N)}(\mathbb{R},\mathbb{R})$.

(*iii*) The classes $\mathcal{E}_{\{M\}}$ resp. $\mathcal{E}_{(M)}$ are closed under pointwise multiplication, in fact we need only condition (alg) (see e.g. [38, Proposition 2.0.8]).

(*iv*) The Denjoy-Carleman (see e.g. [36, 19.11 Theorem] or [15, Theorem 1.3.8.]) tells us that condition (nq) is satisfied, if and only if $\mathcal{E}_{[M]}$ contains functions with compact support ($\mathcal{E}_{[M]}$-test functions).

We call a class $\mathcal{E}_{[\star]}$, $\star \in \{M, \omega\}$, *not quasi-analytic* if $\mathcal{E}_{[\star]}$ contains functions with compact support, otherwise we will call it *quasi-analytic*. So (*iii*) above precisely means that if M is a weight sequence with (lc), then (nq) holds if and only if $\mathcal{E}_{[M]}$ is not quasi-analytic. Recall: In the third chapter in [5], it was shown that for $\omega \in \mathcal{W}$ with (ω_1) and (ω_{nq}) both classes $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ are not quasi-analytic.

3 Properties of weight functions - consequences and relations

The aim of this chapter is to study all introduced conditions of a function $\omega \in \mathcal{W}$ in detail. This chapter has to be understood as a preparation for the fifth chapter below, where we transfer conditions from a function ω to conditions of it's associated *weight matrix* $\mathcal{M}$.

3.1 Remarks about the convexity condition (ω_4)

We study the precise meaning of (ω_4). For this we recall the history of ultradiff. classes defined by weight functions ω (for a good and short survey see also e.g. the introductions in [5] and [4]). First for a smooth function f (with compact support) the decay property of its Fourier transform $\hat{f}$ was analyzed and it was weighted with terms $\exp(l \cdot \omega(\cdot))$ with $l > 0$ (see e.g. [34] and [5]). For this definition property (ω_4) was not used necessarily (as in [34]). In [5, 3.3 Lemma, 3.4. Proposition] the "old" definitions were transformed (for non-quasi-analytic weights) into the "new" now frequently used definitions, where all derivatives of f occur and are weighted with terms $\exp(-l \cdot \varphi^*_\omega(\frac{1}{l}\cdot))$ with $l > 0$. Finally in the fourth chapter in [5] the general classes were introduced and studied (as above in section 2.4., see 2.4.2 resp. 2.4.3).

Now for the important characterizing results [5, 3.3 Lemma, 3.4. Proposition] (ω_4) was needed: In the proof there $\varphi^{**}_\omega = \varphi_\omega$ was used which can be only satisfied if the function φ_ω itself is convex. So this condition is necessary that the new definitions using the Legendre-Fenchel-Young-conjugate are generalizations of the older ones (on the other hand in [5] the sub-additivity for ω, which was always assumed in the previous papers, was replaced by the more general condition (ω_1)).
Nevertheless the function φ^*_ω is convex in any case, we summarize the whole situation in the following lemma:

Lemma 3.1.1. *Let $\omega : [0,\infty) \to [0,\infty)$ be a function with properties (ω_0) and (ω_3). Then we obtain: For $x \geq 0$ we get $\varphi^*_\omega(x) = \sup_{y \geq 0}\{xy - \varphi_\omega(y)\}$, the function $x \mapsto \varphi^*_\omega(x)$ is well-defined, increasing, convex, $\varphi^*_\omega(0) = 0$ and the mapping $x \mapsto \frac{\varphi^*_\omega(x)}{x}$ is increasing and tending to infinity for $x \to \infty$. Finally $\varphi^{**}_\omega \leq \varphi_\omega$ holds with equality if and only if condition (ω_4) holds in addition.*

Proof. Since by assumption $\omega(x) = 0$ for $x \in [0,1]$, we get $\varphi(x) = \omega(\exp(x)) = 0$ for $x \in (-\infty, 0]$, hence $\varphi^*_\omega(x) := \sup_{y \in \mathbb{R}}\{xy - \varphi_\omega(y)\} = \sup_{y \geq 0}\{xy - \varphi_\omega(y)\}$. Since (ω_3) holds, i.e. $\frac{y}{\varphi_\omega(y)} \to 0$ for $y \to 0$, we see that $\varphi^*_\omega(x)$ is finite for any $x \geq 0$. Moreover $\varphi^*_\omega(0) = \sup_{y \geq 0}\{-\varphi_\omega(y)\} = 0$, because ω is increasing and $\varphi_\omega(0) = \omega(\exp(0)) = \omega(1) = 0$. All further statements are satisfied by definition and convexity. □

Remark 3.1.2. (i) *The main problem of property (ω_4) is the following: It is not stable w.r.t. relation $\sim$ since it is a convexity condition.*

(ii) *During this whole PhD-thesis, except the cases where it is mentioned explicitly, we will deal with weight functions in the set* $\mathcal{W}$, *so we assume* (ω_0), (ω_3) *(the assumptions in 3.1.1) and also* (ω_4).

(iii) *As a general statement we can say: In this chapter* 3 *and also in chapter* 5 *below we are going to transfer the introduced increasing conditions from* ω *first to* φ_ω, *then to* φ^*_ω *and finally to the (family of) sequences* $M^l := (M^l_j)_j$, $M^l_j := \exp(\frac{1}{l}\cdot\varphi^*_\omega(l\cdot j))$ *which will be considered and studied in chapter* 5. *For this approach, transferring conditions from* ω *to it's associated weight matrix* $\mathcal{M}$, *only the assumptions from 3.1.1 are needed for* ω, *so* (ω_0) *and* (ω_3). *More precisely we only need the convexity and increasing properties for the mapping* $t\mapsto\varphi^*_\omega(t)$ *there. Also the first part of the comparison theorem 6.2.1 resp. 6.2.6 are valid.*

(iv) *But* (ω_4) *is in fact needed, if one wants to prove characterization results (e.g. for property* (ω_6) *in 5.2.1, for property* (ω_7) *in 5.4.1, for relation* $\sim$ *in 6.5.1 and 6.5.2): More precisely it is necessary if one wants to transfer properties of the associated weight matrix* $\mathcal{M}$ *to* ω, *for this we have to consider in the proofs the "double conjugate"* φ^{**}_ω *and we really need* $\varphi^{**}_\omega=\varphi_\omega$.

(v) *A problem could also be if one has to deal with explicit properties of the associated weight matrices. If one has two functions,* $\omega\in\mathcal{W}$ *and* σ *with only* (ω_0) *and* (ω_3) *(the assumptions in 3.1.1) and assume* $\omega\sim\sigma$. *Then the proof in 3.2.1 shows* $\mathcal{E}_{[\omega]}=\mathcal{E}_{[\sigma]}$ *and the proof of 5.3.1 shows the equivalence of the associated weight matrices. Then properties for the weight matrix associated to* σ *can be transferred to the weight matrix associated to* ω, *in this situation one is able to apply the characterizing results to obtain the desired properties for* ω. *Finally, by 3.2.2 and* $\omega\sim\sigma$, *we can transfer the property from* ω *also to the weight* σ.

3.2 Equivalence of weight functions

Recall: For two weight functions ω_1 and ω_2 we write $\omega_1\preceq\omega_2$ if $\omega_2(t)=O(\omega_1(t))$ (for $t\to\infty$) and we call two weights equivalent and write $\omega_1\sim\omega_2$ if $\omega_1\preceq\omega_2$ and $\omega_2\preceq\omega_1$. The following result is well-known, it was mentioned e.g. in [5, 3.2. Remark]. For convenience of the reader we give the full proof:

Lemma 3.2.1. *Let* $\omega_1,\omega_2\in\mathcal{W}$ *be given, if* $\omega_1\preceq\omega_2$, *then we get* $\mathcal{E}_{\{\omega_1\}}\subseteq\mathcal{E}_{\{\omega_2\}}$ *resp.* $\mathcal{E}_{(\omega_1)}\subseteq\mathcal{E}_{(\omega_2)}$. *If* $\omega_1\sim\omega_2$, *then* $\mathcal{E}_{\{\omega_1\}}=\mathcal{E}_{\{\omega_2\}}$ *and* $\mathcal{E}_{(\omega_1)}=\mathcal{E}_{(\omega_2)}$, *so one can change to an equivalent weight, without changing the associated function space of Roumieu- and Beurling-type.*

Proof. $\omega_1\preceq\omega_2$ implies the existence of a constant $C\ge 1$ such that $\omega_2(t)\le C\cdot\omega_1(t)+C$ for all $t\ge 0$. Thus we obtain for the function φ_ω also $\varphi_{\omega_2}(t)\le C\cdot\varphi_{\omega_1}(t)+C$ for all $t\in\mathbb{R}$ and so we can estimate:

$$\varphi_{\omega_2}(y)\le C\cdot\varphi_{\omega_1}(y)+C\Longrightarrow C\cdot\sup_{y\ge 0}\left\{\frac{x}{C}\cdot y-\varphi_{\omega_1}(y)\right\}\le\sup_{y\ge 0}\{x\cdot y-\varphi_{\omega_2}(y)\}+C$$
$$\Longrightarrow C\cdot\varphi^*_{\omega_1}\left(\frac{x}{C}\right)\le\varphi^*_{\omega_2}(x)+C.$$

The second implication above is by definition clearly an equivalence, but for the first one the converse conclusion is not true because the supremum can be attained at different values. Similarly one gets $D\cdot\varphi^*_{\omega_2}\left(\frac{x}{D}\right)\le\varphi^*_{\omega_1}(x)+D$ for all $x\ge 0$ and a constant $D\ge 1$ if one assumes $\omega_2\preceq\omega_1$. □

"Good and nice" conditions and assumptions on a weight ω shall be stable w.r.t the relation $\sim$, so we summarize:

Lemma 3.2.2. *Let $\omega : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $\lim_{x\to\infty} \omega(x) = +\infty$.*
Then properties $(\omega_1), (\omega_2), (\omega_3), (\omega_5)$, (ω_6) and (ω_7) are stable under relation $\sim$. Moreover property $\omega(x) = 0$ for $x \in [0,1]$ in (ω_0) can be assumed w.l.o.g.

Proof. Properties (ω_2), (ω_3) and (ω_5) are clearly stable w.r.t. relation $\sim$ (follows by definition). For (ω_1) we point out: Assume that τ satisfies (ω_1) and $\tau \sim \sigma$, then σ also satisfies (ω_1): For all $t \geq 0$ on the one hand we have $\tau(2t) \leq C \cdot \tau(t) + C \leq C \cdot C_1 \cdot \sigma(t) + C \cdot C_1 + C$, on the other hand $\frac{1}{C_2} \cdot \sigma(2t) - 1 \leq \tau(2t)$. Hence combining these two inequalities we get $\sigma(2t) \leq C \cdot C_1 \cdot C_2 \cdot \sigma(t) + C \cdot C_1 \cdot C_2 + C \cdot C_2 + C_2 = D \cdot \sigma(t) + D$ for all $t \geq 0$.

For the stability of property (ω_6) we point out: Assume that τ has (ω_6) and $\sigma \sim \tau$. Then recall (3.4.2): By iterating property (ω_6) we get that for each $n \in \mathbb{N}$ and $t \geq 0$ we obtain $2^n \cdot \tau(t) \leq \tau(H^n \cdot t) + (2^n - 1) \cdot H$. By the equivalence $\sigma \sim \tau$ we have that there exist constants $C_1 \geq 1$ and $C_2 > 0$ large enough such that for all $t \geq 0$ we obtain $\frac{1}{C_1} \cdot \tau(t) - C_2 \leq \sigma(t) \leq C_1 \cdot \tau(t) + C_2$. Then take $n \in \mathbb{N}$ minimal such that $C_1 \leq 2^n$ is satisfied, and with this special choice we obtain for all $t \geq 0$:

$$\begin{aligned} 2 \cdot \sigma(t) &\leq 2 \cdot C_1 \cdot \tau(t) + 2 \cdot C_2 \underbrace{\leq}_{(\star)} \frac{1}{C_1} \cdot \tau(H_1 \cdot t) + H_2 + 2 \cdot C_2 \\ &\leq \sigma(H_1 \cdot t) + H_2 + 3 \cdot C_2 \leq \sigma(H_3 \cdot t) + H_3. \end{aligned}$$

$(\star)$ holds because $2 \cdot C_1^2 \cdot \tau(t) \leq 2^{2n+1} \cdot \tau(t) \leq \tau(H^{2n+1} \cdot t) + (2^{2n+1} - 1) \cdot H$, hence $2 \cdot C_1 \cdot \tau(t) \leq \frac{1}{C_1} \cdot \tau(H_1 \cdot t) + H_2$ for all $t \geq 0$.

Condition (ω_7): Assume that τ has (ω_7) and let σ with $\sigma \sim \tau$. Then on the one hand $\tau(t^2) \leq C \cdot \tau(H \cdot t) + C \leq B_1 \cdot C \cdot \sigma(H \cdot t) + B_2 + C$ and on the other hand $\frac{1}{B_1} \cdot \sigma(t^2) - B_2 \leq \tau(t^2)$ for constants $B_1, B_2 > 1$ large enough and all $t \geq 0$. By combining both inequalities we obtain (ω_7) for σ:
$\sigma(t^2) \leq B_1^2 \cdot C \cdot \sigma(H \cdot t) + 2B_2 \cdot B_1 + C \cdot B_1$.

We finally remark: The assumption $\omega(x) = 0$ for all $x \in [0,1]$ can be assumed w.l.o.g. If ω is given with all assumptions of (ω_0) except $\omega(1) = a > 0$, then we can replace ω by the weight σ defined via $\sigma(x) := \omega(x) - a$ for all $x \geq 1$ and $\sigma(x) = 0$ for $x \in [0,1]$. Then σ satisfies of course (ω_0) and $\sigma \sim \omega$, because $\lim_{x\to\infty} \omega(x) = \lim_{x\to\infty} \sigma(x) = +\infty$. □

Note: A weight $\omega \in \mathcal{W}$ is in general not equivalent to the weight σ defined by $\sigma(x) = \omega(x + a)$ for some $a > 0$. $\sigma \preceq \omega$ is clear, but for the converse direction one would need property (ω_1) for ω, in fact we use (3.3.1) and calculate as follows:

$$\frac{\sigma(x)}{\omega(x)} = \frac{\omega(x+a)}{\omega(x)} \underbrace{\leq}_{(\star)} \frac{\omega(2^n \cdot x)}{\omega(x)} \underbrace{\leq}_{(3.3.1)} \frac{L^n \cdot \omega(x) + \sum_{i=1}^n L^i}{\omega(x)} = L^n + \underbrace{\frac{\sum_{i=1}^n L^i}{\omega(x)}}_{\to 0, x\to\infty}$$

$(\star)$ holds for $x \geq \frac{2^n}{2^n - 1}$: Take $n \in \mathbb{N}$ minimal such that $a \leq 2^n$ is satisfied, then $x + a \leq x + 2^n$ and $x + 2^n \leq 2^n \cdot x \Leftrightarrow 2^n \leq x \cdot (2^n - 1) \Leftrightarrow \frac{2^n}{2^n - 1} \leq x$.

3.3 Important consequences from (ω_1)

Property (ω_1) is much more general than sub-additivity for ω and has important consequences. It can be stated in two different (equivalent) ways: First we can say that there exists a constant $C \geq 1$ and $t_0 > 0$, such that $\omega(2t) \leq C \cdot \omega(t)$ holds for all $t \geq t_0$ (w.l.o.g. can take $t_0 = 1$)! Iterating this estimate we get $\omega(2^k \cdot t) \leq C^k \cdot \omega(t)$ for all $t \geq t_0$ and $k \in \mathbb{N}$.

The second equivalent formulation is:

$$\exists\, L \geq 1\, \forall\, t \geq 0 : \omega(2t) \leq L \cdot \omega(t) + L,$$

and by iterating this inequality we can prove:

$$\forall\, n \in \mathbb{N}:\ \omega(2^n \cdot t) \leq L^n \cdot \omega(t) + \sum_{i=1}^{n} L^i. \tag{3.3.1}$$

For $n = 1$ there is nothing to show, for $n \mapsto n+1$ we get: $\omega(2^{n+1} \cdot t) \leq L \cdot \omega(2^n \cdot t) + L \leq L \cdot \left(L^n \cdot \omega(t) + \sum_{i=1}^{n} L^i\right) + L = L^{n+1} \cdot \omega(t) + \sum_{i=2}^{n+1} L^i + L$.

Both conditions are equivalent since $\lim_{t\to\infty} \omega(t) = +\infty$, but with the second method we obtain very important formulas for φ^*_ω. Let $\omega \in \mathcal{W}$, then we get (ω is increasing):

$$\begin{aligned}
\varphi_\omega(y+1) &= \omega(\exp(y+1)) = \omega(\exp(y) \cdot \exp(1)) = \omega\left(2 \cdot \frac{\exp(1)}{2 \cdot \exp(y)}\right) \\
&\leq L \cdot \left(1 + \omega\left(\frac{\exp(1)}{2 \cdot \exp(y)}\right)\right) \underbrace{\leq}_{\frac{\exp(1)}{2} \leq 2 \Leftrightarrow \exp(1) \leq 4} L \cdot (1 + \omega(2 \cdot \exp(y))) \\
&\leq L \cdot (1 + L \cdot (1 + \omega(\exp(y)))) = L + L^2 + L^2 \cdot \omega(\exp(y)) \leq L_1 \cdot (1 + \omega(\exp(y))) \\
&= L_1 \cdot (1 + \varphi_\omega(y)),
\end{aligned}$$

where we have put $L_1 = L^2 + L = L(L+1) > L$.
So we have shown, that $\varphi_\omega(y) \leq L_1 \cdot (1 + \varphi_\omega(y-1))$ for a constant $L_1 \geq 1$ and all $y \in \mathbb{R}$. Since $L_1 > L$ clearly $\omega(2t) \leq L_1 \cdot (1 + \omega(t))$ for all $t \geq 0$ is satisfied (so (ω_1) is still satisfied for the constant L_1), then denote in the following this constant L_1 again by L.

The next proposition is very important, we will use it to prove important inequalities and for the convenience of the reader we will prove it in detail. In the literature it has already been used several times (e.g. in [10]), and in [10, Proposition 2.1.] it is mentioned that a proof of it can be found in [13, 1.1.18.].

Proposition 3.3.1. *Let $\omega \in \mathcal{W}$ be given with additionally condition (ω_1). Then*

$$\exists\, L \geq 1\, \forall\, \lambda > 0\, \forall\, s \in \mathbb{N}\, \exists\, \mu > 0\, \forall\, n \in \mathbb{N}:\ \lambda \cdot \varphi^*_\omega\left(\frac{n}{\lambda}\right) + n \cdot s \leq \mu \cdot \varphi^*_\omega\left(\frac{n}{\mu}\right) + \mu \cdot \sum_{i=1}^{s} L^i \tag{3.3.2}$$

is satisfied.

Proof. First we show that there exists a constant $L \geq 1$ such that for all $x \geq 0$ we get

$$L \cdot \varphi^*_\omega(x) + L \cdot x \leq \varphi^*_\omega(L \cdot x) + L. \tag{3.3.3}$$

This holds by the following calculation:

$$\begin{aligned}\varphi^*_\omega(L\cdot x) &= \sup\{(L\cdot x)\cdot y-\varphi_\omega(y): y\ge 0\} = L\cdot\sup\left\{x\cdot y-\frac{1}{L}\cdot\varphi_\omega(y): y\ge 0\right\}\\ &\ge L\cdot\sup\{x\cdot y-(1+\varphi_\omega(y-1)): y\ge 0\}\ge L\cdot\sup\{x\cdot y-(1+\varphi_\omega(y-1)): y\ge 1\}\\ &= L\cdot\sup\{x\cdot(y-1)+x-1-\varphi_\omega(y-1): y\ge 1\}\\ &= L\cdot x-L+L\cdot\sup\{x\cdot(y-1)-\varphi_\omega(y-1): y\ge 1\} = L\cdot x-L+L\cdot\varphi^*_\omega(x).\end{aligned}$$

We summarize: $(\omega_{\text{sub}})\Rightarrow(\omega_1)\Rightarrow$ (3.3.3) holds, where (ω_{sub}) denotes the sub-additivity condition which will be studied in detail below.
We prove by induction:

$$\exists\, L\ge 1:\ \forall\, s\in\mathbb{N}:\ \forall\, x\ge 0:\ \ L^s\cdot\varphi^*_\omega(x)+s\cdot L^s\cdot x\le\varphi^*_\omega(L^s\cdot x)+\sum_{i=1}^{s}L^i,$$

and one can take for the constant $L\ge 1$ above the constant which appears in (ω_1).

The case $s=1$ is precisely inequality (3.3.3).
The case $s\mapsto s+1$: By multiplying the inequality for the case s (I.H.) with $L\ge 1$ we get $L^{s+1}\cdot\varphi^*_\omega(x)+s\cdot L^{s+1}\cdot x\le L\cdot\varphi^*_\omega(L^s\cdot x)+\sum_{i=2}^{s+1}L^i$. Using now the case $s=1$ for the point $L^s\cdot x$, then we can estimate $L\cdot\varphi^*_\omega(L^s\cdot x)\le L+\varphi^*_\omega(L^{s+1}\cdot x)-L^{s+1}\cdot x$. Combining these two inequalities we get:

$$L^{s+1}\cdot\varphi^*_\omega(x)+s\cdot L^{s+1}\cdot x\le L\cdot\varphi^*_\omega(L^s\cdot x)+\sum_{i=2}^{s+1}L^i\le L+\varphi^*_\omega(L^{s+1}\cdot x)-L^{s+1}\cdot x+\sum_{i=2}^{s+1}L^i$$

$$\Longrightarrow L^{s+1}\cdot\varphi^*_\omega(x)+s\cdot L^{s+1}\cdot x+L^{s+1}\cdot x\le\varphi^*_\omega(L^{s+1}\cdot x)+\sum_{i=1}^{s+1}L^i.$$

Let $\lambda>0$, then we use this formula for the point $x=\frac{n}{\lambda}$, $n\in\mathbb{N}$ arbitrary, and put $\mu=\frac{\lambda}{L^s}>0$, $s\in\mathbb{N}$ arbitrary. Clearly $\mu\le\lambda$ and we get: $L^s=\frac{\lambda}{\mu}$, $x=\frac{n}{\lambda}$, $s\cdot L^s\cdot x=s\cdot n\cdot\frac{L^s}{\lambda}=\frac{s\cdot n}{\mu}$ and $L^s\cdot x=n\cdot\frac{L^s}{\lambda}=\frac{n}{\mu}$. With this special choice we have shown (after multiplying the inequality with $\mu>0$!):

$$\exists\, L\ge 1\,\forall\,\lambda>0\,\forall\, s\in\mathbb{N}\,\exists\,\mu>0\,\forall\, n\in\mathbb{N}:\ \lambda\cdot\varphi^*_\omega\left(\frac{n}{\lambda}\right)+n\cdot s\le\mu\cdot\varphi^*_\omega\left(\frac{n}{\mu}\right)+\mu\cdot\sum_{i=1}^{s}L^i.$$

□

Remarks:

- (*i*) The proof of the previous proposition shows that in (3.3.2) one can replace "$\forall\,\lambda>0\,\forall\, s\in\mathbb{N}\,\exists\,\mu>0$" also by "$\forall\,\mu>0\,\forall\, s\in\mathbb{N}\,\exists\,\lambda>0$", because there is a unique correspondence between λ and μ, which is given by $\mu=\frac{\lambda}{L^s}$.
- (*ii*) This inequality is used in [10, §2] to prove closedness under composition for classes of ultradifferentiable functions defined by (non-quasi-analytic) weight functions ω.
- (*iii*) If we divide (3.3.2) by $n\in\mathbb{N}_{>0}$ then this implies:

$$\exists\, L\ge 1\,\forall\, m>0\,\forall\, s\in\mathbb{N}\,\exists\, l(=L^s\cdot m):\quad\liminf_{p\to\infty}\frac{1}{lp}\cdot\varphi^*_\omega(lp)-\frac{1}{mp}\cdot\varphi^*_\omega(mp)\ge s. \tag{3.3.4}$$

With an analogous calculation as before we will prove in 9.3.8 the following generalization which will be needed there:

Proposition 3.3.2. *Let a family of functions $\{\sigma_l : l \in \Lambda\}$ be given, where Λ a partially ordered index set and such that each $\sigma_l \in \mathcal{W}$. Assume now that for all $l \in \Lambda$ there exists $n \in \Lambda$ with $\sigma_n(2t) = O(\sigma_l(t))$ for $t \to \infty$, which means that for each $l \in \Lambda$ there exists $n \in \Lambda$ $(n < l)$ and a constant $L \geq 1$ such that $\sigma_n(2t) \leq L \cdot (1 + \sigma_l(t))$ for all $t \geq 0$. Then we obtain*

$$\forall\, l \in \Lambda\, \forall\, s \in \mathbb{N}\, \exists\, n \in \Lambda\, \exists\, L \geq 1\, \forall\, a > 0\, \forall\, j \in \mathbb{N} :$$
$$\exp\left(\frac{1}{a} \cdot \varphi^*_{\sigma_l}(aj)\right) \cdot \exp(s)^j \leq \exp\left(\frac{\sum_{i=1}^s L^i}{L^s \cdot a}\right) \cdot \exp\left(\frac{1}{L^s \cdot a} \cdot \varphi^*_{\sigma_n}(L^s \cdot a \cdot j)\right).$$

If for all $l \in \Lambda$ there exists $n \in \Lambda$ with $\sigma_l(2t) = O(\sigma_n(t))$ for $t \to \infty$, then we get an analogously result ("Beurling-type").

If each σ_l has (ω_1), then clearly the previous proposition is satisfied for both cases with the choice $l = n$.

3.4 Important consequences from (ω_6)

For the second comparison result [4, 16. Corollary], the following condition was the important characterizing one

$$(\omega_6) :\ \exists\, H \geq 1 :\ \forall\, t \geq 0 :\ 2 \cdot \omega(t) \leq \omega(H \cdot t) + H,$$

which means $\exists\, H \geq 1 : \limsup_{t\to\infty} \frac{\omega(t)}{\omega(Ht)} \leq \frac{1}{2}$ for $\omega \in \mathcal{W}$. Furthermore this condition is equivalent to $\exists\, \tilde{H} \geq 1 :\ \exists\, t_0 > 0\ \forall\, t \geq t_0 : 2 \cdot \omega(t) \leq \omega(\tilde{H}t)$ (since $\omega(t) \to \infty$ for $t \to \infty$) and so (ω_6) is not condition $\omega(t) = O(\omega(2t))$ for $t \to \infty$, which means $\limsup_{t\to\infty} \frac{\omega(t)}{\omega(2t)} < \infty$ (and which is always satisfied, because ω is increasing)!
We point out: H. Komatsu has shown in [16, Proposition 3.6] that if $M \in \mathcal{LC}$, then property (ω_6) for the associated function ω_M (for the definition see (4.0.1) below) is equivalent to the fact, that M satisfies (mg)!

Assume that (ω_6) is satisfied for $0 < H < 1$. W.l.o.g. we can assume that $\omega(t) = 0$ for $0 \leq t \leq t_0$. Let now $t_1 > t_0$ be arbitrary but fixed and put $\omega(t_1) = c > 0$. Hence $2^n \cdot \omega(t_1) = 2^n \cdot c > 0$ and there exists $N \in \mathbb{N}$ depending on t_1, t_0 and H such that $H^N \cdot t_1 \leq t_0$, which means $\omega(H^n \cdot t_1) = 0$ for all $n \geq N$. This implies now $2^n \cdot c \leq (2^n - 1) \cdot H$ for all $n \geq N$ and so

$$\frac{2^n \cdot c}{2^n \cdot H} \leq 1 - \frac{H}{2^n \cdot H} \Longrightarrow c < \frac{c}{H} \leq 1 - \frac{1}{2^n}$$

for all $n \geq N$. Thus $\omega(t_1) = c < 1$ and so $\omega(t) \leq 1$ for all $t \geq 0$ (if t_1 is tending to infinity, then N has to tend to infinity) which cannot hold for $\omega \in \mathcal{W}$.

We show now

Proposition 3.4.1. *Assume that $\omega \in \mathcal{W}$ satisfies additionally (ω_6). Then we obtain:* $\exists\, H \geq 1\, \forall\, l > 0\, \forall\, n \in \mathbb{N}\, \exists\, m > 0\ (m = \frac{l}{2^n} < l)\ \forall\, p \in \mathbb{N} :$

$$\frac{1}{l} \cdot \varphi^*_\omega(l \cdot p) \leq \frac{1}{m} \cdot \varphi^*_\omega(m \cdot p) + n \cdot p \cdot \log(H) + \left(\frac{1}{m} - \frac{1}{l}\right) \cdot H. \tag{3.4.1}$$

Proof. By iterating (ω_6) we get

$$2^n \cdot \omega(t) \leq \omega(H^n \cdot t) + (2^n - 1) \cdot H \tag{3.4.2}$$

for all $t \geq 0$ and $n \in \mathbb{N}$: For $n = 1$ there is nothing to show and for $n \mapsto n+1$ we calculate:

$$\begin{aligned} 2 \cdot 2^{n-1} \cdot \omega(t) \underbrace{\leq}_{I.H.} 2 \cdot \omega(H^{n-1} \cdot t) + 2 \cdot (2^{n-1} - 1) \cdot H \underbrace{\leq}_{(\omega_6)} \omega(H^n \cdot t) + H + 2^n \cdot H - 2 \cdot H \\ = \omega(H^n \cdot t) + (2^n - 1) \cdot H. \end{aligned}$$

We transfer now condition (ω_6) to a condition for the function φ^*_ω: For this we divide by 2 and replace $H \cdot t$ by t to get $\omega(\frac{t}{H}) \leq \frac{1}{2} \cdot \omega(t) + \frac{H}{2}$. Furthermore write $H = \exp(h)$, so $h = \log(H) \geq 0$, to get $\omega\left(\frac{\exp(s)}{\exp(h)}\right) \leq \frac{1}{2} \cdot \omega(\exp(s)) + \frac{H}{2}$, hence

$$\varphi_\omega(s - h) \leq \frac{1}{2} \cdot \varphi_\omega(s) + \frac{H}{2}$$

and we have this for all $s \in \mathbb{R} \Leftrightarrow t > 0$. Now apply the *Legendre-Fenchel-Young-conjugate* on both sides at the point $x \geq 0$:
The left hand side gives, if we put $y' = y - h$:

$$\sup_{y \in \mathbb{R}} \{x \cdot y - \varphi_\omega(y - h)\} = \sup_{y' \in \mathbb{R}} \{x \cdot y' - \varphi_\omega(y')\} + h \cdot x = \sup_{y' \geq 0} \{x \cdot y' - \varphi_\omega(y')\} + h \cdot x = \varphi^*_\omega(x) + h \cdot x,$$

and the right hand side gives

$$\sup_{y \geq 0} \left\{x \cdot y - \frac{1}{2} \cdot \varphi_\omega(y) - \frac{H}{2}\right\} = \frac{1}{2} \cdot \sup_{y \geq 0} \{(2x) \cdot y - \varphi_\omega(y)\} - \frac{H}{2} = \frac{1}{2} \cdot \varphi^*_\omega(2x) - \frac{H}{2}.$$

Combining both sides we obtain:

$$\exists\, H \geq 1\ \forall\, x \geq 0 : \varphi^*_\omega(2x) \leq 2 \cdot \varphi^*_\omega(x) + 2 \cdot x \cdot \log(H) + H.$$

Using this we can prove now by induction:

$$\exists\, H \geq 1\ \forall\, n \in \mathbb{N}\ \forall\, x \geq 0 : \quad \varphi^*_\omega(2^n \cdot x) \leq 2^n \cdot \varphi^*_\omega(x) + n \cdot 2^n \cdot x \cdot \log(H) + (2^n - 1) \cdot H. \tag{3.4.3}$$

The case $n = 1$ is satisfied by the calculation before. Now $n \mapsto n+1$:

$$\begin{aligned} &\varphi^*_\omega(2 \cdot 2^n \cdot x) \leq 2 \cdot \varphi^*_\omega(2^n \cdot x) + 2^{n+1} \cdot x \cdot \log(H) + H \\ &\underbrace{\leq}_{I.H.} 2^{n+1} \cdot \varphi^*_\omega(x) + 2 \cdot n \cdot 2^n \cdot x \cdot \log(H) + 2 \cdot (2^n - 1) \cdot H + 2^{n+1} \cdot x \cdot \log(H) + H \\ &= 2^{n+1} \cdot \varphi^*_\omega(x) + (n+1) \cdot 2^{n+1} \cdot x \cdot \log(H) + (2^{n+1} - 2 + 1) \cdot H. \end{aligned}$$

We consider (3.4.3) and put in this equation $x = m \cdot p$, $p \in \mathbb{N}$, $m > 0$, and $l = 2^n \cdot m > m$, then $2^n \cdot x = 2^n \cdot m \cdot p = l \cdot p$. Thus

$$\varphi^*_\omega(l \cdot p) \leq \frac{l}{m} \cdot \varphi^*_\omega(m \cdot p) + n \cdot l \cdot p \cdot \log(H) + \left(\frac{l}{m} - 1\right) \cdot H,$$

and if we divide both sides by $l > 0$, then we have (for $l = 2^n \cdot m$):
$\exists\, H \geq 1\ \forall\, l > 0\ \forall\, n \in \mathbb{N}\ \exists\, m > 0 (m = \frac{l}{2^n} < l)\ \forall\, p \in \mathbb{N}$:

$$\frac{1}{l} \cdot \varphi^*_\omega(l \cdot p) \leq \frac{1}{m} \cdot \varphi^*_\omega(m \cdot p) + n \cdot p \cdot \log(H) + \left(\frac{1}{m} - \frac{1}{l}\right) \cdot H.$$

□

Remarks:

(*i*) The proof above shows that in (3.4.1) the expression "$\forall\, l > 0\ \forall\, n \in \mathbb{N}\ \exists\, m > 0$" can be also replaced by "$\forall\, m > 0\ \forall\, n \in \mathbb{N}\ \exists\, l > 0$", because there is a unique correspondence between l and m which is given by $l = 2^n \cdot m$.

(*ii*) If we divide both sides in (3.4.1) by $p \in \mathbb{N}_{>0}$, then this turns into

$$\exists\, H \geq 1\ \forall\, l > 0\ \forall\, n \in \mathbb{N}\ \exists\, m \in \mathbb{N}:\ \limsup_{p\to\infty} \frac{1}{lp}\cdot\varphi^*_\omega(l\cdot p) - \frac{1}{mp}\cdot\varphi^*_\omega(m\cdot p) \leq n\cdot\log(H). \tag{3.4.4}$$

3.5 Important consequences from (ω_2) and (ω_5)

With 3.2.1 condition (ω_2) has the following meaning: It is known (see e.g. [4]) that the classical Gevrey classes $\mathcal{E}_{[G^s]}$ of parameters $s \geq 1$ defined by the sequences $G^s := (p!^s)_p$ are equal to the classes $\mathcal{E}_{[\omega_s]}$, by using the weight functions $\omega_s(t) = t^{1/s}$. Thus the class of real analytic functions $\mathcal{C}^\omega$ is exactly $\mathcal{E}_{\{\mathrm{id}\}}$ (using the case $s = 1$).
Hence condition (ω_2) means that $\mathrm{id} \preceq \omega$ holds and so $\mathcal{E}_{\{\mathrm{id}\}} \subseteq \mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(\mathrm{id})} \subseteq \mathcal{E}_{(\omega)}$. This tells us that the real-analytic functions are contained in the Roumieu-class $\mathcal{E}_{\{\omega\}}$ resp. restrictions of entire functions in the Beurling-class $\mathcal{E}_{(\omega)}$.

Sometimes for the Beurling-case condition (ω_5) is considered and which is much stronger than (ω_2). This condition means precisely $\mathrm{id} \lhd \omega$ and has the following meaning:

Lemma 3.5.1. *Let $\omega, \sigma \in \mathcal{W}$, then we obtain:*

(1) *If $\omega \lhd \sigma$, then the following inclusion holds: $\mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{(\omega)}$.*

(2) *If ω has additionally (ω_5), then the class of real-analytic functions $\mathcal{C}^\omega$ is contained in the Beurling-class, so $\mathcal{E}_{\{\mathrm{id}\}} \subseteq \mathcal{E}_{(\omega)}$.*

Proof. (1) First condition $\sigma(t) = o(\omega(t))$ for $t \to \infty$ means

$$\forall\, C > 0\ \exists\, D_C > 0\ \forall\, t \geq 0 : \sigma(t) \leq C \cdot \omega(t) + D_C \Leftrightarrow \frac{1}{C}\cdot\sigma(t) - \frac{D_C}{C} \leq \omega(t),$$

which implies $\frac{1}{C}\cdot\varphi_\sigma(t) - \frac{D_C}{C} \leq \varphi_\omega(t)$.
Applying now the conjugate-operator we get $\left(\frac{1}{C}\cdot\varphi_\sigma(\cdot) - \frac{D_C}{C}\right)^*(x) \geq \varphi^*_\omega(x)$ and the left hand side of this equation gives for all $x \geq 0$:

$$\left(\frac{1}{C}\cdot\varphi_\sigma(\cdot) - \frac{D_C}{C}\right)^*(x) = \sup_{y\geq 0}\left\{x\cdot y - \left(\frac{1}{C}\cdot\varphi_\sigma(y) - \frac{D_C}{C}\right)\right\} = \frac{D_C}{C} + \frac{1}{C}\cdot\varphi^*_\sigma(C\cdot x).$$

Thus this condition turns into:

$$\forall\, C > 0\ \exists\, D_C > 0\ \forall\, x \geq 0 : \varphi^*_\omega(x) \leq \frac{1}{C}\cdot\varphi^*_\sigma(C\cdot x) + \frac{D_C}{C},$$

and this implies immediately the desired inclusion.

(2) This is an immediate consequence of (1): condition (ω_5) means now that $\mathcal{E}_{\{\mathrm{id}\}} \subseteq \mathcal{E}_{(\omega)}$. □

Conditions (ω_2) and (ω_5) imply important inequalities (where we put $0^0 = 1$):

Lemma 3.5.2. *Let $\omega \in \mathcal{W}$, then we obtain:*

(1) *If ω satisfies also (ω_1) and (ω_2), then*

$$\exists\, C > 0\, \exists\, D > 0\, \forall\, x \geq 0 : \quad x^x \leq D \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right). \tag{3.5.1}$$

(2) *If ω satisfies also (ω_1) and (ω_5), then*

$$\forall\, C > 0\, \exists\, D_C > 0\, \forall\, x \geq 0 : \quad x^x \leq D_C \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right). \tag{3.5.2}$$

Proof. For the weight $\omega(t) = t$ we have $\varphi_\omega(t) = \exp(t)$, so define for $x, y \geq 0$ the function $f_x(y) = x \cdot y - \exp(y)$, then $f_x^{'}(y) = x - \exp(y) = 0 \Leftrightarrow y = \log(x)$. Hence $f_x(\log(x)) = x \cdot \log(x) - x = \varphi^*_\omega(x)$ for all $x \geq 0$ and so (ω_5) turns into

$\forall\, C > 0\, \exists\, D_C > 0\, \forall\, x \geq 0 :$

$$x \cdot \log(x) \leq x + \frac{1}{C} \cdot \varphi^*_\omega(C \cdot x) + \frac{D_C}{C} \Leftrightarrow x^x \leq \exp\left(\frac{D_C}{C}\right) \cdot \exp(x) \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right), \tag{3.5.3}$$

while (ω_2) means that

$$\exists\, C > 0\, \exists\, D > 0\, \forall\, x \geq 0 : \quad x^x \leq \exp\left(\frac{D}{C}\right) \cdot \exp(x) \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right). \tag{3.5.4}$$

We use now property (ω_1), more precisely consequence (3.3.2): We have

$$\exp(x) \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right) = \exp\left(x + \frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right)$$
$$= \exp\left(\frac{1}{C_1} \cdot \left(C_1 \cdot x + L \cdot \varphi^*_\omega\left(C_1 \cdot \frac{x}{L}\right)\right)\right) \leq \exp\left(\frac{1}{C_1}\right) \cdot \exp\left(\frac{1}{C_1} \cdot \left(\varphi^*_\omega(C_1 \cdot x)\right)\right),$$

with $C_1 = L \cdot C \geq C$, where $L \geq 1$ is the constant appearing in (ω_1). So (ω_2) gives by (3.5.4)

$$\exists\, C > 0\, \exists\, D > 0\, \forall\, x \geq 0 : \quad x^x \leq D \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right),$$

and property (ω_5) implies by (3.5.3)

$$\forall\, C > 0\, \exists\, D_C > 0\, \forall\, x \geq 0 : \quad x^x \leq D_C \cdot \exp\left(\frac{1}{C} \cdot \varphi^*_\omega(C \cdot x)\right).$$

Remark: (3.5.1) and (3.5.2) are often used for the case where $x \geq 0$ is replaced by $n \in \mathbb{N}$ (e.g. in [1]). □

3.6 Important consequences of (ω_7)

Next we study condition (ω_7), which seems to be new in the literature:

Lemma 3.6.1. *Let $\omega \in \mathcal{W}$, if ω satisfies also*

$$(\omega_7) :\Leftrightarrow \exists\, H > 0\, \exists\, C > 0\, \forall\, t \geq 0 : \omega(t^2) \leq C \cdot \omega(H \cdot t) + C,$$

then there exist $C, H > 0$ such that for all $x \geq 0$ we get

$$C \cdot \varphi^*_\omega\left(\frac{x}{C}\right) \leq \varphi^*_\omega\left(\frac{x}{2}\right) + \log(H) \cdot x + C.$$

Proof. First we translate (ω_7) into a property for the function $\varphi_\omega = \omega \circ \exp$: Put $t = \exp(s)$, then $t^2 = \exp(2s)$ and for $h = \log(H)$ we get $H \cdot t = \exp(\log(H)) \cdot \exp(s) = \exp(h+s)$ and see that (ω_7) means $\varphi_\omega(2s) \le C \cdot \varphi_\omega(h+s) + C$. Now we calculate the function φ^*_ω, so apply the *Legendre-Fenchel-Young-conjugate* on both sides. For $x \ge 0$ the left hand side gives

$$(\varphi_\omega(2\cdot))^*(x) = \sup_{y\ge 0}\{x \cdot y - \varphi_\omega(2y)\} = \sup_{y'\ge 0}\left\{\frac{x}{2} \cdot y' - \varphi_\omega(y')\right\} = \varphi^*_\omega\left(\frac{x}{2}\right),$$

where we have put $y' = 2y$. For the right hand side we get:

$$\begin{aligned}
&(C \cdot \varphi_\omega(h + \cdot) + C)^*(x) = \sup_{y\in\mathbb{R}}\{x \cdot y - C \cdot \varphi_\omega(h+y) - C\} \\
&= C \cdot \sup_{z\in\mathbb{R}}\left\{\frac{x}{C} \cdot (z-h) - \varphi_\omega(z)\right\} - C = C \cdot \sup_{z\in\mathbb{R}}\left\{\frac{x}{C} \cdot z - \varphi_\omega(z)\right\} - h \cdot x - C \\
&= C \cdot \sup_{z\ge 0}\left\{\frac{x}{C} \cdot z - \varphi_\omega(z)\right\} - h \cdot x - C = C \cdot \varphi^*_\omega\left(\frac{x}{C}\right) - h \cdot x - C,
\end{aligned}$$

where we have put $z := h + y$.
Hence we have transformed (ω_7) into a condition for φ^*_ω: There exist $C, H > 0$ such that for all $x \ge 0$ we get

$$C \cdot \varphi^*_\omega\left(\frac{x}{C}\right) \le \varphi^*_\omega\left(\frac{x}{2}\right) + \log(H) \cdot x + C. \tag{3.6.1}$$

□

3.7 Non-quasi-analyticity for ω

Recall: A weight function ω is called a *non quasi-analytic weight* if it satisfies

$$(\omega_{\text{nq}}) :\Leftrightarrow \int_1^\infty \frac{\omega(t)}{t^2} dt < \infty \Leftrightarrow \int_1^\infty \frac{\omega(t)}{1+t^2} dt < \infty,$$

otherwise it is called *quasi-analytic*. For the equivalence above we remark that for $t \ge 1$ we have $\frac{1+t^2}{2} \le t^2 \le 1 + t^2$. From the definition it's clear, that (ω_{nq}) is stable w.r.t. equivalence $\sim$ of weight functions.

A stronger condition than (ω_{nq}) is

$$(\omega_{\text{snq}}) :\Leftrightarrow \exists\, C > 0 :\ \forall\, y > 0 : \int_1^\infty \frac{\omega(y \cdot t)}{t^2} dt \le C \cdot \omega(y) + C,$$

because by assuming (ω_{snq}) for the particular value $y = 1$, we get $\int_1^\infty \frac{\omega(t)}{t^2} dt \le C \cdot \omega(1) + C < +\infty$. A further consequence is: If ω is only assumed to be an increasing continuous function, then (ω_{nq}) implies (ω_2), because for all $t \ge 1$ we have:

$$\frac{\omega(t)}{t} = \omega(t) \cdot \int_t^\infty \frac{1}{s^2} ds \underbrace{\le}_{\omega \text{ is incr.}} \int_t^\infty \frac{\omega(s)}{s^2} ds,$$

thus $\lim_{t\to\infty} \frac{\omega(t)}{t} = 0$.

In [5, Chapters 2,3] it is shown that for $\omega \in \mathcal{W}$ with (ω_1) and (ω_{nq}) there exist functions with compact support in the Beurling-class $\mathcal{E}_{(\omega)}$ (and hence in the Roumieu-class $\mathcal{E}_{\{\omega\}}$), so both classes are not quasi-analytic.

3.8 Sub-additivity for ω

Instead of (ω_1) sometimes it is assumed that ω is sub-additive:

$$(\omega_{\text{sub}}) :\Leftrightarrow \ \omega(t+s) \le \omega(t) + \omega(s) \qquad \forall\, s,t \ge 0$$

which implies $\omega(2t) = \omega(t+t) \le 2\cdot\omega(t)$ for all $t \ge 0$ and so clearly (ω_1). If one assumes w.l.o.g. $\omega(t) = 0$ for $0 \le t \le t_0$, then one has (ω_{sub}) only for all $t, s \ge t_0$.
We prove now the following very useful result about convex resp. concave functions:

Lemma 3.8.1. (1) *If $f : [0,+\infty) \to [0,+\infty)$ is a positive, concave function, which satisfies $f(0) \ge 0$, then f is already sub-additive on $[0,+\infty)$.*

(2) *If $f : [0,+\infty) \to [0,+\infty)$ is convex and additionally $f(0) = 0$, then $f(x)+f(y) \le f(x+y)$ for all $x, y \ge 0$.*

Proof. (1) Take $x \ge 0$ arbitrary and $y = 0$, then for all $1 \ge t \ge 0$ we get

$$f(t\cdot x) = f(t\cdot x + (1-t)\cdot 0) \ge t\cdot f(x) + (1-t)\cdot \underbrace{f(0)}_{\ge 0} \ge t\cdot f(x),$$

hence $f(x)+f(y) = f((x+y)\cdot\frac{x}{x+y}) + f((x+y)\cdot\frac{y}{x+y}) \ge \frac{x}{x+y}\cdot f(x+y) + \frac{y}{x+y}\cdot f(x+y) = \frac{x+y}{x+y}\cdot f(x+y) = f(x+y)$ for all $x, y \ge 0$.

(2) As above consider the case $y = 0$, then $f(t\cdot x) = f(t\cdot x + (1-t)\cdot 0) \le t\cdot f(x) + (1-t)\cdot \underbrace{f(0)}_{=0} = t\cdot f(x)$ for all $x \ge 0$ and $0 \le t \le 1$. So for all $x, y \ge 0$ we get:

$$\begin{aligned} f(x)+f(y) &= f\left((x+y)\cdot\frac{x}{x+y}\right) + f\left((x+y)\cdot\frac{y}{x+y}\right) \\ &\le \frac{x}{x+y}\cdot f(x+y) + \frac{y}{x+y}\cdot f(x+y) = f(x+y). \end{aligned}$$

As a special case (2) holds for the function φ^*_ω whenever we have $\omega \in \mathcal{W}$, more precisely whenever ω satisfies (ω_0) and (ω_3) - see 3.1.1 and 3.1.2 for more details! □

In [11, Lemma 3.3.] it is shown that (ω_{sub}) implies for the sequence $(m^l_j)_j$, $m^l_j := \frac{\exp(\frac{1}{l}\cdot\varphi^*_\omega(jl))}{j!}$, the following inequality:

$$\forall\, j,k \in \mathbb{N}\ \forall\, l > 0 :\ m^l_j \cdot m^l_k \le m^l_{j+k}, \tag{3.8.1}$$

which is exactly (alg) for the sequence m^l with $C = 1$. This result is also used in [10, Proposition 2.1.] to prove closedness under composition for classes defined by (non-quasi-analytic) weight functions ω.
For convenience of the reader we give the important proof of (3.8.1) in detail:

Lemma 3.8.2. *Assume that $\omega \in \mathcal{W}$ is a sub-additive weight, then we get*

$$\forall\, j,k \in \mathbb{N}\ \forall\, l > 0 :\ m^l_j \cdot m^l_k \le m^l_{j+k}.$$

Proof. First we calculate for fixed $l > 0$ and $j \in \mathbb{N}$:

$$\begin{aligned}
&\exp\left(\frac{1}{l}\cdot\varphi^*_\omega(j\cdot l)\right) = \exp\left(\frac{1}{l}\cdot\sup\{(j\cdot l)\cdot y - \varphi_\omega(y) : y \geq 0\}\right)\\
&= \exp\left(\sup\left\{j\cdot y - \frac{1}{l}\cdot\varphi_\omega(y) : y\geq 0\right\}\right) = \exp\left(\sup\left\{j\cdot\log(s) - \frac{1}{l}\cdot\omega(s) : s\geq 1\right\}\right)\\
&= \sup\left\{\exp\left(j\cdot\log(s) - \frac{1}{l}\cdot\omega(s)\right) : s\geq 1\right\} = \sup\left\{s^j\cdot\exp\left(-\frac{1}{l}\cdot\omega(s)\right) : s\geq 1\right\}\\
&= \sup\left\{s^j\cdot\exp\left(-\frac{1}{l}\cdot\omega(s)\right) : s\geq 0\right\},
\end{aligned}$$

where we have put $y = \log(s)$ and the last equality holds if $\omega(s) = 0$ for $0 \leq s \leq 1$ (w.l.o.g). Note that $s^j \leq 1$ for $s \in [0,1]$ and all $j \in \mathbb{N}$. Using this we get

$$\begin{aligned}
m^l_j\cdot m^l_k &= \frac{\sup_{s\geq 1}\{s^j\cdot\exp(-\frac{1}{l}\cdot\omega(s))\}}{j!}\cdot\frac{\sup_{t\geq 1}\{t^k\cdot\exp(-\frac{1}{l}\cdot\omega(t))\}}{k!}\\
&\leq \sup_{s,t\geq 1}\frac{s^j\cdot t^k}{j!\cdot k!}\cdot\exp\left(-\frac{1}{l}\cdot\omega(s+t)\right) \leq \frac{1}{(j+k)!}\cdot\sup_{s,t\geq 1}(s+t)^{j+k}\cdot\exp\left(-\frac{1}{l}\cdot\omega(s+t)\right)\\
&\leq m^l_{j+k}.
\end{aligned}$$

The first inequality holds by the calculation above and because $\omega(s+t) \leq \omega(s)+\omega(t)$ for all $s,t \geq 1$. The second inequality holds because we have $(s+t)^{j+k} = \sum_{i=0}^{j+k}\binom{j+k}{i}\cdot s^i\cdot t^{j+k-i}$, hence for $i = j \leq j+k$ we get in the sum the term $\binom{j+k}{j}\cdot s^j\cdot t^k = \frac{(j+k)!}{j!\cdot k!}\cdot s^j\cdot t^k$. □

A similar computation as in 3.8.2 gives the following result:

Lemma 3.8.3. *Assume that $\omega \in \mathcal{W}$ is sub-additive, then we obtain*

$$\forall\, j,k \in \mathbb{N}\ \forall\, l > 0:\ m^l_j\cdot m^l_k \leq m^{2l}_1\cdot m^{2l}_{j+k-1}. \tag{3.8.2}$$

Proof. As seen in 3.8.2 we get $m^l_j = \frac{\sup\{s^j\cdot\exp(-\frac{1}{l}\cdot\omega(s)):s\geq 1\}}{j!}$ for each $j \in \mathbb{N}$ and $l > 0$. So for (3.8.2) we have to show

$$\begin{aligned}
&\frac{\sup\{s^j\cdot\exp(-\frac{1}{l}\cdot\omega(s)) : s\geq 1\}}{j!}\cdot\frac{\sup\{t^k\cdot\exp(-\frac{1}{l}\cdot\omega(t)) : t\geq 1\}}{k!}\\
&\leq \sup\left\{a\cdot\exp\left(-\frac{1}{2l}\cdot\omega(a)\right) : a\geq 1\right\}\cdot\frac{\sup\{b^{j+k-1}\cdot\exp(-\frac{1}{2l}\cdot\omega(b)) : b\geq 1\}}{(j+k-1)!}.
\end{aligned}$$

We have to show that for each chosen $s,t \geq 1$ we can find $a,b \geq 1$ with

$$\exp\left(\frac{1}{2l}\cdot(\omega(a)+\omega(b))\right)\cdot\frac{s^j\cdot t^k}{j!\cdot k!} \leq \frac{b^{j+k-1}\cdot a}{(j+k-1)!}\cdot\exp\left(\frac{1}{l}\cdot(\omega(s)+\omega(t))\right).$$

This is satisfied, since for arbitrary given $s,t \geq 1$ and $j,k \in \mathbb{N}_{>0}$ on the left hand side the choice $a = b := s+t$ on the right hand is sufficient. On the one hand $\frac{(j+k-1)!}{j!\cdot k!}\cdot s^j\cdot t^k \leq (s+t)^{j+k}$ holds again clearly by the binomial formula, on the other hand $\exp(\frac{1}{2l}\cdot(\omega(a)+\omega(b))) = \exp(\frac{2}{2l}\cdot\omega(s+t)) \leq \exp(\frac{1}{l}\cdot(\omega(s)+\omega(t)))$ by the sub-additivity of ω. □

We introduce now the following new properties for ω, which are very closely related to the sub-additivity:

$$\sup_{\lambda\geq 1}\limsup_{t\to\infty}\frac{\omega(\lambda\cdot t)}{\lambda\cdot\omega(t)} < \infty \tag{3.8.3}$$

which is precisely condition (α_1) in [34, 1.1. Proposition] and

$$(\omega_{1'}):\ \exists\, D>0:\exists\, t_0>0:\forall\,\lambda\geq 1:\forall\, t\geq t_0:\omega(\lambda\cdot t)\leq D\cdot\lambda\cdot\omega(t). \tag{3.8.4}$$

We recall: Condition $(\omega_{1'})$ was used in the literature several times (e.g. in [10] and in [1]), and it is precisely condition (α_0) in [10, §2,§3] where it played the key-role there to characterize closedness under composition for function spaces introduced by a *non-quasi-analytic weight function* ω, see [10, 3.8. Theorem].

On the one hand condition $(\omega_{1'})$ implies clearly (3.8.3). The converse implication is not true in general: For this we consider the weight $\omega(t):=\log(t)\cdot t$ for $t\geq 1$ and $\omega(t)=0$ for $t\in[0,1]$. It's clear that $\omega\in\mathcal{W}$ holds, moreover (ω_1) and (ω_6), but not (ω_2) and (ω_7). We have for $\lambda\geq 1$, $t>1$:

$$\frac{\omega(\lambda\cdot t)}{\lambda\cdot\omega(t)}=\frac{\log(\lambda)+\log(t)}{\log(t)}.$$

This expression is monotone decreasing in t and satisfies $\lim_{t\to\infty}\frac{\log(\lambda)+\log(t)}{\log(t)}=1$ for arbitrary $\lambda\geq 1$, which proves condition (3.8.3). But $(\omega_{1'})$ cannot hold: For this remark that $\lim_{\lambda\to\infty}\frac{\log(\lambda)+\log(t)}{\log(t)}=+\infty$ for any $t>1$ and $\frac{\log(\lambda)+\log(t)}{\log(t)}\leq D\Leftrightarrow\lambda\leq t^{D-1}$ for some fixed $D>1$. Hence for increasing λ we also have to increase t to dominate the quotient by a fixed constant.

The following result was also mentioned several times in the literature (e.g. in [10] and in [1]), we prove it in detail:

Lemma 3.8.4. *Assume that $\omega:[0,\infty)\to[0,\infty)$ is a continuous increasing function. Then ω satisfies property $(\omega_{1'})$ if and only if there exists a sub-additive function $\sigma:[0,\infty)\to[0,\infty)$ with $\omega\sim\sigma$. All possible further properties for ω except the convexity condition (ω_4) (if $\omega\neq\sigma$) can be transferred to σ automatically.*

Proof. Assume that there exists an equivalent weight σ, which is sub-additive. By the equivalence there exist constants $D_1,D_2\geq 1$ such that $\omega(t)\leq D_1\cdot\sigma(t)$ and $\sigma(t)\leq D_2\cdot\omega(t)$ for all $t\geq t_0$, $t_0>0$ sufficiently large. First take $\lambda\in\mathbb{N}_{>0}$ arbitrary, then for all $t\geq t_0$ we get

$$\omega(\lambda\cdot t)\underbrace{\leq}_{\omega\sim\sigma}D_1\cdot\sigma(\lambda\cdot t)\leq D_1\cdot\lambda\cdot\sigma(t)\underbrace{\leq}_{\omega\sim\sigma}D_1\cdot D_2\cdot\lambda\cdot\omega(t),$$

because σ is sub-additive.
If $\lambda\geq 1$ is an arbitrary real number, then we choose $n=n_\lambda\in\mathbb{N}$ such that $n\leq\lambda<n+1$. Hence for arbitrary $t\geq t_0$ we have

$$\omega(\lambda\cdot t)\leq\omega((n+1)\cdot t)\leq D_1\cdot D_2\cdot(n+1)\cdot\omega(t)\leq(\underbrace{2\cdot D_1\cdot D_2}_{=C_1})\cdot\lambda\cdot\omega(t),$$

because $2\cdot\lambda\geq n+1$ and since by assumption $2\cdot\lambda\geq 2\cdot n$ and $n+1\leq 2\cdot n\Leftrightarrow 1\leq n$. Note that the constant C_1 is independent of λ, it depends only on D_1 and D_2 which means only on ω and σ and the estimate holds for all $t\geq t_0$. Thus $(\omega_{1'})$ is satisfied for ω.

For the converse direction we point out that in [32, Lemma 1] the following is shown: If for a continuous function $f:[0,\infty)\to[0,\infty)$ there exists a constant $C>0$ such that

$$f(s)\leq C\cdot\max\left\{1,\frac{s}{t}\right\}\cdot f(t) \tag{3.8.5}$$

for all s and t (so C doesn't depend on s and t), then f is equivalent (w.r.t. relation $\sim$) to its least concave majorant F. If we assume that ω has now property $(\omega_{1'})$, then we put $s := \lambda \cdot t \geq t$ and ω satisfies (3.8.5) for all $t \geq t_0$, so ω is equivalent to its least concave majorant, denote it by σ. By the first part of 3.8.1 a positive concave function f, which satisfies $f(0) \geq 0$, is automatically sub-additive.

All further additional conditions except the convexity condition (ω_4) for the function $t \mapsto \varphi_\sigma(t) = \sigma(\exp(t))$ can be transferred from ω to σ by 3.2.2 automatically.

□

We close this section with the following very important remark:

Remark 3.8.5. *The possible loss of* (ω_4) *for the sub-additive weight* σ *(if* $\sigma \neq \omega$*) does not effect the important characterizing theorem 8.7.4, more precisely the proofs of* (3.8.1)*, 5.1.2 and 8.7.1 stay valid for functions* σ *with only* (ω_0) *and* (ω_3)*, see 3.1.1 and 3.1.2 for more details.*

3.9 Algebraic structure on the set of all weight functions

We show in this section:

Proposition 3.9.1. *The set of all weight functions is a commutative semi-group w.r.t. pointwise addition and as neutral element we get the zero-function* θ*.*

For the zero-function we point out: By definition $\varphi_\theta(x) = 0$ for all x, hence $\varphi^*_\theta(x) = +\infty$ for all x and so this weight leads to the class of all smooth functions $\mathcal{E}$.
The function $\omega(x) = +\infty$ for all $x \geq 0$ leads to the trivial space $\{0\}$, since $\varphi^*_\omega(x) = -\infty$ for all x.

If both weights ω_1 and ω_2 satisfy (ω_0), then also $\omega_3 := \omega_1 + \omega_2$. We have $\omega_3(0) = 0$, and it is clearly a continuous and increasing function on $[0, +\infty)$, $\lim_{x\to\infty} \omega_3(x) = +\infty$ and $\omega_3(x) = 0$ for all $x \in [0, 1]$ w.l.o.g. Furthermore we get $\max\{\omega_1(x), \omega_2(x)\} \leq \omega_3(x) \leq 2 \cdot \max\{\omega_1(x), \omega_2(x)\}$ for all $x \geq 0$.

(ω_1): If $\omega_1(2t) \leq C \cdot \omega_1(t) + C$ and $\omega_2(2t) \leq D \cdot \omega_2(t) + D$ for all $t \geq 0$, then $\omega_3(2t) = \omega_1(2t) + \omega_2(2t) \leq C \cdot \omega_1(t) + D \cdot \omega_2(t) + C + D \leq \max\{C, D\} \cdot \omega_3(t) + C + D \leq (C + D) \cdot \omega_3(t) + (C + D)$. If ω_1 and ω_2 both satisfy (ω_{sub}), then ω_3, too.
(ω_2): Similarly we get $\omega_3(t) \leq (C + D) \cdot t + (C + D)$.
(ω_3): Holds, because $\frac{\log(t)}{\omega_3(t)} \leq \frac{\log(t)}{\max\{\omega_1(t),\omega_2(t)\}}$.
(ω_4): Clearly we have $\varphi_{\omega_3}(t) = \omega_3(\exp(t)) = \omega_1(\exp(t)) + \omega_2(\exp(t)) = \varphi_{\omega_1}(t) + \varphi_{\omega_2}(t)$, and the sum of two convex functions is still convex.
(ω_5): $\frac{\omega_3(t)}{t} \leq \frac{2\cdot\max\{\omega_1(t),\omega_2(t)\}}{t} \to 0$ for $t \to +\infty$.

If both ω_1 and ω_2 satisfy (ω_6), then ω_3, too: Let ω_1 satisfy (ω_6) for the constant $H_1 \geq 1$ and ω_2 for the constant $H_2 \geq 1$. Then $2 \cdot \omega_3(t) = 2 \cdot \omega_1(t) + 2 \cdot \omega_2(t) \leq \omega_1(H_1 \cdot t) + H_1 + \omega_2(H_2 \cdot t) + H_2 \leq \omega_3(\max\{H_1, H_2\} \cdot t) + H_1 + H_2 \leq \omega_3((H_1 + H_2) \cdot t) + (H_1 + H_2)$ for all $t \geq 0$, because both weight functions are increasing. A similar calculation shows this also for condition (ω_7).

If both ω_1 and ω_2 satisfy (ω_{snq}), then ω_3, too, in particular we have $\int_1^\infty \frac{\omega_3(y\cdot t)}{t^2} dt = \int_1^\infty \frac{\omega_1(y\cdot t)}{t^2} dt + \int_1^\infty \frac{\omega_2(y\cdot t)}{t^2} dt \leq C \cdot \omega_1(y) + C + D \cdot \omega_2(y) + D \leq \max\{C, D\} \cdot \omega_3(y) + C + D \leq$

$(C+D)\cdot\omega_3(y)+(C+D)$ for all $y>0$. The same is clearly true for (ω_{nq}) by the linearity of the integral.

If both ω_1 and ω_2 satisfy $(\omega_{1'})$, then ω_3, too: Let ω_1 satisfy $(\omega_{1'})$ for the constant D_1 and all $t\ge t_0^1$ and ω_2 for the constant D_2 and all $t\ge t_0^2$, then for arbitrary $\lambda\ge 1$ we have $\omega_3(\lambda\cdot t)\le\max\{D_1,D_2\}\cdot\lambda\cdot\omega_3(t)$ for all $t\ge t_0^3:=\max\{t_0^1,t_0^2\}>0$ and so ω_3 is still equivalent to a sub-additive weight function.

3.10 An explicit example of a weight function

An important concrete example for a weight function is

$$\omega_s(t):=\max\{0,\log(t)^s\},$$

where $s>1$ is a real parameter and one can extend this weight to $t\in\mathbb{R}$ by putting $\omega_s(t):=\max\{0,\log(|t|)^s\}$. It was mentioned in [4, 20. Example] that this weight defines function classes $\mathcal{E}_{[\omega]}$ for which we cannot find a weight sequence M (with several properties), such that the considered function spaces coincide, i.e. $\mathcal{E}_{[\omega]}=\mathcal{E}_{[M]}$ is not possible and so one gets really new function classes. We prove in this section:

Proposition 3.10.1. *Each function ω_s, $s>1$, satisfies $\omega_s\in\mathcal{W}$, (ω_1), (ω_2), (ω_5), (ω_7), (ω_{snq}) and $(\omega_{1'})$, but NOT (ω_6).*

Proof. This weight function is equivalent (w.r.t. $\sim$) to the sub-additive weight $\omega_s(t):=\log(1+|t|)^s$, for this use the rule of *de l'Hopital:* $\frac{\log(1+t)^s}{\log(t)^s}=\left(\frac{\log(1+t)}{\log(t)}\right)^s$ and $\lim_{t\to\infty}\frac{\log(1+t)}{\log(t)}=\lim_{t\to+\infty}\frac{t}{1+t}=1$.
More precisely ω_s satisfies $(\omega_{1'})$ for all $s>1$ by direct calculation: For all $\lambda\ge 1$ and $t\ge 1$ we have

$$\frac{\log(\lambda\cdot t)^s}{\lambda\cdot\log(t)^s}=\frac{(\log(\lambda)+\log(t))^s}{\lambda\cdot\log(t)^s}=\frac{1}{\lambda}\cdot\left(\frac{\log(\lambda)}{\log(t)}+1\right)^s=\left(\frac{\log(\lambda)}{\lambda^{1/s}\cdot\log(t)}+\frac{1}{\lambda^{1/s}}\right)^s.$$

Note that $\frac{\log(\lambda)}{\lambda^{1/s}\cdot\log(t)}\to 0$ for $t\to\infty$ and each $\lambda\ge 1$ fixed resp. also for $\lambda\to\infty$ and each $t\ge 1$ (use *de l'Hopital* to get $\lim_{\lambda\to\infty}\frac{\log(\lambda)}{\lambda^{1/s}}=\lim_{\lambda\to\infty}\frac{\lambda^{-1}}{s^{-1}\cdot\lambda^{1/s-1}}=s\cdot\lim_{\lambda\to\infty}\frac{1}{\lambda^{1/s}}=0$).
$\omega_s(x)=0$ for $x\in[0,1]$ for all $s>1$, and it is clearly an even (if it is extend to $\mathbb{R}$), increasing (on $[0,\infty)$) and continuous function with $\lim_{t\to\infty}\omega_s(t)=\infty$.
(ω_3): $\lim_{t\to\infty}\frac{\log(t)}{\log(t)^s}=\lim_{t\to\infty}\log(t)^{s-1}=\infty$, because $s-1>0$ (here we need $s>1$!)
(ω_4): $\varphi_{\omega_s}(t)=\omega_s(\exp(t))=t^s$ for all $s>1$ and $t\ge 0$, hence a convex function.
So we have shown $\omega_s\in\mathcal{W}$ and it satisfies $(\omega_{1'})$ for any $s>1$.

Moreover $\omega_{s_1}(t)\le\omega_{s_2}(t)$ for all t if and only if $s_1\le s_2$ and also $\omega_{s_1}(t)=o(\omega_{s_2}(t))$ for $t\to\infty$ if $s_1<s_2$. So $\omega_{s_2}\lhd\omega_{s_1}$ holds which means $\mathcal{E}_{\{\omega_{s_2}\}}\subseteq\mathcal{E}_{(\omega_{s_1})}$.

(ω_1): We have $\log(2t)^s=(\log(2)+\log(t))^s=\sum_{k=0}^{s}\binom{s}{k}(\log(t)^k\cdot\log(2)^{s-k}\le C\cdot\log(t)^s$ is satisfied for $t>1$ for a constant C (the parameter s is fixed!). Alternatively we see, that $\frac{\omega(2t)}{\omega(t)}=\frac{\log(2t)^s}{\log(t)^s}=\left(\frac{\log(2)+\log(t)}{\log(t)}\right)^s=\left(1+\frac{\log(2)}{\log(t)}\right)^s$ which tends to 1 for $t\to\infty$ for all $s>1$.
(ω_2): $(\log(t))^s\le C\cdot t+C$ holds for all $s>1$ for a constant $C\ge 1$ (depending on s).

ω_s satisfies the stronger property (ω_5) for all $s > 1$, too: Consider $\lim_{t\to+\infty} \frac{\omega_s(t)}{t} = \lim_{t\to+\infty} \frac{\log(t)^s}{t}$, then by the rule of *de l'Hopital* this expression is equal to $\lim_{t\to+\infty} \frac{s\cdot\log(t)^{s-1}\cdot 1/t}{1} = s\cdot\lim_{t\to+\infty} \frac{\log(t)^{s-1}}{t}$. Either $s-1 < 0$, then the limes is equal to 0, or we can iterate the procedure: For all $s > 1$, there exists a k_s such that $s-k_s-1 < 0$, hence the limes above is equal to $s\cdot(s-1)\dots(s-k_s)\cdot\lim_{t\to+\infty} \frac{\log(t)^{s-k_s-1}}{t} = 0$.

ω_s satisfies (ω_{snq}) for all $s > 1$, because by [29, 1.3. Proposition] condition (ω_{snq}) is equivalent to the existence of a constant $K > 1$ such that $\limsup_{t\to\infty} \frac{\omega(K\cdot t)}{\omega(t)} = \limsup_{t\to\infty} \frac{(\log(K)+\log(t))^s}{\log(t)^s} < K$, which is clearly satisfied, because, as above, the left hand side tends to 1 for arbitrary $K > 1$ if $t \to \infty$.

But ω_s doesn't satisfy the important comparison condition (ω_6) for any $s > 1$: Assume that there would exist some constant $H \geq 1$ with $2\cdot\log(t)^s \leq (\log(H\cdot t))^s + H$ for all $t \geq 0$, then $2 \leq \left(1 + \frac{\log(H)}{\log(t)}\right)^s + \frac{H}{(\log(t))^s}$ should be satisfied for all $t \geq 0$. But for $t \to \infty$ the right hand side tends to 1 for any $s > 1$ (and any $H \geq 1$), a contradiction.

Finally we prove property (ω_7) for each ω_s: For $t \geq 1$ we get $2^s\cdot\log(t)^s \leq C\cdot(\log(H)+\log(t))^s + C$, which is clearly satisfied (e.g. take $H = 1$ and $C = 2^s$).

So let $s > 1$ be arbitrary but fixed, we calculate as follows: For $t \geq 0$ we get $\varphi_{\omega_s}(t) = \omega_s(\underbrace{\exp(t)}_{\geq 1}) = (\log(\exp(t)))^s = t^s$. Thus $\varphi^*_{\omega_s}(x) = \sup\{x\cdot y - \varphi_{\omega_s}(y) : y > 0\} = \sup\{x\cdot y - y^s : y > 0\}$ and put $f_{x,s}(y) := x\cdot y - y^s$. Since $s > 1$, $\lim_{y\to+\infty} f_{x,s}(y) = -\infty$ and we calculate the maximal point.
$f'_{x,s}(y) = x - s\cdot y^{s-1}$, so $y = (\frac{x}{s})^{1/(s-1)}$ is the critical point, with other words

$$\varphi^*_{\omega_s}(x) = f_{x,s}\left(\left(\frac{x}{s}\right)^{\frac{1}{s-1}}\right) = x\cdot\left(\frac{x}{s}\right)^{\frac{1}{s-1}} - \left(\frac{x}{s}\right)^{\frac{s}{s-1}} = x^{\frac{s}{s-1}}\cdot\underbrace{\left(\frac{1}{s^{\frac{1}{s-1}}} - \frac{1}{s^{\frac{s}{s-1}}}\right)}_{=:R(s)}.$$

We point out that $R(s) > 0 \Leftrightarrow s^s > s$ which holds for all $s > 1$. Furthermore

$$R(s) = \underbrace{\frac{1}{s^{\frac{1}{s-1}}}}_{\to 1} - \underbrace{\frac{s^{\frac{1}{s}}}{s}}_{\to 0} \to 1, \qquad \text{for } s \to +\infty.$$

We point out that $t = \frac{s}{s-1}$ is the dual index related to $s > 1$, because $\frac{1}{t} + \frac{1}{s} = \frac{s-1}{s} + \frac{1}{s} = 1$! From now on let $s > 1$ be arbitrary but fixed, then we introduce the family of weight sequences with the parameter $l > 0$ (see the fifth chapter below for more details):

$$M^l_p := \exp\left(\frac{1}{l}\cdot\varphi^*_{\omega_s}(l\cdot p)\right) = \exp\left(l^{\frac{1}{s-1}}\cdot p^{\frac{s}{s-1}}\cdot R(s)\right) \tag{3.10.1}$$

for all $p \in \mathbb{N}$ and note that $\frac{s}{s-1} - 1 = \frac{1}{s-1}$. Hence for each $l > 0$ we get $M^l_0 = 1$, the sequence $(M^l_p)_p$ is weakly log. convex (since $p \mapsto \log(M^l_p)$ is convex) and furthermore $(M^l_p)^{1/p} = \exp\left(l^{\frac{1}{s-1}}\cdot p^{\frac{1}{s-1}}\cdot R(s)\right) \to +\infty$ for $p \to +\infty$.
Since ω_s satisfies (ω_{nq}) we get by the first part of 5.1.3 that on the one hand also each associated function ω_{M^l} satisfies (ω_{nq}) and on the other hand (by using Komatsu's

version of the Denjoy-Carleman-Theorem [16, Lemma 4.1.]) that each M^l has (nq), i.e. it is a non quasi-analytic weight sequence.

$(M^l_p)_p$ satisfies for each $l>0$ condition (β_3) which means:

$$(\beta_3):\ \exists\, Q\in\mathbb{N}:\ \liminf_{j\to\infty}\frac{\mu^l_{Qj}}{\mu^l_j}>1, \tag{3.10.2}$$

where we have put $\mu^l_j=\frac{M^l_j}{M^l_{j-1}}$.
If $x\mapsto\varphi^*_\omega(x)$ is $\mathcal{C}^1$, then for any $Q\in\mathbb{N}$, $Q\ge 2$ and all $j\in\mathbb{N}$:

$$\varphi^*_\omega(Qj)-\varphi^*_\omega(Qj-1)-(\varphi^*_\omega(j)-\varphi^*_\omega(j-1))\ge\varphi^{*'}_\omega(Qj-1)-\varphi^{*'}_\omega(j).$$

In fact we have

$$\frac{\mu^l_{Qj}}{\mu^l_j}=\exp\left(\underbrace{R(s)}_{>0}\cdot l^{1/(s-1)}\cdot\Big((Qj)^{s/(s-1)}-(Qj-1)^{s/(s-1)}-j^{s/(s-1)}+(j-1)^{s/(s-1)}\Big)\right)>1$$

for a $Q\in\mathbb{N}$ depending on s. This holds, since for $\frac{s}{s-1}>1$ we look at $f(x):=x^{s/(s-1)}$, with $f'(x)=\frac{s}{s-1}\cdot x^{s/(s-1)-1}=\frac{s}{s-1}\cdot x^{1/(s-1)}$. Note that $\frac{1}{s-1}>0$ and so we have $\lim_{x\to+\infty}f'(Qj-1)-f'(j)=+\infty$ for any $Q\ge 2$. Note that in fact $(n\cdot x)^{1/l}-x^{1/l}=n^{1/l}\cdot x^{1/l}-x^{1/l}=x^{1/l}\cdot(n^{1/l}-1)\to+\infty$ for $x\to+\infty$ and for all $l>0$ and all $n\in\mathbb{N}$, $n\ge 2$. More precisely this shows also that $(M^l_p)_p$ satisfies stronger condition (β_1) (see [33]) for each $l>0$, too:

$$(\beta_1):\ \exists\, Q\in\mathbb{N}(Q\ge 2):\ \liminf_{j\to\infty}\frac{\mu^l_{Qj}}{\mu^l_j}>Q. \tag{3.10.3}$$

For each $l>0$ the sequence $(M^l_p)_p$ doesn't have (mg), we can show this using two different arguments: First we can use the second part of 5.1.3 to conclude, that M^l cannot have moderate growth because ω_s doesn't satisfy (ω_6).

Second we prove non moderate growth by direct calculation:

$$\begin{aligned}\left(\frac{M^l_{j+k}}{M^l_j\cdot M^l_k}\right)^{1/(j+k)}&=\Big(\exp\Big(R(s)\cdot l^{1/(s-1)}\cdot\Big((j+k)^{s/(s-1)}-j^{s/(s-1)}-k^{s/(s-1)}\Big)\Big)\Big)^{1/(j+k)}\\&=\exp\left(R(s)\cdot l^{1/(s-1)}\cdot\frac{1}{j+k}\cdot\Big((j+k)^{s/(s-1)}-j^{s/(s-1)}-k^{s/(s-1)}\Big)\right)\\&=\exp\left(R(s)\cdot l^{1/(s-1)}\cdot\left((j+k)^{1/(s-1)}-\frac{j^{s/(s-1)}}{j+k}-\frac{k^{s/(s-1)}}{j+k}\right)\right).\end{aligned}$$

In the above expression one has to take the supremum over all $(j,k)\in\mathbb{N}^2\backslash\{0,0\}$, but consider now the special case $j = k$ (the diagonal), then we get

$$\begin{aligned}
&\exp\left(R(s)\cdot l^{1/(s-1)}\cdot\left((2j)^{1/(s-1)}-\frac{j^{s/(s-1)}}{2j}-\frac{j^{s/(s-1)}}{2j}\right)\right)\\
&=\exp\left(R(s)\cdot l^{1/(s-1)}\cdot\left((2j)^{1/(s-1)}-\frac{j^{1/(s-1)}}{2}-\frac{j^{1/(s-1)}}{2}\right)\right)\\
&=\exp\left(R(s)\cdot l^{1/(s-1)}\cdot\left((2j)^{1/(s-1)}-j^{1/(s-1)}\right)\right)\\
&=\exp\left(\underbrace{R(s)}_{>0}\cdot l^{1/(s-1)}\cdot\left(\underbrace{j^{1/(s-1)}}_{\to+\infty,\text{ for } j\to+\infty}\cdot\underbrace{(2^{1/(s-1)}-1)}_{>0\Leftrightarrow s>1}\right)\right).
\end{aligned}$$

Nevertheless we have will show in 5.1.2 (inequality (5.1.2)) that both $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ are closedness under taking derivatives, more precisely the so called *matrix generalized moderate-growth conditions* of Roumieu- resp. Beurling-type are satisfied:

$$(\mathcal{M}_{\{\mathrm{mg}\}}) : \forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, C \geq 1\ \forall\, j,k \in \mathbb{N} : M^l_{j+k} \leq C^{j+k}\cdot M^n_j \cdot M^n_k$$

$$(\mathcal{M}_{(\mathrm{mg})}) : \forall\, n \in \Lambda\ \exists\, l \in \Lambda\ \exists\, C \geq 1\ \forall\, j,k \in \mathbb{N} : M^l_{j+k} \leq C^{j+k}\cdot M^n_j \cdot M^n_k.$$

□

4 Weight function versus associated function ω_M

For an arbitrary weight sequence $M := (M_p)_p$ we define the *associated function* $\omega_M : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ by

$$\omega_M(t) := \sup_{p \in \mathbb{N}} \log\left(\frac{|t|^p \cdot M_0}{M_p}\right) \quad \text{for } t > 0, \qquad \omega_M(0) := 0. \tag{4.0.1}$$

An easy consequence is, that ω_M is an even, positive (take $p = 0$) and increasing function, because $\omega_M(t) \geq \log(t) + \log(M_0) - \log(M_1)$ for all $t \neq 0$ holds by definition. One can see that $\omega_M(t) \geq p \cdot \log(t) + \log(M_0) - \log(M_p)$ for arbitrary p, hence ω_M is increasing faster than $p \cdot \log(t)$ for any $p \in \mathbb{N}$ if $t \to \infty$ (and so $\lim_{t\to\infty} \omega_M(t) = +\infty$): For this take $q \in \mathbb{N}$ arbitrary, then for all $p \in \mathbb{N}$ we get $\lim_{t\to\infty} \frac{p\cdot\log(t)-\log(M_p)}{q\cdot\log(t)} = \frac{p}{q} - \lim_{t\to\infty} \frac{\log(M_p)}{q\cdot\log(t)} = \frac{p}{q}$. Thus for $p \geq q$ one gets the desired property!
Property (ω_3) holds, since

$$\frac{\omega_M(t)}{\log(t)} = \frac{\sup_{p\in\mathbb{N}}\{p \cdot \log(t) - \log(M_p)\}}{\log(t)} = \sup_{p\in\mathbb{N}} \left\{p - \frac{\log(M_p)}{\log(t)}\right\} \to \infty$$

for $t \to \infty$, because for each $p \in \mathbb{N}$ we can find $t_0 \in \mathbb{R}_{>0}$ such that for all $t \geq t_0$ we have $p - \frac{\log(M_p)}{\log(t)} \geq \frac{p}{2} \Leftrightarrow t^{p/2} \geq M_p$.

If e.g. $(M_p)_p$ is increasing, then ω_M vanishes for $0 \leq t \leq t_M$, where t_M depends on $M = (M_p)_p$: Suppose that $\omega_M(t) > 0$ for all $t > 0$, then for a fixed $t' > 0$ (very small) we would have $p \cdot \log(t') + \log(M_0) > \log(M_p) \Leftrightarrow (t')^p \cdot M_0 > M_p$ for at least one $p \in \mathbb{N}$ (which is depending on t'). Since $t' > 0$ is arbitrary small, this can only hold, if at least one M_p is arbitrary small (for $p = 0$ we would have $t^0 \cdot M_0 > M_0 \Rightarrow 1 > 1$, a contradiction!). If one assumes $(1 =)M_0 \leq M_p$ for all p, then $\omega_M(t) = 0$ for $0 \leq t \leq 1$, because then $p \cdot \log(t) \leq 0$ and $\log(M_p) - \log(M_0) \leq 0$ for all $p \in \mathbb{N}$.

For the regularizing algorithm in [25, Chapitre I.] the condition $\lim_{p\to\infty} M_p^{1/p} = \infty$ was assumed and this is equivalent to $\frac{1}{p} \cdot \log(M_p) \to +\infty$. It guarantees, that $\omega_M(t) < +\infty$ for all $t < +\infty$, because $p \cdot \log(t) - \log(M_p) \leq C_t$ for a constant $C_t > 0$, all $p \in \mathbb{N}$, $p \geq 1$, (and w.l.o.g. $t \geq 1$), if and only if $\log(t) - \frac{1}{p} \cdot \log(M_p) \leq \frac{C_t}{p}$. Since the right hand side tends to 0 for $p \to +\infty$ and since this should hold for all $t \in \mathbb{R}_{\geq 1}$, we need $\lim_{p\to\infty} \frac{1}{p} \cdot \log(M_p) = +\infty$.

If $M = (M_p)_p$ is weakly log. convex with $\lim_{p\to\infty}(M_p)^{1/p} = \infty$, then (ω_4) holds automatically by definition. In fact the mapping $t \mapsto \varphi_{\omega_M}(t) = \omega_M(\exp(t)) = \sup_{p\in\mathbb{N}}\{p \cdot t - \log(M_p)\}$ is a convex function by Mandelbrojt's formula [25, 1.8.III.]: For fixed t the supremum for $p \mapsto p \cdot t - \log(M_p)$ is attained at $p_t \in \mathbb{N}$ such that $\mu_{p_t} \leq t < \mu_{p_t+1}$ is satisfied (and $2 \cdot \varphi_{\omega_M}(t) \leq \varphi_{\omega_M}(2t)$ for all t, if $M_p \geq 1$ for all p).

Recall: H. Komatsu pointed out in [16, Definition 3.1.], that if the sequence $\left(\frac{M_p}{M_0}\right)^{1/p}$ is bounded by below by a positive constant, then ω_M is an increasing convex function in the variable $\log(t)$ and it is increasing faster than $\log(t^p)$ for any $p \in \mathbb{N}_{>0}$ as $t \to \infty$. This condition is clearly satisfied, if $(M_p)_p$ is assumed to be weakly log. convex, since in this case $(M_p^{1/p})_p$ is an increasing sequence. If $\left(\frac{M_p}{M_0}\right)^{1/p} \geq C$ for all $p \in \mathbb{N}$ and a constant $C > 0$, then $\frac{M_0}{M_p} \leq \frac{1}{C^p}$ and so $\frac{|t|^p \cdot M_0}{M_p} \leq \frac{|t|^p}{C^p}$ for all $p \in \mathbb{N}$. Hence for $|t| \leq C$ the associated function vanishes (and the supremum is attained for $p = 0$)!

If $M \in \mathcal{LC}$, then one has for ω_M a very useful integral-representation formula:

$$\omega_M(t) = \int_{\mu_1}^{t} \frac{\mu(\lambda)}{\lambda} d\lambda, \tag{4.0.2}$$

where $\mu(\lambda) := |\{p \in \mathbb{N} : \mu_p \leq \lambda\}|$ and $\mu_p := \frac{M_p}{M_{p-1}}$ for all $p \geq 1$, $\mu_0 := 1$. To show this formula one has to use [25, 1.8.III] and [25, 1.8.V.], see also [38, Lemma 8.2.4] resp. [38, Proposition 8.2.5]. This integral formula has already been used several times for important proofs, see e.g. in [16] and [4].

If $\liminf_{p\to\infty}(m_p)^{1/p} \geq c > 0$ with $m_p := \frac{M_p}{p!}$, then by using *Stirling's formula* ($p! \sim \left(\frac{p}{\exp(1)}\right)^p \cdot \sqrt{2\pi p}$) we see that this is equivalent to the following condition

$$\exists\, C_1, C_2 > 0\ \forall\, p \in \mathbb{N} : (C_1 \cdot (p+1))^p \leq C_2 \cdot M_p, \tag{4.0.3}$$

which should be compared with condition $(M0)$ in [4]. We can use this condition and the same proof as in [4, 12. Lemma $(iv) \Rightarrow (v)$] to show property (ω_2) for ω_M.
If the stronger condition $\lim_{p\to\infty}(m_p)^{1/p} = +\infty$ holds then this is equivalent to

$$\forall\, C_1 > 0\ \exists\, C_2 > 0\ \forall\, p \in \mathbb{N} : (C_1 \cdot (p+1))^p \leq C_2 \cdot M_p. \tag{4.0.4}$$

The analogous proof as in [4, 12. Lemma $(iv) \Rightarrow (v)$] shows that this condition implies property (ω_5) for ω_M.

We summarize our observations:

Lemma 4.0.2. *Let $M = (M_p)_p$ be a sequence with $M \in \mathcal{LC}$. Then the associated function ω_M is an element of $\mathcal{W}$.*
If in addition $\liminf(m_p)^{1/p} > 0$ holds, then (ω_2) for ω_M is valid. If in addition $\lim(m_p)^{1/p} = +\infty$ holds, then we also get (ω_5) for ω_M.

The next theorem is very important, it gives a connection between a weight function ω and the associated function ω_M of a weight sequence M, which is defined via the weight ω:

Theorem 4.0.3. *Let $\omega \in \mathcal{W}$ be given, then $\omega \sim \omega_M$ holds for the sequence $M = (M_p)_p$ defined by $(M_p^1 =)M_p := \exp(\varphi^*_\omega(p))$, for all $p \in \mathbb{N}$.*

Important Remark: The proof of this theorem doesn't hold if we only assume conditions (ω_0) and (ω_3) for ω.

Proof. By the properties for ω we see, that $M = (M_p)_p$ is weakly log. convex (the function ($p \mapsto \log(M_p) = \varphi^*_\omega(p)$ is convex), $M_0 = 1$ and $M_p^{1/p} \to +\infty$ for $p \to +\infty$

because $M_p^{1/p} = \exp(\frac{1}{p} \cdot \varphi^*_\omega(p))$. We point out, that by the integral representation formula ω_M vanishes on $[0, \mu_1]$, where $\mu_1 = \frac{M_1}{M_0} \geq 1$. Hence we can restrict on $t \geq \mu_1$ and first we get:

$$\begin{aligned}
\omega_M(t) \underbrace{=}_{\text{Def. of } \omega_M} & \sup_{p\in\mathbb{N}} (p \cdot \log(t) - \log(M_p)) = \sup_{p\in\mathbb{N}} (p \cdot \log(t) - \varphi^*_\omega(p)) \\
& \leq \sup_{x\geq 0} (x \cdot \log(t) - \varphi^*_\omega(x)) = \varphi^{**}_\omega(\log(t)) \\
& \underbrace{=}_{\omega\in\mathcal{W}} \varphi_\omega(\log(t)) = \omega(\exp(\log(t))) = \omega(t).
\end{aligned}$$

If (ω_4) would be not valid then we would only get $\varphi^{**}_\omega(\log(t)) \leq \varphi_\omega(\log(t))$ in the previous estimate. But to prove the converse direction $\omega(t) = O(\omega_M(t))$ for $t \to \infty$, more precisely we will show $\omega(t) \leq 2 \cdot \omega_M(t)$ for all $t \geq \mu_2 = \frac{M_2}{M_1} \geq \mu_1$, we are going to compare $\sup_{p\in\mathbb{N}} (p \cdot \log(t) - \varphi^*_\omega(p))$ with $\sup_{x\geq 0} (x \cdot \log(t) - \varphi^*_\omega(x)) = \varphi^{**}_\omega(\log(t))$ and so we need really equality, i.e. φ_ω has to be convex.

By Mandelbrojt's formula [25, 1.8.III.] for the associated function the supremum for $p \mapsto p \cdot \log(t) - \underbrace{\varphi^*_\omega(p)}_{\log(M_p)}$ for fixed t is attained at $p_t \in \mathbb{N}$ such that $\log(\mu_{p_t}) \leq \log(t) < \log(\mu_{p_t+1}) \Leftrightarrow \mu_{p_t} \leq t < \mu_{p_t+1}$ is satisfied. Furthermore $p \mapsto p \cdot \log(t) - \varphi^*_\omega(p)$ is a concave function for all $t \geq 1$ (because $p \mapsto \varphi^*_\omega(p)$ is convex) and it's the restriction of the function $f_t : x \mapsto x \cdot \log(t) - \varphi^*_\omega(x)$ (which is concave for all $t \geq 1$, too) to all positive integers.

To get $\omega \sim \omega_M$ we have to show now $\sup_{t\geq 1} \frac{\sup_{x\in\mathbb{R}} f_t(x)}{\sup_{x\in\mathbb{N}} f_t(x)} < +\infty$.

For $x > 0$ we have $\frac{f_t(x)}{x} = \log(t) - \frac{\varphi^*(x)}{x}$, hence $\lim_{x\to\infty} \frac{f_t(x)}{x} = -\infty$ for all $t > 0$. Or with other words: $x \mapsto f_t(x)$ is decreasing faster than lin. decr. for arbitrary $t > 0$.

First we show that the supremum of $p \mapsto p \cdot \log(t) - \underbrace{\varphi^*_\omega(p)}_{\log(M_p)}$ is not attained (only) at $p = 0$ for all $t \geq \mu_1$: For $p = 0$ we get $0 \cdot \log(t) - \log(M_0) = \log(1) = 0$ and because for $p = 1$ we have $\log(t) - \log(M_1) = \log(t) - \varphi^*_\omega(1)$, this value is larger than 0 (or equal to 0) if and only if $\log(t) \geq \varphi^*_\omega(1) \Leftrightarrow t \geq \exp(\varphi^*_\omega(1)) = M_1 = \frac{M_1}{M_0} = \mu_1$. If we need, that the value at $p = 2$ is larger or equal than the value at $p = 1$, we need $2\log(t) - \log(M_2) \geq \log(t) - \log(M_1) \Leftrightarrow \log(t) \geq \log(\frac{M_2}{M_1}) \Leftrightarrow t \geq \frac{M_2}{M_1} = \mu_2$. Since $(M_p)_p$ is weakly log. convex, the sequence $(\mu_p)_p$ is increasing, hence for $t \geq \mu_2 \geq \mu_1$ the supremum is not attained (only) at $p = 1$ and $p = 0$. So in the following calculation we consider only $t \geq \mu_2$!

Next we show that the supremum of $x \mapsto f_t(x)$, where $x \geq 0$, has to attain either at p_t or between p_t and $p_t + 1$ resp. at p_t or between $p_t - 1$ and p_t: Otherwise assume that the supremum would be attained at x_t for $x_t > p_t + 1$ (resp. $x_t < p_t - 1$), then, since f_t is concave, all points $f_t(x)$ such that $x \in [p_t, x_t]$ (resp. $x \in [x_t, p_t]$) have to lie above the line which is connecting $f_t(p_t)$ and $f_t(x_t)$. But $f_t(p_t + 1) \leq f_t(p_t), f_t(p_t + 1) < f_t(x_t)$ (resp. $f_t(p_t - 1) \leq f_t(p_t), f_t(p_t - 1) < f_t(x_t)$), hence $f_t(p_t + 1)$ resp. $f_t(p_t - 1)$ lies below this line! A contradiction to the concavity of f_t!

We point out, that $f_t(0) = 0$ for all $t \geq 0$ ($\varphi^*_\omega(0) = 0$), and proceed now as follows: We distinguish two cases, first assume that the supremum of $x \mapsto f_t(x)$ for $x \in [0, +\infty)$ is attained at x_t, where we have $p_t < x_t < p_t + 1$. Consider now the connecting line between $(0,0)$ and $(p_t, f_t(p_t))$ which is given by the equation $\overline{x} \mapsto \frac{f_t(p_t)}{p_t} \cdot \overline{x}$. Then $f(x_t)$ has to lie below this line, otherwise we would have $f_t(x_t) > \frac{f_t(p_t)}{p_t} \cdot x_t$. But this would imply that $f_t(x_t) \cdot p_t > f_t(p_t) \cdot x_t$, and so finally $\frac{f_t(x_t)}{x_t} \cdot p_t > f_t(p_t)$. But $\overline{x} \mapsto \frac{f_t(x_t)}{x_t} \cdot \overline{x}$ is nothing else but the straight line passing through $(0,0)$ and $(x_t, f_t(x_t))$, and the previous inequality implies, that the point $f_t(p_t)$ would lie below this line. But this is a contradiction to the concavity of f_t (the points 0 and $f_t(x_t)$ lie on the concave curve!).

Hence we have shown that $f_t(x_t) \leq \frac{f_t(p_t)}{p_t} \cdot \overline{x}$ for all $\overline{x}$ between x_t and $p_t + 1$. The maximal point of the line $\overline{x} \mapsto \frac{f_t(p_t)}{p_t} \cdot \overline{x}$ on this interval is clearly attained at $p_t + 1$, so $f_t(x_t) \leq \frac{f_t(p_t)}{p_t} \cdot (p_t + 1) = f_t(p_t) + \frac{f_t(p_t)}{p_t}$. Hence

$$\frac{f_t(x_t)}{f_t(p_t)} \leq \frac{f_t(p_t) + \frac{f_t(p_t)}{p_t}}{f_t(p_t)} = 1 + \frac{1}{p_t} \leq 2, \quad \forall\, t \geq \mu_1.$$

Now consider the case, where the supremum is attained at x_t, where we have $p_t - 1 < x_t < p_t$. Consider again the connecting line between $(0,0)$ and $(p_t - 1, f_t(p_t - 1))$ which is given by the equation $\overline{x} \mapsto \frac{f_t(p_t-1)}{p_t-1} \cdot \overline{x}$. Then $f(x_t)$ has to lie below this line, otherwise we would have $f_t(x_t) > \frac{f_t(p_t-1)}{p_t-1} \cdot x_t$. But this would imply that $f_t(x_t) \cdot (p_t - 1) > f_t(p_t - 1) \cdot x_t$, and so finally $\frac{f_t(x_t)}{x_t} \cdot (p_t - 1) > f_t(p_t - 1)$. But $\overline{x} \mapsto \frac{f_t(x_t)}{x_t} \cdot \overline{x}$ is nothing else but the straight line passing through $(0,0)$ and $(x_t, f_t(x_t))$, and the previous inequality implies, that the point $f_t(p_t - 1)$ would lie below this line. But this is a contradiction to the concavity of f_t (the points 0 and $f_t(x_t)$ lie on the concave curve!).

Thus we have shown that $f_t(x_t) \leq \frac{f_t(p_t-1)}{p_t-1} \cdot \overline{x}$ for all $\overline{x}$ between x_t and p_t. The maximal point of the line $\overline{x} \mapsto \frac{f_t(p_t-1)}{p_t-1} \cdot \overline{x}$ on this interval is clearly attained at p_t, so $f_t(x_t) \leq \frac{f_t(p_t-1)}{p_t-1} \cdot p_t \leq \frac{f_t(p_t)}{p_t-1} \cdot p_t$ by the definition of p_t. Hence

$$\frac{f_t(x_t)}{f_t(p_t)} \leq \frac{p_t}{p_t - 1} \leq 2, \quad \forall\, t \geq \mu_2,$$

because for the last inequality we need $\frac{p_t}{p_t-1} \leq 2 \Leftrightarrow p_t \leq 2p_t - 2 \Leftrightarrow 2 \leq p_t$, and this holds for $t \geq \mu_2$ as pointed out before, hence the theorem is shown! □

We can summarize the previous result: $\omega_M(t) \leq \omega(t)$ for all $t \geq \mu_1$ and $\omega(t) \leq 2 \cdot \omega_M(t)$ for all $t \geq \mu_2$ holds, this is equivalent to the existence of constants $C_1, C_2 \geq 1$ such that $\omega_M(t) \leq C_1 \cdot (\omega(t) + 1)$ and $\omega(t) \leq C_2 \cdot (\omega_M(t) + 1)$ for all $t \geq 0$ because $\lim_{t \to \infty} \omega(t) = \lim_{t \to \infty} \omega_M(t) = +\infty$!

An immediate consequence of 4.0.3 is the following:

Corollary 4.0.4. *If $\omega \in \mathcal{W}$, then $\mathcal{E}_{\{\omega_M\}} = \mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(\omega_M)} = \mathcal{E}_{(\omega)}$, where $M = (M_p)_p$ with $M_p := \exp(\varphi^*_\omega(p))$.*

Proof. By 4.0.3 we get $\omega \sim \omega_M$, since M is normalized and $M \in \mathcal{LC}$, the associated function satisfies $\omega_M \in \mathcal{W}$ and properties (ω_3) and (ω_4) are always satisfied by definition (see 4.0.2), so we can use now 3.2.1. □

Proposition 4.0.5. *Let M and N be two weight sequences with $M, N \in \mathcal{LC}$. Then we get:*

(1) *Assume that ω_M satisfies also condition (ω_1), then $M \preceq N \Longrightarrow \omega_M \preceq \omega_N$.*

(2) *Assume that N satisfies additionally condition moderate growth* (mg), *then $\omega_N \preceq \omega_M \Longrightarrow N \preceq M$.*

Proof. (1) $M \preceq N$ means that there exists a constant $D > 0$ such that $M_p \leq D^p \cdot N_p$ for all $p \in \mathbb{N}$. Thus we get for all $t > 0$:

$$\begin{aligned}\omega_M(t) &= \sup_{p\in\mathbb{N}}(p \cdot \log(t) - \log(M_p)) \geq \sup_{p\in\mathbb{N}}(p \cdot \log(t) - \log(D^p \cdot N_p)) \\ &= \sup_{p\in\mathbb{N}}(p \cdot \underbrace{(\log(t) - \log(D))}_{\log(t/D)} - \log(N_p)) = \omega_N\left(\frac{t}{D}\right).\end{aligned}$$

This calculation tells us: $\omega_N\left(\frac{t}{D}\right) \leq \omega_M(t)$ or equivalently $\omega_N(t) \leq \omega_M(Dt)$ for all $t \geq 0$, hence $\omega_N(t) = O(\omega_M(Dt))$. Since ω_M has by assumption (ω_1) there exists a constant $C \geq 1$, such that $\omega_M(2t) \leq C \cdot \omega_M(t) + C$ for all $t \geq 0$. An iterated application of (ω_1) (see (3.3.1)) leads to $\omega_M(2^n \cdot t) \leq C^n \cdot \omega_M(t) + \tilde{C}$ for a constant $\tilde{C} \geq 1$ and all $t \geq 0$. Choose now $n \in \mathbb{N}$ minimal such that $D \leq 2^n$, hence $\omega_N(t) \leq \omega_M(D \cdot t) \leq \omega_M(2^n \cdot t) \leq C^n \cdot \omega_M(t) + \tilde{C}$ for all $t \geq 0$, and so $\omega_N(t) = O(\omega_M(t))$.

(2) Because M and N both are assumed to be weakly log. convex we can use [16, Proposition 3.2.] to obtain for all $p \in \mathbb{N}$:

$$M_p = \sup_{t>0} \frac{t^p}{\exp(\omega_M(t))}, \quad \text{resp.} \quad N_p = \sup_{t>0} \frac{t^p}{\exp(\omega_N(t))}.$$

By assumption we have $\omega_M(t) = O(\omega_N(t))$, hence there exists a constant $C_1 \geq 1$ such that $\omega_M(t) \leq C_1 \cdot \omega_N(t) + C_1$ for all $t \geq 0$. Furthermore by condition moderate growth (mg) for $N = (N_p)_p$ we obtain by [16, Proposition 3.6.] that property (ω_6) for ω_N holds. So there exists a constant $H \geq 1$ such that for all $t \geq 0$ we get $2 \cdot \omega_N(t) \leq \omega_N(Ht) + H$. We can estimate for all $p \in \mathbb{N}$ as follows:

$$\begin{aligned}M_p &= \sup_{t>0} \frac{t^p}{\exp(\omega_M(t))} \underbrace{\geq}_{\omega_M \preceq \omega_N} \sup_{t>0} \frac{t^p}{\exp(C_1 \cdot \omega_N(t) + C_1)} = \frac{1}{\exp(C_1)} \cdot \sup_{t>0} \frac{t^p}{\exp(C_1 \cdot \omega_N(t))} \\ &\underbrace{\geq}_{(\star)} \frac{1}{\exp(C_1)} \cdot \sup_{t>0} \frac{t^p}{\exp(\omega_N(H^n t) + (2^n - 1) \cdot H)} \\ &= \frac{1}{\exp(C_1 + (2^n - 1) \cdot H)} \cdot \sup_{t>0} \frac{t^p}{\exp(\omega_N(H^n t))} \\ &\underbrace{=}_{t \mapsto \frac{t}{H^n}} \frac{1}{\exp(C_1 + (2^n - 1) \cdot H)} \cdot \left(\frac{1}{H^n}\right)^p \cdot \sup_{t>0} \frac{t^p}{\exp(\omega_N(t))} \\ &= \frac{1}{\exp(C_1 + (2^n - 1) \cdot H)} \cdot \left(\frac{1}{H^n}\right)^p \cdot N_p.\end{aligned}$$

$(\star)$ is satisfied, because an iterated application of (ω_6) (see (3.4.2)) gives $C_1 \cdot \omega_N(t) \leq 2^n \cdot \omega_N(t) \leq \omega_N(H^n \cdot t) + (2^n - 1) \cdot H$ for $n \in \mathbb{N}$ chosen minimal such that $C_1 \leq 2^n$ holds. This shows now $N \preceq M$. □

5 The weight matrix $\mathcal{M}$ associated to a given weight function ω

5.1 Definitions and basic properties for $\mathcal{M}$

A new technique and a central idea of this work is the following: To a given weight function $\omega \in \mathcal{W}$ we associate a *matrix/family* of weight sequences $\mathcal{M} := \{M^l = (M_j^l)_j : l > 0\}$ by setting

$$M_j^l := \exp\left(\frac{1}{l} \cdot \varphi_\omega^*(l \cdot j)\right) \qquad m_j^l := \frac{M_j^l}{j!} \qquad \mu_j^l := \frac{M_j^l}{M_{j-1}^l}$$

for $j \in \mathbb{N}$ and $l > 0$ (which plays the role of a parameter). Recall: For $l = 1$ we obtain the sequence, which is considered in the previous chapter in 4.0.3 and which was used in the second comparison theorem in [4, 16. Corollary] (with a misprint in assertion (3) there)!
The goal of this section is to transfer properties of ω into properties of the associated matrix $\mathcal{M} := \{M^l = (M_j^l)_j : l > 0\}$ (by using the derived results from the previous chapters), and these conditions can be assumed axiomatically for abstract matrices.
Note that for this approach we only need properties (ω_0) and (ω_3) for ω and not necessarily the convexity condition (ω_4), since we only use the convexity resp. increasing properties for the mapping φ_ω^*, see 3.1.1 and 3.1.2 for more details.

We are going to summarize now properties for $\mathcal{M}$ and start with the following lemma:

Lemma 5.1.1. *Let $\omega \in \mathcal{W}$, then $M^l \in \mathcal{LC}$ for each $l > 0$. So each sequence M^l is increasing, normalized ($M_1^l \geq M_0^l = 1$), weakly log. convex and $(M_j^l)^{1/j} \to \infty$ for $j \to \infty$.*
Moreover we have $M_j^{l_1} \leq M_j^{l_2} \Leftrightarrow m_j^{l_1} \leq m_j^{l_2}$ for all $j \in \mathbb{N}$ and $0 < l_1 \leq l_2$.

Proof. $(M_j^l)_j$ is an increasing sequence for each $l > 0$, because

$$M_{j+1}^l \geq M_j^l \Leftrightarrow \exp\left(\frac{1}{l} \cdot (\varphi_\omega^*(l \cdot (j+1)) - \varphi_\omega^*(l \cdot j))\right) \geq 1 \Leftrightarrow \varphi_\omega^*(l \cdot (j+1)) \geq \varphi_\omega^*(l \cdot j), \ \forall\, j \in \mathbb{N}$$

and this holds because $x \mapsto \varphi_\omega^*(x)$ is an increasing function. Moreover we have $M_0^l = \exp(\frac{1}{l} \cdot \varphi_\omega^*(0)) = \exp(0) = 1$ for all $l > 0$!

M^l is weakly log. convex for each $l > 0$. First because the sequence $\mu^l = (\mu_j^l)_j$, defined by $\mu_j^l = \frac{M_j^l}{M_{j-1}^l}$, is increasing:

$$\begin{aligned}\mu_{j+1}^l \geq \mu_j^l &\Leftrightarrow \exp\left(\frac{1}{l} \cdot (\varphi_\omega^*(l \cdot (j+1)) - \varphi_\omega^*(l \cdot j))\right) \geq \exp\left(\frac{1}{l} \cdot (\varphi_\omega^*(l \cdot j) - \varphi_\omega^*(l \cdot (j-1)))\right) \\ &\Leftrightarrow \varphi_\omega^*(l \cdot (j+1)) - \varphi_\omega^*(l \cdot j) \geq \varphi_\omega^*(l \cdot j) - \varphi_\omega^*(l \cdot (j-1))\end{aligned}$$

holds for all $j \in \mathbb{N}$, because $x \mapsto \varphi^*_\omega(x)$ is an increasing and convex function (faster than linear incr.). A second possibility to see the log. convexity is, that the mapping $j \mapsto \log(M^l_j) = \frac{1}{l} \cdot \varphi^*_\omega(l \cdot j)$ is a convex function for each $l > 0$ fixed.
We have $(M^l_j)^{1/j} \to +\infty$, for $j \to \infty$, because $(M^l_j)^{1/j} = \exp(\frac{1}{lj} \cdot \varphi^*_\omega(lj))$. By definition we immediately get that $(M^m_j)^{1/j} = M^{m \cdot j}_1$.
Finally for $M^{l_1}_j \le M^{l_2}_j$ for $0 < l_1 \le l_2$ and all $j \in \mathbb{N}$ we point out, that this condition is equivalent to $\frac{1}{l_1} \cdot \varphi^*_\omega(l_1 \cdot j) \le \frac{1}{l_2} \cdot \varphi^*_\omega(l_2 \cdot j)$. If $j = 0$ then we have clearly equality ($\varphi^*_\omega(0) = 0$), if $j > 0$ we can divide the equation by j and then we use the fact that the mapping $x \mapsto \frac{\varphi^*_\omega(x)}{x}$ is increasing. □

Some immediate consequences of 5.1.1:

(i) $M^{l_1}_j \le M^{l_2}_j$ for $0 < l_1 \le l_2$ and all $j \in \mathbb{N}$ has the consequence, that one can restrict in the *Roumieu-case* in (2.4.2) in the semi-norms $\| \cdot \|_{\omega,K,l}$ introduced in (2.4.1) to $l \in \mathbb{N}_{>0}$. In the *Beurling-case* in (2.4.3) one has to consider all $l > 0$ and the small numbers $0 < l \le 1$ are important. Hence in this case in the defining seminorms $\| \cdot \|_{\omega,K,l}$ the number $l > 0$ can be exchanged by $\frac{1}{l}$, $l \in \mathbb{N}_{>0}$, and so we obtain countable many seminorms in both cases! - For the more general situation of arbitrary weight matrices $\mathcal{M}$ see also 7.3.1, 7.3.5, 7.3.2 and 7.3.6 below.

(ii) Since each M^l is weakly log. convex, we obtain for each $l > 0$ a characteristic function $\theta_{M^l} = \theta_l$ (see (chf) above).

Lemma 5.1.2. *Let $\omega \in \mathcal{W}$, then we get the following two very important inequalities for the sequences M^l. For all $j, k \in \mathbb{N}$ and $n_1, n_2 > 0$ we have*

$$M^{n_1}_j \cdot M^{n_2}_k \le M^l_{j+k} \tag{5.1.1}$$

where we have put $l := \max\{n_1, n_2\}$ and for all $j, k \in \mathbb{N}$ and $l > 0$ we get

$$M^l_{j+k} \le M^{2l}_j \cdot M^{2l}_k. \tag{5.1.2}$$

Important Remarks:

(i) The second inequality (5.1.2) has an important consequence: For $\omega \in \mathcal{W}$ each space $\mathcal{E}_{[\omega]}$ is automatically closed under taking derivatives. Furthermore it implies clearly by definition also $m^l_{j+k} \le m^{2l}_j \cdot m^{2l}_k$.

(ii) We point out that for the proofs of 5.1.1 and 5.1.2 we only need properties (ω_0) and (ω_3) for ω and not necessarily (ω_4), since only the convexity for the mapping $x \mapsto \varphi^*_\omega$ is used, see 3.1.1 and 3.1.2 for more details.

Proof. Since φ^*_ω is convex, $\frac{\varphi^*_\omega(x)}{x}$ is increasing and $\varphi^*_\omega(0) = 0$, we get $\varphi^*_\omega(x) + \varphi^*_\omega(y) \le \varphi^*_\omega(x+y)$ for all $x, y \ge 0$ by 3.8.1. So for (5.1.1) we calculate:

$$\begin{aligned} M^{n_1}_j \cdot M^{n_2}_k &= \exp\left(\frac{1}{n_1} \cdot \varphi^*_\omega(n_1 \cdot j)\right) \cdot \exp\left(\frac{1}{n_2} \cdot \varphi^*_\omega(n_2 \cdot k)\right) \\ &\le \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot j)\right) \cdot \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot k)\right) = \exp\left(\frac{1}{l} \cdot (\varphi^*_\omega(l \cdot j) + \varphi^*_\omega(l \cdot k))\right) \\ &\le \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot (j+k))\right) = M^l_{j+k}. \end{aligned}$$

For (5.1.2) we point out: We use the convexity of φ^*_ω, because a convex-combination between the points $2lj$ and $2lk$ for $t=\frac{1}{2}$ leads to $\frac{1}{l}\cdot\varphi^*_\omega(lj+lk)\le\frac{1}{2l}\cdot\varphi^*_\omega(2lj)+\frac{1}{2l}\cdot\varphi^*_\omega(2lk)$ and so

$$M^l_{j+k}=\exp\left(\frac{1}{l}\cdot\varphi^*_\omega(l\cdot(j+k))\right)\le\exp\left(\frac{1}{2l}\cdot\varphi^*_\omega(2lj)\right)\cdot\exp\left(\frac{1}{2l}\cdot\varphi^*_\omega(2lk)\right)=M^{2l}_j\cdot M^{2l}_k.$$

Finally (5.1.2) shows for the case $k=1$ and $j\in\mathbb{N}$ arbitrary, that the function spaces are always closed under taking derivatives: $M^l_{j+1}\le D_l\cdot M^{2l}_j$ hold for all $j\in\mathbb{N}$ and each $l>0$, where have put $D_l:=M^{2l}_1=\exp(\frac{1}{2l}\cdot\varphi^*_\omega(2l))$. □

Lemma 5.1.3. *Let $\omega\in\mathcal{W}$, then for each $l>0$ we get $\omega\sim\omega_l\sim\omega_{M^l}$, hence $\mathcal{E}_{\{\omega\}}=\mathcal{E}_{\{\omega_l\}}=\mathcal{E}_{\{\omega_{M^l}\}}$ resp. $\mathcal{E}_{(\omega)}=\mathcal{E}_{(\omega_l)}=\mathcal{E}_{(\omega_{M^l})}$, where $\omega_l(x):=\frac{\omega(x)}{l}$.*

Moreover we have the following equivalences:

(1) *(a) ω satisfies additionally* (ω_{nq})
  (b) M^l is non-quasi-analytic (nq) *for each $l>0$.*
  (c) M^l is non-quasi-analytic (nq) *for some $l>0$.*

(2) *(a) ω satisfies additionally* (ω_6)
  (b) M^l satisfies moderate growth (mg) *for each $l>0$*
  (c) M^l satisfies moderate growth (mg) *for some $l>0$.*

Proof. We are going to use 4.0.3, so we really need $\omega\in\mathcal{W}$. As shown in 4.0.3 we have $\omega_{M^1}\sim\omega$. For each $l>0$ and all $x\ge0$ we get:

$$\frac{1}{l}\cdot\varphi^*_\omega(l\cdot x)=\frac{1}{l}\cdot\sup_{y\ge0}\{(l\cdot x)\cdot y-\varphi_\omega(y)\}=\sup_{y\ge0}\left\{x\cdot y-\frac{1}{l}\cdot\varphi_\omega(y)\right\}=\varphi^*_{\omega_l}(x),$$

where we have put $\omega_l(x):=\frac{\omega(x)}{l}$ for all $x\ge0$ and $l>0$. Note that $\varphi_{\omega_l}(x)=\omega_l(\exp(x))=\frac{\omega(\exp(x))}{l}=\frac{1}{l}\cdot\varphi_\omega(x)$ holds.
So $\exp(\varphi^*_{\omega_l}(j))=M^l_j$ and clearly $\omega_l\sim\omega$ and $\omega_l\in\mathcal{W}$ holds for each $l>0$. Hence we can apply 4.0.3 to the weight ω_l and the associated function ω_{M^l} to obtain the equivalences

$$\omega_{M^1}\sim\omega\sim\omega_l\sim\omega_{M^l}$$

for each $l>0$. By 3.2.1 this implies immediately that for each $l>0$ we have now $\mathcal{E}_{\{\omega\}}=\mathcal{E}_{\{\omega_{M^l}\}}$ resp. $\mathcal{E}_{(\omega)}=\mathcal{E}_{(\omega_{M^l})}$.

(1) $(a)\Rightarrow(b)$ Since (ω_{nq}) is stable w.r.t. $\sim$ we get: If $\omega\in\mathcal{W}$ with (ω_{nq}), then also ω_{M^l} has (ω_{nq}) for each $l>0$. Hence by Komatsu's version of the D.-C.-Theorem [16, Lemma 4.1] the sequence M^l is non-quasi-analytic (nq) for each $l>0$, too!
$(b)\Rightarrow(c)$ is obvious.
$(c)\Rightarrow(a)$ If there exists a number $l_0>0$ such that M^{l_0} is non quasi-analytic, then again by [16, Lemma 4.1] the associated function $\omega_{M^{l_0}}$ has property (ω_{nq}). Since $\omega_{M^l}\sim\omega_{M^n}$ for all $n,l>0$ we get (ω_{nq}) for ω_{M^l} and ω_l for each $l>0$. Then use again [16, Lemma 4.1] to see, that M^l has (nq) for each $l>0$!

(2) $(a)\Rightarrow(b)$ If $\omega\in\mathcal{W}$ with (ω_6), then ω_{M^l} has also (ω_6) for each $l>0$, because (ω_6) is stable under $\sim$ by 3.2.2. So the sequences M^l satisfy additionally moderate growth (mg) for each $l>0$, which holds now by [16, Proposition 3.6]!

$(b) \Rightarrow (c)$ is clear.
$(c) \Rightarrow (a)$ If there exists a number $l_1 > 0$ such that M^{l_1} has moderate growth (mg), then by [16, Proposition 3.6] the associated function $\omega_{M^{l_1}}$ has (ω_6). By the equivalence it follows immediately that ω_{M^l} and ω_l satisfy now (ω_6) for each $l > 0$, and so by [16, Proposition 3.6] once again we obtain that M^l has moderate growth (mg) for each $l > 0$, too. □

Lemma 5.1.4. *Let $\omega \in \mathcal{W}$, then we obtain:*

(1) *Property (ω_1) holds if and only if for all $C_1 > 0$ and $l_1 > 0$ there exist $l_2, C_2 > 0$ (resp. also for all $l_2 > 0$ and $C_1 > 0$ there exist $l_1, C_2 > 0$) such that*

$$C_1^j \cdot M_j^{l_1} \le C_2 \cdot M_j^{l_2} \tag{5.1.3}$$

holds for all $j \in \mathbb{N}$. More precisely (ω_1) implies (5.1.3) for $l_2 = L^s \cdot l_1$, where $s \in \mathbb{N}$ is chosen minimal such that $\exp(s) \ge C_1$ *is satisfied, and $L \ge 1$ the constant appearing in property (ω_1).*

(2) *If in addition (ω_6) holds, then we obtain that for all $l > 0$ and all $n \in \mathbb{N}$ there exists $l_n = \frac{l}{2^n} < l$ and constants $C, h > 0$ such that for all $j \in \mathbb{N}$ we obtain*

$$M_j^l \le C \cdot h^j \cdot M_j^{l_n}. \tag{5.1.4}$$

More precisely the estimate holds for the choice $h = H^n$ and $C = \exp\left(\left(\frac{1}{l_n} - \frac{1}{l}\right) \cdot H\right)$.

Remark: For $(\Rightarrow)$ in (1) and (2) we only need (ω_0) and (ω_3) for ω.

Proof. (1) $(\Rightarrow)$: Follows immediately by (3.3.2), apply the function exp to that inequality and use the previous definitions.
$(\Leftarrow)$: If (5.1.3) is satisfied, then for all $C_1 > 0$ and $l_1 > 0$ there exist $l_2, C_2 > 0$ (resp. for all $l_2 > 0$ and $C_1 > 0$ there exist $l_1, C_2 > 0$) such that for all $j \in \mathbb{N}$ we have $\frac{C_1^j}{M_j^{l_2}} \le \frac{C_2}{M_j^{l_1}}$. Hence by multiplying with $t^j > 0$ for any $j \in \mathbb{N}$ and applying log we get
$\omega_{M^{l_2}}(C_1 \cdot t) = \sup_{j\in\mathbb{N}} \log\left(\frac{(C_1 \cdot t)^j}{M_j^{l_2}}\right) \le C_2 \cdot \sup_{j\in\mathbb{N}} \log\left(\frac{t^j}{M_j^{l_1}}\right) = C_2 \cdot \omega_{M^{l_1}}(t)$.
So for all $t \ge 0$ we have $\omega_{M^{l_2}}(C_1 \cdot t) \le C_2 \cdot \omega_{M^{l_1}}(t)$. Since C_1 was arbitrary (in both cases), consider $C_1 = 2$ and use 5.1.3 (gives $\omega \sim \omega_{M^l}$ for each l) to get (ω_1) for ω.

(2) Follows by applying the function exp to (3.4.1) and by definition of the sequences M^l. □

5.2 Important consequences of property (ω_6)

In this section we will characterize property (ω_6) in terms of properties for the associated matrix $\mathcal{M}$. The first proposition shows, that one can get property (ω_6) if all spaces $\mathcal{E}_{[M^l]}$ coincide:

Proposition 5.2.1. *Let $\omega \in \mathcal{W}$ and assume $\mathcal{E}_{[M^l]} = \mathcal{E}_{[M^m]}$ for each $l, m > 0$. Then ω satisfies also condition (ω_6).*

Remark: For the proof of this proposition we really need $\omega \in \mathcal{W}$.

Proof. First we consider the *Roumieu-case:* By assumption $\mathcal{E}_{\{M^l\}} = \mathcal{E}_{\{M^m\}}$ for each $l, m > 0$. Because the sequences M^l are weakly log. convex and normalized for each

$l > 0$, the equality of the spaces implies $M^l \approx M^m$ which means in particular $M^l \preceq M^m$ and $M^m \preceq M^l$ for each $l, m > 0$ (see e.g. [38, Theorem 3.1.3], which follows by [40, Theorem 1]).
This means now: There exist constants $C_1, C_2 > 0$ such that for all $p \in \mathbb{N}$ we have $M^l_p \le C_1^p \cdot M^m_p$ and $M^m_p \le C_2^p \cdot M^l_p$ for each $l, m > 0$ fixed. We summarize this in the following equation:

$$\forall\, l, m > 0\ \exists\, D > 0\ \forall\, p \in \mathbb{N}:\ \frac{1}{l} \cdot \varphi^*_\omega(lp) \le p \cdot D + \frac{1}{m} \cdot \varphi^*_\omega(mp). \tag{5.2.1}$$

(5.2.1) is a quite strong condition: Dividing both sides by $p \ge 1$ yields to the fact that for all $l, m > 0$ there exists a constant $D > 0$ with $\frac{1}{lp} \cdot \varphi^*_\omega(lp) - \frac{1}{mp} \cdot \varphi^*_\omega(mp) \le D$ for all $p \ge 1$, so

$$\forall\, l, m > 0:\ \limsup_{p \to \infty} \frac{1}{lp} \cdot \varphi^*_\omega(lp) - \frac{1}{mp} \cdot \varphi^*_\omega(mp) < +\infty.$$

Replace now in (5.2.1) "for all $p \in \mathbb{N}$" by "for all $y \ge 0$", then we can calculate for $l, m > 0$ arbitrary but fixed as follows:

$$\begin{aligned}\left(\frac{1}{l} \cdot \varphi^*_\omega(l\cdot)\right)^*(x) &= \sup_{y \ge 0}\left\{x \cdot y - \frac{1}{l} \cdot \varphi^*_\omega(l \cdot y)\right\} = \frac{1}{l} \cdot \sup_{y \ge 0}\{(x \cdot l) \cdot y - \varphi^*_\omega(l \cdot y)\} \\ &\underbrace{=}_{y' = ly} \frac{1}{l} \cdot \sup_{y' \ge 0}\left\{(x \cdot l) \cdot \frac{y'}{l} - \varphi^*_\omega(y')\right\} = \frac{1}{l} \cdot \varphi^{**}_\omega(x) = \frac{1}{l} \cdot \varphi_\omega(x).\end{aligned}$$

Similarly we calculate for $\left(\cdot D + \frac{1}{m} \cdot \varphi^*_\omega(m\cdot)\right)^*$:

$$\begin{aligned}\left(\cdot D + \frac{1}{m} \cdot \varphi^*_\omega(m\cdot)\right)^*(x) &= \sup_{y \ge 0}\left\{(x - D) \cdot y - \frac{1}{m} \cdot \varphi^*_\omega(m \cdot y)\right\} \\ &= \frac{1}{m} \cdot \sup_{y' \ge 0}\{(x - D) \cdot m \cdot \frac{y'}{m} - \varphi^*_\omega(y')\} = \frac{1}{m} \cdot \varphi^{**}_\omega(x - D) \\ &= \frac{1}{m} \cdot \varphi_\omega(x - D),\end{aligned}$$

where we have put $y' := m \cdot y$. Then we use 4.0.3: We have $\omega_m(t) \le 2 \cdot \omega_{M^m}(t)$ for all $t \ge \mu^m_2$ and each $m > 0$. Recall the notation: $M^m_p = \exp(\frac{1}{m} \cdot \varphi^*_\omega(mp)) = \exp(\varphi^*_{\omega_m}(p))$ where we have set $\omega_m(t) = \frac{\omega(t)}{m}$ by 5.1.3. So we have for $x - D \ge \log(\mu^m_2)$, where we have put as usual $\mu^m_j = \frac{M^m_j}{M^m_{j-1}}$:

$$\begin{aligned}\frac{1}{l} \cdot \varphi_\omega(x) &= \sup_{y \ge 0}\left\{x \cdot y - \frac{1}{l} \cdot \varphi^*_\omega(l \cdot y)\right\} \ge \sup_{p \in \mathbb{N}}\left\{x \cdot p - \frac{1}{l} \cdot \varphi^*_\omega(l \cdot p)\right\} \\ &\underbrace{\ge}_{(5.2.1)} \sup_{p \in \mathbb{N}}\left\{x \cdot p - p \cdot D - \frac{1}{m} \cdot \varphi^*_\omega(mp)\right\} \ge \frac{1}{2} \cdot \sup_{y \ge 0}\left\{x \cdot y - y \cdot D - \frac{1}{m} \cdot \varphi^*_\omega(my)\right\} \\ &= \frac{1}{2m} \cdot \varphi_\omega(x - D).\end{aligned}$$

Hence we have shown: Consider a pair (l, m) arbitrary but fixed, then there exist constants $D_1, D_2 > 0$ with
$\frac{1}{l} \cdot \varphi_\omega(x) \ge \frac{1}{2m} \cdot \varphi_\omega(x - D_1)$ for $x \ge \log(\mu^m_2) + D_1$ and

$\frac{1}{m} \cdot \varphi_\omega(x) \geq \frac{1}{2l} \cdot \varphi_\omega(x - D_2)$ for all $x \geq \log(\mu_2^l) + D_2$.
Since $\varphi_\omega(x) = \omega(\exp(x))$, we have $\frac{1}{l} \cdot \omega(e^x) \geq \frac{1}{2m} \cdot \omega(e^{(x-D_1)}) = \frac{1}{2m} \cdot \omega(\frac{e^x}{e^{D_1}}) = \frac{1}{2m} \cdot \omega(\frac{e^x}{C_1})$ if we put $C_1 = \exp(D_1) \geq 1$ and analogously for D_2.
Hence we have $\frac{1}{l} \cdot \omega(y) \geq \frac{1}{2m} \cdot \omega(\frac{y}{C_1})$ for all $y \geq C_1 \cdot \mu_2^m$ (note: $y = \exp(x) \geq \exp(D_1 + \log(\mu_2^m)) = \exp(D_1) \cdot \mu_2^m$) and we summarize again:

$$\forall\, l, m > 0\; \exists\, C_1, C_2 \geq 1 : \quad \omega\left(\frac{y}{C_1}\right) \leq \frac{2m}{l} \cdot \omega(y) \quad \text{and} \quad \omega\left(\frac{y}{C_2}\right) \leq \frac{2l}{m} \cdot \omega(y), \quad (5.2.2)$$

where the first inequality holds for all $y \geq C_1 \cdot \mu_2^m$ and the second one for all $y \geq C_2 \cdot \mu_2^l$.

So take for example $m = 4l$ in the second inequality, then $\frac{m}{2l} = 2$ and $2 \cdot \omega(\frac{y}{C_2}) \leq \omega(y)$ for a constant $C_2 \geq 1$ and all $y \geq C_2 \cdot \mu_2^l$, or equivalently $2 \cdot \omega(y^{'}) \leq \omega(C_2 \cdot y^{'})$ for all $y^{'} \geq \mu_2^l$ with $y^{'} := \frac{y}{C_2}$. This implies clearly condition (ω_6) for ω.

In fact it's sufficient to take $m > 2l$, then the second inequality of (5.2.2) yields $\frac{m}{2l} \cdot \omega(y) \leq \omega(C_2 \cdot y)$. Since $\frac{m}{2l} > 1$ we can iterate this inequality, so for $n \in \mathbb{N}$ minimal with $\left(\frac{m}{2l}\right)^n \geq 2$ we get $2 \cdot \omega(y) \leq \left(\frac{m}{2l}\right)^n \cdot \omega(y) \leq \omega(C_2^n \cdot y)$, which holds for all $y \geq \mu_2^l$ and we still get property (ω_6) for ω.

The *Beurling-case:* For this we use the second chapter in [6] to obtain that $\mathcal{E}_{(M^l)} = \mathcal{E}_{(M^m)}$ implies $M^l \approx M^m$ for each $l, m > 0$ and we obtain again (5.2.1)! Then use the previous proof for the Roumieu-case.
Note that by assumption (see 5.1.1) each sequence $M^l \in \mathcal{LC}$, so all assumptions in [6] are satisfied! □

5.1.3, 5.1.4 (the second part (5.1.4) there) and 5.2.1 together give the following important characterization:

Proposition 5.2.2. *Let $\omega \in \mathcal{W}$, then the following are equivalent:*

(1) *ω has additionally* (ω_6)

(2) $\mathcal{E}_{[M^{l_1}]} = \mathcal{E}_{[M^{l_2}]}$ *is valid for all* $l_1, l_2 > 0$

(3) $M^{l_1} \approx M^{l_2}$ *for all* $l_1, l_2 > 0$

(4) *M^l has moderate growth* (mg) *for some/for each* $l > 0$.

Proof. (1) ⇒ (2) Let $l_1 \leq l_2$, then by 5.1.1 we see that we have always $M^{l_1} \leq M^{l_2}$, hence $\mathcal{E}_{[M^{l_1}]} \subseteq \mathcal{E}_{[M^{l_2}]}$. On the other hand we can use (5.1.4): Choose $n \in \mathbb{N}$ minimal with $\frac{l_2}{2^n} \leq l_1$, then $\mathcal{E}_{[M^{l_2}]} \underbrace{\subseteq}_{(5.1.4)} \mathcal{E}_{[M^{l_2/(2^n)}]} \subseteq \mathcal{E}_{[M^{l_1}]}$ and (2) holds.
(2) ⇒ (1) Is exactly 5.2.1.
(2) ⇔ (3) Holds by [38, Theorem 3.1.3] and [40, Theorem 1] for the Roumieu-case and by the second chapter in [6] for the Beurling-case. Note that by 5.1.1 we have $M^l \in \mathcal{LC}$ for each $l > 0$ and so the necessary assumptions are satisfied for both cases as pointed out before.
(1) ⇔ (4) Holds by 5.1.3. □

By using 5.2.2 we obtain the following corollary:

Corollary 5.2.3. *Let $\omega \in \mathcal{W}$, if (ω_6) doesn't hold then for each $l > 0$ there exists $m > 0$ with $m < l$ (resp. for each $m > 0$ there exists a number $l > 0$ with $l > m$) such that*

$$\sup_{p\ge 1}\left(\frac{M_p^l}{M_p^m}\right)^{1/p} = +\infty, \tag{5.2.3}$$

or equivalently $\mathcal{E}_{[M^m]} \subsetneq \mathcal{E}_{[M^l]}$.

Proof. Assume that (ω_6) doesn't hold but such that there exists $l_0 > 0$ with $\mathcal{E}_{[M^m]} = \mathcal{E}_{[M^{l_0}]}$ for each $m \le l_0$. But then we can compute as in 5.2.1, instead "for each $l, m > 0$" use now "for each $m > 0$ with $0 < m < l_0$, and then (5.2.2) is satisfied for each $m > 0$ with $0 < m < l_0$, and we would still get property (ω_6) for ω (e.g. take $m = \frac{l_0}{4}$ in the first inequality of (5.2.2)), which is a contradiction!
Hence (5.2.3) is now satisfied for $l = 4m$, because $\mathcal{E}_{[M^m]} \subsetneq \mathcal{E}_{[M^l]}$. □

To obtain property (ω_1) we prove the following:

Proposition 5.2.4. *Let $\omega \in \mathcal{W}$ and assume $\mathcal{E}_{[M^l]} = \mathcal{E}_{[\omega]}$ for each $l > 0$ and additionally now*

$$\exists\, m > 0\ \exists\, l > 0\ (m > l):\ \liminf_{p\to\infty} \frac{1}{mp}\cdot\varphi^*_\omega(mp) - \frac{1}{lp}\cdot\varphi^*_\omega(lp) > 0. \tag{5.2.4}$$

Then ω satisfies also property (ω_1).

Proof. By (5.2.4) for such $l, m > 0$ inequality (5.2.1) holds also for $0 < C_1 < 1$ where $D := \log(C_1)$: We get $\frac{1}{l}\cdot\varphi^*_\omega(lp) \le p\cdot\log(C_1) + \frac{1}{m}\cdot\varphi^*_\omega(mp)$, then divide both sides by $p \ge 1$ ($p = 0$ is clearly satisfied for all $C_1 > 0$!), hence

$$0 < -\log(C_1) \le \frac{1}{mp}\cdot\varphi^*_\omega(mp) - \frac{1}{lp}\cdot\varphi^*_\omega(lp).$$

We point out: This condition is weaker than (3.3.2), which was a consequence of property (ω_1), because in (3.3.2) on the right hand side a term with $s \in \mathbb{N}$ occurs. But (5.2.4) is not satisfied automatically for each $\omega \in \mathcal{W}$, because in this case $\frac{\varphi^*_\omega(x)}{x}$ is increasing and tending to infinity for $x \to +\infty$, but in the limit the difference can be equal to zero.

We show: Condition (ω_1) is satisfied. For this we put for $l, m > 0$ coming from (5.2.4) now in (5.2.2) $C = \frac{2m}{l} > 1$, then $\omega(y) \le C\cdot\omega(C_1\cdot y)$ for all $y \ge C_1\cdot\mu_2^m$. An iterated application yields $\omega(y) \le C^n\cdot\omega(C_1^n\cdot y) \le C^n\cdot\omega(\frac{y}{2})$ for $n \in \mathbb{N}$ minimal, such that $C_1^n \le \frac{1}{2}$. □

5.3 Consequences of (ω_2) and (ω_5)

Let two weight functions σ, τ be given, then we can translate both relations $\preceq$ and $\lhd$ into properties for their associated weight matrices, in this section we don't need necessarily condition (ω_4) for ω, only (ω_0) and (ω_3). More precisely we prove the following lemma:

Lemma 5.3.1. *Let $\sigma, \tau \in \mathcal{W}$ be given and denote the associated matrices by $M_j^l := \exp(\frac{1}{l}\cdot\varphi^*_\sigma(lj))$ and $N_j^l := \exp(\frac{1}{l}\cdot\varphi^*_\tau(lj))$. Then we obtain the following:*

(*a*) *If $\tau \preceq \sigma$ then there exists a constant $C \ge 1$ such that for all $l > 0$ we can find a constant $\tilde{C}_l \ge 1$ such that*

$$N_j^{l/C} \le \tilde{C}_l \cdot M_j^l \tag{5.3.1}$$

holds for all $j \in \mathbb{N}$.

(5.3.1) *implies $\mathcal{N}\{\preceq\}\mathcal{M}$ (resp. $\mathcal{N}(\preceq)\mathcal{M}$):*

(*i*) *For each $l > 0$ we can find $n > 0$ (resp. for each $n > 0$ there exists $l > 0$) such that $N^l \preceq M^n$ or equivalently*

(*ii*) *For each $l > 0$ we can find $n > 0$ (resp. for each $n > 0$ there exists $l > 0$) such that $\mathcal{E}_{[N^l]} \subseteq \mathcal{E}_{[M^n]}$.*

If σ has additionally property (ω_1), then $(i) \Leftrightarrow (ii)$ implies that for each $l > 0$ we can find $n > 0$ and $C \ge 1$ (resp. for each $n > 0$ there exists $l > 0$ and $C \ge 1$) such that

$$N_j^l \le C \cdot M_j^n. \tag{5.3.2}$$

for all $j \in \mathbb{N}$.

(*b*) *If $\tau \lhd \sigma$, then for all $C > 0$ and all $l > 0$ there exists a constant $\tilde{D}_{C,l} \ge 1$ with*

$$N_j^l \le \tilde{D}_{C,l} \cdot M_j^{Cl} \tag{5.3.3}$$

for all $j \in \mathbb{N}$. If τ has additionally (ω_1), then this implies now $\mathcal{N} \lhd \mathcal{M}$:

(*i*) *$N^{l_1} \lhd M^{l_2}$ for each $l_1, l_2 > 0$ or equivalently*

(*ii*) *$\mathcal{E}_{\{N^{l_1}\}} \subseteq \mathcal{E}_{(M^{l_2})}$ for all $l_1, l_2 > 0$.*

The converse direction that $(i) \Leftrightarrow (ii)$ implies (5.3.3) is always true.

Proof. (*a*) First $(i) \Rightarrow (ii)$ is clear by definition, for $(ii) \Rightarrow (i)$ we use in the Roumieu-case [38, Theorem 3.1.3] and [40, Theorem 1], in the Beurling-case the arguments given in [6] (note that each sequence $M^l \in \mathcal{LC}$ and so satisfies the required assumptions there).
We prove that $\tau \preceq \sigma$ implies (5.3.1), and for this we use 3.2.1: $\tau \preceq \sigma$ implies that there exists a constant $C \ge 1$ such that for all $x \ge 0$ we get $C \cdot \varphi_\tau^*(\frac{x}{C}) \le \varphi_\sigma^*(x) + C$. Then let $l > 0$ and $j \in \mathbb{N}$ be arbitrary, we use this inequality now for the case $x = l \cdot j$ and moreover divide it by $l > 0$ to obtain

$$\frac{C}{l} \cdot \varphi_\tau^*\left(\frac{lj}{C}\right) \le \frac{1}{l} \cdot \varphi_\sigma^*(lj) + \frac{C}{l}.$$

To this inequality we apply the exponential function and so we have shown (5.3.1) with $\tilde{C}_l = \exp(\frac{C}{l})$ (note: $\tilde{C}_l \to \infty$ for $l \to 0$).

(5.3.1) implies clearly (i), (i) implies (5.3.2) if σ satisfies in addition (ω_1): We can absorb by (3.3.2) (see 5.1.4 and (5.1.3)) the exponential growth on the right hand side to obtain (5.3.1).

(*b*) $(i) \Leftrightarrow (ii)$: First $N^{l_1} \lhd M^{l_2}$ for all $l_1, l_2 > 0$ implies immediately $\mathcal{E}_{\{N^{l_1}\}} \subseteq \mathcal{E}_{(M^{l_2})}$. Conversely, if $\mathcal{E}_{\{N^{l_1}\}} \subseteq \mathcal{E}_{(M^{l_2})}$, then similarly as in [38, Theorem 3.1.3] and [40, Theorem 1] (for each $l_1 > 0$ there exists a characteristic function $\theta_{N^{l_1}}$ by (chf)), we get $N^{l_1} \lhd M^{l_2}$ for all $l_1, l_2 > 0$.

Now assume that $\tau \lhd \sigma$. Then for all $C > 0$ there exists a constant $D_C \geq 1$ such that $\varphi^*_\tau(x) \leq \frac{1}{C} \cdot \varphi^*_\sigma(C \cdot x) + \frac{D_C}{C}$ for all $x \geq 0$ holds by the calculation in 3.5.1. We use again this inequality for $x = l \cdot j$, divide it by $l > 0$ and finally apply the exponential function to show (5.3.3) for the constant $\tilde{D}_{C,l} = \exp(\frac{D_C}{Cl})$.
Then we have to use (ω_1) for τ to obtain for all $k \in \mathbb{N}$ (for $h > 1$ large): $h^k \cdot N^{l_1}_k \leq D_{h,l} \cdot N^l_k \leq D_{h,l} \cdot \tilde{D}_{C,l} \cdot M^{Cl}_k$. The number l depends on l_1 and h and will be much larger than l (see (3.3.2) (see 5.1.4 and (5.1.3)), but by (5.3.3)we can choose $C > 0$ in such a way that $C \cdot l = l_2$ for some given l_2 (small). Hence we have shown $N^{l_1} \lhd M^{l_2}$ for all $l_1, l_2 > 0$.
$N^{l_1} \lhd M^{l_2}$ implies clearly (5.3.3), more precisely we can take $h = 1$. □

We use this result for the weight $\sigma = \mathrm{id}$ to obtain:

Lemma 5.3.2. *Let $\omega \in \mathcal{W}$, then*

(1) *If ω has also (ω_1) and (ω_2), then for all $l > 0$ there exists $D_l > 0$ such that for all $j \in \mathbb{N}$ we obtain*
$$j^j \leq D_l \cdot M^l_j.$$

(2) *If ω has also (ω_1) and (ω_5), then for all $l > 0$, all $1 > D > 0$ small and for all $j \in \mathbb{N}$ we obtain*
$$j^j \leq D \cdot M^l_j.$$

Proof. We use (3.5.1) resp. (3.5.2) for $x = l \cdot j$, $l > 0$ and $j \in \mathbb{N}$ arbitrary, and take the l-th root to obtain the desired properties directly. □

An immediate consequence of the previous lemma together with Stirling's formula is the following:

(1) For $\omega \in \mathcal{W}$ with (ω_1) and (ω_2) we get
$$\forall\, l > 0 : \ \liminf_{k \to \infty} (m^l_k)^{1/k} > 0,$$
hence each space $\mathcal{E}_{\{M^l\}}$ contains the class of all real analytic functions.

(2) For $\omega \in \mathcal{W}$ with (ω_1) and (ω_5) we get
$$\forall\, l > 0 : \ \liminf_{k \to \infty} (m^l_k)^{1/k} = +\infty,$$
hence each space $\mathcal{E}_{(M^l)}$ contains the class of all real analytic functions.

5.4 Consequences of (ω_7)

Lemma 5.4.1. *Let $\omega \in \mathcal{W}$, then property (ω_7) holds if and only if*
$$\forall\, l > 0\ \exists\, C > 0\ \exists\, n > l\ \forall\, j \in \mathbb{N} : M^l_{2j} \leq C^j \cdot M^n_j$$
or alternatively
$$\forall\, n > 0\ \exists\, C > 0\ \exists\, l < n\ \forall\, j \in \mathbb{N} : M^l_{2j} \leq C^j \cdot M^n_j.$$
This implies that then for all $l > 0$ there exists $n > 0$ (resp. for all $n > 0$ there exists $l > 0$) with $M^l \lhd M^n$ and so $\lim_{p \to \infty} \left(\frac{M^l_p}{M^n_p}\right)^{1/p} = 0$.

Remark: For ($\Rightarrow$) we don't need necessarily property (ω_4).

Proof. ($\Rightarrow$) Recall that in 3.6.1 we have shown (3.6.1): There exist $C, H > 0$ such that for all $x \geq 0$ we get

$$C \cdot \varphi^*_\omega\left(\frac{x}{C}\right) \leq \varphi^*_\omega\left(\frac{x}{2}\right) + \log(H)\cdot x + C.$$

Now we use this inequality for $x = 2lj$, $l > 0$ and $j \in \mathbb{N}$ to obtain after dividing by l the following: $\frac{C}{l}\cdot\varphi^*_\omega(\frac{2lj}{C}) \leq \frac{1}{l}\cdot\varphi^*_\omega(lj) + 2j\cdot\log(H) + \frac{C}{l}$. Finally we apply the function exp to see:

$$\exists\, C, H > 0 :\ \forall\, l > 0\ \forall\, j \in \mathbb{N} : M_{2j}^{l/C} \leq \exp\left(\frac{C}{l}\right)\cdot (H^2)^j \cdot M^l_j.$$

($\Leftarrow$) Conversely assume that one of the properties is satisfied, then after applying log we obtain for all $j \in \mathbb{N}$:

$$\frac{1}{l}\cdot\varphi^*_\omega(2lj) \leq j\cdot\log(C) + \frac{1}{n}\cdot\varphi^*_\omega(nj).$$

Now proceed by standard arguments: Replace $j \in \mathbb{N}$ by $y \geq 0$ and apply the *Legendre-Fenchel-Young-conjugate* to both sides for $x \geq 0$. The left hand side gives

$$\begin{aligned}\left(\frac{1}{l}\cdot\varphi^*_\omega(2l\cdot)\right)^*(x) &= \sup_{y\geq 0}\left\{xy - \frac{1}{l}\cdot\varphi^*_\omega(2ly)\right\} = \frac{1}{l}\cdot\sup_{z\geq 0}\left\{\frac{lxz}{2l} - \varphi^*_\omega(z)\right\}\\ &= \frac{1}{l}\cdot\varphi^{**}_\omega\left(\frac{x}{2}\right) = \frac{1}{l}\cdot\omega(\sqrt{e^x})\end{aligned}$$

and the right hand side

$$\begin{aligned}&\left(\cdot\log(C) + \frac{1}{n}\cdot\varphi^*_\omega(n\cdot)\right)^*(x) = \sup_{y\geq 0}\left\{xy - y\log(C) - \frac{1}{n}\cdot\varphi^*_\omega(n\cdot y)\right\}\\ &= \frac{1}{n}\cdot\sup_{z\geq 0}\left\{\frac{z}{n}(x-\log(C))n - \varphi^*_\omega(z)\right\} = \frac{1}{n}\cdot\varphi^{**}_\omega(x-\log(C)) = \frac{1}{n}\cdot\omega\left(\frac{e^x}{C}\right).\end{aligned}$$

We summarize:

$$\begin{aligned}&\frac{1}{l}\cdot\omega(\sqrt{e^x}) = \sup_{y\geq 0}\left\{xy - \frac{1}{l}\cdot\varphi^*_\omega(2ly)\right\} \geq \sup_{j\in\mathbb{N}}\left\{xj - \frac{1}{l}\cdot\varphi^*_\omega(2lj)\right\}\\ &\geq \sup_{j\in\mathbb{N}}\left\{xj - j\cdot\log(C) - \frac{1}{n}\cdot\varphi^*_\omega(nj)\right\} \underbrace{\geq}_{(\star)} \frac{1}{2}\cdot\sup_{y\geq 0}\left\{xy - y\cdot\log(C) - \frac{1}{n}\cdot\varphi^*_\omega(ny)\right\}\\ &= \frac{1}{n}\cdot\omega\left(\frac{e^x}{C}\right).\end{aligned}$$

($\star$) holds again by 4.0.3 and 5.1.3: $\omega_n = \frac{1}{n}\cdot\omega \sim \omega_{M^n}$, note that we have $\omega_n(t) \leq 2\cdot\omega_{M^n}(t)$ for all $t \geq \frac{M^n_2}{M^n_1} = \mu^n_2$. Hence for $\frac{\exp(x)}{C} \geq \mu^n_2$ we have shown $\omega(\frac{t}{C}) \leq \frac{n}{l}\cdot\omega(\sqrt{t})$, where we have put $t := \exp(x)$ and we are done.

So we shown the equivalence, for the last argument use now (5.1.1) for $j = k$: Let $n_1, n_2 > 0$ be arbitrary but fixed, then $M^{n_1}_j \cdot M^{n_2}_j \leq M^{n_3}_{2j} \leq C_1 \cdot C^j_2 \cdot M^{n_4}_j$ for $n_3 = \max\{n_1, n_2\}$ and $n_4 = C\cdot n_3$. Hence the sequence $L^{n_1,n_2}_j := M^{n_1}_j \cdot M^{n_2}_j$ satisfies $L^{n_1,n_2} \preceq M^{n_4}$ but by definition $M^{n_i} \lhd L^{n_1,n_2}$ for $i = 1, 2$. $\square$

Remarks:

(*i*) By 3.6.1 property (ω_7) implies that for all $l > 0$ there exists $m > 0$ (or for all $m > 0$ there exists $l > 0$) with $M^m \lhd M^l$ and so $\mathcal{E}_{\{M^m\}} \subseteq \mathcal{E}_{(M^l)}$. Hence by (5.2.3) and 5.2.2 we see, that (ω_7) is an obstruction to property (ω_6).

(*ii*) As we have shown in 3.10.1, the weight function $\omega_s = \max\{0, \log(t)^s\}$ doesn't satisfy (ω_6) for any $s > 1$, but (ω_7) holds.

(*iii*) An iterated application of 5.4.1 gives the following: We can estimate a "chain" of weight sequences by a single sequence. More precisely for all $i \in \mathbb{N}_{>0}$ and all $l_1, \dots, l_i > 0$ there exists some $n > 0$ such that $M^{l_1}_j \cdots M^{l_i}_j \le C^j \cdot M^n_j$ holds for all $j \in \mathbb{N}$ for a constant $C \ge 1$ (and also with "Beurling-type-order" of quantifiers).

5.5 Remarks about strong log. convexity

As shown in 5.1.1, each sequence M^l is weakly log. convex. What about strong log. convexity, i.e. that the sequence $m^l_j := \frac{M^l_j}{j!}$ is log. convex?
As shown above in (3.8.1) for sub-additive $\omega \in \mathcal{W}$ each sequence m^l satisfies $m^l_j \cdot m^l_k \le m^l_{j+k}$ for all $l > 0$ and $j, k \in \mathbb{N}$. This is not strong log. convexity directly, but a weaker condition (see e.g. [38, Lemma 2.0.6.]).

For strong log. convexity one would need $(m^l_j)^2 \le m^l_{j-1} \cdot m^l_{j+1}$ for all $j \ge 1$ (for a certain $l > 0$). Recall the identity $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot j)) = \sup_{s \ge 1}\{s^j \cdot \exp(-\frac{1}{l} \cdot \omega(s))\}$ (used in the proof of (3.8.1)) and introduce the function $f_{j,l}$ by $f_{j,l}(s) := s^j \cdot \exp(-\frac{1}{l} \cdot \omega(s))$, $j \ge 1$ and $l > 0$. We study this function on $[1, +\infty)$ (or even on $[0, +\infty)$, because $\omega(s) = 0$ for $s \in [0, 1]$ and $s^j \le 1$ for all $s \in [0, 1]$ and $j \ge 1$).
We are searching for the point s_0, where the supremum of $f_{j,l}$ is attained! We differentiate: $f'_{j,l}(s) = j \cdot s^{j-1} \cdot \exp(-\frac{1}{l} \cdot \omega(s)) + s^j \cdot \exp(-\frac{1}{l} \cdot \omega(s)) \cdot (-\frac{1}{l} \cdot \omega'(s))$. So the first derivative vanishes either for $s = 0$ (which is not the interesting case), otherwise divide the equation by $s^{j-1} > 0$ and $\exp(-\frac{1}{l} \cdot \omega(s)) > 0$ to obtain now $j - \frac{s}{l} \cdot \omega'(s) = 0$. Hence we are searching for $s \in [0, +\infty)$, such that the equation $j \cdot l = s \cdot \omega'(s)$ holds (where we have assumed throughout this calculation that ω should be at least $\mathcal{C}^1$).

Example: For $\omega(s) = s^\alpha$, $0 < \alpha < 1$, (which leads to the Gevrey-classes with index $\frac{1}{\alpha}$) we need $jl = s \cdot \alpha \cdot s^{\alpha-1}$, hence for $s_0 = \left(\frac{jl}{\alpha}\right)^{1/\alpha}$. So at this point the supremum is attained and $f_{j,l}(s_0) = \left(\frac{jl}{\alpha}\right)^{j/\alpha} \cdot \exp(-\frac{1}{l} \cdot \omega(s_0)) = \left(\frac{jl}{\alpha}\right)^{j/\alpha} \cdot \exp(-\frac{j}{\alpha}) = \left(\frac{jl}{\exp(1) \cdot \alpha}\right)^{j/\alpha}$.

5.6 Limits of sequences M^l

In all previous calculations we have assumed $0 < l < +\infty$ for the parameter l. In this section we will study the limit cases $l \to 0$ resp. $l \to +\infty$. First we point out: Let $\omega \in \mathcal{W}$, then for each $p \in \mathbb{N}_{>0}$ arbitrary but fixed we have

$$M^{+\infty}_p := \lim_{l \to +\infty} M^l_p = \lim_{l \to +\infty} \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot p)\right) = \exp(+\infty) = +\infty$$

and $M_0^{+\infty} = \lim_{l\to+\infty} \exp\left(\frac{1}{l}\cdot\varphi^*_\omega(0)\right) = \lim_{l\to+\infty}\exp(0) = 1$. In this calculation we have only used the standard properties of φ^*_ω, see [5, 1.3. Remark, 1.5. Lemma].
On the other side assume that φ^*_ω is $\mathcal{C}^1$ (can be done w.l.o.g. by [5, Lemma 1.7.]). Then

$$M_p^l = \exp\left(\frac{1}{l}\cdot\varphi^*_\omega(l\cdot p)\right) = \exp\left(\frac{\varphi^*_\omega(l\cdot p)-\varphi^*_\omega(0)}{l}\right) \underbrace{=}_{lp=m} \exp\left(p\cdot\frac{\varphi^*_\omega(m)-\varphi^*_\omega(0)}{m-0}\right),$$

hence for all $p\in\mathbb{N}$:

$$M_p^0 := \lim_{l\to 0} M_p^l = \lim_{m\to 0}\exp\left(p\cdot\frac{\varphi^*_\omega(m)-\varphi^*_\omega(0)}{m-0}\right) = \exp(p\cdot(\varphi^*_\omega)'(0)) = \left(\exp((\varphi^*_\omega)'(0))\right)^p. \tag{5.6.1}$$

It's clear, that $M^0 = (M_p^0)_p$ is again an increasing sequence and furthermore weakly log. convex, because the function $p\mapsto\log(M_p^0) = p\cdot(\varphi^*_\omega)'(0)$ is clearly convex. Since φ^*_ω is an increasing function, we have $(\varphi^*_\omega)'(0)\ge 0$, hence $M_p^0 \ge \exp(p\cdot 0) = 1$ for all $p\in\mathbb{N}$.
Example: $\varphi^*_\omega(x) = x^{s/(s-1)}$, $s>1$, then $M_p^{+\infty} = +\infty$ and $M_p^0 = \exp(p\cdot 0) = 1$ for all $p\in\mathbb{N}$. The same holds for the function $\varphi^*_\omega(x) = x^2\cdot\log(x)$ for $x\ge 1$ and $\varphi^*_\omega(x) = 0$ for $0\le x\le 1$.

5.7 Spaces of few functions $\mathcal{E}_{((\omega))}$ resp. $\mathcal{E}_{((M))}$

Let $\omega\in\mathcal{W}$, then on non-empty open subsets $U\subseteq\mathbb{R}^n$ we can define the space $\mathcal{E}_{((\omega))}(U)$ by

$$\forall\, K\subseteq U \text{ compact } \exists\, C>0\ \forall\, l>0\ \forall\, x\in K\ \forall\, \alpha\in\mathbb{N}^n: \left|f^{(\alpha)}(x)\right| \le C\cdot\exp\left(\frac{1}{l}\cdot\varphi^*_\omega(l|\alpha|)\right).$$

One can see immediately: $\mathcal{E}_{((\omega))}\subseteq\mathcal{E}_{(\omega)}\subseteq\mathcal{E}_{\{\omega\}}$ and a function f belongs to $\mathcal{E}_{((\omega))}$ if it is in the Beurling class $\mathcal{E}_{(\omega)}$ and for the occurring constants $C_l>0$ we have $\sup_{l>0} C_l < +\infty$.

But $M^{l_1}\le M^{l_2}$ whenever $l_1\le l_2$ and by (5.6.1) above we have $M_p^0 = \left(\exp((\varphi^*_\omega)'(0))\right)^p$. Hence $\left|f^{(\alpha)}(x)\right| \le C\cdot\left(\exp((\varphi^*_\omega)'(0))\right)^{|\alpha|}$ should also hold and we see, that f must belong to the real-analytic functions, because we can take $\varrho := \exp((\varphi^*_\omega)'(0))$ now and $\varrho\ge\exp(0)=1$ by the properties of $(\varphi^*_\omega)'$. So we have shown $\mathcal{E}_{((\omega))}\subseteq\mathcal{E}_{\{\text{id}\}}$ (recall: the real analytic functions have the weight $\omega(t)=t!$), hence the spaces $\mathcal{E}_{((\omega))}$ are really small. If one assumes additionally (ω_5) for ω, then one has $\mathcal{E}_{((\omega))}\subseteq\mathcal{E}_{\{\text{id}\}}\subseteq\mathcal{E}_{(\omega)}\subseteq\mathcal{E}_{\{\omega\}}$.

The space $\mathcal{E}_{((\omega))}$ contains not many functions, but it is still infinite dimensional, because it contains all polynomials. Furthermore the functions exp, sin and cos are always elements of $\mathcal{E}_{((\omega))}$. If $(\varphi^*_\omega)'(0)=0$ like in the example above, then we have $\left|f^{(\alpha)}(x)\right|\le C$ for all $\alpha\in\mathbb{N}$, and the constant C is depending only on K. So we see: The polynomials, exp and sin are in $\mathcal{E}_{((\omega))}$ too, but for $a>1$ arbitrary we have $(\exp(ax))^{(k)} = a^k\cdot\exp(x)$, and so the function $x\mapsto\exp(ax)$ is not an element of $\mathcal{E}_{((\omega))}$ anymore and similarly for $x\mapsto\sin(ax)$ (resp. take $a>\exp((\varphi^*_\omega)'(0))$ in the more general case).

One can use the same proof as in [5, 4.4 Proposition] to show, that for $\omega\in\mathcal{W}$ with (ω_1) the class $\mathcal{E}_{((\omega))}$ is closed under pointwise multiplication.

Difference to the weight sequence case: Also for weight sequences $M = (M_p)_p$ one can consider for non-empty open subsets $U \subseteq \mathbb{R}^n$ the spaces $\mathcal{E}_{((M))}(U)$ defined by

$$\forall\, K \subseteq U \text{ compact } \exists\, C > 0\, \forall\, h > 0\, \forall\, x \in K\, \forall\, \alpha \in \mathbb{N}^n : \left|f^{(\alpha)}(x)\right| \leq C \cdot h^k \cdot M_k.$$

But in this case in the estimate C depends again only on the compact set K, not on $h > 0$ and $p \in \mathbb{N}$. In particular (one dimensional case) for $k = 1$ we have $\left|f^{'}(x)\right| \leq C \cdot h \cdot M_1$ for all $h > 0$, hence $\sup_{x \in K} \left|f^{'}(x)\right| = 0$ for all $K \subseteq U$ compact (and also for all higher derivatives). For $k = 0$ one obtains $|f(x)| \leq C \cdot M_0$, hence $\mathcal{E}_{((M))}(U)$ consists of all constant functions on U. So this space is only one-dimensional and an immediate consequence is $\mathcal{E}_{((\omega))} \neq \mathcal{E}_{((M))}$.

6 A new version of the comparison theorem

6.1 Introduction and first observations

The aim of this chapter is the following: We use the sequences M^l introduced in the previous chapter and prove new comparison results for classes of ultradifferentiable functions defined by weight sequences M and weight functions ω. These comparison theorems will motivate the definition of classes of ultradifferentiable functions defined by a weight matrix $\mathcal{M}$ in the following chapter below.

(i) All results in this chapter are also valid for globally defined classes by modification of the proofs, see 2.4.1.

(ii) For the first part of 6.2.1 and 6.2.6 we don't need necessarily property (ω_4), see 3.1.1 and 3.1.2 for more details.

Recall the definitions of such classes: For compact sets K with smooth boundary the space

$$\mathcal{E}_{\omega,l}(K) := \{f \in \mathcal{E}(K) : \|f\|_{\omega,K,l} < +\infty\}$$

is a Banach-space, where $\|f\|_{\omega,K,l} := \sup_{k\in\mathbb{N},x\in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R})}}{\exp(\frac{1}{l}\cdot\varphi^*_\omega(l\cdot k))}$ and $f^{(k)}(x)$ denotes the k-th order Fréchet derivative of f at x. Moreover we have the following representations as locally convex vector spaces:

$$\mathcal{E}_{\{\omega\}}(U) = \varprojlim_{K\subseteq U} \varinjlim_{l>0} \mathcal{E}_{\omega,l}(K) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{\omega\}}(K)$$

resp.

$$\mathcal{E}_{(\omega)}(U) = \varprojlim_{K\subseteq U} \varprojlim_{l>0} \mathcal{E}_{\omega,l}(K) = \varprojlim_{K\subseteq U} \mathcal{E}_{(\omega)}(K),$$

where the limits run over all compact sets $K \subseteq U$ and $l > 0$.
For the weight sequence case the situation is analogous: For compact sets K with smooth boundary the space

$$\mathcal{E}_{M,h}(K) := \{f \in \mathcal{E}(K) : \|f\|_{M,K,h} < +\infty\}$$

is a Banach-space, $\|f\|_{M,K,h} := \sup_{k\in\mathbb{N},x\in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R})}}{h^k\cdot M_k}$ and we have the topological vector space representations

$$\mathcal{E}_{\{M\}}(U) = \varprojlim_{K\subseteq U} \varinjlim_{h>0} \mathcal{E}_{M,h}(K) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{M\}}(K)$$

resp.

$$\mathcal{E}_{(M)}(U) = \varprojlim_{K\subseteq U} \varprojlim_{h>0} \mathcal{E}_{M,h}(K) = \varprojlim_{K\subseteq U} \mathcal{E}_{(M)}(K),$$

where the limits run over all compact sets $K \subseteq U$ and $h > 0$.

We start with some easy observations:

Lemma 6.1.1. *Let $\omega \in \mathcal{W}$, then on each compact subset $K \subseteq \mathbb{R}^n$ we get*

$$\mathcal{E}_{\{\omega\}}(K) \subseteq \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(K)$$

and for $U \subseteq \mathbb{R}^n$ non-empty open we get

$$\varprojlim_{l\in\Lambda} \mathcal{E}_{(M^l)}(U) \subseteq \mathcal{E}_{(\omega)}(U) \subseteq \varprojlim_{l>0} \mathcal{E}_{\{M^l\}}(U).$$

All inclusions are continuous.

Proof. For the first part: If $f \in \mathcal{E}_{\{\omega\}}(K)$, then there exist $C, l > 0$, with $|f^{(\alpha)}(x)| \leq C \cdot M^l_{|\alpha|}$ for all $\alpha \in \mathbb{N}^n$ and $x \in K$. Then clearly $f \in \mathcal{E}_{\{M^l\}}(K)$ with $C > 0$ and $h = 1$ in the defining estimate on K and more precisely in the loc. convex vector space representations we get $\mathcal{E}_{\omega,l}(K) \subseteq \mathcal{E}_{M^l,1}(K)$ for each K compact and $l > 0$. We can restrict of course the limit(s) to $l \in \mathbb{N}_{>0}$ and we will show in 6.2.4 that if we assume additionally (ω_1), then the converse inclusion also holds.

For the second part: If $f \in \mathcal{E}_{(\omega)}(U)$, then on an arbitrary but fixed compact set $K \subseteq U$ the defining estimate holds in such a way that for all $l > 0$ there exists $C_l > 0$, such that $|f^{(\alpha)}(x)| \leq C_l \cdot M^l_{|\alpha|}$ for all $\alpha \in \mathbb{N}^n$ and $x \in K \subseteq U$. Hence $f \in \mathcal{E}_{\{M^l\}}(K)$ for all $l > 0$ with $C_l > 0$ and $h = 1$, so more precisely $f \in \varprojlim_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(U)$ and $\mathcal{E}_{(\omega)}(U) \subseteq \varprojlim_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(U)$ for each $U \subseteq \mathbb{R}^n$ non-empty open.
If $f \in \varprojlim_{l\in\Lambda} \mathcal{E}_{(M^l)}(U)$, then for each $l > 0$ and each $h > 0$ there exists a constant $C_{l,h} > 0$ such that $\sup_{x\in K} |f^{(\alpha)}(x)| \leq C_{l,h} \cdot h^{|\alpha|} \cdot M^l_{|\alpha|}$ holds for all $\alpha \in \mathbb{N}^n$, where $K \subseteq U$ is an arbitrary but fixed compact set. Hence for the choice $h = 1$ we obtain clearly $f \in \mathcal{E}_{(\omega)}(U)$.

We will show in 6.2.6, that for the first inclusion in fact equality is satisfied, if we have property (ω_1) for $\omega \in \mathcal{W}$, too. For the second inclusion the converse inclusion "$\supseteq$" cannot be true in general, because if 6.2.1 holds then $\mathcal{E}_{\{M^m\}} = \mathcal{E}_{\{M^l\}}$ for all $m, l > 0$ and so we would get $\mathcal{E}_{(\omega)} = \mathcal{E}_{\{\omega\}}$. But we will show in 6.2.7, that if $\omega \in \mathcal{W}$ satisfies in addition (ω_1) and (ω_7) (then (ω_6) cannot be satisfied), we have equality instead of both inclusions.
Of course we can restrict the intersections to all $l' := \frac{1}{l}$, $l' \in \mathbb{N}_{>0}$. □

6.2 First central result

In this section we will prove the following important theorem:

Theorem 6.2.1. *Let $\omega \in \mathcal{W}$ be given such that conditions (ω_1) and (ω_6) are satisfied. Then we get:*

(1) *$\mathcal{E}_{[M^l]}(U) = \mathcal{E}_{[\omega]}(U)$ holds for each $l > 0$ and for all $U \subseteq \mathbb{R}^n$ non-empty open sets as locally convex vector spaces.*

(2) *Moreover in this situation for each $l > 0$ the following holds: $\omega \sim \omega_l \sim \omega_{M^l}$, where $\omega_l(x) := \frac{\omega(x)}{l}$, and so as locally convex vector spaces we get $\mathcal{E}_{[M^l]}(U) = \mathcal{E}_{[\omega]}(U) = \mathcal{E}_{[\omega_{M^l}]}(U)$ again for all $U \subseteq \mathbb{R}^n$ non-empty open.*

$M^l \in \mathcal{LC}$ for each $l > 0$ and each function $\omega_{M^l}, \omega_l \in \mathcal{W}$ with additionally (ω_1) *and* (ω_6).

Finally $M^l \approx M^m$ holds for all $l, m > 0$ and M^l satisfies moderate growth (mg), *hence also closedness under taking derivatives* (dc), *for each $l > 0$.*

(3) *Let ω be as in* (1) *with additionally condition* (ω_2), *then for each $l > 0$ the sequence M^l satisfies also $\liminf_{p\to\infty}(m_p^l)^{1/p} > 0$ with $m_p^l := \frac{M_p^l}{p!}$ or equivalently (by using Stirling's formula)*

$$\exists\, C, D > 0\, \forall\, p \in \mathbb{N}: \quad (C \cdot (p+1))^p \le D \cdot M_p^l, \tag{6.2.1}$$

which should be also compared with condition $(M0)$ *in [4]. Moreover condition* (β_3) *holds also for each sequence M^l separately. Finally for each $l > 0$ we have that the functions ω_l and ω_{M^l} have condition* (ω_2), *too.*

If ω is as in (1) *with additionally condition* (ω_5), *then for each $l > 0$ the sequence M^l satisfies also $\lim_{p\to\infty}(m_p^l)^{1/p} = +\infty$ or equivalently*

$$\forall\, C_1 > 0\, \exists\, C_2 > 0\, \forall\, p \in \mathbb{N} : (C_1 \cdot (p+1))^p \le C_2 \cdot M_p^l. \tag{6.2.2}$$

In this case for each $l > 0$ we have that the functions ω_l and ω_{M^l} have condition (ω_5), *too.*

(4) *Let ω be like in* (1) *with additionally property* (ω_{nq}). *Then the sequences M^l have* (nq) *for each $l > 0$, too.*

We split the proof of Theorem 6.2.1 in several statements and start with the first one:

Lemma 6.2.2. *Assume that $\omega \in \mathcal{W}$ satisfies* (ω_6). *Then for all $m > 0$ and each $K \subseteq U$ compact the inclusion $\mathcal{E}_{\omega,m}(K) \subseteq \mathcal{E}_{M^1,H^n}(K)$ holds, where $n \in \mathbb{N}$ is chosen minimal, such that $m \le 2^n$ and $H \ge 1$ is the constant coming from* (ω_6).

Proof. Let $m > 0$ be arbitrary but fixed and estimate as follows:

$$\begin{aligned}
&\exp\left(\frac{1}{m}\cdot\varphi_\omega^*(m\cdot j)\right) = \exp\left(\varphi_\omega^*(m\cdot j)\right)^{1/m} \underbrace{=}_{(\star)} \left(\sup_{t\ge 0}\frac{t^{m\cdot j}}{\exp(\omega(t))}\right)^{1/m}\\
&= \sup_{t\ge0}\frac{t^j}{\exp(\omega(t))^{1/m}} = \sup_{t\ge0}\frac{t^j}{\exp(\frac{1}{m}\cdot\omega(t))} \underbrace{\le}_{(\star\star)} \sup_{t\ge0}\frac{t^j}{\exp(\omega(C_m\cdot t) - \frac{D_m}{m})}\\
&= \exp\left(\frac{D_m}{m}\right)\cdot\sup_{t\ge0}\frac{t^j}{\exp(\omega(C_m\cdot t))} \underbrace{=}_{t\mapsto\frac{t}{C_m}} \exp\left(\frac{D_m}{m}\right)\cdot\left(\frac{1}{C_m}\right)^j\cdot\sup_{t\ge0}\frac{t^j}{\exp(\omega(t))}\\
&= \exp\left(\frac{D_m}{m}\right)\cdot\underbrace{\left(\frac{1}{C_m}\right)^j}_{=h^j}\cdot\underbrace{\exp\left(\varphi_\omega^*(j)\right)}_{=:M_j^1}.
\end{aligned}$$

The second equality $(\star)$ holds for arbitrary $m > 0$ because:

$$\begin{aligned}
\sup_{t\ge0}\frac{t^m}{\exp(\omega(t))} &= \exp\left(\sup_{t\ge0}\left(m\cdot\log(t) - \omega(t)\right)\right) \underbrace{=}_{x=\log(t)} \exp\left(\sup_{x\in\mathbb{R}}\left(m\cdot x - \omega(\exp(x))\right)\right)\\
&= \exp\left(\sup_{x\in\mathbb{R}}\left(m\cdot x - \varphi_\omega(x)\right)\right) = \exp\left(\sup_{x\ge0}\left(m\cdot x - \varphi_\omega(x)\right)\right) = \exp\left(\varphi_\omega^*(m)\right).
\end{aligned}$$

For ($\star\star$) we point out: One would need that for all $m > 0$ there exist constants $C_m, D_m > 0$ such that for all $t \geq 0$ we get:

$$\omega(C_m \cdot t) - \frac{1}{m} \cdot D_m \leq \frac{1}{m} \cdot \omega(t) \Leftrightarrow m \cdot \omega(C_m \cdot t) \leq \omega(t) + D_m. \qquad (6.2.3)$$

If $0 < m \leq 1$, then clearly $m \cdot \omega(t) \leq \omega(t)$ for all $t \geq 0$, hence in this situation the choice $C_m = 1$ and $D_m = 0$ is sufficient for (6.2.3).

But for $m > 1$ (in fact $m \in \mathbb{N}_{>0}$ for the Roumieu-case) we need property (ω_6): An iterated application of (ω_6) (see (3.4.2)) implies that for all $n \in \mathbb{N}$ and $t \geq 0$ we get $2^n \cdot \omega(t) \leq \omega(H^n \cdot t) + (2^n - 1) \cdot H$ for a constant $H \geq 1$. So take $n \in \mathbb{N}$ minimal such that $m \leq 2^n$ holds and put for this n now $C_m = \frac{1}{H^n}$, then $m \cdot \omega(C_m \cdot t) = m \cdot \omega(\frac{t}{H^n}) \leq 2^n \cdot \omega(\frac{t}{H^n}) \leq \omega(t) + (2^n - 1) \cdot H$. Then take $D_m = (2^n - 1) \cdot H$ and we have shown (6.2.3).

In terms of the corresponding locally convex vector space representations we have now for all $m > 0$ and each $K \subseteq U$ compact the inclusion

$$\mathcal{E}_{\omega,m}(K) \subseteq \mathcal{E}_{M^1,\frac{1}{C_m}}(K) = \mathcal{E}_{M^1,H^n}(K),$$

where $n \in \mathbb{N}$ is chosen minimal, such that $m \leq 2^n$ and $H \geq 1$ is the constant coming from (ω_6)!

Finally recall that $M^1_j \leq M^l_j$ for all $j \in \mathbb{N}$ and each $l \geq 1$ (by 5.1.1), hence $\mathcal{E}_{\{M^1\}}(U) \subseteq \mathcal{E}_{\{M^l\}}(U)$ for $l \geq 1$. □

In fact we have shown now that $\mathcal{E}_{(\omega)}(U) \subseteq \mathcal{E}_{\{\omega\}}(U) \subseteq \mathcal{E}_{\{M^l\}}(U)$ holds for all $l \geq 1$ with continuous inclusion. For the analogously inclusion in the Beurling-case we need the following result:

Lemma 6.2.3. *Let $\omega \in \mathcal{W}$ with (ω_1), then for $0 < m \leq 1$ and each $K \subseteq U$ compact we obtain the inclusion $\mathcal{E}_{\omega,m}(K) \subseteq \mathcal{E}_{M^1,\frac{1}{2^n}}(K)$, where $n \in \mathbb{N}$ is chosen maximal, such that $L^n \leq \frac{1}{m}$, and $L \geq 1$ is the constant which appears in (ω_1).*

Proof. It sufficient to show $\mathcal{E}_{(\omega)}(U) \subseteq \mathcal{E}_{(M^1)}(U)$ and of course one can use the estimate in 6.2.2 (holds for all $m > 0$). But we have to show now (6.2.3), assuming $0 < m < 1$, for all $h > 0$, in fact for $0 < h \leq 1$ small. So we cannot proceed as in 6.2.2, because there the choice $n = 0$ with $h = \frac{1}{C_m} = H^n = 1$ is sufficient.

For (6.2.3) one would need $\omega(C_m \cdot t) \leq \frac{\omega(t)}{m} + \frac{D_m}{m}$ for all $t \geq 0$. An iterated application of property (ω_1) for ω (see (3.3.1)) gives $\omega(2^n \cdot t) \leq L^n \cdot \omega(t) + \sum_{i=1}^n L^i$ for all $n \in \mathbb{N}$ and $t \geq 0$. Hence take $n \in \mathbb{N}$ maximal, such that $L^n \leq \frac{1}{m} \Leftrightarrow m \leq \frac{1}{L^n}$ holds for $m \to 0$ small, and so for this chosen n and $0 < m \leq 1$ we get

$$\omega(2^n \cdot t) \leq L^n \cdot \omega(t) + \sum_{i=1}^n L^i \leq \frac{\omega(t)}{m} + \sum_{i=1}^n L^i \leq \frac{\omega(t)}{m} + \frac{\sum_{i=1}^n L^i}{m}.$$

Thus the choice $C_m := 2^n$ and $D_m := \sum_{i=1}^n L^i$ is sufficient for this calculation and we summarize: One can let $C_m \to +\infty$ for $m \to 0 \Leftrightarrow n \to +\infty$.

So, if $\omega \in \mathcal{W}$ with additionally (ω_1), then $\mathcal{E}_{(\omega)}(U) \subseteq \mathcal{E}_{(M^1)}(U) \subseteq \mathcal{E}_{(M^l)}(U)$ for all $l \geq 1$, and in terms of the corresponding topological vector space representation for $0 < m \leq 1$

and each $K \subseteq U$ compact we get

$$\mathcal{E}_{\omega,m}(K) \subseteq \mathcal{E}_{M^1,\frac{1}{Cm}}(K) = \mathcal{E}_{M^1,\frac{1}{2^n}}(K),$$

where $n \in \mathbb{N}$ is chosen maximal, such that $L^n \leq \frac{1}{m}$, and $L \geq 1$ is the constant from (ω_1). □

6.2.3 shows that $\mathcal{E}_{(\omega)}(U) \subseteq \mathcal{E}_{(M^l)}(U)$ is valid with continuous inclusion for all $l \geq 1$ and each $U \subseteq \mathbb{R}^n$ non-empty and open.

Now we are going to prove the third step:

Lemma 6.2.4. *Let $\omega \in \mathcal{W}$ with (ω_1). Then for each $m, h > 0$ and each $K \subseteq U$ compact we have $\mathcal{E}_{M^m,h}(K) \subseteq \mathcal{E}_{\omega,m\cdot L^s}(K)$, where $s \in \mathbb{N}$ is chosen minimal, such that* $\exp(s) \geq h$ *and $L \geq 1$ is the constant from (ω_1).*

Proof. First recall that for $\omega \in \mathcal{W}$ we have by 6.1.1 for each $K \subseteq U$ compact set the inclusion $\mathcal{E}_{\{\omega\}}(K) \subseteq \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(K) = \varinjlim_{l\in\mathbb{N}_{>0}} \mathcal{E}_{\{M^l\}}(K)$. Consider $f \in \mathcal{E}_{\{M^m\}}(U)$, then on an arbitrary but fixed compact set $K \subseteq U$ we have that there exist $C, h > 0$ with $|f^{(\alpha)}(x)| \leq C \cdot h^{|\alpha|} \cdot M^m_{|\alpha|} = C \cdot h^{|\alpha|} \cdot \exp(\frac{1}{m} \cdot \varphi^*_\omega(m \cdot |\alpha|))$ for all $\alpha \in \mathbb{N}^n$. Put $|\alpha| = k$, then we estimate for all $k \in \mathbb{N}$ as follows. Use (3.3.2) and take $s \in \mathbb{N}$ minimal, such that $\exp(s) \geq h$ (for $h \in \mathbb{N}$), thus $\exp(sk) \geq h^k$ for all $k \in \mathbb{N}$. Furthermore replace in (3.3.2) $n \leftrightarrow k$, $\lambda \leftrightarrow \frac{1}{m}$ and so $\mu = \frac{1}{m\cdot L^s} = \frac{1}{n}$ (so $n = L^s \cdot m > m$). Thus for all $k \in \mathbb{N}$ we obtain:

$$\begin{aligned} C \cdot h^k \cdot M^m_k &= C \cdot h^k \cdot \exp\left(\frac{1}{m} \cdot \varphi^*_\omega(m \cdot k)\right) \leq C \cdot \exp\left(sk + \frac{1}{m} \cdot \varphi^*_\omega(mk)\right) \\ &\underbrace{\leq}_{(3.3.2)} C \cdot \exp\left(\frac{1}{m \cdot L^s} \cdot \varphi^*_\omega(k \cdot m \cdot L^s) + \frac{1}{m \cdot L^s} \cdot \sum_{i=1}^{s} L^i\right) \\ &= C \cdot \exp\left(\frac{1}{m \cdot L^s} \cdot \sum_{i=1}^{s} L^i\right) \cdot \exp\left(\frac{1}{m \cdot L^s} \cdot \varphi^*_\omega(k \cdot m \cdot L^s)\right) \\ &= C_1 \cdot M^{m\cdot L^s}_k. \end{aligned}$$

The appearing constant $C_1 > 0$ depends now on $L \geq 1$ from (ω_1), on $C, h > 0$ and $m > 0$. More precisely one has in the locally convex vector space representations (can take $h \in \mathbb{N}_{>0}$):

$$\mathcal{E}_{M^m,h}(K) \subseteq \mathcal{E}_{\omega,m\cdot L^s}(K)$$

for each $K \subseteq U$ compact, where $s \in \mathbb{N}$ is chosen minimal, such that $\exp(s) \geq h$. □

6.2.4 shows now $\mathcal{E}_{(M^m)}(U) \subseteq \mathcal{E}_{\{M^m\}}(U) \subseteq \mathcal{E}_{\{\omega\}}(U)$ with continuous inclusion for each $m > 0$ and $U \subseteq \mathbb{R}^n$ non-empty open. In particular we have now together with 6.1.1 for each $K \subseteq U$ compact set the representation

$$\mathcal{E}_{\{\omega\}}(K) = \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(K) = \varinjlim_{l\in\mathbb{N}_{>0}} \mathcal{E}_{\{M^l\}}(K).$$

Similarly as before, this calculation is too weak to obtain $\mathcal{E}_{(M^m)}(U) \subseteq \mathcal{E}_{(\omega)}(U)$ for an arbitrary $m > 0$, because therefore one would need the estimate in 6.2.4 for all $L^s \cdot m > 0$, assuming $0 < h \leq 1$ small. But in this case in the calculation above the choice $s = 0$ would sufficient, because $\exp(0) = 1 \geq h$ for all $0 < h < 1$ small and so $L^0 \cdot m = m$ will not become arbitrary small! To obtain the desired inclusion we need the last lemma:

Lemma 6.2.5. *Let $\omega \in \mathcal{W}$ with (ω_6). Then for each $m > 0$ (for $0 < h < 1$ small) and each $K \subseteq U$ compact $\mathcal{E}_{M^m,h}(K) \subseteq \mathcal{E}_{\omega,m/2^n}(K)$ holds, where $n \in \mathbb{N}$ is chosen maximal, such that $\frac{1}{H^n} \geq h$ and $H \geq 1$ is the constant from (ω_6)!*

Proof. As shown, (ω_6) implies condition (3.4.1):
$\exists\, H \geq 1\, \forall\, l > 0\, \forall\, n \in \mathbb{N}\, \exists\, m > 0 (m = \frac{l}{2^n} < l)\, \forall\, p \in \mathbb{N}:$

$$\frac{1}{l} \cdot \varphi^*_\omega(l \cdot p) \leq \frac{1}{m} \cdot \varphi^*_\omega(m \cdot p) + n \cdot p \cdot \log(H) + \left(\frac{1}{m} - \frac{1}{l}\right) \cdot H.$$

Then we proceed as follows: Take $n \in \mathbb{N}$ maximal, such that $\exp(-n\log(H)) \geq h \Leftrightarrow \frac{1}{H^n} \geq h$ for $0 < h \leq 1$ small, so $\exp(-nk\log(H)) \geq h^k$ for all $k \in \mathbb{N}$ and then we get for all $k \in \mathbb{N}$:

$$C \cdot h^k \cdot M^l_k = C \cdot h^k \cdot \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot k)\right) \leq C \cdot \exp\left(-n \cdot k \cdot \log(H) + \frac{1}{l} \cdot \varphi^*_\omega(l \cdot k)\right)$$
$$\underbrace{\leq}_{(3.4.1)} C \cdot \exp\left(\left(\frac{1}{m} - \frac{1}{l}\right) \cdot H\right) \cdot \exp\left(\frac{1}{m} \cdot \varphi^*_\omega(m \cdot k)\right) = C_2 \cdot M^m_k = C_2 \cdot M^{\frac{l}{2^n}}_k.$$

We point out: The constant $C_2 > 0$ depends only on $C, h > 0$, $l > 0$ and the constant $H \geq 1$, which appears in (ω_6). If $h \to 0$, then $n \to +\infty$, hence $m = \frac{l}{2^n} \to 0$ and so $\mathcal{E}_{(M^l)} \subseteq \mathcal{E}_{(\omega)}$ for all $l > 0$. In the locally convex vector space representations (for $0 < h < 1$ small) we have for each $K \subseteq U$ compact now

$$\mathcal{E}_{M^l,h}(K) \subseteq \mathcal{E}_{\omega,m}(K) = \mathcal{E}_{\omega,\frac{l}{2^n}}(K),$$

where $n \in \mathbb{N}$ is chosen maximal such that $\frac{1}{H^n} \geq h$ holds! □

In fact 6.2.5 shows $\mathcal{E}_{(M^m)}(U) \subseteq \mathcal{E}_{(\omega)}(U)$ with continuous inclusion for each $m > 0$ and each $U \subseteq \mathbb{R}^n$ non-empty open.

We can now prove the first point of 6.2.1: 6.2.2 - 6.2.5 together imply $\mathcal{E}_{\{M^l\}}(U) = \mathcal{E}_{\{\omega\}}(U)$ resp. $\mathcal{E}_{(M^l)}(U) = \mathcal{E}_{(\omega)}(U)$ as locally convex vector spaces for all $l \geq 1$. To obtain this identity for $0 < l < 1$ we use now 5.2.2!

For the second part we use 5.1.1 resp. 5.1.3 and also 4.0.2. Note that properties (ω_1) and (ω_6) are stable under $\sim$ (see 3.2.2).
Since all sequences M^l are weakly log. convex, by [40, Theorem 1] and [38, Theorem 3.1.3] and the first part of the theorem we finally obtain $M^l \approx M^m$ for all $l, m > 0$. Since ω_{M^l} has (ω_6) for each $l > 0$, we can use [16, Proposition 3.6.] and this implies moderate growth (mg) for M^l.
Thus M^l satisfies also closedness under taking derivatives (dc), but one can prove this property directly: As pointed out in 5.1.2 (a consequence of (5.1.2)), the class $\mathcal{E}_{\{\omega\}}$ is always closed under taking derivatives, hence by the first part of the theorem also $\mathcal{E}_{\{M^l\}}$ for each $l > 0$. So fix $l > 0$ and use again [40, Theorem 1] for the sequence M^l (see (chf)): There exists a function $\theta_l = \theta_{M^l} \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R}, \mathbb{R})$ with $|\theta^{(j)}_l(0)| \geq M^l_j$ for all $j \in \mathbb{N}$. On the other side there exist $C, h > 0$ such that $|\theta^{(j)}_l(0)| = |\theta'^{(j-1)}_l(0)| \leq C \cdot h^{j-1} \cdot M^l_{j-1}$.

Let us now consider the third part: To prove (β_3) for M^l we proceed similarly as in [4, 12. Lemma] (1) $\Rightarrow$ (2) for the sequence M^l for each $l > 0$, more precisely we use some

facts of [24]. In the following let $\mathcal{E}^{2\pi}_{(\star)}(\mathbb{R})$ denotes the space of all 2π-periodic functions of Beurling-type, $\star \in \{\omega, M\}$.
First we point out that [24, 3.1. Theorem] $(d) \Rightarrow (c)$, which we will have to use, holds for each M^l separately: $M^l \in \mathcal{LC}$ and moreover, by the second part of 6.2.1, also derivation closedness (dc) is satisfied for each M^l. So the general assumptions in [24] are satisfied for each M^l.
We consider in [24, 3.1. Theorem] (a), (b) and (d) only the periodic case $\mathcal{E}^{2\pi}_{(M)}(\mathbb{R})$ and to show $(d) \Rightarrow (c)$ we use $(d) \Rightarrow (a) \Rightarrow (b)$, which are clearly satisfied, and finally $(b) \Rightarrow (c)$ there. For the last implication for the 2π-periodic Beurling-case only [24, 1.2. Lemma] is then used.
We prove now condition (d) in [24, 3.1. Theorem]: By the first point of 6.2.1, we obtain $\mathcal{E}_{(M^l)}(U) = \mathcal{E}_{(\omega)}(U)$ for each $l > 0$ and each $U \subseteq \mathbb{R}^n$ non-empty and open, hence by the same sheaf-argument which was also used in [4, 14. Theorem] $(2) \Rightarrow (3)$ we get $\mathcal{E}^{2\pi}_{(M^l)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ for each $l > 0$. Moreover, because ω satisfies additionally condition (ω_2), the space $\mathcal{E}_{(\omega)}(U)$ contains restrictions of entire functions (see 3.2.1), hence we can use [28, 3.8. Corollary] to see that $\mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ is isomorphic to a power series space of infinite type (see also [4, 11. Lemma (b)]). Thus $\mathcal{E}^{2\pi}_{(M^l)}(\mathbb{R})$ is isomorphic (via Fourier-series-expansion) to a power series space of infinite type, hence (d) in [24, 3.1. Theorem]. By $(d) \Rightarrow (c)$ we obtain property (β_3) for M^l for each $l > 0$.

With this technique one can prove in this case property moderate growth (mg) directly: For this use now [4, 13. Proposition] $(2) \Rightarrow (3)$. As mentioned before we have $\mathcal{E}^{2\pi}_{(M^l)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega)}(\mathbb{R})$ for each $l > 0$, which implies by [4, 11. Lemma] a growing condition for ω_{M^l}, more precisely (ω_6).

The third part: We see that the additional assumption (ω_2) implies by the second part of 6.2.1 that ω_l and ω_{M^l} satisfy now condition (ω_2) for each $l > 0$, too (this condition is stable w.r.t. $\sim$). By (3.5.1) resp. (3.5.4), where $x \geq 0$ is replaced by $p \in \mathbb{N}$, we see that $p^p \leq D \cdot M^l_p$ holds for a number $l > 0$ a constant $D > 0$ and all $p \in \mathbb{N}$. So we have shown property (6.2.1) and this condition implies by [4, 12. Lemma $(4) \Rightarrow (5)$] also (ω_2) for ω_{M^l} (see also 4.0.2). Moreover by the second part $M^l \approx M^m$ for all $l, m > 0$, thus we obtain property (6.2.1) for each M^l since this condition is clearly stable w.r.t. $\approx$.
Precisely the same arguments can be applied if we have the stronger assumption (ω_5) on ω, we have to use (3.5.2) resp. (3.5.3) and get (6.2.2).

The fourth part is a consequence of 5.1.3, so the first comparison theorem 6.2.1 is shown!

We can also show now the following result, which summarizes the representation for both classes for $\omega \in \mathcal{W}$ with (ω_1) and condition (ω_6) is not necessarily satisfied. This is related to 6.1.1 and 6.2.4:

Proposition 6.2.6. *Let $\omega \in \mathcal{W}$ be given with (ω_1). Then for compact sets $K \subseteq \mathbb{R}^n$ resp. non-empty open sets $U \subseteq \mathbb{R}^n$ we have the representations*

$$\mathcal{E}_{(\omega)}(U) = \varprojlim_{l>0} \mathcal{E}_{(M^l)}(U) \qquad \text{and} \qquad \mathcal{E}_{\{\omega\}}(K) = \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(K).$$

Of course one can restrict the inductive limit to all $l \in \mathbb{N}_{>0}$ resp. the projective limit to all $l' \in \mathbb{N}_{>0}$ with $l' := \frac{1}{l}$.

Proof. The representation for the Roumieu case was already shown in 6.1.1 and 6.2.4. For the Beurling case we have shown $\bigcap_{l>0}\mathcal{E}_{(M^l)}(U)\subseteq\mathcal{E}_{(\omega)}(U)$ in 6.1.1 by assuming only $\omega\in\mathcal{W}$.

On the other hand assume $f\in\mathcal{E}_{(\omega)}(U)$, which means that for all $l>0$ we can find a constant $C_l>0$ such that $\sup_{x\in K}|f^{(\alpha)}(x)|\le C_l\cdot M^l_{|\alpha|}$ holds for all $\alpha\in\mathbb{N}^n$ on each compact set $K\subseteq U$. Fix now a compact set K and since we have assumed (ω_1) we get $h^j\cdot M^l_j\le C_h\cdot M^{l_1}_j$ for all large $h\ge 1$. So we summarize: $C_l\cdot M^l_j\le C_l\cdot C_h\cdot\frac{1}{h^j}\cdot M^{l_1}_j$, where $l_1=L^s\cdot l$, $s\in\mathbb{N}$ chosen minimal such that $\exp(s)\ge h$ and $L\ge 1$ is the constant appearing in (ω_1). Hence we get also $f\in\bigcap_{l>0}\mathcal{E}_{(M^l)}(U)$.

□

An immediate consequence of 6.2.6 together with 5.4.1 is now the following:

Corollary 6.2.7. *If $\omega\in\mathcal{W}$ has additionally (ω_1) and (ω_7), then on each compact set $K\subseteq\mathbb{R}^n$ and $U\subseteq\mathbb{R}^n$ non-empty open we have the following representations:*

$$\mathcal{E}_{(\omega)}(U)=\varprojlim_{l>0}\mathcal{E}_{(M^l)}(U)=\varprojlim_{l>0}\mathcal{E}_{\{M^l\}}(U)\quad\textit{and}\quad\mathcal{E}_{\{\omega\}}(K)=\varinjlim_{l>0}\mathcal{E}_{\{M^l\}}(K)=\varinjlim_{l>0}\mathcal{E}_{(M^l)}(K).$$

The next corollary proves the converse direction of the third chapter in [5]:

Corollary 6.2.8. *Let $\omega\in\mathcal{W}$ be given with (ω_1). If there exists a function f in $\mathcal{E}_{\{\omega\}}$ resp. $\mathcal{E}_{(\omega)}$ with compact support, i.e. the classes are not quasi-analytic, then ω satisfies property (ω_{nq}), too.*

Proof. We use 6.2.6 and consider the Roumieu-case: If we have a function f with compact support K, then we can conclude that there exists a number $l_0>0$ such that $f\in\mathcal{E}_{\{M^{l_0}\}}$ with compact support K. So $\mathcal{E}_{\{M^{l_0}\}}$ is not quasi-analytic and by the classical D.-C.-Theorem for weight sequences we have $\sum_{p\ge1}\frac{1}{\left(M^{l_0}_p\right)^{1/p}}<+\infty$, i.e. condition (nq) for the sequence M^{l_0}. Now use Komatsu's version of the D.-C.-Theorem [16, Lemma 4.1] and 5.1.3, hence all sequences M^l are not quasi-analytic and ω_{M^l} and ω_l satisfy (ω_{nq}) for each $l>0$. In particular ω itself is not quasi-analytic, i.e. condition (ω_{nq}) is satisfied.
For the Beurling-case we proceed analogously (the function f is an element of the intersection of all spaces $\mathcal{E}_{(M^l)}$). □

The following Corollary is a consequence of 6.2.1 and 5.2.2:

Corollary 6.2.9. *Let $\omega\in\mathcal{W}$ with (ω_1). Then the following are equivalent:*

(1) *ω satisfies additionally (ω_6).*

(2) *$\mathcal{E}_{[\omega]}=\mathcal{E}_{[M^l]}$ for each $l>0$.*

(3) *M^l satisfies condition moderate growth* (mg) *for each/for some $l>0$.*

Proof. (1) ⇒ (2) holds by the first part of 6.2.1.
(2) ⇒ (1) By assumption we have $\mathcal{E}_{\{M^l\}}=\mathcal{E}_{\{M^n\}}$ all $l,n>0$ resp. for the Beurling-case, hence we can use 5.2.2 to get property (ω_6), too. Note that for this implication property (ω_1) is not necessary.
(1) ⇔ (3) Follows immediately by 5.2.2 resp. 5.1.3. □

An important consequence of 5.2.2 and 5.2.3 is now the following:

Corollary 6.2.10. *Let $\omega \in \mathcal{W}$ be given with* (ω_1)*. Then in the Beurling-class $\mathcal{E}^{\text{global}}_{(\omega)}(\mathbb{R},\mathbb{R})$ there doesn't exist functions θ as considered in* (chf)*, more precisely there doesn't exist $\theta \in \mathcal{E}^{\text{global}}_{(\omega)}(\mathbb{R},\mathbb{R})$ such that for any $l>0$ we can have $|\theta^{(j)}(0)| \ge M^l_j$ for all $j \in \mathbb{N}$.*

Proof. Assume that there would exist a function $\theta \in \mathcal{E}^{\text{global}}_{(\omega)}(\mathbb{R},\mathbb{R})$ with $|\theta^{(j)}(0)| \ge M^l_j$ for some $l>0$ and all $j \in \mathbb{N}$. Then for each $i>0$ there would exist a constant $C_i>0$ such that $\sup_{x\in K}|\theta^{(j)}(x)| \le C_i \cdot M^i_j$ for all $j \in \mathbb{N}$ and $K \subseteq \mathbb{R}$ compact. Both inequalities together would finally imply

$$\left(\frac{M^l_j}{M^i_j}\right)^{1/j} \le C_i^{1/j}.$$

We distinguish now two cases: First, if ω doesn't satisfy condition (ω_6), we use 5.2.3 to obtain a contradiction: The left hand side above is unbounded for some $i<l$ for $j\to\infty$, one can take for this argument e.g. $i=\frac{l}{4}$. But the right hand side tends to 1 for $j \to \infty$ which gives a contradiction.
On the other hand, if (ω_6) is satisfied for ω in addition, then Theorem 6.2.1 is valid, hence $\mathcal{E}_{(\omega)} = \mathcal{E}_{(M^l)}$ for each $l>0$. But now we are in the "single weight sequence situation": Each M^l is weakly log. convex and the space $\mathcal{E}^{\text{global}}_{(M^l)}(\mathbb{R},\mathbb{R})$ doesn't contain such functions θ (as mentioned in [38, Proposition 3.1.2]). □

We close this section with the following observation:
Let $\omega \in \mathcal{W}$ with (ω_1) and put now $\omega_2(x) := \frac{1}{2}\cdot\omega(x)$ (in the notation of 5.1.3). Then we have, by the same calculation as in 5.1.3, for all $x \ge 0$:

$$\begin{aligned}\varphi^*_{\omega_2}(x) &= \sup\{x\cdot y - \varphi_{\omega_2}(y) : y \ge 0\} = \sup\left\{x\cdot y - \frac{1}{2}\cdot\varphi_\omega(y) : y \ge 0\right\} \\ &= \frac{1}{2}\cdot\sup\{(2x)\cdot y - \varphi_\omega(y) : y\ge 0\} = \frac{1}{2}\cdot\varphi^*_\omega(2x).\end{aligned}$$

Of course one can iterate this procedure, we denote $\omega_{2^n}(x) := \frac{1}{2^n}\cdot\omega(x)$ and so

$$M^{(n)^l}_j := \exp\left(\frac{1}{l}\cdot\varphi^*_{\omega_{2^n}}(l\cdot j)\right) = \exp\left(\frac{1}{2^n\cdot l}\cdot\varphi^*_\omega((2^n\cdot l)\cdot j)\right) = M^{2^n\cdot l}_j$$

for each $n \in \mathbb{N}$ $(\omega_1 = \omega)$. Clearly for each $n \in \mathbb{N}$ we have that $\omega_{2^n} \sim \omega$ and $\omega_{2^n} \in \mathcal{W}$ with property (ω_1). We can use 6.2.6 to obtain on each compact set $K \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^n$ non-empty open the following representations as locally convex vector spaces for each $n \in \mathbb{N}$:

$$\varinjlim_{l>0}\mathcal{E}_{\{M^l\}}(K) = \mathcal{E}_{\{\omega\}}(K) = \mathcal{E}_{\{\omega_{2^n}\}}(K) = \varinjlim_{l>0}\mathcal{E}_{\{M^{(n)^l}\}}(K)$$

resp.

$$\varprojlim_{l>0}\mathcal{E}_{(M^l)}(U) = \mathcal{E}_{(\omega)}(U) = \mathcal{E}_{(\omega_{2^n})}(U) = \varprojlim_{l>0}\mathcal{E}_{(M^{(n)^l})}(U).$$

6.3 Second central result

Lemma 6.3.1. *Let $\omega \in \mathcal{W}$ be given, then we obtain the following properties:*

(1) *Assume now that there exists $l_0>0$ such that M^{l_0} satisfies moderate growth (mg). Then ω and for each $l>0$ the functions ω_l and ω_{M^l} have property (ω_6) and each sequence M^l has property moderate growth (mg).*

(2) *Assume that there exists* $l_1 > 0$ *such that* M^{l_1} *satisfies condition* (β_3)*. Then* ω *and for each* $l > 0$ *the functions* ω_l *and* ω_{M^l} *have property* (ω_1)*.*

(3) *If both* (1) *and* (2) *are satisfied, then the first two points of the comparison theorem 6.2.1 are valid.*

(4) *If both* (1) *and* (2) *are satisfied and assume additionally that there exists* $l_2 > 0$ *such that* M^{l_2} *satisfies* (6.2.1)*. Then* ω *and for each* $l > 0$ *the functions* ω_l *and* ω_{M^l} *have property* (ω_2)*, and so also the third part of 6.2.1 is valid, hence* M^l *has* (β_3) *for each* $l > 0$*, too.*

Proof. (1) This item was already shown in 5.1.3.
(2) By assumption there exists $l_1 > 0$ such that condition (β_3) holds for M^{l_1}. Because M^{l_1} is also by the assumptions on ω normalized, weakly log. convex and $\lim_{j\to\infty}(M_j^{l_1})^{1/j} = +\infty$ we can use [4, 12. Lemma] (2) $\Rightarrow$ (3) $\Rightarrow$ (4), which implies now property (ω_1) for $\omega_{M^{l_1}}$. Hence again by 5.1.3 we see that ω_l and ω_{M^l} have (ω_1) for each $l > 0$, too.
(3) If both (1) and (2) are satisfied, then the weight function ω satisfies all assumptions, which were used in the proofs of the first two parts of 6.2.1.
(4) By the first two parts of 6.2.1 we obtain $M^l \approx M^m$ for all $l, m > 0$, hence each M^l has property (6.2.1), since this condition is stable w.r.t. $\approx$. We use now 4.0.2 to obtain property (ω_2) for each ω_{M^l} and so also for ω and ω_l. So also the third part of 6.2.1 is valid, hence condition (β_3) for each sequence M^l follows. □

The following theorem is the converse statement related to 6.2.1:

Theorem 6.3.2. *Let* M *be a given sequence of positive real numbers such that* $M \in \mathcal{LC}$ *and furthermore conditions* (β_3) *and moderate growth* (mg) *are satisfied.*
Then we get the following properties:

(1) $\omega_M \in \mathcal{W}$ *with* (ω_1) *and* (ω_6)*.*

(2) *We get* $\mathcal{E}_{[\omega_M]}(U) = \mathcal{E}_{[N^l]}(U) = \mathcal{E}_{[M]}(U)$ *for each* $l > 0$ *and* $U \subseteq \mathbb{R}^n$ *non-empty open as locally convex vector spaces, where* $N^l := (N_p^l)_p$ *is defined by* $N_p^l := \exp(\frac{1}{l} \cdot \varphi^*_{\omega_M}(lp))$ *for all* $p \in \mathbb{N}$ *and each* $l > 0$*.*

   Moreover $N^1 = M$*,* $N^l \in \mathcal{LC}$*,* N^l *has moderate growth* (mg) *for each* $l > 0$*, hence closedness under taking derivatives* (dc)*, and* $\omega_{N^l} \sim \omega_M$*. We get* $\omega_{N^l} \in \mathcal{W}$ *with* (ω_1) *and* (ω_6)*, finally* $M \approx N^l$ *holds for each* $l > 0$*.*

(3) *If* M *satisfies additionally condition* (6.2.1)*, i.e.*

$$\liminf_{p\to\infty}(m_p)^{1/p} > 0,$$

   which implies $\lim_{p\to\infty} M_p^{1/p} = +\infty$*, then the functions* ω_M *and* ω_{N^l} *have additionally property* (ω_2)*. Furthermore each sequence* N^l *satisfies also condition* (β_3) *and* (6.2.1)*. If* M *satisfies the stronger condition* (6.2.2)*, i.e.*

$$\lim_{p\to\infty}(m_p)^{1/p} = \infty,$$

   then the functions ω_M *and* ω_{N^l} *have additionally property* (ω_5) *and each* N^l *has* (6.2.2)*.*

(4) *If M is not-quasi-analytic* (nq)*, then each sequence N^l is not-quasi-analytic* (nq) *and ω_{N^l} has property* (ω_{nq}) *for each $l > 0$, too.*

Proof. (1) By assumptions on the weight sequence M it follows that $\omega_M \in \mathcal{W}$ (see 4.0.2). We use [4, 12. Lemma] (2) $\Rightarrow$ (3) $\Rightarrow$ (4) to get (ω_1) by condition (β_3). Since M has additionally moderate growth (mg), the function ω_M satisfies now (ω_6) by [16, Proposition 3.6], too.

(2) By (1) we see that we can take $\omega = \omega_M$ in Theorem 6.2.1, so by the first two parts we obtain: $\mathcal{E}_{[\omega_M]}(U) = \mathcal{E}_{[N^l]}(U)$ for each $l > 0$ and $U \subseteq \mathbb{R}^n$ non-empty open, where we have put $N_p^l := \exp(\frac{1}{l} \cdot \varphi^*_{\omega_M}(lp))$. Each sequence N^l is now normalized, weakly log. convex, $\lim_{p\to\infty}(N_p^l)^{1/p} = +\infty$. Since M is weakly log. convex and normalized, we have for all $p \in \mathbb{N}$

$$M_p = \sup_{t\geq 0} \frac{t^p}{\exp(\omega_M(t))} = \exp(\varphi^*_{\omega_M}(p)) =: N_p^1,$$

where the first equality holds by [16, Proposition 3.2] and the second by the same calculation as in 6.2.2 for ω_M instead of ω. Thus we obtain $\mathcal{E}_{[M]} = \mathcal{E}_{[N^1]} = \mathcal{E}_{[\omega_M]} = \mathcal{E}_{[N^l]}$ for each $l > 0$. This implies finally $M \approx N^l$ for each $l > 0$ because all occurring weight sequences satisfy the necessary conditions (we can use [38, Theorem 3.1.3], a consequence of [40, Theorem 1], resp. [6]). Since for M it was assumed to have moderate growth (mg), each N^l satisfies this condition, too (recall: property moderate growth (mg) is stable under $\approx$).
We have $\omega_{N^l} \in \mathcal{W}$ for each $l > 0$ by the properties of the sequences N^l (see 4.0.2). By 5.1.3 (again for ω_M instead of ω there) we have $\omega_{N^l} \sim \omega_M$ for each $l > 0$, thus condition (ω_1) for each ω_{N^l} is satisfied. As mentioned above N^l has moderate growth (mg) for each $l > 0$, hence (ω_6) holds for each ω_{N^l} again by [16, Proposition 3.6].

(3) Finally for the third part we proceed similarly as in the third part of 6.2.1. First, by assumption (6.2.1) is satisfied for M hence by 4.0.1 we see hat ω_M has now additionally (ω_2). By the second part of this theorem we see that $\omega_{N^l} \sim \omega_M$, thus ω_{N^l} satisfies now condition (ω_2) for each $l > 0$, too.
We use [4, 12. Lemma] (1) $\Rightarrow$ (2) for the sequence N^l for each $l > 0$. First we point out that [24, 3.1. Theorem] $(d) \Rightarrow (c)$ holds for each N^l: Recall that each N^l is normalized, weakly log. convex, $(N_j^l)^{1/j} \to +\infty$ for $j \to \infty$ and finally, by the second part also derivation closedness (dc) is satisfied!
By the second point we obtain $\mathcal{E}_{(N^l)}(U) = \mathcal{E}_{(\omega_M)}(U)$ for each $l > 0$ and each $U \subseteq \mathbb{R}^n$ non-empty and open, hence again by the sheaf-argument which was also used in [4, 14. Theorem] (2) $\Rightarrow$ (3) we get $\mathcal{E}^{2\pi}_{(N^l)}(\mathbb{R}) = \mathcal{E}^{2\pi}_{(\omega_M)}(\mathbb{R})$ for each $l > 0$.
Because ω_M satisfies condition (ω_2), the space $\mathcal{E}_{(\omega_M)}(U)$ contains the entire functions (see 3.2.1), hence we can use [28, 3.8. Corollary] to see, that $\mathcal{E}^{2\pi}_{(\omega_M)}(\mathbb{R})$, and so $\mathcal{E}^{2\pi}_{(N^l)}(\mathbb{R})$, is isomorphic (via Fourier-series-expansion) to a power series space of infinite type. So by [24, 3.1. Theorem] $(d) \Rightarrow (c)$ we obtain property (β_3) for N^l for each $l > 0$. Property (6.2.1) for each N^l follows now by the same argument as for the third part in 6.2.1 for the weight ω_M instead of ω. If (ω_5) holds then proceed analogously.

(4) Since M is not-quasi-analytic (nq), we get by [16, Lemma 4.1] that ω_M has (ω_{nq}). In (2) we have shown $\omega_{N^l} \sim \omega_M$ for each $l > 0$ and so we get (ω_{nq}) for ω_{N^l}, too. That each N^l is not-quasi-analytic can be seen either by $M \approx N^l$ which is shown in (2) or again by [16, Lemma 4.1]. □

6.4 Final statements concerning the comparison results

To each sequence M we can associate the function ω_M. If $M \in \mathcal{LC}$, then $\omega_M \in \mathcal{W}$ (see 4.0.2) and so one can consider $M \mapsto \omega_M \mapsto N^l$ for $l > 0$, where we have put $N^l_p := \exp(\frac{1}{l} \cdot \varphi^*_{\omega_M}(lp))$. One has, by using [16, 3.2. Proposition] as in 6.3.2, the following formula:

$$M_p = \sup_{t \geq 0} \frac{t^p}{\exp(\omega_M(t))} = \exp(\varphi^*_{\omega_M}(p)) =: N^1_p.$$

But by 5.2.2 we see: $N^1 = M \approx N^l$ for each $l > 0$ holds if and only if ω_M has (ω_6) and this is equivalent to moderate growth (mg) for M (use [16, 3.6. Proposition]). So condition (mg) for M is really necessary for the "stability" in the sense that all M^l are equivalent w.r.t. $\approx$.
Moreover $M \approx N$ doesn't imply necessarily that $\omega_M \sim \omega_N$ holds and also the converse direction is in general not clear (see 4.0.5 for more details).

On the other hand to each weight function $\omega \in \mathcal{W}$ we can associate a family of weight sequences M^l via $M^l_p := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lp))$. For $\omega \in \mathcal{W}$ we have $M^l \in \mathcal{LC}$ for each $l > 0$ by using 5.1.1. If $l > 0$ is arbitrary but fixed one can consider now $\omega \mapsto M^l \mapsto \omega_{M^l}$, and by 5.1.3 we obtain $\omega \sim \omega_{M^l}$.
But by 5.2.2 we see, that $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$ if and only if ω has property (ω_6), too.

In the rest of this section each equality between classes of ultradifferentiable functions should be understood on the level of locally convex vector spaces. We recall also the definition of the set of sequences

$$\mathcal{LC} := \{N \in \mathbb{R}^{\mathbb{N}}_{>0} : N \text{ is normalized, weakly log. convex, } \lim_{k \to \infty} (N_k)^{1/k} = +\infty\}$$

and prove the following important characterizing result:

Theorem 6.4.1. *Let $\omega \in \mathcal{W}$ be given with (ω_1) and denote by M^l the sequences $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$, $l > 0$ and $j \in \mathbb{N}$. TFAE:*

- *(i) There exists $N \in \mathcal{LC}$ with $\mathcal{E}_{[\omega]} = \mathcal{E}_{[N]}$.*
- *(ii) ω has also property (ω_6).*
- *(iii) There exists $N \in \mathcal{LC}$ such that for each $l > 0$ we have $\mathcal{E}_{[M^l]} = \mathcal{E}_{[N]}$ or equivalently $N \approx M^l$.*

Moreover in this situation we have now

- *(a) $\mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ for each $l > 0$, N and each M^l satisfies moderate growth* (mg)*, too.*

  So one can replace (up to equivalence w.r.t. $\approx$) in (i) the sequence N by some M^l.

- *(b) If ω has in addition (ω_2), then $\liminf_{p \to \infty} (n_p)^{1/p} > 0$ and $\liminf_{p \to \infty} (m^l_p)^{1/p} > 0$, for $n_p := \frac{N_p}{p!}$ and $m^l_p := \frac{M^l_p}{p!}$. Finally N and each M^l have in addition property (β_3) and we obtain $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ for each $l > 0$, where ω_N denotes the associated function of N.*

Proof.
$(i) \Rightarrow (ii)$ Let $N \in \mathcal{LC}$ with $\mathcal{E}_{[\omega]}(\mathbb{R}) = \mathcal{E}_{[N]}(\mathbb{R})$ and distinguish between the Roumieu- and the Beurling-case.
The Roumieu-case (see e.g. [38, Theorem 3.1.3]): By using characteristic functions $\theta_l \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}^{\text{global}}_{\{\omega\}}(\mathbb{R},\mathbb{R})$ (see (chf)) we get $M^l \preceq N$ for each $l>0$. On the other side we use a characteristic function $\theta_N \in \mathcal{E}^{\text{global}}_{\{N\}}(\mathbb{R},\mathbb{R})$ to obtain $N \preceq M^{l_0}$ for some $l_0>0$ and together now $N \approx M^{l_0}$ is satisfied.
For the Beurling-case we use the arguments from the second chapter in [6] and the analogously proof of 6.5.2, see also remark 6.5.3. Note that both spaces $\mathcal{E}_{(N)}(\mathbb{R},\mathbb{R})$ and $\mathcal{E}_{(\omega)}(\mathbb{R},\mathbb{R})$ are Fréchet-spaces and $N, M^l \in \mathcal{LC}$ for each $l>0$, so the assumptions in [6] are satisfied. Hence we have shown also $N \approx M^{l_1}$ for some $l_1>0$.
But more precisely we have now $M^l \preceq M^{l_0} \approx N$ resp. $M^l \preceq M^{l_1} \approx N$, so $\mathcal{E}_{\{M^l\}} \subseteq \mathcal{E}_{\{M^{l_0}\}}$ resp. $\mathcal{E}_{(M^l)} \subseteq \mathcal{E}_{(M^{l_1})}$, for each $l>0$.
We show now: This implies already condition (ω_6). Assume that (ω_6) is not satisfied, then by 5.2.3 we would get for the choice $l=4l_0$ resp. $l=4l_1$ that $\mathcal{E}_{\{M^{l_0}\}} \subsetneq \mathcal{E}_{\{M^l\}}$ resp. $\mathcal{E}_{(M^{l_1})} \subsetneq \mathcal{E}_{(M^l)}$ holds. But this is a contradiction!

So ω has to satisfy also property (ω_6) and the comparison theorem 6.2.1 is valid. By the first part of this theorem $\mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ holds for each $l>0$, by the second part $M^l \approx M^n$ for all $l,n>0$. Since now $\mathcal{E}_{[M^l]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[N]}$ for each $l>0$ we obtain also $M^l \approx N$ (as already shown above).
Finally N has to satisfy moderate growth (mg), because by the second part of 6.2.1 the sequences M^l have moderate growth (mg) and this property is clearly stable w.r.t. relation $\approx$.

$(ii) \Rightarrow (i)$ If ω has in addition (ω_6), then by the first part of 6.2.1 we get $\mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ for each $l>0$. So we can put $N = M^l$ for some $l>0$ and more precisely $\mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ for $N \in \mathcal{LC}$ is equivalent to $N \approx M^l$ for each $l>0$: This follows by [38, Theorem 3.1.3] resp. in the Beurling-case by the second chapter in [6], because $M^l \in \mathcal{LC}$ for each $l>0$ by 5.1.1.

$(ii) \Rightarrow (iii)$: Each M^l satisfies moderate growth (mg) by 5.2.2. By (i) and the first part of 6.2.1 we have $\mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$, hence $M^l \approx N$ for each $l>0$ and this finally implies (mg) for N since this condition is clearly stable w.r.t. $\approx$.

$(iii) \Rightarrow (ii)$: By assumption $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$ holds, hence ω has property (ω_6) by 5.2.2.

(a) Holds by the above arguments.
(b) If ω has now in addition property (ω_2), then we can use also the third part of 6.2.1 and so we get condition (β_3) and $\liminf_{p\to\infty}(m^l_p)^{1/p} > 0$, with $m^l_p := \frac{M^l_p}{p!}$, for each M^l. Since we have already shown $N \approx M^l$, we also get $\liminf_{p\to\infty}(n_p)^{1/p} > 0$ for $n_p := \frac{N_p}{p!}$. By item (i) above we can use the proof of the third part of 6.2.1 also to derive property (β_3) for N.
For the last property we show $\omega \sim \omega_N$: On the one hand ω has (ω_1) by assumption, hence by 5.1.3 we get (ω_1) for ω_{M^l} for each $l>0$. On the other hand condition (β_3) implies (ω_1) for ω_N (see [4, 12. Lemma] $(2) \Rightarrow (3) \Rightarrow (4)$). Moreover we have already shown $N \approx M^l$ and so we can use the first part of 4.0.5 to obtain $\omega_N \sim \omega_{M^l}$ for each

$l > 0$. This implies finally $\omega_N \sim \omega_{M^l} \sim \omega$ where the second equivalence holds by 5.1.3. □

As an immediate consequence of the previous theorem we obtain:

Corollary 6.4.2. *Let $\omega \in \mathcal{W}$ be given with (ω_1), then the following conditions are equivalent:*

(i) There exists a sequence $N \in \mathcal{LC}$ with $\mathcal{E}_{\{\omega\}} = \mathcal{E}_{\{N\}}$ and $\mathcal{E}_{(\omega)} = \mathcal{E}_{(N)}$.

(ii) There exists a sequence $N \in \mathcal{LC}$ with $\mathcal{E}_{\{\omega\}} = \mathcal{E}_{\{N\}}$.

(iii) There exists a sequence $N \in \mathcal{LC}$ with $\mathcal{E}_{(\omega)} = \mathcal{E}_{(N)}$.

Proof. $(i) \Rightarrow (ii), (iii)$ are clear, $(ii), (iii) \Rightarrow (i)$ holds by the proof of 6.4.1. □

In the following theorem we don't start with a weight function ω as in 6.4.1, but with a weight sequence N:

Theorem 6.4.3. *Let $N \in \mathcal{LC}$ with property (β_3) be given, then the following conditions are equivalent:*

(i) There exists a weight function $\omega \in \mathcal{W}$ with (ω_1) such that $\mathcal{E}_{[\omega]} = \mathcal{E}_{[N]}$ holds.

(ii) N satisfies moderate growth (mg), too.

(iii) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]}$ holds for the associated function ω_N of N.

Moreover we get in this situation:

(a) $\omega, \omega_N \in \mathcal{W}$ both satisfy also properties (ω_1) and (ω_6).

(b) $\omega \sim \omega_N$.

(c) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ for each $l > 0$, where $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$. Finally $N \approx M^l$ for all $l > 0$ and each M^l has moderate growth (mg), too.

(d) If we assume that N satisfies in addition $\liminf_{p\to\infty}(n_p)^{1/p} > 0$, with $n_p := \frac{N_p}{p!}$, then for ω, ω_N we get also property (ω_2) and each M^l has (β_3), too.

Proof. $(i) \Rightarrow (ii)$ Holds by $(i) \Rightarrow (ii)$ in 6.4.1 (also with weaker assumptions on N).
$(ii) \Rightarrow (iii)$ The sequence N satisfies by assumption all properties of 6.3.2. Hence by the first two parts there we have $\omega_N \in \mathcal{W}$ and properties (ω_1), (ω_6) for ω_N. Finally $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[N]}$ holds.
$(iii) \Rightarrow (i)$ Holds with the choice $\omega = \omega_N$.

(a) By item (i) and $(i) \Rightarrow (ii)$ in 6.4.1 the weight ω has also property (ω_6).
(b) For $l > 0$ and $j \in \mathbb{N}$ we put $M^l_j = \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$. On the one hand ω has (ω_1) by assumption, hence by 5.1.3 we get (ω_1) for ω_{M^l} for each $l > 0$. On the other hand condition (β_3) implies (ω_1) for ω_N (see [4, 12. Lemma] (2) ⇒ (3) ⇒ (4)). Moreover, since ω satisfies all necessary assumptions, by the first two parts of 6.2.1 we have for each $l > 0$ that $\mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$, $N \approx M^l$ and M^l has moderate growth (mg), too.
Hence by the first part of 4.0.5 we obtain $\omega_N \sim \omega_{M^l}$ for each $l > 0$, and so $\omega_N \sim \omega_{M^l} \sim \omega$ where the second equivalence holds by 5.1.3.
(c) Is now obvious.

(d) If also $\liminf_{p\to\infty}(n_p)^{1/p} > 0$ holds, then by the third part of 6.3.2 we get (ω_2) for ω_N, too. Thus by $\omega \sim \omega_N$ we get property (ω_2) for ω. Finally can use the third part of 6.2.1 to get (β_3) for each sequence M^l. □

Again as above we formulate an immediate consequence:

Corollary 6.4.4. *Let $N \in \mathcal{LC}$ with (β_3), then the following conditions are equivalent:*

(i) *There exists a weight function $\omega \in \mathcal{W}$ with (ω_1) such that $\mathcal{E}_{\{\omega\}} = \mathcal{E}_{\{N\}}$ and $\mathcal{E}_{(\omega)} = \mathcal{E}_{(N)}$ holds.*

(ii) *There exists a weight function $\omega \in \mathcal{W}$ with (ω_1) such that $\mathcal{E}_{\{\omega\}} = \mathcal{E}_{\{N\}}$ holds.*

(iii) *There exists a weight function $\omega \in \mathcal{W}$ with (ω_1) such that $\mathcal{E}_{(\omega)} = \mathcal{E}_{(N)}$ holds.*

Proof. $(i) \Rightarrow (ii), (iii)$ are clear, $(ii), (iii) \Rightarrow (i)$ holds by the proof of 6.4.3. □

We are going to formulate and prove some further consequences:

Proposition 6.4.5. *Let $N \in \mathcal{LC}$ be given and assume that there exists a weight function $\omega \in \mathcal{W}$ with (ω_1) and a number $l > 0$ such that $\mathcal{E}_{[N]} = \mathcal{E}_{[M^l]}$, where we have put of course $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$.*
Then we obtain the following equivalences:

(i) $\mathcal{E}_{[N]} = \mathcal{E}_{[\omega]}$.

(ii) *ω satisfies additionally condition (ω_6).*

(iii) *N satisfies moderate growth* (mg), *too.*

(iv) *$\mathcal{E}_{[N]} = \mathcal{E}_{[M^l]}$ for each $l > 0$.*

(v) *$N \approx M^l$ for each $l > 0$.*

Proof. $(ii) \Rightarrow (iii), (iv), (v)$ We use 5.2.2: (ω_6) is equivalent to $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$ and to moderate growth (mg) for each sequence M^l. Note that by assumption $N \approx M^l$ for some $l > 0$ and moderate growth is stable w.r.t. $\approx$.
$(iv) \Leftrightarrow (v)$ is clear (since each occurring sequence satisfies the necessary conditions).
$(v) \Rightarrow (ii), (iii)$ is clear, because by assumption $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$ and so again by 5.2.2 we get (ω_6) and moderate growth for each M^l, hence also for N itself, too.
$(iii) \Rightarrow (ii)$ If N has moderate growth (mg) and by assumption $\mathcal{E}_{[N]} = \mathcal{E}_{[M^l]}$ holds for some $l > 0$, then $N \approx M^l$. Hence M^l has moderate growth (mg), too, and by [16, 3.6. Proposition] we get (ω_6) for ω_{M^l} and so by 5.1.3 we get (ω_6) for ω.
$(i) \Leftrightarrow (ii)$ This is exactly 6.4.1. □

Proposition 6.4.6. *Let $N \in \mathcal{LC}$ with condition (β_3) be given and let $\omega \in \mathcal{W}$ with additionally property (ω_1). If we write again $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$ then the following conditions are equivalent:*

(a) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[\omega]}$.

(b) *For each $l > 0$ we get $\omega_N \sim \omega_{M^l}$.*

Moreover in this situation we obtain the following equivalences:

(c) ω has also property (ω_6).

(d) N has also moderate growth (mg).

(e) $N \approx M^l$ for each $l > 0$.

(f) $\mathcal{E}_{[\omega_N]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]} = \mathcal{E}_{[N]}$ for each $l > 0$.

Proof. $(a) \Rightarrow (b)$ By 4.0.2 and the assumptions on N we see, that $\omega_N \in \mathcal{W}$. By [4, 12. Lemma $(ii) \Rightarrow (iv)$] we get also (ω_1) for ω_N. Hence, by 6.5.1 (Roumieu-case) resp. 6.5.2 (Beurling-case - for this condition (ω_1) for ω and ω_N is not needed necessarily!) we obtain $\omega_N \sim \omega$. By 5.1.3 we finally get $\omega \sim \omega_{M^l}$ for each $l > 0$.
$(b) \Rightarrow (a)$ Again by 5.1.3 we have $\omega \sim \omega_{M^l}$ for each $l > 0$ and so $\omega_N \sim \omega$.
$(c) \Leftrightarrow (d)$ We have $\omega_N \sim \omega$ and so we get: If (c) holds, then the function ω_N has also (ω_6) (by 3.2.2 this condition is stable w.r.t $\sim$) and so moderate growth (mg) for N holds by [16, 3.6. Proposition]. If (d) is valid, condition (mg) for N implies by [16, 3.6. Proposition] also (ω_6) for ω_N. Hence ω has also (ω_6).
$(f) \Rightarrow (e)$ is clear by the third equality (each occurring sequence satisfies the necessary conditions).
$(f) \Rightarrow (c)$ The third equality implies also $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$, hence by 5.2.2 we get (ω_6) for ω.
$(e) \Rightarrow (c)$ This implies again $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$.
$(c) \Rightarrow (e), (f)$ We can use the first part of 6.2.1, which gives the second equality in (f). By the second part of 6.2.1 each sequence M^l satisfies now also moderate growth (mg). Since we have moderate growth for N (by (d)) and $\omega_N \sim \omega_{M^l}$ for each $l > 0$ (by (b) above) we can use the second part of 4.0.5 to obtain $N \approx M^l$ for each $l > 0$, hence (e). Finally this shows also the third equality in (f). $\square$

6.5 $\mathcal{E}_{[\sigma]} = \mathcal{E}_{[\tau]} \Leftrightarrow \sigma \sim \tau$

In this section we will prove that for weights $\omega \in \mathcal{W}$ with additionally (ω_1) our equivalence relation for weights $\sim$ characterizes equality of the associated function spaces $\mathcal{E}_{[\omega]}$.

(*i*) The results in this section are still valid for globally defined classes by obvious modifications of the proofs (see 2.4.1).

(*ii*) For the proofs in this section we will need $\omega \in \mathcal{W}$. More precisely condition (ω_4) which guarantees $\varphi_\omega^{**} = \varphi_\omega$ is really necessary, see also 3.1.1 and 3.1.2 for more details.

We will distinguish between the Roumieu- and the Beurling-case and first we start now with the Roumieu-case.

Proposition 6.5.1. *Assume that $\tau, \sigma \in \mathcal{W}$ are given and furthermore σ should satisfy (ω_1). Then we obtain:*

(*i*) *If $\sigma \preceq \tau$, then $\mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{\{\tau\}}$.*

(*ii*) *If $\mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\tau\}}(\mathbb{R},\mathbb{R})$, then $\sigma \preceq \tau$.*

Proof. (*i*) As shown in 3.2.1, $\sigma \preceq \tau$ implies $\mathcal{E}_{[\sigma]} \subseteq \mathcal{E}_{[\tau]}$ and for this direction we don't need property (ω_1) for σ.

(*ii*) Now the converse direction: Assume $\mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\tau\}}(\mathbb{R},\mathbb{R})$.
Then we can use property (chf) because M^l is weakly log. convex for each $l > 0$. Take $m > 0$ arbitrary but fixed and consider the function $\theta_m \in \mathcal{E}^{\text{global}}_{\{M^m\}}(\mathbb{R},\mathbb{R})$ for $M_j^m := \exp(\frac{1}{m} \cdot \varphi_\sigma^*(mj))$. Then $M_j^m \le |\theta_m^{(j)}(0)|$ for all $j \in \mathbb{N}$, but since we have also (ω_1) for σ we can use 6.2.4 to get $\theta_m \in \mathcal{E}^{\text{global}}_{\{M^m\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\tau\}}(\mathbb{R},\mathbb{R})$.
So there exist $C, l > 0$ such that for all $j \in \mathbb{N}$ we get $|\theta_m^{(j)}(0)| \le C \cdot \exp\left(\frac{1}{l} \cdot \varphi_\tau^*(lj)\right)$ and we summarize:

$$\exists\, m > 0\ \exists\, C, l > 0\ \forall\, j \in \mathbb{N}: \quad \frac{1}{m} \cdot \varphi_\sigma^*(mj) \le \log(C) + \frac{1}{l} \cdot \varphi_\tau^*(lj). \tag{6.5.1}$$

If one replaces in (6.5.1) "for all $j \in \mathbb{N}$" by "for all $y \ge 0$", then we can apply in (6.5.1) the *Legendre-Fenchel-Young-conjugate* and calculate as follows:

$$\left(\frac{1}{m} \cdot \varphi_\sigma^*(m\cdot)\right)^*(x) = \frac{1}{m} \cdot \sup_{y \ge 0}\{(xm) \cdot y - \varphi_\sigma^*(m \cdot y)\} \underbrace{=}_{y' = y \cdot m} \frac{1}{m} \cdot \sup_{y' \ge 0}\{x \cdot y' - \varphi_\sigma^*(y')\}$$
$$= \frac{1}{m} \cdot \varphi_\sigma^{**}(x) = \frac{1}{m} \cdot \varphi_\sigma(x) = \frac{\sigma(\exp(x))}{m}.$$

Similarly we get $\left(\log(C) + \frac{1}{l} \cdot \varphi_\tau^*(l\cdot)\right)^*(x) = \frac{\tau(\exp(x))}{l} - \log(C)$ and we use again 4.0.3 to get

$$\frac{\sigma(\exp(x))}{m} = \frac{1}{m} \cdot \sup_{y \ge 0}\{(xm) \cdot y - \varphi_\sigma^*(m \cdot y)\} \ge \frac{1}{m} \cdot \sup_{p \in \mathbb{N}}\{(xm) \cdot p - \varphi_\sigma^*(m \cdot p)\}$$
$$\underbrace{\ge}_{(6.5.1)} \sup_{p \in \mathbb{N}}\left\{x \cdot p - \frac{1}{l} \cdot \varphi_\tau^*(lp)\right\} - \log(C) \underbrace{\ge}_{4.0.3} \frac{1}{2} \cdot \sup_{y \ge 0}\left\{x \cdot y - \frac{1}{l} \cdot \varphi_\tau^*(ly)\right\} - \log(C)$$
$$= \frac{\tau(\exp(x))}{2l} - \log(C),$$

where this calculation holds for all $x \geq \log(\nu_2^l)$, with $\nu_2^l := \frac{N_2^l}{N_1^l}$ and we have set $N_p^l = \exp(\frac{1}{l} \cdot \varphi_\tau^*(lp))$. Recall 4.0.3 and also 5.1.3: We have $\tau_l(t) \leq 2 \cdot \omega_{N^l}(t)$ for all $t \geq \nu_2^l$ and $N_p^l = \exp(\frac{1}{l} \cdot \varphi_\tau^*(lp)) = \exp(\varphi_{\tau_l}^*(p))$, where we have put $\tau_l(t) = \frac{\tau(t)}{l}$ for each $l > 0$. If we put $y = \exp(x)$, then we obtain:

$$\exists\, m > 0\, \exists\, C, l > 0\, \forall\, y \geq \nu_2^l: \ \frac{1}{m} \cdot \sigma(y) \geq \frac{1}{2l} \cdot \tau(y) - \log(C) \Leftrightarrow \tau(y) \leq \frac{2l}{m} \cdot \sigma(y) + 2l \cdot \log(C),$$

hence $\tau(t) = O(\sigma(t))$ for $t \to \infty$. □

Now we prove the same result for the Beurling-case:

Proposition 6.5.2. *Let $\tau, \sigma \in \mathcal{W}$ be given, then we get:*

(i) If $\sigma \preceq \tau$, then $\mathcal{E}_{(\sigma)} \subseteq \mathcal{E}_{(\tau)}$.

(ii) If $\mathcal{E}_{(\sigma)}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R}, \mathbb{R})$, then $\sigma \preceq \tau$.

Remark 6.5.3. *To prove (ii) we use the same trick and technique as introduce in the second chapter in [6]. There in the original proof the (complex-valued) functions $f_t(x) := \exp(itx)$ for $x \in \mathbb{R}$ and $t \geq 0$ were considered. Note that $f_t \in \mathcal{E}_{(\omega)}^{\text{global}}(\mathbb{R}, \mathbb{C})$ for any $\omega \in \mathcal{W}$ and $t \geq 0$ because $\sup_{x \in \mathbb{R}} |f_t^{(k)}(x)| = t^k$ and by 5.1.1 we have $\lim_{k \to \infty} (M_k^l)^{1/k} = +\infty$ for each $l > 0$.*
Instead of these functions we also can consider the real-valued functions $\tilde{f}_t(x) := \sin(tx) + \cos(tx)$ for $x \in \mathbb{R}$ and $t \geq 0$ with $\sup_{x \in \mathbb{R}} |\tilde{f}_t^{(k)}(x)| \leq 2t^k$, thus $f_t \in \mathcal{E}_{(\omega)}^{\text{global}}(\mathbb{R}, \mathbb{R})$ for any $\omega \in \mathcal{W}$, and finally $|\tilde{f}_t^{(k)}(0)| = t^k$ for each $k \in \mathbb{N}$.

Proof. *(i)* As mentioned above, $\tau \preceq \sigma$ implies $\mathcal{E}_{(\sigma)} \subseteq \mathcal{E}_{(\tau)}$.
(ii) Now the converse direction: Assume $\mathcal{E}_{(\sigma)}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R}, \mathbb{R})$ and to prove this item we use now remark 6.5.3.

We put $M_j^l := \exp(\frac{1}{l} \cdot \varphi_\sigma^*(lj))$ and $N_j^l := \exp(\frac{1}{l} \cdot \varphi_\tau^*(lj))$ and we have $\mathcal{E}_{(\sigma)}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R}, \mathbb{R})$ with continuous inclusion by the closed graph theorem - note that all occurring spaces are Fréchet in this situation because there are only countable many semi-norms (see (2.4.3)). The continuity of the inclusion mapping implies: For each compact interval $I \subseteq \mathbb{R}$ and all $l_1 > 0$ there exist numbers $C, l_2 > 0$ and another compact interval $J \subseteq \mathbb{R}$ with

$$\sup_{x \in I, k \in \mathbb{N}} \frac{|f^{(k)}(x)|}{N_k^{l_1}} \leq C \cdot \sup_{x \in J, k \in \mathbb{N}} \frac{|f^{(k)}(x)|}{M_k^{l_2}}$$

for all $f \in \mathcal{E}_{(\sigma)}(\mathbb{R}, \mathbb{R})$.
We apply this now to a compact interval I containing the point 0 and the functions $\tilde{f}_t(x) := \sin(tx) + \cos(tx)$ for $x \in \mathbb{R}$ and $t \geq 0$. So the previous inequality yields

$$\sup_{k \in \mathbb{N}} \frac{t^k}{N_k^{l_1}} = \sup_{k \in \mathbb{N}} \frac{|\tilde{f}_t^{(k)}(0)|}{N_k^{l_1}} \leq \sup_{x \in I, k \in \mathbb{N}} \frac{|\tilde{f}_t^{(k)}(x)|}{N_k^{l_1}} \leq C \cdot \sup_{x \in J, k \in \mathbb{N}} \frac{|\tilde{f}_t^{(k)}(x)|}{M_k^{l_2}} \leq 2C \cdot \sup_{k \in \mathbb{N}} \frac{t^k}{M_k^{l_2}}$$

and so

$$\sup_{k \in \mathbb{N}} \frac{t^k}{N_k^{l_1}} \leq 2C \cdot \sup_{k \in \mathbb{N}} \frac{t^k}{M_k^{l_2}} \Longleftrightarrow \exp(\omega_{N^{l_1}}(t)) \leq 2C \cdot \exp(\omega_{M^{l_2}}(t))$$

holds for all $t \geq 1$. After applying log to this inequality we can use 5.1.3, which shows $\omega_{N^l} \sim \tau$ and $\omega_{M^n} \sim \sigma$ for all $l, n > 0$, and implies $\tau(t) = O(\sigma(t))$ for $t \to \infty$. Alternatively the above inequality implies (by using [16, 3.2. Proposition]):

$$\forall\, l_1 > 0\; \exists\, D, l_2 > 0\; \forall\, k \in \mathbb{N}: \quad M_k^{l_2} \leq D \cdot N_k^{l_1}.$$

Then apply log to this inequality and by the definition of the sequences get immediately again (6.5.1) and then use the proof of the above Roumieu-case. □

The previous results imply the following corollary:

Corollary 6.5.4. *Let $\omega \in \mathcal{W}$ be given with (ω_1) and (ω_6) (so the assumptions of 6.2.1* (1) *and* (2) *are satisfied), and assume that $\sigma \in \mathcal{W}$ is another weight function with (ω_1). Then we obtain $\mathcal{E}_{[\sigma]} = \mathcal{E}_{[\omega]} = \mathcal{E}_{[M^l]}$ (for some/each $l > 0$) if and only if $\sigma \sim \tau$ is satisfied.*

In the last result we are going to mix the Roumieu- with the Beurling-case:

Proposition 6.5.5. *Assume that two weights $\tau, \sigma \in \mathcal{W}$ are given, such that σ has in addition (ω_1). Then we get:*

(*i*) *$\sigma \lhd \tau$ implies $\mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{(\tau)}$.*

(*ii*) *If $\mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R},\mathbb{R})$, then $\sigma \lhd \tau$.*

Proof. (*i*) If we have $\sigma \lhd \tau$, i.e. $\tau(t) = o(\sigma(t))$ for $t \to +\infty$, then $\mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{(\tau)}$ holds by 3.5.1.
(*ii*) Now the converse direction: Assume $\mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R},\mathbb{R})$ and proceed analogously as in the proof of the previous proposition 6.5.1.
We use again characteristic functions: Take $m > 0$ arbitrary but fixed and consider (see (chf)) the function $\theta_m \in \mathcal{E}^{\text{global}}_{\{M^m\}}(\mathbb{R},\mathbb{R})$ for $M_j^m := \exp(\frac{1}{m} \cdot \varphi^*_\sigma(mj))$. Then $M_j^m \leq |\theta_m^{(j)}(0)|$ for all $j \in \mathbb{N}$, but since we have also (ω_1) for σ we can use 6.2.4 to get $\theta_m \in \mathcal{E}^{\text{global}}_{\{M^m\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\sigma\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(\tau)}(\mathbb{R},\mathbb{R})$.
So for all $l > 0$ there exists $C_l > 0$ such that for all $j \in \mathbb{N}$ we get $|\theta_m^{(j)}(0)| \leq C_l \cdot \exp(\frac{1}{l} \cdot \varphi^*_\tau(lj))$. More precisely we get now:

$$\exists\, m > 0\; \forall\, l > 0\; \exists\, C_l > 0\; \forall\, j \in \mathbb{N}: \quad \frac{1}{m} \cdot \varphi^*_\sigma(mj) \leq \log(C_l) + \frac{1}{l} \cdot \varphi^*_\tau(lj). \tag{6.5.2}$$

Analogously as before we replace in (6.5.2) "for all $j \in \mathbb{N}$" by "for all $y \geq 0$", then we can apply the *Legendre-Fenchel-Young-conjugate* and calculate as follows: The left hand side gives again

$$\left(\frac{1}{m} \cdot \varphi^*_\sigma(m\cdot)\right)^*(x) = \frac{\sigma(\exp(x))}{m},$$

and the right hand side turns into $\left(\log(C_l) + \frac{1}{l} \cdot \varphi^*_\tau(l\cdot)\right)^*(x) = \frac{\tau(\exp(x))}{l} - \log(C_l)$. Then we use once again 4.0.3 to get

$$\frac{\sigma(\exp(x))}{m} \geq \frac{\tau(\exp(x))}{2l} - \log(C_l), \tag{6.5.3}$$

where this calculation holds for all $x \geq \log(\nu_2^l)$, with $\nu_2^l := \frac{N_2^l}{N_1^l}$ and we have set again $N_p^l = \exp(\frac{1}{l} \cdot \varphi^*_\tau(lp))$. We get $\tau_l(t) \leq 2 \cdot \omega_{N^l}(t)$ for all $t \geq \nu_2^l$ and in the Beurling-case we are interested in l small, so $0 < l \leq 1$.

If we put $y = \exp(x)$, then we obtain:

$$\begin{aligned} &\exists\, m > 0\; \forall\, l > 0\; \exists\, C_l > 0\; \forall\, y \geq \nu_2^l : \\ &\frac{1}{m} \cdot \sigma(y) \geq \frac{1}{2l} \cdot \tau(y) - \log(C_l) \Leftrightarrow \tau(y) \leq \frac{2l}{m} \cdot \sigma(y) + 2l \cdot \log(C_l), \end{aligned}$$

hence $\tau(y) = o(\sigma(y))$ for $y \to +\infty$ follows. □

7 Spaces of ultradifferentiable functions defined by a weight matrix

7.1 Introduction and motivation

Recall: Let $\omega \in \mathcal{W}$, then we can introduce a family of sequences $\{M^l : l \in \mathbb{R}_{>0}\}$ by $M^l_j = \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(lj)\right)$.
We have shown in 6.2.4 that for $\omega \in \mathcal{W}$ with additionally property (ω_1) the representation

$$\mathcal{E}_{\{\omega\}}(K) = \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(K) = \varinjlim_{l \in \mathbb{N}_{>0}} \mathcal{E}_{\{M^l\}}(K)$$

holds for all K compact (arbitrary but fixed). Recall: The inclusion "$\subseteq$" holds by 6.1.1 also without property (ω_1). By the first part of 6.2.6 we get for the Beurling-space

$$\mathcal{E}_{(\omega)}(U) = \varprojlim_{l>0} \mathcal{E}_{(M^l)}(U) = \varprojlim_{l>0} \mathcal{E}_{(M^{1/l})}(U)$$

for all non-empty open sets U and furthermore, again by 6.1.1, we have $\mathcal{E}_{(\omega)}(U) \subseteq \varprojlim_{l>0} \mathcal{E}_{\{M^{1/l}\}}(U)$.
As pointed out there, the converse inclusion cannot be true in general, one would need the following: For arbitrary but fixed $l \in \mathbb{N}_{>0}$ and $h > 0$ the inequality $h^k \cdot M^l_k \leq C_m \cdot M^m_k$ should be satisfied for all $k \in \mathbb{N}$ and (important!) all $m > 0$. But e.g. if the comparison theorem 6.2.1 is valid then one would get by the first part there $\mathcal{E}_{(\omega)} = \mathcal{E}_{\{\omega\}}$, a contradiction. More precisely 6.2.3 shows that for $\omega \in \mathcal{W}$ with (ω_1) we have $\mathcal{E}_{(\omega)}(U) \subseteq \mathcal{E}_{(M^l)}(U)$ for all $l \geq 1$ and each U non-empty open. But for the converse direction one would need property (ω_6), see 6.2.5.

Main technique and a new idea: We are going to generalize this situation/definitions to an abstract family of weight sequences, a so-called *weight matrix* $\mathcal{M} := \{M^l \in \mathbb{R}^{\mathbb{N}}_{>0} : l \in \Lambda\}$, where Λ is a directed partially ordered set (and l plays the role of a parameter). By using this new method we can describe classes defined by a single weight sequence M ("constant matrices") and by a weight function ω simultaneously but also more classes, see 7.3.3 for more details.

7.2 Most general definitions and conditions

In this section we are going to introduce several important conditions of weight matrices, which we will have to use in the chapters below frequently. For these definitions we are considering the most general situation: Let $\mathcal{M}$ be an arbitrary (large - uncountable) set of weight sequences (for an important example see e.g. the definition [23, 1.5.]), then for $n, m \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^n$ non-empty and open, and for all $K \subseteq U$ compact we put

$$\mathcal{E}_{\{\mathcal{M}\}}(K,\mathbb{R}^m) := \bigcup_{M\in\mathcal{M}} \mathcal{E}_{\{M\}}(K,\mathbb{R}^m) \qquad \mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) := \bigcap_{K\subseteq U}\bigcup_{M\in\mathcal{M}} \mathcal{E}_{\{M\}}(K,\mathbb{R}^m) \tag{7.2.1}$$

and

$$\mathcal{E}_{(\mathcal{M})}(K,\mathbb{R}^m) := \bigcap_{M\in\mathcal{M}} \mathcal{E}_{(M)}(K,\mathbb{R}^m) \qquad \mathcal{E}_{(\mathcal{M})}(U,\mathbb{R}^m) := \bigcap_{M\in\mathcal{M}} \mathcal{E}_{(M)}(U,\mathbb{R}^m). \tag{7.2.2}$$

We are going to introduce now some conditions on a weight matrix in detail. First put $m^\circ := (m^\circ_k)_k$ defined by

$$m^\circ_k := \max\left\{m_j\cdot m_{\alpha_1}\cdots m_{\alpha_j} : \alpha_i\in\mathbb{N}_{>0}, \sum_{i=1}^j \alpha_i = k\right\}, \qquad (m_0)^\circ := 1,$$

with $m_k := \frac{M_k}{k!}$. Now consider the following conditions *of Roumieu-type:*

($\mathcal{M}_{\{\text{fil}\}}$) $\forall\, N^1,N^2\in\mathcal{M}\;\exists\, M\in\mathcal{M}:\; N^1,N^2\le M$

($\mathcal{M}_{\{\text{dc}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, C>0\;\exists\, N\in\mathcal{M}\;\forall\, j\in\mathbb{N}: M_{j+1}\le C^{j+1}\cdot N_j$

($\mathcal{M}_{\{\text{mg}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, C>0\;\exists\, N^1,N^2\in\mathcal{M}\;\forall\, j,k\in\mathbb{N}: M_{j+k}\le C^{j+k}\cdot N^1_j\cdot N^2_k$

($\mathcal{M}_{\{\text{alg}\}}$) $\forall\, N^1,N^2\in\mathcal{M}\;\exists\, C>0\;\exists\, M\in\mathcal{M}\;\forall\, j,k\in\mathbb{N}: N^1_j\cdot N^2_k\le C^{j+k}\cdot M_{j+k}$

($\mathcal{M}_{\{\text{L}\}}$) $\forall\, C>0\;\forall\, M\in\mathcal{M}\;\exists\, D>0\;\exists\, N\in\mathcal{M}\;\forall\, k\in\mathbb{N}: C^k\cdot M_k\le D\cdot N_k$

($\mathcal{M}_{\{\text{BR}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, N\in\mathcal{M}: M\lhd N$

($\mathcal{M}_{\{\text{strict}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, N\in\mathcal{M}:\;\sup_{k\in\mathbb{N}_{>0}}\left(\frac{N_k}{M_k}\right)^{1/k}=+\infty$

($\mathcal{M}_{\{\text{diag}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, C>0\;\exists\, N\in\mathcal{M}\;\forall\, j\in\mathbb{N}: M_{2j}\le C^j\cdot N_j$

($\mathcal{M}_{\{\text{FdB}\}}$) $\forall\, M\in\mathcal{M}\;\exists\, N\in\mathcal{M}: m^\circ\preceq n$,
for $m=(m_k)_k$, $n=(n_k)_k$ with $m_k:=\frac{M_k}{k!}$ resp. $n_k:=\frac{N_k}{k!}$.
We will also write $N := M^{\{\text{FdB}\}}$.

Analogously we introduce the *Beurling-type-conditions:*

($\mathcal{M}_{(\text{fil})}$) $\forall\, N^1,N^2\in\mathcal{M}\;\exists\, M\in\mathcal{M}:\; M\le N^1,N^2$

($\mathcal{M}_{(\text{dc})}$) $\forall\, N\in\mathcal{M}\;\exists\, C>0\;\exists\, M\in\mathcal{M}\;\forall\, j\in\mathbb{N}: M_{j+1}\le C^{j+1}\cdot N_j$

($\mathcal{M}_{(\text{mg})}$) $\forall\, N^1,N^2\in\mathcal{M}\;\exists\, C>0\;\exists\, M\in\mathcal{M}\;\forall\, j,k\in\mathbb{N}: M_{j+k}\le C^{j+k}\cdot N^1_j\cdot N^2_k$

($\mathcal{M}_{(\text{alg})}$) $\forall\, M\in\mathcal{M}\;\exists\, C>0\;\exists\, N^1,N^2\in\mathcal{M}\;\forall\, j,k\in\mathbb{N}: N^1_j\cdot N^2_k\le C^{j+k}\cdot M_{j+k}$

($\mathcal{M}_{(\text{L})}$) $\forall\, C>0\;\forall\, N\in\mathcal{M}\;\exists\, D>0\;\exists\, M\in\mathcal{M}\;\forall\, k\in\mathbb{N}: C^k\cdot M_k\le D\cdot N_k$

($\mathcal{M}_{(\text{BR})}$) $\forall\, M\in\mathcal{M}\;\exists\, N\in\mathcal{M}: N\lhd M$

($\mathcal{M}_{(\text{strict})}$) $\forall\, M\in\mathcal{M}\;\exists\, N\in\mathcal{M}\;:\;\sup_{k\in\mathbb{N}_{>0}}\left(\frac{M_k}{N_k}\right)^{1/k}=+\infty$

$(\mathcal{M}_{(\mathrm{diag})})$ $\forall\, N \in \mathcal{M}\; \exists\, C > 0\; \exists\, M \in \mathcal{M}\; \forall\, j \in \mathbb{N} : M_{2j} \le C^j \cdot N_j$

$(\mathcal{M}_{(\mathrm{FdB})})$ $\forall\, N \in \mathcal{M}\; \exists\, M \in \mathcal{M} : m^\circ \preceq n.$

We will also write $M := N^{(\mathrm{FdB})}$.

Conventions: We will write $(\mathcal{M}_{[\star]})$ for either $(\mathcal{M}_{\{\star\}})$ or $(\mathcal{M}_{(\star)})$ but not mixing the cases if statements involve more than one $(\mathcal{M}_{[\star]})$ symbol. We will call a matrix $\mathcal{M}$ *constant*, if $\mathcal{M} = \{M\}$ or more generally if $M \approx N$ for all $M, N \in \mathcal{M}$ is satisfied.

If one has two matrices $\mathcal{M}$ and $\mathcal{N}$, then we introduce the relations $\{\preceq\}$, $(\preceq)$ as follows: First write

$$\mathcal{M}\{\preceq\}\mathcal{N} \;:\Leftrightarrow \forall\, M \in \mathcal{M}\; \exists\, N \in \mathcal{N}\; \exists\, C > 0\; \forall\, j \in \mathbb{N} : M_j \le C^j \cdot N_j.$$

It means that for each sequence $M \in \mathcal{M}$ we can find $N \in \mathcal{N}$ with $M \preceq N$ and so $\mathcal{E}_{[M]} \subseteq \mathcal{E}_{[N]}$. If $\mathcal{M}\{\preceq\}\mathcal{N}$ then of course $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{\{\mathcal{N}\}}$ holds. Analogously define $(\preceq)$ by

$$\mathcal{M}(\preceq)\mathcal{N} \;:\Leftrightarrow \forall\, N \in \mathcal{N}\; \exists\, M \in \mathcal{M}\; \exists\, C > 0\; \forall\, j \in \mathbb{N} : M_j \le C^j \cdot N_j.$$

It means that for each $N \in \mathcal{N}$ we can find $M \in \mathcal{M}$ with $M \preceq N$ and so $\mathcal{E}_{[M]} \subseteq \mathcal{E}_{[N]}$. Hence this condition implies $\mathcal{E}_{(\mathcal{M})} \subseteq \mathcal{E}_{(\mathcal{N})}$.
We write

$$\mathcal{M}\{\approx\}\mathcal{N} \;:\Leftrightarrow \mathcal{M}\{\preceq\}\mathcal{N}, \mathcal{N}\{\preceq\}\mathcal{M}$$

resp.

$$\mathcal{M}(\approx)\mathcal{N} \;:\Leftrightarrow \mathcal{M}(\preceq)\mathcal{N}, \mathcal{N}(\preceq)\mathcal{M}.$$

Finally we introduce the (not reflexive!) relation $\lhd$ as follows:

$$\mathcal{M} \lhd \mathcal{N} \;:\Leftrightarrow \forall\, M \in \mathcal{M}\; \forall\, N \in \mathcal{N} : M \lhd N$$

or equivalently

$$\forall\, M \in \mathcal{M}\; \forall\, N \in \mathcal{N}\; \forall\, h > 0\; \exists\, C > 0\; \forall\, j \in \mathbb{N} : M_j \le C \cdot h^j \cdot N_j.$$

Hence $\mathcal{M} \lhd \mathcal{N}$ implies $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{(\mathcal{N})}$.
One can also consider the following condition:

$$\mathcal{M}(\preceq\}\mathcal{N} \;:\Leftrightarrow \exists\, M \in \mathcal{M}\; \exists\, N \in \mathcal{N} : M \preceq N$$

which is weaker than $(\preceq)$ and $\{\preceq\}$ and implies $\mathcal{E}_{(\mathcal{M})} \subseteq \mathcal{E}_{\{\mathcal{N}\}}$.

In this context we introduce also (write $m_k := \frac{M_k}{k!}$):

$(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ $\exists\, M \in \mathcal{M} :\; \liminf_{k\to\infty}(m_k)^{1/k} > 0 \iff \exists\, M \in \mathcal{M} :$ (6.2.1) is valid

$(\mathcal{M}_{\mathcal{H}})$ $\forall\, M \in \mathcal{M} :\; \liminf_{k\to\infty}(m_k)^{1/k} > 0 \iff \forall\, M \in \mathcal{M} :$ (6.2.1) is valid

$(\mathcal{M}_{(\mathrm{C}^\omega)})$ $\forall\, M \in \mathcal{M} :\; \lim_{k\to\infty}(m_k)^{1/k} = +\infty \iff \forall\, M \in \mathcal{M} :$ (6.2.2) is valid

If $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ holds then the class of real-analytic-functions is contained in $\mathcal{E}_{\{\mathcal{M}\}}$, if $(\mathcal{M}_{(\mathrm{C}^\omega)})$ then the real-analytic functions are contained in $\mathcal{E}_{(\mathcal{M})}$. If $(\mathcal{M}_{\mathcal{H}})$ is satisfied, then the restrictions of entire functions are contained in $\mathcal{E}_{(\mathcal{M})}$. Of course $(\mathcal{M}_{\mathcal{H}})$ implies $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$.
Recall [35, 2.15. Theorem]: If $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ holds we can replace in the definition for the class $\mathcal{E}_{\{\mathcal{M}\}}$ the matrix $\mathcal{M}$ by $\mathcal{M}^{\mathrm{lc}}$, where $M \in \mathcal{M} \Leftrightarrow M^{\mathrm{lc}} \in \mathcal{M}^{\mathrm{lc}}$. If $(\mathcal{M}_{(\mathrm{C}^\omega)})$ holds the same can be done for the Beurling-case $\mathcal{E}_{(\mathcal{M})}$.

7.3 One parameter weight matrices and basic definitions

From now on, except the cases where it is mentioned explicitly, we will restrict ourselves to the following case: Let the weight matrix $\mathcal{M}$ be given by

$$\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\},$$

where Λ is a partially ordered set (e.g. $\Lambda = \mathbb{R}_{>0}$ as the most important example) and we put also $m_p^l := \frac{M_p^l}{p!}$ and $\mu_p^l := \frac{M_p^l}{M_{p-1}^l}$. So in this situation $\mathcal{M}$ can be regarded as a one parameter family of weight sequences M^l. Nevertheless in some cases it is important to consider more general "large matrices" and the conditions in the most general sense of the previous section, e.g. for certain important projective representations for Roumieu-classes - see [23] and also chapter 11 below.

Conventions:

(1) In this special situation we replace in each condition in the previous section M by M^l, N by M^n and N^i by M^{n_i} for $i = 1, 2$. E.g. in $(\mathcal{M}_{\{\mathrm{mg}\}})$ we replace "$\forall\ M \in \mathcal{M}$" by "$\forall\ l \in \Lambda$" and "$\exists\ N^1, N^2 \in \mathcal{M}$" by "$\exists\ n_1, n_2 \in \Lambda$" resp. for all other conditions.

(2) If $\Lambda = \mathbb{R}_{>0}$ resp. $\Lambda = \mathbb{N}_{>0}$ (as the most important examples) then $\mathbb{R}_{>0}$ resp. $\mathbb{N}_{>0}$ is always regarded with its natural order $\leq$.

Definition 7.3.1. *Notation: We call the matrix $\mathcal{M}$ arbitrary, if*

$$(\mathcal{M}) :\Leftrightarrow\ \forall\, l \in \Lambda :\ M^l \text{ is normalized, increasing, } M^{l_1} \leq M^{l_2} \text{ for } l_1 \leq l_2.$$

We call an arbitrary matrix $\mathcal{M}$ standard log. convex, if

$$(\mathcal{M}_{\mathrm{sc}}) :\Leftrightarrow (\mathcal{M}) \text{ and } \forall\, l \in \Lambda :\ M^l \in \mathcal{LC},$$

i.e. for each $l \in \Lambda$ the sequence M^l is (weakly) log. convex and $\lim_{j\to\infty}(M_j^l)^{1/j} = +\infty$.

Let $\omega \in \mathcal{W}$, then the associated matrix $\mathcal{M}$ defined by $M_j^l = \exp\left(\frac{1}{l} \cdot \varphi_\omega^*(lj)\right)$ is by 5.1.1 always a standard log. convex matrix, so $(\mathcal{M}_{\mathrm{sc}})$ holds (in fact we only need (ω_0) and (ω_3) for ω).
For arbitrary weight matrices $\mathcal{M}$, i.e. $(\mathcal{M})$, we introduce the ultradifferentiable function classes defined by a weight matrix of Roumieu- resp. Beurling-type $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ as follows:
Let $n, m \in \mathbb{N}_{>0}$, let $U \subseteq \mathbb{R}^n$ be non-empty and open, for all $K \subseteq U$ compact we put

$$\mathcal{E}_{\{\mathcal{M}\}}(K, \mathbb{R}^m) := \bigcup_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(K, \mathbb{R}^m) \qquad \mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^m) = \bigcap_{K\subseteq U} \bigcup_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(K, \mathbb{R}^m) \quad (7.3.1)$$

and

$$\mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^m) := \bigcap_{l\in\Lambda} \mathcal{E}_{(M^l)}(K, \mathbb{R}^m) \qquad \mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^m) := \bigcap_{l\in\Lambda} \mathcal{E}_{(M^l)}(U, \mathbb{R}^m). \quad (7.3.2)$$

For compact sets K with smooth boundary we introduce the Banach-space

$$\mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) := \{f \in \mathcal{E}(K, \mathbb{R}^m) : \|f\|_{\mathcal{M},K,l,h} < +\infty\}, \quad (7.3.3)$$

where we have set

$$\|f\|_{\mathcal{M},K,l,h} := \sup_{k\in\mathbb{N},x\in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R}^m)}}{h^k \cdot M_k^l} = \|f\|_{M^l,K,h}. \tag{7.3.4}$$

With this definition one has the topological representation for a compact set $K \subseteq \mathbb{R}^n$ (with smooth boundary):

$$\mathcal{E}_{\{\mathcal{M}\}}(K,\mathbb{R}^m) := \lim_{\to l\in\Lambda} \lim_{\to h>0} \mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m) = \lim_{\to l\in\Lambda} \lim_{\to h>0} \mathcal{E}_{M^l,h}(K,\mathbb{R}^m),$$

and so for $U \subseteq \mathbb{R}^n$ non-empty open

$$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) := \varprojlim_{K\subseteq U} \varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{M^l,h}(K,\mathbb{R}^m). \tag{7.3.5}$$

Similarly we get for the Beurling-case

$$\mathcal{E}_{(\mathcal{M})}(U,\mathbb{R}^m) := \varprojlim_{K\subseteq U} \varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{M^l,h}(K,\mathbb{R}^m). \tag{7.3.6}$$

In the above description of the steps $\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m)$ instead of compact sets K with smooth boundary one can also consider open $K \subseteq U$ with $\overline{K}$ compact in U resp. one can work with *Whitney jets* on compact K.
Finally for $U \subseteq \mathbb{R}^n$ non-empty open we can also define such classes by global estimates (no compact set $K \subseteq U$ is involved) as follows:

$$\mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \exists\, l \in \Lambda\, \exists\, h > 0 : \|f\|_{\mathcal{M},U,l,h} < +\infty\}$$

resp.

$$\mathcal{E}^{\text{global}}_{(\mathcal{M})}(U,\mathbb{R}^m) := \{f \in \mathcal{E}(U,\mathbb{R}^m) : \forall\, l \in \Lambda\, \forall\, h > 0\ \|f\|_{\mathcal{M},U,l,h} < +\infty\},$$

where $\|f\|_{\mathcal{M},U,l,h}$ denotes the semi-norm from (7.3.4) and "$\sup_{k\in\mathbb{N},x\in K}$" there is replaced by "$\sup_{k\in\mathbb{N},x\in U}$". Similarly as in 2.4.1 for the single weight sequence- resp. weight function case we point out:

Remark 7.3.2. *We will always deal with classes of ultradiff. functions which are defined locally, but it turns out that many important theorems below are also valid for globally defined classes. More precisely we point out:*

(i) *In many proofs we will deal with certain inequalities concerning only the denominator of the quotient in the defining semi-norm* (7.3.4)*. So in the numerator there we can use instead of* $f_k := \sup_{x\in K,k\in\mathbb{N}}$ *also the sequence* $f_k := \sup_{x\in U,k\in\mathbb{N}}$.

(ii) *In globally defined classes introduced by* $(\mathcal{M}_{\text{sc}})$ *weight matrices we still can find the important functions* $\theta_M \in \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ *resp.* $\tilde{\theta}_M \in \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ *from* (chf) *resp.* (2.4.7)*, see also the remark below.*

(iii) *Note that for globally defined classes the top. vector space representations* (7.3.5) *resp.* (7.3.6) *become more easier since no projective limit over all compact sets* $K \subseteq U$ *is involved.*

For formal calculations one can deal also with the following abstract spaces of sequences of positive real numbers:

$$\mathcal{F}_{\mathcal{M},l,h} := \left\{(f_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \exists\, C>0 : \forall\, k\in\mathbb{N} : |f_k| \le C\cdot h^k \cdot M_k^l\right\}$$

with

$$\mathcal{F}_{\{\mathcal{M}\}} := \bigcup_{l\in\Lambda,h>0} \mathcal{F}_{\mathcal{M},l,h} = \bigcup_{l\in\Lambda,h>0} \mathcal{F}_{M^l,h} \qquad \mathcal{F}_{(\mathcal{M})} := \bigcap_{l\in\Lambda,h>0} \mathcal{F}_{\mathcal{M},l,h} = \bigcap_{l\in\Lambda,h>0} \mathcal{F}_{M^l,h}.$$

Convention: We will write $\mathcal{E}_{[\mathcal{M}]}$ if we consider either $\mathcal{E}_{\{\mathcal{M}\}}$ or $\mathcal{E}_{(\mathcal{M})}$ but not mixing the cases if statements involve more than one $\mathcal{E}_{[\mathcal{M}]}$ symbol resp. the same for $\mathcal{F}_{[\mathcal{M}]}$. We will write $\mathcal{E}_{[\mathcal{M}]}(U)$ instead of $\mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R})$.
Since $M^l \le M^n$ for $l \le n$ by assumption we have the following: If e.g. $\Lambda = \mathbb{R}_{>0}$, then all occurring limits are countable. In the Roumieu-case we can restrict ourselves to $\Lambda = \mathbb{N}_{>0}$; take then a countable exhaustion with compact sets $(K_j)_{j\in\mathbb{N}}$ of U, take $h\in\mathbb{N}$ and $l\in\Lambda$.
In the Beurling-case we take again a countable exhaustion of U, moreover $h=\frac{1}{h_1}$, $h_1\in\mathbb{N}_{>0}$ and $l=\frac{1}{l_1}$, $l_1\in\mathbb{N}_{>0}$. In fact $\mathcal{E}_{(\mathcal{M})}(U,\mathbb{R}^m)$ is then a *Fréchet-space.* We will study the topology of these spaces in detail in 9.1.1 below.

Recall some known facts for $(\mathcal{M}_{\text{sc}})$ weight matrices (the assumption $\lim_{j\to\infty}(M_j^l)^{1/j} = +\infty$ is not needed necessarily):

(i) Each sequence M^l satisfies $M_j^l \cdot M_k^l \le M_{j+k}^l$ for all $j,k\in\mathbb{N}$, i.e. condition (alg) with $C=1$, and the mapping $j\mapsto (M_j^l)^{1/j}$ is increasing (for proofs see e.g. [38, Lemma 2.0.4, Lemma 2.0.6]).

(ii) Each class $\mathcal{E}_{\{M^l\}}$ resp. $\mathcal{E}_{(M^l)}$ is closed under pointwise multiplication (see e.g. [38, Proposition 2.0.8]).

(iii) Recall (chf) resp. (2.4.7): For each $l\in\Lambda$ there exists a *characteristic function* $\theta_l := \theta_{M^l} \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ with $|\theta_l^{(j)}(0)| \ge M_j^l$ resp. $\tilde{\theta}_l := \tilde{\theta}_{M^l} \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{C}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ with $|\tilde{\theta}_l^{(j)}(0)| \ge M_j^l$ for all $j\in\mathbb{N}$ and $\tilde{\theta}_l^{(j)} = (\sqrt{-1})^j\cdot s_j^l$, with $s_j^l := \sum_{k=0}^{\infty} M_k^l \cdot (2\mu_k^l)^{j-k} \ge M_j^l$ for arbitrary $l\in\Lambda$ and $j\in\mathbb{N}$.

In the next step we study the introduced conditions of the last section for $\mathcal{M}=\{M^l : l\in\Lambda\}$ in detail. Recall: We can replace in each condition there M by M^l, N by M^n and N^i by M^{n_i} for $i=1,2$.
First observations: For arbitrary matrices, i.e. $(\mathcal{M})$, we can assume w.l.o.g. $n_1=n_2$ in $(\mathcal{M}_{\{\text{mg}\}})$ resp. $(\mathcal{M}_{(\text{mg})})$ and in $(\mathcal{M}_{\{\text{alg}\}})$ resp. $(\mathcal{M}_{(\text{alg})})$, since we have $M^l\le M^n$ for $l\le n$. Moreover both conditions $(\mathcal{M}_{\{\text{fil}\}})$ and $(\mathcal{M}_{(\text{fil})})$ are satisfied automatically.

If $\mathcal{M}$ is an arbitrary matrix, i.e. $(\mathcal{M})$, together with condition $(\mathcal{M}_{[\text{dc}]})$, then the space $\mathcal{E}_{[\mathcal{M}]}$ is automatically closed under taking derivatives. If $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then by [38, Theorem 3.1.3] (Roumieu-case) and [6] (Beurling-case) also the converse conclusion is valid. Of course both $\mathcal{E}_{\{\mathcal{M}\}}$ and $\mathcal{E}_{(\mathcal{M})}$ are also closed under taking derivatives, if one assumes that each sequence M^l satisfies closedness under taking derivatives (dc) (and which implies both $(\mathcal{M}_{\{\text{dc}\}})$ and $(\mathcal{M}_{(\text{dc})})$).

Clearly $(\mathcal{M}_{[\text{mg}]})$ implies $(\mathcal{M}_{[\text{dc}]})$ for the special case $k=1$, $(\mathcal{M}_{\{\text{mg}\}})$ resp. $(\mathcal{M}_{(\text{mg})})$ are called the *matrix generalized moderate growth conditions* of Roumieu- resp. Beurling-type. They will play the key-role in the proof of *Cartesian closedness* for classes of ultradifferentiable functions defined by a weight matrix and they are very important in general. Condition $(\mathcal{M}_{(\mathrm{mg})})$ in the most general situation in the previous section should be also compared with [23, 1.6. Theorem (3)].

If $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix, then both $(\mathcal{M}_{\{\mathrm{alg}\}})$ and $(\mathcal{M}_{(\mathrm{alg})})$ are clearly satisfied for the choice $n_1 = n_2 = l$. Condition $(\mathcal{M}_{[\mathrm{alg}]})$ implies the fact that $\mathcal{E}_{[\mathcal{M}]}$ is closed under pointwise multiplication (e.g. see [38, Proposition 2.0.8]).

For a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix condition $(\mathcal{M}_{\{\mathrm{strict}\}})$ resp. $(\mathcal{M}_{(\mathrm{strict})})$ means now that for all $n_1 \in \Lambda$ we can find a number $n_2 \in \Lambda$ (resp. for all $n_2 \in \Lambda$ there exists $n_1 \in \Lambda$) such that $\mathcal{E}_{[M^{n_1}]} \subsetneq \mathcal{E}_{[M^{n_2}]}$ holds. For this use [38, Theorem 3.1.3] and [40, Theorem 1]) resp. for the Beurling case use the second chapter in [6].
Of course we can iterate $(\mathcal{M}_{\{\mathrm{strict}\}})$ resp. $(\mathcal{M}_{(\mathrm{strict})})$ and obtain an increasing (resp. decreasing) "chain" of indices $n_1, n_2, n_3, \dots$. These conditions have the consequence that the spaces $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ cannot be described by a single weight sequence, see also 7.3.3 below. If they are satisfied we will say that the matrix $\mathcal{M}$ is *not constant*. Otherwise, if $\mathcal{M} = \{M\}$ or more general if $\mathcal{M} = \{M^l : l \in \Lambda\}$ and $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 \in \Lambda$, we will call the matrix *constant*.

The meaning and importance of $(\mathcal{M}_{\{\mathrm{BR}\}})$ is, that the Roumieu-tpye-matrix-space $\mathcal{E}_{\{\mathcal{M}\}}$ can also be written as the intersection of the Beurling spaces $\mathcal{E}_{(M^l)}$. If $(\mathcal{M}_{(\mathrm{BR})})$ is satisfied, then the Beurling-type-matrix-space $\mathcal{E}_{(\mathcal{M})}$ can also be written as the intersection of the Roumieu spaces $\mathcal{E}_{\{M^l\}}$. It's clear, that constant weight matrices never can satisfy $(\mathcal{M}_{\{\mathrm{BR}\}})$ resp. $(\mathcal{M}_{(\mathrm{BR})})$.
For more details and general consequences and effects of these conditions on the topological vector space description see 9.1.1 below.
For arbitrary weight matrices $(\mathcal{M}_{[\mathrm{BR}]})$ implies $(\mathcal{M}_{[\mathrm{strict}]})$, for $(\mathcal{M}_{\mathrm{sc}})$ weight matrices we also have that $(\mathcal{M}_{[\mathrm{diag}]})$ implies $(\mathcal{M}_{[\mathrm{BR}]})$: One gets $M^l_j \cdot M^l_j \le M^l_{2j} \le C^j \cdot M^n_j$ for a constant $C \ge 1$, then $\left(\frac{M^l_j}{M^n_j}\right)^{1/j} \le C \cdot \left(\frac{1}{M^l_j}\right)^{1/j} \to 0$, hence $M^l \lhd M^n$.

Conditions $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and $(\mathcal{M}_{(\mathrm{FdB})})$ and their consequences will be studied in detail below, see 8.3.1, 8.3.5, 8.6.1. They characterize very important stability properties for the associated function spaces.

We summarize now the consequences if the weight matrix $\mathcal{M} = \{M^l : l > 0\}$ is coming from a weight $\omega \in \mathcal{W}$ defined by $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$. Of course the introduced conditions in the previous section were motivated by the results of the fifth chapter, were we have transformed properties of ω into properties of $\mathcal{M}$. First, as pointed out before, $\mathcal{M}$ is automatically a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix and moreover we get:

(1) Both conditions $(\mathcal{M}_{\{\mathrm{fil}\}})$ and $(\mathcal{M}_{(\mathrm{fil})})$ are satisfied automatically by 5.1.1. Conditions $(\mathcal{M}_{\{\mathrm{mg}\}})$ resp. $(\mathcal{M}_{(\mathrm{mg})})$ and $(\mathcal{M}_{\{\mathrm{alg}\}})$ resp. $(\mathcal{M}_{(\mathrm{alg})})$ are satisfied by 5.1.2: In $(\mathcal{M}_{\{\mathrm{mg}\}})$ resp. $(\mathcal{M}_{(\mathrm{mg})})$ we can take $n_1 = n_2 = 2l$, $C = 1$, and in $(\mathcal{M}_{\{\mathrm{alg}\}})$ resp. $(\mathcal{M}_{(\mathrm{alg})})$ choose $l = \max\{n_1, n_2\}$ and again $C = 1$.

(2) If one assumes in addition that (ω_1) is valid, then $(\mathcal{M}_{\{\mathrm{L}\}})$ and $(\mathcal{M}_{(\mathrm{L})})$ hold: We use (3.3.2), then we get in both conditions $n = L^s \cdot l$, where $L \ge 1$ comes from property (ω_1) and $s \in \mathbb{N}$ is chosen minimal such that $\exp(s) \ge C$ (see also the proof of (5.1.3)).

(3) Both conditions $(\mathcal{M}_{\{\text{strict}\}})$ resp. $(\mathcal{M}_{(\text{strict})})$ hold if and only if (ω_6) is not satisfied, see 5.2.2. We can take $n_1 = 4n_2$ by 5.2.3 and moreover in this case the assumptions of the comparison theorem 6.2.1 are violated.

(4) If additionally (ω_7) holds, then both conditions $(\mathcal{M}_{\{\text{diag}\}})$ and $(\mathcal{M}_{(\text{diag})})$ are satisfied and they imply $(\mathcal{M}_{\{\text{BR}\}})$ resp. $(\mathcal{M}_{(\text{BR})})$ - see 5.4.1 for more details.

(5) Let $\tau, \sigma \in \mathcal{W}$ with (ω_1) and denote the associated matrices by $\mathcal{M}, \mathcal{N}$. Now recall 5.3.1 and the consequences there: If $\tau \preceq \sigma$, then both $\mathcal{M}\{\preceq\}\mathcal{N}$ and $\mathcal{M}\{\preceq\}\mathcal{N}$ hold (and so the weaker condition $(\preceq\}$, too). Moreover, if $\tau \lhd \sigma$, then $\mathcal{M} \lhd \mathcal{N}$ is valid.

(6) Finally we recall 5.3.2: By using *Stirlings formula* we see that the first part there has the meaning that if the matrix is obtained by a weight function $\omega \in \mathcal{W}$ with (ω_1) and (ω_2), then $(\mathcal{M}_{\mathcal{H}})$ is satisfied. The second part means, that for $\omega \in \mathcal{W}$ with (ω_1) and (ω_5) condition $(\mathcal{M}_{(\mathrm{C}^\omega)})$ holds.

The next statements are partially based on joint work with *Armin Rainer,* see [35, 4.6. Proposition] for complete proofs: If both $\mathcal{M}$ and $\mathcal{N}$ are $(\mathcal{M}_{\text{sc}})$ weight matrices with $\Lambda = \mathbb{R}_{>0}$ then our relations $\{\preceq\}$, $(\preceq)$, $\lhd$ and $(\preceq\}$ are characterized in terms of inclusions of the associated function spaces:

(*i*) $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R},\mathbb{R})$ implies $\mathcal{M}\{\preceq\}\mathcal{N}$, for this we have to use again [38, Theorem 3.1.3] and [40, Theorem 1], see also the proof of 6.5.1.

(*ii*) $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(\mathcal{N})}(\mathbb{R},\mathbb{R})$ implies $\mathcal{M}(\preceq)\mathcal{N}$, for this we can use the proof of [6] (both spaces are Fréchet by assumption on Λ), see also the proof of 6.5.2.

(*iii*) $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{(\mathcal{N})}(\mathbb{R},\mathbb{R})$ implies also $\mathcal{M} \lhd \mathcal{N}$, again by [38, Theorem 3.1.3] and [40, Theorem 1], see also the above proof of 6.5.5.

(*iv*) Finally $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R},\mathbb{R})$ implies also $\mathcal{M}(\preceq\}\mathcal{N}$, see [35, 4.6. Proposition (3)] for more details. Note that by assumption $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$ is a Fréchet-space and $\mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R},\mathbb{R})$ is a countable projective limit of *Silva-spaces*, see (2)(*a*) in 9.1.1 below for more details.

We close this section with the following statement which shows that by using the very general definitions of weight matrices one is able to describe classes of ultradiff. functions which can be described neither by a single weight sequence M nor by a weight function ω. The result is based on joint work with *Armin Rainer*, see also [35, 5.19., 5.22. Theorem, 5.25. Remark]. Recall: For shortness we have put $\mathcal{E}_{[\star]}(\mathbb{R}) := \mathcal{E}_{[\star]}(\mathbb{R},\mathbb{R})$ for $\star \in \{\mathcal{M}, \omega, M\}$.

Proposition 7.3.3. *There exist weight matrix spaces $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R})$ resp. $\mathcal{E}_{(\mathcal{M})}(\mathbb{R})$, which don't coincide with $\mathcal{E}_{\{M\}}(\mathbb{R})$, $\mathcal{E}_{(M)}(\mathbb{R})$, $\mathcal{E}_{\{\omega\}}(\mathbb{R})$, $\mathcal{E}_{(\omega)}(\mathbb{R})$ for any single weight sequence N and any single weight function $\omega \in \mathcal{W}$ with (ω_1).*

Proof. Consider a $(\mathcal{M}_{\text{sc}})$ weight matrix $\mathcal{M} = \{M^l : l > 0\}$ with $(\mathcal{M}_{(\mathrm{C}^\omega)})$ and with the following properties:

(1) each sequence M^l has moderate growth (mg)

(2) M^l are pairwise not equivalent (w.r.t. $\approx$)

(3) condition (β_3) is satisfied for each sequence M^l.

An explicit example for such a weight matrix is the so-called *Gevrey-matrix*

$$\mathcal{G} := \{G^s = (p!^s)_p, s > 1\}$$

and the associated function spaces are denoted by $\mathcal{E}_{\{\mathcal{G}\}}$ resp. $\mathcal{E}_{(\mathcal{G})}$, see also [35, 5.22. Theorem].

We prove analogously as in 6.4.1: First we consider a weight sequence N such that $\mathcal{E}_{[\mathcal{M}]}(\mathbb{R}) = \mathcal{E}_{[N]}(\mathbb{R})$ holds and N is at least weakly log. convex and $\lim_{k\to\infty}(N_k)^{1/k} = +\infty$. The Roumieu-case (see e.g. [38, Theorem 3.1.3]): Using characteristic functions $\theta_l \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, see (chf), we get $M^l \preceq N$ for each $l \in \Lambda$. On the other side we use a characteristic function $\theta_N \in \mathcal{E}^{\text{global}}_{\{N\}}(\mathbb{R},\mathbb{R})$ to obtain $N \preceq M^{l_0}$ for some $l_0 \in \Lambda$ and together now $N \approx M^{l_0}$ is satisfied. For the Beurling-case we use the second chapter in [6] and the matrix-generalization of 6.5.2, see also [35, 4.6. Proposition]. (Note that $\Lambda = \mathbb{R}_{>0}$, hence $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$ is a Fréchet-space). This proves $N \approx M^{l_1}$ for some $l_1 \in \Lambda$. But then we get for the Beurling-case:

$$\mathcal{E}_{(N)}(\mathbb{R}) = \mathcal{E}_{(\mathcal{M})}(\mathbb{R}) = \bigcap_{l\in\Lambda} \mathcal{E}_{(M^l)}(\mathbb{R}) \subsetneq \mathcal{E}_{(M^{l_1})}(\mathbb{R}) = \mathcal{E}_{(N)}(\mathbb{R})$$

which gives a contradiction. Note that the inclusion is really strict, because the sequences are pair-wise non-equivalent.
For the Roumieu-case consider a compact set $K \subseteq \mathbb{R}$ and so we have:

$$\begin{aligned}\mathcal{E}_{\{N\}}(\mathbb{R}) = \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}) &= \bigcap_{K\subseteq\mathbb{R}} \mathcal{E}_{\{\mathcal{M}\}}(K) = \bigcap_{K\subseteq\mathbb{R}} \bigcup_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(K) \\ &\supseteq \bigcup_{l\in\Lambda} \bigcap_{K\subseteq\mathbb{R}} \mathcal{E}_{\{M^l\}}(K) = \bigcup_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(\mathbb{R}) \supsetneq \mathcal{E}_{\{M^{l_0}\}}(\mathbb{R}) = \mathcal{E}_{\{N\}}(\mathbb{R})\end{aligned}$$

which gives again a contradiction (as above the inclusion is strict).
Finally note that by condition $(\mathcal{M}_{(C^\omega)})$ both classes $\mathcal{E}_{[\mathcal{M}]}(\mathbb{R})$ contain the real analytic functions $\mathcal{C}^\omega(\mathbb{R})$ and so the case for a general weight sequence N follows by [35, 2.15. Theorem].

Assume now that there exists a weight function $\omega \in \mathcal{W}$ which satisfies (ω_1) and such that $\mathcal{E}_{[\mathcal{M}]}(\mathbb{R}) = \mathcal{E}_{[\omega]}(\mathbb{R}) = \mathcal{E}_{[\mathcal{N}]}(\mathbb{R})$, where we denote by $\mathcal{N}$ the matrix associated to ω, which is given by $N^l_p := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lp))$ for $l > 0$ and $p \in \mathbb{N}$. Recall: The second equality holds by 6.2.6.
Since $\mathcal{N}$ is automatically a $(\mathcal{M}_{\text{sc}})$ weight matrix by 5.1.1, we get $\mathcal{M}\{\approx\}\mathcal{N}$ resp. $\mathcal{M}(\approx)\mathcal{N}$ (see proofs of 6.5.1 and 6.5.2).
$\mathcal{M}\{\approx\}\mathcal{N}$ implies, that for each l_1 there exists l_2 with $M^{l_1} \preceq N^{l_2}$ and for each l_3 there exists l_4 such that $N^{l_3} \preceq M^{l_4}$. Note that $\omega_{N^l} \in \mathcal{W}$ by 5.1.1 and 4.0.2, moreover by 5.1.3 we have $\omega \sim \omega_{N^l}$ for each $l > 0$, hence property (ω_1) for ω_{N^l} holds, too.
We recall now the first two items of 6.3.2: By 4.0.2 and assumption on $\mathcal{M}$ we have $\omega_{M^l} \in \mathcal{W}$. Since in addition each sequence M^l satisfies (β_3) we can use (2) $\Rightarrow$ (4) in [4, 12. Lemma] to obtain (ω_1) for each ω_{M^l}, too.
Each sequence M^l satisfies also moderate growth (mg), hence each ω_{M^l} has (ω_6), too (by [16, 3.6. Proposition]). Thus each ω_{M^l} satisfies the assumptions of (1) in 6.2.1 and we get for each $l > 0$:

$$\mathcal{E}_{[\omega_{M^l}]}(\mathbb{R}) = \mathcal{E}_{[M^l]}(\mathbb{R}).$$

So we have that for each l_1 there exist l_2, l_3 such that $N^{l_1} \preceq M^{l_2} \preceq N^{l_3}$, which implies

$$\mathcal{E}_{[N^{l_1}]}(\mathbb{R}) \subseteq \mathcal{E}_{[M^{l_2}]}(\mathbb{R}) = \mathcal{E}_{[\omega_{M^{l_2}}]}(\mathbb{R}) \subseteq \mathcal{E}_{[N^{l_3}]}(\mathbb{R}).$$

Now we use again the proofs of 6.5.1 in the Roumieu- and 6.5.2 in the Beurling-case to obtain in both cases $\omega_{M^{l_2}} \sim \omega$. But this would imply

$$\mathcal{E}_{[M^{l_2}]}(\mathbb{R}) = \mathcal{E}_{[\omega_{M^{l_2}}]}(\mathbb{R}) = \mathcal{E}_{[\omega]}(\mathbb{R}) = \mathcal{E}_{[\mathcal{M}]}(\mathbb{R})$$

which is a contradiction in both cases to our assumption (2): The sequences M^l are pair-wise not equivalent.

For the remaining cases we point out: If N is a weight sequence which is at least weakly log. convex and $\lim_{k\to\infty}(N_k)^{1/k} = +\infty$, then $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}) = \mathcal{E}_{(N)}(\mathbb{R})$ and $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}) = \mathcal{E}_{(\omega)}(\mathbb{R})$ resp. $\mathcal{E}_{(\mathcal{M})}(\mathbb{R}) = \mathcal{E}_{\{N\}}(\mathbb{R})$ and $\mathcal{E}_{(\mathcal{M})}(\mathbb{R}) = \mathcal{E}_{\{\omega\}}(\mathbb{R})$ would imply $\mathcal{M} \lhd \mathcal{N}(\preceq\}\mathcal{M}$ resp. $\mathcal{M}(\preceq\}\mathcal{N} \lhd \mathcal{M}$, which is impossible.
Of course $\mathcal{N}$ should denote here either $\{N\}$ or $\{N^l : N^l_p := \exp(\frac{1}{l}\cdot\varphi^*_\omega(lp)),\ l>0, p\in\mathbb{N}\}$, in both cases one gets $(\mathcal{M}_{\text{sc}})$ weight matrices. For the general single sequence case $\mathcal{N} = \{N\}$ we use again [35, 2.15. Theorem] together with property $(\mathcal{M}_{(\mathrm{C}^\omega)})$ as before. □

7.4 Non-quasi-analyticity for ultradifferentiable function spaces defined by a weight matrix

Let $\mathcal{M}$ be an arbitrary (large) set of weight sequences. We call $\mathcal{M}$ *Roumieu-non-quasi-analytic* if

$$(\mathcal{M}_{\{\text{nq}\}}) \;:\Leftrightarrow \mathcal{E}_{\{\mathcal{M}\}} \text{ contains functions with compact support}$$

and we call it *Beurling-non-quasi-analytic* if

$$(\mathcal{M}_{(\text{nq})}) \;:\Leftrightarrow \mathcal{E}_{(\mathcal{M})} \text{ contains functions with compact support,}$$

where the spaces can be considered in the most general notation (7.2.1) resp. (7.2.2). Otherwise we call the weight matrix $\mathcal{M}$ *Roumieu- resp. Beurling-quasi-analytic.*
An immediate consequence is of course that $(\mathcal{M}_{(\text{nq})})$ implies $(\mathcal{M}_{\{\text{nq}\}})$. Now we formulate the following easy consequence of the important Denjoy-Carleman-Theorem for the single weight sequence case (see e.g. [15, Theorem 1.3.8.] or [36, 19.11 Theorem]):

Lemma 7.4.1. *Let $\mathcal{M}$ be an arbitrary set of weight sequences such that each $M\in\mathcal{M}$ is weakly log. convex (which includes the case where $\mathcal{M} = \{M^l : l\in\Lambda\}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix), let $U\subseteq\mathbb{R}^n$ be a non-empty open subset, then the following are equivalent:
$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R})$ contains functions with compact support, i.e. $(\mathcal{M}_{\{\text{nq}\}})$ holds, if and only if there exists a sequence $M\in\mathcal{M}$ (resp. there exists an index $l_0\in\Lambda$) such that M (resp. M^{l_0}) is not quasi-analytic, i.e. has property* (nq).

Proof. On the one hand, if some $M\in\mathcal{M}$ is not quasi-analytic then, by the well-known Denjoy-Carleman-theorem, the class $\mathcal{E}_{\{M\}}$ contains functions with compact support, and so $\mathcal{E}_{\{\mathcal{M}\}}$, too.
On the other hand let $U\subseteq\mathbb{R}^n$ be non-empty open and assume that $\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R})$ contains a function f with compact support $K\subseteq U$. Then there exist some $C,h>0$ and

$M \in \mathcal{M}$ such that $\sup_{x\in K} |f^{(\alpha)}(x)| \le C \cdot h^{|\alpha|} \cdot M_{|\alpha|}$ for all $\alpha \in \mathbb{N}^n$. Since this estimate is clearly valid also on whole U, we have $f \in \mathcal{E}_{\{M\}}(U,\mathbb{R})$ and then, again by the Denjoy-Carleman-theorem, the sequence M is not quasi-analytic, i.e. has property (nq). □

We summarize some further remarks for the non-quasi-analyticity of ultradifferentiable function classes defined by weight matrices. For this let $\mathcal{M}$ be now always an arbitrary set of weakly log. convex weight sequences (includes for $\mathcal{M} = \{M^l : l \in \Lambda\}$ the $(\mathcal{M}_{\text{sc}})$ weight matrices).

- *(i)* Even if each $M \in \mathcal{M}$ is not quasi-analytic, i.e. (nq) holds, we cannot conclude that $\mathcal{M}$ has $(\mathcal{M}_{(\text{nq})})$. Note that the intersection of a large set of non-quasi-analytic classes can be quasi-analytic, see e.g. [23] but also 11.1.1 below. But maybe the intersection of non-quasi-analytic classes is again non-quasi-analytic if the matrix is given by a one parameter family $\mathcal{M} = \{M^l : l \in \Lambda\}$. This is true if the matrix is obtained by a weight function ω, see also item (v) below.
- *(ii)* Both conditions $(\mathcal{M}_{\{\text{nq}\}})$ and $(\mathcal{M}_{(\text{nq})})$ are clearly satisfied, if we assume that there exists a (at least) weakly log. convex and non quasi-analytic sequence N with $N \preceq M$ for each $M \in \mathcal{M}$. In this case immediately all M are then not quasi-analytic (nq), too.
- *(iii)* On the other hand $\mathcal{M}$ is both Roumieu- and Beurling-quasi-analytic, so both $(\mathcal{M}_{\{\text{nq}\}})$ and $(\mathcal{M}_{(\text{nq})})$ are not satisfied, if e.g. there exists a (at least) weakly log. convex and quasi-analytic sequence N with $M \preceq N$ for each $M \in \mathcal{M}$. In this case all $M \in \mathcal{M}$ are then quasi-analytic, i.e. doesn't satisfy (nq), too.
- *(iv)* If $\mathcal{M}$ is constant, i.e. $\mathcal{M} = \{M\}$, or more general if $M \approx N$ for each $M, N \in \mathcal{M}$, then the Denjoy-Carleman-theorem states that $(\mathcal{M}_{\{\text{nq}\}})$ if and only if $(\mathcal{M}_{(\text{nq})})$ and both conditions are equivalent to the fact that each sequence $M \in \mathcal{M}$ is not-quasi-analytic, i.e. (nq).
- *(v)* Let $\omega \in \mathcal{W}$ be given with properties (ω_1) and (ω_{nq}). Then the associated weight matrix $\mathcal{M}$ defined by $M^l_j = \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$ (which is automatically $(\mathcal{M}_{\text{sc}})$ by 5.1.1) has both conditions $(\mathcal{M}_{(\text{nq})})$ and $(\mathcal{M}_{\{\text{nq}\}})$. More precisely in the second and third chapter in [5] it is shown that in this case now both classes $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ contain functions with compact support. Moreover by the first part of 5.1.3, each sequence M^l is then not quasi-analytic (nq), too.

8 Characterization of stability properties for the classes $\mathcal{E}_{[\mathcal{M}]}$

8.1 The general $\mathcal{E}_{[\mathcal{M}]}$-case - introduction

The results in this chapter are partially based on joint work with *Armin Rainer,* see [35].
We start with the following definition:

Definition 8.1.1. *Let $\mathcal{A}$ be a sub-class of smooth functions, then*

(1) *$\mathcal{A}$ is called closed under composition, if for each $f, g \in \mathcal{A}$ we have $f \circ g \in \mathcal{A}$, wherever the composition is defined.*

(2) *$\mathcal{A}$ is called closed under inversion, if for each $f \in \mathcal{A}$ with $f(x) \neq 0$ for all x it follows that $\frac{1}{f} \in \mathcal{A}$, too.*

(3) *$\mathcal{A}$ is called holomorphically closed, if $h \in \mathcal{H}(\mathbb{C})$ and $f \in \mathcal{A}$ imply $h \circ f \in \mathcal{A}$.*

(4) *$\mathcal{A}$ is called analytically closed, if $h \in \mathcal{C}^\omega$ and $f \in \mathcal{A}$ imply $h \circ f \in \mathcal{A}$.*

(5) *$\mathcal{A}$ is called closed under solving ODE's, if the initial value problem $x'(t) = f(t, x(t))$, $x(0) = x_0$, with $f \in \mathcal{A}$ implies already $x \in \mathcal{A}$, wherever the smooth solution x exists.*

The aim of this chapter and general remarks:

(1) We are going to characterize for a $(\mathcal{M}_{\text{sc}})$ weight matrix $\mathcal{M}$ with index set $\Lambda = \mathbb{R}_{>0}$ these introduced closedness properties for $\mathcal{A} = \mathcal{E}_{[\mathcal{M}]}$ in terms of properties of the weight matrix $\mathcal{M}$. Moreover we are also going to characterize the $\mathcal{E}_{[\mathcal{M}]}$-inverse and $\mathcal{E}_{[\mathcal{M}]}$-implicit function theorem.

(2) For convenience we will work in the proofs with *Fréchet-derivatives.* Recall: $f^{(k)}(x)$ is the k-th order *Fréchet-derivative* of f at x and we denote the norm of it by $\|f^{(k)}(x)\|_{L^k(E,F)}$, where E, F are (finite-dimensional) Banach-spaces.

(3) All statements in this chapter are formulated for locally defined classes, but they are also valid for globally defined classes by obvious modifications of the proofs below, see 7.3.2.

(4) The strategy in this chapter for the proof of the main characterizing theorem is the following: First we will generalize the proofs of the different steps from the constant weight matrix case $\mathcal{M} = \{M\}$ to the more general (also non-constant) one and afterwards we put all the different results together.

(5) We summarize the literature of proofs for the different steps in the case $\mathcal{M} = \{M\}$: For closedness under composition see e.g. [2, Theorem 4.7.]; closedness under inversion for the Roumieu-case was treated in [37] and in [39] (see also

for analytically closedness there), for the Beurling-case in [6]; closedness under solving ODE's was considered in [19] (for both cases) and in [42] (only for the Roumieu-case); the inverse/implicit function theorem can be found in [18] (for both cases) and in [41] (only for the Roumieu-case). For closedness under composition for the classes $\mathcal{E}_{[\omega]}$ defined by non-quasi-analytic weight functions $\omega \in \mathcal{W}$ with (ω_1) we refer to [10].

8.2 Definitions and first observations

First we have the following result which is clear by the structure of the considered function classes:

Lemma 8.2.1. *Assume that $\mathcal{M}$ is an arbitrary weight matrix, i.e.* $(\mathcal{M})$*, and such that each sequence m^l is (strongly) log. convex, i.e.* (slc)*. Then both classes $\mathcal{E}_{\{\mathcal{M}\}}$ and $\mathcal{E}_{(\mathcal{M})}$ are closed under composition.*

Proof. Since the sequences m^l are log. convex for each $l \in \Lambda$, it is well-known that the classes $\mathcal{E}_{\{M^l\}}$ resp. $\mathcal{E}_{(M^l)}$ are closed under composition for each $l \in \Lambda$ (see e.g. [2, Theorem 4.7.]). Now recall the structure and definition of weight matrix spaces (see (7.3.5) resp. (7.3.6)).
The converse implication is not true in general, it doesn't hold already for the case $\mathcal{M} = \{M\}$, see the counter example in [35, 3.3.-3.6.]). □

We recall now some definitions for weight sequences: A sequence $M = (M_k)_k$ of positive real numbers has the *Faà-di-Bruno-property*, if

$$\text{(FdB)} :\Leftrightarrow m^\circ \preceq m,$$

where we have put $m^\circ = (m^\circ_k)_k$ with $m^\circ_k := \max\{m_j \cdot m_{\alpha_1} \cdots m_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \sum_{i=1}^j \alpha_i = k\}$, $m^\circ_0 := 1$ and finally $m_j := \frac{M_j}{j!}$. This property is clearly stable w.r.t. $\approx$ and it's well-known that strong log. convexity (slc) implies (FdB), see e.g. [21, 2.9. Lemma]. Note that (slc) is not stable w.r.t. $\approx$ since it is a convexity condition. A sequence $M = (M_j)_j$ is said to be *almost increasing*, if

$$\exists\, C \geq 1\; M_j \leq C \cdot M_k \;\; \forall\, j,k \in \mathbb{N} \text{ with } j \leq k.$$

Of course this condition implies immediately $M_j \leq C \cdot M_{j+1}$ for a constant C and each $j \in \mathbb{N}$, but the converse implication is not true except the case $C = 1$, i.e. if the sequence M is increasing.
For this chapter we will need the following condition:

$$\text{(ai)} :\Leftrightarrow\; \exists\, C \geq 1\; \forall\, 1 \leq j \leq k: \; (m_j)^{1/j} \leq C \cdot (m_k)^{1/k}$$

and (ai) means exactly that the sequence $(m_j^{1/j})_j$ is almost increasing. If the mapping $j \mapsto (m_j)^{1/j}$ is increasing, this property holds clearly with $C = 1$. We point out that (ai) is stable w.r.t. relation $\approx$: Assume condition (ai) for M, consider $N \approx M$ and let $1 \leq j \leq k$ be given, then $(n_j)^{1/j} \leq C_1 \cdot (m_j)^{1/j} \leq C_1 \cdot C \cdot (m_k)^{1/k} \leq C_1 \cdot C \cdot C_2 \cdot (n_k)^{1/k}$ holds for some constants $C_1, C_2 \geq 1$.
We introduce the following condition on a weight matrix $\mathcal{M}$ (in the most general sense):

$$(\mathcal{M}_\circ) :\Leftrightarrow\; \forall\, M \in \mathcal{M}\; \exists\, D > 0\; \forall\, j,k \in \mathbb{N} : m_j \cdot m_k \leq D \cdot m_{j+k},$$

with $m_k := \frac{M_k}{k!}$. If $\mathcal{M} = \{M^l : l \in \Lambda\}$, then replace m by m^l and "$\forall\ M \in \mathcal{M}$" by "$\forall\ l \in \Lambda$". If the weight matrix $\mathcal{M} = \{M^l : l > 0\}$ is obtained by a weight function $\omega \in \mathcal{W}$ via $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(lj))$ (in fact we need only properties (ω_0) and (ω_3) for ω - see 3.8.5), then $(\mathcal{M}_\circ)$ is satisfied with $D = 1$ whenever ω is sub-additive (see (3.8.1) and 3.8.2). Some brief remarks on condition $(\mathcal{M}_\circ)$:

(1) In [10, § 2,3] it was shown that for $\omega \in \mathcal{W}$ with (ω_1), (ω_{nq}) and which satisfies property $(\omega_{1'})$ (i.e. ω is equivalent to a sub-additive weight - see 3.8.4), then the classes $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ both are closed under composition.

(2) Property $(\mathcal{M}_\circ)$ implies immediately that for all $l \in \Lambda$ there exists a constant $D > 0$ such that for all $j, k \in \mathbb{N}$ we have also $M^l_j \cdot M^l_k \le D \cdot M^l_{j+k}$, hence both $(\mathcal{M}_{\{\text{alg}\}})$ and $(\mathcal{M}_{(\text{alg})})$. On the other hand, if $(\mathcal{M}_{[\text{alg}]})$ holds, then for all $l \in \Lambda$ there exist $n \in \Lambda$ and $C \ge 1$ (resp. for all $n \in \Lambda$ there exist $l \in \Lambda$ and $C \ge 1$) such that $m^l_j \cdot m^l_k \le C^{j+k} \cdot m^n_{j+k}$ for all $j, k \in \mathbb{N}$. But this is too weak to obtain $(\mathcal{M}_\circ)$, since we have exponential increase on the right hand side. Even if $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then we get $M^l_j \cdot M^l_k \le M^l_{j+k}$ for all $l \in \Lambda$ and $j, k \in \mathbb{N}$, which is again too weak.

(3) Since $m^l_1 = M^l_1 \ge 1$ for each $l \in \Lambda$ we can consider in $(\mathcal{M}_\circ)$ the case $k = 1$, then we obtain $m^l_j \le m^l_j \cdot m^l_1 \le D \cdot m^l_{j+1}$. So for all $l \in \Lambda$ there exists $D > 0$ such that $m^l_j \le D \cdot m^l_{j+1}$ for all $j \in \mathbb{N}$.

(4) If (ai) is satisfied with $C = 1$ for each sequence m^l, i.e. if the mapping $k \mapsto (m^l_k)^{1/k}$ is increasing, then property $(\mathcal{M}_\circ)$ holds with $D = 1$: For arbitrary $j, k \ge 1$ we have $(m^l_j)^{1/j} \le (m^l_{j+k})^{1/(j+k)}$, $(m^l_k)^{1/k} \le (m^l_{j+k})^{1/(j+k)}$, hence

$$m^l_j \cdot m^l_k \le (m^l_{j+k})^{j/(j+k)} \cdot (m^l_{j+k})^{k/(j+k)} = m^l_{j+k}.$$

The remaining cases ($j = k = 0$, $j = 0$, k arb. resp. j arb. and $k = 0$) are obvious by normalization.

This proof doesn't hold for $C > 1$, because then we would get $D = C^{j+k}$ and so exponential growth in j, k.

Finally (3) in 8.2.3 below shows: Even if we assume that $(\mathcal{M}_\circ)$ holds with $D = 1$, then we only get (ai) for each sequence m^l separately with $C > 1$ (by using Stirling's formula).

Moreover we recall for $\mathcal{M} = \{M^l : l \in \Lambda\}$:

$$(\mathcal{M}_{\{\text{dc}\}}) :\Leftrightarrow\ \forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, C > 0\ \forall\, j \in \mathbb{N} : M^l_{j+1} \le C^{j+1} \cdot M^n_j$$

and

$$(\mathcal{M}_{(\text{dc})}) :\Leftrightarrow\ \forall\, n \in \Lambda\ \exists\, l \in \Lambda\ \exists\, C > 0\ \forall\, j \in \mathbb{N} : M^l_{j+1} \le C^{j+1} \cdot M^n_j$$

which follow immediately for matrices obtained from $\omega \in \mathcal{W}$ by (5.1.2) in 5.1.2. The *matrix generalized Faà-di-Bruno-formulas* of Roumieu- resp. Beurling-type for a weight matrix $\mathcal{M}$ are defined as follows:

$$(\mathcal{M}_{\{\text{FdB}\}}) :\Leftrightarrow\ \forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, C > 0\ \forall\, k \in \mathbb{N} :\ \left(m^l\right)^\circ_k \le C^k \cdot m^n_k$$

$$(\mathcal{M}_{(\text{FdB})}) :\Leftrightarrow\ \forall\, n \in \Lambda\ \exists\, l \in \Lambda\ \exists\, C > 0\ \forall\, k \in \mathbb{N} :\ \left(m^l\right)^\circ_k \le C^k \cdot m^n_k$$

where we put $(m^l)^\circ_k := \max\{m^l_j \cdot m^l_{\alpha_1} \cdots m^l_{\alpha_j} :\ \alpha_i \in \mathbb{N}_{>0}, \sum_{i=1}^j \alpha_i = k\}$, $(m^l)^\circ_0 := 1$. Finally we have to introduce new important conditions, the matrix generalizations of (ai) of Roumieu- resp. Beurling-type in the most general sense:

$$(\mathcal{M}_{\{\text{ai}\}}) :\Leftrightarrow\ \forall\, M \in \mathcal{M}\ \exists\, N \in \mathcal{M}\ \exists\, H > 0:\ (m_q)^{1/q} \le H \cdot (n_p)^{1/p},\ 1 \le q \le p,$$

$$(\mathcal{M}_{(\text{ai})}) :\Leftrightarrow\ \forall\, N \in \mathcal{M}\ \exists\, M \in \mathcal{M}\ \exists\, H > 0:\ (m_q)^{1/q} \le H \cdot (n_p)^{1/p},\ 1 \le q \le p,$$

where we have put $m_k := \frac{M_k}{k!}$, $n_k := \frac{N_k}{k!}$. It's obvious that if $((m_j)^{1/j})_j$ is almost increasing for each $M \in \mathcal{M}$, then both $(\mathcal{M}_{\{\text{ai}\}})$ and $(\mathcal{M}_{(\text{ai})})$ are satisfied. If $\mathcal{M} = \{M^l : l \in \Lambda\}$ then replace e.g. in the first condition m by m^l and n by m^n, "$\forall\, M \in \mathcal{M}$" by "$\forall\, l \in \Lambda$" and "$\exists\, N \in \mathcal{M}$" by "$\exists\, n \in \Lambda$".

Remark 8.2.2. *We point out that $(\mathcal{M}_{[\text{ai}]})$, $(\mathcal{M}_{[\text{FdB}]})$ and $(\mathcal{M}_{[\text{ai}_K]})$ are stable w.r.t. relation $[\approx]$. For example assume $(\mathcal{M}_{\{\text{ai}\}})$ for the matrix $\mathcal{M}$ and let $\mathcal{N}$ with $\mathcal{M}\{\approx\}\mathcal{N}$. So for each $N^1 \in \mathcal{N}$ there exists $M^1 \in \mathcal{M}$ with $N^1 \preceq M^1 \Leftrightarrow n^1 \preceq m^1$ and for each $M^2 \in \mathcal{M}$ there exists $N^2 \in \mathcal{N}$ with $m^2 \preceq n^2$. So let $1 \le q \le p$ be given and then $\frac{1}{D} \cdot (n^1_q)^{1/q} \le (m^1_q)^{1/q} \le H \cdot (m^2_p)^{1/p} \le H \cdot (C^p \cdot n^2_p)^{1/p} = HC(n^2_p)^{1/p}$ holds, i.e. $(\mathcal{M}_{\{\text{ai}\}})$ for the matrix $\mathcal{N}$.*

We connect all these conditions in the following preparation result, and for this recall the last important condition in this chapter:

$$(\mathcal{M}_{\mathcal{H}}) :\Leftrightarrow\ \forall\, l \in \Lambda:\ \liminf_{j\to\infty} (m^l_j)^{1/j} > 0.$$

Condition $(\mathcal{M}_{\mathcal{H}})$ has the consequence that $\mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}_{(\omega)}(U)$ for all $U \subseteq \mathbb{R}^n$, i.e. the restrictions of entire functions belong already to the class $\mathcal{E}_{(\omega)}$. It is satisfied for a matrix coming from $\omega \in \mathcal{W}$ with (ω_1) and (ω_2), see for this the first part of 5.3.2 and use *Stirlings formula*. If $(\mathcal{M}_{\mathcal{H}})$ is satisfied, we can assume that for all $j \in \mathbb{N}_{>0}$ we have $(m^l_j)^{1/j} \ge C_l > 0$ for a (small) constant C_l depending on $l \in \Lambda$ (put $C_l := \inf(m^l_j)^{1/j} > 0$).

Lemma 8.2.3. *Let $\mathcal{M} := \{M^l \in \mathbb{R}^{\mathbb{N}}_{>0} : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and assume that for each $l \in \Lambda$ there exists a constant $H \ge 1$ with $(M^l_j)^{1/j} \le H \cdot (M^l_k)^{1/k}$ for all $1 \le j \le k$, i.e. each sequence $\left((M^l_j)^{1/j}\right)_j$ is almost increasing (this assumption is only needed for* (3), (4)*). Then we obtain the following implications:*

(1) $(\mathcal{M}_\circ)$, $(\mathcal{M}_{[\text{dc}]}) \Longrightarrow (\mathcal{M}_{[\text{FdB}]})$.

(2) $(\mathcal{M}_{[\text{ai}]}), (\mathcal{M}_{[\text{dc}]}) \Longrightarrow (\mathcal{M}_{[\text{FdB}]})$.

(3) $(\mathcal{M}_\circ) \Longrightarrow$ *property* (ai) *holds for each sequence m^l.*

(4) $(\mathcal{M}_{[\text{FdB}]}),(\mathcal{M}_{\mathcal{H}}) \Longrightarrow (\mathcal{M}_{[\text{ai}]})$.

Proof. We prove this lemma for the Roumieu-case, the Beurling-case is completely analogous.
(1) First we use property $(\mathcal{M}_{\{\text{dc}\}})$, hence for each $l \in \Lambda$ there exist $n \in \Lambda$ and $C > 0$ such that $m^l_{j+1} \le C^{j+1} \cdot m^n_j$ holds for all $j \in \mathbb{N}$. This implies immediately $m^l_{\alpha_i} \le C^{\alpha_i} \cdot m^n_{\alpha_i - 1}$ for $i = 1, \ldots, j$, hence $m^l_{\alpha_1} \cdots m^l_{\alpha_j} \le C^k \cdot m^n_{\alpha_1 - 1} \cdots m^n_{\alpha_j - 1}$. Now use property $(\mathcal{M}_\circ)$ (iterated application) to obtain $m^n_{\alpha_1 - 1} \cdots m^n_{\alpha_j - 1} \le D^{j-1} \cdot m^n_{\alpha_1 + \cdots + \alpha_j - j} = D^{j-1} \cdot$

m^n_{k-j} for a constant $D>0$ depending on $n\in\Lambda$. By applying once again properties $(\mathcal{M}_{\{\mathrm{dc}\}})$ and $(\mathcal{M}_\circ)$ we finally obtain

$$\begin{aligned} m^l_j\cdot m^l_{\alpha_1}\cdots m^l_{\alpha_j} &\le m^l_j\cdot C^k\cdot m^n_{\alpha_1-1}\cdots m^n_{\alpha_j-1}\le C^k\cdot D^{j-1}\cdot m^l_j\cdot m^n_{k-j}\\ &\le C^k\cdot D^{j-1}\cdot C^j\cdot m^n_{j-1}\cdot m^n_{k-j}\le C^k\cdot D^{j-1}\cdot C^j\cdot D\cdot m^n_j\cdot m^n_{k-j}\\ &\le C^k\cdot D^{j-1}\cdot C^j\cdot D\cdot D\cdot m^n_k\le \tilde{C}^k\cdot m^n_k, \end{aligned}$$

hence $(\mathcal{M}_{\{\mathrm{FdB}\}})$ is satisfied.
(2) Let in the following $\alpha_1,\dots,\alpha_j\in\mathbb{N}_{>0}$, $j\ge 1$, with $\alpha_1+\dots+\alpha_j=k$. Then first property $(\mathcal{M}_{\{\mathrm{ai}\}})$ implies that for all $l\in\Lambda$ there exists $n\in\Lambda$ and $H>0$ with

$$m^l_{\alpha_1}\cdots m^l_{\alpha_j}\le H^{\alpha_1+\dots+\alpha_j}\cdot(m^n_k)^{\alpha_1/k}\cdots(m^n_k)^{\alpha_j/k}\le H^k\cdot m^n_k.$$

But since $m^l_0=1$ for each $l\in\Lambda$ this estimate holds also for the cases if $\alpha_i=0$ for some $1\le i\le j$ and if $\alpha_1=\dots=\alpha_j=0$ (and so $k=0$). Furthermore by property $(\mathcal{M}_{\{\mathrm{dc}\}})$ we get that for all $l\in\Lambda$ we can find $C>0$ and $n\in\Lambda$ such that $m^l_j\cdot m^l_{\alpha_1}\cdots m^l_{\alpha_j}\le m^l_j\cdot C^{\alpha_1+\dots+\alpha_j}\cdot m^n_{\alpha_1-1}\cdots m^n_{\alpha_j-1}$. So we can estimate as follows:

$$\begin{aligned} m^l_j\cdot m^l_{\alpha_1}\cdots m^l_{\alpha_j} &\le m^l_j\cdot C^k\cdot m^n_{\alpha_1-1}\cdots m^n_{\alpha_j-1}\le m^l_j\cdot C^k\cdot H^k_1\cdot m^{n_1}_{k-j}\le H^k_2\cdot m^{n_2}_j\cdot m^{n_2}_{k-j}\\ &\le H^k_3\cdot m^{n_3}_k. \end{aligned}$$

We have put $n_2=\max\{n_1,l\}$ and in the last step we have used again $(\mathcal{M}_{\{\mathrm{ai}\}})$. So we have shown $(\mathcal{M}_{\{\mathrm{FdB}\}})$.

(3) Let $n\in\Lambda$ be arbitrary but from now on fixed. We iterate condition $(\mathcal{M}_\circ)$ for the choice $j=k\in\mathbb{N}_{>0}$ to obtain for all $l\in\mathbb{N}_{>0}$ the estimate $(m^n_j)^l=m^n_j\cdots m^n_j\le D^l\cdot m^n_{lj}$. Take now the lj-th root, put $k=lj$ and then we get $(m^n_j)^{1/j}\le D^{1/j}\cdot(m^n_k)^{1/k}$.
More general, we consider for some $j\le k$ a number $l\in\mathbb{N}$ such that $j\cdot l\le k<(l+1)\cdot j$ and then, since $(M^n_j)^{1/j}\le H\cdot(M^n_k)^{1/k}$ for a constant $H\ge 1$ and all $1\le j\le k$ we get (by using *Stirling's formula*):

$$(k!\cdot m^n_k)^{1/k}\ge\frac{1}{H}\cdot((lj)!\cdot m^n_{lj})^{1/(lj)}\ge\frac{C}{H}\cdot(lj)\cdot(m^n_j)^{1/j}\ge\frac{C}{H}\cdot\frac{(l+1)}{2}\cdot j\cdot(m^n_j)^{1/j},$$

which proves the claim.
(4) We use in $(\mathcal{M}_{\{\mathrm{FdB}\}})$ the case $\alpha_1=\dots=\alpha_l=j$ to obtain $m^x_l\cdot m^x_j\cdots m^x_j=m^x_l\cdot(m^x_j)^l\le C^{lj}\cdot m^y_{lj}$ for $x,y\in\Lambda$ related via $(\mathcal{M}_{\{\mathrm{FdB}\}})$. Then take the lj-th root to obtain $(m^x_l)^{1/(lj)}\cdot(m^x_j)^{1/j}\le C\cdot(m^y_{lj})^{1/(lj)}$ and so the desired property for $k=lj$. Note that by property $(\mathcal{M}_{\mathcal{H}})$ there exists a constant $C_x\le 1$ depending only on $x\in\Lambda$ such that $(m^x_l)^{1/(lj)}\ge C_x>0$ for all $j,l\in\mathbb{N}_{>0}$. Now proceed as above again for the general case: For some $j\le k$ we take $l\in\mathbb{N}$ with $j\cdot l\le k<(l+1)\cdot j$ and then similarly as in (3) we get (again by using *Stirling's formula*):

$$(k!\cdot m^y_k)^{1/k}\ge\frac{1}{H}\cdot((lj)!\cdot m^y_{lj})^{1/(lj)}\ge C_2\cdot(lj)\cdot(m^x_j)^{1/j}\ge C_2\cdot\frac{(l+1)}{2}\cdot j\cdot(m^x_j)^{1/j},$$

hence $(\mathcal{M}_{\{\mathrm{ai}\}})$. □

Very similar to $(\mathcal{M}_{\{\mathrm{ai}\}})$ resp. $(\mathcal{M}_{(\mathrm{ai})})$ are the following conditions:

$$(\mathcal{M}_{\{\mathrm{ai}_K\}}):\Leftrightarrow\ \forall\, M\in\mathcal{M}\ \exists\, N\in\mathcal{M}\ \exists\, H>0:\ (m_q)^{1/(q-1)}\le H\cdot(n_p)^{1/(p-1)},\ 2\le q\le p,$$

$(\mathcal{M}_{(\mathrm{ai}_K)}) :\Leftrightarrow \forall\, N \in \mathcal{M}\ \exists\, M \in \mathcal{M}\ \exists\, H > 0:\ (m_q)^{1/(q-1)} \le H \cdot (n_p)^{1/(p-1)},\ 2 \le q \le p,$

where $n_k := \frac{N_k}{k!}$, $m_k := \frac{M_k}{k!}$. Again, if $\mathcal{M} = \{M^l : l \in \Lambda\}$ then replace e.g. in the first condition m by m^l and n by m^n, "$\forall\, M \in \mathcal{M}$" by "$\forall\, l \in \Lambda$" and "$\exists\, N \in \mathcal{M}$" by "$\exists\, n \in \Lambda$". Note that H. Komatsu has used in [19] resp. [18] condition

$$(\mathrm{ai}_K) :\Leftrightarrow\ \exists\, H > 0:\ (m_q)^{1/(q-1)} \le H \cdot (m_p)^{1/(p-1)},\ 2 \le q \le p$$

for $m_p := \frac{M_p}{p!}$ to prove for the single weight sequence Roumieu case $\mathcal{E}_{\{M\}}$ closedness under solving ODE's resp. the $\mathcal{E}_{\{M\}}$-inverse/implicit function theorem. For the Beurling-case $\mathcal{E}_{(M)}$ he assumed additional assumptions on M.
We have the following result:

Lemma 8.2.4. *Let $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $(\mathcal{M}_{\mathcal{H}})$. Then we obtain $(\mathcal{M}_{[\mathrm{ai}_K]}) \Rightarrow (\mathcal{M}_{[\mathrm{ai}]})$, which was already mentioned in [18] for the case $\mathcal{M} = \{M\}$.*
If in addition $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix and condition $(\mathcal{M}_{[\mathrm{diag}]})$ holds, then we have $(\Leftarrow)$, too.

Proof. First recall: Property $(\mathcal{M}_{\mathcal{H}})$ is satisfied, hence $\inf_{j\to\infty}(m^l_j)^{1/j} \ge C_l > 0$ for each $l \in \Lambda$, $C_l \le 1$ small depends only on l and note that maybe $C_l \to 0$ for l decreasing.

$(\mathcal{M}_{[\mathrm{ai}_K]}) \Rightarrow (\mathcal{M}_{[\mathrm{ai}]})$: We distinguish three cases. First, for $2 \le q \le p$, we estimate as follows:

$$\frac{m^l_q}{H^{q-1}\cdot C_n^{q-1}} \le \left(\frac{m^n_p}{C_n^{p-1}}\right)^{(q-1)/(p-1)} = \left(\frac{m^n_p}{C_n^p}\right)^{(q-1)/(p-1)} \cdot C_n^{(q-1)/(p-1)}$$
$$\le \left(\frac{m^n_p}{C_n^p}\right)^{q/p} \cdot C_n^{(q-1)/(p-1)} \le \frac{1}{C_n^q} \cdot \left(m^n_p\right)^{q/p}.$$

The first inequality holds by $(\mathcal{M}_{[\mathrm{ai}_K]})$, the second because $1 \le \frac{m^n_p}{C_n^p}$ for each p and $\frac{q-1}{p-1} \le \frac{q}{p} \Leftrightarrow q \le p$. Finally take the q-th root.
If $1 = q = p$, then for $(\mathcal{M}_{[\mathrm{ai}]})$ nothing is to prove. Finally, if $1 = q$ and $2 = p$ we need $M^l_1 = m^l_1 \le H \cdot (m^n_2)^{1/2} \Leftrightarrow 2 \cdot (M^l_1)^2 \le H^2 \cdot M^n_2$, which can be achieved by choosing the constant $H \ge 1$ large enough.

$(\mathcal{M}_{[\mathrm{ai}]}) \Rightarrow (\mathcal{M}_{[\mathrm{ai}_K]})$: By assumption we have $m^i_q \le H^q \cdot (m^l_p)^{q/p}$ for a constant $H \ge 1$ and all $1 \le q \le p$. Since $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix, we have $M^l_j \cdot M^l_k \le M^l_{j+k}$ for all $j,k \in \mathbb{N}$ and $l \in \Lambda$, hence $m^l_j \cdot m^l_k \le 2^{j+k} \cdot m^l_{j+k}$ holds. Moreover $(\mathcal{M}_{[\mathrm{diag}]})$ means $(2j)! \cdot m^l_{2j} = M^l_{2j} \le C^j \cdot M^n_j = C^j \cdot j! \cdot m^n_j$ with the corresponding order of quantifiers for the indices, hence $m^l_{2j} \le C^j \cdot m^n_j$ is satisfied. Using these remarks we can estimate as follows for $2 \le q \le p$:

$$\frac{m^l_q}{H^q \cdot C_n^q} \le \left(\frac{m^n_p}{C_n^p}\right)^{q/p} \le \left(\frac{m^n_p}{C_n^p}\right)^{2\cdot(q-1)/(p-1)} = \left(\frac{1}{C_n}\right)^{2p(q-1)/(p-1)} \cdot \left((m^n_p)^2\right)^{(q-1)/(p-1)}$$
$$\le \left(\frac{1}{C_n}\right)^{2p(q-1)/(p-1)} \cdot (2^{2p} \cdot m^n_{2p})^{(q-1)/(p-1)}$$
$$\le \left(\frac{1}{C_n}\right)^{2p(q-1)/(p-1)} \cdot (2^{2p} \cdot D^p \cdot m^{n_1}_p)^{(q-1)/(p-1)} \le H_1^{q-1} \cdot (m^{n_1}_p)^{(q-1)/(p-1)}.$$

for some constant $H_1 > 0$ large enough. The first inequality holds by assumption, the second since $\frac{q}{p} \le \frac{2(q-1)}{p-1} \Leftrightarrow 2p - q \le p \cdot q$ holds clearly for all $2 \le q \le p$ and $1 \le \frac{m_p^n}{C_n^p}$ for each p. Finally we have to take the $(q-1)$-th root to get $(\mathcal{M}_{[\mathrm{ai}_K]})$. □

Remark 8.2.5. *(i) Conditions* $(\mathcal{M}_{\{\mathrm{diag}\}})$ *and* $(\mathcal{M}_{(\mathrm{diag})})$ *are quite strong conditions, nevertheless they are both satisfied for a matrix associated to a weight function* $\omega \in \mathcal{W}$ *with* (ω_7) *(see 5.4.1).*

(ii) T. Yamanaka has shown in [41] the $\mathcal{E}_{\{M\}}$*-implicit/inverse function theorem, where he used only* (ai). *Moreover in [42] he was able to prove closedness under solving ODE's for* $\mathcal{E}_{\{M\}}$ *assuming* (ai). *But in fact Yamanaka worked in his both papers with the class* $\mathcal{E}_{\{N\}}$*, where he used the translated sequence* $(M_{p-1})_{p\ge 0}$ *- see inequality* (1.4) *in [42] - and where we have put* $N_p := M_{p-1}$*. Then we get*

$$\exists\, H \ge 1 : \left(\frac{N_{q+1}}{q!}\right)^{1/q} \le H \cdot \left(\frac{N_{p+1}}{p!}\right)^{1/p} \Leftrightarrow \left(\frac{N_q}{(q-1)!}\right)^{1/(q-1)} \le H \cdot \left(\frac{N_p}{(p-1)!}\right)^{1/(p-1)},$$

for $2 \le q \le p$*, which implies* (ai_K) *for the sequence* $n_p := \frac{N_p}{p!}$*, because*

$$\frac{(q-1)!^{1/(q-1)}}{(p-1)!^{1/(p-1)}} \le \frac{q!^{1/(q-1)}}{p!^{1/(p-1)}} \Leftrightarrow p^{1/(p-1)} \le q^{1/(q-1)}. \tag{8.2.1}$$

Note that the types $\alpha\{M_p\}$ *and* $\beta\{M_p\}$ *introduced in [42] coincide, if condition* (dc) *is satisfied. More precisely, in our notation, we have then* $\mathcal{E}_{[M]} = \mathcal{E}_{[N]}$.

(iii) So, in the general weight matrix case, we will have to use conditions $(\mathcal{M}_{\{\mathrm{dc}\}})$ *resp.* $(\mathcal{M}_{(\mathrm{dc})})$. *Because then we can assume* $(\mathcal{M}_{\{\mathrm{ai}\}})$ *resp.* $(\mathcal{M}_{(\mathrm{ai})})$ *for the translated matrix* $\mathcal{M}^d := \{M^l_{+1} : M^l \in \mathcal{M}\}$, $M^l_{+1} := (M^l_{k+1})_k$*, and obtain* $(\mathcal{M}_{\{\mathrm{ai}_K\}})$ *resp.* $(\mathcal{M}_{(\mathrm{ai}_K)})$ *for the original matrix* $\mathcal{M}$*. Finally, by* $(\mathcal{M}_{[\mathrm{dc}]})$*, we get* $\mathcal{E}_{[\mathcal{M}]} = \mathcal{E}_{[\mathcal{M}^d]}$*,* $\mathcal{M}^d$ *still has* $(\mathcal{M}_{[\mathrm{dc}]})$ *and so we can work also with the matrix* $\mathcal{M}^d$*. More precisely by definition we obtain immediately that* $(\mathcal{M}_{\{\mathrm{dc}\}})$ *implies* $\mathcal{M}\{\approx\}\mathcal{M}^d$ *and* $(\mathcal{M}_{(\mathrm{dc})})$ *implies* $\mathcal{M}(\approx)\mathcal{M}^d$.

8.3 Characterization of closedness under composition

With the introduced notation and conditions we can prove the first important theorem (see also [35, 3.1. Proposition] for the constant matrix case $\mathcal{M} = \{M\}$):

Theorem 8.3.1. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we get:*

(a) *If condition $(\mathcal{M}_{\{\mathrm{FdB}\}})$ is satisfied, then $\mathcal{E}_{\{\mathcal{M}\}}$ is closed under composition.*

(b) *If $(\mathcal{M}_{(\mathrm{FdB})})$ is satisfied, then $\mathcal{E}_{(\mathcal{M})}$ is closed under composition.*

Proof. (a) As pointed out at the beginning of this chapter we use *Fréchet-derivatives* and so we get: Let E, F, G be (finite-dimensional) Banach-spaces, $U \subseteq E$ and $V \subseteq F$ be open subsets. Assume now that we have $f \in \mathcal{E}_{\{\mathcal{M}\}}(V,G)$ and $g \in \mathcal{E}_{\{\mathcal{M}\}}(U,V)$. Let $K \subseteq U$ be a compact subset, then there exist $C_1, h_1 > 0$ and $l_1 \in \Lambda$ such that $\|f^{(k)}(y)\|_{L^k(F,G)} \le C_1 \cdot h_1^k \cdot M_k^{l_1} = C_1 \cdot h_1^k \cdot k! \cdot m_k^{l_1}$ for all $y \in g(K)$ and $k \in \mathbb{N}$ (means $\|f\|_{\mathcal{M},g(K),l_1,h_1} < +\infty$). Furthermore there exist $C_2, h_2 > 0$ and $l_2 \in \Lambda$ such that $\|g^{(k)}(x)\|_{L^k(E,F)} \le C_2 \cdot h_2^k \cdot M_k^{l_2} = C_2 \cdot h_2^k \cdot k! \cdot m_k^{l_2}$ for all $x \in K$ and $k \in \mathbb{N}$ (means $\|g\|_{\mathcal{M},K,l_2,h_2} < +\infty$). We estimate now as follows, where the sum below is taken over all multi-indices $\alpha = (\alpha_1, \dots, \alpha_j) \in (\mathbb{N}_{>0})^j$ with $\alpha_1 + \dots + \alpha_j = k$:

$$\frac{\|(f \circ g)^{(k)}(x)\|_{L^k(E,G)}}{k!} \le \sum_{j=1}^{k} \sum_{\alpha=(\alpha_1,\dots,\alpha_j)\in(\mathbb{N}_{>0})^j} \frac{\|f^{(j)}(g(x))\|_{L^j(F,G)}}{j!} \cdot \prod_{i=1}^{j} \frac{\|g^{(\alpha_i)}(x)\|_{L^{\alpha_i}(E,F)}}{\alpha_i!}$$

$$\le \sum_{j=1}^{k} \sum_{\alpha=(\alpha_1,\dots,\alpha_j)\in(\mathbb{N}_{>0})^j} C_1 \cdot h_1^j \cdot m_j^{l_1} \cdot \prod_{i=1}^{j} C_2 \cdot h_2^{\alpha_i} \cdot m_{\alpha_i}^{l_2}$$

$$\underbrace{\le}_{(\star)} \sum_{j=1}^{k} \sum_{\alpha=(\alpha_1,\dots,\alpha_j)\in(\mathbb{N}_{>0})^j} C_1 \cdot h_1^j \cdot m_j^{l} \cdot \prod_{i=1}^{j} C_2 \cdot h_2^{\alpha_i} \cdot m_{\alpha_i}^{l}$$

$$= C_1 \cdot h_2^k \cdot \sum_{j=1}^{k} \sum_{\alpha=(\alpha_1,\dots,\alpha_j)\in(\mathbb{N}_{>0})^j} (h_1 \cdot C_2)^j \cdot m_j^l \cdot \prod_{i=1}^{j} m_{\alpha_i}^l$$

$$\underbrace{\le}_{(\mathcal{M}_{\{\mathrm{FdB}\}})} C_1 \cdot h_2^k \cdot \sum_{j=1}^{k} \sum_{\alpha=(\alpha_1,\dots,\alpha_j)\in(\mathbb{N}_{>0})^j} m_k^i \cdot (h_1 \cdot C_2 \cdot D)^j$$

$$= C_1 \cdot h_2^k \cdot m_k^i \cdot \sum_{j=1}^{k} \binom{k-1}{j-1} (h_1 \cdot C_2 \cdot D)^j = (C_1 C_2 D h_1) \cdot h_2^k \cdot m_k^i \cdot \sum_{j=1}^{k} \binom{k-1}{j-1} (h_1 C_2 D)^{j-1}$$

$$= (C_1 C_2 D h_1) \cdot h_2^k \cdot (1 + h_1 C_2 D)^{k-1} \cdot m_k^i \le (C_1 C_2 D h_1) \cdot (h_2 \cdot (1 + h_1 C_2 D))^k \cdot m_k^i.$$

$(\star)$ holds for $l := \max\{l_1, l_2\}$ and so we have shown $\|f \circ g\|_{\mathcal{M},K,i,h_3} < +\infty$ with $h_3 := h_2 \cdot (1 + h_1 C_2 D)$.

(b) We can use for the Beurling-case precisely the same estimate as in the above Roumieu-case, of course we use $(\mathcal{M}_{(\mathrm{FdB})})$ instead of $(\mathcal{M}_{\{\mathrm{FdB}\}})$. For arbitrary $i \in \Lambda$ and $t > 0$ (both small) we choose $s > 0$ in such a way that $t = \sqrt{s} + s$ holds and put then $h_2 := \sqrt{s}$ and $h_1 := \frac{\sqrt{s}}{C_2 \cdot D}$. It follows that $h_2 \cdot (1 + h_1 C_2 D)) = \sqrt{s} \cdot (1 + \sqrt{s}) = \sqrt{s} + s = t$ and so we have shown $\|f \circ g\|_{\mathcal{M},K,i,t} < +\infty$. □

Remark: The proof of Theorem 8.3.1 still holds if $\mathcal{M}$ is an arbitrary (large) set of sequences and $\mathcal{E}_{(\mathcal{M})}$ resp. $\mathcal{E}_{\{\mathcal{M}\}}$ are considered like in the most general definition (7.2.2) resp. (7.2.1) with the following assumptions on $\mathcal{M}$: We have to assume $(\mathcal{M}_{(\mathrm{FdB})})$ and $(\mathcal{M}_{(\mathrm{fil})})$ in the Beurling- resp. $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and $(\mathcal{M}_{\{\mathrm{fil}\}})$ in the Roumieu-case.

We concentrate now on the Roumieu-case, the next result is analogous to [39, Theorem 2, Theorem 3], see also [35, 4.9. Theorem]:

Lemma 8.3.2. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with additionally conditions $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ and $(\mathcal{M}_{\{\mathrm{dc}\}})$. Then the following statements are equivalent:*

(1) *$\mathcal{E}_{\{\mathcal{M}\}}$ is closed under composition.*

(2) *$\mathcal{E}_{\{\mathcal{M}\}}$ is holomorphically resp. analytically closed.*

(3) *$\mathcal{E}_{\{\mathcal{M}\}}$ is inverse closed.*

(4) *Property $(\mathcal{M}_{\{\mathrm{ai}\}})$ holds.*

(5) *Property $(\mathcal{M}_{\{\mathrm{FdB}\}})$ holds.*

Property $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ is only needed for (1) $\Rightarrow$ (2), *$(\mathcal{M}_{\{\mathrm{dc}\}})$ only for* (4) $\Rightarrow$ (5).

Proof. (1) $\Rightarrow$ (2) By property $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ the class of all real analytic function $\mathcal{C}^\omega$ is contained in $\mathcal{E}_{\{\mathcal{M}\}}$, so (2) holds by closedness under composition as assumed for the class $\mathcal{E}_{\{\mathcal{M}\}}$.
(2) $\Rightarrow$ (3) is clearly satisfied since $x \mapsto \frac{1}{x}$ is a holomorphic (resp. real-analytic) mapping for $x \neq 0$.
(3) $\Rightarrow$ (4) To prove this implication we use the technique as in [39, Theorem 1, $(c) \Rightarrow (a)$]: By assumption for each $l \in \Lambda$ we can find a characteristic function $\tilde{\theta}_l \in \mathcal{E}^{\mathrm{global}}_{\{M^l\}}(\mathbb{R},\mathbb{C}) \subseteq \mathcal{E}^{\mathrm{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ with $\tilde{\theta}_l^{(j)}(0) = (\sqrt{-1})^j \cdot s^l_j$ and $s^l_j \geq M^l_j$ for all $j \in \mathbb{N}$ (see (2.4.7)). So let $l \in \Lambda$ be arbitrary but from now on fixed, since $\|\tilde{\theta}_l\|_\infty < +\infty$ we can choose a real number H such that $H > 1 + \|\tilde{\theta}_l\|_\infty$ holds. Then $H - \|\tilde{\theta}_l\|_\infty > 1$, so $H - \tilde{\theta}_l$ doesn't vanish (on $\mathbb{R}$) and moreover $H - \tilde{\theta}_l \in \mathcal{E}^{\mathrm{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$. Hence by assumption (inverse closedness) we also have $x \mapsto \frac{1}{H-\tilde{\theta}_l(x)} \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$. If we write $\psi(x) := \frac{1}{H-\tilde{\theta}_l(x)}$ we can find an index $n \in \Lambda$ and numbers $C, h > 0$ such that

$$\|\psi\|_{\mathcal{M},[-1,1],n,h} \leq C.$$

Then use the *Faà-di-Bruno-formula* for the composed function $x \mapsto \psi(x)$, more precisely $x \mapsto (H - \tilde{\theta}_l(x)) \mapsto \frac{1}{H-\tilde{\theta}_l(x)}$, at the point $x = 0$ as follows: For any $k \geq 1$ we get

$$\frac{\psi^{(k)}(0)}{k!} = \sum_{j=1}^{k} \sum_{\alpha_1+\dots+\alpha_j=k, \alpha_i \in \mathbb{N}_{>0}} \frac{1}{(H-\tilde{\theta}_l(0))^{j+1}} \cdot \prod_{i=1}^{j} \underbrace{\frac{\tilde{\theta}_l^{(\alpha_i)}(0)}{\alpha_i!}}_{=(\sqrt{-1})^{\alpha_i} \cdot s^l_{\alpha_i} \cdot (\alpha_i!)^{-1}}$$

$$= (\sqrt{-1})^k \cdot \sum_{j=1}^{k} \sum_{\alpha_1+\dots+\alpha_j=k, \alpha_i \in \mathbb{N}_{>0}} \frac{1}{(H-\tilde{\theta}_l(0))^{j+1}} \cdot \prod_{i=1}^{j} \frac{s^l_{\alpha_i}}{\alpha_i!}.$$

So we obtain:

$$C\cdot h^k\cdot m_k^n\ge\frac{|\psi^{(k)}(0)|}{k!}=\sum_{j=1}^k\sum_{\alpha_1+\dots+\alpha_j=k,\alpha_i\in\mathbb{N}_{>0}}\frac{1}{(H-\tilde{\theta}_l(0))^{j+1}}\cdot\prod_{i=1}^j\frac{s^l_{\alpha_i}}{\alpha_i!}$$
$$\ge\sum_{j=1}^k\sum_{\alpha_1+\dots+\alpha_j=k,\alpha_i\in\mathbb{N}_{>0}}\frac{1}{(H-\tilde{\theta}_l(0))^{j+1}}\cdot\prod_{i=1}^j\underbrace{\frac{M^l_{\alpha_i}}{\alpha_i!}}_{=m^l_{\alpha_i}}\ge\frac{1}{(H-\tilde{\theta}_l(0))^{k+1}}\cdot\prod_{i=1}^j m^l_{\alpha_i}.$$

Note: The last inequality holds, since $\tilde{\theta}_l(0)=s^l_0\ge M^l_0=1$ and $H-\tilde{\theta}_l(0)>1$, so each occurring summand is a positive real number.
If $\alpha_1=\dots=\alpha_j=p\ge 1$, then we have shown $(m^l_p)^j\le C_1\cdot h_1^{jp}\cdot m^n_{pj}$ and so $(m^l_p)^{1/p}\le C_2\cdot(m^n_{pj})^{1/(pj)}$, the desired property for j and $k=jp$. For the more general case $1\le p\le k$ we choose $j\in\mathbb{N}$ with $jp\le k<(j+1)p$ and proceed as in the above proof for (3) resp. (4) in 8.2.3 (the mapping $k\mapsto(M^l_k)^{1/k}$ is increasing for each $l\in\Lambda$ by assumption).
(4) $\Rightarrow$ (5) By assumption we have property $(\mathcal{M}_{\{\mathrm{dc}\}})$, hence by (2) in 8.2.3 we get $(\mathcal{M}_{\{\mathrm{FdB}\}})$.
(5) $\Rightarrow$ (1) This is exactly (a) in 8.3.1. □

By using the same trick as in (3) $\Rightarrow$ (4) in 8.3.2 we can prove:

Lemma 8.3.3. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $(\mathcal{M}_{\{\mathrm{dc}\}})$ and assume that the class $\mathcal{E}_{\{\mathcal{M}\}}$ is closed under solving ODE's. Then condition $(\mathcal{M}_{\{\mathrm{ai}\}})$ is satisfied.*

Proof. By assumption for each $l\in\Lambda$ there exists a characteristic function $\tilde{\theta}_l\in\mathcal{E}^{\mathrm{global}}_{\{M^l\}}(\mathbb{R},\mathbb{C})\subseteq\mathcal{E}^{\mathrm{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ (see 2.4.7), so $|\theta_l^{(j)}(0)|\ge M^l_j$ for all $j\in\mathbb{N}$. We consider now the following ODE:

$$x'(t)=\tilde{\theta}'_l(t)\cdot x^2(t),\qquad x(0)=x_0>0.$$

Since $(\mathcal{M}_{\{\mathrm{dc}\}})$ holds we see that the right hand side belongs to $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}^2,\mathbb{C})$ (in variables $x,t\in\mathbb{R}$). This ODE has the solution $x(t)=\frac{1}{C-\tilde{\theta}_l(t)}$ with the constant $C=\frac{1}{x_0}+\tilde{\theta}_l(0)$. We want to have $C>1+\|\tilde{\theta}_l\|_\infty$, so we choose now x_0 in the ODE small enough to have $x_0<\frac{1}{1+\|\tilde{\theta}_l\|_\infty-\tilde{\theta}_l(0)}=1$. By assumption (closedness under solving ODE's) also the solution $t\mapsto\frac{1}{\tilde{\theta}_l(t)+C}$ is now an element of class $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ and finally we proceed analogously as in (3) $\Rightarrow$ (4) in 8.3.2 to obtain property $(\mathcal{M}_{\{\mathrm{ai}\}})$. □

More general we are able to prove for both cases simultaneously:

Lemma 8.3.4. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $\Lambda=\mathbb{R}_{>0}$ and $(\mathcal{M}_{[\mathrm{dc}]})$. Assume that the class $\mathcal{E}_{[\mathcal{M}]}$ is closed under solving ODE's. Then $\mathcal{E}_{[\mathcal{M}]}$ is inverse closed and consequently condition $(\mathcal{M}_{[\mathrm{ai}]})$ is satisfied.*

Proof. Let $f\in\mathcal{E}_{[\mathcal{M}]}(\mathbb{R},\mathbb{R})$ be given, such that $f(x)\neq 0$ for all $x\in\mathbb{R}$ and put $g:=\frac{1}{f}$. Then $g'=\frac{f'}{f^2}$ holds and so g satisfies the following ODE

$$x'(t)=-f'(t)\cdot x^2(t),\qquad x(0)=\frac{1}{f(0)}.$$

Since $\mathcal{E}_{[\mathcal{M}]}$ is derivation closed by $(\mathcal{M}_{[\mathrm{dc}]})$, we see that the right hand side belongs to class $\mathcal{E}_{[\mathcal{M}]}$, hence by closedness under solving ODE's the solution $g=\frac{1}{f}$, too.

In the Roumieu-case property $(\mathcal{M}_{\{\mathrm{ai}\}})$ follows from (3) $\Rightarrow$ (1) in 8.3.2, in the Beurling-case property $(\mathcal{M}_{(\mathrm{ai})})$ follows from 8.3.6. $\square$

We concentrate now on the Beurling-case $\mathcal{E}_{(\mathcal{M})}$ and we start with the following theorem (see also [35, 4.11. Theorem]):

Theorem 8.3.5. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$-weight matrix with $\Lambda = \mathbb{R}_{>0}$ and properties $(\mathcal{M}_{(\mathrm{dc})})$ and $(\mathcal{M}_{\mathcal{H}})$, then the following are equivalent:*

(1) *$\mathcal{E}_{(\mathcal{M})}$ is closed under composition.*

(2) *$\mathcal{E}_{(\mathcal{M})}$ is holomorphically closed.*

(3) *Condition $(\mathcal{M}_{(\mathrm{ai})})$ is satisfied.*

(4) *Condition $(\mathcal{M}_{(\mathrm{FdB})})$ is satisfied.*

Remark: Condition $(\mathcal{M}_{\mathcal{H}})$ will be used only for (1) $\Rightarrow$ (2), $(\mathcal{M}_{(\mathrm{dc})})$ only for (3) $\Rightarrow$ (4) and $\Lambda = \mathbb{R}_{>0}$ only for (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Holds clearly by condition $(\mathcal{M}_{\mathcal{H}})$, since the restrictions of entire functions are contained in $\mathcal{E}_{(\omega)}$.

(2) $\Rightarrow$ (3) We use for this a trick, which is mentioned on [10, page 405]: By assumption we have $\Lambda = \mathbb{R}_{>0}$ and by (7.3.6) the space $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{C})$ is now a Fréchet-Algebra which is holomorphically closed. So we can use [6, 4.1. Theorem $(f) \Rightarrow (a)$] and [30, Theorem 1] to obtain: $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{C})$ is also a *locally m-convex algebra*, i.e. there exists an equivalent system of semi-norms $(q_n)_{n\in\mathbb{N}}$ such that $q_n(f^m) \le (q_n(f))^m$ for all $n, m \in \mathbb{N}$ and $f \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{C})$. Then for each $x \in \Lambda$ and $h > 0$ there exist numbers $C, B, a, h_1 > 0$, $n \in \mathbb{N}$, and finally an index $y \in \Lambda$ such that for all $f \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{C})$ and for all $m \in \mathbb{N}$ we obtain:

$$\|f^m\|_{\mathcal{M},[-1,1],x,h} \le C \cdot q_n(f^m) \le C \cdot q_n(f)^m \le C \cdot B^m \cdot \|f\|^m_{\mathcal{M},[-a,a],y,h_1}, \qquad (8.3.1)$$

where the first and the third inequality hold by the equivalence of the semi-norm-systems, the second inequality holds by the multiplicativity of the system $(q_n)_n$. We apply this estimate now (as in [10, page 405] resp. [6, 4.1. Theorem $(b) \Rightarrow (c)$]) to the function $f_t(s) = \exp(its)$, $s \in \mathbb{R}$, $t \ge 0$ and for $h = 1$. Hence $f_t^{(l)}(s) = (it)^l \cdot \exp(its) = (it)^l \cdot f_t(s)$ and similarly we get $(f_t^m)^{(l)}(s) = (itm)^l \cdot \exp(itms)$.
Note that $\lim_{j\to\infty}(M_j^l)^{1/j} = +\infty$ (which holds for $(\mathcal{M}_{\mathrm{sc}})$ weight matrices by definition) for all $l \in \Lambda$ is sufficient to have $f_t \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{C})$ for all $t \ge 0$. With this choice we obtain for all $m \in \mathbb{N}$:

$$\begin{aligned}\sup_{l\in\mathbb{N}} \frac{(tm)^l}{M_l^x} &= \sup_{s\in[-1,1],l\in\mathbb{N}} \frac{\left|(f_t^m)^{(l)}(s)\right|}{M_l^x} = \|f_t^m\|_{\mathcal{M},[-1,1],x,1} \\ &\le C \cdot B^m \cdot \|f_t\|^m_{\mathcal{M},[-a,a],y,h_1} = C \cdot B^m \cdot \left(\sup_{s\in[-a,a],l\in\mathbb{N}} \frac{\left|f_t^{(l)}(s)\right|}{h_1^l \cdot M_l^y}\right)^m \\ &= C \cdot B^m \cdot \left(\sup_{l\in\mathbb{N}} \frac{t^l}{h_1^l \cdot M_l^y}\right)^m .\end{aligned}$$

Hence by definition of the weight sequences M^x and the associated function this translates into:

$$\exp(\omega_{M^x}(tm)) \le C \cdot B^m \cdot \exp\left(\omega_{M^y}\left(\frac{t}{h_1}\right)\right)^m .$$

Let $j \le k$ and assume that $k = j \cdot l$ for some $l \in \mathbb{N}$. Then we obtain by above:

$$\begin{aligned}(\exp(\omega_{M^x}(t)))^{1/k} &= \left(\exp\left(\omega_{M^x}\left(\frac{lt}{l}\right)\right)\right)^{1/k} \le C^{1/k} \cdot B^{1/j} \cdot \left(\exp\left(\omega_{M^y}\left(\frac{t}{lh_1}\right)\right)\right)^{1/j} \\ &\le D \cdot \left(\exp\left(\omega_{M^y}\left(\frac{t}{lh_1}\right)\right)\right)^{1/j}\end{aligned}$$

for a constant $D > 0$, and finally by using [16, Proposition 3.2.]

$$\begin{aligned}(M_k^x)^{1/k} &= \sup_{t \ge 0} \frac{t}{(\exp(\omega_{M^x}(t)))^{1/k}} \ge \sup_{t \ge 0} \frac{t}{D \cdot \left(\exp(\omega_{M^y}(\frac{t}{lh_1}))\right)^{1/j}} \\ &= \frac{lh_1}{D} \cdot \sup_{s \ge 0} \frac{s}{(\exp(\omega_{M^y}(s)))^{1/j}} = \frac{lh_1}{D} \cdot (M_j^y)^{1/j},\end{aligned}$$

where we have put $s := \frac{t}{lh_1}$.
In the general case we proceed as for (3) resp. (4) in 8.2.3: Consider for $j \le k$ a number $l \in \mathbb{N}_{>0}$ with $jl \le k < (l+1)j$. Since each M^x is log. convex the mapping $k \mapsto (M_k^x)^{1/k}$ is increasing and so we get:

$$(M_k^x)^{1/k} \ge (M_{jl}^x)^{1/(jl)} \ge \frac{lh_1}{D} \cdot (M_j^y)^{1/j} \ge \frac{h_1(l+1)}{2D} \cdot (M_j^y)^{1/j}.$$

In the last step we have to use *Stirling's formula* and get that there exists a constant $A \ge 1$ such that $(m_j^y)^{1/j} \le A \cdot (m_k^x)^{1/k}$, for $j \le k$, this is property $(\mathcal{M}_{(\mathrm{ai})})$.

(3) $\Rightarrow$ (4) Since we have assumed $(\mathcal{M}_{(\mathrm{dc})})$ we can use (2) in 8.2.3.
(4) $\Rightarrow$ (1) Is exactly (b) in 8.3.1. □

Another application of the trick used in (2) $\Rightarrow$ (3) in the previous result gives the following:

Lemma 8.3.6. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and assume that $\mathcal{E}_{(\mathcal{M})}$ is inverse closed. Then property $(\mathcal{M}_{(\mathrm{ai})})$ is satisfied.*

Proof. The proof is inspired by [6, 5.2. Proposition] and completely analogous. We write in the following $\mathcal{A} := \mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{C})$ and denote by $B\mathcal{A}$ the sub-algebra of $\mathcal{A}$ consisting of the bounded functions of $\mathcal{A}$. We endow $B\mathcal{A}$ with the topology generated by all semi-norms $\|\cdot\|_{\mathcal{M},K,l,h}$ together with the single norm $\|\cdot\|_B$ defined by $\|f\|_B := \sup\{|f(x)| : x \in \mathbb{R}\}$. The space $B\mathcal{A}$ is a Fréchet-algebra (note that $\mathcal{A}$ is a Fréchet-Algebra, because $\Lambda = \mathbb{R}_{>0}$).
By assumption $\mathcal{A}$ is inverse-closed and the invertible functions of $B\mathcal{A}$ are such functions, which are bounded below, i.e. $|f(x)| \ge c > 0$ for all $x \in \mathbb{R}$. For such a function f and some g with $\|f - g\|_B < \frac{c}{2}$ we obtain $|c - \|g\|_B| \le |\|f\|_B - \|g\|_B| \le \|f - g\|_B < \frac{c}{2}$ which implies $|g(x)| \ge \frac{c}{2} > 0$ and so g is invertible, too. Hence the set of invertible functions in $B\mathcal{A}$ is open and so, by the same argument as given in [6, 5.2. Proposition] we use [43, Theorem 13.17], hence the algebra $B\mathcal{A}$ is *locally m-convex.*
Now we have, as for (8.3.1) in (2) $\Rightarrow$ (3) in 8.3.5, that for each $x \in \Lambda$ and $h > 0$ there exist numbers $C, B > 0$ and a semi-norm p such that for all $f \in B\mathcal{A}$ and for all $m \in \mathbb{N}$ we obtain (as in inequality (8.3.1)):

$$\|f^m\|_{\mathcal{M},[-1,1],x,h} \le C \cdot B^m \cdot p(f)^m, \tag{8.3.2}$$

where p is either a semi-norm $\|\cdot\|_{\mathcal{M},[-a,a],y,h_1}$ for some $a, h_1 > 0$ and $y \in \Lambda$, or $p = \|\cdot\|_B$. Of course we apply this inequality again to the functions $f_t(s) := \exp(its)$ for $t, s \in \mathbb{R}$ and since we have $\|f_t\|_B = 1 \leq \|f_t\|_{\mathcal{M},[-a,a],y,h_1} = \sup_{l\in\mathbb{N}} \frac{t^l}{h_1^l \cdot M_l^y} = \exp(\omega_{M^y}(\frac{t}{h_1}))$ for any t, we can replace in (8.3.2) the semi-norm p in any case by some $\|\cdot\|_{\mathcal{M},[-a,a],y,h_1}$. Then proceed analogously as in (2) $\Rightarrow$ (3) in 8.3.5 to obtain property $(\mathcal{M}_{(\mathrm{ai})})$. $\square$

8.4 The $\mathcal{E}_{[\mathcal{M}]}$-inverse/implicit function theorem

The next step is to prove the $\mathcal{E}_{[\mathcal{M}]}$-inverse/implicit function theorem. The strategy will be to prove the $\mathcal{E}_{[\mathcal{M}]}$-inverse function theorem, the $\mathcal{E}_{[\mathcal{M}]}$-implicit function theorem follows then by standard techniques. More precisely we will generalize [41] also to the non-constant weight matrix case.

Theorem 8.4.1. *Let $\mathcal{M} := \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$-weight matrix.*

(1) *If $(\mathcal{M}_{\{\text{ai}\}})$ holds and $(\mathcal{M}_{\{\text{dc}\}})$, then the $\mathcal{E}_{\{\mathcal{M}\}}$-inverse/implicit function theorem is valid.*

(2) *If $(\mathcal{M}_{(\text{ai})})$ holds and $(\mathcal{M}_{(\text{dc})})$ and finally $(\mathcal{M}_{(\mathcal{C}^\omega)})$, then the $\mathcal{E}_{(\mathcal{M})}$-inverse/implicit function theorem is valid.*

Proof. We will follow and generalize the proof of [41], there global classes (estimates) are considered. We will prove a matrix version of it with local estimates. First we will have to recall the *Lagrange expansion theorem*: Let the equation

$$y = a + x \cdot \varphi(y) \tag{8.4.1}$$

be given, where φ is analytic around the point $a \in \mathbb{R}$. We denote by $\gamma : x \mapsto y$ the solution of (8.4.1), which is analytic by the classical implicit function theorem. Furthermore let f be a smooth function, which is analytic around a, too, then we obtain the following formula for $n \geq 1$:

$$(f \circ \gamma)^{(n)}(0) = \left(\varphi^n \cdot f'\right)^{(n-1)}(a).$$

A proof of this formula can be found in [14, Chapitre IX., no. 195, p. 481-484].

Now we are able to prove a matrix version of [41, Theorem 2] and we have to distinguish between the Roumieu- and the Beurling-case! In the following let E and F always be real Banach-spaces, $U \subseteq E$ and $V \subseteq F$ open subsets and $f : U \to V$ an infinitely differentiable function.

The Roumieu-case: We assume that $f \in \mathcal{E}_{\{\mathcal{M}\}}$ and for a point $x_0 \in U$ we assume that $(f'(x_0))^{-1} \in L(F; E)$ exists. By the inverse mapping theorem for smooth functions we know: There exist U_0, V_0 open subsets in E resp. F such that we have $x_0 \in U_0 \subseteq U$ resp. $f(x_0) \in V_0 \subseteq V$, and $f : U_0 \to V_0$ is a $\mathcal{E}$-diffeomorphism. So it remains to prove that on V_0 we have $f^{-1} \in \mathcal{E}_{\{\mathcal{M}\}}$, too.
Let $g := f^{-1} : V_0 \to U_0$ and let $a \in U_0$, $K \subseteq U_0$ compact be fixed with $a \in K \subseteq U_0$. Put $b := f(a)$ and $S := f'(a)$, $T := S^{-1} = g'(b)$. Furthermore we put for $x \in U_0$:

$$\phi(x) := x - (T \circ f)(x).$$

Hence, for $y \in V_0$, we have $g(y) = T(y) + g(y) - T(y) = T(y) + g(y) - (T \circ f)(g(y)) = T(y) + (\phi \circ g)(y)$. Then we remark, that $\phi'(a) = \text{id} - (T \circ f')(a) = \text{id} - T \circ S = 0$, thus $g'(b) = T + \phi'(g(b)) \cdot g'(b) = T + \phi'(a) \cdot g'(b) = T$. For $p \geq 2$ we can use the *Faà-di-Bruno-formula* to get a recursion formula for $g^{(p)}(b)$:

$$g^{(p)}(b) = \operatorname{sym}\left(p! \cdot \sum_{j=2}^{p} \phi^{(j)}(a) \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot \left(\frac{g^{(i)}(b)}{i!}\right)^{q_i}\right), \tag{8.4.2}$$

where sym denotes the symmetrization of a multi-linear operator and for $q = (q_1, \ldots, q_{p-j+1}) \in \mathbb{N}^{p-j+1}$ we have set $|q| := q_1 + \cdots + q_{p-j+1}$ resp. $\|q\| := q_1 + 2q_2 + \cdots + (p-j+1)q_{p-j+1}$ (for a proof of this formula see e.g. [41, Theorem 1]). So we immediately get the following estimate:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \leq p! \sum_{j=2}^{p} \|\phi^{(j)}(a)\|_{L^j(E;E)} \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \left(\frac{\|g^{(i)}(b)\|_{L^i(F;E)}}{i!} \right)^{q_i}. \tag{8.4.3}$$

By assumption $f \in \mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}_{\{\mathcal{M}^d\}}$ holds where $\mathcal{M}^d = \{M^l_{+1} : M^l \in \mathcal{M}\}$, $M^l_{+1} := (M^l_{k+1})_k$ (note that $(\mathcal{M}_{\{\mathrm{dc}\}})$ is assumed), hence for all compact sets $K \subseteq U_0$ there exist $C, h > 0$ and $l \in \Lambda$ such that for all $x \in K$ and $j \geq 1$ we get

$$\|f^{(j)}(x)\|_{L^j(E;F)} \leq C \cdot h^{j-1} \cdot M^l_{j-1},$$

and finally for $j \geq 2$

$$\|\phi^{(j)}(x)\|_{L^j(E;E)} = \|(T \circ f^{(j)})(x)\|_{L^j(E;F)} \leq C \cdot \|T\|_{L(F;E)} \cdot h^{j-1} \cdot M^l_{j-1}.$$

In the following let the compact set K with $a \in K \subseteq U_0$ be fixed and $C, h > 0$, $l \in \Lambda$ be given by f by the above estimates. Moreover introduce the constant $A \geq \max\left\{\|g(b)\|_E, \|g'(b)\|_{L(F;E)}\right\}$ and for $N \in \mathbb{N}$, $N \geq 2$, we define for $t \in \mathbb{R}$ the following sum:

$$\psi^l_N(t) := C \cdot A \cdot \sum_{j=2}^{N} \frac{h^{j-1} \cdot M^l_{j-1}}{j!} \cdot t^j.$$

Now consider the equation $s = \frac{t - \psi^l_N(t)}{A}$ and we want to solve it with respect to the variable t. Therefore we put $g^l_N(s) := \sum_{j=1}^{\infty} c^l_j \cdot s^j$ and consider now the equation $s = \frac{g^l_N(s) - \psi^l_N(g_N(s))}{A}$ for $s \in \mathbb{R}$ near 0, more precisely we have set $g^l_N(s) = t$ for all small s.

For this we apply the *Lagrange expansion theorem* as follows: We put $f := \mathrm{id}$, $x := s$, $y := t$, $a := 0$ and $\varphi(t) := \frac{A}{1 - \frac{\psi^l_N(t)}{t}} = \frac{t}{s}$ which is analytic near $t = 0$. Then (8.4.1) is satisfied and we obtain for $n \geq 1$:

$$\frac{\partial^n g^l_N(s)}{\partial s^n}\bigg|_{s=0} = \left(\frac{d^{n-1}}{dt^{n-1}} \left(\frac{A}{1 - \psi^l_N(t)/t} \right)^n \right)\bigg|_{t=0}.$$

Hence the coefficients c^l_i, $i \geq 1$, in the above series can be expressed in the following way:

$$c^l_i = \frac{1}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\frac{A}{1 - \psi^l_N(t)/t} \right)^i \right)\bigg|_{t=0}. \tag{8.4.4}$$

On the other side we see that $g^l_N(s) = A \cdot s + (\psi^l_N \circ g^l_N)(s)$, hence for $2 \leq p \leq N$ we can use again the *Faà-di-Bruno-formula* to obtain

$$(g^l_N)^{(p)}(0) = p! \cdot \sum_{j=2}^{p} (\psi^l_N)^{(j)}(0) \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot \left(\frac{(g^l_N)^{(i)}(0)}{i!} \right)^{q_i}. \tag{8.4.5}$$

Since for all $i \in \mathbb{N}_{>0}$ we get $(g_N^l)^{(i)}(0) = i! \cdot c_i^l$ and for $2 \le j \le N$ we have $(\psi_N^l)^{(j)}(0) = C \cdot A \cdot h^{j-1} \cdot M_{j-1}^l$ (and $(\psi_N^l)^{(j)}(0) = 0$ for $j > N$) it follows that for $2 \le p \le N$ we obtain

$$(g_N^l)^{(p)}(0) = p! \cdot \sum_{j=2}^{p} C \cdot A \cdot h^{j-1} \cdot M_{j-1}^l \cdot \sum_{|q|=j, \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i!} \cdot (c_i^l)^{q_i}.$$

For $2 \le i \le N$ we show now for the coefficients c_i^l, which are all positive real numbers, the following estimate:

$$(\forall\, l \in \Lambda)\ \exists\, n \in \Lambda\ \exists\, H \ge 1 :\ c_i^l < A(4A(CAh+1)Hh)^{i-1} \cdot \frac{M_{i-1}^n}{i!}, \tag{8.4.6}$$

where $n \in \Lambda$ and $H \ge 1$ are related to given $l \in \Lambda$ via property $(\mathcal{M}_{\{\mathrm{ai}\}})$. To do so we are going to calculate for $2 \le i \le N$ in the following way:

$$\begin{aligned}
0 < c_i^l &\underbrace{=}_{(1)} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} \left(CA \cdot \sum_{r=1}^{i-1} \frac{M_r^l}{(r+1)!} \cdot (ht)^r \right)^s \right)^i \right) \Bigg|_{t=0} \\
&\le \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} \left(CA \cdot \sum_{r=1}^{i-1} \frac{M_r^l}{r!} \cdot (ht)^r \right)^s \right)^i \right) \Bigg|_{t=0} \\
&\underbrace{\le}_{(\mathcal{M}_{\{\mathrm{ai}\}})} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} \left(CA \cdot \sum_{r=1}^{\infty} \left(H \cdot \left(\frac{M_{i-1}^n}{(i-1)!} \right)^{1/(i-1)} \cdot ht \right)^r \right)^s \right)^i \right) \Bigg|_{t=0} \\
&= \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{s=0}^{\infty} (CA)^s \cdot \sum_{r=0}^{\infty} \binom{r+s-1}{r} \cdot \left(H \cdot \left(\frac{M_{i-1}^n}{(i-1)!} \right)^{1/(i-1)} \cdot ht \right)^{r+s} \right)^i \right) \Bigg|_{t=0} \\
&\underbrace{=}_{(2)} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \left(\sum_{r'=0}^{\infty} CA \cdot (CA+1)^{r'-1} \cdot \left(H \cdot \left(\frac{M_{i-1}^n}{(i-1)!} \right)^{1/(i-1)} \cdot ht \right)^{r'} \right)^i \right) \Bigg|_{t=0} \\
&\le \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}} \sum_{r'=0}^{\infty} \binom{i+r'-1}{r'} \left((CA+1) \cdot H \cdot \left(\frac{M_{i-1}^n}{(i-1)!} \right)^{1/(i-1)} \cdot ht \right)^{r'} \right) \Bigg|_{t=0} \\
&\underbrace{=}_{(3)} \frac{A^i}{i!} \cdot \binom{2i-2}{i-1} \cdot ((CA+1) \cdot (Hh))^{i-1} \cdot (i-1)! \cdot \frac{M_{i-1}^n}{(i-1)!} \\
&\le A^i \cdot 4^{i-1} \cdot ((CA+1) \cdot (Hh))^{i-1} \cdot \frac{M_{i-1}^n}{i!}.
\end{aligned}$$

Recall that the number H in the above calculation is precisely the constant appearing in $(\mathcal{M}_{\{\mathrm{ai}\}})$.
For (1) we have used geometric series expansion and (8.4.4). Furthermore we remember: If $2 \le i \le N$, then $(\psi_N^l)^{(i)}(0) = C \cdot A \cdot h^{i-1} \cdot M_{i-1}^l$ holds and because we consider the $(i-1)^{\mathrm{th}}$-derivative on the right hand side above at the point $t = 0$, we can break off the summation over the index r at $i-1$.
For (2) we have put $r' := r + s$.
(3) follows, because only the term for $r' = i-1$ survives. Finally the last inequality holds, because $4^{i-1} = 2^{2i-2} = \sum_{k=0}^{2i-2} \binom{2i-2}{k} \ge \binom{2i-2}{i-1}$.

In the next step we want to compare $\|g^{(p)}(b)\|_{L^p(F;E)}$ with $(g_N^l)^{(p)}(0)$ for $1 \leq p \leq N$. First we have $(g_N^l)^{'}(0) = c_1^l = A$ (use $i = 1$ in (8.4.4)) and so we see:

$$\|g^{'}(b)\|_{L(F;E)} = \|T\|_{L(F;E)} \leq A = (g_N^l)^{'}(0). \tag{8.4.7}$$

For $2 \leq p \leq N$ we have $(\psi_N^l)^{(p)}(0) = C \cdot A \cdot h^{p-1} \cdot M_{p-1}^l$ and so

$$\|\phi^{(p)}(a)\|_{L^p(E;E)} \leq C \cdot \|T\|_{L(F;E)} \cdot h^{p-1} \cdot M_{p-1}^l \leq C \cdot A \cdot h^{p-1} \cdot M_{p-1}^l = (\psi_N^l)^{(p)}(0). \tag{8.4.8}$$

Finally we get by induction the following estimate for $2 \leq p \leq N$:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \underbrace{\leq}_{(8.4.3),(8.4.5)(8.4.7),(8.4.8)} (g_N^l)^{(p)}(0) = p! \cdot c_p^l \underbrace{<}_{(8.4.6)} A(4A(CA+1)Hh)^{p-1} \cdot M_{p-1}^n.$$

But $N \in \mathbb{N}$, $N \geq 2$, was chosen arbitrary, hence for all $p \geq 2$ we obtain:

$$\|g^{(p)}(b)\|_{L^p(F;E)} \leq A(4A(CA+1)Hh)^{p-1} \cdot M_{p-1}^n.$$

Because $\|g(b)\|_E \leq A$ resp. $\|g^{'}(b)\|_{L(F;E)} = \|T\|_{L(F;E)} \leq A$ we have shown: $g \in \mathcal{E}_{\{M\}}$.

In fact we have shown now the $\mathcal{E}_{\{\mathcal{M}\}}$-*inverse function theorem* and we immediately get also the $\mathcal{E}_{\{M\}}$-*implicit function theorem:* For this we apply the $\mathcal{E}_{\{\mathcal{M}\}}$-*inverse function theorem* to the inverse of the mapping $(x, y) \mapsto (x, f(x, y))$ for $f \in \mathcal{E}_{\{\mathcal{M}\}}$.

The Beurling-case: One cannot transfer the proof of the Roumieu-case directly, because we are interested in small $h > 0$ and the constant C_h is depending on h. For $h \to 0$ we have $C_h \to \infty$ and so $4A(C_h A + 1)Hh$ will not become arbitrary small. To avoid this problem we modify the Roumieu-case proof at several steps and we use an analogous trick as used for the proof of the (single weight sequence) Beurling-case in [18].
First we point out that by $(\mathcal{M}_{(\mathrm{dc})})$ we have $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}_{(\mathcal{M}^d)}$.
We use the same notation for f, T and g as in the Roumieu-case. Let $h > 0$ and $l \in \Lambda$ be arbitrary (small) but from now on fixed. By assumption there exist a constant $C_{l,h} > 0$ (large) such that for $j \geq 2$

$$\|\phi^{(j)}(x)\|_{L^j(E;E)} = \|(T \circ f^{(j)})(x)\|_{L^j(E;F)} \leq C_{l,h} \cdot \|T\|_{L(F;E)} \cdot \left(\frac{h}{2}\right) h^{j-1} \cdot M_{j-1}^l.$$

Then choose $p_{l,h} \in \mathbb{N}$ large enough such that $C_{l,h} \cdot \frac{1}{2^{p_{l,h}}} \leq 1$ holds. With this choice we immediately get

$$\|\phi^{(j)}(x)\|_{L^j(E;E)} \leq C_{l,h} \cdot \|T\|_{L(F;E)} \cdot h^{j-1} \cdot M_{j-1}^l \ \text{ for } 2 \leq j \leq p_{l,h}$$

$$\|\phi^{(j)}(x)\|_{L^j(E;E)} \leq \|T\|_{L(F;E)} \cdot h^{j-1} \cdot M_{j-1}^l \ \text{ for } j > p_{l,h}.$$

In the next step we put (for sufficiently large N depending on l and h)

$$\psi_N^l(t) := C_{l,h} \cdot A \cdot \sum_{j=2}^{p_{l,h}} \frac{h^{j-1} \cdot M_{j-1}^l}{j!} \cdot t^j + A \cdot \sum_{j=p_{l,h}+1}^{N} \frac{h^{j-1} \cdot M_{j-1}^l}{j!} \cdot t^j,$$

where $A \geq \max\{\|g(b)\|_E, \|g^{'}(b)\|_E\}$ is the constant as in the Roumieu-case.

Now, as in the Roumieu-case, we apply *Lagrange expansion theorem* for the equation $s = \frac{t - \psi_N^l(t)}{A}$ and introduce again g_N^l. Note that $(\psi_N^l)^{(j)}(0) = C_{l,h} \cdot A \cdot h^{j-1} \cdot M_{j-1}^l$ for

$2 \le j \le p_{l,h}$, $(\psi^l_N)^{(j)}(0) = A \cdot h^{j-1} \cdot M^l_{j-1}$ for $p_{l,h} < j \le N$ and finally $(\psi^l_N)^{(j)}(0) = 0$ for $j > N$.
For the estimate of the coefficients c^l_i we use our additional assumption $(\mathcal{M}_{(\mathcal{C}^\omega)})$, i.e. $\lim_{j\to\infty}(m^l_j)^{1/j} = +\infty$ for each $l \in \Lambda$, which has the following consequence : For all $l \in \Lambda$, for each $C > 0$ (large) and $p_0 \in \mathbb{N}$ we can find $i_0 \in \mathbb{N}$ large enough such that

$$C \cdot m^l_p \le \left((m^l_i)^{1/i}\right)^p$$

holds for all $1 \le p \le p_0$ and $i \ge i_0$. Combining this with property $(\mathcal{M}_{(\mathrm{ai})})$ we obtain that for each $C > 0$ (large) and $p_0 \in \mathbb{N}$ we can find $i_0 \in \mathbb{N}$ large enough such that

$$C \cdot m^l_p \le \left(H \cdot (m^{l_1}_i)^{1/i}\right)^p \quad (\star)$$

holds for all $1 \le p \le p_0$ and $i \ge i_0$, where the indices $l, l_1 \in \Lambda$ are related by $(\mathcal{M}_{(\mathrm{ai})})$ with the constant $H \ge 1$ there.
Now we apply $(\star)$ to $C = C_{l,h}$ and $p_0 = p_{l,h}$ and calculate for $2 \le i \le N$ with $i \ge i_0$ (so one has to consider N large enough) as follows:

$$\begin{aligned}
0 < c^l_i &= \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}}\left(\sum_{s=0}^{\infty}\left(CA \cdot \sum_{r=1}^{p_0} \frac{M^l_r}{(r+1)!}\cdot (ht)^r + A \cdot \sum_{r=p_0+1}^{i-1} \frac{M^l_r}{(r+1)!}\cdot (ht)^r\right)^s\right)^i\right)\Bigg|_{t=0} \\
&\le \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}}\left(\sum_{s=0}^{\infty}\left(CA \cdot \sum_{r=1}^{p_0} \frac{M^l_r}{r!}\cdot (ht)^r + A \cdot \sum_{r=p_0+1}^{i-1} \frac{M^l_r}{r!}\cdot (ht)^r\right)^s\right)^i\right)\Bigg|_{t=0} \\
&\underbrace{\le}_{(\mathcal{M}_{(\mathrm{ai})}),(\star)} \frac{A^i}{i!} \cdot \left(\frac{d^{i-1}}{dt^{i-1}}\left(\sum_{s=0}^{\infty}\left(A \cdot \sum_{r=1}^{\infty}\left(H \cdot \left(\frac{M^{l_1}_{i-1}}{(i-1)!}\right)^{1/(i-1)} \cdot ht\right)^r\right)^s\right)^i\right)\Bigg|_{t=0} \\
&= \underbrace{\ldots}_{\text{see Roumieu-case}} \le A^i \cdot 4^{i-1} \cdot ((A+1)\cdot(Hh))^{i-1} \cdot \frac{M^{l_1}_{i-1}}{i!}.
\end{aligned}$$

Hence for given $h_1 > 0$ small, we can choose h small enough to have $4A(A+1)Hh \le h_1$.
Recall: $(\psi^l_N)^{(j)}(0) = C_{l,h} \cdot A \cdot h^{j-1} \cdot M^l_{j-1}$ for $2 \le j \le p_{l,h}$, $(\psi^l_N)^{(j)}(0) = A \cdot h^{j-1} \cdot M^l_{j-1}$ for $p_{l,h} < j \le N$ and finally $(\psi^l_N)^{(j)}(0) = 0$ for $j > N$. Thus, by definition of the function ϕ we obtain for $p \in \mathbb{N}$ with $2 \le p \le N$ and $l \in \Lambda$ small (and $N \in \mathbb{N}$ suff. large dep. on $l \in \Lambda$ and $h > 0$) analogously to (8.4.8):

$$\|\phi^{(p)}(a)\|_{L^p(E;E)} \le (\psi^l_N)^{(p)}(0). \tag{8.4.9}$$

Thus we can proceed and estimate as in the Roumieu-case and note that we are only interested in large N since one can absorb the cases $2 \le p \le p'$, $p' \in \mathbb{N}$ fixed, by a sufficiently large constant. Another possibility is to put $\psi^l_N(t) := C_{l,h} \cdot A \cdot \sum_{j=2}^{p_{l,h}} \frac{h^{j-1}\cdot M^l_{j-1}}{j!} \cdot t^j$ for $N \le p_{l,h}$ (only the first sum in the above definition of $\psi^l_N(t)$) and use then the same arguments and estimates as before. □

8.5 Closedness under solving ODE's

In the next step we want to prove closedness under solving ODE's for classes $\mathcal{E}_{[\mathcal{M}]}$ and we first we summarize some remarks:

(i) Proofs of this result for the single weight sequence Roumieu-case can be found e.g. in [19] or in [42]. In [19] also the single weight sequence Beurling-case was treated, more precisely [17, Lemma 6] was used to transfer closedness under solving ODE's from the Roumieu- to the Beurling-case.

(ii) We use in the following the proof of [42] and transfer it to the Roumieu-matrix-case. If one wants to apply an analogous trick as in [19] to transfer the proof from the Roumieu- to the Beurling-type weight matrix case, we would have to use 9.4.3. But it's hopeless to get property (ai) for the constructed sequences L there, even if we assume $(\mathcal{M}_{[\mathrm{ai}]})$ for the matrix $\mathcal{M}$, and similarly for any other properties (see 9.2.1 and the remark there).

(iii) The theorem in [42], and so also our general weight-matrix version 8.5.1 below, uses in the assumption (8.5.2) for f global estimates on open subsets $W \subseteq E$ in a real Banach space E. But if $E = \mathbb{R}^n$ for some $n \in \mathbb{N}_{>0}$ (the finite dimensional case) then we can replace this assumption by the hypothesis that f in (8.5.2) satisfies the estimates on all compact sets $K \subseteq W$ locally. It's clear that global estimates imply locally ones. On the other hand, if $U \subseteq \mathbb{R}^n$ is non-empty open and $f \in \mathcal{E}_{[\mathcal{M}]}(U)$ satisfies (8.5.2) locally on all compact sets $K \subseteq U$, then we put $W := \overset{o}{K}$ (the interior of K) and so f satisfies now (8.5.2) (resp. the Beurling-type inequalities) on the whole open set W and we can use then Theorem 8.5.1.

Now we formulate and prove the main theorem of this section:

Theorem 8.5.1. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with index set $\Lambda = \mathbb{R}_{>0}$.*

(1) *The Roumieu-case: If additionally properties $(\mathcal{M}_{\{\mathrm{dc}\}})$ and $(\mathcal{M}_{\{\mathrm{ai}\}})$ hold, then $\mathcal{E}_{\{\mathcal{M}\}}$ is closed under solving ODE's. More precisely we are going to prove the following: For real Banach-spaces E_1 and E_2 we consider an infinitely differentiable function $f : E_1 \times \mathbb{R} \times E_2 \supseteq W \to E_1$, W an open subset, and consider the initial value problem*

$$\begin{cases} x'(t) = f(x(t), t, \lambda) \\ x(0) = x_0, \quad x_0 \in E_1. \end{cases} \tag{8.5.1}$$

If there exist $C, h > 0$ and an index $l \in \Lambda$ such that

$$\|f^{(i,j,k)}(x, t, \lambda)\|_{L^{i,j,k}(E_1,\mathbb{R},E_2;E_1)} \leq C \cdot h^{i+j+k} \cdot M^l_{i+j+k-1} \tag{8.5.2}$$

for all $(i, j, k) \in \mathbb{N}^3$ and $(x, t, \lambda) \in W$, then the solution x is also an element of class $\mathcal{E}_{\{\mathcal{M}\}}$ wherever it exists.

(2) *The Beurling-case: If $\mathcal{M}$ has $(\mathcal{M}_{(\mathrm{dc})})$ and $(\mathcal{M}_{(\mathrm{ai})})$, then we get:*

   (a) *If the matrix $\mathcal{M}$ is not constant and also property $(\mathcal{M}_{(\mathrm{BR})})$ holds (which cannot be satisfied for a constant matrix $\mathcal{M} = \{M\}$!), then $\mathcal{E}_{(\mathcal{M})}$ is closed under solving ODE's, too.*

(b) *If the matrix is constant, so $\mathcal{M} = \{M\}$ and if in addition condition $(\mathcal{M}_{(\mathcal{C}^\omega)})$ holds, i.e. $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$, $m_k := \frac{M_k}{k!}$, then $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}_{(M)}$ is closed under solving ODE's, too.*

Proof. We prove a "matrix-generalized" result of [42], so we are dealing again with Fréchet-derivatives.
(1) We can start now with the proof of the *Roumieu-case*, where we can restrict ourselves by assumption to $\Lambda = \mathbb{N}_{>0}$ and recall that by $(\mathcal{M}_{\{\mathrm{dc}\}})$ we have $\mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}_{\{\mathcal{M}^d\}}$.
Now we formulate our central problem: For this let E_1, E_2 be two real Banach-spaces, then let $f : E_1 \times \mathbb{R} \times E_2 \supseteq W \to E_1$ be an infinitely differentiable function such that: $\exists\, l \in \Lambda\ \exists\, C, h > 0\ \forall\, (i,j,k) \in \mathbb{N}^3\ \forall\, (x,t,\lambda) \in W :$

$$\|f^{(i,j,k)}(x,t,\lambda)\|_{L^{i,j,k}(E_1,\mathbb{R},E_2;E_1)} \le C \cdot h^{i+j+k} \cdot M^l_{i+j+k-1}. \tag{8.5.3}$$

Let the following ODE be given:

$$\begin{cases} x'(t) = f(x(t),t,\lambda) \\ x(0) = x_0, \quad x_0 \in E_1. \end{cases} \tag{8.5.4}$$

We remark that (8.5.4) defines a more general initial value problem than the ODE in [19] because we have introduced here an additional parameter λ. First we simplify the above given initial value problem as in the original proof in [42]: Put $f_1(x(t),t,\lambda,x_0) := f(x(t)+x_0,t,\lambda)$ to get

$$\begin{cases} x'(t) = f_1(x(t),t,\lambda,x_0) \\ x(0) = 0. \end{cases} \tag{8.5.5}$$

Then define $y := (x,\xi)$, $\mu := (\lambda,x_0)$ and finally $g(y,\mu) := (f_1(x,\xi,\lambda,x_0),1) = (f(x+x_0,\xi,\lambda),1)$. Thus we obtain the following new ODE:

$$\begin{cases} y'(t) = g(y(t),\mu) \\ y(0) = 0, \end{cases} \tag{8.5.6}$$

where $y : \mathbb{R} \to E_1 \times \mathbb{R} =: E_3$ is the unknown function and $\mu \in E_2 \times E_1 =: E_4$ is a parameter. If f satisfies (8.5.3) with respect to (x,t,λ), then g satisfies (8.5.3) with respect to (y,μ). If the solution y of (8.5.6) satisfies (8.5.3) with respect to (t,μ), then the solution x of (8.5.4) satisfies (8.5.3) with respect to (t,λ,x_0).

From now on assume w.l.o.g. that $M^l_{-1} = M^l_0 = 1$, $M^l_1 \ge 2$ holds for all $l \in \Lambda$.

Consider the initial value problem (8.5.6), let E_3 and E_4 be real Banach-spaces, U a ball of radius R centered at $0 \in E_3$ and let V be an open set of E_4. Let $g : E_3 \times E_4 \supseteq U \times V \to E_3$, $(y,\mu) \mapsto g(y,\mu)$, be an infinitely differentiable function with the following property: $\exists\, C, h \ge 1$ and $l \in \Lambda = \mathbb{N}_{>0}$ such that for all $(i,j) \in \mathbb{N}^2$ and $(y,\mu) \in U \times V$ we have the estimate

$$\left\|g^{(i,j)}(y,\mu)\right\|_{L^{i,j}(E_3,E_4;E_3)} \le C \cdot h^{i+j} \cdot M^l_{i+j-1}. \tag{8.5.7}$$

(8.5.7) has the following consequences: The function $y \mapsto g(y,\mu)$ is bounded in the norm $\|\cdot\|_{E_3}$ by the constant C and it is *Lipschitz-continuous* on U with respect to y and uniformly for $\mu \in V$. If $|t| \le \frac{R}{C}$, then g is bounded by R, so $g(y,\mu) \in U$. Thus for each $\mu \in V$ and $t \in \mathbb{R}$ with $|t| \le \frac{R}{C}$ the classical existence theorem for ODE's implies

the local existence of a unique solution y where $y : \mathbb{R} \times E_4 \to E_3$, $(t, \mu) \mapsto y(t, \mu)$, is defined for $|t| \leq \frac{R}{C}$ and $\mu \in V$. It's well known that y is infinitely differentiable in (t, μ), thus to prove theorem 8.5.1 we have to estimate the derivatives of y.
More precisely the aim is to prove: There exist constants $B, \xi > 0$ and an index $n \in \Lambda = \mathbb{N}_{>0}$ such that for all $(t, \mu) \in I_T \times V = [-T, T] \times V$ (for $T > 0$ small enough dep. on f) we have

$$\left\| y^{(i,j)}(t, \mu) \right\|_{L^{i,j}(\mathbb{R}, E_4; E_3)} \leq B \cdot \xi^{j+k} \cdot M^n_{i+j-1} \tag{8.5.8}$$

for all $(i, j) \in \mathbb{N}^2$.
We start now with the proof of (8.5.8) and in the first step we estimate $y^{(0,j)}$, $j \in \mathbb{N}$ arbitrary.
Let I be a compact interval such that $0 \in I$ and let $\mathcal{C}(I, E_3)$ be the space of all continuous functions from I to E_3. The space $\mathcal{C}(I, E_3)$ endowed with the topology induced by the maximum norm becomes a Banach-spaces and we put $\mathcal{C}(I, U) := \{y \in \mathcal{C}(I, E_3) : y(I) \subseteq U\}$. Let $(y, \mu, t) \in \mathcal{C}(I, U) \times V \times I$, then we define the following integral

$$\alpha(y, \mu)(t) := \int_0^t g(y(\tau), \mu) d\tau$$

and obtain a function $\alpha : \mathcal{C}(I, U) \times V \to \mathcal{C}(I, E_3)$. Let $T \in \mathbb{R}$ be given with $0 < T < \frac{R}{C}$ and put $I_T := [-T, T]$. We define the set $\mathcal{B}_{E_3,R} := \{y \in E_3 : \|y\|_{E_3} \leq R\}$ and so $\mathcal{B}^\circ_{E_3,R} := \{y \in E_3 : \|y\|_{E_3} < R\} = U$. We want to apply *Banach's fix point theorem* to the mapping α.

Claim: $\alpha : \mathcal{C}(I_T, \mathcal{B}_{E_3,R}) \times V \to \mathcal{C}(I_T, \mathcal{B}_{E_3,R})$ resp. $\alpha : \mathcal{C}(I_T, U) \times V \to \mathcal{C}(I_T, U)$ holds.

Therefore we estimate

$$\left\| \int_0^t g(y(\tau), \mu) d\tau \right\|_{E_3} \leq \int_0^t \|g(y(\tau), \mu)\|_{E_3} d\tau \underbrace{\leq}_{(8.5.7)} C \cdot h^0 \cdot M^l_{-1} \cdot T < C \cdot \left(\frac{R}{C}\right) = R,$$

hence

$$\|\alpha(y, \mu)(t)\|_{E_3} = \left\| \int_0^t g(y(\tau), \mu) d\tau \right\|_{E_3} < R. \tag{8.5.9}$$

Claim: α is a contraction map.

If $(y, z, \mu) \in \mathcal{B}_{E_3,R} \times \mathcal{B}_{E_3,R} \times V$, then we get by (8.5.7):

$$\|g(y, \mu) - g(z, \mu)\|_{E_3} \leq C \cdot h \cdot M^l_0 \cdot \|y - z\|_{E_3} = C \cdot h \cdot \|y - z\|_{E_3}.$$

Thus for $(y, z, \mu) \in \mathcal{C}(I_T, \mathcal{B}_{E_3,R}) \times \mathcal{C}(I_T, \mathcal{B}_{E_3,R}) \times V$ and $T \in \mathbb{R}$ with $0 < T < (C \cdot h)^{-1}$ we can estimate:

$$\|\alpha(y, \mu) - \alpha(z, \mu)\|_{E_3} \leq C \cdot h \cdot \|y - z\|_{E_3} \cdot T < \|y - z\|_{E_3}, \tag{8.5.10}$$

which proves the claim.

We summarize: By (8.5.10) α is for $T \in \mathbb{R}_{>0}$ small enough a contraction map and by (8.5.9) it is a self-mapping on $\mathcal{C}(I_T, \mathcal{B}_{E_3,R}) \subseteq \mathcal{C}(I_T, E_3)$, which is a closed subset of a Banach-space. So we can use *Banach's fix point theorem* to obtain for each $\mu \in V$

a unique function $\hat{y}(\mu) \in \mathcal{C}(I_T, \mathcal{B}_{E_3,R})$ with $\alpha(\hat{y}(\mu), \mu) = \hat{y}(\mu)$. This function is the unique solution of (8.5.6), for $t \in I_T$ we have $\hat{y}(\mu)(t) = y(t, \mu)$.

In the next step we estimate the derivatives of α.
Claim: α satisfies (8.5.3) with respect to the variables y and μ.

For this we have to introduce some further notation: Let $\gamma : U \times V \to L^{i,j}(E_3, E_4; E_3)$ be a continuous mapping and let $y \in \mathcal{C}(I_T, U)$, $y_1, \dots, y_i \in \mathcal{C}(I_T, E_3)$, $\mu \in V$ and $\mu_1, \dots, \mu_j \in E_4$. Then for all $t \in I_T$ we put

$$\rho(y,\mu)(y_1,\dots,y_i;\mu_1,\dots,\mu_j)(t) := \int_0^t \gamma(y(\tau),\mu)(y_1(\tau),\dots,y_i(\tau);\mu_1,\dots,\mu_j)d\tau.$$

The integral on the right side is well-defined for all $t \in I_T$, furthermore we have $\rho(y,\mu)(y_1,\dots,y_i;\mu_1,\dots,\mu_j) \in \mathcal{C}(I_T,E_3)$ and $\rho(y,\mu) \in L^{i,j}(\mathcal{C}(I_T,E_3),E_4;\mathcal{C}(I_T,E_3)) =: \mathcal{L}^{i,j}$. We apply this notation to $g^{(i,j)}$ and get

$$\alpha^{(i,j)}(y,\mu)(y_1,\dots,y_i;\mu_1,\dots,\mu_j)(t) = \int_0^t g^{(i,j)}(y(\tau),\mu)(y_1(\tau),\dots,y_i(\tau);\mu_1,\dots,\mu_j)d\tau.$$

Now we estimate as follows:

$$\left\|\alpha^{(i,j)}(y,\mu)\right\|_{\mathcal{L}^{i,j}} \le T \cdot \sup_{a\in U, b\in V}\left\{\left\|g^{(i,j)}(a,b)\right\|_{L^{i,j}(E_3,E_4;E_3)}\right\} \underbrace{\le}_{(8.5.7)} T\cdot C\cdot h^{i+j}\cdot M^l_{i+j-1},$$

which proves the claim.

Claim: The solution y of (8.5.6) satisfies (8.5.8) in the variable μ uniformly in t.

Therefore we define now $\beta(y,\mu) := y - \alpha(y,\mu)$. The fixed point $\hat{y}(\mu)$ of α satisfies clearly $\beta(\hat{y}(\mu),\mu) = 0$, hence $\hat{y}(\mu)$ is determined via the implicit equation $\beta(y,\mu) = 0$. We remark that $\beta^{(1,0)}(y,\mu) \in L(\mathcal{C}(I_T,E_3),\mathcal{C}(I_T,E_3)) =: \mathcal{L}^{1,0}$, so it is a linear operator and $\beta^{(1,0)}(y,\mu) = \mathrm{id}_{\mathcal{L}^{1,0}} - \alpha^{(1,0)}(y,\mu)$. We have chosen $T < (C\cdot h)^{-1}$, hence again by (8.5.7) we get

$$\left\|\alpha^{(1,0)}(y,\mu)\right\|_{\mathcal{L}^{1,0}} \le T\cdot C\cdot h\cdot M^l_0 < 1.$$

This implies $\left\|\beta^{(1,0)}(y,\mu)\right\|_{\mathcal{L}^{1,0}} > 0$ and we can compute the inverse operator $(\beta^{(1,0)}(y,\mu))^{-1}$ in the Banach-algebra $\mathcal{L}^{1,0}$ as follows:

$$\left(\beta^{(1,0)}(y,\mu)\right)^{-1} = \left(\mathrm{id}_{\mathcal{L}^{1,0}} - \alpha^{(1,0)}(y,\mu)\right)^{-1} = \sum_{j=0}^{\infty}\left(\alpha^{(1,0)}(y,\mu)\right)^j.$$

Because α satisfies (8.5.3) we obtain that β satisfies (8.5.3) with respect to y and μ, too. We can apply now the first part of 8.4.1, which implies the fact that the fixpoint $\hat{y}(\mu)$ shares also property (8.5.3). If y is the solution of (8.5.6), then $\hat{y}(\mu)(t) = y(t,\mu)$ holds for all $t \in I_T$, which implies that y satisfies (8.5.3) in the variable μ uniformly in t. With other words there exist constants $A, \eta > 0$ and an index $l_1 \in \Lambda = \mathbb{N}_{>0}$ such that for all $j \in \mathbb{N}$ and $(t,\mu) \in I_T \times V$ we have

$$\left\|y^{(0,j)}(t,\mu)\right\|_{L^j(E_4;E_3)} \le A\cdot\eta^j\cdot M^{l_1}_{j-1}. \tag{8.5.11}$$

In the next step of the proof we have to estimate $y^{(i,j)}$ for $i \geq 1$.
For the following computations we assume w.l.o.g. $A \geq \max\{1, C\}$ and $\eta \geq \max\{1, 2h\}$, where C, h are the constants from (8.5.7). Furthermore we put

$$p_i^l := l \cdot \left(\frac{M_{i-1}^l}{i!} \right)^{1/(i-1)} \quad \text{for } l \in \Lambda = \mathbb{N}_{>0},\ i \in \mathbb{N}, i \geq 2.$$

For such l, i we define the functions $G_i^l, Y_{i,0}^l : \mathbb{R} \to \mathbb{R}$ for $|s|$ small (the domain of s is depending on l and i) in the following way:

$$G_i^l(s) := A \cdot \sum_{j=0}^{\infty} \left(p_i^l \cdot \eta \cdot s \right)^j = A \cdot (1 - p_i^l \cdot \eta \cdot s)^{-1}$$

$$Y_{i,0}^l(s) := A \cdot \Big(1 + \eta \cdot s \cdot \sum_{j=0}^{\infty} (p_i^l \cdot \eta \cdot s)^j \Big) = A \cdot \big(\eta \cdot s \cdot (1 - p_i^l \cdot \eta \cdot s)^{-1} + 1 \big).$$

For $k \in \mathbb{N}$ we obtain

$$(G_i^l)^{(k)}(s) = A \cdot (p_i^l \cdot \eta)^k \cdot \Big(\sum_{j=k}^{\infty} j \cdot (j-1) \cdots (j-k+1) \cdot (p_i^l \cdot \eta \cdot s)^{j-k} \Big),$$

hence $(G_i^l)^{(k)}(0) = A \cdot k! \cdot \eta^k \cdot (p_i^l)^k$ holds.
We introduce now the following ODE:

$$\begin{cases} (Y_i^l)'(t, \sigma) = G_i^l(Y_i^l(t, \sigma) - A + \sigma) \\ Y_i^l(0, \sigma) = Y_{i,0}^l(\sigma), \end{cases} \tag{8.5.12}$$

where σ will be regarded as a complex parameter.
The strategy of the proof will be to compare y, which is the solution of (8.5.6), with the solution Y_i^l of (8.5.12).

First we are going to solve (8.5.12): We set $\varrho_i^l(\sigma) := Y_{i,0}^l(\sigma) - A + \sigma$, then the solution of (8.5.12) is given by

$$Y_i^l(t, \sigma) = \frac{1 - \big((1 - p_i^l \cdot \eta \cdot \varrho_i^l(\sigma))^2 - 2 \cdot A \cdot p_i^l \cdot \eta \cdot t \big)^{1/2}}{p_i^l \cdot \eta} + A - \sigma.$$

We prove this by direct calculation: First we compute the initial value

$$Y_i^l(0, \sigma) = \frac{1 - (1 - p_i^l \cdot \eta \cdot \varrho_i^l(\sigma))}{p_i^l \cdot \eta} + A - \sigma = \underbrace{\varrho_i^l(\sigma)}_{Y_{i,0}^l(\sigma) - A + \sigma} + A - \sigma = Y_{i,0}^l(\sigma).$$

Now we compute the derivative with respect to t:

$$\begin{aligned} (Y_i^l)'(t, \sigma) &= \frac{(2 \cdot A \cdot p_i^l \cdot \eta) \cdot \big((1 - p_i^l \cdot \eta \cdot \varrho_i^l(\sigma))^2 - 2 \cdot A \cdot p_i^l \cdot \eta \cdot t \big)^{-1/2}}{2 \cdot p_i^l \cdot \eta} \\ &= \frac{A}{\sqrt{(1 - p_i^l \cdot \eta \cdot \varrho_i^l(\sigma))^2 - 2 \cdot A \cdot p_i^l \cdot \eta \cdot t}}. \end{aligned}$$

On the other hand we have

$$G_i^l(Y_i^l(t,\sigma)-A+\sigma) = \frac{A}{1-p_i^l\cdot\eta\cdot(Y_i^l(t,\sigma)-A+\sigma)} = \frac{A}{\sqrt{(1-p_i^l\cdot\eta\cdot\varrho_i^l(\sigma))^2-2\cdot A\cdot p_i^l\cdot\eta\cdot t}}.$$

Claim:

$$\exists\, n_1\in\Lambda\;\forall\,(t,\mu)\in I_T\times V:\; \left\|y^{(0,j)}(t,\mu)\right\|_{L^j(E_4;E_3)} \le (Y_{k,0}^{n_1})^{(j)}(0) = (Y_k^{n_1})^{(0,j)}(0,0) \tag{8.5.13}$$

holds for all $k\ge 2$ and $j\in\mathbb{N}$ with $0\le j\le k$.

First we recall that $(\mathcal{M}_{\{\mathrm{ai}\}})$ implies by (8.2.1) (which was already mentioned for the single weight sequence case in [42] and see $(ii),(iii)$ in 8.2.5 for more details) the following inequality:

$$\forall\, l\in\Lambda\;\exists\, n\in\Lambda\;\exists\, H\ge 1:\;\left(\frac{M_{j-1}^l}{j!}\right)^{1/(j-1)} \le H\cdot\left(\frac{M_{k-1}^n}{k!}\right)^{1/(k-1)}\quad\text{for}\quad 2\le j\le k. \tag{8.5.14}$$

Recall: Since we have $\Lambda=\mathbb{N}_{>0}$, for given $l\in\Lambda$ we can take $n=H=\tilde{n}:=\max\{n,H\}$ on the right hand side in $(\mathcal{M}_{\{\mathrm{ai}\}})$, moreover clearly $p^{l_1}\ge p^{l_2}$ holds for $l_1\ge l_2$. So we can replace in (8.5.14) on the right hand side n and H by $\tilde{n}:=\max\{H,n\}$ and write in the following again n instead of $\tilde{n}$. Hence for $2\le j\le k$ we get

$$\frac{M_{j-1}^l}{j!}\le n^{j-1}\cdot\left(\frac{M_{k-1}^n}{k!}\right)^{(j-1)/(k-1)} = (p_k^n)^{j-1} = (p_k^n)^j\cdot(p_k^n)^{-1}. \tag{8.5.15}$$

Obviously (8.5.15) is still valid for all $k\ge 2$ and $j\in\mathbb{N}_{>0}$, because for $j=1$ in this case we have $1=M_0^l\le(p_k^n)^0=1$. By assumption we have $M_2^l\ge 2\Leftrightarrow m_2^l\ge 1$ for each $l\in\Lambda$ and so we get by (8.5.14) for $k\ge 2$:

$$1\le\frac{M_2^l}{2}\le n\cdot\left(\frac{M_{k-1}^n}{k!}\right)^{1/(k-1)} = p_k^n. \tag{8.5.16}$$

We summarize:

$$\forall\, l\in\Lambda\;\exists\, n\in\mathbb{N}_{>0}:\;\frac{M_{j-1}^l}{j!}\underbrace{\le}_{(8.5.15)}(p_k^n)^j\cdot(p_k^n)^{-1}\underbrace{\le}_{(8.5.16)}(p_k^n)^j\;\text{ for } 2\le j\le k. \tag{8.5.17}$$

Moreover (8.5.17) is still satisfied for all $k\ge 2$ and $j\in\mathbb{N}$, because if $j=0$, then nothing is to show and if $j=1$, this is exactly (8.5.16).

By assumption we have $A\ge C$ and $\eta\ge 2h$, thus by (8.5.17):

$$\forall\, l\in\Lambda\;\exists\, n\in\mathbb{N}_{>0}:\; C\cdot(2h)^j\cdot M_{j-1}^l\le A\cdot\eta^j\cdot(p_k^n)^j\cdot j! = (G_k^n)^{(j)}(0)\;\text{ for }\;0\le j\le k, k\ge 2. \tag{8.5.18}$$

For $j\in\mathbb{N}$ consider $(Y_{k,0}^l)^{(j)}(s)$ and we get:

$$(Y_{k,0}^n)^{(j)}(0) = A\cdot j!\cdot\eta^j\cdot(p_k^n)^{j-1}\underbrace{\ge}_{(8.5.15)} A\cdot\eta^j\cdot M_{j-1}^l\;\text{ for }\;1\le j\le k, k\ge 2$$

where the indices $l, n \in \Lambda$ are related via (8.5.14). Let $\tilde{l} := \max\{l, l_1\}$, then this implies finally: $\exists\, n_1 \in \Lambda\, \forall\, (t,\mu) \in I_T \times V$:

$$\left\| y^{(0,j)}(t,\mu) \right\|_{L^j(E_4;E_3)} \underbrace{\leq}_{(8.5.11)} A \cdot \eta^j \cdot M^{\tilde{l}}_{j-1} \leq (Y^{n_1}_{k,0})^{(j)}(0) = (Y^{n_1}_k)^{(0,j)}(0,0), \qquad (8.5.19)$$

for $1 \leq j \leq k$, $k \geq 2$. Note that for the index l_1, which is coming from (8.5.11), we have $M^{l_1} \leq M^{\tilde{l}}$ and $\tilde{l}$ and n_1 are related via (8.5.14) resp. (8.5.17) above. (8.5.19) is still valid for $j = 0$ because in this case we have $\|y(t,\mu)\|_{E_3} \leq A \cdot \eta^0 \cdot M^{\tilde{l}}_{-1} = A = Y^{n_1}_{k,0}(0) = Y^{n_1}_k(0,0)$ (for arbitrary $n_1 \in \Lambda$ and $k \in \mathbb{N}$, $k \geq 2$), which proves the claim.

Claim: $\exists\, n_1 \in \Lambda\, \forall\, (t,\mu) \in I_T \times V$:

$$\left\| y^{(i,j)}(t,\mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} \leq (Y^{n_1}_k)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0} \text{ for } i+j \leq k, k \geq 2, \qquad (8.5.20)$$

and n_1 is the index arising in (8.5.19).

To prove this claim first we have to recall some definitions and notations from Yamanaka, we refer to section 5 in [42] for more details and complete proofs. Let E and F be real Banach spaces, $U \subseteq E$ open and $g : E \supseteq U \to F$ be infinitely differentiable. Furthermore let $V \subseteq \mathbb{R}$ be open and a point $b \in V$ be given and let $G : \mathbb{R} \supseteq V \to \mathbb{R}$ be a smooth function. Then we write

$$g \underset{(U,p,b)}{\lll} G :\Longleftrightarrow \left\| g^{(q)}(x) \right\|_{L^q(E;F)} \leq G^{(q)}(b) \text{ for all } x \in U, q \in \mathbb{N} \text{ with } 0 \leq q \leq p.$$

We write

$$g \underset{(U,p,b)}{\ll} G :\Leftrightarrow g' \underset{(U,p-1,b)}{\lll} G',$$

which is clearly a weaker condition than $g \underset{(U,p,b)}{\lll} G$. An easy consequence of the *Faà-di-Bruno-Formula* (8.4.2) is the following result, see [42, Lemma 5.1.]:

Let E_1, E_2 and E_3 be three real Banach-spaces, $U_1 \subseteq E_1$ and $U_2 \subseteq E_2$ open and $f : E_2 \supseteq U_2 \to E_3$, $g : E_1 \supseteq U_1 \to U_2$ be two infinitely differentiable mappings. Let $V_1, V_2 \subseteq \mathbb{R}$ open and $F : V_2 \to \mathbb{R}$, $G : V_1 \to V_2$ be two smooth mappings such that $f \underset{(U_2,p,b)}{\lll} F$ and $g \underset{(U_1,p,a)}{\ll} G$ holds for a point $a \in V_1$ with $V_2 \ni b = G(a)$. Then we obtain $f \circ g \underset{(U_1,p,a)}{\lll} F \circ G$.

Let $E_1, \ldots, E_n$ and F be real Banach-spaces, put $E := E_1 \times \cdots \times E_n$ and consider an open set $U \subseteq E$. Let $g : E \supseteq U \to F$ be an infinitely differentiable function and $V \subseteq \mathbb{R}^n$. Now fix a point $a := (a_1, \ldots, a_n) \in V$ and consider a smooth mapping $G : \mathbb{R}^n \supseteq V \to \mathbb{R}$. Take $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, then we write

$$g \underset{(U,\alpha,a)}{\lll_c} G :\Longleftrightarrow \left\| g^{(\beta)}(x) \right\|_{L^\beta(E;F)} \leq G^{(\beta)}(a) \text{ for all } \beta, \beta_i \leq \alpha_i,\ 1 \leq i \leq n, x \in U,$$

where $L^\beta(E;F) = L^{\beta_1,\ldots,\beta_n}(E_1,\ldots,E_n;F) \cong \oplus_{i=1}^n L^{\beta_i}(E_i;F)$. The letter c in the definition is standing of course for componentwise. Similarly we write

$$g \underset{(U,\alpha,a)}{\ll_c} G :\Longleftrightarrow \left\| g^{(\beta)}(x) \right\|_{L^\beta(E;F)} \leq G^{(\beta)}(a) \text{ for all } \beta \in \mathbb{N}^n \backslash \{0\}, \beta_i \leq \alpha_i,\ 1 \leq i \leq n, x \in U.$$

With this notation we can formulate the following result (see [42, Lemma 5.4.]):
Let $E_1, \ldots, E_n$, F and G be real Banach-spaces. Set $E := E_1 \times \cdots \times E_n$ and let $U_1 \subseteq E$ and $U_2 \subseteq F$ be two open sets. Let $h_1 : U_1 \to U_2$ and $h_2 : U_2 \to G$ be infinitely differentiable functions and $V_1 \subseteq \mathbb{R}^n$, $V_2 \subseteq \mathbb{R}$ be two open subsets. Furthermore let $H_1 : \mathbb{R}^n \supseteq V_1 \to V_2$ and $H_2 : V_2 \to \mathbb{R}$ be two smooth functions and $a = (a_1, \ldots, a_n) \in V_1$ a point such that $b = H_1(a)$ holds. We assume that for $\alpha \in \mathbb{N}^n$ the following holds: $h_2 \underset{(U_2,|\alpha|,b)}{\lll} H_2$ and $h_1 \underset{(U_1,\alpha,a)}{\ll_c} H_1$. Then we get:

$$h_2 \circ h_1 \underset{(U_1,\alpha,a)}{\lll_c} H_2 \circ H_1.$$

We prove now the claim and first recall that y is a local solution of (8.5.6), hence smooth by the classical existence theorem for ODE's and note that (8.5.13) implies (8.5.20) for $i = 0$ and $j \leq k$, $k \geq 2$. In the following let $k \in \mathbb{N}$, $k \geq 2$. Suppose now that $\alpha_1, \alpha_2 \in \mathbb{N}$ are given with $\alpha_1 < k$ and $\alpha_1 + \alpha_2 = k$ and that (8.5.20) is valid for all $(i, j) \in \mathbb{N}^2$ with $0 \leq i \leq \alpha_1$ and $0 \leq j \leq \alpha_2$. Put in the following $\alpha := (\alpha_1, \alpha_2)$ and $E_5 := E_3 \times E_4$. Hence g can be viewed as a Fréchet-infinitely differentiable function from $U \times V \subseteq E_5$ into the space E_3. Let $z \in U \times V$, $z := (y, \mu)$, then we denote by $g^{(p)}$ the p-th Fréchet-derivative of g with respect to the variable z, and we get (polarization formula)

$$\left\| g^{(k)}(z) \right\|_{L^k(E_5;E_3)} \leq 2^k \cdot \sup_{i+j=k} \left\| g^{(i,j)}(y, \mu) \right\|_{L^{i,j}(E_3,E_4;E_3)}.$$

Hence for $0 \leq j \leq k$ and $z \in U \times V$ we estimate

$$\left\| g^{(j)}(z) \right\|_{L^j(E_5;E_3)} \underbrace{\leq}_{(8.5.7)} 2^j \cdot C \cdot h^j \cdot M_{j-1}^l \leq 2^j \cdot C \cdot h^j \cdot M_{j-1}^{\tilde{l}} \underbrace{\leq}_{(8.5.18)} (G_k^{n_1})^{(j)}(0),$$

where $\tilde{l}$ and n_1 are related again via (8.5.14) resp. (8.5.17). In our introduced notation this means nothing else but $g \underset{(U\times V,k,0)}{\lll} G_k^{n_1}$. Now put $z(t, \mu) := (y(t, \mu), \mu)$ and $Z_k^l(\tau, \sigma) := Y_k^l(\tau, \sigma) - A + \sigma$. Because $y(t, \mu) \in \mathcal{B}_{E_3,R}$ holds for all $t \in I_T$ and $\mu \in V$, one has $z : I_T \times V \to U \times V$, and by componentwise differentiation we see

$$\begin{aligned} \left\| z^{(i,j)}(t, \mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3,E_4)} &= \max \left\{ \left\| y^{(i,j)}(t, \mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)}, \left\| \mu^{(i,j)} \right\|_{E_4} \right\} \\ &\leq \left\| y^{(i,j)}(t, \mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} + \left\| \mu^{(i,j)} \right\|_{E_4}. \end{aligned}$$

If $i + j > 0$, then $(Z_k^l)^{(i,j)}(\tau, \sigma) = (Y_k^l)^{(i,j)}(\tau, \sigma) + \sigma^{(i,j)}$. If $0 \leq i \leq \alpha_1$ and $0 \leq j \leq \alpha_2$, then (8.5.20) holds, hence for $(i, j) \in \mathbb{N}^2$ such that $0 \leq i \leq \alpha_1$, $0 \leq j \leq \alpha_2$ and $i+j > 0$ we obtain:

$$\left\| z^{(i,j)}(t, \mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3,E_4)} \leq (Z_k^{n_1})^{(i,j)}(\tau, \sigma)|_{\tau=\sigma=0} \text{ for } (t, \mu) \in I_T \times V.$$

By using again the introduced notation this means exactly $z \underset{(I_T\times V,\alpha,0)}{\ll_c} Z_k^{n_1}$, and because we have already shown $g \underset{(U\times V,k,0)}{\lll} G_k^{n_1}$ we get by [42, Lemma 5.4.]: $g \circ z \underset{(I_T\times V,\alpha,0)}{\lll_c} G_k^{n_1} \circ Z_k^{n_1}$, i.e.

$$\left\| (g \circ z)^{(i,j)} \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} = \left\| \partial_t^i \partial_\mu^j g(y(t, \mu), \mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} \leq \partial_t^i \partial_\mu^j G_k^{n_1}(Y_k^{n_1}(\tau, \sigma) - A + \sigma)|_{\tau=\sigma=0}$$

for $(i,j) \in \mathbb{N}^2$ with $0 \le i \le \alpha_1$ and $0 \le j \le \alpha_2$. We remark: y is a solution of (8.5.6) and $Y_k^{n_1}$ of (8.5.12), so it follows that (8.5.20) holds now for $0 \le i \le \alpha_1 + 1$ and $0 \le j \le \alpha_2$. Thus we have shown that (8.5.20) holds for $(i,j) \in \mathbb{N}^2$ with $i+j \le k$ and arbitrary $k \in \mathbb{N}$, $k \ge 2$.

In the next step of the proof let $l \in \Lambda$ be arbitrary but fixed and we are going to estimate $(Y_k^l)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0}$ for $i \ge 1$. For this we start to estimate $(Y_k^l)^{(i,0)}(\tau,\sigma)|_{\tau=0}$ for $i \ge 1$:

$$\begin{aligned}(Y_k^l)^{(i,0)}(\tau,\sigma)|_{\tau=0} &= -(p_k^l\eta)^{-1}(-2Ap_k^l\eta)^i 2^{-1}(2^{-1}-1)\cdots(2^{-1}-i+1)(1-p_k^l\eta\varrho_k^l(\sigma))^{1-2i} \\ &= (-1)^{i+1}(p_k^l\eta)^{i-1}(2A)^i(-1)^i(-2^{-1})\cdots(i-1-2^{-1})(1-p_k^l\eta\varrho_k^l(\sigma))^{1-2i} \\ &= (p_k^l\eta)^{i-1}\cdot(2A)^i\frac{\Gamma(i-2^{-1})}{2\pi^{1/2}}(1-p_k^l\eta\varrho_k^l(\sigma))^{1-2i},\end{aligned}$$

where $\Gamma(x) := \int_0^\infty e^{-t}\cdot t^{x-1}dt$ denotes the *Gamma-function.* So the last equality holds because

$$\Gamma(i-2^{-1}) = (i-1-2^{-1})\cdot\Gamma(i-1-2^{-1}) = \cdots = (i-1-2^{-1})\cdots(-2^{-1})\cdot\Gamma(-2^{-1})$$

and $\pi^{1/2} = \Gamma(2^{-1}) = \Gamma(-2^{-1}+1) = -2^{-1}\cdot\Gamma(-2^{-1})$, thus $\Gamma(-2^{-1}) = -2\cdot\pi^{1/2}$. Hence we have shown:

$$(Y_k^l)^{(i,0)}(\tau,\sigma)|_{\tau=0} = (p_k^l\eta)^{i-1}\cdot(2A)^i\cdot\frac{\Gamma(i-2^{-1})}{2\pi^{1/2}}\cdot(1-p_k^l\eta\varrho_k^l(\sigma))^{1-2i} \qquad (8.5.21)$$

for all $i \in \mathbb{N}$, $i \ge 1$.

We study now the expression $(1-p_k^l\cdot\eta\cdot\varrho_k^l(\sigma))^{1-2i}$, which appears in (8.5.21), for $i \in \mathbb{N}$, $i \ge 1$, in detail. First we see that

$$\begin{aligned}1-p_k^l\cdot\eta\cdot\varrho_k^l(\sigma) &= 1-p_k^l\cdot\eta\cdot(Y_{k,0}^l(\sigma)-A+\sigma) \\ &= 1-p_k^l\cdot\eta\cdot\big((A\cdot\eta\cdot\sigma\cdot(1-p_k^l\cdot\eta\cdot\sigma)^{-1}+A)-A+\sigma\big) \\ &= 1-p_k^l\cdot\eta\cdot\sigma - A\cdot p_k^l\cdot\eta^2\cdot\sigma\cdot(1-p_k^l\cdot\eta\cdot\sigma)^{-1}.\end{aligned}$$

So for $\sigma \in \mathbb{C}$ with $|\sigma|$ small enough we can estimate as follows:

$$\begin{aligned}|1-p_k^l\cdot\eta\cdot\varrho_k^l(\sigma)| &\ge 1-p_k^l\cdot\eta\cdot|\sigma| - A\cdot p_k^l\cdot\eta^2\cdot|\sigma|\cdot(1-p_k^l\cdot\eta\cdot|\sigma|)^{-1} \\ &\underbrace{\ge}_{A,\eta\ge1} 1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma| - A\cdot p_k^l\cdot\eta^2\cdot|\sigma|\cdot(1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma|)^{-1} \\ &= 1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma| - (1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma|)^{-1}+1.\end{aligned}$$

Now we want to find a sufficient condition for σ such that

$$|1-p_k^l\cdot\eta\cdot\varrho_k^l(\sigma)| \ge \frac{1}{2} \qquad (8.5.22)$$

is satisfied. For this we set $x := 1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma|$, then by the above calculation $|1-p_k^l\cdot\eta\cdot\varrho_k^l(\sigma)| \ge x-x^{-1}+1$ holds. Because $x-x^{-1}+1 \ge 1/2 \Rightarrow 2x^2+x-2 \ge 0$ we are interested in solving the quadratic equation $2x^2+x-2=0$. We get the solutions $x_{1,2} = \frac{-1\pm\sqrt{17}}{4}$, so for $x \ge \frac{-1+\sqrt{17}}{4}$ inequality (8.5.22) is satisfied. We can take now $x \ge \frac{4}{5}$, which means $1-A\cdot p_k^l\cdot\eta^2\cdot|\sigma| \ge 4/5 \Leftrightarrow |\sigma| \le (5\cdot A\cdot p_k^l\cdot\eta^2)^{-1}$.

Now take $\sigma \in \mathbb{C}$ such that $|\sigma| \le (5 \cdot A \cdot p_k^l \cdot \eta^2)^{-1}$ holds and put

$$\omega_{l,k,i} : \mathbb{C} \to \mathbb{C}, \quad \omega_{l,k,i}(\sigma) := (1 - p_k^l \cdot \eta \cdot \varrho_k^l(\sigma))^{1-2i}. \tag{8.5.23}$$

Claim:

$$\forall\, j \in \mathbb{N} : \quad \left|\omega_{l,k,i}^{(j)}(0)\right| < j! \cdot (2^{2i-1}) \cdot (8 \cdot A \cdot p_k^l \cdot \eta^2)^j. \tag{8.5.24}$$

By (8.5.22) we get $|\omega_{l,k,i}(\sigma)| \le 2^{2i-1}$ and we can use *Cauchy's integral formula* to estimate $\omega_{l,k,i}^{(j)}$ at the point 0 for $j \in \mathbb{N}$ in the following way:

$$\begin{aligned}\left|\omega_{l,k,i}^{(j)}(0)\right| &\le \frac{j!}{2\pi} \cdot \int_{\{\sigma \in \mathbb{C}: |\sigma| = r\}} |\omega_{l,k,i}(\sigma)| \cdot |\sigma|^{-j-1} d\sigma \\ &\le j! \cdot (2^{2i-1}) \cdot r^j < j! \cdot (2^{2i-1}) \cdot (8 \cdot A \cdot p_k^l \cdot \eta^2)^j,\end{aligned}$$

where we have set the radius $r := (5 \cdot A \cdot p_k^l \cdot \eta^2)^{-1}$.
Furthermore for $j \in \mathbb{N}$ we have $0 < \omega_{l,k,i}^{(j)}(0)$ which holds by it's *Maclaurin series expansion* as pointed out in [42].

Claim:

$$(Y_k^l)^{(i,j)}(0,0) \le 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^{i+j} \cdot (i+j)! \cdot (p_k^l)^{i+j-1} \tag{8.5.25}$$

holds for $i \in \mathbb{N}$, $i \ge 1$. For this we estimate as follows:

$$\begin{aligned} 0 < (Y_k^l)^{(i,j)}(0,0) &\underbrace{=}_{(8.5.21)} (p_k^l \eta)^{i-1} \cdot (2A)^i \cdot \frac{\Gamma(i - 2^{-1})}{2\pi^{1/2}} \cdot \omega_{l,k,i}^{(j)}(0) \\ &\underbrace{\le}_{(8.5.24)} (\underbrace{p_k^l \eta}_{\le p_k^l \eta^2})^{i-1} \cdot (2A)^i \cdot \frac{\Gamma(i - 2^{-1})}{2\pi^{1/2}} \cdot 2^{2i-1} \cdot j! \cdot (8A p_k^l \eta^2)^j \\ &\le \underbrace{2^{3(i+j)-2}}_{=8^{i+j} \cdot 2^{-2}} \cdot \pi^{-1/2} \cdot A^{i+j} \cdot \underbrace{\Gamma(i - 2^{-1})}_{\le i! = \Gamma(i+1)} \cdot j! \cdot (p_j^l \eta^2)^{i+j-1} \\ &\le 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^{i+j} \cdot (i+j)! \cdot (p_k^l)^{i+j-1}, \end{aligned}$$

which proves (8.5.25). The last inequality in the previous calculation holds because $\eta \ge 1$ and we have used $1 \le \frac{(i+j)!}{i! \cdot j!} = \binom{i+j}{j}$, which holds for all $i, j \in \mathbb{N}$.

With this estimate we can finish the proof: Let $k \ge 2$ and put $i + j = k$ in (8.5.25), then we obtain

$$(Y_k^l)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0} \le 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^k \cdot k! \cdot (p_k^l)^{k-1} \le 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2 l)^k \cdot M_{k-1}^l. \tag{8.5.26}$$

For the last equality recall that by definition we have $(p_k^l)^{k-1} = l^{k-1} \cdot \frac{M_{k-1}^l}{k!}$. We use now this estimate precisely for the index n_1 coming from (8.5.20) (instead of l arbitrary), hence for $i + j = k$, $i \ge 1$, we obtain for all $(t, \mu) \in I_T \times V$:

$$\begin{aligned}\left\|y^{(i,j)}(t,\mu)\right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} &\underbrace{\le}_{(8.5.20)} (Y_k^{n_1})^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0} \\ &\underbrace{\le}_{(8.5.26)} 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2 n_1)^{i+j} \cdot M_{i+j-1}^{n_1}.\end{aligned}$$

But since $k \in \mathbb{N}$, $k \geq 2$, is arbitrary and we have already shown (8.5.11) (recall: $\tilde{l} := \max\{l, l_1\}$ and $\tilde{l}$ and n_1 were related via (8.5.14) resp. (8.5.17)) we have shown (8.5.8) and are done. Note that the case $i = 1$, $j = 0$ is satisfied by assumptions (8.5.6) and (8.5.7).

The Beurling-case:
(2)(a) For this we change the proof of the above Roumieu-case a little bit. First note that by $(\mathcal{M}_{(\mathrm{dc})})$ we have $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}_{(\mathcal{M}^d)}$. Then for E_1 and E_2 real Banach-spaces we consider an infinitely differentiable function $f : E_1 \times \mathbb{R} \times E_2 \supseteq W \to E_1$ such that for all $h > 0$ and $l \in \Lambda$ there exists a constant $C_{l,h} > 0$ with

$$\|f^{(i,j,k)}(x,t,\lambda)\|_{L^{i,j,k}(E_1,\mathbb{R},E_2;E_1)} \leq C_{l,h} \cdot h^{i+j+k} \cdot M^l_{i+j+k-1} \tag{8.5.27}$$

for all $(i,j,k) \in \mathbb{N}^3$ and $(x,t,\lambda) \in W$. Then consider again the ODE

$$\begin{cases} x'(t) = f(x(t),t,\lambda) \\ x(0) = x_0, \quad x_0 \in E_1 \end{cases} \tag{8.5.28}$$

and, as in the above Roumieu-case, we simplify it to

$$\begin{cases} y'(t) = g(y(t),\mu) \\ y(0) = 0, \end{cases} \tag{8.5.29}$$

where $y : \mathbb{R} \to E_1 \times \mathbb{R} =: E_3$ is the unknown function and $\mu \in E_2 \times E_1 =: E_4$ is a parameter. If f satisfies (8.5.27) with respect to (x,t,λ), then g satisfies (8.5.27) with respect to (y,μ).

Consider now the initial value problem (8.5.29), let E_3 and E_4 be real Banach-spaces, U again a ball of radius R centered at $0 \in E_3$ and let V be an open set of E_4. Let $g : E_3 \times E_4 \supseteq U \times V \to E_3$, $(y,\mu) \mapsto g(y,\mu)$, be an infinitely differentiable function with the following property: For all $l \in \Lambda$ and $h > 0$ there exists a constant $C_{l,h}$ such that for all $(i,j) \in \mathbb{N}^2$ and $(y,\mu) \in U \times V$ we have the estimate

$$\left\|g^{(i,j)}(y,\mu)\right\|_{L^{i,j}(E_3,E_4;E_3)} \leq C_{l,h} \cdot h^{i+j} \cdot M^l_{i+j-1}. \tag{8.5.30}$$

Note that (8.5.30) is much stronger than (8.5.7) above and we consider in (8.5.30) now the choice $l = h = 1$ and put $C := C_{1,1}$. For this case we use the arguments of the Roumieu-case. First the function $y \mapsto g(y,\mu)$ is bounded in the norm $\|\cdot\|_{E_3}$ by the constant C and it is *Lipschitz-continuous* on U with respect to y and uniformly for $\mu \in V$. If $|t| \leq \frac{R}{C}$, then $g(y,\mu) \in U$ and so for each $\mu \in V$ and $t \in \mathbb{R}$ with $|t| \leq \frac{R}{C}$ the classical existence theorem for ODE's implies the local existence of a unique solution y where $y : \mathbb{R} \times E_4 \to E_3$, $(t,\mu) \mapsto y(t,\mu)$, is defined for $|t| \leq \frac{R}{C}$ and $\mu \in V$. It's well known that y is infinitely differentiable in (t,μ), thus we have to show in this case:
For all indices $n \in \Lambda$ and numbers $\xi > 0$ there exist a constant $B_{n,\xi} > 0$ such that for all $(t,\mu) \in I_T \times V = [-T,T] \times V$ ($T > 0$ small dep. on f) the estimate

$$\left\|y^{(i,j)}(t,\mu)\right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} \leq B_{n,\xi} \cdot \xi^{j+k} \cdot M^n_{i+j-1} \tag{8.5.31}$$

is satisfied for all $(i,j) \in \mathbb{N}^2$. Again first we estimate $y^{(0,j)}$, $j \in \mathbb{N}$ arbitrary.

Let I be a compact interval such that $0 \in I$, let $(y,\mu,t) \in \mathcal{C}(I,U) \times V \times I$ and recall the following integral

$$\alpha(y,\mu)(t) := \int_0^t g(y(\tau),\mu)d\tau$$

to obtain a function $\alpha : \mathcal{C}(I,U) \times V \to \mathcal{C}(I,E_3)$. Let $T \in \mathbb{R}$ be given with $0 < T < \frac{R}{C} = \frac{R}{C_{1,1}}$ and put $I_T := [-T,T]$. In the next step we apply again *Banach's fix point theorem* to the mapping α. One shows as in the Roumieu-case that $\alpha : \mathcal{C}(I_T, \mathcal{B}_{E_3,R}) \times V \to \mathcal{C}(I_T, \mathcal{B}_{E_3,R})$ resp. $\alpha : \mathcal{C}(I_T,U) \times V \to \mathcal{C}(I_T,U)$, α is a contraction map. So we can apply *Banach's fix point theorem* and finally for all $l \in \Lambda$ and $h > 0$ we find a constant $C_{l,h} > 0$ such that for all $(i,j) \in \mathbb{N}^2$ we obtain

$$\left\|\alpha^{(i,j)}(y,\mu)\right\|_{\mathcal{L}^{i,j}} \le T \cdot \sup_{a \in U, b \in V} \left\{ \left\|g^{(i,j)}(a,b)\right\|_{L^{i,j}(E_3,E_4;E_3)} \right\} \underbrace{\le}_{(8.5.30)} T \cdot C_{l,h} \cdot h^{i+j} \cdot M^l_{i+j-1}.$$

So we can show that the solution y of (8.5.29) satisfies (8.5.27) in the variable μ uniformly in t, i.e. for all $l \in \Lambda$ and $h > 0$ there exist a constants $A_{l,h} > 0$ such that for all $j \in \mathbb{N}$ and $(t,\mu) \in I_T \times V$ we have (analogously to (8.5.11) above):

$$\left\|y^{(0,j)}(t,\mu)\right\|_{L^j(E_4;E_3)} \le A_{l,h} \cdot h^j \cdot M^l_{j-1}. \tag{8.5.32}$$

In the next step of the proof we have to estimate $y^{(i,j)}$ for $i \ge 1$.
First we recall that $(\mathcal{M}_{(\mathrm{ai})})$ implies by (8.2.1) the following inequality:

$$\forall\, l \in \Lambda\; \exists\, n \in \Lambda\; \exists\, H \ge 1 : \; \frac{1}{H} \cdot \left(\frac{M^n_{j-1}}{j!}\right)^{1/(j-1)} \le \left(\frac{M^l_{k-1}}{k!}\right)^{1/(k-1)} \quad \text{for} \quad 2 \le j \le k. \tag{8.5.33}$$

Recall: Since we have $\Lambda = \mathbb{R}_{>0}$, for given $l \in \Lambda$ we can replace n and $\frac{1}{H}$ in (8.5.33) by $\tilde{n} := \min\{n, \frac{1}{H}\}$ (hence $\tilde{n} \le n$ and $\tilde{n} \le \frac{1}{H} \Leftrightarrow H \le \frac{1}{\tilde{n}}$). We write again n instead of $\tilde{n}$ and put for such related indices l and n (recall $\Lambda = \mathbb{R}_{>0}$)

$$p^l_i := \frac{1}{n} \cdot \left(\frac{M^l_{i-1}}{i!}\right)^{1/(i-1)} \quad \text{for } l \in \mathbb{R}_{>0},\; i \in \mathbb{N}, i \ge 2.$$

Hence for $2 \le j \le k$ we get

$$\frac{M^n_{j-1}}{j!} \le \frac{1}{n^{j-1}} \cdot \left(\frac{M^l_{k-1}}{k!}\right)^{(j-1)/(k-1)} = (p^l_k)^{j-1} = (p^l_k)^j \cdot (p^l_k)^{-1}. \tag{8.5.34}$$

Obviously (8.5.34) is still valid for all $k \ge 2$ and $j \in \mathbb{N}_{>0}$, because for $j = 1$ in this case we have $1 = M^l_0 \le (p^l_k)^0 = 1$. By assumption we have $M^n_2 \ge 2 \Leftrightarrow m^n_2 \ge 1$ for each $n \in \Lambda$ and so we get by (8.5.33) for $k \ge 2$:

$$1 \le \frac{M^n_2}{2} \le \frac{1}{n} \cdot \left(\frac{M^l_{k-1}}{k!}\right)^{1/(k-1)} = p^l_k. \tag{8.5.35}$$

We summarize:

$$\forall\, l \in \Lambda\; \exists\, n \in \Lambda : \; \frac{M^n_{j-1}}{j!} \underbrace{\le}_{(8.5.34)} (p^l_k)^j \cdot (p^l_k)^{-1} \underbrace{\le}_{(8.5.35)} (p^l_k)^j \;\text{ for } 2 \le j \le k. \tag{8.5.36}$$

Moreover (8.5.36) is still satisfied for all $k \geq 2$ and $j \in \mathbb{N}$, because if $j = 0$, then nothing is to show and if $j = 1$, this is exactly (8.5.35).

In the following computations let $l \in \Lambda$ be arbitrary (small) but fixed and $h = 1$ and let the indices n and l be related by (8.5.33). Take now $\eta \geq 2 = 2h$ and $A_{n,\eta} \geq \max\{1, C_{n,1}\}$, where $C_{n,1}$ is the constant related to n and $h = 1$ via (8.5.27) resp. (8.5.30) and $A_{n,\eta}$ is the constant appearing in (8.5.32) for given $\eta > 0$ and $n \in \Lambda$ (both small).

For $l \in \Lambda$ and $i \in \mathbb{N}$, $i \geq 2$ we define again the functions $G_i^l, Y_{i,0}^l : \mathbb{R} \to \mathbb{R}$ for $|s|$ small (the domain of s is depending on l and i) in the following way:

$$G_i^l(s) := A_{n,\eta} \cdot \sum_{j=0}^{\infty} \Big(p_i^l \cdot \eta \cdot s\Big)^j = A_{n,\eta} \cdot (1 - p_i^l \cdot \eta \cdot s)^{-1}$$

$$Y_{i,0}^l(s) := A_{n,\eta} \cdot \Big(1 + \eta \cdot s \cdot \sum_{j=0}^{\infty} (p_i^l \cdot \eta \cdot s)^j\Big) = A_{n,\eta} \cdot \big(\eta \cdot s \cdot (1 - p_i^l \cdot \eta \cdot s)^{-1} + 1\big).$$

So for $k \in \mathbb{N}$ we obtain again $(G_i^l)^{(k)}(0) = A_{n,\eta} \cdot k! \cdot \eta^k \cdot (p_i^l)^k$ and recall the ODE

$$\begin{cases} (Y_i^l)'(t,\sigma) = G_i^l(Y_i^l(t,\sigma) - A_{n,\eta} + \sigma) \\ Y_i^l(0,\sigma) = Y_{i,0}^l(\sigma), \end{cases} \tag{8.5.37}$$

with solution $Y_i^l(t,\sigma) = \frac{1 - \left((1 - p_i^l \cdot \eta \cdot \varrho_i^l(\sigma))^2 - 2 \cdot A_{n,\eta} \cdot p_i^l \cdot \eta \cdot t\right)^{1/2}}{p_i^l \cdot \eta} + A_{n,\eta} - \sigma$ and σ will be regarded as a complex parameter.

Claim:

$$\forall\, l \in \Lambda\ \forall\, (t,\mu) \in I_T \times V : \ \Big\| y^{(0,j)}(t,\mu) \Big\|_{L^j(E_4;E_3)} \leq (Y_{k,0}^l)^{(j)}(0) = (Y_k^l)^{(0,j)}(0,0) \tag{8.5.38}$$

holds for all $k \geq 2$ and $j \in \mathbb{N}$ with $0 \leq j \leq k$.

By assumption we have $A_{n,\eta} \geq C_{n,1}$ and $\eta \geq 2$, thus by (8.5.36):

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda : \ C_{n,1} \cdot 2^j \cdot M_{j-1}^n \leq A_{n,\eta} \cdot \eta^j \cdot (p_k^l)^j \cdot j! = (G_k^l)^{(j)}(0) \ \text{ for } \ 0 \leq j \leq k, k \geq 2. \tag{8.5.39}$$

For $j \in \mathbb{N}$ consider $(Y_{k,0}^l)^{(j)}(s)$ and we get:

$$(Y_{k,0}^l)^{(j)}(0) = A_{n,\eta} \cdot j! \cdot \eta^j \cdot (p_k^l)^{j-1} \underbrace{\geq}_{(8.5.34)} A_{n,\eta} \cdot \eta^j \cdot M_{j-1}^n \ \text{ for } \ 1 \leq j \leq k, k \geq 2$$

where the indices $l, n \in \Lambda$ are related via (8.5.33). This finally implies: $\forall\, l \in \Lambda\ \forall\, (t,\mu) \in I_T \times V$:

$$\Big\| y^{(0,j)}(t,\mu) \Big\|_{L^j(E_4;E_3)} \underbrace{\leq}_{(8.5.32)} A_{n,\eta} \cdot \eta^j \cdot M_{j-1}^n \leq (Y_{k,0}^l)^{(j)}(0) = (Y_k^l)^{(0,j)}(0,0), \tag{8.5.40}$$

for $1 \leq j \leq k$, $k \geq 2$. Note that the indices l and n are related via above inequalities (8.5.33) resp. (8.5.36). But (8.5.40) is still valid for $j = 0$ because in this case we have $\|y(t,\mu)\|_{E_3} \leq A_{n,\eta} \cdot \eta^0 \cdot M_{-1}^l = A_{n,\eta} = Y_{k,0}^l(0) = Y_k^l(0,0)$ (for arbitrary $l \in \Lambda$ and $k \in \mathbb{N}$, $k \geq 2$), which proves the claim.

Claim: $\forall\, l \in \Lambda\ \forall\, (t,\mu) \in I_T \times V$:

$$\left\| y^{(i,j)}(t,\mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} \leq (Y_k^l)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0} \ \text{ for } i+j \leq k, k \geq 2. \tag{8.5.41}$$

To prove this claim we use again the definitions and notations from Yamanaka [42], [42, Lemma 5.1.], [42, Lemma 5.4.] and the induction argument as for the Roumieu-case. First, or $0 \leq j \leq k$ and $z \in U \times V$ we have

$$\left\| g^{(j)}(z) \right\|_{L^j(E_5;E_3)} \underbrace{\leq}_{(8.5.30)} 2^j \cdot C_{n,1} \cdot M_{j-1}^n \underbrace{\leq}_{(8.5.39)} (G_k^l)^{(j)}(0),$$

i.e. $\underset{(U\times V,k,0)}{g \lll G_k^l}$ for each $l \in \Lambda$. Then we show $\underset{(I_T\times V,\alpha,0)}{z \ll_c Z_k^l}$ for each $l \in \Lambda$, and because we have already shown $\underset{(U\times V,k,0)}{g \lll G_k^l}$ we get by [42, Lemma 5.4.] $\underset{(I_T\times V,\alpha,0)}{g \circ z \lll_c G_k^l \circ Z_k^l}$ for each $l \in \Lambda$. Thus we have shown that (8.5.41) holds for $(i,j) \in \mathbb{N}^2$ with $i+j \leq k$ and arbitrary $k \in \mathbb{N}$, $k \geq 2$.

In the following we write $A := A_{n,\eta}$, then we obtain as in the Roumieu-case

$$(Y_k^l)^{(i,0)}(\tau,\sigma)|_{\tau=0} = (p_k^l\eta)^{i-1} \cdot (2A)^i \cdot \frac{\Gamma(i-2^{-1})}{2\pi^{1/2}} \cdot (1 - p_k^l \eta \varrho_k^l(\sigma))^{1-2i} \tag{8.5.42}$$

for all $i \in \mathbb{N}$, $i \geq 1$. Since $A, \eta \geq 1$ we also get

$$|1 - p_k^l \cdot \eta \cdot \varrho_k^l(\sigma)| \geq \frac{1}{2} \tag{8.5.43}$$

for $|\sigma| \leq (5 \cdot A \cdot p_k^l \cdot \eta^2)^{-1}$. For such $\sigma \in \mathbb{C}$ we introduce again

$$\omega_{l,k,i} : \mathbb{C} \to \mathbb{C}, \quad \omega_{l,k,i}(\sigma) := (1 - p_k^l \cdot \eta \cdot \varrho_k^l(\sigma))^{1-2i}. \tag{8.5.44}$$

and so

$$\forall\, j \in \mathbb{N} : \quad \left| \omega_{l,k,i}^{(j)}(0) \right| < j! \cdot (2^{2i-1}) \cdot (8 \cdot A \cdot p_k^l \cdot \eta^2)^j. \tag{8.5.45}$$

Using this we finally obtain

$$(Y_k^l)^{(i,j)}(0,0) \leq 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^{i+j} \cdot (i+j)! \cdot (p_k^l)^{i+j-1} \tag{8.5.46}$$

for $i \in \mathbb{N}$, $i \geq 1$.
With this estimate we can finish the proof: Let $k \geq 2$ and put $i + j = k$ in (8.5.46), then we obtain

$$(Y_k^l)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0} \leq 2^{-2} \cdot \pi^{-1/2} \cdot (8A\eta^2)^k \cdot k! \cdot (p_k^l)^{k-1} = 2^{-2} \cdot \pi^{-1/2} \cdot n \cdot \left(\frac{8A\eta^2}{n}\right)^k \cdot M_{k-1}^l. \tag{8.5.47}$$

For the last equality above recall that by definition we have $(p_k^l)^{k-1} = \left(\frac{1}{n}\right)^{k-1} \cdot \frac{M_{k-1}^l}{k!}$. Thus for l arbitrary we obtain for all $(t,\mu) \in I_T \times V$ (where we put $i+j=k$, $i \geq 1$):

$$\left\| y^{(i,j)}(t,\mu) \right\|_{L^{i,j}(\mathbb{R},E_4;E_3)} \underbrace{\leq}_{(8.5.41)} (Y_k^l)^{(i,j)}(\tau,\sigma)|_{\tau=\sigma=0}$$

$$\underbrace{\leq}_{(8.5.47)} 2^{-2} \cdot \pi^{-1/2} \cdot n \cdot \left(\frac{8A\eta^2}{n}\right)^{i+j} \cdot M_{i+j-1}^l.$$

But since $k \in \mathbb{N}$, $k \geq 2$, is arbitrary and we have already shown (8.5.32) and finally the case $i = 1$, $j = 0$ is satisfied by assumptions 8.5.29 and 8.5.30 we have (8.5.31) and are done.

More precisely we have shown that the solution y belongs to the intersection of all Roumieu-classes $\mathcal{E}_{\{M^l\}}$. But by our additional assumption $(\mathcal{M}_{(\mathrm{BR})})$ on the weight matrix $\mathcal{M}$ we can use (9.1.3) in 9.1.1 below, i.e. the Beurling space $\mathcal{E}_{(\mathcal{M})}$ can also be written as intersection of the Roumieu-classes $\mathcal{E}_{\{M^l\}}$.

(2)(b) We use the same proof and trick as in [19] to reduce the Beurling- to the Roumieu-case. Therefore we have to use Lemma 8.5.2 below which is a little variation (resp. generalization) of [17, Lemma 6].
First, by assumption on M, we have $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$, $m_k := \frac{M_k}{k!}$, and (dc) hence $\mathcal{E}_{(M)} = \mathcal{E}_{(M^d)}$ holds with $M^d = (M_{k+1})_k$. As pointed out before in $(ii), (iii)$ in 8.2.5, condition (ai) for M^d translates into (ai$_K$) for M. Assume now that the right hand side f in (8.5.1) satisfies the Beurling-type estimates for the class $\mathcal{E}_{(M)}$. We put now $A_k := \max\{\sup\{\|f^{(a,b,c)}(x,t,\lambda)\|_{L^{a,b,c}(E_1,\mathbb{R},E_2;E_1)} : (x,t,\lambda) \in W\} : \ a+b+c = k\}$ (see (8.5.2)), hence by definition $A \lhd M$ follows. Then, by applying 8.5.2, we can find now a sequence N with $A \leq N \lhd M$ and N has property (ai$_K$), too. So f in (8.5.1) satisfies the Roumieu-type estimates for the class $\mathcal{E}_{\{N\}}$ and we can use the Roumieu-case (1) of 8.5.1 to conclude that the solution x of (8.5.1) satisfies the estimates for the class $\mathcal{E}_{\{N\}}$, too. Finally $N \lhd M$ tells us that x belongs to the class $\mathcal{E}_{(M)}$, too. □

Lemma 8.5.2. *Let L be an arbitrary sequence of positive real numbers and let M be a an arbitrary sequence of positive real numbers weight with additionally $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$, where $m_k := \frac{M_k}{k!}$. Assume now that $L \lhd M$ holds. Then there exists a sequence N with $L \leq N \lhd M$ and $\lim_{k\to\infty}(n_k)^{1/k} = +\infty$, where $n_k := \frac{N_k}{k!}$. If M satisfies in addition property* (ai$_K$) *resp.* (ai)*, then N has* (ai$_K$) *resp.* (ai)*, too.*

Proof. (ai$_K$): We start as in [17, Lemma 6] and we introduce a sequence $\overline{L} = (\overline{L}_p)_p$ defined by $\overline{L}_p := \inf_{h>0}\{C_h \cdot h^p \cdot M_p\}$, where $C_h > 0$ is the least real number coming from relation $L \lhd M$. By definition we get $L \leq \overline{L} \lhd M$. Since the infimum is attained at some $h > 0$ we obtain

$$\left(\frac{M_p}{\overline{L}_p}\right)^2 \leq \frac{M_{p-1}}{\overline{L}_{p-1}} \cdot \frac{M_{p+1}}{\overline{L}_{p+1}},$$

i.e. the sequence $\left(\frac{M_p}{\overline{L}_p}\right)_{p\geq 0}$ is log. convex. Now put

$$c_p := \left(\frac{M_p}{\overline{L}_p}\right)^{1/(p-1)} \qquad \text{for } p \geq 2,$$

then by 8.5.3 we see that $c_p \leq \left(\frac{M_1}{\overline{L}_1}\right)^{1/p} \cdot c_{p+1}$ holds for $p \geq 2$. Since $\overline{L} \lhd M$ we also get $\lim_{p\to\infty} c_p = +\infty$. We define the sequence $N = (N_p)_{p\geq 0}$ by

$$\left(\frac{N_p}{p!}\right)^{1/(p-1)} = (n_p)^{1/(p-1)} := \max\left\{(m_p)^{1/(2(p-1))}, \max\left\{\frac{(m_q)^{1/(q-1)}}{c_q} : 2 \leq q \leq p\right\}\right\}$$

for $p \geq 2$ and $n_0 = N_0 := L_0$, $n_1 = N_1 := L_1$. Then, by definition we immediately get (for the case $q = p$ in the second maximum) $n_p \geq \frac{m_p}{c_p^{p-1}} \Leftrightarrow N_p \geq \frac{M_p}{c_p^{p-1}} = M_p \cdot \frac{\overline{L}_p}{M_p} = \overline{L}_p$ for all $p \geq 2$, moreover $N_0 = L_0$ and $N_1 = L_1$ by definition, hence $L \leq N$ follows.

Next we are going to show $N \lhd M$: For $p \geq 2$ we get

$$\begin{aligned}\left(\frac{N_p}{M_p}\right)^{1/(p-1)} &= \left(\frac{n_p}{m_p}\right)^{1/(p-1)} \\ &= \max\left\{\frac{1}{(m_p)^{1/(2(p-1))}}, \max\left\{\frac{1}{(m_p)^{1/(p-1)}}\cdot\frac{(m_q)^{1/(q-1)}}{c_q} : 2 \leq q \leq p\right\}\right\},\end{aligned}$$

since $(m_p)^{1/p} \to \infty$ we also have $(m_p)^{1/(p-1)} \to \infty$ for $p \to \infty$ and because also $\lim_{p\to\infty} c_p = +\infty$ we get that for each $\varepsilon > 0$ (small) there exists a natural number q_ε such that $\frac{1}{c_q} \leq \varepsilon$ for all $q \geq q_\varepsilon$. Thus we get

$$\left(\frac{n_p}{m_p}\right)^{1/(p-1)} \leq \max\left\{\frac{1}{(m_p)^{1/(2(p-1))}}, C\cdot\varepsilon, \frac{1}{(m_p)^{1/(p-1)}}\cdot\frac{(m_q)^{1/(q-1)}}{c_q} : 2 \leq q \leq q_\varepsilon\right\} \leq C\cdot\varepsilon$$

for $p \in \mathbb{N}$ sufficiently large enough. C is precisely the constant coming from (ai_K) since $\frac{(m_q)^{1/(q-1)}}{(m_p)^{1/(p-1)}} \leq C$ holds for $2 \leq q \leq p$.

Finally we have: By definition $(n_p)^{1/(p-1)} \geq \sqrt{(m_p)^{1/(p-1)}} \to \infty$ for $p \to \infty$, so also $(n_p)^{1/p} \to \infty$ for $p \to \infty$ and property (ai_K) for $n = (n_p)_p$ follows with constant $\sqrt{C}$, whenever we have this property for $m = (m_p)_p$ with constant C. More precisely we can use 8.5.4 below.

(ai): We proceed as before and consider again the sequence $\overline{L} = (\overline{L}_p)_p$ defined by $\overline{L}_p := \inf_{h>0}\{C_h \cdot h^p \cdot M_p\}$, where $C_h > 0$ is the least real number coming from $L \lhd M$. Then put

$$c_p := \left(\frac{M_p}{\overline{L}_p}\right)^{1/p} \quad \text{for } p \geq 1,$$

and so (see e.g. [38, Lemma 2.0.4]) we have $c_p \leq \left(\frac{M_0}{\overline{L}_0}\right)^{1/(p+1)} \cdot c_{p+1}$ for all $p \geq 1$. Since $\overline{L} \lhd M$ we immediately get $\lim_{p\to\infty} c_p = +\infty$. We define the sequence $N = (N_p)_{p\geq 0}$ by

$$\left(\frac{N_p}{p!}\right)^{1/p} = (n_p)^{1/p} := \max\left\{(m_p)^{1/(2p)}, \max\left\{\frac{(m_q)^{1/q}}{c_q} : 1 \leq q \leq p\right\}\right\}$$

for $p \geq 1$ and $n_0 = N_0 := L_0$. Then $L \leq N$ follows as in the previous case by definition. Relation $N \lhd M$ follows by

$$\left(\frac{N_p}{M_p}\right)^{1/p} = \left(\frac{n_p}{m_p}\right)^{1/p} = \max\left\{\frac{1}{(m_p)^{1/(2p)}}, \max\left\{\frac{1}{(m_p)^{1/p}}\cdot\frac{(m_q)^{1/q}}{c_q} : 1 \leq q \leq p\right\}\right\},$$

and since $(m_p)^{1/p} \to \infty$ for $p \to \infty$, $\lim_{p\to\infty} c_p = +\infty$. For each $\varepsilon > 0$ (small) there exists a natural number q_ε such that $\frac{1}{c_q} \leq \varepsilon$ for all $q \geq q_\varepsilon$ and so

$$\left(\frac{n_p}{m_p}\right)^{1/p} \leq \max\left\{\frac{1}{(m_p)^{1/(2p)}}, C\cdot\varepsilon, \frac{1}{(m_p)^{1/p}}\cdot\frac{(m_q)^{1/q}}{c_q} : 1 \leq q \leq q_\varepsilon\right\} \leq C\cdot\varepsilon$$

for $p \in \mathbb{N}$ sufficiently large enough. C is precisely the constant coming from (ai) for m, since $\frac{(m_q)^{1/q}}{(m_p)^{1/p}} \leq C$ holds for $1 \leq q \leq p$.

By definition we have $(n_p)^{1/p} \geq \sqrt{(m_p)^{1/p}} \to \infty$ for $p \to \infty$ and property (ai) for $n = (n_p)_p$ follows with constant $\sqrt{C}$, whenever we have this property for $m = (m_p)_p$ with constant C. More precisely we can use again 8.5.4 below. □

Lemma 8.5.3. *Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ be a sequence of positive real numbers and assume that M is log. convex. Then for $k \geq 2$ we get $(M_k)^{1/(k-1)} \leq (M_1 \cdot M_{k+1})^{1/k}$, i.e. if $M_1 = 1$ holds then the sequence $\left((M_k)^{1/(k-1)}\right)_{k\geq 2}$ is increasing.*

Proof. We prove analogously as in [38, Lemma 2.0.4]: We want to show $\log(M_1) + \log(M_{k+1}) \geq \frac{k}{k-1} \cdot \log(M_k)$ for all $k \geq 2$, use induction on k. For $k = 2$ we need $\log(M_1) + \log(M_3) \geq 2 \cdot \log(M_2)$, whereas log. convexity implies also $2 \cdot \log(M_2) \leq \log(M_1) + \log(M_3)$.
$k \mapsto k+1$: Log. convexity and I.H. imply $2 \cdot \log(M_k) \leq \log(M_{k+1}) + \log(M_{k-1}) \leq \log(M_{k+1}) + \frac{k-1}{k-2} \cdot \log(M_k) + \log(M_1)$, hence $\log(M_{k+1}) + \log(M_1) \geq \log(M_k) \cdot \left(2 - \frac{k-2}{k-1}\right) = \frac{k}{k-1} \cdot \log(M_k)$. □

Lemma 8.5.4. *Let $M^1, M^2 \in \mathbb{R}_{>0}^{\mathbb{N}}$ be given and assume that both sequences have* (ai) *resp.* (ai_K) *with constants C_1, C_2. Then both sequences $N_k = (N_k)_k$ and $L = (L_k)_k$ defined by $N_k := \max\{M_k^1, M_k^2\}$ resp. $L_k := \min\{M_k^1, M_k^2\}$ have property* (ai) *resp.* (ai_K) *with constant $C = \max\{C_1, C_2\}$, too.*

Proof. We put $n_k := \max\{m_k^1, m_k^2\}$ resp. $l_k := \min\{m_k^1, m_k^2\}$ for $n_k := \frac{N_k}{k!}$ resp. for m_k^i, $i = 1, 2$ and we treat condition (ai).
So consider $j, k \in \mathbb{N}$ with $1 \leq j \leq k$ and we obtain $(n_j)^{1/j} = \max\{(m_j^1)^{1/j}, (m_j^2)^{1/j}\}$. Assume e.g. that $(n_j)^{1/j} = (m_j^1)^{1/j}$ holds, then $(m_j^1)^{1/j} \leq C_1 \cdot (m_k^1)^{1/k}$ and so $(n_j)^{1/j} = (m_j^1)^{1/j} \leq C_1 \cdot (m_k^1)^{1/k} \leq \max\{C_1 \cdot (m_k^1)^{1/k}, C_1 \cdot (m_k^2)^{1/k}\} = C_1 \cdot (n_k)^{1/k}$. The analogous estimate holds if $(n_j)^{1/j} = (m_j^2)^{1/j}$ with the constant C_2 and so N satisfies (ai) with constant $C = \max\{C_1, C_2\}$.

Let $1 \leq j \leq k$, $j, k \in \mathbb{N}$, be given and by definition we get $(l_k)^{1/k} = \min\{(m_k^1)^{1/k}, (m_k^2)^{1/k}\}$. Now we argue as follows:
Assume that $(l_j)^{1/j} = (m_j^1)^{1/j}$, then by (ai) and definition we get $(m_j^1)^{1/j} \leq C_1 \cdot (m_k^1)^{1/k}$ and $(m_j^1)^{1/j} \leq (m_j^2)^{1/j} \leq C_2 \cdot (m_k^2)^{1/k}$. This implies $(l_j)^{1/j} \leq C \cdot (m_k^i)^{1/k}$ for $i = 1, 2$ with again $C = \max\{C_1, C_2\}$ and so $(l_j)^{1/j} \leq C \cdot \min\{(m_k^1)^{1/k}, (m_k^2)^{1/k}\} = C \cdot (l_k)^{1/k}$. Of course the analogous argument holds if we assume that $(l_j)^{1/j} = (m_j^2)^{1/j}$.

For property (ai_K) we prove analogously and consider $j, k \in \mathbb{N}$ with $2 \leq j \leq k$. □

We close this section with the following *remarks:*

(i) To reduce the proof of the Beurling-case to the Roumieu-case for the single weight sequence case Komatsu has assumed in [19] the following condition: $\frac{M_p}{p \cdot M_{p-1}} = \frac{m_p}{m_{p-1}} \to \infty$ for $p \to \infty$, which is needed to apply [17, Lemma 6]. But $\frac{m_p}{m_{p-1}} \to \infty$ for $p \to \infty$ implies $\lim_{p\to\infty}(m_p)^{1/p} = +\infty$ (see also remark on page 7 above 2.13. Lemma in [35]): There exists $k_0 \in \mathbb{N}$ with $m_{k_0} \geq 1$ (e.g. $m_1 = 1$), and so for arbitrary large $C > 0$ there exists $k_1 \in \mathbb{N}$ with $k_1 \geq k_0$ and $m_k \geq C \cdot m_{k-1}$ for all $k \geq k_1$. Hence, by iteration and after applying k-th root we get $(m_k)^{1/k} \geq (m_{k_0})^{1/k} \cdot C^{1-k/k_0} \geq C^{1/2}$, where the last inequality holds for $k \geq 2 \cdot k_1$. This shows that our assumption is weaker than Komatsu's one.

(ii) But if e.g. $k \mapsto (m_k)^{1/k}$ is increasing, then also the converse implication holds and both assumptions are equivalent: We get $(m_{k+1})^{1/(k+1)} \geq (m_k)^{1/k} \Rightarrow m_{k+1} \geq (m_k)^{(k+1)/k} = m_k \cdot (m_k)^{1/k}$. Hence $\frac{m_{k+1}}{m_k} \geq (m_k)^{1/k} \to \infty$ for $k \to \infty$.

8.6 Final characterizing theorem

Now we are able to combine all these results of the previous sections to obtain the final characterizing theorem where the classes $\mathcal{E}_{[\mathcal{M}]}$ are considered for functions between finite dimensional real Banach spaces.

Theorem 8.6.1. (*a*) *The Beurling-case: Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and assume conditions $(\mathcal{M}_{\mathcal{H}})$ and $(\mathcal{M}_{(\text{dc})})$, then the following are equivalent:*

(1) *$\mathcal{E}_{(\mathcal{M})}$ is closed under composition.*

(2) *$\mathcal{E}_{(\mathcal{M})}$ is holomorphically closed.*

(3) *Property $(\mathcal{M}_{(\text{ai})})$ holds.*

(4) *Property $(\mathcal{M}_{(\text{FdB})})$ holds.*

If we assume instead $(\mathcal{M}_{\mathcal{H}})$ the stronger condition $(\mathcal{M}_{(\mathcal{C}^\omega)})$, then $(1)-(4)$ are equivalent to

(5) *$\mathcal{E}_{(\mathcal{M})}$ is closed under inversion.*

(6) *The $\mathcal{E}_{(\mathcal{M})}$-inverse function theorem is valid.*

(7) *The $\mathcal{E}_{(\mathcal{M})}$-implicit function theorem is valid.*

If the matrix is constant, i.e. $\mathcal{M} = \{M\}$, then all conditions $(1)-(8)$ above are equivalent to

(8) *$\mathcal{E}_{(\mathcal{M})}$ is closed under solving ODE's.*

For non-constant matrices we have to assume in addition property $(\mathcal{M}_{(\text{BR})})$ (which is only needed for $(3) \Rightarrow (8)$), and then all eight conditions are equivalent.

(*b*) *The Roumieu-case: Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{N}_{>0}$ and conditions $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$, $(\mathcal{M}_{\{\text{dc}\}})$, then the following are equivalent:*

(1) *$\mathcal{E}_{\{\mathcal{M}\}}$ is closed under composition.*

(2) *$\mathcal{E}_{\{\mathcal{M}\}}$ is holomorphically resp. analytically closed.*

(3) *Property $(\mathcal{M}_{\{\text{ai}\}})$ holds.*

(4) *Property $(\mathcal{M}_{\{\text{FdB}\}})$ holds.*

(5) *$\mathcal{E}_{\{\mathcal{M}\}}$ is closed under inversion.*

(6) *The $\mathcal{E}_{\{\mathcal{M}\}}$-inverse function theorem holds.*

(7) *The $\mathcal{E}_{\{\mathcal{M}\}}$-implicit function theorem holds.*

(8) *$\mathcal{E}_{\{\mathcal{M}\}}$ is closed under solving ODE's.*

Proof. *Beurling-case:*
Conditions $(1)-(4)$ are equivalent by 8.3.5.
$(3) \Rightarrow (6)$ Holds by the second part of 8.4.1, here condition $(\mathcal{M}_{(\mathcal{C}^\omega)})$ is used.
$(6) \Rightarrow (7)$ Holds, as remarked in 8.4.1, by applying the $\mathcal{E}_{(\mathcal{M})}$-inverse function theorem to the inverse of the mapping $(x, y) \mapsto (x, f(x, y))$.
$(7) \Rightarrow (5)$ Let $f \in \mathcal{E}_{(\mathcal{M})}$ with $f(x) \neq 0$ for all x and consider the implicit equation $f \cdot g = f \cdot \frac{1}{f} = 1$.
$(5) \Rightarrow (3)$ holds by 8.3.6.

(8) $\Rightarrow$ (5) If $\mathcal{E}_{(\mathcal{M})}$ is closed under solving ODE's, then (5) holds by 8.3.4.
(3) $\Rightarrow$ (8) For the constant weight matrix case $\mathcal{M} = \{M\}$ we have to use 8.5.2 and (2)(b) in 8.5.1, for the non-constant case we use part (2)(a) of 8.5.1.

Roumieu-case:
(1) $-$ (5) are equivalent by 8.3.2.
(3) $\Rightarrow$ (6) holds by the first part of 8.4.1.
(6) $\Rightarrow$ (7) holds, as remarked in 8.4.1, by applying the $\mathcal{E}_{\{\mathcal{M}\}}$-inverse function theorem to the inverse of the mapping $(x, y) \mapsto (x, f(x, y))$.
(7) $\Rightarrow$ (5) Let $f \in \mathcal{E}_{\{\mathcal{M}\}}$ with $f(x) \neq 0$ for all x and consider the implicit equation $f \cdot g = f \cdot \frac{1}{f} = 1$.
(3) $\Rightarrow$ (8) holds by (1) in 8.5.1.
(8) $\Rightarrow$ (5) holds by 8.3.4, resp. one can also use 8.3.3 to show (8) $\Rightarrow$ (3). □

Final important remarks:

(i) To guarantee all stability properties in the literature often strong log. convexity (slc) is assumed. We summarize: 8.5.3 proves that strong log. convexity implies (ai_K) (with $H = 1$), e.g. [38, Lemma 2.0.4] proves that strong log. convexity implies (ai) (with $C = 1$) and finally e.g. in [21, 2.9. Lemma] it was shown that (slc) implies (FdB). So strong log. convexity is the strongest condition and is easy to check, but note:

(ii) In [35, 3.3.-3.6.] we have shown by constructing an explicit counterexample M that (slc) is really stronger than (FdB), more precisely there is no strong log. convex weight sequence N with $\mathcal{E}_{[M]} = \mathcal{E}_{[N]}$.

(iii) As pointed out at the end in [35, 3.6. Example] we have $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$ and so by [35, 2.15. Theorem] we have $\mathcal{E}_{[M]} = \mathcal{E}_{[M^{\mathrm{lc}}]}$. Since the constructed weight M satisfies also (dc) finally we can apply [35, 3.2. Theorem] to conclude that the sequence M^{lc} has both (FdB) and (ai) but nevertheless we cannot find any strong log. convex N with $\mathcal{E}_{[N]} = \mathcal{E}_{[M^{\mathrm{lc}}]}$.

(iv) (slc) is as a convexity condition NOT stable w.r.t. relation $\approx$, but as pointed out in remark 8.2.2, the characterizing conditions (FdB) and (ai) used in 8.6.1 and also (ai_K) are stable w.r.t. this relation.

8.7 The case if $\mathcal{M}$ is obtained by a weight function ω

In this section we study the situation where the weight matrix $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$ by $\mathcal{M} := \{M^l : l > 0\}$ with $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot j))$ and put again $m^l_j := \frac{M^l_j}{j!}$.
In this "special" case we can show more equivalences in the final theorem 8.6.1 above. First recall that each $\omega \in \mathcal{W}$ generates by 5.1.1 a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix $\mathcal{M}$, in fact properties (ω_0) and (ω_3) for ω are sufficient.
By (1) in 8.2.3 and 3.8.1 it follows, that a sub-additive weight $\omega \in \mathcal{W}$ generates a weight matrix $\mathcal{M}$, which has both conditions $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and $(\mathcal{M}_{(\mathrm{FdB})})$. More precisely we can prove:

Proposition 8.7.1. *Let $\omega \in \mathcal{W}$ be sub-additive, then we obtain the following inequality: For each $l > 0$ and all $k, j \geq 1$ with $\alpha_1 + \dots + \alpha_j = k$ we get*

$$m^l_j \cdot m^l_{\alpha_1} \cdots m^l_{\alpha_j} \leq D^j_l \cdot m^{2l}_k.$$

More precisely both conditions $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and $(\mathcal{M}_{(\mathrm{FdB})})$ are satisfied for the choice $n = 2l$.
Alternatively we also obtain a Childress-type-inequality: For $n \geq 1$ and $k_i \geq 1$ with $k := \sum_{i=1}^n k_i$ and $n = \sum_{i=1}^n i \cdot k_i$ we obtain for each $l > 0$

$$m^l_k \cdot (m^l_1)^{k_1} \cdots (m^l_n)^{k_n} \leq D^k_l \cdot m^{2l}_n.$$

Important remark: The proof of this proposition still works if we assume only (ω_0) and (ω_3) for ω, see also 3.1.2 and 3.8.5.

Proof. First, by the consequence of (5.1.2), we have $m^l_{\alpha_i} \leq D_l \cdot m^{2l}_{\alpha_i - 1}$ for $i = 1, \dots, j$ where we have put $D_l := M^{2l}_1 = \exp(\frac{1}{2l} \cdot \varphi^*_\omega(2l))$, hence $m^l_{\alpha_1} \cdots m^l_{\alpha_j} \leq D^j_l \cdot m^{2l}_{\alpha_1 - 1} \cdots m^{2l}_{\alpha_j - 1}$. Since ω is sub-additive, we can use (3.8.1) to obtain $m^{2l}_{\alpha_1 - 1} \cdots m^{2l}_{\alpha_j - 1} \leq m^{2l}_{\alpha_1 + \dots + \alpha_j - j} = m^{2l}_{k-j}$. By applying once again (3.8.1) we finally obtain

$$m^l_j \cdot m^l_{\alpha_1} \cdots m^l_{\alpha_j} \leq D^j_l \cdot m^{2l}_{k-j} \cdot m^{2l}_j \leq D^j_l \cdot m^{2l}_k.$$

For the *Childress-type-inequality* we estimate analogously as follows:

$$m^l_k \cdot (m^l_1)^{k_1} \cdots (m^l_n)^{k_n} \leq m^{2l}_k \cdot (D_l \cdot m^{2l}_{1-1})^{k_1} \cdots (D_l \cdot m^{2l}_{n-1})^{k_n} = m^{2l}_k \cdot D^k_l \cdot (m^{2l}_0)^{k_1} \cdots (m^{2l}_{n-1})^{k_n}$$
$$\leq m^{2l}_k \cdot D^k_l \cdot m^{2l}_0 \cdot m^{2l}_{1 \cdot k_2} \cdots m^{2l}_{(n-1) \cdot k_n} \leq m^{2l}_k \cdot D^k_l \cdot m^{2l}_{n-k} \leq D^k_l \cdot m^{2l}_n.$$

□

The next result is important for the characterization of the closedeness properties in terms of a condition for ω, see [10, 2.3. Proposition] for only non-quasi-analytic weights ω and also [35, 6.3. Theorem (2) ⇒ (3)]:

Lemma 8.7.2. *Let $\omega \in \mathcal{W}$ be a weight function with (ω_1) and (ω_2). If $\mathcal{E}_{\{\omega\}}(\mathbb{R}, \mathbb{C})$ is holomorphically closed, then ω satisfies in addition $(\omega_{1'})$, i.e. there exists a sub-additive weight σ with $\omega \sim \sigma$ and σ satisfies automatically all properties of ω except (ω_4), see 3.8.4.*

Proof. Of course the proof is inspired by [10, 2.3. Proposition] and generalizes this version, so we can prove this statement also for quasi-analytic weights ω. Assume by contradiction that $(\omega_{1'})$ is not satisfied, then there exists an increasing sequence of natural numbers $(k_n)_n$ and an increasing sequence of positive real numbers $(t_n)_n$ with

$$\omega(k_n \cdot t_n) \ge n^2 \cdot k_n \cdot \omega(t_n) \tag{8.7.1}$$

for all $n \in \mathbb{N}$. We introduce the sequence $a = (a_n)_n$ by $a_n := \exp(-n \cdot \omega(t_n))$ and put $f_n(x) := a_n \cdot \exp(i \cdot t_n \cdot x)$ for $x \in \mathbb{R}$. We get $|f_n^{(j)}(x)| = a_n \cdot t_n^j$ for all $j \in \mathbb{N}$ and so we calculate as follows:

$$\begin{aligned}
&\|f_n\|_{\omega,\mathbb{R},l} := \sup_{x\in\mathbb{R},j\in\mathbb{N}} \frac{|f_n^{(j)}(x)|}{\exp(\frac{1}{l}\cdot\varphi^*_\omega(lj))} = a_n \cdot \sup_{j\in\mathbb{N}} \frac{t_n^j}{\exp(\frac{1}{l}\cdot\varphi^*_\omega(lj))}\\
&= a_n \cdot \exp\left(\sup_{j\in\mathbb{N}}\left\{j\cdot\log(t_n) - \frac{1}{l}\cdot\varphi^*_\omega(lj)\right\}\right) = \exp(-n\cdot\omega(t_n))\cdot\exp\left(\sup_{j\in\mathbb{N}}\log\left(\frac{t_n^j}{M^l_j}\right)\right)\\
&= \exp(-n\cdot\omega(t_n))\cdot\exp(\omega_{M^l}(t_n)) \le \exp\left(-\left(n-\frac{1}{l}\right)\cdot\omega(t_n)\right).
\end{aligned}$$

The last inequality holds because $\omega_{M^l}(t) \le \frac{1}{l}\cdot\omega(t) = \omega_l(t)$ for all $t \ge 0$ (see 4.0.3 and 5.1.3).
Hence $\mathcal{F} := \{f_n : n \in \mathbb{N}\}$ is a bounded set in $\mathcal{E}_{\{\omega\}}(\mathbb{R},\mathbb{C})$ (even in $\mathcal{E}_{(\omega)}(\mathbb{R},\mathbb{C})$). Consider the entire mappings $g_k : z \mapsto z^k$, $k \in \mathbb{N}$, then $\mathcal{G} := \{g_k : k \in \mathbb{N}\}$ is a bounded set in $\mathcal{E}_{\{\omega\}}(\mathbb{D},\mathbb{C})$, where we put $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and identify $\mathbb{C} \cong \mathbb{R}^2$. This holds because we have by (ω_2) that the first part of (5.3.2) is valid. Moreover for $r < 1$ we set $\mathbb{D}_r := \{z \in \mathbb{C} : |z| \le r\}$ and we choose $h > 0$ large enough such that $r + \frac{1}{h} < 1$ holds. Then we obtain for arbitrary $k \in \mathbb{N}$:

$$\sup_{j\in\mathbb{N},z\in\mathbb{D}_r} \frac{\left|g_k^{(j)}(z)\right|}{h^j\cdot j!} \le \sup_{j\le k}\binom{k}{j} r^{k-j}\cdot\frac{1}{h^j} \le \left(r+\frac{1}{h}\right)^k.$$

By assumption we have property (ω_2), hence by 5.3.2 we get property $(\mathcal{M}_{\mathcal{H}})$ for the associated weight matrix $\mathcal{M}$. So the restrictions of holomorphic functions $\mathcal{H}$ are contained in $\mathcal{E}_{\{\omega\}}$, which is assumed to be holomorphically closed. Hence we can restrict the composition operator, which was introduced in 4.12. in [35], in the second factor to the class $\mathcal{H}(\mathbb{C},\mathbb{C})$ of all entire functions. Then by the analogous proof of [35, 4.14. Claim] we obtain that the set $\{f_n^k : n,k \in \mathbb{N}\}$ (composition of the set $\mathcal{F}$ with $\mathcal{G}$) is contained and bounded in $\mathcal{E}_{\{\omega\}}(\mathbb{R},\mathbb{C})$. Thus we can find a number $l > 0$ such that

$$\begin{aligned}
+\infty &> \sup_{n,k,j\in\mathbb{N}} \frac{\left|(f_n^k)^{(j)}(0)\right|}{\exp(\frac{1}{l}\cdot\varphi^*_\omega(lj))} = \sup_{n,k,j\in\mathbb{N}} \frac{a_n^k\cdot(t_nk)^j}{\exp(\frac{1}{l}\cdot\varphi^*_\omega(lj))}\\
&= \sup_{n,k\in\mathbb{N}} a_n^k\cdot\exp\left(\sup_{j\in\mathbb{N}}\log\left(\frac{(t_n\cdot k)^j}{M^l_j}\right)\right) = \sup_{n,k\in\mathbb{N}} a_n^k\cdot\exp(\omega_{M^l}(t_n\cdot k))\\
&\underbrace{\ge}_{(\star)} \sup_{n,k\in\mathbb{N}} a_n^k\cdot\exp\left(\frac{1}{2l}\cdot\omega(t_n\cdot k)\right) = \sup_{n,k\in\mathbb{N}}\exp\left(-n\cdot k\cdot\omega(t_n)+\frac{1}{2l}\cdot\omega(t_n\cdot k)\right).
\end{aligned}$$

$(\star)$ holds because $\frac{1}{l}\cdot\omega(t) = \omega_l(t) \le 2\cdot\omega_{M^l}(t)$ for all $t \ge 0$ sufficiently large (see again 4.0.3 and 5.1.3).
But this estimate contradicts the above inequality (8.7.1). □

Remark: Alternatively we can assume in 8.7.2 that the class $\mathcal{E}_{\{\omega\}}$ is closed under composition. In this case we can use 8.3.1 to conclude that property $(\mathcal{M}_{\{\mathrm{FdB}\}})$ is satisfied for the associated weight matrix $\mathcal{M}$. Then one can use [35, 4.14. Claim] directly to obtain that the above set $\{f_n^k : n, k \in \mathbb{N}\}$ is contained and bounded in $\mathcal{E}_{\{\omega\}}(\mathbb{R}, \mathbb{C})$ (see also [35, 6.3. Theorem (2) $\Rightarrow$ (3)]).

In the next step we formulate and prove the same result for the Beurling-case:

Lemma 8.7.3. *If $\omega \in \mathcal{W}$ with (ω_1) and assume that $\mathcal{E}_{(\omega)}(\mathbb{R}, \mathbb{C})$ is holomorphically closed, then ω satisfies $(\omega_{1'})$, too (i.e. ω is equivalent to a sub-additive weight function σ which satisfies automatically all properties of ω except (ω_4), see 3.8.4).*

This result was already mentioned in [10, page 405], for convenience of the reader we are going to give the proof in detail:

Proof. Since by assumption now the Fréchet-Algebra $\mathcal{E}_{(\omega)}(\mathbb{R}, \mathbb{C})$ is holomorphically closed, we can use [6, 4.1. Theorem $(f) \Rightarrow (a)$] and [30, Theorem 1] to obtain: $\mathcal{E}_{(\omega)}(\mathbb{R}, \mathbb{C})$ is a *locally m-convex algebra.* Thus there exists an equivalent system of multiplicative seminorms $(q_n)_n$ on $\mathcal{E}_{(\omega)}(\mathbb{R}, \mathbb{C})$ and we get: There exist constants $C, B, a, k > 0$ and $n \in \mathbb{N}$ such that for all $f \in \mathcal{E}_{(\omega)}(\mathbb{R}, \mathbb{C})$ and for all $m \in \mathbb{N}$ we obtain:

$$\|f^m\|_{\omega,[-1,1],1} \leq C \cdot q_n(f^m) \leq C \cdot q_n(f)^m \leq C \cdot B^m \cdot \|f\|^m_{\omega,[-a,a],k},$$

where the first and the third inequality hold by the equivalence of the semi-norm-systems, the second inequality holds by the multiplicativity of the system $(q_n)_n$. We apply this estimate now (as in [10, page 405] resp. [6, 4.1. Theorem $(b) \Rightarrow (c)$] to the function $f_t(x) = \exp(itx)$, hence $f_t^{(l)}(x) = (it)^l \cdot \exp(itx) = (it)^l \cdot f_t(x)$ and similarly we get $(f_t^m)^{(l)}(x) = (itm)^l \cdot \exp(itmx)$. With this choice we obtain:

$$\sup_{l \in \mathbb{N}} \frac{(tm)^l}{\exp(\varphi^*_\omega(l))} = \sup_{x \in [-1,1], l \in \mathbb{N}} \frac{\left|(f_t^m)^{(l)}(x)\right|}{\exp(\varphi^*_\omega(l))} = \|f^m\|_{\omega,[-1,1],1} \leq C \cdot B^m \cdot \|f\|^m_{\omega,[-a,a],k}$$
$$= C \cdot B^m \cdot \left(\sup_{x \in [-a,a], l \in \mathbb{N}} \frac{\left|f_t^{(l)}(x)\right|}{\exp(\frac{1}{k} \cdot \varphi^*_\omega(k \cdot l))}\right)^m = C \cdot B^m \cdot \left(\sup_{l \in \mathbb{N}} \frac{t^l}{\exp(\frac{1}{k} \cdot \varphi^*_\omega(l \cdot k))}\right)^m.$$

From this we immediately get $\exp(\omega_{M^1}(tm)) \leq C \cdot B^m \cdot (\exp(\omega_{M^k}(t)))^m = C \cdot B^m \cdot \exp(m \cdot \omega_{M^k}(t))$, where we have used the notation of the associated function and the family of sequences M^l. Now we apply 5.1.3 to see that $\omega_{M^1} \sim \omega$ and $\omega_{M^k} \sim \omega_k$, where we put $\omega_k(t) = \frac{1}{k} \cdot \omega(t)$. So there exist constants $A_1, A_2, B_1, B_2 \geq 1$ such that for all $t \geq 0$ we obtain:

$$\exp\left(-\frac{A_2}{A_1} + \frac{\omega(tm)}{A_1}\right) \leq \exp(\omega_{M^1}(tm)) \leq C \cdot B^m \cdot \exp(m \cdot \omega_{M^k}(t))$$
$$\leq C \cdot B^m \cdot \exp\left(B_1 \cdot m \cdot \frac{1}{k} \cdot \omega(t) + B_2 \cdot m\right).$$

Hence after applying the logarithm and multiplication with A_1 we get:

$$\omega(tm) \leq A_1 \cdot \log(C) + A_1 \cdot m \cdot \log(B) + A_1 \cdot B_2 \cdot m + A_2 + A_1 \cdot B_1 \cdot \frac{1}{k} \cdot m \cdot \omega(t).$$

This implies immediately condition $(\omega_{1'})$: Divide the inequality by $m \cdot \omega(t)$ and note that $\omega(t) \to \infty$ for $t \to \infty$. Moreover note that the constants $C, B, k, A_1, A_2, B_1, B_2$ in the calculation above don't depend on m. So ω is equivalent to a sub-additive weight function by property $(\omega_{1'})$. □

We combine all these additional results to obtain the following final theorem:

Theorem 8.7.4. *Let $\omega \in \mathcal{W}$ a weight function with (ω_1) be given. We denote by $\mathcal{M} := \{M^l = (M^l_j)_j : l > 0\}$ the associated weight matrix defined by $M^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\omega(l \cdot j))$.*

(*a*) *The Beurling-case: If ω has in addition property (ω_2), then the following properties are equivalent:*

(1) $\mathcal{E}_{(\omega)}$ *is closed under composition.*
(2) $\mathcal{E}_{(\omega)}$ *is holomorphically closed.*
(3) $\mathcal{M}$ *has property* $(\mathcal{M}_{(\mathrm{ai})})$.
(4) $\mathcal{M}$ *has property* $(\mathcal{M}_{(\mathrm{FdB})})$.
(5) *Property* $(\omega_{1'})$ *holds.*
(6) $\mathcal{M}$ *itself or the weight matrix* $\mathcal{S}$ *associated to an equivalent sub-additive weight function* σ *has property* $(\mathcal{M}_\circ)$.

If instead (ω_2) the stronger property (ω_5) holds then conditions $(1) - (6)$ are equivalent to

(7) $\mathcal{E}_{(\omega)}$ *is closed under inversion.*
(8) *The* $\mathcal{E}_{(\omega)}$*-inverse function theorem is valid.*
(9) *The* $\mathcal{E}_{(\omega)}$*-implicit function theorem is valid.*

If ω satisfies in addition (ω_6), i.e. the associated weight matrix has the form $\mathcal{M} = \{M^l\}$ for some/each $l > 0$, then all nine conditions above are equivalent to

(10) $\mathcal{E}_{(\omega)}$ *is closed under solving ODE's.*

If ω doesn't satisfy property (ω_6), i.e. the associated matrix is not constant, but (ω_7) holds (only needed here for $(3) \Rightarrow (10)$), then also all ten conditions are equivalent.

(*b*) *The Roumieu-case: If ω has in addition (ω_2), then the following are equivalent:*

(1) $\mathcal{E}_{\{\omega\}}$ *is closed under composition.*
(2) $\mathcal{E}_{\{\omega\}}$ *is holomorphically resp. analytically closed.*
(3) $\mathcal{M}$ *has property* $(\mathcal{M}_{\{\mathrm{ai}\}})$.
(4) $\mathcal{M}$ *has property* $(\mathcal{M}_{\{\mathrm{FdB}\}})$.
(5) *Property* $(\omega_{1'})$ *holds.*
(6) $\mathcal{M}$ *itself or the weight matrix* $\mathcal{S}$ *associated to an equivalent sub-additive weight function* σ *has property* $(\mathcal{M}_\circ)$.
(7) $\mathcal{E}_{\{\omega\}}$ *is closed under inversion.*
(8) *The* $\mathcal{E}_{\{\omega\}}$*-inverse function theorem holds*
(9) *The* $\mathcal{E}_{\{\omega\}}$*-implicit function theorem holds.*
(10) $\mathcal{E}_{\{\omega\}}$ *is closed under solving ODE's.*

Before we start with the proof first recall some consequences for the associated weight matrix $\mathcal{M} = \{M^l : l > 0\}$:

(*i*) By the first part of 5.3.2, the additional condition (ω_2) implies $(\mathcal{M}_{\mathcal{H}})$ and so $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$.

(*ii*) By the second part there condition (ω_5) implies $(\mathcal{M}_{(\mathcal{C}^\omega)})$.

(*iii*) Both conditions $(\mathcal{M}_{\{\text{dc}\}})$ and $(\mathcal{M}_{(\text{dc})})$ are satisfied for each $\omega \in \mathcal{W}$ automatically by 5.1.2 as a consequence of (5.1.2) there.

(*iv*) For $\omega \in \mathcal{W}$ condition (ω_6) characterizes if the associated matrix $\mathcal{M}$ is constant or not, see 5.2.2.

(*v*) For $\omega \in \mathcal{W}$ and (ω_7) we obtain property $(\mathcal{M}_{(\text{BR})})$, see 5.4.1.

(*vi*) Let $\mathcal{S} := \{S^l = (S^l_j)_j : l > 0\}$ be the associated weight matrix defined by $S^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\sigma(l \cdot j))$ for the sub-additive weight function $\sigma \sim \omega$ coming from condition $(\omega_{1'})$. Then properties $(\mathcal{M}_{[\text{ai}]})$ resp. $(\mathcal{M}_{[\text{FdB}]})$ in (3) resp. (4) are also satisfied for $\mathcal{S}$, see 8.2.2 and also 5.3.1. Note that 8.7.1 is only valid for sub-additive weights, whereas conditions $(\mathcal{M}_{[\text{ai}]})$ resp. $(\mathcal{M}_{[\text{FdB}]})$ can be also valid for matrices which are obtained by not sub-additive weights. Nevertheless the matrices $\mathcal{S}$ and $\mathcal{M}$ are equivalent w.r.t. both relations $\{\approx\}$ and $(\approx)$. For this recall also 3.1.2: We don't have necessarily condition (ω_4) for σ but this doesn't effect the construction and the proofs of the properties of $\mathcal{S}$.

Proof. *Beurling-case:*
Conditions $(1) - (4)$ are equivalent by 8.3.5.
$(2) \Rightarrow (5)$ holds by 8.7.3.
$(5) \Rightarrow (6)$ Condition $(\omega_{1'})$ means that there exists a sub-additive weight σ with $\omega \sim \sigma$ (see 3.8.4). Now we can switch to σ and use 3.8.2.
$(6) \Rightarrow (3)$ Holds by (3) of 8.2.3.
$(3) \Rightarrow (8)$ Holds by the second part of 8.4.1, here condition (ω_5) is used.
$(8) \Rightarrow (9)$ Holds, as remarked in 8.4.1, by applying the $\mathcal{E}_{(\omega)}$-inverse function theorem to the inverse of the mapping $(x, y) \mapsto (x, f(x, y))$.
$(9) \Rightarrow (7)$ Let $f \in \mathcal{E}_{(\omega)}$ with $f(x) \neq 0$ for all x and consider the implicit equation $f \cdot g = f \cdot \frac{1}{f} = 1$.
$(7) \Rightarrow (3)$ Holds by 8.3.6.
$(10) \Rightarrow (7)$ If the class $\mathcal{E}_{(\omega)}$ is closed under solving ODE's then, by 8.3.4, it is also inverse-closed and so (7) is satisfied.
$(3) \Rightarrow (10)$ For the constant weight matrix case $\mathcal{M} = \{M^l\}$ we have to use 8.5.2 and $(2)(b)$ in 8.5.1, for the non-constant case we use part $(2)(a)$ of 8.5.1.

Roumieu-case:
$(1) - (4)$ and (7) are all equivalent by 8.3.2.
$(2) \Rightarrow (5)$ Holds by 8.7.2. By the remark after 8.7.2 one can also use $(1) \Rightarrow (5)$.
$(5) \Rightarrow (6)$ Condition $(\omega_{1'})$ means that there exists a sub-additive weight σ with $\omega \sim \sigma$ (see 3.8.4). Now we can switch to σ and use 3.8.2.
$(6) \Rightarrow (3)$ Holds by (3) of 8.2.3.
$(3) \Rightarrow (8)$ Holds by the first part of 8.4.1.
$(8) \Rightarrow (9)$ Holds, as remarked in 8.4.1, by applying the $\mathcal{E}_{\{\omega\}}$-inverse function theorem to the inverse of the mapping $(x, y) \mapsto (x, f(x, y))$.
$(9) \Rightarrow (7)$ Let $f \in \mathcal{E}_{(\omega)}$ with $f(x) \neq 0$ for all x and consider the implicit equation $f \cdot g = f \cdot \frac{1}{f} = 1$.
$(3) \Rightarrow (10)$ Holds by the first part of 8.5.1.
$(10) \Rightarrow (3)$ Holds by 8.3.3, alternatively we can also use 8.3.4 to show $(10) \Rightarrow (7)$. □

As pointed out at the beginning of this chapter we cannot conclude for non-constant weight matrices $\mathcal{M}$: If $\mathcal{E}_{[\mathcal{M}]}$ satisfies 8.6.1, then already for each $\mathcal{E}_{[M^l]}$ this theorem is valid. But if the matrix is obtained by special weight functions, then we can also prove this direction:

Corollary 8.7.5. *Let $\omega \in \mathcal{W}$ be a given weight function with (ω_1), $(\omega_{1'})$ and property (ω_2) in the Roumieu resp. (ω_5) in the Beurling-case. Introduce the matrix $\mathcal{S} := \{S^l : l > 0\}$ associated to the sub-additive weight function $\sigma \sim \omega$, $S^l_j := \exp(\frac{1}{l} \cdot \varphi^*_\sigma(l \cdot j))$, and assume that each sequence S^l satisfies condition* (dc). *Then $\mathcal{E}_{[\omega]}$ resp. $\mathcal{E}_{[\sigma]}$ satisfy the closedness properties of 8.7.4 if and only if each $\mathcal{E}_{[S^l]}$ satisfies 8.6.1.*

Proof. If each class $\mathcal{E}_{[S^l]}$ satisfies the closedness properties, then clearly $\mathcal{E}_{[\sigma]}$ (resp. $\mathcal{E}_{[\omega]}$), too.
Conversely, by 8.7.4, we obtain property $(\mathcal{M}_\circ)$ for the weight matrix $\mathcal{S}$ associated to the sub-additive weight σ. By (3) in 8.2.3, each $s^l = (s^l_j)_j$, $s^l_j := \frac{S^l_j}{j!}$, satisfies now (ai). By (ω_2) resp. (ω_5) we get $\liminf_{k\to\infty}(s^l_k)^{1/k} > 0$ resp. $\liminf_{k\to\infty}(s^l_k)^{1/k} = +\infty$ for each $l > 0$ (see (3.5.4) resp. (3.5.3)) and since (dc) also holds by assumption for each S^l we see that 8.6.1 is valid for each sequence ($\leftrightarrow$ constant matrix) S^l separately. □

Remarks:

(*i*) If ω satisfies the assumptions in the previous corollary and in addition property (ω_6), then this new property also holds for σ and in this case the corollary is valid automatically, see 6.2.1 and the above remarks below 8.7.4.

(*ii*) If each sequence $S^l \in \mathcal{S}$ satisfies (dc) but ω (and so σ) doesn't satisfy property (ω_6), then by 5.2.2 all occurring sequences S^l are not equivalent w.r.t. $\approx$. In this case we don't can conclude that also each $M^l \in \mathcal{M}$, where $\mathcal{M}$ is the weight matrix associated to the function ω, satisfies (dc). For this recall also 5.3.1 and (5.1.2) in 5.1.2.

Compare this corollary also with the explicit weight function $\omega_s(t) := \max\{0, \log(t)^s\}$ for arbitrary $s \geq 2$, see the calculations in 3.10.1 and (2) in 9.5.4.

9 More properties and conditions on weight matrix spaces

9.1 Topological structure of $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$

We study now the topological structure of $\mathcal{E}_{\{\mathcal{M}\}}$ and $\mathcal{E}_{(\mathcal{M})}$ (defined locally!) in detail. By 7.3.6 it's clear, that if e.g. $\Lambda = \mathbb{R}_{>0}$, then $\mathcal{E}_{(\mathcal{M})}(U)$, $U \subseteq \mathbb{R}^n$ non-empty open, is a Fréchet-space (countable many semi-norms).
In the next proposition we summarize all properties:

Proposition 9.1.1. *Let $\mathcal{M}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, with index set $\Lambda = \mathbb{R}_{>0}$ and let $n, m \in \mathbb{N}_{>0}$. Then for each compact set $K \subseteq \mathbb{R}^n$ (with smooth boundary) we get:*

(1) *The Beurling-case*

(*a*) *We have always*

$$\varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varprojlim_{n\in\mathbb{N}_{>0}} \mathcal{E}_{\mathcal{M},1/n,1/n}(K, \mathbb{R}^m). \tag{9.1.1}$$

(*b*) *If in addition $(\mathcal{M}_{(\mathrm{L})})$ holds, then*

$$\varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varprojlim_{l\in\Lambda} \mathcal{E}_{\mathcal{M},l,1}(K, \mathbb{R}^m). \tag{9.1.2}$$

(*c*) *If in addition $(\mathcal{M}_{(\mathrm{BR})})$ is satisfied, then*

$$\mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^m) := \varprojlim_{l\in\Lambda} \mathcal{E}_{(M^l)}(K, \mathbb{R}^m) = \varprojlim_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(K, \mathbb{R}^m). \tag{9.1.3}$$

(2) *The Roumieu-case*

(*a*) *We always have*

$$\varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varinjlim_{n\in\mathbb{N}_{>0}} \mathcal{E}_{\mathcal{M},n,n}(K, \mathbb{R}^m), \tag{9.1.4}$$

which is a Silva space, i.e. a countable inductive limit of Banach spaces with compact connecting mappings. Hence this limit is regular, complete, webbed and ultrabornological - see for this e.g. [20, 52.37.].

(*b*) *If in addition property $(\mathcal{M}_{\{\mathrm{L}\}})$ holds, then we get*

$$\varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varinjlim_{l\in\Lambda} \mathcal{E}_{\mathcal{M},l,1}(K, \mathbb{R}^m). \tag{9.1.5}$$

Furthermore also the space $\varinjlim_{l\in\Lambda} \mathcal{E}_{\mathcal{M},l,1}(K, \mathbb{R}^m)$ is a Silva space.

(*c*) *If in addition property* $(\mathcal{M}_{\{\mathrm{BR}\}})$ *is satisfied, then*

$$\mathcal{E}_{\{\mathcal{M}\}}(K,\mathbb{R}^m) := \varinjlim_{l\in\Lambda} \mathcal{E}_{\{M^l\}}(K,\mathbb{R}^m) = \varinjlim_{l\in\Lambda} \mathcal{E}_{(M^l)}(K,\mathbb{R}^m). \tag{9.1.6}$$

Proof. *The Beurling-case:*
(*a*) For this consider

$$\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m) \to \mathcal{E}_{\mathcal{M},\tilde{l},\tilde{l}}(K,\mathbb{R}^m) \to \mathcal{E}_{\mathcal{M},l_1,h_1}(K,\mathbb{R}^m) \tag{9.1.7}$$

where the inclusions are bounded for all $l, l_1, h, h_1 > 0$ with $\tilde{l} := \max\{l,h\}$ and $l_1, h_1 \geq \tilde{l}$.
(*b*) We consider

$$\mathcal{E}_{\mathcal{M},s,h_1}(K,\mathbb{R}^m) \underset{\iota_1}{\to} \mathcal{E}_{\mathcal{M},s,1}(K,\mathbb{R}^m) \underset{\iota_2}{\to} \mathcal{E}_{\mathcal{M},t,h_2}(K,\mathbb{R}^m). \tag{9.1.8}$$

The mapping ι_1 is bounded for all $s \in \Lambda = \mathbb{R}_{>0}$ and $h_1 \leq 1$, the mapping ι_2 is bounded by $(\mathcal{M}_{(\mathrm{L})})$: If $h_2 \geq 1$, then ι_2 is clearly bounded for $t = s$. If $0 < h_2 < 1$ small (in which we are interested in the Beurling-case!), then for each $t \in \Lambda$ and $h_2 > 0$ we can find an index $s \in \Lambda$, $s < t$ and a constant $D > 0$ such that for all $k \in \mathbb{N}$ we get $\left(\frac{1}{h_2}\right)^k \cdot M^s_k \leq D \cdot M^t_k$.
(*c*) This holds because $(\mathcal{M}_{(\mathrm{BR})})$ tells us that for each $l \in \Lambda$ we can find $n \in \Lambda$ with $M^n \lhd M^l$, i.e. for each $l \in \Lambda$ we can find $n \in \Lambda$ such that for each $h > 0$ there exists $C > 0$ with $M^n_j \leq C \cdot h^j \cdot M^l_j$ for all $j \in \mathbb{N}$. With other words for all $l \in \Lambda$ we can find $l_1 \in \Lambda$, such that for each $h, h_1 > 0$ the inclusion

$$\mathcal{E}_{\mathcal{M},l_1,h}(K,\mathbb{R}^m) \to \mathcal{E}_{\mathcal{M},l,h_1}(K,\mathbb{R}^m)$$

is bounded.

The Roumieu-case:
(*a*) (9.1.4) follows of course also by (9.1.7) above.
To prove the Silva-space-property we consider

$$\iota : \mathcal{E}_{\mathcal{M},s,h_1}(K,\mathbb{R}^m) \underset{\iota_1}{\to} \mathcal{E}_{\mathcal{M},t,h_1}(K,\mathbb{R}^m) \underset{\iota_2}{\to} \mathcal{E}_{\mathcal{M},t,h_2}(K,\mathbb{R}^m)$$

and point out: ι_1 is bounded for $s \leq t$ (because then $M^s_k \leq M^t_k$ for all $k \in \mathbb{N}$) and ι_2 is compact for $h_2 > h_1$ and each $t \in \Lambda$ (see [16, Proposition 2.2], [38, Lemma 3.1.8]). Thus the mapping $\iota =: \iota_2 \circ \iota_1$ is compact, too.

(*b*) Now the proof of (9.1.5): Assume additionally condition $(\mathcal{M}_{\{\mathrm{L}\}})$, we show the following: For all $s \in \Lambda$ there exists $t \in \Lambda$, $s < t$, such that the mapping $\iota : \mathcal{E}_{\mathcal{M},s,1}(K,\mathbb{R}^m) \to \mathcal{E}_{\mathcal{M},t,1}(K,\mathbb{R}^m)$ is compact (for K compact set). For this let $s \in \Lambda$ be given and we write

$$\iota : \mathcal{E}_{\mathcal{M},s,1}(K,\mathbb{R}^m) \underset{\iota_1}{\to} \mathcal{E}_{\mathcal{M},s,h_1}(K,\mathbb{R}^m) \underset{\iota_2}{\to} \mathcal{E}_{\mathcal{M},s,h_2}(K,\mathbb{R}^m) \underset{\iota_3}{\to} \mathcal{E}_{\mathcal{M},t,1}(K,\mathbb{R}^m).$$

The first inclusion holds for all $h_1 \geq 1$, the second for $h_2 \geq h_1$ and finally the third by property $(\mathcal{M}_{\{\mathrm{L}\}})$ for some $t > s$: Because $h_2^k \cdot M^s_k \leq D \cdot M^t_k$ is satisfied for all $k \in \mathbb{N}$ for a number $t \in \Lambda$ which depends on s and h_2. We put now $\iota := \iota_3 \circ \iota_2 \circ \iota_1$, the mappings ι_1 and ι_3 are clearly bounded mappings (between Banach spaces), for $h_2 > h_1$ the mapping ι_2 is compact (again see [16, Proposition 2.2], [38, Lemma 3.1.8]). Hence by the ideal property of compact mappings also ι is compact.

Moreover this calculation shows that

$$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) = \varprojlim_{K\subseteq U} \varinjlim_{l\in\Lambda} \mathcal{E}_{\mathcal{M},l,1}(K,\mathbb{R}^m),$$

where we recall that $\mathcal{E}_{\mathcal{M},l,1}(K,\mathbb{R}^m)$ is the Banach-space, which consists of all functions $f \in \mathcal{E}(K,\mathbb{R}^m)$ such that $\|f\|_{\mathcal{M},K,l,1} := \sup_{x\in K,k\in\mathbb{N}} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^n,\mathbb{R}^m)}}{M_k^l} < +\infty$.

(c) Is analogous to (c) in the Beurling-case with other order of quantifiers. So for each $l\in\Lambda$ and $h>0$ we can find $l_1\in\Lambda$, such that for each $h_1>0$ the inclusion

$$\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R}^m) \to \mathcal{E}_{\mathcal{M},l_1,h_1}(K,\mathbb{R}^m)$$

is bounded. □

9.2 Diagonal constructions

An important method to transfer proofs from the single weight sequence M case to the more general (non-constant!) weight matrix case $\mathcal{M}=\{M^l : l\in\Lambda\}$, are "diagonal techniques". Let $\mathcal{M}=\{M^l : l\in\Lambda\}$ be an arbitrary non-constant weight matrix, i.e. $(\mathcal{M})$, with $\Lambda=\mathbb{R}_{>0}$. Now we can consider

(1) $i=(i_n)_{n\ge 0}$ a strictly increasing sequence of natural numbers with $i_0=0$, $i_n\to\infty$ for $n\to\infty$,

(2) and an (strictly) increasing function $\alpha:\mathbb{N}\to\mathbb{N}_{>0}$ with $\lim_{n\to\infty}\alpha(n)=+\infty$.

To this given datas we can introduce two different types of weight sequences, which are associated to a diagonal of $\mathcal{M}$ via $(i_n)_n$ and α as follows:

$$N_k^{\text{roum},\alpha,i} := M_k^{\alpha(n)} \ \text{ for } \ i_n\le k< i_{n+1} \tag{9.2.1}$$

and

$$N_k^{\text{beur},\alpha,i} := M_k^{1/(\alpha(n))} \ \text{ for } \ i_n\le k< i_{n+1}. \tag{9.2.2}$$

This technique of "running through a weight matrix along a diagonal" w.r.t. given datas will be used several times and is very important to transfer proofs of the (single) weight sequence case to the weight matrix case. Note that in the *Roumieu-case* we are interested in large indices $l\in\mathbb{N}_{>0}$, in the *Beurling-case* we are interested in small indices $0<l<1$, alternatively $l=\frac{1}{l'}$, $l'\in\mathbb{N}_{>0}$.

Recall: If a sequence M is normalized and log. convex, then the sequence $(M_j^{1/j})_j$ is increasing and $M_j\cdot M_k\le M_{j+k}$ for all $j,k\in\mathbb{N}$ is satisfied (see e.g. [38, Lemma 2.0.4, Lemma 2.0.6]).
The next result tells us that we can transfer certain properties from the matrix $\mathcal{M}$ to $N^{\text{roum},\alpha,i}$, it will be important for the proof of important projective descriptions for the Roumieu-matrix-case, see 9.4.4 and 11.1.1.

Lemma 9.2.1. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda=\mathbb{N}_{>0}$, furthermore let $i=(i_n)_{n\ge 0}$, α and $N^{\text{roum},\alpha,i}$ be given as defined above. Then we get the following:*

(1) *The sequence* $N^{\text{roum},\alpha,i}$ *satisfies automatically* (alg) *with* $C=1$ *and the mapping* $j\mapsto (N_j)^{1/j}$ *is increasing.*

(2) *Nevertheless in general (for non-constant weight matrices!) the sequence* $N^{\text{roum},\alpha,i}$ *is NOT (weakly) log. convex and as a consequence we see, that* (alg) *is strictly weaker than weakly log. convexity.*

Alternatively we can say: Increasing properties can be transferred to diagonal sequences of Roumieu-type, but convexity conditions are not well-behaved w.r.t. this construction.

Proof. (1) Let $(i_n)_n$ and α be given and put for convenience in the following $N := N^{\text{roum},\alpha,i}$.
By assumption (log. convexity) we have $M^l_j\cdot M^l_k\le M^l_{j+k}$ for all $j,k\in\mathbb{N}$ and $l\in\Lambda$, i.e. condition (alg) with $C=1$. Since $M^l_0=1$ for each $l\in\Lambda$, in (alg) the cases $j=k=0$ and $j\in\mathbb{N}$ arbitrary, $k=0$ (resp. $k\in\mathbb{N}$ arbitrary, $j=0$) are clearly satisfied for the sequence N. So let $j,k\ge 1$ be given and so we can find $n(j),n(k)\in\mathbb{N}$ with $i_{n(j)}\le j<i_{n(j)+1}$ resp. $i_{n(k)}\le k<i_{n(k)+1}$. We get:

$$N_j\cdot N_k=M_j^{\alpha(n(j))}\cdot M_k^{\alpha(n(k))}\le M_j^{\tilde\alpha}\cdot M_k^{\tilde\alpha}\le M_{j+k}^{\tilde\alpha}\le M_{j+k}^{\alpha(n(j+k))}=N_{j+k},$$

where we have put $\tilde\alpha:=\max\{\alpha(n(k)),\alpha(n(j))\}$ and for the last estimate we need only that $n(k),n(j)\le n(j+k)$ is satisfied. But this is quite obvious by definition since $n(j+k)\in\mathbb{N}_{>0}$ with $i_{n(j+k)}\le j+k<i_{n(j+k)+1}$. Similarly we show that the mapping $j\mapsto (N_j)^{1/j}$ is increasing because $j\mapsto (M^l_j)^{1/j}$ is increasing for each $l\in\Lambda$ and so for $j\le k$ we get:

$$(N_j)^{1/j}=\left(M_j^{\alpha(n(j))}\right)^{1/j}\le\left(M_k^{\alpha(n(j))}\right)^{1/k}\le\left(M_k^{\alpha(n(k))}\right)^{1/k}=(N_k)^{1/k},$$

note that $n(j)\le n(k)$ holds for $j\le k$.

Of course the proof of the first part stays valid if we assume only that $\mathcal{M}$ is an arbitrary weight matrix, i.e. $(\mathcal{M})$, and such that each M^l satisfies (alg) and the occurring constants C_l there satisfy $\sup_{l\in\Lambda}C_l\le\tilde C<+\infty$. Then N satisfies (alg) with $\tilde C$.
Similarly for the second part it's sufficient to assume only that $(M^l_j)^{1/j}\le C_l\cdot(M^l_k)^{1/k}$ for a constant $C_l>0$ and all $1\le j\le k$ (almost increasing) and $\sup_{l\in\Lambda}C_l\le\tilde C<+\infty$. Then we get $(N_j)^{1/j}\le\tilde C\cdot(N_k)^{1/k}$ for all $1\le j\le k$.

(2) Even if $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix we can not expect in general (weakly) log. convexity for such a diagonal sequence $N^{\text{roum},\alpha,i}$.
To illustrate this we construct a counter-example and we consider the *Gevrey-matrix* $M^l_j:=(j!)^l$, for $l\in\mathbb{N}_{>0}$. Moreover let $(i_n)_n$ be an arbitrary strictly increasing sequence of natural numbers with $i_0=0$, $i_n\to\infty$ for $n\to\infty$, $i_n-i_{n-1}\ge 2$ for each $n\ge 1$, and finally take $\alpha=\text{id}$ on $\mathbb{N}_{>0}$ and $\alpha(0):=1$. On each block $[i_n,i_{n+1})$ separately the sequence $N:=N^{\text{roum},\alpha,i}$ is of course weakly log. convex, but between i_n-1 and i_n for $n\ge 2$ we get a problem: we would need $(M^n_{i_n})^2\le M^{n-1}_{i_n-1}\cdot M^n_{i_n+1}$ and so

$$((i_n!)^n)^2=(i_n!)^{2n}\le((i_n-1)!)^{n-1}\cdot((i_n+1)!)^n\Leftrightarrow(i_n!)^n\le((i_n-1)!)^{n-1}\cdot(i_n+1)^n$$
$$\Leftrightarrow(i_n)^n\cdot(i_n-1)!\le(i_n+1)^n\Leftrightarrow(i_n-1)!\le\left(\frac{i_n+1}{i_n}\right)^n=\left(1+\frac{1}{i_n}\right)^n\le\left(1+\frac{1}{n}\right)^n$$

for each $n\ge 1$ and the last expression is dominated by $\exp(1)$, a contradiction. □

Remark 9.2.2. *(i) Note that 9.2.1 cannot be transferred to the Beurling-case* $N^{\text{beur},\alpha,i}$*, because here the index* $\frac{1}{\alpha(n)}$ *is decreasing in* n*.*

(ii) Another important property fails in general for non-constant $(\mathcal{M}_{\text{sc}})$ *weight matrices for the Roumieu-case* $N^{\text{roum},\alpha,i}$*: Even if assume that each sequence* $M^l \in \mathcal{M}$ *satisfies property* (dc) *we cannot expect that* $N^{\text{roum},\alpha,i}$ *has also this property: We would need that there exists a constant* $C \geq 1$ *independent of* $n \in \mathbb{N}$ *such that* $N^{\text{roum},\alpha,i}_{i_{n+1}} = M^{\alpha(n+1)}_{i_{n+1}} \leq C^{i_{n+1}} \cdot M^{\alpha(n)}_{i_{n+1}-1} = N^{\text{roum},\alpha,i}_{i_{n+1}-1}$ *holds for all* $n \in \mathbb{N}$*.*
Sow we would need that
$C \geq \left(\frac{M^{\alpha(n+1)}_{i_{n+1}}}{M^{\alpha(n)}_{i_{n+1}-1}}\right)^{1/(i_{n+1})} \geq \left(\frac{M^{\alpha(n+1)}_{i_{n+1}}}{M^{\alpha(n)}_{i_{n+1}}}\right)^{1/(i_{n+1})} \geq \left(\frac{M^{\alpha(n)+1}_{i_{n+1}}}{M^{\alpha(n)}_{i_{n+1}}}\right)^{1/(i_{n+1})}$ *holds and the right hand side can tend to* $+\infty$ *for* $n \to \infty$ *(e.g. this is the case in the previous example for the Gevrey-matrix).*

9.3 Stability of the construction of a weight matrix

In this section let $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ always be a $(\mathcal{M}_{\text{sc}})$ weight matrix. So this includes the cases, where

(*i*) $\mathcal{M} = \{M\}$ is constant and $M \in \mathcal{LC}$ and

(*ii*) $\mathcal{M}$ is associated to a given weight function $\omega \in \mathcal{W}$ by $M_j^l := \exp(\frac{1}{l} \cdot \varphi_\omega^*(lj))$, see 5.1.1 (more precisely conditions (ω_0) and (ω_3) are sufficient, see 3.1.1 and 3.1.2).

Furthermore we recall the *matrix generalized moderate-growth-conditions* of Roumieu- and of Beurling-type, which will play the key-role in the statements below:

$$(\mathcal{M}_{\{\text{mg}\}}) :\Leftrightarrow \ \forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, C \geq 1\ \forall\, j,k \in \mathbb{N} : M_{j+k}^l \leq C^{j+k} \cdot M_j^n \cdot M_k^n$$

$$(\mathcal{M}_{(\text{mg})}) :\Leftrightarrow \ \forall\, n \in \Lambda\ \exists\, l \in \Lambda\ \exists\, C \geq 1\ \forall\, j,k \in \mathbb{N} : M_{j+k}^l \leq C^{j+k} \cdot M_j^n \cdot M_k^n.$$

If the matrix $\mathcal{M}$ is obtained

(*i*) by a weight function $\omega \in \mathcal{W}$ (again conditions (ω_0) and (ω_3) would be sufficient), then in fact both conditions $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{(\text{mg})})$ are satisfied automatically for $n = 2l$ - recall (5.1.2) in 5.1.2 for this, or

(*ii*) if $\mathcal{M} = \{M\}$, then both conditions are satisfied simultaneously if and only if M has condition moderate growth (mg).

(*iii*) Moreover both conditions are clearly satisfied, if $\mathcal{M} = \{M^l : l \in \Lambda\}$, all sequences are pairwise not equivalent and each M^l has moderate growth (e.g. the Gevrey-weight-matrix $\mathcal{G} = \{(j!^s)_j : s > 1\}$).

Let $\mathcal{M} := \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then by 4.0.2 we get that $\omega_{M^l} \in \mathcal{W}$ for each $l \in \Lambda$. On the other hand, by 5.1.1, to each function $\omega \in \mathcal{W}$ we can associate a $(\mathcal{M}_{\text{sc}})$ weight matrix.

So one could consider the following construction:

$$M^l \mapsto \omega_{M^l} \mapsto M^{l;n_1} \mapsto \omega_{M^{l;n_1}} \mapsto\ M^{l;n_1,n_2} \mapsto \dots, \tag{9.3.1}$$

where for $l \in \Lambda$, $n_j > 0$ for all $j \in \mathbb{N}_{>0}$ and $i \in \mathbb{N}$ we put

$$M_i^{l;n_1,\dots,n_{j+1}} := \exp\left(\frac{1}{n_{j+1}} \cdot \varphi^*_{\omega_{M^{l,n_1,\dots,n_j}}}(n_{j+1} \cdot i)\right), \quad M_i^{l;n_1} := \exp\left(\frac{1}{n_1} \cdot \varphi^*_{\omega_{M^l}}(n_1 \cdot i)\right)$$

resp.

$$\omega_{M^{l;n_1,\dots,n_j}}(t) := \sup_{p \in \mathbb{N}} \log\left(\frac{t^p}{M_p^{l,n_1,\dots,n_j}}\right).$$

So one obtains a sequence of weight-matrices with an increasing number of indices ("derived weight matrices" w.r.t. the original matrix $\mathcal{M}$).

Goal of this section and important remark:

(1) We study the effects of this construction in detail and characterize "it's stability", i.e. that the spaces of ultradifferentiable functions of Roumieu- resp. Beurling-type defined by the weight matrix $\mathcal{M}$ are stable as locally convex vector spaces under adjoining more and more indices. Or with other words: We will characterize the situation, in which 9.3.1 yields only weight matrices which are equivalent to $\mathcal{M}$. It will turn out, that only in the first step in 9.3.1, adjoining a second parameter, there can occur a non-stable effect.

(2) All results in this section are also valid for globally defined classes with obvious modifications of the proofs below, see 7.3.2.

First we are going to prove the following important formula:

Lemma 9.3.1. *For each $l \in \Lambda$, $n \in \mathbb{N}_{>0}$ and $i \in \mathbb{N}$ we have*

$$M_i^{l;n} = (M_{i\cdot n}^l)^{1/n}. \tag{9.3.2}$$

Proof. For this we proceed as follows:

$$\begin{aligned} M_j^{l;n} &:= \exp\left(\frac{1}{n}\cdot\varphi^*_{\omega_{M^l}}(nj)\right) = \exp\left(\frac{1}{n}\cdot\sup_{y\ge 0}\{y\cdot(nj)-\varphi_{\omega_{M^l}}(y)\}\right) \\ &= \exp\left(\sup_{y\ge 0}\left\{(y\cdot j)-\frac{1}{n}\cdot\varphi_{\omega_{M^l}}(y)\right\}\right) = \sup_{y\ge 0}\frac{\exp(yj)}{\exp\left(\frac{1}{n}\cdot\varphi_{\omega_{M^l}}(y)\right)} \\ &\underbrace{=}_{\exp(y)=s} \sup_{s\ge 1}\frac{s^j}{\exp\left(\frac{1}{n}\cdot\omega_{M^l}(s)\right)} = \left(\sup_{s\ge 0}\frac{s^{jn}}{\exp(\omega_{M^l}(s))}\right)^{1/n} = (M_{jn}^l)^{1/n}, \end{aligned}$$

where in the last step we have used [16, 3.2. Proposition]. Note that all steps except the last one hold also for $n > 0$ instead of $n \in \mathbb{N}_{>0}$. □

Next we prove an important preparation result which is a generalization of [16, 3.6. Proposition]) and gives an equivalent description of the matrix-generalized moderate-growth conditions:

Proposition 9.3.2. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then we get*

$$(\mathcal{M}_{\{\text{mg}\}}) \Longleftrightarrow \forall\, l\in\Lambda\;\exists\, H\ge 1\;\exists\, n\in\Lambda\;\forall\, t\ge 0: 2\cdot\omega_{M^n}(t)\le\omega_{M^l}(Ht)+H \tag{9.3.3}$$

and

$$(\mathcal{M}_{(\text{mg})}) \Longleftrightarrow \forall\, l\in\Lambda\;\exists\, H\ge 1\;\exists\, n\in\Lambda\;\forall\, t\ge 0: 2\cdot\omega_{M^l}(t)\le\omega_{M^n}(Ht)+H \tag{9.3.4}$$

Proof. The proof is of course inspired by [16, 3.5. Lemma, 3.6. Prop.], we use the same technique and consider the Roumieu-case. The proof of the Beurling-case is completely analogous (with Beurling-type order of quantifiers)!
First remark that $(\mathcal{M}_{\{\text{mg}\}})$ is clearly equivalent to the following condition:
For each $l \in \Lambda$ there exists a constant $H \ge 1$ and $n \in \Lambda$ such that for all $p \in \mathbb{N}$ we get $M_p^l \le H^p \cdot \min_{0\le q\le p} M_q^n \cdot M_{p-q}^n =: N_p^n \cdot H^p$.

By [16, 3.5. Lemma] we have $\omega_{N^n} = 2 \cdot \omega_{M^n}$. Then we proceed analogously as in [16, 3.6. Prop.] to get

$$2 \cdot \omega_{M^n}(t) = \omega_{N^n}(t) = \sup_{p\in\mathbb{N}} \log\left(\frac{t^p}{N^n_p}\right) = \sup_{p\in\mathbb{N}} \log\left(\frac{t^p}{\min_{0\le q\le p} M^n_q \cdot M^n_{p-q}}\right) \le \sup_{p\in\mathbb{N}} \log\left(\frac{t^p \cdot H^p}{M^l_p}\right)$$
$$= \omega_{M^l}(H \cdot t).$$

Conversely we obtain by using again [16, 3.2. Proposition] in the first step:

$$\begin{aligned} N^n_p &= \sup_{t\ge 0} \frac{t^p}{\exp(\omega_{N^n}(t))} = \sup_{t\ge 0} \frac{t^p}{\exp(2\cdot\omega_{M^n}(t))} \ge \sup_{t\ge 0} \frac{t^p}{\exp(\omega_{M^l}(H\cdot t) + H)} \\ &= \frac{1}{\exp(H)} \cdot \sup_{t\ge 0} \frac{t^p}{\exp(\omega_{M^l}(H\cdot t))} \underbrace{=}_{s=H\cdot t} \frac{1}{H^p\cdot\exp(H)} \cdot \sup_{s\ge 0} \frac{s^p}{\exp(\omega_{M^l}(t))} \\ &= \frac{1}{H^p \cdot \exp(H)} \cdot M^l_p. \end{aligned}$$

□

Now we are able to prove the first important result:

Theorem 9.3.3. *Let $\mathcal{M} := \{M^l \in \mathbb{R}^{\mathbb{N}}_{>0} : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix and $n, m \in \mathbb{N}_{>0}$. If additionally property $(\mathcal{M}_{\{\text{mg}\}})$ holds then for each open non-empty set $U \subseteq \mathbb{R}^n$ we get as locally convex vector spaces*

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^m) := \varprojlim_{K\subseteq U} \varinjlim_{l\in\Lambda, h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varprojlim_{K\subseteq U} \varinjlim_{l\in\Lambda, n, h>0} \mathcal{E}_{M^{l,n},h}(K, \mathbb{R}^m),$$

if additionally property $(\mathcal{M}_{(\text{mg})})$ holds then we get as locally convex vector spaces

$$\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^m) := \varprojlim_{K\subseteq U} \varprojlim_{l\in\Lambda, h>0} \mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R}^m) = \varprojlim_{K\subseteq U} \varprojlim_{l\in\Lambda, n, h>0} \mathcal{E}_{M^{l,n},h}(K, \mathbb{R}^m).$$

Proof. First consider the *Roumieu-case:* Since we have always $(\subseteq)$, because $M^l = M^{l;1} \le M^{l_1}$ for each $l_1, l \in \Lambda$ with $l_1 \ge l$ holds as special case of (9.3.2), we only need to show $(\supseteq)$. Moreover (9.3.2) above tells us that for $l \in \Lambda$, $n \in \mathbb{N}_{>0}$ and $j \in \mathbb{N}$ we have

$$M^{l;n}_j = (M^l_{jn})^{1/n}.$$

Hence it suffices to prove that

$$\forall\, l \in \Lambda\ \forall\, n \in \mathbb{N}_{>0}\ \exists\, x \in \Lambda\ \exists\, C \ge 1\ \forall\, j \in \mathbb{N}: \ (M^l_{jn})^{1/n} \le C^j \cdot M^x_j \Leftrightarrow M^l_{jn} \le C^{jn} \cdot (M^x_j)^n,$$

which yields $\mathcal{E}_{M^{l,n},h}(K, \mathbb{R}^m) \subseteq \mathcal{E}_{\mathcal{M},x,C\cdot h}(K, \mathbb{R}^m) = \mathcal{E}_{M^x,C\cdot h}(K, \mathbb{R}^m)$ in the corresponding loc. convex vector space representations.
We use now property $(\mathcal{M}_{\{\text{mg}\}})$ directly: For each $l \in \Lambda$ there exists $D \ge 1$ and $l_1 \in \Lambda$ such that (for the case $j = k$) we have $M^l_{2j} \le D^{2j} \cdot (M^{l_1}_j)^2$. The desired property follows now by iterating this estimate n-times.

For the *Beurling-case* we point out: In this case we have to show $(\subseteq)$, since $M^{l;1} = M^l$ for each $l \in \Lambda$ and so $(\supseteq)$ is valid automatically. We have to prove that

$$\forall\, l \in \Lambda\ \forall\, n > 0\ \exists\, x \in \Lambda\ \exists\, C \ge 1\ \forall\, j \in \mathbb{N}: \ M^x_j \le C^j \cdot M^{l;n}_j,$$

which yields $\mathcal{E}_{\mathcal{M},x,h}(K,\mathbb{R}^m)=\mathcal{E}_{M^x,h}(K,\mathbb{R}^m)\subseteq\mathcal{E}_{M^{l;n},C\cdot h}(K,\mathbb{R}^m)$ in the corresponding loc. convex vector space representations. So we are interested in $n>0$ very small, we put $\frac{1}{n}=n_1\in\mathbb{N}$ (for n_1 large) and then we can estimate for all $l\in\Lambda$ and $j\in\mathbb{N}$ as follows:

$$\begin{aligned}
M_j^{l;1/n_1} &= \sup_{t\geq 0}\frac{t^j}{\exp(n_1\cdot\omega_{M^l}(t))}\underbrace{\geq}_{(\star)}\sup_{t\geq 0}\frac{t^j}{\exp(\omega_{M^{l_1}}(H^k t)+(2^k-1)\cdot H)}\\
&=\frac{1}{\exp((2^k-1)\cdot H)}\cdot\sup_{t\geq 0}\frac{t^j}{\exp(\omega_{M^{l_1}}(H^k t))}\\
&\underbrace{=}_{s=t\cdot H^k}\frac{1}{\exp((2^k-1)\cdot H)}\cdot\left(\frac{1}{H^k}\right)^j\cdot\sup_{s\geq 0}\frac{s^j}{\exp(\omega_{M^{l_1}}(s))}\\
&=\frac{1}{\exp((2^k-1)\cdot H)}\cdot\left(\frac{1}{H^k}\right)^j\cdot M_j^{l_1}.
\end{aligned}$$

The first equality holds by the proof of (9.3.2). $(\star)$ is satisfied because by assumption we have property $(\mathcal{M}_{(\text{mg})})$ which is equivalent to (9.3.4) as shown in 9.3.2. An iterated application of (9.3.4) proves as in (3.4.2) that for all $l\in\Lambda$ we can find $l_1\in\Lambda$ and $H\geq 1$ such that for all $t\geq 0$ we have $n_1\cdot\omega_{M^l}(t)\leq 2^k\cdot\omega_{M^l}(t)\leq\omega_{M^{l_1}}(H^k\cdot t)+(2^k-1)\cdot H$ where $k\in\mathbb{N}$ is chosen minimal such that $n_1\leq 2^k$ holds.
Hence we have shown: To arbitrary $l\in\Lambda$ and $n_1\in\mathbb{N}$ large (so $\frac{1}{n_1}>0$ small) we can find an index $l_1\in\Lambda$ such that $M^{l_1}\preceq M^{l,1/n_1}$ holds. □

Theorem 9.3.3 together with the introduced techniques and notations shows now the following:

(*i*) If we assume more generally that $\mathcal{M}$ is only a $(\mathcal{M}_{\text{sc}})$ weight matrix (without any moderate-growth conditions), then each associated function ω_{M^l} satisfies automatically $\omega_{M^l}\in\mathcal{W}$ (as mentioned at the beginning of this section - see e.g. 4.0.2). Hence the matrix $\mathcal{M}^l:=\{M^{l,n}:n>0\}$ satisfies for each $l\in\Lambda$ separately both conditions $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{(\text{mg})})$ automatically, see 5.1.2. So after the first step in (9.3.1) the construction is *always* stable (as locally convex vector spaces) under adjoining more and more indices.

(*ii*) The construction (9.3.1) and the proofs of (9.3.2), Proposition 9.3.2 and finally Theorem 9.3.3 still hold for "large" (uncountable) weight matrices $\mathcal{M}$ in the sense of definitions (7.2.1) resp. (7.2.2) if we assume for the matrix $\mathcal{M}$ conditions $(\mathcal{M}_{\{\text{fil}\}})$ resp. $(\mathcal{M}_{(\text{fil})})$ and such that each $M\in\mathcal{M}$ satisfies $M\in\mathcal{LC}$.

Moreover in the construction (9.3.1) we also get matrices of weight functions and we have immediately by 4.0.3 and 5.1.3 the following equivalences:

$$\forall\, l\in\Lambda\;\forall\, j\in\mathbb{N}\;\forall\, n_1,\dots,n_j>0:\;\omega_{M^{l;n_1,\dots,n_{j+1}}}\sim\omega_{M^{l;n_1,\dots,n_j}}\sim\dots\sim\omega_{M^l}. \tag{9.3.5}$$

Hence by 3.2.1 this construction is always stable (also in the first step) and we immediately obtain the following representations as top. vector spaces for all non-empty open sets $U\subseteq\mathbb{R}^n$:

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U,\mathbb{R}^m):=\varprojlim_{K\subseteq U}\varinjlim_{l\in\Lambda,h>0}\mathcal{E}_{\omega_{M^l},h}(K,\mathbb{R}^m)=\varprojlim_{K\subseteq U}\varinjlim_{l\in\Lambda,n,h>0}\mathcal{E}_{\omega_{M^{l;n}},h}(K,\mathbb{R}^m) \tag{9.3.6}$$

and

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U,\mathbb{R}^m):=\varprojlim_{K\subseteq U}\varprojlim_{l\in\Lambda,h>0}\mathcal{E}_{\omega_{M^l},h}(K,\mathbb{R}^m)=\varprojlim_{K\subseteq U}\varprojlim_{l\in\Lambda,n,h>0}\mathcal{E}_{\omega_{M^{l;n}},h}(K,\mathbb{R}^m), \tag{9.3.7}$$

where $\omega_{\mathcal{M}} := \{\omega_{M^l} : l \in \Lambda\}$ denotes a one-parameter weight matrix of weight functions in the set $\mathcal{W}$.
The next aim is to prove the converse statement of Theorem 9.3.3:

Proposition 9.3.4. *Let $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix. Assume that*

$$\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) = \varprojlim_{K\subseteq\mathbb{R}} \varinjlim_{l\in\Lambda,n,h>0} \mathcal{E}_{M^{l,n},h}(K,\mathbb{R}),$$

then also property $(\mathcal{M}_{\{\mathrm{mg}\}})$ holds. Moreover if e.g. $\Lambda = \mathbb{R}_{>0}$ and

$$\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R}) = \varprojlim_{K\subseteq\mathbb{R}} \varprojlim_{l\in\Lambda,n,h>0} \mathcal{E}_{M^{l,n},h}(K,\mathbb{R}),$$

then additionally property $(\mathcal{M}_{(\mathrm{mg})})$ is satisfied.

Remark: A second formulation of 9.3.4 would be to assume that there exists a compact set $K \subseteq \mathbb{R}$ such that $\mathcal{E}_{\{\mathcal{M}\}}(K,\mathbb{R}) = \varinjlim_{l\in\Lambda,n,h>0} \mathcal{E}_{M^{l,n},h}(K,\mathbb{R})$ resp. $\mathcal{E}_{(\mathcal{M})}(K,\mathbb{R}) = \varprojlim_{l\in\Lambda,n,h>0} \mathcal{E}_{M^{l,n},h}(K,\mathbb{R})$ holds. In this case we can use precisely the same proof below, in the Roumieu-case replace the occurring function $\theta_{l,n}$ by $\psi_{l,n}(t) := \theta_{l,n}(t - x_0)$, in the Beurling-case the function(s) $\tilde{f}_t$ by $\tilde{g}_t(s) := \tilde{f}_t(s - x_0)$ for a point $x_0 \in K$.

Proof. The proof is analogous to 5.2.1 above.
First *the Roumieu-case:* By assumption $\varprojlim_{K\subseteq\mathbb{R}} \varinjlim_{l\in\Lambda,n,h>0} \mathcal{E}_{M^{l,n},h}(K,\mathbb{R}) \subseteq \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R})$ is satisfied. Note that each sequence $M^{l,n}$ is (weakly) log. convex, hence for each $l \in \Lambda$ and $n > 0$ there exists a function $\theta_{l,n} \in \mathcal{E}^{\mathrm{global}}_{\{M^{l,n}\}}(\mathbb{R},\mathbb{R})$ with $M_j^{l,n} \le |\theta_{l,n}^{(j)}(0)|$ for each $j \in \mathbb{N}$ (see (chf)). By assumption we get now $\theta_{l,n} \in \mathcal{E}^{\mathrm{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ and so we obtain: For all $l_1 \in \Lambda$ and for all $n > 0$ there exists an index $l_2 \in \Lambda$ such that $M^{l_1,n} \preceq M^{l_2} = M^{l_2,1}$. We apply log and translate this into the following condition:

$$\forall\, l_1 \in \Lambda\ \forall\, n > 0\ \exists\, n_2 \in \Lambda\ \exists\, C \ge 1\ \forall\, j \in \mathbb{N}: \ \frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(nj) \le j \cdot \log(C) + \varphi^*_{\omega_{M^{l_2}}}(j). \tag{9.3.8}$$

Replace now in (9.3.8) "for all $j \in \mathbb{N}$" by "for all $y \ge 0$", then we can apply on both sides the *Legendre-Fenchel-Young-conjugate* (for all $x \ge 0$) and calculate as follows: The left hand side gives

$$\begin{aligned}\left(\frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(n\cdot)\right)^*(x) &= \sup_{y\ge0}\left\{x \cdot y - \frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(n \cdot y)\right\} \underbrace{=}_{y'=ny} \frac{1}{n} \cdot \sup_{y'\ge0}\left\{(x \cdot n) \cdot \frac{y'}{n} - \varphi^*_{\omega_{M^{l_1}}}(y')\right\} \\ &= \frac{1}{n} \cdot \varphi^{**}_{\omega_{M^{l_1}}}(x) \underbrace{=}_{(\star)} \frac{1}{n} \cdot \varphi_{\omega_{M^{l_1}}}(x) = \frac{1}{n} \cdot \omega_{M^{l_1}}(\exp(x)).\end{aligned}$$

Note that $(\star)$ holds since by 4.0.2 we get $\omega_{M^l} \in \mathcal{W}$ and so $\varphi^{**}_{\omega_{M^l}}(x) = \varphi_{\omega_{M^l}}(x)$ for arbitrary $l \in \Lambda$ automatically. Similarly we obtain for the right hand side, where we have put $\log(C) = D$:

$$\begin{aligned}(\cdot D + \varphi^*_{\omega_{M^{l_2}}}(\cdot))^*(x) &= \sup_{y\ge0}\{(x - D) \cdot y - \varphi^*_{\omega_{M^{l_2}}}(y)\} = \varphi^{**}_{\omega_{M^{l_2}}}(x - D) \\ &= \varphi_{\omega_{M^{l_2}}}(x - D) = \omega_{M^{l_2}}(\exp(x - D)) = \omega_{M^{l_2}}\left(\frac{\exp(x)}{C}\right).\end{aligned}$$

Then we use 4.0.3 (replace there in the proof ω by $\omega_{M^{l_2}}$): We have $\omega_{M^{l_2}} = \omega_{M^{l_2,1}}$ and for all $x \geq 0$ sufficiently large (x with $\frac{\exp(x)}{C} \geq \mu_2^{l_2}$) we obtain:

$$\frac{1}{n} \cdot \omega_{M^{l_1}}(\exp(x)) = \sup_{y\geq 0}\left\{x \cdot y - \frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(n \cdot y)\right\} \geq \sup_{j\in\mathbb{N}}\left\{x \cdot j - \frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(n \cdot j)\right\}$$
$$\underbrace{\geq}_{(9.3.8)} \sup_{j\in\mathbb{N}}\{x \cdot j - j \cdot D - \varphi^*_{\omega_{M^{l_2}}}(j)\} \geq \frac{1}{2} \cdot \sup_{y\geq 0}\{x \cdot y - y \cdot D - \varphi^*_{\omega_{M^{l_2}}}(y)\}$$
$$= \frac{1}{2} \cdot \varphi^{**}_{\omega_{M^{l_2}}}(x - D) = \frac{1}{2} \cdot \varphi_{\omega_{M^{l_2}}}(x - D) = \frac{1}{2} \cdot \omega_{M^{l_2}}\left(\frac{\exp(x)}{C}\right).$$

So start with an arbitrary $l_1 \in \Lambda$, put (always) $n = 4$, then we obtain an index $l_2 \in \Lambda$ and a constant $C \geq 1$ such that for all t large enough (put $t := \exp(x)$) we have $2 \cdot \omega_{M^{l_2}}\left(\frac{t}{C}\right) \leq \omega_{M^{l_1}}(t)$ and so, by (9.3.3), we are done!

The Beurling-case: By assumption, we have $\bigcap_{l\in\Lambda} \mathcal{E}_{(M^l)}(\mathbb{R},\mathbb{R}) \subseteq \bigcap_{l\in\Lambda,n>0} \mathcal{E}_{(M^{l,n})}(\mathbb{R},\mathbb{R})$, and we use now once again the same trick as in the second chapter in [6] resp. in 6.5.2 and 6.5.3, see also [35, 4.6. Proposition]: Both spaces are Fréchet (because $\Lambda = \mathbb{R}_{>0}$, we can restrict the index to $\frac{1}{l}$, $l \in \mathbb{N}_{>0}$, and all occurring projective limits are countable), by the closed graph theorem the inclusion is continuous.
Hence for each compact set $K_1 \subseteq \mathbb{R}$, $l_1 \in \Lambda$, $n > 0$ and $h > 0$, there exist $C, h_1 > 0$, $l_2 \in \Lambda$ and a compact set $K_2 \subseteq \mathbb{R}$ such that for each $f \in \bigcap_{l\in\Lambda} \mathcal{E}_{(M^l)}(\mathbb{R},\mathbb{R})$ we obtain

$$\sup_{x\in K_1, j\in\mathbb{N}} \frac{|f^{(j)}(x)|}{h^j \cdot M_j^{l_1,n}} \leq C \cdot \sup_{x\in K_2, j\in\mathbb{N}} \frac{|f^{(j)}(x)|}{h_1^j \cdot M_j^{l_2}} = C \cdot \|f\|_{\mathcal{M},K_2,l_2,h_1}.$$

We apply this inequality to the following situation: We consider a compact interval K_1 containing the point 0, put $h = 1$ and take the functions $\tilde{f}_t(x) := \sin(tx) + \cos(tx)$ for $x \in \mathbb{R}$ and $t \geq 0$ (see also 6.5.3). Note that $\tilde{f}_t \in \bigcap_{l\in\Lambda} \mathcal{E}^{\text{global}}_{(M^l)}(\mathbb{R},\mathbb{R})$ for any $t \geq 0$, because $\lim_{k\to\infty}(M_k^l)^{1/k} = +\infty$ for each $l \in \Lambda$. In this situation the previous estimate implies

$$\sup_{j\in\mathbb{N}} \frac{t^j}{h^j \cdot M_j^{l_1,n}} = \sup_{j\in\mathbb{N}} \frac{|\tilde{f}_t^{(j)}(0)|}{h^j \cdot M_j^{l_1,n}} \leq \sup_{x\in K_1, j\in\mathbb{N}} \frac{|\tilde{f}_t^{(j)}(x)|}{h^j \cdot M_j^{l_1,n}} \leq C \cdot \sup_{x\in K_2, j\in\mathbb{N}} \frac{|\tilde{f}_t^{(j)}(x)|}{h_1^j \cdot M_j^{l_2}}$$
$$\leq C \cdot \sup_{j\in\mathbb{N}} \frac{2t^j}{h_1^j \cdot M_j^{l_2}}$$

and so we have

$$\sup_{j\in\mathbb{N}} \frac{t^j}{M_j^{l_1,n}} \leq 2C \cdot \sup_{j\in\mathbb{N}} \frac{t^j}{h_1^j \cdot M_j^{l_2}} \Longrightarrow \exp(\omega_{M^{l_1,n}}(t)) \leq 2C \cdot \exp\left(\omega_{M^{l_2}}\left(\frac{t}{h_1}\right)\right).$$

By using [16, 3.2. Proposition] this implies

$$M_j^{l_1,n} = \sup_{t\geq 0} \frac{t^j}{\exp(\omega_{M^{l_1,n}}(t))} \geq \sup_{t\geq 0} \frac{t^j}{2C \cdot \exp\left(\omega_{M^{l_2}}\left(\frac{t}{h_1}\right)\right)} = \frac{h_1^j}{2C} \cdot M_j^{l_2},$$

and so $M^{l_2} \preceq M^{l_1,n}$. We summarize what we have shown so far: For all $l_1 \in \Lambda$ and $n > 0$ we can find $l_2 \in \Lambda$ such that $M^{l_2} \preceq M^{l_1,n}$ holds or with other words

$$\forall\, l_1 \in \Lambda\ \forall\, n > 0\ \exists\, l_2 \in \Lambda\ \exists\, D \geq 1\ \forall\, j \in \mathbb{N}: \quad \varphi^*_{\omega_{M^{l_2}}}(j) \leq j \cdot \log(D) + \frac{1}{n} \cdot \varphi^*_{\omega_{M^{l_1}}}(nj). \tag{9.3.9}$$

By the same calculation as in the Roumieu-case we obtain: $\omega_{M^{l_2}}(t) \geq \frac{1}{2n} \cdot \omega_{M^{l_1}}\left(\frac{t}{D}\right)$ for t sufficiently large, hence for the choice $n = \frac{1}{4}$ we obtain precisely (9.3.4). □

We summarize 9.3.3 and 9.3.4:

Theorem 9.3.5. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then the construction*

$$M^l \mapsto \omega_{M^l} \mapsto M^{l;n_1} \mapsto \omega_{M^{l;n_1}} \mapsto \; M^{l;n_1,n_2} \mapsto \dots$$

is stable, more precisely it yields equivalent weight matrices, if and only if

(1) *in the Roumieu-case property $(\mathcal{M}_{\{\text{mg}\}})$ holds for $\mathcal{M}$ in addition*

(2) *in the Beurling-case we have $\Lambda = \mathbb{R}_{>0}$ and $(\mathcal{M}_{(\text{mg})})$ in addition.*

Remark 9.3.6. (1) *If the matrix $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$, then both conditions $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{(\text{mg})})$ are satisfied automatically for $n = 2l$ (recall* (5.1.2) *in 5.1.2)! If $\mathcal{M}$ consists only of one sequence M, then the construction is natural, if and only if M has property moderate growth* (mg)*!*

(2) *Moreover recall the second part of 5.1.3 and 5.2.2: For arbitrary $l \in \Lambda$ (but from now on fixed) we have $M^{l,n_1} \approx M^{l,n_2}$ for all $n_1, n_2 > 0$, if and only if M^l has moderate growth* (mg)*!*

If $M^{l,n_1} \approx M^{l,n_2}$ for all $n_1, n_2 > 0$, then ω_{M^l} has in addition property (ω_6), which is equivalent to the fact that each sequence $M^{l,n}$, $n > 0$, has moderate growth (mg)*, too! But in this case, also $M^l = M^{l,1}$ has to satisfy moderate growth.*

Conversely, if M^l has moderate growth (mg)*, then ω_{M^l} has property (ω_6) and so, by $\omega_{M^l} \sim \omega_{M^{l,n}}$ for all $n > 0$, also each $\omega_{M^{l,n}}$ has (ω_6) and each $M^{l,n}$ has moderate growth, too. Finally $M^{l,n_1} \approx M^{l,n_2}$ for all $n_1, n_2 > 0$ holds.*

In the next step we are going to study the classes $\mathcal{E}_{\{\omega_{\mathcal{M}}\}}$ resp. $\mathcal{E}_{(\omega_{\mathcal{M}})}$ defined by the matrix of weight functions $\omega_{\mathcal{M}} := \{\omega_{M^l} : l \in \Lambda\}$ in (9.3.6) resp. (9.3.7). We start with the following:

Proposition 9.3.7. *Let $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix.*

(*i*) *If additionally property $(\mathcal{M}_{\{\text{L}\}})$ holds, then*

$$\forall\, l \in \Lambda\; \exists\, n \in \Lambda: \quad \omega_{M^n}(2t) = O(\omega_{M^l}(t)) \;\; \textit{for}\; t \to \infty.$$

More precisely in this case we get the stronger condition that for all $l \in \Lambda$ and $h > 0$ there exists $n \in \Lambda$ with $\omega_{M^n}(ht) = O(\omega_{M^l}(t))$ for $t \to \infty$.

(*ii*) *If additionally property $(\mathcal{M}_{(\text{L})})$ holds, then*

$$\forall\, l \in \Lambda\; \forall\, h > 0\; \exists\, n \in \Lambda: \quad \omega_{M^l}(ht) = O(\omega_{M^n}(t)) \;\; \textit{for}\; t \to \infty.$$

Proof. By $(\mathcal{M}_{\{\mathrm{L}\}})$ we have that for each $l \in \Lambda$ and each $h > 0$ there exists $n \in \Lambda$ and $D > 0$ such that $M_k^l \cdot h^k \le D \cdot M_k^n$ holds for all $k \in \mathbb{N}$. Thus we get immediately after multiplying this inequality with t^k for any $t > 0$:

$$\frac{(h \cdot t)^k}{M_k^n} \le D \cdot \frac{t^k}{M_k^l} \Longrightarrow \log\left(\frac{(h \cdot t)^k}{M_k^n}\right) \le \log\left(\frac{t^k}{M_k^l}\right) + D_1.$$

Since this holds for all $k \in \mathbb{N}$ we get by definition $\omega_{M^n}(ht) \le \omega_{M^l}(t) + D_1$.
Of course in the case of $(\mathcal{M}_{(\mathrm{L})})$ we only have to use Beurling-type order of quantifiers. Compare this proposition and the proof with the first part of 5.1.4 in the weight function case! □

The next result is the "matrix-type-generalization" of 3.3.2:

Proposition 9.3.8. *Let a family of functions $\{\sigma_l \in \mathcal{W} : l \in \Lambda\}$ be given and assume such that*

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda: \ \sigma_n(2t) = O(\sigma_l(t)), \ \ t \to \infty,$$

which means that for each $l \in \Lambda$ there exists $n \in \Lambda$ and a constant $L \ge 1$ such that $\sigma_n(2t) \le L \cdot (1 + \sigma_l(t))$ for all $t \ge 0$. Then we obtain

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, L \ge 1\ \forall\, s \in \mathbb{N}\ \forall\, a > 0\ \forall\, j \in \mathbb{N}:$$
$$\exp\left(\frac{1}{a} \cdot \varphi^*_{\sigma_l}(aj)\right) \cdot \exp(s)^j \le \exp\left(\frac{\sum_{i=1}^s L^i}{L^s \cdot a}\right) \cdot \exp\left(\frac{1}{L^s \cdot a} \cdot \varphi^*_{\sigma_n}(L^s \cdot a \cdot j)\right).$$

If

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda: \ \sigma_l(2t) = O(\sigma_n(t)), \ \ t \to \infty,$$

then we get an analogously result ("Beurling-type"-estimate).

Proof. The proof is of course completely analogous to the proof of (3.3.2).
First we see that by assumption for all $l \in \Lambda$ there exists $n \in \Lambda$ and $L \ge 1$ with $\sigma_n(4t) \le L \cdot \sigma_l(t) + L$ for all $t \ge 0$ (iterated application), and so $\sigma_n(\exp(1) \cdot t) \le \sigma_n(4t) \le L \cdot \sigma_l(t) + t$. We calculate for all $y \ge 0$ (in fact $y \in \mathbb{R}$):

$$\varphi_{\sigma_n}(y+1) = \sigma_n(\exp(y+1)) = \sigma_n(\exp(1) \cdot \exp(y)) \le L \cdot \sigma_l(\exp(y)) + L = L \cdot (1 + \varphi_{\sigma_l}(y)),$$

Using this we proceed as follows:

$$\begin{aligned}
\varphi^*_{\sigma_n}(L \cdot x) &= \sup\{(L \cdot x) \cdot y - \varphi_{\sigma_n}(y) : y \ge 0\} = L \cdot \sup\left\{x \cdot y - \frac{1}{L} \cdot \varphi_{\sigma_n}(y) : y \ge 0\right\} \\
&\ge L \cdot \sup\{x \cdot y - (1 + \varphi_{\sigma_l}(y-1)) : y \ge 0\} \ge L \cdot \sup\{x \cdot y - (1 + \varphi_{\sigma_l}(y-1)) : y \ge 1\} \\
&= L \cdot \sup\{x \cdot (y-1) + x - 1 - \varphi_{\sigma_l}(y-1) : y \ge 1\} \\
&\underbrace{=}_{y'=y-1} L \cdot x - L + L \cdot \sup\{x \cdot y' - \varphi_{\sigma_l}(y') : y' \ge 0\} = L \cdot x - L + L \cdot \varphi^*_{\sigma_l}(x),
\end{aligned}$$

which we can summarize:

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, L \ge 1\ \forall\, x \ge 0: \ \ L \cdot \varphi^*_{\sigma_l}(x) + L \cdot x \le L + \varphi^*_{\sigma_n}(L \cdot x).$$

We show now:

$$\forall\, l \in \Lambda\ \exists\, n \in \Lambda\ \exists\, L \ge 1\ \forall\, s \in \mathbb{N}\ \forall\, x \ge 0: \ \ L^s \cdot \varphi^*_{\sigma_l}(x) + s \cdot L^s \cdot x \le \varphi^*_{\sigma_n}(L^s \cdot x) + \sum_{i=1}^s L^i.$$

We prove by induction: The case $s=1$ is precisely the above estimate, for $s\mapsto s+1$ we calculate as follows: We multiply the inequality for s with $L\ge 1$ (sufficiently large depending on s and l) to obtain (using again the previous estimate):

$$L^{s+1}\cdot\varphi^*_{\sigma_l}(x)+s\cdot L^{s+1}\cdot x\le L\cdot\varphi^*_{\sigma_n}(L^s\cdot x)+\sum_{i=2}^{s+1}L^i\le\varphi^*_{\sigma_{n_1}}(L^{s+1}\cdot x)+L-L^{s+1}\cdot x+\sum_{i=2}^{s+1}L^i$$

$$\Longrightarrow L^{s+1}\cdot\varphi^*_{\sigma_l}(x)+s\cdot L^{s+1}\cdot x+L^{s+1}\cdot x\le\varphi^*_{\sigma_{n_1}}(L^{s+1}\cdot x)+\sum_{i=1}^{s+1}L^i.$$

Finally put in the above inequality $x=a\cdot j$ for $j\in\mathbb{N}$ and $a>0$ a positive real number, then divide by $L^s\cdot a$ and finally apply the exponential function to get

$$\forall\,l\in\Lambda\;\exists\,n\in\Lambda\;\exists\,L\ge 1\;\forall\,s\in\mathbb{N}\;\forall\,a>0\;\forall\,j\in\mathbb{N}:$$
$$\exp\left(\frac{1}{a}\cdot\varphi^*_{\sigma_l}(aj)\right)\cdot\exp(s)^j\le\exp\left(\frac{\sum_{i=1}^s L^i}{L^s\cdot a}\right)\cdot\exp\left(\frac{1}{L^s\cdot a}\cdot\varphi^*_{\sigma_n}(L^s\cdot a\cdot j)\right).$$

□

Remark: If each ω_{M^l} has (ω_1), then clearly Proposition 9.3.8 is satisfied for both cases with the choice $l=n$.
Propositions 9.3.7 and 9.3.8 together give immediately the following corollary:

Corollary 9.3.9. *Let $\mathcal{M}:=\{M^l\in\mathbb{R}_{>0}^{\mathbb{N}}:l\in\Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with additionally property $(\mathcal{M}_{\{\mathrm{L}\}})$, then we obtain the following inequality:*

$$\forall\,l\in\Lambda\;\exists\,n\in\Lambda\;\forall\,h>0\;\forall\,a\in\mathbb{N}\;\exists\,D>0\;\exists\,b\in\mathbb{N}\;\forall\,j\in\mathbb{N}:M^{l,a}_j\cdot h^j\le D\cdot M^{n,b}_j. \tag{9.3.10}$$

If additionally property $(\mathcal{M}_{(\mathrm{L})})$ holds, then we obtain the following inequality:

$$\forall\,n\in\Lambda\;\exists\,l\in\Lambda\;\forall\,h>0\;\forall\,b>0\;\exists\,D>0\;\exists\,a>0\;\forall\,j\in\mathbb{N}:M^{l,a}_j\cdot h^j\le D\cdot M^{n,b}_j. \tag{9.3.11}$$

Now recall our definitions for all non-empty open sets $U\subseteq\mathbb{R}^n$ (see also (9.3.6) and (9.3.7)):

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U,\mathbb{R}^m):=\varprojlim_{K\subseteq U}\varinjlim_{l\in\Lambda,h>0}\mathcal{E}_{\omega_{M^l},h}(K,\mathbb{R}^m),\mathcal{E}_{(\omega_{\mathcal{M}})}(U,\mathbb{R}^m):=\varprojlim_{K\subseteq U}\varprojlim_{l\in\Lambda,h>0}\mathcal{E}_{\omega_{M^l},h}(K,\mathbb{R}^m)$$

and so we obtain the following:

Theorem 9.3.10. *Let $\mathcal{M}:=\{M^l\in\mathbb{R}_{>0}^{\mathbb{N}}:l\in\Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix, let $n,m\in\mathbb{N}_{>0}$ and U a non-empty open set in $\mathbb{R}^n$. In this situation we obtain:*

(i) If $\mathcal{M}$ has additionally $(\mathcal{M}_{\{\mathrm{L}\}})$, then we get as locally convex vector spaces

$$\mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U,\mathbb{R}^m)=\varprojlim_{K\subseteq U}\varinjlim_{l\in\Lambda,n,h>0}\mathcal{E}_{M^{l;n},h}(K,\mathbb{R}^m).$$

(ii) If additionally the Beurling-type condition $(\mathcal{M}_{(\mathrm{L})})$ is satisfied, then we have as locally convex vector spaces

$$\mathcal{E}_{(\omega_{\mathcal{M}})}(U,\mathbb{R}^m)=\varprojlim_{K\subseteq U}\varprojlim_{l\in\Lambda,n,h>0}\mathcal{E}_{M^{l;n},h}(K,\mathbb{R}^m).$$

Proof. For the Roumieu-case we use (9.3.10), for the Beurling-case (9.3.11). □

We summarize the whole situation in the following important result:

Theorem 9.3.11. *Let* $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ *be a* ($\mathcal{M}_{\text{sc}}$) *weight matrix, let* $n, m \in \mathbb{N}_{>0}$ *and* U *be a non-empty open set in* $\mathbb{R}^n$. *Then the following is satisfied:*

(*i*) *If* $\mathcal{M}$ *satisfies additionally both conditions* ($\mathcal{M}_{\{\text{L}\}}$) *and* ($\mathcal{M}_{\{\text{mg}\}}$), *then we get as locally convex vector spaces*

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^m) = \mathcal{E}_{\{\omega_{\mathcal{M}}\}}(U, \mathbb{R}^m).$$

(*ii*) *If additionally the Beurling-type conditions* ($\mathcal{M}_{(\text{L})}$) *and* ($\mathcal{M}_{(\text{mg})}$) *are satisfied, then we have*

$$\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^m) = \mathcal{E}_{(\omega_{\mathcal{M}})}(U, \mathbb{R}^m).$$

With other words both matrices $\mathcal{M} := \{M^l : l \in \Lambda\}$ *and* $\omega_{\mathcal{M}} := \{\omega_{M^l} : l \in \Lambda\}$ *generate the same space of ultradifferentiable functions.*

Proof. Holds now by applying our Theorems 9.3.3 and 9.3.10. □

Remarks:

(1) If the matrix $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$ with (ω_1), then theorem 9.3.11 is valid (for both cases) with $\mathcal{E}_{[\omega_{\mathcal{M}}]} = \mathcal{E}_{[\omega]}$: Note that Theorem 9.3.10 holds since by 5.1.3 each associated function $\omega_{M^l} \in \mathcal{W}$ has (ω_1) and more precisely all associated functions are equivalent.

(2) Theorem 9.3.11 holds also for the "Gevrey-sequence-matrix" $\mathcal{G} := \{(p!^s)_p : s > 1\}$!

(3) If Theorem 9.3.11 is valid we can replace the given weight matrix $\mathcal{M}$ also by $\omega_{\mathcal{M}}$, consisting of the associated functions ω_{M^l} of the sequences $M^l \in \mathcal{M}$. If all associated functions are equivalent (as in the weight function case - recall 5.1.3), then the new matrix-space consists only of one function, but in general we don't have this property! Nevertheless we obtain the following:

Corollary 9.3.12. *If Theorem 9.3.11 is valid, then we can replace some/all functions* ω_{M^l} *by equivalent ones (w.r.t.* $\sim$*) without changing the associated function space (for both types).*

Assume now that Theorem 9.3.11 is valid and let $\mathcal{N} := \{M^{l,n} \in \mathbb{R}_{>0}^{\mathbb{N}} : (l, n) \in \Lambda \times \mathbb{R}_{>0}\}$ be the matrix consisting of the above defined sequences $M^{l,n}$. If each ω_{M^l} is sub-additive, then first ($\mathcal{M}_\circ$) holds for the matrix $\mathcal{M}$ itself by the same proof as in 3.8.1 for the function ω_{M^l} instead of ω and $l = 1$ in the proof there. Note that we have $M^{l,1} = M^l$ (recall (9.3.2)). Furthermore ($\mathcal{M}_\circ$) holds for each sub-matrix $\mathcal{M}^l := \{M^{l,n} : n \in \mathbb{R}_{>0}\}$: We use again 3.8.1 and ω_{M^l} instead of ω, for arbitrary $l \in \Lambda$. So each $\mathcal{M}^l$ has both properties ($\mathcal{M}_{\{\text{FdB}\}}$) and ($\mathcal{M}_{(\text{FdB})}$) (see 8.7.1), we obtain e.g. closedness under composition by 8.3.1 for each space $\mathcal{E}_{[\omega_{M^l}]}$! This implies of course that $\mathcal{E}_{[\mathcal{M}]}$ is closed under composition, too. For the full version of 8.7.4 for each $\mathcal{E}_{[\omega_{M^l}]}$, we would have to assume additional assumptions on each ω_{M^l}.
If each ω_{M^l} satisfies ($\omega_{1'}$) (equivalent to a sub-additive weight), then we can change to an equivalent family σ_l with $\omega_{M^l} \sim \sigma_l$ for each $l \in \Lambda$ and such that each σ_l is sub-additive. The derived weight matrix $\mathcal{S} := \{S^{l,n} \in \mathbb{R}_{>0}^{\mathbb{N}} : (l, n) \in \Lambda \times \mathbb{R}_{>0}\}$ defined by $S_j^{l,n} := \exp(\frac{1}{n} \cdot \varphi_{\sigma_l}^*(nj))$ is equivalent to the matrix $\mathcal{N}$, more precisely $\mathcal{M}^l\{\approx\}\mathcal{S}^l$ and

$\mathcal{M}^l(\approx)\mathcal{S}^l$ for each $l \in \Lambda$, where $\mathcal{S}^l := \{S^{l,n} : n \in \mathbb{R}_{>0}\}$ (see 5.3.1) and $(\mathcal{M}_\circ)$ is satisfied for each sub-matrix $\mathcal{S}^l$, $l \in \Lambda$, separately.

But the converse implication is not true in general: In the case if some/each ω_{M^l} is not equivalent to a sub-additive weight function this doesn't apply necessarily that $\mathcal{E}_{[\omega_{\mathcal{M}}]}$ is not closed under composition, because in general the associated functions ω_{M^l} are not equivalent. One would need e.g. that $\omega_{M^{l_1}} \sim \omega_{M^{l_2}}$ for each $l_1, l_2 \in \Lambda$ - like if the matrix $\mathcal{M}$ is obtained by a weight function ω (see 5.1.3)!
Related to this question we formulate an immediate consequence of the second part of 4.0.5:

Lemma 9.3.13. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, such that each sequence M^l satisfies property moderate growth* (mg)*, but $M^l \approx M^n$ doesn't hold for any $l, n \in \Lambda$ (e.g. the Gevrey-sequences $M^l_p = (p!)^l$, $l > 1$). Then the associated functions ω_{M^l} are pairwise not equivalent.*

9.4 Connection between classes of Roumieu- and Beurling-type defined by a weight matrix

The goal of this section is the following: We prove a more general matrix-version of [35, 2.12. Proposition], so we want to obtain a inductive representation for the Beurling-case and a projective representation for the Roumieu-case. More precisely we want to write the space $\mathcal{E}_{(\mathcal{M})}$ as union of classes $\mathcal{E}_{[L]}$ and the space $\mathcal{E}_{\{\mathcal{M}\}}$ as intersection of classes $\mathcal{E}_{[L]}$. Such representations (as vector spaces) can be useful to transfer proofs and techniques from one case to the other.
First we consider the Beurling-case and for this we prove a more general matrix-version of [35, 2.3. Lemma], compare this also with (10.6.6) below:

Lemma 9.4.1. *Let $\mathcal{M} := \{M^l : l \in \Lambda = \mathbb{R}_{>0}\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and assume that there exists an arbitrary sequence of positive real numbers L with $L \lhd M^l$ for all $l \in \Lambda$.*

(1) *Then there exist sequences P^1, P^2 with $L \lhd P^1 \lhd P^2 \lhd M^l$ for each $l \in \Lambda$.*

(2) *If moreover we have that $k \mapsto (M^l_k)^{1/k}$ is increasing and $(M^l_k)^{1/k} \to \infty$ for $k \to \infty$ for each $l \in \Lambda$ (if $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then these conditions are satisfied by definition!), then there exist sequences P^1 and P^2 with $(P^i_k)^{1/k} \to \infty$ for $k \to \infty$, $i = 1, 2$, and such that $L \le P^1 \lhd P^2 \lhd M^l$ for each $l \in \Lambda$.*

Proof. (1) We construct the sequence P^1: By assumption $L \lhd M^l$ holds for each $l \in \Lambda$ which precisely means:

$$\forall\, l \in \Lambda\ \forall\, h > 0 \text{ (both small) } \exists\, D_{l,C} > 0 \text{ (large) } \forall\, k \in \mathbb{N}:\ L_k \le D_{l,h} \cdot h^k \cdot M^l_k. \quad (\star)$$

In $(\star)$ we put $h = l = \frac{1}{n}$ for $n \in \mathbb{N}_{>0}$ and let $D_{1/n,1/n}$ be the minimal constant such that this inequality is satisfied! In the following we identify the constant $D_{1/n,1/n}$ with C_n and note that the so obtained sequence $(C_n)_n$ is increasing in n and we can assume $C_n \ge 1$ for all n. Let $A > 1$ be an arbitrary real number (can be chosen very large) but from now on fixed, then for each $n \ge 1$ there exists a minimal number $j_n \in \mathbb{N}_{>0}$ with $(C_n)^{1/(j_n)} \le A$ and we can assume that $(j_n)_{n\ge1}$ is a strictly increasing sequence in n.

Then define the sequence $P^1 = (P^1_j)_j$ as follows: We set $P^1_j := \sqrt{L_j \cdot M^{1/n}_j}$ for $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$, $n \ge 1$, and $P^1_j := \sqrt{L_j \cdot M^1_j}$ for $0 \le j < j_1$.
We show $L \lhd P^1$: For $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$, $n \ge 1$, we obtain:

$$\left(\frac{L_j}{P^1_j}\right)^{1/j} = \left(\frac{\sqrt{L_j}}{\sqrt{M^{1/n}_j}}\right)^{1/j} = \left(\frac{L_j}{M^{1/n}_j}\right)^{1/(2j)} \underbrace{\le}_{(\star)} (C_n)^{1/(2j)} \cdot \frac{1}{\sqrt{n}} \le (C_n)^{1/(2j_n)} \cdot \frac{1}{\sqrt{n}} \le \frac{\sqrt{A}}{\sqrt{n}},$$

which tends to 0 for $j \to \infty \Leftrightarrow n \to \infty$.
We show now $P^1 \lhd M^l$ for each $l \in \Lambda$. Let $l \in \Lambda$ be given but fixed (consider $0 < l < 1$ very small), then for $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$ large enough such that (at least) $\frac{1}{n} \le l$ holds we estimate as follows:

$$\left(\frac{P^1_j}{M^l_j}\right)^{1/j} = \left(\frac{\sqrt{L_j}}{\sqrt{M^l_j}}\right)^{1/j} \cdot \left(\frac{\sqrt{M^{1/n}_j}}{\sqrt{M^l_j}}\right)^{1/j} \le \underbrace{\left(\frac{L_j}{M^l_j}\right)^{1/(2j)}}_{\to 0 \text{ for } j \to \infty} \cdot \underbrace{\left(\frac{M^l_j}{M^l_j}\right)^{1/(2j)}}_{=1}$$

because $L \lhd M^l$ for each $l \in \Lambda$ by assumption.
To construct now the sequence P^2 we replace in the previous construction and definition the sequence L by P^1 (which can be done since we have shown $P^1 \lhd M^l$ for each $l \in \Lambda$).

(b) We start with the construction of the sequence P^1. By assumption the map $k \mapsto (M^l_k)^{1/k}$ is increasing and moreover $(M^l_k)^{1/k} \to \infty$ if $k \to \infty$ for each $l \in \Lambda = \mathbb{R}_{>0}$, so we obtain the following:

$$\forall\, l \in \Lambda\, \forall\, C \in \mathbb{N}_{>0}\, \exists\, k_{l,C} \in \mathbb{N}: \quad (M^l_{k_{l,C}})^{1/(k_{l,C})} > C. \quad (\star\star)$$

We put $l := \frac{1}{n}$ and $C := n$ for $n \in \mathbb{N}_{>0}$ and identify $k_{1/n,n}$ with k_n. Since the mapping $k \mapsto (M^l_k)^{1/k}$ is increasing for each $l \in \Lambda$ we see that if $(\star\star)$ is satisfied for k_n (with n fixed), then also for all $k \ge k_n$. So we can arrange the sequence $(k_n)_{n\ge1}$ in such a way that we obtain a strictly increasing sequence.
Put $P^1_k := \max\left\{\sqrt{M^{1/n}_k}, L_k\right\}$ for $k \in \mathbb{N}$ with $k_n \le k < k_{n+1}$, and for $0 \le k \le k_1$ we put $P^1_k := \max\left\{\sqrt{M^1_k}, L_k\right\}$.
Then of course $P^1 \ge L$ holds by definition.
In the next step we show $P^1 \lhd M^l$ for each $l \in \Lambda$. Let $l \in \Lambda = \mathbb{R}_{>0}$ be given but fixed (consider $1 > l > 0$ very small), then for k large enough with $k_n \le k < k_{n+1}$ with at least $\frac{1}{n} \le l$ we estimate as follows:

$$\left(\frac{P^1_k}{M^l_k}\right)^{1/k} = \max\left\{\left(\frac{\sqrt{M^{1/n}_k}}{M^l_k}\right)^{1/k}, \left(\frac{L_k}{M^l_k}\right)^{1/k}\right\} \le \max\left\{\left(\frac{\sqrt{M^l_k}}{M^l_k}\right)^{1/k}, \left(\frac{L_k}{M^l_k}\right)^{1/k}\right\}$$
$$= \max\left\{(M^l_k)^{-1/(2k)}, \left(\frac{L_k}{M^l_k}\right)^{1/k}\right\} \to 0,$$

for $k \to \infty$ because by assumption $(M^l_k)^{1/k} \to \infty$ for $k \to \infty$ and $L \lhd M^l$ for each $l \in \Lambda$.

Define the sequence P^2 via $P^2_j := \sqrt{P^1_j \cdot M^{1/n}_j}$ for $j \in \mathbb{N}$ with $j_n \leq j < j_{n+1}$, $n \geq 1$, and $P^2_j := \sqrt{P^1_j \cdot M^1_j}$ for $0 \leq j < j_1$ analogously like the sequence P^1 in (a) (note that we have already shown $P^1 \lhd M^l$ for each $l \in \Lambda$).

Finally we have by $(\star\star)$ and definition that $(P^1_{k_n})^{1/(k_n)} \geq \sqrt{\left(M^{1/n}_{k_n}\right)^{1/(k_n)}} > \sqrt{n} \to \infty$ for $n \to \infty$. Since for k with $k_n \leq k < k_{n+1}$ the mapping $k \mapsto (M^{1/n}_k)^{1/k}$ is increasing (for each $n > 0$ separately but fixed) we have $(P^1_k)^{1/k} \to \infty$ for $k \to \infty$. By $P^1 \lhd P^2$ we have also $\lim_{k\to\infty}(P^2_k)^{1/k} = +\infty$. □

Remark 9.4.2. *Of course we can replace in the above formulation of the theorem and the proof* M^l_k *by* $m^l_k := \frac{M^l_k}{k!}$, L_k *by* $l_k := \frac{L_k}{k!}$.

We use 9.4.1 to prove the following

Proposition 9.4.3. *Let* $\mathcal{M} := \{M^l : l \in \Lambda = \mathbb{R}_{>0}\}$ *be an arbitrary weight matrix, i.e.* $(\mathcal{M})$, *then we obtain the following inductive representation (as vector spaces) for all* $n, m \in \mathbb{N}_{>0}$ *and* $K \subseteq \mathbb{R}^n$ *compact:*

$$\mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^m) = \bigcup_{L \lhd \mathcal{M}} \mathcal{E}_{\{L\}}(K, \mathbb{R}^m) = \bigcup_{L \lhd \mathcal{M}} \mathcal{E}_{(L)}(K, \mathbb{R}^m).$$

If moreover $k \mapsto (M^l_k)^{1/k}$ *is increasing and* $(M^l_k)^{1/k} \to \infty$ *for* $k \to \infty$ *for each* $l \in \Lambda$ *(e.g. if* $\mathcal{M}$ *is a* $(\mathcal{M}_{\text{sc}})$ *weight matrix), then we can restrict the unions to sequences* L *with* $L \lhd \mathcal{M}$ *and* $(L_k)^{1/k} \to \infty$ *for* $k \to \infty$.

Proof. Inclusions $(\supseteq)$ are clear by definition.
Conversely for $(\subseteq)$: Let $f \in \mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^m)$, then define the (arbitrary) sequence of positive numbers $L_k := \sup_{x \in K} \|f^{(k)}(x)\|_{L^k(\mathbb{R}^n, \mathbb{R}^m)}$, and by definition $L \lhd M^l$ holds now for all $l \in \Lambda$. We use (1) of 9.4.1 to conclude that there exist sequences P^1, P^2 with $L \lhd P^1 \lhd P^2 \lhd M^l$ for each $l \in \Lambda$, hence $f \in \mathcal{E}_{\{P^1\}}(K, \mathbb{R}^m)$ and $f \in \mathcal{E}_{(P^2)}(K, \mathbb{R}^m)$. If the additional assumptions on $\mathcal{M}$ are valid, then by (2) of 9.4.1 we get also $(P^i_k)^{1/k} \to \infty$ for $k \to \infty$, $i = 1, 2$. □

Next we study the Roumieu-matrix-case:

Proposition 9.4.4. *Let* $\mathcal{M} := \{M^l : l \in \Lambda = \mathbb{N}_{>0}\}$ *be an arbitrary weight matrix, i.e.* $(\mathcal{M})$, *then we obtain the following projective representation (as vector spaces) for all* $n, m \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^n$ *non-empty open:*

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^m) = \bigcap_{\mathcal{M} \lhd L} \mathcal{E}_{(L)}(U, \mathbb{R}^m) = \bigcap_{\mathcal{M} \lhd L} \mathcal{E}_{\{L\}}(U, \mathbb{R}^m).$$

Moreover we get in this situation for $\mathcal{M}$:

- (a) *If we assume in addition that there exists an index* $l_0 \in \Lambda$ *with* $\liminf_{j\to\infty}(m^{l_0}_j)^{1/j} > 0$, *i.e. condition* $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$, *then we can replace the matrix* $\mathcal{M}$ *by* $\mathcal{M}^{\text{lc}} := \{(M^l)^{\text{lc}} : l \in \Lambda\}$, *where* $(M^l)^{\text{lc}}$ *denotes the weakly log. convex minorant of* M^l, *and the intersections on the right hand side can be restricted to all weakly log. convex sequences* L *with* $\mathcal{M} \lhd L$, $(L_k)^{1/k} \to \infty$ *for* $k \to \infty$.
- (b) *If* $k \mapsto (M^l_k)^{1/k}$ *is increasing for each* $l \in \Lambda$ *(e.g. satisfied for* $\mathcal{M}$ *a* $(\mathcal{M}_{\text{sc}})$ *weight matrix automatically), then we can restrict the intersections also to all weakly log. convex weight sequences* L *with* $\mathcal{M} \lhd L$ *and* $(L_k)^{1/k} \to \infty$ *for* $k \to \infty$.

Proof. We prove a matrix-version of [22, 2.4. Theorem] and proceed as follows:
Recall that $\mathcal{M} \lhd L$ means $M^l \lhd L$ for each $l \in \Lambda$, hence
$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) \subseteq \bigcap_{\mathcal{M}\lhd L}\mathcal{E}_{(L)}(U,\mathbb{R}^m) \subseteq \bigcap_{\mathcal{M}\lhd L}\mathcal{E}_{\{L\}}(U,\mathbb{R}^m)$ holds by definition.
For the converse direction we show $\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) \supseteq \bigcap_{\mathcal{M}\lhd L}\mathcal{E}_{\{L\}}(U,\mathbb{R}^m)$. Let $f \notin \mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m)$, so there exists some compact set $K \subseteq U$ such that for all $l \in \Lambda = \mathbb{N}_{>0}$ and $C > 0$ there exists $j_{l,C}$ with $f_{j_{l,C}} > C^{j_{l,C}} \cdot M^l_{j_{l,C}}$, where we have put $f_k := \sup_{x\in K}\left\|f^{(k)}(x)\right\|_{L^k(\mathbb{R}^n,\mathbb{R}^m)}$. We choose now $C = l = n \in \mathbb{N}_{>0}$ and identify $j_{n,n}$ with j_n to obtain a strictly increasing sequence $(j_n)_n$ with:

$$\left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} > n, \tag{9.4.1}$$

for all $n \in \mathbb{N}_{>0}$. We show now that there exists a weight sequence L with $\mathcal{M} \lhd L$ and such that $f \notin \mathcal{E}_{\{L\}}(U,\mathbb{R}^m)$.
We proceed now similarly as in [22, 2.4. Theorem]: W.l.o.g. we have $j_1 > 0$, and we consider (for $n \geq 1$) the sequences $(n)_n$ and e.g. $(n^{-1/2})_n$. Now introduce the sequence $\beta = (\beta_n)_n$ via

$$\beta_n := \frac{1}{\sqrt{n}} \cdot \left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} > \sqrt{n} \to \infty \tag{9.4.2}$$

for $n \to \infty$ and clearly $\beta_n > 1$ for all $n \geq 1$. We pass now to a subsequence γ of β with additional property $\gamma_{n+1} \geq \gamma_n^{j_n}$.
In the next step we define the piecewise affine function ϕ as follows:
Put $\phi(0) = 0$ and $\phi(j) = j_n \cdot \log(\gamma_n)$ for $j = j_n$, $n \geq 1$. For $j \in \mathbb{N}$ with $j_n < j < j_{n+1}$, $n \geq 1$ the function ϕ should be the affine line connecting the points $(j_n, j_n \cdot \log(\gamma_n))$ and $(j_{n+1}, j_{n+1} \cdot \log(\gamma_{n+1}))$. More precisely, this affine straight line is given by

$$g_n : x \mapsto \underbrace{\frac{j_{n+1}\cdot\log(\gamma_{n+1}) - j_n\cdot\log(\gamma_n)}{j_{n+1}-j_n}}_{=:d_n}\cdot(x-j_n)+j_n\cdot\log(\gamma_n) = d_n\cdot x + \underbrace{\frac{j_{n+1}\cdot j_n\cdot\log(\gamma_n/\gamma_{n+1})}{j_{n+1}-j_n}}_{=:c_n}$$

and so for j with $j_n < j < j_{n+1}$ we put $\phi(j) = g_n(j)$ where $n \geq 1$. Finally for $0 < j < j_1$ we put $\phi(j) = j \cdot \log(\gamma_1)$.

The first observation is that $c_n \leq 0$ for all n and moreover the sequence $(d_n)_{n\geq 1}$ is increasing, because

$$\begin{aligned}\log(\gamma_{n+1}) \leq d_n = \frac{j_{n+1}\cdot\log(\gamma_{n+1}) - j_n\cdot\log(\gamma_n)}{j_{n+1}-j_n} &\leq \frac{j_{n+1}}{j_{n+1}-j_n}\cdot\log(\gamma_{n+1}) \leq \frac{\log(\gamma_{n+2})}{j_{n+1}-j_n}\\ &\leq \log(\gamma_{n+2})\end{aligned}$$

by the choice of the sub-sequence γ. Finally d_0, the slope on $[0, j_1]$ is given by $d_0 = \log(\gamma_1)$ and so $d_0 \leq \log(\gamma_2) \leq d_1$. This implies that $n \mapsto d_n$ is increasing, hence $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a convex function. Because $c_n \leq 0$ the mapping $x \mapsto \frac{\phi(x)}{x}$ is increasing.

Define now the weight sequence $L = (L_j)_j$, or with other words the (constant) weight matrix $\mathcal{L} := \{L\}$ (consisting only of this sequence L) via

$$L_j := \exp(\phi(j)) \cdot M^n_j$$

for $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$, $n \ge 1$, and $L_j = \exp(\phi(j)) \cdot M^1_j$ for $0 \le j < j_1$.
We show $M^l \lhd L$ for each $l \in \Lambda$: We have for arbitrary $l \in \Lambda = \mathbb{N}_{>0}$ (large):

$$\left(\frac{M^l_{j_n}}{L_{j_n}}\right)^{1/(j_n)} = \underbrace{\frac{1}{\gamma_n}}_{\to 0} \cdot \underbrace{\left(\frac{M^l_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)}}_{\le 1},$$

which holds for $l \le n$ (since then $M^l \le M^n$) and because $\gamma_n \to \infty$ for $n \to \infty$. Because the mapping $j \mapsto \frac{\phi(j)}{j}$ is increasing and $\frac{\phi(j_n)}{j_n} = \log(\gamma_n) \to \infty$ for $n \to \infty$ we are done.

It remains to show, that $f \notin \mathcal{E}_{\{L\}}(U, \mathbb{R}^m)$:

$$\left(\frac{f_{j_n}}{L_{j_n}}\right)^{1/(j_n)} = \left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} \cdot \frac{1}{\exp\left(\frac{\phi(j_n)}{j_n}\right)} = \left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} \cdot \frac{1}{\gamma_n} \to \infty$$

for $n \to \infty$ by the definition of the sequences γ and β.

Unfortunately the second part of 9.2.1 shows the following for non-constant weight matrices $\mathcal{M}$: By using the above proof we cannot expect that the sequence L is weakly log. convex, even if each M^l is assumed to be weakly log. convex (in the constant case this can be done, see the proof of [22, 2.4. Theorem]). To obtain weakly log. convexity for L one can either assume additional properties (see (a)) or change the above proof (see (b)).

(a) If we assume in addition for the weight matrix $\mathcal{M}$ that there exists an index $l_0 \in \Lambda$ with $\liminf_{j\to\infty}(M^{l_0}_j)^{1/j} \ge \varepsilon > 0$, then also $\liminf_{j\to\infty}(M^l_j)^{1/j} \ge \varepsilon$ for all $l \ge l_0$ and in this situation $\lim_{j\to\infty}(L_j)^{1/j} = +\infty$ holds automatically: Recall that $j \mapsto \frac{\phi(j)}{j}$ is increasing and $\frac{\phi(j_n)}{j_n} = \log(\gamma_n) \to \infty$ for $n \to \infty$.
The same argument stays valid if we replace M^{l_0} by $m^{l_0} = (m^{l_0}_j)_j$, $m^{l_0}_j = \frac{M^{l_0}_j}{j!}$, and L by $l = (l_j)_j$, $l_j = \frac{L_j}{j!}$, because $l_j = \frac{L_j}{j!} = \exp(\phi(j)) \cdot \frac{M^n_j}{j!} = \exp(\phi(j)) \cdot m^n_j$ then holds for $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$. Now we can use [35, 2.15. Theorem] to conclude that we can replace L and each M^l by it's (weakly) log. convex minorant without changing the associated function space.

(b) We start with the same proof as above and use again the sequence β defined in (9.4.2) via

$$\beta_n := \frac{1}{\sqrt{n}} \cdot \left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} > \sqrt{n} \to \infty.$$

But now pass to a subsequence γ of β with additional property $\gamma_{n+1} \ge \gamma_n^{j_n} \cdot \frac{M^n_{j_n}}{M^{n-1}_{j_{n-1}}}$.
In the next step we define the piecewise affine function ϕ as follows:
Put $\phi(0) = \log(M^1_{j_1}) \ge \log(M^1_0) = 0$ and $\phi(j) = j_n \cdot \log(\gamma_n) + \log(M^n_{j_n})$ for $j = j_n$, $n \ge 1$. For $j \in \mathbb{N}$ with $j_n < j < j_{n+1}$ the function ϕ should be the affine line connecting the points $(j_n, j_n \cdot \log(\gamma_n) + \log(M^n_{j_n}))$ and $(j_{n+1}, j_{n+1} \cdot \log(\gamma_{n+1}) + \log(M^{n+1}_{j_{n+1}}))$. More

precisely, this affine straight line is given by

$$g_n : x \mapsto \underbrace{\frac{j_{n+1}\cdot\log(\gamma_{n+1}) - j_n\cdot\log(\gamma_n) + \log(M^{n+1}_{j_{n+1}}) - \log(M^n_{j_n})}{j_{n+1}-j_n}}_{=:d_n}\cdot(x-j_n)$$

$$+ j_n\cdot\log(\gamma_n) + \log(M^n_{j_n})$$

$$= d_n\cdot x + \underbrace{\frac{j_{n+1}\cdot j_n\cdot\log(\gamma_n/\gamma_{n+1}) + j_{n+1}\cdot\log(M^n_{j_n}) - j_n\cdot\log(M^{n+1}_{j_{n+1}})}{j_{n+1}-j_n}}_{=:c_n}$$

and so for j with $j_n < j < j_{n+1}$ we put $\phi(j) = g_n(j)$ where $n \geq 1$. Finally for $j \in \mathbb{N}$ with $0 < j < j_1$ we put $\phi(j) = j\cdot\log(\gamma_1) + \log(M^1_{j_1})$.

The sequence $(d_n)_{n\geq 1}$ is increasing, because

$$\log(\gamma_{n+1}) \leq \frac{j_{n+1}\cdot\log(\gamma_{n+1}) - j_n\cdot\log(\gamma_n)}{j_{n+1}-j_n} \leq d_n$$

$$= \frac{j_{n+1}\cdot\log(\gamma_{n+1}) - j_n\cdot\log(\gamma_n) + \log(M^{n+1}_{j_{n+1}}/M^n_{j_n})}{j_{n+1}-j_n}$$

$$\leq \frac{j_{n+1}\cdot\log(\gamma_{n+1}) + \log(M^{n+1}_{j_{n+1}}/M^n_{j_n})}{j_{n+1}-j_n} \leq \frac{\log(\gamma_{n+2})}{j_{n+1}-j_n} \leq \log(\gamma_{n+2})$$

by the choice of the sub-sequence γ. Finally d_0, the slope on $[0, j_1]$ is given by $d_0 = \log(\gamma_1)$ and so $d_0 \leq \log(\gamma_2) \leq d_1$. This implies that $n \mapsto d_n$ is increasing, hence $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a convex function.
Moreover $c_n \leq 0 \leq \log(M^1_{j_1})$ for each $n \geq 1$, because $\log(\frac{\gamma_n}{\gamma_{n+1}}) < 0$ and

$$j_{n+1}\cdot\log(M^n_{j_n}) - j_n\cdot\log(M^{n+1}_{j_{n+1}}) \leq 0 \Leftrightarrow \frac{1}{j_n}\cdot\log(M^n_{j_n}) \leq \frac{1}{j_{n+1}}\cdot\log(M^{n+1}_{j_{n+1}})$$

$$\Leftrightarrow (M^n_{j_n})^{1/j_n} \leq (M^{n+1}_{j_{n+1}})^{1/j_{n+1}},$$

which holds because $k \mapsto (M^l_k)^{1/k}$ is increasing for each $l \in \Lambda$ separately and $M^{l_1} \geq M^{l_2}$ for $l_1 \geq l_2$.

Define now the weight sequence $L = (L_j)_j$, or with other words the (constant) weight matrix $\mathcal{L} := \{L\}$ (consisting only of this sequence L) via

$$L_j := \exp(\phi(j)).$$

Observations: ϕ is a convex mapping and so L is a (weakly) log. convex sequence. Since $c_n \leq 0$ for all $n \geq 1$ we see that $j \mapsto \frac{\phi(j)}{j}$ is increasing on $[j_1, +\infty)$. For $j = j_n$, $n \geq 1$, we obtain $L_{j_n} = \exp(\phi(j_n)) = \gamma_n^{j_n}\cdot M^n_{j_n}$, hence $(L_{j_n})^{1/j_n} = \exp(\frac{\phi(j_n)}{j_n}) = \gamma_n\cdot\left(M^n_{j_n}\right)^{1/j_n} \to \infty$ for $n \to \infty$.
We show $M^l \lhd L$ for each $l \in \Lambda$: For arbitrary $l \in \Lambda = \mathbb{N}_{>0}$ (very large but fixed) we get:

$$\left(\frac{M^l_{j_n}}{L_{j_n}}\right)^{1/(j_n)} = \underbrace{\frac{1}{\gamma_n}}_{\to 0}\cdot\underbrace{\left(\frac{M^l_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)}}_{\leq 1},$$

which holds for $l \le n$ (since then $M^l \le M^n$) and because $\gamma_n \to \infty$ for $n \to \infty$. Note that $\frac{\phi(j_n)}{j_n} = \log(\gamma_n) + \frac{\log(M^n_{j_n})}{j_n} = \log(\gamma_n) + \log((M^n_{j_n})^{1/j_n}) \to \infty$ for $n \to \infty$ and $j \mapsto \frac{\phi(j)}{j}$ is increasing on $[j_1, +\infty)$.
Finally this implies immediately also $(L_j)^{1/j} \to \infty$ for $j \to \infty$.
It remains to show $f \notin \mathcal{E}_{\{L\}}(U, \mathbb{R}^m)$: $\left(\frac{f_{j_n}}{L_{j_n}}\right)^{1/(j_n)} = \left(\frac{f_{j_n}}{M^n_{j_n}}\right)^{1/(j_n)} \cdot \frac{1}{\gamma_n} \to \infty$ for $n \to \infty$ by the definition of the sequences γ and β. □

Important remarks:

(i) In (b) in 9.4.4 we have assumed that $k \mapsto (M^l_k)^{1/k}$ is increasing for each $l \in \Lambda$. Note that this condition implies immediately (alg) for each sequence M^l with $C = 1$ separately, i.e. $M^l_j \cdot M^l_k \le M^l_{j+k}$ for all $j, k \in \mathbb{N}$: For arbitrary $j, k \ge 1$ we have $(M^l_j)^{1/j} \le (M^l_{j+k})^{1/(j+k)}$, $(M^l_k)^{1/k} \le (M^l_{j+k})^{1/(j+k)}$, hence

$$M^l_j \cdot M^l_k \le (M^l_{j+k})^{j/(j+k)} \cdot (M^l_{j+k})^{k/(j+k)} = M^l_{j+k}.$$

The remaining cases ($j = k = 0$, $j = 0$, k arb. resp. j arb. and $k = 0$) are obvious by normalization. Recall: (alg) for each M^l tells us that each class $\mathcal{E}_{[M^l]}$ separately is closed under pointwise multiplication.

(ii) Moreover 9.4.4 generalizes [1, 4.6. Corollary] to the more general weight matrix situation. [1, 4.6. Corollary] proves the previous result if $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$ with (ω_1). For this recall 6.2.6 and (5.3.3) in 5.3.1.

9.5 Embedding of $\mathcal{E}_{\{\mathcal{M}\}}$ into the Gevrey-matrix-space fails

In this section let $\mathcal{M}$ be again always a $(\mathcal{M}_{\text{sc}})$ weight matrix. We introduce some definitions (see [26, § 1]): An arbitrary sequence of positive real numbers $M = (M_k)_k$ satisfies *moderate growth* (also called stability under ultradifferential operators, or "separativity condition" in [26]), if

$$(\text{mg}) :\Leftrightarrow \ \exists\, C \ge 1\ \forall\, j, k \in \mathbb{N} : M_{j+k} \le C^{j+k} \cdot M_j \cdot M_k,$$

it satisfies *weak moderate growth* (or "weak separativity" introduced in [26]), if

$$(\text{wmg}) :\Leftrightarrow \ \exists\, N = (N_k)_k\ \exists\, C \ge 1\ \forall\, j, k \in \mathbb{N} : M_{j+k} \le C^j \cdot M_j \cdot N_k,$$

and finally recall *derivation closedness*

$$(\text{dc}) :\Leftrightarrow \ \exists\, C \ge 1\ \forall\, j \in \mathbb{N} : M_{j+1} \le C^{j+1} \cdot M_j.$$

The last condition is called "differentiability condition" (D) in [26] with a misprint there. By putting $N_k := C^k \cdot M_k$ we see that (mg) implies (wmg) and the case $k = 1$ in (wmg) implies clearly (dc).

The following definitions and notations are motivated by [26]. Note that in this paper all considered weight sequences M are assumed to belong to $\mathcal{LC}$ and $M^l \in \mathcal{LC}$ for each $l \in \Lambda$ by definition if $\mathcal{M}$ is assumed to be a $(\mathcal{M}_{\text{sc}})$ weight matrix. We denote by

$$\mathcal{G} := \{G^t = (p!^t)_p : t > 1\},$$

the so-called *Gevrey-weight-matrix.* We put in the following $a^l_n := \log(M^l_n)$ and by assumption (see e.g. 4.0.2) we can introduce the function

$$H_l(t) := \sup_{n\in\mathbb{N}}\{n\cdot t - a^l_n\}, \qquad H_l(\log(t)) = \omega_{M^l}(t) \text{ for } t\geq 1.$$

Note that by normalization we have $M^l_0 \leq M^l_1$ for each $l\in\Lambda$, hence $\omega_{M^l}(t) = 0$ for $0\leq t\leq 1$ (see e.g. (4.0.2)) and so $H|_{(-\infty,0]} = 0$. Since $M^l_j = \sup_{t\geq 1}\frac{t^j}{\exp(\omega_{M^l}(t))}$ holds by [16, 3.2. Proposition], we get

$$a^l_j = \log(M^l_j) = \sup_{t\geq 1}\{j\cdot\log(t)-\omega_{M^l}(t)\} \underbrace{=}_{s=\log(t)} = \sup_{s\geq 0}\{j\cdot s-\omega_{M^l}(\exp(s))\} = \sup_{s\geq 0}\{j\cdot s-H_l(s)\}.$$

In the above calculation we can also take $\sup_{t\geq 0}$ and $\sup_{s\in\mathbb{R}}$. With this preparation we obtain the following theorem (for the analogous result see [26, Theorem 1]):

Theorem 9.5.1. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then the following conditions are equivalent:*

- (i) *$\mathcal{M}$ satisfies condition $(\mathcal{M}_{\{\text{mg}\}})$,*
- (ii) $\forall\, l\in\Lambda\;\exists\, n\in\Lambda\;\exists\, C\geq 1\;\forall\, j\in\mathbb{N}: M^l_{2j}\leq C^{2j}\cdot(M^n_j)^2$,
- (iii) $\forall\, l\in\Lambda\;\exists\, n\in\Lambda:\; \sup_{j\in\mathbb{N}}\frac{a^l_{2j}-2a^n_j}{j} < +\infty$,
- (iv) *Condition* (9.3.3) *is satisfied,*
- (v) $\forall\, l\in\Lambda\;\exists\, n\in\Lambda\;\exists\, C>0:\;\forall\, t\in\mathbb{R}: 2\cdot H_n(t)\leq H_l(t+C)+C.$

Proof. $(i)\Leftrightarrow(iv)$ is already shown in 9.3.2.
$(i)\Rightarrow(ii)$ is clear: Take $j=k$ in $(\mathcal{M}_{\{\text{mg}\}})$ to get (ii) for all $j\in\mathbb{N}$.
$(ii)\Leftrightarrow(iii)$: We apply log to $M^l_{2j}\leq C^{2j}\cdot(M^n_j)^2$ and see that $a^l_{2j}\leq(2j)\cdot C_1+2\cdot a^n_j$, and also the converse direction by applying exp.
$(iv)\Leftrightarrow(v)$ This holds by definition - recall $\omega_{M^l}(t) = H_l(\log(t))$ for $t\geq 1$ (in fact for $t\geq 0$) and each $l\in\Lambda$, hence $2\cdot\omega_{M^n}(t)\leq\omega_{M^l}(H\cdot t)+H \Leftrightarrow 2\cdot H_n(\log(t))\leq H_l(\log(H)+\log(t))+H$ for all $t\geq 0$.

$(v)\Rightarrow(iii)$ By the above calculation we have:

$$\begin{aligned} a^l_{2j} &= \sup_{s\in\mathbb{R}}\{(2j)\cdot s - H_l(s)\} \leq \sup_{s\in\mathbb{R}}\{(2j)\cdot s - 2\cdot H_n(s-C)+C\} \\ &= 2\cdot\sup_{s\geq 0}\{j\cdot s - H_n(s-C)\}+C = 2jC + 2\cdot\sup_{s'\in\mathbb{R}}\{j\cdot s' - H_n(s')\} - C \\ &= 2jC + 2a^n_j + C, \end{aligned}$$

where we have put $s' := s-C$. So we have shown that for all $j\in\mathbb{N}$ we get $\frac{a^l_{2j}-2a^n_j}{j}\leq 2C+\frac{C}{j}$, hence (iii) is satisfied.
$(iii)\Rightarrow(v)$ Similarly we calculate as follows:

$$\begin{aligned} H_l(t) &= \sup_{j\in\mathbb{N}}\{j\cdot t - a^l_j\} \geq \sup_{j\in\mathbb{N}}\{(2j)\cdot t - a^l_{2j}\} \geq \sup_{j\in\mathbb{N}}\{(2j)\cdot t - 2jC - 2a_j\} \\ &= 2\cdot\sup_{j\in\mathbb{N}}\{j\cdot(t-C) - a^n_j\} = 2\cdot H_n(t-C). \end{aligned}$$

The first inequality holds, because we consider only even positive integers $2n$. $\square$

If we assume some further properties for $\mathcal{M}$, then we can prove also some more useful equivalences for $(\mathcal{M}_{\{\text{mg}\}})$ (analogous result to Appendix B in [26]):

Theorem 9.5.2. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with additionally $(\mathcal{M}_{\{\text{diag}\}})$ and put $\tilde{a}_l^k := a_{k+1}^l - a_k^l = \log(M_{k+1}^l) - \log(M_k^l) = \log\left(\frac{M_{k+1}^l}{M_k^l}\right) = \log(\mu_{k+1}^l)$, then the following conditions are equivalent:*

(i) $\mathcal{M}$ *satisfies condition* $(\mathcal{M}_{\{\text{mg}\}})$,

(ii) $\forall\, l \in \Lambda\; \exists\, n \in \Lambda\; \exists\, C \geq 1\; \forall\, j \in \mathbb{N} : M_{2j}^l \leq C^{2j} \cdot (M_j^n)^2 \Longleftrightarrow \frac{a_{2j}^l}{2j} \leq \frac{a_j^n}{j} + \log(C)$,

(iii) $\forall\, l \in \Lambda\; \exists\, n \in \Lambda\; \exists\, C \geq 1\; \forall\, j \in \mathbb{N} : \tilde{a}_{2j}^l \leq \tilde{a}_j^n + C$

(iv) $\forall\, l \in \Lambda\; \exists\, n \in \Lambda\; \exists\, C \geq 1\; \forall\, j \in \mathbb{N} : \tilde{a}_j^l \leq \frac{a_j^n}{j} + C$

(v) $\forall\, l \in \Lambda\; \exists\, n \in \Lambda\; \exists\, C \geq 1\; \forall\, j, k \in \mathbb{N}\; \forall\, i \leq k : a_{i+j}^l \leq i \cdot \tilde{a}_k^n + a_j^n + (i+j) \cdot C$.

Note that for matrices obtained by a weight function $\omega \in \mathcal{W}$, condition $(\mathcal{M}_{\{\text{dc}\}})$ is satisfied automatically by (5.1.2) in 5.1.2 and condition $(\mathcal{M}_{\{\text{diag}\}})$ holds, if and only if property (ω_7) is satisfied, see 5.4.1. Note that $(\mathcal{M}_{\{\text{diag}\}})$ implies condition *(ii)* (because $M_j^n \geq 1$ for each $n \in \Lambda$ and $j \in \mathbb{N}$).

Proof. *(i)* $\Leftrightarrow$ *(ii)* was already shown in Theorem 9.5.1.
(i) $\Rightarrow$ *(v)* We use for this implication an estimate, which will be used again in the implications below: Since $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, we have for each $l \in \Lambda$ and $j \in \mathbb{N}$ the following inequality: $\frac{a_j^l}{j} \leq \tilde{a}_j^l \Leftrightarrow a_j^l \leq j \cdot (a_{j+1}^l - a_j^l) = j \cdot a_{j+1}^l - j \cdot a_j^l \Leftrightarrow (j+1) \cdot a_j^l \leq j \cdot a_{j+1}^l \Leftrightarrow \frac{a_j^l}{j} \leq \frac{a_{j+1}^l}{j+1} \Leftrightarrow (M_j^l)^{1/j} \leq (M_{j+1}^l)^{1/(j+1)}$. Moreover this implies also that the mapping $j \mapsto \tilde{a}_j^l = \log(\mu_{j+1}^l)$ is increasing for each $l \in \Lambda$ and so we are done.
For the next steps we use inequality $(B.1)$ in Appendix B in [26] for each $l \in \Lambda$ separately:

$$j \cdot \tilde{a}_j^l \leq \sum_{i=j}^{2j-1} \tilde{a}_i^l = a_{2j}^l - a_j^l \leq j \cdot \tilde{a}_{2j}^l \tag{9.5.1}$$

holds for all $j \in \mathbb{N}$ (because by log. convexity $j \mapsto \tilde{a}_j^l$ is increasing).
(ii) $\Rightarrow$ *(iv)* This holds by the following estimate

$$\tilde{a}_j^l \leq \frac{a_{2j}^l}{j} - \frac{a_j^l}{j} \leq \frac{a_{2j}^l}{j} \leq \frac{2 \cdot a_j^n}{j} + 2 \cdot \log(C) \leq \frac{a_j^{n_1}}{j} + C_1.$$

For the first inequality we have used the first part of (9.5.1), the second inequality holds since $a_j^l = \log(M_j^l) \geq 0$ for all l, j. The third inequality holds by assumption property *(ii)* and for the fourth we finally first have used the log. convexity and then $(\mathcal{M}_{\{\text{diag}\}})$:
$2 \cdot a_j^n \leq a_{2j}^n \leq a_j^{n_1} + j \cdot \tilde{C}$.
(iv) $\Rightarrow$ *(ii)* By log. convexity we obtain $a_{2j}^l \leq (2j) \cdot \tilde{a}_{2j}^l$ and by assumption *(iv)* we get $(2j) \cdot \tilde{a}_{2j}^l \leq (2j) \cdot (\frac{a_{2j}^n}{2j} + C) = a_{2j}^n + (2j) \cdot C$. So we can estimate as follows:

$$\frac{a_{2j}^l}{2j} \leq \frac{a_{2j}^n}{2j} + C \leq \frac{a_j^{n_1}}{j} + C_1$$

where for the the second inequality we need $a_{2j}^n \leq 2 \cdot a_j^{n_1} + (2j) \cdot \tilde{C}$ and this holds again by $(\mathcal{M}_{\{\text{diag}\}})$ (need here in fact only $M_{2j}^n \leq \exp(\tilde{C})^{2j} \cdot (M_j^{n_1})^2$, $C_1 = \tilde{C} + C$).

$(iii) \Rightarrow (ii)$ We get

$$\frac{a^l_{2j}}{2j} \le \tilde{a}^l_{2j} \le \tilde{a}^n_j + C \le \frac{a^n_{2j}}{j} - \frac{a^n_j}{j} + C \le \frac{a^n_{2j}}{j} + C \le \frac{a^{n_1}_j}{j} + C_1.$$

The first inequality holds by log. convexity, the second by assumption (iii), the third by the first part of (9.5.1), finally the last inequality is exactly $(\mathcal{M}_{\{\text{diag}\}})$ again.
$(iv) \Rightarrow (iii)$ We get

$$\tilde{a}^l_{2j} \le \frac{a^n_{2j}}{2j} + C \le \frac{a^{n_1}_j}{j} + C_1 \le \tilde{a}^{n_1}_j + C_1.$$

The first inequality holds by assumption (iv), the second by $(ii) \Leftrightarrow (iv)$ and finally the third is again exactly obtained by log. convexity.
$(v) \Rightarrow (ii)$ In (v) we take $i = j = k$, then we obtain $a^l_{2j} \le n \cdot \tilde{a}^n_j + a^n_j + (2j) \cdot C$. Hence we have

$$\frac{a^l_{2j}}{2j} \le \frac{\tilde{a}^n_j}{2} + \frac{a^n_j}{2j} + C \le \frac{a^n_{2j}}{2j} - \frac{a^n_j}{2j} + \frac{a^n_j}{2j} + C = \frac{a^n_{2j}}{2j} + C \le \frac{a^{n_1}_j}{j} + C_1.$$

The second inequality holds by the first part of (9.5.1), the second by $(\mathcal{M}_{\{\text{diag}\}})$ (need here again in fact only $M^n_{2j} \le \exp(\tilde{C})^{2j} \cdot (M^{n_1}_j)^2$, $C_1 = \tilde{C} + C$). □

Analogously we can formulate (with the same proof as in Theorem 9.5.1 and "Beurling-type-order" of quantifiers):

Theorem 9.5.3. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then the following conditions are equivalent:*

- (i) *$\mathcal{M}$ satisfies condition $(\mathcal{M}_{(\text{mg})})$,*
- (ii) $\forall\, n \in \Lambda\, \exists\, l \in \Lambda\, \exists\, C \ge 1\, \forall\, j \in \mathbb{N} : M^l_{2j} \le C^{2j} \cdot (M^n_j)^2$,
- (iii) $\forall\, n \in \Lambda\, \exists\, l \in \Lambda :\ \sup_{j\in\mathbb{N}} \frac{a^l_{2j} - 2a^n_j}{j} < +\infty$,
- (iv) *Condition* (9.3.4) *is satisfied,*
- (v) $\forall\, n \in \Lambda\, \exists\, l \in \Lambda\, \exists\, C > 0 :\ \forall\, t \in \mathbb{R} : 2 \cdot H_n(t) \le H_l(t + C) + C$.

By using Theorem 9.5.1 the aim would be to prove an analogous result to [26, Theorem 2] where the following was shown: If M is a (weakly) log. convex weight sequence with $\lim_{j\to\infty}(M_j)^{1/j} = +\infty$ and property (mg), then there exists $s > 1$ with $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{G^s\}}$.
- The general weight matrix-type result would be: If $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix with $(\mathcal{M}_{\{\text{mg}\}})$, then

$$\forall\, l \in \Lambda\, \exists\, t > 1\, \exists\, h > 0\, \forall\, j \in \mathbb{N} : M^l_j \le h^j \cdot j!^t, \tag{9.5.2}$$

or alternatively $\forall\, l \in \Lambda\ \exists\, t > 1 : M^l \preceq G^t$. This would mean that $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{\{\mathcal{G}\}}$ is satisfied.

But for non-constant weight matrices we have the following very important counterexample:

Example 9.5.4. *There exist (non-constant) $(\mathcal{M}_{\text{sc}})$ weight matrices $\mathcal{M} = \{M^l : l \in \Lambda\}$ with additionally property $(\mathcal{M}_{\{\text{mg}\}})$ (and so $(\mathcal{M}_{\{\text{dc}\}})$), but such that $\mathcal{E}_{\{\mathcal{G}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$. Moreover we can obtain the following properties:*

(1) *Each sequence M^l doesn't satisfy* (mg) *and* (wmg)*, but each M^l has property* (dc)*.*

(2) *Each M^l doesn't satisfy* (mg)*, but each M^l has* (wmg) *and* (dc)*.*

(3) *Each M^l doesn't satisfy* (mg)*,* (wmg) *and* (dc)*.*

(4) *There exist* $(\mathcal{M}_{\text{sc}})$ *weight matrices $\mathcal{M} = \{M^l : l > 0\}$ with $\mathcal{E}_{\{\mathcal{G}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$, but such that both* $(\mathcal{M}_{\{\text{mg}\}})$ *and* $(\mathcal{M}_{\{\text{dc}\}})$ *don't hold. Moreover each M^l doesn't satisfy* (mg)*,* (wmg) *and* (dc)*.*

Proof. (1) Put $M^l_k := l^{k^2}$ for $l, k \in \mathbb{N}$, $l \geq 2$ (sequence M^2 was already considered in [21, 5.4. Example]). Then immediately $M^{l_1} \leq M^{l_2}$ for $l_1 \leq l_2$, each sequence M^l is clearly normalized and log. convex ($k \mapsto \log(M^l_k) = k^2 \cdot \log(l)$ is convex) and of course $\lim_{k\to\infty}(M^l_k)^{1/k} = \lim_{k\to\infty} l^k = +\infty$ for each $l \geq 2$. So we have shown property $(\mathcal{M}_{\text{sc}})$ and moreover also condition $(\mathcal{M}_{\{\text{mg}\}})$ is satisfied: For given $l \in \mathbb{N}$, $l \geq 2$, we can take e.g. $n := l^2$ and then we have for all $j, k \in \mathbb{N}$:

$$l^{j^2+k^2+2jk} = l^{(j+k)^2} = M^l_{j+k} \leq M^n_j \cdot M^n_k = n^{j^2} \cdot n^{k^2} = (l^2)^{j^2} \cdot (l^2)^{k^2} = l^{2j^2} \cdot l^{2k^2} = l^{2j^2+2k^2}$$

$$\Longleftrightarrow l^{2jk} \leq l^{k^2+j^2} \Leftrightarrow 1 \leq l^{k^2+j^2-2jk} = l^{(k-j)^2}.$$

On the other hand apply log to (9.5.2) and we have for $a^l_k = \log(M^l_k) = k^2 \cdot \log(l)$ and each $l \geq 2$

$$\lim_{k\to\infty} \frac{a^l_k}{k \cdot \log(k)} = \lim_{k\to\infty} \frac{k \cdot \log(l)}{\log(k)} = +\infty,$$

so for $G^t := (j!^t)_j$ we get get $G^t \preceq M^l$ for any $l \geq 2$ and $t > 1$. But the converse relation $M^l \preceq G^t$ cannot be valid for any $l \geq 2$, $t > 1$, since $\lim_{k\to\infty} \frac{k\cdot\log(k)}{a^l_k} = 0$. Hence we have shown $\mathcal{E}_{\{\mathcal{G}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$. Note that all occurring sequences G^t and M^l are weakly log. convex and compare this calculation with $(i) \Leftrightarrow (i')$ in [26, Theorem 2]. Moreover note that by Stirling's formula we have $\log(j!^s) \sim (js) \cdot \log(j)$.

Each sequence M^l doesn't have (mg): For moderate growth $M^l_{j+k} \leq C^{j+k} \cdot M^l_j \cdot M^l_k$ should be satisfied for a constant $C > 0$ and all $j, k \in \mathbb{N}$. But for the special choice $j = k$ (compare this with condition (3) in [26, Theorem 1]) we get for the left hand side $M^l_{2k} = l^{4k^2}$ and for the right hand side $(M^l_k)^2 = l^{2k^2}$, hence for each $l \geq 2$ we obtain

$$\left(\frac{M^l_{2k}}{(M^l_k)^2}\right)^{1/(2k)} = l^{2k^2/(2k)} = l^k \to \infty$$

for $k \to \infty$. But also (wmg) cannot be satisfied, for this we check that condition (e') in [26, Theorem 4] doesn't hold:

$$\frac{a^l_j}{j^2} = \frac{\log(M^l_j)}{j^2} = \frac{j^2 \cdot \log(l)}{j^2} = \log(l)$$

doesn't tend to 0 for $j \to \infty$ for all $l \geq 2$.
But this shows automatically that (dc) is satisfied since condition (γ') in [26, Theorem 5] holds: $\frac{a^l_j}{j^2}$ is bounded for $j \to \infty$ for each $l \geq 2$.

(2), (3) Let the matrix $\mathcal{M} = \{M^l : l > 0\}$ be given by $M^l_j = \exp(l^{1/(s-1)} \cdot j^{s/(s-1)} \cdot R(s))$, $R(s) = \left(\frac{1}{s^{1/(s-1)}} - \frac{1}{s^{s/(s-1)}}\right)$, obtained by the weight function $\omega_s(x) := \max\{0, \log(x)^s\}$ for $s > 1$ a real parameter (see the above section 1.10. for the precise calculation).

Since $\mathcal{M}$ is obtained by a weight function it is always a $(\mathcal{M}_{\text{sc}})$ weight matrix (see 5.1.1) and moreover we get by convexity of the function φ^*_ω property $(\mathcal{M}_{\{\text{mg}\}})$ (with the choice $C = 1$ and $n = 2l$ - see (5.1.2) in 5.1.2). But (mg) cannot be satisfied for any sequence M^l, because ω doesn't satisfy (ω_6) (see the second part of 5.1.3) or by direct calculation (see above section 1.10. for more details). These arguments hold for any parameter $s > 1$.

Analogously we get as above in (1):

$$\frac{a^l_j}{j\cdot\log(j)} = \frac{\log(M^l_j)}{j\cdot\log(j)} = \frac{\log(M^l_j)}{j\cdot\log(j)} = \frac{l^{1/(s-1)}\cdot j^{s/(s-1)}\cdot R(s)}{j\cdot\log(j)} = l^{1/(s-1)}\cdot R(s)\cdot \underbrace{\frac{j^{1/(s-1)}}{\log(j)}}_{\to\infty,\ j\to\infty}$$

because by *de l'Hopital* we have $\lim_{j\to\infty}\frac{j^{1/(s-1)}}{\log(j)} = \lim_{j\to\infty}\frac{1}{s-1}\cdot\frac{j^{(2-s)/(s-1)}}{1/j} = \frac{1}{s-1}\cdot \lim_{j\to\infty} j^{1/(s-1)} = +\infty$. This calculation shows that for all $l > 0$ and $t > 1$ we get $G^t \preceq M^l$, but $M^l \preceq G^t$ cannot be satisfied for any $t > 1$ and $l > 0$.
Similarly as before we use now the characterizations in [26, Theorems 4,5] and we calculate:

$$\frac{a^l_j}{j^2} = \frac{\log(M^l_j)}{j^2} = \frac{l^{1/(s-1)}\cdot j^{s/(s-1)}\cdot R(s)}{j^2} = R(s)\cdot l^{1/(s-1)}\cdot j^{(2-s)/(s-1)}.$$

If $s \geq 2$, then the expression is bounded for $j \to \infty$ and all $l > 0$ but doesn't tend to 0, hence each M^l satisfies (dc) but not (wmg). If $s > 2$, then in fact the expression tends to 0, hence each M^l satisfies both (dc) and (wmg).

But if $1 < s < 2$, then the above expression tends to $+\infty$ for $j \to \infty$ for each $l > 0$, hence each sequence M^l satisfies neither (dc) nor (wmg).
(4) Let $\mathcal{M} := \{M^l : l \in \mathbb{N}, l \geq 2\}$ with $M^l_k := l^{k^k}$ for $k \geq 1$ and $M^l_0 := l$. Of course $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, furthermore we have immediately for all $l \geq 2$

$$\frac{\log(M^l_j)}{j\cdot\log(j)} = \frac{j^j\cdot\log(l)}{j\cdot\log(j)} = \log(l)\cdot\frac{j^{j-1}}{\log(j)} \to \infty,\ \text{ for } j\to\infty,$$

which shows $\mathcal{E}_{\{\mathcal{G}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$ and similarly

$$\frac{\log(M^l_j)}{j^2} = \log(l)\cdot\frac{j^{j-2}}{\log(j)} \to \infty,\ \text{ for } j\to\infty$$

which shows, using again [26, Theorems 4,5], that neither (wmg) nor (dc) can be satisfied.
But now, for property $(\mathcal{M}_{\{\text{mg}\}})$, we proceed as follows: Let $l \in \mathbb{N}$, $l \geq 2$, be arbitrary but fixed and then for each $n \in \mathbb{N}$ we can find a constant $C > 0$ such that $C\cdot l \geq n$ and so $M^{C\cdot l} \geq M^n$. We obtain:

$$\begin{aligned}\left(\frac{M^l_{2j}}{(M^n_j)^2}\right)^{1/(2j)} &= \left(\frac{l^{(2j)^{2j}}}{(n^{j^j})^2}\right)^{1/(2j)} = \left(\frac{l^{2^{2j}\cdot j^{2j}}}{n^{2\cdot j^j}}\right)^{1/(2j)} \geq \left(\frac{l^{2^{2j}\cdot j^{2j}}}{(C\cdot l)^{2\cdot j^j}}\right)^{1/(2j)}\\ &= \left(\frac{l^{2\cdot j^j\cdot(2^{2j-1}\cdot j^j-1)}}{C^{2\cdot j^j}}\right)^{1/(2j)} = \frac{l^{j^{j-1}\cdot(2^{2j-1}\cdot j^j-1)}}{C^{j^{j-1}}} = \left(\frac{l^{2^{2j-1}\cdot j^j-1}}{C}\right)^{j^{j-1}} \to +\infty\end{aligned}$$

for $j \to \infty$, hence $(\mathcal{M}_{\{\mathrm{mg}\}})$ cannot be satisfied because (ii) in 9.5.1 is violated. But clearly for $n = l$ and $C = 1$ in the previous calculation we see that also (mg) cannot be satisfied for any M^l, $l \geq 2$.
Analogously we proceed for $(\mathcal{M}_{\{\mathrm{dc}\}})$: Let $l \in \mathbb{N}$, $l \geq 2$, be arbitrary but fixed and assume that we could find some (very large) $n > l$ and a constant $C \geq 1$ with $M^l_{k+1} = l^{(k+1)^{(k+1)}} \leq C^k \cdot n^{k^k}$ for all $k \in \mathbb{N}$. But for arbitrary $n > l$ we can find a number $A \in \mathbb{N}$ such that $n \leq A \cdot l$ and so we would get

$$l^{(k+1)^{(k+1)}} \leq C^k \cdot (A \cdot l)^{k^k} \Rightarrow l^{(k+1)^k \cdot (k+1)/(k^k) - 1} \leq C^{1/(k^{k-1})} \cdot A$$
$$\Rightarrow l^{(1+1/k)^k \cdot (k+1) - 1} \leq C^{1/(k^{k-1})} \cdot A.$$

Finally note that $\left(1 + \frac{1}{k}\right)^k \cdot (k+1) \to +\infty$ and so we obtain a contradiction. □

Remark 9.5.5. (i) *Example 9.5.4 is very important in the following sense: The embedding-result by Matsumoto cannot be transferred to the non-constant matrix-case! The reason: The matrix generalized moderate-growth-condition of Roumieu-type* $(\mathcal{M}_{\{\mathrm{mg}\}})$ *for* $\mathcal{M} = \{M^l : l \in \Lambda\}$ *is much more general than assuming moderate growth* (mg)*) itself for some/each sequence* M^l.

(ii) *By [26, Theorem 2] for each constant* $(\mathcal{M}_{\mathrm{sc}})$ *weight matrix* $\mathcal{M} = \{M\}$, *we have that if* (mg) *holds then there exists* $s > 1$ *with* $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{G^s\}}$. *So beyond Gevrey sequence spaces we cannot expect property* (mg), *which would be necessary/convenient to introduce a theory of pseudo-differential operators of ultradifferential classes, see remarks in [27, §1].*

(iii) *But example 9.5.4 shows, that also beyond the Gevrey-matrix of Roumieu-type* $\mathcal{E}_{\{\mathcal{G}\}}$ *there exist non-constant* $(\mathcal{M}_{\mathrm{sc}})$ *weight matrices* $\mathcal{M}$ *with property* $(\mathcal{M}_{\{\mathrm{mg}\}})$ *(and each* $M^l \in \mathcal{M}$ *doesn't have* (mg)*). So it seems possible that one can use property* $(\mathcal{M}_{\{\mathrm{mg}\}})$ *and transfer the theory/proofs of [27] to the general weight matrix case (also to classes beyond the Gevrey-matrix space* $\mathcal{E}_{\{\mathcal{G}\}}$*).*

By using Theorem 9.5.1 $(i) \Leftrightarrow (ii)$ we can also show: There exist $(\mathcal{M}_{\mathrm{sc}})$ weight matrices with arbitrary large sequences M^l and satisfying $(\mathcal{M}_{\{\mathrm{mg}\}})$. For this we start with $M^l_k := l^{k^2}$ from above (for $l, k \in \mathbb{N}$, $l \geq 2$) and put $\exp(M^l_k) := \exp(l^{k^2})$. We can iterate this procedure and write $\exp^i(M^l_k) := \exp(\exp^{i-1}(M^l_k))$ for $i \in \mathbb{N}$, where $\exp^i = \underbrace{\exp \circ \cdots \circ \exp}_{i-\text{times}}$, $\exp^0 = \mathrm{id}$.
Let now $i \in \mathbb{N}$ be given with $i \geq 1$, we show that for each i the matrix $(\exp^i(M^l_k))_{l,k \in \mathbb{N}^2}$ satisfies condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ with the choice $n := l^4$. It suffices to prove (ii) in Theorem 9.5.1 and we get by applying log:

$$\exp^i(M^l_{2k})) \leq \left(\exp^i(M^n_k)\right)^2 \Leftrightarrow \exp^{i-1}(M^l_{2k})) \leq 2 \cdot \exp^{i-1}(M^n_k).$$

But this holds by induction on i: For $i = 1$ we get

$$M^l_{2k} \leq M^n_k \Leftrightarrow M^l_{2k} \leq M^{l^4}_k \Leftrightarrow l^{(2k)^2} \leq (l^4)^{k^2} \Leftrightarrow l^{4k^2} \leq l^{4k^2}$$

and for $i \mapsto i + 1$ this inequality stays valid by applying exp to M^l_{2k} resp. $M^{l^4}_k$.

The matrix obtained by $M^l_k := l^{k^2}$ has some other special properties: First each sequence m^l, $m^l_k := \frac{M^l_k}{k!}$, is log. convex (so (slc) holds for each sequence m^l):

$$(m^l_k)^2 \leq m^l_{k-1} \cdot m^l_{k+1} \Leftrightarrow \frac{l^{2k^2}}{k!^2} \leq \frac{l^{(k-1)^2}}{(k-1)!} \cdot \frac{l^{(k+1)^2}}{(k+1)!} \Leftrightarrow \frac{k+1}{k} \cdot l^{2k^2} \leq l^{2k^2+2} \Leftrightarrow \frac{k+1}{k} \leq l^2$$

and the last inequality holds for all $k \geq 1$ and $l \geq 2$, because then $\frac{k+1}{k} \leq 2 \leq l^2$. Moreover $(\mathcal{M}_{\{\mathrm{L}\}})$ holds, because for given $h > 0$ and $l \in \mathbb{N}$, $l \geq 2$, we can put $H := \max\{h, l\}$ and then for all $k \geq 1$ we get $h^k \cdot M^l_k = h^k \cdot l^{k^2} \leq H^k \cdot H^{k^2} = H^{k^2+k} \leq H^{2k^2} = (H^2)^{k^2}$. Hence $(\mathcal{M}_{\{\mathrm{L}\}})$ holds if we choose $n = H^2$.
Another nice property is that for each $l_1, l_2 \in \mathbb{N}$ we can find $n \in \mathbb{N}$ with $M^{l_1}_k \cdot M^{l_2}_k = M^n_k$ for all $k \in \mathbb{N}$, because take clearly $n := l_1 \cdot l_2$.

Conversely we have the following:

Proposition 9.5.6. *Let $\mathcal{M} := \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix and assume that $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{\{\mathcal{G}\}}$ holds. Then already each "derived multi-index weight matrix" from $\mathcal{M}$ in the sense of construction* (9.3.1) *shares this property.*

Proof. First recall (9.3.2): For each $l \in \Lambda$ and $i \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$ we obtain $M^{l,n}_i = (M^l_{in})^{1/n}$. So (since M^l is weakly log. convex) for each $l \in \Lambda$ there exists $t_l > 1$ with $M^l \preceq G^{t_l}$. Hence $M^l_j \leq h^j_l \cdot (j!)^{t_l}$ for a number $h_l > 0$ and all $j \in \mathbb{N}$, and so we get of course $(M^l_{in})^{1/n} \leq h^i_l \cdot ((in)!)^{t_l/n}$ for all $l \in \Lambda$ and $i \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$.
Let $n \in \mathbb{N}_{>0}$ be arbitrary but from now on fixed, we choose $k \in \mathbb{N}$ minimal such that $n \leq 2^k$ holds and then for all $i \in \mathbb{N}$ we have $((2^k i)!)^{t_l} \leq 4^{t_l c_k i} \cdot i!^{2^k t_l}$, where for $k \geq 2$ we have put $c_k = 2^{k-1} + 2 \cdot c_{k-1}$, $c_1 = 1$. This holds by iteration and moderate growth (mg) for each sequence $i!^{t_l}$, more precisely $(2i)!^{t_l} \leq 2^{2it_l} \cdot (i)!^{2t_l} = 4^{it_l} \cdot (i)!^{2t_l}$ and so $(2^k i)!^{t_l} = ((2 \cdot 2^{k-1} i)!)^{t_l} \leq 4^{2^{k-1} i t_l} \cdot ((2^{k-1} i)!)^{2t_l}$ (see also the remark below this proof).
Hence for each $l \in \Lambda$ and $i \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$ we get:

$$\begin{aligned} M^{l,n}_i &= (M^l_{in})^{1/n} \leq h^i_l \cdot ((in)!)^{t_l/n} \leq h^i_l \cdot ((i2^k)!)^{t_l/n} \leq h^i_l \cdot 4^{(c_k i \cdot t_l)/n} \cdot i!^{(2^k t_l)/n} \\ &= (h_l \cdot 4^{(c_k \cdot t_l)/n})^i \cdot (i!)^{(2^k t_l)/n}, \end{aligned}$$

which shows precisely that $M^{l,n} \preceq G^{(2^k t_l)/n}$ for $k \in \mathbb{N}$ chosen minimal such that $n \leq 2^k$ holds. □

Conclusion: If the original weight matrix is contained in the Gevrey-matrix, then also each "derived multi-index-matrix" defined by construction (9.3.1). For this implication we don't need necessarily property $(\mathcal{M}_{\{\mathrm{mg}\}})$ for $\mathcal{M}$, the inclusion holds by the special structure of the the Gevrey-matrix $\mathcal{G}$: Each sequence $G^t \in \mathcal{G}$ has moderate growth (mg) and furthermore for each $t_1, t_2 > 1$ there exists $t_3 > 1$ resp. also for each $t_3, t_2 > 1$ there exists $t_1 > 1$ such that $G^{t_1}_j \cdot G^{t_2}_j = G^{t_3}_j$ for all $j \in \mathbb{N}$.

Remark about the recursion and the precise structure of the sequence $(c_k)_k$ in 9.5.6:
We have $c_k = 2^{k-1} + 2c_{k-1}$ and $c_{k-1} = 2^{k-2} + 2c_{k-2}$, hence after multiplying the second equation by 2 we obtain: $c_k - 2c_{k-1} = 2^{k-1} + 2c_{k-1} - 2^{k-1} - 4c_{k-2}$, hence for $k \geq 2$ we have $b_k = 4b_{k-1} - 4b_{k-2}$ and $b_0 = 1$, $b_1 = 4$ with $b_k := c_{k+1}$.
This new (linear) recursive definition gives for the formal power series $A(z) := \sum_{k \geq 0} b_k \cdot z^k$ the equation $A(z) - 4z - 1 = 4z(A(z) - 1) - 4z^2 A(z) \Leftrightarrow A(z)(4z^2 - 4z + 1) = 1 \Leftrightarrow A(z) = \frac{1}{(1-2z)^2} = \sum_{k \geq 0} (k+1) 2^k z^k$. This means that for all $k \geq 0$ we have $b_k = (k+1)2^k$, hence $c_k = k2^{k-1}$ for all $k \geq 1$.

We close this section with some concrete examples for 9.5.6:

Proposition 9.5.7. *There exist non-constant $(\mathcal{M}_{\mathrm{sc}})$ weight matrices $\mathcal{M} = \{M^l : l \in \Lambda\}$ such that $\mathcal{E}_{\{\mathcal{M}\}} \subsetneq \mathcal{E}_{\{\mathcal{G}\}}$ and with the following properties:*

(1) *Each sequence* M^l *satisfies* (mg).

(2) *Condition* $(\mathcal{M}_{\{\mathrm{mg}\}})$ *holds for* $\mathcal{M}$ *but each sequence* M^l *doesn't satisfy* (mg).

(3) *Each sequence* M^l *doesn't have* (mg) *and also* $(\mathcal{M}_{\{\mathrm{mg}\}})$ *is not satisfied.*

Nevertheless in all three examples each M^l *has properties* (wmg) *and* (dc).

Proof. (1) Let $l \geq 2$, we put $M_0^l = M_1^l = M_2^l := 1$, for $k \geq 3$ we put $M_k^l := l^{k \cdot \log(\log(k))}$. Then each M^l is log. convex, because $k \mapsto \log(M_k^l) = k \cdot \log(\log(k)) \cdot \log(l)$ is convex (note that $\mu_3^l = \frac{M_3^l}{M_2^l} = M_3^l \geq \mu_2^l = 1 \Leftrightarrow l^{3\log(\log(3))} \geq 1 \Leftrightarrow 3\log(\log(3)) \geq 0 \Leftrightarrow 3 \geq \exp(1)$).
Normalization and $M_j^{l_1} \leq M_j^{l_2}$ for $l_1 \leq l_2$ and all $j \in \mathbb{N}$ is clear, moreover $\lim_{j\to\infty}(M_j^l)^{1/j} = \lim_{j\to\infty} l^{\log(\log(j))} = +\infty$ for each $l \geq 2$. Hence $\mathcal{M} := \{M^l : l \geq 2\}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix.
Finally we obtain for all $l \geq 2$

$$\frac{\log(M_j^l)}{j \cdot \log(j)} = \frac{j \cdot \log(\log(j)) \cdot \log(l)}{j \cdot \log(j)} = \log(l) \cdot \underbrace{\frac{\log(\log(j))}{\log(j)}}_{\to 0, j\to\infty}$$

which holds again by the rule of *de l'Hopital:* $\log(\log(x))' = \frac{1}{\log(x)} \cdot \frac{1}{x}$, hence $\lim_{j\to\infty} \frac{\log(\log(j))}{\log(j)} = \lim_{j\to\infty} \frac{1/(\log(j)) \cdot 1/j}{1/j} = \lim_{j\to\infty} \frac{1}{\log(j)} = 0$.
This shows that for each $l \geq 2$ we can find some $t_l > 1$ with $M^l \preceq G^{t_l}$, but the converse implication $G^t \preceq M^l$ doesn't hold for any $t > 1$ and $l \geq 2$.
Moreover, each sequence M^l satisfies (mg), we check for this that condition (3) in [26, Theorem 1] holds:

$$\begin{aligned}\left(\frac{M_{2k}^l}{(M_k^l)^2}\right)^{1/(2k)} &= \left(\frac{l^{2k\cdot\log(\log(2k))}}{l^{2k\cdot\log(\log(k))}}\right)^{1/(2k)} = \left(l^{2k\cdot(\log(\log(2k))-\log(\log(k))}\right)^{1/(2k)} \\ &= l^{\log(\log(2k)/\log(k))} \to l^{\log(1)} = l^0 = 1\end{aligned}$$

for $k \to \infty$, because $\frac{\log(2k)}{\log(k)} = 1 + \frac{\log(2)}{\log(k)} \to 1$ for $k \to \infty$.

(2) Consider the matrix $\mathcal{M} := \{M^l : l \geq 2\}$ defined as follows: First consider the sequence $b_k := 2^{2^{2^k}}$ for $k \in \mathbb{N}$. If $j \in \mathbb{N}$ is given with $b_k \leq j < b_{k+1}$, then we put

$$M_j^l := l^{j\cdot\log(\log(b_k))} = l^{j\cdot\log(2^{2^k}\cdot\log(2))} = l^{j\cdot(2^k\cdot\log(2)+\log(\log(2))}$$

and $M_0^l = M_1^l = M_2^l = M_3^l = 1$. Remark: If $b_0 \leq j < b_1$, so if $4 \leq j < 16$, we have $M_j^l = l^{j\cdot(\log(2)+\log(\log(2)))} \geq 1$.
By definition it's now clear, that $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix.
We show $\mathcal{E}_{\{\mathcal{M}\}} \subsetneq \mathcal{E}_{\{\mathcal{G}\}}$: Let $l \geq 2$ be arbitrary but fixed, then for $j \in \mathbb{N}$ with $b_k \leq j < b_{k+1}$, $k \in \mathbb{N}$, we obtain:

$$\begin{aligned}\frac{\log(M_j^l)}{j \cdot \log(j)} &= \log(l) \cdot \frac{j \cdot (2^k \cdot \log(2) + \log(\log(2))}{j \cdot \log(j)} = \log(l) \cdot \left(\frac{2^k \cdot \log(2)}{\log(j)} + \frac{\log(\log(2))}{\log(j)}\right) \\ &\leq \log(l) \cdot \left(\frac{2^k \cdot \log(2)}{\log(b_k)} + \frac{\log(\log(2))}{\log(b_k)}\right) = \log(l) \cdot \left(\frac{2^k \cdot \log(2)}{2^{2^k} \cdot \log(2)} + \frac{\log(\log(2))}{2^{2^k} \cdot \log(2)}\right).\end{aligned}$$

The last expression tends now to 0 for $j \to \infty$ (so for $k \to \infty$). Similarly we get $\log(M_j^l) = o(j^2)$ for $j \to \infty$, hence by the characterizing results [26, Theorems 4,5] both conditions (wmg) and (dc) are satisfied for each sequence M^l separately.
(mg) cannot be satisfied for any M^l, $l \geq 2$: For this we consider the sub-sequence $(j_k)_k$ with $j_k := b_{k+1} - 1$ for each $k \in \mathbb{N}$ and so $j_k < b_{k+1} \leq 2j_k < b_{k+2}$ for each $k \in \mathbb{N}$. With this choice we have

$$\left(\frac{M_{2j_k}^l}{(M_{j_k}^l)^2}\right)^{1/(2j_k)} = \left(\frac{l^{2j_k \cdot (2^{k+1} \cdot \log(2) + \log(\log(2))}}{l^{2j_k \cdot (2^k \cdot \log(2) + \log(\log(2))}}\right)^{1/(2j_k)} = l^{(2^{k+1} - 2^k) \cdot \log(2)} = l^{2^k \cdot \log(2)} \to +\infty$$

for $k \to \infty$.
To prove property $(\mathcal{M}_{\{\mathrm{mg}\}})$ we distinguish two cases: For given $j \in \mathbb{N}$ with $b_k \leq j < b_{k+1}$ either $b_k \leq 2j < b_{k+1}$ or $b_{k+1} \leq 2j < b_{k+2}$ holds: For this note that $j < b_{k+1} \Leftrightarrow 2j < 2b_{k+1}$ and $2b_{k+1} < b_{k+2}$ for any $k \in \mathbb{N}$. If $b_k \leq j < 2j < b_{k+1}$, then by definition $M_{2j}^l = (M_j^l)^2$ holds and the desired property is satisfied for $n = l$. For the second case, if $b_{k+1} \leq 2j < b_{k+2}$, then we put $n := l^2$ and obtain

$$\begin{aligned}\left(\frac{M_{2j}^l}{(M_j^n)^2}\right)^{1/(2j)} &= \left(\frac{l^{2j \cdot (2^{k+1} \cdot \log(2) + \log(\log(2))}}{(l^2)^{2j \cdot (2^k \cdot \log(2) + \log(\log(2))}}\right)^{1/(2j)} \\ &= l^{2^{k+1} \cdot \log(2) + \log(\log(2)) - 2^{k+1} \cdot \log(2) - 2 \cdot \log(\log(2))} = l^{-\log(\log(2))}.\end{aligned}$$

So we have shown condition (ii) in Theorem 9.5.1 and for a given number $l \geq 2$ with the choice $n := l^2$ we have shown $(\mathcal{M}_{\{\mathrm{mg}\}})$ in any case.

(3) Consider the matrix $\mathcal{M} := \{M^l : l \geq 2\}$ defined as follows: First consider the sequence $b_k := 2^{2^{2^{2^k}}}$, for $k \in \mathbb{N}$. If now $j \in \mathbb{N}$ is given with $b_k \leq j < b_{k+1}$, then we put

$$M_j^l := l^{j \cdot \log(\log(b_k))} = l^{j \cdot \log(2^{2^{2^k}} \cdot \log(2))} = l^{j \cdot (2^{2^k} \cdot \log(2) + \log(\log(2))}$$

and put also $M_0^l = M_1^l = \cdots = M_{15}^l = 1$ (note: $b_0 = 2^4 = 16$).
By definition it's clear, that $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix.
$\mathcal{E}_{\{\mathcal{M}\}} \subsetneq \mathcal{E}_{\{\mathcal{G}\}}$ follows as in (2) above: Let $l \geq 2$ be arbitrary but fixed, then for $j \in \mathbb{N}$ with $b_k \leq j < b_{k+1}$ we see that

$$\begin{aligned}\frac{\log(M_j^l)}{j \cdot \log(j)} &= \log(l) \cdot \frac{j \cdot (2^{2^k} \cdot \log(2) + \log(\log(2))}{j \cdot \log(j)} = \log(l) \cdot \left(\frac{2^{2^k} \cdot \log(2)}{\log(j)} + \frac{\log(\log(2))}{\log(j)}\right) \\ &\leq \log(l) \cdot \left(\frac{2^{2^k} \cdot \log(2)}{\log(b_k)} + \frac{\log(\log(2))}{\log(b_k)}\right) = \log(l) \cdot \left(\frac{2^{2^k} \cdot \log(2)}{2^{2^{2^k}} \cdot \log(2)} + \frac{\log(\log(2))}{2^{2^{2^k}} \cdot \log(2)}\right).\end{aligned}$$

The last expression tends very fast to 0 for $j \to \infty$ (so for $k \to \infty$). Similarly $\log(M_j^l) = o(j^2)$ for $j \to \infty$, hence by [26, Theorems 4,5] both conditions (wmg) and (dc) are satisfied for each M^l.

But now we prove that condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ cannot be satisfied: We prove that condition (ii) in Theorem 9.5.1 doesn't hold. For this we proceed as follows: Let $l \in \mathbb{N}$, $l \geq 2$, be arbitrary but from now on fixed, then for each $n \in \mathbb{N}$ (large) we can find a constant $C > 0$ such that $l^C \geq n$ and so $M^{l^c} \geq M^n$. Now consider as in (2) the sub-sequence

$(j_k)_k$ defined by $j_k := b_{k+1} - 1$ for each $k \in \mathbb{N}$ and so $j_k < b_{k+1} \le 2j_k < b_{k+2}$ for each $k \in \mathbb{N}$. Then we obtain:

$$\left(\frac{M^l_{2j_k}}{(M^n_{j_k})^2}\right)^{1/(2j_k)} = \left(\frac{l^{2j_k\cdot(2^{2^{k+1}}\cdot\log(2)+\log(\log(2)))}}{n^{2j_k\cdot(2^{2^k}\cdot\log(2)+\log(\log(2)))}}\right)^{1/(2j_k)} \ge \left(\frac{l^{2j_k\cdot(2^{2^{k+1}}\cdot\log(2)+\log(\log(2)))}}{l^{C\cdot 2j_k\cdot(2^{2^k}\cdot\log(2)+\log(\log(2)))}}\right)^{1/(2j_k)}$$

$$= l^{(2^{2^k})^2\cdot\log(2)+\log(\log(2))-C\cdot(2^{2^k}\cdot\log(2)+\log(\log(2)))} = l^{2^{2^k}\cdot\log(2)\cdot(2^{2^k}-C)}\cdot l^{\log(\log(2))\cdot(1-C)}.$$

The last expression tends very fast to $+\infty$ for $k \to +\infty$, note that $\log(\log(2)) < 0$. Clearly the same argument for $n = l$ and $C = 1$ in the previous calculation shows that (mg) cannot be satisfied for any M^l, $l \ge 2$. □

9.6 Stability under applying ultradifferential operators

Recall: If $\mathcal{M} = \{M^l : l \in \Lambda\}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then both classes $\mathcal{E}_{\{\mathcal{M}\}}$ and $\mathcal{E}_{(\mathcal{M})}$ are automatically closed under pointwise multiplication (more precisely already each class $\mathcal{E}_{[M^l]}$). Moreover we have $(\mathcal{M}_{\{\text{dc}\}})$ resp. $(\mathcal{M}_{(\text{dc})})$ if and only if the classes $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ are closed under taking derivatives. Hence the classes defined by a $(\mathcal{M}_{\text{sc}})$ weight matrix with this additional property are differential algebras.
The goal of this section and important remark:

(1) If we assume instead $(\mathcal{M}_{[\text{dc}]})$ the stronger condition $(\mathcal{M}_{[\text{mg}]})$ then we are going to have closedness w.r.t. more general types of differential operators, too.

(2) For the constant weight matrices $\mathcal{M} = \{M\}$ item (1) is already known and for this condition (udo) is used which is equivalent to moderate growth (mg).

(3) All results in the section are also valid for globally defined classes and globally conditions $(\mathcal{M}_{\{\text{udo}\}})$ resp. $(\mathcal{M}_{(\text{udo})})$ with obvious modification of the proofs and definitions below, see remark 7.3.2.

An *ultradifferential operator* of $\mathcal{E}_{\{\mathcal{M}\}}$- resp. of $\mathcal{E}_{(\mathcal{M})}$-type is a differential operator of the form $P(\partial) := \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial^\alpha$, where each $a_\alpha : \mathbb{R}^n \to \mathbb{C}$ is a smooth function and with the following growth conditions:

$$(\mathcal{M}_{\{\text{udo}\}}) :\Leftrightarrow \forall\, K \subseteq \mathbb{R}^n \text{ comp. } \forall\, l \in \Lambda\, \forall\, L > 0\, \exists\, C > 0 :\ \forall\, \alpha, \beta \in \mathbb{N}^n :$$
$$\sup_{x\in K}\left|a^{(\beta)}_\alpha(x)\right| \le C \cdot \frac{L^{|\alpha+\beta|}}{M^l_{|\alpha|}}$$

resp.

$$(\mathcal{M}_{(\text{udo})}) :\Leftrightarrow \forall\, K \subseteq \mathbb{R}^n \text{ comp. } \exists\, l \in \Lambda\, \exists\, L > 0\, \exists\, C > 0 :\ \forall\, \alpha, \beta \in \mathbb{N}^n :$$
$$\sup_{x\in K}\left|a^{(\beta)}_\alpha(x)\right| \le C \cdot \frac{L^{|\alpha+\beta|}}{M^l_{|\alpha|}}.$$

An immediate consequence is of course $(\mathcal{M}_{\{\text{udo}\}})\Rightarrow(\mathcal{M}_{(\text{udo})})$. In the special case for constant weight matrices $\mathcal{M} = \{M\}$ and if each a_α is a complex constant, then for $\beta = 0$ we obtain the well-known classical definition of ultradifferential operators which was used and mentioned in the literature several times.

Note: If $P(\partial)$ is a UDO of $\mathcal{E}_{\{\mathcal{M}\}}$-type and $\mathcal{N}\{\preceq\}\mathcal{M}$, i.e. for all $N^l \in \mathcal{N}$ there exists $M^{l_1} \in \mathcal{M}$ with $N^l \preceq M^{l_1}$, then $P(\partial)$ is also a UDO of $\mathcal{E}_{\{\mathcal{N}\}}$-type. The analogous situation holds for the Beurling-case where $\mathcal{M}$ and $\mathcal{N}$ are related via $\mathcal{N}(\preceq)\mathcal{M}$.

Examples:

(1) Consider $a_j := A \cdot \frac{B^j}{M_j^{l_0}}$ for arbitrary constants $A, B > 0$ and some arbitrary $l_0 \in \Lambda$. Then $P(\partial) := \sum_{j=0}^{\infty} a_j \cdot \partial^j$ satisfies clearly $(\mathcal{M}_{(\mathrm{udo})})$, but not $(\mathcal{M}_{\{\mathrm{udo}\}})$.

(2) Let $\Lambda = \mathbb{N}_{>0}$ and consider $a_j := A \cdot \frac{1}{{k_j}^j \cdot M_j^{k_j}}$, where $A > 0$ is an arbitrary constant and $(k_j)_j$ be an increasing sequence of natural numbers with $\lim_{j\to\infty} k_j = +\infty$, then $P(\partial) := \sum_{j=0}^{\infty} a_j \cdot \partial^j$ satisfies $(\mathcal{M}_{\{\mathrm{udo}\}})$ (differential operator associated to a diagonal of the matrix $\mathcal{M}$!).

We are going to prove now the following:

Proposition 9.6.1. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$.*

(1) *If additionally $(\mathcal{M}_{\{\mathrm{mg}\}})$ is satisfied, then $\mathcal{E}_{\{\mathcal{M}\}}$ is closed under applying ultradifferential operators of $\mathcal{E}_{\{\mathcal{M}\}}$-type with constant coefficients.*

(2) *If additionally $(\mathcal{M}_{(\mathrm{mg})})$ is satisfied, then $\mathcal{E}_{(\mathcal{M})}$ is closed under applying ultradifferential operators of $\mathcal{E}_{(\mathcal{M})}$-type with constant coefficients.*

In both cases $P(\partial)$ is a bounded operator, more precisely we obtain in the topological vector space representation: For each $l \in \Lambda$ and $h > 0$ we have $P(\partial)(\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R})) \subseteq \mathcal{E}_{\mathcal{M},n,hD}(K,\mathbb{R})$, where l and n are related by condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ and D comes also from this condition. The analogous situation is valid for the Beurling-case, where the indices l and n are related by $(\mathcal{M}_{(\mathrm{mg})})$.

Proof. For both cases we consider the following estimate: Let a compact set $K \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ be given but from now on fixed, then take a function $f \in \mathcal{E}_{\{\mathcal{M}\}}$ resp. $f \in \mathcal{E}_{(\mathcal{M})}$. On this compact set K we obtain for the summand $a_\alpha \cdot \partial^\alpha$, which occurs in $P(\partial)$ and which we apply to f, the following estimate:

$$\sup_{x\in K}\left|(a_\alpha \cdot \partial^\alpha f)^{(\beta)}(x)\right| = \sup_{x\in K}\left|a_\alpha \cdot f^{(\alpha+\beta)}(x)\right| \le C \cdot \frac{L^{|\alpha|}}{M^n_{|\alpha|}} \cdot C_1 \cdot h^{|\alpha+\beta|} \cdot \underbrace{M^l_{|\alpha+\beta|}}_{\le D^{|\alpha+\beta|}\cdot M^n_{|\alpha|}\cdot M^n_{|\beta|}}$$
$$\le C \cdot C_1 \cdot (L\cdot h\cdot D)^{|\alpha|} \cdot (h\cdot D)^{|\beta|} \cdot M^n_{|\beta|}.$$

Now distinguish between both cases:
(1) In the Roumieu-case we have to choose $C_1, h > 0$ and $l \in \Lambda$ sufficiently large (depending on the given function f and compact set K), from this choice we obtain an index $n \in \Lambda$ and a constant $D > 0$ coming from $(\mathcal{M}_{\{\mathrm{mg}\}})$. According to this given data we use property $(\mathcal{M}_{\{\mathrm{udo}\}})$: We apply this condition exactly for this index n and we take $L > 0$ small enough to guarantee $L \cdot h \cdot D < 1$, because then the infinite sum over $\alpha \in \mathbb{N}^n$ is converging.

(2) In the Beurling-case the situation is analogous: First we point out that if $(\mathcal{M}_{(\mathrm{udo})})$ holds for a certain index $l_0 \in \Lambda$, then also for all $l \le l_0$. To a given $n \in \Lambda$ (small) we obtain by property $(\mathcal{M}_{(\mathrm{mg})})$ an index $l \in \Lambda$ and a constant $D > 0$ (and since we are only

interested in small indices we can assume that n is compatible with the index coming from property $(\mathcal{M}_{(\mathrm{udo})})$). Furthermore we get by this property constants $C, L > 0$ and according to this data we can choose $h > 0$ sufficiently small to get $L \cdot h \cdot D < 1$ and $h \cdot D < h'$ for h' small. Finally the constant C_1 is now depending on n and h. □

If we consider the more general situation in $(\mathcal{M}_{\{\mathrm{udo}\}})$ resp. $(\mathcal{M}_{(\mathrm{udo})})$, where the coefficients of the ultradifferential operator are non-constant smooth functions, then we can prove the following result:

Proposition 9.6.2. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$.*

(1) *If additionally $(\mathcal{M}_{\{\mathrm{mg}\}})$ is satisfied, then $\mathcal{E}_{\{\mathcal{M}\}}$ is closed under applying ultradifferential operators of $\mathcal{E}_{\{\mathcal{M}\}}$-type.*

(2) *If additionally $(\mathcal{M}_{(\mathrm{mg})})$ is satisfied and $P(\partial)$ is an ultradifferential operator of $\mathcal{E}_{(\mathcal{M})}$-type, then*

$$f \in \mathcal{E}_{(\mathcal{M})}(U) \Longrightarrow P(\partial)(f) \in \varprojlim_{l \in \Lambda} \mathcal{E}_{\{M^l\}}(U)$$

holds for each non-empty open set $U \subseteq \mathbb{R}^n$.

Also here in both cases $P(\partial)$ is a bounded operator, more precisely we obtain in the topological vector space representation: For each $l \in \Lambda$ and $h > 0$ we have $P(\partial)(\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R})) \subseteq \mathcal{E}_{\mathcal{M},n,\tilde{h}}(K,\mathbb{R})$, where l and n are related by condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ and $\tilde{h} = \left(1 + \frac{L}{h}\right) \cdot L \cdot h \cdot D$. D comes also from condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ and L from $(\mathcal{M}_{\{\mathrm{udo}\}})$ chosen maximal such that $LhD < 1$. The analogous situation is valid for the Beurling-case, where the indices l and n are related by $(\mathcal{M}_{(\mathrm{mg})})$.

Proof. For both cases we obtain the following estimate: Let a compact set $K \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ be given but from now on fixed, take a function $f \in \mathcal{E}_{\{\mathcal{M}\}}$ resp. $f \in \mathcal{E}_{(\mathcal{M})}$. Then we obtain for each $\beta \in \mathbb{N}^n$ on K for the summand $a_\alpha(x)\partial^\alpha$ occurring in $P(\partial)$ and which we apply to f the following estimate:

$$\begin{aligned}
\sup_{x\in K}\left|(a_\alpha \cdot \partial^\alpha f)^{(\beta)}(x)\right| &= \sup_{x\in K}\left|\left(a_\alpha \cdot f^{(\alpha)}\right)^{(\beta)}(x)\right| \le \sup_{x\in K}\sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left|a_\alpha^{(\gamma)}(x)\right| \cdot \left|f^{(\alpha+\beta-\gamma)}(x)\right| \\
&\le \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma} C \cdot \frac{L^{|\alpha+\gamma|}}{M^n_{|\alpha|}} \cdot C_1 \cdot h^{|\alpha+\beta-\gamma|} \cdot M^l_{|\alpha+\beta-\gamma|} \\
&\le C \cdot C_1 \cdot \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma} L^{|\alpha|} \cdot L^{|\gamma|} \cdot h^{|\alpha+\beta|} \cdot h^{-|\gamma|} \cdot \frac{1}{M^n_{|\alpha|}} \cdot \underbrace{M^l_{|\alpha+\beta|}}_{\le D^{|\alpha+\beta|} \cdot M^n_{|\alpha|} \cdot M^n_{|\beta|}} \\
&\le (C \cdot C_1) \cdot (h \cdot D)^{|\beta|} \cdot (L \cdot h \cdot D)^{|\alpha|} \cdot M^n_{|\beta|} \cdot \sum_{\gamma=0}^{\beta}\binom{\beta}{\gamma}\left(\frac{L}{h}\right)^{|\gamma|} \\
&\le (C \cdot C_1) \cdot \left(\left(1 + \frac{L}{h}\right) \cdot L \cdot h \cdot D\right)^{|\beta|} \cdot (L \cdot h \cdot D)^{|\alpha|} \cdot M^n_{|\beta|}.
\end{aligned}$$

The arguments are now precisely the same as in 9.6.1. In the Roumieu-case we have to take L small enough (by $(\mathcal{M}_{\{\mathrm{udo}\}})$) to guarantee $L \cdot h \cdot D < 1$, in the Beurling-case we have to take h small enough. But note that in the Beurling-case we cannot reach that the expression $\left(1+\frac{L}{h}\right)\cdot L\cdot h\cdot D$ becomes arbitrary small for $h\to 0$. So we get only $P(\partial)(f)\in\varprojlim_{l\in\Lambda}\mathcal{E}_{\{M^l\}}(U)\supsetneq\varprojlim_{l\in\Lambda}\mathcal{E}_{(M^l)}(U)=:\mathcal{E}_{(\mathcal{M})}(U)$ for each non-empty open set $U\subseteq\mathbb{R}^n$.

For non-constant matrices: If we are in the (natural) situation where above we have $=$ instead of $\supsetneq$, then $\mathcal{E}_{(\mathcal{M})}$ is closed under applying ultradifferential operators of $\mathcal{E}_{(\mathcal{M})}$-type in the most general sense. For this we assume condition $(\mathcal{M}_{(\mathrm{BR})})$, see (1)(*c*) in 9.1.1. If the weight matrix $\mathcal{M}=\{M^l:l>0\}$ is coming from a weight function $\omega\in\mathcal{W}$ with (ω_1), then we have to assume in addition (ω_7) to guarantee this desired property, see 5.4.1.

Finally note that condition $(\mathcal{M}_{(\mathrm{BR})})$ never can't be satisfied for constant matrices $\mathcal{M}=\{M\}$, because the relation $\lhd$ is not reflexive. For constant matrices $\mathcal{M}=\{M\}$ we only get $P(\partial)(f)\in\varprojlim_{l\in\Lambda}\mathcal{E}_{\{M^l\}}(U)=\mathcal{E}_{\{M\}}(U)$. □

Remark: If $\mathcal{M}$ is an arbitrary (large) set of weight sequences and if we consider the classes $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ in the most general situation defined in (7.2.1) resp. (7.2.2), then we get: 9.6.1 and 9.6.2 stay valid if each $M\in\mathcal{M}$ is assumed to be increasing and if we have conditions $(\mathcal{M}_{\{\mathrm{fil}\}})$ resp. $(\mathcal{M}_{(\mathrm{fil})})$ for $\mathcal{M}$, too. Of course conditions $(\mathcal{M}_{[\mathrm{udo}]})$ and $(\mathcal{M}_{[\mathrm{mg}]})$ are also considered in the most general sense (e.g. replace "for all $l\in\Lambda$" by "for all $M\in\mathcal{M}$").

If in the Roumieu-case condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ is not satisfied we are still able to prove the following more general result:

Lemma 9.6.3. *Let $\mathcal{M}=\{M^l:l\in\Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and assume the following condition for the matrix $\mathcal{M}$:*

$$(\mathcal{M}_{\{\mathrm{wmg}\}}):\ \forall\,l\in\Lambda\ \exists\,n\in\Lambda\ \exists\,C\geq 1\ \exists\,R^l=(R^l_k)_k\ \forall\,j,k\in\mathbb{N}:\ M^l_{j+k}\leq C^{j+k}\cdot M^n_j\cdot R^l_k,$$

where R^l is some (large weakly log. convex) weight sequence and this condition is the matrix-generalization of property (wmg) *introduced in [26, § 1].*

Then a ultrdifferential operator $P(\partial)$ of $\mathcal{E}_{\{\mathcal{M}\}}$-type maps the class $\mathcal{E}_{\{\mathcal{M}\}}$ into the class $\mathcal{E}_{\{\mathcal{R}\}}$, where $\mathcal{R}:=\{R^l:l\in\Lambda\}$, and it is again a bounded operator.

Note that $(\mathcal{M}_{\{\mathrm{wmg}\}})$ is much more general than $(\mathcal{M}_{\{\mathrm{mg}\}})$.

Proof. The proof follows immediately by the same calculations and estimates as in 9.6.1 resp. 9.6.2, e.g. we have

$$\sup_{x\in K}\left|(a_\alpha\cdot\partial^\alpha f)^{(\beta)}(x)\right|=\sup_{x\in K}\left|a_\alpha\cdot f^{(\alpha+\beta)}(x)\right|\leq C\cdot\frac{L^{|\alpha|}}{M^n_{|\alpha|}}\cdot C_1\cdot h^{|\alpha+\beta|}\cdot\underbrace{M^l_{|\alpha+\beta|}}_{\leq D^{|\alpha+\beta|}\cdot M^n_{|\alpha|}\cdot R^l_{|\beta|}}$$
$$\leq C\cdot C_1\cdot(L\cdot h\cdot D)^{|\alpha|}\cdot(h\cdot D)^{|\beta|}\cdot R^l_{|\beta|}.$$

□

In the next step we are going to prove now the converse direction for the Roumieu-case:

Proposition 9.6.4. *Let $\mathcal{M}=\{M^l:l\in\Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $\Lambda=\mathbb{N}_{>0}$. If the space $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ is closed under applying ultradifferential operators of $\mathcal{E}_{\{\mathcal{M}\}}$-type with constant coefficients, then additionally condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ is satisfied for $\mathcal{M}$.*

Proof. We prove that $\neg(\mathcal{M}_{\{\mathrm{mg}\}})$ implies the fact that the class $\mathcal{E}_{\{\mathcal{M}\}}$ cannot be closed under applying ultradifferential operators of $\mathcal{E}_{\{\mathcal{M}\}}$-type. The technique of this proof should be also compared with 10.10.7.

Before we start with the proof we show the following claim: Ultradifferential operators $P(\partial)$ of $\mathcal{E}_{\{\mathcal{M}\}}$-type are well-defined on $\mathcal{E}_{\{\mathcal{M}\}}$, more precisely we prove $P(\partial)(\mathcal{E}_{\{\mathcal{M}\}}) \subseteq \mathcal{E}$. So let $f \in \mathcal{E}_{\{\mathcal{M}\}}$ be given, then $P(\partial)(f)(x) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x) f^{(\alpha)}(x)$ and we estimate analogously as in 9.6.1 resp. 9.6.2 on an arbitrary but fixed compact set $K \subseteq \mathbb{R}^n$ for a fixed summand $a_\alpha(x)\cdot f^{(\alpha)}(x)$ (and the case $\beta = 0$) as follows:

$$\sup_{x\in K}\left|a_\alpha(x)\cdot f^{(\alpha)}(x)\right| \le C\cdot\frac{L^{|\alpha|}}{M^n_{|\alpha|}}\cdot C_1\cdot h^{|\alpha|}\cdot M^l_{|\alpha|} = C_2\cdot(L\cdot h)^{|\alpha|}\cdot\frac{M^l_{|\alpha|}}{M^n_{|\alpha|}}.$$

The index l and $h > 0$ (both large) are depending on f and K, but according to this given data we choose by $(\mathcal{M}_{\{\text{udo}\}})$ the constant L small enough to guarantee $L\cdot h < 1$ and the index $n \in \Lambda$ large enough (at least $n = l$) to get $\frac{M^l_{|\alpha|}}{M^n_{|\alpha|}} \le 1$ for all $\alpha \in \mathbb{N}^n$. Hence the sum is absolutely convergent and we are done. The same estimate still holds for the Beurling-case: Here we get by $(\mathcal{M}_{(\text{udo})})$ a constant L (large) and an index n (small) and we to choose $h > 0$ small enough to guarantee $L\cdot h < 1$ and $l \in \Lambda$ small to get at least $l = n$.

Now $\neg(\mathcal{M}_{\{\text{mg}\}})$ means:

$$\exists\, l\in\Lambda\ \forall\, n\in\Lambda\ \forall\, C\ge 1\ \exists\, j_{n,C},k_{n,C}\in\mathbb{N} : M^l_{j_{n,C}+k_{n,C}} > C^{j_{n,C}+k_{n,C}}\cdot M^n_{j_{n,C}}\cdot M^n_{k_{n,C}}.$$

Since by assumption we have $\Lambda = \mathbb{N}_{>0}$ we can consider the case $n = C \in \mathbb{N}_{>0}$ above and identify $j_{n,n} \leftrightarrow j_n$ resp. $k_{n,n} \leftrightarrow k_n$. The sequence $(j_n)_{n\ge1}$ is increasing with $j_n \to \infty$ for $n \to \infty$ and $k_n \ge 1$ for all $n \in \mathbb{N}_{>0}$.
We prove by contradiction: Assume now that the class $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ would be closed under applying ultradifferential operators of $\mathcal{E}_{\{\mathcal{M}\}}$-type $P(\partial) := \sum_{j\ge0} a_j\partial^j$, then for each function $f \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ we would have: For each compact set $K \subseteq \mathbb{R}$ there would exist $C_1, C_2 > 0$ and $l_1 \in \Lambda$ such that for each $k \in \mathbb{N}$:

$$\sup_{x\in K}\left|\left(P(\partial)(f)\right)^{(k)}(x)\right| \le C_1\cdot C_2^k\cdot M^{l_1}_k. \tag{9.6.1}$$

We apply this situation to our special function $\tilde{\theta}_l \in \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{C})$ with $\tilde{\theta}^{(j)}_l(0) = (\sqrt{-1})^j\cdot s^l_j$, $s^l_j := \sum_{k=0}^\infty M^l_k(2\mu^l_k)^{j-k} \ge M^l_j$ for all $j\in\mathbb{N}$ and $\left|\tilde{\theta}^{(j)}_l(t)\right| \le s^l_j = \left|\tilde{\theta}^{(j)}_l(0)\right|$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$ (see (2.4.7)). The arising index l is precisely the index coming from $\neg(\mathcal{M}_{\{\text{mg}\}})$. Moreover we introduce an UDO of $\mathcal{E}_{\{\mathcal{M}\}}$-type (with constant coefficients) as follows: Let $A > 0$ be an arbitrary but fixed real constant, for $j = j_n$ we put $b_j := \frac{A}{n^j\cdot M^n_j}$, for $j \in \mathbb{N}$ with $j_n \le j < j_{n+1}$ we put $b_j := b_{j_n}$ and for $0 \le j \le j_1$ we put $b_j = b_{j_1}$. A second UDO of $\mathcal{E}_{\{\mathcal{M}\}}$-type is obtained by putting $a_j := (\sqrt{-1})^{3j}\cdot b_j$ and so clearly $|a_j| = |b_j| = b_j$ for all $j \in \mathbb{N}$.
By hypothesis we have (9.6.1) with $k = 0$ and we get for a compact set K containing the point 0 and the UDO $P(\partial) := \sum_{j=1}^\infty a_j\cdot\partial^j$:

$$C_1 \ge \sup_{t\in K}|P(\partial)(\tilde{\theta}_l)(t)| \ge \left|P(\partial)(\tilde{\theta}_l)(0)\right| = \left|\sum_{j=0}^\infty a_j\cdot\tilde{\theta}^{(j)}_l(0)\right| = \left|\sum_{j=0}^\infty(\sqrt{-1})^{3j}\cdot b_j\cdot\underbrace{\tilde{\theta}^{(j)}_l(0)}_{=(\sqrt{-1})^j\cdot s^l_j}\right|$$

$$= \left|\sum_{j=0}^\infty b_j\cdot s^l_j\right| = \sum_{j=0}^\infty b_j\cdot s^l_j \ge \sum_{j=0}^\infty b_j\cdot\left|\tilde{\theta}^{(j)}_l(x)\right|.$$

The last inequality holds for arbitrary $x \in \mathbb{R}$ and since $|a_j| = |b_j| = b_j$ for all $j \in \mathbb{N}$ we have shown that in this situation the sum $\sum_{j=0}^{\infty} a_j \cdot \tilde{\theta}_l^{(j)}(x) = P(\partial)(\tilde{\theta}_l)(x)$ is absolutely convergent for any $x \in \mathbb{R}$ and so we can interchange summation and differentiation. Now we can estimate on a compact set K containing the point 0 for each $k_n \geq 1$, $n \in \mathbb{N}_{>0}$:

$$\begin{aligned}
\sup_{t \in K} \left|(P(\partial)(\tilde{\theta}_l))^{(k_n)}(t)\right| &\geq \left|(P(\partial)(\tilde{\theta}_l))^{(k_n)}(0)\right| = \left|\sum_{j=0}^{\infty} a_j \tilde{\theta}_l^{(j+k_n)}(0)\right| \\
&= \left|\sum_{j=0}^{\infty} (\sqrt{-1})^{3j} b_j \underbrace{\tilde{\theta}_l^{(j+k_n)}(0)}_{=(\sqrt{-1})^{j+k_n} s^l_{j+k_n}}\right| = \left|(\sqrt{-1})^{k_n} \cdot \sum_{j=0}^{\infty} b_j \cdot s^l_{j+k_n}\right| = \sum_{j=0}^{\infty} b_j \cdot s^l_{j+k_n} \\
&\geq \sum_{j=0}^{\infty} b_j \cdot M^l_{j+k_n} \underbrace{\geq}_{j=j_n} b_{j_n} \cdot M^l_{j_n+k_n} \underbrace{>}_{\neg(\mathcal{M}_{\{\mathrm{mg}\}})} b_{j_n} \cdot n^{j_n+k_n} \cdot M^n_{j_n} \cdot M^n_{k_n} \\
&= \frac{A}{n^{j_n} \cdot M^n_{j_n}} \cdot n^{j_n+k_n} \cdot M^n_{j_n} \cdot M^n_{k_n} = A \cdot n^{k_n} \cdot M^n_{k_n}.
\end{aligned}$$

Note that each term $b_j \cdot M^l_{j+k_n}$ is a strict positive real number, so this estimate holds for arbitrary $n \geq 1$. But by assumption this would imply for each $n \geq 1$:

$$A \cdot n^{k_n} \cdot M^n_{k_n} \leq C_1 \cdot C_2^{k_n} \cdot M^{l_1}_{k_n} \Longrightarrow \left(\frac{A}{C_1}\right)^{1/k_n} \cdot \frac{n}{C_2} \cdot \left(\frac{M^n_{k_n}}{M^{l_1}_{k_n}}\right)^{1/k_n} \leq 1,$$

for some constants $C_1, C_2 \geq 1$ and an index $l_1 \in \Lambda$ which gives a contradiction for $n \to \infty$ (note that $M^n_k \geq M^{l_1}_k$ for each $k \in \mathbb{N}$ whenever $n \geq l_1$). $\square$

We close this section with the following summary: Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be always a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix and $P(\partial)$ a ultradifferential operator of $\mathcal{E}_{\{\mathcal{M}\}}$- resp. $\mathcal{E}_{(\mathcal{M})}$-type.

(*i*) $P(\partial) : \mathcal{E}_{[\mathcal{M}]} \to \mathcal{E}$.

(*ii*) If $\mathcal{M}$ has in addition $(\mathcal{M}_{\{\mathrm{wmg}\}})$, then $P(\partial) : \mathcal{E}_{\{\mathcal{M}\}} \to \mathcal{E}_{\{\mathcal{R}\}}$ for some (larger) $(\mathcal{M}_{\mathrm{sc}})$ weight matrix $\mathcal{R}$ which depends on the given matrix $\mathcal{M}$ and is related to it via $(\mathcal{M}_{\{\mathrm{wmg}\}})$.

(*iii*) If $\mathcal{M}$ has in addition $(\mathcal{M}_{[\mathrm{mg}]})$ and $P(\partial)$ is an UDO for the particular case with constant coeff., then $P(\partial) : \mathcal{E}_{[\mathcal{M}]} \to \mathcal{E}_{[\mathcal{M}]}$.

(*iv*) If $\mathcal{M}$ has in addition $(\mathcal{M}_{[\mathrm{mg}]})$ and $P(\partial)$ is a general UDO for the particular case, then $P(\partial) : \mathcal{E}_{\{\mathcal{M}\}} \to \mathcal{E}_{\{\mathcal{M}\}}$ and $P(\partial) : \mathcal{E}_{(\mathcal{M})} \to \varprojlim_{l \in \Lambda} \mathcal{E}_{\{M^l\}} \supsetneq \mathcal{E}_{(\mathcal{M})}$ (see 9.6.2).

9.7 Remarks ad Gelfand-Shilov-spaces

Let $\mathcal{M}$ and $\mathcal{N}$ be two arbitrary weight matrices, i.e. $(\mathcal{M})$, then inspired by the definitions given in [9, Definition 2.1.] we are going to introduce the *Gelfand-Shilov-sapces* defined by weight matrices as follows: First the Roumieu-type space

$$\mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}(\mathbb{R}^n) := \big\{f \in \mathcal{E}(\mathbb{R}^n) :\ \exists\, C, h_1, h_2 > 0 :\ \exists\, l_1, l_2 \in \Lambda :\ \forall\, \alpha, \beta \in \mathbb{N}^n : \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| \le C \cdot h_1^{|\alpha|} \cdot h_2^{|\beta|} \cdot M^{l_1}_{|\alpha|} \cdot N^{l_2}_{|\beta|}\big\},$$

and similarly one could also consider the Beurling-type space

$$\mathbb{S}_{(\mathcal{M},\mathcal{N})}(\mathbb{R}^n) := \big\{f \in \mathcal{E}(\mathbb{R}^n) :\ \forall\, h_1, h_2 > 0\, \forall\, l_1, l_2 \in \Lambda : \exists\, C > 0 :\ \forall\, \alpha, \beta \in \mathbb{N}^n : \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| \le C \cdot h_1^{|\alpha|} \cdot h_2^{|\beta|} \cdot M^{l_1}_{|\alpha|} \cdot N^{l_2}_{|\beta|}\big\}.$$

Now we can formulate the analogous result to [9, Theorem 2.3.]:

Theorem 9.7.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two given $(\mathcal{M}_{\mathrm{sc}})$ weight matrices. Then we get: If in addition $\liminf_{k\to\infty}(m^l_k \cdot n^l_k)^{1/k} > 0$ for some $l \in \Lambda$, where we have put as usual $m^l_k := \frac{M^l_k}{k!}$, $n^l_k := \frac{N^l_k}{k!}$, and both matrices satisfy condition $(\mathcal{M}_{\{\mathrm{mg}\}})$, then the following are equivalent:*

(*i*) $f \in \mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}$.

(*ii*) *There exist constants $C, h_1, h_2 > 0$ and indices $l_1, l_2 \in \Lambda$ such that for all $\alpha, \beta \in \mathbb{N}^n$ we have:*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \cdot f(x)| \le C \cdot h_1^{|\alpha|} \cdot M^{l_1}_{|\alpha|} \qquad \sup_{x\in\mathbb{R}^n} \left|\partial^\beta f(x)\right| \le C \cdot h_2^{|\beta|} \cdot N^{l_2}_{|\beta|}.$$

(*iii*) *There exist constants $C, h_1, h_2 > 0$ and indices $l_1, l_2 \in \Lambda$ such that for all $\alpha, \beta \in \mathbb{N}^n$ we have:*

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \cdot f(x)| \le C \cdot h_1^{|\alpha|} \cdot M^{l_1}_{|\alpha|} \qquad \sup_{\xi\in\mathbb{R}^n} \left|\xi^\beta \cdot \hat{f}(\xi)\right| \le C \cdot h_2^{|\beta|} \cdot N^{l_2}_{|\beta|}.$$

Remark: The assumption $\liminf_{k\to\infty}(m^l_k \cdot n^l_k)^{1/k} > 0$ for some $l \in \Lambda$ is clearly satisfied if both matrices $\mathcal{M}$ and $\mathcal{N}$ have $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$.

Proof. We use the same proof as in [9, Theorem 2.3.]. First the implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ hold by definition of the spaces and the property $\widehat{\mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}} = \mathbb{S}_{\{\mathcal{N},\mathcal{M}\}}$, see 9.7.2 below.

For $(ii) \Rightarrow (i)$ we proceed as follows: We start with the Roumieu-case and estimate as in the original proof in [9, Theorem 2.3.] (where the L^2-norm instead the supremum norm is used):

$$\begin{aligned}
\|x^\alpha \partial^\beta f(x)\|^2_{L^2} &= \int_{\mathbb{R}^n} \left(x^{2\alpha}\partial^\beta f(x)\right) \partial^\beta f(x) dx \\
&\le \sum_{\gamma\le 2\alpha} \sum_{\gamma\le\beta} \binom{2\alpha}{\gamma}\binom{\beta}{\gamma} \gamma! \|\partial^{2\beta-\gamma} f(x)\|_{L^2} \cdot \|x^{2\alpha-\gamma} f(x)\|_{L^2} \\
&\le C^2 \cdot \sum_{\gamma\le 2\alpha} \sum_{\gamma\le\beta} \binom{2\alpha}{\gamma}\binom{\beta}{\gamma} \gamma! h^{2(|\alpha|+|\beta|-|\gamma|)} M^l_{|2\alpha-\gamma|} N^l_{|2\beta-\gamma|}.
\end{aligned}$$

For this estimate we have used integration by parts, the Leibniz rule and the Schwarz inequality, finally in the last step we have used our assumption (*ii*) and we can put $h := \max\{h_1, h_2\}$ and also $l := \max\{l_1, l_2\}$. Now we can estimate as follows:

$$\|x^\alpha \partial^\beta f(x)\|_{L^2}^2 \le C^2 h^{2(|\alpha|+|\beta|)} M^l_{|2\alpha|} N^l_{|2\beta|} \sum_{\gamma\le 2\alpha}\sum_{\gamma\le\beta}\binom{2\alpha}{\gamma}\binom{\beta}{\gamma}\underbrace{\frac{\gamma!}{M^l_{|\gamma|}N^l_{|\gamma|}}}_{\le C_1 (\star)}$$
$$\le C_2^2 (2hH)^{2(|\alpha|+|\beta|)} (M^{l'}_{|\alpha|})^2 (N^{l'}_{|\beta|})^2.$$

In the first estimate we have used the previous estimate and then recall that by assumption each sequence M^l resp. N^l is weakly log. convex. For $(\star)$ we point out: W.l.o.g. we can assume that $l \in \Lambda$ is chosen large enough to guarantee $\liminf_{k\to\infty}(m^l_k \cdot n^l_k)^{1/k} > 0$. Finally in the second estimate we have used condition $(\mathcal{M}_{\{\mathrm{mg}\}})$ (the indices l and l' are related by this condition). Hence we have shown $f \in \mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}$.

(*iii*) $\Rightarrow$ (*ii*) First we point out that for arbitrary $h > 0$ and $l \in \Lambda$ we obtain that $|\xi^\beta \cdot \hat{f}(\xi)| \le C \cdot h^{|\beta|} \cdot N^l_{|\beta|}$ implies immediately by definition $|\hat{f}(\xi)| \le C \cdot \exp\left(-\omega_{N^l}\left(\frac{|\xi|}{h}\right)\right)$ $(\star)$. So we can estimate as follows where $l \in \Lambda$ is the index coming from assumption (*iii*):

$$\left|\partial^\beta f(x)\right| \le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\exp(ix\xi)\cdot\xi^\beta\cdot\hat{f}(\xi)\right| d\xi \underbrace{\le}_{(\star)} C_1 \int_{\mathbb{R}^n} |\xi|^{|\beta|}\cdot\exp\left(-\omega_{N^l}\left(\frac{|\xi|}{h}\right)\right) d\xi$$
$$\le C_1 \cdot \sup_{\xi\in\mathbb{R}^n}\left\{|\xi|^{|2\beta|}\cdot\exp\left(-\omega_{N^l}\left(\frac{|\xi|}{h}\right)\right)\right\}^{1/2}\cdot\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\cdot\omega_{N^l}\left(\frac{|\xi|}{h}\right)\right) d\xi$$
$$\underbrace{=}_{(\star\star)} C_1 \cdot h^{|\beta|}\cdot (N^l_{2|\beta|})^{1/2}\cdot\int_{\mathbb{R}^n}\exp\left(-\frac{1}{2}\cdot\omega_{N^l}\left(\frac{|\xi|}{h}\right)\right) d\xi$$
$$\le C_2 \cdot h^{|\beta|}\cdot (N^l_{2|\beta|})^{1/2} \underbrace{\le}_{(\mathcal{M}_{\{\mathrm{mg}\}})} C_2\cdot (hH)^{|\beta|}\cdot N^{l'}_{|\beta|}.$$

The first inequality is exactly the Fourier-inversion-formula. For $(\star\star)$ we have used the fact that each occurring weight sequence is an element of the set $\mathcal{LC}$, hence we can apply [16, Proposition 3.2.]. Moreover note that the integral is finite by 4.0.2. □

In order to finish the proof of the previous result we have to show $\widehat{\mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}} = \mathbb{S}_{\{\mathcal{N},\mathcal{M}\}}$ and we formulate the following lemma:

Lemma 9.7.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{M}_{\mathrm{sc}})$ weight matrices with property $(\mathcal{M}_{\{\mathrm{dc}\}})$. Then $\widehat{\mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}}(\mathbb{R},\mathbb{R}) \subseteq \mathbb{S}_{\{\mathcal{N},\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is satisfied.*

Proof. The following proof was communicated by *Armin Rainer* for the case if both weight matrices are constant. Let $g \in \mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}(\mathbb{R},\mathbb{R})$, then first partial integration gives

$$\left|\xi^p \cdot \hat{g}^{(q)}(\xi)\right| \le (2\pi)^{q-p}\cdot\sum_{k=0}^{p}\frac{p!}{k!}\binom{q}{p-k}\int_{\mathbb{R}}\left|x^{q-p+k} g^{(k)}(x)\right| dx,$$

hence

$$\begin{aligned}\left|\xi^p\cdot\hat{g}^{(q)}(\xi)\right| &\le C\cdot(2\pi)^q\cdot\sum_{k=0}^{p}\frac{p!}{k!}\binom{q}{p-k}h_1^{q-p+k+2}\cdot M^{l_1}_{q-p+k+2}\cdot h_2^k\cdot N^{l_2}_k\cdot\int_{\mathbb{R}}\frac{1}{(1+|x|)^2}dx\\ &\le C\cdot(2\pi)^q\cdot\sum_{k=0}^{p}\frac{p!}{k!}\binom{q}{p-k}h^{q+k+2}\cdot M^{l_1}_{q+2}\cdot N^{l_2}_p\cdot\int_{\mathbb{R}}\frac{1}{(1+|x|)^2}dx\\ &\le C\cdot(4\pi)^q\cdot h^{q+p}\cdot M^{l_1}_{q+2}\cdot N^{l_2}_p\cdot\frac{2h^3}{h-1},\end{aligned}$$

where we have put $h:=\max\{h_1,h_2\}$ and w.l.o.g. $h_i\ge 1$ for $i=1,2$. Note that weakly log. convexity and normalization of each occurring sequence implies that each M^l resp. N^l is already increasing. Moreover $\sum_{k=0}^{p}h^k=\frac{h^{p+1}-1}{h-1}$ for $h>1$ and finally we use property $(\mathcal{M}_{\{\mathrm{dc}\}})$ to obtain $C\cdot(4\pi)^q\cdot h^{q+p}\cdot M^{l_1}_{q+2}\cdot N^{l_2}_p\cdot\frac{2h^3}{h-1}\le C_1\cdot h_3^{q+p}\cdot M^{l_3}_q\cdot N^{l_2}_p$. $\square$

By using the above Theorem 9.7.1 we can also formulate the analogous result to [9, Corollary 2.4.]:

Corollary 9.7.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $(\mathcal{M}_{\mathrm{sc}})$ weight matrices. Then we get If in addition* $\liminf_{k\to\infty}(m^l_k\cdot n^l_k)^{1/k}>0$ *for some $l\in\Lambda$ and both matrices $\mathcal{M}$ and $\mathcal{N}$ satisfy condition $(\mathcal{M}_{\{\mathrm{mg}\}})$, then the following are equivalent:*

(*i*) $f\in\mathbb{S}_{\{\mathcal{M},\mathcal{N}\}}$.

(*ii*) *There exist $C,h_1,h_2>0$ and indices $l_1,l_2\in\Lambda$ such that*

$$\sup_{x\in\mathbb{R}^n}|f(x)|\cdot\exp(\omega_{M^{l_1}}(h_1\cdot|x|))\le C\qquad\sup_{\xi\in\mathbb{R}^n}\left|\hat{f}(\xi)\right|\cdot\exp(\omega_{N^{l_2}}(h_2\cdot|\xi|))\le C.$$

Important remarks: Let both matrices $\mathcal{M}:=\{M^l:l>0\}$ and $\mathcal{N}:=\{N^l:l>0\}$ are coming from weight functions $\omega,\sigma\in\mathcal{W}$ with (ω_1).

(1) To guarantee the assumption $\liminf_{k\to\infty}(m^l_k\cdot n^l_k)^{1/k}>0$ for some $l>0$ in Theorem 9.7.1 resp. Corollary 9.7.3 we can assume for both weights property (ω_2) (see 5.3.2). All further properties for $\mathcal{M}$ resp. $\mathcal{N}$ are satisfied automatically, see 5.1.1 and (5.1.2) in 5.1.2.

(2) But we point out that we cannot replace in (*ii*) in Corollary 9.7.3 the occurring associated functions ω_{M^l} resp. ω_{N^l} by ω resp. σ itself. On the one hand $\omega_{M^l}\sim\omega$ and $\omega_{N^l}\sim\sigma$ for all $l>0$ holds by 5.1.3, but nevertheless we would have to absorb a (large) constant $D>0$ in an expression $D\cdot\omega(\cdot)$ occurring in the exponential function. Hence we would have to apply an iterated application of (ω_6), more precisely see (3.4.2), and so we would need also (ω_6) for ω (resp. σ).

(3) This should be also compared with the results in [31] (where also the Beurling-case was treated), more precisely with the characterizing result [31, Theorem 3.1.]. There it was shown that Gelfand-Shilov classes defined by weight functions coincide with classes defined by (single) weight sequences if and only if the weight functions are "admissible functions" in the sense of [31, Definition 3.1.]. Condition (*b*) there is precisely property (ω_6) in our notation, hence this situation is consistent with 5.2.2. Nevertheless this condition is not needed in our previous results since in our proofs we don't need necessarily that $M^{l_1}\approx M^{l_2}$ resp. $N^{l_1}\approx N^{l_2}$ holds for all $l_1,l_2>0$.

9.8 Constructing a new matrix by pointwise multiplication of elements of $\mathcal{M}$

Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and we can define the (two-parameter) matrix $\mathcal{L} := \{L^{n_1,n_2} : n_1, n_2 \in \Lambda\}$ by

$$L_k^{n_1,n_2} := M_k^{n_1} \cdot M_k^{n_2} \qquad n_1, n_2 \in \Lambda.$$

More precisely we can construct new matrices (with more indices) by pointwise multiplication of all sequences which belong to $\mathcal{M}$. We write now L^n instead of $L^{n,n}$ and by using this notation we obtain the following result:

Lemma 9.8.1. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a weight matrix, then we obtain:*

(1) *If $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then $\mathcal{E}_{\{M^{n_i}\}} \subseteq \mathcal{E}_{(L^{n_1,n_2})}$ holds for $i = 1, 2$ and the obtained matrix $\mathcal{L}$ is again a $(\mathcal{M}_{\text{sc}})$ weight matrix. Moreover one gets the following representation as vector spaces*

$$\mathcal{E}_{\{\mathcal{M}\}} := \bigcup_{l \in \Lambda} \mathcal{E}_{\{M^l\}} \subseteq \bigcup_{n_1,n_2 \in \Lambda} \mathcal{E}_{(L^{n_1,n_2})} \subsetneq \bigcup_{n_1,n_2 \in \Lambda} \mathcal{E}_{\{L^{n_1,n_2}\}} = \bigcup_{n \in \Lambda} \mathcal{E}_{\{L^n\}} =: \mathcal{E}_{\{\mathcal{L}\}}.$$

If $\mathcal{M}$ is in addition not-quasi-analytic of Roumieu-type, i.e. $(\mathcal{M}_{\{\text{nq}\}})$, then $\mathcal{L}$ has this property, too.

(2) *If $\mathcal{M}$ is an arbitrary matrix, i.e. $(\mathcal{M})$, then each property $(\mathcal{M}_{\{\text{mg}\}})$-$(\mathcal{M}_{\{\text{strict}\}})$ and $(\mathcal{M}_{\circ})$ can be transferred to $\mathcal{L}$ (resp. for the Beurling-type-conditions).*

(3) *If $\mathcal{M}$ is an arbitrary matrix, i.e. $(\mathcal{M})$, which satisfies $(\mathcal{M}_{\{\text{diag}\}})$, then also $\mathcal{L}$ has property $(\mathcal{M}_{\{\text{diag}\}})$ (resp. for $(\mathcal{M}_{(\text{diag})})$).*

(4) *If $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ matrix with property $(\mathcal{M}_{\{\text{diag}\}})$, then both matrices $\mathcal{M}$ and $\mathcal{L}$ have also $(\mathcal{M}_{\{\text{strict}\}})$ and finally one gets the representation (as vector spaces)*

$$\mathcal{E}_{\{\mathcal{M}\}} := \bigcup_{l \in \Lambda} \mathcal{E}_{\{M^l\}} = \bigcup_{n_1,n_2 \in \Lambda} \mathcal{E}_{(L^{n_1,n_2})} = \bigcup_{n_1,n_2 \in \Lambda} \mathcal{E}_{\{L^{n_1,n_2}\}} = \bigcup_{n \in \Lambda} \mathcal{E}_{\{L^n\}} =: \mathcal{E}_{\{\mathcal{L}\}}.$$

Proof. (1) First we see by definition that $\left(\frac{M_k^{n_1}}{L_k^{n_1,n_2}}\right)^{1/k} = \frac{1}{(M_k^{n_2})^{1/k}} \to 0$ for $k \to \infty$, so $M^{n_1} \lhd L^{n_1,n_2}$ for each $n_1, n_2 \in \Lambda$ and of course also $M^{n_2} \lhd L^{n_1,n_2}$, furthermore $L^{n_1,n_2} = L^{n_2,n_1}$. This implies $\mathcal{E}_{\{M^{n_i}\}} \subseteq \mathcal{E}_{(L^{n_1,n_2})}$ for $i = 1, 2$.
Each sequence L^{n_1,n_2} is clearly normalized and increasing, furthermore $(L_k^{n_1,n_2})^{1/k} = (M_k^{n_1})^{1/k} \cdot (M_k^{n_2})^{1/k} \to \infty$ for $k \to \infty$. We put $\lambda_k^{n_1,n_2} := \frac{L_k^{n_1,n_2}}{L_{k-1}^{n_1,n_2}} = \frac{M_k^{n_1}}{M_{k-1}^{n_1}} \cdot \frac{M_k^{n_2}}{M_{k-1}^{n_2}} = \mu_k^{n_1} \cdot \mu_k^{n_2}$ and see immediately: L^{n_1,n_2} is weakly log. convex and $\lambda_k^{n_1,n_2} \geq \mu_k^{n_i}$ for all $k \in \mathbb{N}$, for $i = 1, 2$.
For the last equality in the representation above we point out: For arbitrary weight matrices the inequalities $L^{n_1,n_2} \leq L^{n,n} \leq L^{n_3,n_4}$ hold by definition for $n := \max\{n_1, n_2\}$ and all $n_3, n_4 \in \Lambda$ with $n \leq n_3, n_4$. So we can restrict ourselves to the diagonal $L^{n,n}$.
If in addition $\mathcal{M}$ is not quasi-analytic, then by 7.4.1 there exists some $l_0 \in \Lambda$ such that M^{l_0} has (nq). Thus also the matrix $\mathcal{L}$ has $(\mathcal{M}_{\{\text{nq}\}})$.

(2) Properties $(\mathcal{M}_{\{\mathrm{mg}\}})$ and $(\mathcal{M}_{\{\mathrm{alg}\}})$ are satisfied for $\mathcal{L}$ if they hold for $\mathcal{M}$. For property $(\mathcal{M}_{\{\mathrm{mg}\}})$ we have:

$$L^{n_1,n_2}_{j+k} = M^{n_1}_{j+k} \cdot M^{n_2}_{j+k} \underbrace{\leq}_{(\mathcal{M}_{\{\mathrm{mg}\}})} C_1^{j+k} \cdot M^x_j \cdot M^v_k \cdot C_2^{j+k} \cdot M^y_j \cdot M^w_k$$
$$= (C_1 \cdot C_2)^{j+k} \cdot (M^x_j \cdot M^y_j) \cdot (M^v_k \cdot M^w_k) = C_3^{j+k} \cdot L^{x,y}_j \cdot L^{v,w}_k$$

and for $(\mathcal{M}_{\{\mathrm{alg}\}})$ we get

$$L^{x,y}_j \cdot L^{v,w}_k = (M^x_j \cdot M^y_j) \cdot (M^v_k \cdot M^w_k) \underbrace{\leq}_{(\mathcal{M}_{\{\mathrm{alg}\}})} C_1^{j+k} \cdot M^{n_1}_{j+k} \cdot C_2^{j+k} \cdot M^{n_2}_{j+k} = C_3^{j+k} \cdot L^{n_1,n_2}_{j+k}.$$

If $(\mathcal{M}_{\{\mathrm{L}\}})$ is satisfied for $\mathcal{M}$, then for $\mathcal{L}$, too:

$$C^k \cdot L^{n_1,n_2}_k = C^k \cdot M^{n_1}_k \cdot M^{n_2}_k \underbrace{\leq}_{(\mathcal{M}_{\{\mathrm{L}\}})} D \cdot M^{n_3}_k \cdot M^{n_2}_k = D \cdot L^{n_3,n_2}_k.$$

For property $(\mathcal{M}_{\{\mathrm{strict}\}})$ we point out: Let $n_3, n_4 \in \Lambda$ be arbitrary given, then there exist $n_1, n_2 \in \Lambda$ with $\sup_{k\geq 1}\left(\frac{L^{n_1,n_2}_k}{L^{n_3,n_4}_k}\right)^{1/k} = \sup_{k\geq 1}\left(\frac{M^{n_1}_k \cdot M^{n_2}_k}{M^{n_3}_k \cdot M^{n_4}_k}\right)^{1/k} = +\infty$. This holds, since by $(\mathcal{M}_{\{\mathrm{strict}\}})$ for $\mathcal{M}$ for all $n_3, n_4 \in \Lambda$ there exist $n_1, n_2 \in \Lambda$ with $\sup_{k\geq 1}\left(\frac{M^{n_1}_k}{M^{n_3}_k}\right)^{1/k} = \sup_{k\geq 1}\left(\frac{M^{n_2}_k}{M^{n_4}_k}\right)^{1/k} = +\infty$.

If property $(\mathcal{M}_{\circ})$ is satisfied for $\mathcal{M}$, then also for $\mathcal{L}$, because put $l^{n_1,n_2}_k := \frac{L^{n_1,n_2}_k}{k!} = \frac{M^{n_1}_k \cdot M^{n_2}_k}{k!} = m^{n_1}_k \cdot M^{n_2}_k = M^{n_1}_k \cdot m^{n_2}_k$ to obtain a sequence $(l^{n_1,n_2}_k)_k$ and for this we get for each $n_1, n_2 \in \Lambda$ and all $j, k \in \mathbb{N}$ the following estimate:

$$l^{n_1,n_2}_k \cdot l^{n_1,n_2}_j = m^{n_1}_k \cdot m^{n_1}_j \cdot M^{n_2}_k \cdot M^{n_2}_j \leq D \cdot m^{n_1}_{j+k} \cdot D \cdot M^{n_2}_{j+k} = D^2 \cdot l^{n_1,n_2}_{j+k}.$$

Similarly all these calculations also hold for the Beurling-type-conditions.
(3) For $n_1, n_2 \in \Lambda$ arbitrary and all $j \in \mathbb{N}$ we have $L^{n_1,n_2}_{2j} = M^{n_1}_{2j} \cdot M^{n_2}_{2j} \leq C_1^j \cdot C_2^j \cdot M^{n_3}_j \cdot M^{n_4}_j = (C_1 \cdot C_2)^j \cdot L^{n_3,n_4}_j$ by property $(\mathcal{M}_{\{\mathrm{diag}\}})$ for $\mathcal{M}$, so $(\mathcal{M}_{\{\mathrm{diag}\}})$ still holds for $\mathcal{L}$ (and similarly for $(\mathcal{M}_{(\mathrm{diag})})$).

(4) By assumption on $\mathcal{M}$ property $(\mathcal{M}_{\{\mathrm{alg}\}})$ for this matrix holds and so we obtain $L^{n_1,n_2}_k = M^{n_1}_k \cdot M^{n_2}_k \leq C^{2k} \cdot M^{n_3}_{2k}$ for all $k \in \mathbb{N}$ and a number $n_3 \in \Lambda$. By $(\mathcal{M}_{\{\mathrm{diag}\}})$ we get $M^{n_3}_{2k} \leq D^k \cdot M^{n_4}_k$ hence by combining both conditions we have now $L^{n_1,n_2}_k \leq (C^2 \cdot D)^k \cdot M^{n_4}_k$ for all $k \in \mathbb{N}$. This implies $\mathcal{E}_{(L^{n_1,n_2})} \subseteq \mathcal{E}_{\{L^{n_1,n_2}\}} \subseteq \mathcal{E}_{\{M^{n_4}\}}$. Furthermore we have $\mathcal{E}_{\{M^{n_i}\}} \subseteq \mathcal{E}_{(L^{n_1,n_2})}$ for $i = 1, 2$ as seen in (1).
This calculation also shows that both $(\mathcal{M}_{\{\mathrm{alg}\}})$ and $(\mathcal{M}_{\{\mathrm{diag}\}})$ together imply now $(\mathcal{M}_{\{\mathrm{strict}\}})$: Note that in this case $L^{n_1,n_2} \preceq M^{n_4}$ holds and by construction $\left(\frac{M^{n_4}_k}{M^{n_i}_k}\right)^{1/k} \geq \frac{1}{D} \cdot \left(\frac{L^{n_1,n_2}_k}{M^{n_i}_k}\right)^{1/k} \to \infty$ for $k \to \infty$ for a constant $D > 0$ and $i = 1, 2$. Thus $\mathcal{E}_{\{M^{n_i}\}} \neq \mathcal{E}_{\{M^{n_4}\}}$ for $i = 1, 2$ and so condition $(\mathcal{M}_{\{\mathrm{strict}\}})$ for $\mathcal{M}$ follows.

By (2) we have also $(\mathcal{M}_{\{\text{strict}\}})$ for $\mathcal{L}$ and so we obtain the representation as vector spaces:

$$\mathcal{E}_{\{\mathcal{M}\}} = \bigcup_{l\in\Lambda} \mathcal{E}_{\{M^l\}} = \bigcup_{n_1,n_2\in\Lambda} \mathcal{E}_{(L^{n_1,n_2})} = \bigcup_{n_1,n_2\in\Lambda} \mathcal{E}_{\{L^{n_1,n_2}\}} = \mathcal{E}_{\{\mathcal{L}\}}.$$

□

Some remarks if the matrix $\mathcal{M} = \{M^l : l > 0\}$ is obtained by a weight function $\omega \in \mathcal{W}$ via $M^l_j := \exp\left(\frac{1}{l} \cdot \varphi^*_\omega(lj)\right)$:

(*i*) Both conditions $(\mathcal{M}_{\{\text{alg}\}})$ and $(\mathcal{M}_{\{\text{diag}\}})$ together imply $(\mathcal{M}_{\{\text{strict}\}})$ and so they are in fact an obstruction to property (ω_6), see 5.2.2 and 5.2.3. Hence the comparison theorem 6.2.1 cannot be valid in this case.

(*ii*) Property $(\mathcal{M}_{\{\text{diag}\}})$ is satisfied for $\mathcal{M}$ if ω satisfies in addition (ω_7), see 3.6.1 and 5.4.1.

(*iii*) For the weight function $\omega_s = \max\{0, \log(t)^s\}$, $s > 1$, we get $\varphi^*_{\omega_s}(x) = x^{s/(s-1)}$ (see the calculation in 3.10.1) and so condition $(\mathcal{M}_{\{\text{diag}\}})$ holds: After applying log we obtain $n_1^{1/(s-1)} \cdot 2^{s/(s-1)} \cdot j^{s/(s-1)} \le j \cdot \log(C) + n_2^{1/(s-1)} \cdot j^{s/(s-1)}$, which holds for some $n_2 \ge 2^s \cdot n_1$.

As special weight matrices $\mathcal{M} = \{M^l : l \in \Lambda\}$ one can think about conditions

$$\forall\, n_1, n_2 \in \Lambda\ \exists\, n_3 \in \Lambda\ \exists\, C \ge 1\ \forall\, j \in \mathbb{N} : M^{n_1}_j \cdot M^{n_2}_j = C^j \cdot M^{n_3}_j$$

resp.

$$\forall\, n_3 \in \Lambda\ \exists\, n_1, n_2 \in \Lambda\ \exists\, C \ge 1\ \forall\, j \in \mathbb{N} : M^{n_1}_j \cdot M^{n_2}_j = C^j \cdot M^{n_3}_j.$$

Recall: For the associated weight matrix obtained by the above weights $\omega_s = \max\{0, \log(t)^s\}$, $s > 1$, we get

$$M^l_j := \exp\left(\frac{1}{l} \cdot \varphi^*_{\omega_s}(l \cdot j)\right) = \exp\left(l^{\frac{1}{s-1}} \cdot j^{\frac{s}{s-1}} \cdot R(s)\right),$$

see (3.10.1) and the calculations in 3.10.1. But this shows that both introduced conditions are satisfied since we obtain the equation $n_3 = \left(n_1^{1/(s-1)} + n_2^{1/(s-1)}\right)^{s-1}$.

10 Convenient setting for the non-quasi-analytic $\mathcal{E}_{\{\mathcal{M}\}}$-case by using curves

The goal of this chapter is to transfer the proofs and definitions in [21] from the (single) weight sequence case $\mathcal{M} = \{M\}$ to the more general non-constant weight matrix case $\mathcal{M} = \{M^l : l \in \Lambda\}$. A special case are of course the matrices obtained by weight functions $\omega \in \mathcal{W}$ with (ω_1) and in the first step we want to apply the above comparison results 6.2.1 resp. 6.3.2 to transfer the proofs of [21] and more general of [22] directly from the weight sequence case M to the weight function case ω.

10.1 Using the comparison theorems for special weight matrices $\mathcal{M}$ obtained by a weight function ω

For the *non-quasi-analytic Roumieu case* $\mathcal{E}_{\{M\}}$ in [21] for the considered weight sequence the following conditions were assumed: W.l.o.g. $m_0 \le m_1 \Leftrightarrow M_0 \le M_1$, strong log. convexity (slc) (for closedness under composition, solving ODE's) and derivation closedness (dc). Furthermore it was assumed that the real analytic functions are (strict) contained in the class $\mathcal{E}_{\{M\}}$, which in this situation is equivalent to $\lim_{p\to\infty}(m_p)^{1/p} = \infty \Leftrightarrow \lim_{p\to\infty}\left(\frac{M_p}{p!}\right)^{1/p} = \infty$. Such a sequence M was called a *"DC-weight-sequence"*, see definition [21, p. 3], which is then automatically increasing. Furthermore throughout this paper it was assumed that $(M_p)_p$ should be non-quasi-analytic, i.e. (nq)! For the proof of the Cartesian closedness theorem [21, 5.3. Theorem], property moderate growth (mg) played the key role.

So assume now that a weight sequence M is given with

(*i*) $1 = M_0 \le M_1 \Leftrightarrow 1 = m_0 \le m_1$ (w.l.o.g.),

(*ii*) strong log. convexity, i.e. (slc),

(*iii*) $\lim_{p\to\infty}(m_p)^{1/p} = \lim_{p\to\infty}\left(\frac{M_p}{p!}\right)^{1/p} = \infty$,

(*iv*) strong regularity $:\Longleftrightarrow$ (snq) and (mg).

The last condition (*iv*) guarantees that (mg) and (nq) hold, the second implies (lc). In [33, 1.1 Proposition] it was shown, that for weight sequences M, which satisfy (snq) and weakly log. convexity (lc), the strong non-quasi-analyticity-condition (snq), which is called (γ_1) there, is equivalent to (β_1). So we get (β_1) and this condition is much stronger than (β_3).

First, on the one hand such a sequence M is a "DC-weight-sequence" in the sense of definition in [21, p. 3] with (nq) and (mg).

On the other hand the third condition (iii) together with *Stirling's formula*, i.e. $(p!)^{1/p} \sim \frac{p}{\exp(1)} \cdot (2\pi p)^{1/(2p)}$, imply that M has also property (6.2.1). Thus all assumptions for 6.3.2 hold, so $\omega_M \in \mathcal{W}$ with (ω_1), (ω_6) and finally (ω_{nq}). By the second part of 6.3.2 we have $\mathcal{E}_{[\omega_M]}(U) = \mathcal{E}_{[M]}(U)$ for all $U \subseteq \mathbb{R}^n$ non-empty open.
Conclusion: For weight sequences M satisfying $(i)-(iv)$ the comparison theorem 6.3.2 is valid, there is no difference between the weight function and the weight sequence case and we can transfer the proofs of [21] directly to the weight function (with weight ω_M) case!

On the other side we can start with a weight function $\omega \in \mathcal{W}$ which satisfies (ω_1), (ω_2), (ω_6) and furthermore properties (ω_{nq}) and $(\omega_{1'})$. Hence then the first comparison theorem 6.2.1 is valid: By the first point there we get $\mathcal{E}_{[M^l]}(U) = \mathcal{E}_{[\omega]}(U)$ for each $l > 0$ and U non-empty open, moreover the second part implies $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 > 0$. The space $\mathcal{E}_{\{\omega\}}$ is closed under composition by property $(\omega_{1'})$, see [10, §2,§3] or more general 8.6.1 resp. 8.7.4.
Moreover for each $l > 0$ the sequence M^l is normalized, weakly log-convex, $\lim_{p\to\infty}(M^l_p)^{1/p} = +\infty$ and (nq) holds (see 5.1.1 and 5.1.3).
Finally, by (ω_2), the space $\mathcal{E}_{\{\omega\}}$ and so $\mathcal{E}_{\{M^l\}}$ contains the real-analytic functions. In this case we have by the third part of 6.2.1 that each M^l has (6.2.1), furthermore conditions (β_3) and moderate growth (mg) (so also (dc)) are valid.
If the real analytic functions are really smaller than $\mathcal{E}_{\{\omega\}} = \mathcal{E}_{\{M^l\}}$, then $\lim_{p\to\infty}(m^l_p)^{1/p} = +\infty$ holds for each $l > 0$. This yields property (10.6.1), hence the $\mathcal{E}_{\{M^l\}}$-special-curve-lemma holds (see 10.6.2 and also 10.6.1 below).
Attention: A sequence M^l is not necessarily a DC-weight-sequence in the sense of [21], since we don't have strong log. convexity for M^l. But nevertheless each $\mathcal{E}_{\{M^l\}}$ satisfies (the Roumieu-version of) Theorem 8.6.1, because $\mathcal{E}_{[M^l]} = \mathcal{E}_{[\omega]}$ holds and more precisely we obtain: By Theorem 8.7.4 (see also the remarks below this result) we get $(\mathcal{M}_{\{\text{FdB}\}})$ and $(\mathcal{M}_{(\text{FdB})})$ for the associated matrix $\mathcal{M}$. Since $m^{l_1} \approx m^{l_2}$ for all $l_1, l_2 > 0$ we see that (FdB) holds for each sequence m^l separately, see also 8.2.2. The further properties in 8.6.1 are satisfied by the assumptions on ω, see also the remark below 8.7.4.

So also in this case we can use the proofs of [21] directly for some sequence M^l, in the matrix obtained by ω and transfer them to $\mathcal{E}_{\{\omega\}}$.
For the more general proofs of [22], where also quasi-analytic weight sequences and the Beurling-case were considered, we can skip the non-quasi-analyticity conditions (nq) for M resp. (ω_{nq}) for ω.

10.2 A brief introduction into calculus of smooth mappings beyond Banach spaces

For more details, notation and the complete proofs we refer to the first chapter in [20], moreover one can find a brief summary also in [21, 7. Appendix].
Let E be a locally convex vector space, then a curve $c : \mathbb{R} \to E$ is called smooth (always denoted by $\mathcal{E}$), if all derivatives exist and are continuous (see [20, 1.2. Definition]). The space of all such curves will be denoted by $\mathcal{E}(\mathbb{R}, E)$ and one can show that this space does not depend on the locally convex topology on E, only on its associated bornology (the system of bounded sets in E), see [20, 1.8. Corollary].
Moreover we define:

(*i*) The c^∞-topology on E is the final topology w.r.t. all smooth curves $c : \mathbb{R} \to E$ and it's open sets will be denoted by c^∞-open (see [20, 2.12. Definition]). This topology is studied in detail in the fourth section in the first chapter in [20].

(*ii*) E is called convenient, if E is c^∞-complete. In [20, 2.14. Theorem] this notation is characterized by seven different equivalent conditions, e.g. E is convenient if and only if

E is Mackey-complete (c^∞-completeness is a bornological concept), resp.

$c : \mathbb{R} \to E$ is smooth if and only if $\alpha \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\alpha \in E^*$, resp.

the space E_B is Banach w.r.t. the Minkowski-functional $\|\cdot\|_B$ for each closed bounded absolutely convex set $B \subseteq E$.

(*iii*) In [20, 3.7. Lemma] it is shown that E is convenient, if and only if $\mathcal{E}(\mathbb{R}, E)$ is convenient.

Let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. A mapping $f : E \supseteq U \to F$ (defined on U) is called smooth, if it maps smooth curves in U to smooth curves in F. The space of all such smooth mappings is denoted by $\mathcal{E}(U, F)$ with the initial topology w.r.t. all mappings $c^* : \mathcal{E}(U, F) \to \mathcal{E}(\mathbb{R}, F)$ for each $c \in \mathcal{E}(\mathbb{R}, U)$ (see [20, 3.11. Definition]).
A central result of this calculus is called *Cartesian closedness*, see [20, 3.12. Theorem]:

Theorem 10.2.1. *Let E, F and G be convenient vector spaces, $U \subseteq E$ and $V \subseteq F$ be c^∞-open.*
Then a mapping $f : U \times V \to G$ is $\mathcal{E}$, if and only if it's canonically associated mapping $f^\vee : U \to \mathcal{E}(V, G)$ exists and is $\mathcal{E}$.

In [20, 3.14. Corollary] it is pointed out that this concept of smooth mappings is a generalization of the usual definition of smooth mappings on open sets in $\mathbb{R}^n$.

Another central result is the *uniform-boundedness-principle* - UBP - see [20, 5.18. Theorem]:

Theorem 10.2.2. *Let E_i, $1 \le i \le n$, be convenient vector spaces and F be a locally convex vector space. Then the bornology on the space $L(E_1, \dots, E_n; F)$ consists of all pointwise bounded sets.*
Alternatively we can say: A mapping into the space $L(E_1, \dots, E_n; F)$ is smooth if and only if all composites with evaluations at points in $E_1 \times \dots \times E_n$ are smooth.

10.3 Basic definitions

Let $\mathcal{M} := \{M^l \in \mathbb{R}_{>0}^{\mathbb{N}} : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$. The following definitions are of course inspired by [21, 3.1. Definition]. A smooth curve $c : \mathbb{R} \to E$, where E is a convenient vector space, is called (*weakly*) $\mathcal{E}_{\{\mathcal{M}\}}$, if for each continuous linear functional $\alpha \in E^*$ the curve $\alpha \circ c : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{E}_{\{\mathcal{M}\}}$, which means that for each $\alpha \in E^*$ and each compact set there exists $l \in \Lambda$ and $h > 0$ such that $\left\{\frac{(\alpha \circ c)^{(k)}(x)}{h^k \cdot M_k^l} : k \in \mathbb{N}, x \in K\right\}$ is a bounded set in $\mathbb{R}$. Attention: The index $l \in \Lambda$ is also depending on the linear functional α, in [21, 3.1. Definition] such an index of course doesn't occur!

The curve $c : \mathbb{R} \to E$ is called *strongly* $\mathcal{E}_{\{\mathcal{M}\}}$ if c is smooth and for all $K \subseteq \mathbb{R}$ compact there exist $l \in \Lambda$ and $h > 0$ such that $\left\{\frac{c^{(k)}(x)}{h^k \cdot M_k^l} : \ k \in \mathbb{N}, x \in K\right\}$ is a bounded set in E. If c is smooth and there exist $l \in \Lambda$ and $h > 0$ such that $\left\{\frac{c^{(k)}(x)}{h^k \cdot M_k^l} : \ k \in \mathbb{N}, x \in \mathbb{R}\right\}$ is a bounded set in E, then we call c *strongly uniform* $\mathcal{E}_{\{\mathcal{M}\}}$.

Remarks:

(*i*) If $\Lambda = \mathbb{N}_{>0}$ we can take in both above definitions $l = h \in \mathbb{N}_{>0}$.

(*ii*) One could also consider the definitions where $\mathcal{M}$ is an arbitrary set of weight sequences and where we have to replace above "$\exists\, l \in \Lambda$" by "$\exists\, M \in \mathcal{M}$".

(*iii*) Let $\omega \in \mathcal{W}$ be given with (ω_1) and furthermore property (ω_{nq}). If we would define that a curve $c : \mathbb{R} \to E$ with compact support is $\mathcal{E}_{\{\omega\}}$ if and only if $\alpha \circ c$ is $\mathcal{E}_{\{\omega\}}$ in the sense of [5, 3.1. Definition] (by using the decay property of it's Fourier transform) for each $\alpha \in E^*$, then we see as in the third chapter in [5] that this means that $\alpha \circ c$ satisfies the global estimates using the sequences M^l from chapter 5. By using the new comparison theorems (see 6.2.6) and the introduced notations we see that this means that c is already weakly $\mathcal{E}_{\{\omega\}}$ as defined above.

Let U be a c^∞-open subset of a convenient vector space E and let F be another convenient vector space. A mapping $f : U \to F$ is then called $\mathcal{E}_{\{\mathcal{M}\}}$, if f is smooth in "the convenient sense", i.e. f maps smooth curves in U to smooth curves in F (see [20, 3.11. Definition]), and if $f \circ c$ is a $\mathcal{E}_{\{\mathcal{M}\}}$-curve in F for every $\mathcal{E}_{\{\mathcal{M}\}}$-curve c in U. We denote this structure by $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$, so

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F) := \left\{f \in \mathcal{E}(U,F) : \ \forall\, \alpha \in F^* \ \forall\, c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U) : \ \alpha \circ f \circ c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})\right\}. \tag{10.3.1}$$

An immediate consequence of the definitions is: The composition of $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$-mappings is again a $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$-mapping. The space $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ is equipped with the initial locally convex structure w.r.t. the mappings

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F) \overset{\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(c,\alpha)}{\longrightarrow} \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}), \tag{10.3.2}$$

$f \mapsto \alpha \circ f \circ c$, $\alpha \in F^*$ and $c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U)$. The space $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, for the definition see (7.3.1), carries the "usual" locally convex topology defined in (7.3.5). We show now (for the constant matrix case $\mathcal{M} = \{M\}$ already mentioned in [21]):

Lemma 10.3.1. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, let E, F be convenient vector spaces and $U \subseteq E$ be c^∞-open, then the space $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ is convenient, too.*

Proof. First we point out that the space $\prod_{c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U), \alpha \in F^*} \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is convenient. This holds because by (2)(a) in 9.1.1 for $K \subseteq \mathbb{R}$ compact the space $\mathcal{E}_{\{\mathcal{M}\}}(K,\mathbb{R})$ is a *Silva-space*, hence complete (see [20, 52.37.]) and so also c^∞-complete, i.e. convenient. $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is the (countable) projective limit of c^∞-complete spaces, hence by [20, 2.15. Theorem]) it is c^∞-complete, too. And this implies finally also the c^∞-completeness of $\prod_{c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U), \alpha \in F^*} \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$.

To prove the lemma we show now: $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ is c^∞-closed in the product

$\prod_{c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U),\alpha\in F^*}\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(\mathbb{R},\mathbb{R})$.
For this we consider in the following the canonical embedding
$\iota : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F) \to \prod_{c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U),\alpha\in F^*}\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ and the projection mappings $\text{pr}_{c,\alpha} : \prod_{c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U),\alpha\in F^*}\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, hence $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(c,\alpha) = \text{pr}_{c,\alpha}\circ\iota$.
Let $(f_n)_n$ be a sequence in $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ and assume that $\iota(f_n) \to \tilde{g} = (\tilde{g}_{c,\alpha})_{c,\alpha}$ is *Mackey-convergent,* where $\tilde{g} \in \prod_{c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U),\alpha\in F^*}\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ and $\iota(f_n) = (\alpha\circ f_n\circ c)_{c,\alpha}$. This means that there exists a sequence $(\mu_n)_n$ with $\mu_n \to \infty$ for $n\to\infty$ and $\{\mu_n\cdot(\iota(f_n)-\tilde{g}) : n\in\mathbb{N}\}$ is bounded in $\prod_{c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U),\alpha\in F^*}\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, or $\{\mu_n\cdot(\alpha\circ f_n\circ c-\tilde{g}_{c,\alpha}) : n\in\mathbb{N}\}$ is bounded in $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$.
We have to show that $\tilde{g}\in\iota\left(\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)\right)$ holds and first we put
$\alpha(g(x)) := g_{\text{const}_x,\alpha}(0) := \lim_{n\to\infty}(\alpha\circ f_n\circ c)(0)$ for all $\alpha\in F^*$ and $x\in U$, where $c = \text{const}_x$. Hence the mapping $g(x) : F^*\to\mathbb{R}$, $\alpha\mapsto\alpha(g(x))$, is linear, so $g(x)\in\prod_{F^*}\mathbb{R}$ and more precisely $g(x)$ is an element of the bi-dual of F for each $x\in U$. The canonical embedding $\delta : F\hookrightarrow\prod_{F^*}\mathbb{R}$ is bornological and, because F is assumed to be convenient, also $\delta(F)$ is c^∞-closed. Since $(f_n\circ c)(0)\in F$ holds for each $n\in\mathbb{N}$, Mackey-convergence implies pointwise convergence and $\delta(F)$ is Mackey-complete. Finally, by definition of the mapping g we finally obtain $g(x)\in F$, too.
Now prove $\tilde{g}\in\iota\left(\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)\right)$, we show $\iota(g) = \tilde{g}$. For this we take a curve $c\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U)$ and $\alpha\in F^*$ and we show that $g_{\text{const}_{c(t)},\alpha}(0) = \tilde{g}_{c,\alpha}(t)$ holds for each t. On the one hand we have $\tilde{g}_{c,\alpha}(t) := \lim_{n\to\infty}(\alpha\circ f_n\circ c)(t)$, whereas on the other hand $\alpha(g(c(t))) := g_{\text{const}_{c(t)},\alpha}(0) := \lim_{n\to\infty}(\alpha\circ f_n\circ\text{const}_{c(t)})(0) = \lim_{n\to\infty}(\alpha\circ f_n\circ c)(t)$ holds.

□

Analogously we could also define weakly and strongly $\mathcal{E}_{(\mathcal{M})}$-curves, replace in the Roumieu-case definition "there exists $l\in\Lambda$" by "for all $l\in\Lambda$ (resp. for all $M\in\mathcal{M}$ in the most general sense) and "there exists $h>0$" by "for all $h>0$". Then we could put

$$\mathcal{E}_{(\mathcal{M}),\text{curve}}(U,F) := \left\{f\in\mathcal{E}(U,F) :\ \forall\,\alpha\in F^*\ \forall\,c\in\mathcal{E}_{(\mathcal{M})}(\mathbb{R},U) :\ \alpha\circ f\circ c\in\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})\right\}. \tag{10.3.3}$$

So by definition the composition of $\mathcal{E}_{(\mathcal{M}),\text{curve}}$-mappings is again a $\mathcal{E}_{(\mathcal{M}),\text{curve}}$-mapping. The space $\mathcal{E}_{(\mathcal{M}),\text{curve}}(U,F)$ is equipped with the initial locally convex structure w.r.t. the mappings

$$\mathcal{E}_{(\mathcal{M}),\text{curve}}(U,F) \overset{\mathcal{E}_{(\mathcal{M}),\text{curve}}(c,\alpha)}{\longrightarrow} \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R}), \tag{10.3.4}$$

$f\mapsto\alpha\circ f\circ c$, $\alpha\in F^*$ and $c\in\mathcal{E}_{(\mathcal{M})}(\mathbb{R},U)$. The space $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$, for the definition see (7.3.2), carries the "usual" locally convex topology defined in (7.3.6) and $\mathcal{E}_{(\mathcal{M}),\text{curve}}(U,F)$ is also convenient by the analogous proof as in 10.3.1.

Important remarks:

(1) If $\mathcal{M}$ satisfies the assumptions of 10.6.2 and 8.3.1, then the structure $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ is a generalization of the classical definitions (on the level of Banach-spaces), for this see 10.7.1 below. More precisely, to guarantee this conditions, we have to assume that $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix with index set $\Lambda = \mathbb{N}_{>0}$ and additionally property $(\mathcal{M}_{\{\text{FdB}\}})$ (so closedness under composition 8.3.1 is valid) and the assumption (10.6.4) for the $\mathcal{E}_{\{\mathcal{M}\}}$-special-curve lemma 10.6.2 which has the consequence that *Roumieu-non-quasi-analyticity,* i.e. condition $(\mathcal{M}_{\{\text{nq}\}})$, is satisfied.

(2) We point out: Even if E is a Banach-space the topology on $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E)$ obtained by the classical definition doesn't coincide with the topology obtained

by the above definition. For this consider $T \hookrightarrow \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, E)$, where T denotes the sub-space of all constant functions (curves). Then the initial topology with respect to all continuous linear functionals $\alpha \in E^*$ restricted to T yields the weak topology, whereas the topology generated by the semi-norms $\|c\|_{\mathcal{M},K,l,h} := \sup_{x\in K, k\in\mathbb{N}} \frac{\|c^{(k)}(x)\|_E}{h^k \cdot M_k^l}$ yields the norm topology.

(3) In the Beurling-case $\mathcal{E}_{(\mathcal{M}),\mathrm{curve}}$ it's not clear if this concept/definition is a generalization of the classical definition, since we don't have 10.7.1 even for the case $\mathcal{M} = \{M\}$. So in nearly all results of this chapter we will concentrate on the Roumieu-case $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$ but in some results it's easy to transfer the proofs from the Roumieu- also to the Beurling-case.

10.4 Compare the different definitions of ultra-differentiable curves

First we obtain for the Beurling-case the following result:

Lemma 10.4.1. *Let $\mathcal{M}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, then in the Beurling-case $\mathcal{E}_{(\mathcal{M})}$ there is no difference between strongly and weakly $\mathcal{E}_{(\mathcal{M})}$-curves, both definitions coincide.*

Proof. A smooth curve $c : \mathbb{R} \to E$ into a locally convex vector space E is weakly $\mathcal{E}_{(\mathcal{M})}$, if and only if for all continuous linear functionals $\alpha \in E^*$, for all $K \subseteq \mathbb{R}$ compact, all $l \in \Lambda$ and each $h > 0$ the set $\left\{\frac{(\alpha\circ c)^{(k)}(x)}{h^k\cdot M_k^l} : x \in K, k \in \mathbb{N}\right\}$ is bounded in $\mathbb{R}$. This is equivalent that for all $K \subseteq \mathbb{R}$, all $l \in \Lambda$, all $h > 0$ and all $\alpha \in E^*$ the set $\alpha\left(\left\{\frac{c^{(k)}(x)}{h^k\cdot M_k^l} : x \in K, k \in \mathbb{N}\right\}\right)$ is bounded in $\mathbb{R}$ ($(\alpha \circ c)^{(k)} = \alpha \circ c^{(k)}$), and this is equivalent to: For all $K \subseteq \mathbb{R}$ and all $l \in \Lambda$, $h > 0$ the set $\left\{\frac{c^{(k)}(x)}{h^k\cdot M_k^l} : x \in K, k \in \mathbb{N}\right\}$ is bounded in E. The last expression is now exactly the condition for c to be a strongly $\mathcal{E}_{(\mathcal{M})}$-curve. □

Furthermore the proof of 10.4.1 shows that a curve $c : \mathbb{R} \to E$ is $\mathcal{E}_{(\mathcal{M})}$ if and only if c is $\mathcal{E}_{(M^l)}$ for each $l \in \Lambda$ in the sense of [21, 3.1.Definition] (with "for all $\varrho > 0$" instead of "there exists $\varrho > 0$"). Moreover it holds also for the case where $\mathcal{M}$ is an arbitrary set of sequences and we replace "for all $l \in \Lambda$" by "for all $M \in \mathcal{M}$".
The next result is analogous to [21, 3.3. Lemma]:

Lemma 10.4.2. *Let $\mathcal{M}$ be an arbitrary matrix, i.e. $(\mathcal{M})$. Let E be a convenient vector space such that there exists a Baire-vector-space-topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$.*
Then a curve $c : \mathbb{R} \to E$ is weakly $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if c is strongly $\mathcal{E}_{\{\mathcal{M}\}}$.

Proof. Let c be $\mathcal{E}_{\{\mathcal{M}\}}$ and K a compact set in $\mathbb{R}$, so consider for $h, C > 0$ and $l \in \Lambda$ the sets

$$A_{l,h,C} := \left\{\alpha \in E^* : \frac{|(\alpha \circ c)^{(k)}(x)|}{h^k \cdot M_k^l} \le C,\ \forall\, k \in \mathbb{N}, x \in K\right\}.$$

These sets are closed in E^* for the Baire-topology and $\bigcup_{l,h,C} A_{l,h,C} = E^*$ holds. Then, by the *Baire-property* of E^*, there exist $l_0 \in \Lambda$, $h_0, C_0 > 0$ such that the interior

$A^{\circ}_{l_0,h_0,C_0} =: U$ is non-empty! Let $\alpha_0 \in U$, then for each $\alpha \in E^*$ there exists $\varepsilon > 0$, such that we get $\varepsilon \cdot \alpha \in U - \alpha_0 \Leftrightarrow \varepsilon \cdot \alpha + \alpha_0 \in U$.
Thus for all $x \in K$ and $k \in \mathbb{N}$ we get

$$|(\alpha \circ c)^{(k)}(x)| \leq \frac{1}{\varepsilon} \cdot \left(|((\varepsilon \cdot \alpha) + \alpha_0) \circ c)^{(k)}(x)| + |(\alpha_0 \circ c)^{(k)}(x)| \right) \leq \frac{2 \cdot C_0}{\varepsilon} \cdot h_0^k \cdot M_k^{l_0}.$$

This implies: The set $\left\{ \frac{c^{(k)}(x)}{h_0^k \cdot M_k^{l_0}} : k \in \mathbb{N}, x \in K \right\}$ is weakly bounded (in E), hence bounded.

□

Compare all definitions - important remarks

In the following let $\mathcal{M} = \{M^l : l \in \Lambda\}$ always be an arbitrary weight matrix, i.e. $(\mathcal{M})$.

First we see by definition that a curve $c : \mathbb{R} \to E$ is strongly uniform $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if there exists an index $l \in \Lambda$ such that c is strongly uniform $\mathcal{E}_{\{M^l\}}$ in the sense of [21, 3.1. Definition]. - The reason for this is that no $\alpha \in F^*$ and no compact set $K \subseteq \mathbb{R}$ is involved. But for weakly and strongly curves one has to be careful, since linear functionals α and compact sets K occur and they are depending on the index $l \in \Lambda$. We summarize the situation:

(1) Ad weakly curves:

 (a) By definition $c : \mathbb{R} \to E$ is (weakly) $\mathcal{E}_{\{\mathcal{M}\}}$ means that for each $\alpha \in E^*$ we have $\alpha \circ c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{R})$, more precisely: For each $\alpha \in E^*$ and each compact set $K \subseteq \mathbb{R}$ we can find an index $l \in \Lambda$ and a number $h > 0$ such that $\left\{ \frac{(\alpha \circ c)^{(k)}(x)}{h^k \cdot M_k^l} : k \in \mathbb{N}, x \in K \right\}$ is bounded (in $\mathbb{R}$).

 (b) If c is (weakly) $\mathcal{E}_{\{M^l\}}$ for some $l \in \Lambda$ in the sense of [21, 3.1. Definition] then for this l and all $\alpha \in E^*$ we obtain that $\alpha \circ c$ has to be in the class $\mathcal{E}_{\{M^l\}}(\mathbb{R}, \mathbb{R})$. More precisely: There exists an index $l \in \Lambda$ such that for each $\alpha \in E^*$ and each compact K in $\mathbb{R}$ there exists a number $h > 0$ such that $\left\{ \frac{(\alpha \circ c)^{(k)}(x)}{h^k \cdot M_k^l} : k \in \mathbb{N}, x \in K \right\}$ is bounded (in $\mathbb{R}$). Hence the quantifiers are in different order than in (a) above. In fact the index $l \in \Lambda$ depends in the $\mathcal{E}_{\{\mathcal{M}\}}$-definition now on the chosen functional α and compact set K.

So obviously $(b) \Rightarrow (a)$ holds. If we have for the matrix $M^{l_1} \approx M^{l_2}$ for each $l_1, l_2 \in \Lambda$ (constant matrix case), then also $(a) \Rightarrow (b)$ holds since in this situation "$\exists\, l \in \Lambda\ \exists\, h > 0$" is equivalent to "$\forall\, l \in \Lambda\ \exists\, h > 0$" and then we can interchange the all-quantifiers in (a).

(2) Ad strongly curves:

 (a) A curve $c : \mathbb{R} \to E$ is strongly $\mathcal{E}_{\{\mathcal{M}\}}$ if for each compact set K there exists an index $l \in \Lambda$ and a number $h > 0$ such that the set $\left\{ \frac{c^{(k)}(x)}{h^k \cdot M_k^l} : k \in \mathbb{N}, x \in K \right\}$ is bounded (in E), whereas

 (b) the curve c is strongly $\mathcal{E}_{\{M^l\}}$ for some $l \in \Lambda$ in the sense of [21, 3.1. Definition], if for each compact set K there exists a number $h > 0$ such that the set $\left\{ \frac{c^{(k)}(x)}{h^k \cdot M_k^l} : k \in \mathbb{N}, x \in K \right\}$ is bounded (in E).

So again we have changed quantifiers because in (a) the index l depends on the compact set K and $(b) \Rightarrow (a)$ holds always. The converse direction is true if $M^{l_1} \approx M^{l_2}$ for each $l_1, l_2 \in \Lambda$ as above. Moreover we obtain here $(b) \Rightarrow (a)$ if the curve c is defined on one fixed compact set $K \subseteq \mathbb{R}$, because in this case in both definitions "for each K compact" does not occur, hence the definitions coincide (localized version). In fact this holds if c is a curve with compact support (possible only in non-quasi-analytic classes!).

We summarize: The introduced structures in (a) and (b) in (1) resp. (2) are generalizations of the original definition of strongly resp. weakly curves in [21, 3.1. Definition] in the case for a constant matrix $\mathcal{M} = \{M\}$ (or more general if all occurring sequences in the matrix are equivalent, i.e. $M^{l_1} \approx M^{l_2}$ for all $l_1, l_2 \in \Lambda$). Another possibility, which is a generalization too, would be the following: In the case $\Lambda = \mathbb{N}_{>0}$ we can say that a curve $c : \mathbb{R} \to E$ is

called (weakly) $\widetilde{\mathcal{E}}_{\{\mathcal{M}\}}$ if for each $\alpha \in E^*$ and $K \subseteq \mathbb{R}$ compact there exists a number $h > 0$ such that the set $\left\{\frac{(\alpha \circ c)^{(k)}(x)}{h^k \cdot M_k^k} : k \in \mathbb{N}, x \in K\right\}$ is bounded in E,

and a similar definition for strongly $\widetilde{\mathcal{E}}_{\{\mathcal{M}\}}$-curves. Of course we could replace the term M_k^k above also by $M_k^{r_k}$, where $(r_k)_k$ is a sequence of natural numbers with $r_k \to \infty$ for $k \to \infty$ ("concept of diagonals of a weight matrix").

(3) For a given matrix $\mathcal{M} := \{M^l : l \in \Lambda\}$ it could sometimes also be important to extend the argumentation to a strictly larger matrix $\mathcal{N} := \{N^l : l \in \Lambda\}$ with $\mathcal{M} \lhd \mathcal{N}$, which means $M^l \lhd N^n$ for each $l, n \in \Lambda$, because then we obtain: If a curve $c : \mathbb{R} \to E$ is weakly/strongly $\mathcal{E}_{\{\mathcal{M}\}}$, then it is of course weakly/strongly $\mathcal{E}_{\{\mathcal{N}\}}$. But more precisely the curve c is already weakly/strongly $\mathcal{E}_{\{N^n\}}$ for each $n \in \Lambda$ in the sense of [21, 3.1. Definition] - in fact for this we need "only" $M^l \preceq N^n$ for each $l \in \Lambda$ and each $n \in \Lambda$:

$\forall\, K\, \exists\, h > 0\, \exists\, l \in \Lambda \Rightarrow \forall\, K\, \exists\, h > 0\, \forall\, n \in \Lambda \Rightarrow \forall\, K\, \forall\, n \in \Lambda\, \exists\, h > 0 \Rightarrow \forall\, n \in \Lambda\, \forall\, K\, \exists\, h > 0$.

By the previous summary we see that it's important to study the situation where $\mathcal{M} = \{M^l : l \in \Lambda\}$ is not-constant, i.e. condition $(\mathcal{M}_{\{\text{strict}\}})$ for $\mathcal{M}$ is satisfied. We prove now the following important result:

Proposition 10.4.3. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with a countable index set $\Lambda = \mathbb{N}_{>0}$ and such that condition $(\mathcal{M}_{\{\text{strict}\}})$ is satisfied.*
Then there exist locally convex vector spaces E and curves $c : \mathbb{R} \to E$ which are weakly $\mathcal{E}_{\{\mathcal{M}\}}$ but for any index $\tilde{l} \in \Lambda$ the curve c is neither strongly nor weakly $\mathcal{E}_{\{M^{\tilde{l}}\}}$ (in the sense of [21, 3.1. Definition]).

Proof. By assumption and $(\mathcal{M}_{\{\text{strict}\}})$ we have that for each $l \in \Lambda$ we can find another index l_1, $l_1 > l$, such that $\mathcal{E}_{\{M^l\}} \subsetneq \mathcal{E}_{\{M^{l_1}\}}$. Of course for this l_1 we can find an index $l_2 > l_1$ with $\mathcal{E}_{\{M^{l_1}\}} \subsetneq \mathcal{E}_{\{M^{l_2}\}}$ and so on.
So let $l \in \Lambda$ be arbitrary but from now on fixed and set $E = \mathbb{R}^{\mathbb{N}}$. Consider a curve $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$, $c(t) = (c_i(t))_{i \in \mathbb{N}} = (c_0(t), c_1(t), \dots)$, with the following property: Put $l_0 = l$, the curve c_0 is strongly $\mathcal{E}_{\{M^l\}}$, and each curve c_i, where $i \geq 0$, is $\mathcal{E}_{\{M^{l_i}\}}$. But

c_{i+1}, where $i \geq 0$, is not $\mathcal{E}_{\{M^{l_i}\}}$ in the sense of [21, 3.1. Definition] and the "chain" of indices $(l_i)_{i\geq 0}$ is coming from property $(\mathcal{M}_{\{\text{strict}\}})$ as explained before.
We show now: The curve c is weakly $\mathcal{E}_{\{\mathcal{M}\}}$ but there doesn't exist any number $\tilde{l} \in \Lambda$ such that c is strongly $\mathcal{E}_{\{M^{\tilde{l}}\}}$.
Each continuous linear functional $\alpha \in (\mathbb{R}^{\mathbb{N}})^* = \mathbb{R}^{(\mathbb{N})}$ only depends on finitely many coordinates. So for each $\alpha \in \mathbb{R}^{(\mathbb{N})}$ there exists a maximal coordinate $i \in \mathbb{N}$: For this note that $M^l \leq M^{l_1} \leq M^{l_2} \leq \cdots \leq M^{l_i}$ and among these coordinates we can find a maximal $h > 0$ in the occurring quotients. Then for this index l_i and maximal $h > 0$ we have $\alpha \circ c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, hence c is weakly $\mathcal{E}_{\{\mathcal{M}\}}$.
Assume that there would exist a number $\tilde{l} \in \Lambda$ such that c is strongly $\mathcal{E}_{\{M^{\tilde{l}}\}}$. By assumption $\lim_{i\to\infty} l_i = +\infty$ (because $l_{i+1} > l_i$), hence for this index $\tilde{l} \in \Lambda$ we can find a number (coordinate) $i_0 \in \mathbb{N}$ with $l_i \geq l_{i_0} > \tilde{l}$ for all $i \geq i_0$. But then we obtain a contradiction: Since $\mathcal{E}_{\{M^{\tilde{l}}\}} \subseteq \mathcal{E}_{\{M^{l_{i_0}}\}}$ we have that all curves c_{i_0+k} are not $\mathcal{E}_{\{M^{l_{i_0}}\}}$, and so not strongly $\mathcal{E}_{\{M^{\tilde{l}}\}}$, for any $k \geq 1$.

If there would exist a number $\tilde{l} \in \Lambda$ such that c is weakly $\mathcal{E}_{\{M^{\tilde{l}}\}}$, then for each $\alpha \in \mathbb{R}^{(\mathbb{N})}$ we would get that $\alpha \circ c \in \mathcal{E}_{\{M^{\tilde{l}}\}}(\mathbb{R},\mathbb{R})$. But to the given $\tilde{l} \in \Lambda$ we choose a linear functional α depending on at least $i_0 + 1$ many coordinates where $l_{i_0} > \tilde{l}$ to obtain also a contradiction. □

Moreover 10.4.3 shows: If we assume that c is defined only on one fixed compact set K (e.g. if c has compact support $K \subseteq \mathbb{R}$), then in general we have weakly $\mathcal{E}_{\{\mathcal{M}\}}$-curves, which are not strongly. A similar result with curves defined on whole $\mathbb{R}$ and without using condition $(\mathcal{M}_{\{\text{strict}\}})$ is the following result, which is analogous to [21, 3.2. Example]:

Proposition 10.4.4. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with a countable index set $\Lambda = \mathbb{N}_{>0}$, then there are weakly $\mathcal{E}_{\{\mathcal{M}\}}$-curves, which are not strongly $\mathcal{E}_{\{\mathcal{M}\}}$.*

Proof. By assumption we see that for each $l \in \Lambda$ there exists a function $\theta_l \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ such that $|\theta_l^{(j)}(0)| \geq M_j^l$ for all $j \in \mathbb{N}$ and $l \in \Lambda$, see (chf). Now we follow [21, 3.2. Example]:
For each $l \in \Lambda$ the set $\left\{\frac{\theta_l^{(k)}(t)}{h_0^k \cdot M_k^l} : k \in \mathbb{N}, t \in K\right\}$ is unbounded for any compact set $K \subseteq \mathbb{R}$, which contains the point 0, and each $0 < h_0 < 1$ separately. This holds since $\frac{|\theta_l^{(k)}(0)|}{h_0^k \cdot M_k^l} \geq \frac{1}{h_0^k} \to \infty$ for $k \to \infty$. We consider now the curve

$$c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}\times\Lambda}, \quad c(t) = (c(t)_{n,l})_{(n,l)\in\mathbb{N}\times\Lambda}, \quad c(t)_{n,l} := \theta_l(n \cdot t) \text{ for } n \in \mathbb{N},\ l \in \Lambda,$$

and we show that this curve is weakly $\mathcal{E}_{\{\mathcal{M}\}}$, but not strongly.
The curve is weakly $\mathcal{E}_{\{\mathcal{M}\}}$, since each $\alpha \in (\mathbb{R}^{\mathbb{N}\times\mathbb{N}})^* = \mathbb{R}^{(\mathbb{N}\times\mathbb{N})}$ depends only on finitely many coordinates and we can choose $h > 0$ and $l \in \Lambda$ large enough (depending on α).
On the other side assume by contradiction that for each compact set $L \subseteq \mathbb{R}$ we can find $h > 0$ and $l \in \Lambda$ such that the set $\left\{\frac{c^{(k)}(t)}{h^k \cdot M_k^l} : t \in L, k \in \mathbb{N}\right\}$ is bounded. So consider a compact neighborhood L of the point 0, an arbitrary number $h > 0$ and an arbitrary index $l_0 \in \Lambda$, then the set $\left\{\frac{c^{(k)}(t)}{h^k \cdot M_k^{l_0}} : t \in L, k \in \mathbb{N}\right\}$ has (n,l)-th coordinate unbounded if $n \cdot h_0 > h \Leftrightarrow \frac{n^k}{h^k} > \frac{1}{h_0^k}$ for some fixed $0 < h_0 < 1$ and with $l \geq l_0$, so $M^{l_0} \leq M^l$.

More precisely the (n,l)-th coordinate (at the point $t=0$) gives:

$$\frac{n^k\cdot|\theta_l^{(k)}(0)|}{h^k\cdot M_k^{l_0}}\ge\frac{|\theta_l^{(k)}(0)|}{h_0^k\cdot M_k^{l_0}}\ge\frac{|\theta_l^{(k)}(0)|}{h_0^k\cdot M_k^{l}}\ge\frac{1}{h_0^k},$$

and finally the last expression tends to infinity for $k\to\infty$. □

The next result is analogous to [22, 4.6. Example], it shows that the additional assumption about the Baire-vector-space-topology on the dual in 10.4.2 cannot be dropped:

Lemma 10.4.5. *Let $\mathcal{M}=\{M^l:l\in\Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with the assumptions from 10.7.1, i.e. a countable index set $\Lambda=\mathbb{N}_{>0}$, moreover $(\mathcal{M}_{\{\text{FdB}\}})$ (for closedness under composition) and finally the assumption in 10.6.2 below (which implies Roumieu-non-quasi-analyticity, i.e. $(\mathcal{M}_{\{\text{nq}\}})$).*
Then there exist locally convex vector spaces E and functions $f:\mathbb{R}^2\to E$ which are $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$, but there is no reasonable topology on $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E)$ such that the associated mapping $f^\vee:\mathbb{R}\to\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E)$ is strong $\mathcal{E}_{\{\mathcal{M}\}}$.

For a "reasonable topology" on $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E)$ we assume only that all point-evaluations $\text{ev}_t:\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E)\to E$ are bounded linear mappings.

Proof. Consider $f:\mathbb{R}^2\to\mathbb{R}^{\mathbb{N}_{>0}}$ defined by $f(s,t):=(\theta_l(s\cdot t))_{l\in\Lambda}$, $\theta_l\in\mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R})$ with $|\theta_l^{(k)}(0)|\ge M_k^l$ for all $k\in\mathbb{N}$ (see (chf)). The mapping f is clearly $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$, since we have by our assumptions Theorem 10.7.1 and so the structure $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ is a generalization of the finite dimensional definitions. Moreover $\text{pr}_i\circ f\circ c$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for each $i\in\mathbb{N}_{>0}$ and each linear functional on $\mathbb{R}^{\mathbb{N}_{>0}}$ depends only on finitely many coordinates. If $f^\vee:\mathbb{R}\to\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}^{\mathbb{N}_{>0}})$ would be strong $\mathcal{E}_{\{\mathcal{M}\}}$, then there would exist $h>0$ and some $n\in\Lambda$ such that the set

$$\left\{\frac{(f^\vee)^{(k)}(0)}{h^k\cdot M_k^n}:k\in\mathbb{N}\right\}$$

would be bounded in $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}^{\mathbb{N}_{>0}})$. But if we apply the bounded linear function ev_t for $t=2h$ then we get

$$\frac{|(f^\vee)^{(k)}(0)(2h)|}{h^k\cdot M_k^n}=\left(\frac{(2h)^k\cdot|\theta_l^{(k)}(0)|}{h^k\cdot M_k^n}\right)_{l\in\Lambda}\ge\left(\frac{2^k\cdot M_k^l}{M_k^n}\right)_{l\in\Lambda}.$$

Finally the coordinates are unbounded for $k\to\infty$ whenever $l\ge n$. □

10.5 Elimination of existence quantifiers in the definition of $\mathcal{E}_{\{\mathcal{M}\}}$

In this section we are going to study one of the most important techniques in this work: We want to prove projective representations for the Roumieu-matrix class $\mathcal{E}_{\{\mathcal{M}\}}$ by using abstract families of sequences of positive real numbers. This technique is very important since we want to get rid of *both* existence quantifiers in the definitions (more precisely in the above definition for $\mathcal{E}_{\{\mathcal{M}\}}$-curves $c:\mathbb{R}\to E$). So we want to show an analogous result to [21, 3.4. Lemma] (resp. [22, 4.8. Lemma]). Furthermore we are going to prove also analogous results for the Beurling-matrix-case $\mathcal{E}_{(\mathcal{M})}$ (see also [22, 4.7. Lemma]). To do so we will have to prove important variations resp. generalizations of [20, 9.2. Lemma] (for the Roumieu-case) and for the Lemma between 4.7. Lemma and 4.8. Lemma on page 17 in [22] (for the Beurling-case).

First we start with some definitions because we have to introduce several classes of sequences of positive real numbers. For all sequences which we will consider it's no restriction to assume $r_0 = 1$ resp. $s_0 = 1$ (normalization).

$$\mathcal{R}_{\text{roum}} := \{r = (r_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : r_k \cdot t^k \to 0 \text{ for } k \to \infty \text{ for each } t > 0\}$$

$$\mathcal{R}_{\text{roum,sub}} := \{r = (r_k)_k \in \mathcal{R}_{\text{roum}} : r_{j+k} \le r_k \cdot r_j \; \forall\, j,k \in \mathbb{N}\},$$

$$\mathcal{R}_{\text{beur}} := \{r = (r_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : r_k \cdot t^k \to 0 \text{ for } k \to \infty \text{ for some } t > 0\},$$

$$\mathcal{R}_{\text{beur,sub}} := \{r = (r_k)_k \in \mathcal{R}_{\text{beur}} : r_{j+k} \le r_k \cdot r_j \; \forall\, j,k \in \mathbb{N}\}.$$

Moreover we introduce (recall the definition $m_k^l := \frac{M_k^l}{k!}$ for all $k \in \mathbb{N}$ and $l \in \Lambda$):

$$\int_{\text{roum}}^{\mathcal{M}} := \{s = (s_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \forall\, l \in \Lambda\; \exists\, C_l > 0\; \forall\, k \in \mathbb{N} : s_k \cdot m_k^l \le C_l^k\},$$

$$\int_{\text{roum,FdB}}^{\mathcal{M}} := \{s = (s_k)_k \in \int_{\text{roum}}^{\mathcal{M}} :\; \exists\, \hat{s} \in \int_{\text{roum}}^{\mathcal{M}}\; \exists\, D > 0\; \forall\, k \in \mathbb{N} : s_k \le D^k \cdot (\hat{s}_o)_k\},$$

where for a sequence $s \in \int_{\text{roum}}^{\mathcal{M}}$ we have put

$$(\hat{s}_o)_k := \min\{s_j \cdot s_{\alpha_1} \cdots s_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k\} \quad (\hat{s}_o)_0 := 1.$$

Similarly we can introduce the Beurling-type-sets:

$$\int_{\text{beur}}^{\mathcal{M}} := \{(s_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \exists\, l \in \Lambda\; \exists\, C_l > 0\; \forall\, k \in \mathbb{N} : s_k \cdot m_k^l \le C_l^k\},$$

$$\int_{\text{beur,FdB}}^{\mathcal{M}} := \{(s_k)_k \in \int_{\text{beur}}^{\mathcal{M}} :\; \exists\, \hat{s} \in \int_{\text{beur}}^{\mathcal{M}}\; \exists\, D > 0\; \forall\, k \in \mathbb{N} : s_k \le D^k \cdot (\hat{s}_o)_k\}.$$

All these sets are stable w.r.t. mappings $(r_k)_k \mapsto (B^k \cdot r_k)_k$ for each $B > 0$, for $\mathcal{R}_{\text{beur}}$ and $\mathcal{R}_{\text{beur,sub}}$ we point out that one has to replace then t by $\frac{t}{B}$.

First we can show a "localized version" of [21, 3.4. Lemma], which has the following form:

Proposition 10.5.1. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and let $c : \mathbb{R} \to E$ with E a Banach-space, then the following assertions are equivalent:*

(1) *c is $\mathcal{E}_{\{\mathcal{M}\}}$.*

(2) *For each compact set $K \subseteq \mathbb{R}$ there exists $l \in \Lambda$ such that for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum}}$ the set*

$$\left\{\frac{c^{(k)}(a) \cdot r_k}{M_k^l} : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E .

(3) *For each compact set $K \subseteq \mathbb{R}$ there exists $l \in \Lambda$ such that for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ there exists a number $\varepsilon > 0$ such that the set*

$$\left\{\frac{c^{(k)}(a) \cdot r_k \cdot \varepsilon^k}{M_k^l} : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E.

Proof. (1) ⇒ (2): Let c be $\mathcal{E}_{\{\mathcal{M}\}}$, take an arbitrary but fixed compact set $K \subseteq \mathbb{R}$ the index $l \in \Lambda$ large enough (depending on c and chosen K) and consider an arbitrary sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$. Then we can estimate for $h > 0$ sufficiently large (where we use 10.4.2):

$$\left\|\frac{c^{(k)}(a)}{M_k^l} \cdot r_k\right\|_E = \left\|\frac{c^{(k)}(a)}{M_k^l \cdot h^k}\right\|_E \cdot \underbrace{\left|r_k \cdot h^k\right|}_{\to 0},$$

where $a \in K$.

(2) ⇒ (3): Take $\varepsilon = 1$.

(3) ⇒ (1): Let $K \subseteq \mathbb{R}$ be an arbitrary but fixed compact set and we put $a_k := \sup_{a \in K}\left\|\frac{c^{(k)}(a)}{M_k^l}\right\|_E$, where $l \in \Lambda$ depends on K and c and is the index coming from (3). Like as in [21, 3.4. Lemma] we can use [20, 9.2.(4) ⇒ (1)]: Then there exists a number $h > 0$ such that $\sup_{k \in \mathbb{N}} \frac{a_k}{h^k} < +\infty$. Since K was arbitrary we have that c is $\mathcal{E}_{\{\mathcal{M}\}}$ (with both $l \in \Lambda$ and $h > 0$ depending on the given compact set K). □

To eliminate also the second existence quantifier we are going to show two versions, a "weak version" which will be used in combination with property $(\mathcal{M}_{\{\mathrm{FdB}\}})$ for $\mathcal{M}$ and a "strong version" for more general weight matrices $\mathcal{M}$. We start with the weak version and prove the following generalized variation of [20, 9.2. Lemma]:

Lemma 10.5.2. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $\Lambda = \mathbb{N}_{>0}$ and property $(\mathcal{M}_{\{\mathrm{FdB}\}})$. For a given formal power series $\sum_{k \geq 0} a_k^l \cdot t^k = \sum_{k \geq 0} \frac{b_k}{k! \cdot m_k^l} \cdot t^k$ with real coefficients $a_k^l := \frac{b_k}{M_k^l}$ ($\Leftrightarrow b_k = a_k^l \cdot M_k^l = a_k^l \cdot k! \cdot m_k^l$) the following conditions are equivalent:*

(1) *There exists an index $l \in \Lambda$ such that the series $\sum_{k \geq 0} a_k^l \cdot t^k$ has positive radius of convergence.*

(2) *$\sum_{k \geq 0} \frac{b_k \cdot r_k \cdot s_k}{k!}$ converges absolutely for all sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and $(s_k)_k \in \int^{\mathcal{M}}_{\mathrm{roum}}$.*

(3) *The sequence $\left(\frac{b_k \cdot r_k \cdot s_k}{k!}\right)_k$ is bounded for all sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and $(s_k)_k \in \int^{\mathcal{M}}_{\mathrm{roum}}$.*

(4) *For each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and for each sequence $(s_k)_k \in \int^{\mathcal{M}}_{\mathrm{roum,FdB}}$ there exists a number $\varepsilon > 0$ such that $\left(\frac{b_k \cdot r_k \cdot s_k}{k!} \cdot \varepsilon^k\right)_k$ is bounded.*

Proof. (1) ⇒ (2): For the given series (with $l \in \Lambda$ coming from (1)) and arbitrary sequences $(r_k)_k$ and $(s_k)_k$ as considered in (2) we can write

$$\sum_{k \geq 0} \frac{b_k \cdot r_k \cdot s_k}{k!} = \sum_{k \geq 0} a_k^l \cdot m_k^l \cdot r_k \cdot s_k = \sum_{k \geq 0} (a_k^l \cdot t^k) \cdot \underbrace{(s_k \cdot m_k^l)}_{\leq C_l^k} \cdot \frac{r_k}{t^k} \leq \sum_{k \geq 0} (a_k^l \cdot t^k) \cdot \underbrace{r_k \cdot \left(\frac{C_l}{t}\right)^k}_{\to 0, \text{for } k \to \infty},$$

hence the first sum converges for $t > 0$ sufficiently small.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are clearly satisfied.

(4) $\Rightarrow$ (1): We prove by contradiction - assume that each series $\sum_{k\ge 0} a_k^l \cdot t^k$ would have radius of convergence 0. So we would get $\sum_{k\ge 0} |a_k^l| \cdot \left(\frac{1}{n^2}\right)^k = +\infty$ for each $n \in \mathbb{N}_{>0}$ and each $l \in \Lambda = \mathbb{N}_{>0}$. Consider now the case $n = l$ (*the diagonal!*) and then we can find an increasing sequence $(k_n)_{n\ge 0}$ with $k_0 = 1$, $\lim_{n\to\infty} k_n = +\infty$ such that

$$\forall\, n \in \mathbb{N}_{>0} : \sum_{k=k_{n-1}}^{k_n-1} |a_k^n| \cdot \left(\frac{1}{n^2}\right)^k \ge 1.$$

We put now

$$r_k := \left(\frac{1}{n^2}\right)^k \quad \text{for} \;\; k_{n-1} \le k \le k_n - 1, n \in \mathbb{N}_{>0},$$

and show $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$: For $k_{n-1} \le k \le k_n - 1$ by definition $r_k \cdot t^k = \left(\frac{t}{n^2}\right)^k$, and so $r_k \cdot t^k \to 0$ for $k \to \infty$ and all $t > 0$. Clearly $(r_k)_k$ is also log. sub-additive. In addition one can see that $(\sqrt{r_k})_k \in \mathcal{R}_{\text{roum,sub}}$ and this has the following consequence: For all $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that for all $k \ge k_\varepsilon$ we get $\sqrt{r_k} \cdot \frac{1}{\varepsilon^k} \le 1 \Leftrightarrow \sqrt{r_k} \le \varepsilon^k$.

Since we have assumed property $(\mathcal{M}_{\{\text{FdB}\}})$ the following holds: For each $l \in \Lambda$ we can find $l_1 \in \Lambda$, $l_1 \ge l$, such that $(m^l)^\circ \preceq m^{l_1}$ is satisfied, with $(m_k^l)^\circ := \max\{m_j^l \cdot m_{\alpha_1}^l \cdots m_{\alpha_j}^l : \alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k\}$, $(m_0^l)^\circ := 1$. Since $m^{l_1} \le m^{l_2}$ for $l_1 \le l_2$ we can associate to each $l \in \Lambda$ the index $\min(l) := \min\{n \in \Lambda : (m^l)^\circ \preceq m^n\}$. Because $(m^{l_1})^\circ \le (m^{l_2})^\circ$ for $l_1 \le l_2$, we also have $\min(l_1) \le \min(l_2)$ and $\lim_{l\to\infty} \min(l) = +\infty$. On the other hand we can define $\max(n) := \max\{l \in \Lambda : \min(l) \le n\}$, which is well-defined (since clearly $\min(0) = 0$), $\max(n_1) \le \max(n_2)$ for $n_1 \le n_2$ and finally $\lim_{n\to\infty} \max(n) = +\infty$.

With this preparation we are able to define the sequence $s = (s_k)_k$ as follows: Put $s_k := \frac{1}{m_k^{n(k)}}$, where $n(k) = n$ for $k_{n-1} \le k \le k_n - 1$. We show $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum,FdB}}$: Take some arbitrary $l \in \Lambda$, for $k_{n-1} \le k \le k_n - 1$ we get $s_k \cdot m_k^l = \frac{m_k^l}{m_k^{n(k)}} = \frac{M_k^l}{M_k^{n(k)}}$ and for all $k \in \mathbb{N}$ we can estimate $\frac{M_k^l}{M_k^{n(k)}} \le C_l^k$ with some constant $C_l > 0$, because $\lim_{k\to\infty} n(k) = +\infty$. This proves $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum}}$. Define

$$\hat{s}_k := \frac{1}{m_k^{\max(n(k))}} \qquad \text{for} \;\; k_{n-1} \le k \le k_n - 1,$$

and then we can find a constant $D_l > 0$ such that $\hat{s}_k \cdot m_k^l \le D_l^k$ for each $l \in \Lambda$ and $k \in \mathbb{N}$ because $\lim_{k\to\infty} \max(n(k)) = +\infty$. This proves $(\hat{s}_k)_k \in \int^{\mathcal{M}}_{\text{roum}}$, too. But for $\alpha_1 + \cdots + \alpha_j = k$ we obtain for $k \in \mathbb{N}$ with $k_{n-1} \le k \le k_n - 1$:

$$\begin{aligned} s_k = \frac{1}{m_k^{n(k)}} &\le C \cdot h^k \cdot \frac{1}{m_j^{\max(n(k))} \cdot m_{\alpha_1}^{\max(n(k))} \cdots m_{\alpha_j}^{\max(n(k))}} \\ &\le C \cdot h^k \cdot \frac{1}{m_j^{\max(n(j))} \cdot m_{\alpha_1}^{\max(n(\alpha_1))} \cdots m_{\alpha_j}^{\max(n(\alpha_j))}} = C \cdot h^k \cdot \hat{s}_j \cdot \hat{s}_{\alpha_1} \cdots \hat{s}_{\alpha_j}, \end{aligned}$$

which precisely shows $s \preceq \hat{s}_o$. The first inequality holds by $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and the definition of the function max, the second inequality because $j, \alpha_1, \dots, \alpha_j \leq k$. Moreover the following holds:

$$\sum_{k\geq 1} \frac{|b_k| \cdot r_k \cdot s_k}{k!} = \sum_{n\geq 1} \sum_{k=k_{n-1}}^{k_n-1} \frac{|b_k| \cdot r_k \cdot s_k}{k!} = \sum_{n\geq 1} \sum_{k=k_{n-1}}^{k_n-1} |a_k^n| \cdot \left(\frac{1}{n^2}\right)^k \geq \sum_{n\geq 1} 1 = +\infty,$$

because by definition we have $\frac{|b_k|}{k!} \cdot s_k = \frac{|b_k|}{M_k^n} = |a_k^n|$ for $k_{n-1} \leq k \leq k_n - 1$, recall: $n(k) = n$ for $k \in [k_{n-1}, k_n - 1]$.

Finally we show that the sequence $\left(\frac{b_k}{k!} \cdot \sqrt{r_k} \cdot s_k \cdot (2\cdot\varepsilon)^k\right)_k$ cannot be bounded for any $\varepsilon > 0$. First we get

$$\sum_{k\geq 1} \frac{|b_k|}{k!} \cdot \sqrt{r_k} \cdot s_k \cdot \varepsilon^k \geq \sum_{k\geq k_\varepsilon} \frac{|b_k|}{k!} \cdot \sqrt{r_k} \cdot s_k \cdot \underbrace{\varepsilon^k}_{\geq\sqrt{r_k}} \geq \sum_{k\geq k_\varepsilon} \frac{|b_k|}{k!} \cdot r_k \cdot s_k = +\infty.$$

If the sequence would be bounded for some ε, then for all $k \in \mathbb{N}$ we would get $\frac{b_k}{k!} \cdot \sqrt{r_k} \cdot s_k \cdot \varepsilon^k \leq \frac{C}{2^k}$, hence $\sum_{k\geq 0} \frac{|b_k|}{k!} \cdot \sqrt{r_k} \cdot s_k \cdot \varepsilon^k \leq \sum_{k\geq 0} \frac{C}{2^k} = 2C$, a contradiction. □

We use 10.5.2 to show the following important characterization:

Proposition 10.5.3. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ matrix with index-set $\Lambda = \mathbb{N}_{>0}$ and property $(\mathcal{M}_{\{\mathrm{FdB}\}})$ and let $c : \mathbb{R} \to E$ be a curve with E a Banach-space. Then the following conditions are equivalent:*

(1) *c is $\mathcal{E}_{\{\mathcal{M}\}}$.*

(2) *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and for each sequence $(s_k)_k \in \mathfrak{f}_{\mathrm{roum}}^{\mathcal{M}}$ the set*

$$\left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E.

(3) *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and moreover for each sequence $(s_k)_k \in \mathfrak{f}_{\mathrm{roum,FdB}}^{\mathcal{M}}$, there exists a number $\varepsilon > 0$ such that the set*

$$\left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E.

Proof. (1) $\Rightarrow$ (2): Let c be $\mathcal{E}_{\{\mathcal{M}\}}$, then we can estimate as follows (where we use 10.4.2):

$$\left\|\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k\right\|_E = \left\|\frac{c^{(k)}(a)}{k! \cdot m_k^l \cdot h^k}\right\|_E \cdot \left|r_k \cdot h^k\right| \cdot \underbrace{s_k \cdot m_k^l}_{\leq C_l^k} \leq \underbrace{\left|r_k \cdot (C_l \cdot h)^k\right|}_{\to 0}$$

for $a \in K$, $l \in \Lambda$ and $h > 0$ large enough (depending on K and f) and for arbitrary sequences $(r_k)_k$ and $(s_k)_k$ as considered in (2).

(2) $\Rightarrow$ (3): Take $\varepsilon = 1$.
(3) $\Rightarrow$ (1): For this we use the previous result 10.5.2, more precisely (4) $\Rightarrow$ (1) - Let $K \subseteq \mathbb{R}$ be an arbitrary compact set but fixed and we put $b_k := \sup_{x \in K} \left\| c^{(k)}(x) \right\|_E$. Then there exists a number $h > 0$ and $l \in \Lambda$ such that $\sup_{k \in \mathbb{N}} \frac{b_k}{M_k^l \cdot h^k} < +\infty$, hence c is by definition $\mathcal{E}_{\{\mathcal{M}\}}$. □

With this preparation we are able to prove the very important analogous result to [21, 3.5. Lemma]:

Proposition 10.5.4. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with index set $\Lambda = \mathbb{N}_{>0}$ and $(\mathcal{M}_{\{\text{FdB}\}})$, let E be a convenient vector space and finally let $\mathcal{B}$ be a family of bounded linear functionals on E which together can detect bounded sets, which means that $B \subseteq E$ is bounded in E if and only if $\alpha(B)$ is bounded in $\mathbb{R}$ for all $\alpha \in \mathcal{B}$.*
Then we obtain: A curve $c : \mathbb{R} \to E$ is $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if $\alpha \circ c : \mathbb{R} \to \mathbb{R}$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for all $\alpha \in \mathcal{B}$.

Proof. Recall, as in the original proof in [21, 3.5. Lemma], that for smooth curves this holds by [20, 2.1, 2.11]. Now use (1) $\Leftrightarrow$ (2) in 10.5.3 to obtain: For $\alpha \in E^*$ the mapping $\alpha \circ c$ is $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if for all sets K in $\mathbb{R}$ compact the set

$$\left\{ \frac{(\alpha \circ c)^{(k)}(a)}{k!} \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N} \right\}$$

is bounded in $\mathbb{R}$ for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum}}$ and for each sequence $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum,FdB}}$. And this is equivalent to the fact that for each compact K the set $\left\{ \frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N} \right\}$ is bounded in E for all sequences $(r_k)_k \in \mathcal{R}_{\text{roum}}$, $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum,FdB}}$.
Finally we can replace in the above equivalences E^* by $\mathcal{B}$, because $\mathcal{B}$ detects bounded sets. □

Important remark: 10.5.4 shows, that we can replace E^* in the definition of (weak) $\mathcal{E}_{\{\mathcal{M}\}}$-curves also by E', the space of all *bounded linear functionals* on E.

We will need 10.5.3 in the "weak version" as shown before for the proof of 10.7.1, there condition $(\mathcal{M}_{\{\text{FdB}\}})$ will play the key-role (closedness under composition!). But for some general applications the following "strong version" is sufficient, more precisely condition $(\mathcal{M}_{\{\text{FdB}\}})$ is not needed necessarily and we obtain projective representations for more general classes of weight matrices $\mathcal{M}$. For this we have to introduce the following families of positive real numbers:

$$\mathcal{S}^{\mathcal{M}}_{\text{roum}} := \{(s_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \forall\, l \in \Lambda\ \exists\, C_l > 0\ \forall\, k \in \mathbb{N} : s_k \cdot M^l_k \leq C^k_l\}$$

$$\mathcal{S}^{\mathcal{M}}_{\text{roum,sub}} := \{(s_k)_k \in \mathcal{S}_{\text{roum}} : \exists\, D > 0\ \forall\, j, k \in \mathbb{N} : s_{j+k} \leq D \cdot s_j \cdot s_k\}.$$

Clearly both $\mathcal{S}^{\mathcal{M}}_{\text{roum}}$ and $\mathcal{S}^{\mathcal{M}}_{\text{roum,sub}}$ are stable w.r.t. the mappings $s_k \mapsto C^k \cdot s_k$ for each $C > 0$ and $\mathcal{S}^{\mathcal{M}}_{\text{roum}} \subseteq \int^{\mathcal{M}}_{\text{roum}}$. We have that $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{roum}}$ if and only if $(k! \cdot s_k)_k \in \int^{\mathcal{M}}_{\text{roum}}$. Moreover, if one assumes for a $(\mathcal{M}_{\text{sc}})$ weight matrix $\mathcal{M}$ e.g. in addition property $(\mathcal{M}_{\{\text{diag}\}})$ (related to property (ω_7) if the matrix is obtained by a weight function ω - see 5.4.1), then $\mathcal{S}^{\mathcal{M}}_{\text{roum}}$ is also stable w.r.t. mappings $s_k \mapsto M^l_k \cdot s_k$ for each $l \in \Lambda$. This holds, because we estimate $M^{l_1}_k \cdot M^{l_2}_k \leq M^{l_3}_{2k} \leq C^{2k} \cdot M^{l_4}_k$ for all $k \in \mathbb{N}$ with $l_3 = \max\{l_1, l_2\}$.

By using these new families of sequences we can reformulate 10.5.2 without the additional assumption $(\mathcal{M}_{\{\mathrm{FdB}\}})$:

Lemma 10.5.5. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with $\Lambda = \mathbb{N}_{>0}$. For a given formal power series $\sum_{k\ge 0} a^l_k \cdot t^k = \sum_{k\ge 0} \frac{b_k}{M^l_k} \cdot t^k$ with real coefficients $a^l_k := \frac{b_k}{M^l_k}$ the following conditions are equivalent:*

1. *There exists an index $l \in \Lambda$ such that the series $\sum_{k\ge 0} a^l_k \cdot t^k$ has positive radius of convergence.*
2. *$\sum_{k\ge 0} b_k \cdot r_k \cdot s_k$ converges absolutely for all sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$.*
3. *The sequence $(b_k \cdot r_k \cdot s_k)_k$ is bounded for all sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum,sub}}$.*
4. *For each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum,sub}}$ there exists a number $\varepsilon > 0$ such that $(b_k \cdot r_k \cdot s_k \cdot \varepsilon^k)_k$ is bounded.*

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we obtain the same equivalences where in conditions (3) and (4) we replace the set $\mathcal{S}^{\mathcal{M}}_{\mathrm{roum,sub}}$ by $\mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$.

Proof. The proof for (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is completely analogous as in 10.5.2, for (4) $\Rightarrow$ (1) we prove again by contradiction for the same sequence $r = (r_k)_k$, and for $s = (s_k)_k$ we put $s_k := \frac{1}{M^n_k}$ if $k_{n-1} \le k \le k_n - 1$. In the case if $\mathcal{M}$ is $(\mathcal{M}_{\mathrm{sc}})$ to prove the correct properties for s, i.e. $s \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum,sub}}$, we use 9.2.1 and so we need in this case only $M^l_j \cdot M^l_k \le M^l_{j+k}$ for each $j, k \in \mathbb{N}$ and $l \in \Lambda$ and $M^l \le M^n$ for $l \le n$, but this holds by assumption on $\mathcal{M}$ (each sequence M^l is (weakly) log. convex). If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, then $s \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$ is clear by definition. Note that $\lim_{n\to\infty} k_n = +\infty$ holds. □

By using this result we can prove new versions of 10.5.3 resp. 10.5.4:

Proposition 10.5.6. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\mathrm{sc}})$ matrix with index-set $\Lambda = \mathbb{N}_{>0}$ and let $c : \mathbb{R} \to E$ be a curve with E a Banach-space, then the following conditions are equivalent:*

1. *c is $\mathcal{E}_{\{\mathcal{M}\}}$.*
2. *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$ the set*
$$\left\{c^{(k)}(a) \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}\right\}$$
is bounded in E .
3. *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and moreover for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum,sub}}$, there exists a number $\varepsilon > 0$ such that the set*
$$\left\{c^{(k)}(a) \cdot r_k \cdot s_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}\right\}$$
is bounded in E.

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we obtain the same equivalences where in condition (3) we replace the set $\mathcal{S}^{\mathcal{M}}_{\text{roum,sub}}$ by $\mathcal{S}^{\mathcal{M}}_{\text{roum}}$.

Proof. (1) ⇒ (2): Let c be $\mathcal{E}_{\{\mathcal{M}\}}$, then we can estimate as follows (where we use 10.4.2):

$$\left\|c^{(k)}(a)\cdot r_k\cdot s_k\right\|_E=\left\|\frac{c^{(k)}(a)}{M^l_k\cdot h^k}\right\|_E\cdot\left|r_k\cdot h^k\right|\cdot\underbrace{s_k\cdot M^l_k}_{\le C^k_l}\le\underbrace{\left|r_k\cdot(C_l\cdot h)^k\right|}_{\to 0}$$

for $a\in K$, $l\in\Lambda$ and $h>0$ sufficiently large (depending on K and f) and arbitrary sequences $(r_k)_k$ and $(s_k)_k$ as considered in (2).
(2) ⇒ (3): Take $\varepsilon=1$.
(3) ⇒ (1): For this we use the previous result 10.5.5, more precisely (4) ⇒ (1) - Let K be an arbitrary compact set in $\mathbb{R}$ but from now on fixed, then we put $b_k:=\sup_{x\in K}\left\|c^{(k)}(x)\right\|_E$. Hence there exists a number $h>0$ and $l\in\Lambda$ such that $\sup_{k\in\mathbb{N}}\frac{b_k}{M^k_l\cdot h^k}<+\infty$, hence c is by definition $\mathcal{E}_{\{\mathcal{M}\}}$. □

By using 10.5.6 instead of 10.5.3 we can immediately prove:

Proposition 10.5.7. *Let $\mathcal{M}=\{M^l:l\in\Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, resp. a $(\mathcal{M}_{\text{sc}})$ weight matrix, with index set $\Lambda=\mathbb{N}_{>0}$, let E be a convenient vector space and finally let $\mathcal{B}$ be a family of bounded linear functionals on E which together can detect bounded sets ($B\subseteq E$ is bounded in E if and only if $\alpha(B)$ is bounded in $\mathbb{R}$ for all $\alpha\in\mathcal{B}$). Then we obtain: A curve $c:\mathbb{R}\to E$ is $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if $\alpha\circ c:\mathbb{R}\to\mathbb{R}$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for all $\alpha\in\mathcal{B}$.*

Now we prove analogous results for the Beurling-case, we start again with the "weak version" where we have to assume in addition property $(\mathcal{M}_{(\text{FdB})})$ (for closedness under composition!). We will need the "weak version" for the proof of 12.2.8 (see also 12.2.5 below), but for many general applications $(\mathcal{M}_{(\text{FdB})})$ is not needed and so we will also study the "strong version".

Lemma 10.5.8. *Let $\mathcal{M}=\{M^l:l\in\Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda=\mathbb{R}_{>0}$ and additionally $(\mathcal{M}_{(\text{FdB})})$. For a given formal power series $\sum_{k\ge 0}a^l_k\cdot t^k=\sum_{k\ge 0}\frac{b_k}{M^l_k}\cdot t^k$ with real coefficients $a^l_k:=\frac{b_k}{M^l_k}$ the following conditions are equivalent:*

(1) *The series $\sum_{k\ge 0}a^l_k\cdot t^k$ has infinite radius of convergence for each $l\in\Lambda$.*

(2) *For each sequence $(r_k)_k\in\mathcal{R}_{\text{beur,sub}}$ and moreover for each sequence $(s_k)_k\in\int^{\mathcal{M}}_{\text{beur,FdB}}$ the sequence $\left(\frac{b_k}{k!}\cdot r_k\cdot s_k\cdot\delta^k\right)_k$ is bounded for each number $\delta>0$.*

Proof. (1) ⇒ (2): Let $(r_k)_k$ and $(s_k)_k$ be given sequences as considered in (2), then

$$\sum_{k\ge 0}\frac{b_k}{k!}\cdot r_k\cdot s_k\cdot\delta^k=\sum_{k\ge 0}a^l_k\cdot\underbrace{(m^l_k\cdot s_k)}_{\le C^k_l}\cdot(r_k\cdot t^k)\cdot\left(\frac{\delta}{t}\right)^k\le\sum_{k\ge 0}a^l_k\cdot\underbrace{(r_k\cdot t^k)}_{\to 0}\cdot\left(\frac{\delta\cdot C_l}{t}\right)^k$$

is absolutely convergent for each $\delta>0$, where $l\in\Lambda$ was chosen such that $s_k\cdot m^l_k\le C^k_l$ holds for all $k\in\mathbb{N}$. The index is depending on $(s_k)_k\in\int^{\mathcal{M}}_{\text{beur}}$ and $t>0$ was chosen in such a way that $r_k \cdot t^k \to 0$ for $k \to \infty$ since $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$. Hence $\left(\frac{b_k}{k!} \cdot r_k \cdot s_k \cdot \delta^k\right)_k$ is bounded for each $\delta > 0$.

(2) $\Rightarrow$ (1): We prove by contradiction - assume that there would exist an index $l \in \Lambda$ such that the series $\sum_{k\geq 0} a_k^l \cdot t^k$ would have finite radius of convergence. Then there would exist a number $h > 0$ such that $\sum_{k\geq 0} |a_k^l| \cdot n^k = +\infty$ for each $n > h$. Put now $r_k := \frac{1}{n^k}$ for some $n > h$ and $s_k := \frac{1}{m_k^l}$.
$r = (r_k)_k$ is clearly an element in $\mathcal{R}_{\text{beur,sub}}$.
For $s = (s_k)_k$ we point out: By $(\mathcal{M}_{(\text{FdB})})$ we have that for all $l \in \Lambda$ there exists $l_1 \in \Lambda$, $l_1 \leq l$, and $D > 0$ such that for all $\alpha_1 + \cdots + \alpha_j = k$ we obtain

$$s_k := \frac{1}{m_k^l} \leq D^k \cdot \frac{1}{m_j^{l_1} \cdot m_{\alpha_1}^{l_1} \cdots m_{\alpha_j}^{l_1}} =: \hat{s}_j \cdot \hat{s}_{\alpha_1} \cdots \hat{s}_{\alpha_j}.$$

But it's clear that the sequence $\hat{s}_j := \frac{1}{m_j^{l_1}}$ belongs to $\int_{\text{beur}}^{\mathcal{M}}$, hence $s \in \int_{\text{beur,FdB}}^{\mathcal{M}}$ and so both $r = (r_k)_k$ and $s = (s_k)_k$ are sequences as considered in (2). But then, by hypothesis, there would exist a constant $C > 0$ such that for all $k \in \mathbb{N}$ we would get:

$$C > \frac{b_k}{k!} \cdot s_k \cdot r_k \cdot (2n^2)^k = \frac{b_k}{k! \cdot m_k^l} \cdot r_k \cdot (2n^2)^k = a_k^l \cdot r_k \cdot n^{2k} \cdot 2^k = a_k^l \cdot n^k \cdot 2^k.$$

Hence $\sum_{k\geq 0} |a_k^l| \cdot n^k \leq C \cdot \sum_{k\geq 0} \frac{1}{2^k} = 2C$, a contradiction! $\square$
We use the previous result to show the following important characterization:

Proposition 10.5.9. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and additionally $(\mathcal{M}_{(\text{FdB})})$. Let $c : \mathbb{R} \to E$ be a curve and E be a Banach-space, then the following conditions are equivalent:*

(1) *c is $\mathcal{E}_{(\mathcal{M})}$.*

(2) *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur}}$ and for each sequence $(s_k)_k \in \int_{\text{beur}}^{\mathcal{M}}$ the set*

$$\left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E.

(3) *For each compact set $K \subseteq \mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$ and moreover for each sequence $(s_k)_k \in \int_{\text{beur,FdB}}^{\mathcal{M}}$ the set*

$$\left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot \delta^k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E for each number $\delta > 0$.

Proof. (1) $\Rightarrow$ (2): Let c be $\mathcal{E}_{(\mathcal{M})}$ and $(r_k)_k$, $(s_k)_k$ given by (2), then we can estimate as follows (where we use 10.4.1):

$$\left\|\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k\right\|_E = \left\|\frac{c^{(k)}(a)}{k! \cdot m_k^l \cdot h^k}\right\|_E \cdot \left|r_k \cdot h^k\right| \cdot \underbrace{s_k \cdot m_k^l}_{\leq C_l^k} \leq \left\|\frac{c^{(k)}(a)}{M_k^l \cdot h^k}\right\|_E \cdot \underbrace{\left|r_k \cdot (C_l h)^k\right|}_{\to 0}$$

for $a \in K$. We have chosen $l \in \Lambda$ depending on given $(s_k)_k \in \int_{\text{beur}}^{\mathcal{M}}$ such that $s_k \cdot m_k^l \leq C_l^k$ and $h > 0$ depending on given $(r_k)_k \in \mathcal{R}_{\text{beur}}$ such that $r_k \cdot (Ch)^k \to 0$ for $k \to \infty$.
(2) $\Rightarrow$ (3): Replace in (2) the sequence $(r_k)_k$ by $(r_k \cdot \delta^k)_k$.
(3) $\Rightarrow$ (1): For this we use the previous result 10.5.10, (2) $\Rightarrow$ (1) - Let K in $\mathbb{R}$ be an arbitrary compact set but from now on fixed and we put $b_k := \sup_{x \in K} \left\|c^{(k)}(x)\right\|_E$. Then for each $h > 0$ and each $l \in \Lambda$ we have that $\sup_{k \in \mathbb{N}} \frac{b_k}{M_l^k \cdot h^k} < +\infty$, hence c is by definition $\mathcal{E}_{(\mathcal{M})}$. □

We formulate and prove now analogously as in the Roumieu-case the "strong version". For this first we introduce the families

$$\mathcal{S}_{\text{beur}}^{\mathcal{M}} := \{(s_k)_k \in \mathbb{R}_{>0}^{\mathbb{N}} : \exists\, l \in \Lambda\, \exists\, C_l > 0\, \forall\, k \in \mathbb{N} : s_k \cdot M_k^l \leq C_l^k\}$$

$$\mathcal{S}_{\text{beur,sub}}^{\mathcal{M}} := \{(s_k)_k \in \mathcal{S}_{\text{beur}} : \exists\, D > 0\, \forall\, j, k \in \mathbb{N} : s_{j+k} \leq D \cdot s_j \cdot s_k\}.$$

We clearly have again that both classes are stable w.r.t. mappings $s_k \mapsto C^k \cdot s_k$ for arbitrary $C > 0$ and moreover $\mathcal{S}_{\text{beur}}^{\mathcal{M}} \subseteq \int_{\text{beur}}^{\mathcal{M}}$ and $s \in \mathcal{S}_{\text{beur}}^{\mathcal{M}}$ if and only if $(k! \cdot s_k)_k$ is an element of $\int_{\text{beur}}^{\mathcal{M}}$.
By working with these new families of sequences we can skip condition $(\mathcal{M}_{(\text{FdB})})$ in the previous results and get the following variation:

Lemma 10.5.10. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$. For a given formal power series $\sum_{k \geq 0} a_k^l \cdot t^k = \sum_{k \geq 0} \frac{b_k}{M_k^l} \cdot t^k$ with real coefficients $a_k^l := \frac{b_k}{M_k^l}$ the following conditions are equivalent:*

(1) *The series $\sum_{k \geq 0} a_k^l \cdot t^k$ has infinite radius of convergence for each $l \in \Lambda$.*

(2) *For each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$ and moreover for each sequence $(s_k)_k \in \mathcal{S}_{\text{beur,sub}}^{\mathcal{M}}$ the sequence $(b_k \cdot r_k \cdot s_k \cdot \delta^k)_k$ is bounded for each number $\delta > 0$.*

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we obtain the same equivalence where in condition (2) we replace the set $\mathcal{S}_{\text{roum,beur}}^{\mathcal{M}}$ by $\mathcal{S}_{\text{beur}}^{\mathcal{M}}$.

Proof. The proof of 10.5.10 is precisely the same as for 10.5.8, in (2) $\Rightarrow$ (1) we put $s_k := \frac{1}{M_k^l}$ (where $l \in \Lambda$ is the index as in (2) $\Rightarrow$ (1) in 10.5.8 arising by the contradiction argument).
Hence $(s_k)_k \in \mathcal{S}_{\text{beur,sub}}^{\mathcal{M}}$ holds whenever $\mathcal{M}$ is $(\mathcal{M}_{\text{sc}})$, again by weak log. convexity for the sequence M^l. If $\mathcal{M}$ is only assumed to be arbitrary, i.e. $(\mathcal{M})$, then by definition $(s_k)_k \in \mathcal{S}_{\text{beur}}^{\mathcal{M}}$ holds immediately. □
So we are able to prove a strong (more general) version of 10.5.9:

Proposition 10.5.11. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$, let $c : \mathbb{R} \to E$ be a curve and E be a Banach-space. Then the following conditions are equivalent:*

(1) *c is $\mathcal{E}_{(\mathcal{M})}$.*

(2) *For each compact set K in $\mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur}}$ and for each sequence $(s_k)_k \in \mathcal{S}_{\text{beur}}^{\mathcal{M}}$ the set*

$$\left\{c^{(k)}(a) \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E.

(3) *For each compact set K in $\mathbb{R}$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$ and moreover for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{beur,sub}}$ the set*

$$\left\{c^{(k)}(a) \cdot r_k \cdot s_k \cdot \delta^k : a \in K, k \in \mathbb{N}\right\}$$

is bounded in E for each number $\delta > 0$.

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we obtain the same equivalences where in condition (3) *we replace the set $\mathcal{S}^{\mathcal{M}}_{\text{roum,beur}}$ by $\mathcal{S}^{\mathcal{M}}_{\text{beur}}$.*

Proof. (1) $\Rightarrow$ (2): Let c be $\mathcal{E}_{(\mathcal{M})}$ and $(r_k)_k$, $(s_k)_k$ as considered in (2), then we can estimate as follows (where we use 10.4.1):

$$\left\|c^{(k)}(a) \cdot r_k \cdot s_k\right\|_E = \left\|\frac{c^{(k)}(a)}{M^l_k \cdot h^k}\right\|_E \cdot \left|r_k \cdot h^k\right| \cdot \underbrace{s_k \cdot M^l_k}_{\le C^k_l} \le \left\|\frac{c^{(k)}(a)}{M^l_k \cdot h^k}\right\|_E \cdot \underbrace{\left|r_k \cdot (C_l h)^k\right|}_{\to 0}$$

for $a \in K$. We have chosen $l \in \Lambda$ depending on given $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{beur}}$ such that $s_k \cdot M^l_k \le C^k_l$ and $h > 0$ depending on given $(r_k)_k \in \mathcal{R}_{\text{beur}}$ such that $r_k \cdot (Ch)^k \to 0$ for $k \to \infty$.
(2) $\Rightarrow$ (3): Replace in (2) the sequence $(r_k)_k$ by $(r_k \cdot \delta^k)_k$.
(3) $\Rightarrow$ (1): For this we use the previous result 10.5.10, (2) $\Rightarrow$ (1) - Let K in $\mathbb{R}$ be an arbitrary compact set but from now on fixed and we put $b_k := \sup_{x \in K} \left\|c^{(k)}(x)\right\|_E$. Then for each $h > 0$ and each $l \in \Lambda$ we have that $\sup_{k \in \mathbb{N}} \frac{b_k}{M^k_l \cdot h^k} < +\infty$, hence c is by definition $\mathcal{E}_{(\mathcal{M})}$. □

We close this section with the following important remark:

Remark 10.5.12. (*i*) *The assumption that the weight matrix $\mathcal{M} = \{M^l : l \in \Lambda\}$ is a standard log. convex weight matrix, i.e. $(\mathcal{M}_{\text{sc}})$, is not very strong but in fact even this assumption can be skipped in 10.5.5, 10.5.6, 10.5.7, 10.5.10 and 10.5.11.*

(*ii*) *As we have seen in the proofs it's sufficient to assume that $\mathcal{M}$ is an arbitrary weight matrix, i.e. $(\mathcal{M})$ holds. The advantage is that we can formulate 10.5.7 also without the assumption that the matrix is $(\mathcal{M}_{\text{sc}})$!*

(*iii*) *More general one can say that the more assumptions we have for the matrix $\mathcal{M}$, the more special projective representations for the associated function class of Roumieu-type $\mathcal{E}_{\{\mathcal{M}\}}$ are available. The choice of the "right description" depends now on the particular application in the proofs of the theorems.*

10.6 Special curve lemma for the non-quasi-analytic $\mathcal{E}_{[\mathcal{M}]}$-case

The goal of this section is to transfer the special-curve-lemma [21, 3.6. Lemma] to the general (also not constant) weight matrix-case $\mathcal{E}_{[\mathcal{M}]}$. We will formulate the main result in this section in the most general setting, where $\mathcal{M}$ is an arbitrary (very large) set of weight sequences, because we will have to use it in Chapter 11 below where such large matrices will be considered. So the classes $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ will be considered in the most general definition (7.2.1) resp. (7.2.2).
First we start with a preparation result which deals with an explicit construction of certain weight sequences:

Lemma 10.6.1. *Let M be a weakly log. convex and non-quasi-analytic, i.e.* (nq) *holds, weight sequence with additional* $\lim_{k\to\infty}(m_k)^{1/k}=+\infty$. *Then there exists a non-quasi-analytic weakly log. convex sequence N with $N\lhd M$ and finally* $\lim_{k\to\infty}(n_k)^{1/k}=+\infty$ *(where we have put $m_k:=\frac{M_k}{k!}$ and $n_k:=\frac{N_k}{k!}$).*

Proof. Take $f\in\mathcal{E}_{(M)}(\mathbb{R},\mathbb{R})$ with compact support (exists by the Denjoy-Carleman-Theorem - see e.g. [38, Chapter 4] for a collection of different proofs) to obtain a sequence $L_k:=\sup_{x\in K}|f^{(k)}(x)|$, where $K:=\operatorname{supp}(f)$. Then, by definition, we get $L\lhd M$ and use [35, 2.3. Lemma] to obtain sequences N and P, with $L\le N\lhd P\lhd M$ and finally $\lim_{k\to\infty}(n_k)^{1/k}=\lim_{k\to\infty}(p_k)^{1/k}+\infty$ for $n_k:=\frac{N_k}{k!}$, $p_k:=\frac{P_k}{k!}$.
By the second part of [35, 2.15. Theorem] we get $f\in\mathcal{E}_{(P)}(\mathbb{R},\mathbb{R})=\mathcal{E}_{(P^{\mathrm{lc}})}(\mathbb{R},\mathbb{R})$, where P^{lc} denotes the weakly log. convex minorant of P and the Denjoy-Carleman-Theorem in [15, Theorem 1.3.8.] implies that P^{lc} satisfies property (nq), i.e. is not quasi-analytic. Moreover $P^{\mathrm{lc}}\le P\lhd M$ and because $\mathcal{C}^\omega\subsetneq\mathcal{E}_{(P)}=\mathcal{E}_{(P^{\mathrm{lc}})}$ we finally get $\lim_{k\to\infty}\left(\frac{P^{\mathrm{lc}}_k}{k!}\right)^{1/k}=+\infty$. Note that the inclusion is strict because the class $\mathcal{E}_{(P)}$ doesn't coincide with $\mathcal{C}^\omega$ since $f\in\mathcal{E}_{(P)}(\mathbb{R},\mathbb{R})\backslash\mathcal{C}^\omega(\mathbb{R},\mathbb{R})$ (the real-analytic functions are quasi-analytic!). □

Remark: If a weight matrix $\mathcal{M}=\{M^l:l>0\}$ is obtained by a weight function $\omega\in\mathcal{W}$ by $M^l_j:=\exp(\frac{1}{l}\cdot\varphi^*_\omega(lj))$, where ω should satisfy additionally conditions (ω_1), (ω_5) and (ω_{nq}), then the assumptions of Lemma 10.6.1 are satisfied for any sequence M^l. More precisely we can use the second part of 5.3.2 to see that $\lim_{k\to\infty}(m^l_k)^{1/k}=+\infty$ holds for all $l>0$, where $m^l_k:=\frac{M^l_k}{k!}$ and in fact property $(\mathcal{M}_{(\mathcal{C}^\omega)})$ holds, moreover by the first part of 5.1.3 we see that each M^l has property (nq).

Inspired by 10.6.1 we can introduce now important conditions: From now on let $\mathcal{M}$ be an arbitrary set of weight sequences, such that each $M\in\mathcal{M}$ is weakly log. convex (which includes the case where $\mathcal{M}:=\{M^l:l\in\Lambda\}$ is assumed to be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix). It will turn out that the results in this section are valid in this most general case and we introduce now several conditions on $\mathcal{M}$.
First

$$\exists\, M\in\mathcal{M}:\ \lim_{k\to\infty}(m_k)^{1/k}=+\infty, \tag{10.6.1}$$

with $m_k:=\frac{M}{k!}$ and M is not quasi-analytic, i.e. has property (nq). This condition is equivalent to

$$\exists\, M\in\mathcal{M}\ \exists\, N\in\mathbb{R}^{\mathbb{N}}_{>0}:N\preceq M, \tag{10.6.2}$$

such that N should be weakly log. convex, non-quasi-analytic and $\lim_{k\to\infty}(n_k)^{1/k}=+\infty$ with $n_k:=\frac{N_k}{k!}$. (10.6.1) ⇒ (10.6.2) holds clearly for $N=M$. Conversely (10.6.2)

implies immediately $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$ and moreover M is non-quasi-analytic (nq), since N has this condition by assumption.
Alternatively one can assume

$$\exists\, M \in \mathcal{M}\ \exists\, N \in \mathbb{R}_{>0}^{\mathbb{N}} : N \lhd M, \tag{10.6.3}$$

such that N should be weakly log. convex, non-quasi-analytic and $\lim_{k\to\infty}(n_k)^{1/k} = +\infty$. Of course we have (10.6.3) $\Rightarrow$ (10.6.2), but, by 10.6.1, we also have (10.6.1) $\Rightarrow$ (10.6.3). So in fact all three conditions are equivalent and they have the consequence that $\mathcal{M}$ is *Roumieu-non-quasi-analytic*, i.e. $(\mathcal{M}_{\{\mathrm{nq}\}})$ holds and the space $\mathcal{E}_{\{\mathcal{M}\}}$ defined in (7.2.1) contains functions with compact support.
To prove the special curve-lemma for the Roumieu-case it's sufficient to assume the following weaker condition:

$$\exists\, M \in \mathcal{M}\ \exists\, N \in \mathbb{R}_{>0}^{\mathbb{N}} : N \lhd M, \tag{10.6.4}$$

such that N should be only normalized, weakly log. convex and non-quasi-analytic. This implies $(\mathcal{M}_{\{\mathrm{nq}\}})$, too. If $\mathcal{M} = \{M^l : l \in \Lambda\}$, then in each condition above we replace "$\exists\, M \in \mathcal{M}$" by "$\exists\, l_0 \in \Lambda$".

A little bit different is the situation in the Beurling-case, we introduce:

$$\exists\, N \in \mathbb{R}_{>0}^{\mathbb{N}}\ \forall\, M \in \mathcal{M} :\ N \preceq M, \tag{10.6.5}$$

such that N should be weakly log. convex, non-quasi-analytic and $\lim_{k\to\infty}(n_k)^{1/k} = +\infty$ with $n_k := \frac{N_k}{k!}$. Alternatively one can assume

$$\exists\, N \in \mathbb{R}_{>0}^{\mathbb{N}}\ \forall\, M \in \mathcal{M} :\ N \lhd M, \tag{10.6.6}$$

such that N should be weakly log. convex, non-quasi-analytic and $\lim_{k\to\infty}(n_k)^{1/k} = +\infty$. Of course we have (10.6.6) $\Rightarrow$ (10.6.5), and, by 10.6.1, we obtain also (10.6.5) $\Rightarrow$ (10.6.6). So in fact both conditions are equivalent and they imply *Beurling-non-quasi-analyticity*, i.e. $(\mathcal{M}_{(\mathrm{nq})})$ for $\mathcal{M}$ and the space $\mathcal{E}_{(\mathcal{M})}$ defined in (7.2.2) contains functions with compact support.
To prove the special-curve-lemma for the Beurling-case it's sufficient to assume the following, by 10.6.1, weaker condition

$$\exists\, P, N \in \mathbb{R}_{>0}^{\mathbb{N}}\ \forall\, M \in \mathcal{M} :\ P \lhd N \preceq M, \tag{10.6.7}$$

and such that N^1 should be only normalized, weakly log. convex and non-quasi-analytic. This condition implies $(\mathcal{M}_{(\mathrm{nq})})$, too. If $\mathcal{M} = \{M^l : l \in \Lambda\}$, then in each condition above we replace "$\forall\, M \in \mathcal{M}$" by "$\forall\, l \in \Lambda$".

Finally we point out: (10.6.6) implies (10.6.3) and (10.6.7) implies (10.6.4), hence the Beurling-type-conditions imply the Roumieu-type-conditions.
With this preparation we are able to formulate and prove now the $\mathcal{E}_{\{\mathcal{M}\}}$- resp. $\mathcal{E}_{(\mathcal{M})}$-version of the special-curve-lemma.

Recall notation: A sequence $(x_n)_n$ in a locally convex vector space E is called *Mackey-convergent* to a point x, if there exists an increasing sequence of positive real numbers $(\lambda_n)_n$ with $\lim_{n\to\infty}\lambda_n = +\infty$ such that $(\lambda_n \cdot (x_n - x))_n$ is bounded. If the sequence $\lambda := (\lambda_n)_n$ is fixed, then $(x_n)_n$ is called λ-*converging*.

Proposition 10.6.2. *Let $\mathcal{M}$ be a an arbitrary set of weakly log. convex weight sequences and assume that additionally* (10.6.4) *is valid.*
Then there exist sequences of real numbers $(\lambda_k)_k$ with $\lambda_k \to 0$, $(t_k)_k$ with $t_k \to t_\infty$ and $(s_k)_k$ with $s_k > 0$, with the following property: For $\left(\frac{1}{\lambda}\right)$-converging sequences $(x_n)_n$ and $(v_n)_n$ in a convenient vector space E, there exists a strongly uniform $\mathcal{E}_{\{\mathcal{M}\}}$-curve $c : \mathbb{R} \to E$ with $c(t_k + t) = x_k + t \cdot v_k$ for $|t| \leq s_k$.
If we have the stronger condition (10.6.7) *then we obtain also a strongly uniform $\mathcal{E}_{(\mathcal{M})}$-curve.*

Remarks: More precisely in 10.6.2 one has

(*i*) in the Roumieu-case a strongly uniform $\mathcal{E}_{\{M\}}$-curve c, where $M \in \mathcal{M}$ is precisely the sequence from (10.6.4),

(*ii*) in the Beurling-case a strongly uniform $\mathcal{E}_{(N)}$-curve c, where N is precisely the sequence from (10.6.7).

Proof. The *Roumieu-case:* We have (10.6.4), so we can replace in the original proof of [21, 3.6. Lemma] the sequence $(k! \cdot \overline{M}_k)_k$ by the sequence $(N_k)_k$, the sequence $(k! \cdot M_k)_k$ by $(M_k)_k$ coming from (10.6.4) and obtain a uniform strong $\mathcal{E}_{\{M\}}$-curve c, which is then clearly $\mathcal{E}_{\{\mathcal{M}\}}$.

The *Beurling-case:* If we assume (10.6.7), then replace in the original proof of [21, 3.6. Lemma] the sequence $(k! \cdot \overline{M}_k)_k$ by the sequence $(P_k)_k$ and the sequence $(k! \cdot M_k)_k$ by $(N_k)_k$ coming from (10.6.7) and obtain a uniform strong $\mathcal{E}_{(N)}$-curve c, which is then clearly $\mathcal{E}_{(M)}$ for all $M \in \mathcal{M}$ and so finally $\mathcal{E}_{(\mathcal{M})}$.

We remark: In the proof of [21, 3.6. Lemma] it was used that the sequence $\overline{M}$ is increasing. But in fact we need only that the sequence $(k! \cdot \overline{M}_k)_k = (N_k)_k$ is increasing (which follows by weakly log. convexity and normalization), because we estimate in the proof by $k \cdot (k-1)! \cdot \overline{M}_{k-1} = k \cdot N_{k-1} \leq 2^k \cdot N_k$ for all $k \geq 1$ and so one has the estimate $c^{(k)}(t) \subseteq \overline{C} \cdot \frac{5}{2} \cdot T^{-k} \cdot (2\varrho)^k \cdot N_k \cdot B$ in the Roumieu and $c^{(k)}(t) \subseteq \underbrace{\overline{C} \cdot \frac{5}{2} \cdot \frac{1}{\varrho}}_{=C_\varrho} \cdot T^{-k} \cdot (2\varrho)^k \cdot N_k \cdot B$ in the Beurling-case (since we are interested in $\varrho \geq 1$ in the Roumieu and in $0 < \varrho < 1$ in the Beurling-case). □

Note that for the proof of 10.6.2 we have to use the analogous result to [21, 3.7. Lemma]:

Lemma 10.6.3. *Let $c : \mathbb{R}\backslash\{0\} \to E$ be a strongly $\mathcal{E}_{[\mathcal{M}]}$-curve, which should mean that c is smooth and for all bounded $K \subseteq \mathbb{R}\backslash\{0\}$ there exists $M \in \mathcal{M}$ and $h > 0$ (resp. for all $M \in \mathcal{M}$ and all $h > 0$) such that we have that*

$$\left\{ \frac{c^{(k)}(x)}{h^k \cdot M_k} : k \in \mathbb{N}, x \in K \right\}$$

is a bounded set in E. Then c has a unique extension to a strongly $\mathcal{E}_{[\mathcal{M}]}$-curve on $\mathbb{R}$.

Proof. The curve c has a unique extension to a smooth curve by [20, 2.9.], the strong $\mathcal{E}_{[\mathcal{M}]}$-condition extends by continuity. □

The next corollary is analogous to [21, 3.8. Corollary]:

Lemma 10.6.4. *Let $\mathcal{M}$ be a weight matrix with the same assumptions as in 10.6.2 (for the particular case). Furthermore let E be a locally convex vector space, then*

(1) *The final topology on E with respect to all strongly $\mathcal{E}_{[\mathcal{M}]}$-curves coincides with the Mackey-closure-topology.*

(2) *E is convenient, if and only if for any strongly $\mathcal{E}_{[\mathcal{M}]}$-curve $c : \mathbb{R} \to E$ there exists a strongly $\mathcal{E}_{[\mathcal{M}]}$-curve $c_1 : \mathbb{R} \to E$ with $c_1' = c$ (existence of anti-derivatives!).*

Proof. (1) This holds by the $\mathcal{E}_{[\mathcal{M}]}$-special-curve-lemma 10.6.2 which we can use by the assumptions on $\mathcal{M}$. More precisely 10.6.2 proves that the final topologies generated by all Mackey-converging sequences and by all $\mathcal{E}_{[\mathcal{M}]}$-curves coincide.
(2) We proceed as in (2) in [21, 3.8. Corollary]: To prove that E is convenient we show that E is c^∞-closed in its completion. Let $(x_n)_n$ be a Mackey-convergent sequence in the completion of E with $x_n \to x_\infty$ for $n \to \infty$. By 10.6.2 there exists a stronlgy uniform $\mathcal{E}_{[\mathcal{M}]}$-curve c in the completion which passes in finite time through a sub-sequence of $(x_n)_n$ with velocity 0. The proof in [21, 3.6. Lemma] shows that the derivatives $c^{(k)}(t)$ are multiples of the x_n, hence are elements of E. Then c' is a $\mathcal{E}_{[\mathcal{M}]}$-curve and the anti-derivative c of c' lies in E by assumption. So $x_\infty \in c(\mathbb{R}) \subseteq E$ and E is c^∞-closed.

Conversely, if E is convenient, then by [20, 2.14. Theorem] every smooth curve c has a smooth anti-derivative c_1 in E (so $c_1' = c$). We can estimate for each $M \in \mathcal{M}$ and $h > 0$ as follows:

$$\frac{c_1^{(k+1)}(t)}{h^{k+1} \cdot M_{k+1}} = \frac{M_k}{h \cdot M_{k+1}} \cdot \frac{c^{(k)}(t)}{h^k \cdot M_k} \leq \frac{1}{h \cdot M_1} \cdot \frac{c^{(k)}(t)}{h^k \cdot M_k},$$

which holds since by assumption each $M \in \mathcal{M}$ is (weakly) log. convex and so $M_j \cdot M_k \leq M_{j+k}$ for all $j, k \in \mathbb{N}$. So if c is (strongly) $\mathcal{E}_{[\mathcal{M}]}$, then c_1, too. □

We compare now the introduced conditions for an abstract weight matrix $\mathcal{M}$ with known facts for the special case where the weight matrix $\mathcal{M} := \{M^l : l > 0\}$ is obtained by a weight function $\omega \in \mathcal{W}$ by $M_j^l := \exp(\frac{1}{l} \cdot \varphi_\omega^*(lj))$.

First recall [5, 1.6. Lemma]: Let $\omega \in \mathcal{W}$ be given with properties (ω_1) and (ω_{nq}), then there exists another weight function σ with the same properties as ω and moreover $\omega = o(\sigma(t))$ for $t \to \infty$ (i.e. $\sigma \lhd \omega$) and so $\mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{(\omega)}$. Note: Condition (ω_{nq}) for ω is not necessary to construct σ, but if we have this additional condition we can transfer it to σ, too.
For such weight functions σ, ω we denote the associated weight sequences by $N_j^l := \exp(\frac{1}{l} \cdot \varphi_\sigma^*(lj))$ resp. $M_j^l := \exp(\frac{1}{l} \cdot \varphi_\sigma^*(lj))$ and get the associated weight matrices $\mathcal{N} := \{N^l : l > 0\}$ resp. $\mathcal{M} := \{M^l : l > 0\}$. By 5.3.1 we see that $N^{l_1} \lhd M^{l_2}$ holds for each $l_1, l_2 > 0$, this means exactly $\mathcal{N} \lhd \mathcal{M}$.
By the first part of (5.1.3) both matrices $\mathcal{M}$ and $\mathcal{N}$ satisfy $(\mathcal{M}_{\{\text{L}\}})$ and $(\mathcal{M}_{(\text{L})})$, too. By 5.1.1 and 5.1.3 we obtain that all occurring sequences M^l and N^l are weakly log. convex and not-quasi-analytic (nq).
We summarize: For a given weight function $\omega \in \mathcal{W}$ with properties (ω_1) and (ω_{nq}) we consider the associated weight matrix $\mathcal{M}$ and can construct explicitly a new weight σ with associated weight matrix $\mathcal{N}$. For these two weight matrices we obtain condition (10.6.4), in fact here we have not only a single sequence N but a whole new weight matrix $\mathcal{N} := \{N^l : l > 0\}$, consisting only of sequences N^l with the required relation of

(10.6.4). To obtain the stronger condition (10.6.7) for the Beurling-case we can repeat this procedure, i.e. switch to a function τ with $\sigma = o(\tau(t))$ for $t \to \infty$.

Note: In the third chapter in [5] it was shown that for weight functions $\omega \in \mathcal{W}$ with properties (ω_1) and (ω_{nq}) both classes $\mathcal{E}_{\{\omega\}}$ and $\mathcal{E}_{(\omega)}$ contain functions with compact support, i.e. they are not quasi-analytic.

Moreover we recall 5.4.1 and the consequences: If $\omega \in \mathcal{W}$ is given with (ω_1), (ω_{nq}) and finally (ω_7), then by applying 5.4.1 we get: For all $l > 0$ we can find $m_l > 0$ (with $m_l > l$) such that $M^l \lhd M^{m_l}$ and so $\lim_{k\to\infty} \left(\frac{M_k^{m_l}}{M_k^l}\right)^{1/k} = +\infty$. In this case condition (10.6.4) is satisfied for the sequence $N = M^l$ (each sequence M^l is not-quasi-analytic and weakly log. convex) and we can take then l_0 with some $l_0 \geq m_l$. Hence we see: The desired new sequence N in (10.6.4) still belongs to the original given weight matrix $\mathcal{M}$ which is associated to ω.

More general we see that these arguments stay valid in the following situation: Condition (10.6.4) always holds for a sequence $N \in \mathcal{M}$ whenever

- (i) $\mathcal{M}$ is an arbitrary set of weakly log. convex weight sequences (which includes the case where $\mathcal{M} = \{M^l : l \in \Lambda\}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix),
- (ii) $\mathcal{M}$ satisfies
$$(\mathcal{M}_{\{\text{BR}\}}) : \forall\, N \in \mathcal{M}\ \exists\, M \in \mathcal{M} :\ N \lhd M$$
- (iii) There exists at least one sequence $M \in \mathcal{M}$ which is not quasi-analytic, i.e. (nq).

10.7 Testing along strong $\mathcal{E}_{\{\mathcal{M}\}}$-curves and consequences

The next result is analogous to [21, 3.9. Theorem]:

Theorem 10.7.1. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with index set $\Lambda = \mathbb{N}_{>0}$, property $(\mathcal{M}_{\{\text{FdB}\}})$ and finally (10.6.4), i.e. the assumption in 10.6.2 for the Roumieu-case hence Roumieu-non-quasi-analyticity $(\mathcal{M}_{\{\text{nq}\}})$. Let $U \subseteq E$ be a c^∞-open subset in a convenient vector space E and let F be a Banach-space. For a mapping $f : U \to F$ the following statements are equivalent:*

1. *f is $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$.*
2. *f is $\mathcal{E}_{\{\mathcal{M}\}}$ along strongly $\mathcal{E}_{\{\mathcal{M}\}}$-curves.*
3. *f is smooth and for each closed bounded absolutely convex subset B in E and each $x \in U \cap E_B$ there exist $r, h, C > 0$ and $l \in \Lambda$ such that*
$$\left\|\frac{d^k(f \circ i_B)(a)}{h^k \cdot M_k^l}\right\|_{L^k(E_B,F)} \leq C$$
for all $a \in U \cap E_B$ with $\|a - x\|_B \leq r$ and all $k \in \mathbb{N}$.
4. *f is smooth and for each closed bounded absolutely convex subset B in E and each compact $K \subseteq U \cap E_B$ there exist $h, C > 0$ and $l \in \Lambda$ such that*
$$\left\|\frac{d^k(f \circ i_B)(a)}{h^k \cdot M_k^l}\right\|_{L^k(E_B,F)} \leq C$$

for all $a \in K$ *and all* $k \in \mathbb{N}$.

Some important remarks concerning this result:

(i) In fact the proof of 10.6.2 shows that in condition (2) we can restrict ourselves to all strongly $\mathcal{E}_{\{M^{l_0}\}}$-curves, where $l_0 \in \Lambda$ is precisely the index which arises in (10.6.4). If we assume the stronger condition (10.6.7), then we can restrict to all strong $\mathcal{E}_{\{N\}}$-curves, where N is precisely the sequence from (10.6.7). This remark should be also compared with [23, 2.7. Theorem].

(ii) The assumptions of the special-curve-lemma 10.6.2 are only needed for (2) $\Rightarrow$ (3), condition $(\mathcal{M}_{\{\mathrm{FdB}\}})$ is only needed for (4) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) is clear.
For (2) $\Rightarrow$ (3) we proceed as in the original proof (see [21, 3.9. Theorem]) via contradiction. W.l.o.g. let $E = E_B$ be a Banach-space. For each $v \in E$ and $x \in U$ there exists the iterated directional derivative $d_v^k f(x)$ since f is $\mathcal{E}_{\{\mathcal{M}\}}$ along affine lines. To show the smoothness of f it is enough to show that $d_{v_n}^k f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey convergent sequences $(x_n)_n$ and $(v_n)_n$ with $v_n \to 0$ by [20, 5.20].
For contradiction assume that there exist $k \in \mathbb{N}$ and sequences $(x_n)_n$ and $(v_n)_n$ with $\|d_{v_n}^k f(x_n)\| \to \infty$. We pass to a subsequence and assume that $(x_n)_n$ and $(v_n)_n$ are $\left(\frac{1}{\lambda}\right)$-converging for the sequence $(\lambda_n)_n$, which comes from the $\mathcal{E}_{\{\mathcal{M}\}}$-special-curve-lemma. Then there exists a strong $\mathcal{E}_{\{\mathcal{M}\}}$-curve $c : \mathbb{R} \to E$ with $c(t+t_n) = x_n + t \cdot v_n$ for t near 0 and for each $n \in \mathbb{N}$ separately and for $(t_n)_n$ coming from the $\mathcal{E}_{\{\mathcal{M}\}}$-special-curve-lemma 10.6.2. But then we get $\|(f \circ c)^{(k)}(t_n)\| = \|d_{v_n}^k f(x_n)\| \to \infty$, which is a contradiction! Consequently f is smooth.

Assume that there exists $x \in U$ such that for all $l \in \Lambda$ and $r, h, C > 0$ there exists $a = a(l,r,h,C) \in U$ and $k = k(l,r,h,C) \in \mathbb{N}$ with $\|a - x\| \le r$ but such that

$$\|d^k f(a)\|_{L^k(E,F)} > C \cdot h^k \cdot M_k^l.$$

Again, as in the original proof by [20, 7.13.], we have

$$\|d^k f(a)\|_{L^k(E,F)} \le (2e)^k \cdot \sup_{\|v\| \le 1} \|d_v^k f(a)\|.$$

Then for all $h > 0$, $n \in \mathbb{N}_{>0}$ and $l \in \Lambda$ we take $r = \frac{1}{nhl}$ and $C = n$. Then there exist $a_{l,n,h} \in U$ with $\|a_{l,n,h} - x\| \le \frac{1}{nhl} = r$ and furthermore $v_{l,n,h}$ with $\|v_{l,n,h}\| \le 1$ and $k_{l,n,h} \in \mathbb{N}$ such that now

$$\frac{(2e)^{k_{l,n,h}}}{h^{k_{l,n,h}} \cdot M_{k_{l,n,h}}^l} \cdot \|d_{v_{l,n,h}}^{k_{l,n,h}} f(a_{l,n,h})\| > n. \qquad (\star)$$

Put $K := \{a_{l,n,h} : l \in \Lambda, n, h \in \mathbb{N}\} \cup \{x\}$, then K is compact and $(\star)$ contradicts now the following claim similar to the proof in the weight sequence case [21, 3.9. Theorem]: For all $K \subseteq E$ compact there exist numbers $\tilde{C}, h > 0$ and an index $l \in \Lambda$ such that for all $k \in \mathbb{N}$ and $x \in K$ we get $\sup_{\|v\| \le 1} \|d_v^k f(x)\| \le \tilde{C} \cdot h^k \cdot M_k^l$.
Otherwise, there would exist a compact subset $K \subseteq E$ such that for all $n \in \mathbb{N}_{>0}$ and all $l \in \Lambda$ there are $k_{l,n} \in \mathbb{N}$, $x_{l,n} \in K$ and vectors $v_{l,n}$ with $\|v_{l,n}\| = 1$ such that we would get

$$\|d_{v_{l,n}}^{k_{l,n}} f(x_{l,n})\| \ge M_{k_{l,n}}^l \cdot \left(\frac{1}{\lambda_n^2}\right)^{k_{l,n}+1},$$

where we have put $\tilde{C} = h = \frac{1}{\lambda_n^2}$ and the sequence $(\lambda_n)_n$ comes from the $\mathcal{E}_{\{\mathcal{M}\}}$-special-curve-lemma 10.6.2! Note that by construction in 10.6.2 the sequence $(\lambda_j)_j$ doesn't depend on the index $l \in \Lambda$.

In the next step we identify $n \in \mathbb{N}_{>0}$ with $(n,n) \in \Lambda \times \mathbb{N}_{>0}$ and so we have (for $l = n \in \mathbb{N}_{>0}$ - the diagonal): There exists a strongly (uniform) $\mathcal{E}_{\{\mathcal{M}\}}$-curve $c : \mathbb{R} \to E$ with $c(t_n + t) = x_n + t \cdot \lambda_n \cdot v_n$ for t near 0 and by the chain-rule we get $(f \circ c)^{(k)}(t_n) = \lambda_n^k \cdot d_{v_n}^k f(x_n)$. Thus we have:

$$\left(\frac{\|(f \circ c)^{(k_n)}(t_n)\|}{M_{k_n}^n}\right)^{1/(k_n+1)} = \left(\lambda_n^{k_n} \cdot \frac{\|d_{v_n}^{k_n} f(x_n)\|}{M_{k_n}^n}\right)^{1/(k_n+1)} > \frac{1}{\lambda_n^{(k_n+2)/(k_n+1)}} \to +\infty,$$

for $n \to \infty$ which is a contradiction to $f \circ c \in \mathcal{E}_{\{\mathcal{M}\}}$ (since $n \in \Lambda$ is arbitrary!).

(3) $\Rightarrow$ (4): The compact set K can be covered by finitely many balls.

For step (4) $\Rightarrow$ (1) one has to use the *Faà-di-Bruno-formula* and 10.5.3 (in fact the "strong version" where we have used condition $(\mathcal{M}_{\{\mathrm{FdB}\}})$) and proceed similarly as in [21, 3.9. Theorem]. We have to show: $f \circ c$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for each $\mathcal{E}_{\{\mathcal{M}\}}$-curve $c : \mathbb{R} \to E$. By condition (3) in 10.5.3 it's sufficient to prove: For each compact interval $I \subseteq \mathbb{R}$, each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and each sequence $(s_k)_k \in \int_{\mathrm{roum,FdB}}^{\mathcal{M}}$, there exists a number $\varepsilon > 0$ such that the set

$$\left\{\frac{(f \circ c)^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot \varepsilon^k : a \in I, k \in \mathbb{N}\right\}$$

is bounded in E.

Fix now an arbitrary compact interval I and sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and $(s_k)_k \in \int_{\mathrm{roum,FdB}}^{\mathcal{M}}$ and replace in condition (2) in 10.5.3 the sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ by $r_k \cdot (2D)^k$, where $D > 0$ is the constant arising in the inequality $s_k \le D^k \cdot (\hat{s}_o)_k$ (since $s \in \int_{\mathrm{roum,FdB}}^{\mathcal{M}}$). Then, by assumption, for each $\alpha \in E^*$ the set $\left\{\frac{(\alpha \circ c)^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot (2D)^k : a \in I, k \in \mathbb{N}\right\}$ is bounded in $\mathbb{R}$ for all sequences $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and $(s_k)_k \in \int_{\mathrm{roum}}^{\mathcal{M}}$.
So the set $\left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot (2D)^k : a \in I, k \in \mathbb{N}\right\}$ is contained in some closed absolutely convex set $B \subseteq E$. Hence we have that $c^{(k)} : I \to E_B$ is smooth and so the set

$$K_k := \left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot (2D)^k : a \in I\right\}$$

is compact in E_B for each $k \in \mathbb{N}$ separately. So each sequence $(x_n)_n$ in the set

$$K := \left\{\frac{c^{(k)}(a)}{k!} \cdot r_k \cdot s_k : a \in I, k \in \mathbb{N}\right\} = \bigcup_{k \in \mathbb{N}} \frac{1}{(2D)^k} K_k$$

has a cluster point in $K \cup \{0\}$: either there is a subsequence in one K_k, or $(2D)^{k_n} \cdot x_{k_n} \in K_{k_n} \subseteq B$ for $k_n \to \infty$, thus $x_{k_n} \to 0$ in E_B and $K \cup \{0\}$ is compact in E_B.

We use now the *Faà-di-Bruno-formula*, and recall: Since $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$, we have $r_{j+k} \le r_j \cdot r_k$ for all $j, k \in \mathbb{N}$, hence by iteration we obtain $r_k \le r_{\alpha_1} \cdot r_{\alpha_2} \cdots r_{\alpha_j}$ for

$k=\alpha_1+\dots+\alpha_j$, $\alpha_i\ge 1$. Moreover we have that for each $s\in\int^{\mathcal{M}}_{\text{roum,FdB}}$ we can find a sequence $\hat{s}\in\int^{\mathcal{M}}_{\text{roum}}$ with $s\preceq\hat{s}_\circ$.
So we take $l\in\Lambda$ and $h\ge 1$ large enough (depending on the given function f) and estimate as follows for all $a\in I$ and $k\in\mathbb{N}$, $k\ge 1$:

$$\left\|\frac{(f\circ c)^{(k)}(a)}{k!}\cdot r_k\cdot s_k\right\|_F$$

$$\le\sum_{j\ge 0}\sum_{\alpha\in\mathbb{N}^j_{>0},\sum_{i=1}^j\alpha_i=k}D^k\cdot\frac{\left\|d^jf(c(a))\right\|_{L^j(E_B,F)}\hat{s}_j}{j!}\underbrace{\frac{\|c^{(\alpha_1)}(a)\|_B r_{\alpha_1}\hat{s}_{\alpha_1}}{\alpha_1!}\cdots\frac{\|c^{(\alpha_j)}\|_B r_{\alpha_j}\hat{s}_{\alpha_j}}{\alpha_j!}}_{\le(2D)^{-\alpha_1}\cdots(2D)^{-\alpha_j}=\frac{1}{(2D)^k}}$$

$$\le\left(\frac{1}{2}\right)^k\cdot\sum_{j\ge 0}\sum_{\alpha\in\mathbb{N}^j_{>0},\sum_{i=1}^j\alpha_i=k}\underbrace{\frac{\left\|d^jf(c(a))\right\|_{L^j(E_B,F)}}{j!\cdot m^l_j}}_{\le C\cdot h^j}\cdot\underbrace{\hat{s}_j\cdot m^l_j}_{\le C_1^j}$$

$$\le C\cdot\left(\frac{1}{2}\right)^k\cdot\sum_{j\ge 0}\sum_{\alpha\in\mathbb{N}^j_{>0},\sum_{i=1}^j\alpha_i=k}(h\cdot C_1)^j\le(C\cdot h\cdot C_1)\cdot\left(\frac{1}{2}\right)^k\cdot\sum_{j\ge 0}\binom{k-1}{j-1}(h\cdot C_1)^{j-1}$$

$$=(C\cdot h\cdot C_1)\cdot\left(\frac{1}{2}\right)^k\cdot(1+C_1\cdot h)^{k-1}.$$

Note that $\hat{s}\in\int^{\mathcal{M}}_{\text{roum}}$, hence $\hat{s}_j\cdot m^l_j\le C_1^j$ for a constant $C_1\ge 1$ and all $j\in\mathbb{N}$. So we have shown now property (3) in 10.5.3 for $\varepsilon=\frac{2}{(1+C_1\cdot h)}$ and are done. □

The next result is analogous to [21, 3.10. Corollary]:

Corollary 10.7.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two weight matrices both with the same assumptions as in 10.7.1. Furthermore let E,F be convenient vector spaces and $U\subseteq E$ a c^∞-open subset. Then we obtain:*

1. *If we have $\mathcal{M}\{\preceq\}\mathcal{N}$, then $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)\subseteq\mathcal{E}_{\{\mathcal{N}\},\text{curve}}(U,F)$.*
2. *If one assumes for the matrix $\mathcal{M}$ additionally condition $(\mathcal{M}_{\{C^\omega\}})$, then $\mathcal{E}_{\{\text{id}\}}(U,F)\subseteq\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)\subseteq\mathcal{E}(U,F)$, where $\mathcal{E}$ denotes the space of all smooth functions and $\mathcal{E}_{\{\text{id}\}}=\mathcal{C}^\omega$ denotes the space of all real analytic functions.*

Proof. The inclusions $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)\subseteq\mathcal{E}_{\{\mathcal{N}\},\text{curve}}(U,F)\subseteq\mathcal{E}$ hold by 10.7.1, more precisely condition (3) in this theorem is applied to the mappings $\alpha\circ f$ for all $\alpha\in F^*$. For the second part w.l.o.g. assume $F=\mathbb{R}$. If $f\in\mathcal{E}_{\{\text{id}\}}(U,F)$, then for each closed absolutely convex bounded $B\subseteq E$ the mapping $f\circ i_B:U\cap E_B\to\mathbb{R}$ is given by its locally convergent Taylor-series by [20, 10.1]. Hence condition (3) in 10.7.1 is satisfied for the constant matrix $\mathcal{I}=\{(k!)_{k\ge 0}\}$ and so also for all weight matrices $\mathcal{M}$ which satisfy additionally $(\mathcal{M}_{\{C^\omega\}})$. □

Corollary 10.7.3. *Let $\mathcal{M}$ be a weight matrix with the assumptions as in 10.7.1, then we obtain:*

1. *Multilinear mappings between convenient vector spaces are $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ if and only if they are bounded.*

(2) *If in addition condition* $(\mathcal{M}_{\{\mathrm{dc}\}})$ *holds and* $f : E \supseteq U \to F$ *is* $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$*, then the derivative* $df : U \to L(E,F)$ *is* $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$ *and also* $\hat{df} : U \times E \to F$ *is* $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$*, where the space* $L(E,F)$ *of all bounded linear mappings between* E *and* F *is equipped with the topology of uniform convergence on bounded sets.*

(3) *The chain-rule holds.*

Proof.
(1) If f is multilinear and $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$, then it is smooth by 10.7.1 and hence bounded by [20, 5.5. Lemma]. Conversely, if f is multilinear and bounded, then it is smooth by [20, 5.5. Lemma]. Furthermore $f \circ i_B$ is multilinear and continuous and all derivatives of high order vanish. So condition (3) in 10.7.1 is satisfied, hence f is $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$.
(2) f is smooth, so by [20, 5.5. Lemma] the map $df : U \to L(E,F)$ exists and it is smooth, too. Let $c : \mathbb{R} \to U$ be a (weakly) $\mathcal{E}_{\{\mathcal{M}\}}$-curve. We want to show now that $t \mapsto df(c(t)) \in L(E,F)$ is $\mathcal{E}_{\{\mathcal{M}\}}$. By [20, 5.18.] it is sufficient to show that $t \mapsto c(t) \mapsto \alpha(df(c(t)) \cdot v) \in \mathbb{R}$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for each $\alpha \in F^*$ and $v \in E$. We use now again 10.7.1 and show that $x \mapsto \alpha(df(x) \cdot v)$ satisfies the conditions of this theorem. More precisely we use condition (3) of 10.7.1 applied to $\alpha \circ f$ in the following way:
For all $B \subseteq E$ closed, bounded, absolutely convex subset of E, for all $x \in U \cap E_B$, there exist $r, h, C > 0$ and $l \in \Lambda = \mathbb{N}_{>0}$, such that for all $a \in U \cap E_B$ with $\|a - x\|_B \le r$ and all $k \in \mathbb{N}$ we obtain

$$\left\| d^k(\alpha \circ f \circ i_B)(a) \right\|_{L^k(E_B,\mathbb{R})} \le C \cdot h^k \cdot M_k^l.$$

For $v \in E$ and B as assumed above and B should contain the vector v, we calculate as follows for all $k \in \mathbb{N}$:

$$\begin{aligned}
&\left\| d^k(d(\alpha \circ f)(\cdot)(v) \circ i_B)(a) \right\|_{L^k(E_B,\mathbb{R})} = \left\| d^{k+1}(\alpha \circ f \circ i_B)(a)(v, \dots) \right\|_{L^k(E_B,\mathbb{R})} \\
&\le \left\| d^{k+1}(\alpha \circ f \circ i_B)(a) \right\|_{L^{k+1}(E_B,\mathbb{R})} \cdot \|v\|_{E_B} \le C \cdot h^{k+1} \cdot M_{k+1}^l \\
&\underbrace{\le}_{(\mathcal{M}_{\{\mathrm{dc}\}})} C \cdot h^{k+1} \cdot h_1^{k+1} \cdot M_k^n = C_1 \cdot h_2^k \cdot M_k^n.
\end{aligned}$$

(3) The chain-rule is valid for all smooth f. □

10.8 $\mathcal{E}_{\{\mathcal{M}\}}$-Uniform-boundedness-principle and consequences by using curves

We start with the analogous result to [21, 4.1. Theorem], more precisely it is the $\mathcal{E}_{[\mathcal{M}],\text{curve}}$-analogue result to the smooth UBP [20, 5.26. Theorem].

Theorem 10.8.1. *Let $\mathcal{M} := \{M^l : l \in \Lambda\}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, with a countable index set $\Lambda = \mathbb{N}_{>0}$. Let E, F, G convenient vector spaces, furthermore let $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, G)$ is bounded if and only if $\text{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*
If e.g. $\Lambda = \mathbb{R}_{>0}$, the same statement holds also for the Beurling-case $\mathcal{E}_{(\mathcal{M}),\text{curve}}$.

Proof. For $x \in U$ and $\alpha \in G^*$ the linear mapping $\alpha \circ \text{ev}_x = \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(x, \alpha) : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, G) \to \mathbb{R}$ is continuous and so ev_x is bounded. Hence we get: If T is bounded, then $\text{ev}_x \circ T$, too.
Conversely, we assume that $\text{ev}_x \circ T$ is bounded for each $x \in U$. For an arbitrary closed absolutely convex and bounded subset $B \subseteq E$ consider the Banach-space E_B and consider for each $\alpha \in G^*$, each $\mathcal{E}_{\{\mathcal{M}\}}$-curve $c : \mathbb{R} \to U$, $t \in \mathbb{R}$ and compact set $K \subseteq \mathbb{R}$ the analogous diagram as considered in [21, 4.1. Theorem]. Recall: For an arbitrary weight matrix $\mathcal{M}$, so $(\mathcal{M})$ holds, and $U \subseteq \mathbb{R}^n$ non-empty open we have the topological vector space description

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}) = \varprojlim_{K \subseteq U} \varinjlim_{l \in \Lambda} \varinjlim_{h>0} \mathcal{E}_{M^l,h}(K, \mathbb{R}) = \varprojlim_{K \subseteq U} \varinjlim_{n \in \mathbb{N}_{>0}} \mathcal{E}_{M^n,n}(K, \mathbb{R}),$$

see (7.3.5) for more details.
To transfer the original proof [21, 4.1. Theorem] to the $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$-case, we only need that $\varinjlim_{l \in \Lambda} \varinjlim_{h>0} \mathcal{E}_{M^l,h}(K, \mathbb{R})$ is webbed (see the diagram and the arguments in [21, 4.1. Theorem]). But this is satisfied, since $\varinjlim_{l \in \Lambda} \varinjlim_{h>0} \mathcal{E}_{M^l,h}(K, \mathbb{R}) = \varinjlim_{n \in \mathbb{N}_{>0}} \mathcal{E}_{M^n,n}(K, \mathbb{R})$ is a countable inductive limit of Banach-spaces (a (LB)-space) and all (LF)-spaces are webbed (see also 9.1.1). Thus the UBP is valid for the $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$-case, since we can use the closed graph theorem (see e.g. [20, 52.10.]) to get that the mapping $E_B \longrightarrow \varinjlim_{n \in \mathbb{N}_{>0}} \mathcal{E}_{M^n,n}(K, \mathbb{R})$ is continuous. This implies the boundedness of T.

The same proof still works for the Beurling-case $\mathcal{E}_{(\mathcal{M}),\text{curve}}$: Since the index set can be chosen to be countable ($l = \frac{1}{n}, n \in \mathbb{N}_{>0}$), we obtain a Fréchet-space (see (7.3.6) for more details), which is always webbed. □

In the rest of this section we will prove several important consequences from 10.8.1 and we will concentrate on the Roumieu-case. The next result is analogous to [21, 4.2. Corollary]:

Corollary 10.8.2. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1. Then the following properties are satisfied:*

(1) *For convenient vector spaces E and F, on $L(E, F)$ the following bornologies coincide which are induced by:*

  (a) *The topology of uniform convergence on bounded sets of E.*

  (b) *The topology of pointwise convergence.*

(c) *The embedding* $L(E,F) \subseteq \mathcal{E}(E,F)$.

(d) *The embedding* $L(E,F) \subseteq \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E,F)$.

(2) *Let* E, F, G *be convenient vector spaces and* $U \subseteq E$ *be a* c^∞*-open subset. A mapping* $f : U \times F \to G$*, which is linear in the second variable is* $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ *if and only if* $f^\vee : U \to L(E,G)$ *is well-defined and* $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$.

Proof. (1) The first three topologies on $L(E,F)$ have the same bounded sets as it was shown in [20, 5.3., 5.18.]. The inclusion $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E,F) \to \mathcal{E}(E,F)$ is bounded by the consequence 10.7.2 in the previous section and [20, 5.26.] (smooth UBP).
For the last point (d) we have to show now that $L(E,F) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E,F)$ is bounded, where $L(E,F)$ is endowed with the topology of uniform convergence on bounded sets; and this holds by the UBP 10.8.1.

(2) In the $\mathcal{E}$-case this holds by [21, (7.3.6.)]. If f is $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$, then consider a curve $c : \mathbb{R} \to U$ which is assumed to be (weakly) $\mathcal{E}_{\{\mathcal{M}\}}$. We prove, that the mapping $t \mapsto f^\vee(c(t))$ is $\mathcal{E}_{\{\mathcal{M}\}}$. By [20, 5.18.] and by 10.5.4 (note that $(\mathcal{M}_{\{\text{FdB}\}})$ holds by assumption) we have to show that $t \mapsto \alpha\left(f^\vee(c(t))(v)\right) = \alpha\left(f(c(t),v)\right)$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for each $\alpha \in G^*$ and $v \in F$. But this is clearly satisfied, because of the definition for f to be $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$.

Conversely, assume that $f^\vee : U \to L(F,G)$ is $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$. We have to show that $f : U \times F \to G$ is $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ and to do so we prove property (3) in 10.7.1 for f.
By composing with some $\alpha \in G^*$ we can assume w.l.o.g. $G = \mathbb{R}$. We use the same formula as in [21, 4.2. Corollary] and calculate for all $k \in \mathbb{N}$ as follows:

$$\left\|d^k f(x,w_0)\right\|_{L^k(E_B \times F_{B'},\mathbb{R})} \le \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^{k} \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}$$

$$\le \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'}\mathbb{R}))} \cdot \|w_0\|_{B'} + \sum_{i=1}^{k} \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}$$

$$\le C \cdot h^k \cdot M_k^l \cdot \|w_0\|_{B'} + \sum_{i=1}^{k} C \cdot h^{k-1} \cdot M_{k-1}^l = C \cdot h^k \cdot M_k^l \cdot \|w_0\|_{B'} + k \cdot C \cdot h^{k-1} \cdot M_{k-1}^l$$

$$\le C \cdot h^k \cdot M_k^l \cdot \|w_0\|_{B'} + C \cdot h_1^k \cdot h^k \cdot M_k^l \le C \cdot (\|w_0\|_{B'} + 1) \cdot h_2^k \cdot M_k^l.$$

First we have used (3) in 10.7.1 for $L(i_{B'},\mathbb{R}) \circ f^\vee : U \to L(F_{B'},\mathbb{R})$. Then we have put $h_2 := h_1 \cdot h$ for $h, h_1 \ge 1$ and $l \in \Lambda$ large enough and note that each M^l is weakly log. convex and normalized, hence automatically increasing. □

The next result, which is analogous to [21, 4.7. Lemma], doesn't need condition $(\mathcal{M}_{\{\text{FdB}\}})$ necessarily, hence we can use the "strong version" 10.5.6 resp. 10.5.7.

Lemma 10.8.3. *Let* $\mathcal{M} = \{M^l : l \in \Lambda\}$ *be an arbitrary matrix, i.e.* $(\mathcal{M})$*, resp. a* $(\mathcal{M}_{\text{sc}})$ *weight matrix, with* $\Lambda = \mathbb{N}_{>0}$ *and let* E *be a convenient vector space. The flip of variables induces the following vector space isomorphism:* $L(E, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})) \cong \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E')$.

Proof. Let $c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},E')$, consider for each $x \in E$ the curve $\tilde{c}(x) := \text{ev}_x \circ c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$. Note that $\tilde{c}$ is a linear and by UBP 10.8.1 we obtain that $\tilde{c}$ is bounded, because $\text{ev}_t \circ \tilde{c} = \tilde{c}(t) = \text{ev}_t \circ c = c(t) \in E'$.

Conversely let $\alpha \in L(E, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}))$, then consider for each $t \in \mathbb{R}$ the associated linear functional $\tilde{\alpha}(t) := \mathrm{ev}_t \circ \alpha \in E' = L(E,\mathbb{R})$. The bornology of E' is generated by $\mathcal{B} := \{\mathrm{ev}_x : x \in E\}$ and moreover $\mathrm{ev}_x \circ \tilde{\alpha} = \tilde{\alpha}(x) = \mathrm{ev}_x \circ \alpha = \alpha(x) \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$. Hence $\tilde{\alpha} : \mathbb{R} \to E'$ is $\mathcal{E}_{\{\mathcal{M}\}}$ by 10.5.7. □

The next result is analogous to [21, 4.8. Lemma]:

Lemma 10.8.4. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1. We denote in the following with $\lambda_{\{\mathcal{M}\}}(\mathbb{R})$ the c^∞-closure of the sub-space generated by $\{\mathrm{ev}_t : t \in \mathbb{R}\}$ in $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})'$. Moreover consider the mapping $\delta : \mathbb{R} \to \lambda_{\{\mathcal{M}\}}(\mathbb{R})$ defined by $t \mapsto \mathrm{ev}_t$. Then the following is satisfied:*
For every convenient vector space G the $\mathcal{E}_{\{\mathcal{M}\}}$-curve δ induces a bornological isomorphism

$$L(\lambda_{\{\mathcal{M}\}}(\mathbb{R}), G) \cong \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, G).$$

Proof. $\lambda_{\{\mathcal{M}\}}(\mathbb{R})$ is a convenient vector space, because it is a closed subspace in $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})'$, which is convenient by 10.3.1. Moreover the mapping δ introduced above is $\mathcal{E}_{\{\mathcal{M}\}}$: This holds because for each $x \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ we have by definition $\mathrm{ev}_x \circ \delta = x$, and we can use now the more general version 10.5.7 to conclude that δ is $\mathcal{E}_{\{\mathcal{M}\}}$. This implies that the mapping $\delta^* : L(\lambda_{\{\mathcal{M}\}}(\mathbb{R}), G) \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, G)$ is well-defined and linear.

δ^* is injective: Each bounded linear mapping $\lambda_{\{\mathcal{M}\}}(\mathbb{R}) \to G$ is uniquely determined on $\delta(\mathbb{R}) := \{\mathrm{ev}_t : t \in \mathbb{R}\}$.
δ^* is surjective: Consider some $f \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, G)$, then, by definition, $\alpha \circ f \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ for each $\alpha \in G^*$. Introduce now $\tilde{f} : \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})' \to \prod_{G^*}\mathbb{R}$ by $\varphi \mapsto (\varphi(\alpha \circ f))_{\alpha \in G^*}$, and this mapping is well-defined, bounded and linear.
ev_t is mapped to $\tilde{f}(\mathrm{ev}_t) = \delta(f(t))$, where $\delta : G \to \prod_{G^*}\mathbb{R}$ denotes the bornological embedding given by $x \mapsto (\alpha \circ x)_{\alpha \in G^*}$. This map induces a bounded linear mapping $\tilde{f} : \lambda_{\{\mathcal{M}\}}(\mathbb{R}) \to G$ with finally $\tilde{f} \circ \delta = f$, which shows the surjectivity of δ.
But moreover δ^* is a bornological isomorphism: Both mappings δ^* and $(\delta^*)^{-1}$ are bounded, which holds by the UBP 10.8.1 and 10.8.2. □

By using these results we can prove the analogous result to [21, 4.9. Corollary]:

Corollary 10.8.5. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1. Then we obtain the following vector-space-isomorphisms:*

(1) $\mathcal{E}(\mathbb{R}, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})) \cong \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathcal{E}(\mathbb{R},\mathbb{R}))$

(2) $\mathcal{C}^\omega(\mathbb{R}, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})) \cong \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathcal{C}^\omega(\mathbb{R},\mathbb{R}))$

(3) $\mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R}, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})) \cong \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathcal{E}_{\{\mathcal{N}\}}(\mathbb{R},\mathbb{R}))$.

Proof. Let $\mathcal{X} \in \{\mathcal{E}, \mathcal{C}^\omega, \mathcal{E}_{\{\mathcal{N}\}}\}$, then we get:

$$\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathcal{X}(\mathbb{R},\mathbb{R})) \cong L(\lambda_{\{\mathcal{M}\}}(\mathbb{R}), \mathcal{X}(\mathbb{R},\mathbb{R})) \cong \mathcal{X}(\mathbb{R}, L(\lambda_{\{\mathcal{M}\}}(\mathbb{R}),\mathbb{R})) \cong \mathcal{X}(\mathbb{R}, \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}))$$

The first isomorphism holds by 10.8.4 for $G = \mathcal{X}(\mathbb{R},\mathbb{R})$, the third one again for $G = \mathbb{R}$. For the second isomorphism we have used 10.8.3 for $E = \lambda_{\{\mathcal{M}\}}(\mathbb{R})$ resp. [20, 3.13.4, 5.3., 11.15]. □

We close this section with the analogous result to [21, 4.10. Theorem], where for the proof we will have to use the previous results:

Theorem 10.8.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be given and both matrices as considered in 10.7.1. Moreover let E, F be convenient vector spaces and $U \subseteq E$ resp. $V \subseteq F$ be c^∞-open subsets. Then we obtain the following bornological isomorphisms:*

(1) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{E}_{\{\mathcal{N}\},\text{curve}}(V, F)) \cong \mathcal{E}_{\{\mathcal{N}\},\text{curve}}(V, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F))$

(2) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{E}(V, F)) \cong \mathcal{E}(V, \mathcal{E}_{\{\mathcal{N}\},\text{curve}}(U, F))$

(3) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{C}^\omega(V, F)) \cong \mathcal{C}^\omega(V, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F))$

(4) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, L(V, F)) \cong L(V, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F))$

(5) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, l^\infty(X, F)) \cong l^\infty(X, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F))$

(6) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{L}\text{ip}^k(X, F)) \cong \mathcal{L}\text{ip}^k(X, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F))$

X denotes in (5) resp. (6) a so-called l^∞-space, i.e. a set together with a bornology which is induced by a family of real valued functions on X (see [12, 1.2.4]), resp. a $\mathcal{L}ip^k$-space (see [12, 1.4.1]). For the definition of the spaces $l^\infty(X, F)$ resp. $\mathcal{L}ip^k(X, F)$ we refer to [12, 3.6.1., 4.4.1], for $l^\infty(X, F)$ see also [20, 2.15. Theorem].

Proof. All mappings and their inverse mappings are given by $f \mapsto \tilde{f}$, where $\tilde{f}(x)(y) := f(y)(x)$. All occurring spaces are convenient and satisfy the $\mathcal{S}$-UBP, where $\mathcal{S}$ denotes the set of all point-evaluations (see definitions, 10.8.1 resp. [20, 2.14.3, 5.18, 5.24, 5.25, 5.26, 11.11, 11.12] and [12]).
$\tilde{f}$ takes values in the corresponding spaces, because $\tilde{f}(x) = \text{ev}_x \circ f$. It remains to show that $\tilde{f}$ itself is of the corresponding class. First note that $f \mapsto \tilde{f}$ is bounded, because we have

$$(\text{ev}_x \circ (\tilde{\ }))(f) = \text{ev}_x(\tilde{f}) = \tilde{f}(x) = \text{ev}_x \circ f = (\text{ev}_x)_*(f)$$

and now apply the UBP.

(1), (2) That $\tilde{f}$ is of the appropriate class holds because we compose with appropriate curves $c_1 : \mathbb{R} \to U$, $c_2 : \mathbb{R} \to V$ and functionals $\alpha \in F^*$ and then we are precisely in the special (real-valued) case (3) in 10.8.5.

(3) We write again $\mathcal{X} \in \{\mathcal{E}, \mathcal{C}^\omega\}$ and proceed as follows: To show that $\tilde{f}$ is in the appropriate class, we have to compose with $c_1 \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, U)$ and $\mathcal{X}(c_2, \alpha) : \mathcal{C}^\omega(V, F) \to \mathcal{X}(\mathbb{R}, \mathbb{R})$ for all $\alpha \in F^*$ and $c_2 \in \mathcal{X}(\mathbb{R}, V)$.
So by (1) and (2) in 10.8.5 the mapping

$$\mathcal{X}(c_2, \alpha) \circ \tilde{f} \circ c_1 = (\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(c_1, \alpha) \circ f \circ c_2)\tilde{\ } : \mathbb{R} \to U \to \mathcal{C}^\omega(V, F) \to \mathcal{X}(\mathbb{R}, \mathbb{R})$$

is $\mathcal{E}_{\{\mathcal{M}\}}$, because the mapping $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(c_1, \alpha) \circ f \circ c_2 : \mathbb{R} \to V \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F) \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{R})$ is $\mathcal{X}$.

(4) In this case $\tilde{f}$ is of the appropriate class since $L(U, F)$ is a c^∞-closed subspace of $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F)$ which consists of all linear $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$-mappings.

(5), (6) The arguments for the proofs of these items are completely the same as given in [21, 4.10. Theorem (5), (6)]. □

10.9 Some projective descriptions for $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$

The next result is analogous to [21, 4.3. Proposition]:

Proposition 10.9.1. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1, let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. Then we obtain the bornological identity*

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F) = \varprojlim_s \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},F),$$

where s runs through all strongly $\mathcal{E}_{\{\mathcal{M}\}}$-curves $s : \mathbb{R} \to U$ and the connecting mappings are given by g^* for all reparametrizations $g \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ of curves s.

Proof. By 10.7.1 we see that $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$, $\varprojlim_s \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},F)$ and $\varprojlim_c \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},F)$ coincide as linear spaces, where c in the third limit runs through all weakly $\mathcal{E}_{\{\mathcal{M}\}}$-curves $c : \mathbb{R} \to U$.
Each $(f_c)_c \in \varprojlim_c \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},F)$ determines a unique function $f : U \to F$ as follows: We put $f(x) := (f \circ \text{const}_x)(0)$ with $f \circ c = f_c$ for all $c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U)$. Finally, again by 10.7.1, we have that $f \in \mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ if and only if $f_c \in \mathcal{E}_{\{\mathcal{M}\}}$ for all $c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},U)$.
By definition $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},F)$ carries the the initial structure w.r.t. α_* for all $\alpha \in F^*$, hence we can assume w.l.o.g. $F = \mathbb{R}$. The identity $\varprojlim_c \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}) \to \varprojlim_s \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is clearly continuous, the identity in the converse direction is bounded by the UBP 10.8.1, because $\varprojlim_s \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is convenient as projective limit of convenient spaces. □

The following definitions are analogous to [21, 4.5. Definition]:
Let E and F be two real Banach-spaces and $A \subseteq E$ a convex subset. We denote by $\mathcal{E}(A,F)$ the vector space of all sequences $(f^k)_k \in \prod_{k\in\mathbb{N}} \mathcal{C}(A, L^k(E,F))$ such that

$$f^k(y)(v) - f^k(x)(v) = \int_0^1 f^{k+1}(x+t(y-x))(y-x,v)dt$$

is satisfied for all $k \in \mathbb{N}$, $x,y \in A$ and $v \in E^k$.
Let $r = (r_k)_k$ and $s = (s_k)_k$ be arbitrary sequences of positive real numbers. We introduce now the normed spaces

$$\mathcal{E}_{(s),(r)}(A,F) := \left\{ f = (f^k)_k \in \mathcal{E}(A,F) : \|f\|_{(s),(r)} < +\infty \right\},$$

where we have put

$$\|f\|_{(s),(r)} := \sup \left\{ \frac{\|f^k(a)(v_1,\dots,v_k)\|}{s_k \cdot r_k \cdot \|v_1\| \cdots \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E \right\}.$$

The space $\mathcal{E}_{(s),(r)}(A,F)$ is a Banach-space, because it is closed in $l^\infty(\mathbb{N}, l^\infty(A, L^k(E,F)))$ by the mapping $(f^k)_k \mapsto (k \mapsto \frac{1}{s_k \cdot r_k} \cdot f^k)$.

The next result is analogous to [21, 4.6. Theorem]:

Theorem 10.9.2. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1, let E and F be real Banach-spaces and $U \subseteq E$ open. Then we obtain the following:*
The space $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ is bornologically isomorphic to the following constructions, so we obtain the same vector space and the same bounded sets:

(1) $\varprojlim_{K} \varinjlim_{l,h,W} \mathcal{E}_{\mathcal{M},l,h}(W,F)$

(2) $\varprojlim_{K} \varinjlim_{l,h} \mathcal{E}_{\mathcal{M},l,h}(K,F)$

(3) $\varprojlim_{K} \varprojlim_{s,r} \mathcal{E}_{(s),(r)}(K,F)$

(4) $\varprojlim_{c,I} \varinjlim_{l,h} \mathcal{E}_{\mathcal{M},l,h}(I,F)$

All involved inductive limits are regular, hence bounded sets of these limits are contained and bounded in some step. In the limits

(i) *K runs through all compact convex subsets $K \subseteq U$, ordered by inclusion,*

(ii) *W runs through all open subsets with $K \subseteq W \subseteq U$ ordered by inclusion,*

(iii) $h \in \mathbb{N}_{>0}, l \in \Lambda(= \mathbb{N}_{>0})$,

(iv) *r runs through all sequences of the set*

$$(\mathcal{R}_{\text{roum}})^{-1} := \left\{ r = (r_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{r} \in \mathcal{R}_{\text{roum}} \right\},$$

(v) *s runs through all sequences of the set*

$$\left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1} := \left\{ s = (s_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{s} \in \mathcal{S}^{\mathcal{M}}_{\text{roum}} \right\},$$

(vi) *$c \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, U)$ is ordered by reparametrization with $g \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{R})$ and*

(vii) *I runs through all compact intervals in $\mathbb{R}$.*

Proof. First note that in (1) – (4) we obtain smooth functions $f : U \to F$: In (1) – (3) they are given by $x \mapsto f^0(x)$ for appropriately chosen compact set K with $x \in K$, where $f^0 : K \to F$. In (4) they are given by $x \mapsto f_c(t)$ for the curve $c : \mathbb{R} \to U$ with $c = \text{const}_x$ on I and $f_c : I \to F$.
Smoothness of f follows by testing with $\mathcal{E}_{\{\mathcal{M}\}}$-curves, because they factor locally into some compact set K.
By 10.7.1 all representations (1) – (4) describe $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ as vector space.
Obviously the identity is continuous form (1) to (2) and from (2) to (3). The space in (3) is the projective limit of Banach-spaces and so convenient, the inductive limit in (1) is a (LB)-space, hence webbed. So we can apply the so called $\mathcal{S}$-UBP [20, 5.24. Theorem], where $\mathcal{S} = \{\text{ev}_x : x \in U\}$, to conclude that the identity from (3) to (1) is continuous, too.
Hence the representations (1) – (3) yield the same complete bornology on the space $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ and satisfy the $\mathcal{S}$-UBP.
In the next step we show that the inductive limits in (1) and (2) both are regular. Take a bounded set $\mathcal{B}$ in such a limit, then $\mathcal{B}$ is also bounded in (3). More precisely we have: For every compact subset $K \subseteq U$ and sequences $r \in (\mathcal{R}_{\text{roum}})^{-1}$ and $s \in \left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1}$ and for all $h, l \in \mathbb{N}_{>0}$ we have

$$\sup \left\{ \frac{\|f^k(a)(v_1, \ldots, v_k)\|}{s_k \cdot r_k \cdot \|v_1\| \cdots \|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E, f \in \mathcal{B} \right\} < +\infty.$$

So the sequence $b_k := \sup\left\{\frac{\|f^k(a)(v_1,\dots,v_k)\|}{\|v_1\|\cdots\|v_k\|} : k \in \mathbb{N}, a \in A, v_i \in E, f \in \mathcal{B}\right\}$ satisfies $\sup_{k\in\mathbb{N}} \frac{b_k}{r_k \cdot s_k} < +\infty$ for each sequence $r \in (\mathcal{R}_{\text{roum}})^{-1}$ and $s \in \left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1}$, hence we can use the same technique as in (2) $\Rightarrow$ (3) $\Rightarrow$ (1) in 10.5.6 to conclude: There exists $h_0 > 0$ and $l_0 \in \Lambda$ such that $\sup_{k\in\mathbb{N}} \frac{b_k}{M_k^{l_0} \cdot h_0^k} < +\infty$ which implies the fact that the set $\mathcal{B}$ is contained and bounded in the step $\mathcal{E}_{\mathcal{M},l_0,h_0}(K,F)$.
Also the structure in (4) describes the same bornology: The inductive limit in (4) is regular by the same proof as above for the special case $E = \mathbb{R}$ (see (2)(a) in 9.1.1), hence the structure in (4) is convenient and we can apply again the $\mathcal{S}$-UBP. $\square$

10.10 Cartesian closedness for non-quasi-analytic classes $\mathcal{E}_{\{\mathcal{M}\},\mathbf{curve}}$

In this section we are going to prove the central theorem for the convenient setting: By assuming some special properties for the matrix $\mathcal{M} = \{M^l : l \in \Lambda\}$ we will show that the structure $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ is *Cartesian closed*, so we generalize [21, 5.3. Theorem] to the non-constant matrix case. An analogous result is also available for the Beurling-case $\mathcal{E}_{(\mathcal{M}),\text{curve}}$. To do so we will use the notation introduced in [21, 5.1.], for convenience of the reader we will repeat the definitions:
For a subset $K \subseteq \mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, a linear space E and a mapping $f : K \to E$ we put

$$\mathbb{R}^{\langle k \rangle} := \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_i \neq x_j \text{ for } i \neq j\}$$

$$K^\alpha := \{(x^1, \dots, x^n) \in \mathbb{R}^{\alpha_1+1} \times \dots \times \mathbb{R}^{\alpha_n+1} : (x^1_{i_1}, \dots x^n_{i_n}) \in K \text{ for } 0 \leq i_j \leq \alpha_j\}$$

$$K^{\langle \alpha \rangle} := K^\alpha \cap (\mathbb{R}^{\langle \alpha_1 \rangle} \times \mathbb{R}^{\langle \alpha_n \rangle})$$

$$\beta_i(x) = k! \cdot \prod_{0 \leq j \leq k, j \neq i} \frac{1}{x_i - x_j},$$

where $x = (x_0, \dots, x_k) \in \mathbb{R}^{\langle k \rangle}$. Finally we put

$$\delta^\alpha f(x^1, \dots, x^n) = \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_n=0}^{\alpha_n} \beta_{i_1}(x^1) \dots \beta_{i_n}(x^n) f(x^1_{i_1}, \dots, x^n_{i_n})$$

and we have $\delta^0 f = f$ and $\delta^\alpha = \delta_n^{\alpha_n} \circ \dots \circ \delta_1^{\alpha_1}$ with

$$\delta_i^k g(x^1, \dots, x^n) = \delta^k(g(x^1, \dots x^{i-1}, \ , x^{i+1}, \dots, x^n))(x^i).$$

We start with the analogous result to the Lemma in [21, 5.1.], the proof of our matrix-generalized result uses completely the same argumentation:

Lemma 10.10.1. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, let $U \subseteq \mathbb{R}^n$ be open and E a locally convex vector space. For a given mapping $f : U \to E$ the following are equivalent:*

(1) *$f : U \to E$ is $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$*

(2) *For every compact convex subset $K \subseteq U$ and every $\alpha \in E^*$ there exists an index $l \in \Lambda$ and a number $h > 0$ such that the set*

$$\left\{ \frac{|\delta^\beta(\alpha \circ f)(x)|}{h^{|\beta|} \cdot M^l_{|\beta|}} : \beta \in \mathbb{N}^n, x \in K^{\langle \beta \rangle} \right\}$$

is bounded in $\mathbb{R}$. So for compact convex sets K the norm on the Banach-space $\mathcal{E}_{\mathcal{M},l,h}(K, \mathbb{R})$, introduced in (7.3.3)*, is also given by*

$$\|f\|^{\langle\rangle}_{\mathcal{M},K,l,h} := \sup \left\{ \frac{|\delta^\beta f(x)|}{h^{|\beta|} \cdot M^l_{|\beta|}} : \beta \in \mathbb{N}^n, x \in K^{\langle \beta \rangle} \right\} \tag{10.10.1}$$

An analogous result for the Beurling-case is the following:

Lemma 10.10.2. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, let $U \subseteq \mathbb{R}^n$ be open and E a locally convex vector space. For a given mapping $f : U \to E$ the following are equivalent:*

(1) *$f : U \to E$ is $\mathcal{E}_{(\mathcal{M}),\mathrm{curve}}$*

(2) *For every compact convex subset $K \subseteq U$, continuous linear functional $\alpha \in E^*$ and each index $l \in \Lambda$ and real number $h > 0$ the set*

$$\left\{\frac{|\delta^\beta(\alpha \circ f)(x)|}{h^{|\beta|} \cdot M^l_{|\beta|}} : \beta \in \mathbb{N}^n, x \in K^{\langle\beta\rangle}\right\}$$

is bounded in $\mathbb{R}$.

The next result is analogous to [21, 5.2. Lemma] and it is of course related to 10.4.2:

Lemma 10.10.3. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$. Moreover let E be a convenient vector space such that there exists a Baire-vector-space-topology on the dual E^* for which the point evaluations ev_x are continuous for all $x \in E$. Then for a mapping $f : \mathbb{R}^n \to E$ the following are equivalent:*

(1) *$\alpha \circ f$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for all $\alpha \in E^*$.*

(2) *For every convex compact subset $K \subseteq \mathbb{R}^n$ there exists an index $l \in \Lambda$ and a number $h > 0$ such that the set*

$$\left\{\frac{f^{(\beta)}(x)}{h^{|\beta|} \cdot M^l_{|\beta|}} : \beta \in \mathbb{N}^n, x \in K\right\}$$

is bounded in E.

(3) *For every convex compact subset $K \subseteq \mathbb{R}^n$ there exists an index $l \in \Lambda$ and a number $h > 0$ such that the set*

$$\left\{\frac{\delta^\beta f(x)}{h^{|\beta|} \cdot M^l_{|\beta|}} : \beta \in \mathbb{N}^n, x \in K^{\langle\beta\rangle}\right\}$$

is bounded in E.

Proof. (2) $\Rightarrow$ (1) is clear.
(1) $\Rightarrow$ (2): Let K be compact convex set in $\mathbb{R}^n$, so consider for $l \in \Lambda$ and $h, C > 0$ the sets

$$A_{l,h,C} := \left\{\alpha \in E^* : \frac{|(\alpha \circ f)^{(\beta)}(x)|}{h^{|\beta|} \cdot M^l_{|\beta|}} \le C,\ \forall\, \beta \in \mathbb{N}^n, x \in K\right\}.$$

These sets are closed in E^* for the Baire-topology and $\bigcup_{l \in \Lambda, C, h > 0} A_{l,h,C} = E^*$ holds. Then, by the *Baire-property* of E^*, there exist $l_0 \in \Lambda$ and $h_0, C_0 > 0$ such that the interior $A^\circ_{l_0,h_0,C_0} =: U$ is non-empty! Let $\alpha_0 \in U$, then for all $\alpha \in E^*$ there exists $\varepsilon > 0$, such that we get $\varepsilon \cdot \alpha \in U - \alpha_0 \Leftrightarrow \varepsilon \cdot \alpha + \alpha_0 \in U$.
Thus for all $x \in K$ and $\beta \in \mathbb{N}^n$ we get

$$|(\alpha \circ f)^{(\beta)}(x)| \le \frac{1}{\varepsilon} \cdot \Big(|((\varepsilon \cdot \alpha) + \alpha_0) \circ f)^{(\beta)}(x)| + |(\alpha_0 \circ f)^{(\beta)}(x)|\Big) \le \frac{2 \cdot C_0}{\varepsilon} \cdot h_0^{|\beta|} \cdot M^{l_0}_{|\beta|}$$

This implies: The set $\left\{\frac{f^{(\beta)}(x)}{h_0^{|\beta|}\cdot M^{l_0}_{|\beta|}}:\beta\in\mathbb{N}^n, x\in K\right\}$ is weakly bounded in E, hence bounded.
(1) $\Rightarrow$ (3) follows as (1) $\Rightarrow$ (2) and finally (3) $\Rightarrow$ (1) holds by 10.10.1. □

With this notation and the previous preparation results we are able to state and prove the central theorem, *Cartesian closedness for the Roumieu-case* $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$:

Theorem 10.10.4. *Let $\mathcal{M}=\{M^l: l\in\Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and assume also the following conditions:*

(i) $\Lambda=\mathbb{N}_{>0}$,

(ii) $(\mathcal{M}_{\{\text{mg}\}})$,

(iii) $(\mathcal{M}_{\{\text{alg}\}})$.

Let E_1,E_2 and G be convenient vector spaces and $U_i\subseteq E_i$, $i=1,2$, two c^∞-open subsets. Then we get

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U_1\times U_2,G)\cong\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U_1,\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U_2,G)).$$

Proof. First recall that for $U\subseteq\mathbb{R}^n$ non-empty open we have the topological vector space representation

$$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R})=\varprojlim_{K\subseteq U}\varinjlim_{l\in\Lambda}\varinjlim_{h>0}\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R})$$

and in this case by the second part of 9.1.1 the space $\varinjlim_{l\in\Lambda}\varinjlim_{h>0}\mathcal{E}_{\mathcal{M},l,h}(K,\mathbb{R})$ is a *Silva space.*

The following proof is of course related to [21, 5.3. Theorem]!

First consider the case $U_1=U_2=G=\mathbb{R}$. Let $f\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}^2,\mathbb{R})$, then for all $x\in\mathbb{R}$ we have that $f^\vee(x)=f(x,\cdot)\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$. To show now: $f^\vee:\mathbb{R}\to\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is $\mathcal{E}_{\{\mathcal{M}\}}$ and in fact we prove the second condition in 10.10.1 for all $\alpha\in(\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}))^*$.

So take a continuous linear functional $\alpha\in(\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R}))^*$, and such a functional factorizes over some $\varinjlim_{l\in\Lambda}\varinjlim_{h>0}\mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})$ for a compact set $L\subseteq\mathbb{R}$. Consider a compact set $K\subseteq\mathbb{R}$, then, since $f\in\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}^2,\mathbb{R})$, there exist numbers $D,h>0$ and an index $u\in\Lambda$, such that

$$\frac{|\delta^\beta f(x,y)|}{h^{|\beta|}\cdot M^u_{|\beta|}}\le D$$

for all $\beta\in\mathbb{N}^2$ and $(x,y)\in(K\times L)^{\langle\beta\rangle}$. Write $\beta=(\beta_1,\beta_2)$, then we calculate as follows:

$$\left\|\frac{\delta^{\beta_1}f^\vee(x)}{h_1^{\beta_1}\cdot M^s_{\beta_1}}\right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2}=\sup_{\beta_2\in\mathbb{N},y\in L^{\langle\beta_2\rangle}}\left\{\frac{|\delta_2^{\beta_2}\delta_1^{\beta_1}f(x,y)|}{h_1^{\beta_1}\cdot M^s_{\beta_1}\cdot h_2^{\beta_2}\cdot M^t_{\beta_2}}\right\}$$
$$\underbrace{\le}_{(\star)}\sup_{\beta_2\in\mathbb{N},y\in L^{\langle\beta_2\rangle}}\left\{\frac{|\delta^\beta f(x,y)|}{h_1^{\beta_1}\cdot h_2^{\beta_2}\cdot C^{-\beta_1-\beta_2}\cdot M^u_{\beta_1+\beta_2}}\right\}\le\sup_{\beta_2\in\mathbb{N},y\in L^{\langle\beta_2\rangle}}\left\{\frac{|\delta^\beta f(x,y)|}{h^{|\beta|}\cdot M^u_{|\beta|}}\right\}\le D,$$

for $h=\frac{1}{C}\cdot\min\{h_1,h_2\}$, all $\beta_1\in\mathbb{N}$ and $x\in K^{\langle\beta_1\rangle}$. Inequality $(\star)$ holds with a constant $C>0$ by property $(\mathcal{M}_{\{\text{mg}\}})$!

Thus $f^\vee : K \to \mathcal{E}_{\mathcal{M},t,h_2}(L,\mathbb{R})$ is of class $\mathcal{E}_{\{M^s\}}$ (with $h_1 > 0$) and so $\alpha \circ f^\vee$ is $\mathcal{E}_{\{M^s\}}$, hence $\mathcal{E}_{\{\mathcal{M}\}}$.

Converse direction: Assume that $f^\vee : \mathbb{R} \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is $\mathcal{E}_{\{\mathcal{M}\}}$, then $f^\vee : \mathbb{R} \to \varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})$ is $\mathcal{E}_{\{M^s\}}$ for all $L \subseteq \mathbb{R}$ compact subsets and some $s \in \Lambda$ depending on $f^\vee$. The dual space $(\varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R}))^*$ can be equipped with the Baire vector space topology of the countable limit $\varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})^*$ of Banach-spaces. Thus, by 10.10.3, the mapping $f^\vee : \mathbb{R} \to \varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})$ is strongly $\mathcal{E}_{\{M^s\}}$ (for some $s \in \Lambda$). The space $\varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})$ is countable and moreover regular: By $(2)(a)$ in 9.1.1 the connecting mappings are compact and so this inductive limit is a *Silva space*, hence regular (see e.g. [20, 52.37.]).
So for all compact subsets $K \subseteq \mathbb{R}$ there exists $h_1 > 0$ such that the bounded set $\left\{\frac{\partial^{\beta_1} f^\vee(x)}{h_1^{\beta_1}\cdot M^s_{\beta_1}} : \beta_1 \in \mathbb{N}, x \in K\right\}$ is contained and bounded in the step $\mathcal{E}_{\mathcal{M},t,h_2}(L,\mathbb{R})$ of $\varinjlim_{l\in\Lambda} \varinjlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L,\mathbb{R})$ for some $t \in \Lambda$ and $h_2 > 0$. Hence for all $\beta_1 \in \mathbb{N}$ and $x \in K$ we get

$$
\begin{aligned}
+\infty > D &:= \sup_{\beta_1\in\mathbb{N},y\in K} \left\| \frac{\delta^{\beta_1} f^\vee(y)}{h_1^{\beta_1}\cdot M^s_{\beta_1}} \right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2} \geq \left\| \frac{\delta^{\beta_1} f^\vee(x)}{h_1^{\beta_1}\cdot M^s_{\beta_1}} \right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2} \\
&= \sup_{\beta_2\in\mathbb{N},y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta_2^{\beta_2}\delta_1^{\beta_1} f(x,y)|}{h_1^{\beta_1}\cdot M^s_{\beta_1}\cdot h_2^{\beta_2}\cdot M^t_{\beta_2}} \right\} \\
&\underbrace{\geq}_{(\star\star)} \sup_{\beta_2\in\mathbb{N},y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta^{\beta} f(x,y)|}{h^{|\beta|}\cdot M^u_{|\beta|}} \right\}.
\end{aligned}
$$

$(\star\star)$ holds by property $(\mathcal{M}_{\{\mathrm{alg}\}})$: For all $s,t \in \Lambda$ we can find $u \in \Lambda$ and a constant $C > 0$ such that for all $j,k \in \mathbb{N}$ we obtain $M^s_j \cdot M^t_k \leq C^{j+k}\cdot M^u_{j+k}$, hence $h_1^j \cdot M^s_j \cdot h_2^k \cdot M^t_k \leq C^{j+k}\cdot h_1^j \cdot h_2^k \cdot M^u_{j+k} \leq (C\cdot\max\{h_1,h_2\})^{j+k}\cdot M^u_{j+k}$, so put in the above calculation $h := C\cdot\max\{h_1,h_2\}$.

Now we prove the general case by categorial reasoning: Given a $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$-mapping $f : U_1\times U_2 \to G$, we have to show that $f^\vee : U_1 \to \mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(U_2,G)$ is $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$. Therefore we remark that every continuous linear functional on $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(U_2,G)$ factors over some step mapping $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(c_2,\alpha) : \mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(U_2,G) \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$, where c_2 is a $\mathcal{E}_{\{\mathcal{M}\}}$-curve in U_2 and $\alpha \in G^*$. Then we have to check that $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(c_2,\alpha)\circ f^\vee \circ c_1 : \mathbb{R} \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for every $\mathcal{E}_{\{\mathcal{M}\}}$-curve c_1 in U_1. But since $(\alpha\circ f\circ(c_1\times c_2))^\vee = \mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(c_2,\alpha)\circ f^\vee\circ c_1$ this follows from the above $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$-case.

Conversely, if $f^\vee : U_1 \to \mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(U_2,G)$ is $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$, then $(\alpha\circ f\circ(c_1\times c_2))^\vee = \mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}(c_2,\alpha)\circ f^\vee\circ c_1$ is $\mathcal{E}_{\{\mathcal{M}\}}$ for all $\mathcal{E}_{\{\mathcal{M}\}}$-curves $c_1 : \mathbb{R} \to U_1$, $c_2 : \mathbb{R} \to U_2$ and $\alpha \in G^*$. Again by the above $\mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R},\mathbb{R})$-case f is then $\mathcal{E}_{\{\mathcal{M}\},\mathrm{curve}}$. □

Analogously we can prove *Cartesian closedness for the Beurling-case:*

Theorem 10.10.5. *Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, and assume also the following conditions:*

(i) $\Lambda = \mathbb{R}_{>0}$,

(ii) $(\mathcal{M}_{(\mathrm{mg})})$,

(iii) $(\mathcal{M}_{(\mathrm{alg})})$.

Let E_1, E_2 and G be convenient vector spaces and $U_i \subseteq E_i$, $i = 1, 2$, two c^∞-open subsets. Then we obtain:

$$\mathcal{E}_{(\mathcal{M}),\mathrm{curve}}(U_1 \times U_2, G) \cong \mathcal{E}_{(\mathcal{M}),\mathrm{curve}}(U_1, \mathcal{E}_{(\mathcal{M}),\mathrm{curve}}(U_2, G)).$$

Proof. The proof of this theorem should be also compared with the proof of the Beurling case in [22, 5.2. Theorem]. First we point out that by our assumption on Λ the associated function space of Beurling-type is a Fréchet-space.

In the first step we consider again the case $U_1 = U_2 = G = \mathbb{R}$. Let $f \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R}^2, \mathbb{R})$, then for all $x \in \mathbb{R}$ we have that $f^\vee(x) = f(x, \cdot) \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{R})$. To show now: $f^\vee : \mathbb{R} \to \mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{R})$ is $\mathcal{E}_{(\mathcal{M})}$. We prove again the second condition in 10.10.2 for each $\alpha \in (\mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{R}))^*$.

So take a continuous linear functional $\alpha \in (\mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{R}))^*$, and such a functional factorizes over some $\varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L, \mathbb{R})$ for a compact set $L \subseteq \mathbb{R}$. Consider a compact set $K \subseteq \mathbb{R}$, then, since $f \in \mathcal{E}_{(\mathcal{M})}(\mathbb{R}^2, \mathbb{R})$, for all $h > 0$ and indices $u \in \Lambda$ there exists a number $D > 0$, such that

$$\frac{|\delta^\beta f(x, y)|}{h^{|\beta|} \cdot M^u_{|\beta|}} \le D$$

for all $\beta \in \mathbb{N}^2$ and $(x, y) \in (K \times L)^{\langle\beta\rangle}$. Write $\beta = (\beta_1, \beta_2)$, then we calculate as follows:

$$\left\| \frac{\delta^{\beta_1} f^\vee(x)}{h_1^{\beta_1} \cdot M^s_{\beta_1}} \right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2} = \sup_{\beta_2\in\mathbb{N}, y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta_2^{\beta_2} \delta_1^{\beta_1} f(x, y)|}{h_1^{\beta_1} \cdot M^s_{\beta_1} \cdot h_2^{\beta_2} \cdot M^t_{\beta_2}} \right\}$$

$$\underbrace{\le}_{(\star)} \sup_{\beta_2\in\mathbb{N}, y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta^\beta f(x, y)|}{h_1^{\beta_1} \cdot h_2^{\beta_2} \cdot C^{-\beta_1-\beta_2} \cdot M^u_{\beta_1+\beta_2}} \right\} \le \sup_{\beta_2\in\mathbb{N}, y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta^\beta f(x, y)|}{h^{|\beta|} \cdot M^u_{|\beta|}} \right\} \le D,$$

for $h = \frac{1}{C} \cdot \min\{h_1, h_2\}$, all $\beta_1 \in \mathbb{N}$ and $x \in K^{\langle\beta_1\rangle}$. Inequality $(\star)$ holds with a constant $C > 0$ by property $(\mathcal{M}_{(\mathrm{mg})})$!

Thus $f^\vee : K \to \mathcal{E}_{\mathcal{M},t,h_2}(L, \mathbb{R})$ for each $t \in \Lambda$ and $h_2 > 0$ is of class $\mathcal{E}_{(M^s)}$ and so $\alpha \circ f^\vee$ is $\mathcal{E}_{(M^s)}$ for each $s \in \Lambda$, hence $\mathcal{E}_{(\mathcal{M})}$.

Converse direction: Assume that $f^\vee : \mathbb{R} \to \mathcal{E}_{(\mathcal{M})}(\mathbb{R}, \mathbb{R})$ is $\mathcal{E}_{(\mathcal{M})}$, then $f^\vee : \mathbb{R} \to \varprojlim_{l\in\Lambda} \varprojlim_{h>0} \mathcal{E}_{\mathcal{M},l,h}(L, \mathbb{R})$ is $\mathcal{E}_{(M^t)}$ for all compact subsets $L \subseteq \mathbb{R}$ and each $t \in \Lambda$. More precisely we obtain: For all compact subsets $K, L \subseteq \mathbb{R}$, for all $h_1, h_2 > 0$ and $t, s \in \Lambda$ the bounded set $\left\{ \frac{\partial^{\beta_1} f^\vee(x)}{h_1^{\beta_1} \cdot M^s_{\beta_1}} : \beta_1 \in \mathbb{N}, x \in K \right\}$ is contained and bounded in the step

$\mathcal{E}_{\mathcal{M},t,h_2}(L,\mathbb{R})$. Hence for all $\beta_1 \in \mathbb{N}$ and $x \in K$ we get

$$\begin{aligned}
+\infty > D := \sup_{\beta_1\in\mathbb{N}, y\in K} \left\| \frac{\delta^{\beta_1} f^\vee(y)}{h_1^{\beta_1}\cdot M^s_{\beta_1}} \right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2} &\geq \left\| \frac{\delta^{\beta_1} f^\vee(x)}{h_1^{\beta_1}\cdot M^s_{\beta_1}} \right\|^{\langle\rangle}_{\mathcal{M},L,t,h_2} \\
= \sup_{\beta_2\in\mathbb{N}, y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta_2^{\beta_2}\delta_1^{\beta_1} f(x,y)|}{h_1^{\beta_1}\cdot M^s_{\beta_1}\cdot h_2^{\beta_2}\cdot M^t_{\beta_2}} \right\} \\
\underbrace{\geq}_{(\star\star)} \sup_{\beta_2\in\mathbb{N}, y\in L^{\langle\beta_2\rangle}} \left\{ \frac{|\delta^\beta f(x,y)|}{h^{|\beta|}\cdot M^u_{|\beta|}} \right\}.
\end{aligned}$$

$(\star\star)$ holds now by property $(\mathcal{M}_{(\text{alg})})$: For all $u \in \Lambda$ we can find $s,t \in \Lambda$ and a constant $C > 0$ such that for all $j,k \in \mathbb{N}$ we obtain $M^s_j \cdot M^t_k \leq C^{j+k}\cdot M^u_{j+k}$, hence $h_1^j \cdot M^s_j \cdot h_2^k \cdot M^t_k \leq C^{j+k}\cdot h_1^j \cdot h_2^k \cdot M^u_{j+k} \leq (C\cdot\max\{h_1,h_2\})^{j+k}\cdot M^u_{j+k}$, so put in the above calculation $h := C\cdot\max\{h_1,h_2\}$.

Now the general case again by categorial reasoning:
Let a $\mathcal{E}_{(\mathcal{M}),\text{curve}}$-mapping $f : U_1 \times U_2 \to G$ be given, then we have to show that $f^\vee : U_1 \to \mathcal{E}_{(\mathcal{M}),\text{curve}}(U_2,G)$ is $\mathcal{E}_{(\mathcal{M}),\text{curve}}$. We remark that every continuous linear functional on $\mathcal{E}_{(\mathcal{M}),\text{curve}}(U_2,G)$ factorizes over some step mapping $\mathcal{E}_{(\mathcal{M}),\text{curve}}(c_2,\alpha) : \mathcal{E}_{(\mathcal{M}),\text{curve}}(U_2,G) \to \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$, where c_2 is a $\mathcal{E}_{(\mathcal{M})}$-curve in U_2 and $\alpha \in G^*$. Then we have to check that $\mathcal{E}_{(\mathcal{M}),\text{curve}}(c_2,\alpha)\circ f^\vee \circ c_1 : \mathbb{R} \to \mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$ is $\mathcal{E}_{(\mathcal{M})}$ for every $\mathcal{E}_{(\mathcal{M})}$-curve c_1 in U_1. But since $(\alpha\circ f\circ(c_1\times c_2))^\vee = \mathcal{E}_{(\mathcal{M}),\text{curve}}(c_2,\alpha)\circ f^\vee\circ c_1$ this follows from the $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$-case above.

Conversely, if $f^\vee : U_1 \to \mathcal{E}_{(\mathcal{M}),\text{curve}}(U_2,G)$ is $\mathcal{E}_{(\mathcal{M}),\text{curve}}$, then $(\alpha\circ f\circ(c_1\times c_2))^\vee = \mathcal{E}_{(\mathcal{M}),\text{curve}}(c_2,\alpha)\circ f^\vee\circ c_1$ is $\mathcal{E}_{(\mathcal{M})}$ for all $\mathcal{E}_{(\mathcal{M})}$-curves $c_1 : \mathbb{R} \to U_1$, $c_2 : \mathbb{R} \to U_2$ and $\alpha \in G^*$. Again by the $\mathcal{E}_{(\mathcal{M})}(\mathbb{R},\mathbb{R})$-case above, f is then $\mathcal{E}_{(\mathcal{M}),\text{curve}}$. □

Important remarks concerning the Cartesian closedness results 10.10.4 and 10.10.5:

(1) In 10.10.4 and 10.10.5 we can replace $(\mathcal{M}_{\{\text{alg}\}})$ resp. $(\mathcal{M}_{(\text{alg})})$ by the assumption, that $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, because then both conditions are satisfied simultaneously for $C = 1$ and $l = n$.

(2) For the proof we don't need necessarily that $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ resp. $\mathcal{E}_{(\mathcal{M}),\text{curve}}$ is a category, i.e. closedness under composition. More precisely we don't need conditions $(\mathcal{M}_{\{\text{FdB}\}})$ resp. $(\mathcal{M}_{(\text{FdB})})$ for the proof and in both cases $(\Leftarrow)$ holds also without assuming $(\mathcal{M}_{\{\text{mg}\}})$ resp. $(\mathcal{M}_{(\text{mg})})$.

(3) Unfortunately we don't have in this version (by using $\mathcal{E}_{\{\mathcal{M}\}}$- resp. $\mathcal{E}_{(\mathcal{M})}$-curves) that the Beurling-case $\mathcal{E}_{(\mathcal{M}),\text{curve}}$ is a generalization of the finite-dimensional case. Note that we don't know if in the Beurling-case an analogous result to 10.7.1 is valid even for the constant matrix case $\mathcal{M} = \{M\}$.

(4) To guarantee this fact for the Roumieu-case $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$ we have to assume in addition the assumptions of 10.7.1, so more precisely we need:

    $\mathcal{M} := \{M^l : l \in \Lambda\}$ should be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{N}_{>0}$, condition (10.6.4) (for the $\mathcal{E}_{\{\mathcal{M}\}}$-special-curve-lemma 10.6.2), $(\mathcal{M}_{\{\text{FdB}\}})$ for closedness under composition (see e.g. 8.3.1), and finally $(\mathcal{M}_{\{\text{mg}\}})$.

All these required conditions are satisfied if $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$, such that ω satisfies properties (ω_1), (ω_{nq}) and $(\omega_{1'})$ (recall 5.1.1, 5.1.2, the third chapter in [5] and finally 3.8.4, 8.7.1 and 8.3.1).

(5) Finally 10.10.4 and 10.10.5 and the proofs should be also compared with 12.3.2.

By applying 10.10.4 we can use now exactly the same proof as for [21, 5.5. Theorem] to show the following:

Theorem 10.10.6. *Assume that $\mathcal{M} = \{M^l : l \in \Lambda\}$ is an arbitrary weight matrix, i.e. $(\mathcal{M})$, with a countable index set $\Lambda = \mathbb{N}_{>0}$ and moreover properties $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{\{\text{alg}\}})$.*
Let E, F, E_i, F_i, G be convenient vector spaces and let U and V be c^∞-open subsets. Then we get

(1) *The exponential law*
$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(V, G)) \cong \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U \times V, G)$
holds, more precisely $\cong$ holds as a linear $\mathcal{E}_{\{\mathcal{M}\}}$-diffeomorphism of convenient vector spaces.

The following canonical mappings are $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$:

(2) $\text{ev} : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F) \times U \to F$ *given by* $\text{ev}(f, x) = f(x)$.

(3) $\text{ins} : E \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(F, E \times F)$ *given by* $\text{ins}(x)(y) = (x, y)$.

(4) $(\cdot)^\wedge : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(V, G)) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U \times V, G)$.

(5) $(\cdot)^\vee : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U \times V, G) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(V, G))$.

(6) $\prod : \prod_i \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E_i, F_i) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}\left(\prod_i E_i, \prod_i F_i\right)$.

If we have in addition property $(\mathcal{M}_{\{\text{FdB}\}})$, then we also get:

(7) $\text{comp} : \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(F, G) \times \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, F) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U, G)$.

(8) $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(\cdot, \cdot)$:

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(F, F_1) \times \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E_1, E) \to \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E, F), \mathcal{E}_{\{\mathcal{M}\},\text{curve}}(E_1, F_1)),$$

which is given by $(f, g) \mapsto (h \mapsto f \circ h \circ g)$.

We close this section with the following important example which is analogous to [21, 5.4. Example] and shows, that condition $(\mathcal{M}_{\{\text{mg}\}})$ is really needed for Cartesian closedness:

Lemma 10.10.7. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with index set $\Lambda = \mathbb{N}_{>0}$ and which doesn't satisfy condition $(\mathcal{M}_{\{\text{mg}\}})$. Then there exists an function $f \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}^2, \mathbb{C})$ such that the associated mapping $f^\vee : \mathbb{R} \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{C})$ is not $\mathcal{E}_{\{\mathcal{M}\}}$.*

Proof. Since condition $(\mathcal{M}_{\{\text{mg}\}})$ doesn't hold there exists an index $x \in \Lambda$ such that for each $C > 0$ and each index $y \in \Lambda$ there exist $j, k \in \mathbb{N}$ with $M^x_{j+k} > C^{j+k} \cdot M^y_j \cdot M^y_k$.

So for this $x \in \Lambda$ and $C = y = n$, $n \in \mathbb{N}_{>0}$, we obtain sequences $(j_n)_n$ and $(k_n)_n$ with $j_n \to \infty$ for $n \to \infty$ and $k_n \geq 1$ for each $n \in \mathbb{N}_{>0}$ such that

$$\left(\frac{M^x_{j_n+k_n}}{M^n_{j_n} \cdot M^n_{k_n}}\right)^{1/(k_n+j_n)} \geq n.$$

We consider the linear functional $\alpha : \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{C}) \to \mathbb{C}$ which is given by

$$\alpha(f) := \sum_{n\geq 1} (\sqrt{-1})^{3j_n} \cdot \frac{f^{(j_n)}(0)}{M^n_{j_n} \cdot n^{j_n}}.$$

We show that the functional is bounded: For given $f \in \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{C})$ we choose $h > 0$ and $l \in \Lambda$ large enough (dep. on f) and estimate

$$\left|\sum_{n\geq 0} (\sqrt{-1})^{3j_n} \frac{f^{(j_n)}(0)}{M^n_{j_n} \cdot n^{j_n}}\right| \leq \sum_{n\geq 0} \frac{\left|f^{(j_n)}(0)\right|}{h^{j_n} \cdot M^l_{j_n}} \cdot \frac{M^l_{j_n}}{M^n_{j_n}} \cdot \left(\frac{h}{n}\right)^{j_n} \leq \|f\|_{\mathcal{M},[-1,1],l,h} \cdot \sum_{n\geq 0} \frac{M^l_{j_n}}{M^n_{j_n}} \cdot \left(\frac{h}{n}\right)^{j_n} < +\infty.$$

Note that $M^l \leq M^n$ for $l \leq n$ and $\sum_{n\geq 0} \left(\frac{h}{n}\right)^{j_n} < +\infty$ for each $h > 0$.
We will apply now the functional α to our special function $\tilde{\theta}_x \in \mathcal{E}^{\text{global}}_{\{M^x\}}(\mathbb{R}, \mathbb{C}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{C})$ with $\tilde{\theta}^{(j)}_x(0) = (\sqrt{-1})^j \cdot s^x_j$, $s^x_j := \sum_{k=0}^{\infty} M^x_k (2\mu^x_k)^{j-k} \geq M^x_j$ for all $j \in \mathbb{N}$ and $\left|\tilde{\theta}^{(j)}_x(t)\right| \leq s^x_j = \left|\tilde{\theta}^{(j)}_x(0)\right|$ for all $j \in \mathbb{N}$ and $t \in \mathbb{R}$ (see (2.4.7)). The index $x \in \Lambda$ is precisely the index coming from the negation of $(\mathcal{M}_{\{\text{mg}\}})$.
In the next step we define $\psi_x(s,t) := \tilde{\theta}_x(s+t)$ for $s, t \in \mathbb{R}$, and so $\psi_x \in \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R}^2, \mathbb{C})$ with $\psi^{(\beta_1,\beta_2)}_x(0,0) = (\sqrt{-1})^{\beta_1+\beta_2} \cdot s^x_{\beta_1+\beta_2}$ for all $(\beta_1, \beta_2) \in \mathbb{N}^2$.
But we are going to show now: $\alpha \circ \psi^{\vee}_x$ is not $\mathcal{E}_{\{\mathcal{M}\}}$. For this consider $h > 0$ and $l \in \Lambda$ arbitrary but fixed and calculate as follows:

$$\begin{aligned}
\left\|\alpha \circ \psi^{\vee}_x\right\|_{\mathcal{M},[-1,1],l,h} &= \sup_{t\in[-1,1],k\in\mathbb{N}} \frac{|(\alpha \circ \psi^{\vee}_x)^{(k)}(t)|}{h^k \cdot M^l_k} \geq \sup_{k\in\mathbb{N}} \frac{1}{h^k \cdot M^l_k} \cdot \left|\sum_{n\geq 1} (\sqrt{-1})^{3j_n} \cdot \frac{\psi^{(j_n,k)}_x(0,0)}{M^n_{j_n} \cdot n^{j_n}}\right| \\
&= \sup_{k\in\mathbb{N}} \frac{1}{h^k \cdot M^l_k} \cdot \left|\sum_{n\geq 1} (\sqrt{-1})^{3j_n} \cdot \frac{(\sqrt{-1})^{j_n+k} \cdot s^x_{j_n+k}}{M^n_{j_n} \cdot n^{j_n}}\right| = \sup_{k\in\mathbb{N}} \frac{1}{h^k \cdot M^l_k} \cdot \left|(\sqrt{-1})^k \cdot \sum_{n\geq 1} \frac{s^x_{j_n+k}}{M^n_{j_n} \cdot n^{j_n}}\right| \\
&= \sup_{k\in\mathbb{N}} \frac{1}{h^k \cdot M^l_k} \cdot \sum_{n\geq 1} \frac{s^x_{j_n+k}}{M^n_{j_n} \cdot n^{j_n}} \underbrace{\geq}_{k=k_n} \sup_{n\in\mathbb{N}_{>0}} \frac{1}{h^{k_n} \cdot M^l_{k_n}} \cdot \frac{M^n_{k_n}}{M^n_{k_n}} \cdot \frac{s^x_{j_n+k_n}}{M^n_{j_n} \cdot n^{j_n}} \\
&\geq \sup_{n\in\mathbb{N}_{>0}} \frac{M^n_{k_n}}{h^{k_n} \cdot n^{j_n} \cdot M^l_{k_n}} \cdot \frac{M^x_{j_n+k_n}}{M^n_{j_n} \cdot M^n_{k_n}} \geq \sup_{n\in\mathbb{N}_{>0}} \frac{n^{j_n+k_n}}{h^{k_n} \cdot n^{j_n}} \cdot \frac{M^n_{k_n}}{M^l_{k_n}} = +\infty.
\end{aligned}$$

□

10.11 Important summarize - Cartesian closedness versus conditions (mg) and ($\mathcal{M}_{\{\mathbf{mg}\}}$)

As already pointed out in [21, 5.4. Example] and which is a special case of our more general statement 10.10.7, *Cartesian closedness* fails for constant ($\mathcal{M}_{\text{sc}}$) weight matrices $\mathcal{M} = \{M\}$ if the sequence M doesn't satisfy condition moderate growth (mg) (note that moderate growth is related to condition (ω_6) for it's associated function ω_M by [16, Proposition 3.6.]). But in the non-constant weight matrix case $\mathcal{M} = \{M^l : l \in \Lambda\}$ where all sequences are assumed to be pair-wise not equivalent, the much more general condition ($\mathcal{M}_{\{\text{mg}\}}$) is sufficient to prove *Cartesian closedness*.

Recall the important remark 9.5.5: Note that in example 9.5.4 we have constructed ($\mathcal{M}_{\text{sc}}$) weight matrices $\mathcal{M} = \{M^l : l \in \mathbb{N}, l \geq 2\}$ such that $\mathcal{E}_{\{\mathcal{G}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$, each M^l doesn't satisfy (mg) but nevertheless condition ($\mathcal{M}_{\{\text{mg}\}}$) is valid. Moreover the constructed weight matrices there, obtained by the weight function(s) $\omega_s(t) := \max\{0, \log(t)^s\}$, $s > 1$, (see 3.10.1) and $\mathcal{M} = \{(l^{k^2})_{k \geq 0} : l \in \mathbb{N}_{>0}, l \geq 2\}$ satisfy also property ($\mathcal{M}_{\{\text{FdB}\}}$).

But in [26, Theorem 2] it was shown that for each constant ($\mathcal{M}_{\text{sc}}$) weight matrix $\mathcal{M} = \{M\}$ with property (mg) there exists $s > 1$ with $\mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{\{G^s\}}$. So beyond Gevrey sequence spaces we cannot expect *Cartesian closedness* for constant weight matrices $\mathcal{M} = \{M\}$, for non-constant matrices this fact is not true and we can prove *Cartesian closedness* for larger function classes.

If the matrix $\mathcal{M}$ is obtained by a weight function $\omega \in \mathcal{W}$, then both conditions ($\mathcal{M}_{\{\text{mg}\}}$) and ($\mathcal{M}_{(\text{mg})}$) hold automatically by (5.1.2) in 5.1.2.
This situation is consistent with 6.4.1: Let $\omega \in \mathcal{W}$ with (ω_1) be given and assume that there exists a sequence $N \in \mathcal{LC}$ with $\mathcal{E}_{[\omega]} = \mathcal{E}_{[N]}$ as locally convex vector spaces, then N has to satisfy condition moderate growth (mg), too.

11 Convenient setting for certain quasi-analytic classes $\mathcal{E}_{\{\mathcal{Q}\}}$ by using curves

In this chapter we will always consider weight matrices with the following properties

$$\mathcal{Q} := \left\{Q^l : l \in \Lambda = \mathbb{N}_{>0} : \ (\mathcal{M}_{\mathrm{sc}}), \ \forall\, l \in \Lambda : \ Q^l \text{ doesn't satisfy (nq)}\right\},$$

and we will also put $q_k^l := \frac{Q_k^l}{k!}$. First recall a consequence of Lemma 7.4.1: Each such weight matrix $\mathcal{Q}$ is now *Roumieu-quasi-analytic*, so $(\mathcal{M}_{\{\mathrm{nq}\}})$ is not satisfied and the class $\mathcal{E}_{\{\mathcal{Q}\}}$ doesn't contain functions with compact support. The results, the technique and the methods of the proofs in this chapter are motivated by [23]. The goal of this chapter is to transfer the proofs in [23] to the quasi-analytic (non constant) weight matrix case, so we want to derive a convenient setting for such classes by using curves and by using the proofs for the non-quasi-analytic situation in the previous chapter. Recall that only for non-quasi-analytic weight matrices the introduced definitions/structures (by using curves) in the previous chapter are a generalization of the classical definitions (on the level of $\mathbb{R}^n$). To do so we have to prove appropriate projective descriptions for $\mathcal{E}_{\{\mathcal{Q}\}}$.

11.1 Projective descriptions for certain quasi-analytic classes $\mathcal{E}_{\{\mathcal{Q}\}}$

The aim of this section is to obtain for some classes $\mathcal{E}_{\{\mathcal{Q}\}}$ a "nice" projective description. More precisely we are going to prove that some quasi-analytic classes of ultradifferentiable functions defined by such weight matrices $\mathcal{Q}$ can be written as intersection of non-quasi-analytic classes $\mathcal{E}_{\{L\}}$, where the intersection is taken over some (large) set of sequences. Note that the central result 11.1.1 below holds also for globally defined classes $\mathcal{E}_{\{\mathcal{Q}\}}$ by obvious modification of the proof there (see 7.3.2).
First we have to introduce resp. recall some notation. We define the sets

$$\mathcal{L}(\mathcal{Q})_{\mathrm{w}} := \left\{L \in \mathbb{R}_{>0}^{\mathbb{N}} : k \mapsto (L_k)^{1/k} \text{ is increasing}, \mathcal{E}_{\{L\}} \text{ is not q.a.}, \ Q^n \preceq L \ \forall\, n \in \Lambda\right\},$$

$$\mathcal{L}(\mathcal{Q})_{\mathrm{s}} := \left\{L \in \mathbb{R}_{>0}^{\mathbb{N}} : L \in \mathcal{L}(\mathcal{Q})_{\mathrm{w}}, k \mapsto \left(\frac{L_k}{k!}\right)^{1/k} \text{ is increasing}\right\}.$$

Remark: For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{w}}$ we get automatically $\lim_{k\to\infty}(L_k)^{1/k} = +\infty$ (by $Q^l \preceq L$) and L has also (alg) for $C = 1$, i.e. $L_j \cdot L_k \le L_{j+k}$ for all $j, k \in \mathbb{N}$: For arbitrary $j, k \ge 1$ we have $(L_j^l)^{1/j} \le (L_{j+k})^{1/(j+k)}$, $(L_k)^{1/k} \le (L_{j+k}^l)^{1/(j+k)}$, hence

$$L_j \cdot L_k \le (L_{j+k})^{j/(j+k)} \cdot (L_{j+k})^{k/(j+k)} = L_{j+k}.$$

The remaining cases ($j = k = 0$, $j = 0$, k arb. resp. j arb. and $k = 0$) are obvious by normalization. Instead $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{w}}$ we can also introduce (the little bit smaller set)

$$\mathcal{L}(\mathcal{Q})_{\mathrm{lc}} := \left\{L \in \mathbb{R}_{>0}^{\mathbb{N}} : L \text{ is (weakly) log. convex, has (nq)}, \ Q^n \preceq L \ \forall\, n \in \Lambda\right\}.$$

Some further remarks:

(*i*) Notation: $\mathcal{L}(\mathcal{Q})_\star$ will stand for any of such classes of sequences.

(*ii*) Note that if $L \in \mathcal{L}(\mathcal{Q})_\star$, $L = (L_k)_k$, then also $(C^k \cdot L_k)_k \in \mathcal{L}(\mathcal{Q})_\star$ for each $C > 0$. But $\mathcal{L}(\mathcal{Q})_\star$ is not stable w.r.t. relation $\approx$, because log. convexity and the increasing properties $k \mapsto (L_k)^{1/k}$ resp. $k \mapsto \left(\frac{L_k}{k!}\right)^{1/k}$ are not stable w.r.t. this relation.

(*iii*) If $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ holds for the matrix $\mathcal{Q}$, then clearly also $\liminf_{k\to\infty}(l_k)^{1/k} > 0$, $l_k := \frac{L_k}{k!}$, is satisfied for each $L \in \mathcal{L}(\mathcal{Q})_\star$.

(*iv*) If there exists an index $n_0 \in \Lambda$ with $\lim_{k\to\infty}(q_k^{n_0})^{1/k} = +\infty$, which is weaker than condition $(\mathcal{M}_{(\mathrm{C}^\omega)})$ for $\mathcal{Q}$, then $\lim_{k\to\infty}(l_k)^{1/k} = +\infty$ for each $L \in \mathcal{L}(\mathcal{Q})_\star$, too.

Recall now the definitions of two regularizations of an arbitrary weight sequence M: We introduce

$$M^{\mathrm{i}} := (M_k^{\mathrm{i}})_k, \qquad M_k^{\mathrm{i}} := \left(\inf\{(M_j)^{1/j} : j \ge k\}\right)^k \text{ for } k \ge 1, \qquad M_0^{\mathrm{i}} := 1,$$

so the sequence $\left((M_k^{\mathrm{i}})^{1/k}\right)_k$ is the increasing minorant of the sequence $((M_k)^{1/k})_k$. If $k \mapsto (M_k)^{1/k}$ is increasing and $M_0 = 1$ (w.l.o.g.), then $M_k = M_k^{\mathrm{i}}$ follows for all $k \in \mathbb{N}$. Second we define

$$M^{\mathrm{lc}} := (M_k^{\mathrm{lc}})_k, \quad M_k^{\mathrm{lc}} := \inf\left\{(M_j)^{\frac{l-k}{l-j}} \cdot (M_l)^{\frac{k-j}{l-j}} : j \le k \le l, j < l\right\},$$

the (weakly) log. convex minorant of M. If M is log. convex, then $M = M^{\mathrm{lc}}$ holds.

Now we recall the notation of [23, 1.5. Definition], see also [3]: For each sequence Q^l we introduce the associated sequence

$$\check{Q}_k^l := Q_k^l \cdot \prod_{j=1}^k \left(1 - \frac{1}{(Q_j^l)^{1/j}}\right)^k, \ \check{Q}_0^l := 1. \tag{11.1.1}$$

We put also $\check{q}_k^l := \frac{\check{Q}_k^l}{k!}$, hence

$$\check{q}_k^l := q_k^l \cdot \prod_{j=1}^k \left(1 - \frac{1}{(Q_j^l)^{1/j}}\right)^k, \ \check{q}_0^l := 1. \tag{11.1.2}$$

We summarize several properties for $\check{Q}^l$:

(1) First we have by definition $0 < \check{Q}_k^l \le Q_k^l \Leftrightarrow 0 < \check{q}_k^l \le q_k^l$ for all $k \in \mathbb{N}$ and $l \in \Lambda$, moreover $\check{Q}_k^{l_1} \le \check{Q}_k^{l_2}$, $\check{q}_k^{l_1} \le \check{q}_k^{l_2}$ for all $k \in \mathbb{N}$ and $(\check{Q}_k^{l_1})^{1/k} \le (\check{Q}_k^{l_2})^{1/k}$, $(\check{q}_k^{l_1})^{1/k} \le (\check{q}_k^{l_2})^{1/k}$ for all $k \in \mathbb{N}_{>0}$, whenever $l_1 \le l_2$ holds.

(2) See also [23, 1.5. Definition]: We have $(\check{Q}_k^l)^{1/k} = (Q_k^l)^{1/k} \cdot \prod_{j=1}^k \left(1 - \frac{1}{(Q_j^l)^{1/j}}\right)$, hence $\frac{(\check{Q}_{k+1}^l)^{1/(k+1)}}{(\check{Q}_k^l)^{1/k}} = \frac{(Q_{k+1}^l)^{1/(k+1)}}{(Q_k^l)^{1/k}} \cdot \left(1 - \frac{1}{(Q_{k+1}^l)^{1/(k+1)}}\right) = \frac{(Q_{k+1}^l)^{1/(k+1)}-1}{(Q_k^l)^{1/k}}$, or also $(Q_{k+1}^l)^{1/(k+1)} = 1 + (Q_k^l)^{1/k} \cdot \frac{(\check{Q}_{k+1}^l)^{1/(k+1)}}{(\check{Q}_k^l)^{1/k}}$.

By using this recursion we finally obtain $(Q^l_k)^{1/k} = (\check{Q}^l_k)^{1/k} \cdot \left(1 + \sum_{j=1}^k \frac{1}{(\check{Q}^l_j)^{1/j}}\right)$, hence the mapping $k \mapsto \frac{(Q^l_k)^{1/k}}{(\check{Q}^l_k)^{1/k}}$ is increasing for each $l \in \Lambda$ and $\frac{(Q^{l_1}_k)^{1/k}}{(\check{Q}^{l_1}_k)^{1/k}} \geq \frac{(Q^{l_2}_k)^{1/k}}{(\check{Q}^{l_2}_k)^{1/k}}$ for all $k \in \mathbb{N}_{>0}$ whenever $l_1 \leq l_2$ holds.

Finally the previous calculation also shows that the sequence $\left((\check{Q}^l_k)^{1/k}\right)_k$ is increasing, if and only if $(Q^l_k)^{1/k} + 1 \leq (Q^l_{k+1})^{1/(k+1)}$ for all $k \in \mathbb{N}_{>0}$ is satisfied.

(3) See also [23, 1.7. Corollary]: If we assume in addition that the mapping $j \mapsto (q^l_j)^{1/j}$ is increasing for each $l \in \Lambda$, where we have put $q^l_j := \frac{Q^l_j}{j!}$, then w.l.o.g we can assume that $\left((\check{Q}^l_k)^{1/k}\right)_k$ is increasing for each $l \in \Lambda$.

This holds since $(Q^l_{k+1})^{1/(k+1)} - (Q^l_k)^{1/k} = (k+1)!^{1/(k+1)} \cdot (q^l_{k+1})^{1/(k+1)} - k!^{1/k} \cdot (q^l_k)^{1/k} \geq (q^l_k)^{1/k} \cdot \left((k+1)!^{1/(k+1)} - k!^{1/k}\right) \geq q^1_1 \cdot \left((k+1)!^{1/(k+1)} - k!^{1/k}\right) \geq q^1_1 \cdot \exp(-1) \geq \exp(-1)$.

Hence we can replace the original weight matrix $\mathcal{Q}$ by $\tilde{\mathcal{Q}} := \{\tilde{Q}^l = (\tilde{Q}^l_k)_k : \tilde{Q}^l_k := \exp(k) \cdot Q^l_k, Q^l \in \mathcal{Q}\}$, because then each $\tilde{Q}^l \in \tilde{\mathcal{Q}}$ satisfies the same properties as Q^l, $\mathcal{E}_{\{\tilde{\mathcal{Q}}\}} = \mathcal{E}_{\{\mathcal{Q}\}}$ (more precisely $\mathcal{E}_{\{\tilde{Q}^l\}} = \mathcal{E}_{\{Q^l\}}$ for each $l \in \Lambda$) and we have shown $(\tilde{Q}^l_k)^{1/k} + 1 \leq (\tilde{Q}^l_{k+1})^{1/(k+1)}$.

(4) Let $l_1 \leq l_2$, if $k \mapsto \frac{(Q^{l_2}_k)^{1/k}}{(Q^{l_1}_k)^{1/k}}$ is increasing, then the mapping $k \mapsto \frac{(\check{Q}^{l_2}_k)^{1/k}}{(\check{Q}^{l_1}_k)^{1/k}}$, too. This holds since by remark (2) we have

$$\frac{(\check{Q}^{l_2}_k)^{1/k}}{(\check{Q}^{l_1}_k)^{1/k}} \leq \frac{(\check{Q}^{l_2}_{k+1})^{1/(k+1)}}{(\check{Q}^{l_1}_{k+1})^{1/(k+1)}} \Leftrightarrow \frac{(\check{Q}^{l_1}_{k+1})^{1/(k+1)}}{(\check{Q}^{l_1}_k)^{1/k}} \leq \frac{(\check{Q}^{l_2}_{k+1})^{1/(k+1)}}{(\check{Q}^{l_2}_k)^{1/k}}$$
$$\underbrace{\Leftrightarrow}_{(2)} \frac{(Q^{l_1}_{k+1})^{1/(k+1)}}{(Q^{l_1}_k)^{1/k}} - \frac{1}{(Q^{l_1}_k)^{1/k}} \leq \frac{(Q^{l_2}_{k+1})^{1/(k+1)}}{(Q^{l_2}_k)^{1/k}} - \frac{1}{(Q^{l_2}_k)^{1/k}}.$$

(5) If the sequence $\check{q}^l$ satisfies condition (ai) then the sequence q^l, too. For this let $1 \leq j \leq k$ be given, then we get $(\check{q}^l_j)^{1/j} \leq C \cdot (\check{q}^l_k)^{1/k} \Leftrightarrow (q^l_j)^{1/j} \cdot \prod_{i=1}^j \left(1 - \frac{1}{(Q^l_i)^{1/i}}\right) \leq C \cdot (q^l_k)^{1/k} \cdot \prod_{i=1}^k \left(1 - \frac{1}{(Q^l_i)^{1/i}}\right) \Leftrightarrow (q^l_j)^{1/j} \leq C \cdot (q^l_k)^{1/k} \cdot \prod_{i=j+1}^k \underbrace{\left(1 - \frac{1}{(Q^l_i)^{1/i}}\right)}_{\leq 1}$.

Similarly we get: If the mapping $k \mapsto (\check{q}^l_k)^{1/k}$ is increasing then also the mapping $k \mapsto (q^l_k)^{1/k}$.

We use now this notation and the above remarks to prove an analogous result to [23, 1.6. Theorem, 1.7. Corollary]:

Theorem 11.1.1. *Let $\mathcal{Q} := \{Q^n : n \in \Lambda = \mathbb{N}_{>0} : (\mathcal{M}_{\text{sc}}),\ Q^n$ quasi-analytic $\forall\, n \in \Lambda\}$ be given.*
Then we obtain the following properties:

(a) *Assume that the mappings $k \mapsto (\check{Q}^n_k)^{1/k}$ and $k \mapsto \frac{(\check{Q}^{n_2}_k)^{1/k}}{(\check{Q}^{n_1}_k)^{1/k}}$ are increasing for each $n \in \Lambda$ resp. for all $n_1, n_2 \in \Lambda$ with $n_1 \leq n_2$. Then we obtain the following representation (as vector spaces) for all $r, s \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^r$ non-empty and open:*

$$\mathcal{E}_{\{\mathcal{Q}\}}(U, \mathbb{R}^s) = \bigcap_{L \in \mathcal{L}(\mathcal{Q})_{\text{w}}} \mathcal{E}_{\{L\}}(U, \mathbb{R}^s).$$

(*b*) *If* $\mathcal{Q}$ *satisfies the assumptions from* (*a*) *and in addition* $(\mathcal{M}_{\{C^\omega\}})$, *i.e.*

$$\exists\, n_0 \in \Lambda: \liminf_{k\to\infty}(q_k^{n_0})^{1/k} > 0,$$

then we can take in (*a*) *the intersection also over all* $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$.

(*c*) *If* $\mathcal{Q}$ *satisfies the assumptions from* (*a*) *and we assume that* $j \mapsto (\check{q}_j^n)^{1/j}$ *is increasing for each* $n \in \Lambda$, *where* $\check{q}_j^n := \frac{\check{Q}_j^n}{j!}$, *then we can take the intersection also over all* $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{s}}$. *But more precisely we can take the intersection over all*

$$L \in \mathcal{L}(\mathcal{Q}) := \{L \in \mathbb{R}_{>0}^{\mathbb{N}}: \ k \mapsto \left(\frac{L_k}{k!}\right)^{1/k} \text{ is increasing, } L^{\mathrm{lc}} \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}\}$$

and more general over all

$$L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}} := \{L \in \mathbb{R}_{>0}^{\mathbb{N}}: \ L \text{ has (ai)}, \ L^{\mathrm{lc}} \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}\}.$$

The intersection over all $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ *can be also considered if we only assume that each sequence* $\check{q}^n$ *should satisfy* (ai) *with* $\sup_{n\in\Lambda} C_n \le D < +\infty$ *(where* C_n *are the constants occurring in* (ai) *for the index* $n \in \Lambda$*).*

Independently of (*a*), (*b*) *and* (*c*) *we can get additional properties for some of the introduced matrices:*

(*d*) *If* $\mathcal{Q}$ *satisfies* $(\mathcal{M}_{\{C^\omega\}})$, *then property* $(\mathcal{M}_{(\mathrm{fil})})$ *holds for both matrices* $\mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ *and* $\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$, *i.e. for arbitrary sequences* $L^1, L^2 \in \mathcal{L}(\mathcal{Q})_\star$ *there exists a sequence* $L \in \mathcal{L}(\mathcal{Q})_\star$ *with* $L \le L^1, L^2$.

(*e*) *If* $\mathcal{Q}$ *satisfies* $(\mathcal{M}_{\{C^\omega\}})$ *and* $(\mathcal{M}_{\{\mathrm{dc}\}})$, *then both matrices* $\mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ *and* $\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ *satisfy condition* $(\mathcal{M}_{(\mathrm{dc})})$, *too. Recall: This condition means in the most general form*

$$\forall\, L \in \mathcal{L}(\mathcal{Q})_\star \ \exists\, N \in \mathcal{L}(\mathcal{Q})_\star \ \exists\, C \ge 1: \ \forall\, j \in \mathbb{N}: \ N_{j+1} \le C^{j+1}\cdot L_j.$$

Proof. (*a*) We prove a matrix-version of [23, 1.7. Corollary] and proceed as follows: First note that $\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s) \subseteq \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{w}}} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$ holds by definition (and also for any $L \in \mathcal{L}(\mathcal{Q})_\star$, where $\mathcal{L}(\mathcal{Q})_\star$ denotes any class defined before).
For the converse direction let $f \notin \mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s)$, it remains to show now that there exists a sequence $L \in \mathcal{L}(\mathcal{Q})_w$ with $f \notin \mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$. By assumption there exists some compact set $K \subseteq U$ such that for each $n \in \Lambda = \mathbb{N}_{>0}$ we have

$$\limsup_{k\ge 1}\left(\frac{f_k}{Q_k^n}\right)^{1/k} = +\infty,$$

where we have put $f_k := \sup_{x\in K, k\in\mathbb{N}} \left\|f^{(k)}(x)\right\|_{L^k(\mathbb{R}^r,\mathbb{R}^s)}$. We proceed now similarly as in [23, 1.6. Theorem]: Choose an increasing sequence $(a_j)_{j\ge1}$ and a decreasing sequence $(b_j)_{j\ge1}$ such that $\lim_{j\to\infty} a_j = +\infty$, $\lim_{j\to\infty} b_j = 0$ and finally $\sum_{j\ge1}\frac{1}{a_j\cdot b_j} < +\infty$.
For all $n \in \Lambda = \mathbb{N}_{>0}$ and $C > 0$ there exists $k_{n,C}$ with $f_{k_{n,C}} > C^{k_{n,C}}\cdot Q_{k_{n,C}}^n$. We choose now $C = a_j \in \mathbb{N}_{>0}$ and $n = j$ and identify k_{j,a_j} with k_j to obtain a strictly increasing sequence $(k_j)_{j\ge1}$ with

$$\left(\frac{f_{k_j}}{Q_{k_j}^j}\right)^{1/(k_j)} > a_j, \tag{11.1.3}$$

for all $j \in \mathbb{N}_{>0}$. We immediately get $b_j \cdot \left(\frac{f_{k_j}}{\check{Q}^j_{k_j}}\right)^{1/(k_j)} = b_j \cdot \left(\frac{f_{k_j}}{Q^j_{k_j}}\right)^{1/(k_j)} \cdot \left(\frac{Q^j_{k_j}}{\check{Q}^j_{k_j}}\right)^{1/(k_j)} \geq b_j \cdot a_j \cdot \frac{Q^j_1}{\check{Q}^j_1} \geq b_j \cdot a_j \to +\infty$ for $j \to \infty$, where we have used (2) in the above remarks. Now we pass to a sub-sequence to get $k_1 \geq 1$ and to obtain an increasing sequence $\beta = (\beta_j)_j$ with $\lim_{j\to\infty} \beta_j = +\infty$, where we have put $1 < \beta_j := b_j \cdot \left(\frac{f_{k_j}}{\check{Q}^j_{k_j}}\right)^{1/(k_j)}$.
Define now the weight sequence $L = (L_k)_k$, or with other words the (constant) weight matrix $\mathcal{L} := \{L\}$ (consisting only of this sequence L) by

$$L_k := \beta_j^k \cdot \check{Q}^j_k$$

for $k \in \mathbb{N}$ with $k_{j-1} < k \leq k_j$, $j \geq 2$, and $L_k := \beta_1^k \cdot \check{Q}^1_k$ for $0 \leq k \leq k_1$. First observation: We obtain that $k \mapsto (L_k)^{1/k}$ is increasing and $\lim_{k\to\infty}(L_k)^{1/k} = +\infty$. This holds since $k \mapsto (\check{Q}^j_k)^{1/k}$ is increasing by assumption, $(\check{Q}^j_k)^{1/k} \leq (\check{Q}^{j+1}_k)^{1/k}$ for all $k \in \mathbb{N}_{>0}$ and finally $(\beta_j)_j$ is also increasing by construction. Moreover we have $(L_{k_j})^{1/(k_j)} = \beta_j \cdot (\check{Q}^j_{k_j})^{1/(k_j)} \to +\infty$ for $j \to \infty$. But note that the sequence L is in general not weakly resp. strongly log. convex in the non-constant matrix case, even if each sequence $\check{Q}$ is assumed to be weakly resp. strongly log. convex, see 9.2.1 for more details.
Next we show, that $f \notin \mathcal{E}_{\{L\}}(U, \mathbb{R}^s)$:

$$\left(\frac{f_{k_j}}{L_{k_j}}\right)^{1/(k_j)} = \left(\frac{f_{k_j}}{\beta_j^{k_j} \cdot \check{Q}^j_{k_j}}\right)^{1/(k_j)} = \left(\frac{f_{k_j}}{\check{Q}^j_{k_j}}\right)^{1/(k_j)} \cdot \frac{1}{\beta_j} = \frac{1}{b_j} \to \infty$$

for $j \to \infty$ by the definition of the sequence β.

For the non-quasi-analyticity of $\mathcal{E}_{\{L\}}$ we proceed as follows: For $k \in \mathbb{N}$ with $k_{j-1} < k \leq k_j$, $j \geq 2$, we obtain

$$\left(\frac{1}{L_k}\right)^{1/k} = \frac{1}{\beta_j \cdot (\check{Q}^j_k)^{1/k}} = \frac{1}{b_j} \cdot \left(\frac{\check{Q}^j_{k_j}}{f_{k_j}}\right)^{1/(k_j)} \cdot \frac{1}{(\check{Q}^j_k)^{1/k}}.$$

Thus we get for $j \geq 2$:

$$\sum_{k=k_{j-1}+1}^{k_j} \left(\frac{1}{L_k}\right)^{1/k} = \frac{1}{b_j} \cdot \left(\frac{\check{Q}^j_{k_j}}{f_{k_j}}\right)^{1/(k_j)} \cdot \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{(\check{Q}^j_k)^{1/k}} \underbrace{\leq}_{(\star)} \frac{1}{b_j} \cdot \left(\frac{Q^j_{k_j}}{f_{k_j}}\right)^{1/(k_j)} \leq \frac{1}{b_j \cdot a_j}.$$

Note that $(\star)$ holds, because by property (2) in the previous remarks we get

$$(\check{Q}^j_{k_j})^{1/(k_j)} \cdot \sum_{k=k_{j-1}+1}^{k_j} \frac{1}{(\check{Q}^j_k)^{1/k}} \leq (\check{Q}^j_{k_j})^{1/(k_j)} \cdot \left(1 + \sum_{k=1}^{k_j} \frac{1}{(\check{Q}^j_k)^{1/k}}\right) = (Q^j_{k_j})^{1/(k_j)}.$$

Since we have shown that $k \mapsto (L_k)^{1/k}$ is increasing, $L = L^{\mathrm{i}}$ follows. Moreover $\sum_{j\geq 1} \frac{1}{a_j \cdot b_j} < +\infty$ holds by construction, hence we have shown $\sum_{k\geq 1} \frac{1}{(L^{\mathrm{i}}_k)^{1/k}} < +\infty$ and so the second condition of Hörmanders version of the Denjoy-Carleman-theorem [15, Theorem 1.3.8.] is satisfied. The fourth condition there tells us that the sequence

L^{lc}, the log. convex minorant of L, satisfies property (nq) and $\mathcal{E}_{\{L\}}$ is not quasi-analytic, i.e. contains functions with compact support. More precisely also $\mathcal{E}_{\{L^{\mathrm{lc}}\}}$ is not quasi-analytic and of course $\mathcal{E}_{\{L^{\mathrm{lc}}\}} \subseteq \mathcal{E}_{\{L\}}$, hence $f \notin \mathcal{E}_{\{L^{\mathrm{lc}}\}}(U, \mathbb{R}^s)$, too.

Finally we show $Q^n \preceq L$ for each $n \in \Lambda$: So let $n \in \Lambda$ be arbitrary (large) but from now on fixed. By property (2) in the above remarks we get

$$\left(\frac{Q_k^n}{L_k}\right)^{1/k} = \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} \cdot \left(1 + \sum_{i=1}^{k} \frac{1}{(\check{Q}_i^n)^{1/i}}\right) = \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} + \sum_{i=1}^{k} \frac{1}{(\check{Q}_i^n)^{1/i}} \cdot \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k}.$$

Consider the first summand: Since $(\check{Q}_k^{n_1})^{1/k} \le (\check{Q}_k^{n_2})^{1/k}$ for $n_1 \le n_2$ we get for $k \in \mathbb{N}$ with $k_{j-1} < k \le k_j$, whenever $j \ge n$, the following: $\left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} = \left(\frac{\check{Q}_k^n}{\beta_j^k \cdot \check{Q}_k^j}\right)^{1/k} \le \frac{1}{\beta_j}$.

Hence $\lim_{k\to\infty} \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} = 0$, i.e. $\check{Q}^n \lhd L$ holds for each $n \in \Lambda$.

For the second summand we proceed as follows: From now on let $k \in \mathbb{N}$ be sufficiently large, more precisely we need at least $k > k_{n-1}$. Let $i \in \mathbb{N}$ with $k_{n-1} < i \le k$, so there exists some $j \ge n$ with $k_{j-1} < i \le k_j$. Moreover we have some $j_1 \ge j$ with $k_{j_1-1} < k \le k_{j_1}$. So we get

$$\left(\frac{\check{Q}_i^n}{L_i}\right)^{1/i} = \frac{1}{\beta_j} \cdot \left(\frac{\check{Q}_i^n}{\check{Q}_i^j}\right)^{1/i} \ge \frac{1}{\beta_{j_1}} \cdot \left(\frac{\check{Q}_i^n}{\check{Q}_i^j}\right)^{1/i} \ge \frac{1}{\beta_{j_1}} \cdot \left(\frac{\check{Q}_i^n}{\check{Q}_i^{j_1}}\right)^{1/i} \ge \frac{1}{\beta_{j_1}} \cdot \left(\frac{\check{Q}_k^n}{\check{Q}_k^{j_1}}\right)^{1/k} = \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k}.$$

This holds because the sequence $(\beta_j)_j$ is increasing by construction and the mapping $k \mapsto \frac{(\check{Q}_k^{n_2})^{1/k}}{(\check{Q}_k^{n_1})^{1/k}}$ is increasing for all $n_1 \le n_2$ by assumption. Hence we can estimate as follows: Since we have already shown $\check{Q}^n \lhd L$ for each $n \in \Lambda$ we can find a constant C_n such that $\check{Q}^n \le C_n^k \cdot L_k$ for all $k \in \mathbb{N}$ and so

$$\sum_{i=1}^{k} \frac{1}{(\check{Q}_i^n)^{1/i}} \cdot \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} = \underbrace{\left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k}}_{\le C_n} \cdot \sum_{i=1}^{k_{n-1}} \frac{1}{(\check{Q}_i^n)^{1/i}} + \left(\frac{\check{Q}_k^n}{L_k}\right)^{1/k} \cdot \sum_{i=k_{n-1}+1}^{k} \frac{1}{(\check{Q}_i^n)^{1/i}}$$

$$\le \tilde{C}_n + \sum_{i=k_{n-1}+1}^{k} \underbrace{\left(\frac{\check{Q}_i^n}{L_i}\right)^{1/i} \cdot \frac{1}{(\check{Q}_i^n)^{1/i}}}_{=\frac{1}{(L_i)^{1/i}}} \le \tilde{C}_n + \sum_{i=k_{n-1}+1}^{\infty} \frac{1}{(L_i)^{1/i}}.$$

Finally note that $\sum_{i=k_{n-1}+1}^{\infty} \frac{1}{(L_i)^{1/i}} < +\infty$ as shown before and so we are done.

(b) By assumption $\liminf_{k\to\infty} (q_k^n)^{1/k} > 0$ for all $n \ge n_0$, and since we have shown $Q^n \preceq L$ for each $n \in \Lambda$ we also get $\liminf_{k\to\infty} (l_k)^{1/k} > 0$, where $l_k := \frac{L_k}{k!}$. So we can use [35, 2.15. Theorem] to conclude that $\mathcal{E}_{\{L^{\mathrm{lc}}\}}(U, \mathbb{R}^s) = \mathcal{E}_{\{L\}}(U, \mathbb{R}^s)$, where L^{lc} denotes again the log. convex minorant of L and satisfies property (nq) by the Denjoy-Carleman-theorem. Since we have shown $Q^n \preceq L$, we get $\mathcal{E}_{\{Q^n\}}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{E}_{\{L\}}(\mathbb{R}, \mathbb{R}) = \mathcal{E}_{\{L^{\mathrm{lc}}\}}(\mathbb{R}, \mathbb{R})$ for each $n \in \Lambda$. Now each Q^n and L^{lc} are weakly log. convex, hence $Q^n \preceq L^{\mathrm{lc}}$ for each n follows, too. Hence we can replace L by L^{lc} in the projective description.

(c) We consider the proof of (a) and for $l_k := \frac{L_k}{k!}$ we get $l_k := \beta_j^k \cdot \check{q}_k^j$ for $k \in \mathbb{N}$ with $k_{j-1} < k \le k_j$, $j \ge 2$, and $l_k := \beta_1^k \cdot \check{q}_k^1$ for $0 \le k \le k_1$. If we assume that $j \mapsto (\check{q}_j^n)^{1/j}$

is increasing, then $k \mapsto (l_k)^{1/k}$ is also increasing, $\lim_{k\to\infty}(l_k)^{1/k} = +\infty$ follows and so we can show $L^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$ as in (b).
We summarize: In this situation we have now

$$\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s) \subseteq \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s) \subseteq \bigcap_{L\in\mathcal{L}(\mathcal{Q})} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s) = \mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s).$$

If we assume only that each sequence $\check{q}^n$ should satisfy (ai) with $\sup_{n\in\Lambda} C_n \le D < +\infty$ (where C_n are the constants occurring in (ai)), then $l=(l_k)_k$ satisfies (ai) with constant D.

(d) This item of the proof is of course inspired by [23, 1.6. Theorem (2)], we start with the first part. For given sequences $M,N \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$ we define $L=(L_k)_k$ by $L_k := \min\{M_k,N_k\}$. We prove now all desired properties for L^{lc}, i.e. we show $L^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$:
(i) $Q^x \preceq L^{\text{lc}}$ for all $x\in\Lambda$: First $Q^x \preceq L$ for each $x \in \Lambda$ holds immediately by definition. Now proceed as in (a): By our assumption on the matrix $\mathcal{Q}$ (property $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$) we get also $\liminf_{k\to\infty}(l_k)^{1/k} > 0$, where $l_k := \frac{L_k}{k!}$. We use [35, 2.15. Theorem] to conclude that $\mathcal{E}_{\{L^{\text{lc}}\}}(U,\mathbb{R}^s) = \mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$ and so $\mathcal{E}_{\{Q^x\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{L\}}(\mathbb{R},\mathbb{R}) = \mathcal{E}_{\{L^{\text{lc}}\}}(\mathbb{R},\mathbb{R})$ for each $x\in\Lambda$. But each Q^x and L^{lc} are weakly log. convex, hence $Q^x \preceq L^{\text{lc}}$ for each x follows, too.
(ii) By definition $L^{\text{lc}} \le L \le M,N$ holds.
(iii) L^{lc} has property (nq): Since both mappings $k\mapsto (M_k)^{1/k}$ and $k \mapsto (N_k)^{1/k}$ are increasing we also have that $k \mapsto (L_k)^{1/k}$ is increasing and

$$\sum_{k\ge1}\frac{1}{(L_k)^{1/k}} \le \sum_{k\ge1}\frac{1}{M_k^{1/k}} + \sum_{k\ge1}\frac{1}{N_k^{1/k}} < +\infty.$$

Hence $L=L^{\text{i}}$ and we have shown the second condition of [15, Theorem 1.3.8.]. This implies that $\mathcal{E}_{\{L\}}$ is not quasi-analytic (contains functions with compact support) and L^{lc}, the log. convex minorant of L, satisfies property (nq).

Now we prove the second part: If $M,N\in\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$, then we have to show that we can find $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ with $L \le M,N$.
For this we put $m_k := \frac{M_k}{k!}$, $n_k := \frac{N_k}{k!}$ and define $l_k := \min\{m_k,n_k\}$ with $l_k := \frac{L_k}{k!}$ or also equivalently $L_k := \min\{M_k,N_k\}$. Hence by definition $L\le M,N$ follows immediately.
By applying 8.5.4 we obtain that (ai) is satisfied for the sequence $l=(l_k)_k$, too.
We have to prove now $L^{\text{lc}}\in\mathcal{L}(\mathcal{Q})_{\text{lc}}$:
For this we put $P_k := \min\{M_k^{\text{lc}}, N_k^{\text{lc}}\}$ and first we show $P^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$. The mapping $k\mapsto (P_k)^{1/k}$ is increasing since both mappings $k \mapsto \left(M_k^{\text{lc}}\right)^{1/k}$ and $k\mapsto \left(N_k^{\text{lc}}\right)^{1/k}$ are increasing. Moreover we get

$$\sum_{k\ge1}\frac{1}{(P_k)^{1/k}} \le \sum_{k\ge1}\frac{1}{\left(M_k^{\text{lc}}\right)^{1/k}} + \sum_{k\ge1}\frac{1}{\left(N_k^{\text{lc}}\right)^{1/k}} < +\infty,$$

and the sum is finite because by assumption $M^{\text{lc}},N^{\text{lc}}\in\mathcal{L}(\mathcal{Q})_{\text{lc}}$ holds. So $P=P^{\text{i}}$ and we have shown the second condition of [15, Theorem 1.3.8.], hence $\mathcal{E}_{\{P\}}$ is not quasi-analytic. By definition of P we get $Q^x\preceq P$ for all $x\in\Lambda$ and by assumption property $(\mathcal{M}_{\{\mathrm{C}^\omega\}})$ is satisfied for $\mathcal{Q}$. Hence by applying once again [35, 2.15. Theorem] we get $\mathcal{E}_{\{P^{\text{lc}}\}}(U,\mathbb{R}^s) = \mathcal{E}_{\{P\}}(U,\mathbb{R}^s)$ and so $\mathcal{E}_{\{Q^x\}}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{E}_{\{P\}}(\mathbb{R},\mathbb{R}) = \mathcal{E}_{\{P^{\text{lc}}\}}(\mathbb{R},\mathbb{R})$ which proves $Q^x \preceq P^{\text{lc}}$ for each $x\in\Lambda$. Of course by the D.-C.-Theorem (nq) is satisfied for P^{lc}, too, which proves the first claim.

Now by construction we have $P^{\mathrm{lc}} \le P \le L$ and so by definition we get $P^{\mathrm{lc}} \le L^{\mathrm{lc}}$ (L^{lc} is the largest log. convex minorant of L), hence $L^{\mathrm{lc}} \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ follows.

(e) Let $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ (assume w.l.o.g. $1 = L_0 \le L_1$) be given and put

$$N_k := L_1 \cdot L_{k-1} \text{ for } k \ge 1, \qquad N_0 := 1.$$

(i) By definition it's clear that $N = (N_k)_k$ is weakly log. convex and it supports L via property $(\mathcal{M}_{(\mathrm{dc})})$, i.e. $N_{j+1} \le C^{j+1} \cdot L_j$ holds for a constant $C \ge 1$ and all $j \in \mathbb{N}$. We have to show now: $N \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$.
(ii) $Q^x \preceq N$ holds for each $x \in \Lambda$: Since $\mathcal{Q}$ has $(\mathcal{M}_{\{\mathrm{dc}\}})$, for all $x \in \Lambda$ there exist constants $C, D \ge 1$ and another index x' such that $Q^x_k \le C^k \cdot Q^{x'}_{k-1} \le C^k \cdot D^k \cdot L_{k-1} = (L_1)^{-1} \cdot (CD)^k \cdot N_k$ for all $k \ge 1$.
(iii) N has property (nq): Since L is log. convex, we obtain $L_k^2 \le L_1 \cdot L_{2k-1} = N_{2k}$ for all $k \in \mathbb{N}$, moreover by weakly log. convexity of N the mapping $k \mapsto (N_k)^{1/k}$ is increasing. Because we have property (nq) for L by assumption we obtain

$$\sum_{k\ge2} \frac{1}{(N_k)^{1/k}} \le 2 \cdot \sum_{k\ge1} \frac{1}{(N_{2k})^{1/(2k)}} \le 2 \cdot \sum_{k\ge1} \frac{1}{(L_k)^{1/k}} < +\infty$$

which shows property (nq) for N, too.

Now the second part of this statement: For $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ (w.l.o.g. assume $L_1 = L_0 = 1$) we put $l_k := \frac{L_k}{k!}$ and then define for $n_k := \frac{N_k}{k!}$

$$n_k := l_{k-1} \iff N_k = k \cdot L_{k-1}, \text{ for } k \ge 1, \qquad n_0 = N_0 := 1.$$

(i) Of course by definition N supports L via property $(\mathcal{M}_{(\mathrm{dc})})$, because $N_{j+1} = (j+1) \cdot L_j \le C^{j+1} \cdot L_j$ holds for all $j \in \mathbb{N}$ with a constant $C \ge 1$.
(ii) $Q^x \preceq N$ for each $x \in \Lambda$ follows as in (ii) in the first part of (e): $Q^x_k \le C^k \cdot Q^{x'}_{k-1} \le C^k \cdot D^k \cdot L_{k-1} = k^{-1} \cdot (CD)^k \cdot N_k \le (CD)^k \cdot N_k$ is valid for all $k \ge 1$.
(iii) We prove now (ai) for n: Because (ai) for l is satisfied by assumption, we conclude as follows: There exists a constant $H \ge 1$ such that for all $j, k \in \mathbb{N}$ with $2 \le j \le k$ we have

$$(n_j)^{1/(j-1)} = (l_{j-1})^{1/(j-1)} \le H \cdot (l_{k-1})^{1/(k-1)} = H \cdot (n_k)^{1/(k-1)},$$

and this is precisely condition (ai_K) for the sequence n. We can show now, as in the first part of 8.2.4, that the sequence n has property (ai): First by assumption on the matrix $\mathcal{Q}$ (property $(\mathcal{M}_{\{C^\omega\}})$) we have $\inf_{j\in\mathbb{N}_{>0}} (n_j)^{1/j} \ge C > 0$ (and we can assume from now on $C \le 1$) and so for $2 \le q \le p$ we can estimate as follows:

$$\begin{aligned} \frac{n_q}{H^{q-1} \cdot C^{q-1}} &\le \left(\frac{n_p}{C^{p-1}}\right)^{(q-1)/(p-1)} = \left(\frac{n_p}{C^p}\right)^{(q-1)/(p-1)} \cdot C^{(q-1)/(p-1)} \\ &\le \left(\frac{n_p}{C^p}\right)^{q/p} \cdot C^{(q-1)/(p-1)} \le \frac{1}{C^q} \cdot (n_p)^{q/p}. \end{aligned}$$

The first inequality holds by (ai_K), the second because $1 \le \frac{n_p}{C^p}$ for each p and $\frac{q-1}{p-1} \le \frac{q}{p} \Leftrightarrow q \le p$. Finally take the q-th root.
If $1 = q = p$, then for (ai) nothing is to prove. Finally, if $1 = q$ and $2 = p$ we need $N_1 = n_1 \le (n_2)^{1/2} \Leftrightarrow 2 \cdot (N_1)^2 \le N_2 \Leftrightarrow 2 \cdot L_0^2 \le 2 \cdot L_1$, so $L_1 \ge 1$ is sufficient which holds by assumption (and can be assumed w.l.o.g.).

(iv) It remains to prove $N^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$: For this we introduce the sequence $P = (P_k)_k$ by

$$P_k := L_1 \cdot L^{\text{lc}}_{k-1} \leq L_1 \cdot L_{k-1} \leq k \cdot L_{k-1} \quad \text{for } k \geq 1, \qquad P_0 := 1.$$

We show $P \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$: First P is clearly (weakly) log. convex and since by assumption $L^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$ we get as in *(iii)* in the first part of *(e)* property (nq) for P. Finally by $(\mathcal{M}_{\{\text{dc}\}})$ for $\mathcal{Q}$ we obtain as in *(ii)* in the first part of *(e)* also $Q^x \preceq P$ for all $x \in \Lambda$.
P is (weakly) log. convex and $P \leq N$, hence by definition $P \leq N^{\text{lc}}$ is satisfied and so $N^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$, too. □

Proposition 11.1.2. *The projective representations in 11.1.1 hold bornologically as well.*

Proof. The proof is analogous to [22, 5.3. Remarks] (proof which doesn't use the UBP). By 11.1.1 the representations hold as vector spaces.
Let $U \subseteq \mathbb{R}^r$ be non-empty open and $K \subseteq U$ an arbitrary compact set, moreover let $n \in \Lambda = \mathbb{N}_{>0}$ and $h > 0$ be given. Then the inclusion $\mathcal{E}_{\mathcal{Q},n,h}(K,\mathbb{R}^s) \to \mathcal{E}_{L,h_1}(K,\mathbb{R}^s)$ is continuous (bounded) for some $h_1 > 0$ (depending on n and h) whenever $L \in \mathcal{L}(\mathcal{Q})_\star$.
This implies that the inclusion $\varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{Q},l,h}(K,\mathbb{R}^s) \to \varinjlim_{h_1>0} \mathcal{E}_{L,h_1}(K,\mathbb{R}^s)$ is continuous, hence by definition of the spaces also the inclusion $\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s) \to \mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$.

Conversely, let $\mathcal{B}$ be a bounded set in the space $\varprojlim_{L} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$ where the projective limit is taken over all sequences $L \in \mathcal{L}(\mathcal{Q})_\star$. This means that $\mathcal{B}$ has to be bounded in each space $\mathcal{E}_{\{L\}}(U,\mathbb{R}^s)$.
We have to show now that $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s)$, and we can assume w.l.o.g. that $s = 1$ (by composing with a bounded linear functional). Let $K \subseteq U$ be an arbitrary compact set and we introduce the sequence $b = (b_k)_k$ by $b_k := \sup\left\{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r,\mathbb{R})} : x \in K, f \in \mathcal{B}\right\}$. By assumption we get: For each weight sequence $L \in \mathcal{L}(\mathcal{Q})_\star$ the set $\mathcal{B}$ is bounded in $\mathcal{E}_{\{L\}}(U,\mathbb{R})$. The proof of 11.1.1 shows $b \in \bigcap_{L\in\mathcal{L}(\mathcal{Q})_\star} \mathcal{F}_{\{L\}} = \mathcal{F}_{\{\mathcal{Q}\}}$ and this implies that $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{Q}\}}(K,\mathbb{R}) := \varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{Q},l,h}(K,\mathbb{R})$. Finally, by (7.3.5), the set $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{\mathcal{Q}\}}(K,\mathbb{R})$. □

We close this section with the following remark: Theorem 11.1.1 shows that we can reach several different projective descriptions for $\mathcal{E}_{\{\mathcal{Q}\}}$ (depending on the assumptions for $\mathcal{Q}$). But it will turn out in the next sections, where we are going to introduce a convenient setting for such class by using curves, that the "best" class for our purpose is $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$. Finally we prove for this class the following:

Lemma 11.1.3. *Let $\mathcal{Q}$ be a weight matrix with the basic assumptions in 11.1.1 and such that $\liminf_{k\to\infty}(q^x_k)^{1/k} > 0$ is satisfied for some $x \in \Lambda$. Then $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ is stable w.r.t. relation $\approx$.*

Proof. Let now $M \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ be given and consider N with $N \approx M$. First assume condition (ai) for M and let $1 \leq j \leq k$ be given, then $(n_j)^{1/j} \leq C_1 \cdot (m_j)^{1/j} \leq C_1 \cdot C \cdot (m_k)^{1/k} \leq C_1 \cdot C \cdot C_2 \cdot (n_k)^{1/k}$ holds for some constants $C, C_1, C_2 \geq 1$ with $n_k := \frac{N_k}{k!}$ and $m_k := \frac{M_k}{k!}$ (see also 8.2.2). For this part our additional assumption on the matrix $\mathcal{Q}$ is not needed, but now we use it and by applying [35, 2.15. Theorem] we get $\mathcal{E}_{\{M^{\text{lc}}\}} = \mathcal{E}_{\{M\}} = \mathcal{E}_{\{N\}} = \mathcal{E}_{\{N^{\text{lc}}\}}$, hence $M^{\text{lc}} \approx N^{\text{lc}}$ and which shows $N^{\text{lc}} \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$, too. □

11.2 A concrete example for $\mathcal{E}_{\{\mathcal{Q}\}}$

We want to construct an explicit example for a weight matrix $\mathcal{Q}$ satisfying all conditions in 11.1.1. For this we consider the following recursive construction, see also [23, 1.8., 1.9.]: We don't define the matrix $\mathcal{Q}$ directly, we start with the associated sequences $\check{Q}^l$ defined in (11.1.1). So we introduce the matrix $\check{\mathcal{Q}} := \{\check{Q}^n = (\check{Q}^n_k)_{k\ge 0} : n \in \mathbb{N}_{>0}\}$ recursively by

$$\check{Q}^1_k := k^k \cdot \log(k+\exp(1))^k \quad \text{for } k \ge 1, \quad \check{Q}^1_0 := 1$$

and for $n \ge 2$ we put

$$\check{Q}^n_k := \check{Q}^{n-1}_k \cdot (\log^n(k))^k \text{ for } k \ge \kappa_n, \quad \check{Q}^n_k = \check{Q}^{n-1}_k \text{ for } 0 \le k \le \kappa_n - 1,$$

where κ_n should denote the smallest integer number greater than $(\underbrace{\exp\circ\cdots\circ\exp}_{n-\text{times}})(1)$ and we have put $\log^n := \underbrace{\log\circ\cdots\circ\log}_{n-\text{times}}$.

In [23, 1.9.] it is pointed out that each sequence $\check{Q}^n$ is log. convex, $\check{Q}^n$ satisfies (mg) and finally doesn't satisfy condition (nq). By definition clearly all $\check{Q}^n$ are pairwise not equivalent w.r.t. $\approx$, $\lim_{k\to\infty}(\check{Q}^n_k)^{1/k} = +\infty$ follows for each $n \in \Lambda$ and finally the mapping $k \mapsto \frac{(\check{Q}^{n_2}_k)^{1/k}}{(\check{Q}^{n_1}_k)^{1/k}}$ is increasing for $n_2 \ge n_1$. But moreover by induction and [23, 1.8.] we also see, that each sequence $\check{q}^n = (\check{q}^n_k)_k$, where $\check{q}^n_k := \frac{\check{Q}^n_k}{k!}$, is log. convex (i.e. condition (slc) is satisfied for each $n \in \Lambda$ separately). Hence the mapping $k \mapsto (\check{q}^n_k)^{1/k}$ is increasing for each $n \in \Lambda$ automatically and $\lim_{k\to\infty}(\check{q}^n_k)^{1/k} = +\infty$ for each $n \ge 2$.

Now we can introduce the matrix $\mathcal{Q} := \{Q^n : 2 \le n \in \mathbb{N}_{>0}\}$ by (see item (2) in the remarks in the previous section)

$$(Q^n_k)^{1/k} := (\check{Q}^{n-1}_k)^{1/k} \cdot \left(1 + \sum_{j=1}^{k} \frac{1}{(\check{Q}^{n-1}_j)^{1/j}}\right).$$

By an iterated application of the chain-rule we get $(\log^n)'(x) = \frac{1}{\prod_{i=0}^{n-1}\log^i(x)}$, where $\log^0(x) := x$, and so by comparing the corresponding integrals we can find constants $C_1, C_2 \ge 1$ such that for all $k \ge \kappa_n$ we get:

$$C_1 \cdot \log^n(k) \le \sum_{j=\kappa_n}^{k} \frac{1}{(\check{Q}^{n-1}_j)^{1/j}} \le C_2 \cdot \log^n(k).$$

Hence we obtain

$$C_3 \cdot (\check{Q}^n_k)^{1/k} \le (Q^n_k)^{1/k} \le C_4 \cdot (\check{Q}^n_k)^{1/k}$$

for some constants $C_3, C_4 \ge 1$ and all $k \ge \kappa_n$. Thus we have shown $\check{Q}^n \approx Q^n$ for each $n \in \Lambda$ which implies immediately both $\mathcal{Q}\{\approx\}\check{\mathcal{Q}}$ and $\mathcal{Q}(\approx)\check{\mathcal{Q}}$. Finally all assumptions in $(a)-(e)$ in 11.1.1 are satisfied.

11.3 Convenient setting for some quasi-analytic classes $\mathcal{E}_{\{\mathcal{Q}\}}$ by using curves

In this section, to derive a convenient setting for some quasi-analytic classes of Roumieu-matrix-type $\mathcal{E}_{\{\mathcal{Q}\}}$, we have to use the proven projective descriptions of the previous section and so we consider from now on always weight matrices

$$\mathcal{Q} := \{Q^n : n \in \Lambda = \mathbb{N}_{>0} : \ (\mathcal{M}_{\text{sc}}),\ Q^n \text{ doesn't satisfy (nq)} \ \forall\, n \in \Lambda\},$$

such $(a) - (e)$ in 11.1.1 are satisfied. Instead of $(\mathcal{M}_{\{C^\omega\}})$ we will sometimes also have to use the little bit stronger condition

$$\exists\, n_0 \in \Lambda : \ \lim_{k\to\infty} (q_k^{n_0})^{1/k} = +\infty, \tag{11.3.1}$$

which is much weaker than $(\mathcal{M}_{(C^\omega)})$. Note that the example of the previous section satisfies all these assumptions.
We summarize the consequences: For $r, s \in \mathbb{N}_{>0}$ and non-empty open $U \subseteq \mathbb{R}^r$ we obtain (bornologically)

$$\mathcal{E}_{\{\mathcal{Q}\}}(U,\mathbb{R}^s) = \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\text{lc}}} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s) = \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}} \mathcal{E}_{\{L\}}(U,\mathbb{R}^s),$$

and by (d) in 11.1.1 for both matrices $\mathcal{L}(\mathcal{Q})_{\text{lc}}$ and $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ we get $(\mathcal{M}_{(\text{fil})})$, i.e.

$$\forall\, L^1, L^2 \in \mathcal{L}(\mathcal{Q})_\star \ \exists\, L \in \mathcal{L}(\mathcal{Q})_\star : \ L \le L^1, L^2.$$

By property (e) in 11.1.1 both weight matrices $\mathcal{L}(\mathcal{Q})_{\text{lc}}$ and $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ satisfy also condition $(\mathcal{M}_{(\text{dc})})$, more precisely

$$\forall\, L \in \mathcal{L}(\mathcal{Q})_\star \ \exists\, N \in \mathcal{L}(\mathcal{Q})_\star \ \exists\, C \ge 1 : \ \forall\, j \in \mathbb{N} : \ N_{j+1} \le C^{j+1} \cdot L_j.$$

By using these properties we are able to prove closedness under composition for the considered projective structures:

Lemma 11.3.1. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section (in fact $(\mathcal{M}_{\{C^\omega\}})$ instead of (11.3.1) is sufficient to prove this result). Then we obtain:*

- (a) *The weight matrix $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ satisfies property $(\mathcal{M}_{(\text{FdB})})$ in the most general sense, which means*
$$\forall\, L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}} \ \exists\, N \ \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}} : \quad n^\circ \preceq l, \tag{11.3.2}$$
where we have put $n^\circ = (n_k^\circ)_k$, $n_k^\circ := \max\left\{n_j \cdot n_{\alpha_1} \cdots n_{\alpha_j} : \alpha_i \in \mathbb{N}_{>0}, \sum_{i=1}^j \alpha_i = k\right\}$ and $n_0^\circ := 1$, finally $l_k := \frac{L_k}{k!}$ and $n_k := \frac{N_k}{k!}$. We will also write $L^{(\text{FdB})} := N$.
- (b) *In this situation the class $\mathcal{E}_{\{\mathcal{Q}\}}$ is closed under composition.*

Proof. (a) First we recall: The matrix $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ has clearly always property $(\mathcal{M}_{(\text{ai})})$, more precisely (ai) holds for each sequence in $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$. By assumption we have property $(\mathcal{M}_{\{\text{dc}\}})$ for the matrix $\mathcal{Q}$ and so we get by (e) in 11.1.1 also property $(\mathcal{M}_{(\text{dc})})$ for the matrix $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$. By using (d) in 11.1.1 we finally get property $(\mathcal{M}_{(\text{fil})})$ for $\mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$.
So we can prove the desired property as in (2) in 8.2.3. In the following put $m_k := \frac{M_k}{k!}$, $l_k := \frac{L_k}{k!}$ resp. $n_k := \frac{N_k}{k!}$. To prove (11.3.2) let now $\alpha_1, \dots, \alpha_j \in \mathbb{N}_{>0}$, $j \ge 1$, be given with $\alpha_1 + \cdots + \alpha_j = k$. Then first property (ai) implies that there exists $H \ge 1$ with

$$l_{\alpha_1} \cdots l_{\alpha_j} \le H^{\alpha_1+\cdots+\alpha_j} \cdot (l_k)^{\alpha_1/k} \cdots (l_k)^{\alpha_j/k} \le H^k \cdot l_k$$

for each sequence $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$. But since $l_0 = 1$ (w.l.o.g.) this estimate holds also for the cases if $\alpha_i = 0$ for some $1 \le i \le j$ and if $\alpha_1 = \cdots = \alpha_j = 0$ (and so $k = 0$).

Furthermore by property $(\mathcal{M}_{(\mathrm{dc})})$ we get that for all $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we can find $C \geq 1$ and $M \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ such that $m_{\alpha_1} \cdots m_{\alpha_j} \leq C^{\alpha_1+\dots+\alpha_j} \cdot l_{\alpha_1-1} \cdots l_{\alpha_j-1}$. So we can estimate as follows:

$$n_j \cdot n_{\alpha_1} \cdots n_{\alpha_j} \leq l_j \cdot m_{\alpha_1} \cdots m_{\alpha_j} \leq l_j \cdot C^k \cdot l_{\alpha_1-1} \cdots l_{\alpha_j-1} \leq H_1^k \cdot l_j \cdot l_{k-j} \leq H_2^k \cdot l_k.$$

The first inequality holds now by $(\mathcal{M}_{(\mathrm{fil})})$: We can find a sequence $N \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ with $n \leq l, m \Leftrightarrow N \leq L, M$.

(b) This follows from (a): More precisely by applying property $(\mathcal{M}_{(\mathrm{FdB})})$, the proof of the Beurling-case (b) in Theorem 8.3.1 holds and by 11.1.1 we obtain now closedness under composition for the class $\mathcal{E}_{\{\mathcal{Q}\}}$.

Compare this argumentation also with the remark below Theorem 8.3.1 and also with the fifth remark before Theorem 11.1.1. □

The previous result implies

Corollary 11.3.2. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section, then part (b) (Roumieu-case) in the important characterizing result 8.6.1 is valid, hence all stability properties there are satisfied.*

The next definitions are analogous to [21, 3.1.] and [23, 1.10.] (see there for more details): Let E and F be convenient vector spaces, let $U \subseteq E$ be c^∞-open, then for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we define

$$\begin{aligned} \mathcal{E}_{\{L\},\mathrm{curve}}(U,F) :&= \{f : f \circ c \in \mathcal{E}_{\{L\}}(\mathbb{R},F)\ \forall\, c \in \mathcal{E}_{\{L^{(\mathrm{FdB})}\}}(\mathbb{R},U)\} \qquad (11.3.3)\\ &= \{f : \alpha \circ f \circ c \in \mathcal{E}_{\{L\}}(\mathbb{R},\mathbb{R})\ \forall\, c \in \mathcal{E}_{\{L^{(\mathrm{FdB})}\}}(\mathbb{R},U),\ \forall\, \alpha \in F^*\}. \qquad (11.3.4) \end{aligned}$$

The space $\mathcal{E}_{\{L\},\mathrm{curve}}(U,F)$ is of course supplied with the initial locally convex structure induced by all linear mappings $\mathcal{E}_{\{L\},\mathrm{curve}}(c,\alpha) : f \mapsto \alpha \circ f \circ c \in \mathcal{E}_{\{L\}}(\mathbb{R},\mathbb{R})$, for $\alpha \in F^*$ and $c \in \mathcal{E}_{\{L^{(\mathrm{FdB})}\}}(\mathbb{R},U)$, so

$$\mathcal{E}_{\{L\},\mathrm{curve}}(U,F) = \bigcap_{\alpha\in F^*, c\in\mathcal{E}_{\{L^{(\mathrm{FdB})}\}}(\mathbb{R},U)} \{f : \alpha \circ f \circ c \in \mathcal{E}_{\{L\}}(\mathbb{R},\mathbb{R})\} \qquad (11.3.5)$$

and $\mathcal{E}_{\{L\},\mathrm{curve}}(U,F)$ is convenient as c^∞-closed subspace in the product (see also 10.3.1). This definition should be compared with [21, 3.1. Definition]: Note that each occurring weight sequence there was assumed to be strongly log. convex, thus $L = L^{(\mathrm{FdB})}$ is valid automatically (see e.g. [21, 2.9. Lemma]).

For a weight matrix $\mathcal{Q}$ as assumed at the beginning of this section we put now

$$\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F) := \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}} \mathcal{E}_{\{L\},\mathrm{curve}}(U,F), \qquad (11.3.6)$$

supplied with the initial locally convex structure and so $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F)$ is also convenient. By definition we immediately get that the composition of $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$-mappings is again $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$. By 11.1.1 and 11.3.7 this definition coincides with the classical one if both E and F are finite-dimensional.

We denote by $f^{(k)}(x)$ the k-th-order Fréchet derivative of f at the point x and introduce also the spaces for arbitrary weight sequences L

$$\mathcal{E}^{\mathrm{b}}_{\{L\}}(U,F) := \{f \in \mathcal{E}(U,F) :\ \forall\, B\ \forall\, K \subseteq U \cap E_B\ \exists\, h > 0 :$$

$$\left\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{h^k\cdot L_k}:k\in\mathbb{N},\ x\in K,\ \|v_i\|_B\le 1\right\}\text{is bounded in }F\}$$

$$=\{f\in\mathcal{E}(U,F):\ \forall\, B\,\forall\, K\text{ compact }K\subseteq U\cap E_B\,\exists\, h>0:$$

$$\left\{\frac{d_v^k f(x)}{h^k\cdot L_k}:k\in\mathbb{N},\ x\in K,\ \|v_i\|_B\le 1\right\}\text{is bounded in }F\}.$$

B runs through all closed absolutely convex bounded subsets $B\subseteq E$ and E_B is the (complete) vector space generated by B with the associated *Minkowski functional* $\|v\|_B:=\inf\{\lambda\ge 0: v\in\lambda\cdot B\}$. K is a compact set in E_B w.r.t. $\|\cdot\|_B$. If both E and F are Banach-spaces, then $\mathcal{E}^{\mathrm{b}}_{\{L\}}(U,F)=\mathcal{E}_{\{L\}}(U,F)$ is satisfied, where the latter space is defined by using jets in [22, 4.1.] and see also (12.1.2) for the more general weight matrix case.

We are going to introduce also the following classes, see the definitions given in [22, 4.3.] (and compare also with (12.1.3), (12.1.4) and (12.1.5) below):

$$\mathcal{E}_{\{L\}}(U,F):=\left\{f\in\mathcal{E}(U,F):\forall\,\alpha\in F^*:\forall\, B:\alpha\circ f\circ i_B\in\mathcal{E}_{\{L\}}(U_B,\mathbb{R})\right\},\tag{11.3.7}$$

where B is running again through all closed absolutely convex bounded subsets in E and the mapping $i_B:E_B\to E$ denotes the inclusion of E_B in E and we write $U_B:=i_B^{-1}(U)$. The initial locally convex structure is now induced by all linear mappings

$$\mathcal{E}_{\{L\}}(i_B,\alpha):\mathcal{E}_{\{L\}}(U,F)\longrightarrow\mathcal{E}_{\{L\}}(U_B,\mathbb{R}),\qquad f\mapsto\alpha\circ f\circ i_B.\tag{11.3.8}$$

$\mathcal{E}_{\{L\}}(U,F)\subseteq\prod_{\alpha,B}\mathcal{E}_{\{L\}}(U_B,\mathbb{R})$ is a convenient vector space as c^∞-closed subspaces in the product: Smoothness can be tested by composing with inclusions $E_B\to E$ and $\alpha\in F^*$ as mentioned in [20, 2.14.4, 1.8], see also 11.1.1. Hence we obtain the representation

$$\mathcal{E}_{\{L\}}(U,F):=\{f\in F^U:\ \forall\,\alpha\in F^*\,\forall\, B:\alpha\circ f\circ i_B\in\mathcal{E}_{\{L\}}(U_B,\mathbb{R})\}.\tag{11.3.9}$$

For a weight matrix $\mathcal{Q}$ as defined at the beginning of this section we have now

$$\mathcal{E}_{\{\mathcal{Q}\}}(U,F)\underbrace{=}_{(\star)}\bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}}\mathcal{E}_{\{L\}}(U,F),\tag{11.3.10}$$

where $(\star)$ holds by the definition of the structure on the left hand side in (12.1.3) and on the right hand side in (11.3.7) by applying the characterizing Theorem 11.1.1 to the spaces $\mathcal{E}_{\{Q\}}(U_B,\mathbb{R})$.

First we are going to prove resp. recall some results in [23] for each $L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ resp. $L\in\mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ separately.
We start with the following Lemma, see [23, 2.2 Lemma] and [21, 3.4. Lemma]:

Lemma 11.3.3. *Let E be a Banach space and $c:\mathbb{R}\to E$ a smooth curve. Then for each $L\in\mathcal{L}(\mathcal{Q})_\star$ the following conditions are equivalent:*

(*i*) *The curve c is $\mathcal{E}_{\{L\}}=\mathcal{E}^{\mathrm{b}}_{\{L\}}$.*

(*ii*) *For each sequence $(r_k)_k\in\mathcal{R}_{\mathrm{roum}}$ and each compact set $K\subseteq\mathbb{R}$ the set*

$$\left\{\frac{c^{(k)}(a)}{L_k}\cdot r_k: a\in K, k\in\mathbb{N}\right\}$$

is bounded in E.

(*iii*) *For each sequence* $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$ *and each compact set* $K \subseteq \mathbb{R}$ *there exists some* $\delta > 0$ *such that the set*

$$\left\{ \frac{c^{(k)}(a)}{L_k} \cdot r_k \cdot \delta^k : a \in K, k \in \mathbb{N} \right\}$$

is bounded in E.

Proof. We replace in [23, 2.2. Lemma] resp. [21, 3.4. Lemma] the sequence $(k! \cdot M_k)_k$ by $L = (L_k)_k$ for $L \in \mathcal{L}(\mathcal{Q})_\star$. □

By using this Lemma we get the following consequence, see [23, 2.3. Lemma] and [21, 3.5. Lemma]:

Lemma 11.3.4. *Let* E *be a convenient vector space and* $\mathcal{S}$ *be a family of bounded linear functionals on* E *which together detect bounded sets, i.e.* $B \subseteq E$ *is bounded if and only if* $\alpha(B)$ *is bounded for all* $\alpha \in \mathcal{S}$. *Then for each* $L \in \mathcal{L}(\mathcal{Q})_\star$ *we obtain: A curve* $c : \mathbb{R} \to E$ *is* $\mathcal{E}_{\{L\}}$ *if and only if* $\alpha \circ c : \mathbb{R} \to \mathbb{R}$ *is* $\mathcal{E}_{\{L\}}$ *for all* $\alpha \in \mathcal{S}$.

Proof. We replace in [23, 2.3. Lemma] resp. [21, 3.5. Lemma] the sequence $(k! \cdot M_k)_k$ by $L = (L_k)_k$ for $L \in \mathcal{L}(\mathcal{Q})_\star$. □

But we get also the following result, see [22, 4.8. Lemma]:

Lemma 11.3.5. *Let* E, F *be Banach-spaces,* $U \subseteq E$ *open and* $f : U \to F$ *a* $\mathcal{E}$*-mapping. Then for each* $L \in \mathcal{L}(\mathcal{Q})_\star$ *the following are equivalent:*

(1) f *is* $\mathcal{E}_{\{L\}} = \mathcal{E}^{\text{b}}_{\{L\}}$.

(2) *For each compact* $K \subseteq U$ *and for each sequence* $(r_k)_k \in \mathcal{R}_{\text{roum}}$ *the set*

$$\left\{ \frac{f^{(k)}(a)(v_1, \ldots, v_k)}{L_k} \cdot r_k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1 \right\}$$

is bounded in F .

(3) *For each compact set* $K \subseteq U$ *and for each sequence* $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$ *there exists a number* $\varepsilon > 0$ *such that the set*

$$\left\{ \frac{f^{(k)}(a)(v_1, \ldots, v_k)}{L_k} \cdot r_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1 \right\}$$

is bounded in F.

Proof. We replace in [22, 4.8. Lemma] the sequence $(k! \cdot M_k)_k$ by $L = (L_k)_k$ for $L \in \mathcal{L}(\mathcal{Q})_\star$. □

In the next step we reformulate the special-curve-lemma in our new most general setting (see also [21, 3.6. Lemma], [23, 2.5.] and our abstract matrix-version 10.6.2):

Proposition 11.3.6. *Let* $\mathcal{Q}$ *be a weight matrix as assumed at the beginning of this section with* (11.3.1).
Then for arbitrary sequences $L \in \mathcal{L}(\mathcal{Q})_{\text{lc}}$ *resp.* $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ *we get the following: There exist sequences of real numbers* $(\lambda_k)_k$ *with* $\lambda_k \to 0$, $(t_k)_k$ *with* $t_k \to t_\infty$ *and* $(s_k)_k$ *with* $s_k > 0$, *such that for* $\left(\frac{1}{\lambda}\right)$*-converging sequences* $(x_n)_n$ *and* $(v_n)_n$ *in a convenient vector space* E, *there exists a strongly uniform* $\mathcal{E}_{\{L\}}$*- resp.* $\mathcal{E}_{\{L^{\text{lc}}\}}$*-curve* $c : \mathbb{R} \to E$ *with* $c(t_k + t) = x_k + t \cdot v_k$ *for* $|t| \leq s_k$.

Recall notation: A sequence $(x_n)_n$ in a locally convex vector space E is called *Mackey-convergent* to a point x, if there exists an increasing sequence of positive real numbers $(\lambda_n)_n$ with $\lim_{n\to\infty}\lambda_n = +\infty$ such that $(\lambda_n \cdot (x_n - x))_n$ is bounded. If the sequence $\lambda := (\lambda_n)_n$ is fixed, then $(x_n)_n$ is called λ-*converging*.

Proof. We put $l_k := \frac{L_k}{k!}$ and $l^{\mathrm{lc}}_k := \frac{L^{\mathrm{lc}}_k}{k!}$, then by our assumptions on $\mathcal{Q}$ we get $\lim_{k\to\infty}(l_k)^{1/k} = +\infty$ resp. $\lim_{k\to\infty}(l^{\mathrm{lc}}_k)^{1/k} = +\infty$ for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ resp. $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$. So for an arbitrary sequence $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ resp. $L^{\mathrm{lc}} \in \mathcal{L}(\mathcal{Q})_{\mathrm{lc}}$ condition (10.6.1) holds and by Lemma 10.6.1 we can find a non-quasi-analytic weakly log. convex weight sequence N with $N \lhd L$ resp. $N \lhd L^{\mathrm{lc}}$ and $\lim_{k\to\infty}(n_k)^{1/k} = +\infty$, $n_k := \frac{N_k}{k!}$. So we can replace in the original proof of the special curve lemma [21, 3.6. Lemma] the sequence $(k! \cdot \overline{M}_k)_k$ by $N = (N_k)_k$ and $(k! \cdot M_k)_k$ by $L = (L_k)_k$ resp. $L^{\mathrm{lc}} = (L^{\mathrm{lc}}_k)_k$. $\square$

With these preparations we can formulate the analogous result to [23, 2.7. Theorem] which should be compared also with 10.7.1:

Theorem 11.3.7. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section with* (11.3.1). *Let E be a convenient vector space, $U \subseteq E$ a c^∞-open subset, furthermore let F be a Banach space and a mapping $f : U \to F$ be given.*
(*a*) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ separately we obtain the following implications* (1) ⇒ (2) ⇒ (3) ⇒ (4) *for*

(1) *$f \in \mathcal{E}_{\{L\},\mathrm{curve}}(U, F)$, i.e. $f \circ c \in \mathcal{E}_{\{L\}}(\mathbb{R}, F)$ for each $c \in \mathcal{E}_{\{L^{(\mathrm{FdB})}\}}(\mathbb{R}, U)$.*

(2) *$f|_{U\cap E_B} : E_B \supseteq U \cap E_B \to F$ is $\mathcal{E}_{\{L\}}$ for each closed bounded absolutely convex set B in E.*

(3) *$f \circ c$ is $\mathcal{E}_{\{L\}}$ for all $\mathcal{E}^{\mathrm{b}}_{\{L^{(\mathrm{FdB})}\}}$-curves c.*

(4) *$f \in \mathcal{E}^{\mathrm{b}}_{\{L\}}(U, F)$.*

(*b*) *TFAE:*

(1) *f is $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$, i.e. for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we have $f \in \mathcal{E}_{\{L\},\mathrm{curve}}(U, F)$.*

(2) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we have that $f|_{U\cap E_B} : E_B \supseteq U \cap E_B \to F$ is $\mathcal{E}_{\{L\}}$ for each closed bounded absolutely convex set B in E.*

(3) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we have $f \circ c$ is $\mathcal{E}_{\{L\}}$ for all $\mathcal{E}^{\mathrm{b}}_{\{L^{(\mathrm{FdB})}\}}$-curves c.*

(4) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we have $f \in \mathcal{E}^{\mathrm{b}}_{\{L\}}(U, F)$.*

Proof. (*a*) For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ separately we prove as in [23, 2.7. Theorem].
(1) ⇒ (2) Note that $E_B \to E$ is continuous and linear, hence all $\mathcal{E}_{\{L^{(\mathrm{FdB})}\}}$-curves c into the Banach-space E_B are also $\mathcal{E}_{\{L^{(\mathrm{FdB})}\}}$ into E, and so $f \circ c$ is $\mathcal{E}_{\{L\}}$ by assumption.
(2) ⇒ (3) This is clear since for any $L \in \mathcal{L}(\mathcal{Q})_\star$ we have $\mathcal{E}^{\mathrm{b}}_{\{L\}} \subseteq \mathcal{E}_{\{L\}}$ by definition.
(3) ⇒ (4) We are using the most general version of the special curve-lemma 11.3.6: Let now $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ be given, arbitrary but from now on fixed. As in the original proof we assume w.l.o.g. $E = E_B$, so E is Banach. For each $v \in E$ and $x \in U$ the iterated directional derivative $d^k_v f(x)$ exists, since f is $\mathcal{E}_{\{L\}}$ along affine lines. To show that f is smooth we check that $d^k_{v_n} f(x_n)$ is bounded for each $k \in \mathbb{N}$ and each Mackey-convergent sequences $(x_n)_n$ and $v_n \to 0$. Assume now that there exists $k \in \mathbb{N}$ and sequences $(x_n)_n$ and $(v_n)_n$ with $\|d^k_{v_n} f(x_n)\| \to +\infty$ for $n \to \infty$.

According to our chosen $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ we consider now the sequence $L^{(\text{FdB})} \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$. By passing to a subsequence we may assume that $(x_n)_n$ and $(v_n)_n$ are $(\frac{1}{\lambda_n})$-converging for the sequence $(\lambda_n)_n$ coming from the special-curve-lemma 11.3.6 which we have used for the sequence $L^{(\text{FdB})}$. So there exists a strongly uniform $\mathcal{E}_{\{(L^{(\text{FdB})})^{\text{lc}}\}}$-curve c (hence c is also strongly uniform $\mathcal{E}_{\{L^{(\text{FdB})}\}}$) in E with $c(t + t_n) = x_n + t \cdot v_n$ for t near 0, and for $(t_n)_n$ arising in 11.3.6. But then $\|(f \circ c)^{(k)}(t_n)\| = \|d^k_{v_n} f(x_n)\| \to +\infty$ for $n \to \infty$, a contradiction to our assumption in (3) and consequently f is smooth.

Now assume by contradiction that (4) is violated: Then there would exist a compact set $K \subseteq U$ and some sequence $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ such that for each $n \in \mathbb{N}$ there exist $k_n \in \mathbb{N}$, $x_n \in K$ and finally v_n with $\|v_n\| = 1$ such that

$$\|d^k_{v_n} f(x_n)\| > L_{k_n} \cdot \left(\frac{1}{\lambda_n^2}\right)^{k_n+1},$$

where we have put $C = h = \frac{1}{\lambda_n^2}$ and the sequence $(\lambda_n)_n$ is coming from the special-curve-lemma 11.3.6 which we have used again for the sequence $L^{(\text{FdB})}$. By passing to a subsequence we can assume that the x_n are $\frac{1}{\lambda}$-converging, and so there exists a strongly uniform $\mathcal{E}_{\{(L^{(\text{FdB})})^{\text{lc}}\}}$-curve c (hence c is also strong uniform $\mathcal{E}_{\{L^{(\text{FdB})}\}}$) in E with $c(t_n + t) = x_n + t \cdot \lambda_n \cdot v_n$ for t near 0. So we have $(f \circ c)^{(k)}(t_n) = \lambda_n^k \cdot d^k_{v_n} f(x_n)$ which implies

$$\left(\frac{\|(f \circ c)^{(k_n)}(t_n)\|}{L_{k_n}}\right)^{1/(k_n+1)} = \left(\lambda_n^{k_n} \cdot \frac{\|d^{k_n}_{v_n} f(x_n)\|}{L_{k_n}}\right)^{1/(k_n+1)} > \frac{1}{\lambda_n^{(k_n+2)/(k_n+1)}} \to +\infty$$

for $n \to \infty$. But this is a contradiction to our assumption for this sequence L: More precisely we have shown that $f \circ c \in \mathcal{E}_{\{L\}}$ is not satisfied but c is a strongly uniform $\mathcal{E}_{\{L^{(\text{FdB})}\}}$-curve.

(b) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are valid as before in (a).
So it remains to show (4) $\Rightarrow$ (1): Let $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ be arbitrary, then we have to show that the mapping $f \circ c$ is $\mathcal{E}_{\{L\}}$ for each $\mathcal{E}_{\{L^{(\text{FdB})}\}}$-curve c. For simplicity of the notation we write in the following $N := L^{(\text{FdB})} \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$.
For an arbitrary but fixed $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ we can use now the characterizing result Lemma 11.3.3: By (iii) there it suffices to prove that for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$ and each compact interval $I \subseteq \mathbb{R}$ there exists some $\delta > 0$ such that the set
$\left\{\frac{(f \circ c)^{(k)}(a)}{L_k} \cdot r_k \cdot \delta^k : a \in I, k \in \mathbb{N}\right\}$ is bounded.

So let now $L \in \mathcal{L}(\mathcal{Q})_{\text{ai,lc}}$ be given, arbitrary but from now on fixed. Then by (11.3.2) in 11.3.1 we have $n^\circ \preceq l$ in the notation there.
In the next step we use now 11.3.3 for this particular sequence N (so we consider from now on $\mathcal{E}_{\{N\}}$-curves c) and we replace in (ii) there the sequence $(r_k)_k$ by $(2^k \cdot r_k)_k$. Then for each $\alpha \in E^*$, each $(r_k)_k \in \mathcal{R}_{\text{roum}}$ and each compact interval $I \subseteq \mathbb{R}$ the set $\left\{\frac{(\alpha \circ c)^{(k)}(a)}{N_k} \cdot r_k \cdot 2^k : a \in I, k \in \mathbb{N}\right\}$ is bounded in $\mathbb{R}$.
Hence the set $\left\{\frac{c^{(k)}(a)}{N_k} \cdot r_k \cdot 2^k : a \in I, k \in \mathbb{N}\right\}$ is contained in some closed absolutely convex bounded subset $B \subseteq E$. Hence $c^{(k)} : I \to E_B$ is smooth and the set $K_k :=$

$\left\{\frac{c^{(k)}(a)}{N_k}\cdot r_k\cdot 2^k : a\in I\right\}$ is compact in E_B for each $k\in\mathbb{N}$. So each sequence $(x_n)_n$ in

$$K:=\left\{\frac{c^{(k)}(a)}{N_k}\cdot r_k : a\in I\right\}=\bigcup_{k\in\mathbb{N}}\frac{1}{2^k}\cdot K_k$$

has a cluster point in $K\cup\{0\}$: either there is a subsequence in one K_k or $2^{k_n}\cdot x_{k_n}\in K_{k_n}\subseteq B$ for $k_n\to\infty$, hence $x_{k_n}\to 0$ in E_B. This shows that $K\cup\{0\}$ is compact.

Now we use as in the original proof in [23, 2.7. Theorem] property ($\mathcal{M}_{(\mathrm{FdB})}$) (in the most general sense) and estimate as follows for $a\in I$ and $k\geq 1$:

$$\begin{aligned}\left\|\frac{(f\circ c)^{(k)}(a)}{L_k}\cdot r_k\right\| &\leq D^k\cdot\sum_{j=1}^{k}\sum_{\alpha\in\mathbb{N}^j_{>0},\alpha_1+\dots+\alpha_j=k}\underbrace{\frac{\|d^jf(c(a))\|_{L^j(E_B,F)}}{N_j}}_{\leq C\cdot h^j}\cdot\prod_{i=1}^{j}\underbrace{\frac{\|c^{(\alpha_i)}(a)\|_B}{N_{\alpha_i}}\cdot r_{\alpha_i}}_{\leq 2^{-\alpha_i}}\\ &\leq D^k\cdot\sum_{j=1}^{k}\binom{k-1}{j-1}C\cdot h^j\cdot\frac{1}{2^k}=C\cdot h\cdot\left(\frac{D}{2}\right)^k\cdot\sum_{j=1}^{k}\binom{k-1}{j-1}h^{j-1}\\ &\leq C\cdot h\cdot\left(\frac{D}{2}\right)^k\cdot(1+h)^{k-1}\leq C_1\cdot\left(\frac{D\cdot(1+h)}{2}\right)^k.\end{aligned}$$

So we have shown that the set $\left\{\frac{(f\circ c)^{(k)}(a)}{L_k}\cdot\left(\frac{2}{D\cdot(1+h)}\right)^k\cdot r_k\right\}$ is bounded, note that in the estimate the sequences L and N are precisely related via (11.3.2) and by assumption $f\in\mathcal{E}^{\mathrm{b}}_{\{N\}}(U,F)$ for this particular sequence $N\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$. □

Remarks:

(*i*) Note that we don't have strong log. convexity (slc) for each $L\in\mathcal{L}(\mathcal{Q})_\star$ separately as in [23] and which can be assumed for the constant weight matrix case $\mathcal{M}=\{M\}$ by [23, 1.6. Theorem] there. [21, 2.9. Lemma] shows that (slc) implies (FdB) for each occurring sequence L, so $L=L^{(\mathrm{FdB})}$ is satisfied automatically.

(*ii*) But by 9.2.1 we cannot expect (strong) log. convexity for $L\in\mathcal{L}(\mathcal{Q})_\star$ in the not constant weight matrix case. It would be sufficient to have (FdB) for each $L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ separately because then by 8.3.1 also each space $\mathcal{E}_{\{L\}}$ would be closed under composition. By (2) in 8.2.3 we would need condition (dc) for each $L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ but unfortunately this additional condition will not hold in general in the non-constant matrix case, see also remark 9.2.2.

(*iii*) To prove this important theorem also for the non-constant weight matrix case we have to endow the space $\mathcal{E}_{\{L\},\mathrm{curve}}$ with a little bit different structure, more precisely we have to test by $\mathcal{E}_{\{L^{(\mathrm{FdB})}\}}$-curves instead by $\mathcal{E}_{\{L\}}$-curves. As the proof of (4) ⇒ (1) shows, by using this structure we only need property ($\mathcal{M}_{(\mathrm{FdB})}$) (in the most general sense) to prove part (*b*) of Theorem 11.3.7.

We use 11.3.7 to show the analogous result to [23, 2.8. Corollary] (see also 12.5.1 below for the matrix-type version):

Corollary 11.3.8. *Let $\mathcal{Q}$ be a weight matrix as in 11.3.7 above, let E and F be convenient vector spaces, $U\subseteq E$ be c^∞-open and $f:U\to F$ be a mapping. Then the following statements are equivalent:*

(1) *f is $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$, i.e. for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we get that f is $\mathcal{E}_{\{L\},\mathrm{curve}}$.*

(2) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we get that $f|_{U\cap E_B} : E_B \supseteq U \cap E_B \to F$ is $\mathcal{E}_{\{L\}}$ for each closed bounded absolutely convex set B in E.*

(3) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ the mapping $f \circ c$ is $\mathcal{E}_{\{L\}}$ for all $\mathcal{E}^{\mathrm{b}}_{\{L^{(\mathrm{FdB})}\}}$-curves c.*

(4) *For each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ we get that for each absolutely convex 0-neighborhood $V \subseteq F$ we have $\pi_V \circ f \in \mathcal{E}^{\mathrm{b}}_{\{L\}}(U, F_V)$, where $\pi_V : F \to F_V$ denotes the natural mapping.*

Proof. Each of the statements holds for f if and only if it holds for $\pi_V \circ f$ for each absolutely convex 0-neighborhood $V \subseteq F$, and since F_V is a Banach space (F is assumed to be convenient), we can apply part (b) of Theorem 11.3.7. □

Now recall [22, 4.9. Defintion]:

Definition 11.3.9. *Let E be a convenient vector space. A $\mathcal{E}_{\{L\}}$-Banach-plot in E is a mapping $c : D \to E$, such that $c \in \mathcal{E}_{\{L\}}$ and $D = oE$ denotes the open unit ball in some Banach-space F.*

By using this definition and 11.3.5 we are able to prove now the analogous theorem to [22, 4.10. Theorem] (note that there the considered weight sequence is assumed to be strong log. convex, thus $L = L^{(\mathrm{FdB})}$ holds):

Theorem 11.3.10. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section, let $U \subseteq E$ be a c^∞-open subset in a convenient vector space E. Moreover let F be an arbitrary Banach-space and $f : U \to F$ a mapping, then we obtain or each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$:*

$$f \in \mathcal{E}_{\{\mathcal{Q}\}}(U, F) \Longrightarrow f \circ c \in \mathcal{E}_{\{L\}} \text{ for all } \mathcal{E}_{\{L^{(\mathrm{FdB})}\}} - \text{Banach-plots } c.$$

Proof. Of course we are going to use 11.3.5 and proceed analogously as in [22, 4.10. Theorem] resp. also in (4) ⇒ (1) in 11.3.7. We have to show that $f \circ c$ is $\mathcal{E}_{\{L\}}$ for each $\mathcal{E}_{\{L_{(\mathrm{FdB})}\}}$-Banach-plot $c : G \supseteq D \to E$, where D denotes the open unit ball in an arbitrary Banach-space G. So, by condition (3) in 11.3.5, we have to show: For each compact set $K \subseteq D$ and for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ there exists $\varepsilon > 0$ such that the set

$$\left\{ \frac{(f \circ c)^{(k)}(a)(v_1, \ldots, v_k)}{L_k} \cdot r_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}, \|v_i\|_E \le 1 \right\}$$

has to be bounded in F. So let $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ be arbitrary but from now on fixed and let $L^{(\mathrm{FdB})} \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ be the sequence coming from (11.3.2). We consider an arbitrary $L^{(\mathrm{FdB})}$-Banach-plot c and replace in (2) in 11.3.5 the sequence $(r_k)_k$ by $(2^k \cdot r_k)_k \in \mathcal{R}_{\mathrm{roum}}$. Then for each $\alpha \in E^*$ and each compact set $K \subseteq D$ the set

$$\left\{ \frac{(\alpha \circ c)^{(k)}(a)(v_1, \ldots, v_k)}{L^{(\mathrm{FdB})}_k} \cdot r_k \cdot 2^k : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1 \right\} \tag{11.3.11}$$

is bounded in $\mathbb{R}$. So the set

$$\left\{ \frac{c^{(k)}(a)(v_1, \ldots, v_k)}{L^{(\mathrm{FdB})}_k} \cdot r_k \cdot 2^k : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1 \right\}$$

is contained in some closed absolutely convex bounded subset B of E, hence

$$\frac{\|c^{(k)}(a)\|_{L^k(G,E_B)}}{L_k^{(\mathrm{FdB})}} \cdot r_k \leq \frac{1}{2^k}. \tag{11.3.12}$$

The next arguments are precisely the same as in [22, 4.10. Theorem]: $c(K)$ is compact in E_B since the mapping $c : K \to E_B$ is Lipschitzian: For all $x, y, \in K$ we get $c(x) - c(y) \in \frac{L_1^{(\mathrm{FdB})}\|x-y\|_G}{2r_1} \cdot B$. In the following we estimate for each $k \in \mathbb{N}_{>0}$ (where we use (11.3.2):

$$\begin{aligned}
&\left\|\frac{(f\circ c)^{(k)}(a)}{L_k}\cdot r_k\right\|_{L^k(G,F)} \\
&\leq \sum_{j\geq 0}\sum_{\alpha\in\mathbb{N}^j_{>0},\sum_{i=1}^j\alpha_i=k} D^k\cdot\underbrace{\frac{\left\|f^{(j)}(c(a))\right\|_{L^j(E_B,F)}}{L_j^{(\mathrm{FdB})}}}_{\leq C\cdot h^j}\cdot\underbrace{\prod_{i=1}^j\frac{\|c^{(\alpha_i)}(a)\|_{L^{\alpha_i}(G,E_B)}\cdot r_{\alpha_i}}{L_{\alpha_i!}^{(\mathrm{FdB})}}}_{\leq\frac{1}{2^{\alpha_1}}\cdots\frac{1}{2^{\alpha_j}}=\frac{1}{2^k}} \\
&\leq C\cdot\left(\frac{D}{2}\right)^k\cdot\sum_{j\geq 0}\sum_{\alpha\in\mathbb{N}^j_{>0},\sum_{i=1}^j\alpha_i=k} h^j = (Ch)\cdot\left(\frac{D}{2}\right)^k\cdot\sum_{j\geq 1}\binom{k-1}{j-1}h^{j-1} \\
&\leq (C\cdot h)\cdot\left(\frac{(1+h)}{2}\right)^k.
\end{aligned}$$

Here we have also used $f \in \mathcal{E}_{\{\mathcal{Q}\}} = \bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}} \mathcal{E}_{\{L\}}$ and thus we have shown condition (3) in 11.3.7. □

By using 11.3.10 we can prove the analogous result to [22, 4.11. Theorem], see also 12.2.9:

Theorem 11.3.11. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section, let E, F, G be convenient vector spaces, $U \subseteq E$ and $V \subseteq F$ be c^∞-open and $f : U \to F$, $g : V \to G$ with $f(U) \subseteq V$. Then we obtain:*

$$\text{If } f, g \in \mathcal{E}_{\{\mathcal{Q}\}}, \text{ then } g\circ f \in \mathcal{E}_{\{\mathcal{Q}\}}.$$

Proof. By (11.3.10) we have to show: For all closed absolutely convex bounded subsets $B \subseteq E$ and for all $\alpha \in G^*$ the composite $\alpha \circ g \circ f \circ i_B : U_B \to \mathbb{R}$ is $\mathcal{E}_{\{L\}}$ for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$. By assumption $f\circ i_B \in \mathcal{E}_{\{\mathcal{Q}\}}$ and $\alpha\circ g \in \mathcal{E}_{\{\mathcal{Q}\}}$, so we can use 11.3.10 to obtain the desired result (note that $f \circ i_B$ is a $\mathcal{E}_{\{L\}}$-Banach-plot for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$). □

By using our definitions resp. projective descriptions in 11.3.3, 11.3.5 and 11.3.6 we are also able to prove the $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$-UBP, see [23, 2.9. Theorem] resp. [21, 4.1. Theorem] for the analogous results:

Theorem 11.3.12. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section, let E, F, G be convenient vector spaces and let $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to \mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U, G)$ is bounded if and only if $\mathrm{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*

Proof. First we prove this statement for the class $\mathcal{E}_{\{L\},\mathrm{curve}}$ for each $L \in \mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$ separately: For this we can use the proof of [23, 2.9. Theorem] resp. [21, 4.1. Theorem] directly. We have to replace in the proof $\mathcal{E}_{\{M\}}$-classes by $\mathcal{E}_{\{L\},\mathrm{curve}}$ and instead of

$\mathcal{E}_{\{M\}}$-curves we consider $\mathcal{E}_{\{L^{(\mathrm{FdB})}\}}$-curves. The $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}$-UBP follows now by using the projective definition in (11.3.6): We have $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,G)=\bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}}\mathcal{E}_{\{L\},\mathrm{curve}}(U,G)$ and so the structure on $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,G)$ is initial w.r.t. all inclusions $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,G)\rightarrow\mathcal{E}_{\{L\},\mathrm{curve}}(U,G)$ for all $L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}$. □

Finally we can prove the following statement (see [23, 7.2. Theorem (1)] and also 12.5.2 below):

Theorem 11.3.13. *Let $\mathcal{Q}$ be a weight matrix as assumed at the beginning of this section with* (11.3.1)*, let E and F be convenient vector spaces and $U\subseteq E$ be c^∞-open. Then we obtain the following:*
The structure defined in (11.3.6)*, i.e. $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F)$ (the Roumieu-case), and the structure $\mathcal{E}_{\{\mathcal{Q}\}}(U,F)$ defined by Banach plots in* (12.1.3), (12.1.4) *and* (12.1.5) *coincide as vector spaces with bornology, so*

$$\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F)=\mathcal{E}_{\{\mathcal{Q}\}}(U,F).$$

Proof. We obtain as vector spaces

$$\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F):=\bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}}\mathcal{E}_{\{L\},\mathrm{curve}}(U,F)\underbrace{=}_{(\star\star)}\bigcap_{L\in\mathcal{L}(\mathcal{Q})_{\mathrm{ai,lc}}}\mathcal{E}_{\{L\}}(U,F)\underbrace{=}_{(\star)}\mathcal{E}_{\{\mathcal{Q}\}}(U,F),$$

where $(\star)$ holds as we have already pointed out by the definition of the structure in (12.1.3) and the characterizing theorem 11.1.1 applied to the spaces $\mathcal{E}_{\{Q\}}(U_B,\mathbb{R})$ (where B is a closed absolutely convex bounded subset of E, $U_B:=i_B^{-1}(U)$ and $i_B:E_B\rightarrow E$ denotes the inclusion of E_B in E). The equality $(\star\star)$ in the middle holds as vector spaces by (1) ⇔ (2) in Corollary 11.3.8.
Both spaces $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F)$ and $\mathcal{E}_{\{\mathcal{Q}\}}(U,F)$ are convenient and satisfy the UBP w.r.t. to the set of point evaluations ev_x (see 11.3.12 resp. 12.4.1), hence the identity is a bornological isomorphism. □

Conclusion: Theorem 11.3.13 shows that the structure $\mathcal{E}_{\{\mathcal{Q}\},\mathrm{curve}}(U,F)$ defined in this section and the structure $\mathcal{E}_{\{\mathcal{Q}\}}(U,F)$ defined in the next chapter (with general proofs for quasi- and non-quasi-analytic spaces of Roumieu- and Beurling-type) coincide bornologically. For the proof of *Cartesian closedness* Theorem 12.3.2 property $(\mathcal{M}_{\{\mathrm{mg}\}})$ plays the key-role and so we summarize: To transfer all proofs of the general setting in the next chapter, we have to assume for the matrix

$$\mathcal{Q}:=\{Q^n:n\in\Lambda=\mathbb{N}_{>0}:\ (\mathcal{M}_{\mathrm{sc}}),\ Q^n\ \text{quasi-analytic}\ \forall\, n\in\Lambda\}$$

the following conditions for $q_k^n:=\frac{Q_k^n}{k!}$ and where the associated sequences $\check{Q}^n$ resp. $\check{q}^n$ are defined in (11.1.1) resp. (11.1.2):

(*i*) The mappings $k\mapsto(\check{Q}_k^n)^{1/k}$ and $k\mapsto\frac{(\check{Q}_k^{n_2})^{1/k}}{(\check{Q}_k^{n_1})^{1/k}}$ are increasing for each $n\in\Lambda$ resp. for all $n_1,n_2\in\Lambda$ with $n_1\le n_2$,

(*ii*) $\exists\, n_0\in\Lambda:\ \lim_{k\rightarrow\infty}(q_k^{n_0})^{1/k}=+\infty$, which implies $(\mathcal{M}_{\{C^\omega\}})$,

(*iii*) $(\mathcal{M}_{\{\mathrm{mg}\}})$ which implies $(\mathcal{M}_{\{\mathrm{dc}\}})$,

(*iv*) each sequence $\check{q}^n$ should satisfy (ai) with $\sup_{n\in\Lambda}C_n\le D<+\infty$ and C_n are the constants occurring in (ai) for a certain $n\in\Lambda$.

The example which we have constructed in the previous section satisfies all desired properties.

12 Convenient setting for the general case $\mathcal{E}_{[\mathcal{M}]}$ by using Banach plots

The aim of this chapter is to transfer the results from [22] to ultradifferentiable function classes defined by general (also non-constant) weight matrices $\mathcal{M}$. We will work with *Banach plots* instead of *curves* as used in the previous chapters and then we are able to prove the important theorems, in particular *cartesian closedness*, for quasi- and non-quasi-analytic spaces of ultradifferentiable functions of Roumieu- and Beurling-type defined by a weight matrix $\mathcal{M}$ simultaneously.

12.1 Basic definitions

First recall some definitions of [22, Chapter 3] (see there for more details): Let E, F be Banach-spaces, $K \subseteq E$ compact and $U \subseteq E$ open. We denote by $\mathcal{E}(U, F)$ the space of arbitrarily often Fréchet differentiable mappings $f : U \to F$ and we have $f^{(k)} : U \to L^k_{\text{sym}}(E, F)$, where $L^k_{\text{sym}}(E, F)$ is the space of all symmetric k-linear bounded mappings $\underbrace{E \times \cdots \times E}_{k-\text{times}} \to F$.

Moreover we have iterated uni-directional derivatives $d^k_v f(x) \in F$ defined via

$$d^k_v f(x) := \left(\frac{d}{dt}\right)^k f(x + tv)|_{t=0}$$

and the jet-mapping $j^\infty : \mathcal{E}(U, F) \to J^\infty(U, F) := \prod_{k\in\mathbb{N}} \mathcal{C}(U, L^k_{\text{sym}}(E, F))$ defined by $f \mapsto j^\infty(f) = (f^{(k)})_{k\in\mathbb{N}}$.

On $L^k_{\text{sym}}(E, F)$ consider the operator-norm on $\|\cdot\|_{L^k_{\text{sym}}(E,F)}$ which is given by

$$\|f\|_{L^k_{\text{sym}}(E,F)} := \sup\{\|f(v_1, \dots, v_k)\|_F : \|v_i\|_E \le 1,\ j = 1, \dots, k\}.$$

For an arbitrary subset $X \subseteq E$ and an infinite jet $f = (f^k)_{k\in\mathbb{N}}$ we introduce the *Taylor polynomial* $(T^n_y f)^k : X \to L^k_{\text{sym}}(E, F)$ of order n at the point y as follows:

$$(T^n_y f)^k(x)(v_1, \dots, v_k) := \sum_{j=0}^{n} \frac{1}{j!} f^{j+k}(y)(x - y, \dots, x - y, v_1, \dots, v_k).$$

The *remainder* is given by

$$(R^n_y f)^k(x) := f^k(x) - (T^n_y f)^k(x) = (T^n_x f)^k(x) - (T^n_y f)^k(x)$$

and so $(R^n_y f)^k(x) \in L^k_{\text{sym}}(E, F)$. We put now

$$\|f\|_k := \sup\left\{\|f^k(x)\|_{L^k_{\text{sym}}(E,F)} : x \in K\right\}$$

resp.

$$|||f|||_{n,k} := \sup\left\{(n + 1)! \cdot \frac{\|(R^n_y f)^k(x)\|_{L^k_{\text{sym}}(E,F)}}{\|x - y\|^{n+1}} : x, y \in K, x \neq y\right\}.$$

In [22, 3.1.], equation (1), it was shown that for each $f \in \mathcal{E}(U,F)$ and *convex* subset $X \subseteq U$ we obtain

$$|||j^\infty(f)\big|_X|||_{n,k} \leq \|j^\infty(f)\big|_X\|_{n+k+1}. \tag{12.1.1}$$

We supply $\mathcal{E}(U,F)$ with the seminorms $f \mapsto \|j^\infty(f)|_K\|_k$, where $K \subseteq U$ is a compact set and $k \in \mathbb{N}$. If $K \subseteq E$ is compact and *convex*, then we introduce the space $\mathcal{E}(E \supseteq K, F)$ of *Whitney-jets* on K by

$$\mathcal{E}(E \supseteq K, F) := \left\{ f = (f^k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \mathcal{C}(K, L^k_{\text{sym}}(E,F)) : |||f|||_{n,k} < +\infty \ \forall\, n,k \in \mathbb{N} \right\}$$

and we supply these spaces with both seminorms $\|f\|_k$ and $|||f|||_{n,k}$ for $k,n \in \mathbb{N}$.
We recall now [22, 3.2. Lemma] (without proof):

Lemma 12.1.1. *Let E and F be Banach-spaces and $K \subseteq E$ be a compact convex subset. Then the space $\mathcal{E}(E \supseteq K, F)$ is a Fréchet-space.*

Let $\mathcal{M} := \{M^l : l \in \Lambda\}$ be now an arbitrary weight matrix, i.e. $(\mathcal{M})$.
The next definition is analogous to [22, 4.1.]: Let E and F be Banach spaces and $K \subseteq E$ a compact subset. Then for $l \in \Lambda$ and $h > 0$ we define

$$\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F) := \left\{ (f^m)_m \in \prod_{m\in\mathbb{N}} \mathcal{C}(K, L^m_{\text{sym}}(E,F)) : \|f\|^J_{\mathcal{M},l,h} < +\infty \right\},$$

where

$$\|f\|^J_{\mathcal{M},l,h} := \max\left\{ \sup\left\{ \frac{\|f\|_k}{h^k \cdot M^l_k} : k \in \mathbb{N} \right\}, \sup\left\{ \frac{|||f|||_{n,k}}{h^{n+k-1} \cdot M^l_{n+k+1}} : k,n \in \mathbb{N} \right\} \right\}.$$

For open $U \subseteq E$ and compact $K \subseteq U$ we introduce the space

$$\mathcal{E}_{\mathcal{M},K,l,h}(U,F) := \left\{ f \in \mathcal{E}(U,F) : j^\infty(f)\big|_K \in \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F) \right\},$$

with semi-norm $f \mapsto \left\|j^\infty(f)\big|_K\right\|^J_{\mathcal{M},l,h}$. But this space is not Hausdorff and for infinite dimensional E it's Hausdorff quotient will not always be complete. Note that if K is also assumed to be *convex*, then by (12.1.1) we can take on $\mathcal{E}_{\mathcal{M},K,l,h}(U,F)$ also the semi-norm

$$f \mapsto \sup\left\{ \frac{\|f^{(n)}(x)\|_{L^n_{\text{sym}}(E,F)}}{h^n \cdot M^l_n} : x \in K, n \in \mathbb{N} \right\} =: \|f\|^J_{\mathcal{M},K,l,h}.$$

Thus we see that $\mathcal{E}_{\mathcal{M},K,l,h}(U,F) = \left\{ f \in \mathcal{E}(U,F) : (\|j^\infty(f)\big|_K\|_k)_k \in \mathcal{F}_{\mathcal{M},l,h} \right\}$ with

$$\mathcal{F}_{\mathcal{M},l,h} := \left\{ (f_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \exists\, C > 0 : \forall\, k \in \mathbb{N} : |f_k| \leq C \cdot h^k \cdot M^l_k \right\}.$$

The bounded sets $\mathcal{B}$ in $\mathcal{E}_{\mathcal{M},K,l,h}(U,F)$ are exactly those $\mathcal{B} \subseteq \mathcal{E}(U,F)$ such that $(b_m)_m \in \mathcal{F}_{\mathcal{M},l,h}$ with $b_m := \sup\left\{ \|j^\infty(f)\big|_K\|_m : f \in \mathcal{B} \right\}$.

Let $U \subseteq E$ be convex open and $K \subseteq U$ be convex compact, then we can define:

$$\mathcal{E}_{(\mathcal{M})}(E \supseteq K, F) := \varprojlim_{l\in\Lambda, h>0} \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$$

$$\mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F) := \varinjlim_{l\in\Lambda, h>0} \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$$

and finally

$$\mathcal{E}_{[\mathcal{M}]}(U, F) := \left\{ f \in \mathcal{E}(U, F) : \forall\, K : (f^{(k)}\big|_K) \in \mathcal{E}_{[\mathcal{M}]}(E \supseteq K, F) \right\}.$$

By 6.2.4 and 6.2.6 we have also for each convex and compact set K now the representations $\mathcal{E}_{\{\omega\}}(E \supseteq K, F) = \varinjlim_{l>0} \mathcal{E}_{\{M^l\}}(E \supseteq K, F) = \varinjlim_{l\in\mathbb{N}_{>0}} \mathcal{E}_{\{M^l\}}(E \supseteq K, F)$ and $\mathcal{E}_{(\omega)}(E \supseteq K, F) = \varprojlim_{l>0} \mathcal{E}_{(M^l)}(E \supseteq K, F) = \varprojlim_{l\in\mathbb{N}_{>0}} \mathcal{E}_{(M^{1/l})}(E \supseteq K, F)$. Finally we put

$$\mathcal{E}_{[\mathcal{M}]}(U, F) := \varprojlim_{K\subseteq U} \mathcal{E}_{[\mathcal{M}]}(E \supseteq K, F), \tag{12.1.2}$$

where K runs in the projective limit through all compact and convex subsets of U.
Let $\mathcal{M} = \{M^l : l \in \Lambda\}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with (e.g.) $\Lambda = \mathbb{R}_{>0}$. Then we can restrict in both cases to countable many indices, and in fact to a diagonal of the limit: $\mathcal{E}_{(\mathcal{M})}(E \supseteq K, F) = \varprojlim_{n\in\mathbb{N}_{>0}} \mathcal{E}_{\mathcal{M},1/n,n}(E \supseteq K, F)$ resp. $\mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F) = \varinjlim_{n\in\mathbb{N}_{>0}} \mathcal{E}_{\mathcal{M},n,n}(E \supseteq K, F)$. Recall that by definition we have the bounded inclusions $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F) \to \mathcal{E}_{\mathcal{M},\tilde{l},\tilde{l}}(E \supseteq K, F) \to \mathcal{E}_{\mathcal{M},l_1,h_1}(E \supseteq K, F)$ for each $l, l_1, h, h_1 > 0$ with $\tilde{l} := \max\{l, h\}$ and $l_1, h_1 \geq \tilde{l}$ and compare this with 9.1.1 above.
But a very important difference compared to 9.1.1 arises: As already mentioned in (3) in [22, 4.2. Proposition] the space $\mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F)$ is not a *Silva-space* for infinite dimensional E, because the connecting mappings in the inductive limit $\varprojlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ are not compact any more: The set $\mathcal{B} := \{\alpha \in E' : \|\alpha\| \leq 1\}$ is bounded in $\mathcal{E}_{\mathcal{M},k,k}(E \supseteq K, \mathbb{R})$ for each $k \geq 1$. We have $\|\alpha\|_0 = \sup\{|\alpha(x)| : x \in K\} \leq \sup\{\|x\| : x \in K\}$, $\|\alpha\|_1 = \|\alpha\| \leq 1$ and $\|\alpha\|_m = 0$ for each $m \geq 2$. Moreover $(R^n_y\alpha)^k = 0$ for $n + k \geq 1$ and $(R^0_y\alpha)^0 = \alpha(x - y)$. But $\mathcal{B}$ is not relatively compact in any $\mathcal{E}_{\mathcal{M},k,k}(E \supseteq K, \mathbb{R})$, $k \geq 1$, because it is not even pointwise relatively compact in $\mathcal{C}(K, L(E, \mathbb{R}))$. The space $\mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F)$ will be studied in detail in (3), (4) and (5) in 12.1.2 below.

Moreover we introduce now

$$\mathcal{E}_{(\mathcal{M}),K}(U, F) := \varprojlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},K,l,h}(U, F)$$

$$\mathcal{E}_{\{\mathcal{M}\},K}(U, F) := \varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},K,l,h}(U, F)$$

and so

$$\mathcal{E}_{(\mathcal{M}),K}(U, F) = \left\{ f \in \mathcal{E}(U, F) : (\|j^\infty(f)\big|_K\|_k)_k \in \mathcal{F}_{(\mathcal{M})} \right\}$$

$$\mathcal{E}_{\{\mathcal{M}\},K}(U, F) = \left\{ f \in \mathcal{E}(U, F) : (\|j^\infty(f)\big|_K\|_k)_k \in \mathcal{F}_{\{\mathcal{M}\}} \right\}$$

with $\mathcal{F}_{(\mathcal{M})} = \bigcap_{l\in\Lambda,h>0} \mathcal{F}_{\mathcal{M},l,h}$, $\mathcal{F}_{\{\mathcal{M}\}} = \bigcup_{l\in\Lambda,h>0} \mathcal{F}_{\mathcal{M},l,h}$.
The bounded sets $\mathcal{B} \subseteq \mathcal{E}_{[\mathcal{M}],K}(U, F)$ are exactly those $\mathcal{B} \subseteq \mathcal{E}(U, F)$ for which the sequence $(b_m)_m$, $b_m := \sup\left\{\|j^\infty(f)\big|_K\|_m : f \in \mathcal{B}\right\}$, belongs to $\mathcal{F}_{[\mathcal{M}]}$. Finally we introduce

$$\varprojlim_{K\subseteq U} \mathcal{E}_{[\mathcal{M}],K}(U, F) = \left\{ f \in \mathcal{E}(U, F) : \forall\, K : (\|j^\infty(f)\big|_K\|_m)_m \in \mathcal{F}_{[\mathcal{M}]} \right\}.$$

Recall: Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then by 8.3.1 the class $\mathcal{E}_{[\mathcal{M}]}$ is closed under composition if $\mathcal{M}$ has in addition property $(\mathcal{M}_{[\mathrm{FdB}]})$, it is closed under pointwise multiplication if one has in addition property $(\mathcal{M}_{[\mathrm{alg}]})$.

The next important proposition is the analogous result to [22, 4.2. Proposition]:

Proposition 12.1.2. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$. Then we obtain the following completeness properties:*

(1) *The spaces $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ are Banach-spaces.*

(2) *The spaces $\mathcal{E}_{(\mathcal{M})}(E \supseteq K, F)$ are Fréchet-spaces.*

(3) *The spaces $\mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F)$ are compactly regular (LB)-spaces (which means that compact subsets are contained and compact in some step), so (c^∞)-complete, webbed and ultra-bornological.*

(4) *The spaces $\mathcal{E}_{(\mathcal{M})}(U, F)$ and $\mathcal{E}_{\{\mathcal{M}\}}(U, F)$ are complete.*

(5) *As locally convex vector spaces we have*

$$\mathcal{E}_{(\mathcal{M})}(U,F) = \varprojlim_{K \subseteq U} \mathcal{E}_{(\mathcal{M})}(E \supseteq K, F) = \varprojlim_{K} \mathcal{E}_{(\mathcal{M}),K}(U,F)$$

resp.

$$\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \varprojlim_{K \subseteq U} \mathcal{E}_{\{\mathcal{M}\}}(E \supseteq K, F) = \varprojlim_{K} \mathcal{E}_{\{\mathcal{M}\},K}(U,F).$$

Proof. (1) Let $l \in \Lambda$ and $h > 0$ be given. The injection $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F) \to \prod_{k \in \mathbb{N}} \mathcal{C}(K, L^k_{\mathrm{sym}}(E,F))$ is by definition continuous and $\prod_{k \in \mathbb{N}} \mathcal{C}(K, L^k_{\mathrm{sym}}(E,F))$ is a Banach-space. So a Cauchy-sequence $(f_p)_p$ in $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ has an infinite jet $f_\infty = (f^k_\infty)_k$ as component-wise limit in $\prod_{k \in \mathbb{N}} \mathcal{C}(K, L^k_{\mathrm{sym}}(E,F))$. But this is the limit also with respect to the finer structure of $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$: For $n, k \in \mathbb{N}$ fix and $x \neq y$ we have that $(R^n_y f_p)^k(x)$ converges to $(R^n_y f_\infty)^k(x)$ for $p \to \infty$. We choose $\varepsilon > 0$, $p_0 \in \mathbb{N}$ such that $\|f_p - f_q\|^J_{\mathcal{M},l,h} < \frac{\varepsilon}{2}$ for all $p, q \geq p_0$. For given x, y, n and k we can choose $q > p_0$ with

$$(n+1)! \cdot \frac{\|(R^n_y f_q)^k(x) - R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{h^{n+k+1} \cdot M^l_{n+k+1} \cdot \|x-y\|^{n+1}} < \frac{\varepsilon}{2}$$

and

$$\frac{\|f^n_q(x) - f^n_\infty(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{h^n \cdot M^l_n} < \frac{\varepsilon}{2}.$$

So we can estimate as follows:

$$\begin{aligned}
&(n+1)! \cdot \frac{\|(R^n_y f_p)^k(x) - R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{h^{n+k+1} \cdot M^l_{n+k+1} \cdot \|x-y\|^{n+1}} \\
&< \|f_p - f_q\|^J_{\mathcal{M},l,h} + (n+1)! \cdot \frac{\|(R^n_y f_q)^k(x) - R^n_y f_\infty)^k(x)\|_{L^k_{\mathrm{sym}}(E,F)}}{h^{n+k+1} \cdot M^l_{n+k+1} \cdot \|x-y\|^{n+1}} < \varepsilon,
\end{aligned}$$

and so $\frac{\|\|f_p - f_\infty\|\|_{n,k}}{h^{n+k+1} \cdot M^l_{n+k+1}} \leq \varepsilon$. Analogously one can show $\frac{\|f_p - f_\infty\|_n}{h^n \cdot M^l_n} \leq \varepsilon$, hence $\|f_p - f_\infty\|^J_{\mathcal{M},l,h} \leq \varepsilon$ for all $p \geq p_0$.

(2) Recall: $\mathcal{E}_{(\mathcal{M})}(E \supseteq K, F)$ is a Fréchet-space, since it is a countable projective limit of Banach-spaces (note $\Lambda = \mathbb{R}_{>0}$!). We have $\mathcal{E}_{(\mathcal{M})}(E \supseteq K, F) := \varprojlim_{l\in\Lambda, h>0} \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ and this projective limit is in fact countable: replace h by $h' := \frac{1}{h}$ and take $h' \in \mathbb{N}$, and one can restrict to a countable set $l_n \in \Lambda$, take e.g. $l_n := \frac{1}{n}$ for $n \in \mathbb{N}_{>0}$.

(3) By assumption on the index set we can assume $\Lambda = \mathbb{N}_{>0}$ in the Roumieu-case. We proceed now similarly as in [22, 4.2. Proposition (3)]: To show that the inductive limit is compactly regular it suffices to show that there exists a sequence of increasing 0-neighborhoods $U_n \in \mathcal{E}_{\mathcal{M},n,n}(E \supseteq K, F)$ (set $l = h = n$) such that for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ with $m \geq n$ and for which the topologies of $\mathcal{E}_{\mathcal{M},m,m}(E \supseteq K, F)$ and of $\mathcal{E}_{\mathcal{M},k,k}(E \supseteq K, F)$ coincide on U_n for all $k \geq m$.

In general, for indices $l_1 \geq l_2$ and positive real numbers $h_1 \geq h_2$ we have clearly by definition $\|\cdot\|^J_{\mathcal{M},l_1,h_1} \leq \|\cdot\|^J_{\mathcal{M},l_2,h_2}$. Consider now the ε-Ball $U^{l,h}_\varepsilon(f) := \{g : \|g - f\|^J_{\mathcal{M},l,h} \leq \varepsilon\}$ (and so $U^{l_2,h_2}_\varepsilon \subseteq U^{l_1,h_1}_\varepsilon$ for $l_2 \leq l_1$ and $h_2 \leq h_1$ by definition) in $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ and we restrict in the following ourselves to the diagonal where we put $l = h = n \in \mathbb{N}_{>0}$ and identify $U^{n,n}$ with U^n.
We show: For arbitrary $n \in \mathbb{N}_{>0}$ and $n_2 > n_1 := 2n$, for each $\varepsilon > 0$ and $f \in U^n_1(0)$ we obtain that there exists a number $\delta > 0$ such that $U^{n_2}_\delta(f) \cap U^n_1(0) \subseteq U^{n_1}_\varepsilon(f)$.
Since by assumption $f \in U^n_1(0) = U^{n,n}_1(0)$ we have $\|f\|_a \leq n^a \cdot M^n_a$ resp. $|||f|||_{a,b} \leq n^{a+b+1} \cdot M^n_{a+b+1}$ for all $a, b \in \mathbb{N}$. Furthermore consider $g \in U^{n_2}_\delta(f) \cap U^n_1(0) = U^{n_2,n_2}_\delta(f) \cap U^{n,n}_1(0)$, then $\|g\|_a \leq n^a \cdot M^n_a$, $|||g|||_{a,b} \leq n^{a+b+1} \cdot M^n_{a+b+1}$ and moreover $\|g - f\|_a \leq \delta \cdot n_2^a \cdot M^{n_2}_a$, $|||g - f|||_{a,b} \leq \delta \cdot n_2^{a+b+1} \cdot M^{n_2}_{a+b+1}$ for all $a, b \in \mathbb{N}$. We estimate similarly as in [22, 4.2. Proposition (3)] as follows: For given $\varepsilon > 0$ consider $N \in \mathbb{N}$ (minimal) with $\frac{1}{2^N} < \frac{\varepsilon}{2}$ and put $\delta := \varepsilon \cdot \left(\frac{n_1}{n_2}\right)^{N-1} \cdot \frac{1}{M^{n_2}_N}$.

For $a \geq N$ we have $\frac{1}{2^a} \leq \frac{1}{2^N} < \frac{\varepsilon}{2}$ $(\star)$, so use triangle-inequality to get

$$\|g - f\|_a \leq \|g\|_a + \|f\|_a \leq 2 \cdot n^a \cdot M^n_a = 2 \cdot n_1^a \cdot M^n_a \cdot \frac{1}{2^a} \underbrace{<}_{(\star)} \varepsilon \cdot n_1^a \cdot M^n_a \leq \varepsilon \cdot n_1^a \cdot M^{n_1}_a$$

and the last inequality holds since $n_1 = 2n > n$ and so $M^n_a \leq M^{n_1}_a$ for all $a \in \mathbb{N}$. For $a < N$ we estimate as follows

$$\|g - f\|_a \leq \delta \cdot n_2^a \cdot M^{n_2}_a \leq \varepsilon \cdot n_1^a \cdot \frac{M^{n_2}_a}{M^{n_2}_N} \leq \varepsilon \cdot n_1^a \leq \varepsilon \cdot n_1^a \cdot M^{n_1}_a.$$

Recall: We have $M^n_a \leq M^n_N$ and $\left(\frac{n_1}{n_2}\right)^{N-1} \leq \left(\frac{n_1}{n_2}\right)^a$ since $a < N$ and $\frac{n_1}{n_2} < 1$. $M^{n_1}_a \geq 1$ holds by assumption on $\mathcal{M}$ for each $n_1 \in \Lambda$ and $a \in \mathbb{N}$.
Analogously we can use the same estimates for the seminorms $|||\cdot|||_{a,b}$ instead of $\|\cdot\|_a$ for each $a, b \in \mathbb{N}$.

(4) This holds, since the considered spaces are projective limits of complete spaces. More precisely, in the Beurling-case we have a projective limit of Fréchet-spaces (which are clearly complete), in the Roumieu-case we have a projective limit of (LB)-spaces, which are all compactly regular by (3) and so complete, too.
(5) The mapping $j^\infty|_K : \mathcal{E}_{\mathcal{M},K,l,h}(U, F) \to \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ is by definition well-defined, continuous and linear and it induces mappings $\mathcal{E}_{[\mathcal{M}],K}(U, F) \to \mathcal{E}_{[\mathcal{M}]}(E \supseteq$

$K, F)$ resp. $\varinjlim_{K\subseteq U} \mathcal{E}_{[\mathcal{M}],K}(U,F) \to \varinjlim_{K\subseteq U} \mathcal{E}_{[\mathcal{M}]}(E \supseteq K, F)$, which is injective (for $K = \{x\}$ with $x \in U$).

Conversely, let $f_K^k \in \mathcal{C}(K, L^k_{\text{sym}}(E,F))$ be given, such that for each compact set K there exist $l \in \Lambda$ and $h > 0$ (resp. for each $l \in \Lambda$ and each $h > 0$) we have that $(f_K^k)_{k\in\mathbb{N}} \in \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ and such that $f_K^k|_{K'} = f^k|_{K'}$. We define the infinite jet $(f^k)_{k\in\mathbb{N}} \in J^\infty(U,F)$ by $f^k(x) := f^k_{\{x\}}(x)$, and we have now by definition $f^k|_K = f^k_K$ for each $k \in \mathbb{N}$ and compact set K.
Claim: $f^0 \in \mathcal{E}(U,F)$ and $(f^0)^{(k)} = f^k$ for each $k \in \mathbb{N}$, which means $j^\infty(f^0|_K) = (f^k_K)_k$ for all $k \in \mathbb{N}$ and compact sets K.
We use induction on k and [20, 5.20.]: It suffices to show $d^k_v f^0(x) = f^k(x)(v,\dots,v)$ for each k. For $k = 0$ there is nothing to prove, for $k-1 \mapsto k$ we have:

$$d^k_v f^0(x) := \lim_{t\to 0} \frac{d^{k-1}_v f^0(x+tv) - d^{k-1}_v f^0(x)}{t} = \lim_{t\to 0} \frac{f^{k-1}(x+tv)(v,\dots,v) - f^{k-1}(x)(v,\dots,v)}{t}$$
$$= \lim_{t\to 0} \frac{(R^1_x f)^{k-1}(x+tv)(v,\dots,v)}{t} + f^k(x)(v,\dots,v) = f^k(x)(v,\dots,v).$$

Moreover f^0 defines an element in $\varinjlim_{K\subseteq U} \mathcal{E}_{[\mathcal{M}],K}(U,F)$, because for each compact K we have $f^0 \in \mathcal{E}_{\mathcal{M},K,l,h}(U,F) := \{g \in \mathcal{E}(U,F) : j^\infty(g)|_K \in \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)\}$ for some $l \in \Lambda$ and $h > 0$ (resp. for each $l \in \Lambda$ and each $h > 0$).
That this bijection induces an isomorphism as locally convex vector spaces follows by the fact, that the defined seminorm $\|\cdot\|^J_{\mathcal{M},K,l,h}$ on the space $\mathcal{E}_{\mathcal{M},K,l,h}(U,F)$ is the pull-back of the seminorm $\|\cdot\|^J_{\mathcal{M},l,h}$ on the space $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$. □
Let E, F be convenient vector spaces, $U \subseteq E$ be c^∞-open, then define

$$\begin{aligned}\mathcal{E}^{\text{b}}_{(\mathcal{M})}(U,F) :=&\Big\{f \in \mathcal{E}(U,F) : \forall\, B : \forall\, K \subseteq U \cap E_B : \forall\, l \in \Lambda\; \forall\, h > 0 :\\ &\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1\Big\}\text{is bounded in } F\Big\}\\ =&\Big\{f \in \mathcal{E}(U,F) : \forall\, B : \forall\, K \subseteq U \cap E_B : \forall\, l \in \Lambda\; \forall\, h > 0 :\\ &\Big\{\frac{d^k_v f(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1\Big\}\text{is bounded in } F\Big\}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}^{\text{b}}_{\{\mathcal{M}\}}(U,F) :=&\Big\{f \in \mathcal{E}(U,F) : \forall\, B : \forall\, K \subseteq U \cap E_B : \exists\, l \in \Lambda\; \exists\, h > 0 :\\ &\Big\{\frac{f^{(k)}(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1\Big\}\text{is bounded in } F\Big\}\\ =&\Big\{f \in \mathcal{E}(U,F) : \forall\, B : \forall\, K \subseteq U \cap E_B : \exists\, l \in \Lambda\; \exists\, h > 0 :\\ &\Big\{\frac{d^k_v f(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1\Big\}\text{is bounded in } F\Big\}.\end{aligned}$$

In the previous definitions B runs through all closed absolutely convex bounded subsets in E, the set E_B is the vector space generated by B with the *Minkowski-functional* $\|\cdot\|_B$ which is in this case (since E is convenient) a complete norm. K runs through all sets in $U \cap E_B$ which are compact w.r.t. the norm $\|\cdot\|_B$. For both E, F Banach spaces and

$U \subseteq E$ open we have $\mathcal{E}^b_{[\mathcal{M}]}(U,F) = \mathcal{E}_{[\mathcal{M}]}(U,F)$, where the latter space is introduced in (12.1.2).

Now the most important definition:

$$\mathcal{E}_{[\mathcal{M}]}(U,F) := \left\{ f \in \mathcal{E}(U,F) : \forall\, \alpha \in F^* : \forall\, B : \alpha \circ f \circ i_B \in \mathcal{E}_{[\mathcal{M}]}(U_B,\mathbb{R}) \right\}, \tag{12.1.3}$$

where B is running again through all closed absolutely convex bounded subsets in E and the mapping $i_B : E_B \to E$ denotes the inclusion of E_B in E and we write $U_B := i_B^{-1}(U)$. The initial locally convex structure is now induced by all linear mappings

$$\mathcal{E}_{[\mathcal{M}]}(i_B,\alpha) : \mathcal{E}_{[\mathcal{M}]}(U,F) \longrightarrow \mathcal{E}_{[\mathcal{M}]}(U_B,\mathbb{R}), \qquad f \mapsto \alpha \circ f \circ i_B. \tag{12.1.4}$$

$\mathcal{E}_{[\mathcal{M}]}(U,F) \subseteq \prod_{\alpha,B} \mathcal{E}_{[\mathcal{M}]}(U_B,\mathbb{R})$ are convenient vector spaces as c^∞-closed subspaces in the product: Smoothness can be tested by composing with inclusions $E_B \to E$ and $\alpha \in F^*$ as mentioned in [20, 2.14.4, 1.8], see also 11.1.1. Hence we obtain the representation

$$\mathcal{E}_{[\mathcal{M}]}(U,F) := \{ f \in F^U : \ \forall\, \alpha \in F^*\ \forall\, B : \alpha \circ f \circ i_B \in \mathcal{E}_{[\mathcal{M}]}(U_B,\mathbb{R}) \}. \tag{12.1.5}$$

All definitions here are clearly generalizations of the definitions given in [22, 4.3.] for the cases if the matrix is constant, i.e. $\mathcal{M} := \{M\}$, or more general if all occurring weight sequences are equivalent, so $M^l \approx M^n$ for each $l, n \in \Lambda$.

12.2 Convenient setting by using Banach-plots

First we will show the following generalization of [22, 4.4. Lemma]:

Lemma 12.2.1. *For an arbitrary weight-matrix $\mathcal{M}$, i.e. $(\mathcal{M})$, there is no difference between the spaces $\mathcal{E}_{(\mathcal{M})}$ and $\mathcal{E}^{\mathrm{b}}_{(\mathcal{M})}$, both definitions coincide.*

Proof. Let E, F be convenient vector spaces, $U \subseteq E$ a c^∞-open subset and finally let $f : U \to F$ be $\mathcal{E}$. Then we obtain the following equivalences, where the set B runs always through the family of all closed absolutely convex bounded subsets in E and K runs through all sets in U_B which are compact w.r.t. the norm $\|\cdot\|_B$:

$$\begin{aligned}
& f \in \mathcal{E}_{(\mathcal{M})}(U,F) \\
& \Longleftrightarrow \forall\, \alpha \in F^* \;\forall\, B \;\forall\, K \subseteq U_B \;\forall\, l \in \Lambda \;\forall\, h > 0 : \\
& \left\{ \frac{(\alpha \circ f)^{(k)}(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1 \right\} \text{ is bounded in } \mathbb{R} \\
& \Longleftrightarrow \forall\, B \;\forall\, K \subseteq U_B \;\forall\, l \in \Lambda \;\forall\, h > 0 \;\forall\, \alpha \in F^* : \\
& \alpha\left(\left\{ \frac{f^{(k)}(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1 \right\} \right) \text{ is bounded in } \mathbb{R} \\
& \Longleftrightarrow \forall\, B \;\forall\, K \subseteq U_B \;\forall\, l \in \Lambda \;\forall\, h > 0 : \\
& \left\{ \frac{f^{(k)}(x)(v_1,\dots,v_k)}{h^k \cdot M^l_k} : x \in K, k \in \mathbb{N}, \|v_i\|_B \le 1 \right\} \text{ is bounded in } \mathbb{R} \\
& \Longleftrightarrow f \in \mathcal{E}^{\mathrm{b}}_{(\mathcal{M})}(U,F).
\end{aligned}$$

□

But in general we don't have $\mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}^{\mathrm{b}}_{\{\mathcal{M}\}}$. For this we can use e.g. the constant matrix $\mathcal{M} := \{N\}$ for each, where N is a weakly log. convex sequence and use now [22, 4.6. Example]. For a more general matrix-version see also 10.4.5.

To get $\mathcal{E}_{[\mathcal{M}]} = \mathcal{E}^{\mathrm{b}}_{[\mathcal{M}]}$ in any case (also for the Roumieu-type) we have to assume additional assumptions (see also [22, 4.5. Lemma]):

Lemma 12.2.2. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, let E, F be convenient vector spaces and let $U \subseteq E$ a c^∞-open subset. Assume that there exists a Baire-vector-space-topology on the dual F^* for which the point evaluations ev_x are continuous for all $x \in F$. Then a function $f : U \to F$ is $\mathcal{E}_{[\mathcal{M}]}$ if and only if f is $\mathcal{E}^{\mathrm{b}}_{[\mathcal{M}]}$.*

Proof. The case $\mathcal{E}_{(\mathcal{M})}$ holds by 12.2.1, so consider the case $\mathcal{E}_{\{\mathcal{M}\}}$:
($\Leftarrow$) is clear.
($\Rightarrow$) Let B a closed absolutely convex bounded subset of E, furthermore consider a compact set K in U_B (w.r.t. $\|\cdot\|_B$) and introduce the sets

$$A_{l,h,C} := \left\{ \alpha \in F^* : \frac{|(\alpha \circ f)^{(k)}(x)(v_1,\dots,v_k)|}{h^k \cdot M^l_k} \le C,\ \forall\, k \in \mathbb{N}, x \in K, \|v_i\|_B \le 1 \right\}.$$

These sets are closed in F^* for the Baire-topology and $\bigcup_{l\in\Lambda,h,C>0} A_{l,h,C} = F^*$ holds. Then, by the *Baire-property* of F^*, there exist $l_0 \in \Lambda$, $h_0, C_0 > 0$ such that the interior

$\overset{\circ}{A}_{l_0,h_0,C_0}$ is non-empty! Let $\alpha_0 \in \overset{\circ}{A}_{l_0,h_0,C_0}$, then for all $\alpha \in F^*$ there exists $\varepsilon > 0$, such that we get $\varepsilon \cdot \alpha \in \overset{\circ}{A}_{l_0,h_0,C_0} - \alpha_0 \Leftrightarrow \varepsilon \cdot \alpha + \alpha_0 \in \overset{\circ}{A}_{l_0,h_0,C_0}$.
Thus for all $x \in K$, $k \in \mathbb{N}$ and $\|v_i\|_B \leq 1$ we get

$$\begin{aligned}|(\alpha \circ f)^{(k)}(x)(v_1, \ldots, v_k)| &\leq \frac{1}{\varepsilon} \cdot \Big(|((\varepsilon \cdot \alpha) + \alpha_0) \circ f)^{(k)}(x)| + |(\alpha_0 \circ f)^{(k)}(x)|\Big) \\ &\leq \frac{2 \cdot C_0}{\varepsilon} \cdot h_0^k \cdot M_k^{l_0}.\end{aligned}$$

This implies: The set $\left\{\frac{f^{(k)}(x)(v_1,\ldots,v_k)}{h_0^k \cdot M_k^{l_0}} : k \in \mathbb{N}, x \in K, \|v_i\|_B \leq 1\right\}$ is weakly bounded (in F), hence bounded. But since B was arbitrary we get immediately $f \in \mathcal{E}^{\mathrm{b}}_{\{\mathcal{M}\}}$ (with $h_0 > 0$ and $l_0 \in \Lambda$. □

We formulate now an analogous result to [22, 4.8. Lemma], also here we have a "strong" and a "weak" version. The proofs are completely analogous to 10.5.3 resp. 10.5.6:

Lemma 12.2.3. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with index-set $\Lambda = \mathbb{N}_{>0}$ and property $(\mathcal{M}_{\{\mathrm{FdB}\}})$. Let E, F be Banach-spaces, $U \subseteq E$ open and $f : U \to F$ a $\mathcal{E}$-mapping, then the following are equivalent:*

(1) *f is $\mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}^{\mathrm{b}}_{\{\mathcal{M}\}}$.*

(2) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$ the set*

$$\left\{\frac{f^{(k)}(a)(v_1, \ldots, v_k)}{k!} \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1\right\}$$

is bounded in F.

(3) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum,sub}}$ and moreover for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum,FdB}}$, there exists a number $\varepsilon > 0$ such that the set*

$$\left\{\frac{f^{(k)}(a)(v_1, \ldots, v_k)}{k!} \cdot r_k \cdot s_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1\right\}$$

is bounded in F.

Now the second version:

Lemma 12.2.4. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix with index-set $\Lambda = \mathbb{N}_{>0}$. Let E, F be Banach-spaces, $U \subseteq E$ open and $f : U \to F$ a $\mathcal{E}$-mapping, then the following are equivalent:*

(1) *f is $\mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}^{\mathrm{b}}_{\{\mathcal{M}\}}$.*

(2) *For each compact $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\mathrm{roum}}$ and each $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\mathrm{roum}}$ the set*

$$\left\{f^{(k)}(a)(v_1, \ldots, v_k) \cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1\right\}$$

is bounded in F.

(3) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$ and moreover for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{roum,sub}}$, there exists a number $\varepsilon > 0$ such that the set*

$$\left\{f^{(k)}(a)(v_1,\dots,v_k)\cdot r_k\cdot s_k\cdot\varepsilon^k : a\in K, k\in\mathbb{N}, \|v_i\|_E\le 1\right\}$$

is bounded in F.

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. $(\mathcal{M})$, then we obtain the same equivalences where in condition (3) *we replace the set $\mathcal{S}^{\mathcal{M}}_{\text{roum,sub}}$ by $\mathcal{S}^{\mathcal{M}}_{\text{roum}}$.*

Note that $\mathcal{E}_{\{\mathcal{M}\}} = \mathcal{E}^{\text{b}}_{\{\mathcal{M}\}}$ in (1) in 12.2.3 and 12.2.4 holds by 12.2.2. Now we formulate an analogous result to [22, 4.7. Lemma], the proofs are completely analogous to 10.5.9 resp. 10.5.11:

Lemma 12.2.5. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and additionally $(\mathcal{M}_{(\text{FdB})})$. Let E, F be Banach-spaces, $U \subseteq E$ open and $f : U \to F$ be a $\mathcal{E}$-mapping, then the following are equivalent:*

(1) *f is $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}^{\text{b}}_{(\mathcal{M})}$.*

(2) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur}}$ and for each $(s_k)_k \in \int^{\mathcal{M}}_{\text{beur}}$ the set*

$$\left\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k!}\cdot r_k\cdot s_k : a\in K, k\in\mathbb{N}, \|v_i\|_E\le 1\right\}$$

is bounded in F.

(3) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$, $(s_k)_k \in \int^{\mathcal{M}}_{\text{beur,FdB}}$ and finally for each number $\delta > 0$ the set*

$$\left\{\frac{f^{(k)}(a)(v_1,\dots,v_k)}{k!}\cdot r_k\cdot s_k\cdot\delta^k : a\in K, k\in\mathbb{N}, \|v_i\|_E\le 1\right\}$$

is bounded in F.

Again the second version of this Beurling-type result is the following:

Lemma 12.2.6. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$. Let E, F be Banach-spaces, $U \subseteq E$ open and $f : U \to F$ be a $\mathcal{E}$-mapping, then the following are equivalent:*

(1) *f is $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}^{\text{b}}_{(\mathcal{M})}$.*

(2) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur}}$ and for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{beur}}$ the set*

$$\left\{f^{(k)}(a)(v_1,\dots,v_k)\cdot r_k\cdot s_k : a\in K, k\in\mathbb{N}, \|v_i\|_E\le 1\right\}$$

is bounded in F .

(3) *For each compact set $K \subseteq U$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$, $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{beur,sub}}$ and finally for each number $\delta > 0$ the set*

$$\left\{f^{(k)}(a)(v_1,\dots,v_k)\cdot r_k\cdot s_k\cdot\delta^k : a\in K, k\in\mathbb{N}, \|v_i\|_E\le 1\right\}$$

is bounded in F.

If $\mathcal{M}$ is only assumed to be an arbitrary weight matrix, i.e. ($\mathcal{M}$), then we obtain the same equivalences where in condition (3) we replace the set $\mathcal{S}^{\mathcal{M}}_{\text{beur,sub}}$ by $\mathcal{S}^{\mathcal{M}}_{\text{beur}}$.

Note that $\mathcal{E}_{(\mathcal{M})} = \mathcal{E}^{\text{b}}_{(\mathcal{M})}$ in (1) in 12.2.5 and 12.2.6 holds by 12.2.1.

Definition 12.2.7. *Let E be a convenient vector space. A $\mathcal{E}_{[\mathcal{M}]}$-Banach-plot in E is a mapping $c : D \to E$ such that $c \in \mathcal{E}_{[\mathcal{M}]}$ and $D = oE$ denotes the open unit ball in some Banach-space F.*

By using this definition and the previous projective characterizations we can prove now the analogous theorem to [22, 4.10. Theorem]:

Theorem 12.2.8. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and let $U \subseteq E$ be a c^∞-open subset in a convenient vector space E. Moreover let F be an arbitrary Banach-space and $f : U \to F$ a mapping, then we obtain:*

(*a*) *If $\mathcal{M}$ has in addition $\mathcal{M}_{(\text{FdB})}$, then $f \in \mathcal{E}_{(\mathcal{M})}$ implies $f \circ c \in \mathcal{E}_{(\mathcal{M})}$ for all $\mathcal{E}_{(\mathcal{M})}$-Banach-plots c.*

(*b*) *If $\mathcal{M}$ has in addition $\mathcal{M}_{\{\text{FdB}\}}$, then $f \in \mathcal{E}_{\{\mathcal{M}\}}$ implies $f \circ c \in \mathcal{E}_{\{\mathcal{M}\}}$ for all $\mathcal{E}_{\{\mathcal{M}\}}$-Banach-plots c.*

Proof. For the proof we have to consider the "weak versions" 12.2.3 resp. 12.2.5.
(a) First we consider the case $\mathcal{E}_{(\mathcal{M})}$, for this we use now 12.2.5.
We have to show that $f \circ c$ is $\mathcal{E}_{(\mathcal{M})}$ for each $\mathcal{E}_{(\mathcal{M})}$-Banach-plot $c : G \supseteq D \to E$, where D denotes the open unit ball in an arbitrary Banach-space G. So, by condition (3) in 12.2.5, we have to show: For each compact set $K \subseteq D$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{beur,sub}}$ and moreover for each sequence $(s_k)_k \in \int^{\mathcal{M}}_{\text{beur,FdB}}$ the set

$$\left\{\frac{(f \circ c)^{(k)}(a)(v_1,\dots,v_k)}{k!} \cdot r_k \cdot s_k \cdot \delta^k : a \in K, k \in \mathbb{N}, \|v_i\|_E \le 1\right\}$$

has to be bounded in F for each number $\delta > 0$. So let the number $\delta > 0$, the sequences $(r_k)_k$, $(s_k)_k$, and finally a compact (w.l.o.g. convex) set $K \subseteq D$ be given, arbitrary but from now on fixed. Then for each $\alpha \in E^*$, by assumption and by 12.2.5 applied to the sequence $(r_k \cdot (2 \cdot D \cdot \delta)^k)_k$, where the constant D is coming from $s_k \le D^k \cdot (\hat{s}_o)_k$ (since $(s_k)_k \in \int^{\mathcal{M}}_{\text{beur,FdB}}$), the set

$$\left\{\frac{(\alpha \circ c)^{(k)}(a)(v_1,\dots,v_k) \cdot r_k \cdot s_k \cdot (2 \cdot D \cdot \delta)^k}{k!} : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1\right\} \tag{12.2.1}$$

is bounded in $\mathbb{R}$. So the set

$$\left\{\frac{c^{(k)}(a)(v_1,\dots,v_k) \cdot r_k \cdot s_k \cdot (2 \cdot D \cdot \delta)^k}{k!} : a \in K, k \in \mathbb{N}, \|v_i\|_G \le 1\right\}$$

is contained in some closed absolutely convex bounded subset B of E, hence

$$\frac{\|c^{(k)}(a)\|_{L^k(G,E_B)} \cdot r_k \cdot s_k \cdot \delta^k}{k!} \le \frac{1}{(2D)^k}. \tag{12.2.2}$$

We proceed now as in [22, 4.10. Theorem]. $c(K)$ is compact in E_B since the mapping $c : K \to E_B$ is Lipschitzian: For all $x, y, \in K$ we get $c(x) - c(y) \in \frac{M^l_1 \|x-y\|_G}{2Dr_1\delta} \cdot B$ for each $l \in \Lambda$. Then we estimate for all $\delta > 0$ and $k \in \mathbb{N}$, $k \geq 1$, as follows where the sequences $s = (s_j)_j$ and $\hat{s} = (\hat{s}_j)_j$ are related by $\int^{\mathcal{M}}_{\text{beur,FdB}}$:

$$\begin{aligned}
&\left\| \frac{(f \circ c)^{(k)}(a)}{k!} \cdot r_k \cdot s_k \cdot \delta^k \right\|_{L^k(G,F)} \\
&\leq \sum_{j \geq 0} \sum_{\alpha \in \mathbb{N}^j_{>0}, \sum_{i=1}^j \alpha_i = k} D^k \cdot \frac{\|f^{(j)}(c(a))\|_{L^j(E_B,F)} \cdot \hat{s}_j}{j!} \cdot \underbrace{\prod_{i=1}^j \frac{\|c^{(\alpha_i)}(a)\|_{L^{\alpha_i}(G,E_B)} \cdot r_{\alpha_i} \cdot \hat{s}_{\alpha_i} \cdot \delta^{\alpha_i}}{\alpha_i!}}_{\leq \frac{1}{(2D)^{\alpha_1}} \cdots \frac{1}{(2D)^{\alpha_j}} = \frac{1}{(2D)^k}} \\
&\leq \left(\frac{1}{2}\right)^k \cdot \sum_{j \geq 0} \sum_{\alpha \in \mathbb{N}^j_{>0}, \sum_{i=1}^j \alpha_i = k} \underbrace{\frac{\|f^{(j)}(c(a))\|_{L^j(E_B,F)}}{j! \cdot m^l_j}}_{(\star) \leq C \cdot h^j} \cdot \underbrace{(\hat{s}_j \cdot m^l_j)}_{\leq C_1^j} \\
&\leq (C \cdot h \cdot C_1) \cdot \left(\frac{1}{2}\right)^k \cdot \sum_{j \geq 0} \binom{k-1}{j-1} (h \cdot C_1)^{j-1} = (C \cdot h \cdot C_1) \cdot \left(\frac{1}{2}\right)^k \cdot (1 + C_1 \cdot h)^{k-1} \\
&\leq (C \cdot h \cdot C_1) \cdot \left(\frac{(1 + C_1 \cdot h)}{2}\right)^k .
\end{aligned}$$

We have to choose $l \in \Lambda$ according to the sequence $\hat{s} \in \int^{\mathcal{M}}_{\text{beur}}$, which arises in $\int^{\mathcal{M}}_{\text{beur,FdB}}$, such that $\hat{s}_j \cdot m^l_j \leq C_1^j$ for some constant $C_1 > 0$ and all $j \in \mathbb{N}$. Since $f \in \mathcal{E}_{(\mathcal{M})}$, we obtain the estimate $(\star)$ with this l and arbitrary $h > 0$ for a constant $C = C_{l,h}$ and all $j \in \mathbb{N}$. Finally we can choose $h := \frac{1}{C_1}$ and so the expression at the beginning is bounded by $C = C_{l,1/C_1}$.

(b) Now we consider the case $\mathcal{E}_{\{\mathcal{M}\}}$ and here we use 12.2.3. We recall: By condition (3) there it's sufficient to show that each compact set $K \subseteq D$, for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum,sub}}$ and moreover for each sequence $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum,FdB}}$, there exists a number $\varepsilon > 0$ such that the set

$$\left\{ \frac{(f \circ c)^{(k)}(a)(v_1, \dots, v_k)}{k!} \cdot r_k \cdot s_k \cdot \varepsilon^k : a \in K, k \in \mathbb{N}, \|v_i\|_E \leq 1 \right\} \tag{12.2.3}$$

is bounded in F.
We use the same proof as for the above Beurling-case and replace in (2) in 12.2.3 the sequence $(r_k)_k$ by $((2D)^k \cdot r_k)_k$, where D is the constant arising in $s_k \leq D^k \cdot (\hat{s}_o)_k$ (since $(s_k)_k \in \int^{\mathcal{M}}_{\text{roum,FdB}}$). Then we take $\delta = 1$ in (12.2.1), in (12.2.2) and in the Lipschitz-argument. We can use now precisely the same estimate as for the Beurling-case (for $\delta = 1$) and so we have shown (12.2.3) for $\varepsilon = \frac{2}{(1+C_1 \cdot h)}$. Note that $f \in \mathcal{E}_{\{\mathcal{M}\}}$, hence we have to consider $l \in \Lambda$ and $h > 0$ sufficiently large to obtain estimate $(\star)$ for some constant C. According to this chosen index $l \in \Lambda$ we can estimate $\hat{s}_j \cdot m^l_j \leq C_1^j$ for a constant C_1 and all $j \in \mathbb{N}$, since $\hat{s} \in \int^{\mathcal{M}}_{\text{roum}}$, which arises in $\int^{\mathcal{M}}_{\text{roum,FdB}}$. Compare this calculation with (4) $\Rightarrow$ (1) in 10.7.1. □

By using 12.2.8 we can prove the analogous result to [22, 4.11. Theorem]:

Theorem 12.2.9. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$. Let E, F, G be convenient vector spaces, $U \subseteq E$ and $V \subseteq F$ be c^∞-open and $f : U \to F$, $g : V \to G$ with $f(U) \subseteq V$.*

(a) If $\mathcal{M}$ has in addition $\mathcal{M}_{(\text{FdB})}$, then $f, g \in \mathcal{E}_{(\mathcal{M})}$ implies $g \circ f \in \mathcal{E}_{(\mathcal{M})}$.

(*b*) *If* $\mathcal{M}$ *has in addition* $\mathcal{M}_{\{\mathrm{FdB}\}}$, *then* $f, g \in \mathcal{E}_{\{\mathcal{M}\}}$ *implies* $g \circ f \in \mathcal{E}_{\{\mathcal{M}\}}$.

Proof. By (12.1.3) we have to show: For all closed absolutely convex bounded subsets $B \subseteq E$ and for all $\alpha \in G^*$ the composite $\alpha \circ g \circ f \circ i_B : U_B \to \mathbb{R}$ is $\mathcal{E}_{[\mathcal{M}]}$. By assumption $f \circ i_B \in \mathcal{E}_{[\mathcal{M}]}$ and $\alpha \circ g \in \mathcal{E}_{[\mathcal{M}]}$ hold, so we can use the previous result 12.2.8 to obtain the desired implication. Note that $f \circ i_B$ is a $\mathcal{E}_{[\mathcal{M}]}$-Banach-plot. □

12.3 Exponential laws for both cases by using Banach plots

First we start with the analogous result to [22, 5.1. Lemma]:

Lemma 12.3.1. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, resp. a $(\mathcal{M}_{\text{sc}})$ weight matrix, with $\Lambda = \mathbb{R}_{>0}$, E a Banach-space and $U \subseteq E$ open. Let F be a convenient vector space and $\mathcal{B}$ be family of bounded linear functionals on F which together detect bounded sets. Then we have:*

$$f \in \mathcal{E}_{[\mathcal{M}]}(U,F) \Leftrightarrow \alpha \circ f \in \mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R}) \qquad \forall\, \alpha \in \mathcal{B}.$$

Recall that by remark 10.5.12 we are able to formulate this result also for arbitrary weight matrices.

Proof. For $\mathcal{E}$-curves this follows by [20, 2.1., 2.11.], and so by composing with such curves for $\mathcal{E}$-mappings $f : U \to F$.
In the Roumieu-case we use now (1) $\Leftrightarrow$ (2) in 12.2.4 to get: For arbitrary $\alpha \in F^*$ the mapping $\alpha \circ f$ is $\mathcal{E}_{\{\mathcal{M}\}}$ if and only if for each compact $K \subseteq U$ the set

$$\left\{(\alpha \circ f)^{(k)}(a)(v_1,\dots,v_k)\cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}, \|v_i\|_E \le 1\right\}$$

is bounded in $\mathbb{R}$ for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum}}$ and for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{roum}}$.
But this is equivalent to the fact that $f : U \to F$ is $\mathcal{E}_{\{\mathcal{M}\}}$ if for each compact $K \subseteq U$ the set

$$\left\{f^{(k)}(a)(v_1,\dots,v_k)\cdot r_k \cdot s_k : a \in K, k \in \mathbb{N}, \|v_i\|_E \le 1\right\}$$

is bounded in F for each sequence $(r_k)_k \in \mathcal{R}_{\text{roum}}$ and for each sequence $(s_k)_k \in \mathcal{S}^{\mathcal{M}}_{\text{roum}}$.
Because $\mathcal{B}$ detects bounded sets we can replace now in the above equivalences F^* by $\mathcal{B}$ and obtain the desired equivalence.

For the Beurling-case proceed analogously and use (1) $\Leftrightarrow$ (2) in 12.2.6. □

Now we are able to prove the central theorem *Cartesian closedness* for ultradifferentiable function classes defined by general (also non-constant) weight matrices $\mathcal{M}$ analogously to [22, 5.2. Theorem]. The following theorem should be also compared with 10.10.4 and 10.10.5 and the proofs there.

Theorem 12.3.2. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ (recall: in the Roumieu-case we can restrict ourselves to $\Lambda = \mathbb{N}_{>0}$), let $U_i \subseteq E_i$ be c^∞-open subsets in convenient vector spaces E_i for $i = 1, 2$ and moreover let F be also a convenient vector space. Then we obtain:*

(*a*) *If additionally $(\mathcal{M}_{\{\text{mg}\}})$ is satisfied, then*

$$f \in \mathcal{E}_{\{\mathcal{M}\}}(U_1 \times U_2, F) \Longleftrightarrow f^\vee \in \mathcal{E}_{\{\mathcal{M}\}}(U_1, \mathcal{E}_{\{\mathcal{M}\}}(U_2, F)).$$

(*b*) *If additionally $(\mathcal{M}_{(\text{mg})})$ is satisfied, then*

$$f \in \mathcal{E}_{(\mathcal{M})}(U_1 \times U_2, F) \Longleftrightarrow f^\vee \in \mathcal{E}_{(\mathcal{M})}(U_1, \mathcal{E}_{(\mathcal{M})}(U_2, F)).$$

Some further important remarks concerning this important theorem:

(*i*) In both cases ($\Longleftarrow$) holds also without $(\mathcal{M}_{\{\text{mg}\}})$ resp. $(\mathcal{M}_{(\text{mg})})$.

(*ii*) To prove ($\Longleftarrow$) it's sufficient to assume that $\mathcal{M}$ is arbitrary, i.e. $(\mathcal{M})$, and properties $(\mathcal{M}_{\{\text{alg}\}})$ resp. $(\mathcal{M}_{(\text{alg})})$.

(*iii*) For the proof we don't need necessarily that $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ is a category (closedness under composition).

(*iv*) But in fact if $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix with $(\mathcal{M}_{\{\text{mg}\}})$ and $(\mathcal{M}_{\{\text{FdB}\}})$ resp. with $(\mathcal{M}_{(\text{mg})})$ and $(\mathcal{M}_{(\text{FdB})})$, then by this theorem and 12.2.9 the category $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$ is cartesian closed.

Proof. The proof is completely analogous to [22, 5.2. Theorem]: First we point out that already $\mathcal{E}(U_1 \times U_2, F) \cong \mathcal{E}(U_1, \mathcal{E}(U_2, F))$ holds as shown in [20, 3.12.], so we assume form now on that all occurring mappings are smooth. Let $B \subseteq E_1 \times E_2$ and $B_i \subseteq E_i$, $i = 1, 2$, where B, B_1, B_2 run through all closed absolutely convex bounded subsets. Then we obtain the following equivalences:

$$
\begin{aligned}
& f \in \mathcal{E}_{[\mathcal{M}]}(U_1 \times U_2, F) \\
& \Leftrightarrow \ \forall\, \alpha \in F^* \ \forall\, B: \ \alpha \circ f \circ i_B \in \mathcal{E}_{[\mathcal{M}]}((U_1 \times U_2)_B, \mathbb{R}) \\
& \Leftrightarrow \ \forall\, \alpha \in F^* \ \forall\, B_1, B_2: \ \alpha \circ f \circ (i_{B_1} \times i_{B_2}) \in \mathcal{E}_{[\mathcal{M}]}((U_1)_{B_1} \times (U_2)_{B_2}, \mathbb{R})
\end{aligned}
$$

where the first equivalence holds by definition and the second because each bound set $B \subseteq E_1 \times E_2$ is contained in $B_1 \times B_2$ for some bounded sets $B_i \subseteq E_i$. Hence the inclusion $(E_1 \times E_2)_B \to (E_1)_{B_1} \times (E_2)_{B_2}$ is bounded.
On the other side we obtain:

$$
\begin{aligned}
& f^\vee \in \mathcal{E}_{[\mathcal{M}]}(U_1, \mathcal{E}_{[\mathcal{M}]}(U_2, F)) \\
& \Leftrightarrow \ \forall\, B_1: \ f^\vee \circ i_{B_1} \in \mathcal{E}_{[\mathcal{M}]}((U_1)_{B_1}, \mathcal{E}_{[\mathcal{M}]}(U_2, F)) \\
& \Leftrightarrow \ \forall\, \alpha \in F^* \ \forall\, B_1, B_2: \ \mathcal{E}_{[\mathcal{M}]}(i_{B_2}, \alpha) \circ f^\vee \circ i_{B_1} \in \mathcal{E}_{[\mathcal{M}]}((U_1)_{B_1}, \mathcal{E}_{[\mathcal{M}]}((U_2)_{B_2}, \mathbb{R})).
\end{aligned}
$$

The first equivalence holds again by definition, for the second we have to use 12.3.1 and note that the linear mappings $\mathcal{E}_{[\mathcal{M}]}(i_{B_2}, \alpha)$ generate the bornology.
With these preparations we are able to restrict in the following to the special situation where $U_i \subseteq E_i$ are open sets in Banach-spaces E_i and $F = \mathbb{R}$.

We start now with ($\Longrightarrow$) for both cases:

Let $f \in \mathcal{E}_{[\mathcal{M}]}(U_1 \times U_2, \mathbb{R})$, then clearly $f^\vee$ takes values in the space $\mathcal{E}_{[\mathcal{M}]}(U_2, \mathbb{R})$.

First we show that
Claim: $f^\vee : U_1 \to \mathcal{E}_{[\mathcal{M}]}(U_2, \mathbb{R})$ is $\mathcal{E}$ with $d^j f^\vee = (\partial_1^j f)^\vee$.
$\mathcal{E}_{[\mathcal{M}]}(U_2, \mathbb{R})$ are convenient vector spaces, hence by [20, 5.20.] it suffices to prove that the iterated unidirectional derivatives $d_v^j f^\vee(x)$ exist, are equal to $\partial_1^j f(x, \cdot)(v^j)$, and are separately bounded for x and v in compact subsets. For $j = 1$ and x, v, y fixed we consider the smooth curve $c : t \mapsto f(x + tv, y)$. Then, by the fundamental theorem of calculus, we obtain:

$$
\begin{aligned}
& \frac{f^\vee(x + tv) - f^\vee(x)}{t}(y) - (\partial_1 f)^\vee(x)(y)(v) = \frac{c(t) - c(0)}{t} - c^{'}(0) \\
& = t \int_0^1 s \int_0^1 c^{''}(tsr)drds = t \int_0^1 s \int_0^1 \partial_1^2 f(x + tsrv, y)(v, v)drds.
\end{aligned}
$$

$(\partial_1^2 f)^\vee(K_1)(o(E_1 \times E_1))$ is bounded in $\mathcal{E}_{[\mathcal{M}]}(U_2,\mathbb{R})$ and for each compact set $K_1 \subseteq U_1$ this expression is Mackey-convergent to 0 in $\mathcal{E}_{[\mathcal{M}]}(U_2,\mathbb{R})$ for $t \to 0$. Hence $d_v f^\vee(x)$ exists an is equal to $\partial_1 f(x,\cdot)(v)$.
Now we use induction with the same argumentation as above for $(d_v^j f^\vee)^\wedge : (x,y) \mapsto \partial_1^j f(x,y)(v^j)$ instead of f. So again
$(\partial_1^2 (d_v^j f^\vee)^\wedge)^\vee(K_1)(o(E_1 \times E_1)) = (\partial_1^{j+2} f)^\vee(K_1)(o(E_1), o(E_1), v, \dots, v)$ is bounded, hence also the separated boundedness of $d_v^j f^\vee(x)$ follows. So the claim is proved.

We distinguish now between the Roumieu- and the Beurling-case.
The Beurling-case:
We have to show that the mapping $f^\vee : U_1 \to \mathcal{E}_{(\mathcal{M})}(U_2,\mathbb{R})$ is $\mathcal{E}_{(\mathcal{M})}$. For the proof condition $(\mathcal{M}_{(\mathrm{mg})})$ will play the key-role.
By 12.3.1 it suffices to prove that $f^\vee : U_1 \to \mathcal{E}_{\mathcal{M},l,h}(E_2 \supseteq K_2,\mathbb{R})$ is $\mathcal{E}^b_{(\mathcal{M})} = \mathcal{E}_{(\mathcal{M})}$ for each $K_2 \subseteq U_2$ compact, each $h > 0$ and $l \in \Lambda = \mathbb{R}_{>0}$. This holds, because each $\alpha \in (\mathcal{E}_{(\mathcal{M})}(U_2,\mathbb{R}))^*$ factorizes over $\mathcal{E}_{\mathcal{M},l,h}(E_2 \supseteq K_2,\mathbb{R})$ for some K_2, h and l.
So we have to show that for each compact sets $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, each $h_1, h_2 > 0$ and each $l_1, l_2 \in \Lambda$, the set

$$\left\{ \frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{h_1^{k_1} \cdot M_{k_1}^l} : x_1 \in K_1, k_1 \in \mathbb{N}, \|v_j^1\|_{E_1} \le 1 \right\} \tag{12.3.1}$$

is bounded in the space $\mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2,\mathbb{R})$. Equivalently, for all compact sets K_1, K_2, for all $h_1, h_1 > 0$ and all $l_1, l_2 \in \Lambda$ the set

$$\left\{ \frac{\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1; v_1^2,\dots,v_{k_2}^2)}{h_2^{k_2} \cdot h_1^{k_1} \cdot M_{k_2}^{l_2} \cdot M_{k_1}^{l_1}} : x_i \in K_i, k_i \in \mathbb{N}, \|v_j^i\|_{E_i} \le 1; i = 1,2 \right\} \tag{12.3.2}$$

is bounded in $\mathbb{R}$. Condition $(\mathcal{M}_{(\mathrm{mg})})$ precisely means that for each $l_1, l_2 \in \Lambda$ we can find a constant $C > 0$ and another index $n \in \Lambda$ such that $M_{j+k}^n \le C^{j+k} \cdot M_j^{l_1} \cdot M_k^{l_2}$ holds for all $j, k \in \mathbb{N}$.
Let $x_1 \in K_1$, $k_1 \in \mathbb{N}$, then we obtain the following estimate:

$$\begin{aligned}
&\left\| \frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{h_1^{k_1} \cdot M_{k_1}^{l_1}} \right\|^J_{\mathcal{M},K_2,l_2,h_2} \\
&= \sup \left\{ \frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1; v_1^2,\dots,v_{k_2}^2)\right|}{h_1^{k_1} \cdot h_2^{k_2} \cdot M_{k_1}^{l_1} \cdot M_{k_2}^{l_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \right\} \\
&\underbrace{\le}_{(\mathcal{M}_{(\mathrm{mg})})} \sup \left\{ C^{k_1+k_2} \cdot \frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1; v_1^2,\dots,v_{k_2}^2)\right|}{h_1^{k_1} \cdot h_2^{k_2} \cdot M_{k_1+k_2}^{n}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \right\} \\
&\le \sup \left\{ \frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1; v_1^2,\dots,v_{k_2}^2)\right|}{h^{k_1+k_2} \cdot M_{k_1+k_2}^{n}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1 \right\} < +\infty
\end{aligned}$$

where we have put $h := \frac{1}{C} \cdot \min\{h_1, h_2\}$, C is the constant arising in $(\mathcal{M}_{(\mathrm{mg})})$, and the last expression is bounded because f is $\mathcal{E}_{(\mathcal{M})}$ by assumption. So for arbitrary $h_1, h_2 > 0$ and $l_1, l_2 \in \Lambda$ we can find another index $n \in \Lambda$ and a number $h > 0$ such that the previous estimate is valid. This shows that $f^\vee$ is $\mathcal{E}_{(\mathcal{M})}$.

The Roumieu-case:
By 12.3.1 it suffices to prove that $f^\vee : U_1 \to \varinjlim_{l_2 \in \Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$ is $\mathcal{E}^b_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{\{\mathcal{M}\}}$ for each compact set $K_2 \subseteq U_2$. This holds because each $\alpha \in (\mathcal{E}_{\{\mathcal{M}\}}(U_2,\mathbb{R}))^*$ factorizes over some $\varinjlim_{l_2 \in \Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$.
So we have to prove that for all $K_1 \subseteq U_1$, $K_2 \subseteq U_2$ compact there exist $h_1 > 0$ and some $l_1 \in \Lambda$ such that the set in (12.3.1) is bounded in $\varinjlim_{l_2 \in \Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$.
Equivalently, we have to show that for all K_1, K_2 compact there exist $h_1, h_2 > 0$ and $l_1, l_2 \in \Lambda$ such that the set in (12.3.2) is bounded in $\mathbb{R}$!
We can use now the same estimate as for the above Beurling-case and furthermore use condition $(\mathcal{M}_{\{\mathrm{mg}\}})$. First, because f is $\mathcal{E}_{\{\mathcal{M}\}}$ and by (3) in 12.1.2 we obtain that there exist some $h > 0$ and $n \in \Lambda$, such that the last set

$$\sup\left\{\frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1;v_1^2,\dots,v_{k_2}^2)\right|}{h^{k_1+k_2}\cdot M^n_{k_1+k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \|v_j^2\|_{E_2} \le 1\right\}$$

in the above estimate is bounded. For this index $n \in \Lambda$ we obtain via property $(\mathcal{M}_{\{\mathrm{mg}\}})$ that there exist some other indices $l_1, l_2 \in \Lambda$ and a constant $C > 0$ such that $M^n_{j+k} \le C^{j+k} \cdot M^{l_1}_j \cdot M^{l_2}_k$ holds for all $j, k \in \mathbb{N}$. So we can put in the estimate now $h_i := C \cdot h$ for $i = 1, 2$ to get, that $f^\vee$ is $\mathcal{E}_{\{\mathcal{M}\}}$.

Now we start with $(\Longleftarrow)$ for both cases:

Let $f^\vee : U_1 \to \mathcal{E}_{[\mathcal{M}]}(U_2, \mathbb{R})$ be $\mathcal{E}_{[\mathcal{M}]}$. By 12.4.7 the mapping $f^\vee : U_1 \to \mathcal{E}_{[\mathcal{M}]}(U_2,\mathbb{R}) \to \mathcal{E}(U_2,\mathbb{R})$ is $\mathcal{E}$, hence it remains to show that $f \in \mathcal{E}_{[\mathcal{M}]}(U_1 \times U_2, \mathbb{R})$.

The Beurling-case:
For each compact set $K_2 \subseteq U_2$, each $h_2 > 0$ and each $l_2 \in \Lambda$, the mapping $f^\vee : U_1 \to \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$ is $\mathcal{E}^b_{(\mathcal{M})} = \mathcal{E}_{(\mathcal{M})}$. More precisely this means that for all compact sets $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, each $h_1, h_2 > 0$ and each $l_1, l_2 \in \Lambda$ the set in (12.3.1) is bounded in $\mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$. Because it is contained in the space $\mathcal{E}_{\mathcal{M},K_2,l_2,h_2}(U_2,\mathbb{R}) := \{f \in \mathcal{E}(U_2,\mathbb{R}) : j^\infty(f)|_{K_2} \in \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2,\mathbb{R})\}$ with semi-norm $\|f\|^J_{\mathcal{M},K_2,l_2,h_2} := \|j^\infty(f)|_{K_2}\|^J_{\mathcal{M},l_2,h_2}$, it is also bounded in this space and so the set in (12.3.2) is bounded (in $\mathbb{R}$).
We have assumed that $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix, hence each M^l is (weakly) log. convex which implies $M^l_j \cdot M^l_k \le M^l_{j+k}$ for all $j, k \in \mathbb{N}$. - In fact for the next estimate property $(\mathcal{M}_{(\mathrm{alg})})$ would be sufficient, i.e. for each $n \in \Lambda$ there exist $l_1, l_2 \in \Lambda$ and $C > 0$ such that $M^{l_1}_j \cdot M^{l_2}_k \le C^{j+k} \cdot M^n_{j+k}$ for all $j, k \in \mathbb{N}$.
Let $x_1 \in K$, $k_1 \in \mathbb{N}$ and $\|v^1_j\|_{E_1}$, then we estimate as follows:

$$\begin{aligned}
+\infty &> \left\|\frac{d^{k_1} f^\vee(x_1)(v_1^1,\dots,v_{k_1}^1)}{h_1^{k_1}\cdot M^{l_1}_{k_1}}\right\|^J_{\mathcal{M},K_2,l_2,h_2}\\
&= \sup\left\{\frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1;v_1^2,\dots,v_{k_2}^2)\right|}{h_1^{k_1}\cdot h_2^{k_2}\cdot M^{l_1}_{k_1}\cdot M^{l_2}_{k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \left\|v_j^2\right\|_{E_2} \le 1\right\}\\
&\ge \sup\left\{\frac{\left|\partial_2^{k_2}\partial_1^{k_1} f(x_1,x_2)(v_1^1,\dots,v_{k_1}^1;v_1^2,\dots,v_{k_2}^2)\right|}{h^{k_1+k_2}\cdot M^n_{k_1+k_2}} : x_2 \in K_2, k_2 \in \mathbb{N}, \left\|v_j^2\right\|_{E_2} \le 1\right\}
\end{aligned}$$

where we have put $n := \max\{l_2, l_2\}$ and $h := \max\{h_1, h_2\}$ (resp. put $h := C \cdot \max\{h_1, h_2\}$, where $n \in \Lambda$ and $C > 0$ are coming from property $(\mathcal{M}_{(\mathrm{alg})})$!). So we have shown that f is $\mathcal{E}_{(\mathcal{M})}$.

The Roumieu-case:
For each compact set $K_2 \subseteq U_2$ the mapping $f^\vee : U_1 \to \varinjlim_{l_2\in\Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$ is $\mathcal{E}_{\{\mathcal{M}\}}$. By (3) in 12.1.2 the dual space $\big(\varinjlim_{l_2\in\Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})\big)^*$ can be equipped with the Baire-vector-space-topology of the countable limit of Banach-spaces $\varprojlim_{l_2\in\Lambda} \varprojlim_{h_2>0} (\mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R}))^*$.
Now we can use 12.2.2 to conclude that the mapping $f^\vee : U_1 \to \varinjlim_{l_2\in\Lambda} \varinjlim_{h_2>0} \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$ is $\mathcal{E}^b_{\{\mathcal{M}\}}$.
By (3) in 12.1.2 this inductive limit is countable and compactly regular and so for each compact set $K_1 \subseteq U_1$ there exist $h_1 > 0$ and $l_1 \in \Lambda$ such that the set in (12.3.1) is bounded in $\mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})$ for some $h_2 > 0$ and $l_2 \in \Lambda$. Because it is contained in the space $\mathcal{E}_{\mathcal{M},K_2,l_2,h_2}(U_2, \mathbb{R}) := \{f \in \mathcal{E}(U_2, \mathbb{R}) : j^\infty(f)|_{K_2} \in \mathcal{E}_{\mathcal{M},l_2,h_2}(E_2 \supseteq K_2, \mathbb{R})\}$ with semi-norm $\|f\|^J_{\mathcal{M},K_2,l_2,h_2} := \|j^\infty(f)|_{K_2}\|^J_{\mathcal{M},l_2,h_2}$, it is also bounded in this space and so the set in (12.3.2) is bounded with those given h_1, h_2, l_1, l_2 (in $\mathbb{R}$)!
But now we can use the same estimate as in the above Beurling-case (for precisely those given h_1, h_2, l_1, l_2) to conclude that f is $\mathcal{E}_{\{\mathcal{M}\}}$. We point out: Again alternatively for the estimate we don't need necessarily that each sequence is weakly log. convex, we can also use property $(\mathcal{M}_{\{\mathrm{alg}\}})$. $\square$

By using 12.3.2 we can prove now an analogous result to 10.10.6 above (see [22, 5.5. Corollary]):

Corollary 12.3.3. *Let $\mathcal{M}$ be a weight matrix with the same properties as assumed in 12.3.2 (for the particular case).*
Let E, F, E_i, F_i, G be convenient vector spaces and let U and V be c^∞-open subsets. Then we get

(1) *The exponential law*
$\mathcal{E}_{[\mathcal{M}]}(U, \mathcal{E}_{[\mathcal{M}]}(V, G)) \cong \mathcal{E}_{[\mathcal{M}]}(U \times V, G)$
holds, it is a linear $\mathcal{E}_{[\mathcal{M}]}$-diffeomorphism of convenient vector spaces.

The following mappings are $\mathcal{E}_{[\mathcal{M}]}$:

(2) $\mathrm{ev} : \mathcal{E}_{[\mathcal{M}]}(U, F) \times U \to F$ *given by* $\mathrm{ev}(f, x) = f(x)$.

(3) $\mathrm{ins} : E \to \mathcal{E}_{[\mathcal{M}]}(F, E \times F)$ *given by* $\mathrm{ins}(x)(y) = (x, y)$.

(4) $(\cdot)^\wedge : \mathcal{E}_{[\mathcal{M}]}(U, \mathcal{E}_{[\mathcal{M}]}(V, G)) \to \mathcal{E}_{[\mathcal{M}]}(U \times V, G)$.

(5) $(\cdot)^\vee : \mathcal{E}_{[\mathcal{M}]}(U \times V, G) \to \mathcal{E}_{[\mathcal{M}]}(U, \mathcal{E}_{[\mathcal{M}]}(V, G))$.

(6) $\prod : \prod_i \mathcal{E}_{[\mathcal{M}]}(E_i, F_i) \to \mathcal{E}_{[\mathcal{M}]}\left(\prod_i E_i, \prod_i F_i\right)$.

If the matrix satisfies additionally $(\mathcal{M}_{[\mathrm{FdB}]})$, then we also get

(7) $\mathrm{comp} : \mathcal{E}_{[\mathcal{M}]}(F, G) \times \mathcal{E}_{[\mathcal{M}]}(U, F) \to \mathcal{E}_{[\mathcal{M}]}(U, G)$.

(8) $\mathcal{E}_{[\mathcal{M}]}(\cdot,\cdot) : \mathcal{E}_{[\mathcal{M}]}(F, F_1) \times \mathcal{E}_{[\mathcal{M}]}(E_1, E) \to \mathcal{E}_{[\mathcal{M}]}(\mathcal{E}_{[\mathcal{M}]}(E, F), \mathcal{E}_{[\mathcal{M}]}(E_1, F_1))$ *which is given by* $(f, g) \mapsto (h \mapsto f \circ h \circ g)$.

We close this section with the following remark: We can prove a matrix-generalization of [22, 5.4. Example], so for $(\mathcal{M}_{\mathrm{sc}})$ weight matrices with $\Lambda = \mathbb{R}_{>0}$ and such that $(\mathcal{M}_{\{\mathrm{mg}\}})$ is not satisfied the important Theorem 12.3.2 doesn't hold. In the Roumieu-case this was already shown in 10.10.7, in the Beurling-case we refer to 12.7.5 where we have to use the projective description of 9.4.4.

12.4 $\mathcal{E}_{[\mathcal{M}]}$-uniform boundedness principles and consequences by using Banach plots

In this section we prove the $\mathcal{E}_{[\mathcal{M}]}$-uniform boundedness principles by using Banach plots and we will also give some applications and prove immediate consequences. First we start with our central result which is analogous to 10.8.1 and [22, 6.1. Theorem]:

Theorem 12.4.1. *Let $\mathcal{M}$ be an arbitrary matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$ (so in the Roumieu-case we can restrict to the countable index set $\Lambda = \mathbb{N}_{>0}$). Let E, F, G be convenient vector spaces, furthermore $U \subseteq F$ be c^∞-open. A linear mapping $T : E \to \mathcal{E}_{[\mathcal{M}]}(U, G)$ is bounded if and only if $\mathrm{ev}_x \circ T : E \to G$ is bounded for every $x \in U$.*

Proof. $(\Rightarrow)$ Let $x \in U$ and $\alpha \in G^*$ be given, the linear mapping $\alpha \circ \mathrm{ev}_x = \mathcal{E}_{[\mathcal{M}]}(x, \alpha) : \mathcal{E}_{[\mathcal{M}]}(U, G) \to \mathbb{R}$ is continuous and so ev_x is bounded. Hence, if T is bounded, then also $\mathrm{ev}_x \circ T$.

$(\Leftarrow)$ We assume that $\mathrm{ev}_x \circ T$ is bounded for each $x \in U$.
By the definition of the space $\mathcal{E}_{[\mathcal{M}]}(U, G)$ above (initial structure w.r.t. mappings i_B, B an absolutely convex bounded closed subset in F and $\alpha \in G^*$, see (12.1.3)), it suffices to prove that T is bounded for the special cases where both E and F are Banach-spaces and $G = \mathbb{R}$. But by definition we have that $\mathcal{E}_{[\mathcal{M}]}(U, \mathbb{R}) = \varprojlim_K \mathcal{E}_{[\mathcal{M}]}(F \supseteq K, \mathbb{R})$ holds and by item (2) in 12.1.2 the space $\mathcal{E}_{(\mathcal{M})}(F \supseteq K, \mathbb{R})$ is Fréchet, by item (3) in 12.1.2 the space $\mathcal{E}_{\{\mathcal{M}\}}(F \supseteq K, \mathbb{R})$ is a (LB)-space, hence webbed (by our assumption on Λ!). So we can use the closed graph theorem [20, 52.10] in both cases as in the constant weight matrix case [22, 6.1. Theorem], replace the classes $\mathcal{E}_{[M]}$ by $\mathcal{E}_{[\mathcal{M}]}$ in the diagram there. □

The next result is analogous to [22, 8.3. Proposition] and 10.7.3 (by using curves):

Proposition 12.4.2. *Let $\mathcal{M}$ be a an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$, let E and F be convenient vector spaces and $U \subseteq E$ a c^∞-open subset. Recall notation: We denote by $\mathcal{M}^d := \{M^l_{+1} : M^l \in \mathcal{M}\}$, with $M^l_{+1} = (M^l_{k+1})_k$, and then we obtain:*

(1) *Multilinear mappings between convenient vector spaces are $\mathcal{E}_{[\mathcal{M}]}$ if and only if they are bounded.*

(2) *If $f : E \supseteq U \to F$ is $\mathcal{E}_{[\mathcal{M}]}$, then the derivative $df : U \to L(E, F)$ is $\mathcal{E}_{[\mathcal{M}^d]}$. If $\mathcal{M}$ is a assumed in addition to be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix (more precisely condition $(\mathcal{M}_{[\mathrm{alg}]})$ would be sufficient), then also $\hat{df} : U \times E \to F$ is $\mathcal{E}_{[\mathcal{M}^d]}$, where the space $L(E, F)$ of all bounded linear mappings between E and F is equipped with the topology of uniform convergence on bounded sets.*

(3) *The chain-rule holds.*

Recall: Note that condition $(\mathcal{M}_{[\mathrm{dc}]})$ implies $\mathcal{E}_{[\mathcal{M}]} = \mathcal{E}_{[\mathcal{M}^d]}$.

Proof. (1) We point out: If f is multilinear and $\mathcal{E}_{[\mathcal{M}]}$, then it is smooth and hence bounded by [20, 5.5. Lemma]. Conversely, if f is multilinear and bounded, then it is smooth by [20, 5.5. Lemma]. Furthermore $f \circ i_B$ is multilinear and continuous and all derivatives of high order vanish. So f is $\mathcal{E}_{[\mathcal{M}]}$ by definition.

(2) If f is smooth, so by [20, 3.18.] the map $df : U \to L(E,F)$ exists and it is smooth, too. We have to show now that $(df) \circ i_B : U_B \to L(E,F)$ is $\mathcal{E}_{[\mathcal{M}^d]}$ for all closed absolutely convex bounded subsets $B \subseteq E$. By [20, 5.18.] and 12.3.1 above it is sufficient to show that the mapping $U_B \ni x \mapsto \alpha(df(i_B(x))(v)) \in \mathbb{R}$ is $\mathcal{E}_{[\mathcal{M}^d]}$ for each $\alpha \in E^*$ and $v \in E$.
If $\alpha \circ f$ is $\mathcal{E}_{\{\mathcal{M}\}}$ resp. $\mathcal{E}_{(\mathcal{M})}$, then for all $B \subseteq E$ closed, bounded, absolutely convex subset of E, for each compact set $K \subseteq U_B$ and for some $l \in \Lambda$ and some $h > 0$ (resp. for each $l \in \Lambda$ and each $h > 0$) we obtain that the set

$$\left\{ \frac{\left\| d^k(\alpha \circ f \circ i_B)(a) \right\|_{L^k(E_B,\mathbb{R})}}{h^k \cdot M^l_k} : a \in K, k \in \mathbb{N} \right\}$$

is bounded by some $C > 0$. Now for $v \in E$ and B as assumed above, $v \in B$, we calculate as follows for all $k \in \mathbb{N}$:

$$\begin{aligned}
&\left\| d^k((L(\alpha,v) \circ df) \circ i_B)(a) \right\|_{L^k(E_B,\mathbb{R})} = \left\| d^k((d(\alpha \circ f)()(v)) \circ i_B)(a) \right\|_{L^k(E_B,\mathbb{R})} \\
&= \left\| d^{k+1}(\alpha \circ f \circ i_B)(a)(v,\dots) \right\|_{L^k(E_B,\mathbb{R})} \\
&\le \left\| d^{k+1}(\alpha \circ f \circ i_B)(a) \right\|_{L^{k+1}(E_B,\mathbb{R})} \cdot \|v\|_B \le (C \cdot h) \cdot h^k \cdot M^l_{k+1}.
\end{aligned}$$

This shows that $df : U \to L(E,F)$ is $\mathcal{E}_{[\mathcal{M}^d]}$. If moreover $\mathcal{M}$ is in addition assumed to be a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix, then by (2) in 12.4.3 the mapping $\hat{df} : U \times E \to F$ is also $\mathcal{E}_{[\mathcal{M}^d]}$.
(3) The chain-rule is valid for all smooth f by [20, 3.18.]. □

The next result is analogous to [22, 8.4. Proposition] and to 10.8.2:

Corollary 12.4.3. *Let $\mathcal{M}$ be a an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$. Then the following properties are satisfied:*

(1) *For convenient vector spaces E and F, on $L(E,F)$ the following topologies have the same bounded sets in $L(E,F)$:*

 (*a*) *The topology of uniform convergence on bounded sets of E.*
 (*b*) *The topology of pointwise convergence.*
 (*c*) *The embedding $L(E,F) \subseteq \mathcal{E}(E,F)$.*
 (*d*) *The embedding $L(E,F) \subseteq \mathcal{E}_{[\mathcal{M}]}(E,F)$.*

(2) *If in addition $\mathcal{M}$ is a $(\mathcal{M}_{\mathrm{sc}})$ weight matrix (more precisely condition $(\mathcal{M}_{[\mathrm{alg}]})$ would be sufficient) then we get: Let E, F, G be convenient vector spaces and $U \subseteq E$ be a c^∞-open subset. A mapping $f : U \times F \to G$, which is linear in the second variable is $\mathcal{E}_{[\mathcal{M}]}$ if and only if $f^\vee : U \to L(E,G)$ is well-defined and $\mathcal{E}_{[\mathcal{M}]}$.*

Proof. (1) The first three topologies on $L(E,F)$ have the same bounded sets as it was shown in [20, 5.3., 5.18.]. The inclusion $\mathcal{E}_{[\mathcal{M}]}(E,F) \to \mathcal{E}(E,F)$ is bounded by 12.4.7 and the inclusion $L(E,F) \to \mathcal{E}_{[\mathcal{M}]}(E,F)$ is bounded by the UBP 12.4.1.
(2) In the $\mathcal{E}$-case this holds by [20, 3.12.] (because $L(E,F)$ is closed in $\mathcal{E}(E,F)$). If f is $\mathcal{E}_{[\mathcal{M}]}$, then we have to show now that the mapping $f^\vee \circ i_B$ is $\mathcal{E}_{[\mathcal{M}]}$ into the space $L(F,G)$ for all absolutely convex bounded sets $B \subseteq E$. By [20, 5.18.] and by 12.3.1 we have to show that the mapping $U_B \ni x \mapsto \alpha\left(f^\vee(i_B(x))(v)\right) = \alpha\left(f(i_B(x),v)\right)$ is $\mathcal{E}_{[\mathcal{M}]}$ for each $\alpha \in G^*$ and $v \in F$. But this is clearly satisfied, because of the definition for f to be $\mathcal{E}_{[\mathcal{M}]}$.

Conversely, assume that $f^\vee : U \to L(F,G)$ is $\mathcal{E}_{[\mathcal{M}]}$ and we have to show that $f : U \times F \to G$ is $\mathcal{E}_{[\mathcal{M}]}$. By the first part the inclusion $L(F,G) \to \mathcal{E}_{[\mathcal{M}]}$ is bounded and linear, hence also the mapping $f^\vee : U \to \mathcal{E}_{[\mathcal{M}]}(F,G)$ is $\mathcal{E}_{[\mathcal{M}]}$. Now we can use ($\Longleftarrow$) in 12.3.2 for both cases to conclude that $f : U \times F \to G$ is $\mathcal{E}_{[\mathcal{M}]}$, the linearity in the second variable is clear. Note that here for the use of 12.3.2 we don't need conditions $(\mathcal{M}_{\{\mathrm{mg}\}})$ resp. $(\mathcal{M}_{(\mathrm{mg})})$ (and also $(\mathcal{M}_{\{\mathrm{FdB}\}})$ resp. $(\mathcal{M}_{(\mathrm{FdB})})$ is not necessarily needed!). More precisely we only need conditions $(\mathcal{M}_{\{\mathrm{alg}\}})$ resp. $(\mathcal{M}_{(\mathrm{alg})})$, which are both satisfied automatically by (weakly) log. convexity of each sequence M^l (see remark (ii) in 12.3.2). $\square$

As pointed out after [22, 8.4. Proposition] and similarly as in the proof of 10.8.2, we can show $f^\vee \in \mathcal{E}_{[\mathcal{M}]}(U, L(F,G)) \Rightarrow f \in \mathcal{E}_{[\mathcal{M}]}(U \times F, G)$ in (2) in 12.4.3 also without the use of Cartesian closedness 12.3.2 (by assuming the same properties for the matrix $\mathcal{M}$).
For this we proceed as follows: By composing with some $\alpha \in G^*$ we can assume w.l.o.g. $G = \mathbb{R}$. We use the same formula as in [21, 4.2. Corollary] and calculate for all B, B' closed absolutely convex bounded subsets of E and F, each $K \subseteq U_B$ compact, $x \in K$ and $k \in \mathbb{N}$ as follows:

$$\left\|d^k f(x,w_0)\right\|_{L^k(E_B \times F_{B'},\mathbb{R})} \leq \|d^k(f^\vee)(x)(\dots)(w_0)\|_{L^k(E_B,\mathbb{R})} + \sum_{i=1}^{k} \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}$$

$$\leq \|d^k(f^\vee)(x)\|_{L^k(E_B,L(F_{B'}\mathbb{R}))} \cdot \|w_0\|_{B'} + \sum_{i=1}^{k} \|d^{k-1}(f^\vee)(x)\|_{L^{k-1}(E_B,L(F_{B'},\mathbb{R}))}$$

$$\leq C \cdot h^k \cdot M^l_k \cdot \|w_0\|_{B'} + \sum_{i=1}^{k} C \cdot h^{k-1} \cdot M^l_{k-1} = C \cdot h^k \cdot M^l_k \cdot \|w_0\|_{B'} + k \cdot C \cdot h^{k-1} \cdot M^l_{k-1}$$

$$\leq C \cdot h^k \cdot M^l_k \cdot \|w_0\|_{B'} + C \cdot h_1^k \cdot h^k \cdot M^l_k \leq C \cdot (\|w_0\|_{B'} + 1) \cdot h_2^k \cdot M^l_k.$$

We have used that $L(i_{B'},\mathbb{R}) \circ f^\vee \circ i_B : U_B \to L(F_{B'},\mathbb{R})$ is $\mathcal{E}_{[\mathcal{M}]}$. Note that each M^l is assumed to be increasing and we have put $h_2 := h_1 \cdot h$, and $h_1 \geq 2$ can be chosen arbitrary but fixed (because $k \leq 2^k$ for all $k \in \mathbb{N}$). This shows that f is $\mathcal{E}_{[\mathcal{M}]}$.

The next result is analogous to [22, 8.8. Lemma] and to 10.8.3:

Lemma 12.4.4. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$ and let E, F and G be convenient vector spaces and $V \subseteq F$ be c^∞-open. Then the flip of variables induces the following vector space isomorphism: $L(E, \mathcal{E}_{[\mathcal{M}]}(V,G)) \cong \mathcal{E}_{[\mathcal{M}]}(V, L(E,G))$.*

Proof. Let $f \in \mathcal{E}_{[\mathcal{M}]}(V, L(E,G))$, consider for each $x \in E$ the mapping $\tilde{f}(x) := \mathrm{ev}_x \circ f \in \mathcal{E}_{[\mathcal{M}]}(V,G)$. Note that $x \mapsto \tilde{f}(x)$ is a linear mapping and by UBP 12.4.1 we obtain that $\tilde{f}$ is bounded, because $\mathrm{ev}_x \circ \tilde{f} = \tilde{f}(x) = \mathrm{ev}_x \circ f = f(x) \in L(E,G)$ for all $x \in V$.

Conversely let $\alpha \in L(E, \mathcal{E}_{[\mathcal{M}]}(V,G))$, then consider for each $x \in V$ the associated linear functional $\tilde{\alpha}(x) := \mathrm{ev}_x \circ \alpha \in L(E,G)$. By the first part of 12.4.3, the bornology of $L(E,G)$ is generated by $\mathcal{B} := \{\mathrm{ev}_x : x \in E\}$ and moreover $\mathrm{ev}_x \circ \tilde{\alpha} = \tilde{\alpha}(x) = \mathrm{ev}_x \circ \alpha = \alpha(x) \in \mathcal{E}_{[\mathcal{M}]}(V,G)$. Hence $\tilde{\alpha} : V \to L(E,G)$ is $\mathcal{E}_{[\mathcal{M}]}$ by 12.3.1 (for $(\mathcal{M})$ weight matrices). □

The next result is analogous to [22, 8.9. Lemma] and 10.8.4:

Lemma 12.4.5. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$ and $U \subseteq E$ a c^∞-open subset in a convenient vector space E. We denote in the following with $\lambda_{[\mathcal{M}]}(U)$ the c^∞-closure of the sub-space generated by $\{\mathrm{ev}_x : x \in E\}$ in $\mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R})'$. Moreover consider the mapping $\delta : U \to \lambda_{[\mathcal{M}]}(U)$ defined by $x \mapsto \mathrm{ev}_x$. Then the following is satisfied: For every convenient vector space G the $\mathcal{E}_{[\mathcal{M}]}$-mapping δ induces a bornological isomorphism*

$$L(\lambda_{[\mathcal{M}]}(U), G) \cong \mathcal{E}_{[\mathcal{M}]}(U,G).$$

Proof. $\lambda_{[\mathcal{M}]}(U)$ is a convenient vector space, because it is a c^∞-closed subspace in $\mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R})'$, which is convenient. Moreover the mapping δ introduced above is $\mathcal{E}_{[\mathcal{M}]}$: This holds because for each $x \in \mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R})$ we have by definition $\mathrm{ev}_x \circ \delta = x$, and we can use now 12.3.1 to conclude that δ is $\mathcal{E}_{[\mathcal{M}]}$ (and by composing with all $i_B : U_B \to U$). This implies that the mapping $\delta^* : L(\lambda_{[\mathcal{M}]}(U), G) \to \mathcal{E}_{[\mathcal{M}]}(U,G)$ is well-defined and linear.

δ^* is injective: Each bounded linear mapping $\lambda_{[\mathcal{M}]}(U) \to G$ is uniquely determined on $\delta(U) := \{\mathrm{ev}_x : x \in U\}$.
δ^* is surjective: Consider some $f \in \mathcal{E}_{[\mathcal{M}]}(U,G)$, then, by definition, $\alpha \circ f \in \mathcal{E}_{[\mathcal{M}]}(U,G)$ for each $\alpha \in G^*$. Introduce now $\tilde{f} : \mathcal{E}_{[\mathcal{M}]}(U,\mathbb{R})' \to \prod_{G^*} \mathbb{R}$ by $\varphi \mapsto (\varphi(\alpha \circ f))_{\alpha \in G^*}$, and this mapping is well-defined, bounded and linear.
ev_x is mapped to $\tilde{f}(\mathrm{ev}_x) = \delta(f(x))$, where $\delta : G \to \prod_{G^*} \mathbb{R}$ denotes the bornological embedding given by $x \mapsto (\alpha \circ x)_{\alpha \in G^*}$. This map induces a bounded linear mapping $\tilde{f} : \lambda_{[\mathcal{M}]}(U) \to G$ with finally $\tilde{f} \circ \delta = f$, which shows the surjectivity of δ.
But moreover δ^* is a bornological isomorphism: δ^* and $(\delta^*)^{-1}$ both are bounded, which holds by the UBP 12.4.1 and the first part of 12.4.3. □

By using these results we can prove the analogous result to [22, 8.10. Theorem] resp. to 10.8.6:

Theorem 12.4.6. *Let both $\mathcal{M}$ and $\mathcal{N}$ be arbitrary weight matrices, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$. Moreover let E, F be convenient vector spaces and $U \subseteq E$, $V \subseteq F$, be c^∞-open subsets. Then we obtain the following bornological isomorphisms:*

(1) $\mathcal{E}_{(\mathcal{M})}(U, \mathcal{E}_{(\mathcal{N})}(V,F)) \cong \mathcal{E}_{(\mathcal{N})}(V, \mathcal{E}_{(\mathcal{M})}(U,F))$

(2) $\mathcal{E}_{\{\mathcal{M}\}}(U, \mathcal{E}_{\{\mathcal{N}\}}(V,F)) \cong \mathcal{E}_{\{\mathcal{N}\}}(V, \mathcal{E}_{\{\mathcal{M}\}}(U,F))$

(3) $\mathcal{E}_{(\mathcal{M})}(U, \mathcal{E}_{\{\mathcal{N}\}}(V,F)) \cong \mathcal{E}_{\{\mathcal{N}\}}(V, \mathcal{E}_{(\mathcal{M})}(U,F))$

(4) $\mathcal{E}_{[\mathcal{M}]}(U, \mathcal{E}(V,F)) \cong \mathcal{E}(V, \mathcal{E}_{[\mathcal{M}]}(U,F))$

(5) $\mathcal{E}_{[\mathcal{M}]}(U, \mathcal{C}^\omega(V,F)) \cong \mathcal{C}^\omega(V, \mathcal{E}_{[\mathcal{M}]}(U,F))$

(6) $\mathcal{E}_{[\mathcal{M}]}(U, L(V, F)) \cong L(V, \mathcal{E}_{[\mathcal{M}]}(U, F))$

(7) $\mathcal{E}_{[\mathcal{M}]}(U, l^\infty(X, F)) \cong l^\infty(X, \mathcal{E}_{[\mathcal{M}]}(U, F))$

(8) $\mathcal{E}_{[\mathcal{M}]}(U, \mathcal{L}ip^k(X, F)) \cong \mathcal{L}ip^k(X, \mathcal{E}_{[\mathcal{M}]}(U, F))$
where X denotes in (7) resp. (8) a so-called l^∞-space, i.e. a set together with a bornology which is induced by a family of real valued functions on X (see [12, 1.2.4]), resp. a $\mathcal{L}ip^k$-space (see [12, 1.4.1]). For the definition of the spaces $l^\infty(X, F)$ resp. $\mathcal{L}ip^k(X, F)$ we refer to [12, 3.6.1., 4.4.1].

Proof. All mappings and their inverse mappings are given by $f \mapsto \tilde{f}$, where $\tilde{f}(x)(y) := f(y)(x)$. If we write $\mathcal{X}^1$ and $\mathcal{X}^2$ for any of the occurring function spaces, then we have to show $\mathcal{X}^1(U, \mathcal{X}^2(V, F)) \cong \mathcal{X}^2(V, \mathcal{X}^1(U, F))$.
The fact that $\tilde{f}(x_2) \in \mathcal{X}^1(U, F)$ is obviously satisfied, because $\tilde{f}(x_2) = \mathrm{ev}_{x_2} \circ f : U \to \mathcal{X}^2(V, F) \to F$ holds by definition for all $x_2 \in V$.
Moreover all occurring spaces are convenient and satisfy the $\mathcal{B}$-uniform-boundedness-principle, where $\mathcal{B}$ denotes the set of all point-evaluations (for $\mathcal{E}_{[\mathcal{M}]}$ see the above definitions and the UBP 12.4.1, for all other cases we refer to the proof in [22, 8.10. Theorem]). So the mapping $f \mapsto \tilde{f}$ is bounded and linear in any case, because $f \mapsto \mathrm{ev}_{x_1} \circ \mathrm{ev}_{x_2} \circ \tilde{f} = \mathrm{ev}_{x_2} \circ \mathrm{ev}_{x_1} \circ f$ holds by definition and this mapping is bounded and linear.
It remains to show in any case that $\tilde{f} \in \mathcal{X}^2(V, \mathcal{X}^1(U, F))$, that $\tilde{f}$ is in fact of the appropriate class.

$(1) - (4)$: For $\alpha \in \{(\mathcal{M}), \{\mathcal{M}\}\}$, $\beta \in \{(\mathcal{N}), \{\mathcal{N}\}, \infty\}$ we obtain the following equivalences:

$$\mathcal{X}^\alpha(U, \mathcal{X}^\beta(V, F)) \cong L(\lambda^\alpha(U), \mathcal{X}^\beta(V, F)) \cong \mathcal{X}^\beta(V, L(\lambda^\alpha(U), F)) \cong \mathcal{X}^\beta(V, \mathcal{X}^\alpha(U, F)).$$

For the first and the third isomorphism we have used 12.4.5, for the second isomorphism we have used 12.4.4 resp., if $\beta = \infty$, also [20, 3.13.4, 5.3.].

(5): Follows by (2) and (3) and note that $\mathcal{C}^\omega(U, F) = \mathcal{E}_{\{\mathcal{I}\}}(U, F)$ holds as vector spaces with bornology, where $\mathcal{I}$ is the constant matrix $M^l_p = (p!)_p$ for each $l \in \Lambda$ (see e.g. [22, 7.2. Theorem (2)]).

(6): is exactly 12.4.4.

$(7), (8)$: The arguments for the proofs of these items are completely the same as in [22, 8.10. Theorem (7),(8)]. □

Finally we use the UBP to prove the analogous to [22, 8.1. Proposition]:

Proposition 12.4.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be arbitrary weight matrices, i.e. $(\mathcal{M})$, both with $\Lambda = \mathbb{R}_{>0}$. Moreover let E and F be convenient vector spaces and $U \subseteq E$ a c^∞-open subset. Then we obtain:*

(1) $\mathcal{E}_{(\mathcal{M})}(U, F) \subseteq \mathcal{E}_{\{\mathcal{M}\}}(U, F) \subseteq \mathcal{E}(U, F)$.

(2) $\mathcal{M}\{\preceq\}\mathcal{N}$ *implies* $\mathcal{E}_{\{\mathcal{M}\}}(U, F) \subseteq \mathcal{E}_{\{\mathcal{N}\}}(U, F)$, $\mathcal{M}(\preceq)\mathcal{N}$ *implies* $\mathcal{E}_{(\mathcal{M})}(U, F) \subseteq \mathcal{E}_{(\mathcal{N})}(U, F)$.

(3) $\mathcal{M} \lhd \mathcal{N}$ *implies* $\mathcal{E}_{\{\mathcal{M}\}}(U, F) \subseteq \mathcal{E}_{(\mathcal{N})}(U, F)$.

(4) *If* $U \neq \emptyset$ *and* $E, F \neq \{0\}$*, then*

$(\mathcal{M}_{\{\mathcal{C}^\omega\}}) \Longleftrightarrow \mathcal{C}^\omega(U,F) \subseteq \mathcal{E}_{\{\mathcal{M}\}}(U,F)$

resp.

$(\mathcal{M}_{(\mathcal{C}^\omega)}) \Longleftrightarrow \mathcal{C}^\omega(U,F) \subseteq \mathcal{E}_{(\mathcal{M})}(U,F)$.

All inclusions are bounded.

Proof. All inclusions are valid by the above definitions and moreover they are bounded, because all occurring vector spaces are convenient and satisfy the UBP 12.4.1 resp. [20, 5.26.].
Note that $\mathcal{C}^\omega(U,F) = \mathcal{E}_{\{\mathcal{I}\}}(U,F)$ holds as vector spaces with bornology, where $\mathcal{I}$ is the constant matrix $M^l_p = (p!)_p$ for each $l \in \Lambda$ (see e.g. [22, 7.2. Theorem (2)]). □

12.5 Comparison of the introduced structures $\mathcal{E}_{\{\mathcal{M}\},\mathbf{curve}}$ and $\mathcal{E}_{\{\mathcal{M}\}}$

In this section we want to compare the introduced different structures. We will have to use again the UBP but first we are going to prove an analogous result to [23, 2.8. Corollary]:

Proposition 12.5.1. *Let* $\mathcal{M}$ *be as in 10.7.1, let* $U \subseteq E$ *be* c^∞*-open in a convenient vector space* E*, let* F *be another convenient vector space and consider* $f : U \to F$*. Then the following are equivalent:*

(1) f *is* $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}$.

(2) f *is* $\mathcal{E}_{\{\mathcal{M}\}}$ *along strongly* $\mathcal{E}_{\{\mathcal{M}\}}$*-curves.*

(3) f *is smooth and for each closed bounded absolutely convex subset* B *in* E *and each* $x \in U \cap E_B$ *there exist* $r, h, C > 0$ *and* $l \in \Lambda$ *such that*

$$\left\| \frac{d^k(\pi_V \circ f \circ i_B)(a)}{h^k \cdot M^l_k} \right\|_{L^k(E_B,F_V)} \leq C$$

for all $a \in U \cap E_B$ *with* $\|a - x\|_B \leq r$ *and all* $k \in \mathbb{N}$.

(4) f *is smooth and for each closed bounded absolutely convex subset* B *in* E *and each compact* $K \subseteq U \cap E_B$ *there exist* $h, C > 0$ *and* $l \in \Lambda$ *such that*

$$\left\| \frac{d^k(\pi_V \circ f \circ i_B)(a)}{h^k \cdot M^l_k} \right\|_{L^k(E_B,F_V)} \leq C,$$

where in (3) *and* (4) *the mapping* $\pi_V : F \to F_V$ *denotes the natural mapping and* $V \subseteq F$ *is an absolutely convex* 0*-neighborhood.*

Proof. Each item holds for f if and only if it holds for $\pi_V \circ f$ for each absolutely convex 0-neighborhood $V \subseteq F$, so we can use now 10.7.1. □

By using 12.5.1 we can prove (see also [22, 7.2. Theorem (1)]):

Theorem 12.5.2. *Let $\mathcal{M}$ be a weight matrix as considered in 10.7.1, let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open in E. Then we obtain the following: The structure defined by curves in* (10.3.1) *and* (10.3.2)*, denoted by $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ (Roumieu-case), and the structure defined by Banach-plots defined in* (12.1.3)*, see also* (12.1.4) *and* (12.1.5)*, coincides as vector spaces with bornology, so*

$$\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F) = \mathcal{E}_{\{\mathcal{M}\}}(U,F).$$

Proof. First 12.5.1 proves that $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ and $\mathcal{E}_{\{\mathcal{M}\}}(U,F)$ coincide as vector spaces.
Both spaces $\mathcal{E}_{\{\mathcal{M}\},\text{curve}}(U,F)$ and $\mathcal{E}_{\{\mathcal{M}\}}(U,F)$ are convenient and satisfy the UBP w.r.t. to the set of point evaluations ev_x, see 10.8.1 and 12.4.1. This shows that the identity is a bornological isomorphism. □

12.6 Projective descriptions for both cases

Analogously to the definitions in the previous section 9.1. we can proceed as follows: Let E and F be Banach-spaces, $K \subseteq E$ compact and convex, furthermore let $r = (r_k)_k$ and $s = (s_k)_k$ be two arbitrary sequences of positive real numbers. Then we can introduce the spaces

$$\mathcal{E}_{(s),(r)}(E \supseteq K, F) := \{(f^m)_m \in \prod_{m\in\mathbb{N}} \mathcal{C}(K, L^m_{\text{sym}}(E,F)) : \|f\|^J_{(s),(r)} < +\infty\},$$

where we have put $\|f\|^J_{(s),(r)} := \max\left\{\sup\left\{\frac{\|f\|_k}{r_k \cdot s_k} : k \in \mathbb{N}\right\}, \sup\left\{\frac{|||f|||_{n,k}}{r_{n+k-1}\cdot s_{n+k-1}} : k,n \in \mathbb{N}\right\}\right\}$ and $\|f\|_k$ and $|||f|||_{n,k}$ are defined as in section 9.1.
If $r_k = h^k$ and $s_k = M^l_k$ for some $l \in \Lambda$ with $M^l \in \mathcal{M} = \{M^l : l \in \Lambda\}$ for $\mathcal{M}$ an arbitrary weight matrix, then $\mathcal{E}_{(s),(r)}(E \supseteq K, F)$ coincides of course with the space $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K, F)$ in 9.1. and analogously to (1) in 12.1.2 we see that $\mathcal{E}_{(s),(r)}(E \supseteq K, F)$ is a Banach-space.
So we obtain the analogous result to [22, 8.6. Theorem]:

Theorem 12.6.1. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$, let E and F be Banach spaces and $U \subseteq E$ be an open and convex subset. Then we obtain*

$$\mathcal{E}_{(\mathcal{M})}(U,F) \cong \varprojlim_{K,s,r} \mathcal{E}_{(s),(r)}(E \supseteq K, F)$$

as vector spaces with bornology. In the limit K runs through all compact convex subsets of U, r runs through all sequences of the set $(\mathcal{R}_{\text{beur}})^{-1} := \{r = (r_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{r} \in \mathcal{R}_{\text{beur}}\}$ and finally s runs through all sequences of the set $\left(\mathcal{S}^{\mathcal{M}}_{\text{beur}}\right)^{-1} := \{s = (s_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{s} \in \mathcal{S}^{\mathcal{M}}_{\text{beur}}\}$.

Proof. As in (5) in 12.1.2 we see that $f \in \mathcal{E}_{(s),(r)}(E \supseteq K, F)$ is a smooth function $f : U \to F$. By 12.2.6 we obtain the desired isomorphism as vector spaces.
The identity from the left to the right is continuous, and the space on the right is a projective limit of Banach spaces, hence convenient. Moreover $\mathcal{E}_{(\mathcal{M})}(U,F)$ satisfies the UBP 12.4.1 with respect to the set $\mathcal{B} := \{\text{ev}_x : x \in U\}$, so the identity is also bounded from right to left. □

Now we prove the Roumieu-type-version:

Theorem 12.6.2. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{N}_{>0}$, let E and F be Banach spaces and $U \subseteq E$ be an open and convex subset. Then we obtain*

$$\mathcal{E}_{\{\mathcal{M}\}}(U,F) \cong \varprojlim_{K,s,r} \mathcal{E}_{(s),(r)}(E \supseteq K,F)$$

as vector spaces with bornology. In the limit K runs again through all compact convex subsets of U, r runs through all sequences of the set $(\mathcal{R}_{\text{roum}})^{-1} := \{r = (r_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{r} \in \mathcal{R}_{\text{roum}}\}$ and finally s runs through all sequences of the set $\left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1} := \{s = (s_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0} : \frac{1}{s} \in \mathcal{S}^{\mathcal{M}}_{\text{roum}}\}$.

Proof. We use the same proof as for 12.6.1 but of course we apply 12.2.4 instead of 12.2.6. □

Also in the matrix case situation we can prove the Remark which can be found between 8.7. Theorem and 8.8. Lemma in [22]:

Lemma 12.6.3. *We can show that the identity $\varprojlim_{K,s,r} \mathcal{E}_{(s),(r)}(E \supseteq K,F) \to \mathcal{E}_{\{\mathcal{M}\}}(U,F)$ is bounded without using the UBP 12.4.1, where K runs through all compact convex sets in U, r runs through the set $(\mathcal{R}_{\text{roum}})^{-1}$ and finally s runs through the set $\left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1}$.*

Proof. Let $\mathcal{B}$ be a bounded set in $\varprojlim_{K,s,r} \mathcal{E}_{s,r}(E \supseteq K,F)$. So for each compact convex set $K \subseteq U$, each $r \in (\mathcal{R}_{\text{roum}})^{-1}$ and each $s \in \left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1}$ the set $\mathcal{B}$ is bounded in $\mathcal{E}_{s,r}(E \supseteq K,F)$, which means by definition that

$$\sup\{\|f|_K\|^J_{(s),(r)} : f \in \mathcal{B}\} < +\infty.$$

Because $\varprojlim_{K,s,r} \mathcal{E}_{(s),(r)}(E \supseteq K,F)$ consists of infinite jets of smooth functions we can use equation (3.1.1.) in [22] to estimate again $|||f|||_{n,k}$ by $\|f\|_{n+k+1}$ (defined like above). So we can consider the sequence

$$b_k := \sup\{\|f|_K\|_k : f \in \mathcal{B}\}$$

which satisfies now by definition $\sup_{k\in\mathbb{N}} \frac{b_k}{s_k \cdot r_k} < +\infty$ for each $r \in (\mathcal{R}_{\text{roum}})^{-1}$ and each $s \in \left(\mathcal{S}^{\mathcal{M}}_{\text{roum}}\right)^{-1}$. We can use now 10.5.5 (3) $\Rightarrow$ (1) (recall that by 10.5.12 we don't need necessarily a $(\mathcal{M}_{\text{sc}})$ weight matrix) to conclude that there exists an index $l \in \Lambda = \mathbb{N}_{>0}$ such that the power series $\sum_{k=0}^{\infty} a^l_k \cdot t^k$ with $a^l_k := \frac{b_k}{M^l_k}$ has positive radius of convergence. Hence there exists a number $h > 0$ such that $\sup_{k\in\mathbb{N}} \frac{a^l_k}{h^k} < +\infty$ for some $h > 0$ and this means exactly that the set $\mathcal{B}$ is contained and bounded in $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K,F)$ for the obtained index $l \in \Lambda$ and $h > 0$. - And this gives also an independent proof of the completeness of $\mathcal{E}_{\{\mathcal{M}\}}(U,F)$ and of the regularity of the inductive limit (see (3) and (4) in 12.1.2). □

12.7 Roumieu-type reduced to the Beurling-type

In this section we give some applications of 9.4.4. It turns out that some proofs for spaces of ultradifferentiable functions defined by weight matrices of Roumieu-type can be reduced to the Beurling-case by using the following projective description for $U \subseteq \mathbb{R}^n$ non-empty open:

$$\mathcal{E}_{\{\mathcal{M}\}}(U,\mathbb{R}^m) = \bigcap_{\mathcal{M}\lhd L} \mathcal{E}_{(L)}(U,\mathbb{R}^m) = \bigcap_{\mathcal{M}\lhd L} \mathcal{E}_{\{L\}}(U,\mathbb{R}^m)$$

and the intersections are taken over all arbitrary sequences L with $\mathcal{M} \lhd L$.
If we assume that $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix, then we can restrict the intersections to all weakly log. convex weight sequences L with $\mathcal{M} \lhd L$ and $(L_k)^{1/k} \to \infty$ for $k \to \infty$ (see (b) in 9.4.4).
In 9.4.4 we have shown this projective description on the level of linear spaces, but we are able to prove now an analogue result to [22, 8.2. Theorem]:

Theorem 12.7.1. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$, let E and F be convenient vector spaces and $U \subseteq E$ be a c^∞-open subset. Then we obtain the following projective description for the Roumieu-tpye-space defined by $\mathcal{M}$:*

$$\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \bigcap_{\mathcal{M}\lhd L} \mathcal{E}_{(L)}(U,F) = \bigcap_{\mathcal{M}\lhd L} \mathcal{E}_{\{L\}}(U,F) \tag{12.7.1}$$

holds as vector spaces with bornology, where the intersection is taken over all arbitrary sequences L with $\mathcal{M} \lhd L$. If $\mathcal{M}$ is assumed to be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then the intersection can be taken over all weakly log. convex sequences L with $\mathcal{M} \lhd L$, $(L_k)^{1/k} \to \infty$ for $k \to \infty$.

Proof. By the definition of the spaces (composing with i_B and $\alpha \in F^*$), we can reduce to the case where both E and F are Banach-spaces. By (5) in 12.1.2 and the definitions at the beginning of this chapter we can replace in the proof of 9.4.4 the sequence $(f_k)_k$ there by $(\|j^\infty f|_K\|_k)_k$ and obtain that all occurring spaces coincide as vector spaces.
Each space is convenient (note that projective limits preserve c^∞-completeness) and each space satisfies the UBP 12.4.1 w.r.t. the set of point evaluations: The structure of $\varprojlim_L \mathcal{E}_{[\mathcal{M}]}(U,F)$ is initial w.r.t. the inclusions $\varprojlim_L \mathcal{E}_{[\mathcal{M}]}(U,F) \longrightarrow \mathcal{E}_{[\mathcal{M}]}(U,F)$ for each arbitrary sequence with $\mathcal{M} \lhd L$. This implies that the identity between any two spaces is bounded, hence a bornological isomorphism.

If $\mathcal{M}$ is assumed to be a $(\mathcal{M}_{\text{sc}})$ weight matrix, then the intersection (projective limit) can also be taken only over the set of all weakly log. convex sequences L with $\mathcal{M} \lhd L$, $(L_k)^{1/k} \to \infty$ for $k \to \infty$, because in this case (b) in 9.4.4 is valid. □

We give now a second alternative proof of 12.7.1, which doesn't use 12.4.1 (analogously to [22, 5.3. Remarks]):

Proof. The spaces in the representation (12.7.1) coincide as vector spaces by definition and 9.4.4.
Let $K \subseteq E$ be compact, E and F be Banach spaces and furthermore $l \in \Lambda$ and $h > 0$ be given. Then the inclusion $\mathcal{E}_{\mathcal{M},l,h}(E \supseteq K,F) \to \mathcal{E}_{L,h_1}(E \supseteq K,F)$ is continuous (bounded) for all $h_1 > 0$ whenever $\mathcal{M} \lhd L$ holds (recall: this means $M^l \lhd L$ for all $l \in \Lambda$).
This implies that the inclusion $\varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},l,h}(E \supseteq K,F) \to \varprojlim_{h_1>0} \mathcal{E}_{L,h_1}(E \supseteq K,F)$ is continuous, hence by definition of the spaces also the inclusion $\mathcal{E}_{\{\mathcal{M}\}}(U,F) \to \mathcal{E}_{(L)}(U,F)$.

Conversely, let $\mathcal{B}$ be a bounded set in the space $\varprojlim_L \mathcal{E}_{(L)}(U,F)$ in (12.7.1) and the limit is taken over all sequences L with $\mathcal{M} \lhd L$. This means that $\mathcal{B}$ has to be bounded in each space $\mathcal{E}_{(L)}(U,F)$.
We have to show now that $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{M}\}}(U,F)$, and by definition of the spaces (composition with mapping $\mathcal{E}_{\{\mathcal{M}\}}(i_B,\alpha)$) we can assume w.l.o.g. that E is a Banach-space and $F=\mathbb{R}$. Let $K\subseteq U$ be an arbitrary compact set and we introduce the sequence $b=(b_k)_k$ by $b_k := \sup\{\|j^\infty(f)|_K\|_k : f\in\mathcal{B}\}$. By assumption we get: For each weight sequence L with $\mathcal{M}\lhd L$ the set $\mathcal{B}$ is bounded in $\mathcal{E}_{(L)}(U,F)$. In fact the proof of 9.4.4 shows (since it only deals with a formal series f_k in the numerator of the Denjoy-Carleman-quotient) that we have $b\in\bigcap_L \mathcal{F}_{(L)} = \mathcal{F}_{\{\mathcal{M}\}}$ and this implies that $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{M}\},K}(U,F) := \varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},K,l,h}(U,F)$. Finally, by (5) in 12.1.2, the set $\mathcal{B}$ is also bounded in $\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{\mathcal{M}\},K}(U,F)$. □

We can use 12.7.1 for the case if $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix to prove $(\Longleftarrow)$ in the *Cartesian closedness Theorem* 12.3.2 for the Roumieu-case by using $(\Longleftarrow)$ for the Beurling-case (which is the analogous result to [22, 5.3. Remarks]):

Proposition 12.7.2. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda=\mathbb{R}_{>0}$, let $U_i\subseteq E_i$ for $i=1,2$ be c^∞-open subsets in convenient vector spaces and furthermore F be convenient, too. Assume we have already shown that $f^\vee\in\mathcal{E}_{(L)}(U_1,\mathcal{E}_{(L)}(U_2,F))$ implies $f\in\mathcal{E}_{(L)}(U_1\times U_2,F)$ for each (weakly) log. convex weight sequence L with $\mathcal{M}\lhd L$, $(L_k)^{1/k}\to\infty$, then we obtain also the following implication:*

$$f^\vee\in\mathcal{E}_{\{\mathcal{M}\}}(U_1,\mathcal{E}_{\{\mathcal{M}\}}(U_2,F)) \Longrightarrow f\in\mathcal{E}_{\{\mathcal{M}\}}(U_1\times U_2,F).$$

Proof. Let $f^\vee\in\mathcal{E}_{\{\mathcal{M}\}}(U_1,\mathcal{E}_{\{\mathcal{M}\}}(U_2,F))$, then $f^\vee\in\mathcal{E}_{(L)}(U_1,\mathcal{E}_{(L)}(U_2,F))$ for each weakly log. convex sequence L with $\mathcal{M}\lhd L$, $(L_k)^{1/k}\to\infty$, holds by 12.7.1. By assumption this implies now $f\in\mathcal{E}_{(L)}(U_1\times U_2,F)$, hence again by 12.7.1 we obtain $f\in\mathcal{E}_{\{\mathcal{M}\}}(U_1\times U_2,F)$.

Note that for the proof of $f^\vee\in\mathcal{E}_{(\mathcal{L})}(U_1,\mathcal{E}_{(\mathcal{L})}(U_2,F)) \Longrightarrow f\in\mathcal{E}_{(\mathcal{L})}(U_1\times U_2,F)$ in 12.3.2 for $(\mathcal{M}_{\text{sc}})$ weight matrices $\mathcal{L}$ with $\Lambda=\mathbb{R}_{>0}$ we don't need necessarily property $(\mathcal{M}_{(\text{mg})})$. □
12.7.1 shows also in an alternative way that:

Corollary 12.7.3. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda=\mathbb{R}_{>0}$, let E and F be convenient vector spaces, $U\subseteq E$ c^∞-open. Then we obtain:*

(1) *$\mathcal{E}_{\{\mathcal{M}\}}(U,F)$ is c^∞-complete.*

(2) *Let E, F be Banach spaces and $K\subseteq E$ be convex and compact. Then the inductive limit $\mathcal{E}_{\{\mathcal{M}\}}(E\supseteq K,F) = \varinjlim_{l\in\Lambda,h>0} \mathcal{E}_{\mathcal{M},l,h}(E\supseteq K,F)$ is regular (see (4) in 12.1.2).*

Proof. (1) This holds since by (12.7.1) we have $\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \varprojlim_L \mathcal{E}_{(L)}(U,F)$ as vector spaces with bornology, where the projective limit is taken over all sequences L with $\mathcal{M}\lhd L$, and finally $\varprojlim_L \mathcal{E}_{(L)}(U,F)$ is c^∞-complete.
(2) Is valid by the second proof of 12.7.1. □

With this projective technique we can also give the following independent proof of the UBP 12.4.1 for the Roumieu-matrix-case, which is the analogue result to [22, 6.2. Remark]:

Proposition 12.7.4. *Let $\mathcal{M}$ be an arbitrary weight matrix, i.e. $(\mathcal{M})$, with $\Lambda = \mathbb{R}_{>0}$ and assume that the UBP 12.4.1 holds for the $\mathcal{E}_{(L)}$-case. Then the UBP 12.4.1 is also valid for the $\mathcal{E}_{\{\mathcal{M}\}}$-case.*

Proof. By 12.7.1 we see that $\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \varprojlim_{L} \mathcal{E}_{(L)}(U,F)$ holds as vector spaces with bornology, and the projective limit is taken over all arbitrary sequences L with $\mathcal{M} \lhd L$. The structure of $\mathcal{E}_{\{\mathcal{M}\}}(U,F)$ is initial w.r.t. the inclusions $\varprojlim_{L} \mathcal{E}_{(L)}(U,F) \to \mathcal{E}_{(L)}(U,F)$ for each sequence L (with $\mathcal{M} \lhd L$). $\square$

Finally we prove an analogous result to the second part of [22, 5.4. Example], see also 10.10.7 which we will have to use:

Proposition 12.7.5. *Let $\mathcal{M}$ be a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ and such that $(\mathcal{M}_{\{\text{mg}\}})$ is not satisfied. Then there exists a weakly log. convex sequence $N = (N_k)_k$ with $M^l \lhd N$ for all $l \in \Lambda$ and a function $f \in \mathcal{E}_{(N)}(\mathbb{R}^2, \mathbb{C})$ such that $f^\vee : \mathbb{R} \to \mathcal{E}_{(N)}(\mathbb{R}, \mathbb{C})$ is not $\mathcal{E}_{(N)}$. Consequently N cannot satisfy condition moderate growth* (mg).

Proof. Recall: By (12.7.1) in 12.7.1 we have as vector spaces with bornology

$$\mathcal{E}_{\{\mathcal{M}\}}(U,F) = \bigcap_{\mathcal{M} \lhd L} \mathcal{E}_{(L)}(U,F),$$

where E, F are convenient, $U \subseteq E$ is c^∞-open and the intersection is taken over all weakly log. convex sequences L with $\mathcal{M} \lhd L$, $(L_k)^{1/k} \to \infty$ for $k \to \infty$.
Consider now the function $\psi_x \in \mathcal{E}^{\text{global}}_{\{M^x\}}(\mathbb{R}^2, \mathbb{C}) \subseteq \mathcal{E}^{\text{global}}_{\{\mathcal{M}\}}(\mathbb{R}^2, \mathbb{C})$ from 10.10.7, where $x \in \Lambda$ is precisely the index coming from the violated condition $(\mathcal{M}_{\{\text{mg}\}})$. In 10.10.7 we have shown that $\psi_x^\vee : \mathbb{R} \to \mathcal{E}_{\{\mathcal{M}\}}(\mathbb{R}, \mathbb{C})$ is not $\mathcal{E}_{\{\mathcal{M}\}}$. By 12.7.1 there exist weakly log. convex sequences N^1 and N^2 with $\mathcal{M} \lhd N^i$, and $(N^i_k)^{1/k} \to \infty$ for $i = 1, 2$, such that $\psi_x^\vee : \mathbb{R} \to \mathcal{E}_{(N^2)}(\mathbb{R}, \mathbb{C})$ is not $\mathcal{E}_{(N^1)}$.

As in the original proof of [22, 5.4. Example], we prove an analogous version of the Lemma between 5.4. Example and 5.5. Corollary:
Claim: There exists a weakly log. convex sequence N with $\mathcal{M} \lhd N \le N^i$, $i = 1, 2$.
Proof of the claim: Let $\overline{N} := (\overline{N}_k)_k$ defined by $\overline{N}_k := \min\{N^1_k, N^2_k\}$ and put $N := \overline{N}^{\text{lc}}$, the (weakly) log. convex minorant of $\overline{N}$. Hence by definition $N \le \overline{N} \le N^i$ and $\mathcal{M} \lhd N^i$, $i = 1, 2$, so this implies $\mathcal{M} \lhd \overline{N}$. We prove now $\mathcal{M} \lhd N$:
For the next step in this proof we have to recall the following definition: For an arbitrary sequence $P = (P_k)_k \in \mathbb{R}^{\mathbb{N}}_{>0}$ the globally defined classes of ultradiff. functions of Roumieu- resp. Beurling-type are given by

$$\mathcal{E}^{\text{global}}_{[P]}(\mathbb{R}, \mathbb{R}) := \left\{ f \in \mathcal{E}(\mathbb{R}, \mathbb{R}) : \left(\sup_{x \in \mathbb{R}} \left| f^{(k)}(x) \right| \right)_k \in \mathcal{F}_{[P]} \right\}.$$

By the same arguments as mentioned in the original proof of [22, 5.4. Example] we obtain $\mathcal{E}^{\text{global}}_{(N)}(\mathbb{R}, \mathbb{R}) = \mathcal{E}^{\text{global}}_{(\overline{N})}(\mathbb{R}, \mathbb{R})$.
We have $\mathcal{M} \lhd \overline{N}$, i.e. $M^l \lhd \overline{N}$ for each $l \in \Lambda$, hence $\mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{E}^{\text{global}}_{(\overline{N})}(\mathbb{R}, \mathbb{R}) = \mathcal{E}^{\text{global}}_{(N)}(\mathbb{R}, \mathbb{R})$ for arbitrary $l \in \Lambda$. By assumption (each M^l is weakly log. convex) for

each $l \in \Lambda$ we obtain a function $\theta_l \in \mathcal{E}^{\text{global}}_{\{M^l\}}(\mathbb{R},\mathbb{R})$ (see (chf)) and so $M^l \lhd N$ for all $l \in \Lambda$ follows, i.e. $\mathcal{M} \lhd N$.

So the claim is shown and we can continue with the proof of the proposition: Again by (12.7.1) we have $\psi_x \in \mathcal{E}_{(N)}(\mathbb{R}^2,\mathbb{C})$ and the mapping $\psi_x^\vee$ takes values in $\mathcal{E}_{(N)}(\mathbb{R},\mathbb{C})$ and factorizes over the continuous inclusion $\mathcal{E}_{(N)}(\mathbb{R},\mathbb{C}) \hookrightarrow \mathcal{E}_{(N^2)}(\mathbb{R},\mathbb{C})$. Hence it follows, that the mapping $\psi_x^\vee : \mathbb{R} \to \mathcal{E}_{(N)}(\mathbb{R},\mathbb{C})$ is not $\mathcal{E}_{(N^1)}$ and so also not $\mathcal{E}_{(N)}$.
Finally we point out that by ($\Rightarrow$) in the first part of the Cartesian closedness theorem [22, 5.2. Theorem] we get that N cannot satisfy moderate growth (mg). $\square$

12.8 Manifolds of $\mathcal{E}_{[\mathcal{M}]}$-mappings

In this chapter we will always assume the following conditions for the weight matrix $\mathcal{M}$, they are necessary to develop an infinite dimensional $\mathcal{E}_{[\mathcal{M}]}$-analysis (analogously to the 9[th] chapter in [22] where the constant weight matrix case $\mathcal{M} = \{M\}$ was treated):

(i) $\mathcal{M}$ is a $(\mathcal{M}_{\text{sc}})$ weight matrix with $\Lambda = \mathbb{R}_{>0}$ (can restrict in the Roumieu-case to $\Lambda = \mathbb{N}_{>0}$).

(ii) $(\mathcal{M}_{\{\text{mg}\}})$ in the Roumieu- resp. $(\mathcal{M}_{(\text{mg})})$ in the Beurling-case is satisfied. Note that $(\mathcal{M}_{[\text{mg}]})$ implies automatically $(\mathcal{M}_{[\text{dc}]})$.

(iii) The assumptions of Theorem 8.6.1, where we have characterized all stability properties in terms of properties for $\mathcal{M}$: More precisely in the Roumieu-case assume in addition $(\mathcal{M}_{\{\mathcal{C}^\omega\}})$, in the Beurling-case we need in addition $(\mathcal{M}_{(\mathcal{C}^\omega)})$ and, for non-constant matrices (more important case!), also property $(\mathcal{M}_{(\text{BR})})$.

(iv) A characterizing condition in the characterizing Theorem 8.6.1: So $(\mathcal{M}_{\{\text{ai}\}})$ in the Roumieu- resp. $(\mathcal{M}_{(\text{ai})})$ in the Beurling-case.

The following definitions are analogous to [22, 9.3.] and [22, 9.4.]:

(i) A $\mathcal{E}_{[\mathcal{M}]}$-manifold is a smooth manifold such that all chart changings are $\mathcal{E}_{[\mathcal{M}]}$-mappings, they are endowed with the topology induced by the c^∞-open-topology on the charts. In an analogous way one can also introduce $\mathcal{E}_{[\mathcal{M}]}$-bundles and $\mathcal{E}_{[\mathcal{M}]}$-Lie-groups.

(ii) A mapping between $\mathcal{E}_{[\mathcal{M}]}$-manifolds is called $\mathcal{E}_{[\mathcal{M}]}$, if it maps $\mathcal{E}_{[\mathcal{M}]}$-plots ($\mathcal{E}_{[\mathcal{M}]}$-mappings from the open unit ball of a Banach space to the domain manifold) to $\mathcal{E}_{[\mathcal{M}]}$-plots.

(iii) Let $p : E \to B$ be a $\mathcal{E}_{[\mathcal{M}]}$-vector bundle, the space $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow E)$ of all $\mathcal{E}_{[\mathcal{M}]}$-sections is a convenient vector space induced by

$$\mathcal{E}_{[\mathcal{M}]}(B \leftarrow E) \longrightarrow \prod_\alpha \mathcal{E}_{[\mathcal{M}]}(u_\alpha(U_\alpha), V), \quad s \mapsto \operatorname{pr}_2 \circ \psi_\alpha \circ s \circ u_\alpha^{-1},$$

where $u_\alpha : B \supseteq U_\alpha \longrightarrow u_\alpha(U_\alpha) \subseteq W$ is a $\mathcal{E}_{[\mathcal{M}]}$-atlas for B, modeled on a convenient vector space W and the mappings $\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V$ form a vector bundle atlas over the charts U_α of B.

The next result is analogous to the Lemma before [22, 9.5. Theorem]:

Lemma 12.8.1. *Let D be the open unit ball in a Banach space, then the following holds: A mapping $c : D \to \mathcal{E}_{[\mathcal{M}]}(B \leftarrow E)$ is a $\mathcal{E}_{[\mathcal{M}]}$-plot if and only if $\hat{c} : D \times B \to E$ is $\mathcal{E}_{[\mathcal{M}]}$.*

Proof. By 12.3.1 and the structure of $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow E)$ we can assume that B is c^∞-open in a convenient vector space W and $E = B \times V$. Then $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow B \times V) \cong \mathcal{E}_{[\mathcal{M}]}(B, V)$ and we can use then *Cartesian closedness* 12.3.2. □

Let $U \subseteq E$ be an open neighborhood of $s(B)$, where $s : B \to E$ is a section, and let $q : F \to B$ be another vector bundle. If B is compact (hence finite-dimensional), then the set $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow U)$ of all $\mathcal{E}_{[\mathcal{M}]}$-sections $s' : B \to E$ with $s'(B) \subseteq U$ is c^∞-open in the

convenient vector space $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow E)$, because it is open in the coarser compact-open topology.
An immediate consequence of the previous lemma is now the following: If $U \subseteq E$ is an open neighborhood of $s(B)$, $s : B \to E$ a section, and if $f : U \to F$ is a fiber respecting $\mathcal{E}_{[\mathcal{M}]}$-mapping, where $q : F \to B$ is another vector bundle, then $f_* : \mathcal{E}_{[\mathcal{M}]}(B \leftarrow U) \to \mathcal{E}_{[\mathcal{M}]}(B \leftarrow F)$ is $\mathcal{E}_{[\mathcal{M}]}$ on the open neighborhood $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow U)$ of s in $\mathcal{E}_{[\mathcal{M}]}(B \leftarrow E)$.
The next result is analogous to [22, 9.5. Theorem]:

Theorem 12.8.2. *Let A and B be finite dimensional $\mathcal{E}_{[\mathcal{M}]}$-manifolds, such that A is compact and B is equipped with a $\mathcal{E}_{[\mathcal{M}]}$-Riemannian metric. Then $\mathcal{E}_{[\mathcal{M}]}(A, B)$, the space of all $\mathcal{E}_{[\mathcal{M}]}$-mappings $A \to B$ is a $\mathcal{E}_{[\mathcal{M}]}$-manifold modeled on convenient vector spaces $\mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB)$ of $\mathcal{E}_{[\mathcal{M}]}$-sections of pullback bundles along $f : A \to B$. A mapping $c : D \to \mathcal{E}_{[\mathcal{M}]}(A, B)$ is a $\mathcal{E}_{[\mathcal{M}]}$-plot if and only if $\hat{c} : D \times A \to B$ is $\mathcal{E}_{[\mathcal{M}]}$.*

Proof. By 8.6.1 (closedness under solving ODE's) we get that $\mathcal{E}_{[\mathcal{M}]}$-vector fields have $\mathcal{E}_{[\mathcal{M}]}$-flows. We apply this to the geodesic spray to obtain a $\mathcal{E}_{[\mathcal{M}]}$-exponential mapping $\exp : TB \supseteq U \to B$ of the Riemannian metric, defined on a suitable open neighborhood of the zero section.
By 8.6.1 ($\mathcal{E}_{[\mathcal{M}]}$-inverse function theorem) we get the following: One can assume that U is chosen in such a way that $(\pi_B, \exp) : U \to B \times B$ is a $\mathcal{E}_{[\mathcal{M}]}$-diffeomorphism onto an open neighborhood V of the diagonal $B \times B$.
The convenient vector space $\mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB)$ is canonically isomorphic to the space $\mathcal{E}_{[\mathcal{M}]}(A, TB)_f := \{h \in \mathcal{E}_{[\mathcal{M}]}(A, B) : \pi_B \circ h = f\}$ (via $s \mapsto (\pi^* f) \circ s$) and $h \mapsto (\mathrm{id}_A, h)$.
Introduce now

$$U_f := \{g \in \mathcal{E}_{[\mathcal{M}]}(A, B) : (f(x), g(x)) \in V \;\forall\, x \in A\}$$

and $u_f : U_f \to \mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB)$, where $u_f(g)(x) = (x, \exp^{-1}_{f(x)}(g(x))) = (x, ((\pi_B, \exp)^{-1} \circ (f, g))(x))$. So $u_f : U_F \to \{s \in \mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB) : s(A) \subseteq f^*U = (\pi_B^* f)^{-1}(U)\}$ is a bijection with inverse $u_f^{-1}(s) = \exp \circ (\pi_B^* f) \circ s$, where $U \to B$ is considered as a fiber bundle. The set $u_f(U_f)$ is c^∞-open in $\mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB)$ for the topology which we have described before: A is compact and u_f is $\mathcal{E}_{[\mathcal{M}]}$ because it respects $\mathcal{E}_{[\mathcal{M}]}$-plots as shown in 12.8.1.
We consider the atlas $(U_f, u_f)_{f \in \mathcal{E}_{[\mathcal{M}]}(A,B)}$ for $\mathcal{E}_{[\mathcal{M}]}(A, B)$. The chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq \mathcal{E}_{[\mathcal{M}]}(A \leftarrow g^*TB)$ by

$$(u_f \circ u_g^{-1})(s) = (\mathrm{id}_A, (\pi_B, \exp)^{-1} \circ (f, \exp \circ (\pi_B^* g) \circ s)) = (\tau_f^{-1} \circ \tau_g)_*(s),$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$, $\tau_g : g^*TB \supseteq g^*U \to (g \times \mathrm{id}_B)^{-1}(V) \subseteq A \times B$, is a $\mathcal{E}_{[\mathcal{M}]}$-diffeomorphism which is fiber respecting over A. The chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and it is also $\mathcal{E}_{[\mathcal{M}]}$, because it respects $\mathcal{E}_{[\mathcal{M}]}$-plots by 12.8.1.
For the topology on $\mathcal{E}_{[\mathcal{M}]}(A, B)$ we consider the identification topology from the above given atlas and with the c^∞-topologies on the modeling spaces $\mathcal{E}_{[\mathcal{M}]}(A \leftarrow f^*TB)$, which is finer than the compact-open topology and so Hausdorff.
The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows now that the $\mathcal{E}_{[\mathcal{M}]}$-structure does not depend on the choice of the $\mathcal{E}_{[\mathcal{M}]}$-Riemannian metric on B.
The last statement concerning the $\mathcal{E}_{[\mathcal{M}]}$-plots holds by 12.8.1. □

The next result is analogous to [22, 9.6. Lemma]:

Lemma 12.8.3. *Let A_1, A_2 and B be finite dimensional $\mathcal{E}_{[\mathcal{M}]}$-manifolds with A_1, A_2 compact. Then the composition*

$$\mathcal{E}_{[\mathcal{M}]}(A_2, B) \times \mathcal{E}_{[\mathcal{M}]}(A_1, A_2) \to \mathcal{E}_{[\mathcal{M}]}(A_1, B), \quad (f, g) \mapsto f \circ g$$

is $\mathcal{E}_{[\mathcal{M}]}$.

Proof. The composition maps $\mathcal{E}_{[\mathcal{M}]}$-plots to $\mathcal{E}_{[\mathcal{M}]}$-plots, hence it is $\mathcal{E}_{[\mathcal{M}]}$. □

The next result is analogous to [22, 9.7. Example]:

Lemma 12.8.4. *Consider another $(\mathcal{M}_{\text{sc}})$ weight matrix $\mathcal{N} = \{N^l : l \in \Lambda\}$ such that $\mathcal{E}_{[\mathcal{N}]} \subsetneq \mathcal{E}_{[\mathcal{M}]}$, then composition mapping $\circ$*

$$\mathcal{E}_{[\mathcal{M}]}(S^1, \mathbb{R}) \times \mathcal{E}_{[\mathcal{M}]}(S^1, S^1) \to \mathcal{E}_{[\mathcal{M}]}(S^1, \mathbb{R}) \quad (f, g) \mapsto f \circ g$$

is not $\mathcal{E}_{[\mathcal{N}]}$ w.r.t. the canonical real analytic manifold structures (where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$).

Proof. There exists a function $f \in \mathcal{E}_{[\mathcal{M}]}(S^1, \mathbb{R}) \backslash \mathcal{E}_{[\mathcal{N}]}(S^1, \mathbb{R})$ and f can be regarded as a 2π-periodic function on $\mathbb{R}$. The universal covering space of $\mathcal{E}_{[\mathcal{M}]}(S^1, S^1)$ consists of all $2\pi\mathbb{Z}$-equivariant mappings in $\mathcal{E}_{[\mathcal{M}]}(\mathbb{R}, \mathbb{R})$, i.e. the space of all functions $g + \text{id}_{\mathbb{R}}$ for 2π-periodic functions $g \in \mathcal{E}_{[\mathcal{M}]}$. Hence $\mathcal{E}_{[\mathcal{M}]}(S^1, S^1)$ is a real analytic manifold and $c : t \mapsto (x \mapsto x + t)$ induces a real analytic curve c in $\mathcal{E}_{[\mathcal{M}]}(S^1, S^1)$. But $f_* \circ c$ is not $\mathcal{E}_{[\mathcal{N}]}$, because

$$\frac{(\partial_t^k|_{t=0}(f_* \circ c)(t))(x)}{h^k \cdot N_k^l} = \frac{\partial_t^k|_{t=0} f(x + t)}{h^k \cdot N_k^l} = \frac{f^{(k)}(x)}{h^k \cdot N_k^l}$$

is unbounded for $k \to \infty$ for x in a suitable compact set and for some $h > 0, l \in \Lambda$ resp. for all $h > 0$, $l \in \Lambda$, because by assumption $f \notin \mathcal{E}_{[\mathcal{N}]}$. □

Recall the definitions of relations $\{\preceq\}$, $(\preceq)$ and [35, 4.6. Proposition]: For $\mathcal{N} = \{N^l : l \in \Lambda\}$ and $\mathcal{M} = \{M^l : l \in \Lambda\}$ both $(\mathcal{M}_{\text{sc}})$ weight matrices the strict inclusion $\mathcal{E}_{\{\mathcal{N}\}} \subsetneq \mathcal{E}_{\{\mathcal{M}\}}$ is equivalent to the fact that there exists some $l_0 \in \Lambda$ such that for arbitrary $l \in \Lambda$ we have $\mathcal{E}_{\{N^l\}} \subsetneq \mathcal{E}_{\{M^{l_0}\}}$, or equivalently $\sup_{k \in \mathbb{N}_{>0}} \left(\frac{M_k^{l_0}}{N_k^l}\right)^{1/k} = +\infty$. Analogously for the Beurling-case the strict inclusion $\mathcal{E}_{(\mathcal{N})} \subsetneq \mathcal{E}_{(\mathcal{M})}$ means that there exists some $l_0 \in \Lambda$ such that for arbitrary $l \in \Lambda$ we have $\mathcal{E}_{(N^{l_0})} \subsetneq \mathcal{E}_{(M^l)}$, or equivalently $\sup_{k \in \mathbb{N}_{>0}} \left(\frac{M_k^l}{N_k^{l_0}}\right)^{1/k} = +\infty$.

The next result is analogous to [22, 9.8. Theorem]:

Theorem 12.8.5. *Let A be a compact $\mathcal{E}_{[\mathcal{M}]}$-manifold, then the group $\text{Diff}_{[\mathcal{M}]}(A)$ of all $\mathcal{E}_{[\mathcal{M}]}$-diffeomorphisms of A is an open subset of the $\mathcal{E}_{[\mathcal{M}]}$-manifold $\mathcal{E}_{[\mathcal{M}]}(A, A)$. It is a $\mathcal{E}_{[\mathcal{M}]}$-regular $\mathcal{E}_{[\mathcal{M}]}$-regular Lie-group, i.e. both inversion and composition are $\mathcal{E}_{[\mathcal{M}]}$. It's Lie-algebra consists of all $\mathcal{E}_{[\mathcal{M}]}$-vector fields on A with the negative of the usual bracket as Lie bracket. Finally the exponential mapping is $\mathcal{E}_{[\mathcal{M}]}$ and it is not surjective onto any neighborhood of id_A.*

We recall some definitions: A $\mathcal{E}_{[\mathcal{M}]}$-Lie group G with Lie algebra $\mathfrak{g} = T_e G$ is called $\mathcal{E}_{[\mathcal{M}]}$-regular (for the $\mathcal{E}$-case see [20, 38.4.]), if

(1) for each $\mathcal{E}_{[\mathcal{M}]}$-curve $X \in \mathcal{E}_{[\mathcal{M}]}(\mathbb{R}, \mathfrak{g})$ there exists a $\mathcal{E}_{[\mathcal{M}]}$-curve $g \in \mathcal{E}_{[\mathcal{M}]}(\mathbb{R}, G)$ such that

$$\begin{cases} g(0) = e \\ \partial_t g(t) = T_e(\mu^{g(t)})X(t) = X(t) \cdot g(t), \end{cases}$$

the curve g is uniquely determined by its initial value $g(0)$, if it exists.

(2) If we put $\operatorname{evol}^r_G(X) = g(1)$, where g is the unique solution from above, then $\operatorname{evol}^r_G : \mathcal{E}_{[\mathcal{M}]}(\mathbb{R}) \to G$ is required to be $\mathcal{E}_{[\mathcal{M}]}$, too.

Proof. First we point out that the group $\operatorname{Diff}_{[\mathcal{M}]}(A)$ is c^∞-open in $\mathcal{E}_{[\mathcal{M}]}(A, A)$: This holds because by [20, 43.1.] the $\mathcal{E}$-diffeomorphism-group $\operatorname{Diff}(A)$ is c^∞-open in $\mathcal{E}(A, A)$ and by 8.6.1 (more precisely the $\mathcal{E}_{[\mathcal{M}]}$-inverse function-theorem) we obtain $\operatorname{Diff}_{[\mathcal{M}]}(A) = \operatorname{Diff}(A) \cap \mathcal{E}(A, A)$.
By 12.8.2 we see that $\operatorname{Diff}_{[\mathcal{M}]}(A)$ is a $\mathcal{E}_{[\mathcal{M}]}$-manifold and 12.8.3 shows that composition is $\mathcal{E}_{[\mathcal{M}]}$. We prove now that also the inversion is $\mathcal{E}_{[\mathcal{M}]}$:
Let c be a $\mathcal{E}_{[\mathcal{M}]}$-plot in $\operatorname{Diff}_{[\mathcal{M}]}(A)$, then, by 12.8.2, the mapping $\hat{c} : D \times A \to A$ is $\mathcal{E}_{[\mathcal{M}]}$ and $(\operatorname{inv} \circ c)^\wedge : D \times A \to A$ satisfies the Banach manifold implicit equation $\hat{c}(t, (\operatorname{inv} \circ)^\wedge(t, x)) = x$ for $x \in A$. So we can apply again 8.6.1 (more precisely the Banach $\mathcal{E}_{[\mathcal{M}]}$-implicit function theorem) to see that the mapping $(\operatorname{inv} \circ c)^\wedge$ is locally $\mathcal{E}_{[\mathcal{M}]}$ and so $\mathcal{E}_{[\mathcal{M}]}$. Now, again by 12.8.2, the mapping $\operatorname{inv} \circ c$ is a $\mathcal{E}_{[\mathcal{M}]}$-plot in $\operatorname{Diff}_{[\mathcal{M}]}(A)$, hence $\operatorname{inv} : \operatorname{Diff}_{[\mathcal{M}]}(A) \times \operatorname{Diff}_{[\mathcal{M}]}(A) \to \operatorname{Diff}_{[\mathcal{M}]}(A)$ is $\mathcal{E}_{[\mathcal{M}]}$.
The Lie-Algebra of $\operatorname{Diff}_{[\mathcal{M}]}(A)$ is the convenient vector space of all $\mathcal{E}_{[\mathcal{M}]}$-vector fields on A with the negative of the usual Lie-bracket (see [20, 43.1]).
$\operatorname{Diff}_{[\mathcal{M}]}(A)$ is a $\mathcal{E}_{[\mathcal{M}]}$-regular $\mathcal{E}_{[\mathcal{M}]}$-Lie group: Choose a $\mathcal{E}_{[\mathcal{M}]}$-plot in the space of all $\mathcal{E}_{[\mathcal{M}]}$-curves in the Lie-algebra of all $\mathcal{E}_{[\mathcal{M}]}$-vector fields on A, i.e. $c : D \to \mathcal{E}_{[\mathcal{M}]}(\mathbb{R}, \mathcal{E}_{[\mathcal{M}]}(A \leftarrow TA))$. By 12.8.1 the curve c corresponds now to a $(D \times \mathbb{R})$-time-dependent $\mathcal{E}_{[\mathcal{M}]}$-vector field $\hat{\hat{c}} : D \times \mathbb{R} \times A \to TA$.
By 8.6.1 (closedness under solving ODE's) we see that $\mathcal{E}_{[\mathcal{M}]}$-vector fields have $\mathcal{E}_{[\mathcal{M}]}$-flows and because A is compact we see that $\operatorname{evol}^r(\hat{c}(s))(t) = \operatorname{Fl}_t^{\hat{c}(s)}$ is $\mathcal{E}_{[\mathcal{M}]}$ in all occurring variables.
The exponential mapping is evol^r applied to constant curves in the Lie-algebra, so it consists of flows of autonomous $\mathcal{E}_{[\mathcal{M}]}$-vector fields. The exponential mapping is not surjective onto any $\mathcal{E}_{[\mathcal{M}]}$-neighborhood of the identity, this follows for $A = S^1$ by [20, 43.5]. □

13 Appendix

13.1 Danksagung

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13.2 Curriculum vitae

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Military service	2002-2003
Studies	mathematics-diploma, University of Vienna, 2003-2009 mathematics/physics for teaching profession, University of Vienna, 2003-2010 mathematics-doctorate, University of Vienna, since summer term 2010
Employments	From 1st October to 31st December 2010, research assistant at the University of Vienna, FWF-project P22218, "Perturbations of Polynomials, Generalizations, and Applications", Prof. Dr. Armin Rainer Since 1st January 2011, research assistant at the University of Vienna, PhD-Thesis FWF-project P23028-N13, "Exponential Laws for Classes of Denjoy-Carleman Differentiable Mappings", Prof. Dr. Andreas Kriegl

Summer term 2009 - tutor for the lecture course
"Complex Analysis" of Prof. Dr. Friedrich Haslinger

From 1st September 2010 to 28th February 2011,
1st September 2011 to 29th February 2012,
1st September 2012 to 28th February 2013,
1st September 2013 to 28th February 2014,
teaching assistant for mathematical exercises for
forest managers at the University of Natural Resources and
Applied Life Sciences of Vienna,
Prof. Mag. Dr. Gerald Kuba

Publications	(⋆) Spaces of smooth functions of Denjoy-Carleman-type, Diploma thesis, 2009, online available: `http://othes.univie.ac.at/7715/1/2009-11-18_0304518.pdf` (⋆) Composition in ultradifferentiable classes, 2012, joint work with Armin Rainer, online available: `http://arxiv.org/pdf/1210.5102.pdf` (⋆) The convenient setting for ultradifferentiable mappings of Beurling- and Roumieu-type defined by a weight matrix (in preparation)
Skills	foreign languages: English, French and Czech Microsoft Office knowledge of forestry and fish farming
Other interests/hobbies	natural sciences numismatics cross-country skiing reading anti-littering

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