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“The WKB approximation, harmonic maps to buildings and spectral networks”

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Abstract

The main goal of this thesis is to find a relation between the Riemann-Hilbert WKB problem and harmonic maps from the underlying manifold to Euclidean buildings. The notion of a versal building associated to a spectral cover is introduced. The harmonic map from a Riemann surface to this building encodes many aspects of the geometry of the spectral cover; in particular the spectral network is mapped to the singularities of the versal building.

There are two main conjectures in this thesis: (1) the versal building exists and the metric on this building controls the asymptotic of the Riemann-Hilbert correspondence; (2) the harmonic map to this building encodes the data to construct a 3d Calabi-Yau category such that the Hitchin base is a submanifold of its space of stability conditions. An algorithm to construct this category is provided for locally finite spectral networks.

The main results are: (1) the existence of the universal building for a system of ODEs considered by Berk-Nevin-Roberts is established; (2) a versal building exists for a further, more complicated example; (3) the algorithm to construct the categories is applied in these examples and a relation to buildings is conjectured.

Zusammenfassung

Das Ziel der vorliegenden Arbeit is es, eine Beziehung zwischen den asymptotischen Eigenschaften des Riemann-Hilbert Problems und harmonischen Abbildungen der zu Grunde liegenden Mannigfaltigkeit in Euklidische Gebäude zu finden. Dabei wird das Konzept des zu einer spektralen Überlagerung gehörenden versalen Gebäudes eingeführt. Viele interessante geometrische Eigenschaften der spektralen Überlagerung können von der harmonischen Abbildung in dieses Gebäude abgelesen werden; insbesondere wird in dieser Arbeit argumentiert, dass das Bild des spektralen Netzwerks in den Singularitäten dieses Gebäudes enthalten ist.
Die beiden Hauptvermutungen in dieser Arbeit sind die folgenden: (1) das versale Gebäude existiert und die Metrik darauf bestimmt die Asymptotik der Riemann-Hilbert Korrespondenz; (2) von der harmonischen Abbildung in das versale Gebäude kann eine dreidimensionale Calabi-Yau Kategorie konstruiert werden, sodass die Hitchin Basis eine Teilmannigfaltigkeit des Raums der Stabilitätsbedingungen auf dieser ist. Ein Algorithmus, diese Kategorie zu konstruieren, wird für lokal endliche spektral Netzwerke beschrieben.
Die Hauptresultate sind die folgenden: (1) Ein universelles Gebäude existiert für ein System von gewöhnlichen Differentialgleichungen, das von Berk, Nevin und Roberts betrachtet wurde; (2) das versale Gebäude existiert für ein weiteres, komplizierteres Beispiel; (3) der Algorithmus zur Konstruktion der Kategorie wird in zwei Beispielen angewandt und ein Zusammenhang zum versalen Gebäude wird hergestellt.

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Contents

Chapter 1

Introduction

Portions of this chapter have appeared in [40] and have been submitted for publication to "Communications in Mathematical Physics".

The original motivation for the work described in this thesis was to give a mathematical interpretation to the *spectral networks* recently introduced by physicists [26]. These objects are closely related to *BPS states* which play a prominent role in mathematics [44] and physics [55, 66]. Furthermore, there is a tight connection to the theory of *Stability Hodge Structures* introduced by Kontsevich and Soibelman in [47].

The spectral networks mentioned above are certain combinatorial geometric structures that encode part of the geometry of the spectral cover associated to a point in the Hitchin base B. In the case when

$$B = H^0(X; K_X^2)$$

for X a Riemann surface, they are the traces of the singularites of the leaf space of the associated foliation on the Riemann surface, and can be used to construct a three-dimensional triangluated Calabi-Yau category $\mathcal{T}$ for which B parametrizes part of the stability conditions. For Hitchin systems of higher rank, spectral networks come from the WKB problem of asymptotic analysis of the transport matrices of singularly perturbed ordinary differential equations.

1.1 Main results

Let X be a Riemann surface with universal cover $\widetilde{X}$, $E \to X$ be a holomorphic vector bundle with norm $\|.\|$ and $\varphi \in \mathrm{End}(E, E \otimes \Omega^1_X)$ be a Higgs field. Furthermore, suppose that ∇_0 is a flat holomorphic connection on E, and let

$$\nabla_t := \nabla_0 + t\varphi$$

for $t \in \mathbb{R}_+$. The *complex WKB problem* concerns the question of the asymptotics of the parallel transport operator $T_{PQ}(t)$ for ∇_t from $P \in \widetilde{X}$ to $Q \in \widetilde{X}$ as a function of $t \to \infty$. An interesting quantity that encodes (part of) this asymptotic behavior is the *WKB exponent*

$$\nu^{\mathrm{WKB}}_{PQ} := \limsup_{t \to \infty} \frac{1}{t} \log \|T_{PQ}(t)\| \,. \tag{1.1}$$

In the paper [26], a procedure, called the *non-abelianization map*, for determining the exponent in terms of spectral networks is described, which, in turn, are determined by the mulit-valued one-forms $\phi = (\phi_1, \ldots, \phi_r)$ which are the eigenvalues of the Higgs field φ.
Instead of considering the WKB-exponent ν^{WKB}_{PQ}, we can also use an ultrafilter ω to define an *ultrafilter exponent* ν^{ω}_{PQ}. Recent work of Parreau [58], building upon previous work of Kleiner and Leeb [42], implies that ν^{ω}_{PQ} corresponds to the distance between $h^{\omega}(P)$ and $h^{\omega}(Q)$ for a map h^{ω} from $\widetilde{X}$ to a Euclidean building Cone_{ω}. The first main theorem is the following:

Theorem 1.1. *The map h^{ω} is harmonic in the sense of Gromov-Korevaar-Schoen with differential Re ϕ.*

Furthermore, it is noticed that for any generic harmonic ϕ-map to a building, the image of the spectral network is contained in the singularities of the building. In that sense, the spectral network is contained in the traces on the Riemann surface X of the singularities of the building (this does not characterize the spectral network, however: there is a bigger subset $\mathcal{W}^{\mathrm{ext}}$, the *extended spectral network*, which contains additionally "backward-lines", that maps to the singularities of any building. At the moment, we do not know a complete characterization of the spectral network in terms of buildings).
When trying to understand harmonic ϕ-maps to buildings, it is important to determine the biggest subsets $M \subset \widetilde{X}$ which map into single apartments. These subsets are called "maximal abelian regions" or MARs, for short. We prove several criteria for when a subset $U \subset \widetilde{X}$ includes in a MAR $M \subset \widetilde{X}$. A key-role in these

criteria is played by the notion of a *non-critical path*: this is a path in $\widetilde{X}$ such that the real parts of the differential forms $\phi_1, \ldots, \phi_r$ are distinct when pulled back to this path, and thus can be put in an order. The criterion that will be used most extensively is the following:

Proposition 1.2. *Let $P, Q \in \widetilde{X}$ and let Ω_{PQ} be a region enclosed by two noncritical paths joining P and Q. Then any harmonic ϕ-map sends Ω_{PQ} into a single apartment.*

We look closely at two basic examples: the first one is the most basic example, first considered by Berk, Nevin and Roberts (BNR) [5] in which phenomena specific to higher rank occur. More specifically, in this example a *collision line* contributes to the WKB problem. We carefully analyse this example and prove the following theorem:

Theorem 1.3. *In the BNR example, there exists a* universal *harmonic ϕ-map to a building $\mathcal{B}^\phi$. In particular, the map h^ω, depending on an ultrafilter, factors through a folding map $\mathcal{B}^\phi \to Cone_\omega$ which is an isometry on the image of X. Thus the WKB and ultrafilter exponents are the same, and they are both given by the GMN non-abelianization map.*

The second example considered in detail is called the A_1*-example* (so called because of the category $\mathcal{T}$ that it describes). It is the next most difficult WKB problem that one can consider. We can again prove a theorem similar to Theorem 1.4, only replacing "universal" with "versal".

Theorem 1.4. *In the A1-example, there exists a* versal *harmonic ϕ-map to a building $\mathcal{B}^\phi$. In particular, the map h^ω, depending on an ultrafilter, factors through a folding map $\mathcal{B}^\phi \to Cone_\omega$ which is an isometry on the image of X. Thus the WKB and ultrafilter exponents are the same, and they are both given by the GMN non-abelianization map.*

In the context of this example, an algorithm which should produce the heart of a bounded t-structure for a 3D CY category $\mathcal{T}$ is presented. It is applied to another examples and is shown to agree with the expected category. Furthermore, an interpretation in terms of certain *singular edge lengths* in a versal building $\mathcal{B}^\phi$ is conjectured.

1.2 Motivation

It was suggested by Kontsevich and Soibelman [47] that the moduli space of stability conditions of a Calabi-Yau triangulated category $\mathcal{T}$ should be viewed as part of a "twistor family". The fiber over $0 \in \mathbb{C}$ should correspond to a "Dolbeaut type moduli space" which should, in particular, come with a Hitchin-type fibration. The stability condition on $\mathcal{T}$ should come from a point in the base of this fibration. It is expected that the Hitchin fibration fits into this picture. In order to show that, it is necessary to construct a category and a stability condition on it from a point in the base of the Hitchin fibration. In some examples, the spectral networks of [26] have allowed one to construct such a category, but a general construction is at present not known.
It was already noted, for example in [2, 5], that (extended) spectral networks play an important role in the WKB analysis of singularly perturbed ODEs [69],[61], [62] [17], [34], [25]. The WKB problem corresponds, in turn, to investigating a neighborhood of infinity in the Hitchin and de Rham moduli spaces of flat connections.
The construction of the category was carried out in detail by Bridgeland and Smith [8], building upon [25], in the case when the structure group is $\mathrm{SL}_2\mathbb{C}$. In this case, the category can be interpreted as the Fukaya category of a certain conic bundle [63], something that is also expected more generally [16]. For $\mathrm{SL}_2\mathbb{C}$, a quadratic differential corresponds to a harmonic map to an $\mathbb{R}-$tree [72], namely the leaf space of the foliation induced by the quadratic differential. Important pieces of the Stability Hodge Structure can be related to the geometry of this tree. For example, simple objects in the heart of the category correspond to "shortest edges in the tree bounded by singular points" (at least in the case, where X is noncompact). Rotating the quadratic differential ϕ_2 by an angle θ to $e^{i\theta}\phi_2$ leads to *deformations* of the tree. When an edge degenerates to length zero under this family of rotations, this corresponds precisely to the appearance of a *BPS*-state and subsequently to a *mutation* of the tree. Furthermore, as shown by Simpson in [61], the WKB exponent between two points $P, Q \in \widetilde{X}$ is equal to the distance of the corresponding leaves on this tree. Extending this picture to other structure group was one central motivation for this thesis.
From this point of view, it seems natural to replace trees by Euclidean buildings. Then, one would expect that a harmonic map to a Euclidean building corresponds to a point in the Hitchin base. Constructing such a building for a given point in the Hitchin base should be thought of as the analogue of constructing the leaf space of a foliation. It should be noted that this passage from trees to buildings

has already been suggested elsewhere in the literature [52].

Chapter 2

Buildings and metric spaces

Portions of this chapter have appeared in [40] and have been submitted for publication to “Communications in Mathematical Physics”.

2.1 Spherical and Euclidean buildings

Buildings were invented by Jacques Tits and nowadays have numerous applications ranging from geometry to group theory, Teichmüller theory and the WKB approximation. Standard textbooks on the subjects are [1, 59]. The survey article [60] also treats $\mathbb{R}$-buildings that will be used in consequence.
Roughly speaking, buildings can thought of as (generalized) metric spaces that are obtained by ”gluing together apartments” A. Depending on what type of buildings we study, there are two types of apartments:

1. Spherical: an apartment is a spherical Coxeter complex, S^{r-1}
2. Affine: $A \cong_{met} \mathbb{R}^r$

So, a spherical building is a generalized metric space (the metric takes values in S_r, the symmetric group in r letters) that can be isometrically covered by apartments. Similarly, an affine building is a metric space that can be isometrically covered by charts $\mathbb{R}^r$.

Before going into the technial details, we present some examples:

Example 2.1 (Flag complexes)**.** *Let V be an $n-$dimensional vector space over a field k, let $W = (S_n, (12)(23), \ldots, ((n-1)n)$ be the Coxeter system, where S_n is the symmetric group in n-letters and (ij), $j = i+1$ are $(n-1)$ generators given by consecutive transpositions (recall that a Coxeter system is a group W together with a set of generators S, such that $s_i^2 = e_W$ for all $s \in S$ and $(s_i s_j)^{m_{ij}} = e_W$ for some $m_{ij} \in \mathbb{N} \cup \{\infty\}$). Then, associated with V is a spherical building $\mathcal{B}(V)$ modelled on W, whose* chambers *are complete flags $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$. Clearly, an ordered basis $(e_1, \ldots, e_n)$ gives rise to a flag in an obvious way, namely $V_i = span_{j \leq i}\{e_j\}$. Given any $\sigma \in S_n$, we can permute the basis $(e_{\sigma(1)}, \ldots e_{\sigma(n)})$, thus giving rise to other chambers. An* apartment*, by definition, is the set $A_{\{e_1,\ldots,e_n\}}$ of all flags (or equivalently chambers) obtained by acting with all permutations $\sigma \in S_n$ on the given basis $(e_1, \ldots, e_n)$.*
Given two flags F_1, F_2 in an apartment $A_{\{e_1,\ldots,e_n\}}$ given by $(e_{\sigma(1)}, \ldots e_{\sigma(n)})$ and $(e_{\eta(1)}, \ldots, e_{\eta(n)})$ for $\sigma, \eta \in S_n$, we define the distance from F_1 to F_2 *as*

$$d(F_1, F_2) = \eta^{-1}\sigma \in S_n \, .$$

Roughly speaking, the building axioms require that (and are easy to check in this example):

1. *For any two flags F_1, F_2, we can define a distance by putting them inside a common apartment $A_{\{e_1,\ldots e_n\}}$.*

2. *This distance is independent of the specific apartment in which we measure the distance.*

If k is a finite field, then it is clear that there are only finitely many chambers in $\mathcal{B}(V)$ (as there are only finitely many elements in V). Figure 2.1 shows a picture of $\mathcal{B}(k^3)$ for $k = \mathbb{F}_2$, the field of order two. In this picture, the chambers are the edge joining two vertices. The apartments correspond to hexagons in the picture. Thus, each apartment consists of six flags (which should be clear from the very definition). Note that each apartment is topologically a sphere, the dimension of which is called the rank *of the building (which, in this example, is equal to one).*

Example 2.2. *A more complicated example arises as follows: let (K, ν) be a discrete valuation ring with uniformizing parameter t and let $\mathcal{O}$ be the valuation ring (for example, take $K = k((t))$ the ring of formal power series with its usual valuation). Let V be an n-dimensional vector space over K. Recall that a lattice $L \subset V$ is a free $\mathcal{O}$-submodule of rank n. The* Bruhat-Tits *building $\mathcal{B}(V, K)$ is an* affine building *which is the geometric realization of a simplicial complex, the*

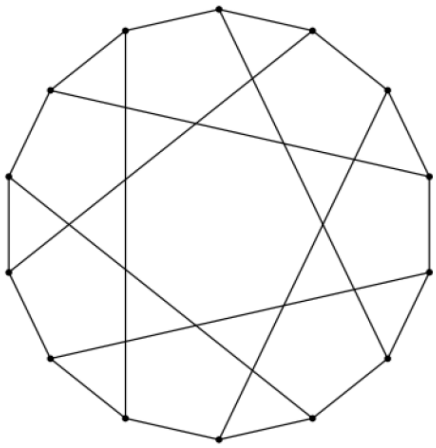

FIGURE 2.1: Building for $\mathbb{F}_2^3$

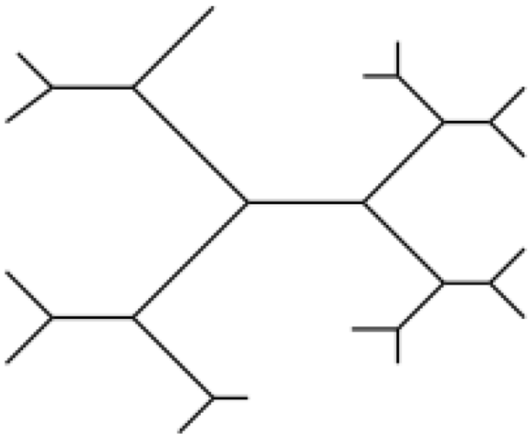

FIGURE 2.2: Building for $SL(3, \mathbb{Q}_2^3)$

k-simplices of which are flags of lattices of the form $L_0 \subset L_1 \subset \cdots \subset L_k \subset t^{-1}L_0$. Just like in the previous example, apartments are obtained by fixing a basis for V and considering only those lattices "generated by the given basis" (for the precise meaning of those words, the reader is referred to [1]). In Figure 2.2 one can see a picture with $K = \mathbb{Q}_2$, and $dim(V) = 3$. Note that in this picture, the tree should actually extend to infinity in all directions.

Example 2.3. *Even more generally, any metric tree $\mathcal{T}$ is an affine building of rank one. In this case, the apartments are isometries $\mathbb{R} \to \mathcal{T}$.*
Note that the set of singular points (i.e. the set of points $p \in \mathcal{T}$ such that no neighborhood of p is contained in a single apartment) can become dense in a general tree.
The most important example of such $\mathbb{R}$-trees that will be considered in this thesis

are the leaf spaces of quadratic differentials on a Riemann surfaces. Recall that a quadratic differential ϕ_2 *on a Riemann surface* X *is a section of* $K_X^{\otimes 2}$*. For a point* $P \in X$ *with* $\phi_2(P) \neq 0$*, we can locally choose a square root* $\lambda = \sqrt{\phi_2}$*. Using* λ *we can define a local coordinate* z *on* X *by*

$$Q \mapsto \int_P^Q \lambda = z(Q)\,.$$

Then, we can define an S^1 *family of foliations, namely straight lines in the coordinate* z *(for now take the foliation lines to have constant real part of* z*). Note that the foliation is independent of the choice of square root.*
More generally, the foliation can also be defined near a zero of ϕ_2*. In the case of a simple zero of* ϕ_2 *it looks as in Figure 2.3. The foliation lines are drawn in black. Pulling back* ϕ_2 *to the universal cover* $\widetilde{X}$*, we can define an equivalence relation* $\sim$ *on* $\widetilde{X}$*:* $P \sim Q$ *if and only if* P *and* Q *lie on the same leaf of the foliation. The set of equivalence classes of this relation is the* leaf space $\mathcal{T}$*, drawn in red in Figure 2.3. It comes naturally equipped with a metric, namely*

$$d(l_1, l_2) = |Re \int_P^Q \lambda|\,. \tag{2.1}$$

Here l_1 *and* l_2 *are two leaves,* $P \in l_1$*,* $Q \in l_2$*. It is then immediate from the definition of the foliation that this is independent of the choice of representatives* P, Q*. Furthermore, there is a canonical map* $h : \widetilde{X} \to \mathcal{T}$*, which is actually harmonic [72], from the universal cover* $\widetilde{X}$ *to this leaf space, mapping a point to the leaf it lies on.*
Note the following important features:

- *The singular points of the tree correspond to the leaves emerging from zeros of* ϕ_2*. These zeros are also the points from which more than two foliation lines emerge. These lines emerging from zeros of* ϕ_2 *constitute the* spectral network *for the case when the Hitchin base consists only of quadratic differentials.*

- *When a foliation line connects two zeros of* ϕ_2*, both these points map to the same point of the leaf space. This correspond to an edge in the tree degenerating to length zero on the one hand, and to a* BPS state *[25] on the other hand.*

For X *a compact Riemann surface, the leaf space is in general not a simplicial tree, i.e. the set of singular points is not necessarily discrete. In the case when*

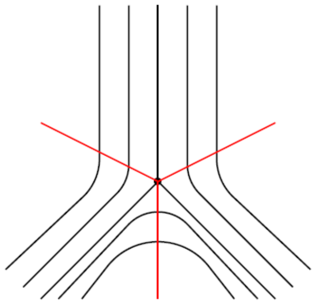

Figure 2.3: Foliation near simple zero

X is a non-compact Riemann surface (corresponding to ϕ_2 having poles), the leaf space is simplicial.
The apartments *in $\mathcal{T}$ are isometries $\mathbb{R} \to \mathcal{T}$. Therefore, by (2.1) any ϕ_2*-non-critical *path γ maps to a single apartment. A ϕ_2 non-critical path $\gamma : [0,1] \to \widetilde{X}$ is a path on $\widetilde{X}$ such that the tangent vector to γ is not tangent to a foliation, i.e*

$$(Re\ \gamma^* \lambda)(t) \neq 0 \quad \forall t \in [0,1] .$$

Non-critical paths will also play an important role later in the case of spectral covers of higher degree.

Example 2.4. *One of the main examples of buildings that will be needed in this thesis, is the* asymptotic cone of a symmetric space*. This example, that will play a prominent role in WKB considerations, will be discussed in more details later, when more technical details on buildings will have been presented.*

After these motivating examples, we now turn to the formal definition of buildings. The reader not familar with buildings should always keep the above examples in mind. Our treatment here follows closely the review [60]. We start by writing down the definition. The rest of this section is then devoted to explaining all the terms appearing therein.

Definition 2.1. An *affine building* (resp. *spherical building* is a triple $(\mathcal{B}, \mathcal{F}, \mathcal{A})$ consiting of a set $\mathcal{B}$, a collection $\mathcal{F}$ of filters on $\mathcal{B}$ (called the *facets* of $\mathcal{B}$) and a collection $\mathcal{A}$ of subsets A of $\mathcal{B}$ called *apartments*, each endowed with a metric d_A, satisfying the following axioms

1. For $A \in \mathcal{A}$, let $\mathcal{F}_a := \{\sigma \in \mathcal{F} | A \in \sigma\}$ be the set of filters contained in A. Then for each apartment A, $((A, d_A), \mathcal{F}_A)$ is isomorphic to a Euclidean (resp. spherical) apartment.

2. For any two facets F, F' in $\mathcal{F}$, there is an apartment A containing $\bar{F}$ and $\bar{F}'$.

3. For any two apartments A and A', their intersection is a union of facets. For any two facets F, F' in $A \cap A'$, there exists an isometry of apartments $A \to A'$ that carries $\mathcal{F}_A$ to $\mathcal{F}_{A'}$ and fixes $\bar{F}$ and $\bar{F}'$ pointwise.

Remark 2.2. From the definition above, it follows immediately that we can define a distance between any two points in a building $\mathcal{B}$: we can put them into a common apartment by 2. Then, 3 tells us that this distance is well defined. The triangle inequality follows as well, although this is not quite as obvious (we refer the interested reader to [60]).

We now explain all the terms used in the definition above. We start with *apartments.*

Let $\mathbb{E}$ be a Euclidean space, i.e. a real vector space V together with a non-degenerate inner product. Let $\mathbb{A}$ be an affine space over V with the metric induced by $\mathbb{E}$. Then, an *affine hyperplane* in $\mathbb{A}$ is an affine subspace of codimension one. A *reflection* of $\mathbb{A}$ is an isometry of order two whose fixed point set is an affine hyperplane H. Note that for each affine hyperplane, there is a unique reflection r_H with fixed point set H (any hyperplane H through the origin defines a reflection by $r : V = H \oplus H^{\perp} \to V, (h, a) \mapsto (h, -a)$; now, take any $p \in H$, then $r_H : v \mapsto (v - p) \mapsto r(v - p) \mapsto r(v - p) + p$, where r is the reflection about $H - p$). The group $\mathrm{Aff}(\mathbb{A})$ of isometries of $\mathbb{A}$ is isomorphic to $V \rtimes \mathrm{GL}(V)$. Then, for any subgroup $W_{aff} \subset \mathrm{Aff}(\mathbb{A})$, denote by W_{lin} its image in $\mathrm{GL}(V)$. A subgroup W_{aff} of the group of affine isometries is called an *affine refelection group* if it is generated by affine reflections such that W_{lin} is finite. Note that W_{lin} naturally acts on the unit sphere $S(\mathbb{E}) = \{v \in \mathbb{E} | \|v\| = 1\}$ where $\|.\|$ denotes the norm on $\mathbb{E}$. We can idenitify W_{lin} with its image in W_{sph} in the isometry group of $S(\mathbb{E})$.

Now, let W_{aff} be an affine reflection group, and let $\mathcal{H}$ be the collection of affine hyperplanes H in $\mathbb{A}$ such that there exists a $w_H \in W_{aff}$ with H the fixed point set of w_H. Denote by $\mathcal{H}_{lin}$ the collection of vector subspaces $U = \{v - w | v, w \in H\}$ and by $\mathcal{H}_{sph}$ the collection of subsets $C \subset S(\mathbb{E})$ such that there exists $H \in \mathcal{H}_{lin}$ such that $C = H \cap S(\mathbb{E})$. An *affine Coxeter complex* (resp. *spherical Coxeter complex*) is a pair $(\mathbb{A}, W_{aff})$ consisting of an affine space $\mathbb{A}$ (resp. a sphere $S(\mathbb{E})$), and an affine reflection group W_{aff} (resp linear reflection group $W_{lin} \simeq W_{sph}$) acting on $\mathbb{A}$) (resp. $S(\mathbb{E})$). These Coxeter complexes are the basic buidling blocks of buildings. It is easy to check that an affine Coxeter complex $(\mathbb{A}, W_{aff})$ is completely determined by the corresponding set of hyperplanes $\mathcal{H}$ (the same is true for a spherical Coxeter complex).

Definition 2.3. Let W_{lin} be a linear reflection group acting on a Euclidean space $\mathbb{E}$. We identify W_{lin} with the spherical reflection group W_{sph} by restricting its action to the unit sphere. Let $\mathcal{H}_{lin}$ and $\mathcal{H}_{sph}$ be as defined in the paragraph above. Elements of $\mathcal{H}_{lin}$ and $\mathcal{H}_{sph}$ will be called *walls*. An *open half apartment* is a connected component of the complement of a wall. Define an equivalence relation on $\mathbb{E}$ (resp. $S(\mathbb{E})$ by saying that $x \sim y$ if the set of open half apartments and walls containing x coincides with the set of open half aprtments and walls containing y.

1. A vectorial *facet* is an equivalence class for the equivalence relation $\sim$.
2. The *support* of a facet is the intersection of walls containing it. The dimension of a facet is the dimension of its support as a manifold.
3. A facet maximal with respect to inclusion is called a *Weyl chamber* (or just a *chamber* in the spherical case)
4. A *panel* is a facet whose support is of codimension one in $\mathbb{E}$ (resp. $S(\mathbb{E})$).

Definition 2.4. Let $(\mathbb{A}, W_{aff})$ be an affine Coxeter complex given by a set of reflection hyperplan $\mathcal{H}$, the elements of which are called *walls*.

- A *sector based at x* in $\mathbb{A}$ is a subset of the form $x + \Delta$, where Δ is a Weyl chamber in $(\mathbb{E}, W_{lin})$.
- A *germ* of Weyl sectors based at x is an equivalence class of Weyl sectors based at x for the following equivalence relation: S and S' are equivalent if $S \cap S'$ is a neighborhood of X in S and S'. The germ of a Weyl sector S at x is denoted $\Delta_x S$.
- A *half-apartment* in $\mathbb{A}$ is the closure of an open half apartment.
- An *enclosure* in $\mathbb{A}$ is the intersection of a collection of half-apartments. The intersection of enclosures is an enclosure. If $Q \subset \mathbb{A}$ then the intersection of enclosures containing Q is called its hull, and denoted hull(Q). A subset Q is said to be *enclosed* or *Finsler convex* if it is equal to its hull.

Let $(\mathbb{A}, W_{aff})$ be an affine Coxeter complex with $\mathcal{H}$ the set of reflection hyperplanes in $\mathbb{A}$. Then it can be shown that W_{aff} is a semi-direct product $W_{aff} \simeq W_{lin} \rtimes T$ with T a translation subgroup of $\mathbb{A}$. The affine Coxeter complex $(\mathbb{A}, W_{aff})$ is called *discrete* if the translation subgroup T is a discrete subgroup of $\mathbb{A}$; otherwise it is called *dense*. In the case that $(\mathbb{A}, W_{aff})$ is discrete, $\mathbb{A}^o_{\mathcal{H}} := \mathbb{A} - \cup_{H\in\mathbb{H}} H$ is an open subset of $\mathbb{A}$ and the connected components of $\mathbb{A}^o_{\mathcal{H}}$ are called *alcoves* or *chambers*. The closures of the chambers tile the affine space. If the model space for our building $\mathcal{B}$ is a discrete affine Coxeter complex, then the structure of the building can be described completely in terms of these alcoves. When T is dense, however, the set $\mathbb{A}^o_{\mathcal{H}}$ is not well behaved and we need the notion of a *filter* to save alcoves.

Definition 2.5. Let A be a set. A *filter* σ on A is a collection of subsets of A satisfying the following conditions:

1. If P is in σ and $P \subset Q$ then Q is in σ
2. If P and Q are in σ, then so in $P \cap Q$.

Example 2.5. *Let $Z \subset A$. Define $\sigma_Z := \{Y \subset A | Z \subset Y\}$. Then σ_Z is a filter. A filter is called* principal *if it is of the form $\sigma_{\{a\}}$ for some $a \in A$.*

Example 2.6. *Let X be a topologcail space and let $x \in X$. Then the collection N_x of subsets of X containing a neighborhood of x is a filter on X which is different from $\sigma_{\{x\}}$ in general.*

Example 2.7. *Let X be a set, and let $F := \{A \subset X | X - A$ is finite$\}$. Then F is filter on X, called the* Frechet *filter. Note that F is not contained in any principal filter.*

The set of filters forms a poset: one says that σ is contained in σ', denoted as $\sigma \leq \sigma'$, if for all $Z \in \sigma'$ we have that $Z \in \sigma$. Subsequently, subsets $Z \subset A$ will often be identified with their corresponding filter σ_Z; in particular we say that a filter σ is contained in a set Z if $\sigma \leq \sigma_Z$, i.e. if $Z \in \sigma$. An arbitrary subset $Z \subset A$ is the union of a family of filters $\{\sigma_\alpha\}_\alpha$ if each σ_α is contained in Z, and for each $x \in Z$ there exists an α such that $\sigma_{\{x\}} \leq \sigma_\alpha \leq \sigma_Z$. For example, a subset of a topological space is open if and only if it is the union of a family of neighborhood filters. The closure $\bar{\sigma}$ of a filter σ in a topological space X is the collection of subsets of X that contains the closure of a set in σ.

Definition 2.6. Let x be a point in a Coxeter complex $\mathbb{A}$ modelled on a Euclidean space $\mathbb{E}$, and let F be a vectorial facet in $\mathbb{E}$. The *facet* $\sigma_{F,x}$ associated to (x, F) is the filter defined by the following condition: Z belongs to $\sigma_{F,x}$ iff it contains a finite intersection of open half apartments and walls containing $U \cap (x+F)$ for some open neighborhood U of x in $\mathbb{A}$. Let $\mathcal{F}_{\mathbb{A},W} = \{\sigma_{F,x} | x \in \mathbb{A},$ and F is a vectorial facet$\}$.

The pair $(\mathbb{A}, \mathcal{F}_{(\mathbb{A},W)})$ consisting of the metric space $\mathbb{A}$ together with the collection of filters $\mathcal{F}_{(\mathbb{A},W)}$ is called the *Euclidean apartment* associated to $(\mathbb{A}, W)$. A metric space A endowed with a collection of filters $\mathcal{F}$ is called a Euclidean apartment if it is isomorphic to $(\mathbb{A}, \mathcal{F}_{(\mathbb{A},W)})$ for some affine Coxeter complex $(\mathbb{A}, W_{aff})$.

Now we have explained all terms in the definition of a building. The next goal is to study maps between buildings.

Definition 2.7. A *(generalized) chamber system* is a set X equipped with a family $\mathcal{F}$ of filters on X. Let $(X, \mathcal{F})$ and $(X', \mathcal{F}')$ be generalized chamber systems. A *morphism of chambers systems* is a map $f : X \to X'$ such that for each filter $\sigma \in \mathcal{F}$, we have that $f_*(\sigma) \in \mathcal{F}'$ (with $f_*(Q) := \{Q' \subset X' | f(Q) \subset Q' \text{ for some } Q \in \sigma\}$.

Definition 2.8. A *pre-building* $(\mathcal{B}, \mathcal{F}, \mathcal{C})$ is a generalized chamber system $(\mathcal{B}, \mathcal{F})$ that is the union of a collection $\mathcal{C}$ of sub-chamber systems called cubicles, each cubicle $C \in \mathcal{C}$ being equipped with a metric d_C, satisfying the following conditions:

1. Each cubicle (C, d_C) is isomorphic to an enclosure in an aparment $\mathbb{A}$ (as a chamber system and as a metric space)

2. For any two cubicles C and C', d_C agrees with $d_{C'}$ on $C \cap C'$.

Definition 2.9. Let $(\mathcal{B}, \mathcal{F}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{F}', \mathcal{A}')$ be pre-buildings. A morphism of generalized chamber systems $f : \mathcal{B} \to \mathcal{B}'$ is an *isometry of pre-buildings* if it restricts to a distance preserving map $f|_A : C \to \mathcal{B}'$ for every cubicle C.
Suppose now that $(\mathcal{B}, \mathcal{F}, \mathcal{A})$ and $(\mathcal{B}', \mathcal{F}', \mathcal{A}')$ are buildings. Then f is

1. An *isometry of buildings* or *strong morphism of buildings* if it is an isometry of pre-buildings.

2. A *folding map* or *weak morphism of buildings* if it has the following property: for every apartment $A \in \mathcal{B}$ there exists a locally finite collection of hyperplanes $\mathcal{H}$ such that f restricts to an isometry on the closure of each connected component of $A - \cup_{H \in \mathcal{H}} H$.

We close this section by stating some properties of buildings that will be needed in later chapters.

Proposition 2.10. *Let $\mathcal{B}$ be an affine building with Weyl group W. Then the follwoing hold:*

1. *Let S and S' be opposite sectors based at a common vertex. Then there is a unique apartment A such that $S \cup S' \subset A$.*

2. *Let S be a sector in $\mathcal{B}$, and let A_1 be an apartment. Suppose that $A \cap S$ is a panel in A_1. Let H be a wall in A_1 containing P. Then there exist apartments $A_2 \neq A_3$ such that $A_1 \cap A_j$ is a half-apartment and $H \cup S \subset A_j$ for $j = 2, 3$.*

3. *Let x be a vertex in an affine building $\mathcal{B}$, and let S and S' be sectors based at x. Then there is an apartment A in $\mathcal{B}$ containing S and the germ $\Delta_x S'$.*

Proof. The reader is referred to [4, 57] □

In this thesis, we restrict our buildings to have a "complete system of apartments", as this will be necessary for some arguments.

Definition 2.11. Let $(\mathcal{B}, \mathcal{F}, \mathcal{A})$ be an affine building. We weill say that $(\mathcal{B}, \mathcal{F}, \mathcal{A})$ is a building *with a complete system of apartments* if for any other system of apartments $\mathcal{A}'$ with $\mathcal{A} \subset \mathcal{A}'$ we have $\mathcal{A} = \mathcal{A}'$.

Remark 2.12. One can easily show [57, 60] that any system of apartments $\mathcal{A}$ is contained in a unique maximal one $\bar{\mathcal{A}}$. Thus, restricting to buildings with complete systems of apartments is not really a restriction (compare this to restricting to manifolds with a maximal atlas). A characterization of buildings with complete apartments can be given as follows: suppose that A is isometric to the standard apartment, and every bounded subset of A is contained in an apartment. Then $A \in \mathcal{A}$. There is an even stronger characterization: a building has a complete system of apartments if and only if every geodesic is contained in a unique apartment [57].

2.2 Asymptotic cones of symmetric spaces

Recall that a locally symmetric space is a Riemannian manifold M such that for any $p \in M$ there is an isometry of a neighborhood of p fixing p, and whose derivative at p is the negative of the identiy. A locally symmetric space is called a symmetric space if each s_p can be extended to a global isometry of M. As a reference for symmetric spaces, we refer to [31]; two surveys that cover the aspects needed in this thesis are [35, 54].

Recall that the connected component G of the identity of the isometry group of a Riemannian manifold M is a Lie group. It is an important theorem in the theory of symmetric spaces that any symmetric space M is a homogeneous space for G, i.e. $M \simeq G/K$ where K denotes the isotropy group of a point $p \in M$.

There is an involution σ of G induced by s_P given by $\sigma(g) = s_P g s_P$, and a

corresponding involution $\mathrm{Ad}(s)$ of the Lie algebra $\mathfrak{g}$ of G. This involution induces a decomposition $\mathfrak{g} \simeq \mathfrak{k} + \mathfrak{p}$ into eigenspaves of eigenvalue $+1$ and -1, respectively. This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$.
Of special importance to us are symmetric spaces of noncompact type, i.e. the restriction of the Killing form to the -1 eigenspace $\mathfrak{p}$ is negative definite.
Recall that a submanifold $N \subset M$ is called *totally geodesic* if any geodesic that intersects N at a point $p \in N$, and such that its tangent vector at p is tangent to N, stays within N.

Definition 2.13. Let M be a symmetric space of non-compact type and N be a totally geodesic submanifold. Then N is called a *k-flat* if it is isometric to $\mathbb{R}^k$ with its Euclidean metric. An *apartment* in M is a flat maximal with respect to inclusion.

Example 2.8. *Let $M = SL_r/SU(r)$. This space can be identified with the space of hermitian metric on $\mathbb{C}^r$. The Cartan decomposition $\mathfrak{sl}_r\mathbb{C} \simeq \mathfrak{su}_r + \mathfrak{h}$ (with $\mathfrak{h}$ being the set of Hermitian matrices) gives, by the exponential map $\exp : \mathfrak{h} \to M$, a diffeomorphism. This space is a symmetric space of non-compact type.*
Furthermore, apartments $A \subset M$ correspond to maximal abelian subalgebras $\mathfrak{a} \subset \mathfrak{h}$ under the exponential map.

The following proposition shows that symmtric spaces share some properties with buildings:

Proposition 2.14 ([54])**.** *Let M be a Riemannian symmetric space. Then the set $\mathcal{A}$ of apartments in M statisfies the following axioms:*

1. *For any two points p and q in M, there is an apartment containing p and q.*

2. *For any two apartments A and A', their intersection is a closed convex set of both. Furthermore, there is an isometry $A \to A'$ fixing $A \cap A'$.*

The following theorem by Kleiner and Leeb shows the precise relationship between buildings and symmetric spaces.

Theorem 2.15 (Kleiner-Leeb)**.** *Fix an ultrafilter ω on $\mathbb{N}$ and M be a non-empty symmetrc space of non-compact type. Then for any sequence $p = \{p_n\}_{n\in\mathbb{N}}$ of base point in M, and any family of scale factors $\mu = \{\mu_n\}_{n\in\mathbb{N}}$, the asymptotic cone $Cone_\omega(M, p, \mu)$ is a thick affine building with a complete system of apartments. Furthermore, if the Coxeter complex associated to M is $(\mathbb{A}, W_{Aff})$ then $Cone_\omega(M, p, \mu)$ is modelled on $(\mathbb{A}, W_{aff})$.*

Now we want to explain what the notion of *asymptotic cone* means [49]. It arises from taking the *Gromov-Hausdorff* limit of the symmetric space, a construction which we will now explain.
Roughly speaking, the asymptotic cone of a metric space (X, d) is obtained by "looking at X from infinity". It is rather straigforward to define the asymptotic cone of a convex subset $X \subset \mathbb{R}^n$: Fix $p \in X$ and consider the family of subsets $X_t = \frac{1}{t}X = \{q \in \mathbb{R}^n | t(q-p) \in X\}$. This gives a nested family of subsets of $\mathbb{R}^n$: for $t_1 \leq t_2, X_{t_2} \subset X_{t_1}$. The asymptotic cone of X with respect to p is $\mathrm{Cone}(X, p) := \bigcap_t X_t$.

Example 2.9. *Let* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -x^3 & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0 \end{cases},$$

let $X = \{(x, y) \in \mathbb{R}^2 | y \geq f(x)\}$ *and let* $p = (0, 0)$*. Then Cone*$(X, p) = \{(x, y)|x \geq 0, y \geq 0\}$*, but the tangent cone at* p *is the entire upper half plane.*

For general metric spaces, $(X, \frac{1}{n}d), n \in \mathbb{N}$ cannot be realized as a nested sequence of subsets. The precise construction of the asymptotic cone is to take an *ultralimit*, which depends on an *ultrafilter*.

Definition 2.16 (Ultrafilters and ultralimits)**.** An *ultrafilter* σ on a set A is a filter on A this is maximal with respect to inclusion. Let ω be a non-principal ultrafilter. Then we say that a family of points $\{x_a\}_{a\in A}$ in a topological space X has ω-limit x, $\lim_\omega x_a = x$, if for each neighborhood U of X, the set $\{a \in A | x_a \in U\}$ belongs to ω.

The examples of interest to us will be $A = \mathbb{N}$ and $A = \mathbb{R}$. Some basic facts about ultrafilters are summarized in the following proposition [10, 51]:

Proposition 2.17. *Let* ω *be a non-principal ultrafilter on* $\mathbb{N}$*. Let* (X, d) *be a metric space and let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X*. Then the following are true:*

1. *If* $\{x_n\}_{n\in\mathbb{N}}$ *is a bounded sequence in* X*, then it has an* ω*-limit in* X*.*
2. *If* $\{x_n\}_{n\in\mathbb{N}}$ *is a convergent sequence (in the sense of metric spaces), then* $\lim_\omega$ *exists and* $\lim_\omega x_n = \lim_{n\to\infty} x_n$*.*
3. *Let* ω *be a non-principal ultrafilter on* $\mathbb{R}$ *and let* $f : \mathbb{R} \to X$ *be a bounded map. Then* $\lim_\omega f \leq \limsup_{t\to\infty} f(t)$

4. *If* $\lim_{t\to\infty} f(t)$ *exists, then we have* $\lim_{t\to\infty} f(t) = \lim_\omega f = \limsup_{t\to\infty} f(t)$

A *family of scale factors* $\{\mu_n\}_{n\in\mathbb{N}}$ (or $\{\mu_t\}_{t\in\mathbb{R}}$) is a sequence (or family) of positive real numbers such that $\mu_n \to \infty$ as $n \to \infty$ (or $\mu_t \to \infty$ as $t \to \infty$).

Definition 2.18. Let (X, d) be a metric space, let $\{p_n\}$ be a sequence of points in X and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a family of scale factors. Fix a non-principal ultrafilter ω on $\mathbb{N}$. The *asymptotic cone* $\mathrm{Cone}_\omega := \mathrm{Cone}_\omega(X, \{p_n\}, \{\mu_n\})$ is the metric space associated to the pre-metric space Cone'_ω:

- the points of Cone'_ω are sequence $\{x_n\}$ in X such that $(1/\mu_n)d(x_n, p_n)$ is bounded.
- The metric is given by

$$d_{\mathrm{Cone}_\omega}(\{x_n\}, \{y_n\}) = \lim_\omega \frac{1}{\mu_n} d(x_n, y_n)\,.$$

Let (X, d) be a symmetric space of non-compact type and let $d_n = \frac{1}{\mu_n} d$. The apartments of the symmetric space give rise to the apartments of the asymptotic cone as follows: Let $f_n : \mathbb{A} \to X$ by an apartment with $f_n(0) = p_n$. Then there is an induced map $[f] : \mathbb{A} \to \mathrm{Cone}_\omega$, $[f](a) = \{f_n(a)\}$.

2.3 Vector valued distance

On symmetric spaces and buildings of rank > 1 it is possible to introduce a more refined distance that takes values in a Euclidean space $\mathbb{E}$, or more precisely in the fundamental Weyl chamber $\mathcal{C}$.
Let $(\mathbb{A}, W_{aff})$ be a Coxeter complex, let W_{sph} denote the spherical part of W_{aff}. Then the subtraction map $\mathbb{A} \times \mathbb{A} \to \mathbb{E}$ gives rise to a map

$$\mathbb{A} \times \mathbb{A} \to (\mathbb{A} \times \mathbb{A})/W_{aff} \to \mathbb{E}/W_{sph}\,.$$

Now, $\mathbb{E}/W_{sph}$ is the same a a fundamental domain for the action of W_{sph} on $\mathbb{E}$; thus we obtain a distance function $\vec{d} : \mathbb{A}\times\mathbb{A} \to \bar{\mathfrak{C}}$. This is called the *vector distance.*

Definition 2.19. Let X be a symmetric space or an affine building, and let x, y be points in X. In light of the definition of buildings and the properties of apartments in symmetric spaces, we know that there exists an apartment $A \subset X$ containing

x and y. Define the *vector valued distance* $\vec{d}(x,y)$ to be the vector valued distance computed in this apartment.

Remark 2.20. It is clear that this is well defined: in the case of buildings it follows immediately from the definition, in the case of symmetric spaces, it follows from the second property.

Remark 2.21. If $X = G/K$ is a symmetric space, the vector valued distance is the Cartan projection: recall that there is a Cartan decomposition $G = KA^+K$. The Cartan projection is the map $\log : A^+ \to \bar{\mathfrak{C}}$.

Example 2.10. *Let $X = SL_r/SU_r$. Then the Cartan decomposition is as follows: let $T \in SL_r\mathbb{C}$. Then $T = UDV$ with U, V unitary and D diagonal with real entries. If $A, B \in X$, and $A = TB = UDVB$ then $\vec{d}(A,B)$ is the vector of logarithms of the diagonal entries of D.*

Proposition 2.22 (Parreau)**.** *Let M be a Riemannian symmetric space of non-compact type, and let $\vec{d}$ denote its vector distance function. Let $Cone_\omega$ denote the affine building obtained from M by passing to its asymptotic cone with respect to some family of base points and scale factors, and an ultrafilter ω on $\mathbb{N}$. Let $[x_n]$ and $[y_n]$ be two points in $Cone_\omega$. Then we have that*

$$\vec{d}_{Cone_\omega}([x_n],[y_n]) = \lim_\omega \vec{d}(x_n,y_n)\ .$$

2.4 The harmonic map to the asymptotic cone

In [58], the action of a group G on the asymptotic cone leads to a (refined) distance on G. In this section this construction is extended by replacing G with a *groupoid*. This allows us to keep track of the points of the Riemann surface later.
A *groupoid* is a category all of whose morphisms are isomorphisms. A basic example is the *fundamental groupoid* $\pi_{\leq 1}(X)$ of a topological space X. By definition, the objects of $\pi_{\leq 1}(X)$ are the points of X and the set of morphisms between P, Q, $\pi_{\leq 1}(P,Q)$, is the set of homotopy classes of paths from P to Q. When X carries some additional geometric structure, the category of representations of $\pi_{\leq 1}(X)$, can have a geometric interpretation. For example, if X is a smooth manifold, then, by the Riemann-Hilbert correspondence, the category of representations of $\pi_{\leq 1}(X)$ is equivalent to the category of flat vector bundles on X. This example will be relevant for the WKB problem, but in this section we work with an abstract groupoid Γ.

Let Γ be a groupoid . Then a finite dimensional complex representation of Γ is a functor $\rho : \Gamma \to \mathrm{Vect}_{\mathbb{C}}$ (here, $\mathrm{Vect}_{\mathbb{C}}$ is the cateogry of finite dimensional complex vector spaces). If $\dim_{\mathbb{C}}(\rho_x) = r$ for all $x \in \mathrm{Ob}(\Gamma)$, then we say that the representation is of *rank* r. Denote by $\mathrm{Rep}(\Gamma, r)$ the set of all representations of Γ of rank r equipped with a trivialization of $\bigwedge^r \rho$. A *hermitian metric* on ρ is the datum of a hermitian metric h_x for all ρ_x that is compatible with the trivialization.

Suppose that $\{\mu_t\}$ is a family of scale factors. Then we say that a map $f : \mathbb{R} \to \mathbb{R}$ is of *exponential growth at infinity with respect to* $\{\mu_t\}$ if there exist constants $C, \eta \in \mathbb{R}$ such that $|f(t)| \le C \exp(\eta\mu_t)$ for $t \gg 0$. The set of function of functions of exponential growth with respect to $\{\mu_t\}$ forms a ring with valuation, where the valuation is the infimum over all η such that there exists a C with $|f(t)| \le C \exp(\eta\mu_t)$ for $t \gg 0$:

$$\nu(f) := \limsup_{t\to\infty} \frac{1}{\mu_t} \log |f(t)| \, .$$

The valuation can be used to measure the growth rate of a family of representations depending on a large parameter.

Definition 2.23. Let Γ be a groupoid and let $\rho : \mathbb{R} \to \mathrm{Rep}(\Gamma, r)$ be a family of representations such that for each $t \in \mathbb{R}$, $h(t)$ is a hermitian metric on $\rho(t)$. Let $\|-\|_t$ denote the operator norm on $\mathrm{Hom}(\rho_x(t), \rho_y(t))$ associated to $h(t)$. Let $\{\mu_t\}_{t\in\mathbb{R}}$ be a family of scale factors. Then ρ is *of exponential type* with respect to $(\{h(-)\}, \{\mu_t\} =$ if for each arrow $\gamma \in \Gamma$, the function $t \mapsto \|\rho_\gamma(t)\|$ is of exponential growth.

Definition 2.24. Let Γ be a groupoid, and let $\rho : \mathbb{R} \to \mathrm{Rep}(\Gamma, r)$ be a family of representations that is of exponential growth with respect to a family of metrics $h(t)$ and a family of scale factors μ_t. Let γ be an arrow in Γ. The *exponent* $\nu(\gamma)$ with respect to (ρ, h, μ) is the number

$$\nu(\gamma) = \limsup_{t\to\infty} \frac{1}{\mu_t} \log\|\rho_\gamma(t)\|_t \, .$$

The *dilation spectrum of* γ is the vector $\vec{\nu}(\gamma) = (\nu_1(\gamma), \dots, \nu_r(\gamma))$ uniquely determined by the condition that $\sum_{i=1}^k \nu_i(\gamma)$ is the WKB exponent of γ with respect to $(\bigwedge^k \rho, h, \mu)$ for each $1 \le k \le r$.

An analogous definition can be made by replacing lim sup by a limit over an ultrafilter. Then the assumption that the family of representations is of exponential type is no longer needed.

Definition 2.25. Fix an ultrafilter ω on $\mathbb{R}$ whose support is a countable set, and a family of scale factors $\{\mu_t\}$. Let $\rho : \mathbb{R} \to \mathrm{Rep}(\Gamma, r)$ be a family of representations, and let γ be a morphism in Γ. The *ultrafilter exponent* of γ, denoted ν_γ^ω is defined by the formula

$$\nu_\gamma^\omega = \lim_\omega \frac{1}{\mu_t} \log\|\rho_\gamma(t)\| \, .$$

One obtains similarly the *ultrafilter dilation exponent* $\vec{\nu}_\gamma^\omega$.

Remark 2.26. For a single fixed arrow γ in Γ and a given k, it is possible to choose the ultrafilter ω such that $\sum_{i=1}^k \vec{\nu}^\omega i, \gamma = \sum_{i=1}^k \vec{\nu}_{i,\gamma}$. Indeed it suffices to choose the ultrafilter subordinate to a subsequence which realizes the lim sup for the k-th exterior power.

Example 2.8 may be restated as follows.

Lemma 2.27. *Let Γ be a groupoid and $\rho : \mathbb{R} \to Rep(\Gamma, r)$ be a family of representations with trivialized determinant, and let h be a metric on this family. Let $\gamma \in Hom_\Gamma(x, y)$, and let $\|A\|_t$ denote the operator norm of a morphism $A : \rho_x \to \rho_y$ with respect to the metric $h_x(t)$ and $h_y(t)$. Define $\alpha_t(\gamma) = (\alpha_{t,1}(\gamma), \ldots, \alpha_{t,r}(\gamma))$ recursively by the formula $\sum_{i=1}^k \alpha_{t,k}(\gamma) = \log\|\bigwedge^k \rho_\gamma(t)\|$. Then we have that $\vec{d}(\rho_\gamma(t)_*(h_x(t)), h_y(t)) = \alpha_t(\gamma)$.*

Proof. Let h and k be hermitian metric on an $r-$dimensional vector space, which we think of as points in the symmetric space $\mathrm{Met}(V)$. Choose a basis $\{e_i\}$ of ρ_y that is orthonormal for both h and k - such a basis exists by the spectral theorem. Let α_i be such that$\|e_i\|_k = e^{\alpha_i}\|e_i\|_h$. Order the e_i such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$. Then the vector distance between h and k is by definition $\vec{d}(h, k) = (\alpha_1, \ldots, \alpha_r)$. Now, let $A : W \to V$ be a linear operator such that $k = A_* h'$ for some hermitian metric h' on W. By the definition, the operator norm $\|A\|$ with respect to h' and h is

$$e^{\alpha_1} = \sup_{\|w\|_{h'}=1} \|Aw\|_h \, .$$

Since the eigenvalues of $\|\bigwedge^k A\|$ are computed from the $k \times k$ minors of A, we have that

$$e^{\sum_{i=1}^k \alpha_i} = \sup_{\|w\|_{h'}=1} \|\bigwedge^k Aw\|_h \, .$$

Thus,

$$\sum_{i=1}^k \alpha_i = \log\|\bigwedge^k A\| \, .$$

By applying this discussion to the case where $W = \rho_x$, $h' = h_x(t), h = h_y(t), V = \rho_y, A = \rho_y(t)$, the lemma follows. □

Let (V, σ) be a complex vector space of dimension r that is equipped with a trivialization $\rho : \bigwedge^r V \to \mathbb{C}$. Let $\mathrm{Met}(V, \sigma)$ be the set of hermitian metric on V that are compatible with σ in the sense that the induced metric on $\bigwedge^r V$ coincides with the standard metric on $\mathbb{C}$ via σ. Observe that $\mathrm{Met}(\mathbb{C}^n, \sigma)$ is canonically isomorphic to $\mathrm{SL}_r\mathbb{C}/SU_r$, where σ is the trivialization given by the standard basis vectors. Thus, we see that Met defines a functor from the groupoid of vector spaces equipped with a volume form, to the groupoid of symmetric spaces of non-compact type (with isometries as morphisms).

Let $\{\mu_t\}$ be a family of scale factors, and ω be an ultrafilter on $\mathbb{R}$ whose support is a countable set. Then we can pass to the asymptotic cone levelwise where the basepoint in each fiber $\mathrm{Met}(\rho_x)$ is given by the family of points $h_x(t)$. This provides us with a family of functors

$$\mathrm{Cone}_{\omega,\mu_t} \circ \mathrm{Met} \circ \rho(t) : \Gamma \to \mathcal{M} \, .$$

Here $\mathcal{M}$ is the category of metric spaces and isometries. By a theorem of Kleiner and Leeb [42], this functor factors through the category of affine buildings and isometries.

Put $X := Ob(\Gamma)$, choose a basepoint $x_0 \in X$, and let $\widetilde{X}$ denote the set of pairs (x, γ) where $x \in X$ and $\gamma : x_0 \to x$. Note that $\Gamma(x_0, x_0)$ acts on $\widetilde{X}$ by composition of paths. Let $\widetilde{X}_\Gamma : \Gamma \to Set$ be the functor defined by $\widetilde{X}_\Gamma(x)$ equal to the set of points in $\widetilde{X}$ over x.

Definition 2.28. Define the *asymptotic limiting map* to be the natural transformation

$$h_\omega : \widetilde{X}_\Gamma \to \mathrm{Cone}_{\omega,\Gamma}$$

given by

$$h_\omega(x)(x, \gamma) := [\{\gamma_* h_{x_0}(t)\}_{t\in\mathbb{R}}] \, .$$

The functor $\mathrm{Cone}_{\omega,\Gamma}$ may be viewed as consisting of the affine building $\mathrm{Cone}_\omega := \mathrm{Cone}_{\omega,\Gamma}(x_0)$ together with the action of $\Gamma(x_0, x_0)$ on it by isometries. The section h_ω then corresponds to a $\Gamma(x_0, x_0)$-equivariant map

$$h_\omega : \widetilde{X} \to \mathrm{Cone}_\omega \, .$$

The following corollary of Proposition 2.22 and Lemma 2.27 says that the asymptotic limiting map computes the ultrafilter dilation exponents.

Corollary 2.29. *Fix an ultrafilter ω on $\mathbb{R}$ whose support is a countable set, and a family of scale factors $\{\mu_t\}$ Let $\rho : \mathbb{R} \to Rep(\Gamma, r)$ be a family of representations.*

Consider the asymptotic limiting map h_ω *of Definition 2.28. For any points* $x, y \in Ob(\Gamma)$ *and a morphism* γ *from* x *to* y*, ket* $\tilde{x}$*,* $\tilde{y}$ *be the source and target of any lifting of the path to an arrow of* $\widetilde{X}$*. Then*

$$\vec{d}_{Cone_\omega}(h_\omega(\tilde{x}), h_\omega(\tilde{y})) = \vec{\nu}_\gamma^\omega \ .$$

Chapter 3

Spectral Covers and Buildings

Portions of this chapter have appeared in [40] and have been submitted for publication to "Communications in Mathematical Physics".

The asymptotic propoerties of the Riemann-Hilbert problem studied in this thesis are closely related to the Riemann surface of an associated meromorphic 1-form. This Riemann surface is called the spectral curve and it gives rise to a ramified cover of X.

3.1 Spectral and Cameral covers

In this section, we want to recall how to construct a spectral cover from a harmonic map to a building, and to formulate the concept of *universal buildings*.
Let φ be a regular semisimple endomorphism of an $n-$dimensional complex vector space V. It is well known that φ can be described in terms of *spectral data*, i.e. a collection of lines $\{L_1, \dots, L_n\}$ (the one-dimensional eigen-spaces of φ) together with a collection of complex numbers λ_i (the eigenvalues). We arrive at the notion of a *spectral curve* Σ [3, 19, 32, 33] together with a line bundle on Σ by considering a family of endomorphisms $\{\varphi_x\}_{x \in X}$ where X is a complex manifold and letting φ_x additionally take values in a "coefficient object" K. Summarizing, we replace the single endomorphism φ by a section $\varphi_x \in H^0(X, \mathrm{End}(E) \otimes K)$ and consider the fiberwise diagonalization. In this thesis, we will content ourselves with the situation in which $dim_{\mathbb{C}}(X) = 1$, $G = \mathrm{SL}_r\mathbb{C}$ and K is a line bundle on X (we could actually restrict to $K = \omega_X$, the canonical bundle on X).

Definition 3.1. Let K be a holomorphic line bundle on a smooth complex curve X.

1. A *K-valued spectral cover* ϕ of X is a pair $(\pi : \Sigma \to X, i : \Sigma \hookrightarrow \mathrm{tot}(K))$, where π is a finite ramified cover and $i : \Sigma \hookrightarrow \mathrm{tot}(K)$ realizes Σ as a closed subscheme of the total space $\mathrm{tot}(K)$.

2. A *K-valued Higgs coherent sheaf* on X is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is a coherent sheaf and φ is a section of $\mathrm{End}(\mathcal{E}) \otimes K$ such that $\varphi \wedge \varphi = 0$.

The spectral cover φ is called *smooth* if the spectral curve Σ is smooth. Subsequently, let $p : \mathrm{tot}(T^*_X) \to X$ be the natural projection and λ be the restriction of the Liouville section $\lambda = pdx$ to Σ, where p is the coordinate along the fiber and x the coordinate along the base curve X.

We will sometimes write "$\phi = \{\phi_1, \ldots, \phi_3\}$" (where r is the degree the cover) in order to emphasize that the spectral cover can be thought of as a multi-valued one-form.
An important construction in the theory of Higgs-bundles is to associate a spectral cover Σ to a Higgs-bundle $(\mathcal{E}, \varphi)$. This is done by associating to a K-valued Higgs bundle its *characteristic polynomial*

$$\mathrm{char}\varphi := \det(\lambda \mathrm{id} - \varphi) = \lambda^r + a_1\lambda^{r-1} + \cdots + a_r \,.$$

Here the coefficent a_k is a section of the line bundle $K^{\otimes k}$. If $(\mathcal{E}, \varphi)$ is an SL_r-Higgs bundle, then $a_1 = \mathrm{Tr}\varphi = 0$. The zeros of the characteristic polynomial thus define a spectral cover Σ. The map associating to a Higgs bundle its spectral cover is called *Hitchin map.*
For the rest of this section, let $G = SL_r\mathbb{C}$ and denote by $W = S_r$ the Weyl group, i.e. the symmetric group in $r-$letters. Furthermore, we fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g} = \mathfrak{sl}_r$. In this case the space of characteristic polynomials can be identified as $\bigoplus_{i=2}^r H^0(X, K^{\otimes i})$.

Definition 3.2. Let X be a smooth complex curve, K a line bundle on X, and let $\phi \in \bigoplus_{i=2}^r H^0(X, K^{\otimes i})$ be a point in the Hitchin base. The *cameral cover* associated to ϕ is the cover $\pi_\phi : \Sigma_\phi \to X$ defined by the following pullback square:

$$\begin{tikzcd} \Sigma_\phi \arrow[r, "\tilde{\phi}"] \arrow[d, "\pi_\phi"] & \mathrm{tot}(K \otimes \mathfrak{t}) \arrow[d, "\sigma"] \\ X \arrow[r, "\phi"] & \mathrm{tot}(\bigoplus_{i=2}^r K^{\otimes i}) \end{tikzcd}$$

Here $\sigma = (\sigma_2, \dots, \sigma_r)$ are the elementary symmetric functions. We use that by Newton's theorem $\mathbb{C}[\mathfrak{t}]^W \simeq \mathbb{C}[\sigma_2, \dots, \sigma_r]$, where $\mathbb{C}[\mathfrak{t}]^W$ is the space of polynomials in the Cartan algebra invariant under the adjoint action of the Weyl group.
Note that there is a *tautological section* $\tilde{\phi} \in H^0(\Sigma_\phi, \pi_\phi^*(K \otimes \mathfrak{t}))$ defined by the diagram. By fixing a linear coordinate system $(x_1, \dots, x_r)$ on $\mathfrak{t}$ we can identify $\tilde{\phi}$ with an ordered sequence of differential forms $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_r)$ with $\sum_{i=1}^r \tilde{\phi}_i = 0$. Thus, intuitively, one can think of the cameral cover as "orderings of the spectral cover $\phi = \{\phi_1, \dots, \phi_r\}$".

3.2 Harmonic maps to buildings

In this section we want to describe a mathematical object from which we can produce a point in the Hitchin base. This object is a $\pi_1(X)$ *equivariant harmonic map* $h : \widetilde{X} \to \mathcal{B}$, where $\widetilde{X}$ is the universal cover of X. This construction was discovered first in [39]. We start by defining differential forms (or rather appropriate combinations thereof):

Definition 3.3. Let $\mathcal{B}$ be a rank $r-1$ affine building with Weyl group $W_{\text{aff}} \simeq W \ltimes T$ with T a subgroup of the translation group $\mathbb{R}^r$, and let $\{f : \mathbb{A} \to \mathcal{B}\}_{f\in\mathcal{A}}$ be an atlas for $\mathcal{B}$. A *differential k-form* η on an open subset $U \subset \mathcal{B}$ is a collection $\{\eta_f\}_{f\in\mathcal{A}}$ of k-forms on $f^{-1}(U) = \mathbb{A} \simeq \mathbb{R}^{r-1}$ such that $(g^{-1} \circ f)^* \eta_g = \eta_f$ on $f^{-1}(g(\mathbb{A}) \cap U)$. Let $T^*_{\mathcal{B}}$ be the sheaf on $\mathcal{B}$ whose sections on U are the differential 1-forms on U, and by $T^*_{\mathcal{B},\mathbb{C}}$ its complexification.

Now, let $(x_1, \dots, x_r)$ be an affine coordinate system on $\mathbb{A}$ such that $\{x_i = x_j\}$ define the reflection hyperplanes for the action of the Weyl group W. Let $dx_1, \dots, dx_r$ denote the differentials of the coordinates. Then, for each $2 \le k \le r$, $\sigma_k(dx_1, \dots, dx_r)$ is $W-$invariant and thus also invariant under the affine Weyl group. In consequence, the differentials $\sigma_k(dx_1, \dots, dx_r)$ give rise to a section $\xi_k \in H^0(\mathcal{B}, \mathrm{Sym}^k(T^*_{\mathcal{B},\mathbb{C}}))$.

Remark 3.4. Intuitively, one can think of the dx_i as defining local one-forms, which, however, are not globally well defined due to the action of the Weyl group. The situation here is analogous to the case of spectral covers, where the one-forms $\phi_1, \ldots, \phi_r$ were not well defined, but the "set of one-forms" $\{\phi_1, \ldots, \phi_r\}$ (i.e. the spectral cover) is well defined.

Let X be a Riemannian manifold and let $h : X \to \mathcal{B}$ be a map. We call a point $x \in X$ *h-regular* if there exists a neighborhood U of x such that $h(U)$ is contained in a single apartment. Denote by X_{reg} the set of $h-$regular points. Now, for a differential form η on $\mathcal{B}$, one can define a pullback $h^*\eta$ which is a section of the complexified cotangent bundle of X. This pullback is defined using an open cover $\{U_\alpha\}_\alpha$ of X_{reg} for which $h(U_\alpha) \subset f_\alpha(\mathbb{A})$ for some apartment f_α. Then $h^*\eta$ is determined by the requirement

$$(h^*\eta)|_{U_\alpha} = (f_\alpha^{-1} \circ h)^*(\eta_{f_\alpha}) \, .$$

We will say that a map $h : X \to \mathcal{B}$ to an $\mathbb{R}$-building is harmonic if it is harmonic in the sense of [48].

Lemma 3.5. *Let X be a smooth complex curve, and suppose that $h : X \to \mathcal{B}$ is a* harmonic *map to a building. Let ξ_k be the harmonic symmetric tensor on $\mathcal{B}$ given locally by $\xi_k = \sigma_k(dx_1, \ldots, dx_r)$. Then there is a* unique *holomorphic section $\phi_k \in H^0(X, \omega_X^{\otimes k})$ that restricts to $h^*(\xi_k)$ on X_{reg}.*

Proof. The idea is that, since h is a harmonic map, the complexified pullbacks $h^*(dx_i)$ are locally defined harmonic 1-forms on X_{reg} whose $(1,0)$ parts are holomorphic. Thus $h^*(\xi_k)$ is a holomorphic section of $\mathrm{Sym}^k(\Omega^1_X)$. Furthermore, since the singularities of a harmonic map are in codimension 2, these holomorphic sections extend to all of X. For the details, see [39]. □

We now come to one of the key definitions.

Definition 3.6. Let X be a smooth complex curve, and let $\phi = (\phi_2, \ldots \phi_r) \in H^0(X, \omega_X^{\otimes i})$ be a point in the $SL_r\mathbb{C}$ Hitchin base. Let $\mathcal{B}$ be an affine building with Weyl group $W_{\mathbb{A}} \simeq W \ltimes T$ where W is the Weyl group of $SL_r\mathbb{C}$ and $T \subset \mathbb{R}^r$ is a translation group. A $\pi_1(X,x)$-equivariant harmonic map $h : \widetilde{X} \to \mathcal{B}$ is a *harmonic ϕ-map* if $\pi^*\phi_k$ coincides with $h^*\xi_k$ on X_{reg} for all k.

Remark 3.7. Intuitively, this means that the local forms $dx_1, \ldots dx_r$ are pulled back to the 1-forms $\phi_1, \ldots, \phi_r$.

3.3 The (uni)versal building

A crucial idea in this thesis is to reverse the construction of the previous section, i.e. try to define a versal or universal building $\mathcal{B}^\phi$ together with a harmonic map $\widetilde{X} \to \mathcal{B}^\phi$ for any spectral cover ϕ such that the spectral cover associated to the harmonic map to the building $\mathcal{B}^\phi$ is the original spectral cover ϕ. Conjecturally, this should even work in families: this means that there should be a space of (uni)versal buildings such that the space of all harmonic maps to all buildings in this space is isomorphic to the locus of smooth spectral covers.

Definition 3.8. Let X be a Riemann surface, $\mathcal{B}$ be a building and $h : \widetilde{X} \to \mathcal{B}$ be a map. The image of h, $\mathrm{im}(h)$, is the pre-building $\bigcup_{A\in\mathcal{A}} \mathrm{hull}(A \cap h(X))$ with the pre-building structure induced from $\mathcal{B}$.

Definition 3.9. With notation as above, a map $h^\phi : \widetilde{X} \to \mathcal{B}^\phi$ is a *(uni)versal ϕ-map* if it is a harmonic ϕ-map and satisfies the following property: for any building $\mathcal{B}$ with a complete system of apartments and with the same vectorial Weyl group, and for any harmonic ϕ-map $h : \widetilde{X} \to \mathcal{B}$ there exists (unique) a folding map of buildings $\psi : \mathcal{B}^\phi \to \mathcal{B}$ such that the following diagram commutes

$$\begin{tikzcd} \widetilde{X} \arrow[r, "h^\phi"] \arrow[dr, "h"'] & \mathcal{B}^\phi \arrow[d, "\psi"] \\ & \mathcal{B} \end{tikzcd}$$

Example 3.1. *In the case $r = 2$, i.e. when the Hitchin base consists of quadratic differentials ϕ on X, the universal building is well known: it is the leaf space of the foliation of ϕ on $\widetilde{X}$ (see Example 2.3) , which is an $\mathbb{R}$-tree $\mathcal{T}^\phi$. The harmonic ϕ-map in this case is given by the natural quotient map $\widetilde{X} \to \mathcal{T}^\phi$.*

Example 3.2. *If the cameral cover decomposes totally, the universal building is a single apartment. It can also be determined in the neighborhood of a simple branch point: locally it is given by trivalent tree times $\mathbb{R}^{r-2}$. Therefore the interesting phenomena appear near "collision points", i.e. points in the building where such trivalent singularities collide. This will examined later in examples.*

Now, we can formulate one of the main conjectures of this paper:

Conjecture 1. Let X be a Riemann surface, and let ϕ be a smooth spectral cover of X. Then there exists a (uni)versal ϕ-map $h^\phi : \widetilde{X} \to \mathcal{B}^\phi$.

The remark above shows the existence of the universal building for $r = 2$. In the higher rank case, some evidence will be provided in Chapters 5 and 6, by constructing examples.

3.4 Some properties of ϕ-maps

In this section, we develop some criteris for when a given region $\Omega \subset X$ is mapped into a single apartment by a ϕ-map. The key-ingredient is the notion of a *non-critical path.*

Definition 3.10. Let X be a Riemann surface and let $\phi = (\phi_2, \ldots, \phi_r) \in H^0(X, \omega_X^{\otimes k})$. Let $\gamma : [0,1] \to X$ be a smooth path, and let $a_k : [0,1] \to \mathbb{C}$ be the function $a_k(t) = (\gamma^*(\phi_k), \partial_t^{\otimes k})$. Then we say that γ is a *ϕ-non-critical path* if the real parts of the roots of the polynomial $\sum_k a_k(t) z^k$ are distinct for all $t \in [0,1]$.

Remark 3.11. Let "$\phi = \{\phi_1, \ldots, \phi_r\}$" be the spectral cover. Then γ being a non-critical path means that the real 1-form $\mathrm{Re}\ \gamma^*\phi_1, \ldots, \mathrm{Re}\ \gamma^*\phi_r$ are distinct. Thus, we can reorder them and assume that $\mathrm{Re}\ \gamma^*\phi_1 > \cdots > \mathrm{Re}\ \gamma^*\phi_r$ along γ.

Lemma 3.12. *Let X be a Riemann surface, and let $\phi \in \oplus_{k=2}^r H^0(X, \omega_X^{\otimes k})$. Let $h : X \to \mathcal{B}$ be a harmonic ϕ-map, and let $X_{reg} \subset X$ denote the locues where h is regular. Let γ be a non-critical path in X_{reg}, and let $s \in (0,1)$. Then there exist $\epsilon > 0$, and sectors S^+ and S^- in $\mathcal{B}$ based at $x := h(\gamma(s))$ such that*

1. *The germs $\Delta_x S^+$ and $\Delta_x S^-$ are opposite.*
2. *$h(\gamma((s-\epsilon, s]) \subset S^-$ and $h(\gamma([s, s+\epsilon)) \subset S^+$.*

Proof. Since $\gamma(s)$ is a regular point, there exists a neighborhood U of $\gamma(s)$, a chart $f : \mathbb{A} \to \mathcal{B}$ (with $\mathbb{A}$ being the standard apartment on which $\mathcal{B}$ is modelled), and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\tilde{h}"] \arrow[d] & \mathbb{A} \arrow[d, "f"] \\ X \arrow[r, "h"] & \mathcal{B} \end{tikzcd}$$

where $\tilde{h}$ is a differentiable map, and $f(a) = x$ for some $a \in \mathbb{A}$. Let $(x_1, \ldots, x_r)$ be standard coordinates on the aparment $\mathbb{A}$, and let $\{\phi_1, \ldots, \phi_r\}$ be the one-forms of the spectral cover. Since h is a ϕ-map, we may choose h such that $\tilde{h}^*(dx_i) = \text{Re } \phi_i$. By continuity, there exists an $\epsilon > 0$ such that $\gamma(J) \subset U$ where $J = (s - \epsilon, s + \epsilon)$. Let $\mathbb{E}$ be the Euclidean space on which $\mathbb{A}$ is modelled. For any $y \in \mathbb{A}$, we can identify the tangent space $T_y\mathbb{A}$ with $\mathbb{E}$. Let $\mathcal{H}_{lin}$ be the set of reflection hyperplanes in $\mathbb{E}$ for the vectorial part of the Weyl group of $\mathcal{B}$. Identifying $\mathbb{E}$ with $\mathbb{A}$ via the map $v \mapsto a + v$, we may think of the x_i as coordinates on $\mathbb{E}$. With respect to these coordinates, $\mathcal{H}$ is the set of hyperplanes $\{x_i = x_j\}$.
The condition that $\gamma|_J$ is ϕ-non-critical is equivalent to the condition that, for all $t \in J$, $(\tilde{h} \circ \gamma)_*(\partial_t) \notin H$ for any $H \in \mathcal{H}_{lin}$. Since J is connected and ψ is continuous, this implies that $\psi(J)$ is contained in a single Weyl chamber $\mathcal{C}$ in $\mathbb{E}$, i.e. in a single connected component of $\mathbb{E} - \bigcup_{in \in \mathcal{H}_{lin}} H$. Let $\mathcal{C}^{op} = \{v | -c \in \mathcal{C}\}$ be the opposite chamber. We have the corresponding sectors based at a: $S^+ = a + \mathcal{C}$ and S^-=a+$\mathcal{C}^{op}$.
Let $f_i = x_i \circ \tilde{h} \circ \gamma : J \to \mathbb{R}$. Then the conclusion of the previous paragraph is that, after reordering coordinates if necessary, we may assume that $f_1'(t) > \cdots > f_r'(t)$ where f_i' is the derivative of f_i. From the forumla

$$f_i(t) = x_i(a) + \int_s^t f_i'(t)dt$$

we see that $(\tilde{h} \circ \gamma)(t) \in S^+$ (resp. $(\tilde{h} \circ \gamma(t)) \in S^-$) for all $t \in [s, s + \epsilon)$ (resp. $t \in (s - \epsilon, s]$). □

Lemma 3.13. *Let X, ϕ be as in the Lemma above. Let $J \subset \mathbb{R}$ be an interval, and let $\gamma : J \to X_{reg}$ be a non-critical path. Suppose that there is an apartment $A \subset \mathcal{B}$ such that $\gamma(J) \subset A$. Let $s \in J$, $x = h(\gamma(s))$, and let $J^- = \{t \in J | t \leq s\}$ and $J^+ = \{t \in J | t \geq s\}$. Then*

1. *There is a pair of opposite sectors S^+, S^- in $\mathbb{A}$ such that $\gamma(J^-) \subset S^-$ and $\gamma(J^+) \subset S^+$.*

2. *For any $t_0 \in J^-$ and $t_1 \in J^+$ we have*

$$\vec{d}(y, z) = (\vec{d}(y, x) + \vec{d}(x, z)$$

where $y = h(\gamma(t_0))$ and $z = h(\gamma(t_1))$. That is, x is in the Finsler convex hull of y and z.

Proof. Only the second statement needs proof. It follows from the first statement. Let $\mathcal{C}'$ be the Weyl chamber in $\mathbb{E}$ that contains $z - x$. Since S^+ and S^- are opposed, we have that $x - y \in \mathcal{C}'$. It follows that $(z-y) = (z-x) + (x-y) \in \mathcal{C}'$. Let w be the element of the spherical Weyl group that carries $\mathcal{C}'$ to the fundamental Weyl chamber $\mathcal{C}$. Then, by definition of the vector distance, we have $\vec{d}(x,z) = w(z-x)$, $\vec{d}(y,x) = w(x-y)$ and $\vec{d}(y,z) = w(y-z)$. Since $w(z-y) = w(z-x) + w(x-y)$, the claim follows. □

Lemma 3.14. *Let X, ϕ be as in Lemma 3.12. Let $J \subset \mathbb{R}$ be an interval, and let $\gamma : J \to X_{reg}$ be a non-critical path. Let $s \in J$, and let $J^- = \{t \in J | t \leq s\}$, and $J^+ = \{t \in J | t \geq s\}$. Suppose that there exist sectors S^- and S^+ based at $x := h(\gamma(s))$ such that $h(\gamma(J^-)) \subset S^-$ and $h(\gamma(J^+)) \subset S^+$. Then the germs $\Delta_x S^-$ and $\Delta_x S^+$ are opposite.*

Proof. According to Proposition 2.10, there exists an apartment $A \subset \mathcal{B}$ containing S^+ and the germ $\Delta_x S^-$. Let S_A^- be the sector in A whose germ at x is $\Delta_x S^-$. Then, by definition of germs, $S_A^- \cap S^-$ is an open neighborhood of x in S^-. It follows that there exists an $\epsilon > 0$ such that $h(\gamma(s-\epsilon, s]) \subset S_A^-$. By Lemma 3.13, S_A^- and S^+ must be opposite sectors in A. Since $\Delta_x S_A^- = \Delta_x S^-$, it follows that $\Delta_x S^-$ and $\Delta_x S^+$ are opposed. □

We are now in a position to prove the main statements of this section:

Proposition 3.15. *Let X be a Riemann surface, and let $\phi \in H^0(X, \omega_X^{\otimes k})$. Let $h : X \to \mathcal{B}$ be a harmonic ϕ-map, and let $X_{reg} \subset X$ denote the locus where h is regular. Let $I = [0,1]$, and let $\gamma : I \to X_{reg}$ be a non-critical path. Then there exists an apartment $A \subset \mathcal{B}$ such that $im(h \circ \gamma) \subset A$.*

Proof. Define a subset

$$K := \{t \in I | \text{there exists an apartment} A \subset \mathcal{B} \text{ such that } h(\gamma([0,t])) \subset A\} \subset I$$

. We must prove that $K = I$. First, note that $K \neq \emptyset$: there exists an apartment containing the point $h(\gamma(0))$, so we have $0 \in K$. Furthermore, note that if $t \in K$, and $s \leq t$, then $s \in K$. We conclude that K is a non-empty interval. Let $t_0 = \sup K$, and let $Q := \gamma(t_0)$. We must prove that $t_0 = 1$. We will do so by contradiction. So assume that $t_0 < 1$.
Since $\gamma(I) \subset X_{reg}$, there is a neighborhood U_Q of Q and an apartment $A_Q \subset \mathcal{B}$ such that $h(U_Q) \subset A_Q$. Passing to a smaller open set, if necessary, we may assume that $U_Q \cap \gamma(I) = \gamma(J)$ for some interval J containing t_0.
Since $t_0 = \sup K$, and J is an open neighborhood of t_0, there exists $t_1 \in J \cap K$. Let $P = \gamma(t_1)$. By definition of K, there exists an apartment $A_P \subset \mathcal{B}$ such that

$h(\gamma([0,t_1])) \subset A_P$. Applying Lemma 3.13, we conclude that there is a sector S_P^- in A_P based at $x := h(P)$ such that $h(\gamma([0,t_1]) \subset S_P^-$.
Since $h(\gamma(J)) \subset A_Q$, we can apply Lemma 3.13 again to conclude that there exist opposed sectors S_Q^+ and S_Q^- in A_Q such that $h(\gamma(J^-)) \subset S_Q^-$ and $h(\gamma(J^+)) \subset S_Q^+$. Here $J^+ = \{t \in J | t \geq t_1\}$ and $J^- = \{t \in J | t \leq t_1\}$. Note that the sectors S_Q^+ and S_Q^- are based at $x := h(P)$.
From Lemma 3.14, it follows that the germs $\Delta_x S_P^-$ and $\Delta_x S_Q^+$ are opposed. Property 1 of Proposition 2.10 states that there is a unique apartment containing a pair of opposite sectors in a building. Let A be the unique apartment in $\mathcal{B}$ containing S_P^- and S_Q^+.
Since J^+ contains an open neighborhood of t_0 in $[0,1]$, and $t_0 < 1$, there exists $t_2 > t_1$ such that $t_2 \in J^+$, and hence there exists $t_2 > t_0$ such that $h(\gamma([t_1,t_2]) \subset S_Q^+$. Since $h(\gamma([0,t_1])) \subset S_P^-$, and $h(\gamma([t_1,t_2])) \subset S_Q^+$, we have that $h(\gamma([0,t_2]) \subset A$. Hence, $t_2 \in K$. But $t_2 > t_0 := \sup K$ by construction, so we have a contradiction. □

Theorem 3.16. *Let X, ϕ be as in the statement of the proposition. Let P and Q be points in X_{reg}, and $\mathcal{P}_{PQ}$ be the set of ϕ-non-critical path in X_{reg} starting at P and ending at Q. Let $\Omega_{PQ} = \bigcup_{\gamma \in \mathcal{P}_{PQ}} im\gamma$ be the union the the images of elements of $\mathcal{P}_{PQ}$. Then $h(\Omega_{PQ})$ is contained in the Finsler convex hull $[h(P), h(Q)]_{Fins}$ of $h(P)$ and $h(Q)$. In particular, there exists an apartment containing $h(\Omega_{PQ})$.*

Proof. Let $R \in \Omega_{PQ}$. By definition of Ω_{PQ}, there exists a ϕ-non-critical path γ starting at P and ending at Q such that $R = \gamma(s)$ for some $s \in I$. By Proposition 3.15, there is an apartment A such that $h(\gamma(I)) \subset A$. So we can apply Lemma 3.13 to conclude that $h(R) \in [h(P), h(Q)]_{\mathrm{Fins}}$.
We know that $[h(P), h(Q)]_{\mathrm{Fins}}$ is contained in the intersection of all apartments containing both $h(P)$ and $h(Q)$. Since this is a non-empty intersection, the proof is complete. □

Lemma 3.17. *Let $h : X \to \mathcal{B}$ be a harmonic ϕ-map, and let $X_{reg} \subset$ be the set of regular points of h. Let Z denote the ramification divisor of the spectral cover defined by ϕ. Then for any $R \in X - Z$ there exists a pair of points P, Q and two paths γ, γ' such that*

1. *γ and γ' start at P and end at Q*
2. *$\gamma^{-1} \circ \gamma'$ bounds a region D containing R that is topologically a disc*
3. *γ and γ' are ϕ-non-critical and contained in X_{reg}*

Proof. First of all, note that the singular set $X - X_{Reg}$ is given by the zeros of the Hopf differential (see e.g. [14]). Since the Hopf differential on a Riemann surface is holomorphic, the singular set is discrete.
Now let $R \in X - Z$. Since R is not a ramification point, there exists a neighborhood U of R on which we can write $\phi = (\phi_1, \ldots, \phi_r)$. Let $\mathcal{F}_{ij}$ be the foliation on U defined by Re $(\phi_i - \phi_j) = 0$. Then we can find a short curve $\gamma_0 : [-1, 1] \to X$ that is everywhere transversal to $\mathcal{F}_{ij}$, and such that $\gamma_0(0) = R$ (take, for example, any vector $v \in T_R X$ that is not in any of the foliations. Fix a Riemannian metric on X. Then there is an $\epsilon > 0$ such that for $|t| < \epsilon$ the path $t \mapsto \exp(tv)$ has the required property).
Let $P = \gamma_0(-1)$ and $Q = \gamma_0(1)$. Let $\mathcal{X}$ be a vector field along $\gamma_0(I)$ that is everywhere normal to $d\gamma(\partial_t)$. Then for $\epsilon > 0$ small enough the paths $\gamma(t) = \exp_{\gamma_0(t)}(\epsilon\mathcal{X}(\gamma_0(t)))$ and $\gamma'(t) = \exp_{\gamma_0(t)}(-\epsilon\mathcal{X}(\gamma_0(t)))$ are non-critical, and satisfy the requirements 1.,2. and 3. (here we have used the fact that the singularities are discrete). □

Lemma 3.18. *Let $h : X \to \mathcal{B}$ be a harmonic map from a Riemann surface X to an affine building. Let $D \subset X$ be a closed disc in X, and A be an apartment in $\mathcal{B}$. Suppose that $h(\partial D) \subset A$. Then $h(D) \subset A$.*

Proof. Consider the function $f : \mathcal{B} \to \mathbb{R}$ which associates to a point X its distance from the apartment A, i.e. $f(x) = d(x, A)$. Since $\mathcal{B}$ is non-positively curved, and A is a convex subset, the function f is convex. Since h is harmonic, and f is convex, $f \circ h$ is a subharmonic function on X (see [24] for a proof in the tree case.) It vanishes on ∂D, since $h(\partial D) \subset A$. It follows that f vanishes on D, i.e. $h(D) \subset A$. □

Proposition 3.19. *Let $h : X \to \mathcal{B}$ be a harmonic ϕ-map, and let $X_{reg} \subset X$ be the set of points at which h is regular. Let Z denote the ramification divisor of the spectral cover defined by ϕ. Then $X - Z \subset X_{reg}$.*

Proof. Let $R \in X - Z$. Then, by Lemma 3.17, there is a disc D containing R such that $\partial D = \gamma^{-1} \circ \gamma'$, where $\gamma, \gamma' : I \to X_{reg}$ are ϕ-non-critical paths. Applying Theorem 3.16, we see that there is an apartment A in $\mathcal{B}$ such that $h(\partial D) \subset A$. Applying Lemma 3.18, we conclude that $h(D) \subset A$. This completes the proof. □

3.5 Spectral networks and the universal building

Spectral networks (see [26]) are certain graphs on a Riemann surface that are closely related to the geometry of the spectral cover. In this section, we want to define spectral networks and state the main conjecture relating spectral networks to the singularities of the universal building.

Let X be a Riemann surface and ϕ be a point in the $\mathrm{SL}_r\mathbb{C}$ Hitchin base. Denote by λ the restriction of the Liouville one form on T^*_X to the spectral cover Σ associated to ϕ. Let $\Sigma^2 = \Sigma \times_X \Sigma$ be the set of pairs of points in the fiber of the spectral curve. Let $\delta\lambda := p_1^*(\lambda) - p_2^*(\lambda)$. For any phase θ we have a vector field $\mathcal{F}_\theta$ on Σ^2 dual to $e^{i\theta}\delta\lambda$. It is singular along the ramification points but there are well-defined outward paths (three of them at a simple branch point).

Definition 3.20. The *(extended) spectral network of phase* θ, $\mathcal{W}_\theta$ (resp. W_θ^{ext}), is the image in X of the smallest subset W^2 of Σ^2 contating the branch points, and closed under the following operations:

- flowing along the vector field $\mathcal{F}_\theta$ in the positive direction (resp. in the positive or negative directions);
- flowing along the outward paths starting from a branch point; and
- collisions: if $y, z \in W^2 \subset \Sigma^2$ and $p_2(y) = p_1(z)$ then $w := (p_1(y), p_2(z))$ should be in W^2, and we start to flow again from there.

One notes that for $P \in W_\theta^{\mathrm{ext}}$ there is a *detour path* [26] $\gamma : [0,1] \to \Sigma$ such that $\pi(\gamma(0)) = \pi(\gamma(1)) = P$ and $\int_\lambda \in e^{i\theta}\mathbb{R}_+$.

Remark 3.21. Note that the extended spectral network contains, in addition to the spectral network $\mathcal{W}_\theta$ defined in [26], the lines starting at collision points, but going in the opposite direction, and further collision lines generated by these "backward lines".

Let P be a branch point of $\Sigma \to X$. Locally, near P, the spectral networks looks like in Figure 3.1: for every angle $\theta \in S^1$, we have three rays emerging from the branch points (owed to the $\mathbb{Z}/2$ monodromy of λ about the branch point).

Suppose that P_1, P_2 are distinct branch points. Then the spectral network curves emerging from these different branch point may intersect. Denote by s_1 and s_2, respectively, the spectral networks emerging from P_1 and P_2 (labelled by two sheets (ij) and (jk) of the spectral cover, respectively). Now, assume that s_1 and s_2 intersect at the *collision point* Q, see Figure 3.2. In this situation, there is a *collision line* s_3 starting at Q. The detour path γ corresponds to the green contour in Figure 3.2.

The following argument shows that for $\mathrm{SL}_3\mathbb{C}$, the extended spectral network maps to the singularities of the building.

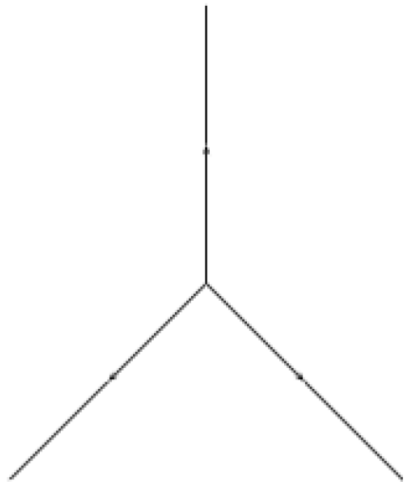

FIGURE 3.1: Spectral Network near a branch point

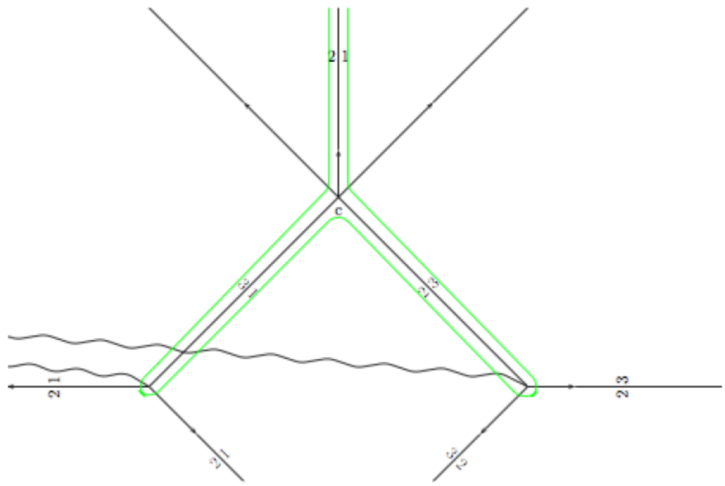

FIGURE 3.2: Spectral Network near a collision point

As we will see later, the three spectral network lines emanating from a branch point map to a reflection hyperplane in the building $\mathcal{B}$. Furthermore, a neighborhood of the branch point maps to $T \times \mathbb{R}$, where T is a trivalent tree (here the vertex of the tree corresponds to the reflection hyperplane described above). Then, a collision point maps to a point in the building at which two half-apartments are attached along two distinct reflection hyperplanes to an apartment containing the image of a neighborhood of the collision point. A straightforward calculation in the link of this vertex shows that, in fact, the building must be singular along the remaining third reflection hyperplane. This hyperplane, in turn, is the image of the (forward and backward) collision line of the spectral network. Using this, one can argue that the image of the *extended* spectral network under any harmonic ϕ-map is contained in the singularities of the building. In fact, we make the following conjecture, for which some evidence is provided later:

Conjecture 2. Let X be a Riemann surface, and let ϕ be a spectral cover. Let $\mathcal{W}^{\text{ext}}$ be the extended spectral network associated to ϕ. Then the support of $\tilde{\mathcal{W}}^{\text{ext}}$ coincides with the inverse image of the singular set $\mathcal{B}^{\phi}$ under the universal map $h^{\phi} : \widetilde{X} \to \mathcal{B}^{\phi}$.

Remark 3.22. At the moment, we do not know how to distinguish forward and backward lines from harmonic maps to buildings. Due to this important distinction in the physical description of $\mathcal{N} = 2$ supersymmetric quantum field theories, and in particular considering the different roles played by these lines in the description of BPS states, this is an important direction of further investigation.

Chapter 4

Singular Perturbation Theory

Portions of this chapter have appeared in [40] and have been submitted for publication to “Communications in Mathematical Physics”.

4.1 Nonabelian Hodge Theory

Suppose that X is a Riemann surface and fix a point $x_0 \in X$. Then the moduli space of local systems comes in different forms:

- the Betti moduli space $M_B = \mathrm{Hom}(\pi_1(X, x_0), \mathrm{SL}_r\mathbb{C})/\mathrm{SL}_r\mathbb{C}$;
- the moduli space of Higgs bundles, also called the *Dolbeault moduli space* $M_{Dol} = \{(\mathcal{E}, \varphi)\}/S$-equiv.;
- the *de Rham moduli space* of vector bundles with integrable algebraic connections $M_{DR} = \{(\mathcal{E}, \nabla)\}/S-$equiv.

Topologically all these three moduli space are isomorphic, and M_B and M_{DR} are even complex analytically isomorphic. Algebraically, however, these three spaces solve very different moduli problems. They are all noncompact and thus investigating the asymptotic behavior of the homeomorphisms mentioned above is an interesting problems.
For the Dolbeault moduli space M_{Dol} we have the *Hitchin fibration* $M_{Dol} \to \mathbb{A}^{r-1}$ which is a map of algebraic varieties. Due to the $\mathbb{C}^*$ -action $t : (\mathcal{E}, \varphi) \mapsto (\mathcal{E}, t\varphi)$ a natural, compatible compactification presents itself: let M^*_{Dol} be the complement of the nilpotent cone, i.e the inverse image of $\mathbb{A}^{r-1} - \{0\}$. Then there is an orbifold

compactification $\bar{M}_{Dol} = M_{Dol} \cup (M^*_{Dol}/\mathbb{C}^*) = (M_{Dol} \times \mathbb{A}^1)^*/\mathbb{C}^*$, where $(M_{Dol} \times \mathbb{A}^1)^*$ is the complement if the nilpotent cone in $M_{Dol} \times \{0\}$. This compactification was discussed in [30] following the general method of [6, 7]. The Hitchin fibration extends to a map $\bar{M}_{Dol} \to \mathbb{WP}^{r-1}$ to a *weighted* projective space compactifying $\mathbb{A}^{r-1}$. The weights come from the different weights of $\mathbb{C}^*$ on the different terms in the characteristic polynomial of Higgs fields.

The de Rham moduli space can be compactified by the same procedure: one starts to look at the family of λ-connections $M_{Hod} \to \mathbb{A}^1$ whose fiber over 0 is M_{Dol} and whose fiber over $\lambda \neq 0$ is M_{DR}. Then we can again define M^*_{Hod} to be the complement of the niplotent cone in M_{Dol} and take the orbifold quotient $\bar{M}_{DR} = M^*_{Hod}/\mathbb{C}^*$, which is compact. It fits into a fiberwise compactification $\bar{M}_{Hod}/\mathbb{A}^1$ such that the family of divisors at infinity is well behaved. The divisors at infinity are all the same, i.e. independent of λ : $\bar{M}_{DR} - M_{DR} = M^*_{Dol}/\mathbb{C}^*$. So the structure at infinity is very similar to the structure at infinity of M_{Dol}.

4.2 The Riemann-Hilbert WKB Problem

From the discussion above, limiting points at infinity of M_{DR} correspond to non-nilpotent Higgs bundles and we can easily write down a family of connections going out to infinity: choose a vector bundle $\mathcal{E}$, a Higgs field φ and an initial connection ∇_0. Then we consider the family of connections, depending on a complex parameter $t \in \mathbb{C}$, defined by

$$\nabla_t = \nabla_0 + t\varphi \,. \tag{4.1}$$

If $(\mathcal{E}, \varphi)$ is semistable and not in the nilpotent cone, then the limit point of this family in the divisor at infinity is

$$(\mathcal{E}, \varphi) \in M^*_{Dol}/\mathbb{C}^* \,.$$

The *Riemann-Hilbert WKB problem* is the problem of finding the WKB exponents, Definition 2.24, for a family $(\mathcal{E}_t, \nabla_t)$ of flat holomorphic vector bundles coming from (4.1) (for an in-depth treatment of this question, see [61, 62]). The main new aspect here is to give a geometric interpretation of the WKB exponents in terms of harmonic maps to buildings.

For the sake of simplicity, we restrict ourselves to the situation where $\mathcal{E}_t = \mathcal{E}$ is independent of the parameter t. A more general family will be of the form $(\mathcal{E}_t, \nabla_t)$ defined for t in a disc around infinity. It will have a limiting point $(\mathcal{E}, \varphi)$ if the vector bundles with $\lambda = t^{-1}$ connection converge to $(\mathcal{E}, \varphi)$ in M_{Hod}. Then the resulting map from the disc to $\bar{M}_{DR}$ will be transverse to the divisor at infinity. The family of connections considered in [25] fit into this more general situation: Gaiotto-Moore-Neitzke start with a harmonic bundle $(\mathcal{E}, \partial, \bar{\partial}, \varphi, \varphi^\dagger)$ and consider the holomorphic bundles $\mathcal{E}_t = (\mathcal{E}, \bar{\partial} + t^{-1}\varphi^\dagger)$ which converge to $(\mathcal{E}, \bar{\partial})$ together with the connections $\nabla_t = \partial + t\varphi$. So, as they vary the family of bundles, the connections have as limit the Higgs bundle underlying the harmonic bundle. We expect that this has the same behavior as the complex WKB problem.

4.3 The local WKB approximation

In this section, we state the result for the "local Riemann-Hilbert WKB problem". By this, we mean the problem of determining the growth rates of the parallel transport operators for "sufficiently small" paths. The result is well known as "classical WKB approximation". For an in-depth discussion of this type of problem, see [70]. The proof given here is tailored to our needs.

Suppose we have an $r \times r$ matrix of functions $a_{ij}(x)$ for $x \in (-c, c) \subset \mathbb{R}$. Suppose $t > 0$ is a large real number, and ϵ and C are constants. Furthermore, suppose we have estimates

$$\mathrm{Re}\ a_{11}(x) > \epsilon t + \mathrm{Re}\ a_{ii}(x)$$

for $i = 2, \ldots, r$, and

$$|a_{ij}(x) > Ct^{1/2}$$

for $i \neq j$.

For the moment, assume that $a_{11}(x) = 0\ \forall x \in (-c, c)$. Then the first condition says that Re $a_{ii}(x) < -\epsilon t$. This assumption will be removed later.

Let $f'(x)$ denote the derivative of $f(x)$. Now, consider the ODE

$$f_i'(x) = \sum_{j=1}^{r} a_{ij}(x) f_j(x)\,.$$

Let $F_i(x)$ be the solution with initial conditions $F_1(0) = 1, F_i(0) = 0$ for $i \geq 2$. For $2 \leq i \leq r$, define $G_i(x) := F_i(x)/F_1(x)$. By differentiation, $G_i(x)$ satisfies

$$G_i'(x) = \frac{F_1(x)F_i'(x) - F_1'(x)F_i(x)}{F_i(x)^2}$$

and thus

$$G_i'(x) = a_{i1}(x) + \sum_{j=2}^{r} a_{ij}(x)G_j(x) - a_{1j}(x)G_j(x)G_i(x)\,.$$

Now, put $M(x) := \sum_{i=2}^{r} |G_i(x)|^2$. We have that

$$M'(x) = \text{Re} \sum_{i=2}^{r} G_i'(x)\bar{G}_i(x) =$$

$$= \text{Re} \sum_{i} a_{i1}(x)\bar{G}_i(x) + \sum_{i=2}^{r}\sum_{j=2}^{r} a_{ij}(x)G_j(x)\bar{G}_i(x) - a_{1j}(x)G_j(x)G_i(x)\bar{G}_i(x) =$$

$$= \text{Re} \sum_{i} a_{i1}(x)\bar{G}_i(x) + a_{ii}(x)|G_i(x)|^2 + \text{Re} \left(\sum_{i\neq j} a_{ij}(x)G_j(x)\bar{G}_i(x) - \sum_{i,j} a_{1j}(x)G_j(x)|G_i(x)|^2 \right).$$

Now, we use that Re $a_{ii}(x) < -\epsilon t$ and $|a_{ij}(x)| C t^{1/2}$. Furthermore,

$$|G_j(x)\bar{G}_i(x)| \leq M(x)$$

as well as

$$|G_j(x)G_i(x)^2| M(x)^{3/2} .$$

Thus, after possibly changing the constant, we obtain

$$M'(x) \leq -\epsilon t M(x) + C t^{1/2}(M(x)^{1/2} + M(x) + M(x)^{3/2}) .$$

Assume that x is a point such that the inequality $1/2 \leq M(x) \leq 1$ is satisfied. Then, for t big enough

$$\epsilon t/2 > 3Ct^{1/2}$$

which is equivalent to

$$t > (4C/\epsilon)^2 .$$

Thus, we conclude that

$$\epsilon t M(x) > C t^{1/2}(M(x)^{1/2} + M(x) + M(x)^{3/2}) .$$

Therefore, for such x, $M'(x) < 0$. Since $M(0) = 0$, it follows that $M(x) \leq 1/2$ for all x.
With $M(x) \leq 1/2$ we obtain $M(x)^{3/2} + M(x) \leq 2M(x)^{1/2}$ and the estimates becomes

$$M'(x) \leq -\epsilon t M(x) + C t^{1/2} M(x)^{1/2} .$$

Now, suppose that $\alpha \leq M(x) \leq 2\alpha$. Then

$$-\epsilon t M(x) + C t^{1/2} M(x)^{1/2} < 0$$

if $\epsilon t\alpha > Ct^{1/2}\alpha^{1/2}$ (for some C). Thus, $M(x) \leq \alpha$ for all x. We can solve for α: whenever

$$\alpha > (Ct^{-1/2}/\epsilon)^2$$

or equivalently $\alpha > C_1 t^{-1}$, we get this condittion.
Therefore, we may finish by concluding that

$$M(x) \leq C_1 t^{-1}$$

or

$$|G(x)| \leq C_2 t^{-1/2}$$

for all x. In terms of the F_i, this means

$$|F_i(x)| \leq C_2 t^{-1/2} |F_1(x)|$$

for $2 \leq i \leq r$.

Note that

$$F_1'(x) = \sum_{j\geq 2} a_{1j}(x) F_j(x) ,$$

which implies that we have

$$|F_1'(x)| \leq C_3 |F_1(x)|$$

by our previous estimate and $|a_{1j}(x)| Ct^{1/2}$. From the equation

$$\frac{d}{dx}|F_1(x)|^2 = \mathrm{Re}\ F_1'(x)\bar{F}_1(x)$$

we obtain

$$-C_3|F_1(x)|^2 \leq \frac{d}{dx}|F_1(x)|^2 \leq C_3|F_1(x)|^2 .$$

Setting $h(x) := \log |F_1(x)|^2$ we get

$$-C_3 \leq h'(x) \leq C_3 .$$

Therefore, for a small enough choice of c depending on our constants up to now but independent of t, we have

$$-\ln(2)/2 \leq h(x) \leq \ln(2)/2$$

for all $x \in [0, c]$. This gives

$$1/2 \leq |F_1(x)| \leq 2$$

for $x \in [0, c]$.

Proposition 4.1. *Let $a_{ij}^v(x)$ be a matrix depending on a parameter $v \in V$ as well as $x \in [0,1]$, and suppose that we have a real valued function $v \mapsto t^v$. Furthermore, let ϵ, C, to be constants such that for all $v \in V$ with $t^v \geq t_0$, we have*

$$Re\ a_{ii}^v(x) \leq Re\ a_{11}^v(x) - \epsilon t^v \quad i \geq 2,$$

and

$$|a_{ij}^v(x) \leq C(t^v)^{1/2} \quad i \neq j\ .$$

Let $T^v(x)$ be the fundamental solution matrix for the linear ODE defined by a_{ij}^v. Define

$$\alpha^v(x) := \int_0^x a_{11}(x) dx\ .$$

Then there exists a small constant c (depending on ϵ), and a t_1, such that for any $v \in V$ with $t^v \geq t_1$ and any $x \in [0, c]$ we have

$$\frac{1}{2} e^{\alpha^v(x) t^v} \leq \|T^v(x)\| \leq C_r e^{\alpha^v(x) t^v}$$

and more precisely the same bound also holds for the upper left coefficient $T_{11}^v(x)$. Here C_r is a constant depending on r.

Proof. By multiplying the solution by $e^{-\alpha^v(x)t^v}$ it suffices to treat the case $a_{11}(x) = 0$ which we now assume.
Suppose $f_i(x)$ is a solution with $f_1(0) = 1$ and $|f_i(0)| \leq 1$. Proceed as before, introducing $g_i(x) := f_i(x)/f_1(x)$. We again get

$$g_i'(x) = a_{i1}(x) + \sum_{j=2}^{r} a_{ij}(x) g_j(x) - a_{1j}(x) g_j(x) g_i(x)\ .$$

Again, put $m(x) := \sum_{i=2}^r |g_i(x)|^2$. We have

$$m'(x) = \mathrm{Re}\ \sum_{i=2}^{r} g_i'(x) \bar{g}_i(x) =$$

$$= \mathrm{Re}\ \sum_i a_{i1}(x) \bar{g}_i(x) + a_{ii} |g_i(x)|^w + \mathrm{Re}\ \left(\sum_{i \neq j} a_{ij}(x) g_j(x \bar{g}_i(x) - \sum_{i,j} a_{1j}(x) g_j(x) |g_i(x)|^2 \right).$$

Using as before Re $a_{ii} \leq -\epsilon t$ and $|a_{ij}(x)| \leq Ct^{1/2}$ for $i \neq j$ together with

$$|g_j(x)\bar{g}_i(x)| \leq m(x)$$

and

$$|g_j(x)g_i(x)^2| \leq m(x)^{3/2} ,$$

we get after increasing the constant that

$$m'(x) \leq -\epsilon t m(x) + Ct^{1/2}(m(x)^{1/2} + m(x) + m(x)^{3/2}) .$$

We have $m(0) \leq r$. Consider a point x with $r \leq m(x) \leq 2r$. Then for t big enough, the second term will have smaller size than the first term and we get $m'(x) \leq 0$. Therefore, $m(x) \leq r$ for all x.
Notice, in particular, that if $f_1(x)$ becomes very small then all of the $f_i(x)$ become small. They cannot all go to zero, by considering the corresponding differential equation on the determinant bundle which contains $\det T(x)$ to satisfy a linear equation. This justifies the division by $f_1(x)$ so the $g_i(x)$ are well defined for all x. Our discussion applies to the sum of the first and any other column of the transport matrix. This gives the bound on the sizes of the other columns of the transport matrix, since we have treated the first column previously. We conclude

$$\|T^v(x)\| \leq C_r$$

which gives back the required estimate of the form

$$\|T^v(x)\| C_r e^{\alpha^v(x)t^v}$$

in the case a_{11} is arbitrary. The previous discussion provided the estimate

$$\frac{1}{2}e^{\alpha^v(x)t^v} \leq T^v_{11}(x)$$

for $x \in [0, c]$. This completes the proof. □

4.4 Dilation exponents

Let E be a C^∞ vector bundle over X. Furthermore, let us consider an $\mathbb{R}_+$−family of flat connections ∇_t on E, and suppose we are also given a hermitian metric h_t on each E_t.
We will now apply the general framework of maps from groupoids to asymptotic

cones discussed in Chapter 2 to this geometric situation. Let $\Gamma = \pi_{\leq 1}(X)$ be the fundamental groupoid whose objects are the points of X and whose morphisms are homotopy classes of paths. Let us furthermore choose a basepoint $x_0 \in X$, so that we have a universal covering $\widetilde{X}$.

Definition 4.2. Let X be a complex manifold, and let $\{(\mathcal{E}_t, \nabla_t)\}_{t\in\mathbb{R}}$ be a family of holomorphic vector bundles on X equipped with integrable connections. Fix a family of scale factors $\{\mu_t\}$ and a hermitian metric h_t on $\mathcal{E}_t$.

1. The family $\{(\mathcal{E}_t, \nabla_t)\}_{t\in\mathbb{R}}$ is said to be of *exponential type* with respect to (h, μ) if the associated monodromy representation if of exponential type in the sense of Definition 2.23.

2. Suppose $\{(\mathcal{E}_t, \nabla_t)\}_{t\in\mathbb{R}}$ is of exponential type, and let γ be a path in X. Then the *WKB exponent* (resp. *WKB dilation spectrum*) of γ with respect to the family $\{(\mathcal{E}_t, \nabla_t), h_t, \mu_t\}_{t\in\mathbb{R}}$ is defined to be the exponent (resp. dilation spectrum) of γ with respect to $\{T_t, h, \mu\}_{t\in\mathbb{R}}$ in the sense of Definition 2.24.

3. Given an ultrafilter ω, the *WKB ultrafilter exponent* of this family with respect to ω is the ultrafilter exponent ν^ω of the family of monodromy representations $T : \mathbb{R} \to \mathrm{Rep}(\pi_{\leq 1}(X), \mathrm{SL}_r\mathbb{C})$, defined in Definition 2.25. The ultrafilter WKB dilation spectrum $\vec{\nu}^\omega$ is defined similarly.

Thus, the WKB exponent is given by (1.1) from Chapter 1. The ultrafilter expression is given by the same expression replacing lim sup by the ultrafilter limit. Furthermore, note that in the case of SL_3, the dilation spectrum is uniquely determined by the exponent of γ and its inverse path γ^{-1}.
The main result in this chapter gives the construction of harmonic ϕ-maps to the asymptotic $Cone_\omega$. Let us use the following notations: Let $(\mathcal{E}, \nabla_0 + t\varphi)$ be as in (4.1), let $h_t = h$ be independent of t and let $\phi = \mathrm{char}\varphi$ be the associated point in the Hitchin base, and write $\phi = (\phi_1, \dots, \phi_r)$ locally. If γ is a path on X, consider its lift $\tilde{\gamma}$ to the universal cover $\widetilde{X}$ and denote its endpoints by P and Q respectively. Recall that γ is ϕ-noncritical if the real parts of the differentials $\gamma^*\mathrm{Re}\phi_i$ are distinct for all $t \in [0, 1]$. Furthermore, let ω be an ultrafilter on $\mathbb{R}$ whose support is a countable discrete set with limit $+\infty$ and scale factors $\mu_t = t$.

Theorem 4.3. *With the above notations, we obtain a limiting map*

$$h^\omega : \widetilde{X} \to Cone_\omega$$

where $Cone_\omega$ *is the asymptotic cone of* $Met(\mathcal{E}_P) \simeq SL_r\mathbb{C}/SU_r$ *with respect to* $(\omega, \{\mu_t\})$*. The following properties hold:*

1. *if γ is any path, then for any ultrafilter ω the ultrafilter dilation spectrum is the refined distance in the building:*

$$\vec{\nu}_{\gamma}^{\omega} = \vec{d}_{Cone_{\omega}}(h(P), h(Q)) \, .$$

2. *if γ is a path such that the* $\limsup$ *in the definition of WKB dilation spectrum - for all exterior powers- are actual limits, then for any ultrafilter ω we have $\vec{\nu}_{\gamma}^{\omega} = \vec{\nu}_{\gamma}$ (i.e. the ultrafilter WKB exponent coincides with the WKB exponent). In particular*

$$\vec{\nu}_{\gamma} = \vec{d}_{Cone_{\omega}}(h^{\omega}(P), h^{\omega}(Q)) \, .$$

3. *if γ is a ϕ-noncritical path, then the previous condition holds: for any ultrafilter ω we have $\vec{\nu}_{\gamma}^{\omega} = \vec{\nu}_{\gamma}$.*

4. *the map h^{ω} maps a noncritical path into the Finsler convex hull of its endpoints $\{P, Q\}$, in particular into any apartment containing P and Q.*

5. *for any ultrafilter ω the limiting map h^{ω} is a continuous ϕ-map regular outside of the branch locus of ϕ, in particular it is a harmonic ϕ-map.*

6. *for an arbitrary path γ and any k, there is some choice of ultrafilter ω, which might however depend on γ and k, such that $(\vec{\nu}_{\gamma}^{\omega})_1 + \cdots + (\vec{\nu}_{\gamma}^{\omega})_k = (\vec{\nu}_{\gamma})_1 + \cdots + (\vec{\nu}_{\gamma})_k$. The terms are the highest components of the vectors in order.*

Proof. Note that the vector bundle $(\mathcal{E}, \nabla_t)$ determines a family of representations $\rho(t) : \Gamma \to \mathrm{Vect}_{\mathbb{C}}$. The family of metrics h_t gives us a family of metrics on the representations ρ_t. Using ∇_t to transport back to a basepoint P, one may view h_t as a family of maps

$$h_t : \widetilde{X} \to \mathrm{Met}(\mathcal{E}_P)$$

for $t \in \mathbb{R}$, sending a homotopy class of paths γ to the metric $T_{\gamma}(t)_{*}(h_{\gamma(0)}$. Choosing a basis of $\mathcal{E}_P$ gives an identification $\mathrm{Met}(\mathcal{E}_P) \simeq \mathrm{SL}_r\mathbb{C}/SU_r$. Applying Definition 2.28, we obtain the asymptotic limiting map to a building

$$h_{\omega} : \widetilde{X} \to \mathrm{Cone}_{\omega} \, .$$

Corollary 2.29 gives the first property: $\vec{\nu}_{\gamma}^{\omega} = \vec{d}_{\mathrm{Cone}_{\omega}}(h(P), h(Q))$. The second property is clear: if a limit exists, then it is equal to the ultrafilter limit for any ultrafilter.

By the local WKB approximation, for a sufficiently short noncritical path γ, the lim sup's in the definition of the WKB dilation spectrum are limits, thus $\vec{\nu}_\gamma = \vec{\nu}^\omega_\gamma$ and these are the vector distances between image points under h^ω. In the notation of the lemma, this may be written as

$$\vec{d}_{\mathrm{Cone}_\omega}(h^\omega(\gamma(0)), h^\omega(\gamma(1)) = (\alpha_1, \ldots, \alpha_r)\,.$$

Therefore, the image of $\tilde{\gamma}$ by h^ω goes into a single apartment. This just follows from a fact about buildings: if x, y, z are three points with

$$\vec{d}(x, y) + \vec{d}(y, z) = \vec{d}(x, z)$$

then x, y, z are in a common apartment, with x and z in opposite chambers centered at y or equivalently, y in the Finsler convex hull of $\{x, z\}$.
We now show that if $\gamma : [0, 1] \to \widetilde{X}_{reg}$ is any (not necessarily short) noncritical path, then $h^\omega \circ \gamma$ maps $[0, 1]$ into a single apartment, and the vector distance which determines the location in this apartment is given by the integrals:

$$\vec{d}_{\mathrm{Cone}_\omega}(h^\omega(\gamma(0)), h^\omega(\gamma(1)) = (\alpha_1, \ldots, \alpha_r)\,.$$

Our path is covered by open intervals each of which go into a single apartment (as a noncritical path in that apartment). Choose a sequence of points $0 = t_1, \ldots, t_n = 1$, so that each triple is in a single neighborhood. Then we show that the path from t_1 to t_i is in a single apartment. Suppose we have done it up to t_{i-1}. Then there is also an apartment A' containing $\gamma([t_{i-2}, t_i])$. Furthermore, the path here is noncritical. Now, take the sector S in A which is based at $x = \gamma(t_{i-1})$ and contains the path up to there. Then, take the sector T in A' which is based at $x = \gamma(t_{i-1})$ and contains the segment $\gamma([t_{i-1}, t_i)$. The claim is that these two sectors have germs at x which are opposite. Therefore, $S \cup T$ is in a single apartment A'', and now this one contains $\gamma([t_1, t_i])$ completing the induction step.
We now show that h^ω is regular outside of the branch locus of ϕ, continuous, and its differential is Re ϕ. For regularity, argue as in Lemma 3.17. Let R be a point that is not on the ramification divisor. Then one can find points P and Q and non-critical paths γ and γ' as in Lemma 3.17. We can furthermore arrange a homotopy through non-critical paths from γ to γ' that sweeps out a disc containing R. Now arguing as in the proof of Theorem 3.16, we see that D is mapped into a single apartment. Then, the map on this disc, into a single apartment, is given by the integrals of the real parts of the 1-forms, because the integrals calculate the vector distance. In particular, our map is a ϕ-map.
It follows, in particular, that h^ω is continuous outside of the ramification points. Note however that h^ω is Lipschitz. Indeed, the WKB exponent satisfies $\nu_\gamma \leq C|\gamma|$,

just from a basic estimate of ODE's. The ultrafilter exponent is necessarily smaller than the WKB exponent since the latter is a lim sup. The exponent can serve as a Finsler metric on Cone_ω, so we get the Lipschitz property. That implies that h^ω is continuous even at the ramification points. Now since h^ω is continuous, regular outside a discrete set, with harmonic differential, it is a harmonic map [48].
The last statement of the theorem is obtained by choosing a sequence $t_i \to \infty$ realizing the lim sup in the definition of the exponent for the k-th exterior power, and letting ω be an ultrafilter supported on this sequence. This completes the proof of Theorem 4.3. □

Definition 4.4. Fix an ultrafilter ω on $\mathbb{R}$ whose support is a countable set, and a family of scale factors $\{\mu_t\}$.

1. Let $\rho : \mathbb{R} \to \mathrm{Rep}(\Gamma, r)$ be a family of representations, and let γ be a morphism in Γ. The *ultrafilter exponent* of γ, ν_γ^ω is given by

$$\nu_\gamma^\omega = \lim_\omega \frac{1}{\mu_t} \log\|\rho_\gamma(t)\| \, .$$

2. Let X be a Riemann surface, and let $(\mathcal{E}, \nabla_0 + t\varphi)$ be a family of integrable connections on a fixed holomorphic vector bundle $\mathcal{E}$ (with trivialized determinant connection). The *WKB ultrafilter exponent* of this family with respect to the ultrafilter ω is the ultrafilter exponent of the family of monodromy representations $T : \mathbb{R} \to \mathrm{Rep}(\pi_{\leq 1}(X), SL_r\mathbb{C})$. One can define the ultrafilter WKB dilation spectrum $\vec{\nu}^\omega$ similarly.

The situation described above was described in [58]: the ultrafilter exponent of a family of representations $\rho : \mathbb{R} \to \mathrm{Rep}(\Gamma, r)$ is computed by the vector distance in the asymptotic cone $\vec{d}_{\mathrm{Cone}_\omega}([\rho_\gamma(t)]_*[h_x(t)], [h_y(t)])$. We are mainly intereseted in the geometric situation in which the family of representations actually comes from a family of integrable connections tending to infinity. The main result, then, is that the translation vector can be interpreted as the distance between two points under a harmonic map.
So, let $(\mathcal{E}, \varphi)$ be a Higgs bundle of rank r on a Riemann surface X, let ∇_0 be an integrable connection on $\mathcal{E}$, let h be a hermitian metric on $\mathcal{E}$ and fix a base point $P \in X$. Then we can identify the universal cover $\widetilde{X}$ with homotopy classes of paths ending at $P \in X$, and we obtain a family of maps

$$h_t : \widetilde{X} \to \mathrm{Met}(\mathcal{E}_P) \, ,$$

for all $t \in \mathbb{R}$, sending γ to $T_\gamma(t)_*(h_{\gamma(0)})$. If we choose a basis for $\mathcal{E}_P$, we get an identification $\mathrm{Met}(\mathcal{E}_P) \simeq \mathrm{SL}_r\mathbb{C}/SU_r$.

The asymptotic cone of 4.3 is a very complicated object and it has to be: it computes WKB exponents for all possible WKB problems. Thus, the theorem above gives a characteization of the WKB exponents in terms of the geometry of a building, but it does not take into account the specific geometric nature of the problem (e.g. the spectral cover $\pi : \Sigma \to X$). The (uni)versal building, on the other hand, is constructed purely in terms of the spectral cover and uses, in particular, the spectral network as a key ingredient.

Remark 4.5. If a (uni)versal ϕ-building exists, there there exist a (unique) folding map of buildings $g : \mathcal{B}^\phi \to \mathrm{Cone}_\omega$ that makes the following diagram commute:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "h^\phi"] \arrow[dr, "h^\omega"] & \mathcal{B}^\phi \arrow[d, "\psi"] \\ & \mathrm{Cone}_\omega \end{tikzcd}$$

In particular, this implies that the distance in the (uni)versal building $\mathcal{B}^\phi$ computes the WKB exponent: for two arbitrary points $P, Q \in \widetilde{X}$, choose an ultrafilter ω_{PQ} that computes the WKB exponent. Then, the commutativity of the diagram above implies the statement.

Chapter 5

The BNR Example

Portions of this chapter have appeared in [40] and have been submitted for publication to "Communications in Mathematical Physics".

In the paper [5], Berk, Nevin and Roberts studied the Stokes phenomena for a certain third order differential equation. Their main discovery was that so-called "new Stokes lines" have to be introduced in order to understand the Stokes phenomena for higher order ODE's. These new Stokes lines arise whenever two (ordinary or new) Stokes lines intersect and are needed to obtain consistent Stokes matrices. Subsequently, the new Stokes lines have been investigated by several authors and nowadays are called "virtual Stokes lines" [2, 34]. In the physics literature [26] they are called "collision lines".

Let $X = \mathbb{C}$ with coordinate x and let p be the canoncial coordinate on the fiber of the total space of the cotangent bundle $\mathrm{tot}(T^*_X \to X) \simeq \mathbb{C}^2$. The spectral cover (or *characteristic curve*) for the differential equation in [5] is the affine variety

$$\Sigma = \{(x,p) \in \mathbb{C}^2 | p^3 - 3p + x = 0\} \subset \mathbb{C}^2\,.$$

The imaginary spectral network associated to Σ is shown in Figure 5.1. Note the following features:

- There are two collision points, which, in fact, are connected via a backward spectral network line (which is not shown in the figure)
- The spectral network curves divide $X = \mathbb{C}$ into 10 regions:

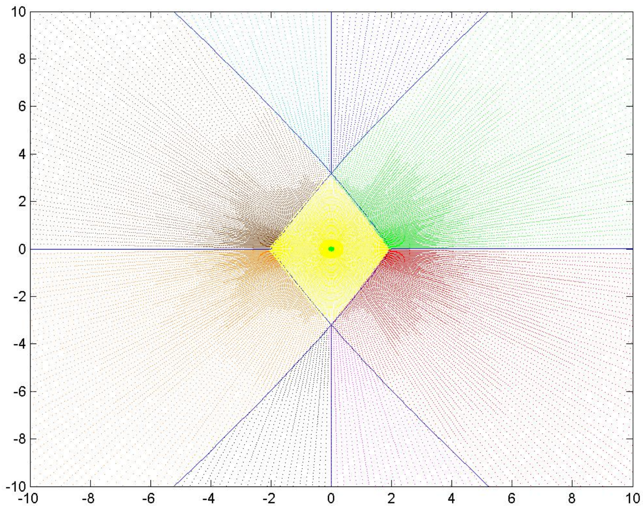

FIGURE 5.1: Spectral Network

4 regions on the outside of the right of the collision line;

4 regions on the outside of the left of the collision line;

2 regions in the square whose vertices are the singularities and the collisions; the two regions are separated by a backward line

The goal of this chapter is to construct the universal building $\mathcal{B}^\phi$, Definition 3.9, for this spectral cover. This, in turn, will prove that the WKB dilation spectrum for any generic Riemann-Hilbert WKB problem with Σ as spectral curve is obtained as the vector distance in $\mathcal{B}^\phi$. Also, the spectral network, Definition 3.20, can be seen as the preimage of the singularities of $h^\phi : X \to \mathcal{B}^\phi$.

The differential equation associated to Σ is obtained by quantization: replace the coordinate along the fiber in the cotangent direction by the differential operator d/dx. In the example at hand, this gives the third order ordinary differential equation (in general, this is not unique, since d/dx and multiplication by x do not commute; in the case at hand, this is no issue)

$$\left(\frac{1}{t^3}\frac{d^3}{dx^3} - \frac{3}{t}\frac{d}{dx} + x\right) f = P_t f = 0 \, .$$

In order to put this equation into the formalism of (4.1), one makes the following redefinitions: let $g = f'$, $h = g'$ and $s \in H^0(X, \mathcal{O}_X^3)$ be given by $s = (f, g, h)$. Then, it is easy to check that the differential equation $P_t f = 0$ is equivalent to the system of first order ordinary differential equations given by $\nabla_t s = 0$ with

$$\nabla_t = d - t \begin{pmatrix} 0 & dx & 0 \\ 0 & 0 & dx \\ -xdx & 3dx & 0 \end{pmatrix},$$

a connection on $\mathcal{O}_X^3$.

In order to faciliate the following lenghty argument, a short outline is presented here:

1. Identify so-called "maximal abelian regions" (MARs): these are maximal subsets of X on which the local WKB approxiamation holds. Heuristic methods (motivated by [26]) are used to construct these regions. For each MAR $M_i \subset X$, there is an associated polytope P_i together with a map $M_i \to P_i$. This polytope is the convex hull of the image of the MAR under a harmonic ϕ-map to $\mathbb{A}$. Two such apartment P_i, P_j will be glued along the image of $M_i \cap M_j$. The resulting pre-building $\mathcal{B}^\phi_{pre}$ thus canonically comes equipped with a map $h^{pre} : X \to \mathcal{B}^\phi_{pre}$. It turns out that the 0-dimensional stratum of the singular locues of $\mathcal{B}^\phi_{pre}$ consists of a single vertex $\{o\}$.

2. A building $\mathcal{B}^\phi$ together with an isometry $i : \mathcal{B}^\phi_{pre} \to \mathcal{B}^\phi$ is constructed. This building is a cone over a spherical building. This spherical building, in turn, is constructed as the "free spherical building" generated by the link of $\{o\}$ in the pre-building.

3. In the third step, it is shown that $i : \mathcal{B}^\phi_{pre}$ satisfies the following universality property: let $\mathcal{B}$ be a building and let $j : \mathcal{B}^\phi_{pre} \hookrightarrow \mathcal{B}$ be an isometric embedding. Then there exists a unique folding map $\psi : \mathcal{B}^\phi \to \mathcal{B}$ with $\psi \circ i = j$.

4. In the final step, it is shown that for every ϕ-map $h : X \to \mathcal{B}$ there is a unique isometric embedding $j : \mathcal{B}^\phi_{pre} \to \mathcal{B}$ with $j \circ h^{pre} = h$. This, in combination with the previous steps, shows immediately that $\mathcal{B}^\phi$ is the universal ϕ-building and that the WKB expoenents are computed by $h^\phi : X \to \mathcal{B}^\phi$.

The key ingredients to Step 4 are showing, using Theorem 3.16, that each connected component of the complement of the spectral network in Figure 5.1 maps to a single apartment under any harmonic ϕ-map h. Then, general axioms of buildings allows one to argue that each MAR, which is a union these regions, is mapped into a single apartment. An important role is also played by the fact that both collision points are mapped to a single point under any harmonic ϕ-map. The interior square consisting of the yellow regions in Figure 5.1 behaves in a special way: under any harmonic ϕ-map it maps into a single apartment, with a fold line along the "caustic" joining the two branch points. This phenomenon will also be seen in the next chapter, where we present a more complicated example.

5.1 Maximal abelian regions and the pre-building

In this section, the first step of the outline above is carried out, namely the pre-building $\mathcal{B}^{\phi}_{pre}$ will be constructed using heuristic methods. We denote by $W_{aff} = W \ltimes \mathbb{R}^2$ the affine Weyl group with spherical part $W = S_3$ and by $(\mathbb{A}, W_{aff})$ the standard appartment.

In figures 5.6-5.15 we see the gluing construction: the left hand side represents a copy of $X = \mathbb{C}$, while the right hand side represents a copy of the standard apartment $\mathbb{A}$. Furthermore, the colored regions are maximal abelian region (MARs); the different colors correspond to the different connected components of the complement of the spectral network.

Now, let us fix an MAR, say MAR1, Figure 5.6, and a local trivialization of the cameral cover over this MAR. Then, we can pull-back the three tautological one-forms from the cameral cover to the MAR to obtain three one-forms ϕ_1, ϕ_2, ϕ_3. After the choice of a basepoint P in the MAR, we obtain a map ψ_P taking values in $\mathbb{R}^2$ defined by

$$Q \mapsto \left(\int_P^Q \text{Re } \phi_1, \int_P^Q \text{Re } \phi_2\right) .$$

The colored regions on the right hand side are the images of the corresponding regions on the left hand side under this map.

Note that the map above actually suffices to recover the more natural map

$$Q \mapsto \left(\int_P^Q \text{Re } \phi_1, \int_P^Q \text{Re } \phi_2, \int_P^Q \text{Re}\phi_3\right)$$

due to the relation $\sum_i \phi_i = 0$.

As described in the outline, we "glue in a convex polytope" for each of the ten MARs and identify points coming from the same point on X:

$$\mathcal{B}^{\phi}_{pre} = \left(\bigsqcup_{i=1}^{10} P_i\right) / \sim$$

where $x \sim y$ if $x = h_i(P)$, $y = h_j(P)$ where h_i, h_j are the maps corresponding to the MARs i and j, respectively.
More formally, the construction can be described as follows:

Definition 5.1. Let $\pi_\phi : \Sigma_\phi \to X$ be a cameral cover of a smooth algebraic curve over $\mathbb{C}$ with Weyl group S_r. A subset $U \subset X$ is ϕ-adapted if it is simply-connected and if Σ_ϕ decomposes over U, i.e.

$$\Sigma_\phi \times_U X \simeq W \times U$$

where W is the Weyl group. A locally finite cover $\{U_\alpha\}_\alpha$ of X is ϕ-adapted if all pair-wise intersections of elements of the cover are ϕ-adapted, and the interiors of the U_α cover the complement of the ramification locus of ϕ.

Let $\mathfrak{U}_\phi$ be the category whose objects are disjoint unions of ϕ-adapted sets in X, and whose morphisms are inclusions. Note that the notion of ϕ-adaptedness is not sufficiently fine to insure that a subset maps into a single apartment (e.g. a neighborhood of a branch point minus a branch cut is ϕ-adapted, but does not map into a single apartment). Therefore, we suppose that we have also chosen a sieve, that is to say a subcategory $\mathfrak{U}_\phi^{\mathrm{ab}} \subset \mathfrak{U}_\phi$ closed under taking subsets. We want to keep "abelian regions" in mind, that is regions that are mapped into a single apartment by any ϕ-map. At the moment, we do not have a general definition. In this example, however, we will apply the criteria developed in Chaper 3 to prove that the MARs mentioned above are abelian. Let $\mathfrak{U}_\phi^{\mathrm{ab}}$ be the sieve associated to this covering.

Construction 5.2. *Let $\pi_\phi : \Sigma_\phi \to X$ be a cameral cover of a smooth algebraic curve over $\mathbb{C}$. Let $\mathfrak{U}_\phi^{ab}$ be our sieve in the category whose objects are disjoint unions of ϕ-adapted sets in X, and whose morphisms are inclusions. We are going to define a coproduct preserving functior $\mathcal{T}$ on $\mathfrak{U}_\phi^{ab}$ taking values in the category of generalized chamber systems. On objects, $\mathcal{T}$ is defined by the formula*

$$\mathcal{T}(U) = \left(\bigsqcup_{p \in \pi_\phi^{-1}(U)} \{p\} \times hull(\psi_P(U)) \right) / \sim$$

where $\sim$ is given by two types of relations:

1. *$(p, v) \sim (wp, w^{-1}v)$ for all v, and for all $p \in \pi_\phi^{-1}(U)$, and all $w \in W$.*
2. *$(p, v) \sim (q, \psi_{\pi_\phi(q)}(q) + v)$ whenever p and q are in the same connected component of $\pi_\phi^{-1}(U)$.*

We can define a pre-building $\mathcal{B}^{\phi}_{pre} = \mathcal{B}^{\phi}_{pre}(\mathfrak{U}^{ab}_{\phi})$ *associated to this sieve (i.e. open cover) in the following way: let* X *be the union of the subsets in the* ϕ*-adapted cover. As a chamber system,* $\mathcal{B}^{\phi}_{pre}$ *is the coequalizer of the natural diagram:*

$$\mathcal{T}(V \times_X V) \rightrightarrows \mathcal{T}(V)\,.$$

The cubicles in $\mathcal{B}^{\phi}_{pre}$ *are given by the images of the sets* $\{p\} \times hull(\psi_P(U))$ *under the natural projection* $\mathcal{T}(V) \to \mathcal{B}^{\phi}_{pre}$.

The following proposition is a straighforward consequence of the construction:

Proposition 5.3. *Let* $\{U_\alpha\}_\alpha$ *be a* ϕ*-adapted cover of* X*, and suppose that for each* α *there exists an apartment* A_α *such that* $h(U_\alpha) \subset A_\alpha$*. Suppose that the sieve* $\mathfrak{U}^{ab}_{\phi}$ *is finer than this covering. Then there exists a unique isometry of pre-buildings that makes the following diagram commute:*

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "h"] & \mathcal{B}^{\phi}_{pre}(\mathfrak{U}^{ab}_{\phi}) \arrow[d] \\ & B \end{tikzcd}$$

We obtain a pre-building that can be completed to a universal ϕ-building by applying this construction to the ϕ-adapted cover consisting of MAR1-MAR10.

Definition 5.4. The BNR-pre-building $\mathcal{B}^{\phi}_{\text{pre}}$ and the map $h^{\text{pre}} : X \to \mathcal{B}^{\phi}_{\text{pre}}$ are obtained by applying Construction 5.2 to the admissible cover $\mathfrak{U}$ consisting of the 10 regions $\{MAR1, \ldots, MAR10\}$ shown in Figures 5.6-5.15.

The goal of the remaining part of this section is to explain the heuristic methods used to obtain the 10 regions MAR1-MAR10, the basis of which are WKB considerations.

Let $\gamma : [0,1] \to X$ be a path with $\gamma(0) = P, \gamma(1) = Q$ and let $\mathcal{W}_{\pi/2}$ the imaginary spectral network. Let $t_k \in [0,1]$ be such that $\gamma(t_k) \in \mathcal{W}_{\pi/2}$. Then, one can define a corresponding *detour integral*

$$D_k = \mathrm{Re}\left(\int_P^{\gamma(t_k)} \phi_1 + \int_{\gamma(t_k)}^{Q} \phi_2\right)\,.$$

Here, $\phi_1(\gamma(t_k)) = \phi(\gamma'(1))$, $\phi_2(\gamma(t_k)) = \phi(\gamma'(0))$ with γ' the path from Definition 3.20.

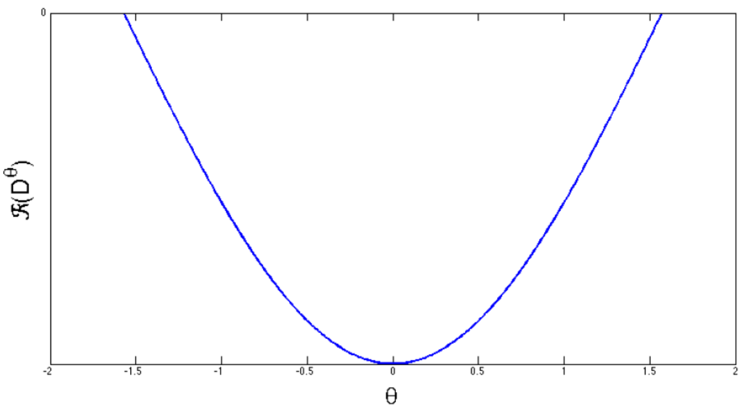

FIGURE 5.2: Detour integral

A detour D_k is said to *dominate along* γ if Re $(D_k) \geq$ Re $I_i =$ Re $\int_\gamma \phi_i$, $i = 1, 2$. Then, we say that two points P and Q are not *simply WKB related by* γ if there is a detour D_k that dominates along γ. Otherwise, P and Q are said to be *simply WKB-related* by γ.
Let $U \subset X$ be connected. Then U is called an *abelian region* if for any path γ whose image is contained in U, the endpoints are simply WKB-related by γ. If M is maximal (with respect to inclusion) and abelian, it is called a *maximal abelian region* (or MAR for short).

Remark 5.5. These detours are related to the ones defined in [26]: note that the detour integral D_k can be rewritten as

$$D_k = \mathrm{Re}\left(\int_P^{\gamma(t_k)} \phi_1 + \int_{\gamma'} \phi + \int_{\gamma(t_k)}^{Q} \phi_2\right) \tag{5.1}$$

with γ' the path from Definition 3.20. The equation 5.1, however, makes sense for spectral networks of all angels θ allowing us to define D_k^θ which is sketched in Figure 5.2. Note, however, that spectral networks that differ by an angle of π are equal (up to orientation). Thus, we have

Heuristic Proposition 5.6. *Let $\gamma : [0, 1] \to X$ be a path. Then no detour dominates along γ if and only if γ is homotopy equivalent to a path that does not intersect two lines $s_1 \neq s_2 \subset \mathcal{W}_{\pi/2}$ such that s_1 is rotated into s_2 as θ goes from $\pi/2$ to $-\pi/2$.*

Remark 5.7. One has to to be careful when a spectral network line sweeps over a branch point, see Figure 5.5.

We can read off an immediate consequence:

Heuristic Corollary 5.8. *Let M be an MAR. Then M is a union of connected components of the complement of the imaginary spectral network.*

Example 5.1. *This example illustrates how MARs look like near branch points. To this end, consider Figure 5.3. The following convention for spectral network lines is used: a line that is labeled (ij), with one letter on each side of the line, means that Re $\int_{P'}^{Q'}(\phi_i - \phi_j) > 0$ if P' lies on the side containing i and Q' is on the side containing j. The branch cut is represented by the curly line.*
In the case of Figure 5.3, P and Q are not simply WKB related by the red path: the detour integral is computed as

$$Re \int_P^R \phi_1 + Re \int_R^Q \phi_2 = Re \int_P^S \phi_1 + Re \int_S^Q \phi_2 \,, \tag{5.2}$$

where the equality is derived from Re $\int_R^S \phi_1 =$ Re $\int_R^S \phi_2$, which, in turn, is a direct consequence of the definition of spectral networks.
In order to determine if the detour domintaes, we have to compare it with both Re $\int_P^Q \phi_{1,2}$:

$$Re \left(\int_P^R \phi_1 + \int_R^Q \phi_2 \right) - Re \int_P^Q \phi_1 = Re \int_S^Q (\phi_2 - \phi_1) > 0 \,.$$

Also,

$$Re \left(\int_P^R \phi_1 + \int_R^Q \phi_2 \right) - Re \int_P^S \phi_2 = Re \int_P^R (\phi_1 - \phi_2) > 0 \,.$$

Every pair of points on the red path lying between R and S (including R and S themselves), on the other hand, are simply WKB-related by the corresponding piece of the red path.

The next example is relevant for the BNR example.

Example 5.2. *Consider Figure 5.4: again, we want to compare the detour $D = \int_P^R \phi_1 + \int_R^Q \phi_2$ with $I = \int_P^Q \phi_1$. Note that*

$$Re \int_R^S (\phi_1 - \phi_2) = 0 \,.$$

This follows from the fact the red contour can be deformed to the green contour. Then, the differential form $\phi_{12} :=$ Re $(\phi_1 - \phi_2)$ vanishes along I, IV and V, whereas the contributions to the integral coming from II and III cancel each other. Then,

$$Re\ (D - I) = \int_R^Q \phi_{12} = \int_S^Q \phi_{12} > 0 \,.$$

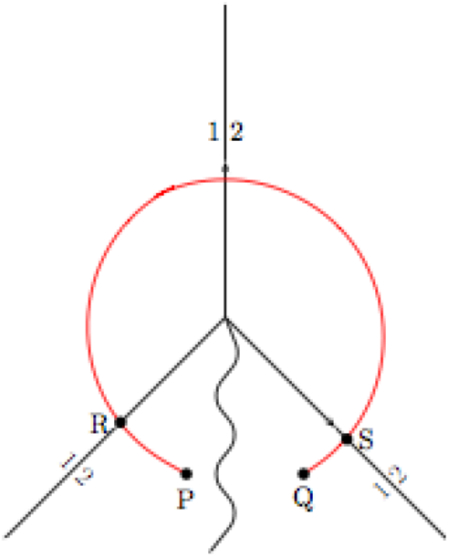

FIGURE 5.3: WKB exponents via spectral networks

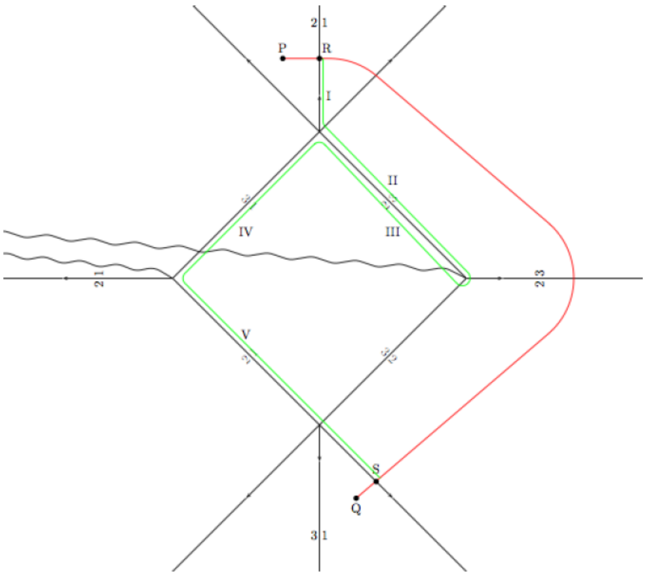

FIGURE 5.4: Maximal Abelian Regions from spectral networks

Analogous computation can be repeated for the other detours.
The final conclusion, then, is that P and Q are not simply WKB related by the red path (later, we will see that they are, in fact, not simply WKB related at all).

This can also be shown using the Heuristic Proposition (see Figure 5.5:).

By doing similar computations, we arrive at the MARs shown in Figure 5.6-5.15. Now, we can apply the gluing construction to arrive at a pre-building $\mathcal{B}^{\phi}_{pre}$.

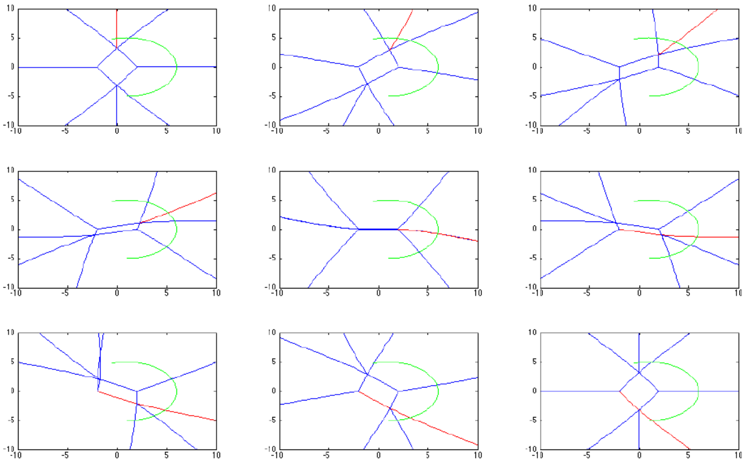

FIGURE 5.5: Spectral Network for various angles θ

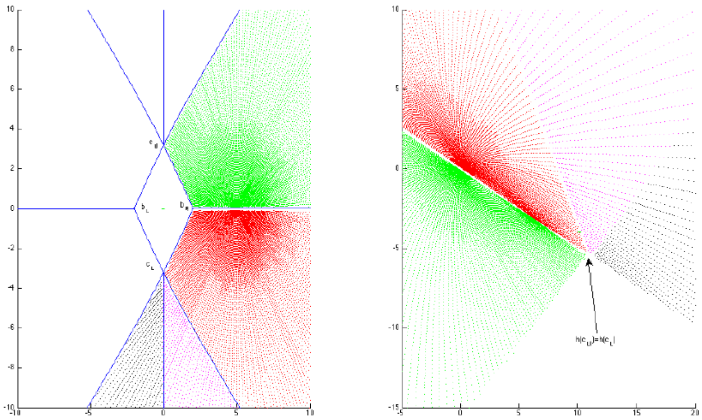

FIGURE 5.6: MAR1

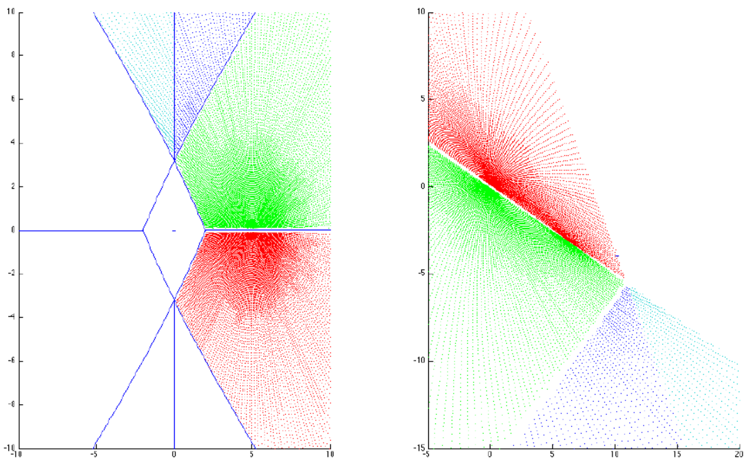

FIGURE 5.7: MAR2

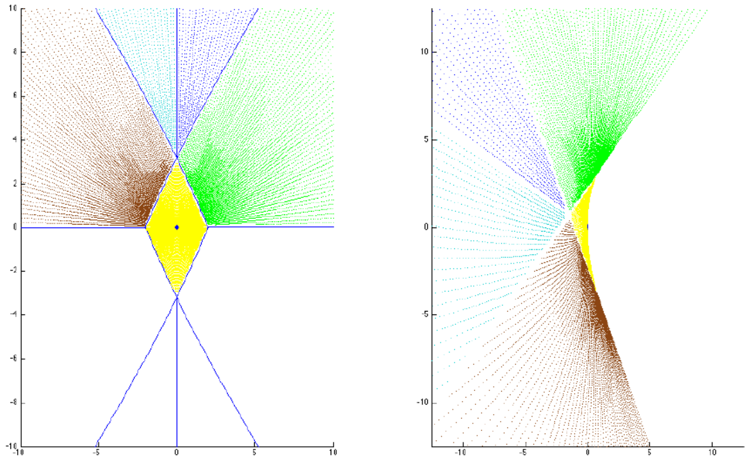

Figure 5.8: MAR3

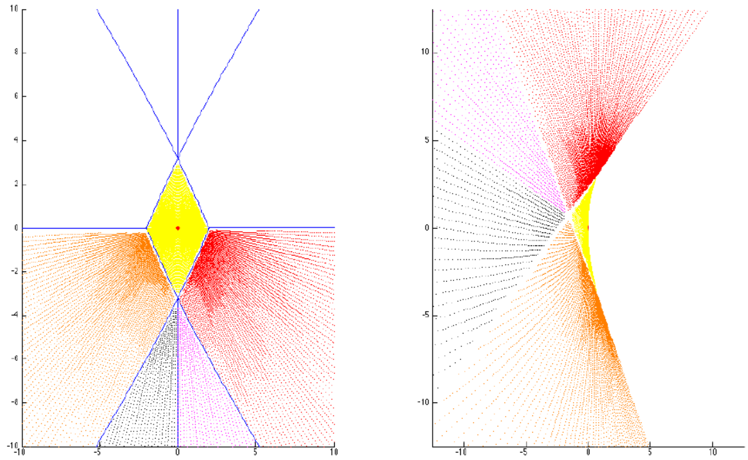

Figure 5.9: MAR4

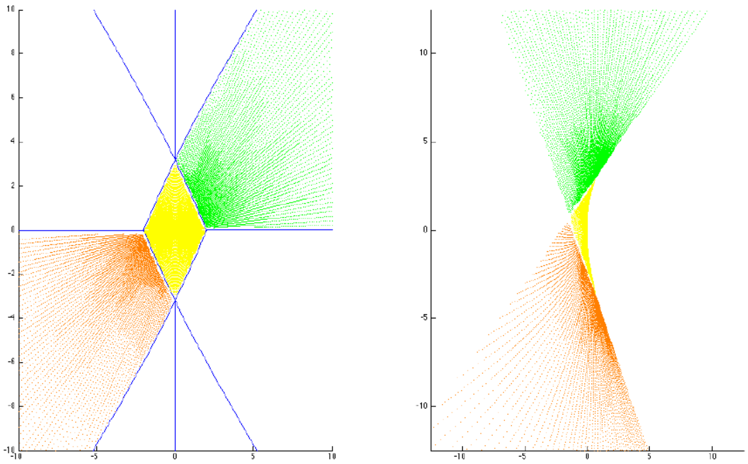

Figure 5.10: MAR5

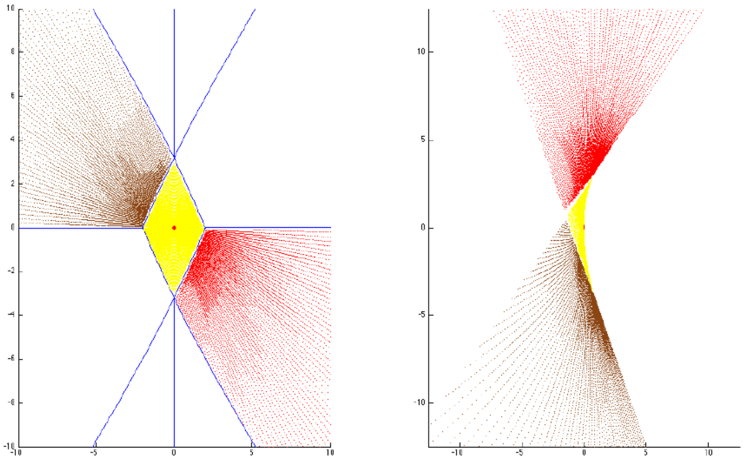

Figure 5.11: MAR6

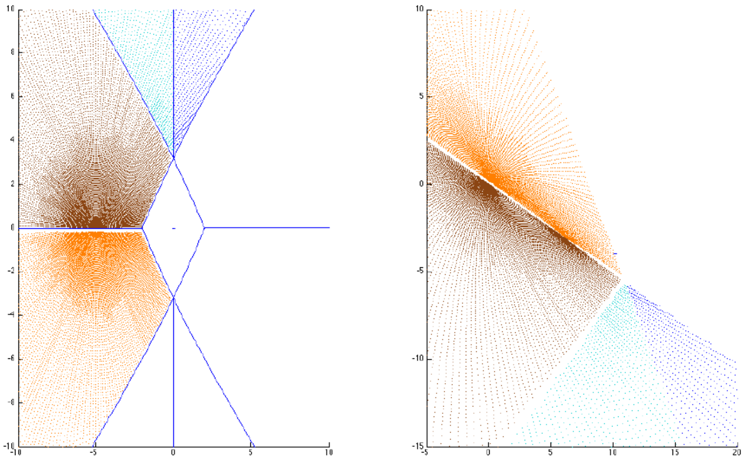

Figure 5.12: MAR7

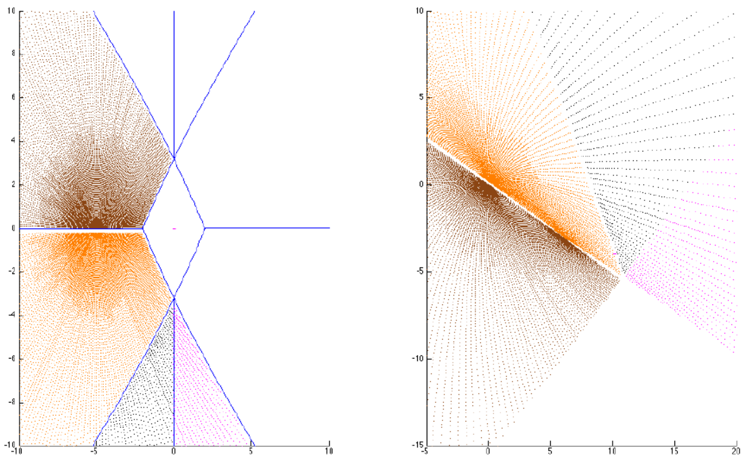

Figure 5.13: MAR8

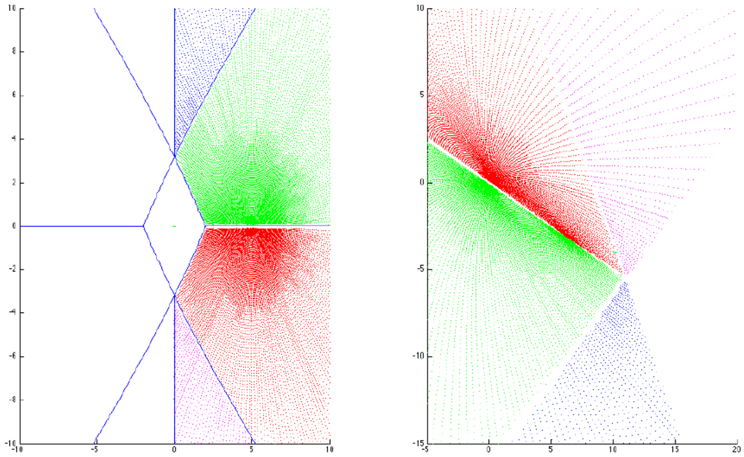

FIGURE 5.14: MAR9

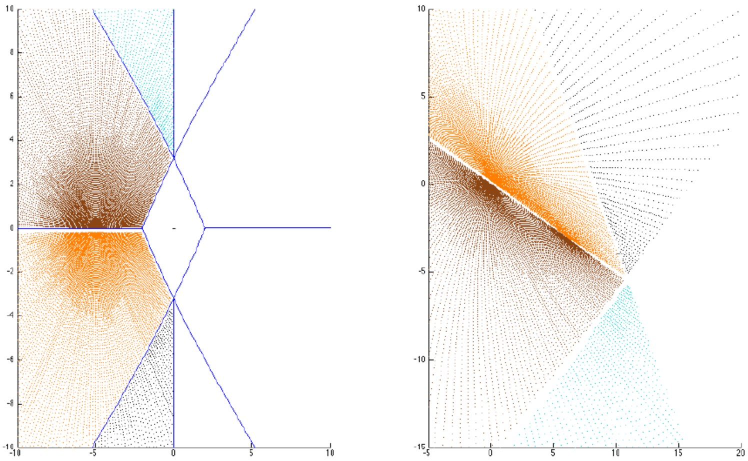

FIGURE 5.15: MAR10

5.2 Constructing the universal building

The goal of this section is to complete the pre-building $\mathcal{B}^{\phi}_{pre}$ constructed in the previous section to a building. We start with some technical lemmas:

Lemma 5.9. *Let A_+ be a closed half apartment in B bounded by a wall H, and let S be a sector with vertex x such that $S \cap H = P$ is a panel of S (in particular $x \in H$), and such that the germ $\Delta_x S$ is not contained in A_+. Then S is opposite to some sector S^- of A_+.*

Proof. Let A be an apartment containing A_+. Consider the link G of B at x. This is a spherical building. Then (keeping only A and S), the situtation is as in Figure 5.16, where $S^- \subset A_+$. From this it is clear that S has to be opposite to S^-; otherwise the girth would be less than six.

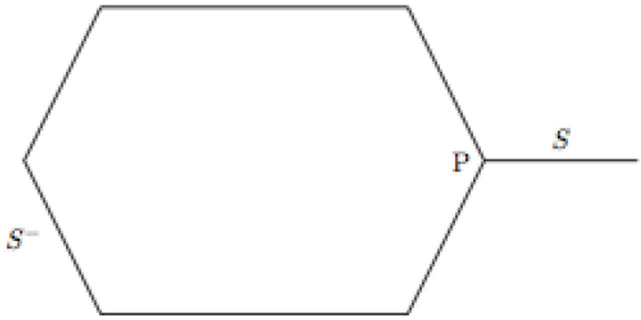

FIGURE 5.16: Part of link near x

□

Lemma 5.10. *Let A_+, H, P and S be as in Lemma 5.9. Then there exists an apartment containing $A_+ \cup S$.*

Proof. Let A be an apartment containing A_+ as a half-apartment. By Lemma 5.10, we can use (1) from Proposition 2.10 to conclude that S and S^- are contained in a common apartment A'. Then $P \cup S^- \subset A \cap A'$, thus $A_+ \subset A \cap A'$, by convexity. □

Proposition 5.11. *In the A_2 case, suppose S_1, S_2, S_3, S_4 are sectors based at a single point x, such that S_i and S_{i+1} share a common face for $i = 1, 2, 3$. Suppose that these successive common faces are distinct. Then S_1, S_2, S_3, S_4 are contained in a common apartment.*

Proof. We show this by induction on i. For $i = 1$ it is easy. Suppose $i \leq 4$ and we have shown it up to $i - 1$, that is to say we have an apartment A' containing $S_1, \ldots, S_{i-1}$. These sectors satisfy the same adjacency condition within A' from which it follows that they are successive sectors arranged around the vertex x. Now, $R := S_i \cap S_{i-1}$ is the face of S_{i-1} which is different from $S_{i-2} \cap S_{i-1}$. Let H be the half-apartment of A' whose boundary contains R, and which contains S_{i-1}. Then, H contains $S_1, \ldots, S_{i-1}$. Indeed, if some previous S_j were not in H then its boundary would have to contain R. (Here is where we use $i - 1 \leq 3$, to say that a previous S_j cannot leave H along the other ray in ∂H.) Now apply the previous lemma: we get an aprtment A containing $H \cup S_i$, so A contains $S_1, \ldots, S_i$. This completes the inductive step. □

Now, the goal is to construct the universal building from $\mathcal{B}^{\phi}_{pre}$. Note that this has a special vertex $\{o\}$ which is the image of *both* collision points (it is the tip of the

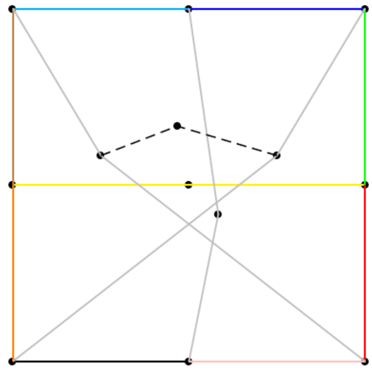

FIGURE 5.17: Link of $\mathcal{B}^{\phi}_{pre}$ at $\{o\}$

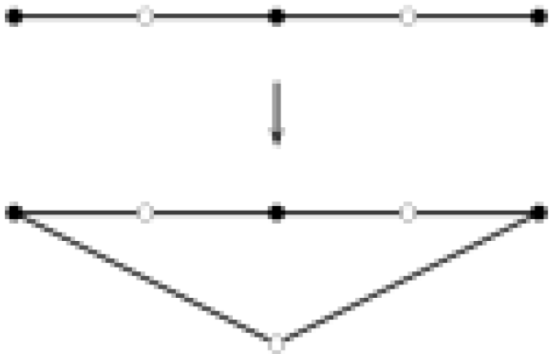

FIGURE 5.18: Completion of 4 chambers to a hexagon

cones in Figure 5.6-5.15). The link G of $\mathcal{B}^{\phi}_{pre}$ at this vertex $\{o\}$ are the full lines in Figure 5.17. Note that this is not a spherical building (e.g. the some grey lines are not contained in a common apartment). Thus, we make the following construction:

Consider G as a bipartite graph (i.e. color the vertices black and white in an alternating way). Then, define a new bipartite graph inductively as follows: take $G_0 = G$, for every sequence of 4 chambers which is not contained in a hexagon, adjoin two new edges (see Figure 5.18 for the general idea and the dashed lines in Figure 5.17 for the first step of the construction in the case at hand). Then, define

$$\mathcal{B}^8 = \bigcup_{n=0}^{\infty} G_n \,.$$

Proposition 5.12. *There is a natural structure of a spherical building of type A_2 on $\mathcal{B}^8$.*

Proof. This follows immediately from [1] Proposition 4.44. This proposition says that a connected bipartite graph in which every vertex is the face of at least two edges is a building if and only if it has diameter m and girth $2m$ for some $2 \leq m \leq \infty$. Our case is that of $m = 3$. $\square$

Corollary 5.13. *There is a natural structure of a building with Weyl group* $W \rtimes \mathbb{R}^2$ *on* $\mathcal{B}^\phi = Cone(\mathcal{B}^8)$*, the cone over the spherical building* $\mathcal{B}^8$*. Here* $W = S_3$ *is the Weyl group of* $SL_3\mathbb{C}$*.*

Proposition 5.14. *The isomoetry of pre-buildings* $i : \mathcal{B}^\phi_{pre} \to \mathcal{B}^\phi$ *has the following universal property: given any building* $\mathcal{B}$ *and an isometric embedding* $j : \mathcal{B}^\phi_{pre} \hookrightarrow \mathcal{B}^\phi$ *there exists a unique folding map of buildings* $\theta : \mathcal{B}^\phi \to \mathcal{B}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{B}^\phi_{pre} \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{B}^\phi \arrow[d, dashed, "\theta"] \\ & \mathcal{B} \end{tikzcd}$$

Proof. The edges in the graph (spherical building) correspond to sectors in the affine building $\mathcal{B}^\phi$. Therefore, the proposition follows from Corollary 5.13 and Proposition 5.11. $\square$

5.3 The universal property and the WKB property

The main theorem in this section is the following:

Theorem 5.15. *Let* $h : X \to \mathcal{B}$ *be a harmonic* ϕ*-map to a building. Then there exists a unique folding map of buildings* $\psi : \mathcal{B}^\phi \to \mathcal{B}$ *such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "h^\phi"] \arrow[dr, "h"'] & \mathcal{B}^\phi \arrow[d, dashed, "\psi"] \\ & \mathcal{B} \end{tikzcd}$$

Furthermore, ψ restricts to an isometry on the image of h^{ϕ}.

Proof. The first part of the theorem follows immediately from the universal property for $i : \mathcal{B}^{\phi}_{pre} \to \mathcal{B}^{\phi}$ (Proposition 6.9), and the universal property of the map $h^{pre} : X \to \mathcal{B}^{\phi}_{pre}$ described in Propostion 5.17 below.
For the second part, observe that every pair of points in the image of h^{ϕ} is already contained in a single apartment in the pre-building $\mathcal{B}^{\phi}_{\text{pre}}$ (see Figure 5.17). This implies that, restricted to the image of h^{pre}, the inclusion $\mathcal{B}^{\phi}_{\text{pre}} \hookrightarrow \mathcal{B}^{\phi}$ is distance preserving in the BNR example. Thus, the statement that ψ restricts to an isometry on the image of h^{ϕ} also follows from Proposition 5.17. □

Corollary 5.16. *The map $h^{\phi} : X \to \mathcal{B}^{\phi}$ computes the dilation spectrum for any WKB problem with spectral cover ϕ.*

Proof. This follows immdediately from Theorem 4.3, Proposition 5.11 and Theorem 5.15. □

The goal in the rest of this section is to prove the universal mapping property for $h^{pre} : X \to \mathcal{B}^{\phi}_{pre}$:

Proposition 5.17. *Let $\mathcal{B}$ be a building, and let $h : X \to \mathcal{B}$ be any harmonic ϕ-map. Then there exists a unique isometry of pre-buildings $\psi : \mathcal{B}^{\phi}_{pre} \to \mathcal{B}$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "h^{pre}"] \arrow[dr, "h"'] & \mathcal{B}^{\phi}_{pre} \arrow[d, dashed, "\exists!\psi"] \\ & \mathcal{B} \end{tikzcd}$$

The strategy for proving this proposition is the following: firstly, we will prove that every connected component of the complement of the spectral network maps to a single sector under any harmonic ϕ-map. Using Proposition 5.11 and Theorem 3.16 we see that the MARs in Figure 5.6-5.15 indeed map to single apartments. This constructs the pre-building $\mathcal{B}^{\phi}_{\text{pre}}$.

Lemma 5.18. *Let R_0 denote the closure of the connected component of the complement of the BNR spectral network that is at the center of the diagram (see Figure 5.19). Let $h : X \to \mathcal{B}$ be a harmonic $\phi-$map to a building $\mathcal{B}$. Then there is a sector S in $\mathcal{B}$ such that $h(U) \subset S$.*

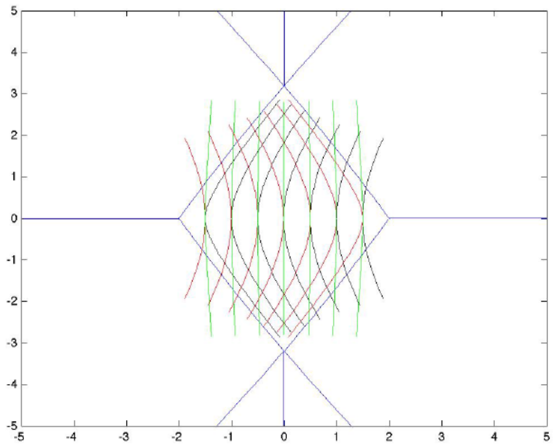

FIGURE 5.19: The caustic region

Proof. Let ϵ be a small positive number, let $P_\epsilon = -2+\epsilon$ and $Q_\epsilon = 2-\epsilon$, and let $\Omega_\epsilon := \Omega_{P_\epsilon Q_\epsilon}$ be the region defined in Theorem 3.16. Then, by this theorem, each of the regions Ω_ϵ is mapped into a single apartment. Since we can make ϵ arbitrarily small, it follows that R_0 is mapped into a single apartment A.
Since h is a ϕ-map, away from the ramification point we can choose coordiantes on the apartment A such that $dh = (\phi_1, \phi_2, \phi_3)$. It follows that the foliation lines are the pullbacks of the hyperplanes defining the apartment. From this one easily sees that R_0 maps into the intersection of sectors S_l and S_r in A based at $h(b_l)$ and $h(b_r)$ respectively. Here b_l is the branch point -2 and b_r is the branch point 2. Furthermore, the segments of the sepctral network lines that constitute the boundary of R_0 map to the bondaries of these sectors. □

Lemma 5.19. *Let $h : X \to \mathcal{B}$ be a harmonic ϕ-map to a building. Let c_u and c_l denote the collision points of the BNR spectral network. Then $h(c_u) = h(c_l)$.*

Proof. We know that the central yellow region maps to a single apartment. So to prove the lemma it suffices to compute the integrals of the 1-forms along a contour (that stays within the central region) from one collision point to the other. More precisely, we can choose a section of the cameral cover over R_0, and use this to write $\phi = (\phi_1, \phi_2, \phi_3)$, and then integrate the real parts of the three 1-forms ϕ_1, ϕ_2, ϕ_3 along a contour joining the collision points. It easily follows from the definition of spectral networks and from the fact that $\sum_i \phi_i = 0$ that this integral is indeed $(0, 0, 0)$. Thus, the vector distance between $h(c_u)$ and $h(c_l)$ is zero and the lemma follows. □

Remark 5.20. The proof of this lemma shows that, indeed $h(x) = h(\bar{x})$ (where $\bar{x}$ is the complex conjugate) for all $x \in R_0$. In that sense the segment $[-2, 2] \subset \mathbb{C}$ is a "fold-line" or caustic for all harmonic ϕ-maps.

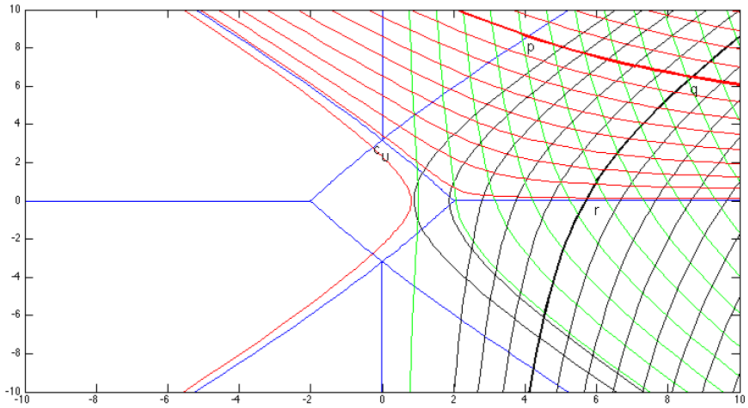

FIGURE 5.20: The foliation lines in a region

Lemma 5.21. *Let $R_0, \ldots, R_9$ denote the closures of connected components of the complement of the BNR spectral network, and let $h : X \to \mathcal{B}$ be a harmonic ϕ-map to a building. Then there exist apartments $A_0, \ldots, A_9$, and sectors $S_i \subset A_i$ such that $h(R_i) \subset S_i$. Furthermore, if two regions R_i and R_j are adjacent (in the sense that their intersection contains an open segment in a spectral network curve), then the corresponding sectors S_i and S_j are adjacent in $\mathcal{B}$, i.e. they share a panel.*

Proof. We have already proven the lemma for R_0. We will describe the proof for one of the regions R_1 through R_8; the other cases are similar. Consider the region R_1 which contains the point q shown in Figure 5.21, and consider the points p and r shown in the same figure (they lie on spectral network lines).
Apply Theorem 3.16 with $P = p$ and $Q = r$. The interior of the region U_{pr} bounded by the two spectral network lines containing p and r on the one side, and the thick red and thick black foliation lines on the other, does not contain any singularities of h, by Proposition 3.19, since it does not contain any ramification point. Using this, it is easy to see that Ω_{PQ}, in the notation of Theorem 3.16 equals U_{pr}. It follows that this entire region is mapped into a single apartment. Since the inverse images of apartments are closed, the closure of this region also maps to a single apartment. Thus, we see that the region R_1 can be exhausted by a family of compact sets, each of which maps to a single apartment. Since we require our buildings to have a complete set of apartments (Definition 2.11), it follows that the entire region R_1 is mapped into a single apartment A. Arguing exactly as in Lemma 5.19, we see that R_1 must in fact be mapped to a single sector S_1 with vertex at $h(c_u)$.
Furthermore, the spectral network lines emanating from c_u (resp. c_l) map to panels in the building based at $h(c_u)$. This immediately implies the last statement of the lemma. ∎

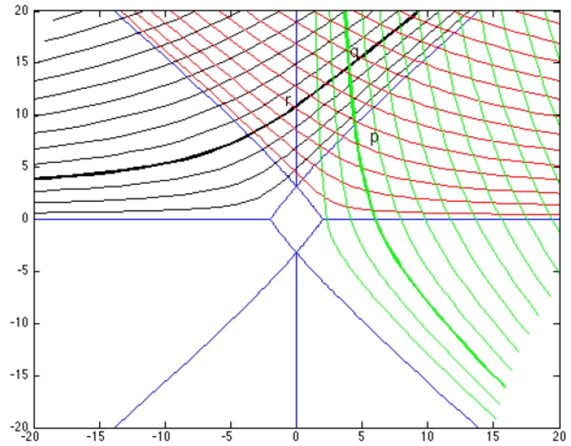

FIGURE 5.21: The foliation lines in another region

Proof of Proposition 5.17. Let $h : X \to \mathcal{B}$ be a ϕ-map. We claim that for each MAR M in X, there exists an apartment A in $\mathcal{B}$ such that $h(M) \subset A$. The proposition follows immediately from this claim and Proposition 5.3.

We now prove that claim. From Lemma 5.21 and 5.18 we see that every sequence of "four adjacent region" on X is mapped to a sequence of four adjacent sectors in the building under and ϕ-map. By Corollary 5.11 every such four sectors are contained in a single apartment. Thus, we have shown that any harmonic ϕ-map carries each of the MARs that consist of at most 4 of the regions R_0 through R_9 into a single apartment.

It remains to prove the claim for MAR3 and MAR4. Note that both of these MARs contain the yellow region R_0 at the center of the picture. Let us consider one of these MARs, say MAR3. The argument for the other MAR is identical. The argument of the previous paragraph shows that there is an apartment A such that the complement of R_0 in MAR3 is mapped to the union of four adjacent sectors in A.

Let ϵ be a small positive number and let $P_\epsilon = -2 + i\epsilon$ be a point just above the branch point b_l and let $Q_\epsilon = 2 + i\epsilon$ be a point just above the branch point b_r. Apply Theorem 3.16 with $P = P_\epsilon$ and $Q = Q_\epsilon$. Then the region $\Omega_{P_\epsilon Q_\epsilon}$ (in the notation of Theorem 3.16) intersects R_0 in the shaded region shown in Figure 5.22. By Theorem 3.16, every point in this region is in the Finsler covex hull of $h(P_\epsilon)$ and $h(Q_\epsilon)$. Since $h(P_\epsilon)$ and $h(Q_\epsilon)$ are containein A, it follows that the entire shaded region is mapped into A. Since ϵ can be made arbitrarily small, and the inverse images of apartments under continuous maps are closed we see that all of R_0 is mapped into A. Thus, the entire maximal abelian region MAR3 is mapped into A. This completes the proof. □

From the construction of $\mathcal{B}^\phi$, the following proposition is obvious:

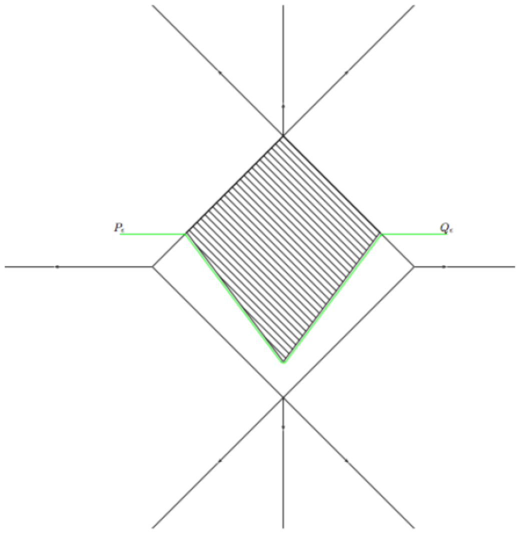

FIGURE 5.22: Showing that MAR3 is mapped into a single apartment

Proposition 5.22. *Let $\mathcal{W}_{BNR}$ denote the BNR spectral network, and let $\mathcal{W}_{BNR}^{ext}$ be the extended spectral network, obtained as the union of the BNR spectral network with the "backward collision line" (the vertical foliation line joining the collision points). Let $h^\phi : X \to \mathcal{B}^\phi$ be the universal harmonic map. Then we have the following:*

1. *The inverse image under h^ϕ of the singular set of $\mathcal{B}^\phi$ contains the spectral network $\mathcal{W}_{BNR}$*
2. *The inverse image under h^ϕ of the singular set of $\mathcal{B}^\phi$ equals the extended spectral network $\mathcal{W}_{BNR}^{ext}$*

Remark 5.23. A point in a building is called singular if no neighborhood of it is contained in an apartment.

Chapter 6

An example with a single BPS state

Portions of this chapter have appeared in [40]

In this chapter we investigate the universal mapping properties of another spectral cover. This example is, from some point of view, even simpler than the BNR example considered in the last chapter, due to the fact that any pair of points lies in a common maximal abelian region. From another point of view, however, this example is more complicated because the versal building is not the affine cone over a spherical building. Thus adding more apartments to satisfy the building axioms becomes more complicated.

The setup is as follows: let $X = \mathbb{A}^1_{\mathbb{C}}$ with coordinate x and let p denote the coordinate along the fiber of the cotangent bundle $T^*_X \cong \mathbb{A}^2_{\mathbb{C}}$. The spectral curve $\Sigma \subset T^*_X$ is given by the equation

$$p^3 + 4px + u = 0\,.$$

Here, $u \in \mathbb{C}$ parametrizes the Hitchin base. The spectral network and the caustic lines can be seen in Figure 6.1 for $u = 1$ and $\theta = \pi/3$ (a *caustic line* consists of points $P \in X$ at which two (and thus three) foliation lines are tangent to each other).

The strategy for investigating the universal mapping properties is as follows:

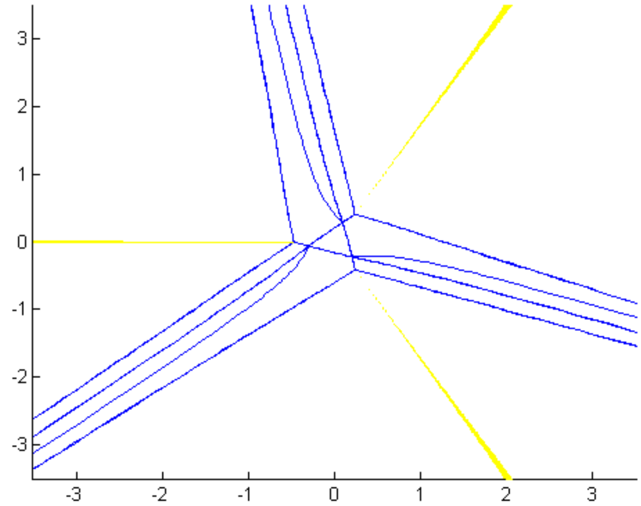

FIGURE 6.1: Spectral network and caustic lines

1. Show that the regions (i.e. connected components of the complement of the spectral network) containing the caustic lines map to single apartments

2. Prove that the complement of the regions containing the caustics maps to a single apartment

3. Use Corollary 5.11 to show that any pair of points is contained in a common maximal abelian region

A consequence of the mapping properties will be that for any harmonic ϕ-map $X \to \mathcal{B}$, the extended spectral network maps to the singularities of $\mathcal{B}$.

6.1 The universal mapping properties

We begin with item (1) of the outline from above:

Lemma 6.1. *Let R_i, $i = 1, 2, 3$ be the regions in Figure 6.1 containing the caustic lines, and let $h : X \to \mathcal{B}$ be a harmonic ϕ-map. Then there exist apartments A_i in $\mathcal{B}$ such that $h(R_i)$ is contained in A_i.*

Proof. Firstly, note that the caustic line is a spectral network line for a different angle θ, namely $\theta = 0$. This implies that it cannot be tangent to any of the foliation lines for the angle $\theta = \pi/3$. Thus, by Proposition 3.15, we know that it maps to a single apartment.

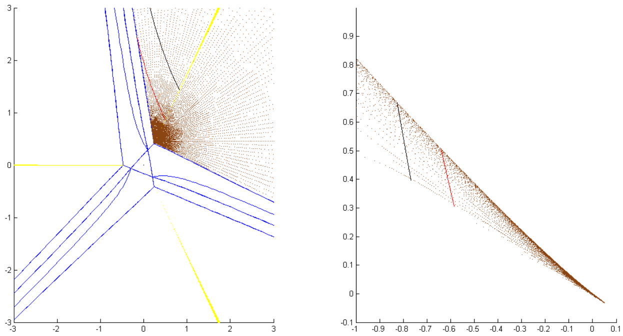

Figure 6.2: Caustic Region

Consider Figure 6.2. Here, the red and black lines are foliation lines starting at a point on the spectral network. The corresponding image in $\mathbb{R}^2$ is shown on the right hand side of Figure 6.2. Now, use Theorem 3.16 for P being the branch point and Q being the intersection of the folitation line from above with the caustic to show that the subset of R_1 which lies to the left of the caustic maps to a single apartment.
The same argument, however, can be applied to the subset of R_1 that lies to the right of the caustic. Thus the claim follows. □

Lemma 6.2. *The connected components of the complement of the spectral network map to a single apartment.*

Proof. We have argued for the regions $R_{1,2,3}$ containing the caustic lines in the previous lemma.
For the other regions, the claim follows easily from Theorem 3.16 together with Figure 6.3. In this figure, the lines are geodesics. □

With these lemmas, we can complete step (2) of the outline.

Lemma 6.3. *Let M be the complement in $\mathbb{C}$ of $R_1 \cup R_2 \cup R_3$ and let $h : \mathbb{C} \to \mathcal{B}$ be a harmonic ϕ-map. Then $h(M)$ is contained in a single apartment.*

Proof. Firstly, note that Lemma 6.1 implies that the two boundary lines of R_i are mapped to a single line in the building (see also Figure 6.2). Thus, the orange sector and the pink sector in Figure 6.3 are neighboring.
Then, we can use Proposition 5.11 to show that both $\{g, r, p, o\}$ and $\{r, p, o, b\}$ map to single apartments (here $\{g, r, p, o\}$ means the union of the green, red, pink and orange region in Figure 6.3; similarly for the other four sectors). This implies that, for $P \in M$ the map $Q \mapsto \mathrm{Re}(\int_P^Q \phi_1, \int_P^Q \phi_2, \int_P^Q \phi_3)$ when restricted to either

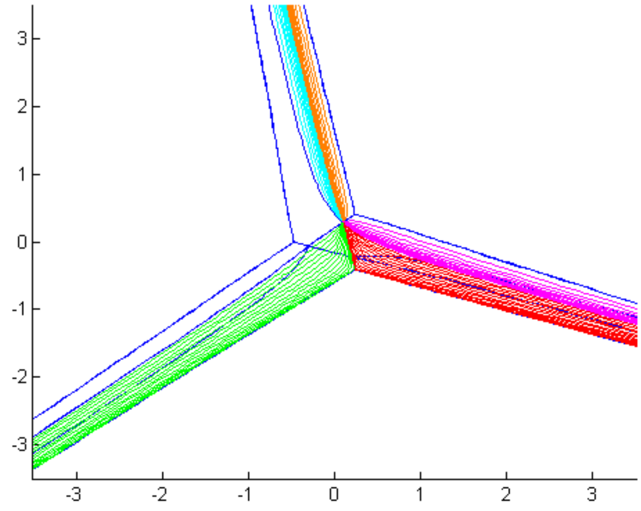

FIGURE 6.3: Other Regions

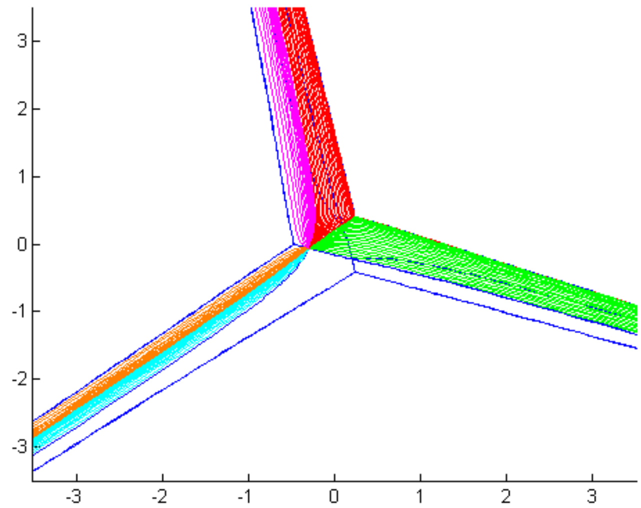

FIGURE 6.4

$\{g, r, p, o\}$ or $\{r, p, o, b\}$ computes the vector distance in $\mathcal{B}$. The same argument can, however, be made for sectors based at another vertex, see Figure 6.4. From this figure it can in particular be seen that the distance between points in the blue and the green sector from Figure 6.3 is given by integrating ϕ_i.
Arguing similarly for the remaining paris of points im M shows that the map

$$Q \mapsto \mathrm{Re}(\int_P^Q \phi_1, \int_P^Q \phi_2, \int_P^Q \phi_3)$$

indeed computes the right distances in $\mathcal{B}$. $\square$

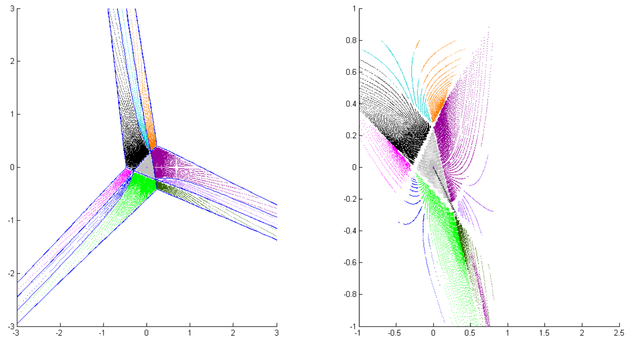

FIGURE 6.5: MAR1

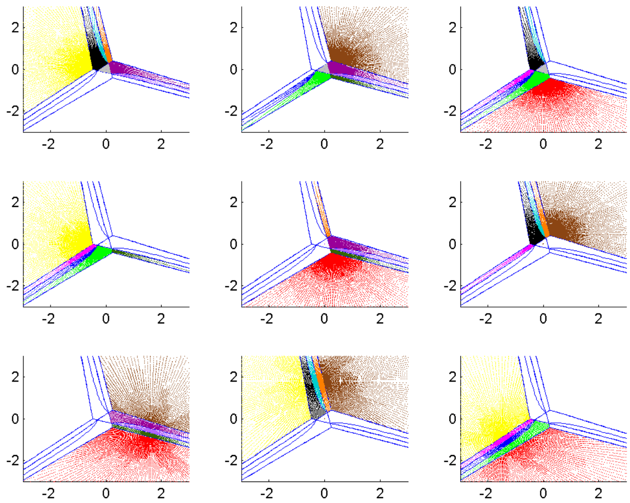

FIGURE 6.6: MARs

From the above lemma, together with Proposition 5.11, is is clear that the Maximal abelian regions are the colored regions in Figure 6.5, 6.6. This gives a good picture of the universal pre-building $\mathcal{B}^{\phi}_{\text{pre}}$: it consists of an apartment A_{base} corresponding to the image of MAR1 together with three half-apartments attached to A_{base} along the image of the spectral network lines emerging from branch points.

Note that any two point on $\widetilde{X}$ are contained in a common MAR and thus the image of $\widetilde{X}$ in $\mathcal{B}^{\phi}_{\text{pre}}$ is a metric space.

Theorem 6.4. *Let $h^{pre} : X \to \mathcal{B}^{\phi}_{pre}$ be the canonical map. Then*

$$\nu^{WKB}_{PQ} = d(h(P), h(Q)) \, .$$

Proof. This follows immediately from the universal mapping properties of h^{pre}. $\square$

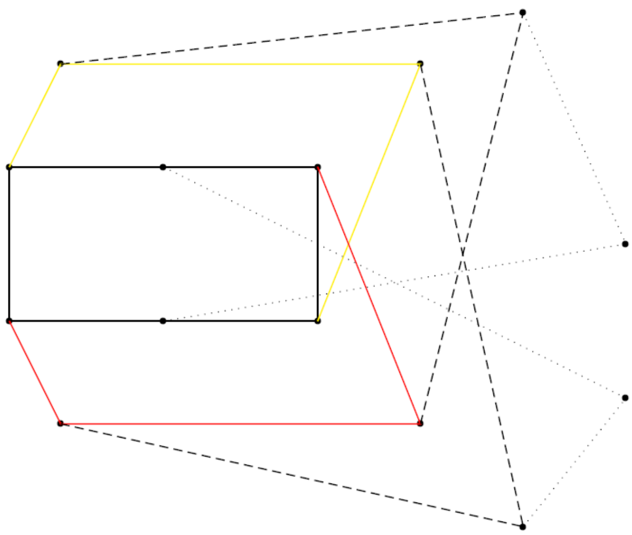

FIGURE 6.7: Link at collision point

We conclude this section with the following

Proposition 6.5. *Let $\mathcal{W}_{A_1}^{ext}$ be the extended spectral network. Then the image of $\mathcal{W}_{A_1}^{ext}$ is contained in the singularities of $\mathcal{B}$ for any harmonic ϕ-map $h : \widetilde{X} \to \mathcal{B}$.*

Sketch of proof. Let $\mathcal{W}_{A_1}^{\mathrm{branch}}$ be the union of the spectral network lines emerging from the branch points of the spectral cover. The images of these lines are precisely the singularities of $\mathcal{B}_{\mathrm{pre}}^{\phi}$.
Now, let us consider the link at the image of a collision point $c \in \widetilde{X}$. The full black lines correspond to the image under h of MAR1, the colored regions correspond to the half apartments attached along the image of the spectral network lines. Counting the distance between the red and yellow edges and recalling that the link of $\mathcal{B}$ at $h(c)$ is a spherical building, implies that the dashed lines have to be there. Similarly, requiring that any two dashed lines have to be in a common hexagon, implies that the dotted lines have to be there. The dotted lines, however, are attached at the vertices that correspond to the collision lines. Thus the forward and backward collision line emerging at c are mapped to the singular locus of $\mathcal{B}$. The same argument shows that the entire extended spectral network maps to the singularities of $\mathcal{B}$. □

6.2 The versal building

In this section we will construct a versal building from $\mathcal{B}_{\mathrm{pre}}^{\phi}$.
Let $h : \widetilde{X} \to \mathcal{B}$ be a harmonic ϕ-map. Note that the image of $\mathcal{W}_{A_1}^{\mathrm{ext}}$ tiles the

apartment A_{base} corresponding to the image of MAR1 by equilateral triangles. By Proposition 6.5, the building $\mathcal{B}$ is singular along the lines of this tiling. The idea for constructing a versal building is to "glue in" more such triangles, similarly to the process for the BNR example. These triangles are added in such a way that the resulting limiting object is a building. To enusre that, the following condition is used [9]:

Proposition 6.6. *Let X be a simply connected piecewise Eulcidean complex of dimension $r \geq 2$ such that the link at every point is a spherical building. Then X is a Euclidean building.*

Recall that a piecewise Euclidean complex is a polyhedeal cell complex together with a complete, geodesic metric d such that each cell is isometric to a convex polyhedral cell in $\mathbb{R}^n$, and

$$d(x,y) = \inf_{\gamma}\{l(\gamma)|\gamma \text{ is a path from x to y}\}\ .$$

We now give the construction of the versal building $\mathcal{B}^{\phi}$:

Construction 6.7. *Let $\mathcal{B}^{\phi}_{pre}$ be the pre-building consisting of A_{base} and three half-apartments attached to it. Let $\Delta \subset \mathbb{R}^2$ be an equilateral triangle with the same side-length as the grey triangle in Figure 6.5. Then we inductively add new triangles with the same side lengths as Δ at all points x for which the the link $lk(x)$ is not a spherical building, according to the rules drawn in Figure 6.8 (the figure can be interpreted as follows: edges correspond to triangles, dashed edges correspond to triangles which are added; dots correspond to reflection hyperplanes and $\circ$'s correspond to newly added reflection hyperplanes):*

1. *If there are four consecutive sectors in $lk(x)$ not contained in a hexagon, we add two new triangles as in the left of Figure 6.8;*

2. *If there is a path of length three in $lk(x)$ not contained in a hexagon and to whose endpoints no new lines are added by rule 1., we add three new triangles as in the middle of Figure 6.8;*

3. *If there is a path of length two in $lk(x)$ not contained in a hexagon to whose endpoints no new lines are added by rule 1. or rule 2., we add four new triangles as in the right of Figure 6.8.*

Let $\mathcal{B}^{\phi}$ be the resulting simplicial complex with metric

$$d(x,y) = \inf_{\gamma}\left\{\sum_i l(\gamma_i)|\gamma(0) = x, \gamma(1) = y, \gamma_i \textit{ is contained in a single triangle } \Delta_i\right\}.$$

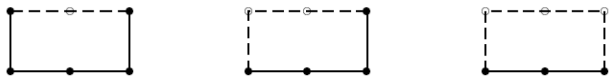

FIGURE 6.8: Gluing construction

We now want to apply Proposition 6.6 to show that $(\mathcal{B}^\phi, d)$ is a Euclidean building:

Proposition 6.8. *Let $\mathcal{B}^\phi$ be the simplicial complex obtained by applying Construction 6.7 to $\mathcal{B}^\phi_{pre}$. Then $(\mathcal{B}^\phi, d)$ is a Euclidean building.*

Proof. Firstly, note that $\mathcal{B}^\phi$ is a piecewise Eucliean simplicial complex. Furthermore, we can assume that $\mathcal{B}^\phi$ is simply connected; otherwise we take its universal cover.

So, we need to show that the link of every point is a spherical building. Note that at points that are not integer translates of the vertices of the grey triangle Δ in Figure 6.5 by vectors corresponding to edges of Δ this condition is satisfied at every step of the induction process: the link is either a hexagon or looks similar to the one in Figure 6.9, possibly containing more (or less) half-apartments.

Thus, we only need to make sure that the links at points which are integer translates of the vertices of the grey triangle are spherical buildings. From the construction rules, it is clear that any two edges will be contained in a common hexagon. So we just have to show that no cycles of length less than six are created.

For that, we want to understand how the link at an endpoint y of a newly added triangle changes if we apply the induction step of Construction 6.7. There are two possibilities: either y was not there before that step, in which case the link at y consists of two segments; or if y was there before the induction step, we simply add some segments to the vertex corresponding to the reflection hyperplane along which we glued the triangles. Thus the bipartite structure of every link is preserved. The rules in Construction 6.7 applied to the link at y also preserves this bipartite structure and so can only generate cycles of even length. Applying rule 1. possibly several times will, then, put any pair of segments into a common hexagon. $\square$

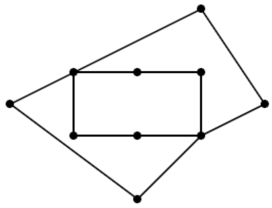

Figure 6.9

Proposition 6.9. *The canonical isomoetry of pre-buildings* $i : \mathcal{B}^{\phi}_{pre} \to \mathcal{B}^{\phi}$ *has the following universal property: given any building* $\mathcal{B}$ *and an isometric embedding* $j : \mathcal{B}^{\phi}_{pre} \hookrightarrow \mathcal{B}^{\phi}$ *there exists a folding map of buildings* $\theta : \mathcal{B}^{\phi} \to \mathcal{B}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathcal{B}^{\phi}_{pre} \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{B}^{\phi} \arrow[d, dashed, "\theta"] \\ & \mathcal{B} \end{tikzcd}$$

Proof. Note that by Proposition 6.5, for all apartments of $\mathcal{B}$ we can define a tiling by equilateral triangles of the same side length as the grey triangle in Figure 6.5. Let $\mathcal{B}^i$ be the pre-building obtained after applying i inductive steps of Construction 6.7. For each $\mathcal{B}^i$, we can define a (not necessarily unique) map $\theta^i : \mathcal{B}^i \to \mathcal{B}$ as follows: Let Δ_i be a triangle that has been added at step i, i.e. $\Delta_i \subset \mathcal{B}^i$, $\Delta_i \not\subset \mathcal{B}^{i-1}$. For every newly attached boundary ∂ of some Δ_i (corresponding to a new vertex at the link at x where Δ_i has been attached), we choose a corresponding vertex in the link of $\theta^{i-1}(x)$ (corresponding to a choice of tangent vector for the image of ∂) such that the resulting map at the link of x is a folding map (when restricted to the link of $\mathcal{B}^{i-1}$ together with the triangles Δ_i that were attached at step i). Then, choose an entire segment, of the same length as the tiling triangle, with the chosen vertex as tangent vector at the link of $\theta^{i-1}(x)$. This construction fixes the image of two boundaries of Δ_i , and thus, by convexity, the image of Δ_i.

□

Theorem 6.10. *Let* $h : X \to \mathcal{B}$ *be a harmonic* ϕ*-map to a building. Then there exists a folding map of buildings* $\psi : \mathcal{B}^{\phi} \to \mathcal{B}$ *such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "h^\phi"] \arrow[dr, "h"'] & \mathcal{B}^\phi \arrow[d, "\psi"] \\ & \mathcal{B} \end{tikzcd}$$

Furthermore, ψ restricts to an isometry on the image of h^ϕ.

Proof. This is an easy consequence of the universal mapping properties together with Proposition 6.9. □

6.3 BPS states and stability condition

As mentioned in the introduction one of the main motivations for this work was to understand BPS states and its relations to harmonic maps to buildings.
Recall that a *BPS web* is a subset of a spectral network of a special phase: it consists of spectral network lines that can be lifted to a closed cycle $\gamma \in H_1(\Sigma, \mathbb{Z})$; as an example consider the union of the red lines in Figure 6.10; the fact that it lifts to a closed cycle in $H_1(\Sigma, \mathbb{Z})$ can easily be seen from Figure 6.11). Furthermore, let us fix the point in the Hitchin base. Conjecturally this should fix a stability condition on an appropriate 3d Calabi-Yau category. The semi-stable objects for this stability condition should be all BPS webs of the spectral networks $\mathcal{W}_\theta$ for all phases $\theta \in S^1$ [26] (actually one should also add the analogues of circles, which have DT-invariant $\Omega = -2$, in the $\mathrm{SL}_2\mathbb{C}$ case, but that is irrelevant for the current example).
Let us note that from [26] we expect that the appropriate 3d Calabi-Yau category $\mathcal{T}$ is the Ginzburg category associated to the quiver $A_1 = \bullet$ (this is due to the fact that, for any parameter $u \in \mathbb{C}$ of the Hitchin base, there is a single BPS state, see Figure 6.10). This corresponds to the fact that

$$H_1(\Sigma, \mathbb{Z}) \simeq \mathbb{Z}\,.$$

Interestingly, the period of the generator of $H_1(\Sigma, \mathbb{Z})$ is related to the side length of the tiling triangle; the notation and the labels of the sheets of the spectral cover is as in Figure 6.11:
From the definition of spectral networks, it follows immeditely that (with $\phi_{ij} = \mathrm{Re}\,(\phi_i - \phi_j)$)

$$\int_P^Q \phi_{13} = \int_P^Q \phi_{23} = \int_P^R \phi_{23} = \int_P^R \phi_{21} =: l\,.$$

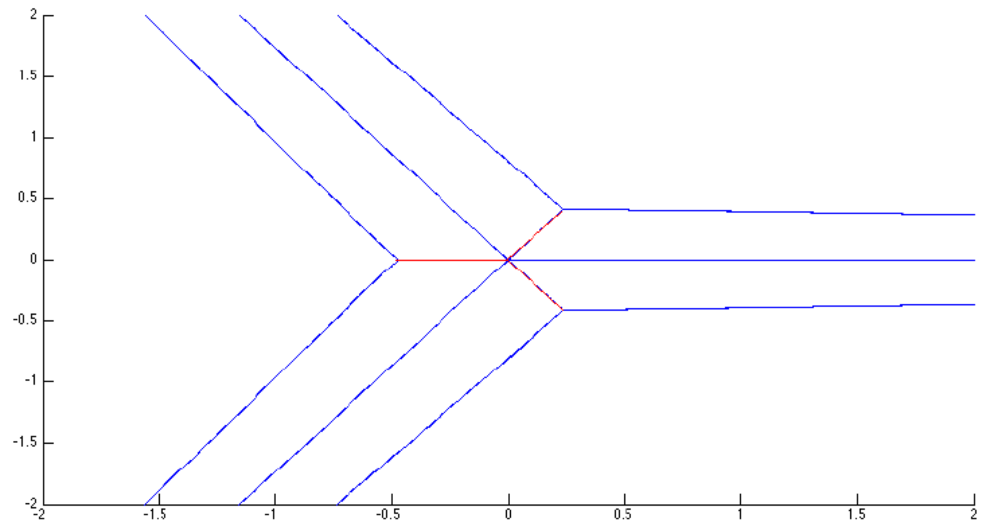

FIGURE 6.10

Similarly,

$$\int_P^Q \phi_{13} = \int_Q^R \phi_{13} = \int_Q^R \phi_{12} = l\,.$$

Using coordinates $(x, y) = \text{Re}\,(\int \phi_1, \int \phi_2)$, we obtain the following relations on the side lengths of the grey triangle Δ from Figure 6.5:

$$\Delta_x^{PQ} = \Delta_y^{PQ} \quad \Delta_x^{QR} = \Delta_y^{QR} + l \quad \Delta_x^{PR} = \Delta_y^{PR} + l\,.$$

On the other hand:

$$l = \int_P^Q \phi_{23} = \left(\int_P^Q + \int_Q^{b_1} + \int_R^Q\right)\phi_{23} + \left(\int_R^{b_2} + \int_R^P\right)\phi_{13}\,,$$

since

$$\int_P^Q \phi_{23} = \int_P^Q \phi_2 + \int_Q^P \phi_3 = \int_P^Q \phi_2 + \left(\int_Q^R + \int_R^P\right)\phi_3 = \int_P^Q \phi_2 + \int_Q^R \phi_3 + \int_R^P \phi_1\,.$$

Adding up the terms in the previous two equations we get

$$l = \text{Re}\int_\gamma \phi\,,$$

where γ is the lift of the BPS state from Figure 6.10 to the spectral cover.

To conclude this section we want to present an algorithm for A_2 spectral covers, in the context of this example, that allows us to deduce the quiver and thus the

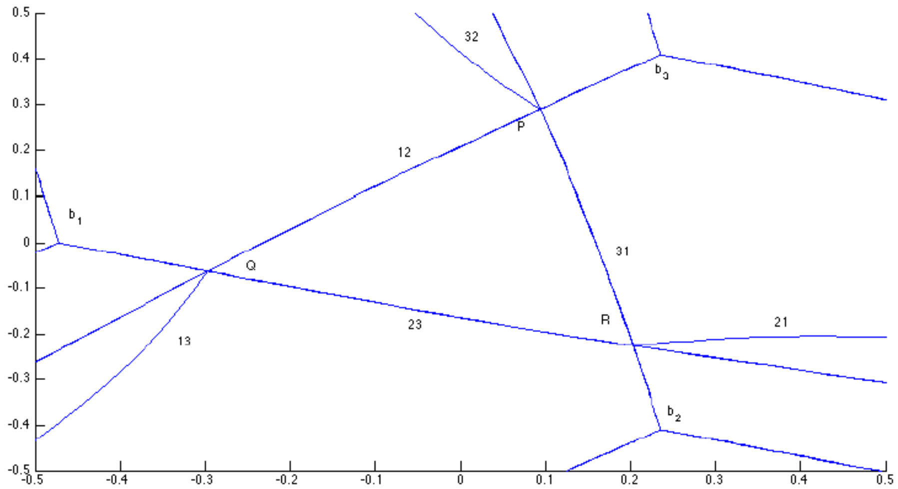

FIGURE 6.11

category. We switch from imaginary spectral networks to real spectral networks, since this is the standard convention for stability conditions.

Construction 6.11. *Let $\Sigma \to X$ be a $SL_3\mathbb{C}$ spectral cover with locally finite spectral network $\mathcal{W}$. For any two collision points P, Q that are connected by a spectral network line, say with labels (ij) and such that no other collision point lies on this line between P and Q, we can associate a cycle $\gamma \in H_1(\Sigma, \mathbb{Z})$ as follows:*
Consider the two open chains on Σ whose two boundary points project to P and Q respectively, and whose endpoints lie on the sheets i and k, $k \neq j$. Then join these two open chains along the spectral network line connecting P and Q to form the closed cycle γ whose orientation is chosen such that

$$Im \int_\gamma \phi \geq 0\,.$$

Then, consider the set of all cycles obtained by this procedure and use the intersection form on Σ to compute an antisymmetric matrix. The corresponding quiver is the one associated to this matrix.

Remark 6.12. In the example at hand, Construction 6.11 produces the cycle corresponding to the BPS state (which is, of course, no surprise since $H_1(\Sigma, \mathbb{Z})$ has a single generator). The construction, however, also makes clear the relation between the central charge and the edge length, since the contribution to the integral from the two open chains is zero. We will apply this algorithm in the next Chapter as well, where the construction is less trivial, but still gives rise to the correct answer.

Chapter 7

An example with two BPS state

In this chapter we treat another example which is quite general already: neither is the pre-building describable as an affine cone over a spherical pre-building, nor are any two points contained in a common maximal abelian regions. It even shows some further phenomenon that we expect in general, namely that the extended spectral network is dense in some regions.Due to these diffictulites we could not yet complete this example and it is still ongoing research. In this chapter, we only present some very preliminary results and speculations.

Let us start by describing the spectral cover we want to consider. As in the previous chapters, let $X = \mathbb{A}^1_{\mathbb{C}}$ with coordinate x and let p denote the coordinate along the fiber of the cotangent bundle $T^*_X \cong \mathbb{A}^2_{\mathbb{C}}$. We consider the spectral curve

$$\Sigma = \{(x,p) | p^3 + 3u_1 p + (x^2 + u_2) = 0\} ,$$

for $u_1, u_2 \in \mathbb{C}$.

7.1 The universal mapping properties

In this section we want to investigate the universal mapping properties for harmonic ϕ-maps for the spectral cover $\Sigma \to X$.
The spectral network together with the caustic lines is shown in Figure 7.1. Notice the following interesting behavior: there are backward spectral network lines joining c_1 and c_2, and c_3 and c_4, respectively. Thus, if we cut out a region $U \subset X$ containing the branch point b_1, b_2 and the collision points c_1, c_2, we have the same

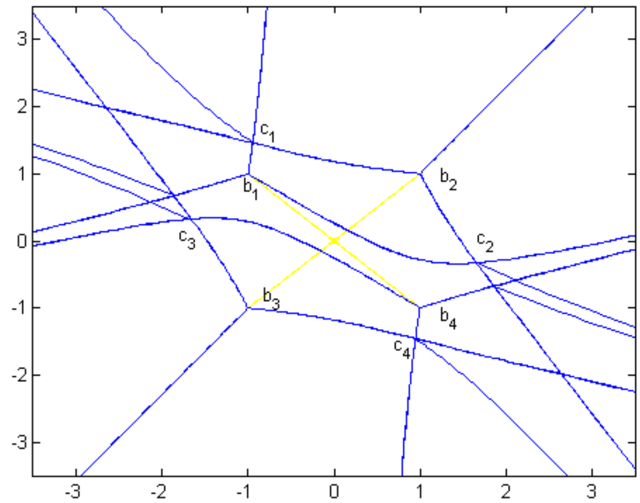

FIGURE 7.1: Spectral network and caustic lines

situation as in the BNR example. Similarly, if we cut out a region containing b_3, b_4 and c_3, c_4 we also have the same situation as in the BNR building. Thus, this example seems to "glue together two BNR buildings". Since, in the end, any two points need to be in a common apartment, this gluing process is of course very nontrivial.

Lemma 7.1. *Let $\mathcal{W}_{A_2}$ be the spectral network. If $R \subset \widetilde{X}$ is a connected component of the complement of $\mathcal{W}_{A_2}$ and $h : \widetilde{X} \to \mathcal{B}$ is a harmonic ϕ-map to a building $\mathcal{B}$, then there exists an apartment $A \subset \mathcal{B}$ such that $h(R) \subset A$.*

Proof. The proof is again based on Theorem 3.16. Consider, for example the sectors in Figure 7.2. It follows immeadiately from Theorem 3.16 that each of them maps to a single apartment.
The argument for the regions containing the caustic lines is also the same as in the BNR example. □

This is only the very first step in investigating the mapping properties. We also applied the procedure of Gaiotto-Moore-Neitzke which allows us to get distances between points that are "farther away". For some examples of these Maximal Abelian Regions, see Figures 7.3, 7.4, 7.5.
We could not reach a complete understanding of the universal mapping properties until now. Constructing a pre-building is hard due to the sheer number of maximal abelian regions (21 were found until now, but there are probably more).

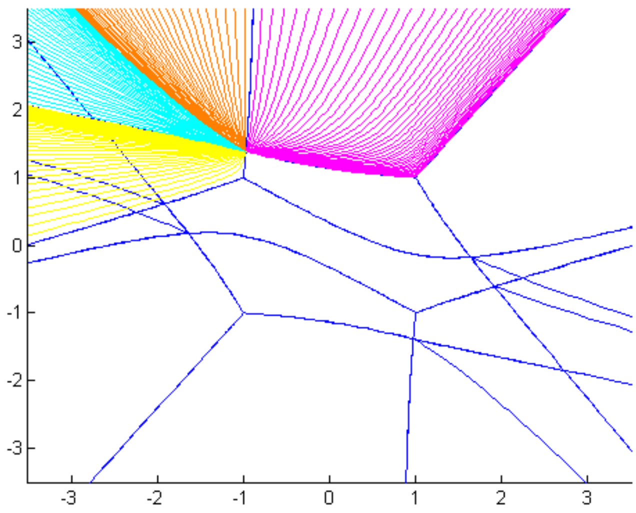

FIGURE 7.2

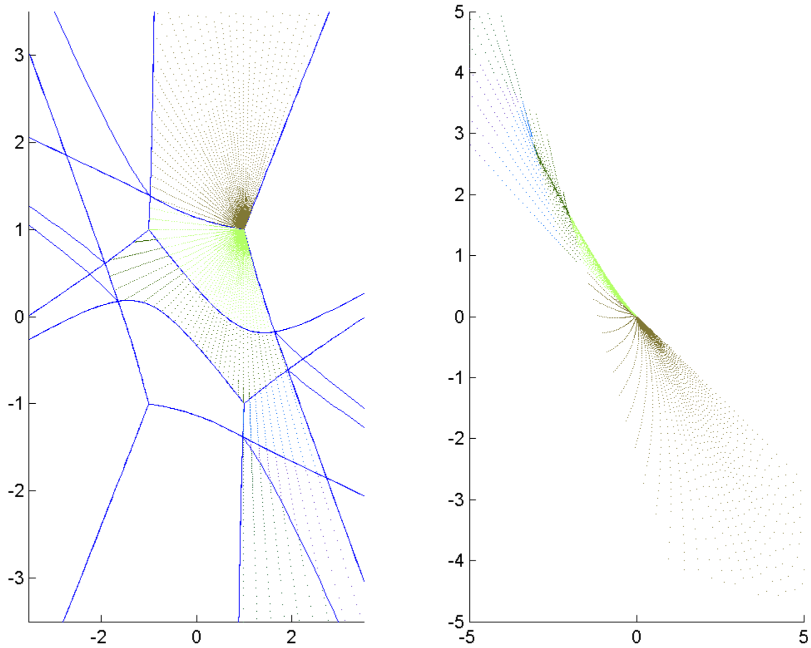

FIGURE 7.3

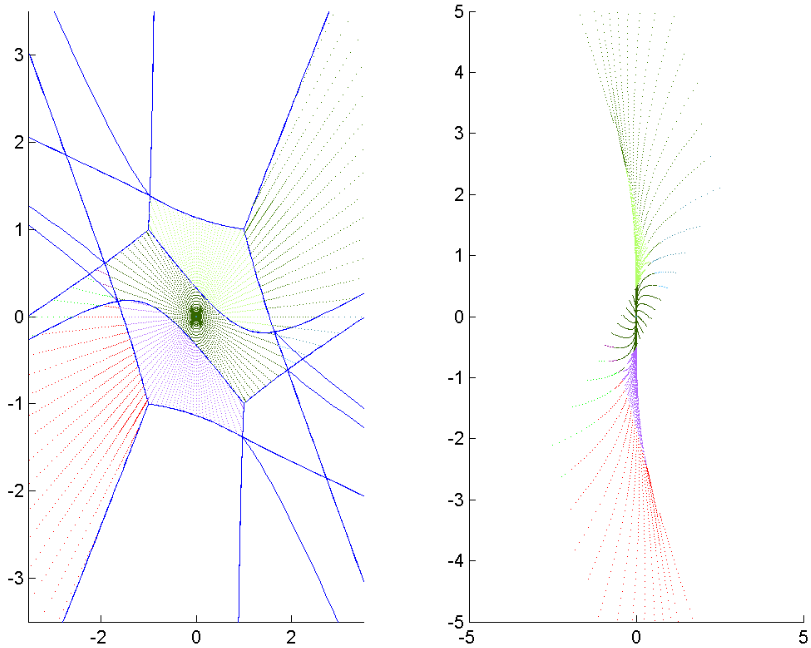

Figure 7.4

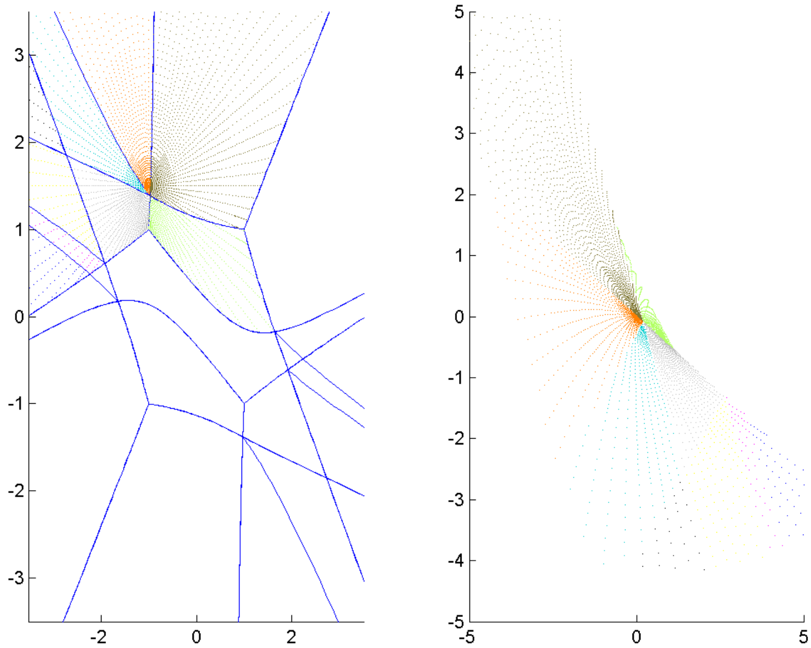

Figure 7.5

7.2 Some speculations about the versal building, BPS states and stability conditions

This example is already much more general than the two previous examples : the extended spectral network is dense in some regions (for generic parameters u_1, u_2 defining the spectral cover Σ). Since the extended spectral network maps to the singular locus of the building, we expect that the versal building $\mathcal{B}^\phi$ is an honest $\mathbb{R}$-building, i.e. it cannot be described by a discrete translation group.
More specifically, looking at Figure 7.4 gives a good idea of what to expect: the dark green rectangle is bounded by the image of spectral network lines. Since the vertices of this rectangle correspond to collision points, there are new singularities emerging from these collision points. These new lines will intersect the boundary of the rectangle again, creating, again, new collision lines. If there is no rational linear combination of the edge lengths of the rectangle, this process will continue indefinitely, thus creating smaller and smaller "Weyl chambers". So, in a sense, this rectangle creates a "tiling" of the apartment by infinitesimally small "Weyl chambers".

From [26], we know what 3d-Calabi-Yau category $\mathcal{T}$ to expect: it should be the Ginzburg category associated to the A_2 quiver: $\bullet \to \bullet$. This is due to the fact that there are, depending on the parameters $u_1, u_2 \in \mathbb{C}$, either two or three BPS states. One is shown in Figgure 7.6. Another one connects the other two branch points. The structure of the third one (which may or may not exists) is shown in Figure 7.7, where the x's are branch points and the $\bullet$'s are collision points.
Note the following interesting feature: we can apply Construction 6.11 which also gives us the corresponding cycles. Furthermore, it immediately gives an equality between the imaginary part of the corresponding period and the side length of the dark green rectangle of Figure 7.4. This is actually the kind of phenomeon that we expect in general.

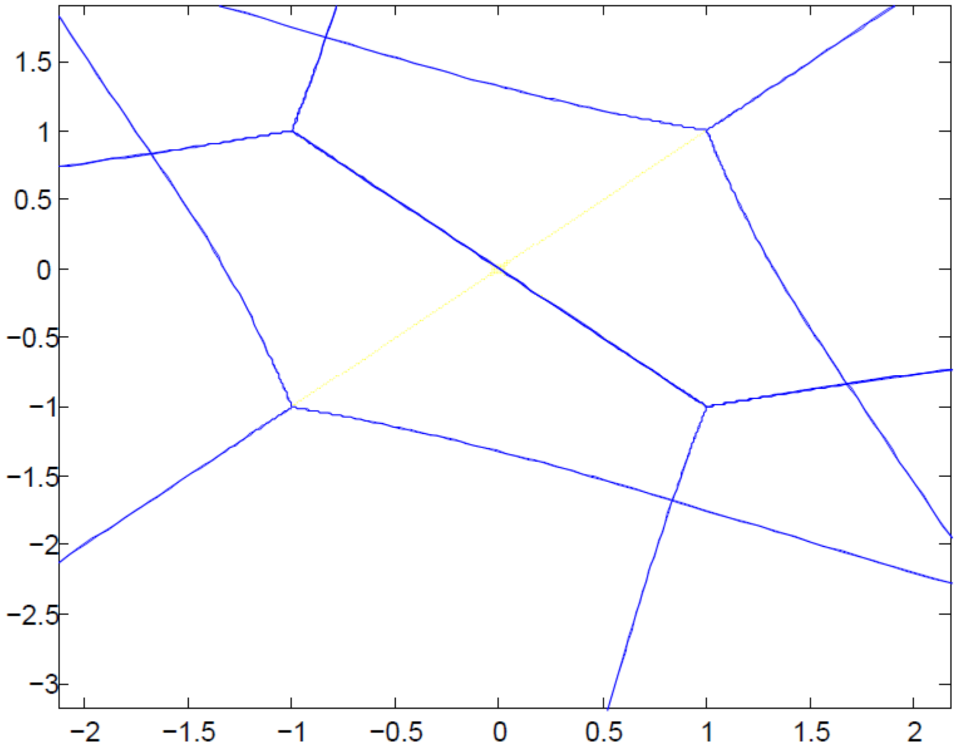

Figure 7.6

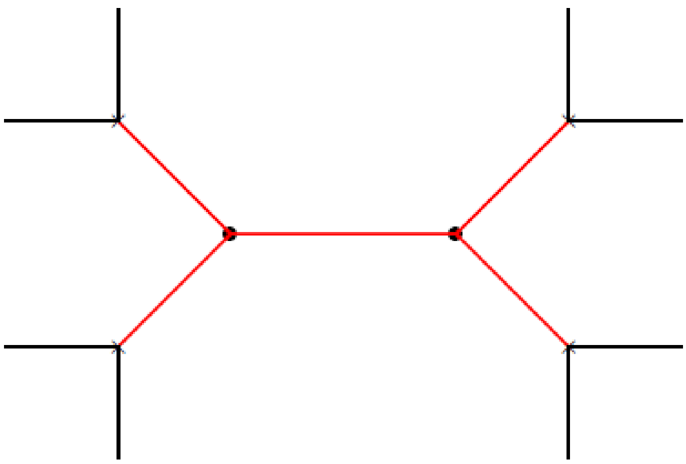

Figure 7.7

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Publications

- R.S. Garavuso, M. Kreuzer and A. Noll, *Fano hypersurfaces and Calabi-Yau supermanifolds*, [arXiv:0812.0097] JHEP03(2009)007 .
- R.S. Garavuso, L. Katzarkov, M. Kreuzer and A. Noll, *Super Landau-Ginzburg mirrors and algebraic cycles*, [arXiv:1101.1368] JHEP 1103 (2011) 017 .
- L. Katzarkov, A. Noll, P. Pandit, C. Simpson, *Harmonic Maps to Buildings and Singular Perturbation Theory*, [arXiv:1311.7101:]

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Scholarships and Awards

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