universität wien

DISSERTATION

Titel der Dissertation

Curvature bounds in low-regularity geometry

Verfasserin

Mag. Nathalie Tassotti

angestrebter akademischer Grad

Doktorin der Naturwissenschaften (Dr. rer. nat.)

Wien, November 2014

Studienkennzahl lt. Studienblatt:	A 791 405
Dissertationsgebiet lt. Studienblatt:	Mathematik
Betreuerin / Betreuer:	Privatdoz. Dr. James D. E. Grant

Acknowledgments

Foremost, I would like to express my sincere gratitude to my supervisor Dr. James D.E. Grant for his support of my Ph. D study and research and for his patience in explaining me things over and over again.

I would also like to thank Prof. P.T.Chruściel for employing me on Grant P23719-N16 of the FWF. Furthermore my work was funded by a Forschungsstipendium from the University of Vienna, and a DOC fellowship from the Austrian Academy of Sciences.

Additionally I would like to thank Partick Breuning, Alexander Lecke and especially Prof. M. Kunzinger who read through the thesis and helped me a lot with various useful corrections.

Additional thanks also to my colleague Alexander Lecke for a lot of encouraging discussions during the last few months.

Furthermore sincere thanks go to Prof. M. Kunzinger for discussions and his help for clarifying parts of the thesis.

I would also like to thank my family for their support not only during my Ph.D, but throughout my whole life.

Last but not least, I would like to thank the people from MoveOn Dance Spectrum, especially Raluca Nicolae, for their permanent moral support through the good and also not so good times and simply for keeping me sane.

Abstract

The main topic of this thesis is the use of curvature bounds for Riemannian manifolds with non-smooth metrics. This thesis is separated in three chapters.

In the first chapter we study scalar curvature bounds. Here we will focus on the positive mass theorem. After providing an overview of well-known results, we will show that the positive mass theorem remains valid for continuous Riemannian metrics that are of regularity $W^{2,n/2}_{\mathrm{loc}}$ on manifolds of dimension $n \leqslant 7$ or spin-manifolds of any dimension. We give a (negative) lower bound on the ADM mass of metrics for which the scalar curvature fails to be non-negative, where the negative part has compact support and has sufficiently small $L^{n/2}$ norm. We show that a Riemannian metric in $W^{2,p}_{\mathrm{loc}}$ for some $p > \frac{n}{2}$ with non-negative scalar curvature in the distributional sense can be approximated locally uniformly by smooth metrics with non-negative scalar curvature. For continuous metrics in $W^{2,n/2}_{\mathrm{loc}}$, there exist smooth approximating metrics with non-negative scalar curvature that converge in L^p_{loc} for all $p < \infty$.

The second chapter consists of the study of metrics that satisfy bounds on the Ricci curvature. In particular, we will study volume comparison results. We will first state the classical volume comparison result for pointwise Ricci curvature bounds due to Bishop and Gromov and we will afterwards provide generalizations for metrics that only satisfy an L^p bound for $p > n/2$ on the part of the Ricci tensor that violates $\mathbf{Ric} \geqslant c(n-1)\mathbf{g}$.

We will develop volume estimates and monotonicity formula based on integral norms of a weighted version of the negative part of Ricci along radial geodesics from each point. The use of *weighted* curvature quantities will lead to sharper formulae in volume comparison calculations that also hold for $p = n/2$.

In the third chapter we will study sequences of manifolds and their convergence properties. We will discuss how a pointwise bound on **Ric** as well as an L^p-bound leads to Gromov-Hausdorff convergence. Furthermore, we will also see that the space of manifolds satisfying the curvature bounds as described in 2 are compact in the Gromov-Hausdorff topology.

We will also investigate harmonic coordinates and their use for proving convergence results in certain Sobolev or Hölder spaces. We will provide a complete detailed proof of $C^{k,\alpha}$ convergence of Riemannian manifolds that satisfy certain bounds on the Ricci curvature.

We will finally consider sequences of manifolds with an L^p bound on the Ricci curvature. We will describe how an additional bound on the full curvature tensor leads to convergence in Hölder spaces.

Zusammenfassung

In dieser Arbeit wird untersucht, wie Beschränkungen an die Krümmung in der Riemannschen Geometrie für nicht-glatte Metriken verwendet werden können. Die Arbeit ist in drei Kapitel eingeteilt.

Im ersten Kapitel untersuchen wir Metriken mit beschränkter skalarer Krümmung. Wir konzentrieren uns auf das "positive mass theorem", das ein wichtiges Resultat in der allgemeinen Relativitätstheorie ist. Nachdem wir einen Überblick über bekannte Resultate gegeben haben, zeigen wir, dass das Theorem auch für stetige Riemannsche Metriken, die im Sobolev Raum $W^{2,n/2}_{\text{loc}}$ liegen, auf Mannigfaltigkeiten mit Dimension $\leqslant 7$ oder auf Spin-Mannigfaltigkeiten beliebiger Dimension gültig bleibt. Wir leiten eine (negative) untere Schranke an die ADM Masse für Metriken her, deren negative skalare Krümmung eine hinreichend kleine $L^{n/2}$ Norm hat und kompakten Träger besitzt. Wir zeigen, dass eine stetige Riemannsche Metrik der Regularität $W^{2,p}_{\text{loc}}$ für $p > n/2$ mit nichtnegativer skalarer Krümmung im distributionellen Sinn lokal gleichmässig durch glatte Metriken mit nichtnegativer skalarer Krümmung approximiert werden kann. Für stetige Metriken in $W^{2,n/2}_{\text{loc}}$ kann man glatte approximierende Metriken mit nichtnegativer skalarer Krümmung finden, die in L^p_{loc} für $p < \infty$ konvergieren.

Das zweite Kapitel beschäftigt sich mit der Untersuchung von Metriken, deren Ricci Krümmung beschränkt ist. Wir werden Volumsvergleiche untersuchen. Zuerst wird das klassische Resultat von Bishop und Gromov für punktweise Schranken an die Ricci Krümmung dargestellt, dieses wird dann für Metriken, die eine L^p Schranke für $p > n/2$ an den Teil des Ricci Tensors, der die Relation $\mathbf{Ric} \geqslant c(n-1)\mathbf{g}$ verletzt, verallgemeinert.

Weiters werden wir Volumsabschätzungen und Monotonie-Formeln, die auf Integralnormen einer gewichteten Version des Negativteils der Ricci Krümmung entlang radialer Geodäten von jedem Punkt aus, herleiten. Die Verwendung dieser gewichteten Krümmungsgrössen führt zu schärferen Abschätzungen, die auch im Fall $p = n/2$ gültg bleiben.

Im dritten Teil untersuchen wir Folgen von Mannigfaltigkeiten und ihre Konvergenzeigenschaften. Wir zeigen wie eine, sowohl punktweise, als auch eine L^p Schranke, an die Ricci Krümmung zu Gromov-Hausdorff Konvergenz führen. Weiters werden wir auch zeigen, dass die Menge der Mannigfaltigkeiten, die die Krümmungseigenschaften von 2 aufweist, kompakt in der Gromov-Hausdorff Topologie ist.

Wir werden auch harmonische Koordinaten und deren Verwendung in Beweisen von Konvergenzresultaten in bestimmten Sobolev oder Hölder Räumen diskutieren. Ein vollsändiger, detaillierter Beweis der $C^{k,\alpha}$ Konvergenz von Riemannschen Mannigfaltigkeiten mit bestimmten Schranken an die Ricci Krümmung wird gegeben.

Schlussendlich betrachten wir auch Folgen von Mannigfaltigkeiten mit einer L^p-Schranke an die Ricci Krümmung. Wir beschreiben, wie eine zusätzliche Schranke an den gesamten Krümmungstensor zu Konvergenz in Hölder Räumen führt.

Contents

Introduction

An important branch of classical Riemannian geometry is the study of Riemannian metrics that satisfy a curvature bound. Of particular significance are metrics where one has a lower and/or upper bound on the sectional curvature or a lower bound on the Ricci tensor or on the scalar curvature. Notable results that arise then include Myers's theorem, the Cartan–Hadamard theorem, the Bishop–Gromov relative volume comparison theorem, the sphere theorem and the Toponogov comparison theorem (see, e.g., [12] for general results and [66] for a recent review on metrics with a lower Ricci curvature bound). The proofs of these results generally rely on the use of the exponential map, and require that the metric have at least C^2 regularity. If one attempts to lower the regularity assumptions, one can often argue that these results hold for metrics that are $C^{1,1}$, where one still has existence and uniqueness of solutions of the geodesic equations. If one attempts to lower the regularity of the metric further, for example to metrics that are $C^{1,\alpha}$ for some $\alpha < 1$, then one encounters metrics for which the geodesic equations do not have unique solutions (see, e.g., [36, 18] for examples in Riemannian and Lorentzian geometry, respectively). Since this implies that the exponential map is no longer a homeomorphism onto a neighbourhood of a point, this poses a genuine obstacle to generalising the standard proofs of classical results to low-regularity metrics.

This thesis is divided into three chapters: In Chapter 1 we will study manifolds that satisfy a bound on the scalar curvature and we will see that this theorem remains valid in a certain low-regularity setting. Chapter 2 discusses volume comparison results for manifolds that satisfy bounds on the Ricci curvature. The final Chapter 3 focusses on convergence of Riemannian manifolds.

0.1. Positive mass theorem

Scalar curvature bounds provide the weakest notion of a curvature bound. They have applications not only in pure mathematics, but also in physics. The result about manifolds with scalar curvature bound that we will study in the following chapter is the positive mass theorem, which has its origin in the study of mass in General Relativity.

This part is organized as follows: In Section 1.1, we will discuss the physical background, the definition of mass and of asymptotic flatness of a Riemannian manifold and state the classical positive mass theorem and also the purely Riemannian version. Furthermore, an overview of the historic development of proofs of this theorem will be provided.

In Section 1.2 we will recall the proofs of the classical positive mass theorem, i.e., for manifolds with smooth metrics. We will sketch both the geometric proof due to Schoen and Yau, [61], for manifolds of dimension $\leqslant 7$ and also the spin approach by Witten, [68], which holds for spin manifolds of arbitrary dimension.

In Section 1.3 we will state some low regularity versions of the positive mass theorem, starting with the result by Miao [51] who studies manifolds that admit corners along hypersurfaces. His method of smoothing and conformally rescaling the approximating metrics to get smooth metrics with nonnegative scalar curvature will be described. Furthermore the Ricci flow technique used by McFeron and Székelyhidi, [49], for the same type of manifolds will be described.

Another result that will be mentioned in this Section is the version by Lee ([44]) who studies Lipschitz metrics with small singular sets. We will see that in his proof he also makes use of Miao's conformal rescaling method.

Section 1.4 is based on [31]. In this Section, we show that the positive mass theorem remains valid for continuous metrics $\mathbf{g}$ that lie in the local Sobolev space $W^{2,n/2}_{\mathrm{loc}}(M)$. We assume that M is either a spin manifold, or that $n \leqslant 7$ in order for the classical positive mass theorem to hold. We assume that the metric $\mathbf{g}$ is smooth outside of the compact set K and satisfies proper asymptotic conditions, although it is also straightforward to generalise our results to the case where the metric lies in an appropriate weighted space outside of K as in [6]. Since $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$, the scalar curvature of $\mathbf{g}$, $s_{\mathbf{g}}$, lies in $L^{n/2}_{\mathrm{loc}}(M)$ and, therefore, is well-defined as a distribution on M. One of our main results is then the following:

THEOREM 0.1.1. *Let $(M, \mathbf{g})$ be a complete, asymptotically flat Riemannian manifold, with $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$ and $s_{\mathbf{g}}$ non-negative in the distributional sense (i.e. $\langle s_{\mathbf{g}}, \varphi \rangle \geqslant 0, \forall \varphi \in \mathscr{D}(M)$). Then the ADM mass $m_{\mathbf{g}}$ is non-negative.*

In fact, our methods allow us to prove a stronger result, for metrics $\mathbf{g}$ for which the scalar curvature is not constrained to be non-negative. Denoting the Sobolev constant of the metric $\mathbf{g}$ by $c_1[\mathbf{g}]$, and the Riemannian measure of a measurable set $E \subseteq M$ by $|E|_{\mathbf{g}}$, we show the following:

THEOREM 0.1.2. *Let $(M, \mathbf{g})$ be a complete, asymptotically flat Riemannian manifold, with $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$ with the properties that $(s_{\mathbf{g}})_-$ is of compact support and that*

$$c_1[\mathbf{g}]\,\|(s_{\mathbf{g}})_-\|_{L^{n/2}(M,\mathbf{g})} < 4\,\frac{n-1}{n-2}.$$

Then the mass of the metric $\mathbf{g}$ satisfies

$$m_{\mathbf{g}} \geqslant -\frac{1}{2(n-1)\omega_{n-1}} \frac{\|(s_{\mathbf{g}})_-\|_{L^{n/2}(M,\mathbf{g})}}{\left(1 - \frac{n-2}{4(n-1)} c_1[\mathbf{g}]\,\|(s_{\mathbf{g}})_-\|_{L^{n/2}(M,\mathbf{g})}\right)^2} \,|\operatorname{supp}(s_{\mathbf{g}})_-|_{\mathbf{g}}^{2/n^*},$$

with $n^* = \frac{2n}{n-2}$. Therefore, as long as the negative part of the scalar curvature of $\mathbf{g}$ is of compact support and has sufficiently small $L^{n/2}$ norm, we have a (negative) lower bound on mass. To our knowledge, this is the first result of this type.

Our approach to proving the positive mass theorem is to construct appropriate smooth approximations to the metric $\mathbf{g}$, and is based on a modification of the approach of Miao [51] (see, also, [49, 44]). In [51], metrics were considered where the singular behaviour was localised on a hypersurface. By an ingenious mollification technique, he smoothed out the metric on a neighbourhood of this hypersurface in a controlled way, while leaving the metric unchanged on the rest of the manifold. The metrics that we consider can be non-smooth on an arbitrary compact subset $K \subset M$, so we cannot directly apply this technique. Nevertheless, we proceed by smoothing the metric on the compact set K but leaving it unchanged on $M \backslash K$. This implies that the smooth approximating metrics $\mathbf{g}_\varepsilon$ have the same asymptotic behaviour, and hence the same mass, as the metric $\mathbf{g}$. The metrics $\mathbf{g}_\varepsilon$ generally no longer have non-negative scalar curvature, but we may perform a conformal transformation to give a new family of smooth metrics $\widehat{\mathbf{g}}_\varepsilon$ that are both asymptotically flat and have non-negative scalar curvature. The classical positive mass theorem then implies that $m_{\widehat{\mathbf{g}}_\varepsilon} \geqslant 0$. Using elliptic estimates, we show that $m_{\widehat{\mathbf{g}}_\varepsilon} \to m_{\mathbf{g}}$ as $\varepsilon \to 0$, thereby implying that $m_{\mathbf{g}} \geqslant 0$.

Section 1.5, which is also part of [30, 31], investigates the convergence properties of the approximating metrics. We will show that the scalar curvatures of the approximating metrics converge in $L^{n/2}_{\mathrm{loc}}(M)$ to the scalar curvature of the rough metric. Furthermore we will investigate how the Sobolev constants of the smooth metrics are related to the Sobolev constant of the non-smooth metric.

In Section 1.6, we study a Dirichlet problem, which comes from the equation of the conformal transformation of scalar curvature. It gives rise to functions v which will then be used as conformal factors we multiply the approximating metrics with. We will study the analytic properties of these factors, and see how small negative curvature influences these properties. Furthermore we also get upper and lower bounds on the solution of the Dirichlet problem.

In the case where the metric $\mathbf{g} \in W^{2,p}_{\mathrm{loc}}$ for some $p > \frac{n}{2}$ (and therefore, by the Sobolev embedding theorem, is continuous) we have the following approximation theorem:

Theorem 0.1.3. *Let $\mathbf{g}$ be a Riemannian metric on an open set $\Omega \subseteq M$ of regularity $W^{2,p}_{\mathrm{loc}}(\Omega)$, $p > \frac{n}{2}$ with non-negative scalar curvature in the distributional sense. Then there exists a family of smooth, Riemannian metrics $\{\widehat{\mathbf{g}}_\varepsilon \mid \varepsilon > 0\}$ with non-negative scalar curvature such that $\widehat{\mathbf{g}}_\varepsilon$ converge locally uniformly to $\mathbf{g}$ as $\varepsilon \to 0$.*

The elliptic estimates that we require to prove this result break down for metrics $\mathbf{g} \in C^0(\Omega) \cap W^{2,n/2}_{\mathrm{loc}}(\Omega)$. However, we can show the following:

Theorem 0.1.4. *Let $\mathbf{g}$ be a Riemannian metric on an open set Ω of regularity $C^0(\Omega) \cap W^{2,n/2}_{\mathrm{loc}}(\Omega)$ with non-negative scalar curvature in the distributional sense. Then there exists* *a family of smooth, Riemannian metrics* $\{\widehat{\mathbf{g}}_\varepsilon \mid \varepsilon > 0\}$ *with non-negative scalar curvature such that* $\widehat{\mathbf{g}}_\varepsilon$ *converge* $\mathbf{g}$ *in* $L^p_{\mathrm{loc}}(\Omega)$ *as* $\varepsilon \to 0$*, for all* $p < \infty$*.*

In general, we do not expect that it will be possible to locally uniformly approximate continuous metrics in $W^{2,n/2}_{\mathrm{loc}}$ with non-negative scalar curvature by smooth metrics with non-negative scalar curvature. Our results suggest an underlying "bubbling off" or non-compactness phenomenon. In light of the critical Sobolev embedding of $W^{2,n/2}$, we expect, however, that the approximating metrics that we construct should converge to $\mathbf{g}$ in appropriate BMO or Orlicz spaces [64]. We have, however, not investigated this possibility.

In the final Section 1.7 we argue that the rigidity statement of the positive mass theorem is unlikely to hold for the metrics studied in [30].

0.2. Volume comparison

We will study manifolds that satisfy Ricci curvature bounds. A Ricci curvature bound is weaker than a sectional curvature bound, but stronger than a bound on the scalar curvature.

For manifolds with sectional curvature bounds, many results have been proven, including the 1/4-pinched Sphere Theorem, the Soul Theorem, and the homotopy finiteness theorem.

The aim became to generalize these results to manifolds that just satisfy a Ricci curvature bound. From 1967 on, examples were constructed that show the differences between sectional and Ricci curvature. It was proven e.g., that the Toponogov comparison theorem does not hold for manifolds that satisfy only a bound on the Ricci curvature.

A detailed review on the comparison geometry of Ricci curvature is provided by [72], for a survey about manifolds with a lower Ricci curvature bound see [66].

In Section 2.1, we will introduce a basic tool for the study of manifolds with Ricci curvature bounds, namely the Bochner formula.

In Section 2.2, the geometry of manifolds with constant curvature will be investigated. We will also study the behavior of the volume element and the Laplace operator in both the constant curvature and in the arbitrary curvature setting. We will use these results to investigate classical volume comparison results in Section 2.3 where we will prove the classical Bishop-Gromov volume comparison result and mention some of the direct applications of this result. We will also investigate mean curvature- and Laplacian comparison.

In Section 2.4, our aim is to generalise the volume comparison results to the case where we have an integral bound on the part of the Ricci tensor that violates $\mathbf{Ric} \geqslant c(n-1)\mathbf{g}$. For simplicity, we only treat the case $c = 0$, but our results may easily be adapted to any $c \in \mathbb{R}$. Results on this topic include [24, 69] and, of particular relevance to us, those of [59].

We will develop volume estimates and monotonicity formula based on integral norms of a weighted version of the negative part of Ricci along radial geodesics from each point.

The novelty of our approach is the use of *weighted* curvature quantities, which lead to sharper formulae in volume comparison calculations.

The main results are the following:

THEOREM 0.2.1. *Let $\widehat{S} \subseteq S_x$, $r > 0$ and $p \geqslant \frac{n}{2}$. Then,*

$$\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant (2p-1)\frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|}\|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))}. \tag{0.2.1}$$

Moreover, if $r < \ell(\theta)$ for all $\theta \in \widehat{S}$, then we have the slightly sharper estimate

$$\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant (2p-1)\frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|}\fint_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\,dt, \tag{0.2.2}$$

where $\fint_0^r f(t)\,dt := \frac{1}{r}\int_0^r f(t)\,dt$.

A special case of our results are those for geodesic balls, where we take $\widehat{S} = S_x$. We then get

THEOREM 0.2.2. *For $0 \leqslant r_0 \leqslant r$ then, for any $p \geqslant \frac{n}{2}$, we have*

$$\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant (2p-1)\frac{V(x,r)^{1-\frac{1}{2p-1}}}{V_0(r)}\fint_0^r \|R_-\|_{L^{2p-1}(B(x,t))}\,dt. \tag{0.2.3}$$

Moreover, for $r < \operatorname{inj} x$, we have

$$\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant (2p-1)\frac{V(x,r)^{1-\frac{1}{2p-1}}}{V_0(r)}\fint_0^r \|R_-\|_{L^{2p-1}(B(x,t))}\,dt.$$

Furthermore, we prove the following monotonicity results:

THEOREM 0.2.3. *Let $p \geqslant \frac{n}{2}$. The quantities*

$$\frac{V(x,r)}{V_0(r)}\exp\left[-(2p-1)\int_0^r\left\{\fint_0^s\left(\fint_{B(x,t)} R_-^{2p-1}\,d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right\} ds\right] \tag{0.2.4}$$

and

$$\frac{V(x,r)}{V_0(r)}\exp\left[-(2p-1)\int_0^r\left(\fint_{B(x,t)} R_-^{2p-1}\,d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right] \tag{0.2.5}$$

are non-increasing functions of r, both of which converge to 1 as $r \to 0$.

0.3. Convergence of Riemannian manifolds

In this chapter we will discuss notions of convergence of Riemannian manifolds. We will start in Section 3.1 with introducing the notions of Hausdorff- and Gromov-Hausdorff convergence and describe how a bound on the Ricci curvature leads to convergence in this topology. We will provide classical examples of which bounds lead to convergence. In particular, we will discuss how a pointwise bound on **Ric** as well as a L^p-bound leads to

Gromov-Hausdorff convergence. Furthermore, we will also see that the set of manifolds satisfying the curvature bounds as described in Part 2 are compact in the Gromov-Hausdorff topology.

In particular, we get

PROPOSITION 0.3.1. *The class of Riemannian manifolds with*

$$\begin{aligned} \operatorname{diam}_M &\leqslant D \\ \left(\fint_{B(x,t)} R_-^{n-1} \, d\mu_{\mathbf{g}} \right)^{1/(n-1)} &\leqslant \varepsilon \end{aligned}$$

is precompact in the Gromov-Hausdorff topology.

In Sections 3.2 and 3.3, we will study a different way of describing the convergence properties of Riemannian manifolds by introducing convergence in certain Sobolev or Hölder spaces. In order to study the best regularity properties for convergence in these spaces, we will introduce harmonic coordinates and the notion of harmonic radius and describe their properties. This Section should provide a review over well-known material, of particular relevance is the work of Anderson and Cheeger [3] on sequences of Riemannian manifolds with Ricci curvature bounded below, and the work of Anderson [1] (see also [37]) on manifolds with Ricci curvature bounded above and below. For the sake of completeness, the proofs of the results are presented, with details added that simplify the understanding of the concept of the proofs. Here we will see how a bound on the harmonic radius leads to convergence of this sequence of manifolds.See also, [14] for a review of work concerning Gromov–Hausdorff limits of sequences of manifolds with Ricci curvature uniformly bounded below.

Section 3.4 provides a historic overview of convergence results for Riemannian manifolds.

Section 3.5 discusses sequences of manifolds with certain integral bounds on the Ricci curvature. We will describe how an additional bound on the full curvature tensor leads to convergence in Hölder spaces. As a final result we get:

PROPOSITION 0.3.2. *Let $n \geqslant 2$, $p > \frac{n}{2}$, $v > 0$, $D < \infty$, $K < \infty$. Then there exists an $\varepsilon = \varepsilon(n, p, K, D)$ such that the class of closed Riemannian manifolds with*

$$\operatorname{Vol}(M) \geqslant v \tag{0.3.1}$$

$$\operatorname{diam}_M \leqslant D \tag{0.3.2}$$

$$\int_M |\mathbf{R}|_{L^p(M)} \leqslant K \tag{0.3.3}$$

$$\left(\fint_{B(x,t)} R_-^{n-1} \, d\mu_{\mathbf{g}} \right)^{1/(n-1)} \leqslant \varepsilon \tag{0.3.4}$$

is precompact in the $C^{0,\alpha}$ topology for $\alpha < 2 - \frac{n}{p}$.

CHAPTER 1

The positive mass theorem

1.1. Background

Let $(\bar{M}, \bar{\mathbf{g}})$ be a 4-dimensional spacetime, i.e. a Lorentzian manifold with the signature of $\bar{\mathbf{g}}$ given by $(-,+,+,+)$. Furthermore let $\bar{\mathbf{g}}$ satisfy the Einstein equations

$$\mathrm{Ric}_{ab} - \frac{1}{2} s \bar{\mathbf{g}}_{ab} = 8\pi T_{ab}, \tag{1.1.1}$$

where Ric_{ab} is the Ricci curvature tensor, s the scalar curvature and T is a symmetric $(0,2)$ tensor, the so-called *stress-energy-momentum tensor.* From a physical point of view, this tensor is the source of the gravitational field. T is divergence free and we assume that it furthermore satisfies the *dominant energy condition (DEC)*

$$T^{00} \geqslant |T^{ab}|, \qquad (a,b = 1,2,3) \qquad T^{00} \geqslant \sum_{i=1}^{3} (-T_{0i}T^{0i})^{1/2}. \tag{1.1.2}$$

For many problems in General Relativity, also including the positive mass theorem, it is not necessary to work with the entire spacetime, it suffices to investigate a spacelike slice. Thus the focus is on the Cauchy data $(M, \mathbf{g}, k)$ where M is a three dimensional Riemannian manifold, with metric $\mathbf{g}$, and k is a symmetric $(0,2)$ tensor, the second fundamental form.

A necessary and sufficient condition for this triple to be a spacelike slice of a spacetime are the following *constraint equations*, which are obtained from the Gauss-Codazzi equations:

$$2\mu = s_{\mathbf{g}} + (\mathrm{tr}_{\mathbf{g}}\, k)^2 - \|k\|^2_{\mathbf{g}}, \qquad J_i = \nabla^j (k_{ij} - (\mathrm{tr}_{\mathbf{g}}\, k)\mathbf{g}_{ij}, \tag{1.1.3}$$

for n a future directed unit normal vector to M, $\mu := T(n,n)$ the energy density, and $J := T(n, \cdot)$ the momentum density.

From the DEC it then follows that

$$\mu \geqslant \|J\|_{\mathbf{g}}.$$

When $\mathrm{tr}_{\mathbf{g}}\, k = 0$, i.e. the slicing is maximal, this condition is equivalent to the condition of nonnegative scalar curvature.

A further condition which is imposed on $(M, \mathbf{g})$ is that of asymptotic flatness. We can define for arbitrary dimensions:

Definition 1.1.1. A smooth complete oriented n-dimensional Riemannian manifold $(M, \mathbf{g})$ is called *asymptotically flat of order* $\tau > 0$, if there exists a compact set $K \Subset M$, such that $M \backslash K$ consists of a finite number of *ends* $N_1, \dots, N_l$ such that each N_i is diffeomorphic to $\mathbb{R}^n \backslash B_r$, where B_r denotes a ball in $\mathbb{R}^n$. The metric has to satisfy the following asymptotic conditions:

$$g_{ij} = \delta_{ij} + O(\rho^{-\tau}), \qquad \partial_k g_{ij} = O(\rho^{-\tau-1}), \qquad \partial_k \partial_l g_{ij} = O(\rho^{-\tau-2}). \tag{1.1.4}$$

for $\rho = |z| \to \infty$ in the, so called, *asymptotic coordinates* $\{z_i^k\}$ on the ends N_k, see [61].

The diffeomorphism $\Phi : M \backslash K \to B_R$ is called *a structure at infinity* if

(1) $(\Phi_* g)_{ij}$ is uniformly equivalent to the flat metric on B_R, the ball of radius R, i.e. there is a $\lambda \geqslant 1$ with

$$\lambda^{-1}|\xi|^2 \leqslant (\Phi_* g)_{ij}\xi^i\xi^j \leqslant \lambda|\xi|^2. \tag{1.1.5}$$

(2) $(\Phi_* g)_{ij} - \delta_{ij} \in W^{1,q}_{-\tau}(E_R)$.

From a structure at infinity one gets coordinates at infinity by setting $x^i = \Phi^i(m)$, $m \in M$.

Equivalently (see [46]) one can describe a manifold $(M, \mathbf{g})$ to be asymptotically flat of order τ if there exists a decomposition $M = M_0 \cup M_\infty$ for a compact M_0 and a diffeomorphism between M_∞ and $\mathbb{R}^n \backslash B_r$ that satisfies the asymptotic conditions (1.1.4), for manifolds with one end, and analogously for more ends.

This definition seems to depend on the choice of coordinates. However, Bartnik showed in [6, Section 3] that an asymptotically flat structure is determined only by the metric, and that two asymptotic structures at infinity differ only by a rigid motion and terms of order $o(r^{1-\tau})$.

1.1.1. Mass in General relativity.

Definition 1.1.2. Let $(\bar{M}, \bar{\mathbf{g}})$ be a spacetime with initial data $(M, \mathbf{g}, k)$ where M is an asymptotically flat Riemannian manifold. Its *mass* or *total energy* is defined as

$$E(\mathbf{g}) := \frac{1}{16\pi} \lim_{R\to\infty} \int_{S_R} \sum_{i,j} (\partial_i g_{ij} - \partial_j g_{ii}) d\Sigma^i_{\mathbf{g}}, \tag{1.1.6}$$

if this limit exists. In this equation $\Sigma_{\mathbf{g}}$ is the surface element of the sphere S_R in $\mathbb{R}^3$. Furthermore, its *linear momentum* is given by

$$P_k(\mathbf{g}) = \frac{1}{8\pi} \lim_{R\to\infty} \int_{S_R} \left(2\sum_i k_{il} - \delta_{il} \sum_j k_{jj} \right) d\Sigma^i_{\mathbf{g}}. \tag{1.1.7}$$

For a detailed derivation of these formulae, see [28].

The appropriate topology to study the mass functional is provided by weighted Hölder spaces.

Both E and P_i are defined on the asymptotic ends, where only the asymptotic behaviour of g_{ij} and k_{ij} plays a role.

The energy E and the linear momentum P_i are collected in a 4-vector (E, P_i). The (total) mass of the spacetime is defined as

$$m(\mathbf{g}) = \sqrt{E^2 - P_i P_j \delta^{ij}}. \tag{1.1.8}$$

In the definition of the energy, (1.1.6), the integral is not necessarily finite. Another problem that can arise is that the mass possibly depends on the choice of coordinates. Both problems can be ruled out by choosing the fall-off conditions correctly. The following theorem, proved in [6] and [19], says that if $\mathbf{g}$ satisfies the fall-off conditions

$$g_{ij} = \delta_{ij} + \gamma_{ij}, \tag{1.1.9}$$

with

$$\gamma_{ij} = O(r^{-\alpha}), \qquad \partial_k \gamma_{ij} = O(r^{-\alpha-1}) \tag{1.1.10}$$

for $\alpha > \frac{1}{2}$, then it holds that:

Theorem 1.1.3. *Let U be an end with a Riemannian metric g_{ij} such that the fall-off conditions* (1.1.10) *with $\alpha > 1/2$ are satisfied. Assume also that the scalar curvature $s_{\mathbf{g}}$ is integrable in U, that is*

$$\int_U |s_{\mathbf{g}}|\, dv < \infty. \tag{1.1.11}$$

Then the energy/mass defined by (1.1.6) *is unique and it is finite.*

The positive energy theorem then states the following:

Theorem 1.1.4 (Positive energy theorem). *Let $(M, \mathbf{g}, k)$ be an asymptotically flat (with possibly many asymptotic ends), complete, initial data set, such that the dominant energy condition* (1.1.2) *holds. Then the mass, as defined by* (1.1.8), *satisfies*

$$m \geqslant \sqrt{P_i P_j \delta^{ij}} \geqslant 0. \tag{1.1.12}$$

at every end. Moreover, $m = 0$ at any end if and only if the initial data correspond to the Minkowski space-time.

This result was first proven by Schoen and Yau in the late 1970s and early 1980s with purely geometric methods, [61], for manifolds of dimension $\leqslant 7$ and later by Witten, [68], using a different approach by investigating manifolds of arbitrary dimension, but that admit a spin structure.

The notion of energy can also be discussed in a purely Riemannian setting, modelling time symmetric initial data. The second fundamental form then plays no role anymore, $k_{ij} = 0$. This Riemannian case of the positive mass theorem has applications in other areas of mathematics, it is, e.g., used in resolving the Yamabe problem (see [46] and references therein). We will mainly focus on this purely Riemannian setting.

Corollary 1.1.5 (Riemannian positive mass theorem). *Let $(M, \mathbf{g})$ be a complete, asymptotically flat, Riemannian manifold. Assume that the scalar curvature is non-negative. Then the energy is non-negative at every end and it is zero at one end if and only if the metric is flat.*

This corollary was proved with the optimal decay conditions for the metric in [6] and [46].

1.1.2. Historic overview. The study of the positive mass theorem started in the 1960ies when Arnowitt, Deser and Misner conjectured that the mass of a spacetime along a spacelike hypersurface is always nonnegative and vanishes only if the spacetime is empty. The proof of this result has a long history, which is outlined in Table 1.

Following Witten in [68], we will now give a brief overview over the different versions of the positive energy conjecture.

In this results, only smooth manifolds (i.e. manifolds with metrics that are smooth) are investigated. In Sections 1.3 and 1.4, we will also investigate manifolds which are of certain lower regularity.

1.2. Classical positive mass theorem

1.2.1. Geometric approach. Schoen and Yau provided the first proof of the Positive Mass/Energy theorem in [61] by using geometric methods. In particular they make use of minimal surface theory to obtain the result. This proof works by contradiction.

	Author	Spacetime
1959/60	Araki Brill, Deser	Minkowski space is the unique stationary point of energy, it is a local minimum
1970	Leibkovitz, Israel	PEC for spherically symmetric initial data set
1971	Misner	PEC for spherically symmetric initial data set
1974	O'Murchadha, York	Time symmetric initial data, spaces with maximal hypersurfaces
1975	Geroch	Spaces with Minkowsi topology admitting a maximal hypersurface
1976	Jang	Spaces with flat initial hypersurfaces
1977	Leite	Spaces whose initial value surface can be embedded isometrically in $\mathbb{R}^4$
1976	Choquet-Bruhat, Marsden	Any space that is in a sufficiently small neighborhood of Minkowski space
1979	Schoen, Yau	Space with maximal slicing
1979	Schoen, Yau	General spacetimes of dimension $\leqslant 7$
1981	Witten	General spacetimes that admit a spin structure
2006	Lohkamp	Manifolds of arbitrary dimension, no topological restrictions (unpublished)

TABLE 1. Proofs for different specific manifolds

Let $(M, \mathbf{g})$ be a Riemannian manifold (a spacelike slice of a Lorentzian manifold), and suppose that the mass on one end M_k is negative, $m < 0$, but the scalar curvature of M is non-negative, $s_{\mathbf{g}} \geqslant 0$. They show that this is not possible.

Schoen and Yau proceed in several steps:

Step 1: Let $\mathbf{g}$ be an asymptotically flat metric on M, with $s_{\mathbf{g}} \geqslant 0$, and $m(\mathbf{g}) < 0$. Then there exists an asymptotically flat metric $\bar{\mathbf{g}}$ which is conformally equivalent to $\mathbf{g}$, which has non-negative scalar curvature $s_{\bar{\mathbf{g}}}$, with $s_{\bar{\mathbf{g}}} > 0$ outside a compact subset of the end M_k and has negative mass.

This metric can be written as

$$g_{ij} = \left(1 + \frac{m}{4r}\right)^4 \delta_{ij} + O\left(\frac{1}{r^2}\right),$$

where δ_{ij} is the Euclidean metric.

Step 2: Construct a complete area minimizing surface (with respect to $\bar{\mathbf{g}}$) S which is properly embedded into M.

Step 3: Show that such a surface can not exist. Indeed, by minimality of S, the trace of the second fundamental form of S, h, has to vanish, i.e. $h_{11} + h_{22} = 0$. By using a second variation argument and integration by parts, it follows that

$$\int_S \mathbf{Ric}(\nu, \nu) + \sum_{i,j=1}^{2} h_{ij}^2 f^2 \leqslant \int_S \|\nabla f\|^2, \tag{1.2.1}$$

for all C^2-functions f with compact support on S. Here ν denotes the normal to S. Using the Gauss curvature equation, $K = K_{12} + h_{11}h_{22} - h_{12}^2$ (where K is the Gaussian curvature of S) and the fact that $h_{11} + h_{22} = 0$, equation (1.2.1) becomes

$$\int_S s_{\bar{\mathbf{g}}} - K + \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2 f^2 \leqslant \int_S \|\nabla f\|^2. \tag{1.2.2}$$

By choosing an appropriate f, it follows that

$$\int_S s_{\bar{\mathbf{g}}} - K + \frac{1}{2}\sum_{i,j=1}^{2} h_{ij}^2 \leqslant 0. \tag{1.2.3}$$

Since $s_{\bar{\mathbf{g}}} \geqslant 0$, and $s_{\bar{\mathbf{g}}} > 0$ outside a compact subset of S, (1.2.3) implies that

$$\int_S K > 0. \tag{1.2.4}$$

But this does not hold. Indeed, by applying a modified version of the Gauss-Bonnet theorem on open Riemannian surfaces, [23, 39] or by applying the classical Gauss-Bonnet theorem and estimating the boundary term ([61], pp. 55), one gets

$$\int_S K \leqslant 0, \tag{1.2.5}$$

which is a contradiction.

This proof can be applied to manifolds up to dimension $n = 7$. In higher dimensions the minimal surface argument breaks down.

1.2.2. Spin approach. In Witten's approach [68], the main idea is to write the mass as a sum of squares. His proof of the positive mass theorem holds true for spin manifolds of arbitrary dimension. For an introduction to spin manifolds see [43].

The Lichnerowicz formula ([47]) for the square of the Dirac operator on a spin manifold is given by

$$D^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi \tag{1.2.6}$$

for sections ψ of a spin bundle with covariant derivative ∇.

Integration by parts of (1.2.6) leads to the Weitzenböck formula

$$\int_M |\nabla\psi|^2 + \frac{s}{4}|\psi| = \text{boundary term.} \tag{1.2.7}$$

Thus, in case the manifold M is a closed Riemannian spin manifold of nonnegative scalar curvature s, which is positive at least somewhere, there are no nontrivial ψ with $\nabla\psi = 0$ on M.

A calculation, see, e.g. [54], shows that the boundary integral is given by the difference $E - |P|$. Thus formula (2.1.3) is the anticipated formula for the mass as a sum of squares. From this, the positive mass theorem follows easily.

In the proof of this result it is required to show the existence of harmonic spinors with appropriate fall-off conditions. Details for these calculations can be found in [54].

1.3. Low regularity versions

Schoen and Yau investigated manifolds with smooth (or at least C^2) metrics. We will now start lowering the regularity of the metric.

1.3.1. Manifolds admitting corners along hypersurfaces.

Approach via conformal transformations

In [51] the author investigates the special case where the metric has a jump across a hypersurface Σ, therefore fails to be C^1 across this surface. We will now give an overview of this result, since [30, 31] use modifications and similar techniques to this paper.

Let $n \geqslant 3$ and let $(M, \mathbf{g})$ be an oriented n-dimensional smooth differentiable manifold without boundary which is asymptotically flat, see Definition 1.1.1. Furthermore, let the classical positive mass theorem be valid on M. Let Σ be the hypersurface ∂K, where K is the compact set in Definition 1.1.1.

DEFINITION 1.3.1. A *metric* $\mathbf{g} = (\mathbf{g}_-, \mathbf{g}_+)$ *admitting corners along* Σ is a Lipschitz metric, such that its restrictions g_- on K and g_+ on $M\backslash\bar{K}$ are in $C^{2,\alpha}$. Furthermore $\mathbf{g}_+, \mathbf{g}_-$ are C^2 up to the boundary and $\mathbf{g}_+|_\Sigma = \mathbf{g}_-|_\Sigma$.

Such a metric $\mathbf{g}$ is called *asymptotically flat* if $(M\backslash K, \mathbf{g}_+)$ is asymptotically flat in the sense of Definition 1.1.1.

The *mass* of $\mathbf{g}$ is defined to be the mass of $\mathbf{g}_+$ as in Definition 1.1.2, whenever it exists.

We denote the scalar curvatures of $\mathbf{g}_-$ and $\mathbf{g}_+$ by $s_{\mathbf{g}_-}$ and $s_{\mathbf{g}_+}$, respectively. Furthermore, the mean curvatures of Σ in $(K, \mathbf{g}_-)$ and $(M\backslash K, \mathbf{g}_+)$ with respect to unit normals pointing outwards from K, are denoted by $H(\Sigma, \mathbf{g}_-)$ and $H(\Sigma, \mathbf{g}_+)$.

The main result of [51] is the following:

THEOREM 1.3.2. [51, Theorem 1] *Let* $\mathbf{g} = (\mathbf{g}_-, \mathbf{g}_+)$ *be an asymptotically flat metric admitting corners along* Σ. *Let the scalar curvatures* $s_{\mathbf{g}_-}$, $s_{\mathbf{g}_+}$ *be nonnegative on* K *and* $M\backslash K$, *respectively. If* $H(\Sigma, \mathbf{g}_-)$ *and* $H(\Sigma, \mathbf{g}_+)$ *furthermore satisfy*

$$H(\Sigma, \mathbf{g}_-) \geqslant H(\Sigma, \mathbf{g}_+), \tag{1.3.1}$$

then the mass of $\mathbf{g}$ *is nonnegative. It is strictly positive if at one point strict inequality holds.*

REMARK 1.3.3. The condition (1.3.1) can be interpreted as the condition that the scalar curvature of $\mathbf{g}$ be nonnegative in a distributional sense, see [51, Section 2] for a detailed explanation.

REMARK 1.3.4. For a C^2 metric $\mathbf{g}$ in a neighborhood of Σ, the Gauss equation holds. Taking the trace of the Gauss equation leads to

$$2K = s_{\mathbf{g}} - 2\mathbf{Ric}(\nu, \nu) + h^2 - |A|^2, \tag{1.3.2}$$

with $\mathbf{Ric}(\nu, \nu)$ being the Ricci curvature of $\mathbf{g}$ along ν, h the mean curvature, A the second fundamental form of Σ and K its Gaussian curvature.

If Σ evolves with speed ν, then the following evolution equation holds:

$$D_\nu h = -\mathbf{Ric}(\nu, \nu) - |A|^2. \tag{1.3.3}$$

Combining (1.3.2) and (1.3.3), we arrive at

$$s_{\mathbf{g}} = 2K - (|A|^2 + h^2) - 2D_\nu h. \tag{1.3.4}$$

This equation will play a role in the estimation of the scalar curvature for approximating metrics, see (2) below.

Proof of Theorem 1.3.2. The proof of Theorem 1.3.2 can be split into different steps:

(1) Smoothing $\mathbf{g}$ across Σ by metrics $\mathbf{g}_\varepsilon$ that are C^2 across Σ.
(2) Estimating $s_{\mathbf{g}_\varepsilon}$.
(3) Modifying $\mathbf{g}_\varepsilon$ by a conformal transformation to obtain C^2 metrics $\hat{\mathbf{g}}_\varepsilon$ which have nonnegative scalar curvatures.

(4) Showing that the masses of $\hat{\mathbf{g}}_\varepsilon$ converge to that of $\mathbf{g}$.
(5) Finally concluding, by using the classical positive mass theorem for $\hat{\mathbf{g}}_\varepsilon$, that $m(\mathbf{g}) \geqslant 0$.

For (1), Miao constructs certain mollifiers in a neighborhood of Σ and is able to show that the smoothed metrics $\mathbf{g}_\varepsilon$ are uniformly close to $\mathbf{g}$ on M, and are equal to $\mathbf{g}$ outside a neighborhood U of Σ.

In order to get the bounds (2), Miao uses (1.3.4) and estimates each term. He then arrives at

$$\begin{aligned} s_{\mathbf{g}_\varepsilon}(x,t) &= O(1), \qquad \text{for } (x,t) \text{ outside of } U && (1.3.5)\\ s_{\mathbf{g}_\varepsilon}(x,t) &= O(1) + \{H(\Sigma,\mathbf{g}_-)(x) - H(\Sigma,\mathbf{g}_+)(x)\}f(t),\\ &\quad \text{for } (x,t) \in U, && (1.3.6)\end{aligned}$$

where $O(1)$ contains quantities that are bounded by constants depending only on $\mathbf{g}$, not on ε and $f(t)$ is a function determined by the chosen mollifier and the size of U, [51, Proposition 3.1.].

For (3) a lemma, due to Schoen and Yau in [61], is used:

LEMMA 1.3.5. [61, Lemma 3.2] *There exists a constant $\varepsilon_0 = \varepsilon_0(\mathbf{g}) > 0$ with the property that if*

$$\|f_-\|_{L^{n/2}(M,\mathbf{g})} \leqslant \varepsilon_0(\mathbf{g}),$$

then the partial differential equation

$$\Delta_{\mathbf{g}} u - fu = g$$

has a unique positive solution on M that satisfies $u(x) = \frac{A}{r^{n-2}} + \omega(r)$ as $r \to \infty$, where A is constant and $\omega(r) = O(r^{1-n})$ as $r \to \infty$.

The proof of this lemma will be investigated more closely in Section 1.6.

Miao then considers for each ε the equation

$$\Delta_{\mathbf{g}_\varepsilon} u_\varepsilon + C_n(s_{\mathbf{g}_\varepsilon})_- u_\varepsilon = 0 \tag{1.3.7}$$

with the additional condition

$$\lim_{\varepsilon\to 0} u_\varepsilon = 1. \tag{1.3.8}$$

It follows from (1.3.6) and the assumptions on $s_{\mathbf{g}_-}$ and $s_{\mathbf{g}_+}$ that

$$\begin{cases} s_{\mathbf{g}_\varepsilon -} = 0, & \text{outside } U,\\ |s_{\mathbf{g}_\varepsilon -}| \leqslant C_0, & \text{inside } U. \end{cases} \tag{1.3.9}$$

Therefore, it is possible to apply Lemma 1.3.5 with f replaced by $-(s_{\mathbf{g}_\varepsilon})_-$, g replaced by 0 and $\mathbf{g}$ by g_ε, for ε small, to obtain a solution to (1.3.7).

Miao is furthermore able to show that $u_\varepsilon \to 1$ in L^∞. In more detail, he gets the following:

PROPOSITION 1.3.6. [51, Proposition 4.1.] $\lim_{\varepsilon\to 0} \|u_\varepsilon - 1\|_{L^\infty(M)} = 0$ *and on a compact set $K \subset M \backslash \Sigma$, also $\|u_\varepsilon\|_{C^{2,\alpha}(K)} \leqslant C_K$. The constant C_K only depends on g and K.*

In the proof Miao uses Moser iteration to obtain an L^∞ bound on $w_\varepsilon = u_\varepsilon - 1$, and he uses Schauder theory to also get the $C^{2,\alpha}$ bound.

Miao then finally defines a new metric, $\hat{\mathbf{g}}_\varepsilon$, by

$$\hat{\mathbf{g}}_\varepsilon = u_\varepsilon^{\frac{4}{n-2}} \mathbf{g}_\varepsilon.$$

By formula (A.2.4) in the Appendix the scalar curvature of $\hat{\mathbf{g}}_\varepsilon$ is non-negative.

To get (4), we use the fact that

$$m(\hat{\mathbf{g}}_\varepsilon) = m(\mathbf{g}_\varepsilon) + (n-1)A_\varepsilon,$$

where A_ε is the term in the expansion $u_\varepsilon = 1 + A_\varepsilon |x|^{2-n} + O(|x|^{1-n})$.

After integration by parts, one obtains

$$m(\mathbf{g}_\varepsilon) = m(\hat{\mathbf{g}}_\varepsilon) + \omega_n \frac{n-1}{n-2} \int_M \left(|\nabla_{\mathbf{g}_\varepsilon} u_\varepsilon|^2 - c_n (s_{\mathbf{g}_\varepsilon})_- u_\varepsilon^2 \right) d\mathrm{Vol}_{\mathbf{g}_\varepsilon}, \tag{1.3.10}$$

where ω_n is the volume of the $n-1$-dimensional unit sphere in $\mathbb{R}^n$.

The term in the integral goes to zero with $\varepsilon \to 0$. Thus

$$\lim_{\varepsilon \to 0} m(\hat{\mathbf{g}}_\varepsilon) = \lim_{\varepsilon \to 0} m(\mathbf{g}_\varepsilon) = m(\mathbf{g}). \tag{1.3.11}$$

To finally conclude that the mass of $\mathbf{g}$ is positive, apply the classical positive mass theorem to each of the $\hat{\mathbf{g}}_\varepsilon$. Indeed,

$$m(\mathbf{g}) = \lim_{\varepsilon \to 0} m(\hat{\mathbf{g}}_\varepsilon) \geqslant 0,$$

which finishes the proof of Theorem 1.3.2.

Miao furthermore proves the following rigidity result:

THEOREM 1.3.7. [51, Theorem 2] *Let $n = 3$ and $\mathbf{g}_-, \mathbf{g}_+$ satisfy all the assumptions in Theorem 1.3.2. If $\mathbf{g}_-$ and $\mathbf{g}_+$ are at least $C^{3,\alpha}_{loc}$, then, if the mass of $\mathbf{g}_+$ is zero, both $\mathbf{g}_-$ and $\mathbf{g}_+$ are flat away from the hypersurface Σ. They induce the same second fundamental form on Σ. $(K, \mathbf{g}_-)$ and $(M \backslash K, \mathbf{g}_+)$ together can be isometrically identified with the Euclidean space $(\mathbb{R}^3, \delta)$.*

In the proof, Miao makes use of the following result by Bray and Finster, [9]:

PROPOSITION 1.3.8. *Let $\{\mathbf{g}_i\}$ be a sequence of C^3, complete, asymptotically flat metrics on M^3 with non-negative scalar curvature and the total masses $\{m_i\}$ converging to a, possibly non-smooth, limit metric $\mathbf{g}$ in C^0. Let U be the interior of the set of points where this convergence of metrics is locally C^3.*

If the metrics $\{\mathbf{g}_i\}$ have uniformly positive isoperimetric constants and their masses $\{m_i\}$ converge to zero, then $\mathbf{g}$ is flat in U.

Details can be found in [51, Section 5].

Approach via Ricci flow

McFeron and Székelyhidi study in [49] the same type of manifolds as Miao, i.e. $(M, \mathbf{g})$ such that the classical positive mass theorem holds, with a hypersurface Σ along which the metric fails to be C^2, but $\mathbf{g}$ is asymptotically flat in C^2 and $H(\Sigma, \mathbf{g}_-) \geqslant H(\Sigma, \mathbf{g}_+)$, as defined in [51].
They modify the proof of Miao by making use of Ricci-flow techniques. Instead of conformally changing the smoothed metrics to get ones with non-negative scalar curvature, they use Ricci-flow, as introduced by Hamilton in [34] to smooth the metric. In this process, the non-negativity of the scalar curvature is preserved.
They use Simon's result, [62], due to which Ricci-flow can be started from a C^0 initial metric. In the next step they make use of Miao's bounds on the scalar curvatures of the $\mathbf{g}_\varepsilon$, (1.3.6), in order to show that the metrics which evolve along the Ricci-flow starting from $\mathbf{g}$ have non-negative scalar curvatures.

Next they prove that the mass is constant along the Ricci-flow, [49, Section 3].

Corollary 1.3.9. [49, Corollary 12] *The mass is preserved under the Ricci flow.*

Note that the proof uses a bound on the derivative of the curvature tensor.

In order to finally conclude the positive mass theorem for manifolds with corners, McFeron and Székelyhidi smooth the metric $\mathbf{g}$ by $\mathbf{g}_\varepsilon$ such that $\mathbf{g} = \mathbf{g}_\varepsilon$ outside a ball B, they are C^0 close to $\mathbf{g}$ and their scalar curvatures are bounded by a fixed constant K. Therefore, by [51, Proposition 3.1], also

$$\int_{\{s_{\mathbf{g}_\varepsilon}<0\}} |s_{\mathbf{g}_\varepsilon}| dV_{\mathbf{g}_\varepsilon} < \varepsilon.$$

They then take a solution $\mathbf{g}(t)$ of the Hamilton-DeTurk flow with initial metric $\mathbf{g}$, see [62]. Let h be the background metric, $h = \mathbf{g}_\varepsilon$ outside a compact set. Using the results by Simon, the $h-$flow of $\mathbf{g}$ can be obtained as the limit of the $h-$flows $\mathbf{g}_\varepsilon(t)$ with initial condition $\mathbf{g}_\varepsilon$. It turns out that these $\mathbf{g}(t)$ are asymptotically flat in $C^{1,\alpha}$, not in C^2 and have non-negative scalar curvature. By [46], this suffices to make the positive mass theorem hold for $\mathbf{g}(t)$.

In the final step they show:

Theorem 1.3.10. [49, Theorem 18] *The C^0-metric $\mathbf{g}$ has mass $m(\mathbf{g}) \geqslant 0$. If it vanishes, i.e. if $m(\mathbf{g}) \equiv 0$, then $\mathbf{g}$ is the flat metric up to a $C^{1,\alpha}$ change of coordinates.*

Sketch of Proof. Let $\mathbf{g}(t)$ be the solution of $h-$flow. Since the metric gets changed only outside a ball, $m(\mathbf{g}_\varepsilon) = m(\mathbf{g})$, for all ε. By the properties of the h-flow, [62, p.3ff], it is possible, for each $\varepsilon > 0$ to find diffeomorphisms $\phi_{\varepsilon,t}$ $(\phi_{\varepsilon,0} = \mathrm{id})$, such that $\phi^*_{\varepsilon,t}\mathbf{g}_\varepsilon(t)$ are solutions of the Ricci flow. Since the mass is constant along the Ricci flow, [49, Corollary 12], it follows that

$$m(\phi^*_{\varepsilon,t}\mathbf{g}_\varepsilon(t)) = m(\mathbf{g}_\varepsilon) = m(\mathbf{g}).$$

On the other hand, $m(\mathbf{g}_\varepsilon(t)) = m(\mathbf{g})$ since, due to [6, Theorem 4.2.], the mass does not depend on the choice of asymptotic coordinates.

Finally using the fact that $\mathbf{g}(t) = \lim \mathbf{g}_\varepsilon(t)$ and Theorem 14 in [49], it follows that $m(\mathbf{g}(t)) \leqslant m(\mathbf{g})$. The metrics $\mathbf{g}(t)$ have non-negative scalar curvature, thus by the classical positive mass theorem, $m(\mathbf{g}(t)) \geqslant 0$, and therefore also $m(\mathbf{g}) \geqslant 0$.

For the rigidity statement, see [49]. □

1.3.2. Lipschitz metrics with small singular sets. Compared to the previous results, where the singular set is a hypersurface, Lee shows in [44], that the positive mass theorem also is valid for metrics that are Lipschitz continuous, with a singular set S that is low-dimensional (i.e. its Minkowski dimension is small).

The m-dimensional Minkowski content is defined as follows:

DEFINITION 1.3.11. Let S be a subset of an n-dimensional Riemannian manifold $(M, \mathbf{g})$, then the *m-dimensional lower Minkowski content* of S is

$$\liminf_{\varepsilon \to 0} \frac{\mathcal{L}^n_g(S_\varepsilon)}{\alpha_{n-m}\varepsilon^{n-m}}$$

where $\mathcal{L}^n_g$ is the Lebesgue measure with respect to $\mathbf{g}$, S_ε is the ε-neighborhood of S, and ω_{n-m} is the volume of the unit ball in $\mathbb{R}^{n-m}$.

In detail, he proves the following results:

THEOREM 1.3.12. [44, Theorem 1.2] *Let M be a smooth manifold such that the classical positive mass theorem holds. Let $\mathbf{g}$ be a complete asymptotically flat Lipschitz metric on M, and let S be a bounded subset with vanishing $n/2$-dimensional lower Minkowski content. If $\mathbf{g}$ has bounded C^2-norm and if on $M \backslash S$ the scalar curvature is nonnegative, then the mass of $\mathbf{g}$ is nonnegative in each end.*

Lee also conjectures, without proving, that the dimension of S need to be less than $n-1$.

Furthermore he shows a $W^{1,p}$ version of this theorem.

THEOREM 1.3.13. [44, Theorem 1.3] *Let M be a smooth manifold such that the positive mass theorem is valid. Let $p > n$, let $\mathbf{g}$ be a complete asymptotically flat $W^{1,p}_{\mathrm{loc}}$ (and hence continuous) metric on M, and let S be a bounded subset whose $\frac{n}{2}(1 - \frac{n}{p})$-dimensional lower Minkowski content is zero. If $\mathbf{g}$ has bounded C^2-norm and nonnegative scalar curvature on the complement of S, then the mass of $\mathbf{g}$ is nonnegative in each end.*

Sketch of proof of Theorem 1.3.12. Let M be an n-dimensional Riemannian manifold with metric $\mathbf{g}$. Let S be the singular set in Theorem 1.3.12. By mollification of $\mathbf{g}$, Lee obtains a smooth metric $\mathbf{g}_\varepsilon$ with $g_\varepsilon = g$ outside a neighborhood of S. By using the hypothesis of a bounded C^2-norm of $\mathbf{g}$, the smoothing should have the property that $\mathbf{g}_\varepsilon$, $\mathbf{g}_\varepsilon^{-1}$, and $\partial \mathbf{g}_\varepsilon$ are bounded independently of ε, but $\partial\partial \mathbf{g}_\varepsilon = O(\varepsilon^{-1})$, with respect to a particular atlas. Thus, $s_{\mathbf{g}_\varepsilon} = O(\varepsilon^{-1})$. By the second hypothesis, i.e. the vanishing $n/2$-dimensional lower Minkowski content, it follows that the volume of S_ε (the ε neighborhood of S) is $o(\varepsilon^{n/2})$. Thus, for this smoothing,

$$\int_{S_{2\varepsilon}} |s_{g_\varepsilon}|^{n/2}\, dg = o(1). \tag{1.3.12}$$

Then Lee conformally deforms $\mathbf{g}_\varepsilon$ to metrics that have nonnegative scalar curvature, without changing the mass too much. By applying the classical positive mass theorem to the smooth metrics with nonnegative scalar curvature it follows that the original manifold $(M, \mathbf{g})$ has mass greater than a small negative number that is $o(1)$ in ε. For $\varepsilon \to 0$ the result follows.

□

1.4. Low regularity metrics with nonnegative scalar curvature in a distributional sense

Let $(M, \mathbf{g})$ be a complete, asymptotically flat, smooth Riemannian manifold of dimension $n \geqslant 3$. We assume that the metric $\mathbf{g}$ is smooth on $M \backslash K$, where K is a compact set, and that it satisfies the asymptotic conditions required for the validity of the smooth positive mass theorem. In addition, we assume that the classical positive mass theorem is valid on M.

Our global regularity assumption throughout is that the metric $\mathbf{g}$ is continuous and lies in the local Sobolev space $W^{2,n/2}_{\mathrm{loc}}(M)$.

Let $\mathscr{D}(M)$ denote the collection of smooth, compactly supported test functions on M. It follows that the map $\mathscr{D}(M) \to \mathbb{R}$ defined by

$$\varphi \mapsto \langle s_{\mathbf{g}}, \varphi \rangle := \int_M s_{\mathbf{g}} \varphi \, d\mu_{\mathbf{g}} \tag{1.4.1}$$

is a well-defined distribution on M.

Definition 1.4.1. A metric $\mathbf{g}$ is said to have *non-negative scalar curvature in the distributional sense* if $s_{\mathbf{g}} \geqslant 0$ in $\mathscr{D}'(M)$, i.e.

$$\langle s_{\mathbf{g}}, \varphi^2 \rangle \geqslant 0, \qquad \forall \varphi \in \mathscr{D}(M). \tag{1.4.2}$$

Remark 1.4.2. On $M \backslash K$, the metric $\mathbf{g}$ is smooth, so the scalar curvature is a pointwise well-defined quantity, and the condition (1.4.2) is equivalent to the condition that $s_{\mathbf{g}} \geqslant 0$ as a smooth function. However, inside the set K the metric $\mathbf{g}$ is not assumed to be C^2, and so (1.4.2) imposes non-negativity of the scalar curvature only in the weak sense.

The main result of [31] is the following.

Theorem 1.4.3. *Let $(M, \mathbf{g})$ be a Riemannian manifold as above with non-negative scalar curvature in the distributional sense. Then the mass of $(M, \mathbf{g})$ is non-negative.*

The proof of this result is a modification of the proof provided by Miao [51] (see also [44, 49]). As outlined in Section 1.3.1, it involves smooth approximations of the metric $\mathbf{g}$. These smooth approximations have to satisfy the following properties:

Lemma 1.4.4. *For all $\varepsilon > 0$, there exists a smooth Riemannian metric $\mathbf{g}_\varepsilon$ and a compact set $K_\varepsilon \subset M$ with the following properties:*

(1) $\mathbf{g}_\varepsilon$ *converges to* $\mathbf{g}$ *locally uniformly and in* $W^{2,n/2}_{\mathrm{loc}}(M)$ *as* $\varepsilon \to 0$*;*
(2) $\mathbf{g}_\varepsilon$ *coincides with the metric* $\mathbf{g}$ *on the set* $M\backslash K_\varepsilon$*;*
(3) K_ε *converges to* K *as* $\varepsilon \to 0$*.*

Remark 1.4.5. By construction, $\mathbf{g}_\varepsilon = \mathbf{g}$ on $M\backslash K_\varepsilon$, so the metrics $\mathbf{g}_\varepsilon$ have the same asymptotic behaviour and mass as $\mathbf{g}$.

Proof of Lemma 1.4.4. Generally, the existence of such a family of smooth approximating metrics follows from density of smooth metrics in $W^{2,n/2}_{\mathrm{loc}}(M) \cap C^0(M)$. More explicitly, we may proceed as follows. We cover K by a finite collection of open coordinate charts $\psi_i \colon O_i \to B(0,1) \subset \mathbb{R}^n$, $i = 1,\dots,m$ with the property that $K \subset \cup_{i=1}^m O_i$ and $M = N \cup (\cup_{i=1}^m O_i)$, where $N := M\backslash K$. Let χ_i, $i = 1,\dots,m$ and χ_N be a smooth squared partition of unity (i.e. $\chi_N^2 + \sum_{i=1}^m \chi_i^2 = 1$) subordinate to the cover of M defined by O_i and N with the property that the functions $\chi_i \circ \psi_i^{-1}$ have compact support contained in the set $B(0,1)$ and that χ_N has support bounded away from ∂N. We decompose the metric using the partition of unity, letting $\mathbf{g}_i := \chi_i \mathbf{g}$, $i = 1,\dots,m$ and $\mathbf{g}_N := \chi_N \mathbf{g}$ be $(0,2)$ tensor fields on M with support contained in O_i and N, respectively. For $i = 1,\dots,m$, we define the $(0,2)$ tensor field $\mathbf{G}_i := \left(\psi_i^{-1}\right)^* \mathbf{g}_i$ on $B(0,1) \equiv \psi_i(O_i)$, which has compact support contained away from the boundary of $B(0,1)$. In terms of the coordinates x_i^α on $\psi_i(O_i) \subset \mathbb{R}^n$, the components $G_{i,\alpha\beta}$ of $\mathbf{G}_i$ are continuous and lie in $W^{2,n/2}_{\mathrm{loc}}(B(0,1))$. Let $\rho \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth, positive mollifier with $\operatorname{supp}\rho \subset B(0,1)$ and $\int_{B(0,1)} \rho = 1$. Let $\rho_\varepsilon(\mathbf{x}) := \varepsilon^{-n} \rho\left(\frac{\mathbf{x}}{\varepsilon}\right)$. We construct smoothed versions of the tensor fields $\mathbf{G}_i$ by taking the scaled convolution of the components $G_{i,\varepsilon;\alpha\beta}(x) := \left(\rho_\varepsilon \star G_{i,\alpha\beta}\right)(x)$ for $x \in B(0,1)$. Since $\mathbf{G}_i$ has support bounded away from $\partial B(0,1)$, it follows that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$, the $G_{i,\varepsilon;\alpha\beta}$ will have support bounded away from $\partial B(0,1)$ for $i = 1,\dots,m$. From the components $G_{i,\varepsilon;\alpha\beta}$, we now reconstruct the smooth $(0,2)$ tensor fields $\mathbf{G}_{i,\varepsilon}$ on $B(0,1)$. Let $\mathbf{g}_{i,\varepsilon} := (\psi_i)^* \mathbf{G}_{i,\varepsilon}$, for $i = 1,\dots,m$. We now define the $(0,2)$ tensor field on M

$$\mathbf{g}_\varepsilon := \chi_N \mathbf{g}_N + \sum_{i=1}^m \chi_i \mathbf{g}_{i,\varepsilon}.$$

(Note that we have not changed the (smooth) metric $\mathbf{g}_N$.) By the convergence properties of smoothing with mollifiers, the $(0,2)$ tensor fields $\mathbf{g}_{i,\varepsilon}$ will converge to $\mathbf{g}_i$ both locally uniformly and in $W^2_{\mathrm{loc}}(M)n/2$ as $\varepsilon \to 0$. Since $\mathbf{g}_i = \chi_i \mathbf{g}$, we therefore have that, both locally uniformly and in $W^2_{\mathrm{loc}}(M)n/2$,

$$\mathbf{g}_\varepsilon \to \chi_N \mathbf{g}_N + \sum_{i=1}^m \chi_i \mathbf{g}_i = \chi_N^2 \mathbf{g} + \sum_{i=1}^m \chi_i^2 \mathbf{g} = \mathbf{g},$$

since $(\chi_1,\dots,\chi_m,\chi_N)$ is a squared partition of unity. Therefore, Condition (1) is satisfied. Since we have not modified the metric $\mathbf{g}_N$, and the $\mathbf{g}_{i,\varepsilon}$ only differ from $\mathbf{g}_i$ on a set of size ε, it follows that the $\mathbf{g}_\varepsilon$ will coincide with $\mathbf{g}$ on an ε-neighbourhood of the set K. We define K_ε to be the closure of this set, which is automatically compact. With this definition of K_ε, Conditions (2) and (3) are satisfied. $\square$

For this approximations it holds that $s_{\mathbf{g}_\varepsilon} \to s_{\mathbf{g}}$ in $L^{n/2}_{\mathrm{loc}}(M)$ as $\varepsilon \to 0$. This will be shown below in 1.5.4.

REMARK 1.4.6. The condition that the metric lies in $C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$ implies that the full curvature tensor lies in $L^{n/2}_{\mathrm{loc}}(M)$ and that the curvature of the metrics $\mathbf{g}_\varepsilon$ converges to that of $\mathbf{g}$ in $L^{n/2}_{\mathrm{loc}}(M)$. This will be proven in Propositions 1.5.1 and 1.5.4 below.

REMARK 1.4.7. For the class of metrics discussed in [**51, 49, 44**], one obtains a pointwise bound on the negative part of $s_{\mathbf{g}_\varepsilon}$. For our metrics, it is an $L^{n/2}$ bound on $[s_{\mathbf{g}_\varepsilon}]_-$ that appears naturally.

In the next step we conformally transform $\mathbf{g}_\varepsilon$ as follows:

Let u_ε denote a solution of the equation

$$\Delta_{\mathbf{g}_\varepsilon} u_\varepsilon + c_n [s_{\mathbf{g}_\varepsilon}]_- u_\varepsilon = 0,$$

where

$$c_n := \frac{n-2}{4(n-1)}.$$

From Lemma 1.3.5 we get that $u_\varepsilon \geqslant 0$. Using these u_ε as conformal factors, we obtain smooth metrics with non-negative scalar curvature. Indeed,

PROPOSITION 1.4.8. *The conformally rescaled metrics $\widehat{\mathbf{g}}_\varepsilon := u_\varepsilon^{4/(n-2)} \mathbf{g}_\varepsilon$ are asymptotically flat and have non-negative scalar curvature. Furthermore, $m(\widehat{\mathbf{g}}_\varepsilon) - m(\mathbf{g}_\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. The scalar curvature of the metric $\widehat{\mathbf{g}}_\varepsilon$ is given by

$$s_{\widehat{\mathbf{g}}_\varepsilon} = \frac{1}{c_n} u_\varepsilon^{-\frac{n+2}{n-2}} \left(-\Delta u_\varepsilon + c_n s_{\mathbf{g}} u_\varepsilon\right) = \frac{1}{c_n} u_\varepsilon^{-\frac{n+2}{n-2}} \left(-\Delta u_\varepsilon + c_n [s_{\mathbf{g}}]_+ u_\varepsilon - c_n [s_{\mathbf{g}}]_- u_\varepsilon\right) = u_\varepsilon^{-\frac{4}{n-2}} [s_{\mathbf{g}}]_+$$

and is therefore non-negative by construction. Moreover, the mass of $(M, \widehat{\mathbf{g}}_\varepsilon)$ is related to that of $(M, \mathbf{g}_\varepsilon)$ by the relation (1.3.10)

$$m(\mathbf{g}_\varepsilon) = m(\widehat{\mathbf{g}}_\varepsilon) + \frac{n-1}{(n-2)\omega_{n-1}} \int_M \left[|\nabla_{\mathbf{g}_\varepsilon} u_\varepsilon|^2_{\mathbf{g}_\varepsilon} - c_n \left[s_{\mathbf{g}_\varepsilon}\right]_- u_\varepsilon^2 \right] d\mu_{\mathbf{g}_\varepsilon}.$$

.

It will be shown in 1.5.8 that the first term in the integral on the right-hand-side converges to 0 as $\varepsilon \to 0$. For the second term, we have

$$\left| \int_M [s_{\mathbf{g}_\varepsilon}]_- \, u_\varepsilon^2 \, d\mu_{\mathbf{g}_\varepsilon} \right| = \left| \int_K [s_{\mathbf{g}_\varepsilon}]_- \, u_\varepsilon^2 \, d\mu_{\mathbf{g}_\varepsilon} \right| \leqslant \|[s_{\mathbf{g}_\varepsilon}]_-\|_{L^{n/2}(K, \mathbf{g}_\varepsilon)} \|u_\varepsilon\|^2_{L^{\frac{2n}{n-2}}(K, \mathbf{g}_\varepsilon)}.$$

Moreover,

$$\begin{aligned} \|u_\varepsilon\|_{L^{\frac{2n}{n-2}}(K, \mathbf{g}_\varepsilon)} &\leqslant \|1\|_{L^{\frac{2n}{n-2}}(K, \mathbf{g}_\varepsilon)} + \|v_\varepsilon\|_{L^{\frac{2n}{n-2}}(K, \mathbf{g}_\varepsilon)} \\ &= |K|_{\mathbf{g}_\varepsilon}^{\frac{n-2}{2n}} + \|v_\varepsilon\|_{L^{\frac{2n}{n-2}}(K, \mathbf{g}_\varepsilon)} \to |K|_{\mathbf{g}}^{\frac{n-2}{2n}} \quad \text{as } \varepsilon \to 0. \end{aligned}$$

From compactness of K, we have $|K|_{\mathbf{g}} < \infty$. Since $\|[s_{\mathbf{g}_\varepsilon}]_-\|_{L^{n/2}(K,\mathbf{g}_\varepsilon)} \to 0$, it follows that $\int_M [s_{\mathbf{g}_\varepsilon}]_- \, u_\varepsilon^2 \, d\mu_{\mathbf{g}_\varepsilon} \to 0$ as $\varepsilon \to 0$. Therefore $m(\widehat{\mathbf{g}}_\varepsilon) - m(\mathbf{g}_\varepsilon) \to 0$ as $\varepsilon \to 0$. $\square$

Since $m(\mathbf{g}_\varepsilon) = m(\mathbf{g})$, we deduce from Proposition 1.4.8 that the masses of the conformally rescaled metrics converge to $m(\mathbf{g})$ as $\varepsilon \to 0$. Since the classical positive mass theorem implies that $m(\widehat{\mathbf{g}}_\varepsilon) \geqslant 0$ for $\varepsilon > 0$, we deduce that $m(\mathbf{g}) \geqslant 0$.

1.5. Properties of the approximating metrics

PROPOSITION 1.5.1. *Let* $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$. *Then* $s_{\mathbf{g}} \in L^{n/2}_{\mathrm{loc}}(M)$.

REMARK 1.5.2. Even though for the proof of the positive mass theorem it is not necessary, this result can be generalized to still hold for $\mathbf{g} \in C^0(M) \cap W^{k,n/k}_{\mathrm{loc}}(M)$ for combinations of the form

$$\sum_{|\alpha|=k} g^{\cdot\cdot} \partial^\alpha g_{\cdot\cdot\cdot}. \tag{1.5.1}$$

We will prove Proposition 1.5.1 in this more general setting.

Proof of Theorem 1.5.1. The result is local, so we may perform the calculations in a local coordinate chart. Since $\mathbf{g} \in W^{k,n/k}_{\mathrm{loc}}(M)$, we have $\partial^k \mathbf{g} \in L^{n/k}_{\mathrm{loc}}(M)$. The Sobolev embedding theorem implies that $\partial \mathbf{g} \in W^{k-1}_{\mathrm{loc}}(M) n/k \subseteq L^n_{\mathrm{loc}}(M)$, etc. Since $\mathbf{g}$ is assumed continuous, we deduce that all of the terms in the expression (1.5.1) lie in $L^{n/k}_{\mathrm{loc}}(M)$, and therefore also $\sum_{|\alpha|=k} g^{\cdot\cdot} \partial^\alpha g_{\cdot\cdot} \in L^{n/k}_{\mathrm{loc}}(M)$.

Setting $k = 2$, we get that the curvature tensor of $\mathbf{g}$, which is schematically of the form $R^{\cdot}{}_{\cdot\cdot\cdot} = g^{\cdot\cdot}\partial^2 g_{\cdot\cdot} + g^{\cdot\cdot}\partial g_{\cdot\cdot}\partial g_{\cdot\cdot}$, lies in $L^{n/2}_{\mathrm{loc}}(M)$. Since the scalar curvature follows from contracting the curvature tensor with the (continuous) inverse metric, it follows that the scalar curvature lies in $L^{n/2}_{\mathrm{loc}}(M)$. $\square$

LEMMA 1.5.3. *There exists* $\rho(\varepsilon) \geqslant 1$ *with the property that*

$$\frac{1}{\rho(\varepsilon)}\mathbf{g}_\varepsilon \leqslant \mathbf{g} \leqslant \rho(\varepsilon)\mathbf{g}_\varepsilon \tag{1.5.2}$$

as bilinear forms on M, *with* $\rho(\varepsilon) \to 1$ *as* $\varepsilon \to 0$.

Proof. By construction, the $\mathbf{g}_\varepsilon$ converge uniformly to $\mathbf{g}$ on compact subsets. Taking the compact set to be K_ε, it follows that there exists $\rho(\varepsilon)$ such that (1.5.2) holds on the set K_ε. Since $\mathbf{g}_\varepsilon$ coincides with $\mathbf{g}$ on $M \backslash K_\varepsilon$, it follows that (1.5.2) holds globally on M for each $\varepsilon > 0$. Uniform convergence of the metrics on K and the fact that $K_\varepsilon \to K$ as $\varepsilon \to 0$ implies that $\rho(\varepsilon) \to 1$ as $\varepsilon \to 0$. $\square$

PROPOSITION 1.5.4. *Let* $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$ *and* $\mathbf{g}_\varepsilon$ *as in Lemma 1.4.4. Then* $s_{\mathbf{g}_\varepsilon} \to s_{\mathbf{g}}$ *in* $L^{n/2}_{\mathrm{loc}}(M)$ *as* $\varepsilon \to 0$.

Proof. Again, this is a local calculation. Since $\mathbf{g} \in W^{2,n/2}_{\text{loc}}(M)$, we have $\partial^2 g_{\varepsilon ..} \to \partial^2 g_{..} \in L^{n/2}_{\text{loc}}(M)$. By the Sobolev embedding theorem, $\partial \mathbf{g} \in L^n_{\text{loc}}(M)$ and $\partial g_{\varepsilon,..} \to \partial g_{..}$ in $L^n_{\text{loc}}(M)$. Finally, since $g_{..}$ is assumed to be continuous, the $g_{\varepsilon,..}$ converge locally uniformly to $g_{..}$ as $\varepsilon \to 0$. We then have

$$\begin{aligned}
\|R[\mathbf{g}_\varepsilon]^{\cdot}{}_{...} - R[\mathbf{g}]^{\cdot}{}_{...}\|_{L^{n/2}} &\sim \|g_\varepsilon^{\cdot\cdot}\partial^2 g_{\varepsilon ..} + g_\varepsilon^{\cdot\cdot}\partial g_{\varepsilon ..}\partial g_{\varepsilon ..} - g^{\cdot\cdot}\partial^2 g_{..} - g^{\cdot\cdot}\partial g_{..}\partial g_{..}\|_{L^{n/2}} \\
&\leqslant \|g_\varepsilon^{\cdot\cdot}\partial^2 g_{\varepsilon,..} - g^{\cdot\cdot}\partial^2 g_{..}\|_{L^{n/2}} + \|g_\varepsilon^{\cdot\cdot}\partial g_{\varepsilon,..}\partial g_{\varepsilon ..} - g^{\cdot\cdot}\partial g_{..}\partial g_{..}\|_{L^{n/2}} \\
&\leqslant \|g_\varepsilon^{\cdot\cdot}\left(\partial^2 g_{\varepsilon ..} - \partial^2 g_{..}\right)\|_{L^{n/2}} + \|\left(g_\varepsilon^{\cdot\cdot} - g^{\cdot\cdot}\right)\partial^2 g_{..}\|_{L^{n/2}} \\
&\qquad + \|g_\varepsilon^{\cdot\cdot}\partial g_{\varepsilon,..}\left(\partial g_{\varepsilon ..} - \partial g_{..}\right)\|_{L^{n/2}} + \|g_\varepsilon^{\cdot\cdot}\left(\partial g_{\varepsilon ..} - \partial g_{..}\right)\partial g_{..}\|_{L^{n/2}} \\
&\qquad + \|\left(g_\varepsilon^{\cdot\cdot} - g^{\cdot\cdot}\right)\partial g_{..}\partial g_{..}\|_{L^{n/2}} \\
&=: I + II + III + IV + V. \qquad (1.5.3)
\end{aligned}$$

We then have

$$\begin{aligned}
I &\leqslant \|g_\varepsilon^{\cdot\cdot}\|_{L^\infty}\|\partial^2 g_{\varepsilon ..} - \partial^2 g_{..}\|_{L^{n/2}}, \\
II &\leqslant \|g_\varepsilon^{\cdot\cdot} - g^{\cdot\cdot}\|_{L^\infty}\|\partial^2 g_{..}\|_{L^{n/2}}, \\
III &\leqslant \|g_\varepsilon^{\cdot\cdot}\|_{L^\infty}\|\partial g_{\varepsilon ..}\|_{L^n}\|\partial g_{\varepsilon ..} - \partial g_{..}\|_{L^n}, \\
IV &\leqslant \|g_\varepsilon^{\cdot\cdot}\|_{L^\infty}\|\partial g_{\varepsilon ..} - \partial g_{..}\|_{L^n}\|\partial g_{..}\|_{L^n}, \\
V &\leqslant \|g_\varepsilon^{\cdot\cdot} - g^{\cdot\cdot}\|_{L^\infty}\|\partial g_{..}\|^2_{L^n}.
\end{aligned}$$

These estimates show that each term on the right-hand-side of (1.5.3) converges to 0 as $\varepsilon \to 0$. Therefore, the curvature of the metrics $\mathbf{g}_\varepsilon$ converges to that of $\mathbf{g}$ in $L^{n/2}_{\text{loc}}(M)$ as $\varepsilon \to 0$. Since $\mathbf{g}_\varepsilon$ converges locally uniformly to $\mathbf{g}$ as $\varepsilon \to 0$, it follows that $s_{\mathbf{g}_\varepsilon} \to s_{\mathbf{g}}$ in $L^{n/2}_{\text{loc}}(M)$ as $\varepsilon \to 0$. ∎

Remark 1.5.5. This result can also be generalized to the case $\mathbf{g} \in C^0(M) \cap W^{k,n/k}_{\text{loc}}(M)$ for combinations of the form $\sum_{|\alpha|=k} g^{\cdot\cdot}\partial^\alpha g_{..}$ by estimating the additional terms via Hölder's inequality.

We recall the following result [61, Lemma 3.1].

Lemma 1.5.6. *For each metric $\mathbf{g}_\varepsilon$, there exists a constant $C > 0$ such that for any function φ with compact support on M we have*

$$\|\varphi\|^2_{L^{\frac{2n}{n-2}}(M,\mathbf{g}_\varepsilon)} \leqslant C\,\|\nabla_{\mathbf{g}_\varepsilon}\varphi\|^2_{L^2(M,\mathbf{g}_\varepsilon)}. \qquad (1.5.4)$$

The smallest such constant will be denoted by $c_1[\mathbf{g}_\varepsilon]$ and referred to as the Sobolev constant *of $\mathbf{g}_\varepsilon$.*

This result will be used together with:

Proposition 1.5.7. *The Sobolev constants of the metrics $\mathbf{g}_\varepsilon$ are related to that of the metric $\mathbf{g}$ by the inequality*

$$\frac{1}{\rho(\varepsilon)^n}\,c_1[\mathbf{g}_\varepsilon] \leqslant c_1[\mathbf{g}] \leqslant \rho(\varepsilon)^n\,c_1[\mathbf{g}_\varepsilon], \qquad t \geqslant 0, \qquad (1.5.5)$$

where ρ is the function introduced in Lemma 1.5.3.

Proof. Let φ be a function with compact support on M. We then have

$$\begin{aligned}
\|\varphi\|^2_{L^{\frac{2n}{n-2}}(M,\mathbf{g})} &= \left(\int_M |\varphi|^{\frac{2n}{n-2}}\, d\mu_{\mathbf{g}}\right)^{\frac{n-2}{n}} \\
&\leqslant \left(\rho(\varepsilon)^{n/2} \int_M |\varphi|^{\frac{2n}{n-2}}\, d\mu_{\mathbf{g}_\varepsilon}\right)^{\frac{n-2}{n}} \\
&\leqslant \rho(\varepsilon)^{\frac{n-2}{2}} c_1[\mathbf{g}_\varepsilon] \left(\int_M |\nabla\varphi|^2_{\mathbf{g}_\varepsilon}\, d\mu_{\mathbf{g}_\varepsilon}\right) \\
&\leqslant \rho(\varepsilon)^{\frac{n-2}{2}} c_1[\mathbf{g}_\varepsilon] \left(\rho(\varepsilon)^{n/2+1} \int_M |\nabla\varphi|^2_{\mathbf{g}}\, d\mu_{\mathbf{g}}\right) \\
&= \rho(\varepsilon)^n c_1[\mathbf{g}_\varepsilon]\, \|\nabla\varphi\|^2_{L^2(M,\mathbf{g})}.
\end{aligned}$$

Therefore $c_1[\mathbf{g}] \leqslant \rho(\varepsilon)^n c_1[\mathbf{g}_\varepsilon]$. Reversing the same argument gives the other part of (1.5.5). $\square$

1.5.1. Small negative curvature.

PROPOSITION 1.5.8. *Let $\mathbf{g} \in C^0(M) \cap W^{2,n/2}_{\mathrm{loc}}(M)$ with non-negative scalar curvature in the distributional sense. Then the negative part of the scalar curvature of the metric $\mathbf{g}_\varepsilon$ satisfies*

$$\left\|[s_{\mathbf{g}_\varepsilon}]_-\right\|_{L^{n/2}(M)} \leqslant \left\|s_{\mathbf{g}_\varepsilon} - s_{\mathbf{g}}\right\|_{L^{n/2}(M)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{1.5.6}$$

In particular, $[s_{\mathbf{g}_\varepsilon}]_- \to 0$ in $L^{n/2}(M)$ as $\varepsilon \to 0$.

Proof. By assumption, given $\varphi \in \mathscr{D}(M)$ with $\varphi \geqslant 0$, we have

$$\begin{aligned}
\int_M s_{\mathbf{g}_\varepsilon}\varphi\, d\mu &= \langle s_{\mathbf{g}}, \varphi\rangle + \int_M (s_{\mathbf{g}_\varepsilon} - s_{\mathbf{g}})\,\varphi\, d\mu \\
&\geqslant \int_M (s_{\mathbf{g}_\varepsilon} - s_{\mathbf{g}})\,\varphi\, d\mu \\
&\geqslant -\left\|s_{\mathbf{g}_\varepsilon} - s_{\mathbf{g}}\right\|_{L^{n/2}(M)} \|\varphi\|_{L^{\frac{n}{n-2}}(M)}.
\end{aligned}$$

The result then follows from Lemma 1.5.4. $\square$

In the case where the metric $\mathbf{g}$ fails to have non-negative scalar curvature, but the negative part of the scalar curvature has small $L^{n/2}$ norm, we may derive a (negative) lower bound on the mass of $(M, \mathbf{g})$.

THEOREM 1.5.9. *Let $\mathbf{g}$ be an asymptotically flat metric with $(s_{\mathbf{g}})_-$ being of compact support and sufficiently small such that*

$$c_1\, c_n \|(s_{\mathbf{g}})_-\|_{L^{n/2}(M)} < 1.$$

Then the mass of the metric $\mathbf{g}$ *satisfies*

$$m(\mathbf{g}_\varepsilon) \geqslant -\frac{n-1}{(n-2)\omega_{n-1}} \frac{c_n \|(s_g)_-\|_{L^{n/2}(M)}}{\left(1 - c_1\, c_n \|(s_{\mathbf{g}})_-\|_{L^{n/2}(M)}\right)^2} \, |\mathrm{supp}\, f|^{2/n^*}\,.$$

Proof. The metric $\hat{\mathbf{g}}$ has non-negative scalar curvature and, hence, non-negative mass. We therefore have

$$\begin{aligned} m(\mathbf{g}) &= m(\hat{\mathbf{g}}) + \frac{n-1}{(n-2)\omega_{n-1}} \int_M \left[|\nabla_{\mathbf{g}} u|^2_{\mathbf{g}} - c_n \left[s_{\mathbf{g}}\right]_- u^2\right] d\mu_{\mathbf{g}} \\ &\geqslant -\frac{n-1}{(n-2)\omega_{n-1}} c_n \|(s_g)_-\|_{L^{n/2}(M)} \|u\|^2_{L^{n^*}(\mathrm{supp}(s_{\mathbf{g}})_-)}. \end{aligned} \tag{1.5.7}$$

Since $u = 1 + v$, the estimate (1.6.10a) below implies that

$$\begin{aligned} \|u\|_{L^{n^*}(\mathrm{supp}(s_{\mathbf{g}})_-)} &\leqslant |\mathrm{supp}\, f|^{1/n^*} + \frac{c_1\, c_n \|(s_{\mathbf{g}})_-\|_{L^{n/2}(M)}}{1 - c_1\, c_n \|(s_{\mathbf{g}})_-\|_{L^{n/2}(M)}} |\mathrm{supp}\, f|^{1/n^*} \\ &= \frac{1}{1 - c_1\, c_n \|(s_{\mathbf{g}})_-\|_{L^{n/2}(M)}} |\mathrm{supp}\, f|^{1/n^*} \end{aligned}$$

Substituting this inequality into (1.5.7) gives the desired result. □

1.6. Dirichlet problem

In this section, $\mathbf{g}$ will denote a *smooth* asymptotically flat, Riemannian metric on the manifold M.

Due to the result of Schoen and Yau, as stated in Lemma 1.3.5, we get a unique solution of

$$\Delta_{\mathbf{g}} u - f u = h$$

provided that

$$\|f_-\|_{L^{n/2}(M,\mathbf{g})} \leqslant \varepsilon_0(\mathbf{g}),$$

This solution satisfies $u(x) = \frac{A}{r^{n-2}} + \omega(r)$ as $r \to \infty$, where A is constant and $\omega(r) = O(r^{1-n})$ as $r \to \infty$.

This follows from (1.5.6), (1.5.5) and the fact that $\rho(\varepsilon) \to 1$ as $\varepsilon \to 0$.

We recall some parts of the proof of Lemma 1.3.5 as it was derived in [61], since we will require some information concerning the constant ε_0. Let Ω be an open subset of M with compact closure, such that $\partial\Omega$ is smooth and contained in the ends of M.

Let $v := u - 1$. We aim at solving Dirichlet problem

$$\Delta_{\mathbf{g}} v - f v = h, \qquad v|\,\partial\Omega = 0 \tag{1.6.1}$$

on the set Ω. Taking a sequence of sets Ω such that $\overline{\Omega}$ form a compact exhaustion of M, then we may extract a subsequence of the corresponding solutions v of the Dirichlet problem that converge to the solution on M with the asymptotic properties given in Lemma 1.3.5. Treating the homogeneous problem with $h = 0$ and using the Sobolev inequality (1.5.4) yields the inequality

$$\int_\Omega |\nabla v|^2_{\mathbf{g}}\, d\mu_{\mathbf{g}} \leqslant c_1[\mathbf{g}]\, \|f_-\|_{L^{n/2}(M,\mathbf{g})} \int_\Omega |\nabla v|^2_{\mathbf{g}}\, d\mu_{\mathbf{g}}.$$

It follows that if the condition

$$c_1[\mathbf{g}]\, \|f_-\|_{L^{n/2}(M,\mathbf{g})} < 1, \tag{1.6.2}$$

is satisfied, then the homogeneous problem has a unique smooth solution $v \equiv 0$ on the region Ω. Fredholm theory then yields the (unique) existence of a solution of the inhomogeneous problem (1.6.1).

Let now u denote a solution of the equation

$$\Delta_{\mathbf{g}} u + c_n[s_{\mathbf{g}}]_- u = 0,$$

where

$$c_n := \frac{n-2}{4(n-1)}.$$

As already described in Section 1.3.1, the metric $\hat{\mathbf{g}} := u^{4/(n-2)}\mathbf{g}$ has scalar curvature satisfying

$$c_n s_{\hat{\mathbf{g}}} u^{\frac{n+2}{n-2}} = -\Delta_{\mathbf{g}} u + c_n s_{\mathbf{g}} u = -\left(\Delta_{\mathbf{g}} u + c_n[s_{\mathbf{g}}]_- u\right) + c_n[s_{\mathbf{g}}]_+ u.$$

Since we are interested in metrics $\hat{\mathbf{g}}$ to have non-negative scalar curvature on Ω, we impose that u satisfy the equation $\Delta_{\mathbf{g}} u + c_n[s_{\mathbf{g}}]_- u = 0$ in Ω with $u = 1$ on $\partial\Omega$.

Letting $u := 1 + v$, then v satisfies the equation

$$\Delta_{\mathbf{g}} v + fv + f = 0 \text{ in } \Omega, \qquad v|\,\partial\Omega = 0, \tag{1.6.3}$$

where $f := c_n[s_{\mathbf{g}}]_- \geqslant 0$.

We will study the general case $p \geqslant n/2$, even though for proving Theorem 1.4.3 we just need $p = n/2$.

Let $\|f\|_{L^p}$ be sufficiently small that

$$c_1\, \|f\|_{L^p} |\Omega|^{\frac{2}{n}-\frac{1}{p}} < 1. \tag{1.6.4}$$

Let $\Omega \subset M$ be an open set with compact closure and smooth boundary $\partial\Omega$. We wish to derive global estimates for a solution, v, of the Dirichlet problem

$$\Delta_{\mathbf{g}} v + fv + f = 0 \quad \text{in } \Omega, \qquad v|\,\partial\Omega = 0, \tag{1.6.5}$$

where $f \geqslant 0$ and we assume that $f \in L^p(\Omega, \mathbf{g})$ for some $p \geqslant \frac{n}{2}$.

Given any measurable set $\Omega \subseteq M$ with $|\Omega| > 0$, we adopt the notation

$$\fint_\Omega f\, d\mu := \frac{1}{|\Omega|} \int_\Omega f\, d\mu$$

for the mean of a function f over the set Ω.

Furthermore, n^* denotes the Sobolev conjugate to n (i.e., $\frac{1}{n^*} = 1 - \frac{1}{n}$).

LEMMA 1.6.1. *If $\|f\|_{L^p}$ satisfies (1.6.4), then the solution v of (1.6.5) satisfies the inequality*

$$\left(\fint_\Omega |v|^{n^*} \, d\mu_{\mathbf{g}} \right)^{1/n^*} \leqslant \frac{c_1 \, \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}{1 - c_1 \, \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}. \tag{1.6.6}$$

Equivalently,

$$\|v\|_{L^{n^*}(\Omega)} \leqslant \frac{c_1 \, \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}{1 - c_1 \, \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}} |\Omega|^{\frac{1}{n^*}}. \tag{1.6.7}$$

Proof. Our proof is a modification of that of Proposition 4.1 in [51]. Multiplying the equation $\Delta v + fv + f = 0$ by v, integrating over Ω, and using Hölder's inequality, we deduce that

$$\begin{aligned}
\|\nabla v\|^2_{L^2(\Omega)} &= \int_\Omega \big(fv^2 + fv\big) \, d\mu_{\mathbf{g}} \\
&\leqslant \|f\|_{L^{n/2}(\Omega)} \|v\|^2_{L^{n^*}(\Omega)} + \|f\|_{L^{\frac{2n}{2n+2}}(\Omega)} \|v\|_{L^{n^*}(\Omega)} \\
&\leqslant \|f\|_{L^p(\Omega)} |\Omega \cap \operatorname{supp} f|^{2/n - 1/p} \|v\|^2_{L^{n^*}(\Omega)} \\
&\quad + \|f\|_{L^{n/2}(\Omega)} |\Omega \cap \operatorname{supp} f|^{1/n^* + 2/n - 1/p} \|v\|_{L^{n^*}(\Omega)}.
\end{aligned} \tag{1.6.8}$$

It follows from the Sobolev inequality that we have

$$\begin{aligned}
\|v\|^2_{L^{n^*}(\Omega)} &\leqslant c \|f\|_{L^p(\Omega)} |\Omega \cap \operatorname{supp} f|^{2/n - 1/p} \|v\|^2_{L^{n^*}(\Omega)} \\
&\quad + \|f\|_{L^{n/2}(\Omega)} |\Omega \cap \operatorname{supp} f|^{1/n^* + 2/n - 1/p} \|v\|_{L^{n^*}(\Omega)}.
\end{aligned} \tag{1.6.9}$$

Therefore, if the condition (1.6.4) is satisfied, we deduce that

$$\|v\|_{L^{n^*}(\Omega)} \leqslant \frac{c\|f\|_{L^p(\Omega)} |\Omega \cap \operatorname{supp} f|^{2/n - 1/p}}{1 - c_1 \|f\|_{L^p(\Omega)} |\Omega \cap \operatorname{supp} f|^{2/n - 1/p}} |\Omega \cap \operatorname{supp} f|^{1/n^*},$$

which is (1.6.6). From this, (1.6.7) follows by the fact that $|\Omega \cap \operatorname{supp} f| \leqslant |\Omega|$. □

In case $p = n/2$, the equations become

LEMMA 1.6.2. *Let v satisfy (1.6.3), with $f := c_n \, [s_{\mathbf{g}}]_-$. Then we have the inequalities*

$$\|v\|_{L^{n^*}(\Omega)} \leqslant \frac{c_1 \, \|f\|_{L^{n/2}(\Omega)}}{1 - c_1 \, \|f\|_{L^{n/2}(\Omega)}} \, |\Omega \cap \operatorname{supp} f|^{1/n^*}, \tag{1.6.10a}$$

$$\|\nabla v\|_{L^2(\Omega)} \leqslant \|f\|_{L^{n/2}(\Omega)} \frac{c_1 \|f\|_{L^{n/2}(\Omega)}}{\big(1 - c_1 \|f\|_{L^{n/2}(\Omega)}\big)^2} \, |\Omega \cap \operatorname{supp} f|^{2/n^*}, \tag{1.6.10b}$$

respectively

$$\left(\fint_{\Omega} |v|^{n^*}\, d\mu_{\mathbf{g}}\right)^{1/n^*} \leqslant \frac{c_1 \|f\|_{L^{n/2}(\Omega)}}{1 - c_1 \|f\|_{L^{n/2}(\Omega)}}, \tag{1.6.11a}$$

$$|\Omega|^{2/n} \fint_{\Omega} |\nabla v|^2\, d\mu_{\mathbf{g}} \leqslant \|f\|_{L^{n/2}(\Omega)} \frac{c_1 \|f\|_{L^{n/2}(\Omega)}}{\left(1 - c_1 \|f\|_{L^{n/2}(\Omega)}\right)^2}. \tag{1.6.11b}$$

Let $\Omega \subset M$ be an open set with compact closure and smooth boundary $\partial\Omega$. We wish to derive global estimates for a solution v of the Dirichlet problem (1.6.5), where $f \geqslant 0$ and $f \in L^p(\Omega, \mathbf{g})$ for some $p \geqslant \frac{n}{2}$. The cases $p > \frac{n}{2}$ and $p = \frac{n}{2}$ behave differently, so we will treat them separately.

1.6.1. Global estimates for $p > \frac{n}{2}$. The main results of this section are lower and upper bounds for the solution v of (1.6.5) on Ω. We begin with the lower bound.

THEOREM 1.6.3. *If $\|f\|_{L^p}$ is sufficiently small such that*

$$c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}} < 1, \tag{1.6.12}$$

then, on Ω, we have

$$v \geqslant -\chi^{\tau} \frac{\left[c_1 \|f\|_p |\Omega|^{\frac{2}{n} - \frac{1}{p}}\right]^{\frac{\sigma}{2}+1}}{1 - c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}, \tag{1.6.13}$$

where $\chi := \frac{n^*}{(2p)^*}, \sigma = \frac{1}{\chi - 1}, \tau = \frac{\chi}{(\chi-1)^2}$.

Notation: For brevity, we let $A_p := c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}$.

Proof of Theorem 1.6.3. Our proof follows, to a large extent, the proof of Theorem 8.15 of [26]. We reproduce the main steps of the proof as we wish to keep explicit control over the constants that appear in the estimates.

In order to get a lower bound on v, we set $\bar{v} = -v$, and note that $\bar{v}$ satisfies the equation

$$\Delta_{\mathbf{g}} \bar{v} + f\bar{v} = f \quad \text{in } \Omega, \qquad \bar{v}|\, \partial\Omega = 0.$$

The weak form of our differential equation is now given by

$$\int_{\Omega} \langle \nabla\Phi, \nabla\bar{v}\rangle_{\mathbf{g}}\, d\mu_{\mathbf{g}} = \int_{\Omega} f\,(\bar{v} - 1)\,\Phi\, d\mu_{\mathbf{g}},$$

where Φ is a test function, which we may take to lie in $H^1_0(\Omega)$. We take $\Phi \geqslant 0$ in which case, using the fact that $f \geqslant 0$, we obtain

$$\int_{\Omega} \langle \nabla\Phi, \nabla\bar{v}\rangle_{\mathbf{g}}\, d\mu_{\mathbf{g}} \leqslant \int_{\Omega} f\bar{v}\Phi\, d\mu_{\mathbf{g}}. \tag{1.6.14}$$

Let $w := \bar{v}_+$. Note that $dw = 0$ a.e. on the set $\bar{v} \leqslant 0$ and $dw = d\bar{v}$ on the set $\bar{v} > 0$. Let $G\colon [0, \infty) \to \mathbb{R}$ be a non-decreasing Lipschitz function with $G(0) = 0$. It follows [52, Theorem 3.1.7] that $\Phi(x) := G(w(x))$ lies in $H^1_0(\Omega)$ and is therefore an admissible test function that we can insert into (A.3.2). Moreover, $\Phi = 0$ on the set $\bar{v} \leqslant 0$. We therefore find that

$$\int_\Omega G'[w] |\nabla w|^2_{\mathbf{g}} \, d\mu_{\mathbf{g}} \leqslant \int_\Omega f w G[w] \, d\mu_{\mathbf{g}}.$$

Let $H\colon [0,\infty) \to [0,\infty)$ be the function defined by

$$G'(t) = (H'(t))^2 , \qquad H(0) = 0.$$

We then have

$$\int_\Omega |\nabla H[w]|^2_{\mathbf{g}} \, d\mu_{\mathbf{g}} \leqslant \int_\Omega f w G[w] \, d\mu_{\mathbf{g}}.$$

Given $\beta \geqslant 1$ and $N > 0$, we take H to be the Lipschitz function

$$H[t] := \begin{cases} t^\beta & 0 \leqslant t \leqslant N, \\ \beta N^{\beta-1} (t - N) & t > N. \end{cases} \tag{1.6.15}$$

In this case, $G[t] \leqslant tG'[t] = tH'[t]^2$ for $t \geqslant 0$. Since $H[w] \in H^1_0(\Omega)$, the Sobolev inequality implies that

$$\|H[w]\|^2 \leqslant c_1 \int_\Omega |\nabla H[w]|^2_{\mathbf{g}} \, d\mu_{\mathbf{g}} \leqslant c_1 \int_\Omega f \left(wH'[w]\right)^2 \, d\mu_{\mathbf{g}}.$$

Taking the limit as $N \to \infty$, we deduce that

$$\|w\|^2_{\beta n^*} \leqslant \beta^2 c_1 \int_\Omega f w^{2\beta} \, d\mu_{\mathbf{g}} \leqslant \beta^2 c_1 \|f\|_p \|w^{2\beta}\|_{\frac{p}{p-1}}.$$

Letting

$$C := \left[c_1 \|f\|_p\right]^{1/2},$$

we therefore have

$$\|w\|_{\beta n^*} \leqslant (C\beta)^{1/\beta} \|w\|_{\beta(2p)^*}.$$

Defining

$$\chi := \frac{n^*}{(2p)^*} > 1,$$

we rewrite this inequality in the form

$$\|w\|_{\beta\chi(2p)^*} \leqslant (C\beta)^{1/\beta} \|w\|_{\beta(2p)^*}$$

Starting with $\beta = \chi^m$, with m a positive integer, and iterating, we have

$$\|w\|_{\chi^{m+1}(2p)^*} \leqslant (C\chi^m)^{1/\chi^m} \|w\|_{\chi^m(2p)^*} \leqslant \cdots \leqslant C^{\sigma_m} \chi^{\tau_m} \|w\|_{n^*},$$

where

$$\sigma_m = \sum_{i=1}^m \frac{1}{\chi^i}, \qquad \tau_m = \sum_{i=1}^m \frac{i}{\chi^i}.$$

Letting $m \to \infty$, we deduce that

$$\sup_\Omega w \leqslant C^\sigma \chi^\tau \|w\|_{n^*},$$

where
$$\sigma = \frac{1}{\chi - 1}, \qquad \tau = \frac{\chi}{(\chi - 1)^2}.$$
We now use the fact that $\sup \bar{v} \leqslant \sup \bar{v}_+ = \sup w$ and
$$\|w\|_{n^*} = \|\bar{v}_+\|_{n^*} = \|v_-\|_{n^*} \leqslant \|v\|_{n^*} \leqslant |\Omega|^{1/n^*} \left(\fint_\Omega |v|^{n^*} \, d\mu_{\mathbf{g}} \right)^{1/n^*}$$
to give
$$\begin{aligned}
\sup_\Omega \bar{v} &\leqslant C^\sigma \chi^\tau |\Omega|^{1/n^*} \left(\fint_\Omega |v|^{n^*} \, d\mu_{\mathbf{g}} \right)^{1/n^*} \\
&= \left[c_1 \|f\|_p |\Omega|^{\frac{2}{n} - \frac{1}{p}} \right]^{\sigma/2} \chi^\tau \left(\fint_\Omega |v|^{n^*} \, d\mu_{\mathbf{g}} \right)^{1/n^*} \\
&\equiv A_p^{\sigma/2} \chi^\tau \frac{A_p}{1 - A_p},
\end{aligned}$$
where we recall that we have defined $A_p = c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}$, and we have substituted the inequality (1.6.6) in the final step. We have thus established (1.6.13). $\square$

We now give an upper estimate for v.

THEOREM 1.6.4. *If $\|f\|_{L^p}$ is sufficiently small such that the condition (1.6.4) holds, then we have*
$$v \leqslant \chi^\tau \frac{c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}{1 - c_1 \|f\|_{L^p} |\Omega|^{\frac{2}{n} - \frac{1}{p}}}, \tag{1.6.16}$$
where $\chi := \frac{n^*}{(2p)^*}$, $\tau = \frac{\chi}{(\chi - 1)^2}$.

Proof. Let $w := v_+ \in H_0^1(\Omega)$. Given $\beta \geqslant 1$ and $N > 0$, it follows that $H[w]$ as defined in (1.6.15) lies in $H_0^1(\Omega)$. The Sobolev inequality and integration by parts then yield
$$\begin{aligned}
\|w^\beta\|_{L^{n^*}(\Omega)}^2 &\leqslant c_1 \int_\Omega |\nabla w^\beta|^2 \, d\mu_{\mathbf{g}} \\
&= -c_1 \frac{\beta^2}{2\beta - 1} \int_\Omega w^{2\beta - 1} \Delta w \, d\mu_{\mathbf{g}} \\
&= c_1 \frac{\beta^2}{2\beta - 1} \int_\Omega w^{2\beta - 1} f \, (1 + w) \, d\mu_{\mathbf{g}} \\
&\leqslant c_1 \beta^2 \|f\|_{L^p(\Omega)} \left[\|w^{2\beta - 1}\|_{L^{p/(p-1)}(\Omega)} + \|w^{2\beta}\|_{L^{p/(p-1)}(\Omega)} \right].
\end{aligned} \tag{1.6.17}$$
In addition, Hölder's inequality yields
$$\|w^{2\beta-1}\|_{L^{\frac{p}{p-1}}(\Omega)} = \|w\|_{L^{(2\beta-1)\frac{p}{p-1}}(\Omega)}^{2\beta-1} \leqslant \|w\|_{L^{2\beta\frac{p}{p-1}}(\Omega)}^{2\beta-1} |\Omega|^{\frac{1}{2\beta}\left(1 - \frac{1}{p}\right)} = \|w\|_{L^{\beta(2p)^*}(\Omega)}^{2\beta-1} |\Omega|^{\frac{1}{\beta(2p)^*}}.$$
We therefore have
$$\|w\|_{L^{\beta n^*}(\Omega)}^{2\beta} \leqslant c_1 \beta^2 \|f\|_{L^p(\Omega)} \|w\|_{L^{\beta(2p)^*}(\Omega)}^{2\beta-1} \left[|\Omega|^{\frac{1}{\beta(2p)^*}} + \|w\|_{L^{\beta(2p)^*}(\Omega)} \right]. \tag{1.6.18}$$

Letting $\chi := \frac{n^*}{(2p)^*} > 1$, we now claim that it follows from (1.6.18) and (1.6.7) that

$$\|w\|_{\chi^m n^*} \leqslant \frac{A_p}{1-A_p}\chi^{\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}}|\Omega|^{\frac{1}{\chi^m n^*}} \tag{1.6.19}$$

for all integers $m \geqslant 1$. We prove this claim by induction. From (1.6.18), taking $\beta = \chi$, we have

$$\|w\|^{2\chi}_{L^{\chi n^*}(\Omega)} \leqslant c_1\chi^2\|f\|_{L^p(\Omega)}\|w\|^{2\chi-1}_{L^{n^*}(\Omega)}\left[|\Omega|^{\frac{1}{n^*}} + \|w\|_{L^{n*}(\Omega)}\right].$$

We now note that

$$\|w\|_{L^{n^*}(\Omega)} = \|v_+\|_{L^{n^*}(\Omega)} \leqslant \|v\|_{L^{n^*}(\Omega)} \leqslant \frac{A_p}{1-A_p}|\Omega|^{\frac{1}{n^*}},$$

where the final inequality follows from (1.6.7). We therefore have

$$\begin{aligned}
\|w\|^{2\chi}_{L^{\chi n^*}(\Omega)} &\leqslant c_1\chi^2\|f\|_{L^p(\Omega)}\left(\frac{A_p}{1-A_p}|\Omega|^{\frac{1}{n^*}}\right)^{2\chi-1}\left[|\Omega|^{\frac{1}{n^*}} + \frac{A_p}{1-A_p}|\Omega|^{\frac{1}{n^*}}\right] \\
&= c_1\chi^2\|f\|_{L^p(\Omega)}|\Omega|^{\frac{2\chi}{n^*}}\frac{A_p^{2\chi-1}}{(1-A_p)^{2\chi}} \\
&= \chi^2|\Omega|^{\frac{2}{(2p)^*}-\frac{2}{n}+\frac{1}{p}}\left(\frac{A_p}{1-A_p}\right)^{2\chi} \\
&= \chi^2|\Omega|^{\frac{1}{\chi n^*}}\left(\frac{A_p}{1-A_p}\right)^{2\chi}.
\end{aligned}$$

Therefore, the claim is established for $m = 1$. Assuming it to be true for some $m \geqslant 1$, taking $\beta = \chi^{m+1}$ in (1.6.18), we have

$$\begin{aligned}
\|w\|^{2\chi^{m+1}}_{L^{\chi^{m+1}n^*}(\Omega)} &\leqslant c_1\chi^{2m+2}\|f\|_{L^p(\Omega)}\|w\|^{2\chi^{m+1}-1}_{L^{\chi^m n^*}(\Omega)}\left[|\Omega|^{\frac{1}{\chi^m n^*}} + \|w\|_{L^{\chi^m n*}(\Omega)}\right] \\
&\leqslant c_1\chi^{2m+2}\|f\|_{L^p(\Omega)}\left[\frac{A_p}{1-A_p}\chi^{\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}}|\Omega|^{\frac{1}{\chi^m n^*}}\right]^{2\chi^{m+1}-1} \\
&\quad\times\left[|\Omega|^{\frac{1}{\chi^m n^*}} + \frac{A_p}{1-A_p}\chi^{\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}}|\Omega|^{\frac{1}{\chi^m n^*}}\right] \\
&\leqslant c_1\chi^{2m+2}\|f\|_{L^p(\Omega)}\left[\frac{A_p}{1-A_p}\chi^{\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}}|\Omega|^{\frac{1}{\chi^m n^*}}\right]^{2\chi^{m+1}-1} \\
&\quad\times|\Omega|^{\frac{1}{\chi^m n^*}}\chi^{\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}}\left[1 + \frac{A_p}{1-A_p}\right] \\
&= c_1\chi^{2m+2}\|f\|_{L^p(\Omega)}\frac{A_p^{2\chi^{m+1}-1}}{(1-A_p)^{2\chi^{m+1}}}|\Omega|^{\frac{2\chi}{n^*}}\chi^{2\chi^{m+1}\left(\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}\right)}, \\
&= \left(\frac{A_p}{1-A_p}\right)^{2\chi^{m+1}}|\Omega|^{\frac{2}{n^*}}\chi^{2\chi^{m+1}\left(\frac{1}{\chi}+\cdots+\frac{m}{\chi^m}+\frac{m+1}{\chi^{m+1}}\right)},
\end{aligned}$$

where we have used the fact that $1 \leqslant \chi$ in the third inequality. Taking the $2\chi^{m+1}$th root, we have therefore established the claim (1.6.19) for $m+1$. Therefore, by induction, (1.6.19) holds for all $m \geqslant 1$.

Taking the limit as $m \to \infty$ in (1.6.19) now yields the inequality (1.6.16). □

1.6.2. Approximations to g. Applying Theorems 1.6.3 and 1.6.4 to the metrics $\mathbf{g}_\varepsilon$, we have the following result.

PROPOSITION 1.6.5. *Let $\mathbf{g} \in W^{2,p}_{\mathrm{loc}}(\Omega)$ with $p > \frac{n}{2}$ have non-negative scalar curvature in the distributional sense, and let $\mathbf{g}_\varepsilon$ be smooth approximating metrics as above. Given any compact subset $K \subset \Omega$, let $\varepsilon(K) > 0$ be such that $K \subset \Omega_\varepsilon$ for all $\varepsilon < \varepsilon(K)$. Then the solutions $v_\varepsilon | K$ for $\varepsilon < \varepsilon(K)$, converge uniformly to zero on K as $\varepsilon \to 0$.*

Proof. Since $\|(s_{g_\varepsilon})_-\|_{L^p(\Omega,\mathbf{g}_\varepsilon)} \to 0$ as $\varepsilon \to 0$, we may assume, without loss of generality, that the condition $c_1[\mathbf{g}_\varepsilon]\,\|(s_{g_\varepsilon})_-\|_{L^p(\Omega,\mathbf{g}_\varepsilon)} |\Omega_\varepsilon|_{\mathbf{g}_\varepsilon}^{\frac{2}{n}-\frac{1}{p}} < 4\frac{n-1}{n-2}$ is satisfied for all $\varepsilon < \varepsilon(K)$. The uniform bounds on v_ε on Ω_ε given by (1.6.13) and (1.6.16) both converge to zero as $\varepsilon \to 0$, therefore implying that $v_\varepsilon| K$ converge uniformly to zero on K. □

We therefore have the following result regarding approximations of $\mathbf{g}$.

THEOREM 1.6.6. *Let $\mathbf{g}$ be a Riemannian metric on an open set Ω of regularity $W^{2,p}_{\mathrm{loc}}(\Omega)$, $p > \frac{n}{2}$ with non-negative scalar curvature in the distributional sense. Then there exists a family of smooth, Riemannian metrics $\{\widehat{\mathbf{g}}_\varepsilon \mid \varepsilon > 0\}$ defined on open sets $\Omega_\varepsilon \subset \Omega$ such that $K_\varepsilon := \overline{\Omega_\varepsilon}$ are a compact exhaustion of Ω, such that:*

- *The $\widehat{\mathbf{g}}_\varepsilon$ are have non-negative scalar curvature;*
- *$\widehat{\mathbf{g}}_\varepsilon$ converge locally uniformly to $\mathbf{g}$ as $\varepsilon \to 0$.*

The use of the sets Ω_ε was necessary since, by smoothing $\mathbf{g}$ on an open set Ω, we will only get locally uniform convergence of $\mathbf{g}_\varepsilon$ to $\mathbf{g}$. We may remove the use of this construction if Ω is contained in a larger open set.

THEOREM 1.6.7. *Let $\mathbf{g}$ be a Riemannian metric on an open set Ω' of regularity $W^{2,p}_{\mathrm{loc}}(\Omega')$, $p > \frac{n}{2}$, that has non-negative scalar curvature in the distributional sense. Let Ω be an open subset of Ω' with compact closure $K \subset \Omega'$, smooth boundary $\partial\Omega$, such that Ω' is an open neighbourhood of K. Then there exists a family of smooth, Riemannian metrics $\{\widehat{\mathbf{g}}_\varepsilon \mid \varepsilon > 0\}$ on K with the following properties:*

- *The $\mathbf{g}_\varepsilon$ are have non-negative scalar curvature;*
- *$\widehat{\mathbf{g}}_\varepsilon$ converge uniformly to $\mathbf{g}$ on K as $\varepsilon \to 0$.*

Proof. Smoothing $\mathbf{g}$ by convolution in charts on Ω' gives, for all sufficiently small ε, a family of smooth Riemannian metrics $\mathbf{g}_\varepsilon$ on the set K. We now solve the Dirichlet problem

$$\Delta_{\mathbf{g}_\varepsilon} u_\varepsilon + c_n (s_{g_\varepsilon})_- u_\varepsilon = 0, \qquad u_\varepsilon | \partial\Omega = 1.$$

The conformally transformed metrics $\widehat{\mathbf{g}}_\varepsilon$ are then metrics with non-negative scalar curvature on the set K. The bounds on v_ε given by (1.6.13) and (1.6.16) now hold on the set Ω, implying that $u_\varepsilon | K$ converge uniformly to 1 on K. Since the $\mathbf{g}_\varepsilon$ converge uniformly to $\mathbf{g}$ on K, it follows that the $\widehat{\mathbf{g}}_\varepsilon$ converge uniformly to $\mathbf{g}$ on K. □

Finally, in the context of the positive mass theorem, our results give the following.

THEOREM 1.6.8. *Let M be a smooth manifold and $\mathbf{g}$ an asymptotically flat, Riemannian metric on M of regularity $W^{2,p}_{\mathrm{loc}}(M)$ and smooth outside of the compact set K. Then there exist smooth metrics $\widehat{\mathbf{g}}_\varepsilon$ on M with non-negative scalar curvature that converge locally uniformly to $\mathbf{g}$ as $\varepsilon \to 0$. In particular, $\mathbf{g}$ can be approximated locally uniformly by smooth metrics with non-negative ADM mass.*

Proof. The only non-trivial point is to note that in the estimates (1.6.13) and (1.6.16), the occurrences of the set Ω may be replaced by $\Omega \cap \operatorname{supp} f$. Since $(s_{g_\varepsilon})_-$ will have compact support, these factors will remain finite, so the v_ε will still converge to zero. □

REMARK 1.6.9. In the region where the metric $\mathbf{g}$ is smooth, the metrics $\mathbf{g}_\varepsilon$ will converge to $\mathbf{g}$ in C^∞.

1.6.3. Breakdown of Moser iteration for $p = n/2$. Finally, we briefly consider the case where the metric $\mathbf{g}$ is of regularity $C^0(\Omega) \cap W^{2,n/2}_{\mathrm{loc}}(\Omega)$. In this case, the Moser iteration arguments used in Section 1.6.1 to derive an L^∞ bound on solutions of the Dirichlet problem break down. In particular, taking $p = \frac{n}{2}$ in the inequality (1.6.17), we deduce that

$$\|w\|^{2\beta}_{L^{\beta n*}(\Omega)} \leqslant c_1 \beta^2 \|f\|_{L^{n/2}(\Omega)} \left[\|w\|^{2\beta-1}_{L^{(\beta-\frac{1}{2})n*}(\Omega)} + \|w\|^{2\beta}_{L^{n*}(\Omega)} \right].$$

Assuming that $c_1\beta^2\|f\|_{L^{n/2}(\Omega)} < 1$, this inequality yields a bound on $\|w\|_{L^{\beta n*}(\Omega)}$ in terms of $\|w\|_{L^{(\beta-\frac{1}{2})n*}(\Omega)}$. However, it is clear that such an iteration process breaks down when β is of the order $\frac{1}{\sqrt{c_1\|f\|_{L^{n/2}(\Omega)}}}$. As such, the Moser iteration argument breaks down after a finite number of iteration processes.[1] This means that we can establish an L^p bound on v, where

$$p \sim \beta n^* \sim n^* / \sqrt{c_1 \|f\|_{L^{n/2}(\Omega)}} < \infty.$$

Nevertheless, we note that $p(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

These observations imply the following result:

THEOREM 1.6.10. *Let Ω' be an open set, $\mathbf{g}$ a continuous Riemannian metric on Ω' of regularity $W^{2,n/2}_{\mathrm{loc}}(\Omega')$ with non-negative scalar curvature in the distributional sense. Let Ω be an open subset of Ω' with compact closure and $\partial\Omega$ smooth. Then, there exists a family of smooth metrics $\widehat{\mathbf{g}}_\varepsilon$ on Ω with non-negative scalar curvature such that $\widehat{\mathbf{g}}_\varepsilon \to \mathbf{g}$ in $L^p_{\mathrm{loc}}(\Omega)$ as $\varepsilon \to 0$, for all $p < \infty$.*

REMARK 1.6.11. It appears in general that the $\widehat{\mathbf{g}}_\varepsilon$ do not converge locally uniformly to $\mathbf{g}$. We note that it is possible to construct functions $f_\varepsilon \geqslant 0$ on the unit ball in $\mathbb{R}^2$ with the property that $f_\varepsilon \to 0$ in L^1 as $\varepsilon \to 0$, but the corresponding solutions of the Dirichlet problem[2]

$$\Delta_0 u_\varepsilon + f_\varepsilon u_\varepsilon = 0 \text{ in } B(0,1), \qquad u_\varepsilon = 1 \text{ on } \partial B(0,1)$$

[1] See, for instance, [60] and [35, Thm. 4.4] for examples of this phenomenon.

[2] Δ_0 denotes the Laplacian $\partial_x^2 + \partial_y^2$.

have the property that $u_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$.[3].

It therefore appears likely that metrics in $C^0 \cap W^{2,n/2}_{\mathrm{loc}}$ with non-negative scalar curvature in the distributional sense cannot, in general, be approximated in C^0 by smooth metrics with non-negative scalar curvature. Similarly, when considering limits of sequences of smooth metrics with non-negative scalar curvature, one might expect to encounter non-compactness or "bubbling off" phenomena. Based on the extremal case of the Sobolev embedding theorem, however, we would expect that such metrics can be approximated in BMO or appropriate Orlicz spaces.

1.7. Rigidity

In [31] it is not possible to achieve a rigidity result. The rigidity part of the classical positive mass theorem asserts that if a metric is asymptotically flat, has non-negative scalar curvature and the ADM mass is zero, then $(M, \mathbf{g})$ is isometric to $\mathbb{R}^n$ with the Euclidean metric. The methods are insufficient to prove such a statement for the class of metrics studied there. The proof of rigidity in Schoen and Yau [61] involves a perturbation of the metric $\mathbf{g}$ by its Ricci tensor. This approach cannot be adapted to our metrics, since the Ricci tensor of the metric $\mathbf{g}$ lies in $L^{n/2}_{\mathrm{loc}}(M)$, so such a perturbation of $\mathbf{g}$ would not preserve the required regularity of the metric. The proof of the rigidity part of the positive mass theorem in the paper of Miao [51] in itself requires more regularity of the metric, in order to apply the results of [9] (which, in turn, use Witten's technique, and therefore works only for spin manifolds). Finally, in the Ricci-flow approach of [49] (perhaps the most promising approach to proving rigidity) it is required that one has an L^∞ bound on the negative part of the scalar curvature of the smooth approximating metrics in order to show that the solution of the h-flow equation starting with the metric $\mathbf{g}$ has non-negative scalar curvature. In our case, we have only an $L^{n/2}$ bound on the negative part of the scalar curvature of the approximating metrics, which is insufficient. It therefore appears that all known ways to prove the rigidity result for non-spin manifolds break down (by some distance) for our class of metrics.

For the case of spin manifolds, the situation is rather better (cf. [6, 7, 45]). In Witten's argument, the rigidity part of the positive mass theorem follows directly from the fact that, if the mass of a metric is zero, then one has a basis of parallel spinor fields. As such, the spin-connection is flat, thus the full curvature tensor of $\mathbf{g}$ is flat. A theorem of Cartan then implies that the manifold is isometric to $\mathbb{R}^{n-k} \times T^k$, where T^k is a flat torus of dimension k.[4] Asymptotic flatness then implies that $k = 0$. As such, the rigidity part of the positive mass theorem for spin manifolds appears to require no additional regularity.

[3] We are grateful to Dr. Jonathan Bevan for pointing this out to us.

[4] This is a smooth result. It is not completely clear to us what regularity conditions a Riemannian metric must satisfy in order vanishing of its Riemann curvature implies the existence of a locally isometry with Euclidean space.

CHAPTER 2

Volume comparison

2.1. Distance functions on Riemannian manifolds

In this section, let $(M, \mathbf{g})$ be a smooth $n-$dimensional Riemannian manifold, Furthermore, let $\mathfrak{X}(M)$ denote the space of smooth vector fields on M.

Definition 2.1.1. Let $f \in C^\infty(M)$ be a smooth function, $X \in \mathfrak{X}(M)$. Then

(1) the *gradient of* f is defined by $\mathbf{g}(\operatorname{grad}(f), X) := X(f)$ for all $X \in \mathfrak{X}(M)$,
(2) the *Hessian of* f is given by $\operatorname{Hess}(f)(X,Y) := \mathbf{g}(\nabla_X(\operatorname{grad} f), Y)$ for all $X, Y \in \mathfrak{X}(M)$
(3) and the *Laplacian of* f is $\Delta f := \operatorname{tr}(\operatorname{Hess} f)$.

Note that the Hessian is symmetric, i.e., $\operatorname{Hess}(f)(X,Y) = \operatorname{Hess}(f)(Y,X)$ ([27, Prop. 6.1]). Indeed,

$$\begin{aligned}\operatorname{Hess}(f)(X,Y) - \operatorname{Hess}(f)(Y,X) &= \mathbf{g}(\nabla_X(\operatorname{grad} f), Y) - \mathbf{g}(\nabla_Y(\operatorname{grad} f), X)\\ &= X(Y(f)) - Y(X(f)) + \mathbf{g}(\operatorname{grad} f, [X,Y]) = 0.\end{aligned}$$

We can now define a certain type of functions is particularly useful in comparison geometry, namely distance functions.

Definition 2.1.2. Fix $p \in M$. Let $r\colon U \subseteq M \to \mathbb{R}$ be a smooth function with $|\operatorname{grad} r| = 1$. This function is called *distance function*.

An example of a distance function is given by $r(x) = d(x,p)$, where $p \in M$ is fixed. Here the set U is a sufficiently small neighborhood of p.

We will denote the smooth vector field $\operatorname{grad} r$ by ∂_r.

Proposition 2.1.3. *Let $U \subseteq M$ be open. A distance function $r\colon U \to \mathbb{R}$ also satisfies $\nabla_{\partial_r}\partial_r = 0$ on U.*

Proof. Indeed, for $X \in \mathfrak{X}(U)$, by using the symmetry of the Hessian of r,

$$\mathbf{g}(\nabla_{\partial_r}\partial_r, X) = \mathbf{g}(\partial_r, \nabla_X\partial_r) = \frac{1}{2}\nabla_X\mathbf{g}(\partial_r,\partial_r) = 0. \tag{2.1.1}$$

□

Proposition 2.1.4. *Let r be a distance function on $(M, \mathbf{g})$. It satisfies the following so called Weitzenböck identity:*

$$|\operatorname{Hess} r|^2 + \mathbf{g}(\partial_r, \operatorname{grad}(\Delta r)) + \mathbf{Ric}(\partial_r, \partial_r) = 0. \tag{2.1.2}$$

Proof. Let $p \in M$, fix a basis of normal coordinates at p, $\{e_1, \dots e_n\}$, i.e. $\mathbf{g}(e_i, e_j) = \delta_{ij}$, $\nabla_{e_i} e_j(p) = 0$.

We calculate at p:

$$\begin{aligned} |\operatorname{Hess} r|^2 &= \sum \mathbf{g}(\nabla_{e_i}\partial_r, \nabla_{e_i}\partial_r) \\ &= \sum \operatorname{Hess} r(e_i, \nabla_{e_i}\partial_r) = (*). \end{aligned}$$

Now we can use the symmetry of Hess to further calculate

$$\begin{aligned} (*) &= \sum \operatorname{Hess} r(\nabla_{e_i}\partial_r, e_i) \\ &= \sum \operatorname{Hess} r(\nabla_{e_i}\partial_r, e_i) - \operatorname{Hess} r(\nabla_{\partial_r} e_i, e_i) \\ &= \sum \operatorname{Hess} r([\partial_r, e_i], e_i) \\ &= \sum \mathbf{g}(\nabla_{[\partial_r, e_i]}(\partial_r), e_i). \end{aligned}$$

Note that $\operatorname{Hess} r(\nabla_{\partial_r} e_i, e_i) = 0$.

The term $\mathbf{g}(\partial_r, \operatorname{grad}(\Delta r))$ can be reformulated as follows:

$$\begin{aligned} \mathbf{g}(\partial_r, \nabla(\Delta r)) &= \mathbf{g}(\partial_r, \operatorname{grad}(\Delta r)) \\ &= \partial_r(\Delta r) \\ &= \partial_r \sum \operatorname{Hess} r(e_i, e_i) \\ &= \partial_r \sum \mathbf{g}(\nabla_{e_i}\partial_r, e_i) \\ &= \sum \partial_r(\mathbf{g}(\nabla_{e_i}\partial_r, e_i)) - \mathbf{g}(\nabla_{e_i}\partial_r, \nabla_{\partial_r} e_i) \\ &= \sum \mathbf{g}(\nabla_{\partial_r}(\nabla_{e_i}\partial_r), e_i). \end{aligned}$$

Here again $\mathbf{g}(\nabla_{e_i}\partial_r, \nabla_{\partial_r} e_i)$ vanishes.

For the final **Ric** term just use the definition of **Ric** and write it as

$$\mathbf{Ric}(\partial_r, \partial_r) = \sum \mathbf{g}(R(e_i, \partial_r)\partial_r, e_i).$$

Summing up the three terms gives

$$\begin{aligned} |\operatorname{Hess} r|^2 + \mathbf{g}(\partial_r, \operatorname{grad}(\Delta r)) + \mathbf{Ric}(\partial_r, \partial_r) &= \sum \mathbf{g}(\nabla_{[\partial_r, e_i]}(\partial_r), e_i) + \sum \mathbf{g}(\nabla_{\partial_r}(\nabla_{e_i}\partial_r), e_i) \\ &+ \sum \mathbf{g}(R(e_i, \partial_r)\partial_r, e_i) = (*). \end{aligned}$$

Due to the definition of R, we can rewrite this expression and get

$$(*) = \sum \mathbf{g}(\nabla_{e_i}(\nabla_{\partial_r}\partial_r), e_i) + \mathbf{g}(\nabla_{\partial_r}\partial_r, \nabla_{e_i} e_i),$$

which, due to (2.1.1), vanishes. □

Remark 2.1.5. This identity not only holds for distance functions but for arbitrary functions that are at least in $C^3(M)$. Note that then the right hand side does not vanish, but consists of a $|\operatorname{grad} f|$ term and we get

$$\frac{1}{2}\Delta|\operatorname{grad} f|^2 = |\operatorname{Hess} f|^2 + \mathbf{g}(\operatorname{grad} f, \nabla(\Delta f)) + \mathbf{Ric}(\operatorname{grad} f, \operatorname{grad} f). \tag{2.1.3}$$

In order to prove this relation, we have to continue reformulating the term

$$\sum \mathbf{g}(\nabla_{e_i}(\nabla_{\operatorname{grad} f} \operatorname{grad} f), e_i) + \mathbf{g}(\nabla_{\operatorname{grad} f} \operatorname{grad} f, \nabla_{e_i} e_i),$$

which vanishes in case of f beng a distance function.

So in the general case,

$$\begin{aligned}\sum \mathbf{g}(\nabla_{e_i}(\nabla_{\operatorname{grad} f} \operatorname{grad} f), e_i) \quad + \quad & \sum \mathbf{g}(\nabla_{\operatorname{grad} f} \operatorname{grad} f, \nabla_{e_i} e_i) = \sum e_i \mathbf{g}(\nabla_{\operatorname{grad} f} \operatorname{grad} f, e_i)\\ &= \sum e_i \operatorname{Hess} f(\operatorname{grad} f, e_i) = \sum e_i \operatorname{Hess} f(e_i, \operatorname{grad} f)\\ &= \sum e_i \mathbf{g}(\nabla_{e_i} \operatorname{grad} f, \operatorname{grad} f) = \frac{1}{2}\sum e_i e_i \mathbf{g}(\operatorname{grad} f, \operatorname{grad} f)\\ &= \frac{1}{2}\Delta|\operatorname{grad} f|^2.\end{aligned}$$

Remark 2.1.6. The Bochner-Weitzenböck formulas (also known as the Bochner technique) are an important tool in geometric analysis. In case of Riemannian geometry, a Weitzenböck formula expresses the Laplace operator in terms of the Levi-Civita connection of the manifold.

This technique was introduced by Bochner in 1946 [8]. Before, in 1923 Weitzenböck established in his book about invariant theory [67] a similar formula for p-forms.

Another property which will be used in Chapter 3 below is the following:

Lemma 2.1.7. [3, Lemma 1.4] *Let M be a Riemannian manifold with $\operatorname{inj}_M > i_0$ and $\mathbf{Ric}_M > -K$, $K > 0$. Let $r = d(x, \cdot)$, be the distance function from $x \in M$. Then it holds that*

$$|\Delta r| \leqslant (n-1)K\coth(Kr), \tag{2.1.4}$$

provided $r < i_0/2$.

Proof. By Laplacian comparison, see 2.3.1 below, we obtain

$$\Delta r \leqslant (n-1)K\coth(Kr) \tag{2.1.5}$$

where $r \leqslant i_0$.

In order to get the other estimate, fix $x \in M$, and let p be a point with $t = d(x,p) < i_0/2$.

Let γ be the minimal geodesic connecting p and x. Set $p_1 = \gamma(2t)$. Then, on $B(p, i_0/2)$, the estimate (2.1.5) is valid for $r(\cdot) = d(x, \cdot)$ and also for $r_1(\cdot) = d(p_1, \cdot)$, since both functions do not reach to the cut locus.

We now want to construct a function ρ that is nonnegative and vanishes on γ. This can be done by setting

$$\rho := r + r_1 - 2t \colon M \to \mathbb{R}.$$

It follows that

$$\rho(\cdot) = d(x, \cdot) + d(p_1, \cdot) - 2t \geqslant d(x, p_1) - 2t \geqslant 0.$$

Thus $\rho \geqslant 0$ and $\rho = 0$ on γ. Hence,

$$\Delta\rho = \Delta(r + r_1) \geqslant 0. \tag{2.1.6}$$

Therefore,

$$\begin{aligned}\Delta r \geqslant -\Delta r_1 &\geqslant -(n-1)K\coth(Kr_1)\\ &\geqslant -(n-1)K\coth(Kr)\end{aligned}$$

on $B(p, i_0/2)$, since $x \mapsto \coth(Kx)$ is decreasing for $K < 0$, $x > 0$. □

2.2. Volume forms and Laplacians

Let $K \in \mathbb{R}$. We define the functions

$$sn_K(r) := \begin{cases} \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}r\right) & K > 0,\\ r & K = 0,\\ \frac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}r\right) & K < 0\end{cases}. \tag{2.2.1}$$

In case $K > 0$, we have to assume $r \in [0, \pi/\sqrt{K}]$, and for $K \leqslant 0$, $r \in [0, \infty)$. Let again $d\theta^{n-1}$ denote the standard volume element on the unit $(n-1)$-sphere in $\mathbb{R}^n$. The metrics

$$\mathbf{g}_K := dr^2 + sn_K(r)^2 d\theta^{n-1}. \tag{2.2.2}$$

are of *constant curvature*, i.e., the $(0, 4)$ form of the curvature tensor takes the form

$$R_{\mathbf{g}_K}(W, X, Y, Z) = K\left(\mathbf{g}_K(W, Y)\mathbf{g}_K(X, Z) - \mathbf{g}_K(W, Z)\mathbf{g}_K(X, Y)\right) \tag{2.2.3}$$

for smooth vector fields W, X, Y, Z.

The Ricci tensor of the metric $\mathbf{g}_K$ is of the form

$$\mathbf{Ric}_{\mathbf{g}_K} = K\,(n-1)\,\mathbf{g}_K.$$

Although, in general, the sectional curvature will depend on the point p and the plane $\sigma \subseteq T_pM$, in the case of our metrics $\mathbf{g}_K$, it follows from (2.2.3) that $K_p(\sigma) = K$ for all $p \in M$ and all two-planes $\sigma \subseteq T_pM$.

For the volume element of these metrics, one can calculate (see e.g., [27])

$$d\mu_{\mathbf{g}_K} = sn_K(r)^{n-1} dr \wedge d\theta^{n-1}.$$

where $d\theta^{n-1}$ is the volume element of the unit sphere in $\mathbb{R}^n$.

The Laplacian of the distance function for these metrics can be calculated to give (for the detailed calculation see, e.g. [27, Section 6]):

$$\Delta_K r = \frac{\partial_r\left(sn_K(r)\right)^{n-1}}{\left(sn_K(r)\right)^{n-1}},, \tag{2.2.4}$$

thus,

$$\Delta_K r = \begin{cases} (n-1)\sqrt{K}\cot\left(\sqrt{K}r\right) & K>0, \\ (n-1)/r & K=0, \\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}r\right) & K<0 \end{cases}. \tag{2.2.5}$$

We now investigate the volume form and the Laplacian for arbitrary metrics which are not necessarily of constant curvature.

Let $\mathbf{h}$ be a general metric on the unit $(n-1)$-sphere, $\mathbf{g}_{S^{n-1}}$ the induced metric on the unit $(n-1)$ sphere in $\mathbb{R}^n$. The volume element of the arbitrary metric $\mathbf{g} = dr^2 + \mathbf{h}(r,\theta)$ is given by

$$\begin{aligned} d\mu_{\mathbf{g}} &= \frac{\sqrt{\det(\mathbf{h})}}{\sqrt{\det(\mathbf{g}_{S^{n-1}})}} dr \wedge d\theta^{n-1} && (2.2.6) \\ &= \bar{\omega}(r,\theta)^{n-1} dr \wedge d\theta^{n-1}, && (2.2.7) \end{aligned}$$

with

$$\bar{\omega}(r,\theta)^{n-1} = \frac{\sqrt{\det(\mathbf{h})}}{\sqrt{\det(\mathbf{g}_{S^{n-1}})}}. \tag{2.2.8}$$

In order for the metric to stay nonsingular at $r=0$ we require that it approaches the flat $\mathbb{R}^n-$metric. Thus $\bar{\omega}(r,\theta) \to r$ as $r \to 0$.

We now calculate Δr in the above metric. Using local coordinates (r,θ) gives

$$\Delta r = \sum_{i,j} g^{ij} \left(\frac{\partial^2 r}{\partial x^i \partial x^j} - \sum_k {\Gamma^k}_{ij} \frac{\partial r}{\partial x^k} \right)$$

Noting that $x^1 = r$, it follows that the first term vanishes, the $k=1$ term remains in the second term. Furthermore, we use the fact that $g^{rr} = 1$, $\partial_i g_{rj} = 0$ for all i,j and the fact that for any invertible, symmetric matrix $\Lambda(x)$, it holds that $\mathrm{tr}(\Lambda^{-1}\frac{d\Lambda}{dx}) = \frac{d\det\Lambda}{dx}/\det\Lambda$. We then arrive at

$$\begin{aligned} \Delta r &= -\sum_{i,j} g^{ij} {\Gamma^r}_{ij} = -\frac{1}{2}\sum_{i,j,k} g^{ij} g^{rk} \left[\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right] \\ &= -\frac{1}{2}\sum_{i,j,k} g^{ij} g^{rr} \left[\partial_i g_{rj} + \partial_j g_{ri} - \partial_r g_{ij}\right] \\ &= \frac{1}{2}\sum_{i,j} g^{ij}\partial_r g_{ij} \equiv \frac{1}{2}\mathrm{tr}\left(\mathbf{g}^{-1}\partial_r \mathbf{g}\right) = \frac{1}{2}\frac{\partial_r(\det\mathbf{g})}{\det\mathbf{g}} = \frac{\partial_r(\det\mathbf{g})^{1/2}}{(\det\mathbf{g})^{1/2}}. \end{aligned}$$

Since $d\mu_{\mathbf{g}} = (\det\mathbf{g})^{1/2}\, d^n x$, we deduce from (2.2.7) that $(\det\mathbf{g})^{1/2} = \bar{\omega}(r,\theta)^{n-1} \cdot (\det\mathbf{h})^{1/2}$ Since $\det\mathbf{h}$ is independent of r, we therefore have

$$\Delta r = \frac{\partial_r \left(\bar{\omega}(r,\theta)\right)^{n-1}}{\left(\bar{\omega}(r,\theta)\right)^{n-1}} = \frac{(n-1)}{\bar{\omega}(r,\theta)}\frac{\partial\bar{\omega}(r,\theta)}{\partial_r}. \tag{2.2.9}$$

2.3. Classical volume comparison results

For all of the following results, we assume that the metric $\mathbf{g}$ is smooth. We will follow [72].

THEOREM 2.3.1 (Laplacian comparison). *Let $(M, \mathbf{g})$ be a complete, n-dimensional Riemannian manifold. If*

$$\mathbf{Ric} \geqslant (n-1)K,$$

then, outside of the cut locus of $p \in M$, the following comparison result holds:

$$\Delta r \leqslant \Delta_K(r). \tag{2.3.1}$$

Proof. By (2.1.2), it holds that

$$|\operatorname{Hess} r|^2 + \frac{\partial}{\partial_r}(\Delta r) + \mathbf{Ric}(\partial_r, \partial_r) = 0.$$

Take $p \in M$, and choose an orthonormal basis in which the matrix $|\operatorname{Hess} r|$ is diagonal. This is possible due to the symmetry of the Hessian. By (2.1.1), $\nabla_{\partial_r}\partial_r = 0$, it follows that $|\operatorname{Hess} r|$ is an $n \times n$ matrix of the form $\begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix}$, where $s = \operatorname{diag}[\lambda_1, \ldots, \lambda_{n-1}]$,with the λ_i the eigenvalues of $|\operatorname{Hess} r|$.

Using the Cauchy-Schwartz inequality it follows that

$$\begin{aligned}(\Delta r)^2 &= (\operatorname{tr}|\operatorname{Hess} r|)^2 \\ &= (\lambda_1 + \ldots \lambda_{n-1})^2 \\ &\leqslant (n-1)(\lambda_1^2 + \ldots + \lambda_{n-1}^2) = (n-1)|\operatorname{Hess} r|^2.\end{aligned}$$

Thus, inserting this in (2.1.2), we obtain

$$\frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial_r}(\Delta r) + \mathbf{Ric}(\partial_r, \partial_r) \leqslant 0, \tag{2.3.2}$$

If $\mathbf{Ric} \geqslant (n-1)K$, then it holds that

$$\frac{(\Delta r)^2}{n-1} + \frac{\partial}{\partial r}(\Delta r) + (n-1)K \leqslant 0. \tag{2.3.3}$$

Setting $u = \Delta r$, this can be rewritten as the following Riccati inequality

$$u' + \frac{u^2}{n-1} + (n-1)K \leqslant 0, \tag{2.3.4}$$

where $'$ denotes differentiation with respect to r. When $\mathbf{g}$ converges to the flat metric of $\mathbb{R}^n$, it follows that $u \to \frac{n-1}{r}$ for $r \to 0$.

Integrating then leads to

$$\Delta r \leqslant \begin{cases} (n-1)\sqrt{K}\cot\left(\sqrt{K}r\right) & K > 0, \\ (n-1)/r & K = 0, \\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}r\right) & K < 0 \end{cases}. \tag{2.3.5}$$

This shows Laplacian comparison. □

Remark 2.3.2. An (in)equality of the form $y'(\leqslant) = -y^2 - K$ is called a Riccati (in)equality/equation. If u and v are solutions of $u' \leqslant -u^2 - K$ and $v' = -v^2 - K$, then it holds that the function $v - u$ is monotonically increasing. If $v(a) \geqslant u(a)$, then $v(t) \geqslant u(t)$ in the whole domain $I = [a, b]$. If additionally for some $t_0 \in I$ equality holds, i.e. $v(t_0) = u(t_0)$, then $v|_{[a,t_0]} = u|_{[a,t_0]}$.

Corollary 2.3.3. *Let $(M, \mathbf{g})$ be a complete, n-dimensional Riemannian manifold with*

$$\mathbf{Ric} \geqslant (n-1)K.$$

then the mean curvatures of geodesic spheres of radius r in M, and in the space of constant curvature K can be compared by

$$h(r) \leqslant h_K(r).$$

Proof. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis for the geodesic sphere of radius r, N the outer normal to this sphere. By definition of the Laplacian we get

$$\begin{aligned} \Delta r &= \operatorname{tr}(\operatorname{Hess} r) = \sum_{i=1}^{n-1} \mathbf{g}(\nabla_{e_i}(\partial_r), e_i) + \mathbf{g}(\nabla_N(\partial_r), N) \\ &= \sum_{i=1}^{n-1} \mathbf{g}(\nabla_{e_i} N, e_i) + \mathbf{g}(\nabla_N N, N) \\ &= \sum_{i=1}^{n-1} \mathbf{g}(\nabla_{e_i} N, e_i) = h(r). \end{aligned} \tag{2.3.6}$$

Thus the result follows from the Laplacian comparison result 2.3.1. □

Another classical comparison result, which will also be used in the proof of the Bishop-Gromov volume comparison theorem below is the volume element comparison, see also [27, 72]:

Proposition 2.3.4. *Let $(M, \mathbf{g})$ be as in Theorem 2.3.1. Then the map $f(r) = \frac{\bar{\omega}(r,\theta)}{sn_K(r)}$ is non-increasing along radial geodesics.*

Proof. Using again formula (2.3.3) and inserting (2.2.9) we see that $\bar{\omega}$ satisfies

$$\frac{1}{\bar{\omega}(r,\theta)} \frac{\partial^2 \bar{\omega}(r,\theta)}{\partial r^2} \leqslant -K.$$

Since by assumption, $\bar{\omega}(r,\theta) \to r$, for $r \to 0$, it follows that $\bar{\omega}(r,\theta) > 0$, for sufficiently small $r > 0$. Hence also

$$\frac{\partial^2 \bar{\omega}(r,\theta)}{\partial r^2} + K\bar{\omega}(r,\theta) \leqslant 0.$$

In the next step we compare $\mathbf{g}$ with the constant curvature metric $\mathbf{g}_K$, in detail, we compare the function $\bar{\omega}$ with the function $sn_K(r)$ defined in equation (2.2.1). This function satisfies

$$\frac{\partial^2 sn_K(r)}{\partial r^2} + K sn_K(r) = 0, \qquad sn_K(0) = 0, \qquad sn_K'(0) = 1.$$

We then have

$$\begin{aligned} \partial_r \left(sn_K(r)\bar{\omega}'(r,\theta) - sn_K'(r)\bar{\omega}(r,\theta) \right) &= sn_K(r)\bar{\omega}''(r,\theta) - sn_K''(r)\bar{\omega}(r,\theta) \\ &\leqslant -K sn_K(r)\bar{\omega}(r,\theta) + K sn_K(r)\bar{\omega}(r,\theta) \\ &= 0. \end{aligned}$$

Hence $sn_K(r)\bar{\omega}'(r\theta) - sn_K'(r)\bar{\omega}(r,\theta)$ is non-increasing. Since at $r = 0$ it is zero, it follows that

$$sn_K(r)\bar{\omega}'(r,\theta) - sn_K'(r)\bar{\omega}(r,\theta) \leqslant 0.$$

Since sn_K, $\bar{\omega}$ are positive for sufficiently small $r > 0$, we get

$$\left(\frac{\bar{\omega}(r,\theta)}{sn_K(r)} \right)' \leqslant 0, \tag{2.3.7}$$

and therefore also

$$f(r) = \frac{\bar{\omega}(r,\theta)}{sn_K(r)} \text{ is non-increasing.} \tag{2.3.8}$$

□

We will prove a global version of the Bishop-Gromov volume comparison theorem, i.e. a version that remains valid beyond the cut locus. We follow the lecture notes [72].

We will make use of the following lemma, which can be found in [72, Lemma 3.2]:

Lemma 2.3.5. *Let f, g be two positive functions. If f/g is non-increasing, then, for $R > r > 0$, $S > s > 0$, $r > s$, $R > S$, it holds that*

$$\frac{\int_r^R f(t)dt}{\int_s^S f(t)dt} \leqslant \frac{\int_r^R g(t)dt}{\int_s^S g(t)dt}. \tag{2.3.9}$$

Let S^{n-1} denote the $n-1$ dimensional unit sphere in $\mathbb{R}^n$, and denote by $V(p,r)$ and $V_K(r)$ the volumes of the geodesic balls around p of radius r in M and in the space of constant curvature K, respectively, we arrive at well-known comparison results:

Theorem 2.3.6. *Let $(M, \mathbf{g})$ be a complete, n-dimensional Riemannian manifold. If*

$$\mathbf{Ric} \geqslant (n-1)K \text{ on } M,$$

then for $R < R'$ it holds that

$$\frac{V(p,R)}{V(p,R')} \geqslant \frac{V_K(R)}{V_K(R')}. \tag{2.3.10}$$

Furthermore, for all $r > 0$,

$$V(p,r) \leqslant V_K(r).$$

Equality for both cases holds if and only if the balls in M and in the space with constant curvature K are actually isometric.

REMARK 2.3.7. The second statement that $V(p,r) \leqslant V_K(r)$ is the original result due to Bishop, who proved this result for r less than the injectivity radius. The result was generalized later by Gromov [33] who showed that the map $r \to V(p,r)/V_K(r)$ is non-increasing and also noted that, with suitable modifications, the result holds for all values of r.

Proof. Denote by $\mathrm{Cut}(p_\theta)$ the distance from p to the cut locus in the direction θ, then

$$V(p,R) = \int_{S^{n-1}} \int_0^{\min(R,\mathrm{Cut}(p_\theta))} \bar{\omega}(r,\theta) dr d\theta, \tag{2.3.11}$$

where $\bar{\omega}(r,\theta)$ is given by (2.2.8). We can now define $\bar{\omega}(r,\theta) \equiv 0$ for $r > \mathrm{Cut}(p_\theta)$ (see also [27, p. 26]).

Therefore,

$$V(p,R) = \int_{S^{n-1}} \int_0^R \bar{\omega}(r,\theta) dr d\theta,$$

By the area comparison result Proposition 2.3.4, we already know that $\frac{\bar{\omega}(r,\theta)}{sn_K(r)}$ is non-increasing as long as $r < \mathrm{Cut}(p_\theta)$. We can use Lemma 2.3.5 to obtain for $R' \geqslant R$,

$$\frac{\int_{0)}^{R)} \bar{\omega}(r,\theta) dr}{\int_0^R sn_K(r) dr} \geqslant \frac{\int_0^{R'} \bar{\omega}(r,\theta) dr}{\int_0^{R'} sn_K(r) dr},$$

where again $\bar{\omega}(r,\theta) \equiv 0$ for $r > \mathrm{Cut}(p_\theta)$.

We furthermore get

$$\int_0^R \bar{\omega}(r,\theta) dr \geqslant \frac{\int_0^R sn_K(r) dr}{\int_0^{R'} sn_K(r) dr} \int_0^{R'} \bar{\omega}(r,\theta) dr.$$

To obtain the final result, we integrate over S^{n-1} and get

$$V(p,R) \geqslant \frac{\int_{S^{n-1}} \int_0^R sn_K(r) dr d\theta}{\int_{S^{n-1}} \int_0^{R'} sn_K(r) dr d\theta} V(p,R') = \frac{V_K(R)}{V_K(R')} V(p,R).$$

Since $\frac{V(p,R)}{V_K(R)} \to 1$ as $R \to 0$, it follows that $V(p,R) \leqslant V_K(R)$. $\square$

By an analogous proof it is possible to show the following more general result, see [72, Theorem 3.1]:

THEOREM 2.3.8. *Let $R \geqslant r \geqslant 0$, $S \geqslant s \geqslant 0$, $r \geqslant s$, $R \geqslant S$. Choose Γ to be an arbitrary measurable subset of the unit tangent sphere about p. Define*

$A^\Gamma_{r,R}(p) := \{x \in M \ : r \leqslant r(x) \leqslant R$ and any minimal geodesic c from p to x satisfies $\dot{c}(0) \in \Gamma\}$.

where $r(x) = d(p,x)$ is the distance function from $p \in M$.

Then,

$$\frac{\mathrm{Vol}\,(A^{\Gamma}_{r,R}(p))}{\mathrm{Vol}\,(A^{\Gamma}_{s,S}(p))} \geqslant \frac{\mathrm{Vol}_K(A^{\Gamma}_{r,R}(p))}{\mathrm{Vol}_K(A^{\Gamma}_{s,S}(p))}, \tag{2.3.12}$$

where the subscript K declares the quantity in the space of constant curvature K. Equality holds if and only if the curvatures along radial geodesics are all equal to K.

2.3.1. Applications.

(1) **Myers' theorem:**

The volume comparison result, Theorem 2.3.10, provides a way of proving Myers' theorem [53] without making use of variational arguments.

THEOREM 2.3.9. *Let M be an $n-$ dimensional complete Riemannian manifold. If for all $X \in \mathfrak{X}(M)$ it holds that $\mathbf{Ric}_M(X,X) \geqslant (n-1)K\mathbf{g}(X,X)$, then every geodesic of length $\geqslant \pi K^{\frac{1}{2}}$ has conjugate points and therefore* $\mathrm{diam}(M) \leqslant \pi K^{\frac{1}{2}}$. *Therefore M is compact.*

Proof. Without loss of generality we assume $K = 1$.

By contradiction, let $p, q \in M$ be two points with $d(p,q) > \pi$, c a minimal geodesic from p to q. Since c is minimal, $c(\pi)$ is not on the cut locus of p, hence the distance function $r(x) = d(p,x)$ is smooth at $c(\pi)$, hence Δr is finite. Using the formula for the Laplacian of r from (2.2.9), thus

$$\Delta r \leqslant (n-1)\cot r.$$

Taking $r \to \pi$, it follows that $\Delta r \leqslant -\infty$, which contradicts the fact that Δr is finite. Hence, the diameter of M, $\mathrm{diam}\, M \leqslant \pi$. ☐

A version of the proof using the Index Lemma, can be found in [15, 1.26], resp. [22].

(2) **Estimating volumes of geodesic balls:**

As a second application we will investigate a result by Yau [71]. It will, together with the Bishop estimate give a both-sided bound on the growth rate of balls in manifolds which are non-compact and have non-negative Ricci curvature.

THEOREM 2.3.10. *For $(M, \mathbf{g})$ a complete, non-compact n-dimensional manifold with* $\mathbf{Ric} > 0$, *the following lower bound on the volume of geodesic balls holds:*

$$V(p,r) \geqslant cr$$

for some $c > 0$.

The original proof of this results uses analytic methods [71]. For the proof with help of volume comparison, see [72, Theorem 3.5].

2.4. Relative volume comparison with integral bounds on the curvature

In this section we will discuss two approaches to studying volume comparison for manifolds that just admit integral bounds on the curvature. In [59] the authors prove a more general version of the classical Bishop-Gromov volume comparison result by imposing an L^p-bound ($p > n/2$) for the part of Ricci curvature that lies below a certain value. In [29] the case $p = \frac{n}{2}$ is covered as well.

NOTATION 2.4.1. Let $(M, \mathbf{g})$ be a Riemannian manifold. Let $x \in M$ and $r(\cdot) := d(x, \cdot)$ denote the distance function from x. Let $S_x \subset T_xM$ denote the sphere of unit tangent vectors at x. For $\theta \in S_x$, let $\gamma_\theta \colon \mathbb{R} \to M$ denote the geodesic with $\gamma_\theta(0) = x$, $\gamma'_\theta(0) = \theta$. Furthermore, we denote the distance along the geodesic γ_θ to its cut-point by

$$\ell(\theta) := \sup\{t > 0 \mid d(x, \gamma_\theta(t)) = t\}.$$

As in Section 2.3, we denote the volume of the ball radius $r \geqslant 0$ centered at $p \in M$ by $V(p, r)$ or $|B(p, r)|$, and the area of the corresponding sphere by $|S(p, r)|$. We denote the corresponding quantities in flat $\mathbb{R}^n$ by $V_0(r) \equiv |B_0(r)| \equiv \frac{1}{n}\omega_{n-1}r^n$, and $|S_0(r)| = \omega_{n-1}r^{n-1}$, where ω_{n-1} denotes the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$. Furthermore, the quantities in the comparison space of constant curvature K are denoted by $V_K(r)$ and $S_K(r)$.

The function $g : M \to [0, \infty)$ is defined by the condition that $g(x)$ is the smallest eigenvalue of the Ricci tensor at x. Furthermore, let

$$k(K, p) := \int_M \max\{-g(x) + (n-1)K, 0\}^p d\mu_{\mathbf{g}}, \tag{2.4.1}$$

$$\bar{k}(K, p) := \frac{1}{\mathrm{Vol}\,(M)} \int_M \max\{-g(x) + (n-1)K, 0\}^p d\mu_{\mathbf{g}}. \tag{2.4.2}$$

The quantities k and $\bar{k}$, respectively, measure the amount, and the averaged amount of curvature respectively, that lies below $(n-1)K$.

The volume element of $\mathbf{g}$ is of the form $d\mu_{\mathbf{g}}(r, \theta) \equiv \omega(r, \theta)\, dr \wedge d\theta^{n-1}$, and analogously $d\mu_{\mathbf{g}_K}(r, \theta) \equiv \omega_K(r, \theta)\, dr \wedge d\theta^{n-1}$ for the constant curvature metric $\mathbf{g}_K$. Note that $\bar{\omega}(r, \theta)^{n-1} = \omega(r, \theta)$. In order to calculate volume integrals correctly, for each $\theta \in S_x$, we adopt the Gromov convention that $\omega(r, \theta) := 0$ for $r \geqslant \ell(\theta)$.

Let h and h_K be the mean curvatures of the geodesic balls in M with respect to $\mathbf{g}$ and $\mathbf{g}_K$ respectively.

For any measurable function f, we let $f_+ := \max\,(f, 0)$ and $f_- := \max\,(-f, 0)$. In particular, $f_- \geqslant 0$.

Finally, given a measurable open set $U \subseteq M$, we denote its Riemannian measure by $|U|$ and define the average

$$\fint_U f\, d\mu_{\mathbf{g}} := \frac{1}{|U|} \int_U f\, d\mu_{\mathbf{g}}.$$

The mean curvatures h and h_K satisfy the following relations due to (2.2.4)and (2.3.6)

$$\frac{\partial}{\partial r}\omega(r,\theta) = h(r,\theta)\,\omega(r,\theta), \tag{2.4.3}$$

$$\frac{d}{dr}\omega_K(r) = h_K(r)\,\omega_K(r). \tag{2.4.4}$$

Furthermore, they satisfy, by (2.3.2),

$$\frac{\partial}{\partial r}h(r,\theta) + \frac{h(r,\theta)^2}{n-1} \leqslant -\mathbf{Ric}(\partial_r,\partial_r), \tag{2.4.5}$$

$$\frac{d}{dr}h_K(r) + \frac{h_K(r)^2}{n-1} = -(n-1)K. \tag{2.4.6}$$

DEFINITION 2.4.2. Given a subset $\widehat{S} \subseteq S_x$ and $r > 0$, we define the cone over $\widehat{S}$ of radius r as the set (see [69, §7])

$$\Gamma(\widehat{S},r) := \left\{ y \in M \,\middle|\, y = \exp_x(t\theta)\,,\ \text{where } \theta \in \widehat{S},\ 0 \leqslant t < r,\ d(x,y) = r \right\}.$$

We define analogous quantities on flat $\mathbb{R}^n$. Given a reference point $x_0 \in \mathbb{R}^n$, let $I\colon T_xM \to T_{x_0}\mathbb{R}^n$ be a fixed linear isometry, by which we may identify the unit spheres $S_x \subset T_xM$ and $S_{x_0} \subset T_{x_0}\mathbb{R}^n$.

Given $\widehat{S} \subseteq S_x$, we will abuse notation and also denote by $\widehat{S}$ the corresponding set $I(\widehat{S})$ in S_{x_0}. We define the corresponding cone

$$\Gamma_0(\widehat{S},r) := \left\{ y \in \mathbb{R}^n \,\middle|\, y = \exp_{x_0}(t\theta)\,,\ \text{where } \theta \in \widehat{S},\ 0 \leqslant t < r,\ d(x,y) = t \right\}.$$

2.4.1. Volume of cones and geodesic balls. In this section, we will investigate volume monotonicity results, thus we start by studying the derivatives of volumes.

REMARK 2.4.3. The derivative of the area of the cone is given by the following:

$$\begin{aligned} \frac{d}{dr}|\Gamma(\widehat{S},r)| &= \frac{d}{dr}\int_{\theta\in\widehat{S}}\int_0^r \omega(t,\theta)\,dt\,d\theta \\ &= \int_{\theta\in\widehat{S}} \omega(r,\theta)\,d\theta, \end{aligned} \tag{2.4.7}$$

and

$$\begin{aligned} \frac{d}{dr}|\Gamma_0(\widehat{S},r)| &= \frac{d}{dr}\int_{\theta\in\widehat{S}}\int_0^r t^{n-1}\,dt\,d\theta = \int_{\theta\in\widehat{S}} r^{n-1}\,d\theta \\ &= \frac{n}{r}\int_{\theta\in\widehat{S}} \frac{r^n}{n}\,d\theta = \frac{n}{r}\int_{\theta\in\widehat{S}}\int_0^r t^{n-1}\,dt\,d\theta \\ &= \frac{n}{r}|\Gamma_0(\widehat{S},r)|. \end{aligned} \tag{2.4.8}$$

In [59] the authors develop volume comparison results based on integral estimates for the difference between mean curvatures,

$$\psi(r,\theta) := h(r,\theta) - h_0(r). \tag{2.4.9}$$

Petersen and Wei prove the following

Lemma 2.4.4. [59, Lemma 2.1] *It holds that*

$$\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant c(n,r)\frac{V(x,r)}{V_0(r)}^{1-\frac{1}{2p}}\left(\int_{B(x,r)} \psi(t,\theta)\, d\mu_{\mathbf{g}}(t,\theta)\right)^{\frac{1}{2p}} V_0(r)^{\frac{1}{2p}}. \tag{2.4.10}$$

with equality for $r < \operatorname{inj} x$.

It is also possible, as in [29], to study, in contrast to (2.4.9), the dimensionless quantity

$$\Psi(r,\theta) := r\,(h(r,\theta) - h_0(r)) \equiv r\,\psi(r,\theta). \tag{2.4.11}$$

The volume comparison analysis in [29] begins with the following result.

Proposition 2.4.5. *Let $\widehat{S}_x \subseteq S_x$ and $r > 0$. Then*

$$r\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant \frac{1}{|\Gamma_0(\widehat{S},r)|}\int_{\Gamma(\widehat{S},r)} \Psi(t,\theta)\, d\mu_{\mathbf{g}}(t,\theta) := \frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\fint_{\Gamma(\widehat{S},r)} \Psi(t,\theta)\, d\mu_{\mathbf{g}}(t,\theta), \tag{2.4.12}$$

with equality if $r \leqslant \ell(\theta)$ for all $\theta \in \widehat{S}$.

Proof. Using (2.4.7), (2.4.8), we get

$$\begin{aligned}\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) &= \frac{d}{dr}\left(\frac{\int_{\theta\in\widehat{S}}\int_0^r \omega(t,\theta)\,dt\,d\theta}{\int_{\theta\in\widehat{S}}\int_0^r t^{n-1}\,dt\,d\theta}\right)\\ &= \frac{1}{|\Gamma_0(\widehat{S},r)|}\int_{\theta\in\widehat{S}}\left[\omega(r,\theta) - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt\right]d\theta.\end{aligned} \tag{2.4.13}$$

Let $\tilde{S}_x := \{\theta \in \widehat{S} \mid r \leqslant \ell(\theta)\}$ and note that $\tilde{S}_x = \widehat{S}$ if $\ell(\theta) < r$ for all $\theta \in \widehat{S}$. If $\theta \in \tilde{S}_x$, then we note that

$$\omega(r,\theta) - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt = \frac{1}{r}\int_0^r t^n \frac{\partial}{\partial t}\left(\frac{\omega(t,\theta)}{t^{n-1}}\right)dt.$$

Indeed,

$$\begin{aligned}
\omega(r,\theta) - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt &= \frac{1}{r}\int_0^r \frac{\partial}{\partial t}(t\omega(t,\theta))\,dt - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt \\
&= \frac{1}{r}\int_0^r t\frac{\partial}{\partial t}\omega(t,\theta) + \omega(t,\theta)\,dt - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt \\
&= \frac{1}{r}\int_0^r t^n\left(\frac{\partial_t\omega(t,\theta)}{t^{n-1}} - \frac{(n-1)\omega(t,\theta)}{t^n}\right)dt \\
&= \frac{1}{r}\int_0^r t^n\frac{\partial}{\partial t}\left(\frac{\omega(t,\theta)}{t^{n-1}}\right)dt.
\end{aligned}$$

Furthermore, using (2.4.3), this expression can be simplified to

$$\begin{aligned}
\frac{1}{r}\int_0^r t^n\frac{\partial}{\partial t}\left(\frac{\omega(t,\theta)}{t^{n-1}}\right)dt &= \frac{1}{r}\int_0^r t^n\frac{\partial}{\partial t}\left(\frac{\omega(t,\theta)}{\omega_0(t)}\right)dt && (2.4.14)\\
&= \frac{1}{r}\int_0^r t\left(h(t,\theta) - \frac{n-1}{t}\right)\omega(t,\theta)\,dt && (2.4.15)\\
&= \frac{1}{r}\int_0^r \Psi(t,\theta)\,\omega(t,\theta)\,dt, && (2.4.16)
\end{aligned}$$

If $\theta \in \widehat{S}\backslash\tilde{S}_x$ then, since $\omega(t,\theta) = 0$ for $t \geqslant \ell(\theta)$,

$$\omega(r,\theta) - \frac{n}{r}\int_0^r \omega(t,\theta)\,dt = -\frac{n}{r}\int_0^{\ell(\theta)} \omega(t,\theta)\,dt \leqslant 0. \tag{2.4.17}$$

Substituting (2.4.16) and (2.4.17) into (2.4.13), we deduce that

$$\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant \frac{1}{r\,|\Gamma_0(\widehat{S},r)|}\int_{\theta\in\tilde{S}_x}\left[\int_0^r \Psi(t,\theta)\,\omega(t,\theta)\,dt\right]d\theta = \frac{1}{r\,|\Gamma_0(\widehat{S},r)|}\int_{\Gamma(\widehat{S},r)} \Psi\,d\mu_{\mathbf{g}},$$

with equality if $\tilde{S}_x = \widehat{S}$, i.e. $r \leqslant \ell(\theta)$ for all $\theta \in \widehat{S}$. □

Applying Proposition 2.4.5 with $\widehat{S} = S_x$ gives the following result for geodesic balls.

Proposition 2.4.6.

$$r\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant \frac{V(x,r)}{V_0(r)} ⨍_{B(x,r)} \Psi(t,\theta)\,d\mu_{\mathbf{g}}(t,\theta), \tag{2.4.18}$$

with equality for $r < \operatorname{inj} x$.

Remark 2.4.7. The expressions (2.4.12) and (2.4.18) appear to be an improvement on previously known formulae developed for volume comparison, in the sense that we have equality until we reach the cut-locus of the point x. They suggest that the quantity Ψ, and its average $⨍\Psi$, are, in fact, the most natural quantities to study in volume comparison.

REMARK 2.4.8. Since $0 \leqslant t \leqslant r$ in the integrals in (2.4.18), we have

$$\Psi(t,\theta) = t\psi(t,\theta) \leqslant t\psi_+(t,\theta) \leqslant r\psi_+(t,\theta).$$

By Hölder's inequality, we therefore deduce from (2.4.18) that

$$\begin{aligned}
\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) &\leqslant \frac{1}{V_0(r)}\int_{B(x,r)} \psi_+ \, d\mu_{\mathbf{g}} \\
&\leqslant \frac{1}{V_0(r)}\|\psi_+\|_{L^{2p}(B(x,r))}\|1\|_{L^{2p/(2p-1)}(B(x,r))} \\
&= \frac{V(x,r)^{1-\frac{1}{2p}}}{V_0(r)}\|\psi_+\|_{L^{2p}(B(x,r))}. \\
&= \left(\frac{V(x,r)}{V_0(r)}\right)^{1-\frac{1}{2p}}\left[V_0(r)^{-\frac{1}{2p}}\|\psi_+\|_{L^{2p}(B(x,r))}\right].
\end{aligned}$$

We thus recover a slightly sharpened version of (2.4.10), which contains a constant $C_1(n,0,r)$ equal to n, which is 1 in our estimate.

2.4.2. Curvature. We now want to relate the functions ψ and Ψ with curvature quantities.

The function $\psi(r,\theta) := h(r,\theta) - h_0(r,\theta)$ satisfies the relation

$$\partial_r\psi(r,\theta) + \frac{1}{n-1}\psi(r,\theta)^2 + \frac{2}{n-1}h_0(r)\psi(r,\theta) = -\left(\mathbf{Ric}(\partial_r,\partial_r) + |\sigma(r,\theta)|^2\right),$$

with $\psi(0) = 0$ (cf., for instance, [59]), where σ denotes the trace-free part of the second fundamental form of the sphere $S(x,r)$. In particular,

$$\partial_r\psi(r,\theta) + \frac{1}{n-1}\psi(r,\theta)^2 + \frac{2}{n-1}h_0(r)\psi(r,\theta) \leqslant -\mathbf{Ric}(\partial_r,\partial_r) =: -\rho_x(r,\theta), \tag{2.4.19}$$

where we denote by $\rho_x(r,\theta)$ the radial component of the Ricci tensor along the minimising geodesic from x to $\exp_x(r\theta)$.

In particular, ρ depends on both x and the point at which the Ricci tensor is evaluated. Since x is fixed, for the rest of this section we will drop the explicit x dependence.

The positive part of ψ may be estimated in terms of the negative part of ρ:

LEMMA 2.4.9.

$$\psi_+' + \frac{1}{n-1}\psi_+^2 + \frac{2}{n-1}h_0\psi_+ \leqslant \rho_-, \tag{2.4.20}$$

with $\psi_+(0) = 0$.

Proof. If $\psi \geqslant 0$, then $\psi_+ = \psi$, so (2.4.19) yields

$$\psi_+' + \frac{1}{n-1}\psi_+^2 + \frac{2}{n-1}h_0\psi_+ \leqslant -\rho \leqslant \rho_-.$$

If $\psi < 0$, then $\psi_+ = 0$, so

$$\psi_+' + \frac{1}{n-1}\psi_+^2 + \frac{2}{n-1}h_0\psi_+ = 0.$$

Since $\rho_- \geqslant 0$, (2.4.20) holds trivially. ☐

The function ψ_+ satisfies the following inequality.

Lemma 2.4.10. *Let $p \geqslant 1$. Let $\varphi\colon [0,\infty) \to \mathbb{R}$ be a non-negative, smooth test function with the property that $\varphi(t) = o(t^{-(n+2p-1)})$ as $t \to 0$. Then, for $0 \leqslant r < \ell(\theta)$,*

$$\begin{aligned}\int_0^r \rho_-(t)\psi_+^{2p-2}(t)\varphi(t)\omega(t)\,dt &\geqslant \frac{1}{2p-1}\psi_+(r)^{2p-1}\varphi(r)\omega(r) \\ &+ \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi_+^{2p}\varphi(t)\omega(t)\,dt + \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^r h_0\psi_+^{2p-1}(t)\varphi(t)\omega(t)\,dt \\ &- \frac{1}{2p-1}\int_0^r \psi_+^{2p-1}\varphi'(t)\omega(t)\,dt. \qquad (2.4.21)\end{aligned}$$

Proof. Multiplying (2.4.20) by $\psi_+^{2p-2}(t)\varphi(t)\omega(t)$ and integrating from $t = 0$ to $t = r$, we have

$$\begin{aligned}\int_0^r \rho_-(t)\psi_+^{2p-2}(t)\varphi(t)\omega(t)\,dt &\geqslant \int_0^r \left[\psi_+' + \frac{1}{n-1}\psi_+^2 + \frac{2}{n-1}h_0\psi_+\right]\psi_+^{2p-2}(t)\varphi(t)\omega(t)\,dt \\ &= \frac{1}{2p-1}\psi_+(t)^{2p-1}\omega(t)\varphi(t)\Big|_{t=0}^{t=r} + \frac{1}{n-1}\int_0^r \left[\psi_+^{2p} + 2h_0\psi_+^{2p-1}\right]\varphi(t)\omega(t)\,dt \\ &- \frac{1}{2p-1}\int_0^r \psi_+^{2p-1}\left[(\psi + h_0)\,\varphi(t) + \varphi'(t)\right]\omega(t)\,dt.\end{aligned}$$

The boundary term at $t = 0$ in the first term of the right-hand-side vanishes due to the asymptotics of φ. Using the fact that $\int_0^r \varphi\,\psi_+^{2p-1}\psi_-\,\omega\,dt \geqslant 0$ and collecting terms then gives (2.4.21). ☐

In [59], the authors take $\varphi(t) = 1$ with $p > n/2$, and use the fact that $h_0(t) \equiv \frac{n-1}{t} \geqslant 0$. Then they deduce from (2.4.21) that there exists a constant $C = C(n,p)$ with the property that

$$\|\psi_+\|^2_{L^{2p}(B(x,r))} \leqslant C\|\rho_-\|_{L^p(B(x,r))}. \qquad (2.4.22)$$

In [5, Lemma 3.1], an alternative approach was adopted and the author showed that, for $p > \frac{n}{2}$, there exists an explicit constant $C(p,n)$ such that for all $r > 0$, one has

$$\psi_+(r,\theta)\,\omega(r,\theta) \leqslant C(n,p)\int_0^r \rho_-(t,\theta)^p\omega(t,\theta)\,dt.$$

Motivated by the expression (2.4.18) and natural scaling properties, we define the scaled curvature quantity

$$R_-(r,\theta) := r\rho_-(r,\theta).$$

Again, note that R depends both on the point x and on (r,θ), but we have suppressed the x dependence for the moment.

Therefore we can prove:

THEOREM 2.4.11. *Let $\widehat{S} \subseteq S_x$, $r > 0$ and $p \geqslant \frac{n}{2}$. Then,*

$$\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant (2p-1)\frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|}\|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))}. \tag{2.4.23}$$

Moreover, if $r < \ell(\theta)$ for all $\theta \in \widehat{S}$, then we have the slightly sharper estimate

$$\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) \leqslant (2p-1)\frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|}\fint_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\,dt, \tag{2.4.24}$$

where $\fint_0^r f(t)\,dt := \frac{1}{r}\int_0^r f(t)\,dt$.

Proof. Take $\varphi(t) = t^{2p-1}$ in (2.4.21). For all fixed $\theta \in \widehat{S}$, we have

$$\begin{aligned}\int_0^r R_-(t,\theta)\Psi_+(t,\theta)^{2p-2}\omega(t,\theta)\,dt \geqslant{}& \frac{1}{2p-1}\Psi_+(r,\theta)^{2p-1}\omega(r,\theta)\\ &+\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r \frac{1}{t}\Psi_+(t,\theta)^{2p}\omega(t,\theta)\,dt\\ &+\left(1-\frac{n-1}{2p-1}\right)\int_0^r \frac{1}{t}\Psi_+(t,\theta)^{2p-1}\omega(t,\theta)\,dt.\end{aligned} \tag{2.4.25}$$

Since $\frac{1}{n-1} - \frac{1}{2p-1} \geqslant 0$, we deduce that[1]

$$\Psi_+(r,\theta)^{2p-1}\omega(r,\theta) \leqslant (2p-1)\int_0^r R_-(t,\theta)\Psi_+(t,\theta)^{2p-2}\omega(t,\theta)\,dt.$$

Integrating over $\Gamma(\widehat{S},r)$, we have

$$\begin{aligned}\int_{\Gamma(\widehat{S},r)} \Psi_+(t,\theta)^{2p-1}\,d\mu_{\mathbf{g}}(t,\theta) &= \int_{\theta\in\widehat{S}}\int_0^{\min(r,\ell(\theta))} \Psi_+(t,\theta)^{2p-1}\,\omega(t,\theta)\,dt\,d\theta\\ &\leqslant (2p-1)\int_{\theta\in\widehat{S}}\int_0^{\min(r,\ell(\theta))}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta.\end{aligned} \tag{2.4.26}$$

[1]Note that, in the case $p = \frac{n}{2}$, the last two terms in (2.4.25) vanish. As such, our estimates seem optimal in this case.

We split the integral into an angular integral over $\widehat{S}_r$ and $\widehat{S}\backslash\widehat{S}_r$. Firstly,

$$\begin{aligned}
&\int_{\theta\in\widehat{S}_r}\int_0^{\min(r,\ell(\theta))}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \\
&\qquad = \int_{\theta\in\widehat{S}_r}\int_0^{r}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \\
&\qquad = \int_0^{r}\int_{\theta\in\widehat{S}_r}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)d\theta\,dt \\
&\qquad = \int_0^{r}\left(\int_{\Gamma(\widehat{S}_r,t)} R_-\Psi_+^{2p-2}\,d\mu_{\mathbf{g}}\right)dt. \qquad (2.4.27)
\end{aligned}$$

In particular, we have

$$\int_{\theta\in\widehat{S}_r}\int_0^{\min(r,\ell(\theta))}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \leqslant r\left(\int_{\Gamma(\widehat{S}_r,r)} R_-\Psi_+^{2p-2}\,d\mu_{\mathbf{g}}\right). \qquad (2.4.28)$$

Secondly,

$$\begin{aligned}
&\int_{\theta\in\widehat{S}\backslash\widehat{S}_r}\int_0^{\min(r,\ell(\theta))}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \\
&\qquad = \int_{\theta\in\widehat{S}\backslash\widehat{S}_r}\int_0^{\ell(\theta)}\left(\int_0^t R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \\
&\qquad \leqslant \int_{\theta\in\widehat{S}\backslash\widehat{S}_r}\int_0^{\ell(\theta)}\left(\int_0^{\ell(\theta)} R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)dt\,d\theta \\
&\qquad = \int_{\theta\in\widehat{S}\backslash\widehat{S}_r}\ell(\theta)\left(\int_0^{\ell(\theta)} R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)d\theta \\
&\qquad \leqslant r\int_{\theta\in\widehat{S}\backslash\widehat{S}_r}\left(\int_0^{\ell(\theta)} R_-(s,\theta)\Psi_+(s,\theta)^{2p-2}\omega(s,\theta)\,ds\right)d\theta \\
&\qquad = r\int_{\Gamma(\widehat{S}\backslash\widehat{S}_r,r)} R_-\Psi_+^{2p-2}\,d\mu_{\mathbf{g}}. \qquad (2.4.29)
\end{aligned}$$

It follows that if $r < \ell(\theta)$ for all $\theta \in \widehat{S}$ (i.e. $\widehat{S}_r = \widehat{S}$), then we may use equation (2.4.27) to deduce that

$$\begin{aligned}\int_{\Gamma(\widehat{S},r)} \Psi_+(t,\theta)^{2p-1}\, d\mu_{\mathbf{g}}(t,\theta) &\leqslant (2p-1)\int_0^r \left(\int_{\Gamma(\widehat{S},t)} R_- \Psi_+^{2p-2}\, d\mu_{\mathbf{g}}\right) dt \\ &\leqslant \int_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))} \|\Psi_+\|^{2p-2}_{L^{2p-1}(\Gamma(\widehat{S},t))}\, dt \qquad (2.4.30)\\ &\leqslant (2p-1)\left(\int_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\, dt\right) \|\Psi_+\|^{2p-2}_{L^{2p-1}(\Gamma(\widehat{S},t))}. \qquad (2.4.31)\end{aligned}$$

Hence,

$$\|\Psi_+\|_{L^{2p-1}(\Gamma(\widehat{S},r))} \leqslant (2p-1)\left(\int_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\, dt\right).$$

From equation (2.4.18), we therefore have

$$\begin{aligned}r\frac{d}{dr}\left(\frac{|\Gamma(\widehat{S},r)|}{|\Gamma_0(\widehat{S},r)|}\right) &= \frac{1}{|\Gamma_0(\widehat{S},r)|}\int_{\Gamma(\widehat{S},r)} \Psi\, d\mu_{\mathbf{g}} \leqslant \frac{1}{|\Gamma_0(\widehat{S},r)|}\int_{\Gamma(\widehat{S},r)} \Psi_+\, d\mu_{\mathbf{g}} \\ &\leqslant \frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|} \|\Psi_+\|_{L^{2p-1}(\Gamma(\widehat{S},r))} \\ &\leqslant (2p-1)\frac{|\Gamma(\widehat{S},r)|^{1-\frac{1}{2p-1}}}{|\Gamma_0(\widehat{S},r)|}\int_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t)}\, dt,\end{aligned}$$

which is (2.4.24).

In the case where there exists $\theta \in \widehat{S}$ such that $\ell(\theta) \leqslant r$, then (2.4.28) and (2.4.29) imply that

$$\begin{aligned}\int_{\Gamma(\widehat{S},r)} \Psi_+(t,\theta)^{2p-1}\, d\mu_{\mathbf{g}}(t,\theta) &\leqslant (2p-1)\, r\left(\int_{\Gamma(\widehat{S},r)} R_- \Psi_+^{2p-2}\, d\mu_{\mathbf{g}}\right) \\ &\leqslant (2p-1)\, r\, \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))} \|\Psi_+\|^{2p-2}_{L^{2p-1}(\Gamma(\widehat{S},r))}\end{aligned}$$

Therefore,

$$\|\Psi_+\|_{L^{2p-1}(\Gamma(\widehat{S},r))} \leqslant (2p-1)\, r\, \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))}.$$

A similar calculation to that above then leads to (2.4.23). □

In the special case of geodesic balls, where we take $\widehat{S} = S_x$, we get:

Theorem 2.4.12. *For $0 \leqslant r_0 \leqslant r$ then, for any $p \geqslant \frac{n}{2}$, we have*

$$\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant (2p-1)\frac{V(x,r)^{1-\frac{1}{2p-1}}}{V_0(r)} \int_0^r \|R_-\|_{L^{2p-1}(B(x,t))}\, dt. \qquad (2.4.32)$$

Moreover, for $r < \operatorname{inj} x$, we have

$$\frac{d}{dr}\left(\frac{V(x,r)}{V_0(r)}\right) \leqslant (2p-1)\frac{V(x,r)^{1-\frac{1}{2p-1}}}{V_0(r)} \int_0^r \|R_-\|_{L^{2p-1}(B(x,t))}\,dt.$$

2.4.3. Comparison and monotonicity results. Petersen and Wei prove the following relative volume comparison theorem.

THEOREM 2.4.13. [59, Theorem 1.1] *Let $x \in M$, $K \leqslant 0$, $p > n/2$, then there exists a constant $c = c(n,p,K,R)$, nondecreasing in R such that*

$$\left(\frac{V(x,R)}{V_K(R)}\right)^{\frac{1}{2p}} - \left(\frac{V(x,r)}{V_K(r)}\right)^{\frac{1}{2p}} \leqslant c(k(K,p))^{\frac{1}{2p}}. \tag{2.4.33}$$

At $r = 0$ it holds that

$$V(x,R) \leqslant (1 + c(k(K,p))^{\frac{1}{2p}})^{2p} V_K(R).$$

The case $k(p,K) = 0$ is the classical relative volume comparison.

Here the case $p = n/2$ is not covered. In [29], an analogous result is shown also in this case.

Petersen and Wei obtain equation (2.4.33) by observing that equation (2.4.10) is a differential inequality of the form

$$\begin{aligned} y' &\leqslant \alpha y^{1-\frac{1}{2p}} f(x),\\ y(0) = 1, &\qquad y > 0, \end{aligned}$$

where $\alpha > 0$ is a constant. Thus after integration and separation of variables they obtain

$$2py^{\frac{1}{2p}}(R) - 2py^{\frac{1}{2p}}(r) \leqslant \alpha \int_r^R f(x)dx.$$

Then by choosing c properly, the result follows.

REMARK 2.4.14. The proof of equation (2.4.22) does not work if $p \leqslant n/2$. It is also necessary to have $p > n/2$ to get convergence of the integral $\int_0^R (V_0(t))^{-1/2p}dt$ that comes up in the proof.

COROLLARY 2.4.15. [59, Cor. 2.4] *Let $(M, \mathbf{g})$ be a complete $n-$dimensional Riemannian manifold with $\operatorname{diam}(M) \leqslant D$ $(D > 0)$, $x \in M$, $r < D$. For $p > n/2$, $K \leqslant 0$, it is possible to find for all $\alpha < 1$ an $\varepsilon = \varepsilon(n,p,K,D,\alpha) > 0$, such that if $\bar{k}(p,K) \leqslant \varepsilon$, then*

$$\alpha\frac{V_K(r)}{V_K(D)} \leqslant \frac{V(x,r)}{\operatorname{Vol} M}. \tag{2.4.34}$$

We will now show volume comparison and monotonicity results which follow from Theorem 2.4.11.

THEOREM 2.4.16. *Let $\widehat{S} \subseteq S_x$, $r > 0$ and $p \geqslant \frac{n}{2}$. Then, for $0 \leqslant r_0 \leqslant r$, we have*

$$\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right)^{\frac{1}{2p-1}} \leqslant \left(\frac{\Gamma(\widehat{S},r_0)}{\Gamma_0(\widehat{S},r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r}\left(\fint_{\Gamma(\widehat{S},t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{\frac{1}{2p-1}} dt\right] \tag{2.4.35}$$

Moreover, if $r < \ell(\theta)$ for all $\theta \in \widehat{S}$, then we have

$$\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right)^{\frac{1}{2p-1}} \leqslant \left(\frac{\Gamma(\widehat{S},r_0)}{\Gamma_0(\widehat{S},r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r}\frac{1}{\Gamma(\widehat{S},s)^{\frac{1}{2p-1}}}\left\{\fint_0^s \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\, dt\right\} ds\right] \tag{2.4.36}$$

$$\leqslant \left(\frac{\Gamma(\widehat{S},r_0)}{\Gamma_0(\widehat{S},r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r}\left\{\fint_0^s\left(\fint_{\Gamma(\widehat{S},t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right\} ds\right] \tag{2.4.37}$$

Proof. From equation (2.4.23) in Theorem 2.4.11, we have

$$\frac{d}{dr}\log\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right) \leqslant (2p-1)\left(\fint_{\Gamma(\widehat{S},r)} R_-^{2p-1} d\mu_{\mathbf{g}}\right)^{\frac{1}{2p-1}}. \tag{2.4.38}$$

Integration then yields (2.4.35).

If $r < \ell(\theta)$ for all $\theta \in \widehat{S}$, then we may use (2.4.24) to deduce that

$$\frac{d}{dr}\log\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right) \leqslant (2p-1)\frac{1}{|\Gamma(\widehat{S},r)|^{\frac{1}{2p-1}}}\fint_0^r \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))} dt.$$

Integration then gives (2.4.36). Since $\Gamma(\widehat{S},t) \leqslant \Gamma(\widehat{S},s)$ for $0 \leqslant t \leqslant s$, (2.4.37) follows from (2.4.36). ☐

The following monotonicity results follow immediately from Theorem 2.4.16.

THEOREM 2.4.17. *The quantity*

$$\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)} \exp\left[-(2p-1)\int_0^r\left(\fint_{\Gamma(\widehat{S},t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right] \tag{2.4.39}$$

is a non-increasing function of r, which converges to 1 as $r \to 0$.

Similarly, the quantities

$$\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)} \exp\left[-(2p-1)\int_0^r\frac{1}{\Gamma(\widehat{S},s)^{\frac{1}{2p-1}}}\left\{\fint_0^s \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},t))}\, dt\right\} ds\right], \tag{2.4.40}$$

$$\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)} \exp\left[-(2p-1)\int_0^r\left\{\fint_0^s\left(\fint_{\Gamma(\widehat{S},t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right\} ds\right] \tag{2.4.41}$$

are non-increasing functions of r, as long as $r < \ell(\theta)$ for all $\theta \in \widehat{S}$, and converge to 1 as $r \to 0$.

Integrating Theorem 2.4.11 in a different way gives the following result.

THEOREM 2.4.18. *If* $0 \leqslant r_0 \leqslant r$*, we have*

$$\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right)^{\frac{1}{2p-1}} - \left(\frac{\Gamma(\widehat{S},r_0)}{\Gamma_0(\widehat{S},r_0)}\right)^{\frac{1}{2p-1}} \leqslant \left(\frac{n}{|\widehat{S}|}\right)^{\frac{1}{2p-1}} \int_{r_0}^{r} s^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,s))}\, ds. \quad (2.4.42)$$

Proof. From equation (2.4.23), we have

$$\frac{d}{dr}\left(\frac{\Gamma(\widehat{S},r)}{\Gamma_0(\widehat{S},r)}\right)^{\frac{1}{2p-1}} \leqslant \frac{1}{\Gamma_0(\widehat{S},r)^{\frac{1}{2p-1}}}\|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))} = \left(\frac{1}{|\widehat{S}|\frac{1}{n}r^n}\right)^{\frac{1}{2p-1}} \|R_-\|_{L^{2p-1}(\Gamma(\widehat{S},r))}.$$

Integration yields (2.4.42). □

In the special case of geodesic balls, where we take $\widehat{S} = S_x$, we get the following results:

THEOREM 2.4.19. *For* $0 \leqslant r_0 \leqslant r$ *then, for any* $p \geqslant \frac{n}{2}$*, we have the relative volume inequalities*

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{2p-1}} \leqslant \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r} \frac{1}{V(x,s)^{\frac{1}{2p-1}}}\left\{\fint_0^s \|R_-\|_{L^{2p-1}(B(x,t))}\, dt\right\} ds\right] \quad (2.4.43)$$

$$\leqslant \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r}\left\{\fint_0^s \left(\fint_{B(x,t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right\} ds\right] \quad (2.4.44)$$

and

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{2p-1}} \leqslant \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{2p-1}} \exp\left[\int_{r_0}^{r}\left(\fint_{B(x,t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right]. \quad (2.4.45)$$

In case $p = \frac{n}{2}$, we obtain

THEOREM 2.4.20. *For* $0 \leqslant r_0 \leqslant r$*, we have*

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{n-1}} \leqslant \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{n-1}} \exp\left[\int_{r_0}^{r}\left\{\fint_0^s \left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}}\right)^{1/(n-1)} dt\right\} ds\right] \quad (2.4.46)$$

and

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{n-1}} \leqslant \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{n-1}} \exp\left[\int_{r_0}^{r}\left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}}\right)^{1/(n-1)} dt\right]. \quad (2.4.47)$$

Integrating gives the following special cases of Theorems 2.4.22 and 2.4.18.[2]

[2] We only state the versions that hold globally.

THEOREM 2.4.21. *For any $p \geqslant \frac{n}{2}$, the quantity*

$$\frac{V(x,r)}{V_0(r)} \exp\left[-(2p-1)\int_0^r \left(\fint_{B(x,t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right] \tag{2.4.48}$$

is a non-increasing function of r, which converges to 1 as $r \to 0$. Moreover, for $0 \leqslant r_0 \leqslant r$, we have

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{2p-1}} - \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{2p-1}} \leqslant \frac{1}{V_0(1)^{\frac{1}{2p-1}}} \int_{r_0}^r s^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,s))}\, ds, \tag{2.4.49}$$

where $V_0(1)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

The following monotonicity results follow immediately from Theorem 2.4.16.

THEOREM 2.4.22. *Let $p \geqslant \frac{n}{2}$. The quantities*

$$\frac{V(x,r)}{V_0(r)} \exp\left[-(2p-1)\int_0^r \left\{\fint_0^s \left(\fint_{B(x,t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right\} ds\right] \tag{2.4.50}$$

and

$$\frac{V(x,r)}{V_0(r)} \exp\left[-(2p-1)\int_0^r \left(\fint_{B(x,t)} R_-^{2p-1}\, d\mu_{\mathbf{g}}\right)^{1/(2p-1)} dt\right] \tag{2.4.51}$$

are non-increasing functions of r, both of which converge to 1 as $r \to 0$.

The special case $p = \frac{n}{2}$ gives

COROLLARY 2.4.23.

$$\frac{V(x,r)}{V_0(r)} \exp\left[-(n-1)\int_0^r \left\{\fint_0^s \left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}}\right)^{1/(n-1)} dt\right\} ds\right] \tag{2.4.52}$$

and

$$\frac{V(x,r)}{V_0(r)} \exp\left[-(n-1)\int_0^r \left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}}\right)^{1/(n-1)} dt\right] \tag{2.4.53}$$

are non-increasing functions of r, both of which converge to 1 as $r \to 0$.

THEOREM 2.4.24. *For $0 \leqslant r_0 \leqslant r$, we have*

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{n-1}} - \left(\frac{V(x,r_0)}{V_0(r_0)}\right)^{\frac{1}{n-1}} \leqslant C_1 \int_{r_0}^r \frac{1}{s^{\frac{n}{n-1}}} \left\{\fint_0^s \|R_-\|_{L^{n-1}(B(x,t))}\, dt\right\} ds \tag{2.4.54}$$

$$\leqslant C_1 \int_{r_0}^r \frac{1}{s^{\frac{n}{n-1}}} \|R_-\|_{L^{n-1}(B(x,s))}\, ds, \tag{2.4.55}$$

where we have defined the constant

$$C_1 = C_1(n) := \frac{1}{V_0(1)^{\frac{1}{n-1}}}.$$

REMARK 2.4.25. If the Ricci tensor of $\mathbf{g}$ is continuous, then

$$\int_0^r s^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,s))}\, ds = O(r^2) \text{ as } r \to 0,$$

for any $p \geqslant \frac{n}{2}$. As such, equation (2.4.49) implies that the (well-defined) quantity

$$\left(\frac{V(x,r)}{V_0(r)}\right)^{\frac{1}{2p-1}} - \frac{1}{V_0(1)^{\frac{1}{2p-1}}} \int_0^r s^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,s))}\, ds$$

is a non-increasing function of r, for any $p \geqslant \frac{n}{2}$. More generally, if $\rho_-(y) = \lambda(x) d(x,y)^\alpha$ as $y \to x$, then

$$r^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,r))}\, ds = O(r^{1+\alpha}) \text{ as } r \to 0,$$

independently of p. In order for this quantity to be integrable, we therefore require $\alpha > -2$. As such, validity of our monotonicity theorems rules out the (standard) critical case where the Ricci tensor has a $\frac{1}{r^2}$ singularity at the point x.

2.4.4. Morrey spaces. The expression on the right-hand-side of (2.4.55) is rather reminiscent of an L^1 version of a Morrey space norm. Recall that the (homogeneous) Morrey space of $\mathbb{R}^n$, $\mathcal{M}_q^p(\mathbb{R}^n)$ with $1 \leqslant q \leqslant p$ consists of those functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n; r > 0} \left[r^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\int_{B(x,r)} |f|^q \, dx \right)^{1/q} \right]$$

is finite (see, for instance, [63, pp. 301]). In our case, let $q \geqslant 1$ and $F\colon M \times M \to \mathbb{R}$ have the property that $F(x,\cdot) \in L^q(M)$ for all $x \in M$. We define the (semi)norm

$$\|F\|_{p,q} := \sup_{x \in M} \left(\int_0^D s^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \|F(x,\cdot)\|_{L^q(B(x,s))}\, ds \right).$$

The quantity that appears inside the integral in the inequality (2.4.49), i.e.

$$s^{-\frac{n}{2p-1}} \|R_-\|_{L^{2p-1}(B(x,s))},$$

would be consistent with an L^1 version of the M^∞_{2p-1} space, rather than an L^∞ version, i.e.

$$\|f\|_{M^\infty_{2p-1,1}} := \sup_{x \in M} \int_0^D r^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\int_{B(x,r) \cap \Omega} |f|^q \, dx \right)^{1/q} dr.$$

PROPOSITION 2.4.26. *Let $R_- \in M^p_{n-1}(M)$ for some $p > n$. Then*

$$\int_0^D \frac{1}{s^{\frac{n}{n-1}}} \|R_-\|_{L^{n-1}(B(x,s))}\, ds$$

is finite, for all $x \in M$.

Proof. $R_- \in M^p_{n-1}(M)$, so $\frac{1}{s^{\frac{n}{n-1}}} \|R_-\|_{L^{n-1}(B(x,s))} \leqslant C s^{-n/p}$ for $0 < s \leqslant D$, so the integral is finite. □

CHAPTER 3

Convergence of Riemannian manifolds

3.1. From volume estimates to convergence

Let $(M_i, \mathbf{g}_i)$ be a sequence of manifolds that satisfy curvature bounds that lead to a Bishop-Gromov type volume comparison result, so, e.g. a pointwise or an L^p-bound on the Ricci curvature, see Chapter 2. We will now describe how such a volume bound leads to convergence in the Gromov-Hausdorff topology.

We define the following quantities:

DEFINITION 3.1.1. Let M be a compact metric space. Then the capacity of M, $\mathrm{Cap}_M(\varepsilon)$ denotes the maximal number of disjoint $\varepsilon/2$-balls in M. The covering of M, $\mathrm{Cov}_M(\varepsilon)$ is defined to be the minimal number of ε-balls it takes to cover M.

Let d_{GH} denote the Gromov-Hausdorff distance, see Definition A.6.5, and let $\mathcal{M}$ denote the set of Riemannian manifolds . In order to show that a collection of Riemannian manifolds with certain control of curvature is precompact in the Gromov-Hausdorff topology, we use the following result due to Gromov (see [58, Proposition 44]):

PROPOSITION 3.1.2. *For a set $\mathcal{C} \subseteq (\mathcal{M}, d_{\mathrm{GH}})$ the following statements are equivalent:*

(1) $\mathcal{C}$ is precompact, i.e. every sequence in $\mathcal{C}$ has a converging subsequence,
(2) there is a function $N_1(\varepsilon) : (0, \alpha) \to (0, \infty)$ such that $\mathrm{Cap}_X(\varepsilon) \leqslant N_1(\varepsilon)$ for $X \in \mathcal{C}$,
(3) there is a function $N_2(\varepsilon) : (0, \alpha) \to (0, \infty)$ such that $\mathrm{Cov}_X(\varepsilon) \leqslant N_2(\varepsilon)$ for $X \in \mathcal{C}$.

3.1.1. Examples. We will now provide some examples of how Proposition 3.1.2 is used for the proof of precompactness results. The main idea in all the examples is to use curvature bounds to estimate the size of balls and the number of balls it takes to cover the manifold.

Pointwise bound on Ric

THEOREM 3.1.3. *The collection of closed $n-$dimensional Riemannian manifolds with* $\mathbf{Ric} \geqslant (n-1)K$ *and* $\mathrm{diam} \leqslant D$ *for* $D > 0$ *is precompact in the Gromov-Hausdorff topology.*

Proof. The aim is to show that M does not contain too many disjoint balls, which will then allow us to apply Proposition 3.1.2. Indeed, let l be the maximal number of disjoint ε balls in M, with centers $\{x_1, \dots x_l\}$. Let $B(x_i, \varepsilon)$ be the ball of least volume. Then, by the Bishop-Gromov result, we get for $r = D$,

$$l \leqslant \frac{\mathrm{Vol} M}{V(x_i, \varepsilon)} \leqslant \frac{V(x_i, D)}{V(x_i, \varepsilon)} \leqslant \frac{V_K(D)}{V_K(\varepsilon)},$$

where $V_K(r)$ denotes, as in 2, the volume of the ball of radius r with respect to the constant curvature metric $\mathbf{g}_K$. The last quantity is bounded, thus, from Proposition 3.1.2 we get the precompactness property. □

L^p $(p > \frac{n}{2})$ bound on the negative part of Ricci

Both Yang [69] and Petersen-Wei [59] study manifolds with L^p bounds on the Ricci curvature for $p > \frac{n}{2}$. We will outline both approaches.

(1) **Yang's approach:**
In [69, 70] the author investigates manifolds with the following bounds:
Let $n \geqslant 3$, $p > n/2$, $0 < \eta < 1$, $D, \rho > 0$, positive constants $\varepsilon(n), \kappa(n, p, \eta)$, $\Omega \subseteq M$,

$$\begin{cases} V(x, r) \geqslant \eta^n n^{-1} \omega r^n & \text{for all } x \in \Omega \subset M, \\ \mathrm{diam}(\Omega) < D, \\ \|\mathbf{R}\|_{L^{n/2}(B(x,r))} \leqslant \varepsilon(n) \eta^{2(n+1)} & \text{for all } x \in \Omega, \\ r^{2-n/p} \|\mathbf{Ric}\|_{L^p(\Omega)} \leqslant \kappa(n, p, \eta). \end{cases} \tag{3.1.1}$$

Yang shows in [69, Theorem 2.1] that sequences of complete Riemannian manifolds that satisfy the bounds (3.1.1) and additionally have $\mathrm{Vol}(\Omega) > v$ for some $v > 0$, contain a subsequence that converges in Gromov-Hausdorff distance to an open C^1 manifold with a continuous Riemannian metric.

He shows in [69, Section 7] that an L^p-bound on the negative part of the Ricci curvature leads to an upper bound on the volume of a geodesic cone, see also [24].

Yang then uses an isoperimetric inequality due to Croke [20]:

Theorem 3.1.4. [20, Theorem 11] *Let M be a Riemannian manifold, $\Omega \subseteq M$ then*

$$\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)^{(n-1)/n}} \geqslant C(n) c_1(\mathbb{R}^n) \left(\frac{\widetilde{\omega}}{\omega_{n-1}}\right)^{1+1/n}, \tag{3.1.2}$$

with c_1 the local isoperimeric constant, $C(n) > 0$. The quantity $\widetilde{\omega}$ is defined as follows: $\widetilde{\omega} = |\widetilde{S}_x|$, where $\widetilde{S}_x \subset S_x$, the space of unit tangent vectors at x, where x is chosen such that $\widetilde{S}_x$ has minimal volume.

Using this result, he gets a volume bound that leads to the following local isoperimetric inequality:

THEOREM 3.1.5. [69, Theorem 7.4] *Let $p > n/2$, $\tau, R > 0$, $x_0 \in M$. Define*

$$\eta = \left(\frac{V(x_0, R)}{n^{-1}\omega_{n-1}R^n}\right)^{-1/n},$$

and

$$r = \frac{\eta}{1+\tau}R.$$

If

$$C(n,p)R^{2p-n}\int_{B(x_0,R+2r)} \mathbf{Ric}\_{}^{p} d\mu_{\mathbf{g}} < \omega_{n-1} \min\left(\tau^{2p-1}\eta^{n-2p}, \frac{2p(n-1)}{n(2p-1)}\frac{\tau\eta^n}{(1+\tau+\eta)^{2p}}\right),$$

then

$$c_1(B(x_0,r)) \geqslant c(n)\left(\frac{\tau\eta}{(1+\tau+\eta)}\right)^{n+1} c_1(\mathbb{R}^n).$$

From there it follows that a sequence of compact Riemannian manifolds that satisfy (3.1.1) has a subsequence that converges to a compact metric space in Hausdorff distance, [69, Cor. 7.7]. Here the number of balls it takes to cover M is estimated by the previously mentioned volume estimate of cones. Note that for this result the bound on $\mathbf{R}$ is not necessary.

In order to prove that the limiting metric space is actually a Riemannian manifold with a continuous metric, Yang uses the so-called local Ricci flow. For a smooth metric $\mathbf{g}_0$ it is defined via the evolution equation

$$\frac{\partial \mathbf{g}}{\partial t} = -2\chi^2 \mathbf{Ric}(\mathbf{g}(t)), \qquad \mathbf{g}(0) = \mathbf{g}_0. \tag{3.1.3}$$

Here χ is a nonnegative compactly supported smooth function. The existence of $T > 0$ such that (3.1.3) has smooth solutions $\mathbf{g}(t)$ for $0 \leqslant t \leqslant T$ under appropriate integral bounds on the Riemann- and Ricci curvature, is shown in [69, Theorem 8.2].

Furthermore the metrics $\mathbf{g}(t)$ regularize the initial metric $\mathbf{g}$. Yang investigates three situations: $\mathbf{R} \in L^{n/2}$, $\mathbf{R} \in L^p$ for $\frac{n}{2} < p < \infty$, and $\mathbf{R} \in L^\infty$.

In the first case, an additional L^p bound for $\mathbf{Ric}$ is needed. Then it is possible to obtain, beside the uniform bound for the time of existence of solutions to the local Ricci flow, also an L^∞ bound on $\mathbf{R}$.

In the second case, an additional isoperimetric inequality is needed to get a uniform time estimate. In the last case, the time estimate is obtained without any additional assumptions.

After having obtained the pointwise bound on the curvature, and therefore a sectional curvature bound, Yang uses the classical Cheeger-Gromov compactness result to obtain convergence. For this result a bound on the injectivity radius is necessary, which Yang obtains by [17, Theorem 4.7].

(2) **Petersen-Wei's approach:**

In [59, Theorem 1.1.], the authors obtain the volume bound

$$\left(\frac{V(x,R)}{V_K(R)}\right)^{\frac{1}{2p}} - \left(\frac{V(x,r)}{V_K(r)}\right)^{\frac{1}{2p}} \leqslant c(k(K,p))^{\frac{1}{2p}}, \tag{3.1.4}$$

for closed n-dimensional Riemannian manifolds with an upper bound on the diameter and a small L^p-norm of $\mathbf{Ric}$.

Hence we can also estimate the capacity of the manifold, and thus an analogous argument to that given in the case of pointwise bounded $\mathbf{Ric}_-$ outlined above, leads to the result.

$M^p_{n-1}(M)$ bound on the negative part of Ric

We use the results of Chapter 2 to get the following result:

PROPOSITION 3.1.6. *Let $D > 0$ be constant. Then for all $\alpha < 1$ we can find $K = K(n, D, \alpha)$ $\left(= -\frac{\log(\alpha)}{(n-1)D}\right)$ such that any closed Riemannian manifold M with*[1]

$$\operatorname{diam} M \leqslant D, \qquad \sup_{x\in M} \sup_{t\geqslant 0} \left(\fint_{B(x,t)} R_-(x,y)^{n-1}\, d\mu_{\mathbf{g}}(y) \right)^{1/(n-1)} \leqslant K.$$

satisfies, for all $x \in M$ and $0 < t \leqslant r \leqslant D$,

$$\alpha \frac{V(x,r)}{V(x,t)} \leqslant \frac{V_0(r)}{V_0(t)}, \tag{3.1.5}$$

respectively,

$$\frac{V(x,r)}{V(x,t)} \leqslant \frac{V_0(r)}{V_0(t)} e^{(n-1)KD}. \tag{3.1.6}$$

Therefore we get

PROPOSITION 3.1.7. *The class of Riemannian manifolds with*

$$\operatorname{diam}_M \leqslant D$$

$$\left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}} \right)^{1/(n-1)} \leqslant \varepsilon$$

is precompact in the Gromov-Hausdorff topology.

[1] Note that we have now reinstated the dependence of R_- on the point x.

Proof. Let $\varepsilon > 0$, and l be the maximal number of disjoint $\frac{\varepsilon}{2}$ balls in M, with centers $\{x_1, \dots, x_l\}$. Let $B(x_i, \frac{\varepsilon}{2})$ be the ball of volume. Then, by (3.1.6), we have

$$l \leqslant \frac{\operatorname{Vol} M}{V\left(x_i, \frac{\varepsilon}{2}\right)} = \frac{V\left(x_i, D\right)}{V\left(x_i, \frac{\varepsilon}{2}\right)} \leqslant \frac{V_0(D)}{V_0\left(\frac{\varepsilon}{2}\right)} e^{(n-1)KD} = \left(\frac{2D}{\varepsilon}\right)^n e^{(n-1)KD} =: N_1(\varepsilon).$$

Applying Proposition 3.1.2 leads to the result. □

3.2. Convergence of Riemannian manifolds in Hölder- and Sobolev spaces

A different notion of convergence of Riemannian manifolds, which has a stronger emphasis on the analytic properties, is the following:

Definition 3.2.1. Let $(M_i, \mathbf{g}_i)$ be a sequence of smooth n-dimensional compact Riemannian manifolds. We say that $M_i \to M$ in $C^{k,\alpha}$ (in $W^{k,p}$) if M is a compact Riemannian manifold, $\mathbf{g}$ a $C^{k,\alpha}$ ($W^{k,p}$) metric, and furthermore if there are diffeomorphisms $f_i : M \to M_i$, for sufficiently large i such that the pulled back metrics $f_i^* \mathbf{g}_i$ converge to $\mathbf{g}$ in $C^{k,\alpha}$ (in $W^{k,p}$) on M. See for example [**2, 3, 37**] etc.

As outlined in [**37**, Remark 1], the appropriate framework for convergence in $C^{k,\alpha}$ is the set of $C^{k+1,\alpha}$ manifolds with $C^{k,\alpha}$ metrics.

Remark 3.2.2. For M a Riemannian manifold, we will denote $f \in C^{k,\alpha}(M)$ briefly by $f \in C^{k,\alpha}$.

In order to obtain (pre)compactness in the $C^{k,\alpha}$ ($W^{k,p}$) topology of manifolds that satisfy certain curvature bounds, an important tool is the use of harmonic coordinates. Since the Ricci tensor becomes a simpler expression in these coordinates, they are particularly useful in the study of manifolds that satisfy bounds on the Ricci curvature. By applying elliptic estimates it is possible to deduce convergence results in the space of $C^{k,\alpha}/W^{k,p}$ manifolds.

3.2.1. Harmonic coordinates. Harmonic coordinates are frequently used in Riemannian geometry. As we will see in Theorem 3.2.7, harmonic coordinates provide optimal regularity. In applications, their important feature of a simple coordinate expression for the Ricci tensor is utilized.

Here we are particularly interested in applying harmonic coordinates for the study of convergence properties of Riemannian manifolds. They are used in proving (pre-)compactness results on the spaces of Riemannian manifolds that satisfy a certain curvature bound and bounds on the injectivity radius, volume of geodesic balls etc.

The notion of harmonic coordinates goes back to DeTurk and Kazdan, [**21**]. They show that these coordinates are optimal as far as regularity issues are concerned. Furthermore it is possible to obtain a-priori bounds for harmonic coordinates as was shown by Jost and Karcher, [**40**] under the additional assumption of a sectional curvature bound. These bounds will be needed in estimates in the proofs below. We will briefly review their results.

DEFINITION 3.2.3. A coordinate chart $(x^1, \dots, x^n)$ on a Riemannian manifold $(M, \mathbf{g})$ is called *harmonic*, if $\Delta_{\mathbf{g}} x^i = 0$, $i = 1, \dots, n$.

It is possible by [21, Lemma 1.1], to relate these coordinates by a straightforward calculation to the Christoffel symbols.

LEMMA 3.2.4. *Let $(x^1, \dots, x^n)$ be a local coordinate chart, and define in this chart $\Gamma^k = \mathbf{g}^{ij}\Gamma^k_{ij}$. This chart is harmonic if and only if $\Delta_{\mathbf{g}} x^l = -\Gamma^l = 0$, $l = 1, \dots, n$.*

As was shown in [21], one can always find harmonic coordinates in a neighborhood about some $p \in M$.

LEMMA 3.2.5. *Let $(M, \mathbf{g})$ be a Riemannian manifold, $k \in \mathbb{N}$, $\alpha \in (0,1)$. Let $\mathbf{g} \in C^{k,\alpha}$ in a local chart about $p \in M$. Then there exists a neighborhood around p on which harmonic coordinates exist. These coordinates are $C^{k+1,\alpha}$-functions of the original coordinates. Additionally, all harmonic coordinate charts defined near p have the same level of regularity.*

Thus, when changing to harmonic coordinates the regularity of tensors behaves as follows by applying the previous result:

COROLLARY 3.2.6. *Let $\mathbf{g} \in C^{k,\alpha}$ in an arbitrary coordinate chart $(x^1, \dots, x^n)$, and let T be a tensor in $C^{l,\beta}$ for $l \geqslant k$, $\beta \geqslant \alpha$ in these coordinates. Then T is of regularity of at least $C^{k,\alpha}$ in harmonic coordinates.*

As a special case of Corollary 3.2.6, when applied to the metric $\mathbf{g}$ itself, we obtain that $\mathbf{g}$ has optimal regularity in harmonic coordinates:

THEOREM 3.2.7. *Let $\mathbf{g} \in C^{k,\alpha}$ in some coordinate chart, then it is also in $C^{k,\alpha}$ in harmonic coordinates.*

REMARK 3.2.8. When changing to geodesic normal coordinates, we just get $\mathbf{g} \in C^{k-2,\alpha}$. In [21, Ex. 2.3], it is shown that this assertion can in general not be improved.

Let ∇ be the connection for $\mathbf{g}$, with Christoffel symbols $\Gamma = \Gamma^l_{ij}$. They involve first derivatives of the metric, so if $\Gamma \in C^k$, we can at most expect that $\mathbf{g} \in C^{k+1}$.
By using the fact that for $\mathbf{g} \in C^1$, the map $T : \mathbf{g} \mapsto \Gamma$ is an overdetermined partial differential operator at $\mathbf{g}$, DeTurk and Kazdan [21] showed the following

THEOREM 3.2.9. *Let $\mathbf{g} \in C^2$. If in some local coordinates Γ is of class $C^{k,\alpha}$, then $\mathbf{g} \in C^{k+1,\alpha}$ in these coordinates.*

In addition to the optimal regularity properties of these coordinates, another advantage of them is that the components of the Ricci tensor take a rather simple form.

PROPOSITION 3.2.10. *Let $(M, \mathbf{g})$ be a Riemannian manifold, let $(x^1, \dots, x^n)$ be coordinates on M. Then the Ricci tensor in these coordinates, $\mathbf{Ric}_{\mathbf{g}}$, is given by*

$$(\mathbf{Ric}_{\mathbf{g}})_{ij} = -\frac{1}{2} g^{rs} \frac{\partial^2 g_{ij}}{\partial x^r \partial x^s} + \frac{1}{2}\left(g_{ri} \frac{\partial \Gamma^r}{\partial x^j} + g_{rj} \frac{\partial \Gamma^r}{\partial x^i} \right) + \dots, \tag{3.2.1}$$

where the dots consist of lower order terms with at most one derivative of the metric.

If the coordinates are harmonic, (3.2.1) *reduces to*

$$(\mathbf{Ric_g})_{ij} = -\frac{1}{2} g^{rs} \frac{\partial^2 g_{ij}}{\partial x^r \partial x^s} + \dots . \tag{3.2.2}$$

See [21, 4.1].

Constructing harmonic coordinates

In [40], Jost and Karcher constructed harmonic coordinates out of almost linear functions. Their results hold for manifolds that satisfy a pointwise bound on the sectional curvature.

In contrast to [21], they also obtain a lower bound on the size of the neighborhood on which it is possible to obtain harmonic coordinates, and they also prove curvature dependent estimates for the Christoffel symbols.

Almost linear functions

Let $(M, \mathbf{g})$ be a compact, $n-$dimensional Riemannian manifold. Let $p \in M$, and $B(p, r)$ a ball around p, that does not meet the cut locus of p. Let $u \in T_pM$ be a unit tangent vector.
The *almost linear function* l_u associated with u is defined as follows:
Let, for $x \neq p \in M$, d the distance in M with respect to $\mathbf{g}$,

$$r(x) := d(p, x), \qquad p(x) := \exp_p(r(x)u), \qquad q(x) := \exp_p(-r(x)u), \qquad f(x) := \frac{1}{2} d(p, x)^2.$$

Then

$$l_u(x) := \frac{d(x, q(x))^2 - d(x, p(x))^2}{4r(x)}.$$

If one additionally assumes the sectional curvature bounds $-\omega^2 \leqslant K \leqslant \kappa^2$, $|K| \leqslant \Lambda^2$ for positive constants ω, κ, Λ, then:

(1) $|\operatorname{grad} l_u(x) - u(x)| \leqslant 2\kappa\Lambda r^2 \frac{2\sinh 2\Lambda r}{\sin 2\kappa r}$,
(2) $|D^2 l_u(x)| \leqslant \left(8\kappa\Lambda \frac{\sinh 2\Lambda r}{\sin 2\kappa r} \omega r \coth \omega r\right) r(x)$,
(3) $|l_u(x) - \langle \operatorname{grad} l, \operatorname{grad} f\rangle(x)| \leqslant \left(\frac{8}{3}\kappa\Lambda \frac{\sinh 2\Lambda r}{\sin 2\kappa r} \omega r \coth \omega r\right) r^3(x)$.

See [40, Satz 2.1.].

Now take an orthonormal basis $u_1, \dots, u_n$ of T_pM and set $l_i := l_{u_i}$. Let $h_i : B(p, r) \to \mathbb{R}$ be the solution of

$$\begin{aligned} \Delta_{\mathbf{g}} h_i &= 0 \qquad \text{on } B(p, r), \\ h_i &= l_i \qquad \text{on } \partial B(p, r). \end{aligned}$$

These h_i are then the harmonic coordinates we searched for. For proving the existence and estimates on these coordinates, see [40, Satz 5.1.].

As we will see below, the existence of harmonic coordinates on balls of a radius that is uniformly bounded below, can also be shown for manifolds that just satisfy a Ricci curvature bound.

3.2.2. Harmonic coordinates for proving compactness results. The basic idea is as follows:

Take a sequence of Riemannian manifolds that satisfy a certain curvature bound, bounds on volume, diameter, or injectivity radius. Show that for these special manifolds one can find a bound on the size of harmonic balls, i.e., that one has harmonic coordinates of certain regularity ($C^{k,\alpha}$ or $W^{k,p}$) on these balls. Then use the Arzela-Ascoli theorem to extract a converging subsequence in the $C^{k,\alpha'}$ respectively, $W^{k,p'}$ ($\alpha' < \alpha, p' < p$) topology. Finally show that the limit manifold satisfies the desired regularity properties.

The key point to perform this process is that one has a uniform lower bound on the size of harmonic balls, i.e. on the radius of balls such that around each $p \in M$ it is possible to find harmonic coordinates. We will recall well known results and also add the proofs. Although the material is not original, our aim is to make these results more accessible by giving detailed proofs.

Harmonic radius

Definition 3.2.11. [**1, 3, 37**] Let $(M, \mathbf{g})$ be an n-dimensional Riemannian manifold. For $\alpha \in (0,1)$, $Q > 1$, the $C^{k,\alpha}$ -harmonic radius at $x \in M$ is the largest number $r_H(x) = r_H(k, \alpha, Q)(x)$ such that on any geodesic ball $B = B(x, r_H)$ there is a harmonic coordinate chart $U = \{u^i\}_{i=1}^n : B \to \mathbb{R}^n$ such that

$$Q^{-1}\delta_{ij} \leqslant g_{ij} \leqslant Q\delta_{ij} \tag{3.2.3}$$

$$\sum_{1 \leqslant |\beta| \leqslant k} r_H^{|\beta|} \sup_x |\partial^\beta g_{ij}(x)| + \sum_{|\beta| = k} r_H^{k+\alpha} \sup_{y \neq z} \frac{|\partial^\beta g_{ij}(y) - \partial^\beta g_{ij}(z)|}{d_x(y,z)^\alpha} \leqslant Q - 1. \tag{3.2.4}$$

The harmonic radius of M is $r_H(M) = \inf_{x \in M} r_H(k, \alpha, Q)(x)$.

Analogously, for $p \in (n, \infty)$, $Q > 1$, the $W^{k,p}$-harmonic radius at $x \in M$, $r_H(x) = r_H(k, p, Q)(x)$, is the largest number such that on any geodesic ball $B = B(x, r_H)$ there is a harmonic coordinate chart $U = \{u^i\}_{i=1}^n : B \to \mathbb{R}^n$ such that

$$Q^{-1}\delta_{ij} \leqslant g_{ij} \leqslant Q\delta_{ij}, \tag{3.2.5}$$

$$\sum_{1 \leqslant |\beta| \leqslant k} r_H^{|\beta| - n/p} \|\partial^\beta g_{ij}\|_{L^p} \leqslant Q - 1, \tag{3.2.6}$$

and $r_H(M) = \inf_{x \in M} r_H(k, p, Q)(x)$.

Remark 3.2.12. The harmonic radius $r_H(x)$ is the radius of the largest geodesic ball around $x \in M$ on which one has harmonic coordinates satisfying (3.2.3) and (3.2.4) or (3.2.5) and (3.2.6) respectively.

Using the results by Jost and Karcher [40], the harmonic radius is positive for a fixed smooth compact Riemannian manifold.

Continuity properties

We will now study what happens to the properties (3.2.5) and (3.2.6) when we consider sequences of Riemannian metrics.

LEMMA 3.2.13. *Let* $\mathbf{g}_i \to \mathbf{g}$ *in the strong* $W^{k,p}$*-topology. Then the bounds* (3.2.5) *and* (3.2.6) *are preserved.*

Proof. Let $\{x^\alpha\}$, $\{y_i^\alpha\}$ be harmonic coordinates for $\mathbf{g}$ resp. $\mathbf{g}_i$. Then

$$\mathbf{g}_i\left(\frac{\partial}{\partial y_i^k}, \frac{\partial}{\partial y_i^l}\right) = \frac{\partial x^a}{\partial y_i^k}\frac{\partial x^b}{\partial y_i^l}\mathbf{g}_i\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right).$$

The third factor converges by assumption in the $W^{k,p}$-topology to $\mathbf{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)$.

Furthermore, $\{y_i^\alpha\} \to \{x^\alpha\}$ in $W^{k+1,p}$, so also in $C^{k,\alpha}$, see A.3.9. Therefore

$$\frac{\partial y}{\partial x} \to \mathrm{Id}, \qquad \frac{\partial x}{\partial y} \to \mathrm{Id}$$

in $C^{k-1,\alpha}$, thus locally uniformly,

$$\mathbf{g}_i\left(\frac{\partial}{\partial y_i^k}, \frac{\partial}{\partial y_i^l}\right) \to \mathbf{g}\left(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right)$$

in C^0 , which gives (3.2.5) with $Q_i \to Q$.

By a similar argument we also obtain (3.2.6). ☐

Following Hebey and Herzlich [37], we will investigate how the harmonic radius behaves when taking limits $\mathbf{g}_i \to \mathbf{g}$. We start with the following observation:

LEMMA 3.2.14. [37, Lemma 8] *Let* $(M, \mathbf{g})$ *be a smooth Riemannian manifold without boundary. Let* $x \in M$, $k \in \mathbb{N}^*$ *and* $p > n$. *Then it holds that for all* $1 < Q \leqslant Q' < \infty$

(1) $r_H(k,p,Q)(x) \leqslant r_H(k,p,Q')(x)$,

(2) for all $Q > 1$, $\lim_{\varepsilon\to 0^+} r_H(k,p,Q+\varepsilon)(x) = r_H(k,p,Q)(x)$.

Thus, with respect to Q*, the* $W^{k,p}$ *harmonic radius is increasing and upper semicontinuous.*

Proof. The first statement just follows by definition, $r_H(k,p,Q)(x)$ is increasing with respect to Q.

It remains to show that $\limsup_{\varepsilon\to 0} r_H(k,p,Q+\varepsilon)(x) \leqslant r_H(k,p,Q)$. So let

$$r < \limsup_{\varepsilon\to 0^+} r_H(k,p,Q+\varepsilon)(x)$$

be fixed.

Then we can find harmonic coordinate charts $U_\varepsilon := \{u_\varepsilon^1 \ldots, u_\varepsilon^n\}$ on $B(x,r)$ which satisfy (3.2.5), (3.2.6) with Q replaced by $Q+\varepsilon$, and r instead of r_H. The sequence $\{U_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{k,p}$ since it satisfies (3.2.6), thus a subsequence converges in $W^{k+1,p}$ by Sobolev embedding, to a limiting chart U.

Define now ta sequence of metrics $\mathbf{g}_\varepsilon$ by $g_{\varepsilon ij} = \mathbf{g}(\frac{d}{du^i_\varepsilon}, \frac{d}{du^j_\varepsilon}$ that converge t $\mathbf{g}$.

Since, by Lemma 3.2.13 the relations (3.2.5), (3.2.6) are preserved under $W^{k,p}$-convergence, it holds that $r_H(k,p,Q)(x) \geqslant r$. But $r < \limsup_{\varepsilon \to 0^+} r_H(k,p,Q+\varepsilon)(x)$ was chosen arbitrarily, therefore, $r_H(k,p,Q)(x) \geqslant \lim_{\varepsilon \to 0^+} r_H(k,p,Q+\varepsilon)(x)$. □

If we now consider a sequence of Riemannian metrics $\{\mathbf{g}_i\}$ on a fixed smooth differentiable manifold without boundary, denote by $r_H(\mathbf{g}_i, Q) = r_H(k,p,Q)(x)$, $x \in M$, the harmonic radius with respect to the metric $\mathbf{g}_i$.

In order to prove that the harmonic radius satisfies certain continuity properties we will need the following result:

Proposition 3.2.15. *Let $\mathbf{g}_i$ be a sequence of Riemannian metrics on M, $\mathbf{g}_i \to \mathbf{g}$ in $W^{k,p}$, with corresponding Laplacians Δ_i and Δ. Let w_i be a sequence of functions that satisfy*

$$\Delta_i w_i = \Delta_i f \qquad \text{on } B = \subseteq M, \tag{3.2.7}$$

$$w_i = 0 \qquad \text{on } \partial B, \tag{3.2.8}$$

with a (with respect to $\mathbf{g}$) harmonic function f.
Then w_i converges to 0 in $W^{k+1,p}$ uniformly on compact subsets $B' \Subset B$.

Proof. By using [26, Theorem 8.16] it follows that

$$\sup_B w_i \leqslant \sup_{\partial B}(w_i)^+ + C\|\Delta_i f\|_{L^p(B)}.$$

Due to the convergence of $\mathbf{g}_i \to \mathbf{g}$, it follows that $\Delta_i \to \Delta$ in $W^{k,p}$, and since $\Delta f = 0$, we obtain that

$$\|\Delta_i f\|_{L^p(B)} \to 0.$$

Therefore, since $w_i = 0$ on ∂B,

$$\lim \|w_i\|_{C^0(B)} \to 0.$$

Applying [26, Theorem 8.33], we can estimate the $C^{1,\alpha}$-norm of w_i to get

$$\|w_i\|_{C^{1,\alpha}(B')} \leqslant C\|w_i\|_{C^0(B)},$$

hence

$$\lim \|w_i\|_{C^{1,\alpha}(B')} \to 0$$

where $\alpha = 1 - \frac{n}{p}$. Note that for $q \in [1, \infty)$, we can estimate $\|w_i\|_{L^q(B')} \leqslant \|w_i\|_{C^{1,\alpha}(B')}$.

We can apply [26, Theorem 9.11] to estimate

$$\|w_i\|_{W^{2,p}(B')} \leqslant C\|w_i\|_{L^p(B')} \leqslant C'\|w_i\|_{C^{1,\alpha}(B')},$$

therefore,

$$\lim \|w_i\|_{W^{2,p}(B')} \to 0.$$

By an induction argument similar to e.g., [57, Appendix] we obtain $\lim \|w_i\|_{W^{k+1,p}(B')} \to 0$. □

We now want to generalize Lemma 3.2.14 to the situation of not having a fixed Riemannian metric, but a sequence of $\mathbf{g}_i$.

It was proven in [37, Lemma 10] for the general $W^{k,p}$-case and in [3, Proposition 1.1] for the $W^{1,p}$-case that the following result holds:

Lemma 3.2.16. [37, Lemma 10], [3, Proposition 1.1] *Let M be a smooth Riemannian manifold without boundary, $\{\mathbf{g}_i\}$ be a sequence of Riemannian metrics on M. For $Q > 1$, $k \in \mathbb{N}^*$, $p > n$ suppose that $\mathbf{g}_i$ converge in $W^{k,p}$ to a $W^{k,p}$ metric $\mathbf{g}$.*

Then the harmonic radius is upper and lower semicontinuous, i.e.,

(1) $r_H(\mathbf{g}, Q) \geqslant \limsup_{i\to\infty} r_H(\mathbf{g}_i, Q)$,
(2) for all $0 < \varepsilon < Q - 1$, $r_H(\mathbf{g}, Q - \varepsilon) \leqslant \liminf_{i\to\infty} r_H(\mathbf{g}_i, Q)$.

Remark 3.2.17. In [1] only the second property is shown in the $C^{1,\alpha}$ case, i.e., that the harmonic radius is lower semi-continuous with respect to $C^{1,\alpha}$ convergence. This is the property that will be needed in the proof of the uniform lower bound and therefore for the precompactness result below.

We will recall the proof in [3], for a more detailed version see [37].

Proof of Lemma 3.2.16. Let $r_i := r_H(\mathbf{g}_i, Q)$. In order to show

$$r_H(\mathbf{g}, Q) \geqslant \limsup_{i\to\infty} r_i,$$

let $U_i : B(x, r_i) \to \mathbb{R}^n$ be harmonic coordinate charts satisfying (3.2.5), (3.2.6). Suppose $\limsup_{i\to\infty} r_i > 0$. Since $\mathbf{g}_i \to \mathbf{g}$ in $W^{k,p}$, a subsequence of the charts U_i converges in $W^{k+1,p}$ to a limiting chart $U : B(x, r) \to \mathbb{R}^n$, for any $r < \limsup_{i\to\infty} r_i$.

Since (3.2.5), (3.2.6) are preserved under $W^{k+1,p}$ convergence, this limit U also satisfies the same conditions. Thus $r_H(\mathbf{g}, Q) \geqslant r$ for any $r < \limsup_{i\to\infty} r_i$, hence $r_H(\mathbf{g}, Q) \geqslant \limsup_{i\to\infty} r_i$.

To show the lower semicontinuity property,

$$r_H(\mathbf{g}, Q - \varepsilon) \leqslant \liminf_{i\to\infty} r_H(\mathbf{g}_i, Q) \qquad \forall \varepsilon \in (0, Q - 1),$$

fix $r < r_H(\mathbf{g}, Q)$. Let $\{x^1, \ldots, x^n\}$ be harmonic coordinates for $\mathbf{g}$ on $B(x, r) = B$. Denote by Δ_i the Laplacian of $\mathbf{g}_i$. View $\mathbf{g}_i$ as metrics on $B(x, r)$. In the coordinates $\{x^1, \ldots, x^n\}$, Δ_i can be written as

$$\Delta_i = (g_i)^{lj} \frac{\partial^2}{\partial x^l \partial x^j} - (g_i)^{lj} (\Gamma_i)^s_{lj} \frac{\partial}{\partial x^s} = \frac{1}{\sqrt{|\det \mathbf{g}_i|}} \frac{\partial}{\partial x^l} \left(g_i^{lj} \sqrt{|\det \mathbf{g}_i|} \frac{\partial}{\partial x^j} \right). \tag{3.2.9}$$

Let now $\{y_i^1, \ldots, y_i^n\}$ be solutions of the problem

$$\Delta_i y_i^l = 0 \qquad \text{on } B \tag{3.2.10}$$

$$y_i^l = x^l \qquad \text{on } \partial B. \tag{3.2.11}$$

We aim at showing that the harmonic coordinates $\{y_i^1, \ldots, y_i^n\}$ converge locally in $W^{k+1,p}$ on uniformly on compact sets $B' \Subset B$ to $\{x^1, \ldots, x^n\}$.

Set therefore $w_i^l = x^l - y_i^l$. This functions satisfy the conditions (3.2.7), (3.2.8) in Proposition 3.2.15 and we obtain that

$$\lim \|w_i^l\|_{W^{k+1,p}(B')} \to 0.$$

Thus for all $B' \Subset B$, $\{y_i^1, \ldots, y_i^n\}$ are harmonic coordinates on B' for $\mathbf{g}_i$ that converge to $\{x^1, \ldots, x^n\}$ in $W^{k+1,p}(B')$. Since the conditions (3.2.5), (3.2.6) are continuous in the strong $W^{k,p}$-topology, $\{y_i^1, \ldots, y_i^n\}$ and $\mathbf{g}_i$ satisfy the same conditions as $\{x^1, \ldots, x^n\}$ and $\mathbf{g}$ with Q replaced by Q_i, $\lim_{i\to\infty} Q_i = Q$.

Therefore, for $\varepsilon > 0$ we get

$$r \leqslant \liminf_{i\to\infty} r_H(\mathbf{g}_i, Q_i) \leqslant \liminf_{i\to\infty} r_H(\mathbf{g}_i, Q+\varepsilon). \tag{3.2.12}$$

Since $r < r_H(\mathbf{g}, Q)$ was arbitrary, the result follows. $\square$

Bounds on r_H

We now want to show that under certain conditions on the curvature we get a lower bound on r_H. We will only investigate the $W^{k,p}$ case, since from a $W^{k,p}$ bound we obtain a $C^{k,\alpha}$ bound by the Sobolev embedding.

We will follow [37, Theorem 11] since their proof includes Anderson-Cheeger's result from [3] and also Anderson's [1] with additional $W^{k,p}$ results for $k \geqslant 2$.

THEOREM 3.2.18. [37, Theorem 11] *Let $(M, \mathbf{g})$ be a smooth Riemannian manifold without boundary, $\Omega \subset M$ open, $Q > 1$, $p > n$, $d > 0$.*

Suppose that for $K \in \mathbb{R}$, $i_0 > 0$, it holds that for all x that are in $\Omega(d)$, i.e. that are of distance at most d from Ω,

$$\mathbf{Ric}_{(M,\mathbf{g})}(x) \geqslant K, \qquad \mathrm{inj}_{(M,\mathbf{g})}(x) \geqslant i_0, \tag{3.2.13}$$

then there exists a constant $c = c(n, Q, p, k, d, i_0, K)$ such that for all $x \in \Omega$ the $W^{1,p}$ harmonic radius satisfies

$$r_H(1, p, Q)(x) \geqslant C. \tag{3.2.14}$$

If, instead of (3.2.13) we have for $k \in \mathbb{N}$, $C(j)$ $(j = 0, \ldots, k)$

$$|D^j \mathbf{Ric}_{(M,\mathbf{g})}(x)| \leqslant C(j), \qquad \mathrm{inj}_{(M,\mathbf{g})}(x) \geqslant i_0, \tag{3.2.15}$$

then there exists a constant $c = c(n, Q, p, k, d, i_0, C(j))$ such that for all $x \in \Omega$ the $W^{k+2,p}$ harmonic radius satisfies

$$r_H(k+2, p, Q)(x) \geqslant C. \tag{3.2.16}$$

This result is proven by contradiction. The idea is as follows: Start with constructing a sequence $(M_i, \mathbf{g}_i, x_i)$ of pointed Riemannian manifolds, such that $x_i \in M_i$, $r_H(k,p,Q)(x_i) = 1$ and $(M_i, \mathbf{g}_i, x_i)$ converges to a Riemannian manifold $(M, \mathbf{g}, x)$ in $W^{k,p}$. In the next step show that $(M, \mathbf{g})$ is isometric to $(\mathbb{R}^n, \delta)$, where δ is the standard Euclidean metric. We finally arrive at a contradiction, since by the continuity of the harmonic radius, (3.2.16), $r_H(k,p,Q)(x) = 1$, but in $\mathbb{R}^n$ it holds that $r_H(k,p,Q)(y) = \infty$ for all $y \in \mathbb{R}^n$.

We will need the following results:

Lemma 3.2.19. *Let* $(M, \mathbf{g})$ *be a Riemannian manifold,* $x \in M$*,* $B \subseteq M$ *a ball around* x*. Then the function*

$$f\colon B \to \mathbb{R}, \qquad x \mapsto \frac{r_H(\mathbf{g}, x)}{\operatorname{inj}_{(B,\mathbf{g})}(x)}$$

can be minimized at a point $z \in B$*,* $z \notin \partial B$*.*

Proof. Consider the set $C := \{q \in B | f(q) \leqslant f(x)\}$, with $C \cap \partial B = \varnothing$. This map f is continuous on C. Furthermore, by definition, C is compact, thus a minimum of f is attained away from the boundary of B.

□

Another result which will be used is the Cheeger-Gromoll splitting theorem:

Theorem 3.2.20 ([16]). *Let* $(M, \mathbf{g})$ *be a complete Riemannian manifold of dimension* $n \geqslant 2$*, with* $\mathbf{Ric}_{\mathbf{g}} \geqslant 0$*. If* M *contains a line, then* M *is isometric to the product* $(\mathbb{R}^k \times N, \delta^k \oplus \mathbf{g}_N)$*, with* $k > 0$*,* N *containing no lines and* δ^k *the standard metric on* $\mathbb{R}^k$*.*

Proof of Theorem 3.2.18. We will prove the result for $k \geqslant 2$, following closely [37], the modifications for the $k = 1$ case will be described in Remark 3.2.21 below.

Assume, that for $Q > 1$, $p > n$, $d > 0$, $i_0 > 0$, $k \in \mathbb{N}$, $C(j) \in \mathbb{R}^+$, we can find a sequence $(M_i, \mathbf{g}_i, x_i)$ of pointed Riemannian manifolds without boundary, and open subsets $\Omega_i \subset M_i$, and points $x_i \in \Omega_i$ such that for all $x \in \Omega_i(d)$ the bounds (3.2.15) hold, but

$$\lim_{i\to\infty} r_H(\mathbf{g}_i, x_i) = r_H(k+2, p, Q)(x) = 0. \tag{3.2.17}$$

We will show in several steps that this assumption leads to a contradiction:

(1) Choose certain optimal basepoints y_i on sets $B_i \subseteq M_i$.
(2) Rescale the metrics $\mathbf{g}_i$ on B_i to metrics $\mathbf{h}_i$ such that for $i \to \infty$ one gets certain convergence results on the Ricci curvature, and $r_H(\mathbf{h}_i, y_i) = 1$.
(3) Show that the harmonic radius with respect to the rescaled metric has a positive lower bound on arbitrarily large balls in M_i, therefore it is possible to construct harmonic coordinates U_i on these balls.
(4) $(B_i, \bar{\mathbf{h}}_i := (U_i^{-1})^* \mathbf{h}_i, y_i)$ converges to a complete Ricci flat Riemannian manifold in $W^{k+2,p}$ uniformly on compact subsets.
(5) This manifold is isometric to $\mathbb{R}^n$ and a contradiction to (3.2.17) can be deduced.

For the detailed calculation we will follow [37, Theorem 11]. Without loss of generality we can set $K = 1$.

Step 1: Fix i. The first idea to choose $x_i \in M_i$ would be to choose it such that $r_H(\mathbf{g}_i, x_i)$ is minimal at x_i. Such points do not necessarily have to exist. To overcome this, consider the sequence $(B_i, \mathbf{g}_i)$, where $B_i = B(x_i, \inf(d, i_0))$ the geodesic ball for $\mathbf{g}_i$. For $x \in B_i$ it holds that $\operatorname{inj}_{(B_i,\mathbf{g}_i)}(x) = d_{\mathbf{g}_i}(x, \partial B_i)$ where $d_{\mathbf{g}_i}$ denotes the distance with respect to $\mathbf{g}_i$. Therefore,

$$\lim_{x\to\partial B_i} \mathrm{inj}_{(B_i,\mathbf{g}_i)}(x) = 0, \qquad \mathrm{inj}_{(B_i,\mathbf{g}_i)}(x_i) = \inf(d, i_0). \tag{3.2.18}$$

We apply Lemma 3.2.19 to find minimizing points y_i for $\frac{r_H(\mathbf{g}_i,x)}{\mathrm{inj}_{(B_i,\mathbf{g}_i)}(x)}$.

The ratio satisfies

$$\frac{r_H(\mathbf{g}_i,y_i)}{\mathrm{inj}_{(B_i,\mathbf{g}_i)}(y_i)} \leqslant \frac{r_H(\mathbf{g}_i,x_i)}{\mathrm{inj}_{(B_i,\mathbf{g}_i)}(x_i)} \to 0 \qquad i\to\infty \tag{3.2.19}$$

by (3.2.17).

Step 2: Let now $r_i := r_H(\mathbf{g}_i, y_i)$. We rescale the metric to $\mathbf{h}_i := \frac{1}{r_i^2}\mathbf{g}_i$ on B_i. Note that in this metric $r_H(\mathbf{h}_i, y_i) = 1$. Furthermore we get

$$\begin{cases} \|\mathbf{Ric}_{(B_i,\mathbf{h}_i)}\|_{C^k} = r_i^2\|\mathbf{Ric}_{(B_i,\mathbf{g}_i)}\|_{C^k} \to 0 \text{ for } i\to\infty, \\ \mathrm{inj}_{(B_i,\mathbf{h}_i)} = \frac{1}{r_i}\,\mathrm{inj}_{(B_i,\mathbf{g}_i)} \to \infty, \\ d_{\mathbf{h}_i}(y_i,\partial B_i) = \frac{1}{r_i} d_{\mathbf{g}_i}(y_i,\partial B_i) \to \infty, \\ r_H(\mathbf{h}_i,y) = \frac{r_H(\mathbf{h}_i,y)}{r_H(\mathbf{h}_i,y_i)} \geqslant \frac{d_{\mathbf{g}_i}(y,\partial B_i)}{d_{\mathbf{g}_i}(y_i,\partial B_i)} \geqslant \frac{d_{\mathbf{h}_i}(y,\partial B_i)}{d_{\mathbf{h}_i}(y_i,\partial B_i)}. \end{cases} \tag{3.2.20}$$

Let now $u_i := \frac{r_i}{\mathrm{inj}_{(B_i,\mathbf{h}_i)}} = \frac{1}{d_{\mathbf{h}_i}(y_i,\partial B_i)}$. Therefore $u_i \to 0$ for $i\to\infty$.

Step 3: We now show that $r_H(\mathbf{h}_i, y) \geqslant \frac{1}{2}$ for $y \in B_{\mathbf{h}_i}(y_i, \frac{1}{2u_i})$.
Indeed, by using the last property in (3.2.20), we obtain

$$r_H(\mathbf{h}_i,y) \geqslant \frac{d_{\mathbf{h}_i}(y,\partial B_i)}{d_{\mathbf{h}_i}(y_i,\partial B_i)}.$$

Choose $y \in B_{\mathbf{h}_i}(y_i, \frac{1}{2u_i})$, so $d_{\mathbf{h}_i}(y_i, y) < \frac{1}{2u_i}$.
For $z \in \partial B_i$ it holds that $d_{\mathbf{h}_i}(z, y_i) \leqslant d_{\mathbf{h}_i}(y, z) + d_{\mathbf{h}_i}(y, y_i)$, since $d_{\mathbf{h}_i}(y_i, z) \geqslant d_{\mathbf{h}_i}(y_i, \partial B_i)$ and $d_{\mathbf{h}_i}(y, y_i) < \frac{1}{2u_i}$,

$$\begin{aligned} d_{\mathbf{h}_i}(y,z) &\geqslant d_{\mathbf{h}_i}(y_i,z) - d_{\mathbf{h}_i}(y_i,y) \\ &> d_{\mathbf{h}_i}(y_i,z) - \frac{1}{2u_i} \\ &= \frac{1}{r_i} d_{\mathbf{g}_i}(y_i,z) - \frac{1}{2u_i}. \end{aligned} \tag{3.2.21}$$

Therefore, because $z \in \partial B_i$, we obtain

$$\begin{aligned} d_{\mathbf{h}_i}(y,\partial B_i) &\geqslant d_{\mathbf{h}_i}(y_i,\partial B_i) - \frac{1}{2u_i} = d_{\mathbf{h}_i}(y_i,\partial B_i) - \frac{d_{\mathbf{h}_i}(y_i,\partial B_i)}{2} \\ &= \frac{1}{2} d_{\mathbf{h}_i}(y_i,\partial B_i). \end{aligned}$$

Hence,

$$r_H(\mathbf{h}_i, y) \geqslant \frac{d_{\mathbf{h}_i}(y, \partial B_i)}{d_{\mathbf{h}_i}(y_i, \partial B_i)} \geqslant \frac{1}{2}. \tag{3.2.22}$$

Now we have that for all $y \in B_{\mathbf{h}_i}\left(y_i, \frac{1}{2u_i}\right)$ the harmonic radius satisfies $r_H(\mathbf{h}_i, y) \geqslant \frac{1}{2}$. Since $u_i \to 0$ as $i \to \infty$, one can choose $R > 0$ arbitrarily large. It is therefore possible to find I such that for all $i > I$ it holds that $r_H(\mathbf{h}_i, z) \geqslant \frac{1}{2}$ for all $z \in B_{\mathbf{h}_i}(y_i, R)$. In other words, it is possible to find harmonic coordinate charts $\{U_i\} \colon \Omega_i \to B(0, \frac{1}{2\sqrt{Q}}) \subseteq \mathbb{R}^n$ on open sets $\Omega_i \subseteq M_i$, centered at $z_i \in \Omega_i$ satisfying (3.2.5), (3.2.6) on these balls. Note that the radius $\frac{1}{2\sqrt{Q}}$ in the Euclidean ball comes from condition (3.2.5). Indeed the condition implies that $d_\delta(x, y) \leqslant \sqrt{Q} d_{\mathbf{h}_i}$.

Step 4: Set now $\bar{\mathbf{h}}_i := (U_i^{-1})^* \mathbf{h}_i$.

We want to show that $(B_i, \bar{\mathbf{h}}_i, y_i)$ converges in $W^{k+2,p}$, uniformly on compact subsets, to a complete Riemannian manifold $(M, \mathbf{h}, y)$.

Since $\bar{\mathbf{h}}_i$ is bounded in $W^{k+2,p}$, by Sobolev embedding a subsequence converges in $C^{k+1,\alpha}$. The expression for the Ricci tensor in these coordinates is given by formula (3.2.2),

$$-2(\mathbf{Ric}_{\bar{\mathbf{h}}_i})_{kl} - A(\bar{\mathbf{h}}_i) = \bar{h}_i^{rs} \frac{\partial^2 (\bar{\mathbf{h}}_i)_{kl}}{\partial x^r \partial x^s}. \tag{3.2.23}$$

where $A(\bar{\mathbf{h}}_i)$ is quadratic in first derivatives of $\bar{\mathbf{h}}_i$.
Since, by (3.2.20) $\|\mathbf{Ric}_{(B_i, \mathbf{h}_i)}\|_{C^k} \to 0$, it follows that

$$(\mathbf{Ric}_{\bar{\mathbf{h}}_i})_{kl} = (U_i^{-1})^* (\mathbf{Ric}_{\mathbf{h}_i})_{kl} \to 0 \qquad \text{in } C^k.$$

Therefore, the sequence $\bar{\mathbf{h}}_i$ is not only bounded in this space, but also, since (3.2.23) includes derivatives of $\bar{\mathbf{h}}_i$ up to second order, converges in $W^{k+2,p}$. Furthermore, also the charts converge to a limiting chart, $U_i \to U$ in $W^{k+3,p}$, which is harmonic as well.

By the third property of (3.2.20), $d_{\mathbf{h}_i}(y_i, \partial B_i) \to \infty$, it follows that $(M, \mathbf{h})$ is complete.

To show that $(M, \mathbf{h})$ is Ricci flat, note that since coordinates $\mathbf{H} := (U^{-1})^* \mathbf{h}$ are harmonic, the Laplacian in these coordinates satisfies

$$A(\mathbf{H}) + H^{rs} \frac{\partial^2 H_{kl}}{\partial x^r \partial x^s} = 0. \tag{3.2.24}$$

Thus, by elliptic regularity, see, e.g. [26, Theorem 8.13] $\mathbf{H}$ is smooth, and, since the left hand side equals the Ricci tensor expression, it follows that $(M, \mathbf{h})$ is a complete, Ricci flat Riemannian manifold.

Step 5: To show that $(M, \mathbf{h})$ is isometric to $(\mathbb{R}^n, \delta)$, we aim to use Theorem 3.2.20. Since M is actually Ricci flat, every geodesic is a line thus we obtain n different linearly independent lines starting from the same point. Applying the splitting theorem n times gives the isometry to $\mathbb{R}^n$.

Step 6: To arrive at the contradiction, observe that $\bar{\mathbf{h}}_i \to \mathbf{h}$ in $W^{k+2,p}$, and by the continuity property of the harmonic radius we also know that $r_H(k+2,p,Q')(y) \leqslant \lim r_H(k+2,p,Q)(y_i)$ for $Q' < Q$. By construction, $r_H(k+2,p,Q)(y_i) = 1$, but $r_H(k+2,p,Q')(y) = \infty$ for all $y \in \mathbb{R}^n$, which is a contradiction. $\square$

REMARK 3.2.21. Anderson-Cheeger [3] prove the $W^{1,p}$-(respectively $C^{0,\alpha}$) version of the theorem. The proof presented above does not work in this case. One of the reasons is that **Ric** does not converge to zero in C^k, but we only get $\mathbf{Ric}_{(M_i,\mathbf{h}_i)} \geqslant K r_H(y_i,\mathbf{h}_i)^2 \to 0$. To overcome this, the authors use the distance function instead of the expression for **Ric**. We will outline the modifications which are necessary to obtain the result.

Instead of the injectivity radius, Anderson-Cheeger work with the following quantity, which allows the result to be phrased purely locally:

DEFINITION 3.2.22. We denote with $s_M(x) := \sup_r\{\min(r, \inf\{\mathrm{inj}(y) : y \in B(x,r) \subseteq M\})\}$ the largest s such that all points $y \in B(x,s) \subseteq M$ have $\mathrm{inj}(y) \geqslant s$.

For closed Riemannian manifolds, this quantity $s_M(x) = \mathrm{inj}(M)$ for all $x \in M$.

The proof of Theorem 3.2.18 in case $k = 1$ works again by contradiction, Steps $1-3$ and Steps $5, 6$ are analogous to the $k \geqslant 2$-case, only the fourth step has to be modified. Without loss of generality we use $K = 1$.

We rescale the metric as above to $\mathbf{h}_i = \frac{1}{r_i^2}\mathbf{g}_i$ and get the same convergence properties as in (3.2.20) with the Ricci condition replaced by

$$\mathbf{Ric}_{(B_i,\mathbf{h}_i)} = r_i^2 \mathbf{Ric}_{(B_i,\mathbf{g}_i)} \geqslant -r_i^2 \to 0 \tag{3.2.25}$$

By an analysis of the distance function, which replaces the previous Step 4, Anderson-Cheeger obtain the convergence result. It is first shown that the Laplacian of an arbitrary distance function has an upper as well as a lower bound depending on the bound on the Ricci curvature. Then a sequence of certain smooth distance functions ρ_i will be defined, such that $|\Delta_i \rho_i| \to 0$. It will be shown that this sequence converges in $W^{2,p}$ to ρ on M. Finally the strong $W^{1,p}$ convergence of $(M_i, \mathbf{g}_i, x_i)$ to a limit manifold $(M, \mathbf{g}, x)$ will be deduced.

LEMMA 3.2.23. *Let $(M_i, \mathbf{h}_i)$ be a sequence of Riemannian manifolds. Let γ_i be a geodesic on $(M_i, \mathbf{h}_i)$, $s_i > 0$, $z_i = \gamma(-s_i)$. Then the distance function*

$$\rho_i(\cdot) = d_{\mathbf{h}_i}(z_i, \cdot) - s_i \tag{3.2.26}$$

is smooth on $B_{\mathbf{h}_i}(x_i, s_i/2)$.

If the metrics $\mathbf{h}_i$ furthermore satisfy (3.2.20) and (3.2.25), with $\mathbf{Ric}_{(M_i,\mathbf{h}_i)} \geqslant K_i \to 0$ then $|\Delta_i \rho_i| \to 0$ as $i \to \infty$.

Proof. To show smoothness we note that $d_{\mathbf{h}_i}(z_i, \cdot)$ is smooth everywhere apart from z_i and the cut locus of z_i, $\mathrm{Cut}(z_i)$. Here it would suffice to have $s_i = \frac{i_0}{2r_i}$.

So for $s_i = \frac{i_0}{2r_i}$ we get a smooth function on $B_{\mathbf{h}_i}(x_i, s_i/2)$.

To show the second part of the claim, i.e. $|\Delta_i \rho_i| \to 0$, we apply (2.1.4) to ρ_i. In order for the estimate to hold, we need $d_{\mathbf{h}_i}(z_i, p) < 1/2 \operatorname{inj}_{(M_i,\mathbf{h}_i)}(z_i)$ so that we do not cross $\operatorname{Cut}(z_i)$. So for $s_i = \frac{i_0}{4r_i}$ we get on $B_{\mathbf{h}_i}(x_i, s_i/2)$:

$$\begin{aligned} |\Delta_i \rho_i| &\leqslant (n-1) K_i \coth(K_i(\rho_i + s_i)) \\ &\leqslant (n-1) r_i \coth(r_i(\rho_i + i_0/(4r_i))) \to 0 \end{aligned} \tag{3.2.27}$$

as $r_i \to 0$ since $\coth(r_i\rho_i + i_0/4) < \infty$. Here (3.2.25) is used. □

LEMMA 3.2.24. *The functions ρ_i are bounded in $W^{2,p}$ on $B' \Subset M_i$. Furthermore, a subsequence of $\{\rho_i\}$ converges strongly in $W^{1,p}$ and weakly in $W^{2,p}$ to a $W^{2,p}$-distance function ρ, uniformly on the compact sets.*

Proof. On each ball $B = B_i \subseteq M_i$ of bounded distance to x_i and sufficiently small radius there exist harmonic coordinates $\{u_i^k\}$. The Laplacian in these coordinates is

$$\Delta_i = \sum g_i^{kl} \frac{\partial^2}{\partial u_i^k \partial u_i^l}. \tag{3.2.28}$$

By elliptic estimates (see, eg., [26, Theorem 9.11]), we obtain

$$\|\rho_i\|_{W^{2,p}(B')} \leqslant C(\|\Delta_i \rho_i\|_{L^p(B)} + \|\rho_i\|_{L^p(B)}) \tag{3.2.29}$$

on $B' \Subset B$. By definition, see (3.2.26), ρ_i is finite on B. Therefore, by (3.2.27), ρ_i stays bounded in $W^{2,p}(B')$. Using now the compactness of the embedding of $W^{2,p}$ in $W^{1,p}$, we obtain that the bounded sequence has a subsequence, also denoted by $\{\rho_i\}$, that converges strongly in $W^{1,p}$ and weakly in $W^{2,p}$. □

REMARK 3.2.25. Since in the proof of 3.2.24 we make use of (3.2.27), we implicitly assume that the metrics satisfy (3.2.20) and (3.2.25), thus the curvature bound is needed.

Next we want to show that this convergence is actually strong:

LEMMA 3.2.26. *The sequence $\{\rho_i\}$ has a subsequence that converges strongly in $W^{2,p}$ on B'.*

Proof. By applying (3.2.29) to $\rho - \rho_i$ we obtain

$$\|\rho - \rho_i\|_{W^{2,p}(B')} \leqslant C(\|\Delta_i(\rho - \rho_i)\|_{L^p(B)} + \|\rho - \rho_i\|_{L^p(B)}). \tag{3.2.30}$$

For the last term we get $\|\rho - \rho_i\|_{L^p(B)} \to 0$ by strong convergence in $W^{1,p}$, 3.2.24. For the other term we estimate,

$$\begin{aligned} \|\Delta_i(\rho - \rho_i)\|_{L^p(B)} &= \|\Delta_i \rho - \Delta\rho + \Delta\rho - \Delta_i \rho_i\|_{L^p(B)} \\ &\leqslant \|(\Delta_i - \Delta)\rho\|_{L^p(B)} + \|\Delta\rho\|_{L^p(B)} + \|\Delta_i \rho_i\|_{L^p(B)}. \end{aligned} \tag{3.2.31}$$

By (3.2.27), the last summand converges to zero. Furthermore,

$$(\Delta_i - \Delta)\rho = \left(g_i^{kl} - g^{kl}\right) \partial_k \partial_l + \ldots \to 0 \tag{3.2.32}$$

in $C^{0,\beta}$, so $\Delta_i \rho \to \Delta \rho$ in L^p.

It remains to show that

$$\Delta \rho = 0 \qquad \text{in } L^p. \tag{3.2.33}$$

Since $\Delta \rho = \lim_i \Delta_i \rho_i$, the result follows by (3.2.27), hence finally we showed that $\rho_i \to \rho$ strongly in $W^{2,p}$. □

The lemmata 3.2.23, 3.2.24 and 3.2.26 are implicitly contained in the proof of [3, Proposition 1.2].

Proposition 3.2.27. [3, Proposition 1.2] *Let $(M_i, \mathbf{g}_i, x_i)$ be a sequence of manifolds satisfying $\mathbf{Ric}_{(M_i,\mathbf{g}_i)} \geqslant K_i \to 0$. Then a subsequence converges in the strong $W^{1,p}$-topology to a limit $W^{1,p}$-manifold $(M, \mathbf{g})$.*

Sketch of Proof. The proof uses the properties of the distance functions ρ_i as described above. Out of certain distance functions ρ_i^l, a system of equations will be constructed. The unknowns are g_i^{kl} with coefficients $\frac{\partial \rho_i^l}{\partial u_i^k} \frac{\partial \rho_i^l}{\partial u_i^l}$. To solve this system, the determinant of the coefficients has to be nonzero.

By the arguments above, each ρ_i^l is close in the $C^{1,\beta}$-(resp. $W^{2,q}$-) topology to a limit distance function on $(M, \mathbf{g})$. Therefore, by choosing Q in (3.2.3) sufficiently close to 1, for the limit distance function, g^{kl} is sufficiently close in the $C^{0,\beta}$-topology to δ^{kl}.

Hence for $B' \subseteq B$ sufficiently small, the distance functions ρ_i^l are close to the correspondingly defined Euclidean distance function on B', which gives nonsingularity of the matrix on B'. So it is possible to solve the system for each i, hence the $(g_i)_{kl}$ on $(M_i, \mathbf{g}_i)$ in the harmonic coordinates $\{u_i^k\}$ converge strongly in $W^{1,p}$ to limit $W^{1,p}$-functions, so also

$$(g_i)_{kl} \to g_{kl} \qquad \text{in } W^{1,p}. \tag{3.2.34}$$

□

For the complete detailed proof, see [3, Proposition 1.2]

The isometry statement and the contradiction are derived as in the $k \geqslant 2$-case.

Remark 3.2.28. Anderson [1] proves the result locally, the assumption of M being compact without boundary simplifies the proof.

In Remark 2.3. he states that if the L^p norm for $p > n/2$ of **Ric** is bounded, then the $C^{0,\alpha}$ norm of the metric is bounded ($\alpha \leqslant 2 - \frac{n}{p}$) and thus we get convergence in $C^{0,\alpha}$.

3.2.3. From harmonic radius bounds to convergence. We have shown in Theorem 3.2.18 that under the appropriate conditions on the curvature, we get a uniform bound on the $C^{k,\alpha}$ ($W^{k+1,p}$) harmonic radius of a manifold. Following [41] (see also [1] and [37, Proposition 12]), we will sketch how to get from the existence of a uniform bound on the harmonic radius to convergence of sequences of Riemannian manifolds.

Remark 3.2.29. The general topological concept of precompactness is given for our setup as follows: Let $n, k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\mathcal{C}$ a set of smooth n-dimensional compact Riemannian manifolds. The set $\mathcal{C}$ is called *precompact* in the $C^{k,\alpha}$-topology, if any sequence in $\mathcal{C}$ possesses a subsequence that converges in the $C^{k,\alpha}$-topology.

Proposition 3.2.30. [37, Proposition 12] *Let $(M_i, \mathbf{g}_i)$ be a sequence of smooth complete Riemannian manifolds of dimension n, that satisfy*

(1) $\mathbf{Ric}_{(M_i,\mathbf{g}_i)} \geqslant K$, *for all i, with $K \in \mathbb{R}$.*
(2) For some $r > 0$, any sequence $\{y_i\}$ of points in M_i possesses correspondingly harmonic charts $U_i : \Omega_i \colon B(0,r) \subseteq \mathbb{R}^n$ where Ω_i is an open neighborhood of y_i.
(3) Any such harmonic chart satisfies for $Q > 1$ and for any i, that

$$Q^{-1}\delta_{kl} \leqslant ((U_i^{-1})^* g_i)_{kl} \leqslant Q\delta_{kl}$$

(4) A subsequence of $((U_i^{-1})^* g_i)_{kl}$ converges either in $C^{k,\alpha}(B(0,r))$ or in $W^{k+1,p}(B(0,r))$, respectively.

Then there exists a complete $C^{k+1,\alpha}$ $(W^{k+2,p})$ manifold $(M, \mathbf{g})$, with a $C^{k,\alpha}$ $(W^{k+1,p})$ metric $\mathbf{g}$ and $x \in M$ such that for any domain $D \subset M$ that contains x, it is possible to find domains $D_i \subset M_i$, $x_i \in M_i$ and diffeomorphisms $\Phi_i \colon D \to D_i$ of class $C^{k+1,\alpha}$ $(W^{k+2,p})$ with

(1) $\lim_{i\to\infty} \Phi_i^{-1}(x_i) = x$,
(2) $(\phi_i^* \mathbf{g}_i) \to \mathbf{g}$ *in* $C^{k,\alpha'}$, $(\alpha' < \alpha)$ $(W^{k+1,p})$ *in any chart of the induced* $C^{k+1,\alpha}$ $(W^{k+2,p})$ *atlas of D.*

Before we will prove this result, we recall:

Proposition 3.2.31. *Let X be an arbitrary metric space. Then X obtains a finite ε-net, i.e., a maximal set of base points $X = \{x_1, \ldots, x_k\}$ such that $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \varnothing$.*

See [10, p. 278].

Remark 3.2.32. For $(M, \mathbf{g})$ a smooth complete Riemannian manifold of dimension n with

$$\mathbf{Ric}_{(M,\mathbf{g})} \geqslant K,$$

$K \in \mathbb{R}$, denote the set of base points as in Proposition 3.2.31 by X. Then each bounded set $D \subset M$, $\mathrm{diam} D = d$, that contains a point of X, can be covered by balls of radius ε around base points in $Y = \{x_1, \ldots, x_l\} \subset X$ $(l \leqslant k)$. The number l can be estimated by using the Bishop-Gromov volume comparison, and depends on n, K, d, ε.

Proposition 3.2.33. *Let $(M, \mathbf{g})$ be a in Remark 3.2.32. Then any bounded set $D \subseteq M$ can be embedded in $\mathbb{R}^N$ were N is determined by l. Furthermore, the image under this embedding is a submanifold in $\mathbb{R}^N$.*

Proof. Choose harmonic coordinates on each ball, $H_i \colon B(x_i, \varepsilon) \to \mathbb{R}^n$ $(i = 1, \ldots l)$. Let $\psi : \mathbb{R}^n \to [0, \infty)$ be a smooth cut-off function with $\psi(t) = 1$ on $B(0, r/2)$, $\psi(t) = 0$ outside $B(0, r)$. Let for $i = 1, \ldots l$, $\psi_i := \psi(H_i)$. These functions are smooth, with compact support.

To embed D in a fixed space $\mathbb{R}^N$, where $N = (n + 1)l$, define $\Psi_D \colon M \to \mathbb{R}^N$ as

$$\Psi_D(y) := (\psi_1(y)H_1(y), \ldots, \psi_l(y)H_l(y), \psi_1(y), \ldots, \psi_l(y)).$$

This map is smooth, any image is contained in a ball of fixed size, depending on n, l, r, which is guaranteed by the use of the cut-off function ψ.

We claim that when restricted to D, Ψ_D is a smooth embedding. Indeed, we first show that Ψ_D is injective. Take $y_1, y_2 \in D$ with $\Psi_D(y_1) = \Psi_D(y_2)$. Choose $k \in \{1, \dots l\}$ such that $\psi_k(y_1) = \psi_k(y_2) \neq 0$. Therefore y_1 and y_2 lie in the support of ψ_k. But then by definition of ψ_k, also $H_k(y_1) = H_k(y_2)$. Since H_k is bijective, we get that $y_1 = y_2$.

The function Ψ_D is also an immersion. To see this, choose $p \in D$, $X \in T_pD$. Then

$$\begin{aligned}(d\Psi_D)_p(X) = (X(\psi_1)H_1(p) + \psi_1(p)(dH_1)_p(X), \dots, X(\psi_l)H_l(p)\\ + \psi_l(p)(dH_l)_p(X), X(\psi_1), \dots, X(\psi_l)).\end{aligned}$$

If now $d\Psi_D = 0$, also $X(\psi_k) = 0$ for all k, and therefore also $\psi_k(p)(dH_k)_p(X) = 0$ for all k. Pick k such that $\psi_k(p) \neq 0$. Then $(dH_k)_p(X)$ has to vanish. Since H_k is a diffeomorphism, it follows that $X = 0$, hence $d\Psi_D$ is injective, thus Ψ_D is an immersion.

On D, Ψ_D is a homeomorphism, thus it is a smooth embedding.

To show that $\Psi_D(D)$ is actually a submanifold in $\mathbb{R}^N$, fix now $1 \leqslant k \leqslant l$, for simplicity let $k = 1$. Then $\Psi_D(H_k^{-1}(B(0, r/2))$ can be represented as a graph over $B(0, r/2) \subseteq \mathbb{R}^n$ by $\Psi_D(H_k^{-1}(x)) = \{(x, f_2F_2, \dots f_lF_l, f_1, \dots, f_l) : x \in B(0, r/2)\}$. The maps F_l and f_l are given by $F_l = H_l \circ H_1^{-1}$, and $f_l = \psi(|F_l|)$, which are smooth.

□

Proof of Proposition 3.2.30. Choose a sequence of pointed manifolds $(M_i, \mathbf{g}_i, x_i)$ that satisfy (1) – (4) . By Proposition 3.2.33, we can set $D = B(x_i, R)$, therefore we get, for arbitrary $R > 0$, for each i an embedding

$$\Psi_R^i \colon B(x_i, R) \to \mathbb{R}^{N(R)}.$$

Note that for $R' > R$, and for fixed i, Ψ_R^i is obtained from Ψ_R^i by the canonical embedding of $\mathbb{R}^{N(R)}$ into $\mathbb{R}^{N(R')}$.

By assumption (4), in any $B(0, r/2)$ a subsequence of each component $((H_m^i)^{-1})^*\mathbf{g}_i$ is bounded in $C^{k,\alpha}$, hence converges in $C^{k,\alpha'}$ $(W^{k+1,p})$, hence also the transition functions $H_m^i \circ (H_n^i)^{-1} =: H_{mn}^i$ converge in $C^{k+1,\alpha'}$ $(W^{k+2,p})$ to H_{mn}. We can perform the same procedure for each of the $m's$, for fixed R there are only finitely many possibilities. Thus by the definition of Ψ_R^i, we obtain a subsequence of the images of the embedding Ψ_R^i that converge in $C^{k+1,\alpha'}$ $(W^{k+1,p})$ as submanifolds of $\mathbb{R}^{N(R)}$ to a $C^{k+1,\alpha}$ $(W^{k+2,p})$ submanifold M_R in $\mathbb{R}^{N(R)}$.

We will now construct the point x. Let therefore $x := \lim_{i\to\infty} \Psi_R^i \circ (H_1^i)^{-1} = \lim_{i\to\infty} \Psi_R^i(x_i)$, $M_R^i := \Psi_R^i(B(x_i, R)) \subseteq \mathbb{R}^{N(R)}$.

Define a metric $\bar{\mathbf{g}}_i := ((\Psi_R^i)^{-1})^*\mathbf{g}_i$ on $\mathbb{R}^{N(R)}$.

Next we construct metrics that converge in $C^{k+1,\alpha}$ $(W^{k+2,p})$ to a limiting metric $\mathbf{g}$ on M_R. Therefore, observe that the projection $\Pi_i \colon M_R^i \to M_R$ induces a $C^{k+1,\alpha}$ $(W^{k+2,p})$ diffeomorphism from $M_R^i \to M_R$.

Consider the metrics

$$((\Pi_i)^{-1})^*\bar{\mathbf{g}}_i = ((\Pi_i)^{-1})^*((\Psi_R^i)^{-1})^*\mathbf{g}_i = ((\Pi_i \circ \Psi_R^i)^{-1})^*\mathbf{g}_i.$$

If necessary passing to a subsequence, these metrics converge to a $C^{k,\alpha}$ ($W^{k+1,p}$) metric $\mathbf{g}$ on M_R by the convergence properties of Ψ_R^i and noting that $\Pi_i \to \mathrm{Id}$.

If $R_j \to \infty$ are increasing numbers, we can use a diagonal sequence argument and obtain a $C^{k+1,\alpha}$ ($W^{k+2,p}$) limit manifold M as an increasing union of M_{R_j} with a $C^{k,\alpha}$ ($W^{k+1,p}$) metric $\mathbf{g}$.

To obtain that the manifold is complete, observe that M is constructed to be an exhaustion of closed and bounded domains that are compact thus, using Hopf-Rinow's theorem, we get the result.

Note that the diffeomorphisms Φ_i are given by $(\Pi_i \circ \Psi_R^i)^{-1}$. □

REMARK 3.2.34. Kasue [41] does not obtain completeness of the limiting manifold, since in his proof he does not work with $D_i \subset M_i$, but with M_i itself, which are assumed to be compact.

Using this proposition, we are in the position to prove the following (pre)compactness theorem [37, 1, 3]:

THEOREM 3.2.35. [37, Main Theorem] *Let $n \in \mathbb{N}$, $K \in \mathbb{R}$, $i, v > 0$. The space of $n-$dimensinal Riemannian manifolds $(M, \mathbf{g})$ with*

$$\mathrm{inj}_{(M,\mathbf{g})}(x) \geqslant i, \qquad \mathrm{Vol}_{(M,\mathbf{g})} \leqslant v \tag{3.2.35}$$

and

$$\mathbf{Ric}_{(M,\mathbf{g})}(x) \geqslant K \tag{3.2.36}$$

is precompact in the $C^{0,\alpha}$-topology for any $\alpha \in (0,1)$.

If, instead of (3.2.36) we have for $k \in \mathbb{N}$, and constants $C(j) > 0$ ($j = 0, \ldots, k$)

$$|D^j \mathbf{Ric}_{(M,\mathbf{g})}(x)| \leqslant C(j), \tag{3.2.37}$$

then it is precompact in the $C^{k+1,\alpha}$-topology for any $\alpha \in (0,1)$.

REMARK 3.2.36. By using [20, Proposition 15], the bounds (3.2.35) are equivalent to a diameter bound $\mathrm{diam} M \leqslant d$.

Proof. Let $(M_i, \mathbf{g}_i)$ be a sequence of manifolds that satisfy (3.2.36) or (3.2.37) with $\lambda, i, v, k, C(j)$ independent of i.

We will show that under this bounds the manifolds satisfy the conditions of Proposition 3.2.30.

In both cases, condition (1) of Proposition 3.2.30 is satisfied. By Theorem 3.2.18, we obtain harmonic coordinates on balls of size which is uniformly bounded below, which therefore gives (2) and (3).

In order to obtain (4), note that the sequences $((U_i^{-1})^* g_i)_{kl}$ are bounded in $C^{0,\alpha}$, respectively, in $C^{k+1,\alpha}$. We can therefore use the Arzela-Ascoli theorem to get a subsequence that converges in $C^{0,\alpha'}$, respectively, in $C^{k+1,\alpha'}$, $(\alpha' < \alpha)$.

By Remark 3.2.36, we obtain an upper bound d on the diameter of the manifolds M_i which is independent of i.

Now we are in the position to apply (3.2.30) with $D = B(x, r)$, $R > d$. We can choose $D_i = M_i$ for i large, and therefore get diffeomorphisms $\Phi_i \colon M \to M_i$ with $\Phi_i^* \mathbf{g}_i \to \mathbf{g}$ in $C^{0,\alpha'}$, or $C^{k+1,\alpha'}$ respectively. □

3.3. Alternative approach via harmonic norm

In [58] the author used a different concept to the harmonic radius to prove precompactness results. Indeed, he works with the $C^{k,\alpha}$-norm of a Riemannian manifold:

Definition 3.3.1. The $C^{k,\alpha}$-norm of an n-dimensional Riemannian manifold $(M, \mathbf{g})$ on scale $r > 0$, $\|(M, \mathbf{g})\|_{C^{k,\alpha},r}$ is the infimum over the positive numbers Q that satisfy the following properties $(s, t > 0)$:

(1) There exist embeddings
$$\phi_t : B(0, r) \subseteq \mathbb{R}^n \to U_t \subseteq M,$$
(2) $|D\phi_t^{-1}| \leqslant e^Q$, and $|D\phi_t| \leqslant e^Q$ on $B(0, r)$
(3) every ball $B(p, r/10e^{-Q})$, $p \in M$, lies in some U_t,
(4) $r^{|l|+\alpha} \|\partial^l \phi_t^* \mathbf{g}\|_{0,\alpha} \leqslant Q \qquad$ for $0 \leqslant |l| \leqslant k$,
(5) the transition functions satisfy $\|\phi_s^{-1} \circ \phi_t\|_{k+1,\alpha} \leqslant (10 + r)Q$ on the domain where they are defined.

Analogously, the $W^{k,p}$ norm of scale r of $(M, \mathbf{g})$, $\|(M, \mathbf{g})\|_{k,p;r}$ is defined to be the infimum over $Q \geqslant 0$ such that there are charts $\phi_t : B(0, r) \subseteq \mathbb{R}^n \to U_t \subseteq M$ with

(1) $|D\phi_t^{-1}| \leqslant e^Q$, and $|d\phi_t| \leqslant e^Q$ on $B(0, r)$
(2) every ball $B(p, r/10e^{-Q})$, $p \in M$, lies in some U_t,
(3) $r^{|j|-n/p} \|\partial^j \mathbf{g}\|_{L^p} \leqslant Q \qquad$ for $0 \leqslant |j| \leqslant k$,
(4) $\phi_t^{-1} : U_t \to B(0, r)$ is harmonic.

In contrast to the harmonic radius, where r is maximised for fixed Q, in the norm-concept Q is minimised for fixed r. As it is shown in [58, Proposition 2.1(iii), Proposition 4.4(iv)], the latter leads to better continuity properties. Whereas for the harmonic radius we only get $r_H(Q_i, k, \alpha)(M_i) \to r_H(Q, k, \alpha)(M)$ for $Q_i \downarrow Q$ by Proposition 3.2.16, the norm gives (both in the $C^{k,\alpha}$ and in the $W^{k,p}$ case) that if $(M_i, \mathbf{g}_i, x_i) \to (M, \mathbf{g}, x)$ in the pointed $C^{k,\alpha}$ (resp. $W^{k,p}$) topology, then for each bounded $B \subset M$ there are bounded sets $B_i \subset M_i$ such that

$$\|(B_i, \mathbf{g}_i)\|_{\alpha;r} \to \|(B, \mathbf{g})\|_{\alpha;r} \qquad \|(B, \mathbf{g}_i)\|_{k,p;r} \to \|(B, \mathbf{g})\|_{k,p;r} \tag{3.3.1}$$

If the manifold is closed, the norm is always finite for all r. In case $M = \mathbb{R}^n$, it vanishes for all r,

3.4. Historic overview

Gromov and Cheeger, [33, 13] were the first to prove a finiteness/convergence theorem for spaces of Riemannian manifolds. Gromov's proof, where details missing, was later on worked out in all detail by Katsuda in [42].

Their result was later on improved independently by Greene-Wu [32] and Peters [56]. The authors show that in the space of compact connected $n-$dimensional Riemannian manifolds with bounded sectional curvature an upper bound on the diameter and a lower bound on the volume, any sequence has a subsequence that converges in the Lipschitz topology to a limit manifold $(M, \mathbf{g})$. The metric $\mathbf{g}$ is of $C^{1,\alpha}$ regularity. Both proofs make use of harmonic coordinates. Peters is able to show that in this space of manifolds, the Hausdorff distance is equivalent to the Lipschitz distance. He uses the same techniques as in [55] to obtain the convergence result.

The first result in which the pointwise bound on the curvature is weakened to an integral bound is due to Gao. In [25] the author weakens the sectional curvature assumption to an $L^{n/2}$ bound on the curvature tensor. Additionally he assumes a pointwise Ricci bound, a lower injectivity radius bound and a upper bound on the diameter. He is then able to show that away from a finite number of points $\{m_1, \dots m_k\}$, a sequence of manifolds satisfying these bounds converges to a manifold $(M, \mathbf{g})$, where $\mathbf{g}$ is a $C^{1,\alpha}$ metric on $M \backslash \{m_1, \dots m_k\}$.
He furthermore shows the following: Let $(M_i, \mathbf{g}_i)$ be a sequence of closed connected Riemannian manifolds that satisfy $|\mathbf{Ric}_{\mathbf{g}_i}| \leqslant H$, $\mathrm{diam}(M_i) \leqslant D$ and $\mathrm{Vol}(M_i) \geqslant V$. If

	Author	Bounds	Convergence
[3]	Anderson, Cheeger	$\mathbf{Ric} \geqslant -\lambda$, $\mathrm{inj} \geqslant i_0$, $\mathrm{Vol} \leqslant V$	in $C^{0,\alpha'}$ to $\mathbf{g} \in C^{0,\alpha}$, $(\alpha' < \alpha)$.
[2]	Anderson	$\lvert\mathbf{Ric}\rvert \leqslant \lambda$, $\mathrm{inj} \geqslant i_0$, $\mathrm{diam} \leqslant D$	$\mathbf{g} \in C^{1,\alpha}$
[33, 42]	Gromov	$\lvert K\rvert \leqslant \Lambda^2$, $\mathrm{diam} < d$, $\mathrm{Vol} \geqslant V_0$	in Lipschitz topology to $\mathbf{g} \in C^0$, $d_{\mathbf{g}} \in C^{1,1}$
[32]	Greene, Wu	$\lvert K\rvert \leqslant \Lambda^2$, $\mathrm{diam} < d$, $\mathrm{Vol} \geqslant V_0$	$\mathbf{g} \in C^{1,\alpha}$
[37]	Hebey, Herzlich	see 3.2.18	$\mathbf{g} \in C^{k,\alpha}$
[41]	Kasue	$\lvert K\rvert \leqslant \Lambda^2$, $\mathrm{diam} \leqslant d$. $\mathrm{inj} \geqslant I_0$	in $C^{1,\alpha'}$ to $\mathbf{g} \in C^{1,\alpha}$, $0 < \alpha' < \alpha < 1$
[55, 56]	Peters	$\lvert K\rvert \leqslant \Lambda^2$, $\mathrm{diam} < d$, $\mathrm{Vol} \geqslant V$	in Lipschitz topology to $\mathbf{g} \in C^{1,\alpha}$
[13]	Cheeger	$\lvert K\rvert \leqslant \Lambda^2$, $\mathrm{diam} \leqslant d$, $\mathrm{Vol} \geqslant V$	Finiteness statement
[69, 70]	Yang	(3.1.1)	$\mathbf{g} \in C^0$ (M a C^1-manifold)

TABLE 1. Important convergence results

in addition there exists a small constant κ depending on H, D and V such that for fixed $\rho > 0$, $\int_{B(x,\rho)} |R(\mathbf{g}_i)|^{n/2} d\mathrm{Vol}_{\mathbf{g}_i} \leqslant \kappa$, then a subsequence converges to a Riemannian manifold $(M, \mathbf{g})$ with $\mathbf{g}$ a $C^{1,\alpha}$ metric. In addition, there are diffeomorphisms $\Phi_i : M \to M_i$ such that $\Phi_i^* \mathbf{g}_i \to \mathbf{g}$ in $C^{1,\alpha}$.

In proving this result, Gao covers the manifolds by harmonic balls (i.e. balls on which one has harmonic coordinates) of controllable size. Since, due to Jost-Karcher [40] harmonic coordinates exist in case of a pointwise bound on the curvature and a lower injectivity radius bound, Gao's main technical achievement is that he manages to prove the existence of harmonic coordinates under just an $L^{n/2}$ bound on the full curvature, a pointwise Ricci bound and a lower injectivity radius bound. This proof can be divided in two parts: first, he shows that if a ball in M_i, $B_i(x,1) \subseteq M_i$ with $|\mathbf{Ric}_{(M_i,\mathbf{g}_i)}| \leqslant H$, $\mathrm{inj}_{(M_i,\mathbf{g}_i)} \geqslant 4$ and $\int_{B(x,\rho)} |R(\mathbf{g}_i)|^{n/2} d\mathrm{Vol}_{\mathbf{g}_i} \to 0$, then the ball can, by using geodesic coordinates $\{y_i^k\}$, be identified with the Euclidean ball and the metrics $\mathbf{g}_i$ converge to the Euclidean metric δ on $B(0,1) \subset \mathbb{R}^n$ in $L^{n/2}$ norm.

In the second step, Gao solves the Dirichlet problem $\Delta F_i = 0$ on $B(0,1)$, $F_i = y_i$ on $\partial B(0,1)$. Via L^p estimates for elliptic equations, Gao shows that the metric tensor in the harmonic coordinates F_i has $C^{1,\alpha}$ regularity. Details of how the harmonic coordinates are constructed can be found in [25, § 3].

3.5. L^p bounds on the curvature

In contrast to Anderson and Anderson-Cheeger, who used pointwise bounds on curvature, we will now investigate manifolds that have just an integral bound on the curvature. We will recall results that state under which bounds it is possible to get compactness and convergence results.

3.5.1. Anderson. The proof of the result by Anderson [1], 3.2.35, remains true when the pointwise bound on $\mathbf{Ric}$ is replaced by an L^p bound $\|\mathbf{Ric}\|_{L^p}$, with the regularity of $\mathbf{g}$ reduced to $C^{0,\alpha}$ for $\alpha \leqslant 2 - \frac{n}{p}$. Indeed, in the proof of the $C^{1,\alpha}$ harmonic radius bound [1, Lemma 2.2], the pointwise bound on $\mathbf{Ric}$ is replaced by $\|\mathbf{Ric}\|_{L^p}$, and $C^{1,\alpha}$ by $C^{0,\alpha}$.

Analogously to the proof of the convergence result in Theorem 3.2.35, we obtain a sequence of manifolds $(M_i, \mathbf{h}_i, x_i)$, where the $\mathbf{h}_i$ are uniformly bounded in $C^{0,\alpha}$, and therefore a subsequence converges in $C^{0,\alpha'}$ to a $C^{0,\alpha}$ limit $(M, \mathbf{g}, x)$ $(\alpha' < \alpha)$.

By the Sobolev embedding result, $C^{0,\alpha} \supseteq W^{2,p}$ for $\alpha \leqslant 2 - \frac{n}{p}$. Hence we will show that $(M_i, \mathbf{h}_i, x_i)$ converge in $W^{2,p}$. As above, write the Ricci tensor in coordinates

$$-2(\mathbf{Ric}_{(M_i,\mathbf{h}_i)})_{kl} - A(\mathbf{h}_i) = h_i^{rs} \frac{\partial^2 h_{ikl}}{\partial x^r \partial x^s}. \tag{3.5.1}$$

where $A(\mathbf{h}_i)$ is quadratic in first derivatives of $\mathbf{h}_i$.

By assumption $\|\mathbf{Ric}_{(M_i,\mathbf{h}_i)}\|_{L^p} \to 0$. The metrics $\mathbf{h}_i$ are uniformly bounded in $W^{2,p}$, so we can assume that $\mathbf{h}_i \to \mathbf{g}$ in $W^{1,2p}$ (for $p < n$). Indeed, for $p < n$, $\frac{1}{2p} > \frac{1}{p} - \frac{1}{n}$, and so by Sobolev embedding $W^{2,p} \subseteq W^{1,2p}$ Therefore $\mathbf{g}$ has a well defined Laplacian $\Delta_{\mathbf{g}}$ and

$\Delta_{\mathbf{h}_i} \to \Delta_{\mathbf{g}}$ in $W^{1,2p}$. For the $p > n$ case, since $2 - n/p > 1$, $\mathbf{g}$ is actually in $C^{1,\beta}$ for $\beta = 1 - n/p$ and therefore $\mathbf{h}_i \to \mathbf{g}$ in $C^{1,\beta}$.

Furthermore, $A(\mathbf{h}_i) \to A(\mathbf{g})$ in L^p, since the first derivatives converge in L^{2p} and A is quadratic in those.

Therefore we can conclude that ${h_i}^{rs} \frac{\partial^2 \mathbf{h}_{ikl}}{\partial x^r \partial x^s} \to g^{rs} \frac{\partial^2 g_{kl}}{\partial x^r \partial x^s}$, in L^p, or, in other words, $\Delta_{\mathbf{h}_i} \mathbf{h}_i \to \Delta_{\mathbf{g}} \mathbf{g}$ in L^p.

Using elliptic estimates, see A.4, we can conclude

$$\begin{aligned} \|(h_i)_{rs} - g_{rs}\|_{W^{2,p}} &\leqslant C\|\Delta_i((h_i)_{rs} - g_{rs})\|_{L^p} + C\|(h_i)_{rs} - g_{rs}\|_{L^p} \\ &\leqslant C\left(\|\Delta_i((h_i)_{rs}) - \Delta(g_{rs})\|_{L^p} + \|(\Delta - \Delta_i) g_{rs}\|_{L^p} + \|(h_i)_{rs} - g_{rs}\|_{L^p}\right). \end{aligned}$$

By the previous observations, this expression converges to zero for $i \to \infty$.

Finally we want to show that $\mathbf{g} \in C^\infty$.

Indeed, the limit metric $\mathbf{g}$ satisfies $-A(\mathbf{g}) = g^{rs} \frac{\partial^2 g_{kl}}{\partial x^r \partial x^s}$. In case $p > n$, observe that $2 - n/p > 1$ so $\mathbf{g}$ is actually in $C^{1,\beta}$ for $\beta = 1 - n/p$. Therefore the highest order coefficients of Δ are in C^1, and $A(\mathbf{g})$ is in $C^{0,\beta}$. Using standard elliptic theory, we see that $\mathbf{g} \in C^{3,\beta}$, and bootstrapping the argument leads to $\mathbf{g} \in C^\infty$. If $n/2 < p < n$, define $q = (2/p - 2/n)^{-1}$. Since $\mathbf{g} \in W^{2,p}$ it follows that $\mathbf{g} \in W^{1,2q} \cap C^{2-n/p}$, therefore $A(\mathbf{g}) \in L^q$. Hence also $\Delta \mathbf{g} \in L^q$ and $\mathbf{g} \in W^{2,q}$. Iterating this procedure, we get $\mathbf{g} \in W^{Q,p}$ with $Q > p + k(p - n/2)$. If k is big enough, $q > n$ and we get $\mathbf{g} \in C^{1,\beta}$, $\beta = 1 - n/Q$. Therefore we carry out the $p > n$ procedure to get $\mathbf{g} \in C^\infty$ also in this case.

The remaining part of the proof works analogously to the proof for a pointwise Ricci bound as in Theorem 3.2.18, i.e. one has to show that $(M, \mathbf{g})$ is isomorphic to $(\mathbb{R}^n, \delta)$ and then to obtain a contradiction.

It is also possible to assume that the derivatives up to some order j of Ricci are bounded in L^p, $p > n/2$. By using that in his case $\|\mathbf{Ric}_{(M_i, \mathbf{h}_i)}\|_{W^{k,p}} \to 0$, we can control the $W^{k+2,p}$ harmonic radius.

3.5.2. Yang. In [70], the author studies manifolds that satisfy (3.1.1) and is able to improve the previously mentioned Gromov-Hausdorff convergence result to the following theorem, [70, Theorem 3.1.]:

THEOREM 3.5.1. *Let $n \geqslant 3$, $p > n/2$, $0 < \eta < 1$. Then there exist constants $\varepsilon(n), \kappa(n, p, \eta) > 0$, such that the following holds:*

Let $M_1, M_2, \ldots$ be a sequence of complete Riemannian manifolds, such that, for open subsets $\Omega_i \subseteq M_i$ $D, \rho > 0, K \geqslant 0$, they satisfy

$$\begin{cases} V(x,\rho) \geqslant \eta^n n^{-1} \omega \rho^n & \text{for all } x \in \Omega_i, \\ \operatorname{diam}(\Omega_i) < D, \\ \|\mathbf{R}\|_{L^{n/2}(B(x,\rho))} \leqslant \varepsilon(n) \eta^{2(n+1)} & \text{for all } x \in \Omega_i, \\ \rho^{2-n/p} \|\mathbf{Ric}\|_{L^p(\Omega_i)} \leqslant \kappa(n,p,\eta)^2. \end{cases} \tag{3.5.2}$$

If for $\varepsilon > 0$, $\Omega_{i,\varepsilon}$, the ε-neighborhood of Ω_i, satisfies $\operatorname{Vol}\Omega_{i,\varepsilon} > v$ for some $v > 0$, then there exists a subsequence, also denoted by $\Omega_{i,\varepsilon}$ and diffeomorphisms $\Phi_i : \Omega \to \Omega_{i,\varepsilon}$ such that $\Phi_i^*\mathbf{g}_i \to \mathbf{g}$ uniformly. For any $x \in \Omega$, the Φ_i can be defined such that locally $\Phi_i^*\mathbf{g}_i \to \mathbf{g}$ weakly in $W^{2,p}$.

Yang shows that under the bounds (3.5.2) uniformly bounded harmonic coordinates exist. Indeed, he first solves the local Ricci flow on $B(x, 2\rho)$ to obtain a family of smooth metrics $\mathbf{g}(t)$, $0 \leqslant t \leqslant T$ on $B(x,\rho)$ that satisfy (3.1.3).

Using [69, Theorem 9.1.] he obtains a value for T, and also that $\|\mathbf{R}(\mathbf{g}(T))\|_{L^\infty}$ is bounded. Therefore, it is possible to use the results by Jost-Karcher to obtain harmonic coordinates $h^1(T), \dots, h^n(T)$ on $B(x,\rho)$ for $\mathbf{g}(T)$. Yang then shows that $h^1(t), \dots, h^n(t)$ remain harmonic coordinates for $0 \leqslant t \leqslant T$, so in particular we get harmonic coordinates for $\mathbf{g}(0) = \mathbf{g}$.

By using the volume estimates of Section 7 in [69], each manifold in the sequence can be covered by a fixed number of harmonic balls. Furthermore, we have L^p-control over the curvature on the entire manifold and also a global Sobolev inequality (see Section 4 in Yang's paper). Hence applying [69, Theorem 12.1.] yields the result.

For the $W^{2,p}$ statement, define Φ_i to have a fixed set of coordinates on $B(x,\rho)$ that are harmonic for $\Phi_i^*\mathbf{g}_i$.

3.5.3. Hiroshima. In [38] the author shows that it is possible to obtain a bound on the $W^{1,p}$ harmonic radius under weaker assumptions than in [3]. Indeed, he shows that if the pointwise bound on **Ric** is weakened to an L^p bound on λ, the lowest non-positive Eigenvalue of **Ric**, Anderson and Cheeger's result remains true without assuming a volume bound.

Hiroshima uses in his proof the results by Yang [69], Gallot [24] and [20] to obtain bounds on the volume of balls. Furthermore he obtains L^p estimates on the Laplacian of the distance function. He then proceeds as in [3] to get, by a contradiction argument, a bound on the harmonic radius. Hiroshima does not use the function $s_M(x) := \sup_r\{\min(r, \inf_{y\in B(x,r)} \operatorname{inj}(y))\}$, but a similar L^p version

$$\kappa(M) := \operatorname{inj}(M)^{p-n} \sup_{x\in M} \kappa(x, \operatorname{inj}(M)),$$

where $\kappa(x,R) = R^{p-n}\int_{B_x(R)} \lambda^p d\mathrm{Vol}$.

3.5.4. Petersen-Wei. The authors show in [59, Theorem 1.4.], that the class of closed $n-$dimensional Riemannian manifolds with

$$\begin{cases} \operatorname{Vol}(M) \geqslant v > 0, \\ \operatorname{diam}(M) \leqslant D < \infty \\ \|\mathbf{R}\|_{L^p} \leqslant \Lambda < \infty, \\ \|\mathbf{Ric}_-\|_{L^p} < \varepsilon \end{cases} \tag{3.5.3}$$

is precompact in the $C^{0,\alpha}$ topology, $\alpha < 2 - \frac{n}{p}$.

The proof uses [57, Theorem 5.1, Theorem 5.4], which state that the class of manifolds with

$$\begin{aligned} V(x,R) \geqslant vr^n > 0 \qquad r \leqslant \tfrac{D}{2} \\ \operatorname{diam}(M) \leqslant D, \\ \|\mathbf{R}\|_{L^p} \leqslant \Lambda, \end{aligned}$$

is precompact in the $C^{0,\alpha}$ topology, $\alpha < 2 - \frac{n}{p}$.

Therefore it remains to show that the volume growth condition is implied by the bounds $\operatorname{Vol}(M) \geqslant v$, $\operatorname{diam}(M) \leqslant D$ and $\|\mathbf{Ric}\|_{L^p} < \varepsilon$. Indeed, from the relative volume comparison (3.1.4), we obtain

$$c\frac{V_K(r)}{V_K(D/2)} \leqslant \frac{V(x,r)}{\operatorname{Vol}(M)}. \tag{3.5.4}$$

Since $\operatorname{Vol}(M) \geqslant V$, we deduce that for $r \leqslant D/2$,

$$\begin{aligned} V(x,r) &\geqslant cV_K(r)\frac{v}{V_K(D)} \\ &\geqslant c(n,p,K,v,D)r^n. \end{aligned}$$

3.5.5. $M^p_{n-1}(M)$ bound. If we further assume that the full curvature $\mathbf{R}$ is bounded in L^p for $p > \frac{n}{2}$ we get the following:

Proposition 3.5.2. *Let $n \geqslant 2$, $p > \frac{n}{2}$, $v > 0$, $D < \infty$, $\Lambda < \infty$. Then there exists $K = K(n,p,\Lambda,D) > 0$ such that the class of closed Riemannian manifolds with*

$$\operatorname{Vol} M \geqslant v \tag{3.5.5a}$$

$$\operatorname{diam} M \leqslant D \tag{3.5.5b}$$

$$\|\mathbf{R}\|_{L^p(M)} \leqslant \Lambda \tag{3.5.5c}$$

$$\sup_{x\in M}\sup_{t>0}\left(\fint_{B(x,t)} R_-(x,y)^{n-1}\,d\mu_{\mathbf{g}}(y)\right)^{1/(n-1)} \leqslant K \tag{3.5.5d}$$

is precompact in the $C^{0,\alpha}$ topology for $\alpha < 2 - \frac{n}{p}$.

In order to prove this theorem, we need the following result due to Petersen [57, Theorem. 5.4], see also [2, 69]:

Proposition 3.5.3. *Let $v, r_0, D, \Lambda \in (0,\infty)$. The class of manifolds that satisfy, for all $x \in M$,*

$$V(x,r) \geqslant vr^n \qquad \text{for } r \leqslant r_0 \tag{3.5.6a}$$

$$\operatorname{diam} M \leqslant D \tag{3.5.6b}$$

$$\|\mathbf{R}\|_{L^p(M)} \leqslant \Lambda \tag{3.5.6c}$$

is precompact in the $C^{0,\alpha}$ topology for $\alpha < 2 - \frac{n}{p}$.

Proof of Proposition 3.5.2. Using Proposition 3.5.3, we need to show that we get the local bound $V(x,r) \geqslant vr^n$ from the bounds (3.5.5). Indeed, from (3.1.5) we have that, for all $x \in M$ and all $r \leqslant D$, and for $\alpha < 1$, we can find $K = -\frac{\log(\alpha)}{(n-1)D}$, such that

$$\begin{aligned}\frac{\operatorname{Vol} M}{V_0(D)} &\leqslant \frac{V(x,r)}{V_0(r)} \exp\left[(n-1)\int_r^D \left\{ \fint_0^s \left(\fint_{B(x,t)} R_-(x,y)^{n-1}\, d\mu_{\mathbf{g}}(y)\right)^{1/(n-1)} dt \right\} ds \right] \\ &\leqslant \frac{V(x,r)}{V_0(r)} \exp\left[(n-1)KD\right]\end{aligned}$$

As in Proposition (3.1.7), observe that the integral on the right hand side is positive, thus

$$f(n,D) = \exp\left[(n-1)\int_{r_0}^D \left\{ \fint_0^s \left(\fint_{B(x,t)} R_-^{n-1}\, d\mu_{\mathbf{g}}\right)^{1/(n-1)} dt \right\} ds \right] \geqslant 1.$$

From there, it follows that

$$\frac{1}{f}\frac{\operatorname{Vol}(M)}{V_0(D)} \leqslant \frac{V(x,r_0)}{V_0(r_0)} \qquad \forall r \leqslant D.$$

Therefore,

$$\begin{aligned}V(x,r) &\geqslant \frac{1}{f}\operatorname{Vol}(M)\frac{V(x,r)}{V_0(D)} \\ &= \frac{1}{f}\operatorname{Vol}(M)(\frac{r}{D})^n,\end{aligned}$$

which is the anticipated estimate $V(x,r) \geqslant vr^n$. Applying the previously mentioned result, the precompactness statement follows. $\square$

APPENDIX A

A.1. Notions of curvature

In this section we will mainly follow [27, 65]. Let $(M, \mathbf{g})$ be a Riemannian manifold with connection ∇. The *curvature tensor* $R \in \mathcal{T}_3^1(M)$ is given by :

$$R(X,Y)Z = \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z \qquad \forall\ X, Y, Z \in \mathfrak{X}(M). \tag{A.1.1}$$

Here $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M.

Alternatively, one can define a $(0,4)$ version of the curvature tensor, also denoted as $R \in \mathcal{T}_4^0(M)$, by

$$R(W,X,Y,Z) = \mathbf{g}(R(W,X)Z,Y). \tag{A.1.2}$$

Locally, in coordinates $(x^1, \dots, x^n)$ we get

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} = \sum_l R^l_{kij} \frac{\partial}{\partial x^l}, \tag{A.1.3}$$

and

$$R_{ijkl} = \sum_m g_{ml} R^m_{kij}. \tag{A.1.4}$$

The coefficients R^l_{kij} can be expressed in terms of the Christoffel symbols, by

$$R^l_{ijk} = \partial_k \Gamma^l_{ji} - \partial_j \Gamma^l_{ki} + \sum_m \Gamma^l_{km} \Gamma^m_{ji} - \sum_m \Gamma^l_{jm} \Gamma^m_{ki}. \tag{A.1.5}$$

Proposition A.1.1. *The curvature tensor has the following properties, for $X, Y, Z, W \in \mathfrak{X}(M)$:*

(1) $R(X,Y)Z = -R(Y,X)Z$
(2) (First Bianchi identity) $R(X,Y)Z + R(Y,Z)X + +R(Z,X)Y = 0$
(3) $R(XY, Z, W) = -R(X,Y,W,Z)$
(4) $R(X,Y,Z,W) = R(Z,W,X,Y)$

Given a point $p \in M$, and a two-dimensional plane $\sigma \subseteq T_pM$, we define the *sectional curvature* of σ at p to be

$$K_p(\sigma) := \frac{\mathbf{R}_p(X,Y,X,Y)}{\mathbf{g}_p(X,X)\mathbf{g}_p(Y,Y) - \mathbf{g}_p(X,Y)^2}, \tag{A.1.6}$$

where $X, Y \in T_pM$ are two vectors at p that span the plane σ.

$K_p(\sigma)$ depends only on the plane $\sigma \subseteq T_pM$, and is independent of the vectors X, Y chosen to span σ.
It can be shown that K_p determines R_p. Thus, the knowledge of all the sectional curvatures determines the curvature tensor.
A Riemannian manifold is said to have constant (or negative, positive) curvature, if its sectional curvature is constant (negative, positive) at all points $p \in M$.

We also define the *Ricci curvature tensor* as the symmetric $(0, 2)$ tensor field on M given by $\mathbf{Ric}(X, Y) := \operatorname{tr}\,[Z \to R(Z, X)Y]$. Hence, for $p \in M$, $x, y \in T_pM$, the Ricci tensor is defined with respect to any orthonormal basis $(e_1, \dots e_n)$ of T_pM by

$$\mathbf{Ric}_p(x, y) = \sum_{i=1}^{n} \mathbf{g}(e_i, R(e_i, x)y). \tag{A.1.7}$$

The *scalar curvature* s, is defined to be the trace of $\mathbf{Ric}$, i.e.,

$$s(p) = \sum_{i=1}^{n} \mathbf{Ric}_p(e_i, e_i). \tag{A.1.8}$$

Furthermore, the *mean curvature* of a geodesic sphere at p with outer normal N is given by

$$h(r) = \sum_{i=1}^{n-1} \mathbf{g}(\nabla_{e_i} N, e_i),$$

where $\{e_1, \dots e_{n-1}\}$ is an orthonormal basis for the geodesic sphere.

A.2. Conformal transformations

Definition A.2.1. Let $u\colon M \to \mathbb{R}$ be a smooth function, $\mathbf{g}$ a Riemannian metric. The metric $\bar{\mathbf{g}} := e^{-2u}\mathbf{g}$ is called *conformal* to $\mathbf{g}$.

An operation which will simplify the calculations below, is the Kulkarni-Nomizu product:

Definition A.2.2. Let A, B be two symmetric $(0, 2)$-tensors. The *Kulkarni-Nomizu product of A and B, $A \odot B$*, is defined as the $(0, 4)$ tensor

$$\begin{aligned} A \odot B(X, Y, Z, W) := {} & A(X, Z)B(Y, W) - A(Y, Z)B(X, W) \\ & -A(X, W)B(Y, Z) + A(Y, W)B(X, Z). \end{aligned}$$

In the following proposition we will investigate how the curvature of $\bar{\mathbf{g}}$ can be expressed in terms of the original metric $\mathbf{g}$. The quantities without a bar come from $\mathbf{g}$, those with a bar are generated by $\bar{\mathbf{g}}$.

Proposition A.2.3. *Let $(M, \mathbf{g})$ be a Riemannian manifold, $\bar{\mathbf{g}} = e^{-2u}\mathbf{g}$. Then*

(1) *the Christoffel symbols transform as*

$$\bar{\Gamma}^i{}_{jk} = g^{il}\left(-\partial_j(u)g_{lk} - \partial_k(u)g_{lj} + \partial_l(u)g_{jk}\right) + \Gamma^i{}_{jk}, \tag{A.2.1}$$

(2) *the* $(1,3)$ *curvature as*

$$\bar{R}^l_{ijk} = g^{lp}\left((\nabla^2 u + du \otimes du - \frac{1}{2}|\nabla u|^2) \odot g\right)_{ijpk} + R^l_{ijk}, \tag{A.2.2}$$

(3) *for the Ricci curvature we get*

$$\overline{\mathbf{Ric}} = (n-2)\left(\nabla^2 u + \frac{1}{n-2}(\Delta u)\mathbf{g} du \otimes du - |\nabla u|^2 \mathbf{g}\right) + \mathbf{Ric}, \tag{A.2.3}$$

(4) *and for the scalar curvature*

$$\bar{s} = e^{2u}\left(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2 + s\right). \tag{A.2.4}$$

The proofs of these formulae are straightforward, but consist of rather lengthy calculations. Therefore we will omit them, details can be found in the lecture notes [65, Lecture 20].

For the calculations relevant to us, we will use a different conformal factor, namely $\bar{\mathbf{g}} = v^{\frac{4}{n-2}}\mathbf{g}$.

Proposition A.2.4. *Let* v *be smooth,* $v > 0$*. Let* $\bar{\mathbf{g}} = v^{\frac{4}{n-2}}\mathbf{g}$*. For the scalar curvature of* $\bar{\mathbf{g}}$ *it holds that*

$$-4\frac{n-1}{n-2}\Delta v + Rv = \bar{R}v^{\frac{n+2}{n-2}}. \tag{A.2.5}$$

Proof. Since $e^{-2u} = v^{\frac{4}{n-2}}$, it follows that

$$u = -\frac{2}{n-2}\ln v.$$

Inserting this in (A.2.4), gives the result. □

A.3. Sobolev spaces

We provide the definitions in $\mathbb{R}^n$ following [26], for a generalization to arbitrary manifolds, see, eg. [4].

A.3.1. L^p Spaces. Let Ω be a bounded domain in $\mathbb{R}^n$. For $p \geqslant 1$, the space $L^p(\Omega)$ denotes the Banach space that consists of equivalence classes of measurable functions on Ω, that are p-integrable, i.e., that satisfy

$$\int_\Omega |f|^p < \infty,$$

where two measurable functions are equivalent if they are equal a.e.

A norm for this space is provided by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p}.$$

In case there is no ambiguity possible, we will denote this norm by $\|f\|_p$.

For $p = \infty$, the space $L^\infty(\Omega)$ is the Banach space of bounded functions on Ω with norm

$$\|f\|_{L^\infty(\Omega)} = \sup_\Omega |f|.$$

An important inequality for L^p spaces is the *Hölder inequality*, i.e. for $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, $1 \leqslant p, q \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|uv\|_1 \leqslant \|u\|_p \|v\|_q. \tag{A.3.1}$$

The number q is called the *conjugated index to p*.

The space $L^p_{\text{loc}}(\Omega)$ consists of measurable functions on Ω that are **locally** p-integrable.

A.3.2. Weak derivatives. For a locally integrable function f on Ω, the α^{th} weak derivative g of f for a multi index α is given by

$$\int_\Omega \phi g dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \phi dx \qquad \text{for all } \phi \in C_0^{|\alpha|}(\Omega). \tag{A.3.2}$$

Then we write $g = D^\alpha f$. The weak derivative is uniquely determined up to sets of measure zero. If all weak derivatives up to order k of f exist, f is called *k-times weakly differentiable*.

A.3.3. Hölder- and Sobolev Spaces.

Definition A.3.1. For $\Omega \subseteq \mathbb{R}^n$ open, and k a nonnegative integer, the *Hölder spaces* $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of the functions whose k-th order partial derivatives are Hölder continuous with exponent α in Ω, i.e., which satisfy

$$\sup_{x \neq y \in \Omega} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha} < \infty. \tag{A.3.3}$$

The spaces $C^k(\Omega)$ equipped with the norms

$$\|f\|_{C^{k,\alpha}} = \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_{C^0} + \sup_{x,y} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha} \tag{A.3.4}$$

are Banach spaces.

Here $\|f\|_{C^0(\Omega)} := \max_{x\in\Omega} |f(x)|$.

Definition A.3.2. For $p \geqslant 1$, and $k \geqslant 0$ an integer, the space $W^{k,p}(\Omega)$ consists of all the functions f, which are weakly p-integrable, such that $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \leqslant k$.

A norm on this space can be defined by

$$\|f\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leqslant k} \|D^\alpha f\|^p dx \right)^{1/p}. \tag{A.3.5}$$

In case no ambiguity can arise, we will denote the norm by $\|f\|_{k,p}$.

This space is again a Banach space.

Taking the closure of $C_0^k(\Omega)$ (i.e., for $f \in C^k(\Omega)$ which have compact support) in $W^{k,p}(\Omega)$, one arrives at the spaces $W_0^{k,p}(\Omega)$. In case of bounded Ω, those spaces do not coincide (see [26, Chapter 7]).

The local version of the $W^{k,p}(\Omega)$ spaces is given by $W_{\text{loc}}^{k,p}(\Omega)$, which consist of functions that belong to $W^{k,p}(\Omega')$ for $\Omega' \Subset \Omega$.

This spaces can be continuously embedded into each other due to the following **Sobolev inequalities**:

THEOREM A.3.3. *Let $\Omega \subset \mathbb{R}^n$, then, for k a nonnegative integer, $1 \leqslant p \leqslant \infty$,*

$$\begin{tikzcd}[column sep=small] & L^{np/(n-kp)}(\Omega) \text{ for } kp < n \\ W_{\text{loc}}^{k,p}(\Omega) \arrow[ur] \arrow[dr] & \\ & C^m(\bar{\Omega}) \text{ for } 0 \leqslant m < k - \frac{n}{p} \end{tikzcd} \tag{A.3.6}$$

In case Ω satisfies the interior cone condition (i.e., there exists a fixed cone K, such that each $x \in \Omega$ is the vertex of a cone which is congruent to K), it furthermore holds that the following embeddings are continuous

$$\begin{tikzcd}[column sep=small] & L^{np/(n-kp)}(\Omega) \qquad \text{for } kp < n \\ W^{k,p}(\Omega) \arrow[ur] \arrow[dr] & \\ & C_B^m(\Omega) \qquad \text{for } 0 \leqslant m < k - \frac{n}{p} \end{tikzcd} \tag{A.3.7}$$

where $C_B^m(\Omega)$ consists of m-times differentiable functions f with $D^\alpha f \in L^\infty(\Omega)$, $|\alpha| \leqslant m$.

Extending this result, it is also possible to show that the embeddings

$$\begin{tikzcd}[column sep=small] & L^q(\Omega) \text{ for } kp < n, q < \frac{np}{n-kp} \\ W_{\text{loc}}^{k,p}(\Omega) \arrow[ur] \arrow[dr] & \\ & C^m(\bar{\Omega}) \text{ for } 0 \leqslant m < k - \frac{n}{p} \end{tikzcd} \tag{A.3.8}$$

are compact and that one can replace $W_{\text{loc}}^{k,p}(\Omega)$ by $W^{k,p}(\Omega)$ for domains Ω that satisfy the interior cone condition.

For proofs of these results in $\mathbb{R}^n$, see [26, Theorem 7.10], where the authors show the continuity of the embedding for $k = 1$, and by iteratively applying the result to higher order derivatives one gets the general case. The compactness statement is proven in Theorem 7.22. of [26].

A proof of the Sobolev embedding theorem for arbitrary manifolds can be found in [4, Theorem 2.20].

Sobolev spaces can also be embedded into certain Hölder spaces by using the following result, known as Morrey's inequality:

THEOREM A.3.4. *For $n < p \leqslant \infty$ let $B(x,r) \subseteq \mathbb{R}^n$, and let $y \in B(x,r)$. Then*

$$|u(x) - u(y)| \leqslant C r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,2r))} \qquad \forall u \in C^1(\mathbb{R}^n).$$

Using this result we obtain the following embedding theorem, which is here stated for $k = 1$, but can be generalized for bigger k by a bootstrapping argument.

THEOREM A.3.5. *There exists a constant $C = C(n,p)$ such that*

$$\|u\|_{C^{0,1-\frac{n}{p}}} \leqslant C\|u\|_{1,p} \qquad \forall u \in W^{1,p}(\mathbb{R}^n).$$

The general result states that if $\frac{k-r-\alpha}{n} = \frac{1}{p}$, with $\alpha \in (0,1)$ it holds that

$$W^{k,p}(\mathbb{R}^n) \subset C^{r,\alpha}(\mathbb{R}^n). \tag{A.3.9}$$

For arbitrary tensor fields, we also introduce the following notation:

Let $\mathbf{e}$ be an arbitrary smooth Riemannian metric on M. We denote by $d\mu_{\mathbf{e}}$ the Riemannian measure defined by the metric $\mathbf{e}$.

Given a tensor field, $\mathbf{T}$, on M, and $p \in [1,\infty]$, we define the L^p-norm of $\mathbf{T}$ on a set $U \subseteq M$ with respect to $\mathbf{e}$ to be

$$\|\mathbf{T}\|_{L^p(U)} := \left(\int_U \|\mathbf{T}\|_{\mathbf{e}}^p \, d\mu_{\mathbf{e}}\right)^{1/p},$$

with the usual extension if $p = \infty$. We will also use L^p norms of tensor fields with respect to different continuous metrics, such as $\mathbf{g}$, in which case we denote the norm by $\|\cdot\|_{L^p(U,\mathbf{g})}$, for example. A tensor field $\mathbf{T}$ is said to lie in $L^p_{\mathrm{loc}}(M)$ if $\|\mathbf{T}\|_{L^p(C)} < \infty$ for each compact set $C \subset M$. (The space $L^p_{\mathrm{loc}}(M)$ is independent of the metric in our class used to define the L^p norms.) Similarly, for $k \geqslant 0$ an integer and $p \in [1,\infty]$, we introduce the Sobolev norm

$$\|\mathbf{T}\|^p_{W^{k,p}(U)} := \sum_{|\alpha| \leqslant k} \int_U |\nabla^\alpha_{\mathbf{e}} \mathbf{T}|^p_{\mathbf{e}} \, d\mu_{\mathbf{e}},$$

where $\nabla_{\mathbf{e}}$ denotes the Levi-Civita connection of the background metric $\mathbf{e}$. Again, a tensor field $\mathbf{T}$ lies in the local Sobolev space $W^{k,p}_{\mathrm{loc}}(M)$ if $\|\mathbf{T}\|_{W^{k,p}(C)} < \infty$ for all compact $C \subset M$.

A.3.4. Weighted Sobolev spaces. Weighted Sobolev spaces are applied when studying the asymptotic behavior of Riemannian manifolds. Thus they are used in the investigation of mass in general relativity.

We follow [46, Section 9], and [6].

DEFINITION A.3.6. Let $(M, \mathbf{g})$ be an asymptotically flat manifold, with asymptotic coordinates $\{z^i\}$ on the end N_1. Let $\rho = |z|$ on N_1, and extend it to a smooth positive function on the whole manifold M.

For $q \geqslant 1$, $\beta \in \mathbb{R}$ the *weighted Lebesgue space*, $L^q_\beta(M)$, is defined to be the set of locally integrable functions f for which the norm

$$\|f\|_{q,\beta} = \left(\int_M |\rho^{-\beta} f|^q \rho^{-n} dV_{\mathbf{g}}\right)$$

is finite. Analogously, for k a nonnegative integer, the *weighted Sobolev space* $W^{k,q}_\beta(M)$ is the set of locally integrable functions f for which $|\nabla^i f| \in L^q_{\beta-i}(M)$, $0 \leqslant i \leqslant k$. A norm on this space is given by

$$\|f\|_{k,q,\beta} = \sum \|D^i f\|_{q,\beta-i}.$$

Furthermore, the weighted C^k space, $C^k_\beta(M)$, consists of C^k-functions f for which the norm

$$\|f\|_{C^k_\beta} = \sum \sup_M \rho^{-\beta-i} |D^i f|.$$

Finally, the *weighted Hölder space*, $C^{k,\alpha}_\beta$ $(0 < \alpha < 1)$, is the set of $f \in C^k_\beta(M)$ with finite norms

$$\|f\|_{C^{k,\alpha}_\beta} = \sum_{|\alpha| \leqslant k} \|f\|_{C^k_\beta} + \sup_{x,y} (\min(\rho(x), \rho(y)))^{-\beta+k+\alpha} \frac{|\nabla^k f(x) - D^k f(y)|}{|x-y|^\alpha}.$$

Note that β determines the order of growth, $f \in C^{k,\alpha}_\beta$ or in $W^{k,q}_\beta$ implies that $f = O(\rho^\beta)$.

For weighted spaces, the Sobolev embedding also holds. Furthermore, also global elliptic regularity results remain true, [6]:

LEMMA A.3.7. *For $q > 1$, $l - k - \alpha > n/q$ and arbitrary $\epsilon > 0$, the embeddings $C^{l,\alpha}_{\beta-\epsilon}(M) \subset W^{l,q}_\beta(M) \subset C^{k,\alpha}_\beta(M)$ are continuous.*

THEOREM A.3.8. *Let $\tau > 0$, $q > 1$. For an asymptotically flat Riemannian manifold $(M, \mathbf{g})$ with $g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}(M)$ the following statements hold:*

(1) For $f \in W^{2,q}_\beta(M)$,

$$\|f\|_{2,q,\beta} \leqslant C\left(\|\Delta f\|_{0,q,\beta-2} + \|f\|_{0,q,\beta}\right).$$

(2) The operator $\Delta : W^{2,q}_\beta(M) \to W^{0,q}_{\beta-2}(M)$ is an isomorphism iff $2 - n < \beta < 0$.

(3) In case $f \in C^0_\beta(M)$, and $\Delta f \in C^{0,\alpha}_{\beta-2}(M)$, then f is actually in $C^{2,\alpha}_\beta(M)$ with

$$\|f\|_{C^{2,\alpha}_\beta} \leqslant C\left(\|\Delta f\|_{C^{0,\alpha}_{\beta-2}} + \|f\|_{C^0_\beta}\right).$$

(4) For $2-n<\beta<0$, $\delta<-2$, $h\in C^{0,\alpha}_{\delta}(M)$, it holds that, if the operator $\Delta+h\colon C^{2,\alpha}_{\beta}(M)\to C^{0,\alpha}_{\beta-2}(M)$ is injective, then it is an isomorphism.

For a proof of this theorem, see [6, 11, 48, 50].

A.4. Elliptic regularity

Let $\Omega\subset\mathbb{R}^n$ be bounded, $\Omega'\subset\subset\Omega$ compact. Let $L=a^{ij}\partial_i\partial_j$ be a strictly elliptic second-order differential operator, with smooth coefficients. Let $u\in C^\infty(\Omega)$. Then the following holds:

THEOREM A.4.1 (Schauder estimates). *If the coefficients a^{ij} satisfy $\|a^{ij}\|_{C^{k,\alpha}(\Omega)}\leqslant Q$, there exists a constant $C=C(k,\alpha,n,Q,\Omega')$ such that*

$$\|u\|_{C^{k+2,\alpha}(\Omega')}\leqslant C(\|Lu\|_{C^{k,\alpha}(\Omega)}+\|u\|_{C^0(\Omega)}). \tag{A.4.1}$$

The proof for $k=0$ can be found in [26, Theorem 6.2], the inductive step is carried out in [57, Theorem A. 1].

Furthermore, we get the following L^p-version:

THEOREM A.4.2 (L^p-estimates). *For coefficients a^{ij} with $\|a^{ij}\|_{W^{k,p}(\Omega)}\leqslant Q$, $V>0$ he following holds: if $\mathrm{Vol}\,(\Omega)\leqslant V$, and $p>n$ for $k=1$ or $p>n/2$ for $k\geqslant 2$ then there exists a constant $C=C(k,p,n,Q,\Omega')$ such that*

$$\|u\|_{W^{k+1,p}(\Omega')}\leqslant C\left(\|Lu\|_{W^{k-1,p}(\Omega)}+\|u\|_{L^p(\Omega)}\right). \tag{A.4.2}$$

If $\Omega=B(0,R)$ is a Euclidean ball with $u\equiv 0$ on $\partial B(0,R)$, then there is a $C=C(n,p,Q,R)$ such that

$$\|u\|_{W^{2,p}(B(0,R))}\leqslant C(\|Lu\|_{L^p(B(0,R))}+\|u\|_{L^p(B(0,R))}). \tag{A.4.3}$$

The proof can also be found in [57, Appendix], who uses the result [26, Theorem 9.11, 9.13]. Another result which will be used is the following:

THEOREM A.4.3. *For $u\in W^{2,,p}(M)$ that solves $Lu=f$ for $L=a^{ij}\partial_i\partial_j$, $a^{ij}\in C^{l,\alpha}$ and $f\in W^{l,,p}(M)$ (or $f\in C^{l,\beta}$). Then $u\in W^{l+2,,p}(M)$ (or $C^{k+2,\beta}$).*

See [26, Theorem 9.19].

A.5. The Arzela-Ascoli theorem

An important tool in the study of convergence and compactness results is the Arzela-Ascoli theorem. We will briefly recall the result and its proof.

DEFINITION A.5.1. Let X be a compact metric space, and $C^0(X)$ be the space of continuous functions $X\to\mathbb{R}$. A sequence of functions $\{f_n\}\subseteq C^0(X)$ is called *bounded* if there exists a positive constant $K<\infty$, such that for all $n\in\mathbb{N}$ and for all $x\in X$ it holds that $|f_n(x)|<K$.

One says that a function $f \subseteq C^0(X)$ is *equicontinuous* if, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

With these definitions in mind we can state the following theorem.

THEOREM A.5.2. *If a sequence $\{f_n\} \subseteq C^0(X)$ is bounded and equicontinuous, then it has a uniformly converging subsequence.*

Proof. By compactness of X, it follows that X is separable. Indeed, for $x \in X$, $n \in \mathbb{N}$, the union $\bigcup_x B_n(x)$ covers X, hence a finite subcollection suffices to cover X. Let S_n be an $1/n$-dense set in X, i.e. each $x \in X$ is within distance of $1/n$ of one of the points in S_n. Then the set $S = \bigcup_n S_n$ is a countable dense subset of X, hence X is separable.

In the next step we construct a subsequence $\{f_{n_k}\}$ of the original sequence that converges pointwise on the set S formed in the above step: Since S is countable, we can write it as $S = \{x_1, x_2, \ldots\}$. The numerical sequence $\{f_n(x_1)\}_{n=1}^{\infty}$ is, by the assumption of boundedness of the whole sequence $\{f_n\}$, bounded, hence using the Bolzano-Weierstrass theorem gives a converging subsequence $\{f_{n1}(x_1)\}_{n=1}^{\infty}$. Now we can insert x_2 in the f_{n1}s and get a bounded sequence $\{f_{n1}(x_2)\}_{n=1}^{\infty}$, and so again we get a converging subsequence $\{f_{n2}(x_2)\}_{n=1}^{\infty}$ (which also converges at x_1, since it is a subsequence of $\{f_{n1}\}$). We can proceed further by the same arguments and if we pick the diagonal sequence $\{f_{nn}\}_{n=1}^{\infty}$ we obtain a subsequence of the original sequence $\{f_n\}$ that converges at each point of S.

As a final step we want to show that $\{f_{nn}\} = \{g_n\}$ is uniformly continuous:
Let therefore $\varepsilon > 0$, and choose $\delta > 0$ such that by equicontinuity we get

$$d(x,y) < \delta \Rightarrow |g_n(x) - g_n(y)| < \varepsilon/3.$$

for all $x, y \in X$, $n \in \mathbb{N}$. Fix $M > 1/\delta$, such that $S_M \subseteq S$ is δ-dense in X.

Since $\{g_n\}$ converges at each point of S_M, we can find a $N > 0$ such that for $m, n > N$, $|g_n(s) - g_m(s)| < \varepsilon/3$ for some $s \in S_M$.

For $x \in X$ arbitrary, it holds that $d(x,s) < \delta$ for some $s \in S_M$. Thus for $m, n > \max(M, N)$ it holds

$$|g_n(x) - g_m(x)| \leqslant |g_n(x) - g_n(s)| + |g_n(s) - g_m(s)| + |g_m(s) - g_m(x)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So $\{g_n\}$ is uniformly Cauchy, hence uniformly convergent. □

A.6. Convergence of metric spaces

In order to study convergence properties of metric spaces, the first thing to do is to introduce a notion of distance between them. There are various possibilities to do so, e.g. Lipschitz distance, or Gromov-Hausdorff distance, which we will briefly describe. Alternatively one can also study how metrics of a sequence of Riemannian manifolds converge. It turns out that in some cases both notions are actually the same (see, e.g., [56]).

In this section we mainly follow the definitions in [10].

A.6.1. Lipschitz distance.

Definition A.6.1. Let X, Y be two metric spaces, $f\colon X \to Y$ an arbitrary map. The *distortion* of f is defined by

$$\mathrm{dis} f := \sup_{x,y \in X} |d_Y(f(x), f(y)) - d_X(x,y)|$$

where d_X, d_Y denote the distances in X and Y respectively.

A sequence $\{X_n\}$ of metric spaces *uniformly converges* to a metric space X if there are homeomorphisms $f_n\colon X_n \to X$ such that $\mathrm{dis} f_n \to 0$ $(n \to \infty)$.

In case the map f is Lipschitz, we define as follows:

Definition A.6.2. Let X, Y be metric spaces, $f\colon X \to Y$ a Lipschitz map. The *dilation* of f is defined by

$$\mathrm{dil} f := \sup_{x,y \in X} \frac{d_Y(f(x), f(y))}{d_X(x,y)}.$$

A homeomorphism is called *bi-Lipschitz*, if both f and f^{-1} are Lipschitz. The Lipschitz distance between X and Y is now defined as

$$d_L(X,Y) := \inf_{f: X \to Y} \log(\max(\mathrm{dil} f, \mathrm{dil} f^{-1})),$$

where the infimum is taken over all bi-Lipschitz maps from X to Y.

A sequence $\{X_n\}$ of metric spaces converges in the Lipschitz distance to a metric space X, if $d_L(X_n, X) \to 0$.

Furthermore, if there are no bi-Lipschitz homeomorphisms between X and Y, $d_L(X,Y) = \infty$.

Two metric spaces are close in the Lipschitz distance, if they are relatively close to each other, i.e., if $\frac{d_Y(f(x),f(y))}{d_X(x,y)}$ is close to 1.

Theorem A.6.3. *d_L is a metric on the space of isometry classes of compact metric spaces.*

See [10, 7.2.4.].

Remark A.6.4. The assumption of compactness is necessary for showing that if the Lipschitz distance between X and Y vanishes, they are actually isometric and vice versa. Moreover one would get a semi-metric when dropping the compactness assumption.

A.6.2. Hausdorff- and Gromov-Hausdorff convergence. Let $X, Y \subseteq M$ subsets of a metric space (M, d_M). The *Hausdorff distance* between X and Y is given by the following expression:

$$d_H(X,Y) = \inf\{\varepsilon > 0 : X \subseteq Y_\varepsilon \text{ and } Y \subseteq X_\varepsilon\},$$

where $X_\varepsilon = \{z \in M : d_M(z, X) < \varepsilon\}$.

The Hausdorff distance is therefore small, if every point in X is close to a point in Y, and vice versa. The Hausdorff distance defines a metric on the closed and bounded subsets of M. This collection is compact, whenever M is compact.

In the 1980ies Gromov used this concept to define a distance between metric spaces.

Definition A.6.5. The *Gromov-Hausdorff distance* between two metric spaces (M, d_M) and (N, d_N) is defined as

$$d_{GH}(M, N) = \inf\{d_H(M', N')\} \tag{A.6.1}$$

where M', N' are subspaces of a metric space (Z, d_Z) and which are isometric to M, N.

A sequence $\{M_i, d_{M_i}\}$ of compact metric spaces converges to a compact metric space (M, d_M) in the Gromov-Hausdorff sense, if $d_{GH}(M_i, M) \to 0$ as $i \to \infty$.

It follows that the Gromov-Hausdorff distance between isometric spaces is zero. It can be shown that d_{GH} is a metric on the space of isometry classes of compact metric spaces [10, Theorem 7.3.30].

If M and N are non-compact, then the *pointed* Gromov-Hausdorff distance is used,

$$d_{GH}((M, m), (N, n)) = \inf\{d_H(M', N') + d_Z(m', n')\},$$

where M', N' are again isometric to M, N in Z and m', n' are the images of m, n in Z under this isometry. Thus if the distance between two pointed spaces is close, they are close in Hausdorff distance and, in addition, the basepoints are close as well.

Definition A.6.6. Let M, N be metric spaces, and $\varepsilon > 0$. An *ε-isometry* is a (not necessarily continuous) map $f\colon M \to N$ such that the distortion of f,

$$\operatorname{dis} f = \sup_{m_1, m_2 \in M} \{|d_N(f(m_1), f(m_2)) - d_M(m_1, m_2)|\} \leqslant \varepsilon.$$

Furthermore, $f(M)$ has to be an ε net in N, i.e., every $n \in N$ has $d_N(n, f(m)) < \varepsilon$ for some $f(m)$, $m \in M$.

It can be shown, [10, Cor. 7.3.28],that if M, N are ε-close in the Gromov-Hausdorff distance, that there exists a 2ε isometry from M to N and conversely, if there exists an ε-isometry between M and N, the metric spaces are 2ε close in the Gromov-Hausdorff distance.

Therefore, a sequence $\{M_i\}_{i=1}^{\infty}$ of metric spaces converges to a metric space M in the Gromov-Hausdorff topology, if and only if there are $\varepsilon_i \to 0$ and maps $f_i\colon M_i \to M$ that are ε_i-isometries.

Another criterion for Gromov-Hausdorff convergence is provided by the following result [10, Proposition 7.4.12]:

Proposition A.6.7. *For compact metric spaces M, $\{M_i\}_{i=1}^{\infty}$, we say that M_i converge to M, $M_i \to M$, in Gromov-Hausdorff if and only if for every $\varepsilon > 0$ there exists a finite (i.e. consisting of finitely many points) ε-net S in M and for each i an ε-net S_i in M_i, such that $S_i \to S$ in Gromov-Hausdorff. For large i, the nets S_i can be chosen to have the same cardinality as S.*

Bibliography

1. M. Anderson, *Convergence and rigidity of manifolds under ricci curvature bounds*, Invent. Math. **102** (1990), 429–445. 6, 66, 69, 70, 76, 79, 82
2. M. T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445. MR MR1074481 (92c:53024) 63, 81, 85
3. M. T. Anderson and J. Cheeger, *C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Differential Geom. **35** (1992), no. 2, 265–281. MR MR1158336 (93c:53028) 6, 37, 63, 66, 69, 70, 74, 76, 79, 81, 84
4. T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR MR1636569 (99i:58001) 89, 92
5. E. Aubry, *Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 675–695. MR 2191529 (2010a:53047) 50
6. R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693. MR MR849427 (88b:58144) 2, 8, 9, 16, 33, 93, 94
7. R. A. Bartnik and P. T. Chruściel, *Boundary value problems for Dirac–type equations, with applications*, (2007). 33
8. S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc. **52** (1946), 776–797. MR 0018022 (8,230a) 37
9. H. Bray and F. Finster, *Curvature estimates and the positive mass theorem*, Comm. Anal. Geom. **10** (2002), no. 2, 291–306. MR MR1900753 (2003c:53047) 15, 33
10. D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR MR1835418 (2002e:53053) 77, 95, 96, 97
11. M. Cantor, *Elliptic operators and the decomposition of tensor fields*, Bull. Amer. Math. Soc. (N.S.) **5** (1981), 235–262. 94
12. I. Chavel, *Riemannian geometry*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006, A modern introduction. MR MR2229062 (2006m:53002) 1
13. J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970), 61–74. MR MR0263092 (41 #7697) 81
14. ———, *Degeneration of Riemannian metrics under Ricci curvature bounds*, Lezioni Fermiane. [Fermi Lectures], Scuola Normale Superiore, Pisa, 2001. MR MR2006642 (2004j:53049) 6
15. J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematical Library, Vol. 9. MR MR0458335 (56 #16538) 44
16. J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443. MR MR0309010 (46 #8121) 71
17. J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), no. 1, 15–53. MR MR658471 (84b:58109) 62
18. P. T. Chruściel and J. D. E. Grant, *On Lorentzian causality with continuous metrics*, Classical Quantum Gravity **29** (2012), no. 14, 145001, 32. MR 2949547 1

19. P.T. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view.*, Topological properties and global structure of space-time, ed. by P.Bergmann, V. de Sabbata **138** (196), 49–59. 9
20. C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 419–435. MR 608287 (83d:58068) 60, 79, 84
21. D. M. DeTurck and J. L. Kazdan, *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 3, 249–260. MR MR644518 (83f:53018) 63, 64, 65
22. M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall. 44
23. R. Finn, *On a class of conformal metrics, with application to differential geometry in the large*, Comment. Math. Helv. **40** (1965), 1–30. 11
24. S. Gallot, *Isoperimetric inequalities based on integral norms of Ricci curvature*, Astérisque (1988), no. 157-158, 191–216, Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987). MR 976219 (90a:58179) 4, 60, 84
25. L. Z. Gao, *Convergence of Riemannian manifolds; Ricci and $L^{n/2}$-curvature pinching*, J. Differential Geom. **32** (1990), no. 2, 349–381. MR 1072910 (92f:53049) 81, 82
26. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364 (2001k:35004) 27, 68, 73, 75, 89, 91, 92, 94
27. J. D. E. Grant, *Global Lorentzian geometry, Part 1*, Lecture Notes. 35, 38, 41, 43, 87
28. ———, *Global Lorentzian geometry, Part 2*, Lecture Notes. 8
29. J. D. E. Grant and N. Tassotti, *Generalized volume comparison.* 45, 47, 54
30. J. D. E. Grant and N. Tassotti, *A positive mass theorem for low-regularity metrics*, Preprint, 2012. 3, 4, 12
31. ———, *A positive mass theorem for low-regularity Riemannian metrics*, (2014). 2, 3, 12, 18, 33
32. R. E. Greene and H. Wu, *Lipschitz convergence of Riemannian manifolds*, Pacific J. Math. **131** (1988), no. 1, 119–141. MR MR917868 (89g:53063) 81
33. M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999, Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR MR1699320 (2000d:53065) 43, 81
34. R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR 664497 (84a:53050) 16
35. Q. Han and F. Lin, *Elliptic partial differential equations*, Courant Lecture Notes in Mathematics, vol. 1, New York University Courant Institute of Mathematical Sciences, New York, 1997. MR 1669352 (2001d:35035) 32
36. P. Hartman, *Ordinary differential equations*, second ed., Birkhäuser, Boston, Mass., 1982. MR 658490 (83e:34002) 1
37. E. Hebey and M. Herzlich, *Harmonic coordinates, harmonic radius and convergence of riemannian manifolds.*, Rend. Mat. Appl., VII. Ser. **17** (1997), no. 4, 569–605 (English). 6, 63, 66, 67, 69, 70, 71, 76, 77, 79, 81
38. T. Hiroshima, *C^α compactness theorem for Riemannian manifolds with bounds on diameter, injectivity radius, and some integral norms of Ricci curvature*, Indiana Univ. Math. J. **44** (1995), no. 2, 397–411 (English). 84
39. A. Huber, *Vollstandige konforme metriken und isolierte singularitaten subharmonischer funktionen*, Comment. Math. Helv. **41** (1966), 105–136. 11
40. J. Jost and H. Karcher, *Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen*, Manuscripta Math. **40** (1982), no. 1, 27–77. MR MR679120 (84e:58023) 63, 65, 67, 82
41. A. Kasue, *A convergence theorem for Riemannian manifolds and some applications*, Nagoya Math. Journal **114** (1989), 21–51. 76, 79, 81

42. A. Katsuda, *Gromov's convergence theorem and its application*, Nagoya Math. J. **100** (1985), 11–48. MR 818156 (87e:53067) 81
43. H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989. MR 1031992 (91g:53001) 12
44. D. A. Lee, *A positive mass theorem for Lipschitz metrics with small singular sets*, Proc. Amer. Math. Soc. **141** (2013), 3997–4004. 2, 3, 17, 18, 20
45. D. A. Lee and P. G. LeFloch, *The positive mass theorem for manifolds with distributional curvature*, (2014). 33
46. J. M. Lee and T. H. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), no. 1, 37–91. MR 888880 (88f:53001) 8, 9, 16, 93
47. A. Lichnerowicz, *Spineurs harmoniques*, C.R. Acad. Sci. Paris **257** (1963), 7–9. 12
48. R. Lockhart, *Fredholm properties of a class of elliptic operators on non-compact manifolds*, Duke Math. J. **48** (1981), 289–312. 94
49. D. McFeron and G. Székelyhidi, *On the positive mass theorem for manifolds with corners*, Comm. Math. Phys. **313** (2012), 425–443. 2, 3, 16, 17, 18, 20, 33
50. R. McOwen, *Behavior of the Laplacian on weighted Sobolev spaces*, Comm. Pure Appl. Math. **32** (1979), 783–795. 94
51. P. Miao, *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys. **6** (2002), no. 6, 1163–1182 (2003). MR 1982695 (2005a:53065) 2, 3, 12, 13, 14, 15, 16, 18, 20, 26, 33
52. C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1966 edition [MR0202511]. MR 2492985 27
53. S. B. Myers, *Riemannian manifolds in the large.*, Duke Math. J. **1** (1935), 39–49 (English). 44
54. T. Parker and C. H. Taubes, *On Witten's proof of the positive energy theorem*, Comm. Math. Phys. **84** (1982), no. 2, 223–238. MR 661134 (83m:83020) 12
55. S. Peters, *Cheeger's finiteness theorem for diffeomorphism classes of Riemannian manifolds*, J. Reine Angew. Math. **349** (1984), 77–82. MR MR743966 (85j:53046) 81
56. ______, *Convergence of Riemannian manifolds*, Compositio Math. **62** (1987), no. 1, 3–16. MR MR892147 (88i:53076) 81, 95
57. P. Petersen, *Convergence theorems in Riemannian geometry*, MSRI Publications **30** (1997), 167–202. 68, 85, 94
58. ______, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR MR2243772 (2007a:53001) 59, 80
59. P. Petersen and G. Wei, *Relative volume comparison with integral curvature bounds*, Geom. Funct. Anal. **7** (1997), no. 6, 1031–1045. MR MR1487753 (99c:53023) 4, 45, 47, 49, 50, 54, 60, 62, 84
60. T. Rivière and P. Strzelecki, *A sharp nonlinear Gagliardo–Nirenberg-type estimate and applications to the regularity of elliptic systems*, Comm. Partial Differential Equations **30** (2005), no. 4-6, 589–604. MR 2153509 (2006d:35085) 32
61. R. Schoen and S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR MR526976 (80j:83024) 2, 8, 9, 10, 11, 14, 22, 24, 33
62. M. Simon, *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. **10** (2002), no. 5, 1033–1074. MR 1957662 (2003j:53107) 16
63. M. E. Taylor, *Partial differential equations III. Nonlinear equations*, second ed., Applied Mathematical Sciences, vol. 117, Springer, New York, 2011. MR 2744149 (2011m:35003) 58
64. N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–483. MR 0216286 (35 #7121) 4
65. J.A. Viaclovsky, *Topics in Riemannian geometry (Lecture notes fall 2007)*, Math 865,. 87, 89
66. G. Wei, *Manifolds with a lower Ricci curvature bound*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 203–227. MR 2408267 (2009j:53039) 1, 4
67. R. Weitzenböck, *Invariantentheorie*, Groningen, 1923. 37

68. E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402. MR 626707 (83e:83035) 2, 9, 10, 12
69. D. Yang, *Convergence of Riemannian manifolds with integral bounds on curvature. I*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 1, 77–105. MR 1152614 (93a:53037) 4, 46, 60, 61, 81, 84, 85
70. ———, *Convergence of Riemannian manifolds with integral bounds on curvature. II*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 2, 179–199. MR 1169351 (93m:53037) 60, 81, 83
71. S.T. Yau, *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.*, Indiana Univ. Math. J. **25** (1976), 659–670 (English). 44
72. S. Zhu, *The comparison geometry of Ricci curvature*, Comparison Geometry, MSRI Publications, 1997, pp. 221–262. 4, 40, 41, 42, 43, 44

Nathalie Tassotti

Curriculum Vitae

Fakultät für Mathematik, Universität Wien,
Oskar Morgenstern Platz 1 1090 Wien, Austria
☎ *+43 (0)1 4277 50636*
✉ *nathalie.tassotti@univie.ac.at*
http://nathalietassotti.wordpress.com

Personal

Full Name	Mag. Nathalie Tassotti
Date of Birth	29 September, 1986, Klagenfurt, Austria
Citizenship	Austrian

Areas of research

Geometric analysis and mathematical relativity.

Education

2010–Present	**Ph.D. studies**, *Universität Wien*.
Jun 2010	**Second part of studies completed with distinction**, *Universität Wien*.
Jun 2010	**Diploma thesis 'Warped Products in General Relativity'**, *Universität Wien*, supervised by Prof. M. Kunzinger.
Feb 2008	**First part of studies completed with distinction**, *Universität Wien*.
2005–2010	**Diploma studies (A405)**, *Fakultät für Mathematik, Universität Wien*.
2005	**Matura (with honours)**.

Awards

2013–2014	**DOC-Stipendium**, *Austrian Academy of Sciences (ÖAW)*, PhD fellowship for the project "Low regularity geometry and curvature bounds"..
Jan–Jun 2012	**Forschungsstipendium**, *University of Vienna*, Austria. PhD fellowship for the project "Low regularity geometry and curvature bounds".
2010	**Leistungsstipendium (merit scholarship)**, *Universität Wien*.

Conferences

27.–30.03.2012 **Workshop on Geometric Analysis**, *Goethe-Universität Frankfurt*, Germany. My participation was funded by a travel grant from the conference organizers.

25.–29.06.2012 **ICTP ESF Conference on Geometric Analysis**, *ICTP, Miramare-Trieste*, Italy. My participation is funded by a travel grant from the conference organizers.

04.–07.09.2012 **International Conference on Geometric Analysis**, *Seillac*, France. My participation is funded by a travel grant from the conference organizers.

7.–11.1.2013 **UK-Japan Winterschool in Nonlinear Analysis**, *King's College, London*, Great Britain. The University of Vienna funded my participation.

7.–9.2.2013 **Relativity Seminar**, *AEI Golm*, Germany. Funded by the University of Vienna.

14.–15.3.2013 **Modern topics in nonlinear PDEs, Oxbridge PDE days**, *Oxford*, Great Britain. Funded by the University of Vienna..

14.–18.7.2014 **Summer School on Geometric Measure Theory and Geometric Analysis**, *Basel*, Switzerland. Funded by the organizers.

Research visits

15.–17.3.2013 **University of Surrey**, *Surrey*, Great Britain. .

24.–26.7.2014 **University of Karlsruhe**, *Karlsruhe*, Germany. Invited by T.Lamm.

Selected recent talks

7.2.2013 **The positive mass theorem for low regularity metrics**, *AEI Golm.*

18.06.2012 **The Cheeger finiteness theorem**, *Fakultät für Mathematik, Universität Wien.*

30.05.2012 **Clifford algebras in the positive mass theorem**, *Gravitationsphysik, Universität Wien.*

20.01.2012 C^{α}**-convergence of Riemannian manifolds**, *Fakultät für Mathematik, Universität Wien.*

Publications

A positive mass theorem for low-regularity Riemannian metrics, in preparation.

A generalized volume comparison theorem, in preparation.

Nathalie Tassotti

Wednesday 5th November, 2014