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Initial data for rotating cosmologies

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Abstract

In order for initial data to evolve to solutions of Einstein's equations, constraint equations have to be satisfied. In this thesis, we make use of the "conformal method" of constructing solutions to the constraints. This leads to the Lichnerowicz equation, whose solutions determine the required conformal transformation. We consider data with positive cosmological constant. We restrict ourselves to axially symmetric and so-called (t, φ)-symmetric data, for which there is a natural definition of angular momentum.

Using theorems by Hebey, Pacard and Pollack we calculate bounds for this angular momentum required for existence or non-existence of solutions. We also consider a recent theorem by Premoselli which gives the most efficient results on this issue. We first apply these general theorems to rotational Bowen-York data on the time-symmetric slice of the Nariai metric as background. Lastly we apply Premoselli's theorem to a background given by the maximal slice of the Kerr-de Sitter spacetime. In particular we construct data by scaling the extrinsic curvature of this slice suitably, which allows "overspinning" of Kerr-de Sitter in a certain sense.

Zusammenfassung

Damit Anfangsdaten zu Lösungen der Einsteingleichungen evolvieren, müssen Zwangsbedingungen erfüllt sein. In dieser Arbeit benutzen wir die "konforme Methode" um Lösungen dieser Zwangsbedingungen zu konstruieren. Dies führt zur Lichnerowicz Gleichung, deren Lösungen die verlangte konforme Transformation lieferen. Wir betrachten Daten mit positiver kosmologischer Konstante. Wir beschränken uns auf axialsymmetrische und sogenannte (t, φ)-symmetrische Daten, für die es eine natürliche Definition des Drehimpulses gibt.

Unter Verwendung von Theoremen von Hebey, Pacard und Pollack berechnen wir Schranken für diesen Drehimpuls, die die Existenz bzw. die Nichtexistenz von Lösungen garantieren. Wir betrachten auch ein vor kurzem von Premoselli bewiesenes Theorem, das die effizientesten Resultate zu diesem Thema liefert. Wir wenden diese allgemeinen Theoreme zuerst auf rotierende Bowen-York Daten auf einem zeitsymmetrischen Schnitt der Nariai Metrik als Hintergrund an. Letztendlich wenden wir Premosellis Theorem auf einen maximalen Schnitt der Kerr-de Sitter Raumzeit an. Insbesondere konstruieren wir Daten indem wir die äußere Krümmung dieses Schnittes geeigent skalieren, was in gewissem Sinn ein "overspinning" von Kerr-de Sitter erlaubt.

Contents

1 Introduction

1.1 The initial value problem with cosmological constant

1.1.1 The basic setup

The framework of this thesis is the initial value problem for the vacuum Einstein equations with positive cosmological constant Λ

$$\tilde{G}_{\alpha\beta} = -\Lambda \tilde{h}_{\alpha\beta} \tag{1.1}$$

on a smooth 4-dimensional manifold $(\tilde{N}, \tilde{h}_{\alpha\beta})$. Here Greek indices take values from 0 to 3, and the metric has signature $(-1, 1, 1, 1)$. We use units in which the gravitational constant G and the speed of light c are equal to 1. This entails that the components $\tilde{h}_{\alpha\beta}$ of the metric are dimensionless and the cosmological constant has dimension $[L^{-2}]$ (inverse length squared). We are in particular interested in constructing initial data which "rotate" in a sense explained below.

In order for the initial value problem to have at least short-time solutions, the constraint equations

$$\tilde{R} + \tilde{K}^2 - \tilde{K}_{ij}\tilde{K}^{ij} = 2\Lambda, \tag{1.2}$$

$$\tilde{D}^i \tilde{K}_{ij} - \tilde{D}_j \tilde{K} = 0. \tag{1.3}$$

have to be satisfied for the initial data $(\tilde{M}, \tilde{g}_{ij}, \tilde{K}_{ij})$ [9, 15] (Latin indices take the values $1, 2, 3$). The initial data consist of a smooth 3-manifold $\tilde{M}$ with metric $\tilde{g}_{ij}$, corresponding covariant derivative $\tilde{D}^i$ and scalar curvature $\tilde{R}$, extrinsic curvature $\tilde{K}_{ij}$ and mean curvature $\tilde{K}$. In most of this work (and unless stated otherwise) we take $\tilde{M}$ to be compact and without boundary.

Below (in Sect. 1.1.2) we first review the well-known conformal method of solving the constraints. We then proceed in Sect. 1.1.3 by sketching the difficulties in solving the Lichnerowicz equation with positive Λ. In particular we mention the existence and non-existence results by Hebey [13, 14], and the breakthrough on this problem achieved very recently by Premoselli [20]. In Sect. 1.2 we briefly describe the achievements of the present thesis, which mainly consist of applications of the above-mentioned results.

1.1.2 The conformal method

The most common procedure of obtaining solutions to the constraints is the conformal method [9, 15] which is just sketched in this subsection but recalled in Sect. 2.1 in some detail. We consider here a well-known ansatz for solving the momentum constraint (1.3), namely the so-called (t, φ)-symmetric data [2] which describe a rather general class of axially symmetric, rotating solutions. We recall their construction in Sect. 2.2. In this setting, we have a "background" manifold (M, g_{ij}, K_{ij}) with metric g_{ij} and a certain divergence-free and trace-free tensor K_{ij}. Then, for any smooth conformal transformation $\phi > 0$

$$\tilde{g}_{ij} = \phi^4 g_{ij} \qquad \tilde{K}_{ij} = \phi^{-2} K_{ij} \tag{1.4}$$

the momentum constraint (1.3) is satisfied. In order for $(\tilde{M}, \tilde{g}_{ij}, \tilde{K}_{ij})$ to satisfy the Hamiltonian constraint (1.2) as well, it remains to solve the Lichnerowicz equation

$$\left(\Delta_g - \frac{R}{8}\right)\phi = -\frac{\Sigma^2}{8\phi^7} - \frac{\Lambda}{4}\phi^5, \tag{1.5}$$

where $\Sigma^2 = K^{ij} K_{ij}$. Unless stated otherwise, we always understand by a "solution" a smooth, positive solution of (1.5) on a compact background manifold. The exceptions which will be discussed for comparison (in Sect. 2.3) are well-known solutions with "punctures" (singularities) on asymptotically flat manifolds.

1.1.3 Solving the Lichnerowicz equation

As mentioned already, there are specific problems with solving (1.5) for $\Lambda > 0$ which we expose here briefly. In particular, given (M, g_{ij}), solutions cannot exist for large momenta Σ as the following heuristic argument shows. Assuming existence, ϕ must have a minimum somewhere, as it is a smooth function on a compact manifold. Using $\Delta\phi|_{\min} \geq 0$ in (1.5), a necessary condition for existence is now that R is positive at this minimum; a more efficient (still necessary) criterion is that the Yamabe constant (cf. Sect. 3) of M is positive. However, this is clearly not sufficient: The rhs must not become "too negative". Increasing now Σ, the first term in (1.5) would blow up unless ϕ_{min} increases correspondingly, but in this case the second term blows up.

For small Σ, however, simple arguments along these lines are inconclusive regarding existence and non-existence, and the problem is subtle. The first rather general existence and non-existence results for this case are due to Hebey, Pacard and Pollack [13]. Roughly speaking, they showed that there exist solutions if $\int_M \Sigma^2$ is small and no solutions if $\int_M \Sigma^{5/6}$ is large. As published, the results require $\Sigma > 0$. This makes them unsuitable for axially symmetric data for which Σ "typically" (i.e. in all examples we are aware of, in particular the ones of Sect. 4) vanishes on the axis. However, the required extension to non-negative Σ is available in Hebey's IAS lecture notes [14]. Still, the results of [13, 14] are not optimal in the sense that there is a "Σ-gap", i.e. a domain of Σ which is not covered.

The breakthrough for this problem came very recently with the work of Premoselli [20]. In his work, Σ is split as $\Sigma = b\ \Sigma_0$ for some constant b and a function Σ_0. Keeping now Σ_0 fixed and changing b, Premoselli's main theorem asserts the existence of a characteristic value $b_\star$ such that there are at least two solutions for $b < b_\star$, precisely one solution for $b = b_\star$ and no solutions for $b > b_\star$. Important concepts which play a role in Premoselli's work are the notions of "minimality" and "stability" of solutions (and of the data generated) under conformal deformations. In particular, for $b < b_\star$ there is always a unique stable, minimal solution. Stability of solutions will be a key tool in the analysis of the KdS example in Sect. 4.2.2.

1.2 Achievements of this thesis

We first elaborate the existence and non-existence criteria of Hebey et al. [14, 13] in the special case where the compact seed manifold M has constant curvature. We obtain bounds which depend on this curvature as well as on the volume and the Yamabe constant of M. We then restrict ourselves further to seed manifolds of topology $\mathbb{S}^2 \times \mathbb{S}$ and we elaborate two classes of examples. Most of the results of this thesis are published in [4]; however, some results are only announced in this paper. The present exposition provides the required supplementary material and further observations.

1.2.1 Bowen-York data

Our first class of examples is called "Bowen-York data" [5, 3] as it is inspired by the synonymous ten-parameter family of solutions of (1.3) in *flat space*. These data consist of a 3-parameter family of boosted black holes, a 3-parameter family of rotating black holes, while the physical significance of the remaining ones is less clear. We now focus on the "rotating" family which we carry over to the "round" $\mathbb{S}^2 \times \mathbb{S}$. In this setting the Lichnerowicz equation (1.5) has two $O(2)$-symmetries, an "axial" and a "toroidal" one, acting on $\mathbb{S}^2$ and $\mathbb{S}$, respectively. Corresponding to the axial symmetry we can consider the ("Komar"-) angular momentum. As to obtaining solutions of (1.5), we first (in Sect. 4.1) apply the results of Hebey et al. [14, 13] which gives an upper bound on $J\Lambda$ as sufficient condition for existence, and a lower bound as sufficient condition for non-existence, with the expected gap in between. We then (in Sect. 4.2) use the results of Premoselli [20], together with some numerics carried out in [4], to obtain the following structure of the axially symmetric solutions in terms of the dimensionless bifurcation parameter $b = 3\Lambda J/2$: For each value of $0 \leq b < b_* \approx 0.238$ we obtain precisely two solutions ϕ_s and ϕ_u, which respect both the axial as well as the toroidal symmetry of the background, and which behave "stably" and "unstably" under conformal deformations. At $b = b_*$ these branches coincide at a unique marginally stable solution, while at $b = 0$ the "stable" branch terminates at $\phi \equiv 0$, and the "unstable" solution reduces to $\phi \equiv 1$. Moreover, from the unstable branch, there bifurcate in general other branches of solutions which break the toroidal symmetry of the background. The number of these branches depends of the circumference of the $\mathbb{S}$-direction of the background metric. At $b = 0$ they reduce to time-symmetric data for the Kottler (Schwarzschild-de Sitter) family of solutions. We conjecture that there are no solutions which break the axial symmetry of the background and of the Lichnerowicz equation.

1.2.2 “overspinning” Kerr de Sitter data

As the second class of examples, we consider as seed manifold a maximal slice of the Kerr-de Sitter (KdS) spacetime, which represents stationary, axisymmetric black holes with angular momentum J and a mass (-parameter) m. We consider the problem of constructing data for dynamic (non-stationary) black holes by rescaling the KdS angular momentum (defined from K_{ij}) to a value $\hat{J}$ different from the one corresponding to the parameters of the background. Turning to the Lichnerowicz equation, Premoselli's theorem [20] now immediately implies that solutions exist for all $0 \leq \hat{J} \leq J$. More interestingly, we can show that solutions also exist for $\hat{J} > J$ provided both J and $|\hat{J} - J|$ are small enough. Hence KdS can be “overspun” in this sense. The key to obtaining this subtle result is the “instability” property of the KdS data under conformal deformations. Together with Premoselli's theorem, this observation also allows us to conclude that there must be “stable” data as well, conformal to the given KdS ones and with the same angular momentum.

2 The Lichnerowicz equation with Bowen-York data

2.1 The Lichnerowicz equation

We start here with recalling key steps of the well-known conformal method for solving the constraints (1.2) and (1.3) cf. e.g. [9, 15]. Up to formula (2.16) we keep the space dimension n arbitrary. We assume that the trace of $\tilde{K}$ is constant. This reduces the constraint equation (1.3) to

$$\tilde{D}^i \tilde{K}_{ij} = 0. \tag{2.1}$$

For some prescribed metric g_{ij}, we will obtain a solution of the constraints by multiplying with a power of a smooth, positive function ϕ

$$g_{ij} = \phi^l \tilde{g}_{ij}, \tag{2.2}$$

where $l \in \mathbb{R}$ will be fixed shortly.

This relation between the metrics implies the following relation between the corresponding Christoffel symbols

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + \frac{l}{2\phi}(\delta^i_k \partial_j \phi + \delta^i_j \partial_k \phi - \tilde{g}_{jk} \tilde{D}^i \phi), \tag{2.3}$$

and for a given trace-free symmetric tensor field L^{ij}

$$\begin{aligned} D_i L^{ij} &= \partial_i L^{ij} + \Gamma^i_{ik} L^{kj} + \Gamma^j_{ik} L^{ik} && (2.4)\\ &= \tilde{D}_i L^{ij} + (\Gamma^i_{ik} - \tilde{\Gamma}^i_{ik}) L^{kj} + (\Gamma^j_{ik} - \tilde{\Gamma}^j_{ik}) L^{ik}. && (2.5)\end{aligned}$$

Inserting (2.3) we obtain the relation

$$\begin{aligned} D_i L^{ij} &= \tilde{D}_i L^{ij} + \frac{(n+2)l}{2\phi} \partial_k \phi L^{kj} && (2.6)\\ &= \phi^{-(n+2)l/2} \tilde{D}_i \left(\phi^{(n+2)l/2} L^{ij}\right). && (2.7)\end{aligned}$$

From this equation it is obvious that if our given tensor L^{ij} is g-divergence free then

$$\tilde{D}_i \left(\phi^{(n+2)l/2} L^{ij}\right) = 0. \tag{2.8}$$

We can now set

$$\tilde{L}^{ij} := \phi^{(n+2)l/2} L^{ij}, \tag{2.9}$$

which implies that

$$D_i L^{ij} = 0 <=> \tilde{D}_i \tilde{L}^{ij} = 0. \tag{2.10}$$

So by giving a symmetric, trace-free, and g-divergence free tensor L^{ij} and multiplying with a conformal factor we get a tensor which satisfies the constraint (2.1). This property extends to K^{ij} when its trace τ is required to be constant:

$$\tilde{K}^{ij} := \tilde{L}^{ij} + \frac{\tau}{n} \tilde{g}^{ij}. \tag{2.11}$$

To fix a convenient value of l we consider the relation between the Ricci scalars R, $\tilde{R}$ of the metrics g^{ij} and $\tilde{g}^{ij}$:

$$\tilde{R} \phi^{-l} = R + \frac{(n-1)l}{\phi} \Delta_g \phi + \frac{(n-1)l[(n-2)l+4]}{4\phi^2} |d\phi|_g^2. \tag{2.12}$$

The optimal choice for l is

$$l = -\frac{4}{n-2} \tag{2.13}$$

since the last term vanishes, and we are left with the simple formula

$$\tilde{R} \phi^{4/(n-2)} = R - \frac{4(n-1)}{(n-2)\phi} \Delta_g \phi. \tag{2.14}$$

We now apply definitions (2.9) and (2.11) to the terms in our constraint equation (1.2)

$$\tilde{K}_{ab} \tilde{K}^{ab} - \tilde{K}^2 = \tilde{g}_{ik} \tilde{g}_{jl} \left(\tilde{L}^{ij} + \frac{\tau}{n} \tilde{g}^{ij} \right) \left(\tilde{L}^{kl} + \frac{\tau}{n} \tilde{g}^{kl} \right) - \tau^2 \tag{2.15}$$

$$= \phi^{nl} |L|_g^2 - \frac{n-1}{n} \tau^2. \tag{2.16}$$

For the rest of chapter 2, we restrict ourselves to dimension $n = 3$. Combining equations (1.2), (2.16) and (2.14) we obtain

$$\Delta_g \phi - \frac{R}{8} \phi = -\frac{L^{ij} L_{ij}}{8\phi^7} + \left(\frac{\tau^2}{12} - \frac{\Lambda}{4} \right) \phi^5, \tag{2.17}$$

This equation is known as the Lichnerowicz equation.

From now on we set $\tau = 0$ so that we have a trace-free $\tilde{K}^{ij}$. This simplifies the equation to

$$\left(\Delta_g - \frac{R}{8}\right)\phi = -\frac{K^{ij}K_{ij}}{8\phi^7} - \frac{\Lambda}{4}\phi^5 \tag{2.18}$$

which is the form mentioned before, cf. (1.5). The transformations (2.2) and (2.9) now take the form

$$\tilde{g}_{ij} = \phi^4 g_{ij}, \tag{2.19}$$
$$\tilde{K}^{ij} = \phi^{-10}K^{ij}. \tag{2.20}$$

2.2 (t, φ)-symmetric data

We now describe the construction of the so-called (t, φ)-symmetric class of data. The terminology goes back to perfect fluid data [2] and indicates that the evolved spacetimes are invariant under simultaneous reversal of time and spin. In the following description we follow in essence [11] (see also [12]). The point is to construct a special class of trace-free, divergence-free extrinsic curvatures K^{ij} from a rather general class of functions ω. We start from an axially symmetric metric g_{ij}; this means that there is a Killing vector η^i

$$D_i\eta_j + D_j\eta_i = 0 \tag{2.21}$$

with closed orbits and zeros aligned along a curve ("axis"). Moreover, we assume η to be hypersurface orthogonal

$$\epsilon_{ijk}\eta^i D^j \eta^k = 0. \tag{2.22}$$

We next choose some axially symmetric function ω, i.e. its Lie derivative with respect to η^i vanishes

$$\pounds_\eta \omega = 0. \tag{2.23}$$

From ω we define a vector field

$$S^i = \frac{1}{\eta}\epsilon^{ijk}\eta_b D_c \omega. \tag{2.24}$$

Here $\eta = \eta^i\eta^j g_{ij}$, ε_{ijk} is totally antisymmetric with $\varepsilon_{123} = 1$, and ϵ_{ijk} is the volume form

$$\epsilon_{ijk} = \sqrt{|det(g)|}\varepsilon_{ijk}, \tag{2.25}$$
$$\epsilon^{ijk} = \frac{1}{\sqrt{|det(g)|}}\varepsilon^{ijk}. \tag{2.26}$$

Using (2.21) and (2.23) it follows for S^i that

$$\pounds_\eta S^i = 0, \tag{2.27}$$
$$S^i \eta_i = 0, \tag{2.28}$$
$$D_i S^i = 0. \tag{2.29}$$

We can now define the tensor K^{ij} based on this vector field

$$K^{ij} = \frac{2}{\eta} S^{(i}\eta^{j)} = \frac{2}{\eta^2}\eta^{(i}\epsilon^{j)kl}\eta_k D_l \omega. \tag{2.30}$$

We require that K_{ij} is smooth everywhere, in particular on the axis, which poses a restriction on ω. The resulting K_{ij} is now easily seen to be trace-free and divergence-free. For the Lichnerowicz equation (2.18) we need $K^{ij}K_{ij}$ which takes the form

$$K^{ij}K_{ij} = 2\frac{D_i\omega D^i\omega}{\eta^2}. \tag{2.31}$$

We mention here the following relation to the axially symmetric spacetime $(\tilde{N}, \tilde{h}_{\alpha\beta})$ which can be constructed out of the data $(g_{ij},\, K_{ij})$ defined above. Denoting the spacetime Killing vector by $\tilde{\zeta}_\alpha$, the spacetime volume form by $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$ and covariant derivative by $\tilde{\nabla}_\alpha$ we can define the twist vector

$$\tilde{\omega}_\alpha = -\frac{1}{2}\tilde{\epsilon}_{\alpha\beta\gamma\delta}\tilde{\zeta}^\beta\tilde{\nabla}^\gamma\tilde{\zeta}^\delta. \tag{2.32}$$

Now Einstein's equations (1.1) imply that $\tilde{\omega}_\alpha$ is curl-free, i.e. $\tilde{\nabla}_{[\alpha}\tilde{\omega}_{\beta]} = 0$. On a simply connected domain, it follows that there exists a potential $\tilde{\omega}$, defined up to a constant by

$$\tilde{\omega}_\alpha = \frac{\partial}{\partial\alpha}\tilde{\omega}. \tag{2.33}$$

It can be shown that the restriction of this $\tilde{\omega}$ to the initial surface $\tilde{M}$ agrees with the function ω from which the data were constructed, if the additive constants are adapted. This justifies the notation.

We can introduce coordinates (r, θ, φ) adapted to the hypersurface orthogonal axial symmetry such that the metric takes the form

$$g = e^{-2q}\left(dr^2 + r^2 d\theta^2\right) + r^2\sin^2\theta d\varphi^2 \tag{2.34}$$

with q satisfying

$$\partial_\varphi q = 0. \tag{2.35}$$

Our Killing vector is therefore $\eta^i = (\partial/\partial\varphi)$ which gives us $\eta = r^2\sin^2\theta$. For the $K^{ij}K_{ij}$-term in (2.18) we obtain from (2.31)

$$\frac{K^{ij}K_{ij}}{8\phi^7} = \frac{D_i\omega D^i\omega}{4\phi^7\eta^2} \tag{2.36}$$

$$= e^{2q}\frac{(r\partial_r\omega)^2 + (\partial_\theta\omega)^2}{4r^6\sin^4\theta\phi^7}. \tag{2.37}$$

If ω is chosen such that K_{ij} is regular, its derivatives in the numerator of (2.37) have to cancel the singular denominator $1/\sin^4\theta$. This will be explicit in the examples below.

We summarize the construction of the data described above as follows:

1. Choose an ω and check K^{ij} defined by (2.30) for regularity.
2. Use ω to calculate $K^{ij}K_{ij}$ via (2.37).
3. Solve the Lichnerowicz equation (2.18) to obtain the conformal factor ϕ .
4. Apply ϕ to (2.19) and (2.20) to get the desired initial data set $(\tilde{M}, \tilde{g}_{ij}, \tilde{K}_{ij})$

We now recall the "Komar-" angular momentum J defined for a two-surface S:

$$J = -\frac{1}{8\pi}\int_S K_{ij}\eta^j dS^i. \tag{2.38}$$

Since the integrand is divergence-free, the integral does not depend on the surface S within a homology class. Moreover, the above expression is conformally invariant. We can also write (2.38) as

$$J = -\frac{1}{8\pi}\int_S S_i n^i dS. \tag{2.39}$$

A simple calculation in adapted coordinates [12] gives still another equivalent expression for J, namely

$$J = \frac{1}{4}\left(\omega(r, \theta = \pi) - \omega(r, \theta = 0)\right), \tag{2.40}$$

which only involves the values of ω at the poles.

2.3 Bowen-York on flat space

In this and the next section we recall and extend the so-called Bowen-York class of data [5, 3]. The simplest example for the conformal method is the construction of time-symmetric Schwarzschild data from the flat background metric

$$g = dr^2 + r^2 d\Omega^2. \tag{2.41}$$

Setting $K^{ij} = \Lambda = R = 0$ the Lichnerowicz equation (2.18) reduces to

$$\Delta_g \phi = 0. \tag{2.42}$$

Clearly the flat metric itself satisfies the vacuum constraint equations, so $\phi = 1$ is an expected result. By the maximum principle, this is the unique result if we restrict ourselves to smooth data which satisfy the asymptotic condition

$$\lim_{r\to\infty} \phi = 1. \tag{2.43}$$

To obtain the Schwarzschild metric we require the singular boundary condition

$$\lim_{r\to 0} r\phi = \frac{m}{2}. \tag{2.44}$$

The unique solution for (2.42) in this case is

$$\phi = 1 + \frac{m}{2r}. \tag{2.45}$$

Applying the coordinate transformation

$$r' = r\left(1 + \frac{m}{2r}\right)^2 \tag{2.46}$$

to these "isotropic coordinates" we obtain

$$\begin{aligned} \tilde{g} &= \phi^4 g & (2.47)\\ &= \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) & (2.48)\\ &= \left(\frac{1}{1 - \frac{2m}{r'}}\right) dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\varphi^2, & (2.49) \end{aligned}$$

which are the time-symmetric Schwarzschild data in standard coordinates.

We now consider more general extrinsic curvatures, still on a flat background, and with the aim of constructing data with $\Lambda = 0$. Then the corresponding momentum constraint and the Lichnerowicz equation (2.18) read

$$\begin{aligned} D_i K^{ij} &= 0 & (2.50)\\ \Delta\phi &= -\frac{K^{ij}K_{ij}}{8\phi^7} & (2.51) \end{aligned}$$

where Δ and D_i now refer to the flat metric. According to [5, 3] there exist solutions of (2.50) which take the form

$$K_{ij} = \frac{3}{2r^2}[P_i n_j + P_j n_i - (\delta_{ij} - n_i n_j)P^i n_i)] \tag{2.52}$$
$$K_{ij} = \frac{6}{r^3}\varepsilon_{kl(i}J^k n^l n_{j)} \tag{2.53}$$
$$K_{ij} = \frac{C}{r^3}(3n_i n_j - \delta_{ij}) \tag{2.54}$$
$$K_{ij} = \frac{3}{2r^4}[-Q_i n_j - Q_j n_i - (\delta_{ij} - 5n_i n_j)Q^i n_i], \tag{2.55}$$

which are known as the Bowen-York solutions. Here n_i is the radial unit vector $n_i = x_i/r$, δ_{ij} is the Kronecker delta function, C is a constant, and P_i, S_i and Q_i are constant vectors when referred to Cartesian coordinates.

The simplest case is (2.54). In this case the Lichnerowicz equation becomes

$$\Delta\phi = -\frac{K^{ij}K_{ij}}{8\phi^7} = -\frac{3C^2}{4r^6\phi^7}. \tag{2.56}$$

which can be solved under the same regularity and boundary conditions as above, namely (2.43) and (2.44).

Since the resulting solutions will be spherically symmetric, the constructed spacetimes will, by Birkhoff's theorem, still be members of the Schwarzschild family, constructed via non-time-symmetric data.

Before turning to the other members of the Bowen-York family, we ask if the above data (2.54) are in the (t,φ)-symmetric class considered in Sect. 2.2; In order to check this, we recall (2.37) and solve

$$K^{ij}K_{ij} = \frac{2\left[(r\partial_r\omega_1)^2 + (\partial_\theta\omega_1)^2\right]}{r^6\sin^4\theta} = \frac{6C^2}{r^6} \tag{2.57}$$

for ω_1, which works indeed:

$$\omega_1 = \frac{\sqrt{3}C}{4}(2\theta - \sin(2\theta)). \tag{2.58}$$

Therefore, it seems that the constant C can be related to the angular momentum (2.40).

Nevertheless, this interpretation fails as the extrinsic curvature K^{ij} constructed from ω_1 via (2.30) is not regular on the axis. To see this explicitly we use Cartesian coordinates since polar coordinates degenerate on the axis themselves. We have

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{2.59}$$
$$\theta = \arccos\left(\frac{z}{r}\right), \tag{2.60}$$
$$\varphi = \arctan\left(\frac{y}{x}\right). \tag{2.61}$$

For our Killing vector this means

$$\eta^i = (0,0,1)^T \longrightarrow \eta^i = (-y,x,0)^T, \tag{2.62}$$
$$\eta = r^2 \sin^2\theta \longrightarrow \eta = x^2+y^2, \tag{2.63}$$

and ω_1 is given by

$$\omega_1 = \frac{\sqrt{3}C}{4}\left[2\theta - \sin(2\theta)\right] = \frac{\sqrt{3}C}{4}\left[2\arccos\left(\frac{z}{r}\right) - \sin\left(2\arccos\left(\frac{z}{r}\right)\right)\right]. \tag{2.64}$$

Applying (2.24) with the flat metric we get

$$S^x = \frac{1}{\eta}\epsilon^{xyz}\eta^y D_z\omega_1 = -\frac{\sqrt{3}Cx\sqrt{x^2+y^2}}{r^4}, \tag{2.65}$$
$$S^y = \frac{1}{\eta}\epsilon^{yxz}\eta^x D_z\omega_1 = -\frac{\sqrt{3}Cy\sqrt{x^2+y^2}}{r^4}, \tag{2.66}$$
$$S^z = \frac{1}{\eta}\left(\epsilon^{zxy}\eta^x D_y\omega_1 + \epsilon^{zyx}\eta^y D_x\omega_1\right) = \frac{\sqrt{3}Cz\sqrt{x^2+y^2}}{r^4}. \tag{2.67}$$

Inserting these formulas into (2.30) gives

$$K^{xx} = \frac{4\sqrt{3}Cxy}{\sqrt{x^2+y^2}r^4}; \qquad K^{yy} = \frac{4\sqrt{3}Cxy}{\sqrt{x^2+y^2}r^4}; \qquad K^{zz} = 0 \tag{2.68}$$

$$K^{xz} = \frac{2\sqrt{3}Czy}{\sqrt{x^2+y^2}r^4}; \qquad K^{yz} = -\frac{2\sqrt{3}Czx}{\sqrt{x^2+y^2}r^4}; \qquad K^{xy} = -\frac{2\sqrt{3}C\sqrt{x^2+y^2}}{r^4}. \tag{2.69}$$

Considering the poles via $\lim_{(x,y)\to 0}$ we have no problems for K^{xx}, K^{yy}, K^{zz} and K^{xy} but discover singularities for K^{xz} and K^{yz}. We conclude that ω_1 does not generate regular data, and hence the data (2.54) are not (t,φ)-symmetric.

We next turn to (2.52). The constant P_i can be interpreted as "boost"; in particular, a boosted Schwarzschild black hole is obtained as follows. We insert (2.52) in (2.51) which gives

$$\Delta_g\phi = -\frac{9}{16r^4}\frac{P^2 + 2(P^i n_i)^2}{\phi^7}. \tag{2.70}$$

Then we solve this equation with the ansatz

$$\phi = 1 + \frac{m}{2r} + \psi \tag{2.71}$$

with a ψ that vanishes at $r=\infty$ and is smooth near $r=0$. Using the methods of [5, 3] it can be proven that there exists a unique positive solution.

Considering finally (2.53), we first show that these data fall into the class of (t, φ)-symmetric data considered in the previous section, with angular momentum $J_i = (0, 0, J)$ in Cartesian coordinates. In fact combining (2.53) with definition (2.30) and (2.40) and switching to polar coordinates, the generating function ω is of the form [11]

$$\omega_{BY} = J(\cos^3\theta - 3\cos\theta). \tag{2.72}$$

In these coordinates, the only non-vanishing component of K^{ij} is

$$K^{r\varphi} = \frac{1}{\eta^2}\eta^\varphi \epsilon^{r\varphi\theta}\eta_\varphi D_\theta \omega = -\frac{3J}{r^4} \tag{2.73}$$

$$K_{r\varphi} = -\frac{3J\sin^2\theta}{r^2} \tag{2.74}$$

From this we get

$$\frac{K^{ij}K_{ij}}{8\phi^7} = \frac{K^{r\varphi}K_{r\varphi}}{4\phi^7} = \tag{2.75}$$

$$= \frac{9\sin^2\theta J^2}{4r^6\phi^7}. \tag{2.76}$$

As desired we got rid of the $1/\sin^4\theta$ term. We now check in detail that the extrinsic curvature generated from ω_{BY} (2.72) is regular. Switching back to Cartesian coordinates we find

$$\omega_{BY} = J\left[\cos^3\theta - 3\cos\theta\right] = J\left[\left(\frac{z}{r}\right)^3 - 3\left(\frac{z}{r}\right)\right]. \tag{2.77}$$

Using formula (2.24) we get

$$S^x = \frac{1}{\eta}\epsilon^{xyz}\eta^y D_z\omega_{BY} = -\frac{3Jx(x^2+y^2)}{r^5}, \tag{2.78}$$

$$S^y = \frac{1}{\eta}\epsilon^{yxz}\eta^x D_z\omega_{BY} = -\frac{3Jy(x^2+y^2)}{r^5}, \tag{2.79}$$

$$S^z = \frac{1}{\eta}\left(\epsilon^{zxy}\eta^x D_y\omega_{BY} + \epsilon^{zyx}\eta^y D_x\omega_{BY}\right) = -\frac{3Jz(x^2+y^2)}{r^5}. \tag{2.80}$$

Inserting again into (2.30) gives

$$K^{xx} = \frac{6Jxy}{r^5}; \qquad K^{yy} = -\frac{6Jxy}{r^5}; \qquad K^{zz} = 0 \tag{2.81}$$

$$K^{xz} = \frac{3Jyz}{r^5}; \qquad K^{yz} = -\frac{3Jxz}{r^5}; \qquad K^{xy} = \frac{3J(-x^2+y^2)}{r^5}. \tag{2.82}$$

None of these expressions have singularities at the poles $\lim_{(x,y)\to 0}$, hence (2.53) indeed corresponds to (t, φ)-symmetric data with angular momentum J_i generated by ω_{BY}.

2.4 Bowen-York on Nariai

We are now going to introduce a new data set via a conformal transformation of the flat metric

$$g = \phi^4(dr^2 + r^2 d\Omega^2), \tag{2.83}$$

with

$$\phi^4 = \frac{1}{\Lambda r^2}, \tag{2.84}$$

where Λ is a constant which will be interpreted "cosmologically" below. Since (2.84) is singular at $r = 0$, the resulting metric (2.83) is only locally conformally flat. Applying the coordinate transformation

$$r = e^{\psi} \tag{2.85}$$

we obtain

$$g = \frac{1}{\Lambda}(d\psi^2 + d\Omega^2). \tag{2.86}$$

Identifying ψ periodically with period a (for an arbitrary constant a) (2.86) becomes the round metric on $\mathbb{S}\times\mathbb{S}^2$ in standard form. The manifold then has Ricci scalar $R = 2\Lambda$ and volume $V = 4\pi a/\Lambda^{3/2}$. For short, we call it "Nariai" henceforth since it is the time-symmetric slice of the Nariai spacetime [18].

Our aim is now to carry over the Bowen-York method, which sets out from a flat background and constructs data with $\Lambda = 0$, to construct data with $\Lambda > 0$, starting from the Nariai background (2.86).

We first note that, by local conformal flatness, to every solution of the momentum constraint (2.1) on a flat background, there exists an equivalent solution on the Nariai background. A complete analysis of the case (2.54) which leads to $K^{ij}K_{ij}$ only depending on ψ can be found in [8].

Analogously to the situation on the flat background, the resulting solutions will be spherically symmetric. Therefore, the constructed spacetimes will, by Birkhoff's theorem, be members of the Schwarzschild-de Sitter family, constructed via non-time-symmetric data.

From now onwards we restrict ourselves to the rotational case, i.e. to (2.53), which will again be interpreted as (t, φ)-symmetric data.

In particular, considering the generating function (2.72) on the background (2.86), we obtain

$$K^{\psi\varphi} = \frac{1}{\eta^2}\eta^{\varphi}\epsilon^{\psi\varphi\theta}\eta_{\varphi}D_{\theta}\omega = -3J\Lambda^{5/2} \tag{2.87}$$

$$K_{\psi\varphi} = -3J\Lambda^{1/2}\sin^2\theta \tag{2.88}$$

as the only non-vanishing term. Alternatively, we can get (2.88) directly from (2.74) via a conformal transformation $K_{\psi\varphi} = \phi^{-2} K_{r\varphi} \frac{\partial r}{\partial \psi}$, using (2.84) and the coordinate transformation (2.85).

As a third alternative, we can adapt the Bowen-York expression (2.53) which now takes the form

$$K_{ij} = 6\Lambda^{3/2} \varepsilon_{kl(i} J^k n^l n_{j)}, \tag{2.89}$$

where the vector J^i reads $\Lambda^{1/2} J(\cos\theta, \sin\theta, 0)$.

Finally, there is yet a fourth way of constructing the data (2.88). For any two orthogonal Killing vectors η^i and ξ^j we define the trace-free tensor

$$K_{ij} = c\eta_{(i}\xi_{j)}, \tag{2.90}$$

where c is a constant. This tensor is also divergence free:

$$\begin{aligned}
D^i(c\eta_{(i}\xi_{j)}) &= c(\xi_j D^i \eta_i + \eta_i D^i \xi_j + \xi_i D^i \eta_j + \eta_j D^i \xi_i) && (2.91)\\
&= c(\xi_j D^i \eta_i - \eta_i D^j \xi_i - \xi_i D^j \eta_i + \eta_j D^i \xi_i) && (2.92)\\
&= c(\xi_j D^i \eta_i + \xi_i D^j \eta_i + \eta_i D^j \xi_i + \eta_j D^i \xi_i) && (2.93)\\
&= 0. && (2.94)
\end{aligned}$$

Considering now the Killing vectors

$$\begin{aligned}
\eta^i &= (\partial/\partial\varphi) && (2.95)\\
\xi^j &= (\partial/\partial\psi) && (2.96)
\end{aligned}$$

on the Nariai background, we see that (2.90) agrees with (2.88) when we set

$$c = -3J\Lambda^{5/2}. \tag{2.97}$$

From (2.88) we can calculate

$$K^{ij}K_{ij} = 2K^{\psi\varphi}K_{\psi\varphi} = 18J^2\Lambda^3 \sin^2\theta. \tag{2.98}$$

which will be used in Sect. 4.1.3.

3 The Yamabe Constant

An important quantity characterizing compact Riemannian manifolds which will be used below is the Yamabe constant. In the next subsection we recall its definition and its properties, following in essence [17]. Then in Sect. 3.2 we apply it to the Nariai metric.

3.1 General properties

First we recall equation (2.14) which for $n = 3$ looks like

$$\tilde{R}\phi^5 = R\phi - 8\Delta_g\phi. \tag{3.1}$$

The "Yamabe problem" consists of obtaining a metric with constant Ricci scalar $\tilde{R}$ via a conformal transformation. We rewrite (3.1) as a non-linear eigenvalue problem which is known as the Yamabe equation

$$\Box\phi = -\lambda\phi^5, \tag{3.2}$$

with

$$\Box = 8\Delta_g - R \tag{3.3}$$

Obviously, we have $\lambda = \tilde{R}$.

We now quote the following result

Theorem 3.1 (Yamabe, Trudinger, Aubin, Schoen)**.** *The Yamabe problem can be solved on any compact manifold M.*

To illustrate key elements of the proof, we consider the functional

$$Q_g(\phi) = E(\phi)/||\phi||_6^2, \tag{3.4}$$

where

$$E(\phi) = \int_M 8|D\phi|^2 + R\phi^2 dv_g, \tag{3.5}$$

$$||\phi||_6 = \left(\int_M |\phi|^6 dv_g\right)^{1/6}. \tag{3.6}$$

We look for the critical points of $Q_g(\phi)$; by partial integration we find

$$\frac{d}{dt}Q_g(\phi+t\psi)|_{t=0} = \frac{2}{||\phi||_6^2}\int_M \left(8\Delta\phi + R\phi - ||\phi||_6^{-6}E(\phi)\phi^5\right)\psi dv_g. \tag{3.7}$$

In order for ϕ to be a critical point of $Q_g(\phi)$, (3.7) has to be zero for any ψ, and therefore the expression in parenthesis has to vanish:

$$8\Delta\phi + R\phi - ||\phi||_6^{-6}E(\phi)\phi^5 = 0. \tag{3.8}$$

So ϕ is a critical point if and only if it satisfies the Yamabe equation with

$$\lambda = E(\phi)/||\phi||_6^6. \tag{3.9}$$

Now Hölder's inequality implies that $|\int_M R\phi^2 dv_g|$ is bounded from above by $||\phi||_6^2$, which implies that $Q_g(\phi)$ is bounded from below. We can therefore define the Yamabe constant as

$$Y(M) = \inf\left(Q_g(\phi) : \phi \text{ a smooth, positive function on } M\right) \tag{3.10}$$

$$= \inf_{\phi\in C_0^\infty, \phi\not\equiv 0} \frac{\int_M 8|D\phi|^2 + R\phi^2 dv_g}{\left(\int_M |\phi|^6 dv_g\right)^{1/3}}. \tag{3.11}$$

An important property of this constant is that it is invariant under conformal transformations which can be seen as follows. For a conformal factor ξ we have

$$g = \xi^{-4}\hat{g}, \tag{3.12}$$

$$dv_g = \xi^{-6}d\hat{v}_g, \tag{3.13}$$

$$\phi = \xi\hat{\phi}, \tag{3.14}$$

For the denominator we get

$$\left(\int_M |\phi|^6 dv_g\right)^{1/3} = \left(\int_M \xi^6|\hat{\phi}|^6\xi^{-6}d\hat{v}_g\right)^{1/3} = \left(\int_M |\hat{\phi}|^6 d\hat{v}\right)^{1/3}. \tag{3.15}$$

For the numerator we consider the conformal Laplace operator in 3 dimensions

$$L := \frac{R}{8} - \Delta_g. \tag{3.16}$$

which satisfies, for any function f

$$L(\xi f) = \xi^5\hat{L}(f). \tag{3.17}$$

Since we work on a compact manifold the numerator results in

$$\int_M 8|D\phi|^2 + R\phi^2 dv_g = \int_M \xi^6\left(8|\hat{D}\hat{\phi}|^2 + \hat{R}\hat{\phi}^2\right)\xi^{-6}d\hat{v}_g = \int_M \left(8|\hat{D}\hat{\phi}|^2 + \hat{R}\hat{\phi}^2\right)d\hat{v}_g. \tag{3.18}$$

If $\phi \in C_0^\infty$ then it must also be true that $\xi^{-1}\phi = \hat{\phi} \in C_0^\infty$, and combining this with (3.15) and (3.18) we get

$$Y(M) = \inf_{\phi\in C_0^\infty,\phi\not\equiv 0} \frac{\int_M 8|D\phi|^2 + R\phi^2 dv_g}{\left(\int_M |\phi|^6 dv_g\right)^{1/3}} = \inf_{\hat{\phi}\in C_0^\infty,\hat{\phi}\not\equiv 0} \frac{\int_M 8|\hat{D}\hat{\phi}|^2 + \hat{R}\hat{\phi}^2 d\hat{v}_g}{\left(\int_M |\hat{\phi}|^6 d\hat{v}_g\right)^{1/3}}. \tag{3.19}$$

We have shown that the Yamabe constant is conformally invariant.

A last point that should be addressed concerning the Yamabe constant is its relation to the Ricci scalar. We rewrite (3.4) in terms of $\tilde{g}$ as

$$Q(\tilde{g}) = \frac{\int_M \tilde{R} d\tilde{v}_g}{(\int_M d\tilde{v}_g)^{1/3}}. \tag{3.20}$$

Since we already know that within a conformal class the functional $Q(\phi)$ is minimized for constant Ricci scalar, we can pull the latter out of the integral. Hence we are left with

$$Y(M) = \inf_{\phi} \tilde{R} \frac{\int_M d\tilde{v}_g}{(\int_M d\tilde{v}_g)^{1/3}} = \inf_{\phi} \tilde{R}\tilde{V}^{2/3}, \tag{3.21}$$

with V being the volume $V = \int_M dv_g$. This expression cannot be simplified further as the solution of the Yamabe problem is not unique in general, i.e. there may exist more than one metric with constant Ricci scalar within a conformal class. In any case, we see that the Yamabe constant is proportional to a constant Ricci scalar.

3.2 The Yamabe constant for Nariai data

We recall the Nariai data (2.86). As shown in [21], for $a \in (2k\pi, 2(k+1)\pi]$ there exist $k+1$ solutions of the Lichnerowicz equation (2.18) which all have constant Ricci scalar $R = 2\Lambda$. One of these is the Nariai metric itself with $\phi_0 = 1$, while the remaining k solutions are Schwarzschild-de Sitter data with fundamental periods $a, \frac{a}{2}, \frac{a}{3}, ..., \frac{a}{k}$. They contain minimal and maximal two surfaces called "black hole" and "cosmological horizons", respectively. In spherical coordinates (r, θ, φ), they look like

$$g_k = \frac{dr^2}{1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.22}$$

where m is a constant determined by Λ, a and k. Since $R = 2\Lambda$ we need to calculate the volume if we want to obtain the Yamabe constant. Is is given as

$$V_n = 8n\pi \int_{r_B(a/n)}^{r_C(a/n)} \frac{r^2 dr}{\sqrt{Z}}, \tag{3.23}$$

with

$$Z = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{3.24}$$

where $n = 1, 2, ..., k$ and r_B and r_C are the black hole horizon and the cosmological horizon, respectively.

This integral can only be obtained numerically. As is shown in [21] for $a \geq 2\pi$ it is within the bounds

$$\int_{r_B(a/n)}^{r_C(a/n)} \frac{r^2 dr}{\sqrt{Z}} \in \frac{\pi}{\Lambda^{3/2}} \left(1, \frac{3^{3/2}}{4}\right). \tag{3.25}$$

As a result we also get bounds for the Yamabe constant

$$Y \in 8\pi^{4/3} \left(1, \frac{3}{4^{2/3}}\right). \tag{3.26}$$

The case $a \leq 2\pi$ is simpler since there is only the solution $\phi = 1$. Therefore the Yamabe constant given by (3.21) takes the value

$$Y(M) = RV^{2/3} = 2\Lambda \left(\frac{4\pi a}{\Lambda^{3/2}}\right)^{2/3} = 2(4\pi a)^{2/3}. \tag{3.27}$$

4 Existence and non-existence of solutions

The core of this thesis is the application of the theorems of Hebey et al [13] and Premoselli [20] on existence and non-existence of solutions of the Lichnerowicz equation (2.18). We first (in Sect. 4.1.1) recall the former theorem in general form, apply it next (in Sect. 4.1.2) to the case of constant Ricci scalar and finally to Bowen-York data in Sect. 4.1.3. In Sect. 4.2 we recall Premoselli's theorem and apply it to Nariai and Kerr-de Sitter.

4.1 The theorems of Hebey, Pacard and Pollack

4.1.1 The general case

In Sect. 1.1.3 of the introduction we gave a heuristic argument for non-existence if the momentum density Σ^2 is large. A quantitative version of this statement is Thm. 2.1 of [13], which we recall below, together with the proof. Following the notation of this paper, we consider an arbitrary dimension n and write the Lichnerowicz equation in the form

$$(\Delta_g - h)\phi = -B\phi^{2^*-1} - \frac{A}{\phi^{2^*+1}}, \tag{4.1}$$

where

$$2^* = \frac{2n}{n-2}. \tag{4.2}$$

Theorem 4.1 (Thm. 2.1 in [13])**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Let also $h, A,$ and B be smooth functions on M with $A \geq 0$ in M. If $B > 0$ in M, and*

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv_g > \int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv_g. \tag{4.3}$$

where $h^+ = \max(0, h)$, then the Lichnerowicz equation (4.1) does not possess any smooth positive solution.

Proof. Assume existence and integrating (4.1) we obtain

$$\int_M B\phi^{2^*-1} dv_g + \int_M \frac{A dv_g}{\phi^{2^*+1}} = \int_M h\phi dv_g \tag{4.4}$$

where the $\Delta\phi$ term vanishes because we have a compact manifold. Reformulating the rhs and applying the Hölder inequality gives

$$\int_M h\phi dv_g = \int_M hB^{\frac{2-n}{2+n}} B^{-\frac{2-n}{2+n}} \phi dv_g \tag{4.5}$$

$$\leq \left(\int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv_g\right)^{\frac{4}{n+2}} \left(\int_M B\phi^{2^*-1} dv_g\right)^{\frac{n-2}{n+2}}. \tag{4.6}$$

Manipulating the expression on the lhs of (4.3) in a similar way yields

$$\int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv_g = \int_M A^{\frac{n+2}{4n}} \phi^{\frac{3n^2+4n-4}{4n^2-8n}} \phi^{-\frac{3n^2+4n-4}{4n^2-8n}} B^{\frac{3n-2}{4n}} dv_g \tag{4.7}$$

$$\leq \left(\int_M B\phi^{2^*-1} dv_g\right)^{\frac{3n-2}{4n}} \left(\int_M \frac{A dv_g}{\phi^{2^*+1}}\right)^{\frac{n+2}{4n}}. \tag{4.8}$$

Combining (4.4), (4.6) and (4.8) we get

$$X + \left(\int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv_g\right)^{\frac{4n}{n+2}} X^{1-n} \leq \left(\int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv_g\right)^{\frac{4}{n+2}} \tag{4.9}$$

for

$$X = \left(\int_M B\phi^{2^*-1} dv_g\right)^{\frac{4}{n+2}}. \tag{4.10}$$

As a last step we estimate the lhs from below by replacing X with the quantity for which this lhs, considered now as polynomial in X, takes its minimal value. This implies that

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M A^{\frac{n+2}{4n}} B^{\frac{3n-2}{4n}} dv_g \leq \int_M (h^+)^{\frac{n+2}{4}} B^{\frac{2-n}{4}} dv_g. \tag{4.11}$$

This has to be fulfilled in order for smooth positive solutions to exist. Hence reversing the inequality gives the required criterion for non-existence.

□

We now specify to the quantities appearing in (2.18), namely

$$h = \frac{R}{8}, B = \frac{\Lambda}{4}, A = \frac{K^{ij}K_{ij}}{8}. \tag{4.12}$$

As a result for $n = 3$ we get the non-existence criterion

$$\left(\frac{3^3}{2^2}\right)^{\frac{5}{12}} \int_M \left(\frac{K^{ij}K_{ij}}{8}\right)^{\frac{5}{12}} \left(\frac{\Lambda}{4}\right)^{\frac{10}{12}} dv_g > \int_M \left(\frac{R}{8}\right)^{\frac{5}{4}} dv_g. \tag{4.13}$$

which will be applied in Sect. 4.1.2.

We now turn to the existence criterion Thm. 3.1 in [13]. We first define the Sobolev norm

$$||\varphi||_{H^1_h} = \left(\int_M (|D\varphi|^2 + h\varphi^2)dv_g\right)^{\frac{1}{2}}. \tag{4.14}$$

and the Sobolev constant S_h as the smallest positive constant so that

$$\int_M |\varphi|^{2^*} \leq S_h \left(\int_M (|D\varphi|^2 + h\varphi^2)dv_g\right)^{\frac{2^*}{2}} \tag{4.15}$$

for all $\varphi \in H^1(M)$.

Theorem 4.2 (Thm. 3.1 in [13])**.** *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Let $h, A,$ and B be smooth functions on M for which $\Delta_g - h$ is coercive, $A > 0$ in M, and $max_M B > 0$. If there exists a smooth positive function φ on M such that*

$$||\varphi||^{2^*}_{H^1_h} \int_M \frac{A}{\varphi^{2^*}} dv_g \leq \frac{C}{(S_h \max_M |B|)^{n-1}} \tag{4.16}$$

and

$$\int_M B\varphi^{2^*} dv_g > 0 \tag{4.17}$$

where $||\cdot||$ is defined in (4.14)*, S_h is defined in* (4.15)*, and*

$$C = \left(\frac{1}{2(n-1)}\right)^{\frac{2^*}{2}} \frac{2^* - 2}{4}, \tag{4.18}$$

then the Lichnerowicz equation (4.1) *has a smooth positive solution.*

For the proof we refer to [13].

To apply the above theorem to the present situation, we specify to $n = 3$ and make the assignments (4.12).

This means that (4.14) and (4.15) take the form

$$||\varphi||_{H^1_h} = \left(\int_M (|D\varphi|^2 + \frac{R}{8}\varphi^2)dv_g\right)^{\frac{1}{2}}, \tag{4.19}$$

$$\int_M |\varphi|^6 \leq S_h \left(\int_M (|D\varphi|^2 + \frac{R}{8}\varphi^2)dv_g\right)^3. \tag{4.20}$$

Comparing this definition of the Sobolev constant with (3.11) we conclude that we can replace it by the Yamabe constant via

$$S_h = \left(\frac{8}{Y}\right)^3. \tag{4.21}$$

Furthermore, we note that for manifolds with positive Yamabe constant, $\Delta_g - h = \Delta_g - R/8$ is indeed coercive, as required in Thm 4.2.

A subtle problem now arises in the attempt to apply the above theorem to (t, φ)-symmetric data. This concerns the requirement $A > 0$ of the theorem, which may be violated on the axis. In fact this happens in the Bowen-York example ω_{BY} (2.72) which gave (2.76). This expression vanishes at the poles, and the same occurs in the Kerr-de Sitter example in Sect. 4.2.2. However, there is an extension of the above theorem, namely theorem 3 in [14], which weakens the above requirement to $A \geq 0, A \not\equiv 0$.

With the above assignments, we can write the existence criterion (4.16) as

$$||\varphi||^6_{H^1_h} \int_M \frac{K^{ij}K_{ij}}{8\varphi^6} dv_g \leq \frac{Y^6}{4 \cdot 8^6 \Lambda^2}. \tag{4.22}$$

In this form it will be applied in Sect. 4.1.2.

We conclude this section with a supplement to Thm. 4.2.

Since the Lichnerowicz equation is conformally invariant, optimal criteria for existence and non-existence should be invariant as well. While the non-existence criterion (4.13) does not appear to be invariant, we can show the following.

Proposition 4.1. *The existence criterion* (4.22) *for solutions of* (2.18) *is conformally invariant.*

Proof. We use the conformal factor $\hat{g}_{ij} = \xi^4 g_{ij}$ and recall the formulas

$$\begin{aligned}
K^{ij} &= \xi^{10}\hat{K}^{ij} && (4.23)\\
K_{kl}g^{ik}g^{jl} &= \xi^{10}\hat{K}_{kl}\hat{g}^{ik}\hat{g}^{jl} && (4.24)\\
K_{kl}g^{ik}g^{jl} &= \xi^{10}\hat{K}_{kl}\xi^{-4}g^{ik}\xi^{-4}g^{jl} && (4.25)\\
K_{kl} &= \xi^{2}\hat{K}_{kl} && (4.26)\\
K^{ij}K_{ij} &= \xi^{12}\hat{K}^{ij}\hat{K}_{ij}. && (4.27)
\end{aligned}$$

We first show invariance of the norm (4.19) by using (3.17)

$$||\varphi||_{H^1_h} = \left(\int_M (|D\varphi|^2 + \frac{R}{8}\varphi^2) dv_g\right)^{\frac{1}{2}} = \left(\int_M \xi^6(|\hat{D}\hat{\varphi}|^2 + \frac{\hat{R}}{8}\hat{\varphi}^2)\xi^{-6} d\hat{v}_g\right)^{\frac{1}{2}} = ||\hat{\varphi}||_{H^1_h}, \tag{4.28}$$

while invariance of the momentum term follows from

$$\int_M \frac{K^{ij}K_{ij}}{8\varphi^6} dv_g = \int_M \frac{\xi^{12}\hat{K}^{ij}\hat{K}_{ij}}{8\xi^6\hat{\varphi}^6}\xi^{-6} d\hat{v} = \int_M \frac{\hat{K}^{ij}\hat{K}_{ij}}{8\hat{\varphi}^6} d\hat{v} \tag{4.29}$$

Since we have already shown invariance of the Yamabe constant Y, we have proven invariance of (4.22).

□

4.1.2 Constant Ricci scalar

We first specify here the non-existence result Thm. 4.1 to obtain the following

Theorem 4.3. *Assume that*

$$\int_M \left(K^{ij}K_{ij}\right)^{5/12} dv_g > 3^{-5/4} R^{5/4} V \Lambda^{-5/6}. \tag{4.30}$$

holds on a manifold (M, g) with volume V and constant Ricci scalar R. Then the Lichnerowicz equation (4.1) for $n = 3$ does not have any smooth, positive solutions.

Proof. Pulling out the Ricci scalar from the integral (4.13) yields

$$\left(\frac{3^3}{2^2}\right)^{5/12} \int_M \left(\frac{K^{ij}K_{ij}}{8}\right)^{5/12} \left(\frac{\Lambda}{4}\right)^{10/12} dv_g > \left(\frac{R}{8}\right)^{5/4} \int_M dv_g \tag{4.31}$$

$$\int_M \left(K^{ij}K_{ij}\right)^{5/12} dv_g > \left(\frac{R}{8}\right)^{5/4} V \left(\frac{2^2}{3^3}\right)^{5/12} \left(\frac{4}{\Lambda}\right)^{5/6} 8^{5/12} \tag{4.32}$$

$$\int_M \left(K^{ij}K_{ij}\right)^{5/12} dv_g > \left(\frac{2^2 2^4 2^3}{2^9 3^3}\right)^{5/12} R^{5/4} V \Lambda^{-5/6} \tag{4.33}$$

which proves the assertion.

□

We can also adapt the existence result Thm. 4.2 to the present setting, which gives

Theorem 4.4. *Assuming that*

$$\int_M K^{ij}K_{ij} dv_g \leq \frac{Y^6}{256\Lambda^2 R^3 V^3}. \tag{4.34}$$

holds on a manifold (M, g) with volume V constant Ricci scalar R and Yamabe constant Y, the Lichnerowicz equation (4.1) for $n = 3$ has a smooth, positive solution.

Proof. We set

$$||\varphi||_{H_h^1} = 1. \tag{4.35}$$

without loss of generality and $\varphi = const.$ with loss of generality, i.e. by this restriction we are likely to miss the optimal bound.

The constant is determined as follows:

$$1 = ||\varphi||_{H_h^1} = \left(\int_M (|D\varphi|^2 + \frac{R}{8}\varphi^2) dv_g\right)^{1/2} = \left(\int_M \frac{R}{8}\varphi^2 dv_g\right)^{1/2} = \left(\frac{RV}{8}\varphi^2\right)^{1/2} \tag{4.36}$$

$$=> \varphi = \left(\frac{8}{RV}\right)^{1/2} \tag{4.37}$$

$$=> \frac{1}{\varphi^6} = \frac{R^3 V^3}{8^3} \tag{4.38}$$

Inserting this into (4.22) gives

$$\int_M K^{ij} K_{ij} \frac{R^3 V^3}{8^4} dv_g \leq \frac{Y^6}{4 \cdot 8^6 \Lambda^2}, \tag{4.39}$$

which immediately implies the claim (4.34).

□

4.1.3 Rotational Bowen-York data

We recall our calculations for the Nariai and Bowen-York data from (2.98) and replace a ($\psi \in (0,a)$) by $a = 2\pi\lambda$. From now onwards, we assume $J \geq 0$ without loss of generality.

Starting again with the bound for non-existence we find

Theorem 4.5. *The Lichnerowicz equation* (2.18) *with rotational Bowen-York data* (2.72) *and angular momentum J has no smooth, positive solution if*

$$J > \frac{2^{6/5}}{3} \left(\frac{2^2}{3^3} \right)^{1/2} \left(\int_0^\pi \sin^{11/6} \theta \right)^{-6/5} \Lambda^{-1} \approx 0.1647 \Lambda^{-1}. \tag{4.40}$$

Proof. We insert into (4.30) and obtain for the left side

$$\int_M \left(K^{ij} K_{ij} \right)^{\frac{5}{12}} dv_g = \frac{1}{\Lambda^{3/2}} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi\lambda} (18 \Lambda^3 J^2 \sin^2 \theta)^{5/12} \sin \theta d\psi d\theta d\varphi \tag{4.41}$$

$$= 18^{5/12} \Lambda^{5/4} J^{5/6} \frac{1}{\Lambda^{3/2}} 2\pi\lambda 2\pi \int_0^\pi \sin^{11/6} \theta \tag{4.42}$$

$$= 4\pi^2 \lambda 18^{5/12} \Lambda^{-1/4} J^{5/6} \int_0^\pi \sin^{11/6} \theta. \tag{4.43}$$

The right side gives

$$3^{-5/4} R^{5/4} V \Lambda^{-5/6} = 3^{-5/4} 8\pi^2 \lambda \Lambda^{-3/2} (2\Lambda)^{5/4} \Lambda^{-5/6} = 3^{-5/4} 8\pi^2 2^{5/4} \lambda \Lambda^{-13/12} \tag{4.44}$$

$$= \left(\frac{2}{3} \right)^{5/4} 8\pi^2 \lambda \Lambda^{-13/12}. \tag{4.45}$$

Combining the two sides again results in

$$J^{5/6}\int_0^\pi \sin^{11/6}\theta > 18^{-5/12}\left(\frac{2}{3}\right)^{5/4} 2\Lambda^{-5/6} \tag{4.46}$$

$$J\left(\int_0^\pi \sin^{11/6}\theta\right)^{6/5} > 18^{-1/2}\left(\frac{2}{3}\right)^{3/2} 2^{6/5}\Lambda^{-1} \tag{4.47}$$

$$J > 18^{-1/2}\left(\frac{2}{3}\right)^{3/2} 2^{6/5}\left(\int_0^\pi \sin^{11/6}\theta\right)^{-6/5}\Lambda^{-1} \tag{4.48}$$

$$J > \frac{2^{6/5}}{3}\left(\frac{2^2}{3^3}\right)^{1/2}\left(\int_0^\pi \sin^{11/6}\theta\right)^{-6/5}\Lambda^{-1}. \tag{4.49}$$

A Mathematica calculation shows

$$\frac{2^{6/5}}{3}\left(\frac{2^2}{3^3}\right)^{1/2}\left(\int_0^\pi \sin^{11/6}\theta\right)^{-6/5} \approx 0.1647 \tag{4.50}$$

which proves the claim.

$\square$

Turning now to the existence criterion, we have the following result.

Theorem 4.6. *The Lichnerowicz equation* (2.18) *with rotational Bowen-York data* (2.72) *and angular momentum J has a smooth, positive solution if*

$$J \leq \frac{1}{6^{1/2}2^3\Lambda\lambda^2} \approx 0.051\frac{1}{\Lambda\lambda^2} \tag{4.51}$$

for $\lambda > 1$.

Proof. Compared to the non-existence result, the left side of (4.34) is now easier to calculate due to the absence of the exponent on the momentum term. We find

$$\begin{aligned}
\int_M K^{ij}K_{ij}dv_g &= \frac{1}{\Lambda^{3/2}}\int_0^{2\pi}\int_0^\pi\int_0^{2\pi\lambda} 18\Lambda^3 J^2\sin^2\theta\sin\theta d\psi d\theta d\varphi && (4.52)\\
&= 18\Lambda^3J^2\frac{1}{\Lambda^{3/2}}2\pi\lambda 2\pi\int_0^\pi \sin^3\theta && (4.53)\\
&= 4\pi^2\lambda 18\Lambda^{3/2}J^2\frac{4}{3} && (4.54)\\
&= 2^5 3\pi^2\lambda\Lambda^{3/2}J^2. && (4.55)
\end{aligned}$$

For the right side we get

$$\begin{aligned}
\frac{Y^6}{256\Lambda^2R^3V^3} &= \frac{Y^6}{2^8\Lambda^2(2\Lambda)^3(\Lambda^{-3/2}8\pi^2\lambda)^3} && (4.56)\\
&= \frac{Y^6}{2^{20}\Lambda^{1/2}\pi^6\lambda^3} && (4.57)
\end{aligned}$$

Combining the two sides results in

$$2^5 3\pi^2\lambda\Lambda^{3/2}J^2 \leq \frac{Y^6}{2^{20}\Lambda^{1/2}\pi^6\lambda^3} \tag{4.58}$$
$$J^2 \leq \frac{Y^6}{2^{25}3\Lambda^2\pi^8\lambda^4} \tag{4.59}$$
$$J \leq \frac{Y^3}{6^{1/2}2^{12}\Lambda\pi^4\lambda^2}. \tag{4.60}$$

We now require $\lambda > 1$ and recall the estimate (3.26) for the Yamabe constant. Since the theorem guaranties smooth positive solutions for J smaller than a certain value we have to consider the lower bound

$$Y \geq 8\pi^{4/3}. \tag{4.61}$$

Inserting gives

$$J \leq \frac{2^9\pi^4}{6^{1/2}2^{12}\Lambda\pi^4\lambda^2} \tag{4.62}$$

$$J \leq \frac{1}{6^{1/2}2^3\Lambda\lambda^2}. \tag{4.63}$$

For $\lambda \leq 1$ we get from (3.27)

$$Y = 8\pi^{4/3}\lambda^{2/3}, \tag{4.64}$$

therefore our bound becomes

$$J \leq \frac{1}{6^{1/2}2^3\Lambda}. \tag{4.65}$$

This finishes the proof.

□

We conclude this section with the following remark. In (4.36) we made the assumption that we have a constant φ which is unlikely to be the optimal choice. We will now make an attempt of improving the inequality (4.22) for the Nariai metric and ω_{BY} by setting

$$\varphi = c\sin^x\theta \tag{4.66}$$

where c and x are constants. This function φ is in general not regular at the poles. However, for the present purposes it suffices that it has finite Sobolev norm, which holds for suitably chosen x as we will see shortly.

We rewrite (4.22) as

$$||\varphi||^6_{H^1_h}\int_M \frac{K^{ij}K_{ij}}{\varphi^6}dv_g \leq A \tag{4.67}$$

with

$$A = \frac{Y^6}{4 \cdot 8^5 \Lambda^2}. \tag{4.68}$$

Inserting (2.98) we get

$$||\varphi||^6_{H^1_h} \int_M \frac{\sin^2\theta}{\varphi^6} dv_g \leq B \tag{4.69}$$

with

$$B = \frac{A}{18 J^2 \Lambda^3}. \tag{4.70}$$

We are now interested in the minimum of the left side of (4.69). We obtain

$$\begin{aligned} ||\varphi||_{H^1_h} &= \left(\int_M (|D\varphi|^2 + \frac{\Lambda}{4}\varphi^2) dv_g \right)^{1/2} && (4.71) \\ &= \left(\int_M (|D(c \sin^x \theta)|^2 + \frac{\Lambda}{4} c^2 \sin^{2x}\theta) dv_g \right)^{1/2} && (4.72) \\ &= \left(\frac{2\pi}{\Lambda^{3/2}} \int_0^\pi (c^2 x^2 \sin^{2x-1}\theta \cos^2\theta + \frac{\Lambda}{4} c^2 \sin^{2x+1}\theta) d\theta \right)^{1/2}, && (4.73) \end{aligned}$$

for $x \neq 0$, which converges for $x > 0$. We further get

$$\begin{aligned} \int_M \frac{\sin^2\theta}{\varphi^6} dv_g &= \int_M \frac{\sin^2\theta}{c^6 \sin^{6x}\theta} dv_g && (4.74) \\ &= \frac{2\pi}{\Lambda^{3/2}} \int_0^\pi \frac{\sin^{3-6x}\theta}{c^6} d\theta && (4.75) \end{aligned}$$

which converges for $x < \frac{2}{3}$. So we can say that for $x \in (0, \frac{2}{3})$ we can choose $\varphi = c \sin^x \theta$ and possibly get a better bound than for $\varphi = c$. However, numerical calculations of this function show that the minimum is achieved at $x = 0$. Therefore $\varphi = c$ yields a less restrictive condition for existence, and we will not consider φ of the form $\varphi = c \sin^x \theta$ any further.

4.2 Premoselli's theorem

As sketched in the introduction already, a breakthrough in the existence theory of the Lichnerowicz equation was achieved by Premoselli [20]. Setting $2^* = 2n/(n-2)$ as before, his general result reads

Theorem 4.7 (Thm. 1.1 in [20]). *Let (M, g) be a n-dimensional closed Riemannian manifold with $3 \leq n \leq 5$, and let h, B, a be smooth functions on M and $\kappa \in \mathbb{R}$ a positive constant. Assume that $\Delta_g - h$ is coercive, $d \geq 0, d \not\equiv 0$, and $max_M B > 0$.*

Then there exist $0 < \kappa_1 \leq \kappa_2 \leq +\infty$ such that the Lichnerowicz equation

$$\Delta_g \phi - h\phi = -B\phi^{2^*-1} - \frac{\kappa d}{\phi^{2^*+1}} \tag{4.76}$$

- *has at least two solutions if $\kappa < \kappa_1$*
- *has no solutions for $\kappa > \kappa_2$*
- *has at least one solution if $\kappa_1 \leq \kappa < \kappa_2$*

In case $B > 0, \kappa_2$ is finite, there holds $\kappa_1 = \kappa_2$, and (4.76) has a unique solution for $\kappa = \kappa_1 = \kappa_2 = \kappa_*$, at least two solutions for $\kappa < \kappa_*$, and no solution for $\kappa > \kappa_*$.

For the complicated proof, we refer to [20].

A key concept both in the proof of this theorem, as well as in its subsequent applications, is the (in)stability of solutions under conformal deformations. The definition below involves the (negative) linearized operator L_ϕ of the Lichnerowicz equation (4.76)

$$L_\phi \gamma = -(\Delta_g - h)\gamma - (2^* - 1)B\phi^{2^*-2}\gamma + (2^* + 1)\frac{\kappa d}{\phi^{2^*+2}}\gamma. \tag{4.77}$$

Definition 4.1. *A solution ϕ of the Lichnerowicz equation (4.76) is called strictly stable (stable, marginally stable, unstable, strictly unstable) if the lowest eigenvalue σ of the linearized operator (which satisfies $L_\phi \zeta = \sigma\zeta$) is positive (non-negative, zero, non-positive, negative) respectively.*

Corollary 4.1 (Proposition 6.1. of [20]). *Under the requirements of Thm. 4.7 and with $B > 0$, there is a unique strictly stable solution $\phi(\kappa)$ for all $\kappa < \kappa_*$ which is "minimal" in the sense that for any positive solution with $\phi \not\equiv \phi_*$ there holds $\phi > \phi_*$. Moreover, the unique solution $\phi_{\kappa_*} = \phi_*$ for $\kappa = \kappa_*$ is marginally stable.*

The final statement of the Corollary means that there exists a positive "principal eigenfunction" $\zeta \in C^\infty(M)$ such that

$$-\Delta_g \zeta + \left[h - (2^* - 1)B\phi_*^{2^*-2} + (2^* + 1)\frac{\kappa_* d}{\phi_*^{2^*+2}}\right]\zeta = 0 \tag{4.78}$$

We can use the stability of solutions to define stability of the initial data themselves, in terms of the lowest eigenvalue of the trivial solution $\phi \equiv 1$ in (4.77). This means that we call initial data strictly stable if $\phi \equiv 1$ is strictly stable, and analogously for the other four definitions. For further properties of this concept of stability cf [4].

In the following applications, we specify to $n = 3$ and adopt the assignments (4.12).

4.2.1 Rotational Bowen-York data

We return to the Nariai data (2.86) on which we put the rotational Bowen-York initial data (2.72). The following description of the solutions is a slightly shortened version of Sect. 3.3.3 of [4].

We recall (2.98) which we rewrite as

$$K^{ij}K_{ij} = 18J^2\Lambda^3 \sin^2\theta = 8b^2\Lambda\sin^2\theta \tag{4.79}$$

with $b = 3J\Lambda/2$. Inserting into the Lichnerowicz equation (2.18) gives

$$\underbrace{-\frac{\partial^2\phi}{\partial\alpha^2}}_{A\phi} - \underbrace{\left(\Delta_2 - \frac{1}{4}\right)\phi}_{B\phi} - \frac{1}{4}\phi^5 - \frac{b^2\sin^2\theta}{\phi^7} = 0, \tag{4.80}$$

Δ_2 being the Laplacian on the 2-sphere $\mathbb{S}^2$. The resulting linearized operator of this equation around ϕ reads

$$\frac{L_\phi}{\Lambda}\gamma = -\left(A + B + \frac{5}{4}\phi^4 - \frac{7b^2\sin^2\theta}{\phi^8}\right)\gamma. \tag{4.81}$$

We now apply Premoselli's theorem to these two equations. We first restrict ourselves to solutions which only depend on θ, i.e. $\phi = \phi(\theta)$, and we assume equatorial symmetry, i.e. $\phi(\theta) = \phi(\pi - \theta)$. Since stable solutions preserve the symmetries of the equation (cf. e.g. Proposition 2 of [4]), this class of solutions will include the unique stable, minimal branch whose existence is guaranteed by the Corollary 4.1 to Premoselli's theorem. For some purposes, in particular numerical calculation, it is useful to regularise this branch near $b = 0$ which can be done by substituting $\psi = b^{-1/4}\phi$. This gives for (4.80)

$$B\phi + \frac{1}{4}\phi^5 + \frac{b^2\sin^2\theta}{\phi^7} = B\psi + \frac{b}{4}\psi^5 + \frac{\sin^2\theta}{\psi^7} = 0. \tag{4.82}$$

Using numerical methods for this case gives 4.1

A numerical analysis now yields the unique stable, minimal branch (green) as well as an unstable branch (red) Fig. 4.1. However, we know from chapter 3.2 that these two branches cannot comprise all solutions since for $b = 0$ there exist Schwarzschild-de Sitter solutions. We conclude that there must exist solutions which also depend on ψ, $\phi = \phi(\psi, \theta)$ (while we still *assume* axial symmetry, i.e. independence of ϕ) . Now from Proposition 2 of [4] it also follows that symmetry breaking, which "generates" such solutions, can only occur when the linearized operator has a zero mode. There exist k such modes for the periods $T \in (2\pi k, 2\pi(k+1)]$. In particular, considering $T = 5\pi$

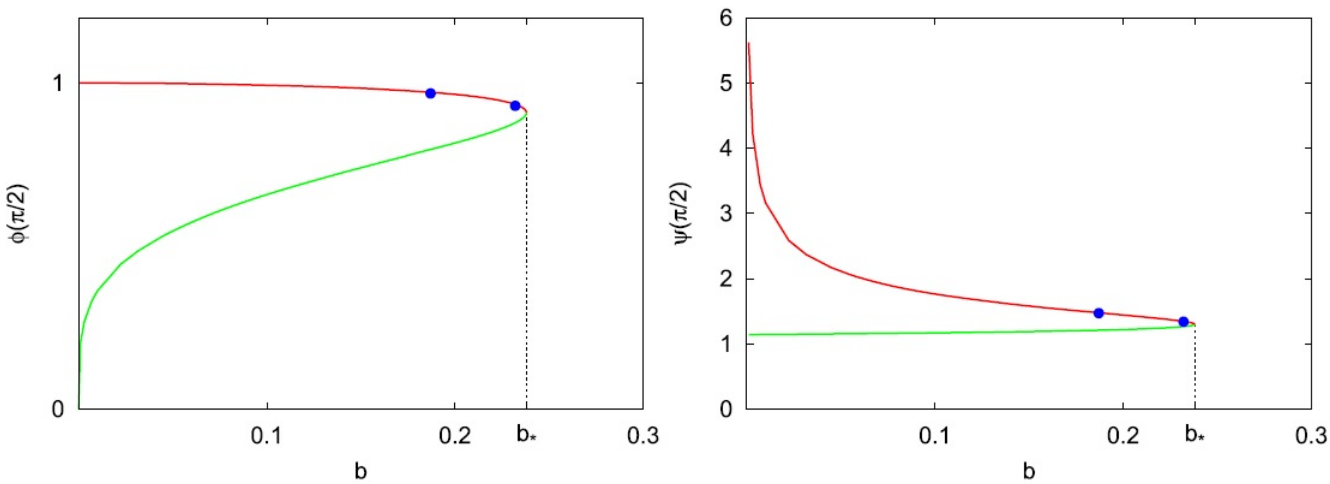

Figure 4.1: The stable (green) and the unstable (red) axially symmetric solutions of (4.82) at the equator. The blue dots are the bifurcation points of the secondary branches for the sample period $T = 5\pi$.

as an example, we get two zero modes which are marked by the blue dots on the unstable principal branch in 4.1. Starting from these points, there emanate secondary branches of solutions $\phi = \phi(\psi, \theta)$ (not shown in the diagrams) which continue till the Schwarzschild-de Sitter solutions at $b = 0$. (For the latter statement we do not have an analytic proof, but numerical support).

Numerics gives us as a bound for the existence of smooth positive solutions $b_* \approx 0.238$, which means

$$J = \frac{2b_*}{3\Lambda} = 0.158\frac{1}{\Lambda}. \tag{4.83}$$

Comparing this to our results (4.40) and (4.51) we see that the present example exceeds the latter (lower) bound, necessary for existence, significantly, while it comes quite close to the former (upper) bound, necessary for non-existence.

4.2.2 Kerr-de Sitter data

In this final section we consider the maximal slice of the Kerr-de Sitter (KdS) data as background, on which we construct data by "spinning up" (via scaling) the extrinsic curvature of KdS.

We first consider the spacetime metric

$$\tilde{h} = \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \sin^2\theta\frac{\Delta_\theta}{\rho^2}\left(\frac{adt - (r^2+a^2)d\varphi}{1+\Lambda a^2/3}\right)^2 - \frac{\Delta_r}{\rho^2}\left(\frac{dt - a\sin^2\theta d\varphi}{1+\Lambda a^2/3}\right)^2 \tag{4.84}$$

where

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2mr, \tag{4.85}$$

$$\Delta_\theta = 1 + \frac{\Lambda}{3} a^2 \cos^2\theta, \tag{4.86}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta. \tag{4.87}$$

This spacetime has been discussed at many places, see e.g. [1, 6, 7, 19]. The key quantity whose zeros determine the location of the horizons is

$$\Delta_r = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2mr = -\frac{\Lambda}{3}(r - r_+)(r - r_-)(r - r_c)(r - r_{--}). \tag{4.88}$$

Below we focus on the generic case in which Δ_r has four distinct zeros, three of which, called $r_- < r_+ < r_C$ are necessarily non-negative. We also include the "extreme" cases in which two positive zeros coincide. In the generic case, the smallest of the non-negative zeros is located at $r = 0$ iff $a = 0$. The largest two zeros, $r_+ \leq r_c$, are called "black hole" and "cosmological" horizons, respectively; Δ_r is positive in the range $r \in (r_+, r_c)$.

We first discuss the case $a = 0$. The condition for extremality reads

$$\Delta_r(r_c) = 0 \tag{4.89}$$

$$\Delta_r'(r_c) = 0. \tag{4.90}$$

which determines the location of the extreme horizon

$$\Delta_r(r_c) = r_c - \frac{\Lambda r_c^3}{3} - 2m = 0 \Rightarrow m = \frac{3r_c - \Lambda r_c^3}{6} \tag{4.91}$$

$$\Delta_r'(r_c) = 1 - \Lambda r_c^2 = 0 \Rightarrow r_c = \frac{1}{\sqrt{\Lambda}}, \tag{4.92}$$

and combining these two equalities gives $m = 1/3\sqrt{\Lambda}$.

This calculation also implies that in the case $a = 0$ the mass parameter has to satisfy

$$0 < m \leq \frac{1}{3\sqrt{\Lambda}}. \tag{4.93}$$

and the region of interest is determined by horizons whose location depends on m and Λ, viz. $0 < r_+(m, \Lambda) \leq r < r_c(m, \Lambda)$.

We now turn to the case $a \neq 0$. We first give a qualitative discussion, and then formulate a precise statement in which we restrict ourselves to small a (compared to $\Lambda^{-1/2}$).

The allowed parameter range for m and a, for which we still obtain two distinct positive roots of $\Delta_r = 0$ with $\Delta_r > 0$ in between, can be understood qualitatively from the following diagram (schematic, not a result of calculation except for the marked points; for a precise diagram of this kind cf. [19]):

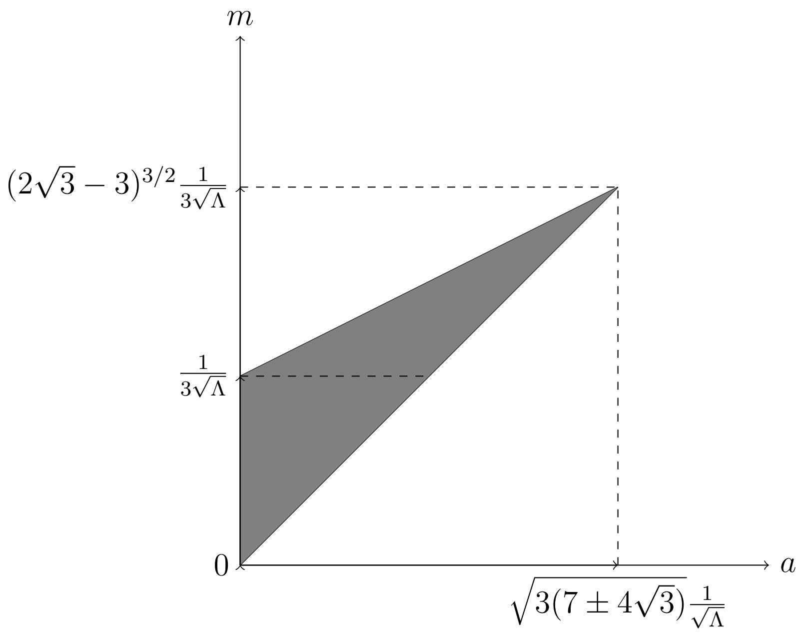

Figure 4.2: Schematic visualization of the allowed values for a and m (shaded region)

The allowed range for r now also depends on a and m, i.e. $0 < r_+(m, a, \Lambda) \leq r < r_c(m, a, \Lambda)$; the dependence of this range for fixed a and Λ but different values of m is illustrated in the following diagrams for which we have set $\Lambda = 1$ and $a = 0.1$, the resulting m bounds being approx. $0.1 < m < 0.337$. Fig. 4.3 shows the resulting function Δ_r for some intermediate value of m.

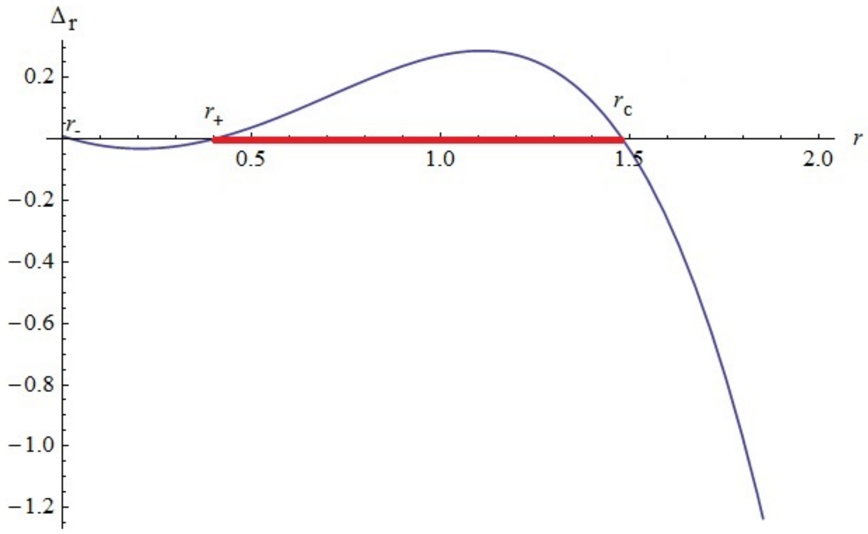

Figure 4.3: Allowed radii between r_+ and r_c marked in red. Values for $\Lambda = 1, a = 0.1$ and $m = 0.2$

As m gets smaller the range for r increases as seen in Fig. 4.4. If m reaches the lower bound, called $m_-(a, \Lambda)$, the two black hole horizons r_+ and r_- coincide.

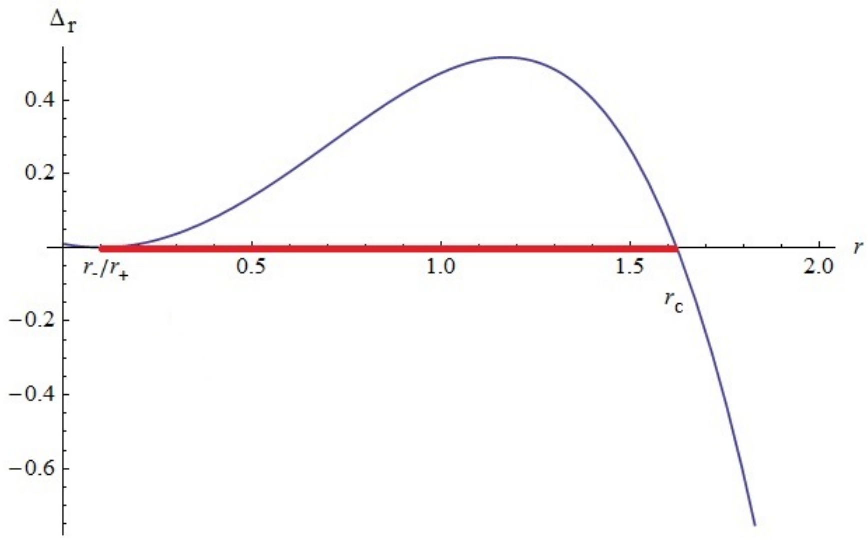

Figure 4.4: Allowed radii between r_+ and r_c marked in red. Values for $\Lambda = 1, a = 0.1$ and $m = 0.1$

For increasing m the range for r decreases as seen in Fig. 4.5. If m reaches the upper bound, called $m_+(a, \Lambda)$, the black hole horizon r_+ and the cosmological horizon r_c coincide. Both cases $m = m_-(a, \Lambda)$ and $m = m_+(a, \Lambda)$ give us an extreme Kerr-de Sitter metric.

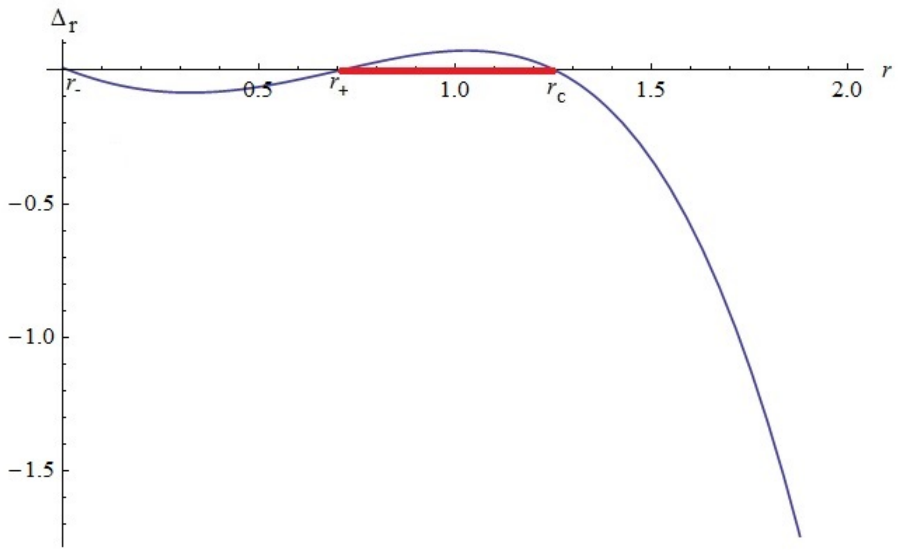

Figure 4.5: Allowed radii between r_+ and r_c marked in red. Values for $\Lambda = 1, a = 0.1$ and $m = 0.3$

From these diagrams it seems that the allowed range of r for any value of m is a subset of the allowed range for $m = m_-$. However, we do not have an analytic proof of this observation, and it will not be needed below.

Restricting ourselves now to small a (compared to $\Lambda^{-1/2}$, the allowed range for m becomes [16]

$$a^2 \left[1 + O(a^4\Lambda^2)\right] < m^2 < \frac{1}{9\Lambda}\left[1 + 2a^2\Lambda + O(a^4\Lambda^2)\right], \tag{4.94}$$

The only information which will be needed below and which is obvious from the previous discussion is

Lemma 4.1. *For $0 < m < \frac{1}{3\sqrt{\Lambda}}$ and sufficiently small $a\sqrt{\Lambda}$, the Kerr-de Sitter metric* (4.84) *has a black hole horizon $r_+(m, a, \Lambda) > 0$ and a cosmological horizon $r_c(m, a, \Lambda) > r_+(m, a, \Lambda)$ in the sense that $\Delta_r(r_+) = 0 = \Delta_r(r_c)$, and $\Delta_r(r) > 0$ for $r \in (r_+, r_c)$.*

We now turn to other properties of the Kerr de Sitter metric needed below.

The fact that its coefficients do not depend on ϕ is a *necessary* condition for axial symmetry; to confirm the latter property we show that the curve $\theta = 0$ is an "axis", i.e. the *unique* location where the Killing vector $\tilde{\eta}^\alpha = \partial/\partial\phi$ vanishes. Defining C via

$$\tilde{\eta} = \tilde{h}_{\alpha\beta}\tilde{\eta}^{\alpha}\tilde{\eta}^{\beta} = \tilde{h}_{\varphi\varphi} = \frac{\sin^2\theta}{\rho^2(1+a^2)^2}[\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta] = \frac{\sin^2\theta}{\rho^2(1+a^2)^2}C \quad (4.95)$$

we have the estimate

$$C = \left(a^2+r^2\right)^2\left(\frac{1}{3}a^2\Lambda\cos^2\theta + 1\right) - a^2\sin^2\theta\left(\left(a^2+r^2\right)\left(1-\frac{\Lambda r^2}{3}\right) - 2mr\right) \geq \quad (4.96)$$

$$\left(a^2+r^2\right)^2 - a^2\left(a^2+r^2\right)\left(1-\frac{\Lambda r^2}{3}\right) + 2mra^2 = \quad (4.97)$$

$$\left(a^2+r^2\right)r^2\left[1+\frac{\Lambda a^2}{3}\right] + 2mra^2 > 0 \quad (4.98)$$

which proves the claim, since as discussed above we restrict ourselves to a domain $0 < r_+(m,a,\Lambda) \leq r \leq r_c(m,a,\Lambda)$.

We next determine the twist potential from (2.32) related to the Killing vector $\tilde{\zeta} = \partial/\partial t$.

The calculation for the case $\Lambda = 0$ can be found in [12], the results are

$$\tilde{\omega}_\theta = -\frac{1}{2}\tilde{\epsilon}_{\theta\phi rt}\tilde{h}^{rr}\tilde{h}^{tt}\partial_r\,\tilde{\zeta}_t - \frac{1}{2}\tilde{\epsilon}_{\theta\phi rt}\tilde{h}^{rr}\tilde{h}^{t\varphi}\partial_r\tilde{\zeta}_\varphi \quad (4.99)$$

$$= J\frac{\partial}{\partial\theta}\left(\cos^3\theta - 3\cos\theta - \frac{a^2\cos\theta\sin^4\theta}{\rho^2}\right), \quad (4.100)$$

$$\tilde{\omega} = \omega = J\left(\cos^3\theta - 3\cos\theta - \frac{a^2\cos\theta\sin^4\theta}{\rho^2}\right). \quad (4.101)$$

where $J = am$. For $\Lambda \neq 0$ the results are identical but J generalizes to

$$J = \frac{am}{\Xi^2}, \quad (4.102)$$

$$\Xi = 1 + \frac{\Lambda a^2}{3}. \quad (4.103)$$

From (2.40), J is the Komar angular momentum.

We now restrict ourselves to the maximal hypersurface given by $t = const.$ in (4.84). The induced metric reads

$$g = \rho^2\left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\sin^2\theta}{\rho^2(1+a^2)^2}\left[\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta\right]d\varphi^2 \quad (4.104)$$

Theorem 4.8. *For a fixed m and small angular momenta the Kerr-de Sitter data given by (4.104) and generated from (4.101) are strictly unstable.*

Proof. Using (2.18) and the linearized operator (4.77) for $n = 3$, we get for the principal eigenfunction ζ and the principal eigenvalue σ,

$$- \Delta_g \zeta + \left[\frac{R^*}{8} - 5\frac{\Lambda}{4}\phi_*^5 + 7\frac{K^{ij}K_{ij}}{8\phi_*^8} \right] \zeta = \sigma\zeta, \tag{4.105}$$

with $K^{ij}K_{ij}$ as in (2.31). We now set $\phi = 1$ since we are interested in the principal eigenvalue of the Kerr-de Sitter data themselves. This simplifies our equation to

$$- \Delta_g \zeta + \left[\frac{R}{8} - 5\frac{\Lambda}{4} + 7\frac{K^{ij}K_{ij}}{8} \right] \zeta = \sigma\zeta. \tag{4.106}$$

Since R is difficult to calculate we replace it using the constraint (1.2) for the maximal case $K = 0$,

$$\frac{R}{8} = \frac{K^{ij}K_{ij}}{8} + \frac{\Lambda}{4}. \tag{4.107}$$

Inserting this into (4.106) yields

$$- \Delta_g \zeta + \left[\frac{K^{ij}K_{ij}}{8} + \frac{\Lambda}{4} - 5\frac{\Lambda}{4} + 7\frac{K^{ij}K_{ij}}{8} \right] \zeta = -\Delta_g\zeta + \left[K^{ij}K_{ij} - \Lambda \right] \zeta = \sigma\zeta. \tag{4.108}$$

This equation can be rewritten as

$$\Delta_g \zeta = (\alpha - \sigma)\,\zeta, \tag{4.109}$$

with

$$\alpha = K^{ij}K_{ij} - \Lambda. \tag{4.110}$$

Using (2.31), we get

$$\alpha = 2\frac{D_i\omega D^i\omega}{\eta^2} - \Lambda \tag{4.111}$$

$$= \frac{2}{\eta^2}\left(\frac{\Delta_r}{\rho^2}(D_r\omega)^2 + \frac{\Delta_\theta}{\rho^2}(D_\theta\omega)^2 \right) - \Lambda. \tag{4.112}$$

Inserting (4.101) gives after some calculation

$$\alpha = \frac{2J^2\left(a^2+1\right)^4\left(AB^2/24 + 4a^4r^2\Delta_r\cos^2\theta\sin^4\theta\right)}{\left(a^2\cos^2\theta + r^2\right)^3 C^2} - \Lambda, \tag{4.113}$$

where

$$A = a^2\Lambda\cos(2\theta) + a^2\Lambda + 6 \tag{4.114}$$

$$B = \sin\theta\left(a^4 - 3a^2r^2 + a^2(a-r)(a+r)\cos(2\theta) - 6r^4\right), \tag{4.115}$$

and C as in (4.95). From this we can conclude that α is negative for

1. m fixed between the bounds $0 < m \leq \frac{1}{3\sqrt{\Lambda}}$,

2. J (and therefore a) small enough, where J is given by (4.102).

We have already seen in figure 4.2 that we can choose our J arbitrarily small for an m between these bounds.

Since the principal eigenfunction ζ is positive we can divide (4.109) by ζ and integrate,

$$\int_M \left(\frac{|D\zeta|^2}{\zeta^2} - \alpha \right) dv_g = -\sigma V. \tag{4.116}$$

We now choose J and m such that α is negative and therefore the lhs is positive. As a result the eigenvalue σ has to be negative. According to definition 4.1 this means our Kerr-de Sitter data are strictly unstable. □

Definition 4.2. *On the background metric* (4.104) *we generate an extrinsic curvature* K_{ij} *via* (2.30) *from the twist potential*

$$\omega = \hat{J} \left(\cos^3\theta - 3\cos\theta - \frac{a^2 \cos\theta \sin^4\theta}{\rho^2} \right), \tag{4.117}$$

where $\hat{J} \in \mathbb{R}^+$.

Note that $\hat{J}$ is no longer defined as in (4.102) but an arbitrary constant. According to (2.40), $\hat{J}$ would be the angular momentum of the new data if we could show that the Lichnerowicz equation (2.18) had a solution. This now follows from Premoselli's theorem under the following restrictions on $\hat{J}^2$ (which can be taken as the parameter κ in (4.76)).

Corollary 4.2. *For* (g_{ij}, K_{ij}) *given in Definition 4.2, the Lichnerowicz equation* (2.18) *has an unstable solution if*

1. *m fixed between the bounds* $0 < m \leq \frac{1}{3\sqrt{\Lambda}}$,

2. *J (and therefore a) small enough, where J is given by* (4.102).

3. *either $\hat{J} \leq J$ or if $\hat{J} \geq J$, then $\hat{J} - J$ small enough.*

Therefore $(\tilde{g}_{ij}, \tilde{K}_{ij})$ *defines new* (t, φ)*-symmetric data.*

Proof. As we have shown in 4.8, the KdS data themselves are unstable. Therefore, setting $\kappa = \hat{J}^2$, Premoselli's theorem 4.7, together with Corollary 4.1, shows the assertion. □

Corollary 4.3. *To all the (unstable) data constructed in Corollary 4.2, there exist stable, minimal data with the same angular momenta, conformal to the above ones. This implies in particular to the Kerr-de Sitter data themselves.*

Proof. The existence of the stable data follows again from Corollary 4.1. These data have the same angular momentum J since (2.38) is conformally invariant. □

We finally remark that the data constructed in Corollary 4.3 can be evolved in time (which is beyond the scope of this thesis). This should constitute an interesting model for a rotating dynamic black hole which might finally “settle down” to a member of the Kerr-de Sitter family. In any case, the angular momentum will be preserved in this evolution.

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