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Abstract

Let M be a finite-dimensional manifold. Denote by $H \subset TM$ a smooth subbundle and by $[H, H] \subset TM$ the subbundle generated by all brackets of first order among sections of H . We say that H is bracket-generating in one step if

$$TM = H + [H, H].$$

The main results of the thesis concern generic one-step bracket-generating distributions of rank four. First of all, we get the classification of their types. These exist in dimensions $5 \leq n \leq 10$, and we focus on the cases $n = 8$ and $n = 9$, which have not been treated in the literature. We prove that there exist two generic types of $(4, 8)$ -distributions and that each type is equivalent to a normal parabolic geometry. Aware of this equivalence, we can read out the local invariants for these structures from the harmonic curvature associated to the corresponding parabolic geometries, thus describing them completely from the point of view of the local geometry. The same tools from parabolic geometry find a nice application in the description of submaximally symmetric models of generic $(4, 8)$ -distributions of hyperbolic type.

Using results of Tanaka and Morimoto about the prolongation procedure, which generalize a certain construction for classical G -structures to the filtered setting, we show that the $(4, 9)$ -case is the lowest dimensional in a countable series of one-step bracket-generating distributions which determine a canonical linear connection on the tangent bundle TM . The torsion and curvature of this connection, then, are local invariants for these structures.

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Introduction

Throughout the text, we consider differentiable manifolds M of finite dimension. The main results of the thesis concern certain smooth subbundles of the tangent bundle TM , which are bracket-generating in one step. A smooth distribution $H \subset TM$ is said to be bracket-generating if it generates, together with all iterated brackets among its sections, the tangent bundle of M . If $[H, H] \subset TM$ is the subbundle generated by all brackets of first order among sections of H , the one-step bracket-generating condition writes as

$$TM = H + [H, H].$$

Let $\dim(M) = n$ and let $H \subset TM$ be a smooth distribution of rank k . Moreover, let $q: TM \rightarrow TM/H$ be the canonical projection. Taking equivalence classes of Lie brackets defines a tensorial map

$$\mathcal{L}: \Lambda^2 H \rightarrow TM/H, \quad (\xi, \eta) \mapsto q([\xi, \eta])$$

If H is bracket-generating in one step, then \mathcal{L} is a surjective homomorphism of vector bundles. If $H \subset TM$ is a smooth distribution of corank one, then \mathcal{L} determines in every point a skew-symmetric bilinear form. As an open condition, one can require that \mathcal{L}_x is non-degenerate at each point $x \in M$. By linear algebra, this hypothesis can be satisfied only for even values of k . The distributions of even rank and corank one satisfying this nondegeneracy condition are one-step bracket-generating and are known as contact structures.

When studying a specific geometric structure, it is natural to focus on those local diffeomorphisms of M preserving the structure. Loosely speaking, we will refer to them as *compatible diffeomorphisms*. Looking at a smooth distribution $H \subset TM$ as a geometric structure, the compatible diffeomorphisms are local diffeomorphisms of M whose differential preserves H . Under certain hypothesis of regularity, the compatible diffeomorphisms form a Lie group. In this sense, one speaks about the automorphism group K of the structure.

From the point of view of the local geometry, we could distinguish between distributions of stable and unstable range. If every small perturbation of H originates via push-forward from a diffeomorphism of M , then H is said to be *stable*. Such distributions have infinite dimensional automorphism group and locally, they all look like a fixed model. This is exactly the content of Pfaff theorem about contact structures, a first example of stable type. The analogous statement for involutive distributions follows from Frobenius Theorem. There is actually not much more in the stable range. Free distributions of length two and another class of distributions, which are in some sense dual to contact structures, complete the list of stable one-step bracket generating distributions (see [2]).

There exist also generic distributions which, from the point of view of the local geometry, behave in a completely different way. First of all they determine local invariants, similar to the curvature of a Riemannian metric. Furthermore, they differ from the stable range for having finite-dimensional automorphism group K . We will speak in this case about *unstable* range. The classical example of this situation was studied in 1910 by E. Cartan in his “five variables paper” ([12]). For generic rank two distributions in dimension five (which are bracket generating in two steps) Cartan constructed what is nowadays called a Cartan geometry related to the exceptional

Lie group of type G_2 . The class of unstable range seems thus to be much more various, so that one can only aim to a case by case description. Many interesting examples in the unstable range are given by bracket-generating distributions. Generic one-step bracket-generating distributions of rank four, the main subject of the thesis, are among them.

1. Classification for the case of rank four

Let $H \subset TM$ be a one-step bracket-generating distribution. Denote by k the rank of H and by n the dimension of the underlying manifold. The *Levi bracket* associated to H

$$\mathcal{L} : \Lambda^2 H \rightarrow TM/H, \quad (\xi, \eta) \mapsto q([\xi, \eta])$$

is a surjective homomorphism of vector bundles. There is a natural action of the Lie group $G = GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R})$ on the set of surjective linear maps $\Lambda^2 \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$, whose orbits are in bijective correspondence with the $GL(k, \mathbb{R})$ -orbits of their kernels in a Grassmannian. Now \mathcal{L}_x defines an orbit \mathcal{O}_x at each point $x \in M$. Denoting by \mathcal{O} a given open orbit, we will suppose that $\mathcal{O}_x = \mathcal{O}$ for all $x \in M$. If this is the case, we speak about *generic* one-step bracket-generating distributions.

A more explicit description of these orbits for $k = 4$ is available and leads to their classification. The wedge product $\Lambda^2 \mathbb{R}^4 \times \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^4 \mathbb{R}^4$ is a symmetric bilinear map onto a one-dimensional vector space, which defines a conformal class of quadratic forms of signature $(3, 3)$ on $\Lambda^2 \mathbb{R}^4$. This shows the existence of a Lie group homomorphism $GL(4, \mathbb{R}) \rightarrow CO(3, 3)$, which restricts to a two-fold covering between the connected components of the identities. Using this homomorphism, the $GL(4, \mathbb{R})$ -orbits of linear subspaces $P \subset \Lambda^2 \mathbb{R}^4$ can be characterized in terms of rank and signature of the restriction of the wedge product to P . The open orbits, in particular, correspond to nondegenerate restrictions. In this way, we get the classification of the open orbits for the case of rank four (Theorem 2) including the bidimensions $(4, n)$ for all $5 \leq n \leq 10$. From our classification, then, one deduces that all cases correspond to well known types of distributions, except for $n = 8, 9$. This motivates a deeper investigation of these two cases.

There are two generic types for $n = 8$. The corresponding generic distributions will be shown to be equivalent to normal, regular parabolic geometries. Local invariants for the structure can thus be deduced from the harmonic curvature of the corresponding parabolic geometries, whose values are described by Kostant's Theorem.

The bidimension $(k, n) = (4, 9)$ is the lowest dimensional case in a whole class of rigid bidimensions, the *dual Darboux bidimensions* $(2s, \binom{2s}{2} - 1)$ of even rank for $s \geq 2$. Each dual Darboux bidimension allows a unique open orbit, hence a unique generic type of distribution. Morimoto's prolongation procedure ([16]) proves that certain bracket-generating distributions come together with a canonical linear connection on TM . We show that Morimoto's result can be applied to any dual Darboux distribution of even rank. In particular, the torsion and the curvature of the canonical linear connection are the basic local invariants for these structures.

2. Prolongation procedure

Morimoto's procedure aims to construct canonical Cartan connections for certain Cartan geometries on filtered manifolds, inspired by the classical theory of G -structures. In Chapter 2, we develop an equivalent prolongation procedure constructing linear connections on TM associated to certain bracket-generating distributions.

Any smooth manifold of dimension n carries a natural frame bundle, which is a principal bundle with structure group $GL(n, \mathbb{R})$. A G -structure is a reduction $\mathcal{F} \rightarrow M$ of the natural frame bundle on M to a closed subgroup $G \subset GL(n, \mathbb{R})$. A G -structure encodes, at the same time, the underlying manifold M and the local diffeomorphisms compatible with the geometric structure on M . The tangent bundle TM can be recovered, via the associated bundle construction, from

the G -structure. The same construction assures the existence of induced linear connections on TM . These can be thought as the linear connections which are compatible with the geometric structure on M and form an affine space. If G satisfies certain algebraic conditions, the existence of a unique choice in the affine space is assured and this can be proved by putting some constraint on the torsion, a so-called *normalization condition*. One usually refers to such choice as the *canonical linear connection* on TM . The remaining components of the torsion, together with the curvature, are invariant under the action of compatible diffeomorphisms of M . A remarkable example of canonical linear connection is given by the Levi-Civita connection ∇ associated to a Riemannian manifold. The existence of metric linear connections on TM comes from the associated bundle construction, which relates TM to the orthonormal frame bundle of M . Requiring that a metric linear connection ∇ is torsion-free is the normalization condition. Morimoto's prolongation procedure, based on the work of Tanaka and inspired by the theory of classical G -structures, aims to get similar results for filtered manifolds. Consider a bracket-generating distribution $H \subset TM$ and the filtration by subbundles

$$\{0\} \subset H = T^{-1}M \subset T^{-2}M \subset \dots \subset T^{-\mu}M = TM$$

generated by iterating brackets among sections of H . Put $T^iM = TM$ for $i \leq -\mu - 1$ and $T^iM = \{0\}$ for $i \geq 0$. The graded vector bundle associated to the filtration is

$$\text{gr}(TM) = \text{gr}_{-\mu}(TM) \oplus \dots \oplus \text{gr}_{-1}(TM), \quad \text{gr}_i(TM) := T^iM/T^{i+1}M$$

Taking equivalence classes of brackets defines the *Levi bracket*

$$\mathcal{L} : \text{gr}(TM) \otimes \text{gr}(TM) \rightarrow \text{gr}(TM),$$

a tensorial map which is homogeneous of degree zero. One first shows that for any $x \in M$, the pair $(\text{gr}(T_xM), \mathcal{L}_x)$ is a nilpotent graded Lie algebra generated by $H_x = \text{gr}_{-1}(T_xM)$, which is said to be the *symbol algebra of H at x* . Furthermore if $x, y \in M$ are mapped to each other by a compatible diffeomorphism, this induces an isomorphism of nilpotent graded Lie algebras between the symbol algebras of H at x and y . We will assume that $(\text{gr}(TM), \mathcal{L})$ is a locally trivial bundle of nilpotent graded Lie algebras and we will speak about *regular pairs* (M, H) . Under this regularity assumption, an adapted frame bundle can be associated to (M, H) . If $(\mathfrak{n}, [\cdot, \cdot])$ denotes the standard fiber of $(\text{gr}(TM), \mathcal{L})$, the set $\text{Aut}_{gr}(\mathfrak{n})$ of grading-preserving isomorphisms of the Lie algebra $(\mathfrak{n}, [\cdot, \cdot])$ is the structure group of the adapted frame bundle.

If we want to interpret a bracket-generating distribution $H \subset TM$ as a geometric structure, we should look for the appropriate notion of compatible linear connection. For linear connections on H , which would be natural to consider, there is indeed no well-defined notion of torsion.

Now if (M, H) is a regular pair, induced linear connections on the associated vector bundle $\text{gr}(TM)$ are available. These are the linear connections compatible with the structure (M, H) . A compatible linear connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(\text{gr}(TM)) \rightarrow \Gamma(\text{gr}(TM))$$

is the direct sum of linear connections ∇^i on $\text{gr}_i(TM)$ satisfying

$$\nabla_\xi^{i+j} \mathcal{L}(X, Y) = \mathcal{L}(\nabla_\xi^i X, Y) + \mathcal{L}(X, \nabla_\xi^j Y)$$

for all $\xi \in \mathfrak{X}(M)$, $X \in \Gamma(\text{gr}_i(TM))$, $Y \in \Gamma(\text{gr}_j(TM))$ and $i, j \leq -1$. Clearly, compatible partial connections can be analogously defined, by allowing derivatives only along directions in a subbundle of TM . Also in the filtered setting one can encode additional geometric structure, namely by considering reductions of the adapted frame bundle to a closed subgroup $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$. In this case, the compatible linear connections may satisfy additional conditions.

The idea is to formulate a notion of graded torsion and graded curvature for partial compatible

connections, via an inductive definition whose base step is requiring that the component in homogeneity zero coincides with the Levi bracket. The inductive definition is enclosed in an inductive procedure in which, at each step, a graded component is defined and then normalized. Fix an integer $k \geq 1$. The k -th step of the inductive procedure can be briefly described as follows. Put $\mathfrak{X}^p = \Gamma(T^p M)$ for $p < 0$ and $E = \text{gr}(TM)$. In particular, $\mathfrak{X}^{-1} = \Gamma(H)$. Suppose that a partial compatible linear connection

$$\nabla : \mathfrak{X}^{-k+1} \times \Gamma(E) \rightarrow \Gamma(E)$$

and an isomorphism $\phi : T^{-k}M \rightarrow E_{-k} \oplus \dots \oplus E_{-1}$ are given. Such ingredients are collected in a *frame form of length $k - 1$* . The set of frame forms of length k extending a given frame form of length $k - 1$ is proved to be an affine space \mathcal{A} . To each element in \mathcal{A} , one can associate a torsion and a curvature in homogeneity k , whose sum will be denoted by C^k . Again, we will assume that the structure group of the adapted frame bundle satisfies some algebraic conditions. Denote by $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{n})$ the Lie algebra of $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$. The direct sum vector space $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$ admits a graded Lie algebra structure. It is possible, indeed, to extend the brackets on \mathfrak{n} and \mathfrak{g}_0 , by defining a bracket $\mathfrak{g}_0 \otimes \mathfrak{n} \rightarrow \mathfrak{n}$ via the evaluation map. The *first prolongation of $(\mathfrak{n}, \mathfrak{g}_0)$* is the kernel in homogeneity one of the map $\partial : L(\mathfrak{n}, \mathfrak{g}_{\leq 0}) \rightarrow L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$ defined by the formula

$$\partial \Phi(X, Y) = [X, \Phi(Y)] - [Y, \Phi(X)] - \Phi([X, Y]).$$

A *normalization condition for the pair $(\mathfrak{n}, \mathfrak{g}_0)$* is a \mathfrak{g}_0 -invariant graded submodule

$$\mathfrak{N} = \bigoplus_{i \geq 1} \mathfrak{N}_i \subset L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$$

such that for all $i \geq 1$:

$$L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_i = \text{Im}(\partial_i) \oplus \mathfrak{N}_i.$$

Assuming the vanishing of the first prolongation and the existence of a normalization condition, one can select a unique element in \mathcal{A} by putting a constraint on C^k . It should be noted that the difference between torsion and curvature, compared to the classical case, is not relevant since, in general, both have to be normalized. In the normalization procedure for filtered manifolds, the homogeneity plays the main role.

The inductive procedure, then, produces a proper compatible linear connection on E and an isomorphism $TM \cong E$. These two ingredients together uniquely define a linear connection $\tilde{\nabla}$ on TM such that $\tilde{\nabla}_\xi(\mathfrak{X}^p) \subset \mathfrak{X}^p$ for all $\xi \in \mathfrak{X}(M)$ and $p < 0$, the canonical linear connection for (M, H) .

3. Overview of the text

In Chapter 1, we give some examples of stable distributions. As a consequence of Frobenius Theorem, unstable distributions are noninvolutive and the study of noninvolutive distributions reduces, in some sense, to the bracket-generating case. After defining bracket-generating distributions of arbitrary depth, we soon focus on the one-step case and describe the Levi bracket in terms of orbits of surjective linear maps. Characterizing the open orbits in the case of rank four in terms of nondegenerate restrictions of the wedge product, finally, leads to the classification of the generic types of rank four.

Chapter 2 starts with a brief review of classical G -structures. We underline the facts which should be taken into account to adapt the construction of canonical connections on classical G -structures to the setting of filtered manifolds. We develop a version of Morimoto's prolongation procedure, using linear connections instead of Cartan connections. We show that the procedure applies to the class of dual Darboux distributions, whose lowest dimensional case $(4, 9)$ appears in the classification of Chapter 1.

In the latter classification, two generic types of $(4, 8)$ -distributions also appear and these are the main subject of Chapter 3. For each type, we describe the model algebra as the negative

graded part of a grading on a real form for $\mathfrak{sl}(5, \mathbb{C})$. Then, we prove that both types are equivalent to normal, regular parabolic geometries, which allows us to deduce the fundamental local invariants by looking at the associated harmonic curvature. The informations obtained from the analysis of the cohomology group $H^2(\mathfrak{g}_-, \mathfrak{g})$, containing the admissible values for the harmonic curvature, find a further application, namely the description of submaximally symmetric models for $(4, 8)$ -distributions of generic hyperbolic type.

CHAPTER 1

Bracket generating distributions

A first distinction in two classes of smooth distributions depends on their behavior under the Lie bracket operator. A smooth distribution $H \subset TM$ is said to be involutive if its space of sections is closed under the Lie bracket. The Frobenius theorem characterizes involutive smooth distributions and shows that, at least locally, they all look like a canonical model depending only on their rank. Conversely, in the noninvolutive range we find examples of local geometry. The noninvolutive case, in some sense, reduces to the study of distributions whose sections, together with their iterated Lie brackets span the whole tangent bundle. As a natural regularity hypothesis, one can assume that the procedure determines a filtration of the tangent bundle by smooth subbundles. Under this assumption, a tensorial map on the graded vector bundle associated to the filtration, the so-called Levi bracket, is induced by the bracket among vector fields. At each point x of the manifold, the Levi bracket endows the associated graded to the tangent space with a nilpotent graded Lie algebra structure, the symbol algebra of H at the point x . In Proposition 1, we show that the symbol algebra is the first basic invariant for the structure. In the rest of the Chapter, we will focus on one-step bracket-generating distributions. In this case, the isomorphism class of the symbol algebra at a given point $x \in M$ can be nicely described in terms of natural actions of general linear groups and related orbits. Assuming that these orbits are open assures that the symbol algebras, locally around x , are isomorphic to a fixed model. In particular, this will lead to the classification for the case of rank four.

1. Involutive distributions

DEFINITION 1. Let M be a smooth manifold and let $H \subset TM$ be a smooth distribution.

- An integral submanifold for H is a smooth submanifold $N \subset M$ such that $T_y N = H_y$ for all $y \in N$.
- H is said to be *integrable* if for each $x \in M$ there exists an integral submanifold for H which contains x .
- H is said to be *involutive* if $[\xi, \eta] \in \Gamma(H)$ for all $\xi, \eta \in \Gamma(H)$.

It is easy to see that an integrable distribution is always involutive. Let $\mathcal{I} \subset \Omega^1(M)$ be a family of one-forms such that for all $x \in M$:

$$H_x = \{\xi \in T_x M \mid \alpha(\xi) = 0 \text{ for all } \alpha \in \mathcal{I}\}$$

From the standard differential calculus formula

$$d\alpha(\xi, \eta) = \xi \cdot \alpha(\eta) - \eta \cdot \alpha(\xi) - \alpha([\xi, \eta])$$

for $\alpha \in \Omega^1(M)$ and vector fields ξ, η on M , we deduce that $d\alpha(\xi, \eta) = -\alpha([\xi, \eta])$ for $\xi, \eta \in \Gamma(H)$ and $\alpha \in \mathcal{I}$. If N is an integral manifold for H and $i: N \rightarrow M$ denotes the immersion, then $i^*\alpha = 0$ for $\alpha \in \mathcal{I}$. Then $i^*d\alpha = di^*\alpha = 0$, so that $\alpha([\xi, \eta]) = 0$ for all $\xi, \eta \in \Gamma(H)$ and $\alpha \in \mathcal{I}$. Less obvious is the fact that involutivity implies integrability, which is proved by the following

THEOREM 1. (*Frobenius*) Let M be a smooth manifold of dimension n and let $H \subset TM$ be a smooth involutive distribution of rank k . Then for each $x \in M$, there exists a local chart (U, u)

for M with $x \in U$ such that $u(U) = V \times W \subset \mathbb{R}^n$ for open subsets $V \subset \mathbb{R}^k$, $W \subset \mathbb{R}^{n-k}$ and for each $a \in W$ the subset $u^{-1}(V \times \{a\}) \subset M$ is an integral manifold for H .

Frobenius Theorem, then, implies that any involutive distribution, locally, looks like a fixed model. Indeed, locally around each point, an involutive distribution H of rank k is the tangent space of the submanifold $\{u^i = a^i, i = r+1, \dots, n\}$, with $a^{r+1}, \dots, a^n \in \mathbb{R}$ constant. Otherwise put,

$$H_x = \{\xi \in T_x M \mid du^i(\xi) = 0 \text{ for all } i = r+1, \dots, n\}$$

for all $x \in U$. In order to find examples of distributions admitting local invariants, then, we shall move to the noninvolutive case.

2. Bracket-generating distributions

Let M be a finite dimensional smooth manifold and $H \subset TM$ a smooth distribution on M . Denote by \mathcal{H}^{-1} the sheaf of smooth sections of H . Define inductively a sequence of sheaves $\{\mathcal{H}^{-k}\}_{k \geq 1}$ by

$$\mathcal{H}^{-k-1} := \mathcal{H}^{-k} + [\mathcal{H}^{-k}, \mathcal{H}^{-1}]$$

Observe that $\mathcal{H}^{-k} \subset \mathcal{H}^{-k-1}$ for any $k \geq 1$ and $[\mathcal{H}^p, \mathcal{H}^q] \subset \mathcal{H}^{p+q}$ for all negative integers p, q . At each point $x \in M$, the sheaf \mathcal{H}^{-k} generates a finite dimensional subspace in $T_x M$, whose dimension in general depends on the point. If it is constant for each integer k , then H generates an increasing sequence of subbundles of TM :

$$(1) \quad \{0\} \subset H = T^{-1}M \subset T^{-2}M \subset \dots \subset T^{-k}M \subset \dots \subset TM$$

DEFINITION 2. Let $H \subset TM$ be a smooth distribution on a smooth manifold M .

- H is said to be *regular* if there exists a sequence $\{T^{-k}M\}_{k \geq 1}$ of subbundles of TM , such that \mathcal{H}^{-k} coincides with the sheaf of smooth sections of $T^{-k}M$ for all k .
- A regular distribution H is said to be *bracket generating* if there exists an integer $\mu \geq 2$ such that $T^{-\mu}M = TM$. The minimal integer μ satisfying this condition is called the *depth* of H and the vector (a_1, \dots, a_μ) with $a_i = \text{rank}(T^{-i}M)$ is called the *small growth vector* of H .

REMARK 1. Frobenius Theorem implies that noninvolutive regular distributions are related to bracket-generating distributions. Indeed, if $H \subset TM$ is a noninvolutive regular distribution, then $H \subsetneq T^{-2}M$. Since M is finite-dimensional, there exists the minimal integer $\mu \geq 2$ such that $T^{-\mu-1}M = T^{-\mu}M$ and the Theorem applies to the involutive subbundle $T^{-\mu}M \subset TM$. Since $H \subset T^{-\mu}M$, we conclude that H is a bracket-generating distribution on each integral submanifold for $T^{-\mu}M$.

Consider now a bracket-generating distribution $H \subset TM$ of depth μ . The sequence (1) ends with $T^{-\mu}M = TM$, so that iterating brackets among sections of H determines a filtration on TM . Put $T^i M = TM$ for $i < -\mu$ and $T^i M = \{0\}$ for $i > -1$. Now define $\text{gr}_i(TM) := T^i M / T^{i+1} M$ for all integers i . The *associated graded bundle* $\text{gr}(TM) = \oplus_i \text{gr}_i(TM)$ reduces to a finite direct sum, indexed by $-\mu \leq i \leq -1$. Denote by $q_i : T^i M \rightarrow \text{gr}_i(TM)$ the natural projection maps. By construction, the Lie bracket of vector fields is compatible with the filtration:

$$[\xi, \eta] \in \Gamma(T^{p+r}M) \quad \text{for all } \xi \in \Gamma(T^p M), \eta \in \Gamma(T^r M).$$

The previous remark assures that $(\xi, \eta) \mapsto q_{p+r}([\xi, \eta])$ defines a map

$$\Gamma(T^p M) \times \Gamma(T^r M) \rightarrow \Gamma(\text{gr}_{p+r}(TM)),$$

whose values only depend on the equivalence classes of ξ, η in $\Gamma(\text{gr}_p(TM))$ and $\Gamma(\text{gr}_r(TM))$, respectively. To see this, take two representatives $\xi_1, \xi_2 \in \Gamma(T^p M)$ of the same equivalence class

in $\Gamma(\text{gr}_p(TM))$ and $\eta \in \Gamma(T^r M)$. Then $\xi_1 = \xi_2 + \zeta$ for $\zeta \in \Gamma(T^{p+1}M)$ and since $[\zeta, \eta]$ is a section of $T^{p+r+1}M$, we have $q_{p+r}([\xi_1, \eta]) = q_{p+r}([\xi_2, \eta])$. The formula above descends to a map $\Gamma(\text{gr}_p(TM)) \times \Gamma(\text{gr}_r(TM)) \rightarrow \Gamma(\text{gr}_{p+r}(TM))$. In order to show that last map is tensorial, first recall the following formula for $\xi, \eta \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$:

$$[f\xi, \eta] = f[\xi, \eta] - (\eta \cdot f)\xi$$

Now $(\eta \cdot f)\xi \in \Gamma(T^p M) \subset \Gamma(T^{p+r+1}M)$ for all negative integers p, r , thus it projects to zero along q_{p+r} . Projecting the formula above along q_{p+r} , leads finally to $q_{p+r}([f\xi, \eta]) = f q_{p+r}([\xi, \eta])$.

DEFINITION 3. The Levi bracket $\mathcal{L} : \text{gr}(TM) \otimes \text{gr}(TM) \rightarrow \text{gr}(TM)$ is the tensorial map characterized by

$$\mathcal{L}(q_p(\xi), q_r(\eta)) = q_{p+r}([\xi, \eta]), \quad \xi \in \Gamma(T^p M), \eta \in \Gamma(T^r M).$$

Since $[\cdot, \cdot]$ satisfies the Jacobi identity, the same holds for \mathcal{L} . For any $x \in M$, the pair $(\text{gr}(T_x M), \mathcal{L}_x)$ is a nilpotent graded Lie algebra generated by $H_x = \text{gr}_{-1}(T_x M)$, which is said to be the *symbol algebra of H at x* .

PROPOSITION 1. Let M, \tilde{M} be smooth manifolds of same dimension. Suppose that H and \tilde{H} are bracket-generating distributions of same rank on M and \tilde{M} , respectively. Let $U \subset M$, $\tilde{U} \subset \tilde{M}$ open subsets and $f : U \rightarrow \tilde{U}$ a diffeomorphism such that $T_x f \cdot H_x = \tilde{H}_{f(x)}$ for all $x \in U$. Then for any $x \in U$, f induces an isomorphism of nilpotent graded Lie algebras

$$\text{gr}(T_x f) : (\text{gr}(T_x M), \mathcal{L}_x) \rightarrow (\text{gr}(T_{f(x)} \tilde{M}), \tilde{\mathcal{L}}_{f(x)}).$$

PROOF. The distributions H, \tilde{H} induce filtrations $\{T^i M\}$ of TM and $\{T^i \tilde{M}\}$ of $T\tilde{M}$, respectively. Denote by $q_i : T^i M \rightarrow \text{gr}_i(TM)$ and by $\tilde{q}_i : T^i \tilde{M} \rightarrow \text{gr}_i(T\tilde{M})$ the natural projection maps. Analogously, the respective Levi brackets are denoted by $\mathcal{L}, \tilde{\mathcal{L}}$. By hypothesis for each point $x \in U$, the differential of f at x is a linear isomorphism $T_x f : T_x M \rightarrow T_{f(x)} \tilde{M}$. By restriction, in particular, it induces an isomorphism between H_x and $\tilde{H}_{f(x)}$. Recall that the pullback of $\xi \in \mathfrak{X}(\tilde{U})$ is the vector field $f^* \xi \in \mathfrak{X}(U)$ defined by $(f^* \xi)_x = T_{f(x)} f^{-1} \cdot \xi_{f(x)} \in T_x U$ for all $x \in U$. Observe that $f^* \xi \in \Gamma(H)$ for $\xi \in \Gamma(\tilde{H})$. Moreover, being f a diffeomorphism, any section of $\Gamma(H)$ is the pullback of some section of $\Gamma(\tilde{H})$. If $\xi, \eta \in \mathfrak{X}(\tilde{U})$, for all $x \in U$ we have that

$$(2) \quad T_x f \cdot [f^* \xi, f^* \eta]_x = [\xi, \eta]_{f(x)} \in T_{f(x)} \tilde{U}$$

Writing last formula for sections $\xi, \eta \in \Gamma(\tilde{H})$ shows that $Tf(T^{-2}M) = T^{-2}\tilde{M}$. From this together with formula (2), $Tf(T^i M) = T^i \tilde{M}$ for all $i = -1, \dots, -\mu$ follows. Since $T_x f$ is compatible with the filtrations, it induces the map

$$\begin{aligned} \text{gr}(T_x f) : \text{gr}(T_x M) &\rightarrow \text{gr}(T_{f(x)} \tilde{M}) \\ \text{gr}(T_x f)(q_i(v)) &= \tilde{q}_i(T_x f \cdot v), \quad \text{for } v \in T_x^i M \end{aligned}$$

between the associated graded vector spaces. Observe that $\text{gr}(T_x f)$ coincides with $T_x f$ on H_x . Clearly, $\text{gr}(T_x f)$ is surjective, since it is the composition of two surjective maps. Moreover, if $v_1, v_2 \in T_x^i M$ are such that $\text{gr}(T_x f)(q_i(v_1)) = \text{gr}(T_x f)(q_i(v_2))$, then $T_x f \cdot (v_1 - v_2) \in T_{f(x)}^{i+1} \tilde{M}$. Now the equality $T_x f(T_x^{i+1} M) = T_{f(x)}^{i+1} \tilde{M}$ implies that $v_1 - v_2 \in T_x^{i+1} M$. Equivalently $q_i(v_1 - v_2) = 0$, hence $\text{gr}(T_x f)$ is injective. For any $\xi \in \Gamma(T^i M)$ and $\eta \in \Gamma(T^j M)$, we have $[f^* \xi, f^* \eta] \in \Gamma(T^{i+j} M)$. One can project (2) onto $\text{gr}_{i+j}(T\tilde{M})$ along \tilde{q}_{i+j} , thus obtaining

$$\text{gr}(T_x f)(q_{i+j}([f^* \xi, f^* \eta]_x)) = \tilde{\mathcal{L}}_{f(x)}(\tilde{q}_i(\xi), \tilde{q}_j(\eta))$$

Putting $v = (f^*\xi)_x \in T_x^i U$ and $w = (f^*\eta)_x \in T_x^j U$ and substituting $\xi_{f(x)} = T_x f \cdot v$ and $\eta_{f(x)} = T_x f \cdot w$ in the right-hand side, last identity rewrites as

$$\mathrm{gr}(T_x f)(\mathcal{L}_x(q_i(v), q_j(w))) = \tilde{\mathcal{L}}_{f(x)}(\mathrm{gr}(T_x f)(v), \mathrm{gr}(T_x f)(w))$$

Therefore, $\mathrm{gr}(T_x f)$ is an isomorphism of nilpotent graded Lie algebras for any $x \in U$, which concludes the proof. \square

The symbol algebra associated to bracket-generating distributions is thus invariant under the action of compatible diffeomorphisms.

EXAMPLE 1. (*Contact structures*) Let V be a real vector space and $\omega : V \times V \rightarrow \mathbb{R}$ a skew-symmetric bilinear form. If ω is nondegenerate, the pair (V, ω) is said to be a *symplectic* vector space. Let $\{e_1, \dots, e_{2n}\}$ be the standard basis of \mathbb{R}^{2n} and $\{e_1^*, \dots, e_{2n}^*\}$ the dual basis. The standard symplectic vector space is the pair $(\mathbb{R}^{2n}, \omega)$, where ω is defined by

$$(3) \quad \omega = \sum_{i=1}^n e_i^* \wedge e_{n+i}^*$$

As a classical result in linear algebra, a skew-symmetric bilinear form on an odd dimensional vector space is degenerate. Moreover, every nondegenerate skew-symmetric bilinear form on \mathbb{R}^{2n} writes as ω with respect to some basis of \mathbb{R}^{2n} . Therefore up to isomorphism, there is a unique structure of a nilpotent graded Lie algebra on $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \mathbb{R}^{2n} \oplus \mathbb{R}$ such that the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is nondegenerate, which is known as the real *Heisenberg algebra* \mathfrak{h}_{2n+1} of dimension $2n+1$.

Let M be a smooth manifold of odd dimension $2n+1$. If $\alpha \in \Omega^1(M)$ is a differential form, then $\mathrm{Ker}(\alpha_x) \subset T_x M$ is an hyperplane for any $x \in M$. Moreover, its differential gives a skew-symmetric bilinear form $d\alpha_x : T_x M \times T_x M \rightarrow \mathbb{R}$ at each $x \in M$. We say that α is a *contact form* on M if, for all $x \in M$, the restriction of $d\alpha_x$ to $\mathrm{Ker}(\alpha_x)$ is nondegenerate. Consider global coordinates $(x_1, \dots, x_n, z, y_1, \dots, y_n)$ on $M = \mathbb{R}^{2n+1}$ and $\alpha \in \Omega^1(M)$ defined by

$$(4) \quad \alpha = dz - \sum_{i=1}^n y_i dx_i$$

Putting $H_x = \mathrm{Ker}(\alpha_x)$ for $x \in M$ defines a smooth distribution $H \subset TM$ of corank one. There is a natural choice of local frame $\{\xi_1, \dots, \xi_{2n}\}$ for H . For $i = 1, \dots, n$, put

$$\xi_i = \frac{\partial}{\partial y_i}, \quad \xi_{n+i} = y_i \frac{\partial}{\partial z} + \frac{\partial}{\partial x_i}$$

Observe that $d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$ and the smooth functions $\{d\alpha(\xi_i, \xi_j) : i, j = 1, \dots, 2n\}$ are constant on M . More precisely, if ω denotes the standard symplectic form on \mathbb{R}^{2n} and $\{e_1, \dots, e_{2n}\}$ the standard basis of \mathbb{R}^{2n} , we have that $d\alpha_x(\xi_i, \xi_j) = \omega(e_i, e_j)$ for all $x \in M$. This shows that (4) defines a contact form, known as the *standard contact form* on \mathbb{R}^{2n+1} . Let M be a smooth manifold of dimension $(2n+1)$ and let $H \subset TM$ be a smooth distribution of corank one. We say that H is a *contact structure* on M if, for any $x \in M$, there exists an open neighborhood $U_x \subset M$ of x and a contact form $\alpha \in \Omega^1(U_x)$ such that $H_y = \mathrm{Ker}(\alpha_y)$ for all $y \in U_x$.

The Levi bracket associated to a smooth distribution $H \subset TM$ of corank one, at each point $x \in M$, gives a skew-symmetric map $\mathcal{L}_x : H_x \times H_x \rightarrow Q_x$, where $Q_x := T_x M / H_x$. We say that \mathcal{L}_x is nondegenerate if for $v \in H_x$, the condition $\mathcal{L}_x(v, w) = 0$ for all $w \in H_x$ implies $v = 0$. Moreover, we say that \mathcal{L} is nondegenerate if \mathcal{L}_x is nondegenerate for all $x \in M$. Suppose that $U \subset M$ is an open subset and H is the kernel of the one-form α on U . At each point $x \in U$, α_x descends to a linear isomorphism $\bar{\alpha}_x : Q_x \rightarrow \mathbb{R}$, which clearly depends smoothly on the point.

Therefore, $\bar{\alpha}$ is a local trivialization of the line bundle Q . The standard differential calculus formula for vector fields ξ, η on M

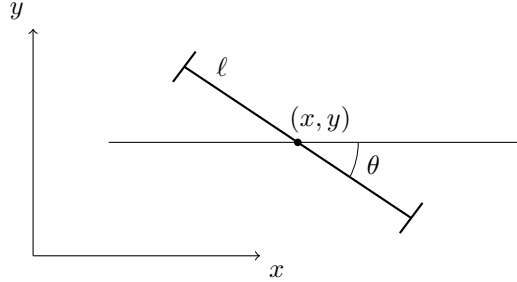
$$d\alpha(\xi, \eta) = \xi \cdot \alpha(\eta) - \eta \cdot \alpha(\xi) - \alpha([\xi, \eta])$$

reduces to $d\alpha(\xi, \eta) = -\alpha([\xi, \eta])$ for $\xi, \eta \in \Gamma(H)$, which shows that

$$(5) \quad (d\alpha_x)|_{\Lambda^2 H_x} = -\bar{\alpha}_x \circ \mathcal{L}_x$$

for any $x \in U$. Now one can easily see that \mathcal{L} is nondegenerate if and only if H is a contact structure. Indeed if H is a contact structure and $x \in M$, then H is the kernel of a contact form α locally around x . The restriction of $d\alpha_x$ to $\Lambda^2 H_x$ is nondegenerate, hence \mathcal{L}_x is nondegenerate by (5). Conversely, suppose that \mathcal{L}_x is nondegenerate for all $x \in M$. Recall that any smooth corank-one distribution can be written as the kernel of a one-form. Then for $y \in M$, consider an open neighborhood $U_y \subset M$ of y and $\alpha \in \Omega^1(U_y)$ such that $H_x = \text{Ker}(\alpha_x)$ for all $x \in U_y$. Now (5) holds on U_y , so that if \mathcal{L}_x is nondegenerate, the same is true for $(d\alpha_x)|_{\Lambda^2 H_x}$ for all $x \in U_y$. Then α is a contact form on U_y and H is a contact structure. The previous argument, then, shows that a contact structure can be equivalently defined as a corank-one distribution, whose symbol algebra at the point x is isomorphic to \mathfrak{h}_{2n+1} for each $x \in M$. As a consequence of a classical result about contact structures, this isomorphism depends smoothly on the point. Indeed, according to Pfaff Theorem, if $H \subset TM$ is a contact structure and $x \in M$, there exists a local coordinate system $(U_x, (x^1, \dots, x^n, z, y_1, \dots, y_n))$ at x such that $H_y = \text{Ker}(\alpha_y)$ for all $y \in U_x$, being $\alpha \in \Omega^1(U_x)$ defined by (4) in the coordinates $(x^1, \dots, x^n, z, y_1, \dots, y_n)$.

EXAMPLE 2. (*Pushcart*) The contact structure in dimension three arises from the description of a simple mechanical system. Consider a pushcart moving on a two-dimensional plane, which can be thought as two wheels connected by an horizontal axis ℓ .

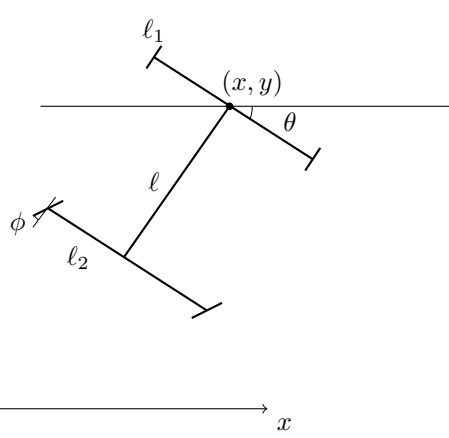


The instant position of the pushcart is described by the coordinates (x, y) on the plane, centered in the midpoint of ℓ and by a third coordinate θ , the angle between ℓ and the x -axis. The movement of the object is allowed along the directions $X = \frac{\partial}{\partial \theta}$ and $Y = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}$. Put $Z = [X, Y] = -\cos \theta \frac{\partial}{\partial x} - \sin \theta \frac{\partial}{\partial y}$. Then, $\{X, Y\}$ generate a distribution of codimension one on $M = \mathbb{R}^2 \times \mathbb{S}^1$, which is noninvolutive since $\{X, Y, Z\}$ are linearly independent at each point of M .

We can rephrase the considerations deduced in Example 1, by saying that contact structures have no local invariants. However, there exist examples of bracket-generating distributions admitting local invariants. The first example was found by Elie Cartan. In his "five variables" paper [12], he investigated distributions of rank two in dimension five, which are bracket-generating of depth two. Again, such a configuration is realized by a mechanical system, describing two balls rolling on each other without grinding (see [1]). The instant position of this system can be described by five real coordinates. The point of contact between the two balls is given by a point on the surface of the first ball. The second bit of information needed to pin down the position of the system is how the second ball sits in space. This can be recorded by a positively oriented

orthonormal basis sitting in its center. We thus conclude that the configuration space of the system is $M = \mathbb{S}^2 \times SO(3)$. One can show that the no-grinding condition defines a rank-two distribution on the five-dimensional manifold M with growth vector $(2, 3, 5)$. Using his method of equivalence, Cartan associated to any such distribution a fundamental quartic invariant, playing here the role of the Riemannian curvature in differential geometry.

Another nice example from classical mechanics is given by a car moving on a plane (a detailed description can be found in [13]). This can be thought as two parallel axes ℓ_1 and ℓ_2 joining the back and front wheels, respectively, whose midpoints are connected by an orthogonal axis ℓ . The instant position is determined by four real coordinates. Indeed, the position of the car with respect to the plane is given by coordinates $(x, y) \in \mathbb{R}^2$ of the midpoint of ℓ_1 on the plane. Moreover, the orientation of the car with respect to the plane is given by a third coordinate θ , the angle between ℓ and the x -axis. Finally, the front wheels are allowed to rotate of an angle ϕ with respect to ℓ .



Therefore, the configuration space is a real manifold M of dimension four. The car can reach every point of the plane, but just running along distinguished directions X, Y generating a rank-two distribution on M with growth vector $(2, 3, 4)$. Bracket-generating distributions with growth vector $(2, 3, 4)$ are also known as *Engel distributions*.

3. Generic one step bracket-generating distributions

The thesis investigates distributions, which are bracket generating in one step. For such distributions of rank k , one can formulate a genericity hypothesis in terms of a smooth action of the general linear group $GL(k, \mathbb{R})$. In particular, Proposition 3 relates the $GL(4, \mathbb{R})$ -orbits to the $CO(3, 3)$ -orbits, which are characterized by Proposition 4 in terms of linear algebra. It thus follows that the generic types of rank four can be classified.

DEFINITION 4. Let M be a smooth manifold of dimension n and $H \subset TM$ a smooth distribution of rank k . Denote by \mathcal{H}^{-1} the set of smooth sections of H . We say that H is *bracket generating in one step* if the set of smooth sections $\mathcal{H}^{-2} = \mathcal{H}^{-1} + [\mathcal{H}^{-1}, \mathcal{H}^{-1}]$, in each point $x \in M$, spans the whole tangent space at x .

Observe that if H is bracket generating in one step, the regularity condition on \mathcal{H}^{-2} is automatically satisfied and the filtration (1) writes as $\{0\} \subset H \subset TM$. In terms of Definition 2, H is a bracket-generating distribution of depth two. According to the growth vector notation, we will also say that H is a (k, n) -distribution. Consider a (k, n) -distribution $H \subset TM$ and

denote by $Q = TM/H$ the quotient bundle. The Levi bracket introduced in Definition 3, in this case, writes as a unique homogeneous component that is a surjective homomorphism of vector bundles $\mathcal{L} : \Lambda^2 H \rightarrow Q$. Such a distribution can only exist for some values of rank k and dimension n , exactly those satisfying $\binom{k}{2} \geq n - k \geq 1$. For values out of this range, indeed, the set of surjective linear maps $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ is empty. For every nonnegative integer m , the matrix-vector multiplication determines a smooth action of the Lie group $GL(m, \mathbb{R})$ on \mathbb{R}^m , known as the standard action. This induces an action of $G = GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R})$ on the set of linear maps $L(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ in a natural way, by putting

$$(6) \quad \begin{aligned} G \times L(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k}) &\longrightarrow L(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k}) \\ (A, B) \cdot F(v, w) &:= B \cdot F(A^{-1} \cdot v, A^{-1} \cdot w) \end{aligned}$$

Two linear maps are in the same G -orbit if and only if they only differ by choices of bases of their domain and codomain. Observe that the vector subspace of surjective linear maps $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ is G -invariant.

PROPOSITION 2. *Let k, n be nonnegative integers such that $1 \leq n-k \leq \binom{k}{2}$. Consider the natural action of $G = GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R})$ on the set of surjective linear maps $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$.*

- (a) *The Levi bracket \mathcal{L} associated to a (k, n) -distribution determines, for every $x \in M$, a G -orbit $\mathcal{O}_x \subset L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$. These orbits are the same for two points $x, y \in M$ if and only if the symbol algebra in x and y are isomorphic.*
- (b) *For $1 \leq n-k < \binom{k}{2}$, the G -orbits in $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ are in one-to-one correspondence with the orbits of the induced action of $GL(k, \mathbb{R})$ on the Grassmannian $Gr(\ell, \Lambda^2 \mathbb{R}^k)$, where $\ell = \binom{k}{2} - n + k$.*

PROOF. (a) Fix $x \in M$ and choose linear isomorphisms $\phi_x : \mathbb{R}^k \rightarrow H_x$, $\psi_x : \mathbb{R}^{n-k} \rightarrow Q_x$. Observe that ϕ_x induces a linear isomorphism $\Lambda^2 \phi_x : \Lambda^2 \mathbb{R}^k \rightarrow \Lambda^2 H_x$, characterized by the formula $\Lambda^2 \phi_x(v \wedge w) := \phi_x(v) \wedge \phi_x(w)$ for all $v, w \in \mathbb{R}^k$. The pair (ϕ_x, ψ_x) defines, together with \mathcal{L}_x , the map $T_x \in L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ via

$$T_x(v \wedge w) = ((\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2 \phi_x)(v \wedge w).$$

The set of all isomorphisms $\mathbb{R}^k \oplus \mathbb{R}^{n-k} \rightarrow \text{gr}(T_x M)$ writes as

$$\{ (\phi_x \circ A, \psi_x \circ B) \mid (A, B) \in G \}.$$

Then, denoting by $S_x : \Lambda^2 \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$ the linear map defined by the pair $(\phi_x \circ A, \psi_x \circ B)$, for all $v, w \in \mathbb{R}^k$ we have:

$$\begin{aligned} (A, B) \cdot S_x(v \wedge w) &= (A, B) \cdot ((\psi_x \circ B)^{-1} \circ \mathcal{L}_x \circ \Lambda^2(\phi_x \circ A))(v \wedge w) \\ &= (A, B) \cdot (B^{-1} \circ (\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2 \phi_x)(Av \wedge Aw) \\ &= T_x(v \wedge w). \end{aligned}$$

hence $(A, B) \cdot S_x = T_x$. We conclude that the G -orbit of T_x is independent on the choice of isomorphism $\mathbb{R}^k \oplus \mathbb{R}^{n-k} \rightarrow \text{gr}(T_x M)$ and thus depends only on \mathcal{L}_x .

An isomorphism $F : \text{gr}(T_x M) \rightarrow \text{gr}(T_y M)$ of nilpotent graded Lie algebras writes as the direct sum $F = (F_1, F_2)$ of linear isomorphisms $F_1 : H_x \rightarrow H_y$ and $F_2 : Q_x \rightarrow Q_y$, satisfying $F_2(\mathcal{L}_x(\xi, \eta)) = \mathcal{L}_y(F_1(\xi), F_1(\eta))$ for all $\xi, \eta \in H_x$. First, suppose that such an isomorphism F is given. Choose linear isomorphisms $\phi_x : \mathbb{R}^k \rightarrow H_x$, $\psi_x : \mathbb{R}^{n-k} \rightarrow Q_x$. By composition, we get linear isomorphisms $F_1 \circ \phi_x : \mathbb{R}^k \rightarrow H_y$ and $F_2 \circ \psi_x : \mathbb{R}^{n-k} \rightarrow Q_y$. For any $v, w \in \mathbb{R}^k$, there exist $\lambda, \theta \in H_y$ such that $\phi_x(v) = F_1^{-1}(\lambda)$ and $\phi_x(w) = F_1^{-1}(\theta)$ and we have that

$$\begin{aligned}
((\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2 \phi_x)(v \wedge w) &= (\psi_x)^{-1}(\mathcal{L}_x(F_1^{-1}(\lambda), F_1^{-1}(\theta))) \\
&= (\psi_x)^{-1}(F_2^{-1}(\mathcal{L}_y(\lambda, \theta))) \\
&= (F_2 \circ \psi_x)^{-1} \circ \mathcal{L}_y \circ \Lambda^2(F_1 \circ \phi_x)(v \wedge w).
\end{aligned}$$

Conversely, suppose that $\mathcal{L}_x, \mathcal{L}_y$ generate the same orbit in $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$. There exist linear isomorphisms $\phi_x : \mathbb{R}^k \rightarrow H_x$, $\phi_y : \mathbb{R}^k \rightarrow H_y$ and $\psi_x : \mathbb{R}^{n-k} \rightarrow Q_x$, $\psi_y : \mathbb{R}^{n-k} \rightarrow Q_y$ such that $(\psi_x)^{-1} \circ \mathcal{L}_x \circ \Lambda^2 \phi_x = (\psi_y)^{-1} \circ \mathcal{L}_y \circ \Lambda^2 \phi_y$. Define $F_1 := \phi_y \circ (\psi_y)^{-1}$ and $F_2 := \phi_x \circ (\psi_x)^{-1}$. Then $F_1 : H_x \rightarrow H_y$ and $F_2 : Q_x \rightarrow Q_y$ are linear isomorphisms such that $F_2 \circ \mathcal{L}_x = \mathcal{L}_y \circ \Lambda^2 F_1$, hence they define an isomorphism $F = (F_1, F_2) : \text{gr}(T_x M) \rightarrow \text{gr}(T_y M)$ of nilpotent graded Lie algebras.

(b) An action of $GL(k, \mathbb{R})$ on $\Lambda^2 \mathbb{R}^k$ is obtained in a natural way, by linearly extending the definition $A \cdot (v_1 \wedge v_2) := A \cdot v_1 \wedge A \cdot v_2$ for $A \in GL(k, \mathbb{R})$ and $v_1, v_2 \in \mathbb{R}^k$ on decomposable elements. This, in turn, induces the natural action on the Grassmannian:

$$\begin{aligned}
GL(k, \mathbb{R}) \times Gr(l, \Lambda^2 \mathbb{R}^k) &\longrightarrow Gr(l, \Lambda^2 \mathbb{R}^k) \\
A \cdot \text{sp}\{\omega_1, \dots, \omega_l\} &:= \text{sp}\{A \cdot \omega_1, \dots, A \cdot \omega_l\}
\end{aligned}$$

If $T \in L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$, then $\text{Ker}(T) \subset \Lambda^2 \mathbb{R}^k$ is a linear subspace of dimension ℓ . Otherwise put, it is an element of $Gr(l, \Lambda^2 \mathbb{R}^k)$. Consider now $S, T \in L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$. We show that S and T lie in the same G -orbit if and only if $\text{Ker}(S)$ and $\text{Ker}(T)$ lie in the same orbit of $GL(k, \mathbb{R})$. First, suppose that there exists $(A, B) \in G$ such that $T = B \circ S \circ \Lambda^2 A^{-1}$. Then, since B is invertible, $\omega \in \Lambda^2 \mathbb{R}^k$ lies in $\text{Ker}(T)$ if and only if $S \circ \Lambda^2 A^{-1}(\omega) = 0$, hence if and only if $\Lambda^2 A^{-1}(\omega) \in \text{Ker}(S)$. Conversely, suppose that $A \in GL(k, \mathbb{R})$ is such that $\Lambda^2 A^{-1}$ restricts to an isomorphism $\text{Ker}(T) \rightarrow \text{Ker}(S)$. Put $V := \text{Ker}(T)$. Then, T and $S \circ \Lambda^2 A^{-1}$ descend to linear isomorphisms $\Lambda^2 \mathbb{R}^k / V \rightarrow \mathbb{R}^{n-k}$, say \bar{T} and $\bar{S} \circ \Lambda^2 A^{-1}$, which define the linear isomorphism $B := \bar{T} \circ (\bar{S} \circ \Lambda^2 A^{-1})^{-1} : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k}$. Observe that by definition, $B \in GL(n-k, \mathbb{R})$ is such that $T = B \circ S \circ \Lambda^2 A^{-1}$, which concludes the proof. \square

Fix $x \in M$. Denote by $\mathcal{O}_x \subset L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ the orbit defined by \mathcal{L}_x as in Proposition 2 (a). Suppose now that \mathcal{O}_x is open. Then $\mathcal{O}_y = \mathcal{O}_x$ for all points y in an open neighborhood of x . Therefore, from the point of view of the local geometry, assuming that \mathcal{L} generates the same open orbit everywhere is not restrictive.

DEFINITION 5. Let k, n be nonnegative integers such that $1 \leq n-k \leq \binom{k}{2}$ and call (k, n) a *bidimension*. Consider the natural action of the Lie group $G = GL(k, \mathbb{R}) \times GL(n-k, \mathbb{R})$ on the set $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ of surjective linear maps.

- (1) The bidimension (k, n) is said to be *rigid* if there exist open G -orbits in $L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$.
- (2) Let be $\mathcal{O} \subset L_s(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ an open orbit. A (k, n) -distribution $H \subset TM$ is said to be *generic of type \mathcal{O}* if $\mathcal{O}_x = \mathcal{O}$ for all $x \in M$, with \mathcal{O}_x denoting the orbit generated by \mathcal{L}_x as in Proposition 2 (a).

Observe that every open orbit \mathcal{O} of bidimension (k, n) is the type of some generic (k, n) -distribution. To see this, first consider a nilpotent graded Lie algebra \mathfrak{n} corresponding to the orbit \mathcal{O} . Let N be the connected and simply connected Lie group with Lie algebra \mathfrak{n} . Then $\mathfrak{g}_{-1} \subset \mathfrak{n}$ generates a left-invariant distribution on N , which is generic of type \mathcal{O} . Therefore, a classification of generic types of (k, n) -distributions consists of a list of open orbits of bidimension (k, n) . These are a special case of rigid Carnot algebras, as defined and classified in [2]. There, it is shown that the list of rigid bidimensions is given by three infinite series and several exceptional cases. Put $p = k + \binom{k}{2}$ for arbitrary integers $k \geq 2$. The following bidimensions are easily seen to be rigid:

- *Darboux bidimensions* $(k, k + 1)$. For these values of rank and dimension, we look at orbits of surjective linear maps $\Lambda^2 \mathbb{R}^k \rightarrow \mathbb{R}$, which are equivalent to nonzero skew-symmetric bilinear forms on \mathbb{R}^k . An open orbit of Darboux bidimension is generated by maximally nondegenerate skew-symmetric bilinear forms, which means nondegenerate for k even and having one-dimensional nullspace for k odd.
- *dual Darboux bidimensions* $(k, p - 1)$. Let be $\ell \subset \Lambda^2 \mathbb{R}^k$ the line generated by a maximally nondegenerate skew-symmetric bilinear form on $(\mathbb{R}^k)^*$. Then ℓ generates an open orbit in $Gr(1, \Lambda^2 \mathbb{R}^k)$ corresponding, in the sense of Proposition 2 (b), to a unique open orbit of dual Darboux bidimension.
- *free bidimensions* (k, p) . For these values, we look at the G -action on the set of isomorphisms between two real vector spaces of dimension $\binom{k}{2}$. It is thus clear that there is a unique orbit of free bidimension (k, p) , whose elements are isomorphisms.

The results from [2] assure that the orbits described above are the unique open orbits of Darboux, dual Darboux and free bidimensions. Throughout the rest of the Chapter, we will focus on the case of rank four. In principle, distributions of rank four and bracket generating in one step may only exist on manifolds of dimension $5 \leq n \leq 10$. The classification in [2] shows that the bidimension $(4, n)$ is rigid for each admissible value of n . However, the same result is recovered through the characterization of open orbits that we are going to prove in Theorem 2. For $n = 5, 9, 10$ we have, respectively, a Darboux, dual Darboux and free bidimension and each of them corresponds to a unique open orbit. On the other hand, for $n = 6, 7, 8$ there exist two distinct open orbits of exceptional bidimension $(4, n)$. In some sense, the open orbits can be characterized as nondegenerate restrictions of a quadratic form on $\Lambda^2 \mathbb{R}^4$ and this will allow us to present the case of rank four in a uniform picture.

PROPOSITION 3. *The wedge product defines a conformal class of quadratic forms of signature $(3, 3)$ on $\Lambda^2 \mathbb{R}^4$. The resulting Lie group homomorphism $GL(4, \mathbb{R}) \rightarrow CO(3, 3)$ restricts to a two-fold covering between the connected components of the identities.*

PROOF. The wedge product $\Lambda^2 \mathbb{R}^4 \times \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^4 \mathbb{R}^4$ is a symmetric bilinear map onto a one-dimensional vector space. Any ordered basis $\{v_1, \dots, v_4\}$ for \mathbb{R}^4 determines a volume form $\text{vol} = v_1 \wedge v_2 \wedge v_3 \wedge v_4$, thus an isomorphism $\Lambda^4 \mathbb{R}^4 \cong \mathbb{R}$ and a real quadratic form $\beta : \Lambda^2 \mathbb{R}^4 \rightarrow \mathbb{R}$ on the real vector space $\Lambda^2 \mathbb{R}^4$ of dimension six. Explicitly for $\phi \in \Lambda^2 \mathbb{R}^4$, define $\beta(\phi)$ as the unique real number such that $\phi \wedge \phi = \beta(\phi) \text{vol}$. A direct computation shows that β is a quadratic form of signature $(3, 3)$. Denote by $CO(3, 3)$ the conformal group of β , which is given by those automorphisms of $\Lambda^2 \mathbb{R}^4$ preserving β up to a nonzero factor scale. For $A \in GL(4, \mathbb{R})$ and $w_i \in \mathbb{R}^4$, the change rule

$$Aw_1 \wedge Aw_2 \wedge Aw_3 \wedge Aw_4 = \det(A)w_1 \wedge w_2 \wedge w_3 \wedge w_4$$

shows that $\beta(A\phi) = \det(A)\beta(\phi)$ for all $A \in GL(4, \mathbb{R})$ and $\phi \in \Lambda^2 \mathbb{R}^4$. Then the $GL(4, \mathbb{R})$ -action on $\Lambda^2 \mathbb{R}^4$ preserves the conformal class of β and it thus defines a group homomorphism $F : GL(4, \mathbb{R}) \rightarrow CO(3, 3)$. The kernel of F is given by those elements in $GL(4, \mathbb{R})$ acting as the identity map on $\Lambda^2 \mathbb{R}^4$. The inclusion $\{-I, I\} \subset \text{Ker}(F)$ follows straightforwardly. Starting with a basis of $\Lambda^2 \mathbb{R}^4$ of decomposable elements, the opposite inclusion can be shown by direct computation, thus proving that $\text{Ker}(F) = \{-I, I\}$. Then F induces an injective Lie group homomorphism $\bar{F} : GL(4, \mathbb{R})/\{-I, I\} \rightarrow CO(3, 3)$. Since \bar{F} lifts to an injective homomorphism between the respective Lie algebras, whose dimensions coincide, it actually restricts to an isomorphism between the connected components of the identities. \square

Sylvester Theorem classifies symmetric bilinear forms, by characterizing their orbits under the action of the general linear group. According to it, any symmetric bilinear form on \mathbb{R}^n is in the orbit of a canonical element. The admissible models are uniquely determined by rank

and signature, which then characterize the orbit. A nondegenerate symmetric bilinear form is said to be a pseudoscalar product. If $n = p + q$ and $\langle \cdot, \cdot \rangle$ is a pseudoscalar product of signature (p, q) on \mathbb{R}^n , the orthogonal group $O(p, q)$ is defined as the stabilizer of $\langle \cdot, \cdot \rangle$. Since $\langle \cdot, \cdot \rangle$ is nondegenerate, the map $v \mapsto \langle v, \cdot \rangle$ induces an isomorphism between \mathbb{R}^n and its dual. Therefore, the system $\{w_1, \dots, w_k\} \subset \mathbb{R}^n$ is linearly independent if and only if the $(k \times k)$ -matrix of products $(\langle w_i, w_j \rangle)$ has nonzero determinant.

PROPOSITION 4. *Put $n = p + q$ for integers $p, q \geq 0$. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate pseudoscalar product of signature (p, q) on \mathbb{R}^n and denote by $O(p, q)$ the corresponding orthogonal group. The $O(p, q)$ -orbits in $Gr(k, \mathbb{R}^{p+q})$ are then characterized in terms of rank and signature of the restriction of $\langle \cdot, \cdot \rangle$ to the subspace in question. Moreover, the open orbits are those of maximal rank.*

PROOF. Let $\langle \cdot, \cdot \rangle$ be a nondegenerate pseudoscalar product of signature (p, q) on \mathbb{R}^n . Denote by I_s the identity matrix of size s and by $I_{s,t}$ the square matrix of the following block form:

$$I_{s,t} = \begin{pmatrix} I_s & 0 \\ 0 & -I_t \end{pmatrix}$$

Let $W \subset \mathbb{R}^n$ be a k -dimensional linear subspace. The restriction $\langle \cdot, \cdot \rangle|_W$ is a symmetric bilinear form on W of signature (p', q') with rank $r := p' + q' \leq k$. Clearly, rank and signature of $\langle \cdot, \cdot \rangle|_{AW}$ are the same for every $A \in O(p, q)$, hence they are constant on the $O(p, q)$ -orbit of W . We will show that they actually characterize the $O(p, q)$ -orbit of W in the Grassmannian $Gr(k, \mathbb{R}^{p+q})$. We claim that there exists a choice of basis for W , which is canonical in the following sense. Such basis can be completed to a basis \mathcal{B} of \mathbb{R}^n whose matrix of scalar products assumes a canonical form, which is uniquely determined by (p', q') . Let $N = W \cap W^\perp$ be the nullspace for $\langle \cdot, \cdot \rangle|_W$, which is a $(k - r)$ -dimensional subspace of W . Choose a basis $\{w_1, \dots, w_{k-r}\}$ of N . By definition, $\langle w_i, w \rangle = 0$ for all $w \in W$ and $i = 1, \dots, k - r$. Moreover, $\langle \cdot, \cdot \rangle$ descends to a nondegenerate pseudoscalar product of signature (p', q') on W/N . The induced product thus writes as $I_{p', q'}$ in a basis $\{\bar{w}_{k-r+1}, \dots, \bar{w}_k\}$ of W/N . First, choose preimages $w_i \in W$ of $\bar{w}_i \in W/N$ for $i = k - r + 1, \dots, k$, thus obtaining an orthonormal system. Then by adding the previously chosen basis for N , complete $\{w_{k-r+1}, \dots, w_k\}$ to a basis $\{w_1, \dots, w_k\}$ of W , whose matrix of scalar products writes in the block form

$$(7) \quad \begin{pmatrix} 0 & 0 \\ 0 & I_{p', q'} \end{pmatrix}$$

where the zero square block on the diagonal has size $k - r$. Observe that if $\langle \cdot, \cdot \rangle$ is nondegenerate, it identifies \mathbb{R}^n with its dual via the mapping $v \mapsto \langle v, \cdot \rangle$. In this picture, N^\perp identifies with the annihilator of N in $(\mathbb{R}^n)^*$, so that $\dim(N^\perp) = n - (k - r)$. Moreover, $N \subset W \subset N^\perp$ and $\langle \cdot, \cdot \rangle$ descends to a nondegenerate pseudoscalar product on N^\perp/N of signature $(p'', q'') = (p - k + r, q - k + r)$. Since $W/N \subset N^\perp/N$, we can complete $\{\bar{w}_{k-r+1}, \dots, \bar{w}_k\}$ to a basis for N^\perp/N , say $\{\bar{w}_{k-r+1}, \dots, \bar{w}_{n-k+r}\}$, such that the scalar product restricted to N^\perp/N writes as

$$\begin{pmatrix} I_{p', q'} & 0 \\ 0 & I_{p'', q''} \end{pmatrix}$$

with $p' + p'' = p$ and $q' + q'' = q$. Taking preimages of \bar{w}_i for $i = k + 1, \dots, n - k + r$, we complete the basis $\{w_1, \dots, w_k\}$ of W to a basis $\{w_1, \dots, w_{n-k+r}\}$ for N^\perp . In particular, $\langle w_i, w_j \rangle = 0$ for all $i = 1, \dots, k - r$ and $j = 1, \dots, n - k + r$, so that $\langle \cdot, \cdot \rangle|_{N^\perp}$ writes as

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{p', q'} & 0 \\ 0 & 0 & I_{p'', q''} \end{pmatrix}$$

in the basis $\{w_1, \dots, w_{n-k+r}\}$. By nondegeneracy of $\langle \cdot, \cdot \rangle$ on \mathbb{R}^n , this can be in turn completed to a basis \mathcal{B} for \mathbb{R}^n such that the scalar product on \mathbb{R}^n writes as

$$\begin{pmatrix} 0 & 0 & 0 & I_{k-r} \\ 0 & I_{p',q'} & 0 & 0 \\ 0 & 0 & I_{p''',q'''} & 0 \\ I_{k-r} & 0 & 0 & 0 \end{pmatrix}$$

and the claim is proved. By construction, W is spanned by the first k basis elements. If $W, W' \subset \mathbb{R}^n$ are k -dimensional subspaces on which $\langle \cdot, \cdot \rangle$ restricts with same rank and signature, we can apply the argument above to each of them. In this way, we find bases $\mathcal{B} = \{w_1, \dots, w_n\}$ and $\mathcal{B}' = \{w'_1, \dots, w'_n\}$ for \mathbb{R}^n such that $\{w_1, \dots, w_k\}$ and $\{w'_1, \dots, w'_k\}$ are bases respectively of W, W' . There exists a unique $A \in GL(n, \mathbb{R})$ such that $Aw_i = w'_i$ for all $i = 1, \dots, n$. By construction, A maps W onto W' . Moreover, since $\langle \cdot, \cdot \rangle$ coincides on $\{w_1, \dots, w_n\}$ and on $\{Aw_1, \dots, Aw_n\}$, A actually lies in $O(p, q)$. Then W and W' are in the same $O(p, q)$ -orbit and the first statement is proved.

Now consider the set \mathcal{V}_k of linearly independent k -tuples of vectors in \mathbb{R}^n . Then \mathcal{V}_k identifies with an open subset of \mathbb{R}^{nk} and inherits a topology from \mathbb{R}^{nk} . The function $f: \mathcal{V}_k \rightarrow \mathbb{R}$, defined by $f(v_1, \dots, v_k) = \det(\langle v_i, v_j \rangle)$, is smooth. There is an obvious projection $\pi: \mathcal{V}_k \rightarrow Gr(k, \mathbb{R}^n)$, which endows $Gr(k, \mathbb{R}^n)$ with the quotient topology. If $W \subset \mathbb{R}^n$ is a k -dimensional subspace, then $\pi^{-1}(W)$ is the set of bases of W . Suppose now that $\langle \cdot, \cdot \rangle|_W$ has maximal rank k . If $\{w_1, \dots, w_k\}$ is a basis of W , then $f(w_1, \dots, w_k) \neq 0$, hence there exists an open neighborhood of (w_1, \dots, w_k) in \mathcal{V}_k such that f does not vanish on it. Any other basis of W writes as $\{Aw_1, \dots, Aw_k\}$ for some $A \in GL(k, \mathbb{R})$. Clearly, $f(w_1, \dots, w_k) \neq 0$. Hence if f is nonzero on (w_1, \dots, w_k) , then it is nonzero on $\pi^{-1}(\pi(w_1, \dots, w_k))$. Thus $\{(w_1, \dots, w_k) \mid f(w_1, \dots, w_k) \neq 0\}$ can be written as $\pi^{-1}(U)$ for some $U \subset Gr(k, \mathbb{R}^n)$, which is open by definition of quotient topology. We thus proved that the orbits of maximal rank are open.

Finally, we show that there are no further open orbits, by proving the following fact. If $W \subset \mathbb{R}^n$ is a k -dimensional linear subspace and $\langle \cdot, \cdot \rangle|_W$ has rank $l < k$, then there exists a k -dimensional linear subspace $W_\delta \subset \mathbb{R}^n$ arbitrary close to W such that $\langle \cdot, \cdot \rangle|_{W_\delta}$ has rank $l+1$. First of all, under our assumption the nullspace of $\langle \cdot, \cdot \rangle|_W$ is nontrivial and we can thus choose a nonzero element $w_1 \in W \cap W^\perp$. Moreover, there exist $w_2, \dots, w_{l+1} \in W$ such that $\mathcal{S} = \{w_1, w_2, \dots, w_{l+1}\} \subset W$ is a linearly independent system and the matrix of pseudoscalar products among elements of \mathcal{S} writes as

$$M = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

with B square matrix of size l with nonzero determinant. Clearly, \mathcal{S} can be completed to a basis $\{w_1, \dots, w_k\}$ for W . By nondegeneracy, there exists $v \in \mathbb{R}^n$ such that $\langle v, w_1 \rangle = 1$. In particular, this implies that $v \notin W$ and therefore, up to apply Gram-Schmidt to the linearly independent system $\{v, w_2, \dots, w_{l+1}\}$, we can assume that $\langle v, w_i \rangle = 0$ for all $i = 2, \dots, l+1$. For any real number $\delta > 0$, $\mathcal{S}_\delta = \{w_1 + \delta v, w_2, \dots, w_{l+1}\}$ defines a linearly independent system, whose matrix of pseudoscalar products writes as

$$M_\delta = \begin{pmatrix} \delta^2 \langle v, v \rangle + 2\delta & 0 \\ 0 & B \end{pmatrix}$$

For any $\delta > 0$, $W_\delta = \text{sp}\{w_1 + \delta v, \dots, w_{l+1}, w_{l+2}, \dots, w_k\} \subset \mathbb{R}^n$ is a k -dimensional subspace. If $\delta > 0$ satisfies $\delta^2 \langle v, v \rangle + 2\delta \neq 0$, then W_δ is such that $\langle \cdot, \cdot \rangle|_{W_\delta}$ has rank $l+1$. Clearly, δ can be chosen sufficiently small such that $(w_1, \dots, w_k), (w_1 + \delta v, w_2, \dots, w_k) \in \mathcal{V}_k$ are arbitrary close, so that W_δ is arbitrary close to W . \square

As a consequence of Proposition 4, the open $CO(3, 3)$ -orbits of linear subspaces in $\Lambda^2\mathbb{R}^4$ are similarly characterized, up to identifying nondegenerate signatures (p', q') and (q', p') . Combining this fact with Proposition 3, we get the classification of generic types of one-step bracket-generating distributions of rank four.

THEOREM 2. *The open $GL(4, \mathbb{R})$ -orbits in $Gr(\ell, \Lambda^2\mathbb{R}^4)$ are those consisting of subspaces for which the restriction of the wedge product is nondegenerate. Hence, there is one generic type in Darboux, dual Darboux and free bidimensions $(4, 5)$, $(4, 9)$ and $(4, 10)$ and there are two generic types in bidimensions $(4, 6)$, $(4, 7)$ and $(4, 8)$.*

PROOF. Let b be a quadratic form on $\Lambda^2\mathbb{R}^4$ in the conformal class defined by the wedge product and let be $CO(3, 3)$ the conformal group of b . By Proposition 4, the $CO(3, 3)$ -orbit of a linear subspace $P \subset \Lambda^2\mathbb{R}^4$ is open if and only if $b|_P$ is nondegenerate. Now we know from Proposition 3 that $GL(4, \mathbb{R})$ -orbits and $CO(3, 3)$ -orbits coincide, so that the first statement is proved. In order to prove the second statement, we shall count the nondegenerate restrictions of the wedge product to a linear subspace $P \subset \Lambda^2\mathbb{R}^4$ of dimension $(10 - n)$ for $n = 5, \dots, 9$, up to identification $(p, s) \equiv (s, p)$ of nondegenerate signatures. Observe that $b|_P$ is nondegenerate if and only if the same holds for the restriction to the orthogonal complement $b|_{P^\perp}$. Therefore, it is enough to consider the following cases:

- (a) orbits of lines ($n = 5, 9$);
- (b) orbits of two-dimensional planes ($n = 6, 8$);
- (c) orbits of three-dimensional planes ($n = 7$).

On the one hand, it is clear that generic Darboux and dual Darboux types are unique, since there is a unique nondegenerate restriction to a line. On the other hand, for each exceptional bidimension there are two distinct open orbits. To see this, suppose that $P \subset \Lambda^2\mathbb{R}^4$ is a linear subspace of dimension $\ell = 2, 3$ such that $b|_P$ is nondegenerate of signature (p, s) , with $p + s = \ell$. Since we identify signatures up to scale, we assume that the pair (p, s) satisfies $p \geq s$. In the case (b), hyperbolic and elliptic orbits are given by signatures $(1, 1)$ and $(2, 0)$, while for (c) they correspond to signatures $(2, 1)$ and $(3, 0)$. \square

The characterization of open $GL(4, \mathbb{R})$ -orbits in $Gr(10 - n, \Lambda^2\mathbb{R}^4)$, in particular, proves the existence of open orbits for all $5 \leq n \leq 9$. We already know that any isomorphism of $\Lambda^2\mathbb{R}^4$ generates the open orbit of free bidimension $(4, 10)$, hence the bidimension $(4, n)$ is rigid for each admissible value of n . We conclude with an overview of generic types and corresponding structures of rank four.

For $n = 5$ we have a Darboux bidimension. General Darboux bidimensions were discussed above. In particular, we know that the open orbit of $(4, 5)$ -type is generated by a nondegenerate skew-symmetric bilinear form on \mathbb{R}^4 . From the description in Example 1, we deduce that generic distributions of $(4, 5)$ -type are contact structures in dimension five. As discussed there, contact structures have infinite-dimensional symmetry group and no local invariants.

The geometry in dimension ten originates open orbits of free bidimension. The Levi bracket associated to a generic $(4, 10)$ -distribution, indeed, determines an isomorphism at each point. Any isomorphism of $\Lambda^2\mathbb{R}^4$ is a graded bracket on the direct sum $\mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4$. The resulting nilpotent graded Lie algebra is also known as the free algebra of length two with four generators. This free algebra can be realized as the negative graded part of a grading on $\mathfrak{g} = \mathfrak{so}(9)$ such that the first cohomology $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneity (see [10, p. 430]). It follows from a general result ([10, Theorem 3.1.14 p. 271]) that any generic $(4, 10)$ -distribution is equivalent to a parabolic geometry. This implies, in particular, that the

automorphism group of the structure is finite-dimensional, as well as the existence of local invariants.

In dimensions $6 \leq n \leq 9$, an explicit description of model brackets comes from the characterization of $GL(4, \mathbb{R})$ -orbits of their kernels. Suppose that $\mathcal{O} \subset L_s(\Lambda^2 \mathbb{R}^4, \mathbb{R}^{n-4})$ is an open orbit. Then, \mathcal{O} corresponds to a unique open orbit in $Gr(10-n, \Lambda^2 \mathbb{R}^4)$, thus to a unique nondegenerate signature (p, s) with $p \geq s$ and $p + s = 10 - n$. If the wedge product restricts to a linear subspace $P \subset \Lambda^2 \mathbb{R}^4$ of dimension $10 - n$ with signature (p, s) , then the canonical projection $\Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^2 \mathbb{R}^4 / P$ onto the quotient is in \mathcal{O} . In this picture, the model algebra of generic type $(4, 9)$ can be immediately deduced. Denote by $V = \mathbb{R}^4$ and observe that the restriction of the wedge product to a line in $\Lambda^2 V$ is nondegenerate if and only if the nonzero elements in this line are nondegenerate as bilinear forms on V^* . A model bracket of generic $(4, 9)$ -type is given by the projection $\Lambda^2 V \rightarrow \Lambda_0^2 V$ onto the kernel of a nondegenerate skew-symmetric bilinear form on V . This generalizes to a description for generic dual Darboux types of even rank. Indeed, denoting by (V, ω) a symplectic real vector space of dimension $2k$, one can analogously define a model bracket of type $(2k, n)$, where $n = 2k + \binom{2k}{2} - 1$. Generic dual Darboux distributions of even rank will be described at the end of next Chapter. There, it will be proved that every such distribution determines a canonical linear connection. In particular, torsion and curvature of the canonical connection are local invariants for the structure and the symmetry group is finite-dimensional.

For the remaining types, we give a simpler description in terms of generalizations of the Heisenberg algebra. Model algebras of hyperbolic and elliptic $(4, 6)$ -types are, respectively, the two-fold product of three-dimensional real Heisenberg algebras and the complex three-dimensional Heisenberg algebra. The locally flat geometries of these types hence are products of two real three-dimensional contact manifolds, respectively three-dimensional complex contact manifolds. In particular, these have infinite-dimensional automorphism group. As shown in [7], for each type there is a tensor whose vanishing is equivalent to local isomorphism to these models, so there are local invariants. However, there exist remarkable examples of finite type. Any generic $(4, 6)$ -distribution endowed with an additional almost complex structure on the subbundle H , which is compatible with the Levi bracket in an appropriate sense, is in fact equivalent to a parabolic geometry (see [10, p. 443-455]). This is related to CR -structures of CR -dimension and codimension two, see [8] and [17]. More precisely, the two-fold product of partially integrable almost CR -structures of hypersurface type is hyperbolic, equivalent to a parabolic geometry of type $(PSU(2, 1) \times PSU(2, 1), P \times P)$. Here, $PSU(2, 1)$ denotes the quotient of $SU(2, 1)$ for its center and $P \subset PSU(2, 1)$ the stabilizer of an isotropic complex line. A complex CR -structure is elliptic and equivalent to a parabolic geometry of type $(SL(3, \mathbb{C}), B)$ (see [10, p. 443-455]), with $B \subset SL(3, \mathbb{C})$ denoting the Borel subgroup.

Generic $(4, 7)$ -types are realized by the imaginary part of an Hermitian form on a real algebra. A real algebra A equipped with a nondegenerate quadratic form k satisfying $k(xy) = k(x)k(y)$ for all $x, y \in A$ is said to be a multiplicative algebra. If A has dimension four, the admissible nondegenerate signatures for a multiplicative form on A are $(4, 0)$ and $(2, 2)$. Both possibilities are realized, namely by the quaternions with their usual norm and by the real (2×2) -matrices with the determinant as the quadratic form. On both algebras there is a well defined notion of conjugation. Then, one can consider A as a right vector space on itself and introduce the concept of Hermitian form $A \times A \rightarrow A$. More precisely, we speak about quaternionic Hermitian forms for $A = \mathbb{H}$ and split-quaternionic Hermitian forms for $A = M_2(\mathbb{R})$. The imaginary part of a Hermitian form is skew-symmetric, thus we can look at it as a real linear map $\Lambda^2 \mathbb{R}^4 \rightarrow \mathbb{R}^3$. Elliptic

and hyperbolic types are realized through the choices $A = \mathbb{H}$ and $A = M_2(\mathbb{R})$, respectively. An alternative nice description in terms of orbits in $Gr(3, \Lambda^2 \mathbb{R}^4)$, involving the Hodge operator, can be found in [14]. Every choice of a scalar product on \mathbb{R}^4 gives an Hodge operator $*$: $\Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^2 \mathbb{R}^4$. The two eigenspaces associated to $*$ are both three-planes in $\Lambda^2 \mathbb{R}^4$ and we look at their $CO(3, 3)$ -orbits. Elliptic and hyperbolic orbits come from two different choices of a scalar product on \mathbb{R}^4 , namely of positive definite and indefinite signature $(2, 2)$. The corresponding generic distributions are known, respectively, as quaternionic and split-quaternionic contact structures. Both structures are equivalent to parabolic geometries. Quaternionic contact structures were introduced by O. Biquard in his work about conformal infinities of quaternionic-Kähler metrics (see [3] and [4]). The model for quaternionic contact structures, arising in the gauge theory of four-dimensional manifolds, is the so-called instanton distribution on $\mathbb{S}^7 \subset \mathbb{H}^2$. The seven-dimensional sphere can be easily seen as an homogeneous space of the semisimple Lie group $Sp(2, 1)$, as follows. For any integer $n \geq 1$, we consider \mathbb{H}^n as a right vector space over \mathbb{H} . We say that a bilinear form $f: \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{H}$ is *quaternionic-Hermitian* if $f(v, wq) = f(v, w)q$ and $f(w, v) = f(v, w)$ for all $v, w \in \mathbb{H}^n$ and $q \in \mathbb{H}$. The standard quaternionic Hermitian form f on \mathbb{H}^3 of signature $(2, 1)$ is defined by

$$f(v, w) = v_1 \bar{w}_1 + v_2 \bar{w}_2 - v_3 \bar{w}_3$$

Let $Sp(2, 1) \subset GL(3, \mathbb{H})$ be the group of automorphisms ϕ of \mathbb{H} such that $f(\phi(v), \phi(w)) = f(v, w)$. Define the null cone as $N = \{ v \in \mathbb{H}^3 \mid v \neq 0, f(v, v) = 0 \}$. Since $f(v, v)$ is a real number for any $v \in \mathbb{H}^3$, we can see $N \subset \mathbb{R}^{12} \cong \mathbb{H}^3$ as a real submanifold of codimension one, which inherits the ambient metric. By putting $x \sim y$ if and only if there exists $\lambda \in \mathbb{H}$ such that $y = \lambda x$ for $x, y \in N$, we define an equivalence relation on N . Let $M = N / \sim$ be the set of quaternionic lines in N and let $\pi: N \rightarrow M$ be the canonical projection. Consider $\mathbb{H}^3 = \mathbb{H}^2 \oplus \mathbb{H}$ and each summand as a real vector space equipped with the standard norm $\| \cdot \|$ on \mathbb{R}^{4n} . Write $q \in \mathbb{H}^3$ as $q = (v, w)$ for $v \in \mathbb{R}^8$ and $w \in \mathbb{R}^4$. Then, $q \in N$ if and only if $\|v\| = \|w\|$. In particular, any $q \in N$ has nonzero w -coordinate and $\|v\| = 1$ for elements of N of the form $q = (v, 1)$. It easily follows that M is isomorphic to the sphere $\mathbb{S}^7 \subset \mathbb{R}^8 \cong \mathbb{H}^2$. Now, we see that M is also isomorphic to the quotient $Sp(2, 1)/P$ with $P \subset Sp(2, 1)$ parabolic. Clearly, the standard action of $Sp(2, 1)$ on \mathbb{H}^3 leaves N invariant, hence it descends to an action on $M \cong \mathbb{S}^7$. The $Sp(2, 1)$ -action on M is easily seen to be transitive. Then, the orbit of a null line $\ell \in M$ coincides with M . A basic result concerning smooth actions of Lie groups, finally, states that the orbit of ℓ is isomorphic to $Sp(2, 1)/P$, where P is the stabilizer of ℓ . The seven-dimensional manifold M , then, is an homogeneous space. The construction above also defines a distribution of rank four on M . Let $\ell \in M$ be a line $\ell \subset N$ and let $x \in N$ be such that $x \in \ell$. Then $\pi(x) = \ell$ and $T_x \pi: T_x N \rightarrow T_\ell M$ is a surjective linear map. Since $\pi(y) = \pi(x)$ for all $y \in \ell$, the map $T_x \pi$ vanishes on $\mathbb{H} \cdot x = \ell$, thus induces an isomorphism $T_x N / \mathbb{H} \cdot x \cong T_\ell M$ between real vector spaces of same dimension, which depends on the choice of $x \in \ell$. Indeed, via the standard metric $\langle \cdot, \cdot \rangle$ of signature $(2, 1)$ on \mathbb{R}^{12} , $T_x N$ identifies with the real orthocomplement of x in \mathbb{R}^{12} :

$$T_x N = \{ v \in \mathbb{R}^{12} \mid \langle x, v \rangle = 0 \}.$$

Denote by ℓ^\perp the quaternionic (with respect to f) orthocomplement of ℓ . Since ℓ is a null line, $\ell \subset \ell^\perp$. Moreover, $\ell^\perp \subset T_x N$ as a real submanifold of codimension three. Then $H_\ell := \ell^\perp / \ell$ defines a subspace $H_\ell \subset T_\ell M$ of dimension four on M , which has the structure of a quaternionic vector space carrying the induced quaternionic Hermitian form of signature one. The distribution of rank four $H \subset TM$ obtained in this way is known as the instanton distribution. The real part of such a quaternionic Hermitian form is the usual norm on \mathbb{H} , while the imaginary part is the bracket of elliptic $(4, 7)$ -type.

All generic distributions of rank four are described in the literature, except for those in dimensions eight and nine, which are studied in detail in the thesis. While in dimension eight we deal with a parabolic geometry, this is not the case in dimension nine, as one can deduce from the classification of parabolic subalgebras of real semisimple Lie algebras.

CHAPTER 2

Canonical linear connections

To any smooth manifold of dimension n one can associate its linear frame bundle, which is a principal bundle with structure group $GL(n, \mathbb{R})$. Classical G -structures are reductions of the frame bundle to a Lie subgroup $G \subset GL(n, \mathbb{R})$, which can be viewed as encoding a geometric structure on the manifold M . It is well known that suitable conditions on G imply the existence of a canonical compatible linear connection on TM , whose torsion satisfies a normalization condition. The Levi-Civita connection on the tangent bundle of a Riemannian manifold is a remarkable example, whose existence and uniqueness follow, in this picture, from algebraic properties of the orthogonal group.

In this Chapter, we develop a similar argument for some bracket-generating distributions $H \subset TM$ to obtain analogous results on existence and uniqueness of linear connections. Our approach is motivated by the fact that if $H \subset TM$ has constant symbol algebra $(\mathfrak{n}, [\cdot, \cdot])$, then it carries an adapted frame bundle $\mathcal{P} \rightarrow M$. This is a principal bundle over M with structure group $Aut_{gr}(\mathfrak{n})$, the group of grading-preserving isomorphisms of the Lie algebra \mathfrak{n} . In this setting, then, one can consider linear connections on $gr(TM)$ induced by the associated bundle construction. We will refer to them as compatible linear connections. We also consider G_0 -reductions of the adapted frame bundle for a closed subgroup $G_0 \subset Aut_{gr}(\mathfrak{n})$, which allows to make more general statements. Some care is needed with relating geometric structures on M to such G_0 -reductions. In order to point out this fact, we will discuss the example of contact structures endowed with a subriemannian metric.

The technical core of the Chapter describes a construction for G_0 -reductions of the adapted frame bundle, inspired by the construction for classical G -structures and resulting in Theorem 3. The original result is due to Morimoto and presented in ([16]) in terms of canonical Cartan connections on principal bundles. Here, we give an equivalent description in terms of induced linear connections. First, we define notions of graded torsion and curvature for compatible linear connections on $gr(TM)$. The key to our proof for G_0 -triples is an inductive procedure, which normalizes the torsion and the curvature homogeneity by homogeneity and constructs an isomorphism $gr(TM) \cong TM$ at the same time. Particularly interesting is that the result leads to invariants for the structures underlying such G_0 -triples, expressed by torsion and curvature of the canonical connection. In the last Section, we show that Theorem 3 applies to dual Darboux distributions of arbitrary even rank, whose lowest dimensional case corresponds to the generic $(4, 9)$ -type appearing in the classification of Theorem 2.

1. Review of classical G -structures

Let \mathcal{P}, M be finite dimensional smooth manifolds, G a Lie group acting smoothly from the right on \mathcal{P} . A *principal bundle with structure group G* (also a *principal G -bundle*) is a smooth submersion $\pi: \mathcal{P} \rightarrow M$ such that $\pi(ug) = \pi(u)$ for all $u \in \mathcal{P}$ and $g \in G$. For $x \in M$, the *fiber over x* is the smooth submanifold $\mathcal{P}_x := \pi^{-1}(x) \subset \mathcal{P}$. We require that the action of G on each fiber is transitive and free, in formulas:

- $\pi(u) = \pi(v)$ if and only if there exists $g \in G$ such that $v = ug$;

- if $u, v \in \mathcal{P}$ are such that $vg = ug$ for some $g \in G$, then $u = v$.

Then for $u \in \mathcal{P}$ such that $\pi(u) = x$, the restriction of the right action $g \mapsto ug$ is a diffeomorphism $r_u : G \rightarrow \mathcal{P}_x$. In particular, its differential at the neutral element $e \in G$ is a linear injection $T_e r_u : \mathfrak{g} \rightarrow T_u \mathcal{P}$. Fix $u \in \mathcal{P}$ such that $\pi(u) = x$. The differential of the projection at u is a surjective linear map $T_u \pi : T_u \mathcal{P} \rightarrow T_x M$. Denote its kernel by $V_u \mathcal{P}$, whose dimension coincides with the dimension of \mathfrak{g} , the Lie algebra of the structure group G . Differentiating $g \mapsto \pi(ug) = \pi(u)$ at e shows that $T_u \pi^{-1}(x) = T_e r_u(\mathfrak{g}) \subset V_u \mathcal{P}$. Since the two vector spaces have the same dimension, we conclude that the tangent space to the fiber over x at u coincides with $V_u \mathcal{P}$. The kernel of $T\pi$ is a smooth subbundle $V\mathcal{P} \subset T\mathcal{P}$, called the *vertical bundle*. A vector field $\xi \in \mathfrak{X}(\mathcal{P})$ such that $T\pi \cdot \xi = 0$ is said to be *vertical*. To every $X \in \mathfrak{g}$, one can associate a vertical vector field $\zeta_X \in \mathfrak{X}(\mathcal{P})$ by putting

$$\zeta_X(u) = T_e r_u(X) = \left. \frac{d}{dt} \right|_{t=0} (u \exp(tX))$$

for all $u \in \mathcal{P}$. We say that $\{ \zeta_X \mid X \in \mathfrak{g} \}$ is the set of *fundamental vector fields*. While the vertical bundle is a canonical subbundle, there is a priori no canonical choice for a subbundle in $T\mathcal{P}$ complementary to it. Clearly, such a choice can be given as the kernel of a fibered projection $T\mathcal{P} \rightarrow V\mathcal{P}$ onto the vertical bundle, which from the discussion above is the same of a \mathfrak{g} -valued one-form ϕ defined on \mathcal{P} . All this motivates the following definition.

DEFINITION 6. A \mathfrak{g} -valued one-form $\phi \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is said to be a *connection form* if

- (1) $\phi(\zeta_X) = X$ for every $X \in \mathfrak{g}$, so that it reproduces the fundamental vector fields;
- (2) ϕ is G -equivariant, that is $(r^g)^* \phi = \text{Ad}(g^{-1}) \cdot \phi$ for all $g \in G$.

Condition (1) means that the restriction of $\phi_{(u)} : T_u \mathcal{P} \rightarrow \mathfrak{g}$ to $V_u \mathcal{P}$ is a left inverse of $T_e r_u$ for each $u \in \mathcal{P}$. Therefore, $T_e r_u \circ \phi_{(u)} : T_u \mathcal{P} \rightarrow V_u \mathcal{P}$ is a projection and the vertical projection of $\xi \in T_u \mathcal{P}$ writes, in this notation, as $\zeta_{\phi(\xi)}(u)$. Observe that the vertical bundle is G -invariant. It is thus natural to require that the same holds for a complementary subbundle. This means that the corresponding one-form ϕ satisfies (2).

Given a principal G -bundle $\pi : \mathcal{P} \rightarrow M$ and a representation $\rho : G \rightarrow GL(V)$ of the Lie group G , one can consider the right action $(u, v) \cdot g = (u \cdot g, \rho(g^{-1})(v))$ on the product $\mathcal{P} \times V$. The corresponding space of orbits is the *vector bundle associated to (π, ρ)* , denoted by $\mathcal{P} \times_\rho V$ or simply by $\mathcal{P} \times_G V$. We denote by $\llbracket u, v \rrbracket \in \mathcal{P} \times_\rho V$ the orbit of $(u, v) \in \mathcal{P} \times V$, so that $\llbracket u \cdot g, \rho(g^{-1})(v) \rrbracket = \llbracket u, v \rrbracket$ for all $g \in G$ and $(u, v) \in \mathcal{P} \times V$. If A, B are representations of the group G , there is a natural representation of G on $A \otimes B$. Let $E = \mathcal{P} \times_G A$, $F = \mathcal{P} \times_G B$ be associated vector bundles. The associated bundle construction is tensorial, meaning that there exists an isomorphism $E \otimes F \cong \mathcal{P} \times_G (A \otimes B)$, where the right term is the space of orbits with respect to the induced G -action on the tensor product. Analogously, any representation of G induces a representation on the dual space in a natural way and the associated bundle construction is compatible with the duality. A linear map $\Phi : A \rightarrow B$ is said to be G -equivariant if $\Phi(g \cdot a) = g \cdot \Phi(a)$ for all $g \in G$ and $a \in A$. Let $\Phi : A \rightarrow B$ be a G -equivariant homomorphism. Then Φ induces an homomorphism $E \rightarrow F$ of vector bundles, hence a smooth section of $E^* \otimes F$.

EXAMPLE 3. (*Frame bundle*) Let M be a smooth manifold of dimension n . For $x \in M$, define $\mathcal{P}_x M$ as the set of linear isomorphisms $\phi : \mathbb{R}^n \rightarrow T_x M$. The disjoint union

$$\mathcal{P} := \bigcup_{x \in M} \mathcal{P}_x$$

together with the obvious projection $\pi : \mathcal{P} \rightarrow M$ is a principal bundle with structure group $GL(n, \mathbb{R})$. The right action of $GL(n, \mathbb{R})$ on \mathcal{P} is given by the composition from the right. Indeed, if $\phi_x : \mathbb{R}^n \rightarrow T_x M$ is a linear isomorphism, then $\phi_x \circ A : \mathbb{R}^n \rightarrow T_x M$ is also a linear isomorphism

lying in \mathcal{P}_x for each $A \in GL(n, \mathbb{R})$. Clearly, the action on each fiber is free and transitive, since every two basis of \mathbb{R}^n are mapped to each other by a unique $A \in GL(n, \mathbb{R})$.

Consider now the standard representation of $G = GL(n, \mathbb{R})$ on \mathbb{R}^n and the associated vector bundle $E = \mathcal{P} \times_G \mathbb{R}^n$. A linear isomorphism $\phi : \mathbb{R}^n \rightarrow T_x M$ is equivalent to a basis of $T_x M$. An n -tuple $x \in \mathbb{R}^n$ such that $v = \phi(x) \in T_x M$ gives the coordinates of v with respect to the basis ϕ and $(\phi, x) \in \mathcal{P} \times \mathbb{R}^n$ as a pair basis-coordinates of $v \in T_x M$ in that basis. For $A \in GL(n, \mathbb{R})$, $A^{-1}x$ are the coordinates of v in the basis $\phi \circ A$ and any basis of \mathbb{R}^n differs from ϕ by right multiplication for elements of $GL(n, \mathbb{R})$. Then, we look at $\mathcal{P} \times \mathbb{R}^n$ as the set of coordinates with respect to a basis. The fact that the pairs $(\phi \circ A, A^{-1}x), (\phi, x)$ are considered equivalent, in this picture, means that there is no preferred choice of basis for a vector space. This shows that there is an isomorphism $TM \cong E$.

EXAMPLE 4. (Orthonormal frame bundle) If TM carries some additional structure, then it is possible to specialize the frame bundle construction. If this procedure selects a smooth subbundle of the frame bundle, whose structure group is a closed subgroup $G \subset GL(n, \mathbb{R})$, it is easy to deal with it. A principal G -bundle obtained in this way is said to be a G -structure. A classical example is given by the orthonormal frame bundle of a Riemannian manifold. If (M, g) is a Riemannian manifold and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product on \mathbb{R}^n , then for $x \in M$ one can consider the set \mathcal{F}_x of linear isometric isomorphisms $\phi : (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \rightarrow (T_x M, g_x)$. Then $\mathcal{F} := \cup_{x \in M} \mathcal{F}_x$ defines a smooth subbundle $\mathcal{F} \subset \mathcal{P}$, which is a principal bundle with structure group $O(n) \subset GL(n, \mathbb{R})$, known as the *orthonormal frame bundle* of (M, g) . Now $\phi \in \mathcal{F}_x$ is equivalent to an orthonormal basis of $T_x M$. Since every two orthonormal bases of \mathbb{R}^n are mapped to each other by a unique orthogonal linear map, $O(n)$ acts on each fiber and the action is free and transitive. Again, we have an isomorphism $TM \cong \mathcal{F} \times_{O(n)} \mathbb{R}^n$.

Let $\pi : \mathcal{P} \rightarrow M$ be a principal G -bundle and $\rho : G \rightarrow GL(V)$ a representation. Let $\alpha \in \Omega^k(\mathcal{P}, V)$ be a V -valued differential form. We say that α is *horizontal* if it annihilates by insertion of a vertical vector field. Moreover, α is said to be ρ -equivariant if $(r^g)^* \alpha = \rho(g^{-1}) \cdot \alpha$ for every $g \in G$, where $(r^g)^*$ denotes the pullback of forms induced by the right G -action on \mathcal{P} . We denote by $\Omega_{\text{hor}}^k(\mathcal{P}, V)^\rho$ the set of horizontal and ρ -equivariant k -forms on \mathcal{P} . For $k = 0$, these are smooth functions $f : \mathcal{P} \rightarrow V$ sending fibers in \mathcal{P} to ρ -orbits in V . We denote by $C^\infty(\mathcal{P}, V)^\rho = \Omega_{\text{hor}}^0(\mathcal{P}, V)^\rho$. Also, we denote by $\Omega^0(E)$ the set of sections of the vector bundle $E \rightarrow M$ and the set of k -forms with values in E by $\Omega^k(E) = \Omega^0(\Lambda^k T^* M \otimes E)$.

PROPOSITION 5. *Let $\pi : \mathcal{P} \rightarrow M$ be a principal G -bundle, $\rho : G \rightarrow GL(V)$ a finite-dimensional representation of the Lie group G and $E = \mathcal{P} \times_\rho V$ the associated vector bundle. For integers $k \geq 0$, denote by $\Omega_{\text{hor}}^k(\mathcal{P}, V)^\rho$ the set of k -forms on \mathcal{P} with values in V , which are horizontal and ρ -equivariant. There is a bijective correspondence*

$$\Omega^k(E) \cong \Omega_{\text{hor}}^k(\mathcal{P}, V)^\rho$$

explicitly given by

$$\sigma_{(\pi(u))}(T_u \pi \cdot \xi_1, \dots, T_u \pi \cdot \xi_k) = \llbracket u, \phi(u)(\xi_1, \dots, \xi_k) \rrbracket$$

for $u \in \mathcal{P}$ and $\xi_1, \dots, \xi_k \in T_u \mathcal{P}$.

PROOF. Consider $\sigma \in \Omega^k(E)$. Then for $u \in \mathcal{P}$, $x \in \pi(u)$ and $\xi_1, \dots, \xi_k \in T_u \mathcal{P}$, we have that $\sigma_{(\pi(u))}(T_u \pi \cdot \xi_1, \dots, T_u \pi \cdot \xi_k)$ lies in E_x , so it coincides with $\llbracket u, v \rrbracket$ for a unique $v \in V$. Therefore, there exists a unique $\phi(u)(\xi_1, \dots, \xi_k) \in V$ such that

$$(8) \quad \sigma_{(\pi(u))}(T_u \pi \cdot \xi_1, \dots, T_u \pi \cdot \xi_k) = \llbracket u, \phi(u)(\xi_1, \dots, \xi_k) \rrbracket$$

Observe that $\phi(u) : (T_u \mathcal{P})^k \rightarrow V$ is alternating, since the same holds for $\sigma_{(\pi(u))}$, and vertical. One easily shows that ϕ depends smoothly on u , so that $\phi \in \Omega_{\text{hor}}^k(\mathcal{P}, V)$. From $\pi \cdot r^g = \pi$ it

follows that $T\pi \cdot Tr^g \cdot \xi = T\pi \cdot \xi$ hence, in turn, that

$$\llbracket u \cdot g, \phi(u \cdot g)(Tr^g \cdot \xi_1, \dots, Tr^g \cdot \xi_k) \rrbracket = \llbracket u, \phi(u)(\xi_1, \dots, \xi_k) \rrbracket.$$

On the other hand, $\llbracket u, \phi(u)(\xi_1, \dots, \xi_k) \rrbracket = \llbracket u \cdot g, \rho(g^{-1})(\phi(u)(\xi_1, \dots, \xi_k)) \rrbracket$ by definition of the associated bundle. From this, we conclude that $\phi(u \cdot g)(Tr^g \cdot \xi_1, \dots, Tr^g \cdot \xi_k) = \rho(g^{-1})\phi(u)(\xi_1, \dots, \xi_k)$, hence that $\phi \in \Omega_{\text{hor}}^k(\mathcal{P}, V)$ is equivariant.

Conversely, an horizontal ρ -equivariant form ϕ on \mathcal{P} defines $\sigma \in \Omega^k(E)$ as follows. For $x \in M$, choose $u \in \mathcal{P}$ such that $\pi(u) = x$ and define $\sigma(x)$ via formula (8). The definition does not depend on the choice of lifts of vector fields, since ϕ is horizontal. Smoothness of ϕ implies that $\sigma(x)$ depends smoothly on x . By equivariance of ϕ , it is also independent on the choice of u in the fiber of x . \square

Denote by $e \in G$ the neutral element. Let $T_e\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be the differential of ρ at e and let $\phi \in \Omega^1(\mathcal{P}, \mathfrak{g})$ be a connection form. For $\tilde{\xi} \in \mathfrak{X}(\mathcal{P})$ and $u \in \mathcal{P}$, by definition $\phi_u(\tilde{\xi}) \in \mathfrak{g}$ and it can be inserted into $T_e\rho$, thus obtaining an endomorphism $T_e\rho(\phi_u(\tilde{\xi})) \in \mathfrak{gl}(V)$.

PROPOSITION 6. *For every ρ -equivariant smooth function $f : \mathcal{P} \rightarrow V$, the formula*

$$D^\phi f = df + (T_e\rho \circ \phi) \cdot f$$

defines a one-form $D^\phi f \in \Omega^1(\mathcal{P}, V)$, which is horizontal and ρ -equivariant.

PROOF. First, we prove that D^ϕ vanishes by insertion of a fundamental vector field. For any $X \in \mathfrak{g}$ and $u \in \mathcal{P}$,

$$\begin{aligned} D^\phi f_u(\zeta_X) &= (\zeta_X)_u f + T_e\rho(\phi_u(\zeta_X)) \cdot f(u) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(u \exp(tX)) + \left. \frac{d}{dt} \right|_{t=0} \rho(\exp(tX)) \cdot f(u) \end{aligned}$$

By ρ -equivariance of f and observing that $\rho(\exp(tX))^{-1} = \rho(\exp(-tX))$, we conclude that the left-hand term rewrites as the negative of the right-hand term, hence that $D^\phi(\zeta_X) = 0$ for every $X \in \mathfrak{g}$ and D^ϕ is horizontal. Now observe that

$$(r^g)^* df = d(r^g)^* f = d(\rho(g^{-1}) \cdot f) = \rho(g^{-1}) \cdot df$$

hence the first summand in $D^\phi f$ is ρ -equivariant. For $u \in \mathcal{P}$ and $\xi \in T_u\mathcal{P}$, we have that

$$\begin{aligned} (r^g)^*((T_e\rho \circ \phi) \cdot f)_u(\xi) &= T_e\rho((r^g)^*\phi_u(\xi)) \cdot (r^g)_u^* f \\ &= T_e\rho(\text{Ad}(g^{-1}) \cdot \phi_u(\xi)) \cdot \rho(g^{-1}) \cdot f_{ug} \end{aligned}$$

where the second equality comes from the equivariance of f and ϕ . Using now that

$$(r^g)^* df = d(r^g)^* f = d(\rho(g^{-1}) \cdot f) = \rho(g^{-1}) \cdot df$$

for all $X \in \mathfrak{g}$, we see that the second term is also equivariant. \square

Let E be a smooth vector bundle over M and $\Omega^0(E)$ the space of its smooth sections. As usual, the space of sections of the tangent bundle is denoted by $\mathfrak{X}(M)$. A linear connection on E is a bilinear map $\nabla : \mathfrak{X}(M) \times \Omega^0(E) \rightarrow \Omega^0(E)$ which is $C^\infty(M)$ -linear in the first variable and satisfies the Leibnitz rule:

$$\nabla_\xi(f\sigma) = (\xi \cdot f)\sigma + f\nabla_\xi\sigma$$

for any choice of $\xi \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $\sigma \in \Omega^0(E)$. It generalizes the usual definition of differential to an operator acting on smooth functions, whose values lie in a vector bundle rather than in a vector space. Observe that ∇ can be equivalently seen as a differential operator $\nabla : \Omega^0(E) \rightarrow \Omega^1(E)$ of first order. If $E = \mathcal{P} \times_\rho V$ is an associated vector bundle, Proposition 5 assures that such an operator can be equivalently described as a map $\Omega_{\text{hor}}^0(\mathcal{P}, V)^\rho \rightarrow \Omega_{\text{hor}}^1(\mathcal{P}, V)^\rho$.

DEFINITION 7. Let $\pi : \mathcal{P} \rightarrow M$ be a principal G -bundle, $\rho : G \rightarrow GL(V)$ a finite-dimensional representation of the Lie group G and $E = \mathcal{P} \times_\rho V$ the associated vector bundle. Let $\sigma \in \Omega^0(E)$ and denote by $f \in C^\infty(\mathcal{P}, V)^\rho$ the ρ -equivariant smooth function corresponding to σ . If $\phi \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is a connection form, we denote by $\nabla^\phi \sigma \in \Omega^1(E)$ the section corresponding to $D^\phi f \in \Omega_{\text{hor}}^1(\mathcal{P}, V)^\rho$. Therefore, we have the commutative diagram

$$\begin{array}{ccc} \Omega_{\text{hor}}^0(\mathcal{P}, V)^\rho & \xrightarrow{D^\phi} & \Omega_{\text{hor}}^1(\mathcal{P}, V)^\rho \\ \uparrow & & \downarrow \\ \Omega^0(E) & \xrightarrow{\nabla^\phi} & \Omega^1(E) \end{array}$$

We say that ∇^ϕ is the *linear connection induced by the connection form ϕ* .

The study of G -structures is related to the problem of existence of canonical linear connections. To any linear connection ∇ on TM , one can associate a torsion

$$T^\nabla(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta], \quad \xi, \eta \in \mathfrak{X}(M)$$

which is a tensorial map $T^\nabla \in \Omega^2(TM)$. If $\nabla, \bar{\nabla}$ are linear connections, then their difference is tensorial in both entries. Fix a point $x \in M$. Up to isomorphism, $(\nabla - \bar{\nabla})_x$ is a linear map $\mathbb{R}^n \rightarrow \mathfrak{gl}(n, \mathbb{R})$, where n denotes the dimension of M . Let $\delta : (\mathbb{R}^n)^* \otimes \mathfrak{gl}(n, \mathbb{R}) \rightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ be the skew-symmetrization map in the first two components, defined on decomposable elements as

$$\delta A(X \wedge Y) = A(X)Y - A(Y)X \quad \text{for } X, Y \in \mathbb{R}^n, A \in (\mathbb{R}^n)^* \otimes \mathfrak{gl}(n, \mathbb{R}).$$

Here, $A(X)Y$ denotes the standard action of $A(X) \in \mathfrak{gl}(n, \mathbb{R})$ on $Y \in \mathbb{R}^n$. By direct computation, we see that $(T^\nabla - T^{\bar{\nabla}})_x = \delta(\nabla - \bar{\nabla})_x$. Together with the fact that δ is a surjective linear map, then, the last identity implies that the space of torsion-free linear connections is an affine space over $\Omega^0(S^2 T^* M \otimes TM)$. Therefore, the torsion-free condition does not uniquely pin down a linear connection on TM . Consider now a Riemannian manifold (M, g) of dimension n . The connection forms on the orthonormal frame bundle take values in $\mathfrak{g} = \mathfrak{o}(n)$. The induced linear connections are compatible with the metric. Explicitely, they satisfy

$$\zeta \cdot g(\xi, \eta) = g(\nabla_\zeta \xi, \eta) + g(\xi, \nabla_\zeta \eta)$$

for all $\zeta, \xi, \eta \in \mathfrak{X}(M)$. If $\nabla, \bar{\nabla}$ are linear connections on TM induced by connection forms on the orthonormal frame bundle, then $(\nabla - \bar{\nabla})_x$ is a linear map $\mathbb{R}^n \rightarrow \mathfrak{o}(n)$. Now $\mathfrak{o}(n) \cong \Lambda^2(\mathbb{R}^n)^*$ canonically and δ restricts to an isomorphism $(\mathbb{R}^n)^* \otimes \mathfrak{o}(n) \rightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. Using this fact, one can modify an arbitrary metric linear connection to another one, which is torsion-free. This proves the existence of the Levi-Civita connection. More in general, suppose that $G \subset GL(n, \mathbb{R})$ is a closed subgroup with Lie algebra \mathfrak{g} , such that:

- the restriction of δ to $(\mathbb{R}^n)^* \otimes \mathfrak{g}$ is injective;
- there exists a G -invariant subspace $\mathfrak{N} \subset \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ such that

$$\Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n = \mathfrak{N} \oplus \text{Im}(\delta)$$

If M is a smooth manifold of dimension n endowed with a G -structure, \mathfrak{N} corresponds to a subbundle of $\Lambda^2 T^* M \otimes TM$. Then, the argument above shows that there exists a unique linear connection on TM with torsion taking values in that subbundle. In this sense, we will refer to it as the canonical linear connection on TM .

Inspired by the classical theory of G -structures, we would like to deal with bracket-generating distributions as geometric structures and investigate the linear connections compatible with these structures. Now for linear connections on H , which would be natural to consider, there is no well-defined notion of torsion in the noninvolutive case. One could then consider linear connections

∇ on TM such that $\nabla_\xi(\Gamma(H)) \subset \Gamma(H)$ for all $\xi \in \mathfrak{X}(M)$ and require that the restriction of the torsion to $H \times H$ coincides with \mathcal{L} . The existence of such linear connections is assured for a class of bracket-generating distributions described in the next Section.

2. Regular pairs and adapted G_0 -structures

As shown in Proposition 1, the symbol algebra associated to a bracket-generating distribution is invariant under the action of compatible diffeomorphisms. We will assume that all symbol algebras are isomorphic to a fixed model $(\mathfrak{n}, [\cdot, \cdot])$. In this setting, a natural frame bundle for H over M is available.

DEFINITION 8. Let M be a finite dimensional smooth manifold, $H \subset TM$ a bracket generating distribution and $(\mathfrak{n} = \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-\mu}, [\cdot, \cdot])$ a finite dimensional nilpotent graded Lie algebra of depth μ generated by \mathfrak{g}_{-1} . We say that the pair (M, H) is *regular of type* $(\mathfrak{n}, [\cdot, \cdot])$ if $(\text{gr}(TM), \mathcal{L})$ is a locally trivial bundle of nilpotent graded Lie algebras modeled on $(\mathfrak{n}, [\cdot, \cdot])$.

Let \mathfrak{h}_{2n+1} be the Heisenberg algebra of dimension $(2n+1)$ and let M be a smooth manifold of dimension $(2n+1)$. In Example 1, we showed that a smooth distribution $H \subset TM$ of corank one is a contact structure if and only if, for each $x \in M$, the symbol algebra at x is isomorphic to \mathfrak{h}_{2n+1} . Moreover, Pfaff Theorem assures that the isomorphism $\text{gr}(T_x M) \rightarrow \mathfrak{h}_{2n+1}$ depends smoothly on x . Therefore, we conclude that a regular pair (M, H) of type \mathfrak{h}_{2n+1} is a contact structure $H \subset TM$ on a smooth manifold of dimension $(2n+1)$.

Another remarkable example is given by the one-step generic case described in the previous Chapter. The hypothesis of genericity, formulated for one step bracket-generating distributions, implies the local triviality of the bundle $(\text{gr}(TM), \mathcal{L})$. Otherwise put, every generic (k, n) -distribution determines a regular pair. Indeed, consider a (k, n) -distribution $H \subset TM$. Put $Q := TM/H$. We can choose local frames for H and Q defined on an open subset $U \subset M$. These are equivalent, respectively, to fibered isomorphisms $\phi: H|_U \rightarrow U \times \mathbb{R}^k$ and $\psi: Q|_U \rightarrow U \times \mathbb{R}^{n-k}$ which, in turn, define a smooth map $T: U \rightarrow L(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ by putting $T_x := \psi_x \circ \mathcal{L}_x \circ \Lambda^2(\phi_x)^{-1}$ for all $x \in U$. If H is generic, then T takes values in an open orbit $\mathcal{O} \subset L(\Lambda^2 \mathbb{R}^k, \mathbb{R}^{n-k})$ of the action (6). Such an open orbit is a smooth submanifold and the projection of the group G onto it, given by acting on some fixed element in \mathcal{O} , is a smooth surjective submersion $G \rightarrow \mathcal{O}$. Choosing a local smooth section of this submersion, we lift T to a smooth map $g: U \rightarrow G$. Otherwise put, there exists a linear map P in the open orbit \mathcal{O} such that $g_x \cdot T_x = P$ for all $x \in U$ and g depends smoothly on x . Write $g = (A, B)$ for A, B smooth functions on U with values in $GL(k, \mathbb{R})$ and $GL(n-k, \mathbb{R})$, respectively. Then $A \cdot \phi: H|_U \rightarrow U \times \mathbb{R}^k$ and $B \cdot \psi: Q|_U \rightarrow U \times \mathbb{R}^{n-k}$ are fibered isomorphisms and $B_x \circ T_x \circ \Lambda^2 A_x^{-1} = P$ for all $x \in U$, which shows that P trivializes the Levi bracket on U .

Let (M, H) be a regular pair of type $(\mathfrak{n}, [\cdot, \cdot])$. For $x \in M$, define \mathcal{F}_x as the set of isomorphisms $u: (\mathfrak{n}, [\cdot, \cdot]) \rightarrow (\text{gr}(T_x M), \mathcal{L}_x)$ of nilpotent graded Lie algebras. Then

$$\mathcal{F} := \cup_{x \in M} \mathcal{F}_x$$

with the obvious projection $\pi: \mathcal{F} \rightarrow M$ defines a principal bundle, the *adapted frame bundle*. Since $(\text{gr}(TM), \mathcal{L})$ is a locally trivial bundle of nilpotent graded Lie algebras, $\pi: \mathcal{F} \rightarrow M$ is a locally trivial principal bundle. The structure group of the adapted frame bundle is the Lie group $\text{Aut}_{gr}(\mathfrak{n})$ of grading-preserving automorphisms of \mathfrak{n} , which acts on \mathcal{F} by composition from the right. Denote by $G_0 = \text{Aut}_{gr}(\mathfrak{n})$ the structure group and by $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{n})$ its Lie algebra. Observe that the notation is consistent with the fact that both G_0 and \mathfrak{g}_0 act on \mathfrak{n} by linear maps of degree zero. Otherwise put, each \mathfrak{g}_i carries the standard actions of G_0 and \mathfrak{g}_0 .

The filtration on M induces a filtration on \mathcal{F} , by putting $T^0\mathcal{F} := V\mathcal{F}$ and $T^i\mathcal{F} := (T\pi)^{-1}(T^iM)$ for $-\mu \leq i \leq -1$. For $u \in \mathcal{F}$ and $g \in G_0$, it follows from $\pi \circ r^g = \pi$ that

$$T_u r^g \cdot T_u^i \mathcal{F} = T_{ug}^i \mathcal{F}$$

so that the filtration on \mathcal{F} is G_0 -invariant. There exist canonical sections θ_i of $L(T^i\mathcal{F}, \mathfrak{g}_i)$. For $u \in \mathcal{F}$ with $\pi(u) = x$, define $(\theta_i)_u : T_u^i \mathcal{F} \rightarrow \mathfrak{g}_i$ by putting $(\theta_i)_u(\tilde{\xi}) := u^{-1}((q_i)_x \cdot T_u \pi \cdot \tilde{\xi})$ for all $\tilde{\xi} \in T_u^i \mathcal{F}$. Equivalently, the following diagram commutes:

$$\begin{array}{ccc} & T_u^i \mathcal{F} & \\ (\theta_i)_u \swarrow & \downarrow q_i \circ T_u \pi & \\ \mathfrak{g}_i & \xrightarrow{u} & \text{gr}_i(T_x M) \end{array}$$

The collection $\theta = (\theta_{-1}, \dots, \theta_{-\mu})$ is known as *soldering form*. Every θ_i can be seen as a G_0 -equivariant \mathfrak{g}_i -valued function on $T^i\mathcal{F}$ with kernel $T^{i+1}\mathcal{F}$.

The associated bundle construction gives isomorphisms $\mathcal{F} \times_{G_0} \mathfrak{g}_i \cong \text{gr}_i(TM)$, which can be explicitly written via the soldering form. For any $\xi \in T_x^i M$, let $\tilde{\xi} \in T_u^i \mathcal{F}$ be a *lift* of ξ , which satisfies $T_u \pi \cdot \tilde{\xi} = \xi$. Then, the identity $(q_i)_x(\xi) = u((\theta_i)_u(\tilde{\xi}))$ follows straightforwardly from the definition of θ_i and we interpret $(u, (\theta_i)_u(\tilde{\xi}))$ as a pair frame-coordinates, where $(\theta_i)_u(\tilde{\xi})$ are the coordinates of $(q_i)_x(\xi)$ in the frame u .

Finally, we observe that the adapted frame bundle construction is functorial. Let be M, \tilde{M} two smooth manifolds of same dimension and $(M, H), (\tilde{M}, \tilde{H})$ regular pairs of same type $(\mathfrak{n}, [\ , \])$. Denote by $\pi : \mathcal{F} \rightarrow M, \tilde{\pi} : \tilde{\mathcal{F}} \rightarrow \tilde{M}$ the adapted frame bundles for H and \tilde{H} . In the same notation of Proposition 1, a compatible diffeomorphism $f : U \rightarrow \tilde{U}$ between open subsets of M, \tilde{M} defines, by composing with $\text{gr}(Tf)$ at the right, a local isomorphism between the adapted frame bundles. Indeed, suppose that $x \in U$ and that $u \in \pi^{-1}(x)$. Then $u : \mathfrak{n} \rightarrow \text{gr}(T_x M)$ is an isomorphism of nilpotent graded Lie algebras, hence also $\text{gr}(T_x f) \circ u : \mathfrak{n} \rightarrow \text{gr}(T_x \tilde{M})$ is an isomorphism of nilpotent graded Lie algebras lying in $\tilde{\pi}^{-1}(f(x))$. Therefore, $\Phi(u) = \text{gr}(T_{\pi(u)} f) \circ u$ defines a fibered isomorphism $\Phi : \pi^{-1}(U) \rightarrow \tilde{\pi}^{-1}(\tilde{U})$ covering f and equivariant with respect to the action of G_0 , hence an isomorphism of G_0 -bundles.

DEFINITION 9. Let $(\mathfrak{n}, [\ , \])$ be a nilpotent graded Lie algebra generated by \mathfrak{g}_{-1} and (M, H) a regular pair of type $(\mathfrak{n}, [\ , \])$. Denote by $\mathcal{P} \rightarrow M$ the adapted frame bundle associated to H and by $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ a closed subgroup.

- (1) An *adapted G_0 -structure* is a smooth subbundle $\mathcal{F} \subset \mathcal{P}$ that is a principal bundle with structure group G_0 .
- (2) A G_0 -triple $(\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ consists of a regular pair (M, H) of type $(\mathfrak{n}, [\ , \])$ and an adapted G_0 -structure $\mathcal{F} \rightarrow M$.

A nice example of additional structure on a bracket-generating distribution is a subriemannian metric.

EXAMPLE 5. (Subriemannian manifolds) Let M be a finite dimensional smooth manifold and $H \subset TM$ a smooth distribution. A subriemannian metric S on H is a smooth assignment, to every $x \in M$, of an inner product $S_x : H_x \times H_x \rightarrow \mathbb{R}$. The pair (H, S) is a *subriemannian structure* on M . A differentiable manifold equipped with a subriemannian structure is called *subriemannian manifold* and denoted by (M, H, S) . In our treatment of subriemannian manifolds, H will be always assumed to be bracket-generating. Let $\mathfrak{n} = \oplus_{p < 0} \mathfrak{g}_p$ be a nilpotent graded Lie algebra generated by \mathfrak{g}_{-1} and let σ be an inner product on \mathfrak{g}_{-1} . The pair (\mathfrak{n}, σ) is said to be a *subriemannian graded Lie algebra*. An isomorphism of nilpotent graded Lie algebras, whose

restriction to the bracket-generating layers is orthogonal, is an *isomorphism of subriemannian graded Lie algebras*. In particular, the automorphism group of the subriemannian graded Lie algebra (\mathfrak{n}, σ) is the closed subgroup $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ of those maps, whose restriction to \mathfrak{g}_{-1} preserves σ . If (M, H, S) is a subriemannian manifold, all symbol algebras of H are subriemannian graded Lie algebras. Suppose that (M, H, S) and $(\tilde{M}, \tilde{H}, \tilde{S})$ are subriemannian manifolds of same dimension, with H, \tilde{H} of same rank. Let be $U \subset M, \tilde{U} \subset \tilde{M}$ open subsets and $f : U \rightarrow \tilde{U}$ a local diffeomorphism, whose differential at the point x restricts to an isometry $H_x \rightarrow \tilde{H}_{f(x)}$ for any $x \in U$. Then, in the notation of Proposition 1, the induced isomorphism $\text{gr}(T_x f)$ between the symbol algebras is an isomorphism of subriemannian graded Lie algebras. Then if $\mathcal{F} \rightarrow (M, H, S)$ and $\tilde{\mathcal{F}} \rightarrow (\tilde{M}, \tilde{H}, \tilde{S})$ are G_0 -triples, $\text{gr}(T_x f)$ induces an isomorphism $\mathcal{F} \rightarrow \tilde{\mathcal{F}}$ of G_0 -bundles.

Care is needed when relating geometric structures to G_0 -triples. This becomes clear by looking at the example of *subriemannian contact structures*. Denote by ω the canonical symplectic form on \mathbb{R}^{2n} given by formula (3), which is the bracket on the Heisenberg algebra $\mathfrak{h}_{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R}$. Then

$$CSp(n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) \mid \exists \lambda \in \mathbb{R} : \omega(Av, Aw) = \lambda \omega(v, w) \text{ for all } v, w \in \mathbb{R}^{2n}\}$$

is the conformal symplectic group. It is easy to verify that $\text{Aut}_{gr}(\mathfrak{h}_{2n+1}) = CSp(n, \mathbb{R})$. The standard scalar product on \mathbb{R}^{2n} endows \mathfrak{h}_{2n+1} with a structure of subriemannian graded Lie algebra. The subset of automorphisms preserving the scalar product is the closed subgroup $G_0 = O(2n) \cap CSp(n, \mathbb{R}) \subset CSp(n, \mathbb{R})$. Suppose that M is a smooth manifold of dimension $(2n + 1)$ and $H \subset TM$ a contact structure. Let $U \subset M$ be an open subset and $\alpha \in \Omega^1(U)$ a contact form on U such that $H_x = \text{Ker}(\alpha_x)$ for all $x \in U$. Then $d\alpha_x : T_x M \times T_x M \rightarrow \mathbb{R}$ is a nondegenerate skew-symmetric bilinear form for any $x \in U$. If $\{\xi_1, \dots, \xi_{2n}\}$ is an orthonormal frame for H on U , then we can write $(d\alpha_x(\xi_i, \xi_j))_{i,j=1,\dots,2n}$ as a skew-symmetric square matrix of size $2n$. By linear algebra, there exists $A : U \rightarrow O(2n)$ and smooth nonzero functions a_1, \dots, a_n such that $(d\alpha(A\xi_i, A\xi_j))_{i,j=1,\dots,2n}$ writes in a block form, with square blocks along the diagonal of the form

$$\begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$$

and zero elsewhere. Observe that $\frac{a_i}{a_j}$ is pointwise invariant with respect to the action of $CSp(n, \mathbb{R})$ for all $i, j = 1, \dots, n$. If M admits a G_0 -triple, locally around each point there exists an orthonormal frame $\{\zeta_1, \dots, \zeta_{2n}\}$ for H such that $d\alpha(\zeta_i, \zeta_j)$ are constant functions. At each point, the frames $\{A\xi_1, \dots, A\xi_{2n}\}$ and $\{\zeta_1, \dots, \zeta_{2n}\}$ differ by a conformal symplectic transformation. Therefore, if there exist $i, j \in \{1, \dots, 2n\}$ such that $\frac{a_i}{a_j}$ is not constant on U , the subriemannian contact structure cannot underlie any G_0 -triple.

3. Tanaka prolongation

Fix a nilpotent graded Lie algebra $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-\mu}$ generated by \mathfrak{g}_{-1} . Let $\mathfrak{der}(\mathfrak{n})$ be the algebra of linear endomorphisms $\phi : \mathfrak{n} \rightarrow \mathfrak{n}$ such that

$$\phi([X, Y]) = [\phi(X), Y] + [X, \phi(Y)]$$

for all $X, Y \in \mathfrak{n}$. The elements of $\mathfrak{der}(\mathfrak{n})$ are also said to be *derivations* of the Lie algebra \mathfrak{n} . Denote by

$$\mathfrak{der}_{gr}(\mathfrak{n}) = \{\phi \in \mathfrak{der}(\mathfrak{n}) \mid \phi(\mathfrak{g}_i) \subset \mathfrak{g}_i \text{ for all } i = -1, \dots, -\mu\}$$

the Lie algebra of grading-preserving derivations of \mathfrak{n} and by $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{n})$ a Lie subalgebra. Tanaka's prolongation is characterized as the graded Lie algebra $\text{pr}(\mathfrak{n}, \mathfrak{g}_0) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{p}_i$ satisfying the following conditions:

- (C1) its nonpositive graded part coincides with $\mathfrak{n} \oplus \mathfrak{g}_0$;
- (C2) if X lying in a nonnegative graded homogeneous component is such that $[X, \mathfrak{g}_{-1}] = \{0\}$, then X is zero;
- (C3) $\text{pr}(\mathfrak{n}, \mathfrak{g}_0)$ is the maximal graded Lie algebra satisfying both previous conditions.

For every pair $(\mathfrak{n}, \mathfrak{g}_0)$, Tanaka showed the existence of a graded Lie algebra $\text{pr}(\mathfrak{n}, \mathfrak{g}_0)$ satisfying (C1), (C2) and (C3) via the following explicit construction. First, define the nonpositive graded components as

- $\mathfrak{p}_i := \{0\}$ for $i < -\mu$;
- $\mathfrak{p}_i := \mathfrak{g}_i$ for $-\mu \leq i \leq -1$;
- $\mathfrak{p}_0 := \mathfrak{g}_0$.

Evidently, the vector space $\bigoplus_{i \leq 0} \mathfrak{p}_i = \mathfrak{n} \oplus \mathfrak{p}_0$ satisfies (C1). For $\varphi \in \mathfrak{p}_0$ and $X \in \mathfrak{n}$, the formula

$$(9) \quad [\varphi, X] := \varphi(X) =: -[X, \varphi]$$

together with the brackets on \mathfrak{n} and \mathfrak{p}_0 , endows the vector space $\bigoplus_{i \leq 0} \mathfrak{p}_i$ with a graded Lie algebra structure. In fact, skew-symmetry of the bracket is imposed and the fact that φ is a derivation of \mathfrak{n} writes as the Jacobi identity for $\varphi \in \mathfrak{p}_0$ and $X, Y \in \mathfrak{n}$. Observe that \mathfrak{p}_0 satisfies (C2), since every derivation of \mathfrak{n} is completely determined by its action on \mathfrak{g}_{-1} . Inductively for $s > 0$, define the vector space

$$\mathfrak{p}_s := \left\{ \varphi \in \bigoplus_{p < 0} \mathfrak{g}_p^* \otimes \mathfrak{p}_{s+p} \mid \varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)], X, Y \in \mathfrak{n} \right\}.$$

Hence $\varphi \in \mathfrak{p}_s$ is a linear map $\varphi : \mathfrak{n} \rightarrow \mathfrak{p}_{s-\mu} + \dots + \mathfrak{p}_{s-1}$ such that $\varphi(\mathfrak{g}_i) \subset \mathfrak{p}_{i+s}$ for all $i \in \{-\mu, \dots, -1\}$. The defining equation shows that φ is completely determined by its restriction to \mathfrak{g}_{-1} , so that \mathfrak{p}_s satisfies (C2) for all $s > 0$. Observe that only brackets defined at the previous inductive steps are involved in the definition of \mathfrak{p}_s . Again, one defines the bracket $[\varphi, X]$ for $\varphi \in \mathfrak{p}_s$ and $X \in \mathfrak{n}$, occurring at the next step of the inductive definition, through the evaluation map. If $X \in \mathfrak{g}_i$, in particular, then $[\varphi, X] \in \mathfrak{p}_{s+i}$. Finally, for $\varphi \in \mathfrak{p}_s, \psi \in \mathfrak{p}_q$ with $p, q \geq 0$ and by induction on $s + q \geq 0$, the formula

$$[\varphi, \psi](X) := [[\varphi, X], \psi] + [\varphi, [\psi, X]] \quad \text{for all } X \in \mathfrak{n}$$

defines a linear map $[\varphi, \psi] \in \bigoplus_{p < 0} \mathfrak{g}_p^* \otimes \mathfrak{p}_{s+q+p}$. The right term contains only brackets between elements in homogeneous components of lower degree, thus defined at a previous step. In particular for $p = q = 0$, the definition coincides with the bracket between derivations in \mathfrak{g}_0 . Inductively, this implies that $[\varphi, \psi] \in \mathfrak{p}_{s+q}$. Finally, every graded Lie algebra satisfying (C1) and (C2) merges into $\text{pr}(\mathfrak{n}, \mathfrak{g}_0)$ via a homomorphism of graded Lie algebras, so that Tanaka construction produces the unique graded Lie algebra satisfying all three axioms.

DEFINITION 10. The vector space \mathfrak{p}_s is the s -th *prolongation* of the pair $(\mathfrak{n}, \mathfrak{g}_0)$. The graded Lie algebra $\text{pr}(\mathfrak{n}, \mathfrak{g}_0)$ is the *Tanaka prolongation of the pair* $(\mathfrak{n}, \mathfrak{g}_0)$. If $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{n})$, one usually refers to it as the *Tanaka prolongation of* \mathfrak{n} , denoted by $\text{pr}(\mathfrak{n})$.

The pair $(\mathfrak{n}, \mathfrak{g}_0)$ is said to be of *finite type* if $\mathfrak{p}_s = \{0\}$ for some $s \geq 0$, otherwise it is of *infinite type*. If $\mathfrak{p}_s = \{0\}$ for some $s \geq 0$, then also $\mathfrak{p}_l = \{0\}$ for all $l \geq s$, as it follows from (C2). Then the full prolongation is finite dimensional if and only if $(\mathfrak{n}, \mathfrak{g}_0)$ is of finite type.

Tanaka prolongation was introduced in [21], a work concerning the geometry of regular differential systems. Results of Tanaka's work apply to the problem of equivalence of transitive G_0 -structures on filtered manifolds, treated by Morimoto in [16]. The geometric counterpart is a prolongation procedure, which is a refinement of Cartan's method of equivalence. It generalizes the Singer-Sternberg prolongation of usual G -structures (see [19] and [20]), corresponding in

Tanaka's picture to the special case of trivial filtration. A description of the geometric prolongation, using a language similar to one used by Singer and Sternberg, can be found in [24]. The thesis only deals with two different cases for the prolongation. In this chapter, we focus on the case of vanishing of the first prolongation. Under this hypothesis, the prolongation procedure selects a unique connection form on the adapted G_0 -structure $\mathcal{F} \rightarrow (M, H)$ together with a unique splitting of the filtration of TM generated by H . Through the associated bundle construction, the selected connection form on $\mathcal{F} \rightarrow (M, H)$ corresponds to a linear connection on $\text{gr}(TM)$.

From the point of view of the parabolic geometry, particularly interesting is the case of $\mathfrak{n} = \mathfrak{g}_-$ arising as the negative part of a grading on a simple Lie algebra. On this second case focuses Tanaka's paper [22], a work inspired by the study of CR -structures. If $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ is a simple Lie algebra, then the G_0 -triple $(\mathcal{F} \rightarrow (M, H), \mathfrak{g}_-)$ is equivalent to a parabolic geometry. Generic $(4, 8)$ -distributions are an example of this situation that will be discussed in the next Chapter.

3.1. Differential map and normalization condition. Let \mathfrak{n} be a nilpotent graded Lie algebra of depth μ generated by \mathfrak{g}_{-1} . Consider a Lie subalgebra $\mathfrak{g}_0 \subset \text{der}_{gr}(\mathfrak{n})$ of grading-preserving derivations of \mathfrak{n} . The direct sum vector space $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$ admits a graded Lie algebra structure. It is possible, indeed, to extend the brackets on \mathfrak{n} and \mathfrak{g}_0 , by defining a bracket $\mathfrak{g}_0 \otimes \mathfrak{n} \rightarrow \mathfrak{n}$ via the evaluation map as in (9). The *differential map* $\partial : L(\mathfrak{n}, \mathfrak{g}_{\leq 0}) \rightarrow L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$ is the \mathfrak{g}_0 -equivariant homomorphism defined by the formula

$$(10) \quad \partial\Phi(X, Y) = [X, \Phi(Y)] - [Y, \Phi(X)] - \Phi([X, Y]).$$

Now ∂ preserves the grading on the spaces involved, thus it splits in the sum of its restrictions ∂_i between homogeneous components of degree i . In particular, the kernel of the component ∂_1 in homogeneity one coincides with the first prolongation of the pair $(\mathfrak{n}, \mathfrak{g}_0)$.

PROPOSITION 7. *Let $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-\mu}$ be a nilpotent graded Lie algebra of depth μ generated by \mathfrak{g}_{-1} , $\mathfrak{g}_0 \subset \text{der}_{gr}(\mathfrak{n})$ a Lie subalgebra of grading-preserving derivations and ∂ the differential map. If the first prolongation of $(\mathfrak{n}, \mathfrak{g}_0)$ vanishes, then ∂ is injective.*

PROOF. Since \mathfrak{n} is generated by \mathfrak{g}_{-1} , any map $\phi_k \in L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_k$ lying in an homogeneous component and satisfying $\partial\phi_k = 0$ is completely determined by its restriction to \mathfrak{g}_{-1} . Consider $\phi \in L(\mathfrak{n}, \mathfrak{g}_{\leq 0})$ such that $\partial\phi = 0$. Write $\phi = \sum_{k \geq 1} \phi_k$ as sum of homogeneous components $\phi_k \in L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_k$. Then

$$\partial\phi = \sum_{k \geq 1} \partial\phi_k.$$

and $\partial\phi = 0$ if and only if $\partial\phi_k = 0$ for all $k \geq 1$. By hypothesis, if $\partial\phi_1 = 0$ then $\phi_1 = 0$. Observe that $\phi_k(\mathfrak{g}_{-1}) = \{0\}$ for $k \geq 2$. Therefore $\partial\phi_k = 0$ implies $\phi_k = 0$ for all $k \geq 2$. \square

Since ∂ is a \mathfrak{g}_0 -equivariant homomorphism, $\text{Im}(\partial) \subset L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$ is a \mathfrak{g}_0 -invariant subspace. Subspaces complementary to it and satisfying the same properties are introduced in the following

DEFINITION 11. A *normalization condition* for the pair $(\mathfrak{n}, \mathfrak{g}_0)$ is a \mathfrak{g}_0 -invariant complement \mathfrak{N} to $\text{Im}(\partial)$ inside $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$, which is compatible with the homogeneity. Explicitely,

$$\mathfrak{N} = \oplus_{i \geq 1} \mathfrak{N}_i$$

is a graded \mathfrak{g}_0 -module such that $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_i = \text{Im}(\partial_i) \oplus \mathfrak{N}_i$ for all $i \geq 1$.

In general, the existence of a normalization condition is not assured. Finding such conditions can be an interesting and involved problem, but we will not deal with the general problem. Here we just mention that, under certain hypothesis on the Lie algebra \mathfrak{g}_0 , the existence of a normalization condition can be proved. Such hypothesis are satisfied in the cases of interest for us, namely dual Darboux distributions and subriemannian bracket-generating structures.

4. Compatible linear connections

In this Section, the main result of this Chapter concerning the existence and uniqueness of canonical linear connections is proved. The argument is summarized in the following main steps. First, we introduce linear connections on $\text{gr}(TM)$ associated to a regular pair (M, H) , which are compatible with the grading and the Levi bracket. Any connection form on the adapted frame bundle induces, via the associated bundle construction, a linear connection of this type. Clearly, compatible linear connections associated to G_0 -triples might satisfy additional conditions. The linear connections of interest in the subriemannian case, for example, are also compatible with the metric on H . For our purposes, it will be convenient to consider partial linear connections and allow derivatives along directions in the subbundles $\{T^{-k}M\}_{k \geq 1}$ generated by H . Such partial linear connections will be extended, via an inductive procedure, to a proper linear connection. The ingredients occurring at each step of the procedure will be encoded in frame forms. A frame form of maximal length, in particular, induces a linear connection on TM preserving H . In Proposition 9, we associate notions of graded torsion and curvature to each frame form. Distinguished subsets of frame forms are affine spaces over the domain of the differential map, which controls the change of both graded torsion and curvature as the frame form varies (Propositions 8 and 10). Finally, two algebraic conditions on the symbol algebra of H and on the Lie algebra of G_0 assure the existence of a canonical linear connection on TM , as proved in Theorem 3.

Let E be a smooth vector bundle over M and $\Gamma(E)$ the space of its smooth sections. Recall that a linear connection on E is a bilinear map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$, which is $C^\infty(M)$ -linear in the first variable and satisfies the Leibnitz rule in the second variable. One can similarly define partial linear connections $\Gamma(\mathcal{V}) \times \Gamma(E) \rightarrow \Gamma(E)$, by admitting only derivatives along directions in a smooth subbundle $\mathcal{V} \subset TM$.

DEFINITION 12. Let M be a smooth filtered manifold, $H \subset TM$ a bracket-generating distribution and $\text{gr}(TM)$ the graded vector bundle associated to the filtration generated by H . Consider a linear connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(\text{gr}(TM)) \rightarrow \Gamma(\text{gr}(TM))$$

We say that ∇ is *compatible with \mathcal{L}* if ∇ is the direct sum of linear connections ∇^i on $\text{gr}_i(TM)$, satisfying

$$\nabla_\xi^{i+j} \mathcal{L}(X, Y) = \mathcal{L}(\nabla_\xi^i X, Y) + \mathcal{L}(X, \nabla_\xi^j Y)$$

for all $\xi \in \mathfrak{X}(M)$, $X \in \Gamma(\text{gr}_i(TM))$, $Y \in \Gamma(\text{gr}_j(TM))$ and $i, j \leq -1$. In particular, such a connection is completely determined by $\nabla^{-1} : \mathfrak{X}(M) \times \Gamma(H) \rightarrow \Gamma(H)$.

Observe that the same definition works if we restrict the first variable to a smooth subbundle $\mathcal{V} \subset TM$. It thus makes sense speaking about partial linear connections on $\text{gr}(TM)$ compatible with \mathcal{L} . Consider now a regular pair (M, H) of type \mathfrak{n} , the filtration $\{T^{-k}M\}_{k \geq 1}$ on TM generated by H and the associated graded bundle $\text{gr}(TM)$. Recall that (M, H) comes together with an adapted frame bundle $\mathcal{P} \rightarrow M$, whose structure group is $\text{Aut}_{gr}(\mathfrak{n})$, such that the associated vector bundle $\mathcal{P} \times_{\text{Aut}_{gr}(\mathfrak{n})} \mathfrak{n}$ is isomorphic to $\text{gr}(TM)$. Moreover, every connection form on \mathcal{P} induces, in the sense of Definition 7, a linear connection on $\text{gr}(TM)$ compatible with \mathcal{L} .

More generally, we consider G_0 -triples $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ as in Definition 9 and linear connections compatible with \mathcal{T} . These are induced by connection forms on \mathcal{F} through the isomorphism $\mathcal{F} \times_{G_0} \mathfrak{n} \cong \text{gr}(TM)$ and in some specific cases, they can be characterized. The compatible linear connections related to subriemannian manifolds (M, H, S) of regular subriemannian type from

Example 5, for instance, also satisfy

$$\xi \cdot S(X, Y) = S(\nabla_\xi X, Y) + S(X, \nabla_\xi Y)$$

for all $X, Y \in \Gamma(H)$ and $\xi \in \Gamma(\mathcal{V})$.

Fix a closed Lie subgroup $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ and a G_0 -triple $(\mathcal{F} \rightarrow (M, H), \mathfrak{n})$. Denote by \mathfrak{g}_0 the Lie algebra of G_0 , which is a subalgebra of $\mathfrak{der}_{gr}(\mathfrak{n})$. Furthermore, let $\text{End}_0(\text{gr}(TM))$ be the space of grading-preserving endomorphisms of $\text{gr}(TM)$, which are compatible with \mathcal{L} . Recall that the associated bundle construction is tensorial. Therefore, $\mathcal{F} \times_{G_0} \mathfrak{g}_0$ is isomorphic to a bundle of subalgebras $E_0 \subset \text{End}_0(\text{gr}(TM))$. In the rest of the Chapter, we will always denote by $E = \text{gr}(TM)$ the associated graded bundle, with $E_i = \text{gr}_i(TM)$ its graded components and by $E_{\leq 0} = E \oplus E_0$ the direct sum of vector bundles. Also, put $E_i = \{0\}$ for $i \geq 1$ and denote by

$$L(E, E_{\leq 0})_k = \bigoplus_{i=-\mu}^{-1} E_i^* \otimes E_{i+k}$$

the space of homomorphisms $\phi: E \rightarrow E_{\leq 0}$ such that $\phi(E_i) \subset E_{i+k}$. Observe that $L(E, E_{\leq 0})_k$ is isomorphic to $\mathcal{F} \times_{G_0} L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_k$. In order to simplify the notation, we put $\mathfrak{X}^p = \Gamma(T^p M)$.

DEFINITION 13. Let \mathfrak{n} be a finite dimensional nilpotent graded Lie algebra of depth μ , $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ a closed Lie subgroup and $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ a G_0 -triple. Let $1 \leq k \leq \mu$ be an integer. A *frame form of length k* for \mathcal{T} is a pair $\theta = (\nabla, \pi)$, consisting of a partial linear connection on E

$$(11) \quad \nabla: \mathfrak{X}^{-k} \times \Gamma(E) \rightarrow \Gamma(E),$$

induced by a connection form on \mathcal{F} , together with a collection $\pi = (\pi_{-1}^k, \dots, \pi_{-\mu}^k)$ of fibered maps $\pi_i^k: T^{i-k}M \rightarrow E_i$ such that π_i^k extends the canonical projection $q_i: T^i M \rightarrow E_i$ for each $i = -1, \dots, -\mu$.

Observe that a frame form of length μ consists of a proper compatible linear connection on E , together with a collection of fibered maps $\{\pi_i: TM \rightarrow E_i, i = -1, \dots, -\mu\}$ extending the canonical projections. Clearly, the latter is equivalent to a splitting of the filtration of TM . Indeed, $\pi_{-\mu} = q_{-\mu}: TM \rightarrow E_{-\mu}$ is surjective and TM writes as direct sum of image and kernel of $\pi_{-\mu}$ as $TM \cong E_{-\mu} \oplus T^{-\mu+1}M$. Applying the same argument for $i = -\mu + 1, \dots, -1$ to the restriction of π_i to $T^i M$, which coincides by hypothesis with q_i , leads to an isomorphism $TM \cong E_{-\mu} \oplus \dots \oplus E_{-1}$.

A frame form of length k canonically has an underlying frame form of length $k-1$. It thus makes sense to talk about extensions of a given frame form of length $k-1$ to a frame form of length k . Frame forms of length μ exist, thus frame forms of arbitrary length exist. Further properties of frame forms are collected in the following

PROPOSITION 8. Let $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ be a closed Lie subgroup and let $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ be a G_0 -triple. Let $k \geq 1$ be an integer.

- (a) The space of all extensions of a given frame form of length $k-1$ to a frame form of length k for \mathcal{T} is an affine space modelled on $\Gamma(L(E, E_{\leq 0})_k)$. Likewise, frame forms of length one for \mathcal{T} form an affine space modelled on $\Gamma(L(E, E_{\leq 0})_1)$.
- (b) A frame form of length μ determines a linear connection on TM preserving H .

PROOF. (a) Fix $k \geq 1$. Consider two frame forms $\theta = (\nabla, \pi)$ and $\tilde{\theta} = (\tilde{\nabla}, \tilde{\pi})$ of length k for \mathcal{T} , both extending a given frame form of length $k-1$. Considering last requirement as an empty condition for $k=1$, the following argument proves the statement for all $k \geq 1$. By hypothesis, the partial linear connections

$$\nabla, \tilde{\nabla}: \mathfrak{X}^{-k} \times \Gamma(E) \rightarrow \Gamma(E)$$

coincide on $\mathfrak{X}^{-k+1} \times \Gamma(E)$. Observe that their difference $\nabla - \tilde{\nabla}$ is tensorial in both entries. Therefore, the homomorphism $(\nabla - \tilde{\nabla})_\xi$ only depends on $q_{-k}(\xi)$. Moreover, since ∇ and $\tilde{\nabla}$ are induced by connection forms on \mathcal{F} , we have that $(\nabla - \tilde{\nabla})_\xi \in E_0$ for any $\xi \in \mathfrak{X}^{-k}$. Putting all together, there exists a smooth section A_0^k of $E_{-k}^* \otimes E_0$ such that

$$(12) \quad \nabla_\xi \lambda - \tilde{\nabla}_\xi \lambda = A_0^k(q_{-k}(\xi))(\lambda).$$

for all $\xi \in \mathfrak{X}^{-k}$ and $\lambda \in \Gamma(E)$. For $i < k$, the maps $\pi_i^k, \tilde{\pi}_i^k : T^{i-k}M \rightarrow E_i$ coincide on $T^{i-k+1}M$. Therefore, their difference factorizes to a fibered homomorphism $E_{i-k} \rightarrow E_i$. Otherwise put, there exists a smooth section A_i^k of $E_{i-k}^* \otimes E_i$ such that

$$(13) \quad (\pi_i^k - \tilde{\pi}_i^k)(\zeta) = A_i^k(q_{i-k}(\zeta))$$

for all $\zeta \in \mathfrak{X}^{i-k}$. The difference between two different choices of frame forms $\theta, \tilde{\theta}$ is thus described by $A^k = (A_0^k, \dots, A_{-\mu}^k)$, which is a smooth section of $L(E, E_{\leq 0})_k$.

Conversely, suppose that $\tilde{\theta} = (\tilde{\nabla}, \tilde{\pi})$ is a frame form of length k for \mathcal{T} and that $A_i^k \in \Gamma(E_{i-k}^* \otimes E_i)$ for $i = -\mu, \dots, 0$ are given, so that $A^k = (A_0^k, \dots, A_{-\mu}^k) \in \Gamma(L(E, E_{\leq 0})_k)$. In particular, $\tilde{\nabla} : \mathfrak{X}^{-k} \times \Gamma(E) \rightarrow \Gamma(E)$ is a partial compatible linear connection and $A_0^k \circ q_{-k}$ is a section of $(T^{-k})^* \otimes E_0$ which vanishes on \mathfrak{X}^{-k+1} . Thus putting

$$\begin{aligned} \nabla : \mathfrak{X}^{-k} \times \Gamma(E) &\rightarrow \Gamma(E), \\ \nabla_\xi \lambda &:= \tilde{\nabla}_\xi \lambda + A_0^k(q_{-k}(\xi))(\lambda) \quad \text{for all } \xi \in \mathfrak{X}^{-k}, \lambda \in \Gamma(E) \end{aligned}$$

defines a partial compatible linear connection, which coincides with $\tilde{\nabla}$ on $\mathfrak{X}^{-k+1} \times \Gamma(E)$. For $i < k$, we have that $A_i^k \circ q_{i-k} \in \Gamma((T^{i-k})^* \otimes E_i)$ vanishes on \mathfrak{X}^{i-k+1} . We thus see that putting

$$\pi_i^k(\zeta) := \tilde{\pi}_i^k(\zeta) + A_i^k(q_{i-k}(\zeta)) \quad \text{for all } \zeta \in \mathfrak{X}^{i-k}$$

defines a tensorial map $T^{i-k} \rightarrow E_i$, which coincides with $\tilde{\pi}_i^k$ on \mathfrak{X}^{i-k+1} . Therefore, $\theta = (\nabla, \pi)$ is a frame form of length k for \mathcal{T} such that θ and $\tilde{\theta}$ have the same underlying frame form of length $k-1$.

(b) By definition, a frame form of length μ consists of a proper compatible linear connection ∇ on E , together with fibered projections $\pi_i^\mu : TM \rightarrow E_i$ for $i = -1, \dots, -\mu$, which will be simply denoted by $\pi_i^\mu = \pi_i$. The projections determine an isomorphism $TM \rightarrow E$ via $\xi \mapsto (\pi_{-1}(\xi), \dots, \pi_{-\mu}(\xi))$, whose inverse will be denoted by $\Phi : E \rightarrow TM$. Now Φ defines a linear connection $\bar{\nabla}$ on TM , by putting for $\xi, \eta \in \mathfrak{X}(M)$:

$$\bar{\nabla}_\xi \eta := \Phi\left(\nabla_\xi \pi(\eta)\right) = \Phi\left(\sum_{j=-1}^{-\mu} \nabla_\xi \pi_j(\eta)\right)$$

Both π and Φ restrict to the identity on H . Then $\bar{\nabla}_\xi \eta = \nabla_\xi \eta$ for all $\eta \in \Gamma(H)$, proving that $\bar{\nabla}_\xi$ preserves $\Gamma(H)$ for all $\xi \in \mathfrak{X}(M)$. \square

In the following, we will write $\theta - \tilde{\theta} = A^k \in \Gamma(L(E, E_{\leq 0})_k)$ for frame forms $\theta, \tilde{\theta}$ of length k satisfying (12) and (13).

REMARK 2. We underline some facts about the homomorphisms occuring in the definition of frame forms. Fix an integer $k \geq 2$. For $i = -1, \dots, -\mu$ and $t = 1, \dots, k-1$, let $\sigma_i^t : T^{i-t}M \rightarrow E_i$ be a fibered map whose restriction to T^iM coincides with the canonical projection q_i . The collection $\{\sigma_i^t\}$ determines surjective fibered homomorphisms

$$\begin{aligned} T^j M &\longrightarrow E_j \oplus \dots \oplus E_{j+k-1} \\ \xi &\mapsto (q_j(\xi), \sigma_{j+1}^1(\xi), \dots, \sigma_{j+k-1}^{k-1}(\xi)) \end{aligned}$$

with kernel $T^{j+k}M$, thus isomorphisms

$$(14) \quad T^j M / T^{j+k} M \cong E_j \oplus \dots \oplus E_{j+k-1}.$$

In particular, we have an isomorphism $T^{-k}M \cong E_{-k} \oplus \dots \oplus E_{-1}$. Then (14) shows that for any $j = -1, \dots, -\mu$ and $X \in \Gamma(E_j)$, there exists $\xi^\sigma \in \mathfrak{X}^j$ satisfying $q_j(\xi^\sigma) = X$ and $\sigma_{j+t}^t(\xi^\sigma) = 0$ for any $1 \leq t \leq k-1$. By construction, ξ^σ is defined by X and σ up to adding $\tilde{\xi} \in \mathfrak{X}^{j+k}$. In particular, this means that ξ^σ is uniquely determined for $-k \leq j \leq -1$.

Now for $i = -1, \dots, -\mu$, choose fibered maps $\pi_i^k : T^{i-k}M \rightarrow E_i$ extending σ_i^{k-1} . Again, these choices determine surjective homomorphisms

$$\begin{aligned} T^j M &\longrightarrow E_j \oplus \dots \oplus E_{j+k-1} \oplus E_{j+k} \\ \xi &\mapsto \left(q_j(\xi), \sigma_{j+1}^1(\xi), \dots, \sigma_{j+k-1}^{k-1}(\xi), \pi_{j+k}^k(\xi) \right) \end{aligned}$$

with kernel $T^{j+k+1}M$, so that for any $X \in E_j$ there exists $\xi^\pi \in \mathfrak{X}^j$ such that $q_j(\xi^\pi) = X$, $\sigma_{j+t}^t(\xi^\pi) = 0$ for all $t = 1, \dots, k-1$ and $\pi_{j+k}^k(\xi^\pi) = 0$. In general, these conditions define $\xi^\pi \in \mathfrak{X}^j$ up to \mathfrak{X}^{j+k+1} . Consider another choice of extensions $\tilde{\pi}_i^k : T^{i-k}M \rightarrow E_i$ of σ_i^{k-1} and let $\xi^{\tilde{\pi}} \in \mathfrak{X}^j$ be such that $q_j(\xi^{\tilde{\pi}}) = X$, $\sigma_{j+t}^t(\xi^{\tilde{\pi}}) = 0$ for all $t = 1, \dots, k-1$ and $\tilde{\pi}_{j+k}^k(\xi^{\tilde{\pi}}) = 0$, corresponding to the second choice of extension. Evidently, we have $\xi^{\tilde{\pi}} - \xi^\pi \in \mathfrak{X}^{j+k}$ by construction.

PROPOSITION 9. *Let $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ be a closed Lie subgroup, \mathfrak{g}_0 the Lie algebra of G_0 and $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ a G_0 -triple. Consider a frame form $\theta = (\nabla, \pi)$ for \mathcal{T} of length k . Denote by $\xi^\pi \in \mathfrak{X}^i$ and $\eta^\pi \in \mathfrak{X}^j$ representatives for $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$, respectively, chosen as in Remark 2.*

- (a) *Consider negative integers $-\mu \leq i, j \leq -1$ such that $i + j + k \leq -1$. For $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$, the following formula:*

$$(15) \quad T_\theta^k(X, Y) = \delta_k^{-i} \nabla_{\xi^\pi} Y - \delta_k^{-j} \nabla_{\eta^\pi} X - \pi_{i+j+k}^k([\xi^\pi, \eta^\pi])$$

defines a tensorial map $E_i \otimes E_j \rightarrow E_{i+j+k}$.

- (b) *Consider negative integers $-\mu \leq i, j \leq -1$ such that $i + j + k = 0$ and $X \in \Gamma(E_i)$, $Y \in \Gamma(E_j)$. For $\ell = -1, \dots, -\mu$ and $Z \in \Gamma(E_\ell)$, the formula*

$$(16) \quad R_\theta^k(X, Y)(Z) = \nabla_{\xi^\pi} \nabla_{\eta^\pi} Z - \nabla_{\eta^\pi} \nabla_{\xi^\pi} Z - \nabla_{[\xi^\pi, \eta^\pi]} Z$$

defines a section $R_\theta^k(X, Y)(Z) \in \Gamma(E_\ell)$. Moreover, $(X, Y) \mapsto R_\theta^k(X, Y)$ defines a tensorial map $E_i \otimes E_j \rightarrow E_0$, where $E_0 \cong \mathcal{F} \times_{G_0} \mathfrak{g}_0$.

The union of all maps defined in (1) and (2), together with the zero maps for integers $i + j + k > 0$, defines fibered maps $T_\theta^k \in L(\Lambda^2 E, E)_k$ and $R_\theta^k \in L(\Lambda^2 E, E_0)_k$ in homogeneity k . We will refer to them as the torsion, respectively the curvature in homogeneity k associated to the frame form θ .

PROOF. If $\theta = (\nabla, \pi)$ is a frame form for \mathcal{T} of length k , then $\nabla : \mathfrak{X}^{-k} \times \Gamma(E) \rightarrow \Gamma(E)$ is a partial compatible linear connection. First of all, observe that the formula (15) is well defined. Indeed, the first term is nonzero only for $i = -k$ and in this case $\xi^\pi \in \mathfrak{X}^{-k}$ is uniquely determined by π . Therefore $\nabla_{\xi^\pi} Y$ makes sense, it lies in E_{i+j+k} and it does not depend on the choice of representatives for X, Y . Analogous considerations hold for the second summand. Since $[\xi^\pi, \eta^\pi] \in \mathfrak{X}^{i+j}$, we can insert it into π_{i+j+k} , thus getting a section of E_{i+j+k} . The representatives $\xi^\pi \in \mathfrak{X}^i$ and $\eta^\pi \in \mathfrak{X}^j$ for X and Y are defined up to \mathfrak{X}^{i+k+1} and \mathfrak{X}^{j+k+1} , respectively, and both $[T^{i+k+1}M, T^jM]$ and $[T^iM, T^{j+k+1}M]$ are contained in $T^{i+j+k+1}M$, where π_{i+j+k} is zero. Therefore, the third summand does not depend on the choice of representatives for X, Y either. Let us now prove that (15) is tensorial. Again by skew-symmetry, it is enough to verify that the

formula is tensorial in the first variable. Apply the transformation $X \mapsto fX$ for $f \in C^\infty(M)$ and choose $f\xi^\pi$ as representative for fX , thus obtaining

$$\begin{aligned} T_\theta^k(fX, Y) &= f\left(\delta_k^{-i}\nabla_{\xi^\pi}Y - \delta_k^{-j}\nabla_{\eta^\pi}X\right) + \\ &\quad - \delta_k^{-j}(\eta^\pi \cdot f)X - \pi_{i+j+k}\left(f[\xi^\pi, \eta^\pi] - (\eta^\pi \cdot f)\xi^\pi\right) \\ &= fT^k(X, Y) - \delta_k^{-j}(\eta^\pi \cdot f)X + (\eta^\pi \cdot f)\pi_{i+j+k}(\xi^\pi) \end{aligned}$$

The sum of the second and third terms is zero for all admissible values of i, j . Indeed, for $j = -k$ the last summand writes as $(\eta^\pi \cdot f)\pi_i(\xi^\pi)$. Now $\pi_i(\xi^\pi) = q_i(\xi^\pi)$ since $\xi^\pi \in \mathfrak{X}^i$ and $q_i(\xi^\pi) = X$, so that the two terms in the last row are equal with opposite sign. Conversely, for $j \neq -k$ both terms are zero. Indeed for $j + k > 0$, ξ^π is chosen as satisfying $\pi_{i+j+k}(\xi^\pi) = 0$, while the same equality follows from the inclusion $T^iM \subset T^{i+j+k+1}M$ for $j + k < 0$. Since the formula is skew-symmetric, this concludes the proof of (a).

For integers $-\mu \leq i, j \leq -1$ such that $i + j = -k$, representatives $\xi^\pi \in \mathfrak{X}^i$ and $\eta^\pi \in \mathfrak{X}^j$ for $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ are uniquely determined by π , as explained in Remark 2. In the following, we will denote $\xi = \xi^\pi$ and $\eta = \eta^\pi$ to simplify the notation. For such values of i, j we have that $\xi, \eta, [\xi, \eta] \in \mathfrak{X}^{-k}$, so that the formula (16) is well defined. Since ∇ is compatible with the grading on E , the same is true for each summand of (16), hence $R_\theta^k(X, Y)(Z) \in \Gamma(E_\ell)$ for all $Z \in \Gamma(E_\ell)$. From the properties of linear connections on general vector bundles, namely C^∞ -linearity in the first variable and Leibnitz rule in the second one, it follows via a direct computation that $R_\theta^k(X, Y)(Z)$ is tensorial in $Z \in \Gamma(H)$. Therefore, (16) defines an endomorphism $\Phi: E \rightarrow E$ which preserves the grading. Moreover, since ∇ is compatible with \mathcal{L} , for $Z_1 \in \Gamma(H)$ and $Z_2 \in \mathfrak{X}^i$ with $i = -1, \dots, -\mu + 1$ we have that $\nabla_{[\xi, \eta]}\mathcal{L}(Z_1, Z_2) = \mathcal{L}(\nabla_{[\xi, \eta]}Z_1, Z_2) + \mathcal{L}(Z_1, \nabla_{[\xi, \eta]}Z_2)$ and that $\nabla_\xi \nabla_\eta \mathcal{L}(Z_1, Z_2)$ writes as the sum

$$\mathcal{L}(\nabla_\xi \nabla_\eta Z_1, Z_2) + \mathcal{L}(\nabla_\eta \nabla_\xi Z_1, Z_2) + \mathcal{L}(\nabla_\xi Z_1, \nabla_\eta Z_2) + \mathcal{L}(Z_1, \nabla_\xi \nabla_\eta Z_2)$$

Since the sum of the second and the third summands is symmetric in $\{\xi, \eta\}$, we have that

$$\begin{aligned} \nabla_\xi \nabla_\eta \mathcal{L}(Z_1, Z_2) - \nabla_\eta \nabla_\xi \mathcal{L}(Z_1, Z_2) &= \mathcal{L}(\nabla_\xi \nabla_\eta Z_1 - \nabla_\eta \nabla_\xi Z_1, Z_2) + \\ &\quad + \mathcal{L}(Z_1, \nabla_\xi \nabla_\eta Z_2 - \nabla_\eta \nabla_\xi Z_2) \end{aligned}$$

hence that

$$\begin{aligned} \mathcal{L}(R_\theta^k(X, Y)Z_1, Z_2) + \mathcal{L}(Z_1, R_\theta^k(X, Y)Z_2) &= \nabla_\xi \nabla_\eta \mathcal{L}(Z_1, Z_2) - \nabla_\eta \nabla_\xi \mathcal{L}(Z_1, Z_2) + \\ &\quad - \nabla_{[\xi, \eta]}\mathcal{L}(Z_1, Z_2) \\ &= R_\theta^k(X, Y)\mathcal{L}(Z_1, Z_2), \end{aligned}$$

for all $Z_1 \in \Gamma(H)$ and $Z_2 \in \mathfrak{X}^i$ and $i = -1, \dots, -\mu + 1$. This shows that Φ is an homomorphism of E which is compatible with \mathcal{L} , hence that $R_\theta^k(X, Y) \in \text{End}_0(\text{gr}(TM))$ for any $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$. The fact that the skew-symmetric \mathbb{R} -bilinear map $(X, Y) \mapsto R_\theta^k(X, Y)$ is tensorial, again, comes from the basic properties defining a linear connection. Consider now the G_0 -triple $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$. Let $\gamma \in \Omega^1(\mathcal{F}, \mathfrak{g}_0)$ be a connection form on \mathcal{F} such that $\nabla = \nabla^\gamma$ is induced by γ in the sense of Proposition 7. Let $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(\mathcal{F})$ be lifts of ξ, η . Observe that $\xi^h = \tilde{\xi} - \zeta_{\gamma(\tilde{\xi})}$ and $\eta^h = \tilde{\eta} - \zeta_{\gamma(\tilde{\eta})}$ are lifts of ξ and η such that $\gamma(\xi^h) = \gamma(\eta^h) = 0$. Lifts satisfying this property are also said to be horizontal. Since $[\xi^h, \eta^h]$ is a lift of $[\xi, \eta]$, we have that an horizontal lift of $[\xi, \eta]$ is given by $[\xi, \eta]^h = [\xi^h, \eta^h] - \zeta_{\gamma([\xi^h, \eta^h])}$. In the sense of Proposition 5, $Z \in \Gamma(E_t)$ corresponds to a unique horizontal and G_0 -equivariant smooth map $f: \mathcal{F} \rightarrow \mathfrak{g}_t$. According to Definition 7, $R_\theta^k(X, Y)Z$ corresponds to

$$(\xi^h \cdot \eta^h - \eta^h \cdot \xi^h - [\xi, \eta]^h) \cdot f = ([\xi^h, \eta^h] - [\xi, \eta]^h) \cdot f = \zeta_{\gamma([\xi^h, \eta^h])} \cdot f$$

In the picture of Proposition 5, $R_\theta^k(X, Y)$ corresponds to the following G_0 -equivariant smooth function $\mathcal{R}(\tilde{\xi}, \tilde{\eta}) : \mathcal{F} \rightarrow \mathfrak{g}_0$ with values in \mathfrak{g}_0 :

$$\mathcal{R}(\tilde{\xi}, \tilde{\eta}) = \gamma([\xi^h, \eta^h]) = \gamma([\tilde{\xi} - \zeta_{\gamma(\tilde{\xi})}, \tilde{\eta} - \zeta_{\gamma(\tilde{\eta})}])$$

Therefore, the endomorphism $R_\theta^k(X, Y)$ lies in a subbundle $E_0 \subset \text{End}_0(\text{gr}(TM))$ isomorphic to $\mathcal{F} \times_{G_0} \mathfrak{g}_0$, which concludes the proof of (b). \square

Let $k \geq 1$ be an integer and put $W_k = L(\Lambda^2 \mathfrak{n}, \mathfrak{n} \oplus \mathfrak{g}_0)_k$. The sum $C_\theta^k = T_\theta^k + R_\theta^k$ of torsion and curvature, defined by formulas (15) and (16), is a smooth section of $L(\Lambda^2 E, E \oplus E_0)_k$, which is isomorphic to the associated vector bundle $\mathcal{F} \times_{G_0} W_k$. We will also briefly refer to C_θ^k as the curvature of a frame form θ of length k . Observe that if $\mathfrak{N} = \bigoplus_{i \geq 1} \mathfrak{N}_i \subseteq \bigoplus_{i \geq 1} W_i$ is a normalization condition for $(\mathfrak{n}, \mathfrak{g}_0)$, by definition $\mathfrak{N}_i \subseteq W_i$ is a G_0 -invariant subspace and we thus have an inclusion of subbundles $\mathcal{F} \times_{G_0} \mathfrak{N}_i \subseteq \mathcal{F} \times_{G_0} W_i$ for all $i \geq 1$. One can thus require that C_θ^k takes values in the subbundle $\mathcal{F} \times_{G_0} \mathfrak{N}_k$.

DEFINITION 14. Let $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ be a closed Lie subgroup and \mathcal{T} a G_0 -triple. Assume that $\mathfrak{N} = \bigoplus_{i \geq 1} \mathfrak{N}_i$ is a normalization condition. A frame form θ of length $k \geq 1$ for \mathcal{T} is said to be *normal* if $C_\theta^k \in \Gamma(\mathcal{F} \times_{G_0} \mathfrak{N}_k) \subseteq \Gamma(\mathcal{F} \times_{G_0} W_k) = \Gamma(\text{End}(\Lambda^2 E, E_{\leq 0})_k)$.

Put $V = L(\mathfrak{n}, \mathfrak{g}_{\leq 0})$ and $W = L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$. Also put $\mathfrak{g}_\ell = \{0\}$ for $\ell > 0$. Recall the differential map $\partial : V \rightarrow W$ defined by (10). Now $V = \bigoplus_k V_k$ is a graded vector space and we write $\Phi^k \in V_k$ as

$$\Phi^k = \sum_{\ell=-\mu+k}^{k-1} \Phi_\ell^k, \quad \Phi_\ell^k \in \mathfrak{g}_{\ell-k}^* \otimes \mathfrak{g}_\ell$$

For $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, we have that $\partial \Phi^k(X, Y) \in \mathfrak{g}_{\leq 0}$ and its \mathfrak{g}_{i+j+k} -component coincides with

$$(17) \quad [\Phi_{i+k}^k(X), Y] + [X, \Phi_{j+k}^k(Y)] - \Phi_{i+j+k}^k([X, Y]).$$

If A, B are G_0 -representations and $F : A \rightarrow B$ is a G_0 -equivariant linear map, then F induces an homomorphism $\mathcal{F} \times_{G_0} A \rightarrow \mathcal{F} \times_{G_0} B$ of vector bundles, hence a tensorial map between the respective spaces of sections. With abuse of notation, we will use the same symbol

$$\partial : \Gamma(\mathcal{F} \times_{G_0} V) \rightarrow \Gamma(\mathcal{F} \times_{G_0} W)$$

to write the differential map and the induced map between sections of the associated vector bundles. If \mathfrak{N} is a normalization condition for $(\mathfrak{n}, \mathfrak{g}_0)$, then $\mathcal{F} \times_{G_0} \mathfrak{N} \subseteq \mathcal{F} \times_{G_0} W$ is complementary to $\text{Im}(\partial)$.

PROPOSITION 10. Let $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ be a closed Lie subgroup and \mathcal{T} a G_0 -triple. Fix an integer $k \geq 1$ and consider two frame forms $\theta, \tilde{\theta}$ of length k for \mathcal{T} with same underlying frame form of length $k-1$. Denote by C^k, \tilde{C}^k the curvatures in homogeneity k of θ and $\tilde{\theta}$, respectively. If $\theta - \tilde{\theta} = A^k$, then $C^k - \tilde{C}^k = \partial A^k$.

PROOF. Suppose that $\theta - \tilde{\theta} = A^k = (A_0^k, A_{-1}^k, \dots, A_{-\mu}^k)$, with $A_s^k \in \Gamma(E_{s-k}^* \otimes E_s)$. Fix indices $-\mu \leq i, j \leq -1$ and elements $X \in \Gamma(E_i), Y \in \Gamma(E_j)$. Our aim is evaluating $(C^k - \tilde{C}^k)(X, Y)$. Recall that $C^k(X, Y)$ is defined in terms of representatives $\xi^\pi \in \mathfrak{X}^i, \eta^\pi \in \mathfrak{X}^j$ for X and Y , respectively, whose choice depends on the projection components $\pi_i = \pi_i^k$ of the frame form θ , as explained in Remark 2. Analogous representatives $\xi^{\tilde{\pi}} \in \mathfrak{X}^i$ and $\eta^{\tilde{\pi}} \in \mathfrak{X}^j$, determined by the projection components $\tilde{\pi}_i = \tilde{\pi}_i^k$ of $\tilde{\theta}$, appear in the formula for $\tilde{C}^k(X, Y)$. By construction, $\xi^\pi - \xi^{\tilde{\pi}} \in \mathfrak{X}^{i+k}$, so $q_{i+k}(\xi^\pi - \xi^{\tilde{\pi}}) \in E_{i+k}$ is well defined and for $i+k < 0$, we get

$$q_{i+k}(\xi^{\tilde{\pi}} - \xi^\pi) = \tilde{\pi}_{i+k}(\xi^{\tilde{\pi}} - \xi^\pi) = -\tilde{\pi}_{i+k}(\xi^\pi) = -\pi_{i+k}(\xi^\pi) + A_{i+k}^k(q_i(\xi^\pi)) = A_{i+k}^k(X)$$

while $q_{i+k}(\xi^{\tilde{\pi}} - \xi^{\pi}) = 0$ for $i+k \geq 0$. Analogously, $q_{j+k}(\eta^{\tilde{\pi}} - \eta^{\pi}) = A_{j+k}^k(Y)$ if $j+k < 0$, while it is equal to zero for $j+k \geq 0$. First, we suppose that i, j satisfy $i+j+k \leq -1$, so that we look at the torsion components T^k, \tilde{T}^k . By skew-symmetry, it will be enough to consider the case $i \geq j$. Evaluating $(T^k - \tilde{T}^k)(X, Y)$ on $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$ gives

$$(18) \quad \begin{aligned} (T^k - \tilde{T}^k)(X, Y) = & \delta_k^{-i}(\nabla_{\xi^{\pi}} Y - \tilde{\nabla}_{\xi^{\tilde{\pi}}} Y) - \delta_k^{-j}(\nabla_{\eta^{\pi}} X - \tilde{\nabla}_{\eta^{\tilde{\pi}}} X) + \\ & -\pi_{i+j+k}([\xi^{\pi}, \eta^{\pi}]) + \tilde{\pi}_{i+j+k}([\xi^{\tilde{\pi}}, \eta^{\tilde{\pi}}]) \end{aligned}$$

Observe that the first summand coincides with $\delta_k^{-i}(\nabla_{\xi^{\pi}} - \tilde{\nabla}_{\xi^{\tilde{\pi}}})(Y)$, since $\xi^{\pi} = \xi^{\tilde{\pi}}$ for $i = -k$, while for all other indices i both terms are zero. By (12), we can rewrite it as $\delta_0^{i+k} A_{i+k}^k(X)(Y)$. Analogously, the second summand is equal to $\delta_0^{j+k} A_{j+k}^k(Y)(X)$. The sum of the third and fourth summands can be determined as follows. First, observe that

$$\begin{aligned} -\tilde{\pi}_{i+j+k}([\xi^{\tilde{\pi}}, \eta^{\tilde{\pi}}]) &= -\tilde{\pi}_{i+j+k}([\xi^{\pi}, \eta^{\pi}] + [\xi^{\tilde{\pi}} - \xi^{\pi}, \eta^{\tilde{\pi}}] + [\xi^{\pi}, \eta^{\tilde{\pi}} - \eta^{\pi}]) \\ &= -\pi_{i+j+k}([\xi^{\pi}, \eta^{\pi}]) + A_{i+j+k}^k(q_{i+j}([\xi^{\pi}, \eta^{\pi}])) + \\ &\quad -q_{i+j+k}([\xi^{\tilde{\pi}} - \xi^{\pi}, \eta^{\tilde{\pi}}]) - q_{i+j+k}([\xi^{\pi}, \eta^{\tilde{\pi}} - \eta^{\pi}]) \end{aligned}$$

Moreover, we have that

$$q_{i+j+k}([\xi^{\tilde{\pi}} - \xi^{\pi}, \eta^{\tilde{\pi}}]) = \begin{cases} 0 & i+k \geq 0 \\ \mathcal{L}(q_{i+k}(\xi^{\tilde{\pi}} - \xi), q_j(\eta^{\pi})) & i+k < 0 \end{cases}$$

and analogously

$$q_{i+j+k}([\xi^{\pi}, \eta^{\tilde{\pi}} - \eta^{\pi}]) = \begin{cases} 0 & j+k \geq 0 \\ \mathcal{L}(q_i(\xi^{\pi}), q_{j+k}(\eta^{\tilde{\pi}} - \eta^{\pi})) & j+k < 0 \end{cases}$$

Therefore

$$\begin{aligned} -\pi_{i+j+k}([\xi^{\pi}, \eta^{\pi}]) + \tilde{\pi}_{i+j+k}([\xi^{\tilde{\pi}}, \eta^{\tilde{\pi}}]) &= -A_{i+j+k}^k(q_{i+j}([\xi^{\pi}, \eta^{\pi}])) + \\ &\quad + q_{i+j+k}([\xi^{\tilde{\pi}} - \xi^{\pi}, \eta^{\tilde{\pi}}]) + q_{i+j+k}([\xi^{\pi}, \eta^{\tilde{\pi}} - \eta^{\pi}]) \\ &= -A_{i+j+k}^k(\mathcal{L}(X, Y)) + \begin{cases} \mathcal{L}(A_{i+k}^k(X), Y) - \mathcal{L}(A_{j+k}^k(Y), X) & i+k < 0, j+k < 0 \\ -\mathcal{L}(A_{j+k}^k(Y), X) & i+k \geq 0, j+k < 0 \\ 0 & i+k \geq 0, j+k \geq 0 \end{cases} \end{aligned}$$

hence from (18) we see that $(T^k - \tilde{T}^k)(X, Y)$ rewrites as

$$\begin{aligned} i = -k, \quad j = -k : & \quad A_0^k(X)(Y) - A_0^k(Y)(X) - A_{-k}^k(\mathcal{L}(X, Y)) \\ i = -k, \quad j+k < 0 : & \quad A_0^k(X)(Y) - \mathcal{L}(A_{j+k}^k(Y), X) - A_j^k(\mathcal{L}(X, Y)) \\ i+k < 0, \quad j+k < 0 : & \quad \mathcal{L}(A_{i+k}^k(X), Y) - \mathcal{L}(A_{j+k}^k(Y), X) - A_{i+j+k}^k(\mathcal{L}(X, Y)) \\ i+k > 0, \quad j+k > 0 : & \quad -A_{i+j+k}^k(\mathcal{L}(X, Y)) \\ i+k > 0, \quad j+k < 0 : & \quad -\mathcal{L}(A_{j+k}^k(Y), X) - A_{i+j+k}^k(\mathcal{L}(X, Y)) \end{aligned}$$

Compare now the last formula with (17) to see that it coincides with $\partial A^k(X, Y)$. The statement is thus proved for the torsion components.

Now fix integers $-\mu \leq i, j \leq -1$ satisfying $i+j+k=0$ and consider $X \in \Gamma(E_i), Y \in \Gamma(E_j)$. Then $i > -k$ and $j > -k$, hence both $T^i M$ and $T^j M$ are contained in $T^{-k+1} M$ and the representative $\xi^{\pi} \in T^i M$ for X is defined by the underlying frame form of length $k-1$ of θ ,

which coincides by hypothesis with the underlying frame form of length $k - 1$ of $\tilde{\theta}$. For these values of i, j , we thus have that $\xi^{\tilde{\pi}} = \xi^{\pi}$ and $\eta^{\tilde{\pi}} = \eta^{\pi}$, which will be simply denoted by ξ, η . Then, $(C^k - \tilde{C}^k)(X, Y) \in E_0$ and its action on E writes as

$$\begin{aligned} R^k(X, Y)Z - \tilde{R}^k(X, Y)Z = & \nabla_{\xi}(\nabla - \tilde{\nabla})_{\eta}Z + (\nabla - \tilde{\nabla})_{\xi}\tilde{\nabla}_{\eta}Z - \tilde{\nabla}_{\eta}(\nabla - \tilde{\nabla})_{\xi}Z + \\ & - (\nabla - \tilde{\nabla})_{\eta}\nabla_{\xi}Z - (\nabla - \tilde{\nabla})_{[\xi, \eta]}Z \end{aligned}$$

Observe that the same hypothesis on the underlying frame forms implies that ∇ and $\tilde{\nabla}$ coincide on $\mathfrak{X}^{-k+1} \times \Gamma(E)$. Then, all terms in the equation above vanish except the last one, which rewrites as $-A_0^k(q_{-k}([\xi, \eta]))$ by (12). This coincides with $\partial A^k(X, Y)$ for the considered values of i, j , as it can be seen by comparing with the explicit formula (17). Indeed, the terms Φ_{i+k}^k and Φ_{j+k}^k showing up there, by definition, are zero for $i + k = -j > 0$ and $j + k = -i > 0$. \square

The results collected in this Section can be now applied at each step of an inductive procedure, leading to the following

THEOREM 3. *Let \mathfrak{n} be a nilpotent graded Lie algebra of depth μ generated by \mathfrak{g}_{-1} . Moreover, let $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ be a closed Lie subgroup with Lie algebra \mathfrak{g}_0 and let $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ be a G_0 -triple. Suppose that $\mathfrak{N} \subset L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$ is a normalization condition and that the first prolongation of the pair $(\mathfrak{n}, \mathfrak{g}_0)$ is trivial. Then, there exists a unique normal frame form of length μ for \mathcal{T} and this defines a canonical linear connection on TM preserving H .*

PROOF. Fix an integer $k \geq 1$ and put $V_k = L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_k$, so that $\mathcal{F} \times_{G_0} V_k \cong \text{End}(E, E_{\leq 0})_k$. Consider an arbitrary frame form $\tilde{\theta}$ of length k for \mathcal{T} and write its curvature \tilde{C}^k in homogeneity k as the sum

$$\tilde{C}^k = \tilde{C}_{\text{Im}(\partial_k)} + \tilde{C}_{\mathfrak{N}_k}$$

of $\tilde{C}_{\text{Im}(\partial_k)} \in \Gamma(\mathcal{F} \times_{G_0} \text{Im}(\partial_k))$ and $\tilde{C}_{\mathfrak{N}_k} \in \Gamma(\mathcal{F} \times_{G_0} \mathfrak{N}_k)$. By hypothesis ∂_k is injective, thus one can look at its left inverse

$$\partial_k^{-1} : \Gamma(\mathcal{F} \times_{G_0} \text{Im}(\partial_k)) \rightarrow \Gamma(\mathcal{F} \times_{G_0} V_k)$$

Now put $P := -\partial_k^{-1} \circ \tilde{C}_{\text{Im}(\partial_k)} \in \Gamma(\mathcal{F} \times_{G_0} V_k) \cong \Gamma(\text{End}(E, E_{\leq 0})_k)$. Then, by Proposition 8 (a), $\theta = \tilde{\theta} + P$ is a frame form of length k with same underlying frame form of length $k - 1$. By Proposition 10, the curvature in homogeneity k of the modified frame form θ is

$$C^k = \tilde{C}_{\mathfrak{N}_k}^k \in \Gamma(\mathcal{F} \times_{G_0} \mathfrak{N}_k),$$

hence θ is normal. Moreover, the normal frame form θ of length k is uniquely determined by C^k . Indeed, suppose that both $\theta, \tilde{\theta}$ are normal frame forms of length k with same underlying frame form of length $k - 1$. Then, denote by $C^k = C_{\mathfrak{N}_k}^k$ and by $\tilde{C}^k = \tilde{C}_{\mathfrak{N}_k}$ the respective curvatures, both lying in $\Gamma(\mathcal{F} \times_{G_0} \mathfrak{N}_k)$. By Proposition 8 (a), there exists $P \in \Gamma(\mathcal{F} \times_{G_0} V_k)$ such that $\theta = \tilde{\theta} + P$ and again by Proposition 10, we have $C^k - \tilde{C}^k = \partial_k P$. The right and left terms of the last equation are sections of two complementary subbundles of $\text{End}(\Lambda^2 E, E_{\leq 0})_k$, therefore the equation is satisfied if and only if $P = 0$ and $C^k = \tilde{C}^k$.

The argument above can be applied at each step of an inductive procedure. First of all, we consider the set of frame forms for \mathcal{T} of length one. For $k = 1$, the argument shows that there exists a unique normal frame form of length one. For $k \geq 2$, we can suppose that a normal frame form of length $k - 1$ is fixed. Then, the argument above shows that this can be uniquely prolonged to a frame form of length k , by normalizing the curvature in homogeneity k . After a finite number of steps, we obtain a unique normal frame form of length μ . This determines, by Proposition 8 (b), a connection on TM preserving H . \square

Consider the G_0 -reduction $\mathcal{F} \rightarrow M$. If $\gamma \in \Omega^1(\mathcal{F}, \mathfrak{g}_0)$ denotes a connection form and $\alpha \in \Omega^1(\mathcal{F}, \mathfrak{n})$ the soldering form, then $\omega = \alpha + \gamma \in \Omega^1(\mathcal{F}, \mathfrak{n} \oplus \mathfrak{g}_0)$ is a Cartan connection for $\mathcal{F} \rightarrow M$. The formula $d\omega + [\omega, \omega]$ defines a two-form on \mathcal{F} with values in $\mathfrak{g}_0 \oplus \mathfrak{n}$, which is said to be the *curvature of the Cartan connection* ω . Let θ be a frame form of length k for the G_0 -triple and let C_θ^k be its curvature in homogeneity k . Through the associated bundle construction, C_θ^k corresponds to the curvature in homogeneity k of a Cartan connection ω of the form above. This explains why we also refer to C_θ^k as the curvature in homogeneity k , denomination which seems to be misleading at a first glance, since C_θ^k is the sum of two tensors defined in Proposition 9 as torsion and curvature of degree k . However, there is a deeper reason to adopt this denomination which concerns a difference between the normalization procedure for classical G -structures and the normalization procedure applied in Theorem 3. Connection forms on classical G -structures induce linear connections on TM . To any such linear connection, one can associate a torsion and a curvature. The curvature does not depend tensorially on the change of linear connection and one always normalizes the torsion. Then, the remaining torsion and the curvature of the canonical linear connection plays the role of a fundamental invariant. In the filtered setting, the distinction between torsion and curvature is much less important than the distinction between homogeneities. There are examples, like CR -structures, in which one can choose between normalizing torsion only or normalizing a mix of torsion and curvature. The same happens with dual Darboux distributions, as we will see in the next Section. However, in general, there are no normalization conditions which concern only torsion and leave the curvature completely free.

By looking at the torsion and curvature of the canonical linear connection, one can determine the isomorphism class of the structures satisfying the hypothesis of Theorem 3. In particular, let $\mathcal{T} = (\mathcal{F} \rightarrow (M, H), \mathfrak{n})$ be a G_0 -triple satisfying the hypothesis of Theorem 3 and let ∇ be the corresponding canonical linear connection. Let $T^\nabla \in \Omega^2(TM)$ be the torsion and let $R^\nabla \in \Omega^2(T^*M \otimes TM)$ be the curvature of ∇ . Then, T^∇ and R^∇ are both identically zero if and only if \mathcal{T} is isomorphic to the homogeneous model of same type $(\mathfrak{n}, \mathfrak{g}_0)$ (Proposition 1.5.2, [10]).

EXAMPLE 6. We conclude by discussing a nice application of last result, namely the existence of canonical subriemannian linear connections associated to structures of generic subriemannian type, which was proved by Morimoto in [15]. Suppose that (\mathfrak{n}, σ) is a subriemannian graded Lie algebra (see Example 5). Denote by $G_0 \subset \text{Aut}_{gr}(\mathfrak{n})$ the closed subgroup of those grading-preserving automorphisms of \mathfrak{n} which restrict to an isometry of $(\mathfrak{g}_{-1}, \sigma)$ and by $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{n})$ the Lie algebra of G_0 . Every map in \mathfrak{g}_0 restricts to an endomorphism of the vector space \mathfrak{g}_{-1} that is skew-symmetric with respect to σ . Morimoto considers subriemannian manifolds (M, H, S) of regular subriemannian type (\mathfrak{n}, σ) , hence G_0 -triples $(\mathcal{F} \rightarrow M, \mathfrak{n})$ and the set of connection forms on $\mathcal{F} \rightarrow M$. The associated vector bundle $\mathcal{F} \times_{G_0} \mathfrak{g}_0$ is isomorphic to the subalgebra E_0 of grading-preserving endomorphisms $F: \text{gr}(TM) \rightarrow \text{gr}(TM)$ whose components on $\text{gr}_{-1}(TM)$ are skew-symmetric with respect to S . We say that ∇ is a *subriemannian* linear connection on TM if $\nabla_\zeta(\Gamma(H)) \subset \Gamma(H)$ for all $\zeta \in \mathfrak{X}(M)$ and satisfies

$$\zeta \cdot S(\xi, \eta) = S(\nabla_\zeta \xi, \eta) + S(\xi, \nabla_\zeta \eta)$$

for all $\xi, \eta \in \Gamma(H)$ and $\zeta \in \mathfrak{X}(M)$. Any connection form on $\mathcal{F} \rightarrow M$ induces a subriemannian linear connection on $\text{gr}(TM)$ and this, together with an isomorphism $\text{gr}(TM) \cong TM$, induces a subriemannian linear connection. If $\nabla, \tilde{\nabla}$ are both subriemannian linear connections, then they both satisfy the formula above, whose left-hand term does not depend on any choice. Then,

$$S((\nabla - \tilde{\nabla})_\zeta \xi, \eta) = -S(\xi, (\nabla - \tilde{\nabla})_\zeta \eta)$$

hence $(\nabla - \tilde{\nabla})_\zeta \in \Gamma(E_0)$ for all $\zeta \in \mathfrak{X}(M)$.

Observe that σ induces inner products on $\Lambda^2 \mathfrak{g}_{-1}$. Moreover, \mathfrak{g}_i identifies with a quotient of

$\mathfrak{g}_{-1} \otimes \mathfrak{g}_{i+1}$ through the bracket. Therefore, \mathfrak{g}_{-2} carries an induced inner product, which in turn induces an inner product on $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2}$, hence on \mathfrak{g}_{-3} and so on. Also $\mathfrak{g}_0 \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ inherits an inner product and they all are \mathfrak{g}_0 -invariant. We thus have a \mathfrak{g}_0 -invariant inner product on $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$. Taking the orthogonal complement $\mathfrak{N} := \text{Im}(\partial)^\perp$, then, defines a normalization condition. Now [15, Prop. 1] proves that the first prolongation of $(\mathfrak{n}, \mathfrak{g}_0)$ vanishes. The existence of a canonical subriemannian linear connection on TM , then, follows from Theorem 3.

5. Dual Darboux distributions

In Chapter 1, we focused on generic types of distributions of rank t in dimension n , which correspond to open orbits of bidimension (t, n) . A classification of such open orbits appears in [2] and contains the class of so-called dual Darboux bidimensions, namely pairs of the form $(t, n(t))$ with $n(t) = t + \binom{t}{2} - 1$ for an integer $t \geq 3$. There exists a unique open orbit of dual Darboux bidimension $(t, n(t))$, generated by linear maps whose kernel is a maximally nondegenerate line in $\Lambda^2 \mathbb{R}^t$. For $t = 2k$ even, the nondegeneracy condition suggests an explicit description of the model algebra. We will refer to it as the dual Darboux algebra \mathfrak{D}_{2k} of dimension $n(2k)$. Observe that the lowest dimensional case \mathfrak{D}_4 corresponds to the generic $(4, 9)$ -type, appearing in the classification of Theorem 2. With analogous terminology, we will refer to one-step bracket-generating distributions of type \mathfrak{D}_{2k} as dual Darboux distributions of even rank.

In the rest of the Chapter, we prove that a canonical linear connection is associated to any dual Darboux distribution of even rank. A simple argument shows that $\mathfrak{g}_0 = \text{der}_{\text{gr}}(\mathfrak{D}_{2k})$ is a reductive Lie algebra. The existence of a normalization condition, in particular, follows from the characterization of finite-dimensional completely reducible representations of reductive Lie algebras. Then, we compute the first prolongation \mathfrak{g}_1 of \mathfrak{D}_{2k} . On the one hand, our computations show that \mathfrak{g}_1 is trivial. On the other hand, from the same computations one can explicitly derive a normalization condition in homogeneity one. In homogeneity two, where also curvature components show up, there are two nice choices of a normalization condition, the first on the torsion and the second one on the curvature. Once a choice is made it encodes, together with the normalization condition in homogeneity one, the admissible values for torsion and curvature of the canonical linear connection from Theorem 3.

Let V be a real vector space and $\omega : V \times V \rightarrow \mathbb{R}$ a skew-symmetric bilinear form. By linear algebra, ω may be nondegenerate only if V has even dimension. If ω is nondegenerate, the pair (V, ω) is said to be a *symplectic vector space*. The automorphisms of V which preserve ω up to scale form the *conformal symplectic group* $CSp(V)$. We will denote by $\text{csp}(V)$ the corresponding Lie algebra. For an integer $k \geq 2$, put $n = 2k + \binom{2k}{2} - 1$. Consider a symplectic vector space (V, ω) of dimension $2k$. The kernel of ω , seen as a linear map $\Lambda^2 V \rightarrow \mathbb{R}$, is a linear subspace $\Lambda_0^2 V \subset \Lambda^2 V$ of codimension one. Since ω is nondegenerate, it uniquely defines an isomorphism $V \rightarrow V^*$ by mapping $v \mapsto \omega(v, -)$. Similarly, the inverse isomorphism $V^* \rightarrow V$ corresponds to a symplectic form on V^* , say $\bar{\omega} \in \Lambda^2 V^*$. Since $\bar{\omega}$ satisfies $\omega(\bar{\omega}) = \nu \neq 0$, it generates a line $\mathbb{R} \cdot \bar{\omega} \subset \Lambda^2 V^*$ such that $\Lambda^2 V^* = \Lambda_0^2 V^* \oplus \mathbb{R} \cdot \bar{\omega}$ is a direct sum of $CSp(V)$ -modules. The projection $\Pi : \Lambda^2 V^* \rightarrow \Lambda_0^2 V^*$ onto the first factor writes, explicitly, as

$$(19) \quad \Pi(v \wedge w) = v \wedge w - \frac{1}{\nu} \omega(v \wedge w) \bar{\omega} \quad \text{for all } v, w \in V.$$

Clearly, $\Lambda_0^2 V^* \cong \Lambda^2 V^* / \mathbb{R} \cdot \bar{\omega}$ and the kernel of Π coincides with $\mathbb{R} \cdot \bar{\omega}$, which is a nondegenerate line in $\Lambda^2 V^*$. Therefore, Π is a surjective linear map of generic dual Darboux type $(2k, n)$. Put $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_{-2} = \Lambda_0^2 V^*$ and $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$. Then $[v, w] := \Pi(v \wedge w)$ defines a bracket $[\cdot, \cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$, which endows \mathfrak{n} with a graded nilpotent Lie algebra structure. We will refer to $(\mathfrak{n}, [\cdot, \cdot])$ as the *dual Darboux algebra* \mathfrak{D}_{2k} of dimension n . The standard action of

$CSp(V)$ on V induces an action on $\Lambda^2 V$ which preserves $\Lambda_0^2 V$. More precisely, $CSp(V)$ could be equivalently defined as the subgroup of $GL(V)$ whose induced action on $\Lambda^2 V$ preserves the decomposition $\Lambda^2 V = \Lambda_0^2 V \oplus \mathbb{R} \cdot \bar{\omega}$. Adopting this definition, $CSp(V)$ is easily seen to be the invariance group of the linear map Π . We thus have that $\text{Aut}_{\text{gr}}(\mathfrak{n}) = CSp(V)$, hence that $\mathfrak{der}_{\text{gr}}(\mathfrak{n}) = \mathfrak{csp}(V)$. Since \mathfrak{D}_{2k} generates an open orbit of dual Darboux bidimension $(2k, n)$ and any such orbit can be realized as type of some $(2k, n)$ -distribution, the existence of generic distributions of type \mathfrak{D}_{2k} is assured. We will refer to a $(2k, n)$ -distribution $H \subset TM$ of type \mathfrak{D}_{2k} as a *dual Darboux distribution of rank $2k$* on M .

Let $\mathfrak{D}_{2k} = (\mathfrak{n}, \Pi)$ be the dual Darboux algebra. Denote by $\mathfrak{g}_0 = \mathfrak{der}_{\text{gr}}(\mathfrak{D}_{2k}) = \mathfrak{csp}(V)$ the algebra of grading-preserving derivations and put $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$. In this section, we shall compute explicitly the differential map $\partial : L(\mathfrak{n}, \mathfrak{g}_{\leq 0}) \rightarrow L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})$ defined by the formula (10). The computations will involve some notions of tensorial calculus that we are now going to introduce. Given a finite-dimensional vector space V with dual vector space V^* , one can consider tensor products of the form $\mathcal{T}_t^s = V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$, containing s copies of V and t copies of V^* . An element of \mathcal{T}_t^s can be thought as a map $V^t \times (V^*)^s \rightarrow \mathbb{R}$ which is linear in each argument (a so-called $(s+t)$ -linear map), and it is said to be a tensor of type $\binom{s}{t}$ (shortly, a $\binom{s}{t}$ -tensor). In particular $\mathcal{T}_1^0 = V^*$, $\mathcal{T}_0^1 = V$ and we put $\mathcal{T}_0^0 = \mathbb{R}$. There are two basic operations on tensors, the tensor product and the contraction. Given a $\binom{s}{t}$ -tensor h and a $\binom{s'}{t'}$ -tensor f , one can define a tensor $h \otimes f$ of type $\binom{s+s'}{t+t'}$. Inserting the first arguments in h , the second arguments in f and then multiplying the resulting values in \mathbb{R} defines a $(s+s'+t+t')$ -linear map, which is said to be the tensor product of h and f and denoted by $h \otimes f$. Furthermore, there is a natural evaluation map $V \otimes V^* \rightarrow \mathbb{R}$, sending $v \otimes \phi$ to $\phi(v)$. This can be extended to a map $\mathcal{T}_t^s \rightarrow \mathcal{T}_{t-1}^{s-1}$ for any choice of s, t both nonzero, usually called contraction, by evaluating a copy of V^* in a copy of V and leaving all other entries untouched.

In abstract index notation, the elements of \mathcal{T}_t^s are denoted by $h^{i_1 \dots i_s}_{j_1 \dots j_t}$, with s upper indices and t lower indices used to indicate their type and their contractions. Hence, we write v^i for a vector, ϕ_j for a linear functional, A_i^j for a $\binom{1}{1}$ -tensor and A_i^i for its contraction. There are isomorphisms $V^* \otimes V \rightarrow L(V, V)$ and $V^* \otimes V \rightarrow L(V^*, V^*)$, explicitly given by $A(v)^j = A_i^j v^i$ for all $v^i \in V$ and $A(\phi)_i = A_i^j \phi_j$ for all $\phi_j \in V^*$. The identity map (on V or on V^*) defines then a canonical element in $V^* \otimes V$, which is usually denoted by δ_i^j . We thus have the identities $\delta_i^j v^i = v^j$ and $\delta_i^j \phi_j = \phi_i$, which will often occur in our computations.

Symmetrization and alternating maps are also fundamental ingredients of tensorial calculus that we will apply. Recall that $S^2 V^* \subset V^* \otimes V^*$ and $\Lambda^2 V^* \subset V^* \otimes V^*$ denote the subspaces of bilinear forms on V which are symmetric, respectively skew-symmetric. The symmetrization map $V^* \otimes V^* \rightarrow S^2 V^*$ associates to any $\binom{0}{2}$ -tensor f the tensor h of same type, defined by $h(v, w) = \frac{1}{2}(f(v, w) + f(w, v))$ for all $v, w \in V$. The alternating map $V^* \otimes V^* \rightarrow \Lambda^2 V^*$ is analogously defined, by subtracting the second summand rather than adding it. In abstract index notation, the symmetrization and the alternation of f writes, respectively, as $\frac{1}{2}(f_{ij} + f_{ji})$ and $\frac{1}{2}(f_{ij} - f_{ji})$. In general, one can symmetrize or alternate over more than two entries of the same type. Denote by $\otimes^t V^*$ the tensor product of t copies of V^* . Let \mathfrak{S}_t be the set of permutations of the set $\{1, \dots, t\}$. A transposition is a permutation which exchanges two elements of $\{1, \dots, t\}$ and fixes all other elements. Any $\sigma \in \mathfrak{S}_t$ can be written as the composition of a finite number ℓ of transpositions. Although, in general, such an expression for σ is not unique, the parity of ℓ is uniquely determined. Otherwise put, σ writes as product of an even number or an odd number of transpositions. For $\sigma \in \mathfrak{S}_t$, we put

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \ell \text{ even} \\ -1 & \text{if } \ell \text{ odd} \end{cases}$$

The symmetrization and skew-symmetrization operators are defined by

$$\begin{aligned} \text{Sym}: \otimes^t V^* &\rightarrow \otimes^t V^*, \\ \text{Sym}(h)_{j_1 \dots j_t} &= \frac{1}{t!} \sum_{\sigma \in \mathfrak{S}_t} h_{j_{\sigma(1)} \dots j_{\sigma(t)}} \end{aligned}$$

and

$$\begin{aligned} \text{Alt}: \otimes^t V^* &\rightarrow \otimes^t V^*, \\ \text{Alt}(h)_{j_1 \dots j_t} &= \frac{1}{t!} \sum_{\sigma \in \mathfrak{S}_t} (-1)^\sigma h_{j_{\sigma(1)} \dots j_{\sigma(t)}} \end{aligned}$$

We define $S^t V^* \subset \otimes^t V^*$ and $\Lambda^t V^* \subset \otimes^t V^*$ as the image of Sym and Alt, respectively. We will also briefly write $h_{(a_1 \dots a_k)} = \text{Sym}(h)_{a_1 \dots a_k}$ and $h_{[a_1 \dots a_k]} = \text{Alt}(h)_{a_1 \dots a_k}$. Finally, consider $\bar{\omega}^{ij} \in \Lambda^2 V$ and $\phi^{ij} \in S^2 V$. The following formulas

$$\begin{aligned} (f_{ij} + f_{ji})\phi^{ij} &= (f_{ij}\phi^{ij} + f_{ji}\phi^{ji}) = 2f_{ij}\phi^{ij}, & (f_{ij} - f_{ji})\phi^{ij} &= (f_{ij}\phi^{ij} - f_{ji}\phi^{ji}) = 0 \\ (f_{ij} - f_{ji})\bar{\omega}^{ij} &= (f_{ij}\bar{\omega}^{ij} - f_{ji}\bar{\omega}^{ji}) = 2f_{ij}\bar{\omega}^{ij}, & (f_{ij} + f_{ji})\bar{\omega}^{ij} &= (f_{ij}\bar{\omega}^{ij} + f_{ji}\bar{\omega}^{ji}) = 0 \end{aligned}$$

describe the behaviour of symmetric and skew-symmetric objects under contraction.

Let $k \geq 2$ be an integer and let (V, ω) be a symplectic vector space of dimension $2k$. Recall that if $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is the vector space underlying the dual Darboux algebra and $\mathfrak{g}_0 = \mathfrak{der}_{\text{gr}}(\mathfrak{n})$, by definition $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 = V^* \otimes \mathfrak{csp}(V)$. Since $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \subset L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1$, the tensor product $V^* \otimes \mathfrak{csp}(V)$ will show up in the explicit computation of the first prolongation of the dual Darboux algebra. Denote by $\omega_{ij} \in \Lambda^2 V^*$ the symplectic form. If $\bar{\omega}^{ps} \in \Lambda^2 V$ is the inverse of ω_{ij} , we have $\omega_{ij} \bar{\omega}^{js} = \delta_i^s$. Up to the isomorphism $V^* \otimes V \cong L(V, V)$, we can write the elements of $\mathfrak{csp}(V)$ as ϕ_j^i satisfying $\phi_j^i \omega_{i\ell} + \omega_{ji} \phi_\ell^i = \lambda \omega_{j\ell}$ for some $\lambda \in \mathbb{R}$. In particular, the symplectic form gives a canonical \mathfrak{g}_0 -isomorphism $\mathfrak{sp}(V) \cong S^2 V^*$. Explicitely, $\phi \in \mathfrak{sp}(V)$ uniquely defines a symmetric bilinear form

$$(20) \quad \begin{aligned} f_\phi: V \times V &\rightarrow \mathbb{R} \\ f_\phi(X, Y) &= \omega(X, \phi(Y)) \quad \text{for } X, Y \in V. \end{aligned}$$

which writes as $\phi_j^i \omega_{i\ell} = \phi_\ell^i \omega_{ij}$. In general, for $\phi_j^i \in \mathfrak{csp}(V)$ we have

$$\lambda = \frac{1}{2k} \lambda \omega_{j\ell} \bar{\omega}^{\ell j} = \frac{1}{2k} (\phi_j^i \delta_i^j + \delta_i^\ell \phi_\ell^i) = \frac{1}{k} \phi_i^i$$

Therefore, the contraction $\phi_j^i \mapsto \phi_i^i$ defines a surjective $\mathfrak{csp}(V)$ -homomorphism $\mathfrak{csp}(V) \rightarrow \mathbb{R}$ with kernel $\mathfrak{sp}(V) \subset \mathfrak{csp}(V)$.

Denote by $\mu: V^* \otimes V^* \otimes V \rightarrow \Lambda^2 V^* \otimes V$ the alternating map in the first two components, explicitly given by $(\mu A)_{ab}^c = A_{[ab]}^c = \frac{1}{2}(A_{ab}^c - A_{ba}^c)$. In particular, we can restrict μ to $V^* \otimes \mathfrak{csp}(V)$. The condition for A_{ab}^c to be in $V^* \otimes \mathfrak{csp}(V)$ writes as

$$(21) \quad A_{ab}^i \omega_{ic} - A_{ac}^i \omega_{ib} = \frac{1}{k} A_{aj}^j \omega_{bc}$$

and multiplying both terms of the last equation by $\bar{\omega}^{ab}$, we obtain

$$(22) \quad A_{ab}^i \bar{\omega}^{ab} \omega_{ic} = A_{ac}^i \omega_{ib} \bar{\omega}^{ab} + \frac{1}{k} A_{aj}^j \omega_{bc} \bar{\omega}^{ab} = -A_{ic}^i + \frac{1}{k} A_{ci}^i.$$

Since $\bar{\omega}^{ab}$ is skew-symmetric, the left-hand term coincides with $A_{[ab]}^t \bar{\omega}^{ab} \omega_{tc}$. Then, last equation expresses the following fact. Up to the isomorphism $V \cong V^*$ given by the symplectic form, inserting $\bar{\omega}$ in μA is a linear combination of the two contractions one can apply to an

element $A \in V^* \otimes \mathfrak{csp}(V)$.

LEMMA 1. *Let $k \geq 2$ be an integer, (V, ω) a symplectic vector space of dimension $2k$ and let $\mathfrak{csp}(V)$ be the corresponding conformal symplectic algebra. There exists a direct sum decomposition*

$$V^* \otimes \mathfrak{csp}(V) = \bigoplus_{i=1}^4 \mathbb{V}_i$$

into $\mathfrak{csp}(V)$ -irreducible components. Writing $A_{ab}^j \in V^* \otimes \mathfrak{csp}(V)$ and putting $C_{abc} = A_{ab}^j \omega_{jc}$, the irreducible components are explicitly described as follows:

- $A_{ab}^j \in \mathbb{V}_1$ if and only if $A_{ab}^j = \frac{1}{2k} \mu_a \delta_b^j$ for $\mu_a \in V^*$;
- $A_{ab}^j \in \mathbb{V}_2$ if and only if $C_{(abc)} = C_{abc}$;
- $A_{ab}^j \in \mathbb{V}_3$ if and only if $A_{ab}^j = \frac{1}{2k+1} (\mu_c \omega_{ba} \bar{\omega}^{cj} - \mu_b \delta_a^j)$ for $\mu_s \in V^*$;
- $A_{ab}^j \in \mathbb{V}_4$ if and only if $C_{abc} = C_{a(bc)}$, $C_{abc} \bar{\omega}^{ab} = 0$ and $C_{(abc)} = 0$.

PROOF. Put $\mathfrak{g}_0 = \mathfrak{csp}(V)$. Define

$$\mathbb{V}_1 := \left\{ A_{ab}^j = \frac{1}{2k} \mu_a \delta_b^j \mid \mu_a \in V^* \right\} \subset V^* \otimes \mathfrak{csp}(V)$$

Take $\mu_s \in V^*$ and observe that $A_{ab}^j = \frac{1}{2k} \mu_a \delta_b^j$ defines an element of $V^* \otimes \mathfrak{csp}(V)$ such that $A_{ai}^i = \mu_a$. The contraction $A_{ab}^i \mapsto A_{ai}^i$ is a surjective \mathfrak{g}_0 -homomorphism $V^* \otimes \mathfrak{csp}(V) \rightarrow V^*$, thus it gives a \mathfrak{g}_0 -invariant decomposition

$$V^* \otimes \mathfrak{csp}(V) = \mathbb{V}_1 \oplus V^* \otimes \mathfrak{sp}(V), \quad \mathbb{V}_1 \cong V^*.$$

Then, recall that $A_{ab}^j \mapsto C_{abc} = A_{ab}^j \omega_{jc}$ defines an isomorphism $V^* \otimes \mathfrak{sp}(V) \cong V^* \otimes S^2 V^*$. Denote by $\text{Sym}: V^* \otimes S^2 V^* \rightarrow S^3 V^*$ the restriction of the symmetrization map. Again, this is a surjective map and gives a \mathfrak{g}_0 -invariant decomposition

$$V^* \otimes S^2 V^* = \mathbb{V}_2 \oplus \text{Ker}(\text{Sym}), \quad \mathbb{V}_2 \cong S^3 V^*.$$

In particular, \mathbb{V}_2 is an irreducible representation of \mathfrak{g}_0 . Now consider the \mathfrak{g}_0 -homomorphism

$$\begin{aligned} i_{\bar{\omega}}: V^* \otimes S^2 V^* &\rightarrow V^* \\ C_{abc} &\mapsto C_{abc} \bar{\omega}^{ab} \end{aligned}$$

Since $\bar{\omega}$ is skew-symmetric, $i_{\bar{\omega}}$ vanishes on \mathbb{V}_2 and descends to a map $\text{Ker}(\text{Sym}) \rightarrow V^*$. Put

$$\mathbb{V}_3 := \left\{ A_{ab}^j = \frac{1}{2k+1} (\mu_t \omega_{ba} \bar{\omega}^{tj} - \mu_b \delta_a^j) \mid \mu_s \in V^* \right\} \subset V^* \otimes \mathfrak{csp}(V)$$

Let be $A_{ab}^j \in \mathbb{V}_3$. Then, $C_{abc} = A_{ab}^j \omega_{jc} = \mu_c \omega_{ba} + \mu_b \omega_{ca}$ is symmetric in $\{b, c\}$, so that $3C_{(abc)} = C_{abc} + C_{bca} + C_{cab}$. One can easily verify that $C_{abc} + C_{cab} + C_{bca} = 0$, so that $C_{abc} \in \text{Ker}(\text{Sym})$. Moreover, $C_{abc} \bar{\omega}^{ab} = \mu_c$. We thus see that the restriction of $i_{\bar{\omega}}$ to $\text{Ker}(\text{Sym})$ is surjective, hence that

$$\text{Ker}(\text{Sym}) = \mathbb{V}_3 \oplus (\text{Ker}(\text{Sym}) \cap \text{Ker}(i_{\bar{\omega}})), \quad \mathbb{V}_3 \cong V^*.$$

Finally, $\mathbb{V}_4 := \text{Ker}(\text{Sym}) \cap \text{Ker}(i_{\bar{\omega}})$ can be seen to be irreducible. More precisely, if λ_i denotes the i -th fundamental weight of $\mathfrak{sp}(V)$ and $\lambda = \lambda_1 + \lambda_2$, then \mathbb{V}_4 is the irreducible representation of highest weight λ . \square

LEMMA 2. *The map*

$$\Lambda^2(\Lambda_0^2 V^*) \longrightarrow S^2 V^*, \quad F_{bcde} \mapsto F_{bcde} \bar{\omega}^{cd}$$

is a surjective \mathfrak{g}_0 -homomorphism. Moreover,

- (1) $F_{bc}^{ij} \omega_{id} \omega_{ej} \in \Lambda_0^2 V^* \otimes \Lambda_0^2 V^*$ for any $F_{bc}^{ij} \in \Lambda_0^2 V^* \otimes \Lambda_0^2 V^*$;
(2) $(F_{bc}^{ij} \omega_{id} \omega_{ej} - F_{de}^{ij} \omega_{ib} \omega_{cj}) \bar{\omega}^{cd} = 2P_{(be)}$, where $P_{be} = F_{bi}^{ij} \omega_{je}$ and $P_{(be)}$ is the symmetrization of P_{be} .

PROOF. An element $F_{bcde} \in \Lambda_0^2 V^* \otimes \Lambda_0^2 V^*$ satisfies

$$(23) \quad \begin{cases} F_{cbde} = -F_{bcde} = F_{bced} \\ F_{bcde} \bar{\omega}^{bc} = 0 \\ F_{bcde} \bar{\omega}^{de} = 0 \end{cases}$$

and lies in $\Lambda^2(\Lambda_0^2 V^*)$ if it satisfies the additional symmetry property $F_{debc} = -F_{bcde}$. If this is the case, then

$$F_{ecdb} \bar{\omega}^{cd} = -F_{dbec} \bar{\omega}^{cd} = -F_{bdce} \bar{\omega}^{cd} = F_{bdce} \bar{\omega}^{dc}$$

holds, showing that $F_{bcde} \bar{\omega}^{cd}$ is a symmetric $\binom{0}{2}$ -tensor for any $F_{bcde} \in \Lambda^2(\Lambda_0^2 V^*)$. To see that $F_{bcde} \mapsto F_{bcde} \bar{\omega}^{cd}$ defines a \mathfrak{g}_0 -homomorphism, it is enough to prove that

$$V^* \otimes V^* \rightarrow \mathbb{R}, \quad \lambda_{cd} \mapsto \lambda_{cd} \bar{\omega}^{cd}$$

is a \mathfrak{g}_0 -homomorphism. But this is clear, since it is the dual map of $\omega: V \otimes V \rightarrow \mathbb{R}$ and \mathfrak{g}_0 is defined as the invariance algebra of ω . For $\phi_{ij} = \phi_{(ij)} \in S^2 V^*$, put

$$F_{bcde} = \phi_{bd} \omega_{ce} - \phi_{cd} \omega_{be} - \phi_{be} \omega_{cd} + \phi_{ce} \omega_{bd}.$$

One easily verifies that F_{bcde} satisfies the conditions characterizing the elements of $\Lambda^2(\Lambda_0^2 V^*)$. Moreover,

$$F_{bcde} \bar{\omega}^{cd} = -\phi_{bd} \delta_e^d - \phi_{cd} \bar{\omega}^{cd} \omega_{be} + 2k \phi_{be} - \phi_{ce} \delta_b^c = (2k - 2) \phi_{be}$$

shows that $F_{bcde} \mapsto F_{bcde} \bar{\omega}^{cd}$ defines a surjective homomorphism $\Lambda^2(\Lambda_0^2 V^*) \rightarrow S^2 V^*$.

Now $F_{bc}^{ij} \in \Lambda_0^2 V^* \otimes \Lambda_0^2 V^*$ satisfies the analogous conditions as in the first and second rows of (23) and, in addition, $F_{bc}^{ij} \omega_{ij} = 0$. For such an element, we compute

$$F_{bc}^{ij} \omega_{id} \omega_{ej} \bar{\omega}^{de} = F_{bc}^{ij} \omega_{id} \delta_j^d = F_{bc}^{ij} \omega_{ij} = 0$$

which is enough to conclude that (1) holds. Finally,

$$\begin{aligned} (F_{bc}^{ij} \omega_{id} \omega_{ej} - F_{de}^{ij} \omega_{ib} \omega_{cj}) \bar{\omega}^{cd} &= F_{bc}^{ij} \omega_{ej} (-\delta_i^c) - F_{de}^{ij} \omega_{ib} (-\delta_j^d) \\ &= -F_{bi}^{ij} \omega_{ej} + F_{je}^{ij} \omega_{ib} \\ &= F_{bi}^{ij} \omega_{je} + F_{ej}^{ji} \omega_{ib} \end{aligned}$$

proves (2). \square

Let $\mathfrak{D}_{2k} = (\mathfrak{n}, \Pi)$ be the dual Darboux algebra, put $\mathfrak{g}_0 = \mathfrak{csp}(V)$ and $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$. The differential map ∂ defined by the formula (10) is an homomorphism of graded \mathfrak{g}_0 -modules. Each component in homogeneity i of ∂ is a \mathfrak{g}_0 -equivariant linear map, denoted by ∂_i . Since \mathfrak{n} is of depth two, we have that $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_\ell = \{0\}$ for all $\ell \geq 3$, hence that the positive graded part of $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})$ reduces to homogeneity one and two and $\partial = \partial_1 + \partial_2$. According to Definition 11, a normalization condition for the pair $(\mathfrak{D}_{2k}, \mathfrak{g}_0)$ is a graded \mathfrak{g}_0 -module $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ such that $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_i = \text{Im}(\partial_i) \oplus \mathfrak{N}_i$ is the direct sum of \mathfrak{g}_0 -modules for $i = 1, 2$. Observe that \mathfrak{g}_0 is reductive and its center acts diagonalizable on both domain and codomain of ∂_i , which are thus completely reducible representations of \mathfrak{g}_0 . This assures the existence of a normalization condition for the dual Darboux algebra which, as we are going to show, can be made explicit. In the next Proposition, we analyze ∂_1 .

PROPOSITION 11. Let $k \geq 2$ be an integer and let (V, ω_{ij}) be a symplectic vector space of dimension $2k$ defining the dual Darboux algebra $\mathfrak{D}_{2k} = (\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}, \Pi)$. Put $\mathfrak{g}_0 = \mathfrak{csp}(V)$ and $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$. The first prolongation of \mathfrak{D}_{2k} is trivial. Moreover, there exists a normalization condition $\mathfrak{N}_1 \subset L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_1$ in homogeneity one. Writing $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_1$ as

$$\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2} = (\Lambda^2 V^* \otimes V) \oplus (V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V)$$

and denoting by $(T_{ij}^t, \tau_{abc}^{de})$ with $T_{ij}^t \in \Lambda^2 V^* \otimes V$ and $\tau_{abc}^{de} \in V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V$ its elements, the following conditions characterize \mathfrak{N}_1 :

- $T_{ij}^t = 0$;
- $\tau_{abs}^{st} \omega_{tc} = \tau_{acs}^{st} \omega_{tb}$;
- $\tau_{abc}^{st} \omega_{sd} \omega_{et} = \tau_{ade}^{st} \omega_{sb} \omega_{ct}$.

PROOF. First, let us analyze the domain and the codomain of the differential map in homogeneity one $\partial_1 : L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1 \rightarrow L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_1$. As we saw in the proof of Lemma 1, the contraction $A_{ab}^i \mapsto A_{ai}^i$ gives a \mathfrak{g}_0 -invariant direct sum decomposition $V^* \otimes \mathfrak{csp}(V) = \mathbb{V}_1 \oplus (V^* \otimes \mathfrak{sp}(V))$ with $\mathbb{V}_1 \cong V^*$. Moreover, recall that the symplectic form determines a \mathfrak{g}_0 -isomorphism $\mathfrak{sp}(V) \cong S^2 V^*$ as in (20). Using these facts, we write the domain of ∂_1 as direct sum of \mathfrak{g}_0 -modules

$$(24) \quad \begin{aligned} L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1 &= (V^* \otimes \mathfrak{csp}(V)) \oplus (\Lambda_0^2 V^* \otimes V) \\ &\cong V^* \oplus (V^* \otimes S^2 V^*) \oplus (\Lambda_0^2 V^* \otimes V). \end{aligned}$$

By (1) of Lemma 2, the map

$$\begin{aligned} \varepsilon : V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V &\rightarrow V^* \otimes \Lambda^2(\Lambda_0^2 V^*), \\ F_{abc}^{ij} &\mapsto F_{abc}^{ij} \omega_{id} \omega_{ej} - F_{ade}^{ij} \omega_{ib} \omega_{cj} \end{aligned}$$

is well defined. Also consider the map $J : V^* \otimes \Lambda^2(\Lambda_0^2 V^*) \rightarrow V^* \otimes S^2 V^*$, defined by applying the identity on the first component and the map from Lemma 2 on the remaining components. In particular, Lemma 2 shows that $J \circ \varepsilon$ is surjective. We thus get the \mathfrak{g}_0 -invariant direct sum decomposition

$$V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V = \mathbb{U} \oplus \text{Ker}(J \circ \varepsilon), \quad \mathbb{U} \cong (V^* \otimes S^2 V^*)$$

and, in turn, an isomorphism of \mathfrak{g}_0 -modules

$$(25) \quad \begin{aligned} L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_1 &= (\Lambda^2 V^* \otimes V) \oplus (V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V) \\ &\cong V^* \oplus (\Lambda_0^2 V^* \otimes V) \oplus (V^* \otimes S^2 V^*) \oplus \text{Ker}(J \circ \varepsilon) \end{aligned}$$

The subspace $\mathfrak{N}_1 := \text{Ker}(J \circ \varepsilon) \subset \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$ is \mathfrak{g}_0 -invariant, since it is the kernel of a \mathfrak{g}_0 -homomorphism. By comparing formulas (24) and (25), we see that the direct sum of $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1$ and \mathfrak{N}_1 is isomorphic to $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_1$.

Write $\Phi = (A, B) \in L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1$ according to the decomposition $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1 = \bigoplus_{i=1}^2 \mathfrak{g}_{-i}^* \otimes \mathfrak{g}_{-i+1}$. Also write $\partial_1 = E \oplus F$ for

$$\begin{aligned} E : L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1 &\rightarrow \Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} \\ E_{(A,B)}(X \wedge Y) &= \mu A(X \wedge Y) - B \circ \Pi(X \wedge Y), \quad X, Y \in \mathfrak{g}_{-1} = V \end{aligned}$$

and

$$\begin{aligned} F : L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_1 &\rightarrow \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2} \\ F_{(A,B)}(X, \Psi) &= A(X) \cdot \Psi + \Pi(X \wedge B(\Psi)) \\ &= T_A(X, \Psi) + S_B(X, \Psi), \quad X \in \mathfrak{g}_{-1} = V, \Psi \in \mathfrak{g}_{-2} = \Lambda_0^2 V. \end{aligned}$$

Here, \cdot denotes the induced action of $A(X) \in \mathfrak{csp}(V)$ on $\Lambda_0^2 V$ and $\Pi: \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ the bracket (19) on the dual Darboux algebra. Now $(\text{Id} \oplus J \circ \varepsilon) \circ \partial_1 = E \oplus (J \circ \varepsilon \circ F)$ is a \mathfrak{g}_0 -homomorphism

$$\begin{aligned} (V^* \otimes \mathfrak{csp}(V)) \oplus (\Lambda_0^2 V^* \otimes V) &\rightarrow (\Lambda^2 V^* \otimes V) \oplus (V^* \otimes S^2 V^*) \\ (A, B) &\mapsto (E_{(A,B)}, J \circ \varepsilon \circ F_{(A,B)}) \end{aligned}$$

between isomorphic \mathfrak{g}_0 -modules. We now prove that $(\text{Id} \oplus J \circ \varepsilon) \circ \partial_1$ is injective, hence an isomorphism. A pair (A, B) satisfies $E_{(A,B)} = 0$ if and only if $\mu A = B \circ \Pi$. Hence, for such a pair (A, B) , we see that B is completely determined by A which also satisfies $\mu A(\bar{\omega}) = 0$. In abstract index notation, we see from (22) that $\mu A(\bar{\omega}) = 0$ explicitly writes as

$$-A_{ie}{}^i + \frac{1}{k} A_{ei}{}^i = 0.$$

In the second equation $J \circ \varepsilon \circ F_{(A,B)} = 0$, we can insert the identity $B(\Psi) = B \circ \Pi(\Psi) = \mu A(\Psi)$ for all $\Psi \in \Lambda_0^2 V$. We now write $J \circ \varepsilon \circ F_{(A,\mu A)} = 0$ in abstract index notation. First, we analyze separately the two summands $F_{(A,\mu A)} = T_A + S_{\mu A}$. Since the induced action is defined as $A(X) \cdot (v^b \wedge v^c) = A(X)(v^b) \wedge v^c + v^b \wedge A(X)(v^c)$, one has

$$T_A(X, \Psi)^{ij} = X^a \Psi^{bc} \left(\delta_c^j A_{ab}{}^i - \delta_c^i A_{ab}{}^j \right)$$

for all $X = (X^a) \in V$ and $\Psi = (\Psi^{bc}) \in \Lambda_0^2 V$. In the equation above, Ψ^{bc} is contracted with the expression between brackets, which is evidently skew-symmetric in the indices $\{i, j\}$ but not in $\{b, c\}$. Since Ψ^{bc} is skew-symmetric, alternating the content of the brackets in $\{b, c\}$ and contracting it with Ψ^{bc} gives the same result. In this way, we get the formula

$$(26) \quad (T_A)_{abc}{}^{ij} = \frac{1}{2} \left(\delta_c^j A_{ab}{}^i - \delta_c^i A_{ab}{}^j - \delta_b^j A_{ac}{}^i + \delta_b^i A_{ac}{}^j \right)$$

which defines an element of $V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V$. Let us now look at the second summand $S_{\mu A}(X, \Psi) = \Pi(X \wedge \mu A(\Psi))$. With respect to our conventions, the trace-free part (19) of $\phi = (\phi^{ij}) \in \Lambda^2 V$ writes as $\Pi(\phi)^{ij} = \phi^{ij} + \frac{1}{2k} \phi^{ap} \omega_{ap} \bar{\omega}^{ij}$. Applying Π to

$$\phi^{ij} = \left(X \wedge \mu A(\Psi) \right)^{ij} = \frac{1}{2} \left(X^i \mu A(\Psi)^j - X^j \mu A(\Psi)^i \right) = \frac{1}{2} \Psi^{bc} \left(X^i A_{[bc]}{}^j - X^j A_{[bc]}{}^i \right)$$

and taking into account, once more, that $\phi^{[ap]} \omega_{ap} = \phi^{ap} \omega_{ap}$, one gets

$$\left(\Pi(X \wedge \mu A(\Psi)) \right)^{ij} = \frac{1}{2} \Psi^{bc} \left[X^i A_{[bc]}{}^j - X^j A_{[bc]}{}^i + \frac{1}{k} \left(X^a A_{[bc]}{}^p \right) \omega_{ap} \bar{\omega}^{ij} \right]$$

thus

$$(27) \quad (S_{\mu A})_{abc}{}^{ij} = \frac{1}{2} \left(\delta_a^i A_{[bc]}{}^j - \delta_a^j A_{[bc]}{}^i + \frac{1}{k} A_{[bc]}{}^p \omega_{ap} \bar{\omega}^{ij} \right)$$

and we get the formula for $F = F_{(A,\mu A)}$:

$$\begin{aligned} F_{abc}{}^{ij} &= (T_A)_{abc}{}^{ij} + (S_{\mu A})_{abc}{}^{ij} \\ &= \frac{1}{2} \left(\delta_c^j A_{ab}{}^i - \delta_c^i A_{ab}{}^j - \delta_b^j A_{ac}{}^i + \delta_b^i A_{ac}{}^j \right) + \\ &\quad + \frac{1}{2} \left(\delta_a^i A_{[bc]}{}^j - \delta_a^j A_{[bc]}{}^i + \frac{1}{k} A_{[bc]}{}^p \omega_{ap} \bar{\omega}^{ij} \right) \end{aligned}$$

This can be inserted in the explicit formula for $J \circ \varepsilon$ deduced from (2) of Lemma 2, thus obtaining

$$\begin{aligned}
 (J \circ \varepsilon \circ F)_{a(be)} &= F_{abi}{}^{ij} \omega_{je} + F_{aei}{}^{ij} \omega_{jb} \\
 &= (1-k)(A_{ab}{}^j \omega_{je} + A_{ae}{}^j \omega_{jb}) + \\
 &\quad + \frac{1}{2}(A_{[ba]}{}^j \omega_{je} + A_{[ea]}{}^j \omega_{jb}) - \frac{1}{2}(A_{[bi]}{}^i \omega_{ae} + A_{[ei]}{}^i \omega_{ab})
 \end{aligned}
 \tag{28}$$

Recall that $A_{ab}{}^j \bar{\omega}^{ab} = 0$. Then

$$\begin{aligned}
 (J \circ \varepsilon \circ F)_{a(be)} \bar{\omega}^{ab} &= (1-k)A_{ae}{}^j(-\delta_j{}^a) + \frac{1}{2}(A_{[ea]}{}^j(-\delta_j{}^a) + A_{[bi]}{}^i \delta_e{}^b + 2kA_{[ei]}{}^i) \\
 &= (k-1)A_{je}{}^j + \frac{1}{2}k(A_{ei}{}^i - A_{ie}{}^i) \\
 &= (\frac{1}{2}k-1)A_{ie}{}^i + \frac{1}{2}kA_{ei}{}^i
 \end{aligned}$$

The matrix of coefficients associated to the homogeneous linear system

$$\begin{cases} -A_{ie}{}^i + \frac{1}{k}A_{ei}{}^i = 0 \\ (\frac{1}{2}k-1)A_{ie}{}^i + \frac{1}{2}kA_{ei}{}^i = 0 \end{cases}$$

has nonzero determinant for any integer $k \geq 2$, so that $A_{ie}{}^i = A_{ei}{}^i = 0$. Therefore, the solutions of the system

$$\begin{cases} \mu A(\bar{\omega}) = 0 \\ J \circ \varepsilon \circ F_{(A, \mu A)} = 0 \end{cases}$$

are $A_{ab}{}^c \in V^* \otimes \mathfrak{csp}(V)$ such that $A_{ie}{}^i = A_{ei}{}^i = 0$. Looking at the decomposition into irreducible components from Lemma 1, we see that there are only two possibilities left for $A_{ab}{}^j$, namely that it lies in \mathbb{V}_2 or in \mathbb{V}_4 . In both cases, $C_{abc} = A_{ab}{}^j \omega_{jc}$ is symmetric in the indices $\{b, c\}$ and therefore $3C_{(abc)} = C_{abc} + C_{bca} + C_{cab}$. Put $2P_{a(bc)} = (J \circ \varepsilon \circ F)_{a(bc)}$. For $A_{ab}{}^c \in \mathbb{V}_4$, we have that $C_{cab} = -C_{bca} - C_{abc}$ and (28) rewrites as

$$\begin{aligned}
 2P_{a(bc)} &= (1-k)(C_{abc} + C_{acb}) + \frac{1}{4}(C_{bac} - C_{abc} + C_{cab} - C_{acb}) \\
 &= 2(1-k)C_{abc} + \frac{1}{4}(C_{bac} - 2C_{abc} + C_{cab}) \\
 &= 2(1-k)C_{abc} + \frac{1}{4}(C_{bac} - 2C_{abc} - C_{bca} - C_{abc}) \\
 &= (2-2k-\frac{3}{4})C_{abc}
 \end{aligned}$$

If $P_{a(bc)} = 0$, then also $P_{a(bc)} + P_{b(ac)} = 0$, which is equivalent to the fact that C_{abc} is skew-symmetric in the indices $\{a, b\}$. Since C_{abc} is also symmetric in $\{b, c\}$, this condition forces C_{abc} to be zero. For $A_{ab}{}^c \in \mathbb{V}_2$, the formula (28) rewrites as $P_{a(bc)} = (1-k)C_{abc}$ and this vanishes if and only if $C_{abc} = 0$. We thus proved that $(\text{Id} \oplus J \circ \varepsilon) \circ \partial_1$ is injective, hence a \mathfrak{g}_0 -isomorphism. From this, it follows that the kernel of ∂_1 , which is by definition the first prolongation of \mathfrak{D}_{2k} , is trivial and that $\mathfrak{N}_1 = \text{Ker}(J \circ \varepsilon) \subset V^* \otimes \Lambda_0^2 V^* \otimes \Lambda_0^2 V$ is a normalization condition in homogeneity one. Finally, the explicit formulas for the elements of \mathfrak{N}_1 follow straightforwardly from the definition of $J \circ \varepsilon$. \square

PROPOSITION 12. Let $k \geq 2$ be an integer and (V, ω) a symplectic vector space of dimension $2k$ defining the dual Darboux algebra $\mathfrak{D}_{2k} = (\mathfrak{n}, \Pi)$. Denote by $\mathfrak{g}_0 = \mathfrak{csp}(V)$ and put $\mathfrak{g}_{\leq 0} = \mathfrak{n} \oplus \mathfrak{g}_0$. There exists a normalization condition $\mathfrak{N}_2 \subset L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_2$ in homogeneity two. Writing the elements of $L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_2$ as

$$(\tau_{abc}^d, T_{ij\ell}^t, \lambda_{abcd}^e) \in (\Lambda^2 V^* \otimes \mathfrak{csp}(V)) \oplus (\Lambda_0^2 V^* \otimes V^* \otimes V) \oplus (\Lambda^2(\Lambda_0^2 V^*) \otimes \Lambda_0^2 V)$$

the following constraints characterize \mathfrak{N}_2 :

- $T_{ijt}^t = 0$;
- $T_{ija}^b \omega_{bc} = -T_{ijc}^b \omega_{ba}$.

Together with the normalization condition \mathfrak{N}_1 in homogeneity one from Proposition 11, we get a normalization condition $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2$ for the pair $(\mathfrak{D}_{2k}, \mathfrak{g}_0)$ on the torsion components.

PROOF. Since $\mathfrak{n} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, an element of $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_2$ is a linear map $\mathfrak{n} \rightarrow \mathfrak{g}_{\leq 0}$ which is zero on \mathfrak{g}_{-1} and restricts to a linear map $\mathfrak{g}_{-2} \rightarrow \mathfrak{g}_0$. Therefore, $L(\mathfrak{n}, \mathfrak{g}_{\leq 0})_2 = \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0$. Moreover, observe that

$$L(\Lambda^2 \mathfrak{n}, \mathfrak{g}_{\leq 0})_2 = (\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}) \oplus (\Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2})$$

Since $\mathfrak{g}_0 = \mathfrak{der}_{\text{gr}}(\mathfrak{D}_{2k})$, restricting $f \in \mathfrak{g}_0$ to \mathfrak{g}_{-2} defines a linear map $\mathfrak{g}_0 \rightarrow \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$. This can be tensorized at the left with the identity of \mathfrak{g}_{-2}^* and then composed from the left with the alternating map $\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2} \rightarrow \Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$, thus obtaining $\chi: \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0 \rightarrow \Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$. Consider $A \in \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0$. By definition, ∂A consists of a \mathfrak{g}_0 -valued component

$$\partial A(X, Y) = -A(\Pi(X \wedge Y)), \quad X, Y \in \mathfrak{g}_{-1}$$

and the following components assuming values, respectively, in \mathfrak{g}_{-1} and \mathfrak{g}_{-2}

$$\begin{aligned} \partial A(\Psi, X) &= A(\Psi) \cdot X, & \Psi \in \mathfrak{g}_{-2}, X \in \mathfrak{g}_{-1} \\ \partial A(\Psi, \Phi) &= A(\Psi) \cdot \Phi - A(\Phi) \cdot \Psi, & \Psi, \Phi \in \mathfrak{g}_{-2} \end{aligned}$$

Putting all together, ∂_2 writes as

$$(29) \quad \begin{aligned} \partial_2: \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0 &\rightarrow (\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}) \oplus (\Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}) \\ A &\mapsto (-A \circ \Pi, A, \chi(A)) \end{aligned}$$

From the description of $\mathfrak{g}_0 = \mathfrak{csp}(\mathfrak{g}_{-1})$ given by formula (20), we see that there is a \mathfrak{g}_0 -invariant decomposition $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1} = \mathfrak{g}_0 \oplus \mathbb{V}$ with $\phi_j^i \in \mathbb{V}$ if and only if $\phi_t^t = 0$ and $\phi_j^i \omega_{i\ell} = -\phi_\ell^i \omega_{ij}$.

Therefore, $(\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_{-2}^* \otimes \mathbb{V}) = \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ and looking at the formula (29), we deduce that

$$\mathfrak{N}_2 = (\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_{-2}^* \otimes \mathbb{V}) \oplus (\Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2})$$

is a normalization condition in homogeneity two. The explicit formulas for the elements of \mathfrak{N}_2 , then, follows straightforwardly from the formulas defining \mathbb{V} . \square

REMARK 3. A normalization condition for a pair $(\mathfrak{n}, \mathfrak{g}_0)$, in general, is not unique. Looking at formula (29), one can easily see that

$$\mathfrak{N}_2 = (\mathbb{R} \cdot \bar{\omega} \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}) \oplus (\Lambda^2 \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2})$$

is also a normalization condition in homogeneity two for \mathfrak{D}_{2k} . In this case, we put a condition on the curvature rather than on the torsion component.

Propositions 11 and 12 show that any dual Darboux distribution of rank $2k$ satisfies the hypothesis of Theorem 3. We thus have the following

COROLLARY 1. Let $H \subset TM$ be a dual Darboux distribution of even rank. Then, there exists a canonical linear connection $\bar{\nabla}$ on TM preserving H .

Let $H \subset TM$ be a dual Darboux distribution of rank $2k$ and let $\bar{\nabla}$ be the canonical linear connection from Corollary 1. As we saw in the proof of Theorem 3, one of the ingredients obtained from the normalization procedure and defining $\bar{\nabla}$ is an isomorphism $TM \cong \text{gr}(TM)$ which is given, together with the canonical projection $q: TM \rightarrow TM/H$, by a fibered projection $\pi_H: TM \rightarrow H$. If $T^{\bar{\nabla}} \in \Omega^2(TM)$ is the torsion of $\bar{\nabla}$, we can project $T^{\bar{\nabla}}$ along π_H onto H and along q onto TM/H to obtain graded torsion components $T^k \in \Gamma(\mathcal{F} \times_{G_0} \mathfrak{N}_k)$. In particular, Propositions 11 and 12 give the explicit formulas for T^1 and T^2 , respectively. The torsion and the curvature of $\bar{\nabla}$ express the local isomorphism of (M, H) to the homogeneous model of dual Darboux type \mathfrak{D}_{2k} , which we are going to describe explicitly.

Let N be the connected and simply connected Lie group with Lie algebra $\mathfrak{D}_{2k} = (\mathfrak{n}, \Pi)$. Clearly, $\mathfrak{g}_{-1} \subset \mathfrak{n}$ generates, via the push-forward along left-translations by elements of N , a left-invariant distribution $D \subset TN$ of type \mathfrak{D}_{2k} . Denote by G_0 the automorphism group of N . Define the group $G_0 \rtimes N$ as the cartesian product of G_0 and N endowed with the following multiplication rule:

$$(\phi, n)(\psi, h) = (\phi \circ \psi, n \cdot \phi(h))$$

Here, \circ denotes the usual composition of group homomorphisms, while \cdot is the group multiplication in N . The operation is easily seen to be associative:

$$\begin{aligned} ((\phi, n)(\psi, h))(\varphi, k) &= (\phi \circ \psi, n \cdot \phi(h))(\varphi, k) \\ &= ((\phi \circ \psi) \circ \varphi, n \cdot \phi(h) \cdot (\phi \circ \psi)(k)) \\ &= (\phi \circ (\psi \circ \varphi), n \cdot \phi(h \cdot \psi(k))) \\ &= (\phi, n)((\psi, h)(\varphi, k)) \end{aligned}$$

The neutral element is (id_N, e_N) and $(\phi^{-1}, \phi^{-1}(n^{-1}))$ is the inverse element of (ϕ, n) . There are obvious surjective projections $pr_1: G_0 \rtimes N \rightarrow G_0$ and $pr_2: G_0 \rtimes N \rightarrow N$, which can be used to define a differentiable structure on $G_0 \rtimes N$ such that the group multiplication defined above is smooth. Otherwise put, $G_0 \rtimes N$ has a natural Lie group structure. The elements of the form (ψ, e_N) for $\psi \in G_0$ form a closed subgroup of $G_0 \rtimes N$ isomorphic to G_0 and

$$r^\psi(\phi, n) := (\phi, n)(\psi, e_N) = (\phi \circ \psi, n)$$

defines a smooth right action of G_0 on $G_0 \rtimes N$ such that $pr_2(r^\psi(\phi, n)) = pr_2((\phi, n))$ for all $\psi \in G_0$. Since $G_0 = \text{Aut}(\mathfrak{n})$, it is easy to see that the action on each fiber of pr_2 is transitive, hence $pr_2: G_0 \rtimes N \rightarrow N$ is a principal bundle with structure group G_0 .

Let G be a Lie group with Lie algebra \mathfrak{g} and denote by $\ell_g: G \rightarrow G$ the left-multiplication by $g \in G$. Recall that G carries the Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$, defined by

$$\omega_g(\xi) = T_g \ell_{g^{-1}} \cdot \xi$$

for all $g \in G$ and $\xi \in T_g G$. For $X \in \mathfrak{g}$, denote by $L_X \in \mathfrak{X}(G)$ the left-invariant vector field on G generated by X . For $g \in G$, denote by ℓ_g^* and $(r^g)^*$ the pull-back along left and right translations, respectively. It is a well known fact that the Maurer–Cartan form satisfies the following conditions:

- (1) $\omega(L_X) = X$ for all $X \in \mathfrak{g}$;
- (2) $\ell_g^* \omega = \omega$ for all $g \in G$;
- (3) $\omega_g: T_g G \rightarrow \mathfrak{g}$ is a linear isomorphism for all $g \in G$;
- (4) $(r^g)^* \omega = \text{Ad}(g^{-1}) \circ \omega$ for all $g \in G$;

Moreover, ω satisfies the Maurer–Cartan equation

$$d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)] = 0$$

for all $\xi, \eta \in \mathfrak{X}(G)$. Consider now the Lie group $G = G_0 \rtimes N$ and the Maurer–Cartan form ω of G from above. Then, $(pr_2: G_0 \rtimes N \rightarrow N, \omega)$ is the homogeneous model for Cartan geometries

of type (G, G_0) and N carries a dual Darboux distribution of rank $2k$. Clearly, the Lie algebra of G coincides with $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}$ and the direct sum decomposition is invariant with respect to the action of $G_0 = \text{Aut}(\mathfrak{n})$. Therefore, if we write $\omega = \gamma + \theta \in \Omega^1(G, \mathfrak{g}_0 \oplus \mathfrak{n})$ according to the decomposition of \mathfrak{g} , we see that each term has to be G_0 -equivariant. From conditions (1) and (3), we see that $\gamma \in \Omega^1(G, \mathfrak{g}_0)$ trivializes the vertical bundle of $pr_2: G_0 \rtimes N \rightarrow N$ and that $\theta \in \Omega^1(G, \mathfrak{n})$ annihilates by insertion of a vertical vector field, hence it induces isomorphisms $T_n N \rightarrow \mathfrak{n}$ for $n \in N$. In particular, $\gamma \in \Omega^1(G, \mathfrak{g}_0)$ is a connection form on G inducing a linear connection on TN preserving D , whose torsion and curvature in positive homogeneity vanish.

CHAPTER 3

Generic rank four distributions in dimension eight

In Chapter 1, we classified the generic types of one-step bracket-generating distributions $H \subset TM$ of rank four. In particular, Theorem 2 proves the existence of two distinct generic types for $\dim(M) = 8$, corresponding to the cases of indefinite and definite nondegenerate quadratic form on a real two-dimensional plane. According to the usual terminology for quadratic forms, we will speak about hyperbolic and elliptic $(4, 8)$ -distributions.

The main subject of the present Chapter is the description of hyperbolic and elliptic $(4, 8)$ -distributions. The model algebra for each type will be described as the negative graded part of a $|2|$ -grading on a real form of $\mathfrak{sl}(5, \mathbb{C})$. In particular, we will underline properties of the model algebra which correspond, via the associated bundle construction, to intrinsic properties of the distributions. An argument of parabolic geometry allows to completely determine the fundamental invariants of the distribution. First, we will show that such structures are equivalent to normal, regular parabolic geometries. As we know from the general theory, to any such parabolic geometry one can associate a harmonic curvature which uniquely determines the fundamental invariants of the underlying structure. Kostant's Theorem describes the target space of the harmonic curvature, thus giving the list of invariants for each generic type of $(4, 8)$ -distributions. Finally, the results from parabolic geometry will be applied to another interesting problem, namely the explicit description of submaximally symmetric models of hyperbolic $(4, 8)$ -distributions.

1. A grading on $\mathfrak{sl}(5, \mathbb{C})$

Consider the complex simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$. The following

$$(30) \quad \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^F & \mathfrak{g}_0 \end{pmatrix}$$

with \mathfrak{g}_0 divided in three square blocks of size two-one-two, describes a $|2|$ -grading $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$. In particular, $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ is the direct sum of two-dimensional subspaces. An element of \mathfrak{g}_0 writes as

$$\begin{pmatrix} A & 0 & 0 \\ 0 & -\text{tr}(A+B) & 0 \\ 0 & 0 & B \end{pmatrix}$$

for some $A, B \in \mathfrak{gl}(2, \mathbb{C})$. We will denote it by $(A, B) \in \mathfrak{g}_0$. Moreover, we will think about elements $e \in \mathfrak{g}_{-1}^E \cong (\mathbb{C}^2)^*$ and $f \in \mathfrak{g}_{-1}^F \cong \mathbb{C}^2$, respectively, as row and column vectors. Elements of \mathfrak{g}_{-1} and \mathfrak{g}_{-2} will be briefly denoted, respectively, by $(e, f) \in \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ and by $Y \in \mathfrak{gl}(2, \mathbb{C})$. Consider $X_j = (e_j, f_j) \in \mathfrak{g}_{-1}$ for $j = 1, 2$. The bracket restricts to a surjective bilinear map

$$(31) \quad \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}, \quad [X_1, X_2] = [(e_1, f_1), (e_2, f_2)] = f_1 \otimes e_2 - f_2 \otimes e_1$$

In particular, observe that

$$[\mathfrak{g}_{-1}^E, \mathfrak{g}_{-1}^E] = 0 = [\mathfrak{g}_{-1}^F, \mathfrak{g}_{-1}^F], \quad \mathfrak{g}_{-2} \cong \mathfrak{g}_{-1}^E \otimes \mathfrak{g}_{-1}^F \cong \mathfrak{gl}(2, \mathbb{C}).$$

Moreover,

$$\mathfrak{g}_0 \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}, \quad [(A, B), (e, f)] = (-\lambda e - eA, \lambda f + Bf)$$

$$\mathfrak{g}_0 \times \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}, \quad [(A, B), Y] = BY - YA$$

where $\lambda = \text{tr}(A + B)$. In particular, $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ is a direct sum of \mathfrak{g}_0 -modules, which can be seen as the direct sum of the standard representation of $\mathfrak{gl}(2, \mathbb{C})$ and of its dual. Moreover, the bracket restricted to the subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is given by the formula

$$\mathfrak{g}_0 \times \mathfrak{g}_0 \rightarrow \mathfrak{g}_0, \quad [(A_1, B_1), (A_2, B_2)] = ([A_1, A_2], [B_1, B_2])$$

which shows that the adjoint representation of \mathfrak{g}_0 coincides with the direct sum of the adjoint representation of $\mathfrak{gl}(2, \mathbb{C})$ and its dual.

Recall the classification for the generic types of $(4, n)$ -distributions from Theorem 2 of Chapter 1. Let V, W be real four-dimensional vector spaces. The generic types in dimension eight are the open orbits of surjective linear maps $\Lambda^2 V \rightarrow W$ for the natural action of $GL(V) \times GL(W)$, equivalent to the open $GL(V)$ -orbits of their kernels in a Grassmannian. Theorem 2 characterizes the open $GL(V)$ -orbits of linear subspaces of $\Lambda^2 V$ in terms of nondegenerate restrictions of the wedge product, thus proving the existence of two distinct open orbits of $(4, 8)$ -type, corresponding to the cases of indefinite and definite nondegenerate quadratic form in dimension two. According to the usual terminology for quadratic forms, we will refer to them as hyperbolic and elliptic types.

Each type will be described in the following sections. First, we describe a model algebra that is a representative for the orbit. Both hyperbolic and elliptic models come from the choice of a $|2|$ -grading on a real form for $\mathfrak{sl}(5, \mathbb{C})$, respectively the split-real form $\mathfrak{sl}(5, \mathbb{R})$ and the real form $\mathfrak{su}(3, 2)$. Then, we will characterize the orbit via some properties which, finally, will be shown to extend to the corresponding distributions. We start dealing with the hyperbolic case.

2. Hyperbolic case

2.1. Model algebra. Since $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ is the split-real form for $\mathfrak{sl}(5, \mathbb{C})$, the grading (30) on the complex simple Lie algebra $\mathfrak{sl}(5, \mathbb{C})$ is the complexification of a grading on \mathfrak{g} admitting a completely analogous description. By replacing \mathbb{C} with \mathbb{R} , the same remarks about the grading (30) from Section 1 hold for the grading on the real algebra \mathfrak{g} . The bracket-generating layer is the direct sum $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ of real two-dimensional \mathfrak{g}_0 -modules, where \mathfrak{g}_0 is a real reductive Lie algebra with two-dimensional center and semisimple part $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$. Both $\mathfrak{g}_{-1}, \mathfrak{g}_{-2} \subset \mathfrak{g}$ are four-dimensional real subspaces. Let $\{v_1, v_2\} \subset \mathbb{R}^2$ be the standard basis and let $\{v_1^*, v_2^*\} \subset (\mathbb{R}^2)^*$ be the dual basis. In the same notation as in (31), we write the elements in \mathfrak{g}_{-1} as $(e, f) \in \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F \cong (\mathbb{R}^2)^* \oplus \mathbb{R}^2$. There are standard bases

$$\{E_1, E_2\} = \{(v_1^*, 0), (v_2^*, 0)\}, \quad \{F_1, F_2\} = \{(0, v_1), (0, v_2)\}$$

for the \mathfrak{g}_0 -submodules $\mathfrak{g}_{-1}^E, \mathfrak{g}_{-1}^F \subset \mathfrak{g}_{-1}$, respectively. The kernel of the bracket $\Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is a linear subspace $P \subset \Lambda^2 \mathfrak{g}_{-1}$ of dimension two and formula (31) shows that

$$P = \text{sp}\{E_1 \wedge E_2\} \oplus \text{sp}\{F_1 \wedge F_2\} = \Lambda^2 \mathfrak{g}_{-1}^E \oplus \Lambda^2 \mathfrak{g}_{-1}^F.$$

Let us look at the restriction of the wedge product to P . As shown in the proof of Proposition 3, the wedge product $\Lambda^2 \mathfrak{g}_{-1} \times \Lambda^2 \mathfrak{g}_{-1} \rightarrow \Lambda^4 \mathfrak{g}_{-1}$ together with a choice of volume form defines a quadratic form β on $\Lambda^2 \mathfrak{g}_{-1}$. Consider the volume form $\text{vol} = E_1 \wedge E_2 \wedge F_1 \wedge F_2$ and the corresponding quadratic form β . Then $\Lambda^2 \mathfrak{g}_{-1}^E, \Lambda^2 \mathfrak{g}_{-1}^F \subset P$ are null lines for β such that $\Lambda^2 \mathfrak{g}_{-1}^E \wedge \Lambda^2 \mathfrak{g}_{-1}^F$ is nonzero, which shows that $\beta|_P$ has signature $(1, 1)$. We thus see that $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ is a model for the hyperbolic $(4, 8)$ -type appearing in the classification of Theorem 2. In particular if (M, H) is a regular pair of type \mathfrak{g}_- , then $H \subset TM$ is an hyperbolic $(4, 8)$ -distribution.

For later use, we explicitly describe a basis of \mathfrak{g}_{-1} . Again, $\{v_1, v_2\} \subset \mathbb{R}^2$ denotes the standard basis and $\{v_1^*, v_2^*\} \subset (\mathbb{R}^2)^*$ the dual basis. The following set gives a basis of \mathfrak{g}_{-1} :

$$(32) \quad \begin{aligned} Y_1 &= (v_1^*, v_1) \\ Y_2 &= (v_1^*, -v_1) \\ Y_3 &= (v_2^*, v_2) \\ Y_4 &= (v_2^*, -v_2) \end{aligned}$$

For $Z \in \mathfrak{g}_{-1}$, denote by $\text{ad}(Z) : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ the restriction of the adjoint action. The basis above, then, satisfies

$$\mathcal{I} = \text{Im}(\text{ad}(Y_1)) = \text{Im}(\text{ad}(Y_2)) \quad \text{and} \quad \mathcal{J} = \text{Im}(\text{ad}(Y_3)) = \text{Im}(\text{ad}(Y_4)).$$

With respect to the choice of coordinates on \mathfrak{g}_{-2} :

$$\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$$

the three-dimensional planes \mathcal{I} and \mathcal{J} are respectively given by $\{y_4 = 0\}$ and $\{y_1 = 0\}$. Moreover, $\mathcal{J} = \text{sp}\{[Y_2, Y_3], [Y_2, Y_4], [Y_3, Y_4]\}$ and the following bracket relations hold:

$$[Y_1, Y_3] = -[Y_2, Y_4], \quad [Y_1, Y_4] = -[Y_2, Y_3].$$

2.2. Characterization of the hyperbolic orbit.

DEFINITION 15. Let V, W be real four-dimensional vector spaces and $F : \Lambda^2 V \rightarrow W$ a surjective linear map. To a nonzero element $X \in V$, we associate the linear map

$$i_X F : V \rightarrow W, \quad i_X F(Y) := F(X \wedge Y) \quad \text{for all } Y \in V.$$

Denote by $\text{Im}(i_X F) \subset W$ the linear subspace given by the image of $i_X F$. We will shortly refer to the dimension of $\text{Im}(i_X F)$ as the *rank of X* with respect to F .

Recall the grading on $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ described in 2.1. In the same notation introduced there, let $\{E_1, E_2\}$ and $\{F_1, F_2\}$ be the standard bases for the \mathfrak{g}_0 -submodules $\mathfrak{g}_{-1}^E, \mathfrak{g}_{-1}^F \subset \mathfrak{g}_{-1}$, respectively. Using formula (31), we easily see that all elements of the basis $\{E_1, E_2, F_1, F_2\}$ for \mathfrak{g}_{-1} are of rank two with respect to the restriction $[\cdot, \cdot] : \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ of the bracket of \mathfrak{g} . For $X \in \mathfrak{g}_{-1}$, we see that $i_X = \text{ad}(X) : \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ is the restriction of the adjoint action of X to \mathfrak{g}_{-1} . The same formula shows that the basis (32) of \mathfrak{g}_{-1} is given by elements of rank three with respect to $[\cdot, \cdot]$. Let V, W be real four-dimensional vector spaces and let $F : \Lambda^2 V \rightarrow W$ be a surjective linear map. Since $i_X F(X) = F(X \wedge X) = 0$ for $X \in V$, the kernel of $i_X F$ is nontrivial, hence every nonzero element $X \in V$ has at most rank three. It is easy to see that if the wedge product is nondegenerate on $\text{Ker} F \subset \Lambda^2 V$, the admissible values for the rank of a nonzero element $X \in V$ are two and three. Indeed, if X_1 is a nonzero vector of V of rank one for F , it can be completed to a basis $\{X_1, \dots, X_4\}$ of V such that $i_{X_1} F(X_2) = i_{X_1} F(X_3) = 0$. Otherwise put, $\mathcal{B} = \{X_1 \wedge X_2, X_1 \wedge X_3\}$ is a basis of $P = \text{Ker} F$. Put $\text{vol} = X_1 \wedge X_2 \wedge X_3 \wedge X_4$ and as in Proposition 3, denote by $\beta : \Lambda^2 V \rightarrow \mathbb{R}$ the quadratic form defined by $\phi \wedge \phi = \beta(\phi) \text{vol}$ for $\phi \in \Lambda^2 V$. Then, $\beta|_P$ computed in the basis \mathcal{B} is zero, hence degenerate, which is a contradiction. The notion of rank allows to formulate the characterization of the hyperbolic orbit in Theorem 4. To prove the Theorem, we will need the following preliminary result.

PROPOSITION 13. Let V be a real four-dimensional vector space, $P \subset \Lambda^2 V$ a linear subspace of dimension two and consider the induced action of $GL(V)$ on the Grassmannian $Gr(2, \Lambda^2 V)$ (see Proposition 2). If P is in the hyperbolic $GL(V)$ -orbit, there exists a basis $\{X_1, \dots, X_4\}$ of V such that $P = \text{sp}\{X_1 \wedge X_2\} \oplus \text{sp}\{X_3 \wedge X_4\}$.

PROOF. If $P \subset \Lambda^2 V$ is a two-dimensional hyperbolic plane, then P contains two isotropic lines ℓ_1, ℓ_2 such that $P = \ell_1 \oplus \ell_2$ and $\omega_1 \wedge \omega_2$ is nonzero for every choice of generators ω_1 and ω_2 for ℓ_1 and ℓ_2 , respectively. Let $\{Y_1, \dots, Y_4\}$ be a basis of V . Then $Q = \text{sp}\{Y_1 \wedge Y_2\} \oplus \text{sp}\{Y_3 \wedge Y_4\}$ is an hyperbolic linear subspace $Q \subset \Lambda^2 V$ of dimension two. By Proposition 4, there exists $\phi \in O(3, 3)$ such that $\phi \cdot Q = P$. Since ϕ preserves the scalar products, it sends the null line generated by $Y_1 \wedge Y_2$ in one of the two null lines contained in P . We can thus suppose that ℓ_1 is generated by $\phi \cdot (Y_1 \wedge Y_2)$ and that ℓ_2 is generated by $\phi \cdot (Y_3 \wedge Y_4)$. Now by Proposition 3, the action of ϕ is the induced action of an element of $GL(4, \mathbb{R})$. Therefore, there exists $A \in GL(4, \mathbb{R})$ such that

$$P = \ell_1 \oplus \ell_2 = \text{sp}\{AY_1 \wedge AY_2\} \oplus \text{sp}\{AY_3 \wedge AY_4\}.$$

Clearly, putting $X_i = AY_i$ for $i = 1, \dots, 4$ we obtain a basis $\{X_1, \dots, X_4\}$ of V proving the statement. \square

THEOREM 4. (*Isotropic subspaces of rank-two elements*) Let V, W be real four-dimensional vector spaces and let $F : \Lambda^2 V \rightarrow W$ be a surjective linear map. If $\text{Ker} F \subset \Lambda^2 V$ is hyperbolic, there exist $Y, Z \in V$ linearly independent of rank two with respect to F such that

$$V = \text{Ker}(i_Y F) \oplus \text{Ker}(i_Z F).$$

Moreover, the rank-two elements in V are exactly the elements of the two subspaces.

PROOF. If $\text{Ker} F \subset \Lambda^2 V$ is hyperbolic, by Proposition 13 there exists a basis $\mathcal{B} = \{X_1, \dots, X_4\}$ of V such that $\text{Ker} F = \text{sp}\{X_1 \wedge X_2\} \oplus \text{sp}\{X_3 \wedge X_4\}$. The identities $F(X_1 \wedge X_2) = 0 = F(X_3 \wedge X_4)$ follow straightforwardly and imply that every X_i is a rank-two element, as well as the direct sum decomposition

$$V = \text{sp}\{X_1, X_2\} \oplus \text{sp}\{X_3, X_4\} = \text{Ker}(i_{X_1} F) \oplus \text{Ker}(i_{X_3} F).$$

The following argument shows that the decomposition is canonical. Every element lying in one of the subspaces $\text{Ker}(i_{X_1} F) = \text{sp}\{X_1, X_2\}$ and $\text{Ker}(i_{X_3} F) = \text{sp}\{X_3, X_4\}$ has evidently rank two. Moreover, every rank-two element of V lies in one of these two subspaces. To see this, first write $S \in V$ as

$$S = aX_1 + bX_2 + cX_3 + dX_4.$$

Then, observe that $\mathcal{E} = \{F(X_1 \wedge X_3), F(X_1 \wedge X_4), F(X_2 \wedge X_3), F(X_2 \wedge X_4)\}$ is a basis of W and $i_S F : V \rightarrow W$ writes with respect to \mathcal{B}, \mathcal{E} as

$$\begin{pmatrix} -c & 0 & a & 0 \\ -d & 0 & 0 & a \\ 0 & -c & b & 0 \\ 0 & -d & 0 & b \end{pmatrix}$$

If S is a rank-two vector, all (3×3) -minors from the matrix above have zero determinant. Imposing this condition leads to the following system of equations on the indeterminates a, b, c, d :

$$\begin{cases} a^2 c = 0 \\ a^2 d = 0 \\ b^2 c = 0 \\ b^2 d = 0 \end{cases}$$

whose solutions (a, b, c, d) must satisfy at least one of the two following systems:

$$\begin{cases} a = 0 \\ b = 0 \end{cases} \quad \text{and} \quad \begin{cases} c = 0 \\ d = 0 \end{cases}$$

If (a, b, c, d) satisfies the first linear system, then $S \in \text{Ker}(i_{X_3}F)$, while $S \in \text{Ker}(i_{X_1}F)$ if (a, b, c, d) satisfies the second linear system, thus proving the statement. \square

PROPOSITION 14. *Let V, W be real vector spaces of dimension four and $F : \Lambda^2 V \rightarrow W$ a surjective linear map of hyperbolic type. Let $Y, Z \in V$ be linearly independent elements of rank two with respect to F such that*

$$V = \text{Ker}(i_Y F) \oplus \text{Ker}(i_Z F)$$

is the direct sum decomposition from Theorem 4 and denote by $[\cdot, \cdot] : \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ the bracket on the hyperbolic model algebra $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \subset \mathfrak{sl}(5, \mathbb{R})$ from 2.1. Then, there exist linear isomorphisms $\phi : \mathfrak{g}_{-1} \rightarrow V$ and $\psi : \mathfrak{g}_{-2} \rightarrow W$ such that

$$\psi^{-1} \circ F \circ \Lambda^2 \phi = [\cdot, \cdot]$$

and ϕ maps $\mathfrak{g}_{-1}^E \cup \mathfrak{g}_{-1}^F$ to $\text{Ker}(i_Y F) \cup \text{Ker}(i_Z F)$.

PROOF. If $F : \Lambda^2 V \rightarrow W$ is hyperbolic, by Proposition 13 there exists a basis $\mathcal{B} = \{X_1, \dots, X_4\}$ of V such that $\text{Ker}(F) = \text{sp}\{X_1 \wedge X_2\} \oplus \text{sp}\{X_3 \wedge X_4\}$. Consider the standard bases of rank-two elements $\{E_1, E_2\}$ and $\{F_1, F_2\}$ for the \mathfrak{g}_0 -submodules $\mathfrak{g}_{-1}^E, \mathfrak{g}_{-1}^F \subset \mathfrak{g}_{-1}$, respectively, from 2.1. Recall that

$$\text{Ker}([\cdot, \cdot]) = \text{sp}\{E_1 \wedge E_2\} \oplus \text{sp}\{F_1 \wedge F_2\} = \Lambda^2 \mathfrak{g}_{-1}^E \oplus \Lambda^2 \mathfrak{g}_{-1}^F.$$

For $i = 1, 2$ put $\tilde{X}_i = E_i$ and $\tilde{X}_{i+2} = F_i$. Now define $\phi : \mathfrak{g}_{-1} \rightarrow V$ by putting $\phi(\tilde{X}_i) = X_i$. By construction, ϕ maps $\mathfrak{g}_{-1}^E \cup \mathfrak{g}_{-1}^F$ to $\text{Ker}(i_Y F) \cup \text{Ker}(i_Z F)$. Furthermore, $\Lambda^2 \phi$ restricts to a linear isomorphism $\text{Ker}([\cdot, \cdot]) \rightarrow \text{Ker}(F)$ and it thus descends to an isomorphism

$$\overline{\Lambda^2 \phi} : \Lambda^2 \mathfrak{g}_{-1} / \text{Ker}([\cdot, \cdot]) \rightarrow \Lambda^2 V / \text{Ker}(F).$$

Clearly, F and $[\cdot, \cdot]$ descend to linear isomorphisms

$$\overline{F} : \Lambda^2 \mathfrak{g}_{-1} / \text{Ker}([\cdot, \cdot]) \rightarrow \mathfrak{g}_{-2}, \quad \overline{[\cdot, \cdot]} : \Lambda^2 V / \text{Ker}(F) \rightarrow W.$$

Then, putting

$$\psi := \overline{F} \circ \overline{\Lambda^2 \phi} \circ \overline{[\cdot, \cdot]}^{-1}.$$

defines a linear isomorphism $\psi : \mathfrak{g}_{-2} \rightarrow W$ such that $\psi^{-1} \circ F \circ \Lambda^2 \phi = [\cdot, \cdot]$. \square

As a Corollary of Proposition 14, any grading-preserving isomorphism of the hyperbolic model algebra $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ is the adjoint action of an element of G_0 , where G_0 is the connected and simply connected Lie group whose Lie algebra is the component $\mathfrak{g}_0 \subset \mathfrak{sl}(5, \mathbb{R})$ of degree zero. Passing to the Lie algebra level, we obtain the following

COROLLARY 2. *Let be $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ and let $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ be the grading described in 2.1. Then, the adjoint action of \mathfrak{g}_0 on \mathfrak{g}_- defines an isomorphism $\mathfrak{g}_0 \cong \text{der}_{\text{gr}}(\mathfrak{g}_-)$.*

Let $H \subset TM$ be a $(4, 8)$ -distribution and let $\mathcal{L} : \Lambda^2 H \rightarrow Q$ be the Levi bracket associated to H . Consider a point $x_0 \in M$. Recall that \mathcal{L} determines, by Proposition 2 (a), an orbit $\mathcal{O}_{x_0} \subset L_s(\Lambda^2 \mathbb{R}^4, \mathbb{R}^4)$ for the natural action of $GL(4, \mathbb{R}) \times GL(4, \mathbb{R})$ on the set $L_s(\Lambda^2 \mathbb{R}^4, \mathbb{R}^4)$ of surjective linear maps. We say that H is *hyperbolic* (respectively *elliptic*) if \mathcal{O}_x is the hyperbolic (respectively elliptic) orbit for all $x \in M$.

In order to discuss the local properties of generic $(4, 8)$ -distributions, it will be useful to introduce the following terminology. Let $H \subset TM$ be a generic $(4, 8)$ -distribution, $U \subset M$ an open subset and let $\xi \in \Gamma(H)$ a local smooth section defined on U . Let be $k \in \{2, 3\}$. We say that ξ is a *section of rank k* on U if $\xi(x)$ has rank k with respect to $\mathcal{L}_x : \Lambda^2 H_x \rightarrow Q_x$ for all $x \in U$. Suppose that ξ is a section of rank k on U and denote by $i_\xi \mathcal{L} : H \rightarrow Q$ the vector bundle homomorphism obtained by inserting ξ as first variable in \mathcal{L} . Then, $\text{Im}(i_\xi \mathcal{L}) \subset Q$ is a smooth subbundle of rank k defined on U .

PROPOSITION 15. *Let $H \subset TM$ be an hyperbolic $(4, 8)$ -distribution. Then, locally around each point $x \in M$, there exist smooth subbundles $E, F \subset H$ of rank two such that $H = E \oplus F$ and $\text{Ker}(\mathcal{L}) = \Lambda^2 E \oplus \Lambda^2 F \subset \Lambda^2 H$.*

PROOF. Denote by $[\cdot, \cdot] : \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ the bracket (31) on the hyperbolic model algebra $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ from 2.1. Let be $x_0 \in M$. By Proposition 14, there exist linear isomorphisms $\phi_{x_0} : \mathfrak{g}_{-1} \rightarrow H_{x_0}$ and $\psi_{x_0} : \mathfrak{g}_{-2} \rightarrow Q_{x_0}$ such that

$$\psi_{x_0}([X, Y]) = \mathcal{L}_{x_0}(\phi_{x_0}(X), \phi_{x_0}(Y)), \quad X, Y \in \mathfrak{g}_{-1}.$$

Let $\{Y_1, \dots, Y_4\}$ be the basis (32) for \mathfrak{g}_{-1} , whose elements are of maximal rank three. Recall that the chosen basis satisfies two linearly independent conditions, namely

$$[Y_1, Y_3] = -[Y_2, Y_4], \quad [Y_1, Y_4] = -[Y_2, Y_3].$$

Let $\{\xi_1, \dots, \xi_4\}$ be a local frame for H , defined around x_0 , such that

$$\phi_{x_0}(Y_i) = \xi_i(x_0), \quad i = 1, \dots, 4.$$

Then, each $\xi_i(x_0) \in H_{x_0}$ is an element of maximal rank three with respect to $i_{\xi_i} \mathcal{L}_{x_0}$. Fix $i \in \{1, \dots, 4\}$. By construction, the set

$$\{\mathcal{L}(\xi_i, \xi_j) \mid j \neq i\}$$

consists of three sections of Q , whose values in x_0 are linearly independent. Therefore, their values at each point of an open neighborhood $U_i \subset M$ of x_0 are linearly independent, so that the set above is a frame for the smooth subbundle $\mathcal{I}_i := \text{Im}(i_{\xi_i} \mathcal{L}) \subset Q$ of rank three defined on U_i . Put $U = \bigcap_{i=1}^4 U_i$. Then, we defined subbundles $\mathcal{I}_1, \dots, \mathcal{I}_4 \subset Q$ of rank three on U .

Several remarks follow straightforwardly from the properties of the basis $\{Y_1, \dots, Y_4\}$. First of all, the identities $\mathcal{I}_1(x_0) = \mathcal{I}_2(x_0)$ and $\mathcal{I}_3(x_0) = \mathcal{I}_4(x_0)$ follow from (33). Moreover, the subset

$$\{\mathcal{L}(\xi_2, \xi_3), \mathcal{L}(\xi_2, \xi_4), \mathcal{L}(\xi_3, \xi_4)\} \subset \Gamma(Q)$$

evaluated at x_0 coincides with a basis for the three-dimensional vector space $\mathcal{I}_3(x_0) = \mathcal{I}_4(x_0)$, hence its elements are linearly independent around x_0 . Up to shrinking U , we can thus suppose that the subset of $\Gamma(Q)$ above defines a smooth subbundle $\mathcal{W} \subset Q$ of rank three on U . Observe that $\mathcal{I}_1 \cap \mathcal{W} \subset Q$ is a smooth subbundle of rank two on U . Indeed, since \mathcal{L} is surjective, the sum $\mathcal{I}_1 + \mathcal{W}$ generates the whole quotient bundle Q on U . Using this fact and Grassmann formula, for $x \in U$ we see that

$$\dim((\mathcal{I}_1)_x \cap \mathcal{W}_x) = \dim((\mathcal{I}_1)_x) + \dim(\mathcal{W}_x) - \dim(Q_x) = 2.$$

Now we modify the initial frame $\{\xi_1, \dots, \xi_4\}$ to another frame for H of rank-two sections, defined locally around x_0 . Consider ζ_1, ζ_2 local smooth sections of $\mathcal{I}_1 \cap \mathcal{W}$ such that

$$(33) \quad \begin{cases} \zeta_1(x_0) = \mathcal{L}_{x_0}(\xi_2, \xi_3) = -\mathcal{L}_{x_0}(\xi_1, \xi_4) \\ \zeta_2(x_0) = \mathcal{L}_{x_0}(\xi_2, \xi_4) = -\mathcal{L}_{x_0}(\xi_1, \xi_3) \end{cases}$$

Since $\{\zeta_1(x_0), \zeta_2(x_0)\}$ is a basis for $\mathcal{I}_1(x_0) \cap \mathcal{W}(x_0)$ we can suppose, up to shrinking U , that $\{\zeta_1, \zeta_2\}$ is a local frame for $\mathcal{I}_1 \cap \mathcal{W}$ on U . There exist smooth functions $(g_{ij}) : U \rightarrow \mathbb{R}$ for $i = 1, 2$ and $j = 1, 2, 3$ such that the conditions $\zeta_1, \zeta_2 \in \Gamma(\mathcal{W})$ write as an homogeneous linear system of two equations

$$\zeta_i = g_{i1} \mathcal{L}(\xi_2, \xi_3) + g_{i2} \mathcal{L}(\xi_2, \xi_4) + g_{i3} \mathcal{L}(\xi_3, \xi_4), \quad i = 1, 2.$$

Evaluating the system above at x_0 , we deduce from (33) the value of the matrix (g_{ij}) at x_0 :

$$(34) \quad (g_{ij}(x_0)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The left-hand minor m of order two of (g_{ij}) has determinant 1 in x_0 . Possibly shrinking U , we can thus suppose that m is invertible on U . Therefore, multiplying both sides of the linear system at the left by m^{-1} and recalling that $\zeta_i \in \mathcal{I}_1$ leads to

$$\left. \begin{array}{l} \mathcal{L}(\xi_2, \xi_3) + A \mathcal{L}(\xi_3, \xi_4) \\ \mathcal{L}(\xi_2, \xi_4) + B \mathcal{L}(\xi_3, \xi_4) \end{array} \right\} \in \mathcal{I}_1$$

with $A, B : U \rightarrow \mathbb{R}$ smooth functions, whose value at x_0 is zero. Now put

$$\tilde{\xi}_2 := \xi_2 + B\xi_3 - A\xi_4$$

thus obtaining a section of H with the property that $\tilde{\mathcal{I}}_2 \subset \mathcal{I}_1$, where $\tilde{\mathcal{I}}_2 := \text{Im}(i_{\tilde{\xi}_2} \mathcal{L}) \subset Q$. Since $\tilde{\xi}_2(x_0) = \xi_2(x_0)$ is an element of rank three, we see that up to shrinking U once more, $\tilde{\xi}_2$ is a section of rank three on U such that $\tilde{\mathcal{I}}_2 = \mathcal{I}_1$ on U .

Denote by $\mathcal{W}' \subset Q$ the subbundle generated by $\{\mathcal{L}(\xi_1, \xi_4), \mathcal{L}(\xi_2, \xi_4), \mathcal{L}(\xi_1, \xi_2)\}$. Starting with the intersection $\mathcal{I}_3 \cap \mathcal{W}'$, one can argue in the same way to get a rank-three section $\tilde{\xi}_4$ of H such that $\tilde{\xi}_4(x_0) = \xi_4(x_0)$ and $\tilde{\mathcal{I}}_4 := \text{Im}(i_{\tilde{\xi}_4} \mathcal{L}) = \mathcal{I}_3$ on U . Without loss of generality, we can thus assume that $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is a frame for H of rank-three sections defined on U , which satisfies (33) and which determines the pair of hyperplanes

$$(35) \quad \mathcal{I}_2 = \mathcal{I}_1, \quad \mathcal{I}_4 = \mathcal{I}_3$$

and

$$\begin{aligned} Q &= \mathcal{I}_1 \oplus \mathcal{R}_{34} \\ &= \mathcal{I}_3 \oplus \mathcal{R}_{12} \end{aligned}$$

hold on U . Here, \mathcal{R}_{ij} denotes the line bundle generated by $\mathcal{L}(\xi_i, \xi_j)$.

For each hyperplane from (35), we find $\zeta_1, \zeta_2 \in \Gamma(H)$ of rank two such that $\text{Im}(i_{\zeta_i} \mathcal{L})$ is contained in the hyperplane for both $i = 1, 2$, as we are going to show. This will be proved now for $\mathcal{I}_2 = \mathcal{I}_1$. The argument should be repeated analogously for the second choice $\mathcal{I}_4 = \mathcal{I}_3$ of hyperplane.

Since $\mathcal{I}_2 = \mathcal{I}_1$, there exist smooth functions $a, b, c, d, e, f : U \rightarrow \mathbb{R}$ such that

$$\left\{ \begin{array}{l} \mathcal{L}(\xi_2, \xi_3) = a \mathcal{L}(\xi_1, \xi_2) + b \mathcal{L}(\xi_1, \xi_3) + c \mathcal{L}(\xi_1, \xi_4) \\ \mathcal{L}(\xi_2, \xi_4) = d \mathcal{L}(\xi_1, \xi_2) + e \mathcal{L}(\xi_1, \xi_3) + f \mathcal{L}(\xi_1, \xi_4) \end{array} \right.$$

holds. Now (33) shows that a, b, d, f vanish at x_0 , while $c(x_0) = e(x_0) = -1$. Therefore, the sections of H

$$\begin{aligned} \bar{\xi}_3 &= \xi_3 + a\xi_1 \\ \bar{\xi}_4 &= \xi_4 + d\xi_1 \end{aligned}$$

are still of rank three around x_0 and $\{\xi_1, \xi_2, \bar{\xi}_3, \bar{\xi}_4\}$ is a local frame for H , satisfying two independent linear conditions, namely

$$(36) \quad \left\{ \begin{array}{l} \mathcal{L}(\xi_2, \bar{\xi}_3) = c \mathcal{L}(\xi_1, \bar{\xi}_4) + b \mathcal{L}(\xi_1, \bar{\xi}_3) \\ \mathcal{L}(\xi_2, \bar{\xi}_4) = f \mathcal{L}(\xi_1, \bar{\xi}_4) + e \mathcal{L}(\xi_1, \bar{\xi}_3) \end{array} \right.$$

To simplify the notation, rename by $\{\xi_1, \dots, \xi_4\} := \{\xi_1, \xi_2, \bar{\xi}_3, \bar{\xi}_4\}$ the local frame for H of rank-three elements around x_0 , which satisfies $\mathcal{I}_2 = \mathcal{I}_1$ and (36) with $c(x_0) = -1 = e(x_0)$ and $b(x_0) = 0 = f(x_0)$. Mapping $\xi_4 \mapsto c\xi_4$ preserves all the properties satisfied by the frame $\{\xi_1, \dots, \xi_4\}$. For this reason, we can assume that the system (36) writes as

$$(37) \quad \left\{ \begin{array}{l} \mathcal{L}(\xi_2, \xi_3) = \mathcal{L}(\xi_1, \xi_4) + b \mathcal{L}(\xi_1, \xi_3) \\ \mathcal{L}(\xi_2, \xi_4) = f \mathcal{L}(\xi_1, \xi_4) + ec \mathcal{L}(\xi_1, \xi_3) \end{array} \right.$$

with $ec(x_0) = 1$. Last system shows that

$$\begin{aligned}\theta_1 &= \xi_2 \wedge \xi_3 - \xi_1 \wedge \xi_4 - b \xi_1 \wedge \xi_3 \\ \theta_2 &= \xi_2 \wedge \xi_4 - f \xi_1 \wedge \xi_4 - ec \xi_1 \wedge \xi_3\end{aligned}$$

is a frame for $\text{Ker}(\mathcal{L}) \subset \Lambda^2 H$. The volume form $\xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4$ induces a smooth section of $S^2(\Lambda^2 H^*)$, whose restriction to $\text{Ker}(\mathcal{L})$ writes in the frame $\{\theta_1, \theta_2\}$ as

$$\begin{pmatrix} -2 & -f+b \\ -f+b & 2ec \end{pmatrix}$$

The determinant of the matrix above is a smooth function on U , which is strictly negative by our hypothesis on the signature. Equivalently,

$$D = ec + \left(\frac{-f+b}{2}\right)^2 > 0$$

at each point of U . Let $H' \subset H$ be the smooth subbundle defined by $\{\xi_1, \xi_2\}$. For $\eta \in \Gamma(H')$ and $\xi \in \Gamma(H)$, $\mathcal{L}(\eta, \xi) \in \mathcal{I}_1$ and its equivalence class in $\mathcal{I}_1/\mathcal{R}_{12}$ only depends on the equivalence class of ξ in H/H' . Therefore, putting

$$\ell_\eta(\xi \bmod H') := \mathcal{L}(\eta, \xi) \bmod \mathcal{R}_{12}, \quad \xi \in \Gamma(H)$$

defines an homomorphism of vector bundles $\ell_\eta : H/H' \rightarrow \mathcal{I}_1/\mathcal{R}_{12}$. Let be $\eta \in \Gamma(H')$ and $x_1, x_2 : U \rightarrow \mathbb{R}$ smooth functions such that $\eta = x_1 \xi_1 + x_2 \xi_2$. Then, compute ℓ_η and substitute (37), thus obtaining

$$\begin{aligned}\xi_3 &\mapsto x_1 \mathcal{L}(\xi_1, \xi_3) + x_2 [\mathcal{L}(\xi_1, \xi_4) + b \mathcal{L}(\xi_1, \xi_3)] \\ \xi_4 &\mapsto x_1 \mathcal{L}(\xi_1, \xi_4) + x_2 [f \mathcal{L}(\xi_1, \xi_4) + ec \mathcal{L}(\xi_1, \xi_3)]\end{aligned}$$

Clearly, $\{\xi_3, \xi_4\}$ and $\{\mathcal{L}(\xi_1, \xi_3), \mathcal{L}(\xi_1, \xi_4)\}$ are frames for H/H' and $\mathcal{I}_1/\mathcal{R}_{12}$, respectively. With respect to these frames, ℓ_η writes as a matrix of functions with determinant

$$\begin{aligned}\det(\ell_\eta) &= (x_1 + bx_2)(x_1 + fx_2) - ecx_2^2 \\ &= x_1^2 + (b+f)x_1x_2 + (bf-ec)x_2^2\end{aligned}$$

The existence of a rank-two $\eta \in \Gamma(H')$ is equivalent to the vanishing of the determinant of a matrix representing ℓ_η . For η nonzero, one coefficient between x_1, x_2 is nonzero. Now $\det(\ell_\eta) = 0$ is a quadratic form in x_1, x_2 and we can thus rewrite it as an equation of degree two in one variable. The discriminant associated to it

$$\frac{\Delta}{4} = ec + \left(\frac{-f+b}{2}\right)^2 = D$$

is strictly positive everywhere and

$$\eta^+ = \Lambda^+ \eta_1 + \eta_2, \quad \eta^- = \Lambda^- \eta_1 + \eta_2, \quad \Lambda^\pm = -\frac{-b+c}{2} \pm \frac{\sqrt{\Delta}}{4}$$

are two local sections of H' of rank two around x_0 . Once we found $\eta_1 \in \Gamma(H')$ of rank two, the kernel of $i_{\eta_1} \mathcal{L} : H \rightarrow Q$ is a smooth subbundle $E := \text{Ker}(i_{\eta_1} \mathcal{L}) \subset H$ of rank two and we can choose a section η_2 such that $\{\eta_1, \eta_2\}$ is a frame for E . Repeating the argument from (35) for $\mathcal{I}_4 = \mathcal{I}_3$, one gets a frame $\{\eta_3, \eta_4\}$ for the subbundles $F := \text{Ker}(i_{\eta_3} \mathcal{L}) \subset H$ of rank-two elements. By construction, $H = E \oplus F$ and $\text{Ker}(\mathcal{L}) = \Lambda^2 E \oplus \Lambda^2 F$ locally around x_0 , which concludes the proof. \square

Let be $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ and let $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ be the grading described in 2.1, so that $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ is the hyperbolic model algebra. The fact that (M, H) is a regular pair of type \mathfrak{g}_- is easily seen to be equivalent to the existence of local decompositions for H as in Proposition 15. First, consider a regular pair (M, H) of type \mathfrak{g}_- . Denote by $\mathcal{F} \rightarrow M$ the adapted frame bundle associated to

(M, H) and by $G_0 = \text{Aut}_{gr}(\mathfrak{g}_-)$ the structure group of $\mathcal{F} \rightarrow M$. Recall that $H \cong \mathcal{F} \times_{G_0} \mathfrak{g}_{-1}$ and that $\mathcal{L}: \Lambda^2 H \rightarrow TM/H$ corresponds to the bracket $[\cdot, \cdot]: \Lambda^2 \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ via the associated bundle construction. Now recall from 2.1 that

$$\text{Ker}([\cdot, \cdot]) = \Lambda^2 \mathfrak{g}_{-1}^E \oplus \Lambda^2 \mathfrak{g}_{-1}^F$$

and observe that, by Corollary 2, \mathfrak{g}_{-1}^E and \mathfrak{g}_{-1}^F are G_0 -submodules of \mathfrak{g}_{-1} . Therefore they correspond, via the associated bundle construction, to smooth subbundles $E, F \subset H$ of rank two such that $H = E \oplus F$ and $\text{Ker}(\mathcal{L}) = \Lambda^2 E \oplus \Lambda^2 F$. Conversely, if subbundles $E, F \subset H$ of rank two satisfying the conditions above exist, they determine a trivialization of $(\text{gr}(TM), \mathcal{L})$ with standard fiber $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$. If this is the case, indeed, using the first condition we write $\Lambda^2 H$ as the direct sum

$$\Lambda^2 H = \Lambda^2 E \oplus E \otimes F \oplus \Lambda^2 F$$

Since $\mathcal{L}: \Lambda^2 H \rightarrow TM/H$ is a surjective homomorphism of vector bundles with kernel $\Lambda^2 E \oplus \Lambda^2 F$, it descends to a fibered isomorphism $TM/H \cong E \otimes F$. Therefore, $(\text{gr}(TM), \mathcal{L})$ is locally trivial with standard fiber $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$.

Aware of this equivalence, we can thus rephrase the result of Proposition 15 by saying that if $H \subset TM$ is an hyperbolic $(4, 8)$ -distribution, then (M, H) is a regular pair of type $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$.

THEOREM 5. *Let $H \subset TM$ be a $(4, 8)$ -distribution. Then H is hyperbolic if and only if locally around each point there exist two smooth subbundles of H of rank two, say E and F , such that $H = E \oplus F$ and satisfying the following properties:*

- $[\xi_1, \xi_2] \in \Gamma(H)$ for every $\xi_1, \xi_2 \in \Gamma(E)$
- $[\eta_1, \eta_2] \in \Gamma(H)$ for every $\eta_1, \eta_2 \in \Gamma(F)$
- The Levi bracket induces an isomorphism $E \otimes F \cong TM/H$

Equivalently, H is hyperbolic if and only if (M, H) is a regular pair of type $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ described in 2.1.

PROOF. Suppose that H is hyperbolic. By Proposition 15, there locally exist smooth subbundles $E, F \subset H$ of rank two satisfying $H = E \oplus F$ and $\text{Ker}(\mathcal{L}) = \Lambda^2 E \oplus \Lambda^2 F$, so that \mathcal{L} descends to an isomorphism $TM/H \cong E \otimes F$. By definition of \mathcal{L} , the second condition just means that the bracket of any two smooth sections of E lies in $\Gamma(H)$ and the analogous statement for F . Conversely, the existence of such a local decomposition for H is equivalent to the fact that (M, H) is a regular pair of hyperbolic type $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$, which concludes the proof. \square

3. Elliptic case

Let $p, q \geq 0$ be integers such that $n = p + q \geq 1$ and let W be a complex vector space of dimension n . A *Hermitian form* on W is a map $\langle \cdot, \cdot \rangle: W \times W \rightarrow \mathbb{C}$ which is linear in the first argument, anti-linear in the second argument and satisfying $\langle z, w \rangle = \overline{\langle w, z \rangle}$ for all $z, w \in W$. Observe that if $\langle \cdot, \cdot \rangle$ is a complex Hermitian form, then $\beta(z) := \langle z, z \rangle \in \mathbb{R}$ for all $z \in W$ defines a quadratic form on the real vector space underlying W . If β is a quadratic form of signature $(2p, 2q)$, we say that $\langle \cdot, \cdot \rangle$ is a *complex Hermitian form of signature (p, q)* . Let $\langle \cdot, \cdot \rangle$ be a complex Hermitian form on $W = \mathbb{C}^n$ of signature (p, q) . The set of linear isomorphisms $\phi: \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\langle \phi(z), \phi(w) \rangle = \langle z, w \rangle$ for all $z, w \in \mathbb{C}^n$ is a group. If we represent any linear isomorphism ϕ preserving $\langle \cdot, \cdot \rangle$ via a matrix $A \in GL_n(\mathbb{C})$ and we require, in addition, that A has determinant one, we select a Lie subgroup of $GL_n(\mathbb{C})$, the so-called *special unitary group of signature (p, q)* . Now if $\{e_1, \dots, e_n\}$ is a basis of \mathbb{C}^n , putting $\mathbb{J}_j^i := \langle e_i, e_j \rangle$ for $i, j = 1, \dots, n$ defines $\mathbb{J} \in M_n(\mathbb{C})$ and $\langle \cdot, \cdot \rangle$ can be written via matrix-vector multiplication as $\langle z, w \rangle = z^t \cdot \mathbb{J} \cdot \bar{w}$ for all $z, w \in \mathbb{C}^n$. The real vector space

$$\mathfrak{su}(p, q) = \{M \in \mathfrak{gl}(n, \mathbb{C}) \mid \text{tr}(M) = 0, M^* \mathbb{J} = -\mathbb{J} M\},$$

endowed with the usual commutator, is a real Lie algebra and coincides with the Lie algebra of the special unitary group of signature (p, q) . Here, M^* denotes the conjugate transpose matrix of M . It is a well known fact that for all integers $p, q \geq 0$ such that $n = p + q \geq 1$, $\mathfrak{su}(p, q)$ is a real form for $\mathfrak{sl}(n, \mathbb{C})$, which is non-isomorphic to the split-real form $\mathfrak{sl}(n, \mathbb{R})$ (see pag. 207, [10]).

Let V be a real vector space. Putting $z(v \otimes q) := v \otimes (zq)$ for all $v \in V$ and $z, q \in \mathbb{C}$ defines a multiplication for complex scalars, hence a structure of complex vector space, on the real vector space $V \otimes_{\mathbb{R}} \mathbb{C}$. We will denote it by $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ and we will refer to it as the *complexification* of V . Clearly, V identifies via $v \mapsto v \otimes 1$ with a real linear subspace of $V^{\mathbb{C}}$ and any real endomorphism of V extends to a unique \mathbb{C} -linear endomorphism of $V^{\mathbb{C}}$.

Let \mathfrak{g} be a real Lie algebra. Its complexification $\mathfrak{g}^{\mathbb{C}}$ admits a natural structure of complex Lie algebra, obtained by extending the bracket on \mathfrak{g} to a \mathbb{C} -bilinear map. Moreover, \mathfrak{g} is semisimple if and only if $\mathfrak{g}^{\mathbb{C}}$ is semisimple. A finite-dimensional *real representation* of \mathfrak{g} is an homomorphism of real Lie algebras $\mathfrak{g} \rightarrow \mathfrak{gl}_{\mathbb{R}}(V)$, where V is a finite-dimensional real vector space. A *complex representation* is a real homomorphism $\mathfrak{g} \rightarrow \mathfrak{gl}_{\mathbb{C}}(W)$, where W is a finite-dimensional complex vector space. Otherwise put, \mathfrak{g} acts on V by \mathbb{C} -linear maps. Any real representation $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}_{\mathbb{R}}(V)$ induces a complex representation $\mathfrak{g} \rightarrow \mathfrak{gl}_{\mathbb{C}}(V^{\mathbb{C}})$ by extending the \mathbb{R} -linear map $\rho(X): V \rightarrow V$, for $X \in \mathfrak{g}$, to a \mathbb{C} -linear endomorphism of $V^{\mathbb{C}}$. If $S \subset V$ is a subspace which is invariant under the action of \mathfrak{g} , then $S^{\mathbb{C}} \subset V^{\mathbb{C}}$ is also \mathfrak{g} -invariant. In particular, if $V^{\mathbb{C}}$ is \mathfrak{g} -irreducible the same holds for V , while the opposite implication is not true in general. Furthermore, there is a bijective correspondence between complex representations of the real Lie algebra \mathfrak{g} and complex representations of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

3.1. Real and complex structures. Let W be a finite-dimensional complex vector space. A *real structure* on W is a conjugate linear map $R: W \rightarrow W$ such that $R^2 = \text{id}_W$. If R is a real structure on W , the induced map $\Lambda^2 R: \Lambda^2 W \rightarrow \Lambda^2 W$ is a real structure on $\Lambda^2 W$. Let V be a finite-dimensional real vector space. The complexification $W = V^{\mathbb{C}}$ carries a natural real structure $R: W \rightarrow W$, characterized by $R(v \otimes z) = v \otimes \bar{z}$. Clearly, $(\Lambda^2 V)^{\mathbb{C}} = \Lambda^2 W$ and the natural real structure on $(\Lambda^2 V)^{\mathbb{C}}$ coincides with the induced real structure $\Lambda^2 R$. From now on, we will simply denote by R both real structures on W and on $\Lambda^2 W$, so that $R(v_1 \wedge v_2) = R(v_1) \wedge R(v_2)$ for all $v_1, v_2 \in W$. The wedge product $\Lambda^2 W \times \Lambda^2 W \rightarrow \Lambda^4 W$ is a symmetric \mathbb{C} -bilinear map. Again, $R(\omega_1 \wedge \omega_2) = R(\omega_1) \wedge R(\omega_2)$ for all $\omega_1, \omega_2 \in \Lambda^2 W$ coincides with the natural real structure on $\Lambda^4 W = (\Lambda^4 V)^{\mathbb{C}}$. By first applying R to the second variable and then forming the wedge product, we thus get a map which is \mathbb{C} -linear in the first variable and conjugate linear in the second and satisfying

$$\omega_2 \wedge R(\omega_1) = R(\omega_1) \wedge \omega_2 = R(\omega_1 \wedge R(\omega_2))$$

for all $\omega_1, \omega_2 \in \Lambda^2 W$.

Suppose now that W has complex dimension four. Recall that any choice of volume form determines an explicit isomorphism $\Lambda^4 W \cong \mathbb{C}$. Let $\mathcal{B} = \{v_1, \dots, v_4\}$ be a basis for V and $\text{vol} = v_1 \wedge \dots \wedge v_4 \in \Lambda^4 V$. We can look at \mathcal{B} as a basis for W and $\text{vol} \in \Lambda^4 W$ such that $R(\text{vol}) = \text{vol}$. Then, putting

$$(38) \quad \mathcal{H}: \Lambda^2 W \times \Lambda^2 W \rightarrow \mathbb{C}, \quad \omega_1 \wedge R(\omega_2) = \mathcal{H}(\omega_1, \omega_2) \text{vol}$$

defines an Hermitian form. Indeed, we see from above that \mathcal{H} is \mathbb{C} -linear in the first variable and conjugate linear in the second. Moreover,

$$\mathcal{H}(\omega_2, \omega_1) \text{vol} = \omega_2 \wedge R(\omega_1) = R(\omega_1 \wedge R(\omega_2)) = \overline{\mathcal{H}(\omega_1, \omega_2)} R(\text{vol}) = \overline{\mathcal{H}(\omega_1, \omega_2)} \text{vol}$$

shows that $\mathcal{H}(\omega_2, \omega_1) = \overline{\mathcal{H}(\omega_1, \omega_2)}$. Since \mathcal{H} is Hermitian, $\mathcal{H}(\omega, \omega) \in \mathbb{R}$ for all $\omega \in \Lambda^2 W$. Therefore, if we look at ω as an element of the real vector space underlying $\Lambda^2 W$, $\mathcal{H}(\omega, \omega)$

defines a real quadratic form. Writing $\omega = \phi \otimes 1 + \psi \otimes i$ for $\phi, \psi \in \Lambda^2 V$, we compute

$$\omega \wedge R(\omega) = (\phi \otimes 1 + \psi \otimes i) \wedge (\phi \otimes 1 - \psi \otimes i) = (\phi \wedge \phi + \psi \wedge \psi) \otimes 1$$

which shows that $\mathcal{H}(\omega, \omega) = \beta(\phi) + \beta(\psi)$ where $\beta: \Lambda^2 V \rightarrow \mathbb{R}$ is the quadratic form defined by vol as in Proposition 3. If $\mathcal{H}(\omega, \omega)$ is nonzero, then β restricts to a definite quadratic form on the real vector space underlying the complex line generated by ω .

Let V be a finite-dimensional real vector space. A *complex structure* on V is an endomorphism $J: V \rightarrow V$ such that $J^2 = -\text{id}_V$. Then, the \mathbb{C} -linear extension $J_{\mathbb{C}}: V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ of J is diagonalizable with eigenvalues $\pm i$. The complexification of V writes as the direct sum

$$V^{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$$

of the eigenspaces of eigenvalue $+i$ and $-i$ for $J_{\mathbb{C}}$, respectively. Elements of the form $v \otimes 1 - J(v) \otimes i$ and $v \otimes 1 + J(v) \otimes i$ for $v \in V$ generate, respectively, $V^{(1,0)}$ and $V^{(0,1)}$. Taking the conjugate defines a real structure $R: V^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ mapping $V^{(1,0)}$ and $V^{(0,1)}$ to each other.

Observe that if J is a complex structure on V , the same is true for $-J$. The eigenspaces of eigenvalue $+i$ and $-i$ for $-J$ coincide, respectively, with the eigenspaces of eigenvalue $-i$ and $+i$ for J .

Let V be a real representation of \mathfrak{g} equipped with a complex structure $J: V \rightarrow V$ such that $J(X \cdot v) = X \cdot J(v)$ for all $X \in \mathfrak{g}$ and $v \in V$. This is equivalent to the fact that both eigenspaces of $J_{\mathbb{C}}$ are invariant under the \mathfrak{g} -action on $V^{\mathbb{C}}$. Suppose that $J: V \rightarrow V$ is a complex structure and $V^{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$ the eigenspace decomposition for $J_{\mathbb{C}}$. Since $J^2 = -\text{id}_V$, the linear map $\Lambda^2 J: \Lambda^2 V \rightarrow \Lambda^2 V$ induced by J via $\Lambda^2 J(v \wedge w) := J(v) \wedge J(w)$ satisfies $(\Lambda^2 J)^2 = \text{id}_{\Lambda^2 V}$ and it is thus diagonalizable with eigenvalues ± 1 . Let us denote by $\Lambda^2 V = \mathcal{E}_1 \oplus \mathcal{E}_{-1}$ the direct sum decomposition into the corresponding eigenspaces. Observe that \mathcal{E}_{-1} is generated by elements of the form

$$J(v) \wedge J(w) - v \wedge w \quad v, w \in V.$$

Moreover, we have the direct sum decomposition

$$(\Lambda^2 V)^{\mathbb{C}} = \Lambda^2 V^{\mathbb{C}} = \Lambda^2 V^{(1,0)} \oplus (V^{(1,0)} \otimes V^{(0,1)}) \oplus \Lambda^2 V^{(0,1)}$$

Now suppose that V has real dimension four. Then $V^{(1,0)}, V^{(0,1)} \subset V^{\mathbb{C}}$ are both two-dimensional complex subspaces and $\ell = \Lambda^2 V^{(0,1)}$ is a complex line whose conjugate $R(\ell)$ coincides with $\Lambda^2 V^{(1,0)}$. Moreover, any element of $\ell \oplus R(\ell)$ is an eigenvector of eigenvalue -1 for $\Lambda^2 J_{\mathbb{C}}$, so that $\ell \oplus R(\ell)$ is contained in the complexification of \mathcal{E}_{-1} . Since evidently $V^{(1,0)} \otimes V^{(0,1)} \subset (\mathcal{E}_1)^{\mathbb{C}}$, we conclude that $(\mathcal{E}_{-1})^{\mathbb{C}} = \ell \oplus R(\ell)$, hence that \mathcal{E}_{-1} has real dimension two.

3.2. Model algebra. In the previous section, we described the model for the symbol algebra of hyperbolic $(4, 8)$ -distributions as the negative part of a grading on $\mathfrak{sl}(5, \mathbb{R})$. An elliptic model can be realized, analogously, inside another real form for $\mathfrak{sl}(5, \mathbb{C})$, namely $\mathfrak{su}(3, 2)$. An appropriate choice of Hermitian form on \mathbb{C}^5 leads to a description of $\mathfrak{su}(3, 2)$, which can be easily compared to the hyperbolic model from the previous section. Write $x \in \mathbb{C}^2$ as $x = (x_1, x_2)$ and $(x, y, z) \in \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^2$. The last summand appearing in the formula

$$\langle (x, y, z), (x', y', z') \rangle = x_1 \bar{z}'_1 + x_2 \bar{z}'_2 + z_1 \bar{x}'_1 + z_2 \bar{x}'_2 + y \bar{y}'$$

is the canonical positive definite Hermitian form on \mathbb{C} and the sum of the remaining terms defines an Hermitian form of signature $(2, 2)$ on $\mathbb{C}^2 \oplus \mathbb{C}^2$. We thus see that the formula defines an Hermitian form on \mathbb{C}^5 of signature $(3, 2)$. Consider the corresponding algebra $\mathfrak{t} = \mathfrak{su}(3, 2)$. Denote by I_2 the identity matrix of size two and by

$$\mathbb{J} = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & 1 & 0 \\ I_2 & 0 & 0 \end{pmatrix}$$

the matrix representing $\langle \cdot, \cdot \rangle$ in the standard basis. Writing the condition $M^*\mathbb{J} = -\mathbb{J}M$ for $M \in \mathfrak{gl}(5, \mathbb{C})$, we get the explicit description of \mathfrak{t} as the set of block matrices

$$\mathfrak{t} = \left\{ \begin{pmatrix} -A^* & -v & B \\ -\bar{w}^t & is & \bar{v}^t \\ C & w & A \end{pmatrix} \mid A \in \mathfrak{gl}(2, \mathbb{C}), s = -2\text{Im}(\text{tr}(A)), B, C \in \mathfrak{u}(2), v, w \in \mathbb{C}^2 \right\}$$

with blocks of same size as in the description of the hyperbolic model. Here, $v^* \in (\mathbb{C}^2)^*$ denotes the element dual to $v \in \mathbb{C}^2$ and A^* the conjugate transpose of $A \in \mathfrak{gl}(2, \mathbb{C})$.

The description above shows that there exist isomorphisms of real vector spaces $\mathfrak{t}_{-1} \cong \mathbb{C}^2$, $\mathfrak{t}_{-2} \cong \mathfrak{u}(2)$ and $\mathfrak{t}_0 \cong \mathfrak{gl}(2, \mathbb{C})$. We thus get a $|2|$ -grading $\mathfrak{t} = \bigoplus_{i=-2}^2 \mathfrak{t}_i$ such that if $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ is the grading on $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ described in the previous section, then

$$\dim_{\mathbb{R}}(\mathfrak{t}_i) = \dim_{\mathbb{R}}(\mathfrak{g}_i) \text{ and } \mathfrak{t}_i^{\mathbb{C}} = \mathfrak{g}_i^{\mathbb{C}} \text{ for all } i = -2, \dots, 2.$$

The natural complex structure on \mathfrak{t}_{-1} , namely the multiplication for $i \in \mathbb{C}$, determines the direct sum decomposition $\mathfrak{t}_{-1}^{\mathbb{C}} = \mathfrak{t}_{-1}^{(1,0)} \oplus \mathfrak{t}_{-1}^{(0,1)}$. Since \mathfrak{t}_0 acts on \mathfrak{t}_{-1} by \mathbb{C} -linear maps, the action is compatible with the complex structure. Then, $\mathfrak{t}_{-1}^{(1,0)}, \mathfrak{t}_{-1}^{(0,1)} \subset \mathfrak{t}_{-1}^{\mathbb{C}}$ are both \mathfrak{t}_0 -submodules.

Recall from Section 1 that the complexification of the graded component $\mathfrak{g}_{-1} \subset \mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ admits a decomposition

$$\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$$

into $\mathfrak{gl}(2, \mathbb{C})$ -irreducible components, namely the subspaces of rank-two elements for the complexification of the bracket, such that $\text{Ker}([\cdot, \cdot]_{\mathbb{C}}) = \Lambda^2 \mathfrak{g}_{-1}^E \oplus \Lambda^2 \mathfrak{g}_{-1}^F$. Since $\mathfrak{t}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^{\mathbb{C}}$ and $\mathfrak{t}_0 \cong \mathfrak{gl}(2, \mathbb{C})$, this is a decomposition of $\mathfrak{t}_{-1}^{\mathbb{C}}$ as the direct sum of \mathfrak{t}_0 -irreducible components. On the other hand, the decomposition of $\mathfrak{t}_{-1}^{\mathbb{C}}$ into eigenspaces for the natural complex structure is also a decomposition into \mathfrak{t}_0 -irreducible components. Since such a decomposition is unique up to the order of the summands, we can suppose that $\mathfrak{g}_{-1}^E = \mathfrak{t}_{-1}^{(1,0)}$ and $\mathfrak{g}_{-1}^F = \mathfrak{t}_{-1}^{(0,1)} = R(\mathfrak{g}_{-1}^E)$, which shows that

$$\text{Ker}([\cdot, \cdot]) = \mathcal{E}_{-1}$$

is the eigenspace of eigenvalue -1 for $\Lambda^2 i$. If $\ell := \Lambda^2 \mathfrak{g}_{-1}^E$, we thus have that

$$\ell \wedge R(\ell) = \Lambda^2 \mathfrak{g}_{-1}^E \wedge \Lambda^2 \mathfrak{g}_{-1}^F$$

is nonzero. Therefore, the Hermitian form (38) is nonzero on ℓ and it thus restricts to an elliptic quadratic form on the two-dimensional real vector space underlying ℓ , which is \mathcal{E}_{-1} . We will refer to $\mathfrak{t}_{-1} \subset \mathfrak{su}(3, 2)$ as the model algebra for the elliptic $(4, 8)$ -type.

In the same notation of (31), we write the elements of \mathfrak{t}_{-1} as $(-\bar{w}^t, w)$ with $w \in \mathbb{C}^2$. Also, we will write $X \in \mathfrak{t}_{-2} \cong \mathfrak{u}(2)$ and $(-A^*, A) \in \mathfrak{t}_0$ with $A \in \mathfrak{gl}(2, \mathbb{C})$. The bracket on \mathfrak{t} restricts to the negative graded components $\mathfrak{t}_{-1} \times \mathfrak{t}_{-1} \rightarrow \mathfrak{t}_{-2}$ as

$$[(-\bar{v}^t, v), (-\bar{w}^t, w)] = -v \otimes \bar{w}^t + w \otimes \bar{v}^t$$

that is the imaginary part of an Hermitian form $\mathfrak{t}_{-1} \times \mathfrak{t}_{-1} \rightarrow (\mathfrak{t}_{-2})_{\mathbb{C}}$, namely

$$\mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathfrak{gl}(2, \mathbb{C}), \quad (v, w) \mapsto w \otimes \bar{v}^t$$

This is equivalent to the fact that

$$[J(-\bar{v}^t, v), J(-\bar{w}^t, w)] = [(-\bar{v}^t, v), (-\bar{w}^t, w)]$$

for all $v, w \in \mathbb{C}^2$, where $J: \mathfrak{t}_{-1} \rightarrow \mathfrak{t}_{-1}$ denotes the natural complex structure. [18] proves that $\pm J$ are the only complex structures on \mathfrak{t}_{-1} satisfying this compatibility condition with the bracket.

COROLLARY 3. *Let $\mathfrak{t} = \mathfrak{su}(3, 2)$ and let $\mathfrak{t} = \bigoplus_{i=-2}^2 \mathfrak{t}_i$ be the grading described in 3.2. Then, the adjoint action of \mathfrak{t}_0 on \mathfrak{t}_- defines an isomorphism $\mathfrak{t}_0 \cong \mathfrak{der}_{gr}(\mathfrak{t}_-)$. In particular, any grading-preserving derivation of \mathfrak{t}_- restricts to an homomorphism of \mathfrak{t}_{-1} which is compatible with the complex structure.*

PROOF. Put $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ and consider the grading $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ from 2.1. Recall that the grading $\mathfrak{t} = \bigoplus_{i=-2}^2 \mathfrak{t}_i$ is such that $\mathfrak{t}_i^{\mathbb{C}} = \mathfrak{g}_i^{\mathbb{C}}$ for all $i = -2, \dots, 2$. The adjoint action of \mathfrak{t}_0 on \mathfrak{t}_- is an injective homomorphism of Lie algebras $\mathfrak{t}_0 \rightarrow \mathfrak{der}_{gr}(\mathfrak{t}_-)$, hence it defines a real representation of \mathfrak{t}_0 which can be naturally extended to a complex representation of \mathfrak{t}_0 . Suppose that $\psi \in \mathfrak{der}_{gr}(\mathfrak{t}_-)$ and that $\phi: \mathfrak{t}_{-1} \rightarrow \mathfrak{t}_{-1}$ is the restriction of ψ . By Corollary 2, the \mathbb{C} -linear extension $\phi_{\mathbb{C}}: \mathfrak{t}_{-1}^{\mathbb{C}} \rightarrow \mathfrak{t}_{-1}^{\mathbb{C}}$ preserves the direct sum decomposition $\mathfrak{t}_{-1}^{\mathbb{C}} = \mathfrak{t}_{-1}^{(1,0)} \oplus \mathfrak{t}_{-1}^{(0,1)}$, which is equivalent to the fact that ϕ is complex linear on \mathfrak{t}_{-1} . Up to isomorphism, ϕ can be thus seen as an element of $\mathfrak{gl}(2, \mathbb{C})$ and we know that $\mathfrak{gl}(2, \mathbb{C}) \cong \mathfrak{t}_0$. On the other hand, ϕ uniquely determines ψ thus showing that $\mathfrak{t}_0 \rightarrow \mathfrak{der}_{gr}(\mathfrak{t}_-)$ is also surjective, hence $\mathfrak{t}_0 \cong \mathfrak{der}_{gr}(\mathfrak{t}_-)$. In particular, the action of $\mathfrak{der}_{gr}(\mathfrak{t}_-)$ on $\mathfrak{t}_{-1} \cong \mathbb{C}^2$ is compatible with the natural complex structure. \square

3.3. Characterization of the elliptic orbit.

THEOREM 6. *(Existence of a complex structure) Let V be a real four-dimensional vector space, $P \subset \Lambda^2 V$ a linear subspace of dimension two and consider the induced action of $GL(V)$ on the Grassmannian $Gr(2, \Lambda^2 V)$ (see Proposition 2). Then P is in the elliptic $GL(V)$ -orbit if and only if there exists a complex structure J on V such that*

$$P = \text{sp}\{Jv \wedge Jw - v \wedge w \mid v, w \in V\}$$

is the eigenspace of eigenvalue -1 for $\Lambda^2 J$.

PROOF. Consider a volume form $\text{vol} \in \Lambda^4 V$, the Hermitian form \mathcal{H} and the real quadratic form $\beta: \Lambda^2 V \rightarrow \mathbb{R}$ defined by vol and the wedge product as in (38).

First, suppose that J is a complex structure on V such that $P = \mathcal{E}_{-1}$ is the eigenspace of eigenvalue -1 for $\Lambda^2 J$. Then $P^{\mathbb{C}} = \Lambda^2 V^{(1,0)} \oplus \Lambda^2 V^{(0,1)}$ and $\ell := \Lambda^2 V^{(0,1)} \subset P^{\mathbb{C}}$ is a complex line such that

$$P^{\mathbb{C}} = \ell \oplus R(\ell)$$

and $\mathcal{H}|_{\ell} = \beta|_P$. For $v \in V$, put $v^{(1,0)} = v \otimes 1 - J(v) \otimes i$. Let $\{v, w\} \subset V$ be linearly independent vectors such that $\{v^{(1,0)}, w^{(1,0)}\}$ is a basis for $V^{(1,0)}$. Then $\omega = v^{(1,0)} \wedge w^{(1,0)} = \phi \otimes 1 + \psi \otimes i \in \ell$ with

$$\phi = J(v) \wedge J(w) - v \wedge w \quad \text{and} \quad \psi = J(v) \wedge w + v \wedge J(w) = J(J(w)) \wedge J(v) - J(w) \wedge v$$

Since $\{v, J(v), w, J(w)\} \subset V$ is a basis of V , $\text{vol} = J(v) \wedge J(w) \wedge v \wedge w \in \Lambda^4 V$ is nonzero and

$$\omega \wedge R(\omega) = (\phi \wedge \phi + \psi \wedge \psi) \otimes 1 = -4 \text{vol} \otimes 1$$

is nonzero. Therefore $\mathcal{H}|_{\ell}$ is nonzero, thus a definite Hermitian form on a line and since $\mathcal{H}|_{\ell} = \beta|_P$, P is elliptic.

Conversely, suppose that P is elliptic. Consider the natural complex structure $\cdot i$ on $V^{\mathbb{C}}$ and the corresponding direct sum decomposition $V^{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$ into eigenspaces for $\cdot i$. Since $V \cap V^{(1,0)} = \{0\}$, the projection $V^{\mathbb{C}} \rightarrow V^{(1,0)}$ descends to an isomorphism $V \cong V^{(1,0)}$, which induces a complex structure $J: V \rightarrow V$ such that $J_{\mathbb{C}} = \cdot i$ and $V^{(1,0)}, V^{(0,1)}$ are the eigenspaces for $J_{\mathbb{C}}$. Denote by $\mathcal{E}_{-1} \subset \Lambda^2 V$ the eigenspace of eigenvalue -1 for $\Lambda^2 J$. From the first part of the proof, we know that $\mathcal{E}_{-1} \subset \Lambda^2 V$ is an elliptic linear subspace of dimension two. Since both P and \mathcal{E}_{-1} are elliptic, by Proposition 4 there exists $\phi \in O(3, 3)$ such that $\phi \cdot \mathcal{E}_{-1} = P$. Now by Proposition 3, the action of ϕ is the induced action of an element of $GL(4, \mathbb{R})$. Therefore, there exists $A \in GL(V)$ such that

$$\Lambda^2 A \cdot \mathcal{E}_{-1} = P.$$

Observe that if J is a complex structure on V and $A \in GL(V)$, then $K = AJA^{-1}$ is a complex structure on V . Recall that the typical element of \mathcal{E}_{-1} writes as $\phi = J(v) \wedge J(w) - v \wedge w$ for some $v, w \in V$. For $\phi \in \mathcal{E}_{-1}$, we thus have

$$\Lambda^2 A(\phi) = AJ(v) \wedge AJ(w) - Av \wedge Aw = K(x) \wedge K(y) - x \wedge y$$

where $x = Av$ and $y = Aw$. Otherwise put, $\Lambda^2 A(\phi) \in \nu_{-1}$ for any $\phi \in \mathcal{E}_{-1}$, where $\nu_{-1} \subset \Lambda^2 V$ denotes the eigenspace of eigenvalue -1 for $\Lambda^2 K$. This shows that $P \subset \nu_{-1}$. Since both spaces have dimension two, we have that $P = \nu_{-1}$, which concludes the proof. \square

COROLLARY 4. *Let V, W be real four-dimensional vector spaces and let $F : \Lambda^2 V \rightarrow W$ be a surjective linear map. Then F is elliptic if and only if there exists a complex structure J on V such that $F(Jv \wedge Jw) = F(v \wedge w)$ for all $v, w \in V$.*

PROOF. The result follows by applying Theorem 6 to $P = \text{Ker}(F)$. \square

3.4. The relation to almost CR-structures. Let M be a smooth manifold of dimension $n = 2k + d$ and let $H \subset TM$ be a smooth distribution of rank $2k$. An *almost complex structure* on H is a vector bundle endomorphism $J : H \rightarrow H$ such that $J^2 = -\text{id}_H$. We will also say that (H, J) is an *almost CR-structure* on M of *CR-dimension* k and *CR-codimension* d . If (H, J) is an almost CR-structure, the complexified bundle $H^{\mathbb{C}} \subset T^{\mathbb{C}}M$ decomposes as the direct sum of complex vector bundles

$$H^{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}.$$

In particular, $H^{(0,1)}$ is generated by the sections of the form $\xi + iJ\xi$ for $\xi \in \Gamma(H)$ and $H^{(1,0)}$ is the conjugate of $H^{(0,1)}$. Put $Q = TM/H$. The Levi bracket $\mathcal{L} : H \times H \rightarrow Q$ uniquely extends to an homomorphism of complex vector bundles $\mathcal{L}_{\mathbb{C}} : H^{\mathbb{C}} \times H^{\mathbb{C}} \rightarrow Q^{\mathbb{C}}$. An almost CR-structure on H is said to be *partially integrable* if $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all $\xi, \eta \in \Gamma(H)$. Observe that partial integrability is equivalent to the fact that the following expression

$$\mathcal{L}_{\mathbb{C}}(\xi + iJ\xi, \eta + iJ\eta) = \mathcal{L}(\xi, \eta) - \mathcal{L}(J\xi, J\eta) + i(\mathcal{L}(J\xi, \eta) + \mathcal{L}(\xi, J\eta))$$

vanishes for all $\xi, \eta \in \Gamma(H)$, hence that $[H^{(0,1)}, H^{(0,1)}] \subset H^{\mathbb{C}}$ and $[H^{(1,0)}, H^{(1,0)}] \subset H^{\mathbb{C}}$. Also, partial integrability is equivalent to the fact that a complex multiple of the map

$$H^{(0,1)} \times H^{(0,1)} \rightarrow Q^{\mathbb{C}}, \quad (\zeta_1, \zeta_2) \mapsto q_{\mathbb{C}}([\zeta_1, \overline{\zeta_2}])$$

is Hermitian. If this is the case, then \mathcal{L} coincides with the imaginary part of this Hermitian map. If (H, J) is a partially integrable almost CR-structure, then both $[\xi, \eta] - [J\xi, J\eta]$ and $[J\xi, \eta] + [\xi, J\eta]$ are smooth sections of H for all $\xi, \eta \in \Gamma(H)$. Therefore, to any such structure we can associate the *Nijenhuis tensor*

$$N : \Lambda^2 H \rightarrow H, \quad (\xi, \eta) \mapsto [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta])$$

A partially integrable almost CR-structure is called *integrable* if the bundles $H^{(0,1)}$ and $H^{(1,0)}$ are involutive and this is equivalent to the vanishing of N .

Let us consider partially integrable almost CR-structures which are one-step bracket-generating. In particular, this makes sense for partially integrable almost CR-structures of CR-dimension k and CR-codimension k^2 for any integer $k \geq 2$. Such structures are investigated in [18], where they are referred to as *free CR-distributions*. For $k = 2$, these are $(4, 8)$ -distributions equipped with a partially integrable almost complex structure. A $(4, 8)$ -distribution carries, locally, such a complex structure if and only if it is elliptic, as we are going to show in the following.

Let $\mathfrak{t} = \mathfrak{su}(3, 2)$ and let $\mathfrak{t} = \bigoplus_{i=-2}^2 \mathfrak{t}_i$ be the grading described in 3.2. As usual, denote by $\mathfrak{t}_- \subset \mathfrak{t}$ the negative part, which is the elliptic model algebra. The notions of regular pair of type \mathfrak{t}_- and of free CR-distribution of CR-dimension two are easily seen to be equivalent.

First, suppose that (M, H) is a regular pair of type \mathfrak{t}_- . Denote by $\mathcal{F} \rightarrow M$ the adapted frame bundle associated to (M, H) and by $T_0 = \text{Aut}_{gr}(\mathfrak{t}_-)$ the structure group of $\mathcal{F} \rightarrow M$. Recall that

$$H \cong \mathcal{F} \times_{T_0} \mathfrak{t}_{-1}$$

and that $\mathcal{L}: \Lambda^2 H \rightarrow TM/H$ corresponds to the bracket $[\cdot, \cdot]: \Lambda^2 \mathfrak{t}_{-1} \rightarrow \mathfrak{t}_{-2}$ via the associated bundle construction. Corollary 3 shows that $\text{Lie}(T_0) = \mathfrak{t}_0$ and we know that $\mathfrak{t}_0 \cong \mathfrak{gl}(2, \mathbb{C})$. Then, the natural complex structure on $\mathfrak{t}_{-1} \cong \mathbb{C}^2$ is T_0 -invariant, thus it corresponds to an almost complex structure $J: H \rightarrow H$, which defines smooth subbundles of rank two

$$H^{(1,0)}, H^{(0,1)} \subset H^{\mathbb{C}} \quad \text{such that} \quad H^{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}.$$

The T_0 -submodules $\mathfrak{t}_{-1}^{(1,0)}, \mathfrak{t}_{-1}^{(0,1)} \subset \mathfrak{t}_{-1}^{\mathbb{C}}$ correspond, via the associated bundle construction, to the subbundles $H^{(1,0)}, H^{(0,1)} \subset H^{\mathbb{C}}$. Since $\text{Ker}([\cdot, \cdot]_{\mathbb{C}}) = \Lambda^2 \mathfrak{t}_{-1}^{(1,0)} \oplus \Lambda^2 \mathfrak{t}_{-1}^{(0,1)}$, we have that

$$\text{Ker}(\mathcal{L}_{\mathbb{C}}) = \Lambda^2 H^{(1,0)} \oplus \Lambda^2 H^{(0,1)} \quad \text{i.e.} \quad \text{Ker}(\mathcal{L}) = \mathcal{E}_{-1},$$

where $\mathcal{E}_{-1} \subset \Lambda^2 H$ denotes the smooth subbundle defined by the eigenspaces of eigenvalues -1 associated to $\Lambda^2 J_x$ for $x \in M$. The condition $\text{Ker}(\mathcal{L}) = \mathcal{E}_{-1}$ evidently means that J is partially integrable.

Conversely, if $J: H \rightarrow H$ is a partially integrable almost complex structure, we have that $\text{Ker}(\mathcal{L}) = \mathcal{E}_{-1}$. In particular, \mathcal{L} descends to an isomorphism $Q \cong \mathcal{E}_1$. Moreover, denoting by $H^{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}$ the induced direct sum decomposition, \mathcal{L} is the imaginary part of an Hermitian form obtained as a complex multiple of

$$H^{(0,1)} \times H^{(0,1)} \rightarrow Q^{\mathbb{C}}, \quad (\zeta_1, \zeta_2) \mapsto q_{\mathbb{C}}([\zeta_1, \overline{\zeta_2}])$$

Any local frame of H , then, determines a local trivialization of $(\text{gr}(TM), \mathcal{L})$ with standard fiber $\mathfrak{t}_- \subset \mathfrak{su}(3, 2)$. We thus showed that regular pairs of type $\mathfrak{t}_- \subset \mathfrak{su}(3, 2)$ and free CR -distributions of CR -dimension two are equivalent.

PROPOSITION 16. *Let $H \subset TM$ be an elliptic $(4, 8)$ -distribution. Then, locally around each point $x \in M$, there exists a partially integrable almost complex structure J on H .*

PROOF. Let $H \subset TM$ be an elliptic $(4, 8)$ -distribution and denote by $\mathcal{L}_{\mathbb{C}}: \Lambda^2 H^{\mathbb{C}} \rightarrow Q^{\mathbb{C}}$ the \mathbb{C} -linear extension of the Levi bracket. Let be $x \in M$. By Corollary 4, there exists a complex structure $J_x: H_x \rightarrow H_x$ such that $\mathcal{L}_x(J_x v, J_x w) = \mathcal{L}_x(v, w)$ for all $v, w \in H_x$. Write as $H_x^{\mathbb{C}} = E_x \oplus F_x$ the decomposition into eigenspaces for $J_x^{\mathbb{C}}$. In particular, we have that $\text{Ker}((\mathcal{L}_{\mathbb{C}})_x) = \Lambda^2 E_x \oplus \Lambda^2 F_x$. Then, $H_x^{\mathbb{C}} = E_x \oplus F_x$ coincides with the decomposition into rank-two elements for $(\mathcal{L}_{\mathbb{C}})_x$ and $(\text{gr}(T_x^{\mathbb{C}} M), (\mathcal{L}_{\mathbb{C}})_x)$ is isomorphic to the complex nilpotent graded Lie algebra $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{C})$ described in Section 1. Starting from this isomorphism, one can prove as in Proposition 15 that complex rank-two smooth subbundles $E, F \subset H^{\mathbb{C}}$ such that $H^{\mathbb{C}} = E \oplus F$ and $\text{Ker}(\mathcal{L}_{\mathbb{C}}) = \Lambda^2 E \oplus \Lambda^2 F$ are defined on a neighborhood of x . On the same neighborhood, one can define an homomorphism $K: H^{\mathbb{C}} \rightarrow H^{\mathbb{C}}$ by requiring that $K(\xi) = i\xi$ for all $\xi \in \Gamma(E)$ and $K(\eta) = -i\eta$ for all $\eta \in \Gamma(F)$. By definition, $K_x = (J_{\mathbb{C}})_x$. Since the real subbundle $H \subset H^{\mathbb{C}}$ has trivial intersection with E , the projection of $H^{\mathbb{C}}$ onto E restricts to an isomorphism $H \cong E$. This induces a complex structure J on H such that $J_{\mathbb{C}} = K$. The expression for $\text{Ker}(\mathcal{L}_{\mathbb{C}})$ just says that J is partially integrable, which concludes the proof. \square

Proposition 16 proves that any elliptic $(4, 8)$ -distribution, locally, is a free CR -distribution of CR -dimension two. The next result, then, easily follows.

THEOREM 7. *Let $H \subset TM$ be a $(4, 8)$ -distribution of generic type. Then H is elliptic if and only if, locally, (M, H) is a free CR -distribution of CR -dimension two. If this is the case and*

$H^{\mathbb{C}} = H^{(1,0)} \oplus H^{(0,1)}$ is the local decomposition determined by the almost complex structure of Proposition 16, then a complex multiple of

$$H^{(0,1)} \times H^{(0,1)} \rightarrow Q^{\mathbb{C}}, \quad (\xi, \eta) \mapsto q_{\mathbb{C}}([\xi, \bar{\eta}])$$

is an Hermitian form, whose imaginary part gives the Levi bracket associated to H and (M, H) is a regular pair of type $\mathfrak{t}_- \subset \mathfrak{su}(3, 2)$.

PROOF. The first implication was proved in Proposition 16. Conversely, the existence of a partially integrable almost complex structure on H is equivalent to the fact that (M, H) is a regular pair of type $\mathfrak{t}_- \subset \mathfrak{su}(3, 2)$, hence H is elliptic. \square

4. Equivalence to parabolic geometries

In this Section, we show that generic $(4, 8)$ -distributions are equivalent to certain parabolic geometries. Let \mathfrak{g} be a semisimple Lie algebra and let $k > 0$ be an integer. A $|k|$ -grading on \mathfrak{g} is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ into a direct sum of subspaces such that

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_i = \{0\}$ for $|i| > k$;
- the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by \mathfrak{g}_{-1} ;
- \mathfrak{g}_{-k} and \mathfrak{g}_k are both nonzero

The choice of grading is equivalent to the choice of a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, which coincides with the sum of the nonnegative graded components (see Section 3 of Appendix A). By definition, the sum \mathfrak{g}_- of the negative graded components is a nilpotent graded subalgebra of \mathfrak{g} generated by \mathfrak{g}_{-1} such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$. Putting $\mathfrak{g}^i = \bigoplus_{j=i}^k \mathfrak{g}_j$ for $i = -k, \dots, k$ defines a filtration of the form $\mathfrak{g}^k \subset \cdots \subset \mathfrak{g}^{-k} = \mathfrak{g}$.

Let G be a Lie group with Lie algebra \mathfrak{g} and let $P \subset G$ be a subgroup with Lie algebra \mathfrak{p} . Then $P \subset G$ is a *parabolic* subgroup for the given $|k|$ -grading on \mathfrak{g} , which means that if we restrict the natural adjoint action $\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g})$ to P , we obtain an action of P on \mathfrak{g} by filtration-preserving maps. In particular, last condition implies that $P \subset G$ is a closed subgroup. Now P writes, according to the direct sum decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$, as the semidirect product of a reductive Levi subgroup $G_0 \subset P$ and a nilpotent normal subgroup P_+ . Via the adjoint action, G_0 acts on \mathfrak{g} by maps preserving the grading on \mathfrak{g} .

Let M be a smooth manifold and $\{T^i M\}_{i=-1}^{-k}$ a filtration on M . For $i = -k, \dots, -1$, suppose that $\text{rank}(T^i M) = \dim(\mathfrak{g}^i/\mathfrak{p})$. Let $p_0: E \rightarrow M$ be a G_0 -bundle. Put $T_u^0 E = \text{Ker}(T_u p_0)$ and $T_u^i E := T_u p_0^{-1}(T^i M)$ for any $u \in E$ and $i = -k, \dots, -1$. Then $T^0 E \subset TE$ is the vertical bundle of $p_0: E \rightarrow M$ and $\{T^i E\}_{i=0}^{-k}$ is a G_0 -invariant filtration of TE . Suppose that $\theta_i \in \Gamma(L(T^i E, \mathfrak{g}_i))$ are partially defined differential forms for $i = -k, \dots, -1$, satisfying:

- $\theta_i(u): T^i E \rightarrow \mathfrak{g}_i$ is surjective with kernel $T_u^{i+1} E$ for all $u \in E$ and $i = -k, \dots, -1$;
- $(r^g)^* \theta_i = \text{Ad}(g^{-1}) \circ \theta_i$ for all $g \in G_0$

and put $\theta = (\theta_{-1}, \dots, \theta_{-k})$. The collection $\mathcal{T} = (\{T^i M\}, p_0: E \rightarrow M, \theta)$ is said to be an *infinitesimal flag structure* of type (G, P) . Suppose now that the bracket of vector fields is compatible with the filtration on M , i.e. $[\xi, \eta] \in \Gamma(T^{i+j} M)$ for all $\xi \in \Gamma(T^i M)$ and $\eta \in \Gamma(T^j M)$. Then, the analogous compatibility condition is satisfied by vector fields on E with respect to the G_0 -invariant filtration of TE introduced above. An infinitesimal flag structure \mathcal{T} of type (G, P) is said to be *regular* if

$$(39) \quad \theta_{i+j}([\lambda, \zeta]) = [\theta_i(\lambda), \theta_j(\zeta)]$$

for all $\lambda \in \Gamma(T^i E)$ and $\zeta \in \Gamma(T^j E)$ with $i, j < 0$ and $i + j \geq -k$.

A remarkable example of regular infinitesimal flag structure of type (G, P) is given, in the language of Definition 9, by a G_0 -triple $(\mathcal{F} \rightarrow (M, H), \mathfrak{g}_-)$ equipped with the soldering form described in Section 2 of Chapter 2. To see this, first observe that the filtration $\{T^i M\}$ originating

from a bracket-generating distribution $H \subset TM$ satisfies the condition of compatibility with the bracket of vector fields by definition. Moreover, since \mathfrak{g}_- is a nilpotent graded Lie algebra, the notion of a regular pair (M, H) of type \mathfrak{g}_- makes sense. This always comes together with an adapted frame bundle $\mathcal{P} \rightarrow (M, H)$ with structure group $\text{Aut}_{gr}(\mathfrak{g}_-)$. There, we showed that for $i = -1, \dots, -k$, sections θ_i of $L(T^i\mathcal{P}, \mathfrak{g}_i)$ with kernel $T^{i+1}\mathcal{P}$, which are $\text{Aut}_{gr}(\mathfrak{g}_-)$ -equivariant, are naturally defined and we called $\theta = (\theta_{-1}, \dots, \theta_{-k})$ the soldering form. It follows straightforwardly from the definition that the soldering form satisfies (39). Recall that if $G_0 \subset \text{Aut}_{gr}(\mathfrak{g}_-)$ is a closed subgroup a G_0 -triple consists, by definition, of a regular pair (M, H) of type \mathfrak{g}_- and an adapted G_0 -structure $\mathcal{F} \rightarrow M$. If $p_0 : \mathcal{F} \rightarrow (M, H)$ is an adapted G_0 -structure, then each θ_i restricts to a G_0 -equivariant section of $L(T^i\mathcal{F}, \mathfrak{g}_i)$ with kernel $T^{i+1}\mathcal{F}$. We thus see that $\mathcal{T} = (\{T^iM\}, p_0 : \mathcal{F} \rightarrow M, \theta)$ is a regular infinitesimal flag structure of type (G, P) .

Let G be a semisimple Lie group and let $P \subset G$ be a parabolic subgroup. A *parabolic geometry* is a Cartan geometry $(p : \mathcal{G} \rightarrow M, \omega)$ of type (G, P) (definitions and properties of Cartan geometries can be found in Section 6 of Appendix A). Any parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ of type (G, P) induces an infinitesimal flag structure of same type. Indeed, since P_+ acts freely on \mathcal{G} , we can form the space of orbits $\mathcal{G}_0 = \mathcal{G}/P_+$ and $p : \mathcal{G} \rightarrow M$ factorizes to a G_0 -bundle $p_0 : \mathcal{G}_0 \rightarrow M$. The remaining data are deduced from the grading on \mathfrak{g} and the Cartan connection ω . Consider the filtration $\mathfrak{g}^k \subset \dots \subset \mathfrak{g}^{-k} = \mathfrak{g}$. By definition, the Cartan connection gives an isomorphism $\omega(u) : T_u\mathcal{G} \rightarrow \mathfrak{g}$ at each $u \in \mathcal{G}$. The filtration of \mathfrak{g} , in turn, defines a filtration of $T\mathcal{G}$ by putting $T^i\mathcal{G} := \omega^{-1}(\mathfrak{g}^i)$. Denote by $\pi : \mathcal{G} \rightarrow \mathcal{G}_0$ the canonical projection. Since the filtration of \mathfrak{g} is P -invariant and ω is P -equivariant, the filtration of $T\mathcal{G}$ satisfies $Tr^g(T^i\mathcal{G}) = T^i\mathcal{G}$ for all $g \in P$ and $i = -k, \dots, k$. It can thus be pushed down along π and p to filtrations $T^0\mathcal{G}_0 \subset \dots \subset T^{-k}\mathcal{G}_0 = T\mathcal{G}_0$ and $T^{-1}M \subset \dots \subset T^{-k}M = TM$. The filtration on \mathcal{G}_0 , in particular, is G_0 -invariant. We can now prove the following

PROPOSITION 17. *Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, P) , where $P \subset G$ is a parabolic subgroup corresponding to the grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ of the Lie algebra \mathfrak{g} of G . Let $G_0 \subset P$ be the Levi subgroup and $p_0 : \mathcal{G}_0 \rightarrow M$ the underlying G_0 -bundle. For each $i = -k, \dots, -1$, the Cartan connection ω descends to $\omega_i^0 \in \Gamma(L(T^i\mathcal{G}_0, \mathfrak{g}_i))$ such that if $\omega^0 = (\omega_{-1}^0, \dots, \omega_{-k}^0)$, then $(p_0 : \mathcal{G}_0 \rightarrow M, \omega^0)$ is an infinitesimal flag structure of type (G, P) .*

PROOF. Fix an index $i \in \{-k, \dots, -1\}$. Let $u_0 \in \mathcal{G}_0$ and $\xi \in T_{u_0}^i\mathcal{G}_0$. Then, choose $u \in \mathcal{G}$ and $\tilde{\xi} \in T_u\mathcal{G}$ such that $\pi(u) = u_0$ and $T_u\pi \cdot \tilde{\xi} = \xi$. By construction, $\tilde{\xi} \in T_u^i\mathcal{G}$ and therefore $\omega(\tilde{\xi}) \in \mathfrak{g}^i = \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_k$. We can thus define $\omega_i^0(\xi)$ as the \mathfrak{g}_i -component of $\omega(\tilde{\xi})$. For u fixed, two different choices of $\tilde{\xi} \in T_u\mathcal{G}$ as above differ by an element of the kernel of $T_u\pi$, which coincides with $T_u^1\mathcal{G}$. We thus see that the choice of $\tilde{\xi}$ does not effect the \mathfrak{g}_i -component of $\omega(\tilde{\xi})$. A different choice of $u \in \pi^{-1}(u_0)$ writes as $u \cdot g$ for some $g \in P_+$. This writes as $g = \exp(Z)$ for some $Z \in \mathfrak{p}_+$. Deriving the identity $\pi \circ r^g = \pi$, we see that also $T_u r^g \cdot \tilde{\xi} \in T_{u \cdot g}^i\mathcal{G}$ is a lift of ξ . The equivariance of ω implies that $\omega(u \cdot g)(T_u r^g \cdot \tilde{\xi}) = e^{\text{ad}(-Z)}\omega(u)(\tilde{\xi})$. Now $\text{ad}(Z)(\mathfrak{g}^i) \subset \mathfrak{g}^{i+1}$ for $Z \in \mathfrak{p}_+$ shows that the \mathfrak{g}_i -component of $\omega_u(\tilde{\xi})$ does not depend on the choice of $u \in \pi^{-1}(u_0)$. We thus get a well-defined map $\omega_i^0 : T^i\mathcal{G}_0 \rightarrow \mathfrak{g}_i$.

If $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}$ is a local smooth section of π , we can restrict $\sigma^*\omega : T\mathcal{G}_0 \rightarrow \mathfrak{g}$ to $T^i\mathcal{G}_0$ and then take its \mathfrak{g}_i -component, thus defining a smooth section of $L(T^i\mathcal{G}_0, \mathfrak{g}_i)$. By construction, this maps $\xi \in T_{u_0}^i\mathcal{G}_0$ to the \mathfrak{g}_i -component of $\omega(\sigma(u_0))(T_{u_0}\sigma \cdot \xi)$, which coincides with $\omega_i^0(\xi)$. We thus showed that each ω_i^0 is smooth.

Observe that $\xi \in T^i\mathcal{G}_0$ satisfies $\omega_i^0(\xi) = 0$ if and only if ξ admits a lift $\tilde{\xi} \in T^i\mathcal{G}$ such that $\omega_u(\tilde{\xi}) \in \mathfrak{g}^{i+1}$ for all $u \in \mathcal{G}$. This is equivalent to the fact that $\tilde{\xi} \in T^{i+1}\mathcal{G}$, hence that $\xi \in T^{i+1}\mathcal{G}_0$. Finally, note that since $P_+ \subset P$ is a normal subgroup, the projection π is G_0 -equivariant. Therefore, given a tangent vector $\xi \in T_{u_0}^i\mathcal{G}_0$, a lift $\tilde{\xi} \in T_u\mathcal{G}$ and $g \in G_0$, we have that $Tr^g \cdot \tilde{\xi}$ is a

lift of $Tr^g \cdot \xi$. In this way, we see that the equivariance of ω with respect to the P -action implies the equivariance of each ω_0^i under the action of G_0 . \square

We say that $(p_0 : \mathcal{G}_0 \rightarrow M, \omega^0)$ from Proposition 17 is the *infinitesimal flag structure of type (G, P) underlying the parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$* .

The curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ associated to the parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ is defined by the formula

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \quad \xi, \eta \in \mathfrak{X}(\mathcal{G}).$$

It is a well known fact that K measures the obstruction to local isomorphism to the homogeneous model of same type (G, P) (see also Section 6 of Appendix A for more details). Now K can be equivalently encoded by the *curvature function* $\kappa_u(X, Y) = K(\omega_u^{-1}(X), \omega_u^{-1}(Y))$ for $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$. Since K vanishes under insertion of a vertical vector field, the curvature function can be seen as a smooth map $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. The adjoint action of P on \mathfrak{g} induces a filtration-preserving action on $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ in a natural way and one can verify that κ is equivariant with respect to this action (see [10, Lemma 1.5.1 p. 72]). A parabolic geometry is said to be *regular* if its underlying infinitesimal flag structure is regular. This turns out to be equivalent to the following condition on κ , which can be thus taken as the definition of regularity for parabolic geometries.

DEFINITION 16. Let $(\mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, P) with curvature function κ , $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ the grading on \mathfrak{g} corresponding to P and $\mathfrak{g}^k \subset \dots \subset \mathfrak{g}^{-k} = \mathfrak{g}$ the induced filtration. Then $(\mathcal{G} \rightarrow M, \omega)$ is regular if and only if $\kappa(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}^{i+j+1}$ for all $i, j < 0$.

We now recall a construction of the Lie algebra cohomology specialized to the setting of graded semisimple Lie algebras. The construction will be applied to select a subclass of parabolic geometries, usually said to be normal. A more general definition of the Lie algebra cohomology and of its properties can be found in Section 5 of Appendix A.

Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ be a graded semisimple Lie algebra and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ a standard parabolic subalgebra. The Killing form of \mathfrak{g} induces isomorphisms $\mathfrak{g} \cong \mathfrak{g}^*$ and $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of P -modules, so that $\Lambda^n(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ identifies with the dual of $C^n(\mathfrak{p}_+, \mathfrak{g}) = \Lambda^n \mathfrak{p}_+^* \otimes \mathfrak{g}$ as P -modules. Also, the Killing form defines an isomorphism $(\mathfrak{g}_-)^* \cong \mathfrak{p}_+$ of \mathfrak{g}_0 -modules. Clearly, $\mathfrak{g}/\mathfrak{p}$ and \mathfrak{g}_- identify as \mathfrak{g}_0 -modules, but only $\mathfrak{g}/\mathfrak{p}$ carries the structure of a \mathfrak{p} -module.

The Lie algebra cohomology differential $\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$, defined by the formula

$$(40) \quad \begin{aligned} \partial\Phi(X_0, \dots, X_n) &= \sum_{i=0}^n (-1)^i [X_i, \Phi(X_0, \dots, \widehat{X}_i, \dots, X_n)] + \\ &\quad + \sum_{i < j} (-1)^{i+j} \Phi([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_n) \end{aligned}$$

for $\Phi \in C^n(\mathfrak{g}_-, \mathfrak{g})$ and $X_0, \dots, X_n \in \mathfrak{g}_-$ is a G_0 -homomorphism such that $\partial \circ \partial = 0$. One can also consider the Lie algebra cohomology differential $\partial_{\mathfrak{p}} : C^n(\mathfrak{p}_+, \mathfrak{g}) \rightarrow C^{n+1}(\mathfrak{p}_+, \mathfrak{g})$, which is an homomorphism of P -modules. Then, $\partial_{\mathfrak{p}}$ dualizes to the boundary operator ∂^* defining the homology $H_*(\mathfrak{p}_+, \mathfrak{g}^*)$ (see Section 5 of Appendix A for the explicit formula for the boundary operator). Using the isomorphisms of P -modules given by the Killing form, ∂^* can be seen as a P -homomorphism

$$\partial^* : \Lambda^{n+1}(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \rightarrow \Lambda^n(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$$

such that $\partial^* \circ \partial^* = 0$. We call ∂^* the *Kostant codifferential*. From the explicit formula for the boundary operator, one can see that ∂^* preserves the natural filtrations on the spaces involved. It thus induces a grading-preserving homomorphism $\text{gr}_0(\partial^*)$ between the associated graded vector spaces. On these spaces, P_+ acts trivially and $\text{gr}_0(\partial^*)$ is thus an homomorphism of G_0 -modules. Actually, the explicit formula for ∂^* shows that ∂^* also preserves the gradings associated to the

natural filtrations, so that $\text{gr}_0(\partial^*)$ and ∂^* are given by the same formula. Now one can define the Kostant Laplacian

$$\square_n := \partial \circ \text{gr}_0(\partial^*) + \text{gr}_0(\partial^*) \circ \partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \rightarrow C^n(\mathfrak{g}_-, \mathfrak{g}),$$

which leads to the Hodge decomposition $C^n(\mathfrak{g}_-, \mathfrak{g}) = \text{Im}(\text{gr}_0(\partial^*)) \oplus \text{Ker}(\square_n) \oplus \text{Im}(\partial)$ (the role played by the Hodge decomposition in the proof of Kostant's Theorem is discussed in 5.1 of Appendix A). This, in turn, shows the existence of a natural identification of G_0 -modules

$$H^n(\mathfrak{g}_-, \mathfrak{g}) \cong \text{Ker}(\square_n) \cong H_n(\mathfrak{p}_+, \mathfrak{g}),$$

which endows the cohomology groups $H^n(\mathfrak{g}_-, \mathfrak{g})$ with the structure of \mathfrak{p} -modules such that the \mathfrak{p}_+ -action is trivial. Since the grading on the spaces of cycles, which is induced by the grading on \mathfrak{g} , is preserved by ∂^* , we have that $H^n(\mathfrak{g}_-, \mathfrak{g})$ carries a natural grading. From the identification above, we can see the grading as given by the homogeneous degree of multilinear maps between graded vector spaces. For $i \in \mathbb{Z}$, we will denote by $H^j(\mathfrak{g}_-, \mathfrak{g})_i$ the graded component of degree i of $H^j(\mathfrak{g}_-, \mathfrak{g})$ and by $H^j(\mathfrak{g}_-, \mathfrak{g})^t = \bigoplus_{i \geq t} H^j(\mathfrak{g}_-, \mathfrak{g})_i$ for $t \in \mathbb{Z}$ the associated filtration.

Let us show how the considerations above apply to the study of parabolic geometries. Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, P) . Then, $p : \mathcal{G} \rightarrow M$ is a principal P -bundle and P acts on \mathfrak{g} by restriction of the adjoint action. We can thus form the associated bundle $\mathcal{A}M = \mathcal{G} \times_P \mathfrak{g}$, the so-called *adjoint tractor bundle*. The P -invariant filtration of \mathfrak{g} induces a filtration by subbundles

$$\mathcal{A}^k M \subset \mathcal{A}^{k-1} M \subset \dots \subset \mathcal{A}^{-k} M = \mathcal{A}M$$

such that $\mathcal{A}^i M = T^i M$ for all $i \leq -1$. The graded vector bundle associated to the filtration will be denoted by $\text{gr}(\mathcal{A}M)$. The Lie algebra differential (40) defines a map

$$\partial : \Lambda^n \text{gr}(T^* M) \otimes \text{gr}(\mathcal{A}M) \rightarrow \Lambda^{n+1} \text{gr}(T^* M) \otimes \text{gr}(\mathcal{A}M)$$

between the associated vector bundles. These are graded vector bundles which depend only on the underlying infinitesimal flag structure, so that they can be viewed as $\mathcal{G}_0 \times_{G_0} C^n(\mathfrak{g}_-, \mathfrak{g})$ and ∂ is homogeneous of degree zero with respect to the considered grading.

Since $TM = \mathcal{G} \times_P \mathfrak{g}_-$ and the associated bundle construction is natural, we also have that $T^*M = \mathcal{G} \times_P \mathfrak{p}_+$. The Kostant codifferential induces an homomorphism

$$\partial^* : \Lambda^{n+1} T^* M \otimes \mathcal{A}M \rightarrow \Lambda^n T^* M \otimes \mathcal{A}M$$

between the associated vector bundles, thus a tensorial operator $\Omega^{j+1}(M, \mathcal{A}M) \rightarrow \Omega^j(M, \mathcal{A}M)$ that will be denoted with the same symbol ∂^* . Recall that the Kostant codifferential preserves the P -invariant filtration on the spaces $\Lambda^n \mathfrak{p}_+ \otimes \mathfrak{g}$ of cycles. Therefore, ∂^* preserves the natural filtrations on the bundles $\Lambda^n T^* M \otimes \mathcal{A}M$ and it thus descends to a homomorphism

$$\text{gr}_0(\partial^*) : \Lambda^{n+1} \text{gr}(T^* M) \otimes \text{gr}(\mathcal{A}M) \rightarrow \Lambda^n \text{gr}(T^* M) \otimes \text{gr}(\mathcal{A}M)$$

between the associated graded bundles, which depends only on the underlying infinitesimal flag structure. The maps ∂ and $\text{gr}_0(\partial^*)$ lead to an analogous of the Hodge decomposition for the vector bundles $\Lambda^n \text{gr}(T^* M) \otimes \text{gr}(\mathcal{A}M)$ as the direct sum of subbundles, which depend only on the underlying infinitesimal flag structure. The P -submodules $\text{Im}(\partial^*) \subset \text{Ker}(\partial^*) \subset \Lambda^n(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ correspond to smooth subbundles $\text{Im}(\partial^*) \subset \text{Ker}(\partial^*) \subset \Lambda^n T^* M \otimes \mathcal{A}M$. The quotient bundles $\text{Ker}(\partial^*)/\text{Im}(\partial^*)$ are associated to representations with trivial P_+ -action, hence they are associated to \mathcal{G}_0 . Explicitly,

$$\text{Ker}(\partial^*)/\text{Im}(\partial^*) \cong \mathcal{G}_0 \times_{G_0} H^n(\mathfrak{g}_-, \mathfrak{g}).$$

Since the curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ associated to the parabolic geometry is a P -equivariant smooth function it corresponds, through the associated bundle construction, to an element of $\Omega^2(M, \mathcal{A}M)$ that will be also denoted by κ .

DEFINITION 17. Let $(\mathcal{G} \rightarrow M, \omega)$ be a parabolic geometry of type (G, P) with curvature function $\kappa \in \Omega^2(M, \mathcal{A}M)$. We say that $(\mathcal{G} \rightarrow M, \omega)$ is *normal* if $\partial^* \kappa = 0$. In this case, the image of κ into the space of sections of $\mathcal{G}_0 \times_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})$ is called *harmonic curvature* and denoted by κ_H .

The equivalence between the categories of normal, regular parabolic geometries and regular infinitesimal flag structures of same type (G, P) is a fundamental result in parabolic geometry. A proof of the equivalence result can be found in [10] (Theorem 3.1.16, p. 277). In the case of interest for us, namely if \mathfrak{g} is simple, the equivalence result can be formulated in the following way.

THEOREM 8. *Let \mathfrak{g} be a simple Lie algebra and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ a $|k|$ -grading such that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$. Suppose that G is a Lie group with Lie algebra \mathfrak{g} , denote by $P \subset G$ the parabolic subgroup corresponding to the grading and by $G_0 \subset P$ the Levi subgroup. Then any G_0 -triple of type \mathfrak{g}_- equipped with the tautological form is the underlying infinitesimal flag structure of a regular, normal parabolic geometry of type (G, P) , which is uniquely determined up to isomorphism.*

Put $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ and $\mathfrak{t} = \mathfrak{su}(3, 2)$. Let $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ and $\mathfrak{t} = \bigoplus_{i=-2}^2 \mathfrak{t}_i$ be the gradings from 2.1 and 3.2. In Example 14 of Appendix A, we verify that $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$ and $H^1(\mathfrak{t}_-, \mathfrak{t})^1 = \{0\}$. Theorem 5 and Theorem 7 assure that the adjoint action restricted to the homogeneous component of degree zero induces respective isomorphisms $\mathfrak{g}_0 \cong \mathfrak{der}_{gr}(\mathfrak{g}_-)$ and $\mathfrak{t}_0 \cong \mathfrak{der}_{gr}(\mathfrak{t}_-)$. Equivalently, $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$ and $H^1(\mathfrak{t}_-, \mathfrak{t})_0 = \{0\}$. Such an isomorphism lifts a Lie group isomorphism between a Levi subgroup and the grading-preserving automorphism group of the model algebra. Then, the fundamental equivalence result implies the following

COROLLARY 5. *Hyperbolic and elliptic $(4, 8)$ -distributions are equivalent to parabolic geometries. More precisely, let (M, H) be a generic $(4, 8)$ -distribution and $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ the grading from 2.1, respectively from 3.2, describing the corresponding generic type. Let $G = \text{Aut}(\mathfrak{g})$ be the set of automorphisms of the Lie algebra \mathfrak{g} , denote by $P = \text{Aut}_{filt}(\mathfrak{g})$ the parabolic subgroup $P \subset G$ of filtration-preserving homomorphisms and by $G_0 \subset P$ the Levi subgroup. Then (M, H) is equivalent to a unique normal, regular parabolic geometry of type (G, P) .*

In the last part of the Section, we show that the assumption on the first cohomology group is related to the Tanaka prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. By definition of grading, \mathfrak{g}_- is a nilpotent graded Lie algebra generated by \mathfrak{g}_{-1} and \mathfrak{g}_0 acts on \mathfrak{g}_- by grading-preserving derivations via the adjoint action. Therefore, if the adjoint action restricts to an injective homomorphism $\mathfrak{g}_0 \rightarrow \mathfrak{der}_{gr}(\mathfrak{g}_-)$, we can look at \mathfrak{g}_0 as a subalgebra $\mathfrak{g}_0 \subset \mathfrak{der}_{gr}(\mathfrak{g}_-)$, hence to the Tanaka prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. We have the following

PROPOSITION 18. *Let \mathfrak{g} be a simple Lie algebra and $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ a $|k|$ -grading on \mathfrak{g} .*

- (1) *For $i \leq 1$ we have that $[\mathfrak{g}_i, \mathfrak{g}_{-1}] = \mathfrak{g}_{i-1}$.*
- (2) *For $i \geq 1$ we have that $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$.*
- (3) *If $i \geq 0$ and $A \in \mathfrak{g}_i$ such that $[A, \mathfrak{g}_{-1}] = \{0\}$, then we have that $A = 0$.*
- (4) *Let $H^1(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{i \in \mathbb{Z}} H^1(\mathfrak{g}_-, \mathfrak{g})_\ell$ be the Lie algebra cohomology of \mathfrak{g}_- with coefficients in \mathfrak{g} and let $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{p}_i$ be the Tanaka prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$. Then $H^1(\mathfrak{g}_-, \mathfrak{g})_\ell = \{0\}$ for all $\ell \geq 1$ if and only if $\mathfrak{g}_\ell \cong \mathfrak{p}_\ell$ for all $\ell \geq 1$. Moreover, $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$ if and only if $\mathfrak{g}_0 \cong \mathfrak{der}_{gr}(\mathfrak{g}_-)$.*

PROOF. (1) By definition of grading, \mathfrak{g}_- is a nilpotent graded Lie algebra generated by \mathfrak{g}_{-1} and this condition writes as $[\mathfrak{g}_i, \mathfrak{g}_{-1}] = \mathfrak{g}_{i-1}$ for $i \leq -1$. Putting $D(X) = jX$ for $X \in \mathfrak{g}_j$ and $j = -k, \dots, k$ defines a derivation $D: \mathfrak{g} \rightarrow \mathfrak{g}$. Now \mathfrak{g} is simple, thus D has to be an inner derivation. This means that there exists $Z \in \mathfrak{g}$ such that $D(X) = [Z, X]$ for all $X \in \mathfrak{g}$. Since

D preserves the grading, $Z \in \mathfrak{g}_0$ is such that $[Z, X] = -X$ for all $X \in \mathfrak{g}_{-1}$, which proves the statement for $i = 0$. Finally, observe that

$$[\mathfrak{g}_1, \mathfrak{g}_{-1}] \oplus \bigoplus_{i \neq 0} \mathfrak{g}_i$$

is an ideal in \mathfrak{g} , which is nontrivial. Being \mathfrak{g} simple, we conclude that this ideal coincides with \mathfrak{g} , hence that $[\mathfrak{g}_1, \mathfrak{g}_{-1}] = \mathfrak{g}_0$ and the statement is proved also for $i = 1$.

(2) The fact that $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$ for $i \geq 1$ is deduced by applying (1) to the opposite grading on \mathfrak{g} .

(3) Let $B = B_{\mathfrak{g}}$ be the Killing form of \mathfrak{g} , $i \geq 0$ and $A \in \mathfrak{g}_i$ such that $[A, \mathfrak{g}_{-1}] = \{0\}$. For any $X \in \mathfrak{g}_{-1}$ and $Y \in \mathfrak{g}_{-i+1}$ we have

$$0 = B([A, X], Y) = B(A, [X, Y])$$

which means that A is orthogonal to $[\mathfrak{g}_{-1}, \mathfrak{g}_{-i+1}]$. But the last space equals \mathfrak{g}_{-i} for (1) and we know that the restriction of B to $\mathfrak{g}_i \times \mathfrak{g}_{-i}$ is nondegenerate. We thus conclude that $A = 0$.

(4) By definition, the nonpositive graded parts of $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ and of \mathfrak{g} coincide. This tells us that \mathfrak{g} satisfies the first of the three conditions defining $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$, namely the condition (C1) from Section 3. Using inductively $[\mathfrak{g}_i, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{i-1}$, we see that $\mathfrak{g}_i \subset \mathfrak{p}_i$ for all $i \geq 0$. Moreover, (3) says that every nonnegative graded component \mathfrak{g}_i satisfies the condition (C2) defining $\text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$. By maximality, we conclude that there always exists an injective homomorphism $\mathfrak{g} \rightarrow \text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ of graded Lie algebras for \mathfrak{g} simple.

Recall that the complex defining $H^1(\mathfrak{g}_-, \mathfrak{g})$ writes as follows:

$$0 \rightarrow C^0(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial} C^1(\mathfrak{g}_-, \mathfrak{g}) \xrightarrow{\partial} C^2(\mathfrak{g}_-, \mathfrak{g}) \rightarrow \dots$$

with $C^0(\mathfrak{g}_-, \mathfrak{g}) = \mathfrak{g}$ and $C^1(\mathfrak{g}_-, \mathfrak{g}) = L(\mathfrak{g}_-, \mathfrak{g})$. The compatibility of ∂ with the natural grading on the spaces $C^j(\mathfrak{g}_-, \mathfrak{g})$ implies that the complex splits in graded components as

$$0 \rightarrow \mathfrak{g}_\ell \xrightarrow{\partial_\ell^0} C^1(\mathfrak{g}_-, \mathfrak{g})_\ell \xrightarrow{\partial_\ell^1} C^2(\mathfrak{g}_-, \mathfrak{g})_\ell \rightarrow \dots$$

and $H^1(\mathfrak{g}_-, \mathfrak{g})_\ell = \text{Ker}(\partial_\ell^1)/\text{Im}(\partial_\ell^0)$. Since ∂_ℓ^0 is injective, we have that $\mathfrak{g}_\ell \cong \text{Im}(\partial_\ell^0)$. Observe that $\phi \in L(\mathfrak{g}_-, \mathfrak{g})_\ell$ is a linear map $\phi: \mathfrak{g}_- \rightarrow \mathfrak{g}$ such that $\phi(\mathfrak{g}_i) \subset \mathfrak{g}_{i+\ell}$ for all $i = -k, \dots, -1$. Hence, ϕ takes values in $\mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{\ell-1} \subset \mathfrak{g}$ and it thus can be equivalently considered as a map $\mathfrak{g}_- \rightarrow \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{\ell-1}$. Now if $\phi = \partial_\ell^0 X$ for some $X \in \mathfrak{g}_\ell$, then $\phi(Y) = [Y, X]$ for all $Y \in \mathfrak{g}_-$. If $\phi \in L(\mathfrak{g}_-, \mathfrak{g})_\ell$ is such that $\partial_\ell^1 \phi = 0$, then it satisfies $\varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)]$ for all $X, Y \in \mathfrak{g}_-$. Hence, by definition $\mathfrak{der}_{gr}(\mathfrak{g}_-) = \text{Ker}(\partial_0^1)$ and the fact that $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$ if and only if \mathfrak{g}_0 and $\mathfrak{der}_{gr}(\mathfrak{g}_-)$ are isomorphic follows straightforwardly from the definitions.

Recall that $\mathfrak{p}_0 = \mathfrak{g}_0$ by definition and for $\ell \geq 1$, the ℓ -th Tanaka prolongation of $(\mathfrak{g}_-, \mathfrak{g}_0)$ is given by

$$(41) \quad \mathfrak{p}_\ell := \left\{ \varphi \in \bigoplus_{p < 0} \mathfrak{g}_p^* \otimes \mathfrak{p}_{\ell+p} \mid \varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)], X, Y \in \mathfrak{g}_- \right\}$$

and $\mathfrak{p}_\ell := (\mathfrak{p}_{\ell-1})_1$ can be equivalently defined as the first prolongation of $\mathfrak{p}_{\ell-1}$ by induction on $\ell \geq 1$. Suppose that $\mathfrak{g}_\ell \cong \mathfrak{p}_\ell$ for all $\ell \geq 1$. Fix $\ell \geq 1$. Using the isomorphisms $\mathfrak{g}_{\ell+p} \cong \mathfrak{p}_{\ell+p}$ for $p < 0$ in (41), we see that $\mathfrak{p}_\ell = \text{Ker}(\partial_\ell^1)$ and since $\mathfrak{g}_\ell \cong \text{Im}(\partial_\ell^0)$, we have that $H^1(\mathfrak{g}_-, \mathfrak{g})_\ell = \{0\}$. Conversely if $H^1(\mathfrak{g}_-, \mathfrak{g})_\ell = \{0\}$ for all $\ell \geq 1$, in particular $H^1(\mathfrak{g}_-, \mathfrak{g})_1 = \{0\}$ and since $\mathfrak{g}_0 = \mathfrak{p}_0$, this implies that $\mathfrak{p}_1 = \text{Ker}(\partial_1^1)$, hence that $\mathfrak{g}_1 \cong \mathfrak{p}_1$. If $\ell \geq 2$, using the isomorphism $\mathfrak{g}_{\ell-1} \cong \mathfrak{p}_{\ell-1}$ we show that

$$\mathfrak{g}_\ell = (\mathfrak{g}_{\ell-1})_1 \cong (\mathfrak{p}_{\ell-1})_1 = \mathfrak{p}_\ell$$

and the result follows by induction. \square

Consider a $|k|$ -grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ on a simple Lie algebra. From (4) of Proposition 18, we see that the grading satisfies $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$ if and only if $\mathfrak{g} \cong \text{pr}(\mathfrak{g}_-, \mathfrak{g}_0)$ as graded Lie algebras. If the same cohomology group also vanishes in homogeneity zero, then $\mathfrak{g} \cong \text{pr}(\mathfrak{g}_-)$. The condition $H^1(\mathfrak{g}_-, \mathfrak{g})^0 = \{0\}$ is usually expressed by saying that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneity. Now suppose that $H^1(\mathfrak{g}_-, \mathfrak{g})^0 = \{0\}$. Let G be a Lie group with Lie algebra \mathfrak{g} , let $P \subset G$ be the parabolic subgroup corresponding to the grading and $G_0 \subset P$ the Levi subgroup. The assumption implies that the adjoint action restricts to an automorphism $G_0 \cong \text{Aut}_{gr}(\mathfrak{g}_-)$. Parabolic geometries of type (G, P) are then completely determined, in the sense of Theorem 8, by a regular filtration by subbundles on the underlying space. Generic (4, 8)–distributions are examples of such geometries.

5. Invariants for generic (4, 8)–distributions

In the previous section, we saw that generic (4, 8)–distributions are equivalent to certain normal, regular parabolic geometries. The parabolic type (G, P) of such geometries is determined by a grading $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ on a real graded simple Lie algebra \mathfrak{g} , described in 2.1 and 3.2, such that $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i=-2}^2 \mathfrak{g}_i^{\mathbb{C}}$ is the grading on $\mathfrak{sl}(5, \mathbb{C})$ described in Section 1. This equivalence allows to deduce the local invariants for these distributions from the curvature of the corresponding parabolic geometries, as we are going to explain.

Let $(\mathcal{G} \rightarrow M, \omega)$ be a regular parabolic geometry of type (G, P) . Recall that the curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ associated to $(p : \mathcal{G} \rightarrow M, \omega)$ is given by the formula

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)], \quad \xi, \eta \in \mathfrak{X}(\mathcal{G}).$$

and the curvature function $\kappa_u(X, Y) = K(\omega_u^{-1}(X), \omega_u^{-1}(Y))$ for $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$. Now the curvature function can be seen as a P –equivariant smooth function $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$. Such a map corresponds, through the associated bundle construction, to an element of

$$\Omega^2(M, \mathcal{A}M) = \Gamma(\Lambda^2(T^*M) \otimes \mathcal{A}M)$$

that we will also denote by κ . The P –invariant filtration $\mathfrak{g}^k \subset \cdots \subset \mathfrak{g}^{-k} = \mathfrak{g}$ induces filtrations by subbundles

$$T^{-1}M \subset T^{-2}M \subset \cdots \subset T^{-k}M = TM$$

and

$$\mathcal{A}^k M \subset \mathcal{A}^{k-1}M \subset \cdots \subset \mathcal{A}^{-k}M = \mathcal{A}M$$

such that $\mathcal{A}^i M = T^i M$ for all $i \leq -1$. The filtrations above induce a filtration of the space of homomorphisms $\Lambda^2 TM \rightarrow \mathcal{A}M$ and, in turn, of the space of its sections. The regularity assumption writes as

$$\kappa(T^i M, T^j M) \subset \mathcal{A}^{i+j+1} M$$

for all $i, j < 0$. We can thus consider the corresponding section $\text{gr}_1(\kappa)$ of the associated graded vector bundle

$$\text{gr}_1(\Lambda^2 T^*M \otimes \mathcal{A}M) = L(\Lambda^2 \text{gr}(TM), \text{gr}(\mathcal{A}M))_1.$$

Consider $i, j < 0$ and $\xi \in T^i M$, $\eta \in T^j M$. Let $\tilde{\xi} \in T^i \mathcal{G}$ and $\tilde{\eta} \in T^j \mathcal{G}$ be lifts for ξ and η , respectively. Denote by $q_s : \mathcal{A}^s M \rightarrow \text{gr}_s(\mathcal{A}M)$ the canonical projections. For $x \in M$, let be $u \in p^{-1}(x)$. Then

$$(42) \quad \text{gr}_1(\kappa)(q_i(\xi), q_j(\eta))(x) := q_{i+j+1}(T_u p \cdot \omega_u^{-1} \kappa_u(\omega(\tilde{\xi}), \omega(\tilde{\eta}))).$$

Observe that since the curvature form is horizontal, the formula above does not depend on the choice of lifts of ξ and η . More precisely, the regularity assumption implies that the right-hand term only depends on $q_i(\xi)$ and on $q_j(\eta)$. Furthermore, the equivariance of κ and ω implies that the right-hand term of the formula does not depend on the choice of $u \in p^{-1}(x)$. Suppose now that the parabolic geometry is regular and normal. From the general theory, we have the

following facts. As in Definition 17, there is a notion of harmonic curvature κ_H associated to the parabolic geometry. This is a smooth section of the bundle

$$\mathrm{Ker}(\partial^*)/\mathrm{Im}(\partial^*) \cong \mathcal{G}_0 \times_{G_0} H^2(\mathfrak{g}_-, \mathfrak{g})$$

which is a fundamental invariant for the geometry. In particular, recall that a normal, regular parabolic geometry of type (G, P) is locally isomorphic to the model of same type, namely the canonical projection $G \rightarrow G/P$ equipped with the Maurer–Cartan form on G , if and only if $\kappa_H = 0$ identically. Now $\mathrm{gr}_1(\kappa)$ is a section of $\mathrm{Ker}(\square_2)$ which coincides, up to the identification $\mathrm{Ker}(\square_2) \cong \mathrm{Ker}(\partial^*)/\mathrm{Im}(\partial^*)$ determined by the Hodge decomposition, with the homogeneous component of degree one of the harmonic curvature κ_H (see [10, Theorem 3.1.12 p. 265]).

PROPOSITION 19. *Let (M, H) be a generic (4, 8)–distribution and let $(p : \mathcal{G} \rightarrow M, \omega)$ be the normal, regular parabolic geometry of type (G, P) equivalent to (M, H) (see Corollary 5). Denote by κ the curvature function and by κ_H the harmonic curvature associated to the parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$. Then, the graded curvature in homogeneity one*

$$\mathrm{gr}_1(\kappa) \in \Gamma((\Lambda^2 H^* \otimes H) \oplus (H^* \otimes Q^* \otimes Q))$$

from (42), where $Q = TM/H$ denotes the quotient bundle, is equivalent to κ_H .

PROOF. Let $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ be the real graded simple Lie algebra describing the model algebra for (M, H) as in 2.1 for the hyperbolic type, respectively as in 3.2 for the elliptic type. In Example 13 of Appendix A, we saw that $H_{\mathbb{C}}^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+) = H_{\mathbb{C}}^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+)_{-1}$. Moreover, $H_{\mathbb{C}}^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+)$ is the complexification of $H_{\mathbb{R}}^2(\mathfrak{g}_-, \mathfrak{g})$ (see Proposition 26 of Appendix A) and it thus follows that $H_{\mathbb{R}}^2(\mathfrak{g}_-, \mathfrak{g}) = H_{\mathbb{R}}^2(\mathfrak{g}_-, \mathfrak{g})_{-1}$, which implies that κ_H reduces to its homogeneous component of degree one. By [10, Theorem 3.1.12 p. 265] this homogeneous component coincides, up to the identification $\mathrm{Ker}(\square_2) \cong \mathrm{Ker}(\partial^*)/\mathrm{Im}(\partial^*)$, with $\mathrm{gr}_1(\kappa)$ which is a section of

$$L(\Lambda^2 \mathrm{gr}(TM), \mathrm{gr}(\mathcal{A}M))_{-1} = (\Lambda^2 H^* \otimes H) \oplus (H^* \otimes Q^* \otimes Q).$$

□

Recall that $H^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+)$ has a natural structure of $\mathfrak{g}_0^{\mathbb{C}}$ –module. By realizing it as a $\mathfrak{g}_0^{\mathbb{C}}$ –submodule of skew–symmetric bilinear maps $\mathfrak{g}_{\mathbb{C}}^- \times \mathfrak{g}_{\mathbb{C}}^+ \rightarrow \mathfrak{g}_{\mathbb{C}}^+$, Kostant’s Theorem makes the decomposition into irreducible components explicit, namely

$$H^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+) = \bigoplus_{w \in W^{\mathfrak{p}}(2)} \mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}}$$

where $w \cdot \lambda_{\mathfrak{g}}$ denotes the affine action of $w \in W^{\mathfrak{p}}(2)$ on the highest root $\lambda_{\mathfrak{g}}$ of \mathfrak{g} and $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}}$ the irreducible representation of lowest weight $-w \cdot \lambda_{\mathfrak{g}}$ (see Theorem 14 of Appendix A). In Example 13 of Appendix A, we compute the set $\{-w \cdot \lambda_{\mathfrak{g}} \mid w \in W^{\mathfrak{p}}(2)\}$ which consists of four distinct weights, represented in the Dynkin diagram notation as follows:

$$(43) \quad \begin{array}{cccc} 0 & 4 & -3 & -1 \\ \circ & \times & \times & \circ \\ -1 & -3 & 4 & 0 \\ \circ & \times & \times & \circ \\ -3 & 3 & 0 & -2 \\ \circ & \times & \times & \circ \\ -2 & 0 & 3 & -3 \\ \circ & \times & \times & \circ \end{array}$$

Observe the symmetry in the diagram (43), which is due to the structure of the representations involved. Using basic notions of representation theory, we can easily deduce the $\mathfrak{g}_0^{\mathbb{C}}$ –irreducible components $H^2(\mathfrak{g}_{\mathbb{C}}^-, \mathfrak{g}_{\mathbb{C}}^+)$ of lowest weights from (43).

Let $\mathfrak{h} \subset \mathfrak{g}^{\mathbb{C}}$ be the standard Cartan subalgebra and $\Delta^0 = \{\alpha_1, \dots, \alpha_4\} \subset \mathfrak{h}^*$ the standard simple

root system. The considered $|2|$ -grading on $\mathfrak{g}^{\mathbb{C}}$ is induced, via the notion of Σ -height, by the choice $\Sigma = \{\alpha_2, \alpha_3\}$ of subset of simple roots. In particular, $\mathfrak{g}_{-1}^{\mathbb{C}}$ decomposes into the direct sum $\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ of two-dimensional complex $\mathfrak{g}_0^{\mathbb{C}}$ -modules with $\Delta(\mathfrak{g}_{-1}^E) = \{-\alpha_1 - \alpha_2, -\alpha_2\}$ and $\Delta(\mathfrak{g}_{-1}^F) = \{-\alpha_3, -\alpha_3 - \alpha_4\}$. The sum $-\alpha_2 + (-\alpha_2 - \alpha_1)$ is the highest weight of $\Lambda^2 \mathfrak{g}_{-1}^E$. This writes, in Dynkin diagram notation, as

$$\begin{array}{c} 1 & -2 & 1 & 0 \\ \circ & \times & \times & \circ \end{array} + \left(\begin{array}{c} 1 & -2 & 1 & 0 \\ \circ & \times & \times & \circ \end{array} + \begin{array}{c} -2 & 1 & 0 & 0 \\ \circ & \times & \times & \circ \end{array} \right) = \begin{array}{c} 0 & -3 & 2 & 0 \\ \circ & \times & \times & \circ \end{array}$$

and its negative is the lowest weight of the one-dimensional module $\Lambda^2(\mathfrak{g}_{-1}^E)^*$. Moreover, the lowest weight of \mathfrak{g}_{-1}^F is $-\alpha_3 - \alpha_4$, which has to be added to the lowest weight of $\Lambda^2(\mathfrak{g}_{-1}^E)^*$:

$$\begin{array}{c} 0 & 3 & -2 & 0 \\ \circ & \times & \times & \circ \end{array} + \begin{array}{c} 0 & 1 & -1 & -1 \\ \circ & \times & \times & \circ \end{array} = \begin{array}{c} 0 & 4 & -3 & -1 \\ \circ & \times & \times & \circ \end{array}$$

to obtain the lowest weight of $\Lambda^2(\mathfrak{g}_{-1}^E)^* \otimes \mathfrak{g}_{-1}^F$, which is therefore the irreducible $\mathfrak{g}_0^{\mathbb{C}}$ -component whose lowest weight is the first one in (43). Analogously, one shows that the irreducible $\mathfrak{g}_0^{\mathbb{C}}$ -component of lowest weight the second weight in (43) is $\Lambda^2(\mathfrak{g}_{-1}^F)^* \otimes \mathfrak{g}_{-1}^E$.

Observe that $S^2 \mathfrak{g}_{-1}^E \otimes (\mathfrak{g}_{-1}^E)^* \otimes \mathfrak{g}_{-1}^F \otimes (\mathfrak{g}_{-1}^F)^*$ contains the irreducible \mathfrak{g}_0 -representation of highest weight

$$2(-\alpha_2) + (\alpha_1 + \alpha_2) + (-\alpha_3) + (\alpha_3 + \alpha_4) = \alpha_1 - \alpha_2 + \alpha_4 = \begin{array}{c} 3 & -3 & 0 & 2 \\ \circ & \times & \times & \circ \end{array}$$

The irreducible \mathfrak{g}_0 -representation of lowest weight the third weight in (43) is thus contained in $S = S^2(\mathfrak{g}_{-1}^E)^* \otimes \mathfrak{g}_{-1}^E \otimes (\mathfrak{g}_{-1}^F)^* \otimes \mathfrak{g}_{-1}^F$. Let us denote it by

$$(S^2(\mathfrak{g}_{-1}^E)^* \otimes \mathfrak{g}_{-1}^E)_0 \otimes \mathfrak{sl}(\mathfrak{g}_{-1}^F) \subset S$$

and describe it explicitly as the joint kernel of two contraction maps, as follows.

Put $V = \mathfrak{g}_{-1}^E$ and $W = \mathfrak{g}_{-1}^F$. Fix bases $\{x_1, x_2\} \subset V$ and $\{y_1, y_2\} \subset W$ and denote by $\{x_1^*, x_2^*\}$, respectively by $\{y_1^*, y_2^*\}$, the dual bases. Denote by $id_V : V \rightarrow V$ and by $id_W : W \rightarrow W$ the identity maps. Define the trace maps

$$(44) \quad \begin{aligned} \text{tr}_V : L(S^2 V \otimes W, V \otimes W) &\rightarrow L(V \otimes W, W), \\ \text{tr}_V(g)(v \otimes w) &= \sum_{i=1}^2 (x_i^* \otimes id_W)(g(x_i \odot v \otimes w)) \end{aligned}$$

for $g \in L(S^2 V \otimes W, V \otimes W)$, $v \in V$, $w \in W$ and

$$(45) \quad \begin{aligned} \text{tr}_W : L(S^2 V \otimes W, V \otimes W) &\rightarrow L(S^2 V, V), \\ \text{tr}_W(g)(v_1 \odot v_2) &= \sum_{j=1}^2 (id_V \otimes y_j^*)(g(v_1 \odot v_2 \otimes y_j)) \end{aligned}$$

for $g \in L(S^2 V \otimes W, V \otimes W)$, $v_1, v_2 \in V$. Furthermore, denote by

$$(46) \quad \begin{aligned} c_V : L(S^2 V, V) &\rightarrow L(V, \mathbb{R}), \\ c_V(a)(v) &= \sum_{i=1}^2 x_i^*(a(x_i \odot v)) \end{aligned}$$

for $a \in L(S^2 V, V)$, $v \in V$. Observe that the definitions of tr_V , tr_W and c_V do not depend on the choice of basis. This can be easily seen, after observing that if the change of basis is described by the invertible matrix A , the respective dual bases differ by multiplication for A^{-1} . Clearly, the three maps introduced above can be equivalently described, as we are going to do in the following, as linear maps between tensor products of copies of V, W and their duals. Elements of such tensor products will be represented in abstract index notation, denoting the copies of V and V^* by roman letters and the copies of W and W^* by greek letters. Let be

$f_{ij\alpha}^{s\beta} \in S = S^2V^* \otimes W^* \otimes V \otimes W$ such that $f_{ji\alpha}^{s\beta} = f_{ij\alpha}^{s\beta}$. Clearly, the trace maps can be equivalently viewed as contractions

$$\begin{aligned} \text{tr}_V: S &\rightarrow V^* \otimes W^* \otimes W, & f_{ij\alpha}^{s\beta} &\mapsto f_{pj\alpha}^{p\beta} \\ \text{tr}_W: S &\rightarrow S^2V^* \otimes V, & f_{ij\alpha}^{s\beta} &\mapsto f_{ij\alpha}^{s\alpha} \\ c_V: S^2V^* \otimes V &\rightarrow V^*, & a_{ij}^s &\mapsto a_{ij}^j. \end{aligned}$$

For $b_{j\alpha}^\beta \in V^* \otimes W^* \otimes W$, putting $f_{ij\alpha}^{s\beta} = \delta_i^s b_{j\alpha}^\beta + \delta_j^s b_{i\alpha}^\beta$ defines an element of S such that $f_{pj\alpha}^{p\beta} = 3b_{j\alpha}^\beta$, which shows that tr_V is surjective.

For $a_{ij}^s \in S^2V^* \otimes V$, putting $f_{ij\alpha}^{s\beta} = a_{ij}^s \delta_\alpha^\beta$ defines an element of S such that $f_{ij\alpha}^{s\alpha} = 2a_{ij}^s$, which shows that tr_W is surjective.

Clearly, the three maps above are equivariant with respect to the action of $\mathfrak{gl}(2, \mathbb{C})$, hence of $\mathfrak{g}_0^{\mathbb{C}}$. The joint kernel of the two maps tr_V and tr_W , in particular, is a $\mathfrak{g}_0^{\mathbb{C}}$ -invariant subspace of S that will be denoted by

$$(47) \quad (S^2V^* \otimes V)_0 \otimes \mathfrak{sl}(W).$$

Clearly, $(S^2W^* \otimes W)_0 \otimes \mathfrak{sl}(V)$ can be defined analogously by exchanging the roles of V and W , thus obtaining the irreducible representation of $\mathfrak{g}_0^{\mathbb{C}}$ whose lowest weight is the fourth weight in (43). We thus completely described all $\mathfrak{g}_0^{\mathbb{C}}$ -irreducible components of $H^2(\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$.

Let us compute explicitly the projection of S onto $(S^2V^* \otimes V)_0 \otimes \mathfrak{sl}(W)$. For $f_{ij\alpha}^{s\beta} \in S$, putting

$$(48) \quad g_{ij\alpha}^{s\beta} = f_{ij\alpha}^{s\beta} - \frac{1}{6}(f_{pj\gamma}^{p\gamma} \delta_i^s + f_{pi\gamma}^{p\gamma} \delta_j^s) \delta_\alpha^\beta$$

defines an element of S such that $g_{ij\alpha}^{i\alpha} = 0$. Now observe that putting

$$(49) \quad a_{ij}^s = -\frac{1}{2}g_{ij\alpha}^{s\alpha}, \quad b_{j\alpha}^\beta = -\frac{1}{3}g_{pj\alpha}^{p\beta}$$

defines $a_{ij}^t \in S^2V^* \otimes V$ and $b_{j\alpha}^\beta \in V^* \otimes W^* \otimes W$ such that $a_{ij}^j = b_{j\alpha}^\alpha = 0$. With the data from (48) and (49), the formula

$$(50) \quad T_{ij\alpha}^{s\beta} = g_{ij\alpha}^{s\beta} + (\delta_i^s b_{j\alpha}^\beta + \delta_j^s b_{i\alpha}^\beta) + a_{ij}^s \delta_\alpha^\beta$$

defines an element of S which satisfies the following identities:

$$T_{pj\alpha}^{p\beta} = g_{pj\alpha}^{p\beta} + 3b_{j\alpha}^\beta + a_{pj}^p \delta_\alpha^\beta = g_{pj\alpha}^{p\beta} - g_{pj\alpha}^{p\beta} = 0$$

and

$$T_{ij\alpha}^{s\alpha} = g_{ij\alpha}^{s\alpha} + (\delta_i^s b_{j\alpha}^\alpha + \delta_j^s b_{i\alpha}^\alpha) + 2a_{ij}^s = g_{ij\alpha}^{s\alpha} - g_{ij\alpha}^{s\alpha} = 0,$$

showing that $T_{ij\alpha}^{s\beta} \in (S^2V^* \otimes V)_0 \otimes \mathfrak{sl}(W)$. Combining (48) and (49), we obtain

$$\begin{aligned} (51) \quad a_{ij}^s \delta_\alpha^\beta &= -\frac{1}{2}g_{ij\gamma}^{s\gamma} \delta_\alpha^\beta \\ &= -\frac{1}{2}(f_{ij\gamma}^{s\gamma} - \frac{1}{6}(f_{pj\gamma}^{p\gamma} \delta_i^s + f_{pi\gamma}^{p\gamma} \delta_j^s)) \delta_\alpha^\beta \\ &= -\frac{1}{2}f_{ij\gamma}^{s\gamma} \delta_\alpha^\beta + \frac{1}{6}(f_{pj\gamma}^{p\gamma} \delta_i^s + f_{pi\gamma}^{p\gamma} \delta_j^s) \delta_\alpha^\beta \end{aligned}$$

and

$$\begin{aligned} \delta_i^s b_{j\alpha}^\beta &= -\frac{1}{3}\delta_i^s g_{rj\alpha}^{r\beta} \\ &= -\frac{1}{3}\delta_i^s (f_{rj\alpha}^{r\beta} - \frac{1}{6}(f_{pj\gamma}^{p\gamma} 2\delta_\alpha^\beta + f_{pr\gamma}^{p\gamma} \delta_j^r \delta_\alpha^\beta)) \\ &= -\frac{1}{3}\delta_i^s f_{rj\alpha}^{r\beta} + \frac{1}{6}f_{pj\gamma}^{p\gamma} \delta_i^s \delta_\alpha^\beta \end{aligned}$$

and symmetrizing last equation in i, j we obtain

$$(52) \quad \delta_i^s b_{j\alpha}^\beta + \delta_j^s b_{i\alpha}^\beta = -\frac{1}{3}(\delta_i^s f_{rj\alpha}^{r\beta} + \delta_j^s f_{ri\alpha}^{r\beta}) + \frac{1}{6}(f_{pj\gamma}^{p\gamma} \delta_i^s + f_{pi\gamma}^{p\gamma} \delta_j^s) \delta_\alpha^\beta$$

Inserting (48), (51) and (52) in (50), we finally obtain the explicit formula

$$(53) \quad T_{ij\alpha}^{s\beta} = f_{ij\alpha}^{s\beta} - \frac{1}{3}(\delta_i^s f_{rj\alpha}^{r\beta} + \delta_j^s f_{ri\alpha}^{r\beta}) + \frac{1}{6}(f_{pj\gamma}^{p\gamma} \delta_i^s + f_{pi\gamma}^{p\gamma} \delta_j^s) \delta_\alpha^\beta - \frac{1}{2} f_{ij\gamma}^{s\gamma} \delta_\alpha^\beta$$

for the projection of S onto $(S^2 V^* \otimes V)_0 \otimes \mathfrak{sl}(W)$. Looking at elements of S as linear maps in $L(S^2 V \otimes W, V \otimes W)$, (53) writes as

$$(54) \quad \begin{aligned} T(v_1 \odot v_2 \otimes w) &= f(v_1 \odot v_2 \otimes w) - \frac{1}{3}(v_1 \otimes \text{tr}_V(f)(v_2 \otimes w) + v_2 \otimes \text{tr}_V(f)(v_1 \otimes w)) + \\ &\quad + \left(\frac{1}{6}(c_V \circ \text{tr}_W(f)(v_1) \cdot v_2 + c_V \circ \text{tr}_W(f)(v_2) \cdot v_1) - \frac{1}{2} \text{tr}_W(f)(v_1 \odot v_2) \right) \otimes w \end{aligned}$$

for $v_1, v_2 \in V$ and $w \in W$.

In Example 13 of Appendix A, we showed that $H_{\mathbb{C}}^2(\mathfrak{g}_{-}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) = H_{\mathbb{C}}^2(\mathfrak{g}_{-}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})_1$. This means that $H_{\mathbb{C}}^2(\mathfrak{g}_{-}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ can be realized as a subset of graded maps $\mathfrak{g}_{-}^{\mathbb{C}} \times \mathfrak{g}_{-}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ of degree one. Since $\mathfrak{g}^{\mathbb{C}}$ is $[2]$ -graded, any such map can be seen as a graded map $\mathfrak{g}_{-}^{\mathbb{C}} \times \mathfrak{g}_{-}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-}^{\mathbb{C}}$ which splits into the sum of two components $\mathfrak{g}_{-1}^{\mathbb{C}} \times \mathfrak{g}_{-1}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-1}^{\mathbb{C}}$ and $\mathfrak{g}_{-1}^{\mathbb{C}} \times \mathfrak{g}_{-2}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-2}^{\mathbb{C}}$. The irreducible representations described above should be interpreted as subspaces of maps of this kind.

We can thus easily deduce the decomposition of the real \mathfrak{g}_0 -module $\mathbb{V} = H^2(\mathfrak{g}_{-}, \mathfrak{g})$ for the real graded Lie algebra \mathfrak{g} associated to a generic (4, 8)-distribution. First, consider the real graded simple Lie algebra $\mathfrak{g} = \oplus_{i=-2}^2 \mathfrak{g}_i$ from 2.1 describing the hyperbolic model. There, we saw that $\mathfrak{g}_j^{\mathbb{C}} = \mathfrak{g}_j \oplus i\mathfrak{g}_j$ for all $j = -2, \dots, 2$. Then $H_{\mathbb{C}}^2(\mathfrak{g}_{-}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) = \mathbb{V} \oplus i\mathbb{V}$, so that the \mathfrak{g}_0 -irreducible components of \mathbb{V} admit a completely analogous description as the $\mathfrak{g}_0^{\mathbb{C}}$ -irreducible components of $\mathbb{V}^{\mathbb{C}}$, obtained by replacing V, W with two-dimensional real vector spaces. As in 2.1, write $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ as the direct sum of real \mathfrak{g}_0 -modules of dimension two. Put $V = \mathfrak{g}_{-1}^E$ and $W = \mathfrak{g}_{-1}^F$. We thus have two distinct components $\Lambda^2 V^* \otimes W$ and $\Lambda^2 W^* \otimes V$ inside $\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$. Moreover, recall that the bracket gives an isomorphism $\mathfrak{g}_{-2} \cong V \otimes W$. We thus see that the components $(S^2 V^* \otimes V)_0 \otimes \mathfrak{sl}(W)$ and $(S^2 W^* \otimes W)_0 \otimes \mathfrak{sl}(V)$ are contained in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$.

Now let $\mathfrak{g} = \oplus_{i=-2}^2 \mathfrak{g}_i$ be the real graded simple Lie algebra from 3.2 describing the elliptic model. Since $\mathfrak{g}_0 \cong \mathfrak{gl}(2, \mathbb{C})$ is the underlying real Lie algebra of a complex Lie algebra, the complexification consists of the sum of two copies of the same Lie algebra. The explicit formulas for the bracket show that conjugating the \mathfrak{g}_0 -representations $\mathfrak{g}_{-1}^{\mathbb{C}}$ and $\mathfrak{g}_{-2}^{\mathbb{C}}$ corresponds to swapping the two summands $\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$. In particular, this shows that the complex cohomology $H_{\mathbb{C}}^2(\mathfrak{g}_{-}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ is given by two pairs of lowest weights such that for each pair, the two weights are different but conjugate to each other. This readily shows that they can only come from the complexification of one complex representation, since real representations always lead to weights which are self-conjugate (see sections 2.3.14 and 2.3.15 of [10] for the details). According to Proposition 3.3.6 in [10], we conclude that $H^2(\mathfrak{g}_{-}, \mathfrak{g}) = \mathbb{V}_1 \oplus \mathbb{V}_2$ is the direct sum of two irreducible \mathfrak{g}_0 -components. Denote by $R : \mathfrak{g}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ the conjugate map. If $T : \mathfrak{g}_j \times \mathfrak{g}_\ell \rightarrow \mathfrak{g}_{j+\ell+1}$ is a bilinear map and $T_{\mathbb{C}} : \mathfrak{g}_j^{\mathbb{C}} \times \mathfrak{g}_\ell^{\mathbb{C}} \rightarrow \mathfrak{g}_{j+\ell+1}^{\mathbb{C}}$ denotes the complex linear extension of T , then $T_{\mathbb{C}} = T_1 + T_2$ for complex linear maps $T_1 : \mathfrak{g}_j^{\mathbb{C}} \times \mathfrak{g}_\ell^{\mathbb{C}} \rightarrow \mathfrak{g}_{j+\ell+1}^{\mathbb{C}}$ and $T_2 : \mathfrak{g}_j^{\mathbb{C}} \times \mathfrak{g}_\ell^{\mathbb{C}} \rightarrow \mathfrak{g}_{j+\ell+1}^{\mathbb{C}}$ such that

$$T_1(v, w) = R(T_2(R(v), R(w)))$$

for all $v, w \in \mathfrak{g}_{-1}^{\mathbb{C}}$. Recall that there is a natural complex structure J on \mathfrak{g}_{-1} , which determines the decomposition $\mathfrak{g}_{-1}^{\mathbb{C}} = \mathfrak{g}_{-1}^{(1,0)} \oplus \mathfrak{g}_{-1}^{(0,1)}$ as direct sum of eigenspaces for $J_{\mathbb{C}}$ of eigenvalues $\pm i$. Both $V = \mathfrak{g}_{-1}^{(1,0)}$ and $W = \mathfrak{g}_{-1}^{(0,1)}$ are complex \mathfrak{g}_0 -submodules of $\mathfrak{g}_{-1}^{\mathbb{C}}$ of dimension two. If $R : \mathfrak{g}_{-1}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-1}^{\mathbb{C}}$ denotes the conjugate map, we have $V = R(W)$. There is one irreducible component of V contained in $\Lambda^2 \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$. Its elements are skew-symmetric bilinear maps $T : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$, whose complexification $T_{\mathbb{C}}$ is the sum of $T_1 : V \times V \rightarrow W$ and $T_2 : W \times W \rightarrow V$ and we have

$$(55) \quad \begin{aligned} 4T(x, y) &= T(2x, 2y) = T_1(x - iJx, y - iJy) + T_2(x + iJx, y + iJy) \\ &= T_1(x - iJx, y - iJy) + R(T_1(x - iJx, y - iJy)) = 2\text{Re}(T_1(x - iJx, y - iJy)) \end{aligned}$$

for all $x, y \in \mathfrak{g}_{-1}$. Analogously, there is a component $T : \mathfrak{g}_{-1} \times \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}$, whose complexification splits into the components $T_1 : \mathfrak{g}_{-1}^{\mathbb{C}} \times \mathfrak{g}_{-2}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-2}^{\mathbb{C}}$ and $T_2 : \mathfrak{g}_{-1}^{\mathbb{C}} \times \mathfrak{g}_{-2}^{\mathbb{C}} \rightarrow \mathfrak{g}_{-2}^{\mathbb{C}}$, which are of type $(S^2 V^* \otimes V)_0 \otimes \mathfrak{sl}(W)$ and $(S^2 W^* \otimes W)_0 \otimes \mathfrak{sl}(V)$, respectively. Recall that the complex structure J on \mathfrak{g}_{-1} is partially integrable and this implies that

$$[x - iJ(x), y + iJ(y)] = 2[x, y] + 2i[x, J(y)] = R([x + iJ(x), y - iJ(y)])$$

for all $x, y \in \mathfrak{g}_{-1}$. We can thus write

$$(56) \quad \begin{aligned} 4T(x_1, [x_2, y]) &= T(2x_1, 2[x_2, y]) \\ &= T_1(x_1 - iJ(x_1), [x_2, y] + i[x_2, J(y)]) + T_2(x_1 + iJ(x_1), [x_2, y] - i[x_2, J(y)]) \\ &= T_1(x_1 - iJ(x_1), [x_2, y] + i[x_2, J(y)]) + R(T_1(x_1 - iJ(x_1), [x_2, y] + i[x_2, J(y)])) \\ &= 2\text{Re}(T_1(x_1 - iJ(x_1), [x_2, y] + i[x_2, J(y)])) \\ &= \text{Re}(T_1(x_1 - iJ(x_1), [x_2 - iJ(x_2), y + iJ(y)])) \end{aligned}$$

for all $x_1, x_2, y \in \mathfrak{g}_{-1}$.

PROPOSITION 20. *Let $H \subset TM$ be an hyperbolic (4, 8)–distribution and $H = E \oplus F$ the local direct sum decomposition from Theorem 5. The following formulas define tensorial maps:*

- (1) $\Lambda^2 E \rightarrow F$, $P_1(\xi_1, \xi_2) = [\xi_1, \xi_2]_F$ for $\xi_1, \xi_2 \in \Gamma(E)$, where $[\xi_1, \xi_2]_F$ denotes the component along F of the bracket.
- (2) $\Lambda^2 F \rightarrow E$, $P_2(\eta_1, \eta_2) = [\eta_1, \eta_2]_E$ for $\eta_1, \eta_2 \in \Gamma(F)$, where $[\eta_1, \eta_2]_E$ denotes the component along E of the bracket.

PROOF. By Theorem 5, H locally writes as the direct sum $H = E \oplus F$ of rank-two subbundles such that the bracket among sections of E , respectively among sections of F , is a section of H . Since $[\xi_1, \xi_2] \in \Gamma(H)$ for $\xi_1, \xi_2 \in \Gamma(E)$, the bracket uniquely writes as the sum

$$[\xi_1, \xi_2] = [\xi_1, \xi_2]_E + [\xi_1, \xi_2]_F$$

of its components along E and F . Clearly, the analogous statement holds for brackets of the form $[\eta_1, \eta_2]$ with $\eta_1, \eta_2 \in \Gamma(F)$. For $p \in C^\infty(M)$, we have that

$$[p\xi_1, \xi_2] = p[\xi_1, \xi_2] - \xi_2(p) \cdot \xi_1$$

and therefore that $[p\xi_1, \xi_2]_F = p[\xi_1, \xi_2]_F$. Since the formula is skew-symmetric, this is enough to prove the statement for (1). The proof of formula (2) is completely analogous. \square

PROPOSITION 21. *Let $H \subset TM$ be an hyperbolic (4, 8)–distribution and $H = E \oplus F$ the local direct sum decomposition from Theorem 5. Choose a linear connection ∇^E on E and define $\Phi : \Gamma(E) \otimes \Gamma(E) \otimes \Gamma(F) \rightarrow \Gamma(Q)$ by*

$$(57) \quad \Phi(\xi_1, \xi_2, \eta) := \mathcal{L}(\nabla_{\xi_1}^E \xi_2, \eta) + \mathcal{L}(\nabla_{\xi_2}^E \xi_1, \eta) - q([\xi_1, [\xi_2, \eta]]) - q([\xi_2, [\xi_1, \eta]])$$

for ξ_1, ξ_2 in $\Gamma(E)$ and $\eta \in \Gamma(F)$. Then there is a well defined totally tracefree part of Φ , which is tensorial in all entries and independent of the choice of ∇^E and thus defines a section T_1 of the

bundle $(S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F)$ which is intrinsically associated to the distribution H . Similarly, by choosing a linear connection ∇^F on F and putting

$$\Psi(\eta_1, \eta_2, \xi) := \mathcal{L}(\nabla_{\eta_1}^F \eta_2, \xi) + \mathcal{L}(\nabla_{\eta_2}^F \eta_1, \xi) - q([\eta_1, [\eta_2, \xi]]) - q([\eta_2, [\eta_1, \xi]])$$

for ξ in $\Gamma(E)$ and $\eta_1, \eta_2 \in \Gamma(F)$, one obtains a section T_2 of the bundle $(S^2 F^* \otimes F)_0 \otimes \mathfrak{sl}(E)$ intrinsically associated to H .

PROOF. Let $\nabla^E : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$ be a linear connection on E and define Φ as in (57). Direct computations show that this is linear over smooth functions in ξ_1 and ξ_2 , while

$$\Phi(\xi_1, \xi_2, f\eta) - f\Phi(\xi_1, \xi_2, \eta) = 2((\xi_1 \cdot f)\mathcal{L}(\xi_2, \eta) + (\xi_2 \cdot f)\mathcal{L}(\xi_1, \eta))$$

for $f \in C^\infty(M)$. Using the isomorphism $Q \cong E \otimes F$ determined by \mathcal{L} as in Theorem 5, we see that there exists a contraction $c^A : E^* \otimes Q \rightarrow F$. Now c^A is a tensorial map and defines, together with Φ , a map

$$\begin{aligned} \Gamma(E^*) \times \Gamma(E) \times \Gamma(E) \times \Gamma(F) &\rightarrow \Gamma(F) \\ (\lambda, \xi_1, \xi_2, \eta) &\mapsto c^A(\lambda \otimes \Phi(\xi_1, \xi_2, \eta)) \end{aligned}$$

which is linear over smooth functions in the E and E^* -entries and symmetric in the E -entries. Hence there is a unique contraction of one E and one E^* component and forming this, we obtain a map $\Psi : \Gamma(E) \times \Gamma(F) \rightarrow \Gamma(F)$ which is linear over smooth functions in the E -component. Choose a frame $\{\theta_1, \theta_2\} \subset \Gamma(E)$ and denote by $\{\theta_1^*, \theta_2^*\} \subset \Gamma(E^*)$ the dual frame. Then, the formula

$$\Psi(\xi, \eta) := \sum_{i=1}^2 c^A(\theta_i^* \otimes \Phi(\theta_i, \xi, \eta)) = \sum_{i=1}^2 (\theta_i^* \otimes id_F)(\Phi(\theta_i, \xi, \eta))$$

can be easily seen to be independent on the choice of frame. For $f \in C^\infty(M)$, we have that

$$\begin{aligned} \Psi(\xi, f\eta) - f\Psi(\xi, \eta) &= \sum_{i=1}^2 (\theta_i^* \otimes id_F)(\Phi(\theta_i, \xi, f\eta) - f\Phi(\theta_i, \xi, \eta)) \\ &= \sum_{i=1}^2 (\theta_i^* \otimes id_F) \left(2((\theta_i \cdot f)\mathcal{L}(\xi, \eta) + (\xi \cdot f)\mathcal{L}(\theta_i, \eta)) \right) \\ &= 2 \left(\sum_{i=1}^2 \theta_i^*(\xi) \theta_i \cdot (f) \right) \eta + 2(\xi \cdot f) \sum_{i=1}^2 \theta_i^*(\theta_i) \eta \\ &= 6(\xi \cdot f) \eta \end{aligned}$$

Therefore, mapping (ξ_1, ξ_2, η) to

$$F^\Phi(\xi_1, \xi_2, \eta) := \Phi(\xi_1, \xi_2, \eta) - \frac{1}{3}(\mathcal{L}(\xi_1, \Psi(\xi_2, \eta)) + \mathcal{L}(\xi_2, \Psi(\xi_1, \eta)))$$

the result will be trilinear over smooth functions, hence it defines $F^\Phi : S^2 E \otimes F \rightarrow Q$ tensorial which depends on ∇^E . Again, the isomorphism $Q \cong E \otimes F$ shows the existence of a contraction $c^B : F^* \otimes Q \rightarrow E$. One can apply c^B to F^Φ , thus obtaining a tensorial map $\chi : S^2 E \rightarrow E$. Choose a frame $\{\alpha_1, \alpha_2\} \subset \Gamma(F)$ and denote by $\{\alpha_1^*, \alpha_2^*\} \subset \Gamma(F^*)$ the dual frame. Then, the formula

$$\chi(\xi_1, \xi_2) := \sum_{j=1}^2 c^B(\alpha_j^* \otimes F^\Phi(\xi_1, \xi_2, \alpha_j))$$

is easily seen to be independent on the choice of frame. Finally, putting

$$G^\Phi(\xi_1, \xi_2, \eta) := F^\Phi(\xi_1, \xi_2, \eta) - \frac{1}{2}\mathcal{L}(\chi(\xi_1, \xi_2), \eta)$$

defines a map in $(S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F)$. Now we can easily see that this map does not depend on the choice of linear connection. Indeed, any other connection on E is of the form

$$\hat{\nabla}_{\xi_1}^E \xi_2 = \nabla_{\xi_1}^E \xi_2 + A(\xi_1, \xi_2)$$

for some tensorial map $A : E \otimes E \rightarrow E$. But this readily says that if $\hat{\Phi}$ is defined by $\hat{\nabla}$, then

$$\hat{\Phi}(\xi_1, \xi_2, \eta) - \Phi(\xi_1, \xi_2, \eta) = \mathcal{L}(A(\xi_1, \xi_2) + A(\xi_2, \xi_1), \eta)$$

and therefore

$$\begin{aligned} \hat{\Psi}(\xi, \eta) - \Psi(\xi, \eta) &= \sum_{i=1}^2 c^A(\theta_i^* \otimes (\hat{\Phi} - \Phi)(\theta_i, \xi, \eta)) \\ &= \sum_{i=1}^2 c^A(\theta_i^* \otimes (\mathcal{L}(A(\theta_i, \xi) + A(\xi, \theta_i), \eta))) \end{aligned}$$

thus

$$\begin{aligned} (F^{\hat{\Phi}} - F^{\Phi})(\xi_1, \xi_2, \eta) &= (\hat{\Phi} - \Phi)(\xi_1, \xi_2, \eta) - \frac{1}{3}(\mathcal{L}(\xi_1, (\hat{\Psi} - \Psi)(\xi_2, \eta)) + \\ &\quad \mathcal{L}(\xi_2, (\hat{\Psi} - \Psi)(\xi_1, \eta))) \end{aligned}$$

and it thus follows that $G^{\hat{\Phi}}(\xi_1, \xi_2, \eta) = G^{\Phi}(\xi_1, \xi_2, \eta)$, so that we obtain a map $T_1 \in (S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F)$. Clearly, one can analogously prove the existence of a map $T_2 \in (S^2 F^* \otimes F)_0 \otimes \mathfrak{sl}(E)$ by exchanging the roles of E and F . \square

Let $(p : \mathcal{G} \rightarrow M, \omega)$ be the normal, regular parabolic geometry of type (G, P) equivalent to an hyperbolic (4, 8)–distribution (M, H) , as in Corollary 5. Denote by $\kappa \in \Omega^2(M, \mathcal{A}M)$ the two–form corresponding to the curvature function and by

$$\text{gr}_1(\kappa) \in \Gamma((\Lambda^2 H^* \otimes H) \oplus (H^* \otimes Q^* \otimes Q))$$

the graded curvature in homogeneity one. We will now to prove that the tensorial maps from Propositions 20 and 21 are (up to multiples) the components of $\text{gr}_1(\kappa)$. Recall from Section 4 that the filtration of \mathfrak{g} induces a filtration of $T\mathcal{G}$ by smooth subbundles, by putting $T^i \mathcal{G} := \omega^{-1}(\mathfrak{g}^i)$. For $u \in \mathcal{G}$, one can restrict the linear isomorphism $\omega_u : T_u \mathcal{G} \rightarrow \mathfrak{g}$ to $T_u^i \mathcal{G}$ and then compose it with the projection onto the quotient $\mathfrak{g}^i \rightarrow \mathfrak{g}_i$. In this way, we obtain a smooth section ω_i of $L(T^i \mathcal{G}, \mathfrak{g}_i)$ which vanishes on $T^{i+1} \mathcal{G}$. The Cartan connection splits, then, into the sum

$$\omega = \omega_{-2} + \dots + \omega_2$$

Each negative graded component vanishes under insertion of vertical vector fields. As observed in Proposition 17, the sum of the nonpositive graded components is equivalent to the sum of a soldering form and a connection form on the adapted frame bundle for (M, H) . In particular if $G_0 \subset P$ denotes the Levi subgroup, ω_0 descends to a G_0 –equivariant one–form ω_0^0 on $\mathcal{G}_0 = \mathcal{G}/P_+$. This corresponds, through the associated bundle construction, to a linear connection on $\text{gr}(TM)$ (see Definition 7 of Chapter 2 for the explicit formula of an induced linear connection).

PROPOSITION 22. *Let $H \subset TM$ be an hyperbolic (4, 8)–distribution and $H = E \oplus F$ the local decomposition from Theorem 5. Let $(p : \mathcal{G} \rightarrow M, \omega)$ be the normal, regular parabolic geometry of type (G, P) equivalent to (M, H) (see Corollary 5) and let κ be the curvature function for the geometry. Then, the following formulas completely describe the graded curvature in homogeneity one:*

- (1) $\text{gr}_1(\kappa)(\xi_1, \xi_2) = -[\xi_1, \xi_2]_F$ for $\xi_1, \xi_2 \in \Gamma(E)$, where $[\xi_1, \xi_2]_F$ denotes the component along F of the bracket. In particular, this component vanishes if and only if the subbundle $E \subset TM$ is involutive.

- (2) $\text{gr}_1(\kappa)(\eta_1, \eta_2) = -[\eta_1, \eta_2]_E$ for $\eta_1, \eta_2 \in \Gamma(F)$, where $[\eta_1, \eta_2]_E$ denotes the component along E of the bracket. In particular, this component vanishes if and only if the subbundle $F \subset TM$ is involutive.
- (3) $\text{gr}_1(\kappa)(\xi_1, \mathcal{L}(\xi_2, \eta)) = -\frac{1}{2}T_1(\xi_1, \xi_2, \eta)$ for $\xi_1, \xi_2 \in \Gamma(E)$ and $\eta \in \Gamma(F)$, where T_1 is the map defined in Proposition 21.
- (4) $\text{gr}_1(\kappa)(\eta_1, \mathcal{L}(\xi, \eta_2)) = -\frac{1}{2}T_2(\eta_1, \xi, \eta_2)$ for $\xi \in \Gamma(E)$ and $\eta_1, \eta_2 \in \Gamma(F)$, where T_2 is the map defined in Proposition 21.

PROOF. Let $\mathfrak{g}_- \subset \mathfrak{sl}(5, \mathbb{R})$ be the model algebra of hyperbolic (4, 8)-type. From Proposition 19, we know that the nontrivial components of $\text{gr}_1(\kappa)$ are associated to the \mathfrak{g}_0 -irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})$. We thus see that the nontrivial components write as $\Lambda^2 E^* \otimes F$, $\Lambda^2 F^* \otimes E$, $(S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F)$ and $(S^2 F^* \otimes F)_0 \otimes \mathfrak{sl}(E)$, according to the local direct sum decomposition of H . Observe that the tensorial maps from Propositions 20 and 21 are of these types. In the following, we will show that such tensorial maps coincide, up to the sign, with the components of $\text{gr}_1(\kappa)$. Due to the symmetry of the formulas, it is enough proving the statement for (1) and (3). Formulas (2) and (4), then, will be deduced from (1) and (3) by exchanging the roles of E and F .

Denote by κ the curvature function. Let $\xi_1, \xi_2 \in \Gamma(E)$ and let $\tilde{\xi}_1, \tilde{\xi}_2 \in \Gamma(\mathcal{G})$ be lifts of ξ_1 and ξ_2 , respectively. Now the component $\Lambda^2 E^* \otimes F$ corresponds to the \mathfrak{g}_{-1}^F -component of $\kappa(\tilde{\xi}_1, \tilde{\xi}_2)$. Since the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ is \mathfrak{g}_0 -invariant and ω_{-1} is \mathfrak{g}_0 -equivariant, we have that $\omega_{-1} = \omega_{-1}^E + \omega_{-1}^F$ with ω_{-1}^E and ω_{-1}^F both \mathfrak{g}_0 -equivariant taking values in \mathfrak{g}_{-1}^E and \mathfrak{g}_{-1}^F , respectively. Since ξ_1 is a section of E , we get $\omega_{-1}(\xi_1) = \omega_{-1}^E(\xi_1)$ and likewise for ξ_2 . First, the \mathfrak{g}_{-1} -component of $\kappa(\tilde{\xi}_1, \tilde{\xi}_2)$ is given by

$$\kappa_{-1}(\tilde{\xi}_1, \tilde{\xi}_2) = d\omega_{-1}(\tilde{\xi}_1, \tilde{\xi}_2) + [\omega_0(\tilde{\xi}_1), \omega_{-1}(\tilde{\xi}_2)] - [\omega_0(\tilde{\xi}_2), \omega_{-1}(\tilde{\xi}_1)]$$

Observe that the second and the third terms take values in \mathfrak{g}_{-1}^E . The first summand rewrites, by using the defining formula for the differential, as

$$\tilde{\xi}_1 \cdot \omega_{-1}(\tilde{\xi}_2) - \tilde{\xi}_2 \cdot \omega_{-1}(\tilde{\xi}_1) - \omega_{-1}([\tilde{\xi}_1, \tilde{\xi}_2]).$$

We thus see that the \mathfrak{g}_{-1}^F -component of $\kappa(\tilde{\xi}_1, \tilde{\xi}_2)$ coincides with $-\omega_{-1}^F([\tilde{\xi}_1, \tilde{\xi}_2])$. This corresponds, through the associated bundle construction, to $-[\xi_1, \xi_2]_F$ and the statement (1) is thus proved. Consider now $\xi_1, \xi_2 \in \Gamma(E)$ and $\eta \in \Gamma(F)$. Choose local lifts $\tilde{\xi}_1, \tilde{\xi}_2$ and $\tilde{\eta}$ for ξ_1, ξ_2 and η , respectively. By definition ω_{-2} vanishes on all of them, $(\omega_{-1})_u(\tilde{\xi}_i) \in \mathfrak{g}_{-1}^E$ for $i = 1, 2$ and $(\omega_{-1})_u(\tilde{\eta}) \in \mathfrak{g}_{-1}^F$ for all $u \in \mathcal{G}$. We have to compute the component in \mathfrak{g}_{-2} of $\kappa(\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}])$, which is given by

$$\kappa_{-2}(\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}]) = d\omega_{-2}(\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}]) + [\omega_0(\tilde{\xi}_1), \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}])] + [\omega_{-1}(\tilde{\xi}_1), \omega_{-1}([\tilde{\xi}_2, \tilde{\eta}])].$$

Since $\omega_{-2}(\tilde{\xi}_1) = 0$, the first term equals $\tilde{\xi}_1 \cdot \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}]) - \omega_{-2}([\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}]])$. Therefore, we can rewrite $\kappa_{-2}(\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}])$ as

$$\tilde{\xi}_1 \cdot \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}]) + [\omega_0(\tilde{\xi}_1), \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}])] - \omega_{-2}([\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}]]) + [\omega_{-1}(\tilde{\xi}_1), \omega_{-1}([\tilde{\xi}_2, \tilde{\eta}])]$$

Now the third term obviously represents $-q([\xi_1, [\xi_2, \eta]])$. Since ω_0 descends to a principal connection form on \mathcal{G}_0 , it gives rise to linear connections on E and F and hence on $Q \cong E \otimes F$. In particular, this shows that $\tilde{\xi}_1 \cdot \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}]) + [\omega_0(\tilde{\xi}_1), \omega_{-2}([\tilde{\xi}_2, \tilde{\eta}])]$ represents $\nabla_{\tilde{\xi}_1}^Q \mathcal{L}(\xi_2, \eta) = \mathcal{L}(\nabla_{\tilde{\xi}_1}^E \xi_2, \eta) + \mathcal{L}(\xi_2, \nabla_{\tilde{\xi}_1}^F \eta)$.

So it remains to interpret the term $[\omega_{-1}(\tilde{\xi}_1), \omega_{-1}([\tilde{\xi}_2, \tilde{\eta}])]$. By Proposition 19, the component $\mathfrak{g}_{-1}^E \otimes \mathfrak{g}_{-1}^F \rightarrow \mathfrak{g}_{-1}$ of κ is identically zero, which implies that

$$0 = d\omega_{-1}(\tilde{\xi}_2, \tilde{\eta}) + [\omega_0(\tilde{\xi}_2), \omega_{-1}(\tilde{\eta})] - [\omega_0(\tilde{\eta}), \omega_{-1}(\tilde{\xi}_2)].$$

Inserting the definition of the exterior derivative and bringing the bracket term to the other side we obtain

$$\omega_{-1}([\tilde{\xi}_2, \tilde{\eta}]) = \tilde{\xi}_2 \cdot \omega_{-1}(\tilde{\eta}) - \tilde{\eta} \cdot \omega_{-1}(\tilde{\xi}_2) + [\omega_0(\tilde{\xi}_{-2}), \omega_{-1}(\tilde{\eta})] - [\omega_0(\tilde{\eta}), \omega_{-1}(\tilde{\xi}_2)]$$

which represents $\nabla_{\xi_2}^F \eta - \nabla_{\eta}^E \xi_2$. Observe that the first term corresponds to the component in \mathfrak{g}_{-1}^F and the second to the one in \mathfrak{g}_{-1}^E . Since $\omega_{-1}(\xi_1) \in \mathfrak{g}_{-1}^E$, we conclude that $[\omega_{-1}(\xi_1), \omega_{-1}([\tilde{\xi}_2, \tilde{\eta}])]$ represents $-\mathcal{L}(\xi_1, \nabla_{\xi_2}^F \eta)$. Putting all together, we see that $\kappa_{-2}(\xi_1, [\tilde{\xi}_2, \tilde{\eta}])$ represents

$$\mathcal{L}(\nabla_{\xi_1}^E \xi_2, \eta) + \mathcal{L}(\xi_2, \nabla_{\xi_1}^F \eta) - q([\xi_1, [\xi_2, \eta]]) - \mathcal{L}(\xi_1, \nabla_{\xi_2}^F \eta).$$

Finally, we use once more that the lowest homogeneous component of κ is harmonic. This implies that $\kappa_{-2}(\xi_1, [\tilde{\xi}_2, \tilde{\eta}])$ is symmetric in the $\tilde{\xi}$'s and totally trace-free. Now the summand

$$\mathcal{L}(\xi_2, \nabla_{\xi_1}^F \eta) - \mathcal{L}(\xi_1, \nabla_{\xi_2}^F \eta)$$

is evidently skew-symmetric in ξ_1, ξ_2 and therefore

$$\frac{1}{2}(\kappa_{-2}(\tilde{\xi}_1, [\tilde{\xi}_2, \tilde{\eta}]) + \kappa_{-2}(\tilde{\xi}_2, [\tilde{\xi}_1, \tilde{\eta}])) = -\frac{1}{2}\Phi(\xi_1, \xi_2, \eta),$$

where Φ is the map defined in (57) of Proposition 21. Clearly, the totally trace-free part of the terms in the last equality coincide, proving (3).

Finally, if (M, H) is locally isomorphic to the homogeneous model, $\text{gr}_1(\kappa)$ is identically zero. The fact that (1) and (2) vanish implies, respectively, that E and F are involutive subbundles of H . \square

Let (M, H) be an elliptic (4, 8)–distribution. By Proposition 16, H carries a partially integrable almost complex structure J , locally defined up to sign around each point. Recall from 3.4 that since J is partially integrable, $[\xi, \eta] - [J\xi, J\eta]$ and $[J\xi, \eta] + [\xi, J\eta]$ are sections of H for all $\xi, \eta \in \Gamma(H)$ and we can thus define the Nijenhuis tensor

$$N: \Lambda^2 H \rightarrow H, \quad (\xi, \eta) \mapsto [\xi, \eta] - [J\xi, J\eta] + J([J\xi, \eta] + [\xi, J\eta])$$

The subbundles $H^{(1,0)}, H^{(0,1)} \subset H^{\mathbb{C}}$, corresponding to anti-holomorphic and holomorphic sections of $H^{\mathbb{C}}$, are conjugate to each other. Furthermore, N vanishes identically if and only if $H^{(1,0)}$ and $H^{(0,1)}$ are involutive subbundles of $H^{\mathbb{C}}$. If this is the case, (H, J) is said to be an integrable CR-structure.

PROPOSITION 23. *Let $H \subset TM$ be an elliptic (4, 8)–distribution and J a local partially integrable almost complex structure on H (see Proposition 16). Let $(p: \mathcal{G} \rightarrow M, \omega)$ be the normal, regular parabolic geometry of type (G, P) equivalent to (M, H) (see Corollary 5) and let κ be the curvature function for the geometry. Then, the following formulas completely describe the graded curvature in homogeneity one:*

- (1) $\text{gr}_1(\kappa)(\xi, \eta) = -\frac{1}{4}N(\xi, \eta)$ for all $\xi, \eta \in \Gamma(H)$, where N denotes the Nijenhuis tensor associated to (H, J) . In particular, this component vanishes if and only if (H, J) is integrable and thus a CR-structure.
- (2) For $\xi_1, \xi_2, \eta \in \Gamma(H)$ put $\zeta_t = \xi_t - iJ(\xi_t) \in \Gamma(H^{(1,0)})$ and $\lambda = \eta + iJ(\eta) \in \Gamma(H^{(0,1)})$. Then

$$\text{gr}_1(\kappa)(\xi_1, \mathcal{L}(\xi_2, \eta)) = -\frac{1}{8} \text{Re}(T_1(\zeta_1, \zeta_2 \otimes \lambda)),$$

where

$$T_1 \in (S^2(H^{(0,1)})^* \otimes H^{(0,1)})_0 \otimes \mathfrak{sl}(H^{(1,0)})$$

is defined by choosing a linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(H) \rightarrow \Gamma(H)$ such that $\nabla J = 0$ (equivalently, $\nabla_\xi J\eta = J\nabla_\xi \eta$ for all $\xi, \eta \in \Gamma(H)$) and taking the totally trace-free part of

$$\Phi(\zeta_1, \zeta_2, \lambda) := \mathcal{L}_\mathbb{C}(\nabla_{\zeta_1} \zeta_2, \lambda) + \mathcal{L}_\mathbb{C}(\nabla_{\zeta_2} \zeta_1, \lambda) - q_\mathbb{C}([\zeta_1, [\zeta_2, \lambda]]) - q_\mathbb{C}([\zeta_2, [\zeta_1, \lambda]])$$

as in Proposition 21.

PROOF. Put $\mathfrak{t} = \mathfrak{su}(3, 2)$ and consider the grading $\mathfrak{t} = \oplus_{i=-2}^2 \mathfrak{t}_i$ from 3.2. The description of $H^2(\mathfrak{t}_-, \mathfrak{t})$, together with the associated bundle construction, assures the existence of two nontrivial components of $\text{gr}_1(\kappa)$. Using the fact that their complexification coincides with the complexification of those described in Proposition 22, they can be easily deduced. Consider the local decomposition

$$H^\mathbb{C} = H^{(1,0)} \oplus H^{(0,1)}$$

as direct sum of rank-two complex vector subbundles $H^{(1,0)}, H^{(0,1)} \subset H^\mathbb{C}$ from Theorem 7. Put $E = H^{(0,1)}$ and $F = H^{(1,0)}$. Let $R : H^\mathbb{C} \rightarrow H^\mathbb{C}$ be the conjugation map. There is a component $\Lambda^2 H \rightarrow H$ of $\text{gr}_1(\kappa)$, whose complexification is the sum of $X_1 : \Lambda^2 E \rightarrow F$ and $X_2 : \Lambda^2 F \rightarrow E$ given by (1) and (2) of Proposition 22 and satisfying

$$X_1(\zeta_1, \zeta_2) = R(X_2(R(\zeta_1), R(\zeta_2)))$$

for all $\zeta_1, \zeta_2 \in \Gamma(E)$. Recall that for $\xi, \eta \in \Gamma(H)$:

$$z = -[\xi + iJ\xi, \eta + iJ\eta] = -([\xi, \eta] - [J\xi, J\eta]) - i([J\xi, \eta] + [\xi, J\eta]) \in \Gamma(H^\mathbb{C})$$

Writing $z = z^{(1,0)} + z^{(0,1)} \in F \oplus E$, we deduce the formula for X_1 from (1) of Proposition 22:

$$X_1(\xi + iJ\xi, \eta + iJ\eta) = z^{(1,0)} = \frac{1}{2}(z - iJ_\mathbb{C}(z)) = -\frac{1}{2}N(\xi, \eta) + \frac{1}{2}iJ(N(\xi, \eta))$$

and by (55) we have

$$\text{gr}_1(\kappa)(\xi, \eta) = \frac{1}{2}\text{Re}(X_1(\xi + iJ\xi, \eta + iJ\eta)) = -\frac{1}{4}N(\xi, \eta)$$

which proves (1).

Observe that a linear connection $\nabla : \mathfrak{X}(M) \times \Gamma(H) \rightarrow \Gamma(H)$ such that $\nabla J = 0$ defines linear connections

$$\Gamma(T^\mathbb{C}M) \times \Gamma(E) \rightarrow \Gamma(E), \quad \Gamma(T^\mathbb{C}M) \times \Gamma(F) \rightarrow \Gamma(F)$$

which will be denoted with the same symbol. These define tensorial maps

$$P_1 \in (S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F), \quad P_2 \in (S^2 F^* \otimes F)_0 \otimes \mathfrak{sl}(E)$$

as in Proposition 21, satisfying

$$P_1(\zeta_1, \zeta_2 \otimes \lambda) = R(P_2(R(\zeta_1), R(\zeta_2) \otimes R(\lambda)))$$

for all $\zeta_1, \zeta_2 \in \Gamma(E)$. There is a second component of $\text{gr}_1(\kappa)$ of type $H^* \otimes Q^* \otimes Q$. By (3) of Proposition 22, the complexification of this component coincides with $-\frac{1}{2}(P_1 + P_2)$. Using formula (56), we thus have

$$\text{gr}_1(\kappa)(\xi_1, \mathcal{L}(\xi_2, \eta)) = -\frac{1}{2} \frac{1}{4} \text{Re}(P_1(\zeta_1, \zeta_2 \otimes \lambda))$$

for all $\xi_1, \xi_2, \eta \in \Gamma(H)$, where $\zeta_1 = \xi_1 - iJ\xi_1$, $\zeta_2 = \xi_2 - iJ\xi_2$ and $\lambda = \eta + iJ\eta$, which proves (2). \square

REMARK 4. Proposition 23 recovers a more general result (Theorem 1, p.16 [18]), describing the fundamental invariants of free CR-distributions of dimension k on manifolds of dimension $2k + k^2$ for integers $k \geq 2$. In [18], it is observed that the case $k = 2$ corresponding to our elliptic $(4, 8)$ -distributions is exceptional, in the following sense. For each integer $k \geq 2$, the symbol algebra for free CR-distributions of dimension k can be realized as the negative graded part of a $|2|$ -grading on $\mathfrak{g} = \mathfrak{su}(k+1, k)$. Furthermore, it is shown that any such structure is equivalent to a normal parabolic geometry and the associated curvature is analyzed. In higher dimensions, all nontrivial components of the curvature are of type $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-2}^* \otimes \mathfrak{g}_{-2}$ and the Nijenhuis tensor of the partially integrable almost complex structure always vanishes. In particular, any free CR-distribution of CR-rank k for $k \geq 3$ is integrable, while this is not the case for $k = 2$.

6. The gap problem for normal, regular parabolic geometries

Fix a Lie group G and a closed subgroup $H \subset G$. Recall that the set of automorphisms of a Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (G, H) forms a group, usually denoted by $\text{Aut}(\mathcal{G}, \omega)$. The set of *infinitesimal symmetries* of $(\mathcal{G} \rightarrow M, \omega)$ is defined as

$$\text{inf}(\mathcal{G}, \omega) = \{ \xi \in \mathfrak{X}(\mathcal{G}) \mid (r^h)^* \xi = \xi \text{ for all } h \in H, \mathcal{L}_\xi \omega = 0 \}$$

The set of infinitesimal symmetries is related to the automorphism group of a Cartan geometry, according to the following fundamental result (see [6] for a proof):

THEOREM 9. *Let M be a connected manifold and let $(\mathcal{G} \rightarrow M, \omega)$ be a Cartan geometry of type (G, H) .*

- (1) *The automorphism group $\text{Aut}(\mathcal{G}, \omega)$ can be made into a Lie group of dimension lower or equal than the dimension of G , whose Lie algebra $\text{aut}(\mathcal{G}, \omega)$ consists of all complete vector fields lying in $\text{inf}(\mathcal{G}, \omega)$.*
- (2) *For $\xi, \eta \in \text{inf}(\mathcal{G}, \omega)$ the formula*

$$\omega([\eta, \xi]) = [\omega(\xi), \omega(\eta)] - \kappa(\omega(\xi), \omega(\eta))$$

holds. Here, κ denotes the curvature function associated to $(\mathcal{G} \rightarrow M, \omega)$. For any point $u \in \mathcal{G}$, the map $\xi \mapsto \omega(\xi(u))$ induces an injection $\text{inf}(\mathcal{G}, \omega) \rightarrow \mathfrak{g}$. Denoting by $\mathfrak{f}(u)$ its image, the Lie bracket on $\text{inf}(\mathcal{G}, \omega)$ is mapped to the operation

$$(X, Y) \mapsto [X, Y]_{\mathfrak{f}(u)} = [X, Y] - \kappa_u(X, Y).$$

In the following, we will always consider Cartan geometries with connected base space M . The *gap problem* is formulated as follows. Given a Lie group G and a closed subgroup $H \subset G$, one aims to determine the submaximal value \mathfrak{S} of $\dim(\text{inf}(\mathcal{G}, \omega))$ over the set of Cartan geometries $(\mathcal{G} \rightarrow M, \omega)$ of type (G, H) . Theorem 9 (1) states that the maximal value for $\dim(\text{inf}(\mathcal{G}, \omega))$ coincides with $\dim(G)$. Moreover, it is a well known fact that the Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$ is such that $\dim(\text{inf}(\mathcal{G}, \omega)) = \dim(G)$ if and only if it is locally isomorphic to the flat model of type (G, H) . Then, the problem is solved by maximizing $\dim(\mathcal{S})$ over the subset of non-flat Cartan geometries of same type, where $\mathcal{S} = \text{inf}(\mathcal{G}, \omega)$ denotes the set of infinitesimal symmetries. [11] investigates the gap problem for normal, regular parabolic geometries. In this setting, non-flatness means that the harmonic curvature is nonzero. The main result of the article is the realization of the submaximally symmetric model, for almost all complex parabolic types (G, P) and the corresponding split real types, as a locally homogeneous space. First, we recall the argument developed in [11]. Then, we describe explicitly the submaximally symmetric models of hyperbolic $(4, 8)$ -distributions. In particular, we will see that the symmetry algebra of such models has dimension 14, while for the flat model it coincides with the dimension of $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$ which is 24.

6.1. Preliminaries from algebra.

DEFINITION 18. (*Refinement of Tanaka prolongation*) Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a graded semisimple Lie algebra and let $\mathfrak{a}_0 \subset \mathfrak{g}_0$ be a Lie subalgebra. Define $\text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) \subset \mathfrak{g}$ graded subalgebra by putting:

- $\mathfrak{a}_{\leq 0} = \mathfrak{g}_{\leq 0}$;
- $\mathfrak{a}_k := \{ X \in \mathfrak{g}_k \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{k-1} \}$ for $k \geq 1$.

Observe that if $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$, then Proposition 18 (4) shows that $\text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0)$ coincides with the Tanaka prolongation $\text{pr}(\mathfrak{g}_-, \mathfrak{a}_0)$.

From the representation theory of parabolic subalgebras, we get the following facts. Let \mathfrak{g} be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$. Recall that a subset $\Sigma \subset \Delta^0$ corresponds to a unique choice of grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ and this, in turn, to a unique choice of standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i$ is the nonnegative graded part for the grading. Now fix a grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$. The subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive. Therefore, it writes as the direct sum $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ of its center $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$ and its semisimple part \mathfrak{g}_0^{ss} . Moreover, $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}''$ such that \mathfrak{h}'' is a Cartan subalgebra. The grading corresponds to the decomposition in eigenspaces of the adjoint action of an element $Z \in \mathfrak{z}(\mathfrak{g}_0)$, which is said to be the *grading element*.

Let be $-\mu \in \mathfrak{h}^*$ and put $-\mu_0 := -\mu|_{\mathfrak{h}''} \in (\mathfrak{h}'')^*$. Suppose that $-\mu$ is \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral. Then, Corollary 7 of Appendix A assures that there exists a unique irreducible \mathfrak{g}_0 -module $\mathbb{V}_{-\mu}^*$ of highest weight $-\mu_0$ such that the diagonalizable action of $\mathfrak{z}(\mathfrak{g}_0)$ is given by $-\mu|_{\mathfrak{z}(\mathfrak{g}_0)}$. This corresponds, by duality, to a unique irreducible \mathfrak{g}_0 -module \mathbb{V}_{μ} of lowest weight μ_0 with $\mathfrak{z}(\mathfrak{g}_0)$ -action given by $\mu|_{\mathfrak{z}(\mathfrak{g}_0)}$. In the following, we will always denote by \mathbb{V}_{μ} the irreducible \mathfrak{g}_0 -module of lowest weight μ_0 . Let $\{\omega_1, \dots, \omega_n\}$ be the fundamental weights for \mathfrak{g} . We know that there exist nonpositive integers a_1, \dots, a_n such that $-\mu = \sum_{i=1}^n a_i \omega_i$. Then, $-\mu_0$ writes as $-\mu_0 = \sum_{i \notin \Sigma} a_i \omega_i$.

Let \mathbb{U} be a finite dimensional representation of \mathfrak{g} . The coboundary operators defining the cohomology complexes $(C^*(\mathfrak{g}_-, \mathbb{U}), \partial)$ and $(C^*(\mathfrak{p}_+, \mathbb{U}^*), \partial_{\mathfrak{p}})$ are \mathfrak{g}_0 -homomorphisms, hence their kernels and images are \mathfrak{g}_0 -modules and the quotient of a \mathfrak{g}_0 -module is a \mathfrak{g}_0 -module (see Section 5.1 of Appendix A for more details). Therefore, the Lie algebra cohomologies $H^n(\mathfrak{g}_-, \mathbb{U})$ and $H^n(\mathfrak{p}_+, \mathbb{U}^*)$ admit natural structures of \mathfrak{g}_0 -modules. Both are realized, via the Hodge decomposition, as subspaces of skew-symmetric bilinear maps $\mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathbb{U}$. The same decomposition shows that the two cohomology groups are reciprocally dual \mathfrak{g}_0 -modules. Kostant's Theorem determines the highest weights of the irreducible components of $H^n(\mathfrak{p}_+, \mathbb{U}^*)$ (see Theorem 14 of Appendix A). By duality, these uniquely correspond to the lowest weights of the irreducible components of $H^n(\mathfrak{g}_-, \mathbb{U})$. We will be mainly interested in the case $n = 2$.

Define the *annihilator* of $\phi \in H^2(\mathfrak{g}_-, \mathbb{U})$ as

$$\mathfrak{a}_0 := \text{ann}(\phi) = \{ X \in \mathfrak{g}_0 \mid X \cdot \phi = 0 \}.$$

The relation

$$X \cdot H \cdot \phi = -[H, X] \cdot \phi + H \cdot X \cdot \phi = 0$$

for $H \in \mathfrak{h}$ and $X \in \mathfrak{a}_0$ shows that $\mathfrak{a}_0 \subset \mathfrak{g}_0$ is an \mathfrak{h} -invariant vector subspace.

Since $\mathfrak{a}_0 \subset \mathfrak{g}_0$ is a subalgebra, we can define $\mathfrak{a}^{\phi} = \text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0)$ for $\phi \in H^2(\mathfrak{g}_-, \mathbb{U})$. We want to estimate

$$\mathfrak{U} := \max\{\dim(\mathfrak{a}^{\phi}) \mid \phi \in H^2(\mathfrak{g}_-, \mathbb{U}), \phi \neq 0\}.$$

Put $I = \{\mu \mid \mathbb{V}_{\mu} \subset H^2(\mathfrak{g}_-, \mathbb{U})\}$, so that

$$H^2(\mathfrak{g}_-, \mathbb{U}) = \bigoplus_{\mu \in I} \mathbb{V}_{\mu}.$$

Write $\phi \in H^2(\mathfrak{g}_-, \mathbb{U})$ as

$$\phi = \sum_{\mu \in I} \phi_\mu, \quad \phi_\mu \in \mathbb{V}_\mu.$$

For $X \in \mathfrak{g}_0$, we have that $X \cdot \phi = \sum_{\mu \in I} X \cdot \phi_\mu$ is the sum of terms lying in different \mathfrak{g}_0 -irreducible components. Therefore $X \cdot \phi = 0$ if and only if $X \cdot \phi_\mu = 0$ for all $\mu \in I$, which tells that $\text{ann}(\phi) \subset \cap_{\mu} \text{ann}(\phi_\mu)$. In particular, $\dim(\mathfrak{a}^\phi) \leq \dim(\mathfrak{a}^{\phi_\mu})$ for all $\mu \in I$ and we can thus compute \mathfrak{U} as

$$\mathfrak{U} = \max\{\mathfrak{U}_\mu \mid \mu \in I\}, \text{ where } \mathfrak{U}_\mu = \max\{\dim(\mathfrak{a}^\phi) \mid \phi \in \mathbb{V}_\mu, \phi \neq 0\}.$$

Therefore, we only need to estimate \mathfrak{U}_μ for a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight $-\mu$. We thus reduced the general problem to the case of estimating the maximum of $\dim(\mathfrak{a}^\phi)$ over the set of those ϕ contained in a given irreducible component $\mathbb{V}_\mu \subset H^2(\mathfrak{g}_-, \mathbb{U})$.

Let $\phi_0 \in \mathbb{V}_\mu$ be a lowest weight vector of weight μ_0 . Now ϕ_0 is defined by μ_0 , hence by μ , up to scale and $\text{ann}(\lambda\phi_0) = \text{ann}(\phi_0)$ for any $\lambda \in \mathbb{C}$. Hence, $\text{ann}(\phi_0) \subset \mathfrak{g}_0$ only depends on μ and the notation $\mathfrak{a}(\mu) = \mathfrak{a}^{\phi_0}$ is consistent.

REMARK 5. Let \mathfrak{g} be a complex simple Lie algebra. The *highest root* of \mathfrak{g} is the highest weight of the adjoint representation of \mathfrak{g} on itself. We will denote it by $\lambda_{\mathfrak{g}}$. Also recall that $W^{\mathfrak{p}}(2)$ is the subset of the Weyl group of elements of length two mapping \mathfrak{g} -dominant and \mathfrak{g} -algebraically integral weights to \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weights (see also Section 4 for more details). From Kostant's Theorem, it follows that

$$I = \{\mu \mid \mathbb{V}_\mu \subset H^2(\mathfrak{g}_-, \mathfrak{g})\} = \{-w \cdot \lambda_{\mathfrak{g}} \mid w \in W^{\mathfrak{p}}(2)\},$$

where \cdot denotes the affine action of the Weyl group on the set of weights. More precisely, I and $W^{\mathfrak{p}}(2)$ are in bijective correspondence. To underline this fact, we adopt the notation $\mathfrak{a}(w) := \mathfrak{a}(\mu)$ for $\mu = -w \cdot \lambda_{\mathfrak{g}}$ and $w \in W^{\mathfrak{p}}(2)$. Let $\Delta^0 = \{\alpha_1, \dots, \alpha_n\}$ be a simple root system for \mathfrak{g} and denote the root vectors by $e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta$. An element of W of length two writes as the product of two reflections with respect to simple roots. We will denote the reflection with respect to the simple root α_j as σ_j and the composition $\sigma_j \circ \sigma_k$ as (jk) . If $w = (jk) \in W^{\mathfrak{p}}(2)$, then

$$\Phi_w = w(-\Delta^+) \cap \Delta^+ = \{\alpha_j, \sigma_j(\alpha_k)\}$$

and if $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}} \subset H^2(\mathfrak{g}_-, \mathfrak{g})$, then

$$\phi_0 = e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes e_{w(-\lambda)}$$

is the unique (up to scale) lowest weight vector in $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}}$. Let $H^2(\mathfrak{g}_-, \mathfrak{g})^1 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ be the submodule of positive homogeneity. Then, $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}} \subset H^2(\mathfrak{g}_-, \mathfrak{g})^1$ if and only if $w \in W_+^{\mathfrak{p}}(2)$, where

$$W_+^{\mathfrak{p}}(2) = \{w \in W^{\mathfrak{p}}(2) \mid Z(-w \cdot \lambda_{\mathfrak{g}}) \geq 1\}.$$

The next result is a consequence of the Borel fixed point Theorem (see [11] for a proof).

LEMMA 3. *Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ and $G_0 \subset G$ the Levi subgroup corresponding to the grading. Let \mathbb{V}_μ be an irreducible G_0 -module and let $\phi_0 \in \mathbb{V}_\mu$ be a lowest weight vector. Then*

$$\dim(\mathfrak{a}^\phi) \leq \dim(\mathfrak{a}^{\phi_0})$$

for all $\phi \in \mathbb{V}_\mu$. Moreover, $\dim(\mathfrak{a}^\phi) = \dim(\mathfrak{a}^{\phi_0})$ if and only if ϕ differs from ϕ_0 for multiplication for a complex scalar multiple. It thus follows that

$$\mathfrak{U}_\mu = \dim(\mathfrak{a}(\mu)).$$

From now on, then, we investigate the structure of $\mathfrak{a}(\mu) = \mathfrak{a}^{\phi_0}$ for a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight $-\mu$. Put

$$J_\mu = \{j \in \Delta^0 \setminus \Sigma \mid \langle \mu, \alpha_j \rangle \neq 0\} = \{j \in \Delta^0 \setminus \Sigma \mid \sigma_j(\mu) \neq \mu\}$$

Since J_μ is a subset of a simple root system for \mathfrak{g}_0^{ss} , the J_μ -height of roots defines a grading on \mathfrak{g}_0^{ss} , which we will denote by $\mathfrak{g}_0^{ss} = \mathfrak{g}_{0,-m} \oplus \dots \oplus \mathfrak{g}_{0,m}$.

PROPOSITION 24. *Let $-\mu$ a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight, let \mathbb{V}_μ be the irreducible G_0 -module of lowest weight μ_0 and let $\phi_0 \in \mathbb{V}_\mu$ be the lowest weight vector. Then $\mathfrak{a}_0 = \text{ann}(\phi_0)$ writes as*

$$\mathfrak{a}_0 = \{H \in \mathfrak{h} \mid \mu(H) = 0\} \oplus \bigoplus_{\gamma \in \Delta(\mathfrak{g}_{0,\leq 0})} \mathfrak{g}_\gamma$$

where $\Delta(\mathfrak{g}_{0,\leq 0}) \subset \Delta(\mathfrak{g}_0)$ denotes the subset of roots with nonpositive J_μ -height.

PROOF. Recall that for the grading defined through the Σ -height of roots, where $\Sigma = \Sigma_{\mathfrak{p}}$, we have that the component of degree zero writes as

$$(58) \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_0)} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Any $\alpha \in \Delta^+(\mathfrak{g}_0)$ writes as linear combination of simple roots from $\Delta^0 \setminus \Sigma$ with nonnegative integer coefficients. Since \mathfrak{a}_0 is \mathfrak{h} -invariant as a vector subspace, \mathfrak{a}_0 writes as the direct sum

$$\mathfrak{a}_0 = \mathfrak{a}_0 \cap \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+(\mathfrak{g}_0)} (\mathfrak{a}_0 \cap \mathfrak{g}_\alpha \oplus \mathfrak{a}_0 \cap \mathfrak{g}_{-\alpha}).$$

By definition, $\mathfrak{a}_0 \cap \mathfrak{h} = \{H \in \mathfrak{h} \mid \mu_0(H) = 0\}$. Recall that $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}''$, where $\mathfrak{h}'' \subset \mathfrak{g}_0^{ss}$ is a Cartan subalgebra which writes as $\mathfrak{h}'' = \{H_\alpha \mid \alpha \in \Delta^0 \setminus \Sigma\}$. Since μ and μ_0 differ for an element which is linear combination of elements of Σ , we have that $\langle \mu - \mu_0, \alpha \rangle = 0$ for $\alpha \in \Delta^0 \setminus \Sigma$, hence that $\mu(H_\alpha) = \langle \mu, \alpha \rangle = \langle \mu_0, \alpha \rangle = \mu_0(H_\alpha)$ and

$$\mathfrak{a}_0 \cap \mathfrak{h} = \{H \in \mathfrak{h} \mid \mu(H) = 0\}.$$

By definition of lowest \mathfrak{g}_0 -weight vector, $X \cdot \phi_0 = 0$ for all $X \in \mathfrak{g}_{-\alpha}$ with $\alpha \in \Delta^+(\mathfrak{g}_0)$, hence $\mathfrak{a}_0 \cap \mathfrak{g}_{-\alpha} = \mathfrak{g}_{-\alpha}$ for the same α . Now for $\alpha \in \Delta^+(\mathfrak{g}_0)$, consider $H_\alpha = [X_\alpha, X_{-\alpha}]$ and observe that

$$\mu(H_\alpha)\phi_0 = H_\alpha \cdot \phi_0 = X_\alpha \cdot X_{-\alpha} \cdot \phi_0 - X_{-\alpha} \cdot X_\alpha \cdot \phi_0 = -X_{-\alpha} \cdot X_\alpha \cdot \phi_0.$$

Therefore $X_\alpha \in \mathfrak{a}_0 \cap \mathfrak{g}_\alpha$ if and only if $\langle \mu, \alpha \rangle = \mu(H_\alpha) = 0$, which means that $\text{ht}_{J_\mu}(\alpha) = 0$. \square

Denote by $\Delta^0 = \{\alpha_1, \dots, \alpha_n\}$ a simple root system for \mathfrak{g} and by $\{H_1, \dots, H_n\}$ the dual basis for \mathfrak{h} . In particular, we know that $\{H_i \mid \alpha_i \in \Sigma\}$ is a basis for $\mathfrak{z}(\mathfrak{g}_0)$ and that H_i acts on the irreducible \mathfrak{g}_0 -module \mathbb{V}_μ by scalar factor $H_i(\mu)$ for all $i \in \Sigma$. In particular, the elements in

$$I_\mu = \{i \in \Sigma \mid \langle \mu, \alpha_i \rangle = 0\}$$

correspond to generators H_i of $\mathfrak{z}(\mathfrak{g}_0)$ acting trivially on \mathbb{V}_μ . We put $I_w := I_{-w \cdot \lambda_{\mathfrak{g}}}$ for \mathfrak{g} simple. For the grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$, any \mathfrak{g}_j is invariant with respect to the adjoint action of \mathfrak{g}_0 and it thus decomposes into \mathfrak{g}_0 -irreducible components. Let $\mathfrak{g}_{1_i} \subset \mathfrak{g}_1$ be a \mathfrak{g}_0 -irreducible component. Now $\mathfrak{g}_{0,0} \subset \mathfrak{g}_0$ is a subalgebra and \mathfrak{g}_{1_i} , considered as $\mathfrak{g}_{0,0}$ -module, might be reducible. Hence, \mathfrak{g}_{1_i} decomposes into $\mathfrak{g}_{0,0}$ -irreducible components. The decompositions above can be described via the notation that we are going to introduce. For $S \subset \{1, \dots, n\}$, put $Z_S = \sum_{i \in S} H_i \in \mathfrak{h}$ and $\tilde{Z}_S = (H_i)_{i \in S}$.

THEOREM 10. *Let $-\mu$ a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight and $\mathfrak{a} = \mathfrak{a}(\mu)$. Then*

- (1) *The positive graded part \mathfrak{a}_+ of \mathfrak{a} writes as direct sum of root spaces*

(2) The decomposition of \mathfrak{g}_1 into \mathfrak{g}_0 -irreducible components writes as

$$\mathfrak{g}_1 = \bigoplus_{i \in \Sigma} \mathfrak{g}_{1_i}, \quad \Delta(\mathfrak{g}_{1_i}) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid H_i(\alpha) = 1\}.$$

(3) For $i \in \Sigma$, the decomposition of \mathfrak{g}_{1_i} into $\mathfrak{g}_{0,0}$ -irreducible components is

$$\mathfrak{g}_{1_i} = \bigoplus_{A \geq 0} \mathfrak{g}_{1_i,A}, \quad \Delta(\mathfrak{g}_{1_i,A}) = \{\alpha \in \Delta(\mathfrak{g}_{1_i}) \mid \tilde{Z}_{J_\mu}(\alpha) = A\},$$

where $A \geq 0$ means that all entries of A are nonnegative with respect to the ordering on Δ .

(4) $\mathfrak{a}_+ = \{0\}$, equivalently $\mathfrak{a} = \mathfrak{g}_- \oplus \mathfrak{a}_0$, if and only if $I_\mu = \emptyset$. Moreover if $I_\mu \neq \emptyset$, then $\Delta(\mathfrak{a}_1) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid Z_{I_\mu}(\alpha) = 1, Z_{J_\mu}(\alpha) = 0\}$.

(5) \mathfrak{a}_1 generates all brackets \mathfrak{a}_j and $\Delta(\mathfrak{a}_j) = \{\alpha \in \Delta(\mathfrak{g}_j) \mid Z_{I_\mu}(\alpha) = j, Z_{J_\mu}(\alpha) = 0\}$ for all $j \geq 1$.

PROOF. (1) Recall that $\mathfrak{a}_j = \{X \in \mathfrak{g}_j \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{j-1}\}$ for all $j \geq 1$ and $\mathfrak{a}_+ = \bigoplus_{j \geq 1} \mathfrak{a}_j$. We prove by induction that \mathfrak{a}_j is \mathfrak{h} -invariant for all $j \geq 0$, which implies the statement. First, the fact that \mathfrak{a}_0 is \mathfrak{h} -invariant follows from the definition. Fix now $j \geq 1$ and suppose that \mathfrak{a}_{j-1} is \mathfrak{h} -invariant. Using the Jacobi identity, then, one can easily see that also \mathfrak{a}_j is \mathfrak{h} -invariant. Indeed, for $H \in \mathfrak{h}, X \in \mathfrak{a}_j$ and $Y \in \mathfrak{g}_{-1}$, the formula

$$[[H, X], Y] = [[H, Y], X] + [H, [X, Y]]$$

shows that $[[H, X], Y] \in \mathfrak{a}_{j-1}$, hence that $[H, X] \in \mathfrak{a}_j$. In particular, each \mathfrak{a}_j is the sum of root spaces corresponding to roots in the subset $\Delta(\mathfrak{a}_j) \subset \Delta(\mathfrak{g}_j)$.

(2) Fix $i \in \Sigma$. Let be $\Delta(\mathfrak{g}_{1_i}) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid H_i(\alpha) = 1\}$ and let $\mathfrak{g}_{1_i} \subset \mathfrak{g}_1$ be the direct sum of root spaces \mathfrak{g}_α corresponding to $\alpha \in \Delta(\mathfrak{g}_{1_i})$. Observe that $\beta \in \Delta(\mathfrak{g}_{1_i})$ writes as $\beta = \alpha_i + \gamma$ with $\gamma \in \Delta^+(\mathfrak{g}_0)$. Therefore, α_i is the lowest root of $\Delta(\mathfrak{g}_{1_i})$. Moreover, if we write β as linear combination of simple roots, α_i is the unique element of Σ showing up. From a basic result of representation theory of semisimple Lie algebras, namely the description of the weights of a finite-dimensional irreducible representation, it is thus clear that $\mathfrak{g}_{1_i} \subset \mathfrak{g}_1$ is a \mathfrak{g}_0 -irreducible component and that $\mathfrak{g}_{1_i} \neq \mathfrak{g}_{1_s}$ for $i \neq s$. Indeed, $\Delta(\mathfrak{g}_{1_i})$ is defined as the set of weights of the irreducible \mathfrak{g}_0^{ss} -representation generated by the lowest weight vector of weight α_i .

(3) The set $\Delta^0(\mathfrak{g}_{0,0}) = \{\alpha_i \in \Delta^0 \setminus \Sigma \mid \langle \mu, \alpha_i \rangle = 0\}$ is a simple root system for the semisimple part of $\mathfrak{g}_{0,0}$. Note that $\alpha + \beta \in \Delta(\mathfrak{g}_{1_i,A})$ for all $\alpha \in \Delta(\mathfrak{g}_{1_i,A})$ and $\beta \in \Delta^0(\mathfrak{g}_{0,0})$. Indeed, $H_i(\alpha + \beta) = H_i(\alpha) = 1$ for $i \in \Sigma$ and $\tilde{Z}_{J_\mu}(\alpha + \beta) = \tilde{Z}_{J_\mu}(\alpha) = A$, since $\langle \mu, \beta \rangle = 0$. Then, $\Delta(\mathfrak{g}_{1_i,A})$ is defined as the set of weights of the irreducible $\mathfrak{g}_{0,0}^{ss}$ -module $\mathfrak{g}_{1_i,A} \subset \mathfrak{g}_{1_i}$ generated by its lowest weight vector.

(4) Denote by B the Killing form of \mathfrak{g} . Recall that for $\gamma \in \Delta \subset \mathfrak{h}^*$, there exists a unique $H_\gamma \in \mathfrak{h}$ such that $\gamma(H) = B(H_\gamma, H)$ for all $H \in \mathfrak{h}$. Each root space \mathfrak{g}_γ is one-dimensional and one can choose generators X_γ of \mathfrak{g}_γ for each $\gamma \in \Delta$ such that $[X_\gamma, X_{-\gamma}] = c_\gamma H_\gamma$, where $c_\gamma = B(X_\gamma, X_{-\gamma}) \neq 0$. Consider now X_α for $\alpha \in \Delta(\mathfrak{g}_1)$. Let us see which conditions X_α must verify to be in \mathfrak{a}_1 . Then, this will be translated in conditions verified by $\alpha \in \Delta(\mathfrak{a}_1)$. If $\beta \in \Delta(\mathfrak{g}_1)$, then $[X_\alpha, X_{-\beta}] \in \mathfrak{g}_0$ and we have three possibilities:

- if $\beta = \alpha$, then $[X_\alpha, X_{-\alpha}] = c_\alpha H_\alpha \in \mathfrak{h}$. By Proposition 24, H_α lies in \mathfrak{a}_0 if and only if $\mu(H_\alpha) = \langle \mu, \alpha \rangle = 0$
- for $\beta \neq \alpha$, we have two distinct cases. If $\alpha - \beta \notin \Delta$, then $[X_\alpha, X_{-\beta}] = 0$, which puts no conditions on α . If $\alpha - \beta \in \Delta$, then $[X_\alpha, X_{-\beta}] = f_{\alpha,\beta} X_{\alpha-\beta}$ with $f_{\alpha,\beta} \neq 0$ and again by Proposition 24, $X_{\alpha-\beta}$ lies in \mathfrak{a}_0 if and only if $\alpha - \beta \in \Delta(\mathfrak{g}_{0,\leq 0})$.

Define $\mathcal{T}_1 = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \langle \mu, \alpha \rangle \neq 0\}$ and $\mathcal{T}_2 = \{\alpha \in \Delta(\mathfrak{g}_1) \mid \exists \beta \in \Delta(\mathfrak{g}_1) : \alpha - \beta \in \Delta(\mathfrak{g}_{0,+})\}$. The argument above shows that $\Delta(\mathfrak{a}_1) = \Delta(\mathfrak{g}_1) \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$.

Consider the $\mathfrak{g}_{0,0}$ -module decomposition from (3). Fix $i \in \Sigma$. \mathfrak{a}_1 is a $\mathfrak{g}_{0,0}$ -module and from Schur's Lemma follows that either $\mathfrak{g}_{1_i,A} \subset \mathfrak{a}_1$ or $\mathfrak{a}_1 \cap \mathfrak{g}_{1_i,A} = \{0\}$. It is enough to verify whether the lowest root of $\mathfrak{g}_{1_i,A}$ lies in $\Delta(\mathfrak{a}_1)$ or not. Suppose that at least one entry of A is strictly positive (we will write this condition as $A > 0$). Let $\alpha \in \Delta(\mathfrak{g}_{1_i,A})$ be the lowest root. Since $A > 0$, $\alpha \neq \alpha_i$, hence α is not the lowest root of \mathfrak{g}_{1_i} . However, since \mathfrak{g}_{1_i} is an irreducible \mathfrak{g}_0 -module containing $\mathfrak{g}_{1_i,A}$, α is a weight of the \mathfrak{g}_0 -module \mathfrak{g}_{1_i} . Hence, the lowest root of $\mathfrak{g}_{1_i,A}$ can be obtained by subtracting a finite number of times simple roots for \mathfrak{g}_0 . In formulas, there exist $j_1, \dots, j_m \in \Delta^0 \setminus \Sigma$ such that

$$\alpha - \alpha_{j_1} - \dots - \alpha_{j_m} = \alpha_i.$$

In particular, since $\text{ht}_\Sigma(\beta) = \text{ht}_\Sigma(\alpha) = 1$, we see that $\beta = \alpha - \alpha_{j_1} \in \Delta(\mathfrak{g}_1)$. Clearly, $\beta \notin \Delta(\mathfrak{g}_{1_i,A})$, since this would be a contradiction with the choice of α as the lowest root. Subtracting α_{j_1} from α , then, modifies the A -component. Otherwise put, $\tilde{Z}_{J_\mu}(\beta) \neq \tilde{Z}_{J_\mu}(\alpha)$, hence $j_1 \in J_\mu$. Together with the fact that $\alpha \in \Delta(\mathfrak{g}_{0,+})$, this implies that $\alpha \in \mathcal{T}_2$, hence $\alpha \notin \Delta(\mathfrak{a}_1)$. We thus showed that $\Delta(\mathfrak{a}_1) \subset \bigcup_{i \in I_\mu} \Delta(\mathfrak{g}_{1_i,0})$. Consider now the case $A = 0$. The lowest root of $\Delta(\mathfrak{g}_{1_i,0})$ is α_i , which evidently does not lie in \mathcal{T}_2 . Therefore, $i \notin I_\mu$ if and only if $\alpha_i \in \mathcal{T}_1$ and we obtain

$$\Delta(\mathfrak{a}_1) = \bigcup_{i \in I_\mu} \Delta(\mathfrak{g}_{1_i,0}),$$

which concludes the proof of (4).

(5) Consider $j \geq 2$ and $\gamma \in \Delta(\mathfrak{a}_j) \subset \Delta(\mathfrak{g}_j)$. By Proposition 18 (2), \mathfrak{g}_1 is bracket-generating in \mathfrak{g}_+ . This implies the existence of $\alpha \in \Delta(\mathfrak{g}_{j-1})$ and $\beta \in \Delta(\mathfrak{g}_1)$ such that $\gamma = \alpha + \beta$. Now $\gamma \in \Delta(\mathfrak{a}_j)$ implies that $\alpha = \gamma - \beta \in \Delta(\mathfrak{a}_{j-1})$ and since \mathfrak{a} is graded, we have that $[\mathfrak{a}_j, \mathfrak{a}_{-(j-1)}] \subset \mathfrak{a}_1$ thus $\beta = \gamma - \alpha \in \Delta(\mathfrak{a}_1)$. For $S_1, S_2 \subset \Delta$, define a subset $S_1 \dot{+} S_2 \subset \Delta$ by putting $S_1 \dot{+} S_2 = \{\alpha + \beta \mid \alpha \in S_1, \beta \in S_2\} \cap \Delta$. Then, the argument above proves that

$$\Delta(\mathfrak{a}_j) \subseteq \Delta(\mathfrak{a}_{j-1}) \dot{+} \Delta(\mathfrak{a}_1)$$

and the characterization of $\Delta(\mathfrak{a}_1)$ from (4) leads to

$$\Delta(\mathfrak{a}_{j-1}) \dot{+} \Delta(\mathfrak{a}_1) \subseteq \{\alpha \in \Delta(\mathfrak{g}_j) \mid Z_{I_\mu}(\alpha) = j, Z_{J_\mu}(\alpha) = 0\} =: S_j.$$

Conversely, take $\gamma \in S_j$ and $\beta \in \Delta(\mathfrak{g}_1)$. If $\alpha = \gamma - \beta \in \Delta$, then $\alpha \in \Delta(\mathfrak{g}_{j-1})$. Since $Z_{\Sigma \setminus I_\mu}(\gamma) = 0$, then $Z_{I_\mu}(\beta) = 1$ and $Z_{J_\mu}(\beta) = 0$, so $\beta \in \Delta(\mathfrak{a}_1)$. Thus, $Z_{I_\mu}(\alpha) = r - 1$ and $Z_{J_\mu}(\alpha) = 0$, so by the induction hypothesis, $\alpha \in \Delta(\mathfrak{a}_{r-1})$ and $\gamma \in \Delta(\mathfrak{a}_r)$. \square

Fix a semisimple Lie group G and a parabolic subgroup $P \subset G$. Let \mathfrak{g} be the Lie algebra of G and $\mathfrak{p} \subset \mathfrak{g}$ the Lie algebra of P . Consider the grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_k$ corresponding to the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. The grading is equivalent to a filtration of \mathfrak{g} denoted by $\{\mathfrak{g}^i\}$. Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a normal, regular parabolic geometry of type (G, P) . By regularity, the curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ associated to the parabolic geometry satisfies $\kappa_u(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and $i, j \leq -1$. The harmonic curvature $\kappa_H : \mathcal{G}_0 \rightarrow H^2(\mathfrak{g}_-, \mathfrak{g})$, then, takes values in the submodule $H^2(\mathfrak{g}_-, \mathfrak{g})^1 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ of positive homogeneity.

Fix $u \in \mathcal{G}$ such that $\kappa_H(u) \neq 0$. Let $\mathcal{S} = \text{inf}(\mathcal{G}, \omega)$ be the set of infinitesimal symmetries and let $\xi \in \mathcal{S}$. The equivariance property $(r^h)^*\xi = \xi$ for $h \in P$ assures that ξ is completely determined by its value in a point. Therefore, the linear map $h : \mathcal{S} \rightarrow \mathfrak{g}$ given by $\xi \mapsto \omega(\xi(u))$, which was considered in (2) of Theorem 9, is an injection. The image of h , denoted by $\mathfrak{f}(u) \subset \mathfrak{g}$, is a vector subspace of \mathfrak{g} , in general not a subalgebra. Let $\xi, \eta \in \mathcal{S}$ and put $X = \omega(\xi(u))$ and $Y = \omega(\eta(u))$, so that $X, Y \in \mathfrak{f}(u)$. From Theorem 9 (2), we know that the bracket of ξ, η in \mathcal{S} (which is by definition the negative of the bracket of vector fields) is mapped by h to

$$(59) \quad [X, Y]_{\mathfrak{f}(u)} = [X, Y] - \kappa_u(X, Y).$$

Since $[\cdot, \cdot]_{\mathfrak{f}(u)}$ is induced by the bracket of vector fields, it defines a bracket on $\mathfrak{f}(u)$. Otherwise put, $\mathfrak{f}(u)$ equipped with the bracket above is a Lie algebra. Since $(p : \mathcal{G} \rightarrow M, \omega)$ is regular, restricting the filtration of \mathfrak{g} to $\mathfrak{f}(u)$ makes $\mathfrak{f}(u)$ into a filtered Lie algebra, whose associated graded algebra is isomorphic to a graded Lie subalgebra of \mathfrak{g} . Indeed, $\mathfrak{f}(u)$ inherits a filtration from \mathfrak{g} by putting $\mathfrak{f}^i(u) := \mathfrak{f}(u) \cap \mathfrak{g}^i$. Clearly, the bracket $[\cdot, \cdot]_{\mathfrak{f}(u)}$ defined by (59) is compatible with the filtration of $\mathfrak{f}(u)$. If $\mathfrak{s}_i(u) = \mathfrak{f}^i(u)/\mathfrak{f}^{i+1}(u)$, then

$$\mathfrak{s}(u) = \bigoplus_{i=-k}^k \mathfrak{s}_i(u) \subset \mathfrak{g}$$

is a graded vector subspace. Since $[\cdot, \cdot]_{\mathfrak{f}(u)}$ preserves the filtration of $\mathfrak{f}(u)$, it induces a graded bracket $\mathfrak{s}_i(u) \otimes \mathfrak{s}_j(u) \rightarrow \mathfrak{s}_{i+j}(u)$ by taking equivalence classes of brackets. The regularity assumption $\kappa_u(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ implies that the second summand in the right-hand term of (59) does not contribute to the induced graded bracket, which therefore coincides with the restriction of $[\cdot, \cdot] : \mathfrak{g}_i \otimes \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}$ to the subspace $\mathfrak{s}(u) \subset \mathfrak{g}$. The graded Lie algebra associated to the filtration of $\mathfrak{f}(u)$ is thus a graded subalgebra $\mathfrak{s}(u) \subset \mathfrak{g}$. In particular, we have the following

LEMMA 4. *The component in homogeneity zero of $\mathfrak{s}(u)$ satisfies $\mathfrak{s}_0(u) \subset \text{ann}(\kappa_H(u)) \subset \mathfrak{g}_0$.*

PROOF. Suppose that $\xi \in \mathcal{S}$ is such that $Y = \omega(\xi(u)) \in \mathfrak{p}$. Then $\xi(u) = \zeta_Y(u)$ is a vertical tangent vector and its flow is given by $\phi_t(u) = \frac{d}{dt}\big|_{t=0} u \cdot \exp(tY)$. Now κ_H is horizontal and P -equivariant, so that

$$0 = (\xi \cdot \kappa_H)(u) = \frac{d}{dt}\bigg|_{t=0} (\kappa_H(\phi_t(u))) = \frac{d}{dt}\bigg|_{t=0} \exp(-tY) \cdot \kappa_H(u) = -Y \cdot \kappa_H(u)$$

Complete reducibility of $H^2(\mathfrak{g}_-, \mathfrak{g})^1$ (where the values of κ_H lie) implies that the action above only depends on $Y \bmod \mathfrak{p}_+$, which is defined up to $\mathfrak{f}^1(u)$ by ξ , thus the statement. \square

Let be $x \in M$ and let $u \in \mathcal{G}$ be such that $p(u) = x$. For $\xi \in \mathcal{S}$ and $h \in P$, we have

$$\text{Ad}(h^{-1}) \circ \omega_u(\xi) = (r^h)^* \omega_u(\xi) = \omega_{uh}(T_u r^h \cdot \xi_u) = \omega_{uh}(\xi_{uh})$$

which shows that $\omega(\xi(u)) \in \mathfrak{g}^i$ if and only if $\omega(\xi(uh)) \in \mathfrak{g}^i$. Therefore, there is an isomorphism $\mathfrak{f}(u) \cong \mathfrak{f}(ug)$ of filtered vector spaces, hence an isomorphism $\mathfrak{s}(u) \cong \mathfrak{s}(ug)$ of graded vector spaces. The Lie algebra structure on $\mathfrak{s}(u)$, up to isomorphism, only depends on the base point $p(u) = x$. In order to emphasize this fact, we introduce the notation $\mathfrak{s}(x)$ for $x \in M$.

DEFINITION 19. A point $x \in M$ is said to be *regular* if it admits a neighborhood $N_x \subset M$ such that $\dim(\mathfrak{s}_j(y)) = \dim(\mathfrak{s}_j(x))$ for all $-k \leq j \leq k$ and all $y \in N_x$.

One can prove that the subset of regular points is open and dense in M (see Lemma 4.2.4, [11]). Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a normal, regular parabolic geometry of type (G, P) . If the geometry is not locally flat, the existence of regular points is assured. In this case, indeed, $N = \{x \in M \mid \kappa_H(u) \neq 0 \ \forall u \in p^{-1}(x)\}$ is a non-empty and open subset of M which, by the density statement above, must contain a regular point.

PROPOSITION 25. [11] *Let $x \in M$ be a regular point and let be $u \in p^{-1}(x)$. Then for all $i \geq 1$, the relation $[\mathfrak{s}_i(u), \mathfrak{g}_{-1}] \subset \mathfrak{s}_{i-1}(u)$ holds. In particular, $\mathfrak{s}_i(u) \subset \mathfrak{p}_i$ for all $i \geq 1$, where \mathfrak{p}_i denotes the homogeneous component of $\mathfrak{p} = \text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0(u))$ of degree i .*

Fix a regular point $x \in M$ and $u \in p^{-1}(x)$. Proposition 25 proves that there exists an inclusion $\mathfrak{s}(u) = \mathfrak{g}_- \oplus \mathfrak{s}_{\geq 0}(u) \subseteq \text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0(u))$ of graded Lie algebras. Lemma 4 tells us that $\mathfrak{s}_0(u) \subset \text{ann}(\kappa_H(u))$, which implies the inclusion

$$\text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0(u)) \subset \mathfrak{a}^{\kappa_H(u)} = \text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \text{ann}(\kappa_H(u)))$$

of graded subalgebras. Therefore, $\mathfrak{s}(u) \subseteq \mathfrak{a}^{\kappa_H(u)}$ is a graded subalgebra and we conclude that

$$\dim(\mathcal{S}) = \dim(\mathfrak{s}(u)) \leq \dim(\mathfrak{a}^{\kappa_H(u)}),$$

hence that $\mathfrak{S} \leq \mathfrak{U} < \dim(\mathfrak{g})$.

Now if $\mathfrak{S} = \mathfrak{U}$, then any submaximally symmetric model is locally homogeneous around a non-flat regular point. Indeed, suppose that $\mathfrak{S} = \mathfrak{U}$. Let $(p : \mathcal{G} \rightarrow M, \omega)$ be a normal, regular parabolic geometry $(p : \mathcal{G} \rightarrow M, \omega)$ of type (G, P) with infinitesimal symmetry algebra \mathcal{S} of maximal dimension \mathfrak{S} and let $x \in M$ be a regular point. Then, there exists an open neighborhood $N_x \subset M$ of x and $\mu \in I = \{\gamma \mid \mathbb{V}_\gamma \subset H^2(\mathfrak{g}_-, \mathfrak{g})^1\}$ such that $\mathfrak{s}(y)$ is isomorphic to the prolongation $\mathfrak{a}(\mu)$ for all $y \in N_x$. In particular, the negative graded part of the graded Lie algebra $\mathfrak{s}(y)$ coincides with \mathfrak{g}_- for all $y \in N_x$. Let $u \in p^{-1}(x)$. Consider the unique connected and simply connected Lie groups $F, F_{\geq 0}, G_-$ with respective Lie algebras $\mathfrak{f}(u), \mathfrak{f}(u)_{\geq 0}$ and \mathfrak{g}_- . Then $F_{\geq 0} \subset F$ is a closed subgroup and $G_- \cong F/F_{\geq 0}$ is an homogeneous space diffeomorphic to $N_x \subset M$.

We saw that for (G, P) given, in general, \mathfrak{U} is an upper bound for \mathfrak{S} . [11] shows that $\mathfrak{S} = \mathfrak{U}$ is verified up to a finite number of exceptions, hence that the upper bound is almost always sharp. We briefly recall the argument.

Consider $\mu \in I = \{\gamma \mid \mathbb{V}_\gamma \subset H^2(\mathfrak{g}_-, \mathfrak{g})^1\}$ and $\phi_0 \in \mathbb{V}_\mu \subset H^2(\mathfrak{g}_-, \mathfrak{g})^1$ a lowest \mathfrak{g}_0 -weight vector. The Hodge decomposition shows that $H^2(\mathfrak{g}_-, \mathfrak{g})^1$ can be seen as a subspace of $\Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$, hence ϕ_0 as a skew-symmetric bilinear map $\mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathfrak{g}$. Suppose now that $\text{Im}(\phi_0) \subset \mathfrak{a} \subset \mathfrak{g}$, where we put $\mathfrak{a} = \mathfrak{a}(\mu)$ to simplify the notation. Since $\Lambda^2 \mathfrak{g}_-^* \otimes \mathfrak{a}$ merges into $\Lambda^2 \mathfrak{a}^* \otimes \mathfrak{a}$, we can consider ϕ_0 as a skew-symmetric bilinear map $\mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{a}$ which vanishes by insertion of elements of $\mathfrak{a} \cap \mathfrak{p}$. Define a Lie algebra \mathfrak{f} , whose underlying space coincides with \mathfrak{a} and equipped with the bracket

$$(60) \quad [\cdot, \cdot]_{\mathfrak{f}} := [\cdot, \cdot]_{\mathfrak{a}} - \phi_0(\cdot, \cdot)$$

Observe that $(\mathfrak{f}_{\geq 0}, [\cdot, \cdot]_{\mathfrak{f}}) = (\mathfrak{a}_{\geq 0}, [\cdot, \cdot]_{\mathfrak{a}})$.

REMARK 6. Let \mathfrak{g} be a complex simple Lie algebra with highest root $\lambda_{\mathfrak{g}}$. The condition $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$ assures that $\text{Im}(\phi_0) \subset \mathfrak{a}$, so that formula (60) defines a map $\mathfrak{a} \times \mathfrak{a} \rightarrow \mathfrak{a}$. By analyzing the set of roots Δ (see Lemma 4.3.2 in [11]), one shows that this condition is satisfied for almost all pairs $(\mathfrak{g}, \mathfrak{p})$ with a finite number of exceptions. The case of interest for us, namely the grading on $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$ from Example 1, is not among these exceptions.

If $\text{Im}(\phi_0) \subset \mathfrak{a}$, one can explicitly construct a normal, regular parabolic geometry with constant harmonic curvature ϕ_0 , which therefore satisfies

$$\mathfrak{S} \geq \dim(\mathcal{S}) = \dim(\mathfrak{s}(u)) = \dim(\mathfrak{a}^{\kappa_H(u)}) = \mathfrak{U}_\mu.$$

LEMMA 5. Let \mathfrak{g} be a complex simple Lie algebra with root system Δ and highest root $\lambda_{\mathfrak{g}}$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra and $w \in W^{\mathfrak{p}}(2)$. Suppose that $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$. Let ϕ_0 be the lowest weight vector of $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}}$ and $\mathfrak{a} = \mathfrak{a}(w)$. Then $\text{Im}(\phi_0) \subset \mathfrak{a}$ and the bracket defined in (60) satisfies the Jacobi identity.

PROOF. Given an orientation on the root system, the set of negative roots writes as the disjoint union $\Delta^- = \Delta^-(\mathfrak{g}_-) \sqcup \Delta^-(\mathfrak{g}_0)$ of those negative roots corresponding to root spaces contained in \mathfrak{g}_- and in \mathfrak{g}_0 , respectively. Since ϕ_0 vanishes outside \mathfrak{g}_- , all root vectors corresponding to $\Delta^-(\mathfrak{g}_0)$ annihilate ϕ_0 .

Let us now verify that $[\cdot, \cdot]_{\mathfrak{f}}$ satisfies the Jacobi identity. From the explicit description of ϕ_0 (see Remark 5), we see that $\text{Im}(\phi_0) \subset \mathfrak{a}$. By Theorem 3.5.1 in [11], we can assume that \mathfrak{a} , whose underlying vector space coincides with the vector space underlying \mathfrak{f} by definition, is prolongation-rigid, namely $\mathfrak{a} = \mathfrak{g}_- \oplus \text{ann}(\phi_0)$. For $X, Y, Z \in \mathfrak{f}$,

$$[[X, Y], Z]_{\mathfrak{f}} = [[X, Y], Z] - \phi_0([X, Y], Z) - [\phi_0(X, Y), Z] + \phi_0(\phi_0(X, Y), Z)$$

where the bracket at the right-hand side is the bracket of \mathfrak{g} . If at least two elements are in \mathfrak{a}_0 , then $[[X, Y], Z]_{\mathfrak{f}} = [[X, Y], Z]$. Suppose that $X, Y \in \mathfrak{g}_-$ and $Z \in \text{ann}(\phi_0)$. The Jacobi identity reduces then to

$$\text{Jac}_{\mathfrak{f}}(X, Y, Z) = [Z, \phi_0(X, Y)] - \phi_0([Z, X], Y) - \phi_0([Y, Z], X) = (Z \cdot \phi_0)(X, Y) = 0.$$

Finally, suppose that $X, Y, Z \in \mathfrak{g}_-$. Now ϕ_0 is ∂ -closed and $(\partial\phi_0)(X, Y, Z) = 0$ rewrites as

$$(61) \quad \begin{aligned} (\partial\phi_0)(X, Y, Z) = & [X, \phi_0(Y, Z)] - [Y, \phi_0(X, Z)] + [Z, \phi_0(X, Y)] + \\ & -\phi_0([X, Y], Z) + \phi_0([X, Z], Y) - \phi_0([Y, Z], X) = 0 \end{aligned}$$

Observe that all terms are brackets of \mathfrak{g} restricted to \mathfrak{a} . Using (61) and the fact that the bracket of \mathfrak{a} satisfies the Jacobi identity, we get

$$(62) \quad \text{Jac}_{\mathfrak{f}}(X, Y, Z) = \phi_0(\phi_0(X, Y), Z) + \phi_0(\phi_0(Y, Z), X) + \phi_0(\phi_0(Z, X), Y)$$

Recall from Remark 5 that $w \in W^{\mathfrak{p}}(2)$ writes as the composition $w = \sigma_j \circ \sigma_k$ of reflections with respect to simple roots and

$$\phi_0 = e_{\alpha_j} \wedge e_{\sigma_j(\alpha_k)} \otimes e_{w(-\lambda)}$$

is the unique (up to scale) lowest weight vector in $\mathbb{V}_{-w \cdot \lambda_{\mathfrak{g}}}$. If $w(-\lambda_{\mathfrak{g}}) \in \{-\alpha_j, -\sigma_j(\alpha_k)\}$, we get the contradiction $\lambda_{\mathfrak{g}} \in \Delta^-$. Therefore $w(-\lambda_{\mathfrak{g}}) \notin \{-\alpha_j, -\sigma_j(\alpha_k)\}$, which implies that each term showing up in (62) is zero and concludes the proof. \square

Let \mathfrak{g} be a complex simple Lie algebra and let $\mathfrak{n} \subset \mathfrak{g}$ be the split-real form, equipped with a grading $\mathfrak{n} = \mathfrak{n}_{-k} \oplus \dots \oplus \mathfrak{n}_k$. The description of real \mathfrak{n}_0 -irreducible components of $H^2(\mathfrak{n}_-, \mathfrak{n})$ is completely analogous to the description of the complex \mathfrak{g}_0 -irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})$. In particular, each \mathfrak{n}_0 -irreducible component admits a lowest weight, whose explicit expression is the same as in Remark 5 for the complex case and the Jacobi identity for the split-real case can be proved analogously as in Lemma 5. The following result proved in [11] holds in both complex and split-real cases.

THEOREM 11. *Let G be a complex or split-real simple Lie group, $P \subset G$ a parabolic subgroup, \mathfrak{g} the Lie algebra of G with highest root $\lambda_{\mathfrak{g}}$. Put $\mu = -w \cdot \lambda_{\mathfrak{g}}$ for $w \in W_+^{\mathfrak{p}}(2)$ and denote by $\phi_0 \in \mathbb{V}_{\mu}$ a lowest \mathfrak{g}_0 -weight vector. If $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$, there exists a normal, regular parabolic geometry $(\mathcal{G} \rightarrow M, \omega)$ of type (G, P) with $\inf(\mathcal{G}, \omega)$ containing a subalgebra isomorphic to \mathfrak{f} and with harmonic curvature ϕ_0 . In particular, $\dim(\text{aut}(\mathcal{G}, \omega)) \geq \dim(\mathfrak{a}(w))$.*

COROLLARY 6. *Let G be a complex or split-real simple Lie group, $P \subset G$ a parabolic subgroup, \mathfrak{g} the Lie algebra of G with highest root $\lambda_{\mathfrak{g}}$. Put $\mu = -w \cdot \lambda_{\mathfrak{g}}$ for $w \in W_+^{\mathfrak{p}}(2)$. If $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$, then $\mathfrak{S}_{\mu} = \mathfrak{U}_{\mu} = \dim(\mathfrak{a}(w))$ and any submaximally symmetric model is locally homogeneous around a non-flat regular point.*

We focus now on the description of the submaximally symmetric models for hyperbolic $(4, 8)$ -distributions.

Submaximally symmetric models of hyperbolic $(4, 8)$ -distributions. By Corollary 5, generic $(4, 8)$ -distributions are equivalent to certain normal, regular parabolic geometries, whose parabolic type (G, P) is determined by a grading $\mathfrak{g} = \bigoplus_{i=-2}^2 \mathfrak{g}_i$ on a real graded simple Lie algebra \mathfrak{g} such that $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i=-2}^2 \mathfrak{g}_i^{\mathbb{C}}$ is the grading on $\mathfrak{sl}(5, \mathbb{C})$ described in Section 1. Aim of this section is the explicit description of the submaximally symmetric models of hyperbolic $(4, 8)$ -distributions, arising from the choice of grading on the split-real form $\mathfrak{sl}(5, \mathbb{R}) \subset \mathfrak{sl}(5, \mathbb{C})$.

Consider the real simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{R})$ and the $|2|$ -grading on \mathfrak{g} described in 2.1. In particular, recall that $\mathfrak{g}_0 \cong \mathfrak{gl}(2, \mathbb{R}) \oplus \mathfrak{gl}(2, \mathbb{R})$ is a reductive Lie algebra and that \mathfrak{g}_{-1} decomposes into the direct sum $\mathfrak{g}_{-1} = E \oplus F$ of two-dimensional real \mathfrak{g}_0 -modules. The explicit description of the lowest weights of the \mathfrak{g}_0 -irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})$, given in Section 5, can be briefly summarized in the following diagram. At the left side, we represent the set of weights

$\{-w \cdot \lambda_{\mathfrak{g}} \mid w \in W_+^p(2)\}$ in the Dynkin diagram notation. For each weight $-w \cdot \lambda_{\mathfrak{g}}$, we write the \mathfrak{g}_0 -irreducible representation of lowest weight $-w \cdot \lambda_{\mathfrak{g}}$ at the right-hand side:

$$\begin{array}{ccc}
 \begin{array}{cccc} 0 & 4 & -3 & -1 \\ \circ & \times & \times & \circ \end{array} & \Lambda^2 E^* \otimes F \\
 \begin{array}{cccc} -1 & -3 & 4 & 0 \\ \circ & \times & \times & \circ \end{array} & \Lambda^2 F^* \otimes E \\
 \begin{array}{cccc} -3 & 3 & 0 & -2 \\ \circ & \times & \times & \circ \end{array} & (S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F) \\
 \begin{array}{cccc} -2 & 0 & 3 & -3 \\ \circ & \times & \times & \circ \end{array} & (S^2 F^* \otimes F)_0 \otimes \mathfrak{sl}(E)
 \end{array}$$

As it follows from Kostant's Theorem, these are the irreducible components of the \mathfrak{g}_0 -submodule $H^2(\mathfrak{g}_-, \mathfrak{g})^1 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$, whose harmonic representatives can be thought as skew-symmetric bilinear graded maps $\mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathfrak{g}_-$ of degree one. The first and second weights in the list correspond to subsets of maps $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$, while the third and fourth ones to subsets of maps $\mathfrak{g}_{-1} \times \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}$. We thus drew a clear picture for $H^2(\mathfrak{g}_-, \mathfrak{g})^1$, which is the space encoding the values of the harmonic curvature of a normal, regular parabolic geometry of type (G, P) corresponding to the grading on the split-real algebra \mathfrak{g} . Lemma 4.3.2 in [11] shows that $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$ for all $w \in W_+^p(2)$ and therefore Corollary 6 applies. The infinitesimal symmetry algebra of the submaximally symmetric models of type (G, P) admits the explicit construction from the previous section, which we briefly recall.

Suppose that $\mu = -w \cdot \lambda_{\mathfrak{g}}$ for some $w \in W_+^p(2)$ and that ϕ_0 is a lowest weight vector in the irreducible \mathfrak{g}_0 -module $V_{\mu} \subset H^2(\mathfrak{g}_-, \mathfrak{g})^1$. The prolongation $\mathfrak{a}(w) = \text{pr}^{\mathfrak{g}}(\mathfrak{g}_-, \text{ann}(\phi_0))$ can be determined, by Theorem 10, by looking at the Dynkin diagram of μ . Put $\mathfrak{f} := \mathfrak{a}(w)$. As shown in Lemma 5, the condition $w(-\lambda_{\mathfrak{g}}) \in \Delta^-$ guarantees that putting

$$[\ , \]_{\mathfrak{f}} := [\ , \] - \phi_0$$

defines a bracket on \mathfrak{f} , hence a Lie algebra structure $(\mathfrak{f}, [\ , \]_{\mathfrak{f}})$. Now any Lie algebra $(\mathfrak{f}, [\ , \]_{\mathfrak{f}})$ of dimension

$$d := \max\{\dim(\mathfrak{a}(w)) \mid w \in W_+^p(2)\},$$

obtained as above is isomorphic to the infinitesimal symmetry algebra of maximal dimension for a non flat normal, regular parabolic geometry of type (G, P) , whose harmonic curvature coincides with ϕ_0 . Therefore, we shall first compute $\dim(\mathfrak{a}(w))$ for all $w \in W_+^p(2)$ and then describe explicitly the homogeneous space resulting from the choice of w such that $\dim(\mathfrak{a}(w)) = d$. Because of the symmetry of the set of such weights, we only need to compute the prolongation for the cases $\Lambda^2 E^* \otimes F$ and $(S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F)$. In both cases the full prolongations have dimension 14, thus they both originate a submaximally symmetric model. In the first case, the first prolongation of $\mathfrak{g}_- \oplus \text{ann}(\phi_0)$ is trivial, while in the second case \mathfrak{a}_1 is one-dimensional and the full prolongation reduces to $\mathfrak{g}_- \oplus \text{ann}(\phi_0) \oplus \mathfrak{a}_1$. Let us now check it directly.

In the same notation of Example 1, we write elements of \mathfrak{g}_- as

$$Y_i = (\psi_i, e_i, f_i) \in \mathfrak{g}_{-2} \oplus E \oplus F \cong \mathfrak{gl}(2, \mathbb{R}) \oplus (\mathbb{R}^2)^* \oplus \mathbb{R}^2$$

for $i = 1, 2$. Recall that the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-2}$ descends to an isomorphism $\mathfrak{g}_{-2} \cong E \otimes F$ and that we write elements of \mathfrak{g}_0 as a pair $(U, V) \in \mathfrak{gl}(E) \oplus \mathfrak{gl}(F)$. In our notation, the \mathfrak{g}_0 -action on \mathfrak{g}_- looks like

$$(63) \quad [(U, V), (\psi_i, e_i, f_i)] = (V\psi_i - \psi_i U, -\lambda e_i - Ue_i, \lambda f_i + Vf_i),$$

where $\lambda = \text{tr}(U + V)$. Also recall that we look at ϕ_0 as a skew-symmetric bilinear map $\mathfrak{g}_- \times \mathfrak{g}_- \rightarrow \mathfrak{g}$. For $X \in \mathfrak{g}_0$, being in the annihilator $\text{ann}(\phi_0)$ writes as

$$(64) \quad [X, \phi_0(Y_1, Y_2)] - \phi_0([X, Y_1], Y_2) - \phi_0(Y_1, [X, Y_2]) = 0$$

for all $Y_1, Y_2 \in \mathfrak{g}_-$.

Lowest weight vector in $\Lambda^2 E^ \otimes F$.* A lowest weight vector $\Phi_0 \in \Lambda^2 E^* \otimes F$ writes as $\Phi_0 = \omega f_0$ for a symplectic form $\omega : E \times E \rightarrow \mathbb{R}$ and a fixed non-zero $f_0 \in F$. If $X \in \mathfrak{ann}(\phi_0)$, then X satisfies (64) for all $Y_1 = e_1, Y_2 = e_2 \in E$. Observe that in this formula, in the first term X acts on an element of F , while in the remaining terms X acts on elements of E . By (63), then, the condition for $X = (U, V) \in \mathfrak{gl}(E) \oplus \mathfrak{gl}(F)$ to be in $\mathfrak{ann}(\phi_0)$ writes as

$$\begin{aligned} 0 &= \lambda \omega(e_1, e_2) f_0 + \omega(e_1, e_2) V f_0 - (\omega(-\lambda e_1 - e_1 U, e_2) + \omega(e_1, -\lambda e_2 - e_2 U)) f_0 \\ &= (3\lambda + \text{tr}(U)) \omega(e_1, e_2) f_0 + \omega(e_1, e_2) V f_0 \end{aligned}$$

for all $e_1, e_2 \in E$ with $\lambda = \text{tr}(U + V)$. A nontrivial solution (U, V) of the last equation satisfies $V f_0 = -(3\lambda + \text{tr}(U)) f_0 = (-3\text{tr}(V) - 4\text{tr}(U)) f_0$. This gives two linear conditions on V , hence $\mathfrak{ann}(\phi_0) \subset \mathfrak{g}_0$ is a six-dimensional subalgebra. Choosing f_0 the first vector of the standard basis for F , one can explicitly describe $\mathfrak{ann}(\phi_0)$ as the set

$$\mathfrak{a}_0 := \mathfrak{ann}(\phi_0) = \left\{ (U, V) \mid U \in \mathfrak{gl}(2, \mathbb{C}), V = \begin{pmatrix} -3a - 4\text{tr}(U) & * \\ 0 & 4\text{tr}(U) + 4a \end{pmatrix}, a \in \mathbb{R} \right\}$$

We can now compute the first prolongation \mathfrak{a}_1 of $(\mathfrak{g}_-, \mathfrak{a}_0)$ using (4) of Theorem 10. For a weight μ , the elements of

$$I_\mu = \{i \in \Sigma \mid \langle \mu, \alpha_i \rangle = 0\}$$

correspond to the crossed nodes in the Dynkin diagram of μ with a zero coefficient. Recall that the weight μ of ϕ_0 is

$$\begin{array}{cccc} 0 & 4 & -3 & -1 \\ \circ & \text{---} \times & \text{---} \times & \text{---} \circ \end{array}$$

so that $I_\mu = \emptyset$. This is equivalent to $\mathfrak{a}_1 = \{0\}$, hence $\mathfrak{a}(w) = \mathfrak{g}_- \oplus \mathfrak{a}_0$ is the vector space underlying \mathfrak{f} , which therefore has dimension 14.

Let us write $[\cdot, \cdot]_{\mathfrak{f}} = [\cdot, \cdot] - \phi_0$ explicitly. Since the deformation term is a map $\phi_0 : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \rightarrow \mathfrak{g}_{-1}$, it only affects the bracket between negative graded components. For every $Y_1, Y_2, Y_3, Y \in \mathfrak{g}_{-1}$, writing $Y_i = (e_i, f_i)$ with $e_i \in E$ and $f_i \in F$, we obtain:

$$\begin{aligned} [Y_1, Y_2]_{\mathfrak{f}} &= (f_1 \otimes e_2 - f_2 \otimes e_1, 0, -\omega(e_1, e_2) f_0) \\ [Y_3, [Y_1, Y_2]_{\mathfrak{f}}]_{\mathfrak{f}} &= (\omega(e_1, e_2) f_0 \otimes e_3, 0, 0) \\ [Y, [Y_3, [Y_1, Y_2]_{\mathfrak{f}}]_{\mathfrak{f}}]_{\mathfrak{f}} &= (0, 0, 0) \end{aligned}$$

The formulas above show that $(\mathfrak{f}_-, [\cdot, \cdot]_{\mathfrak{f}})$ is a nilpotent graded Lie algebra of depth three. In particular, \mathfrak{f}_{-1} determines a left-invariant two-step bracket generating distribution on the connected and simply connected Lie group K , whose Lie algebra is \mathfrak{f}_- . The Baker-Campbell-Hausdorff formula leads to the group multiplication law for K . In fact, for g, h in K , there exist vectors $Y_1, Y_2 \in \mathfrak{f}$ such that $g_1 = e^{Y_1}$ and $g_2 = e^{Y_2}$. Hence $g_1 g_2 = e^Z$, where

$$Z = Y_1 + Y_2 + \frac{1}{2}[Y_1, Y_2]_{\mathfrak{f}} + \frac{1}{12}[Y_1, [Y_1, Y_2]_{\mathfrak{f}}]_{\mathfrak{f}} - \frac{1}{12}[Y_2, [Y_1, Y_2]_{\mathfrak{f}}]_{\mathfrak{f}}$$

is given by the mentioned formula. Explicitly:

$$\begin{aligned} Z &= \left(\psi_1 + \psi_2 + \frac{1}{2}(f_1 \otimes e_2 - f_2 \otimes e_1) + \frac{1}{12}\omega(e_1, e_2) f_0 \otimes (e_1 - e_2), \right. \\ &\quad \left. e_1 + e_2, \right. \\ &\quad \left. f_1 + f_2 - \frac{1}{2}\omega(e_1, e_2) f_0 \right). \end{aligned}$$

Lowest weight vector in $(S^2 E^ \otimes E)_0 \otimes \mathfrak{sl}(F)$.* Let us consider now the lowest weight vector

$$\phi_0 \in (S^2 E^* \otimes E)_0 \otimes \mathfrak{sl}(F).$$

Since E is two-dimensional, $\Lambda^2 E$ is one-dimensional and there exists an isomorphism

$$\psi: E \rightarrow L(E, \Lambda^2 E), \quad v \mapsto (w \mapsto v \wedge w).$$

Therefore, $id_{S^2 E^*} \otimes \psi: S^2 E^* \otimes E \rightarrow S^2 E^* \otimes E^* \otimes \Lambda^2 E$ is also a linear isomorphism, which descends to

$$(S^2 E^* \otimes E)_0 \cong S^3 E^* \otimes \Lambda^2 E.$$

Fix bases $\{e_1, e_2\}$ and $\{f_1, f_2\}$ of E and F . Clearly, $\psi(e_2) = e_1^* \otimes e_1 \wedge e_2$. Suppose that e_1 and f_1 are highest weight vectors of E and F , respectively. Using the isomorphism above and the usual notation for the dual basis, we can write

$$\phi_0 = e_1^* \odot e_1^* \odot e_1^* \odot e_1 \wedge e_2 \otimes f_1^* \otimes f_2.$$

Seen as a map $\phi_0: \mathfrak{g}_{-1} \times \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}$, we have that $\phi_0(e_1, f_1 \otimes e_1) = f_2 \otimes e_2$ and it vanishes on the others elements of the induced basis of $\mathfrak{g}_{-1} \times \mathfrak{g}_{-2}$. With the same notation as before, using (63) the condition (64) for $X = (U, V) \in \mathfrak{a}_0$ writes as

$$(65) \quad V \cdot \phi_0(Y_1, Y_2) - \phi_0(Y_1, Y_2) \cdot U - \phi_0(-Y_1 \cdot (U + \lambda \text{id}), Y_2) - \phi_0(Y_1, V \cdot Y_2 - Y_2 \cdot U) = 0$$

for all $Y_1 \in E$ and $Y_2 \in \mathfrak{g}_{-2} \cong E \otimes F$ with $\lambda = \text{tr}(U + V)$. Now inserting $Y_1 = e_i$ and $Y_2 = f_j \otimes e_k$ for $i, j, k = 1, 2$ in the last equation leads to three nonempty conditions. Clearly, we can write any $M \in \mathfrak{gl}(2, \mathbb{R})$ as $M = M_i^j f_i \otimes e_j$, so this will be done for U and V . First, for $Y_1 = e_1$ and $Y_2 = f_2 \otimes e_1$, (65) reduces to the fourth summand, hence it rewrites as

$$0 = -\phi_0(e_1, V \cdot (f_2 \otimes e_1) - (f_2 \otimes e_1) \cdot U) = V_1^2 e_1^* \otimes e_1 \wedge e_2 \otimes f_2$$

which is satisfied if and only if $V_1^2 = 0$. Secondly, for $Y_1 = e_1$ and $Y_2 = f_1 \otimes e_1$, (65) rewrites as

$$\begin{aligned} 0 &= V \cdot (f_2 \otimes e_2) - (f_2 \otimes e_2) \cdot U + (U_1^1 + \lambda) f_2 \otimes e_2 - V_1^1 f_2 \otimes e_2 + U_1^1 f_2 \otimes e_2 \\ &= V_1^2 f_1 \otimes e_2 - U_2^1 f_2 \otimes e_1 + (V_2^2 - U_2^2 + (U_1^1 + \lambda) - V_1^1 + U_1^1) f_2 \otimes e_2 \end{aligned}$$

where we wrote e_2 rather than $e_1^* \otimes e_1 \wedge e_2$, aware of the fact that they correspond to each other along the isomorphism ψ . Third, for $Y_1 = e_2$, (65) rewrites as

$$\phi_0((U + \lambda \text{id}) \cdot e_2, Y_2) = 0$$

which is a nonempty condition only if $Y_2 = f_1 \otimes e_1$. For this value of Y_2 , we evidently obtain $U_1^2 = 0$. The three conditions

$$\begin{cases} V_1^2 = 0 \\ V_2^2 - U_2^2 + 2U_1^1 - V_1^1 + \lambda = 3U_1^1 + 2V_2^2 = 0 \\ U_1^2 = 0 \end{cases}$$

are linearly independent, thus showing that \mathfrak{a}_0 has dimension five. The system gives an explicit matrix presentation of \mathfrak{a}_0 .

Recall that the weight μ of ϕ_0 is

$$\begin{array}{cccc} -3 & 3 & 0 & -2 \\ \text{O} & \text{---} \times & \text{---} \times & \text{---} \text{O} \end{array}$$

so that $I_\mu = \{\alpha_3\}$ is nonempty. Then (4) of Theorem 10 tells us that

$$\Delta(\mathbf{a}_1) = \{\alpha \in \Delta(\mathfrak{g}_1) \mid Z_{I_\mu}(\alpha) = 1, Z_{J_\mu}(\alpha) = 0\},$$

where

$$J_\mu = \{j \in \Delta^0 \setminus \Sigma \mid \langle \mu, \alpha_j \rangle \neq 0\}.$$

Since $J_\mu = \{\alpha_1, \alpha_4\}$ and $\Delta(\mathfrak{g}_1) = \{\alpha_1 + \alpha_2, \alpha_2, \alpha_3, \alpha_3 + \alpha_4\}$, the simple root α_3 is the unique in $\Delta(\mathfrak{g}_1)$ having Z_{I_μ} -height one and zero Z_{J_μ} -height and we thus have that $\Delta(\mathbf{a}_1) = \{\alpha_3\}$. We thus

see that \mathfrak{a}_1 is one-dimensional and (5) of Theorem 10 shows that $\mathfrak{a}_2 = \{0\}$, hence we conclude that $\mathfrak{a}^{\phi_0} = \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1$ is the full prolongation, which has dimension 14.

Let us write $[\cdot, \cdot]_{\mathfrak{f}} = [\cdot, \cdot] - \phi_0$ explicitly. Since the deformation term is a map $\phi_0 : \mathfrak{g}_{-1} \times \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_{-2}$, it only affects the bracket between negative graded components. The modified bracket between elements $Y_i \in \mathfrak{f}_{-1}$ looks like the following:

$$[Y_1, Y_2]_{\mathfrak{f}} = (f_1 \otimes e_2 - f_2 \otimes e_1 - \phi_0(e_1, \psi_2) + \phi_0(e_2, \psi_1), 0, 0)$$

Since $[Y_1, Y_2]_{\mathfrak{f}}$ has trivial \mathfrak{g}_{-1} -component, $[Y, [Y_1, Y_2]_{\mathfrak{f}}]_{\mathfrak{f}} = (0, 0, 0)$ for all $Y_1, Y_2, Y \in \mathfrak{f}_{-1}$, so that iterating the bracket gives zero. Let K be the connected and simply connected Lie group with Lie algebra \mathfrak{f}_- . Hence \mathfrak{f}_{-1} generates a distribution on K , which is one-step bracket-generating. Again, the Baker-Campbell-Hausdorff formula leads to the group multiplication law for K . Let be $g, h \in K$ and let $X_1, X_2 \in \mathfrak{f}$ be such that $g_1 = e^{X_1}$ and $g_2 = e^{X_2}$. Hence $g_1 g_2 = e^Z$, where

$$Z = \left(\psi_1 + \psi_2 + \frac{1}{2}(f_1 \otimes e_2 - f_2 \otimes e_1 - \phi_0(e_1, \psi_2) + \phi_0(e_2, \psi_1)), \right. \\ \left. e_1 + e_2, \right. \\ \left. f_1 + f_2 \right)$$

Clearly, the formula is polynomial and differs from the formula for the bracket on \mathfrak{g} only for the term $\frac{1}{2}(-\phi_0(e_1, \psi_2) + \phi_0(e_2, \psi_1))$.

Finally, we can see how the construction of a submaximally symmetric model does not work out analogously, in general, for real parabolic types. This can be easily seen for the real parabolic type corresponding to elliptic $(4, 8)$ -distributions. Let be $\mathfrak{t} = \mathfrak{su}(3, 2)$ and $\mathfrak{t} = \oplus_{i=-2}^2 \mathfrak{t}_i$ the grading from 3.2. Recall that $\mathfrak{t}_{-1} \cong \mathbb{C}^2$ and there are only two complex structures J on \mathfrak{t}_{-1} , namely the multiplication for i and its negative, such that $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{t}_{-1}$. Moreover, $H^2(\mathfrak{t}_-, \mathfrak{t})^1 = V_1 \oplus V_2$ is the direct sum of two irreducible complex \mathfrak{t}_0 -modules, with $V_1 \subset \Lambda^2 \mathfrak{t}_{-1}^* \otimes \mathfrak{t}_{-1}$ and $V_2 \subset \mathfrak{t}_{-1}^* \otimes \mathfrak{t}_{-2}^* \otimes \mathfrak{t}_{-2}$. In the following, we see that the bracket modified by subtracting a lowest weight vector $\phi_0 \in V_1$ does not satisfy Jacobi.

Write $X \in \mathfrak{gl}(2, \mathbb{C}) \cong \mathfrak{t}_0$, $v \in \mathbb{C}^2 \cong \mathfrak{t}_{-1}$ and $M \in \mathfrak{u}(2) \cong \mathfrak{t}_{-2}$. The \mathfrak{t}_0 -action on \mathfrak{t}_- writes as

$$\begin{aligned} \mathfrak{t}_0 \times \mathfrak{t}_{-1} &\rightarrow \mathfrak{t}_{-1}, & [X, v] &= X \cdot v + \lambda v \\ \mathfrak{t}_0 \times \mathfrak{t}_{-2} &\rightarrow \mathfrak{t}_{-2}, & [X, M] &= XM - (XM)^* \end{aligned}$$

where $\lambda = -2i\text{Im}(\text{tr}(X))$, \cdot denotes the standard action of $\mathfrak{gl}(2, \mathbb{C})$ on $\mathfrak{t}_{-1} \cong \mathbb{C}^2$ and $(XM)^*$ denotes the conjugate transpose of XM .

A lowest \mathfrak{t}_0 -weight vector $\phi_0 \in V_1$ writes as $\phi_0 = \omega t$ for a non-degenerate skew-symmetric antilinear form $\omega : \mathfrak{t}_{-1} \times \mathfrak{t}_{-1} \rightarrow \mathbb{C}$ and a fixed nonzero vector $t \in \mathfrak{t}_{-1}$. The condition (64) for $X \in \mathfrak{ann}(\phi_0) \subset \mathfrak{t}_0$ writes as

$$\omega(v, w)(X \cdot t + \lambda t) - \omega(X \cdot v + \lambda v, w)t - \omega(v, X \cdot w + \lambda w)t = 0,$$

for all $v, w \in \mathfrak{t}_{-1}$. Choose a basis $\{v, w\} \subset \mathfrak{t}_{-1}$ such that $\omega(v, w) = 1$. By antilinearity, evaluating the equation above on v, w gives

$$(\lambda - \bar{\lambda} - \bar{\lambda} - \text{tr}(\bar{X}))t + X \cdot t = 0$$

which are two \mathbb{C} -linearly independent conditions on X . This shows that $\mathfrak{ann}(\phi_0) \subset \mathfrak{t}_0$ has real dimension 4. Setting $t = (0, 1)^t$, the two conditions on X write as

$$\begin{cases} X_1^2 = 0 \\ X_2^2 = 5i\text{Im}(\text{tr}(X)) + \text{Re}(\text{tr}(X)) \end{cases}$$

giving the explicit description of $\mathfrak{a}_0 := \text{ann}(\phi_0)$ as the set of matrices

$$\mathfrak{a}_0 \cong \left\{ X \in \mathfrak{gl}(2, \mathbb{C}) \mid X = \begin{pmatrix} -4i\text{Im}(\text{tr}(X)) & 0 \\ * & 5i\text{Im}(\text{tr}(X)) + \text{Re}(\text{tr}(X)) \end{pmatrix} \right\}$$

Recall that $\mathfrak{t}_1 \cong (\mathbb{C}^2)^*$. More precisely, we identify

$$\begin{pmatrix} 0 & -\bar{v} & 0 \\ 0 & 0 & v^* \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{t}_1$$

with $v^* \in (\mathbb{C}^2)^*$. In this notation, the bracket $\mathfrak{t}_1 \times \mathfrak{t}_{-1} \rightarrow \mathfrak{t}_0$ writes as $[v^*, w] = -w \otimes v^*$. Therefore, if \mathfrak{a}_1 denotes the first prolongation of $(\mathfrak{t}_-, \mathfrak{a}_0)$, we have that

$$\mathfrak{a}_1 \cong \{ v^* \in (\mathbb{C}^2)^* \mid w \otimes v^* \in \mathfrak{a}_0 \ \forall w \in \mathbb{C}^2 \} = \{0\}$$

Let us write $v^* = (v^1, v^2) \in (\mathbb{C}^2)^*$ and the condition $w \otimes v^* \in \mathfrak{a}_0$ for all $w = (w_1, w_2)^t \in \mathbb{C}^2$ explicitly. For $w = (1, 0)^t$ we obtain

$$\begin{cases} v^2 = 0 \\ v^1 = -4i\text{Im}(v^1) \end{cases}$$

which implies that v^1 is pure imaginary. For $w = (i, 0)^t$, we obtain

$$\begin{cases} iv^2 = 0 \\ iv^1 = -4i\text{Im}(iv^1) = -4i\text{Re}(iv^1) \end{cases}$$

which implies that $v^1 \in \mathbb{R}$. But v^1 can be pure imaginary and real at the same if and only if $v^1 = 0$. We thus conclude that $\mathfrak{a}_1 = \{0\}$, hence that the full prolongation $\mathfrak{a} = \mathfrak{t}_- \oplus \mathfrak{a}_0$ has dimension 12.

Denote by \mathfrak{f} the vector space underlying the graded subalgebra $\mathfrak{a} \subset \mathfrak{t}$. Since $\mathfrak{a}_- = \mathfrak{t}_-$ and $\text{Im}(\phi_0) \subset \mathfrak{t}_{-1}$, we have that $\text{Im}(\phi_0) \subset \mathfrak{a}$. This assures that putting

$$[\ , \]_{\mathfrak{f}} := [\ , \] - \phi_0$$

defines a skew-symmetric bilinear operation $[\ , \]_{\mathfrak{f}} : \mathfrak{f} \times \mathfrak{f} \rightarrow \mathfrak{f}$. Now there are no results from the general theory, as in the split-real case, assuring that $[\ , \]_{\mathfrak{f}}$ satisfies the Jacobi identity and this is indeed not the case. Let us verify it by direct computation. From the proof of Lemma 5, we see that $[\ , \]_{\mathfrak{f}}$ verifies the Jacobi identity if and only if

$$\text{Jac}_{\mathfrak{f}}(X, Y, Z) = \phi_0(\phi_0(X, Y), Z) + \phi_0(\phi_0(Y, Z), X) + \phi_0(\phi_0(Z, X), Y) = 0$$

for all $X, Y, Z \in \mathfrak{t}_{-1}$. Observe that

$$\begin{aligned} \text{Jac}_{\mathfrak{f}}(X, Y, Z) &= \phi_0(\omega(X, Y)t, Z) + \phi_0(\omega(Y, Z)t, X) + \phi_0(\omega(Z, X)t, Y) \\ &= \phi_0(t, \omega(X, Y)Z + \omega(Y, Z)X + \omega(Z, X)Y) \end{aligned}$$

Suppose that $\{X, Y\} \subset \mathfrak{t}_{-1}$ is a basis, so that there exist $a, b \in \mathbb{C}$ such that $Z = aX + bY$. Then

$$\begin{aligned} \omega(X, Y)Z + \omega(Y, Z)X + \omega(Z, X)Y &= \omega(X, Y)(aX + bY) + \omega(Y, aX)X + \omega(bY, X)Y \\ &= (a - \bar{a})\omega(X, Y)X + (b - \bar{b})\omega(X, Y)Y \\ &= 2\text{Im}(a)\omega(X, Y)X + 2\text{Im}(b)\omega(X, Y)Y \end{aligned}$$

Clearly, we can choose a basis $\{X, Y\} \subset \mathfrak{t}_{-1}$ such that $\omega(X, Y) = 1$. Now $t = cX + dY$ for some $c, d \in \mathbb{C}$ which are not both zero and therefore

$$\begin{aligned} \text{Jac}_{\mathfrak{f}}(X, Y, Z) &= \phi_0(t, 2\text{Im}(a)X + 2\text{Im}(b)Y) \\ &= \omega(t, 2\text{Im}(a)X + 2\text{Im}(b)Y)t \\ &= (-2\text{Im}(a)\bar{d} + 2\text{Im}(b)\bar{c})t \end{aligned}$$

This shows that if we choose a, b such that $(-2\operatorname{Im}(a)\bar{d} + 2\operatorname{Im}(b)\bar{c}) \neq 0$, then $Z = aX + bY \in \mathfrak{t}_{-1}$ is a nonzero vector such that $\operatorname{Jac}_{\mathfrak{f}}(X, Y, Z) \neq 0$.

APPENDIX A

Basics on parabolic geometries

Following [10], we recall the basic theory of representations of semisimple Lie algebras and completely reducible representations of parabolic subalgebras. Applying such results in the setting of Cartan geometries is the basic idea of the parabolic geometry theory. Definition and basic properties of Cartan geometries are collected in the last section of this Appendix.

1. Representations of complex semisimple Lie algebras

Let $\mathfrak{g} = (\mathfrak{g}, [\cdot, \cdot])$ be a finite dimensional Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Define $\mathfrak{g}^{(1)} := \mathfrak{g}$ and $\mathfrak{g}^{(k+1)}$ for $k \geq 1$, inductively, by $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. Now \mathfrak{g} is said to be:

- *solvable* if the derived series ends for some $k \in \mathbb{N}$ with $\mathfrak{g}^{(k)} = \{0\}$;
- *semisimple* if it has no nonzero solvable ideals;
- *simple* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and it has no proper ideals.

Let \mathfrak{g} be a finite dimensional Lie algebra. The bracket on \mathfrak{g} defines the *adjoint representation* $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, a finite dimensional representation of \mathfrak{g} on itself. The kernel of ad is a commutative ideal in \mathfrak{g} , hence solvable. Therefore if \mathfrak{g} is semisimple, ad is injective and it induces a Lie algebra isomorphism between \mathfrak{g} and the algebra $\text{der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ of derivations of the Lie algebra \mathfrak{g} . Observe that an ideal of a Lie algebra \mathfrak{g} , by definition, is a vector subspace which is invariant under the adjoint action. It is thus clear that \mathfrak{g} is simple if and only $\dim(\mathfrak{g}) > 1$ and the adjoint action of \mathfrak{g} on itself is irreducible. The symmetric bilinear form on \mathfrak{g} , defined by

$$B(X, Y) := \text{tr}(\text{ad}(X) \circ \text{ad}(Y)), \quad \text{for } X, Y \in \mathfrak{g}$$

is the *Killing form* of \mathfrak{g} . An element $X \in \mathfrak{g}$ is called *semisimple* if $\text{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is diagonalizable. A *Cartan subalgebra* of \mathfrak{g} is a maximal commutative Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, whose elements are semisimple. It turns out that any semisimple Lie algebra \mathfrak{g} contains semisimple elements. Furthermore, any complex semisimple Lie algebra \mathfrak{g} admits a Cartan subalgebra and any two Cartan subalgebras of \mathfrak{g} are conjugated by an inner automorphism of \mathfrak{g} (see Theorem 2.2.2. [10], p. 163 for a proof).

EXAMPLE 7. The Lie algebra of complex trace-free square matrices $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is simple. The subalgebra of diagonal matrices in \mathfrak{g} is the standard choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Now if $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a complex finite dimensional representation of \mathfrak{g} semisimple and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, $\mathcal{F} = \{\rho(H) \mid H \in \mathfrak{h}\} \subset \mathfrak{gl}(V)$ is a family of diagonalizable endomorphisms of V which are mutually commuting. An argument from linear algebra shows that all elements of \mathcal{F} are simultaneously diagonalizable with eigenvalues determined by a linear functional on \mathfrak{h} . If $\lambda : \mathfrak{h} \rightarrow \mathbb{C}$ is a linear functional, then

$$V_\lambda := \{X \in V \mid \rho(H)X = \lambda(H)X \quad \forall H \in \mathfrak{h}\}$$

is a vector subspace of V . If it is nonzero, then it is an eigenspace for \mathcal{F} and the corresponding eigenvalue λ is said to be a *weight* of V . The dimension of V_λ is the *multiplicity* of the weight λ in V . We denote by $\text{wt}(V)$ the set of all such weights, so that the decomposition of V into the direct sum of joint eigenspaces for \mathcal{F} writes as $V = \bigoplus_{\lambda \in \text{wt}(V)} V_\lambda$. Applying this to the adjoint

representation leads to the *root decomposition* of the semisimple Lie algebra \mathfrak{g} . For $\rho = \text{ad}$, the nonzero weights are called *roots* and the set of roots is usually denoted by Δ . The Cartan subalgebra is the eigenspace of weight zero and the *root decomposition* writes as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

One can prove that the restriction of the Killing form B to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate and this allows to define a symmetric complex bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^* . For $\lambda \in \mathfrak{h}^*$, there is a unique $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in \mathfrak{h}$. Thus for $\lambda, \mu \in \mathfrak{h}^*$ we define $\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu)$. For $\alpha \in \Delta$, also $-\alpha \in \Delta$, so that one can make a choice to split $\Delta = \Delta^+ \sqcup \Delta^-$ into the disjoint union of positive and negative subsystems. Since Δ generates \mathfrak{h}^* , this is actually equivalent to the choice of positivity on the subspace $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ of real-valued functionals on \mathfrak{h} . Once a choice is done, one can further select a subset of positive roots with special properties, as follows. The set of *simple roots* $\Delta^0 \subset \Delta^+$ is given by those positive roots, which cannot be written as a sum of two distinct positive roots. Then Δ^0 is a basis for \mathfrak{h}^* and every positive root writes as a linear combination of simple roots, with nonnegative integer coefficients written in terms of $\langle \cdot, \cdot \rangle$. This follows from the consideration that $\langle \beta, \check{\alpha} \rangle$ is an integer for every $\alpha, \beta \in \Delta$, where $\check{\alpha} = 2 \frac{\alpha}{\langle \alpha, \alpha \rangle}$. Observe that $\check{\alpha}$ is well defined, since the complex bilinear form $\langle \cdot, \cdot \rangle$ restricts to a scalar product on \mathfrak{h}_0^* , which coincides with the real vector space generated by the simple roots.

EXAMPLE 8. A root system for the standard Cartan subalgebra of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is given by

$$\Delta = \{ e_i - e_j \mid 1 \leq i, j \leq n, i \neq j \}$$

where $e_i : \mathfrak{h} \rightarrow \mathbb{C}$ denotes the linear functional, extracting the i -th entry of a trace-free diagonal matrix. A usual choice of positive root system is realized by taking all elements of Δ with $i < j$. With respect to these choices, the simple root system is given by

$$\Delta^0 = \{ \alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq n-1 \}$$

Consider $\mathfrak{h}_0 \subset \mathfrak{h}$ the real subalgebra of diagonal matrices with all real entries. Then all simple roots assume real values on \mathfrak{h}_0 , hence every element in the real space generated by them. Now if H_1 and H_2 are diagonal matrices, $\text{tr}(H_1 H_2)$ coincides with the standard scalar product in \mathbb{C}^n between their diagonals. It follows, that the scalar product satisfies $\langle e_i, e_j - e_k \rangle = \delta_{ij} - \delta_{ik}$, thus the scalar product between the simple roots writes as

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } j = i \\ -1 & \text{if } j = i+1 \text{ or } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

For $\alpha \in \Delta$, the *root reflection* $s_\alpha : \mathfrak{h}_0^* \rightarrow \mathfrak{h}_0^*$ is the reflection with respect to the hyperplane orthogonal to α , explicitly defined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha$. The group generated by such reflections is known as *Weyl group*, a subgroup $W = W(\mathfrak{g}, \mathfrak{h}) \subset O(\mathfrak{h}_0^*)$ of the orthogonal group. It is easy to see that W is actually generated by the reflections corresponding to the simple roots. Since every such reflection preserves Δ , the Weyl group can be also seen as a subgroup of bijections of the finite set Δ . Every $w \in W$ can be written as a composition of a finite number of simple reflections, thus it is an involutive endomorphism of \mathfrak{h}_0^* . The *length* of w is the minimal number of simple root reflections required to write w and it is denoted by $l(w)$. Clearly, the determinant of w as an endomorphism of \mathfrak{h}_0^* depends on the length of w , via the identity $\det(w) = (-1)^{l(w)}$. The Weyl group acts on the set $\text{wt}(V)$ of weights of a complex finite dimensional representation of \mathfrak{g} . Understanding the action of W on $\text{wt}(V)$ is the key ingredient to understand the representations of parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$, playing an important role in the description of parabolic geometries.

EXAMPLE 9. Let $\alpha = e_i - e_j \in \Delta$ in the root system A_{n-1} for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, which has been described in the previous example. One easily verifies that $s_\alpha : \mathfrak{h}_0^* \rightarrow \mathfrak{h}_0^*$ permutes e_i with e_j and leaves all remaining e_k 's invariant. The Weyl group of A_{n-1} is thus the permutation group \mathfrak{S}_n of n elements.

2. The theorem of highest weight

2.1. Weights of a finite dimensional representation of \mathfrak{g} . Let V be a finite dimensional representation of a complex semisimple Lie algebra \mathfrak{g} . A linear functional $\lambda \in \mathfrak{h}^*$ is said to be *algebraically integral* if $\langle \lambda, \tilde{\alpha} \rangle$ is an integer for every root $\alpha \in \Delta$. It is actually enough to check this property just for simple roots α , to conclude that the same holds for all roots. The weights of a complex finite dimensional representation of \mathfrak{g} are algebraically integral, in particular they lie in the real subspace $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ of real-valued functionals.

Denote by $\Delta^0 = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathfrak{h}^*$ a simple root system for \mathfrak{h} . Then define the set of *fundamental weights* $\{\omega_1, \dots, \omega_\ell\} \subset \mathfrak{h}^*$ by $\langle \omega_i, \tilde{\alpha}_j \rangle = \delta_{ij}$. We clearly get in this way a new basis of \mathfrak{h}^* ; the transition matrix between $\{\alpha_j\}$ and $\{\omega_i\}$ is called *Cartan matrix*. The set of linear combinations with integer coefficients of fundamental weights coincides with the set of algebraically integral elements, the *weight lattice* of \mathfrak{g} . Then one can represent every weight λ , by writing the integer number $\langle \lambda, \tilde{\alpha}_i \rangle$ on the node of the Dynkin diagram of \mathfrak{g} corresponding to the root α_i . This is exactly the coefficient multiplying ω_i in the expression of λ as linear combination of fundamental weights, hence in this way we draw all elements in the weights lattice.

If λ is a weight of V and w is in the Weyl group, then $w(\lambda)$ is also a weight of V , occurring with the same multiplicity as λ . The Weyl group thus acts on the set of weights. Since it is a finite set, there is a maximal weight λ_0 of V with respect to the chosen total ordering on \mathfrak{h}_0^* . We call λ_0 the *highest weight* of V . The highest weight is algebraically integral and *dominant*, meaning that $\langle \lambda_0, \alpha_i \rangle \geq 0$ for all simple roots α_i .

THEOREM 12 ([10], p. 184). (*Theorem of the highest weight*) If \mathfrak{g} is a finite dimensional complex semisimple Lie algebra, then for any dominant algebraically integral weight $\lambda \in \mathfrak{h}_0^*$ there is a (up to isomorphism) unique finite dimensional irreducible representation with highest weight λ .

By definition, a simple Lie algebra is irreducible under the adjoint representation. The corresponding highest weight is the *highest root* of \mathfrak{g} .

The following remark often applies to the decomposition of tensor products into irreducible components. The *i -th fundamental representation* of the semisimple Lie algebra \mathfrak{g} is the irreducible representation, whose highest weight is ω_i . Let us denote it by V_i and suppose that all fundamental representations V_1, \dots, V_ℓ of \mathfrak{g} are known. Then the irreducible representation of highest weight $a_1\lambda_1 + \dots + a_\ell\lambda_\ell$ is contained in the tensor product of the symmetric tensorial powers $S^{a_1}(V_1) \otimes \dots \otimes S^{a_\ell}(V_\ell)$.

EXAMPLE 10. One easily shows that $\omega_i = \sum_{j=1}^i e_j$ is the i -th fundamental weight and $\Lambda^i \mathbb{C}^n$ is the i -th fundamental representation for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Moreover, the highest root for this classical simple Lie algebra is $\lambda_{\mathfrak{g}} = \sum_{i=1}^{n-1} \alpha_i = e_1 - e_n$. Recalling that the trace-free condition writes as $-e_n = \sum_{i=1}^{n-1} e_i = \omega_{n-1}$, we see that $\lambda_{\mathfrak{g}}$ can be written in terms of the fundamental weights as $\omega_1 + \omega_{n-1}$. Recall that the i -th simple root writes as linear combination of fundamental weights and the coefficients can be read off the i -th column of the Cartan matrix. Using this fact, we compute

$$\begin{aligned}
 s_{\alpha_1}(\omega_1) &= \omega_1 - \alpha_1 = \omega_1 - 2\omega_1 + \omega_2 = -\omega_1 + \omega_2 \\
 s_{\alpha_i}(\omega_i) &= \omega_i - \alpha_i = \omega_i - (-\omega_{i-1} + 2\omega_i - \omega_{i+1}) = \omega_{i-1} - \omega_i + \omega_{i+1} \\
 s_{\alpha_{n-1}}(\omega_{n-1}) &= \omega_{n-1} - \alpha_{n-1} = \omega_{n-1} - 2\omega_{n-1} + \omega_{n-2} = -\omega_{n-1} + \omega_{n-2}
 \end{aligned}
 \tag{66}$$

3. Representations of parabolic subalgebras

3.1. Borel and parabolic subalgebras. A *Borel subalgebra* of a complex semisimple Lie algebra \mathfrak{g} is a maximal solvable subalgebra. The *standard Borel subalgebra* $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ is the direct sum of a Cartan subalgebra and the nilpotent algebra \mathfrak{n}_+ , given by the sum of all positive root spaces. A *parabolic subalgebra* is a subalgebra containing a Borel subalgebra. A *standard parabolic subalgebra* is a subalgebra containing the standard Borel subalgebra. Every parabolic subalgebra is a standard parabolic subalgebra with respect to an appropriate choice of Cartan subalgebra and positive roots. Therefore it is enough to focus on standard parabolic subalgebras \mathfrak{p} , obtained from \mathfrak{b} by adding some negative root spaces. Let us consider a standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and denote by Φ the set of positive roots α such that $\mathfrak{g}_{-\alpha} \subset \mathfrak{p}$. Then \mathfrak{p} is the direct sum

$$\mathfrak{p} = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{-\alpha}$$

Now Φ is completely determined by those simple roots lying in it. Define the subset of simple roots $\Sigma_{\mathfrak{p}} := \Delta^0 \setminus (\Delta^0 \cap \Phi)$. Conversely, given a subset Σ of simple roots, one can consider all positive roots α which are linear combination of elements of $\Delta^0 \setminus \Sigma$. The sum of the corresponding root spaces $\mathfrak{g}_{-\alpha}$, together with \mathfrak{b} , is a standard parabolic subalgebra \mathfrak{p}_{Σ} . The assignments

$$\mathfrak{p} \mapsto \Sigma_{\mathfrak{p}}, \quad \Sigma \mapsto \mathfrak{p}_{\Sigma}$$

are inverse to each other, thus giving a bijective correspondence between subsets of Δ^0 and standard parabolic subalgebras of \mathfrak{g} .

3.1.1. Parabolic subalgebras and gradings on \mathfrak{g} .

DEFINITION 20. Let \mathfrak{g} be a semisimple Lie algebra and let $k > 0$ be an integer. A $|k|$ -grading on \mathfrak{g} is a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of \mathfrak{g} into a direct sum of subspaces such that

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_i = \{0\}$ for $|i| > k$;
- the subalgebra $\mathfrak{g}_{-} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by \mathfrak{g}_{-1} ;
- \mathfrak{g}_{-k} and \mathfrak{g}_k are both nonzero

Fix a subset $\Sigma \subset \Delta^0 = \{\alpha_1, \dots, \alpha_l\}$ and denote by $\mathfrak{p} = \mathfrak{p}_{\Sigma}$ the corresponding parabolic subalgebra. For every $\alpha \in \Delta$, there exist uniquely determined integer coefficients such that $\alpha = \sum_{i=1}^l a_i \alpha_i$. The Σ -height of α , defined as $\text{ht}_{\Sigma}(\alpha) = \sum_{i \in \Sigma} a_i$, induces a grading on \mathfrak{g} . For $0 \neq i \in \mathbb{Z}$, define \mathfrak{g}_i as the sum of those root spaces \mathfrak{g}_{α} such that $\text{ht}_{\Sigma}(\alpha) = i$. Put also \mathfrak{g}_0 equal to the direct sum of the Cartan subalgebra \mathfrak{h} and of those root spaces \mathfrak{g}_{α} , corresponding to roots α of Σ -height zero. Clearly, \mathfrak{g} is the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k,$$

where k denotes the maximal value for the Σ -height on Δ , hence $\mathfrak{g}_k, \mathfrak{g}_{-k} \neq 0$. The Lie bracket on \mathfrak{g} is compatible with the gradation, thus \mathfrak{g} admits a structure of $|k|$ -graded Lie algebra. Therefore \mathfrak{g}_0 is a subalgebra of \mathfrak{g} , every \mathfrak{g}_i is a \mathfrak{g}_0 -module under the adjoint action and the parabolic subalgebra \mathfrak{p} coincides with the direct sum of all non-negative graded components. As consequence of the non-degeneracy of the Killing form, \mathfrak{h} admits a basis $\{H_i \mid i = 1, \dots, l\}$ dual to Δ^0 , such that $\text{sp}(H_i) = [\mathfrak{g}_{-\alpha_i}, \mathfrak{g}_{\alpha_i}]$. We can split the Cartan subalgebra into the direct sum $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, where

$$(67) \quad \begin{aligned} \mathfrak{h}' &= \{ H \in \mathfrak{h} \mid \alpha(H) = 0 \ \forall \ \alpha \in \Delta^0 \setminus \Sigma \}, \\ \mathfrak{h}'' &= \{ H_i \mid \alpha_i \in \Delta^0 \setminus \Sigma \} \end{aligned}$$

Observe that the dual spaces of \mathfrak{h}' and \mathfrak{h}'' are characterized, in term of the inner product, as the subspaces of \mathfrak{h}^* orthogonal to $\Delta^0 \setminus \Sigma$ and to Σ , respectively. Let us now focus on the

subalgebra \mathfrak{g}_0 and denote by $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{g}_0$ its center, an abelian ideal of \mathfrak{g}_0 . Evidently $\mathfrak{h}' \subset \mathfrak{z}(\mathfrak{g}_0)$ and it is easy to see that any solvable ideal of \mathfrak{g}_0 has to be contained in \mathfrak{h}' . Then \mathfrak{g}_0 is a reductive Lie algebra and in particular $\mathfrak{z}(\mathfrak{g}_0) = \mathfrak{h}'$, thus the dimension of the center coincides with the number of elements of Σ . Any reductive Lie algebra writes as the direct sum $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ between its center and its semisimple part $\mathfrak{g}_0^{ss} = [\mathfrak{g}_0, \mathfrak{g}_0]$. Clearly, \mathfrak{h}'' is a Cartan subalgebra of \mathfrak{g}_0^{ss} . The set of positive roots splits

$$(68) \quad \Delta^+ = \Delta^+(\mathfrak{g}_0) \sqcup \Delta^+(\mathfrak{p}_+)$$

where $\Delta^+(\mathfrak{g}_0)$ is a positive root system for \mathfrak{h}'' . Recall that all derivations of a semisimple Lie algebra \mathfrak{g} are inner. Observe that $X \mapsto jX$ for $X \in \mathfrak{g}_j$ and $-k \leq j \leq k$ is a gradation-preserving derivation of \mathfrak{g} . Then there exists an element $Z \in \mathfrak{g}$, the so-called *grading element*, such that $\text{ad}(Z)(X) = jX$ for all $X \in \mathfrak{g}_j$ and $-k \leq j \leq k$. Now Z commutes with all elements of \mathfrak{g}_0 , so that $Z \in \mathfrak{z}(\mathfrak{g}_0)$. Clearly, the grading on \mathfrak{g} corresponds to the decomposition into eigenspaces for the adjoint action of Z . If $\{H_1, \dots, H_n\} \subset \mathfrak{h}$ denotes the basis dual to $\Delta^0 = \{\alpha_1, \dots, \alpha_n\}$ and $\Sigma \subset \Delta^0$ is the subset corresponding to the grading on \mathfrak{g} , then the grading element writes as $Z = \sum_{i \in \Sigma} H_i$. For $\lambda \in \mathfrak{h}^*$, the *homogeneity* of λ is the real number $Z(\lambda)$. The homogeneity of λ , then, can be computed by writing λ as linear combination of simple roots and by taking the sum of the coefficients corresponding to roots in Σ .

3.1.2. Dynkin diagram notation. Let \mathfrak{g} be a complex semisimple Lie algebra. Fix a Cartan subalgebra and a set of positive roots Δ^+ . We represent the parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_\Sigma$, corresponding to the subset $\Sigma \subset \Delta^0$, by putting a cross on every node of the Dynkin diagram of \mathfrak{g} corresponding to a simple root in Σ . Removing the crossed nodes gives the Dynkin diagram for the semisimple Lie algebra \mathfrak{g}_0^{ss} . We also adopt the following notation to encode the weights of the representations of \mathfrak{g}_0^{ss} on the Dynkin diagram. Any such weight λ writes as a linear combination of fundamental weights ω_i defined by a simple root system for \mathfrak{g}_0^{ss} , hence with indices i corresponding to simple roots in $\Delta^0 \setminus \Sigma$. The coefficients of any such linear combination are nonnegative integers. We can thus represent $\lambda = \sum_{i \in \Delta^0 \setminus \Sigma} a_i \omega_i$ by putting the nonnegative integer a_i on the uncrossed node corresponding to the simple root α_i on the Dynkin diagram.

EXAMPLE 11. Let $\Delta^0 = \{\alpha_1, \dots, \alpha_4\}$ be a simple root system for $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$. The subset $\Sigma = \{\alpha_2, \alpha_3\} \subset \Delta^0$ is drawn on the Dynkin diagram as

$$\circ - \times - \times - \circ$$

Then, $\Sigma \subset \Delta^0$ corresponds to a choice of a grading on \mathfrak{g} . The dimension of $\mathfrak{z}(\mathfrak{g}_0)$ coincides with the number of crossed nodes denoting \mathfrak{p} on the Dynkin diagram of \mathfrak{g} , hence in our situation it has dimension two. Removing the crossed nodes from the Dynkin diagram of \mathfrak{g} , then, we obtain the Dynkin diagram of \mathfrak{g}_0^{ss} , which is thus isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

There are four distinct negative roots of Σ -height 1, namely

$$\{ \alpha_2, \alpha_1 + \alpha_2, \alpha_3, \alpha_3 + \alpha_4 \}.$$

Analogously,

$$\{ \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \}$$

are four distinct negative roots of Σ -height 2. Therefore, both \mathfrak{g}_{-1} and \mathfrak{g}_{-2} are complex vector spaces of dimension four and $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is the nilpotent graded Lie algebra generated by \mathfrak{g}_{-1} .

Let $\mathfrak{h} \subset \mathfrak{g}$ be the standard Cartan subalgebra of diagonal matrices, let $e_i: \mathfrak{h} \rightarrow \mathbb{C}$ be the linear functional associating to each $H \in \mathfrak{h}$ the i -th entry on the diagonal of H . Put $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, 4$ and denote by $\Delta^0 = \{\alpha_1, \dots, \alpha_4\}$ the standard simple root system for \mathfrak{g} . The grading

on \mathfrak{g} , seen as an algebra of matrices, can be explicitly described in terms of Δ^0 as

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^E & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^F & \mathfrak{g}_0 \end{pmatrix}$$

with \mathfrak{g}_0 divided in three square blocks of size two-one-two and $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^E \oplus \mathfrak{g}_{-1}^F$ is the direct sum of two-dimensional subspaces.

The highest root $\lambda_{\mathfrak{g}} = \omega_1 + \omega_4$ of \mathfrak{g} writes, in the Dynkin diagram notation, as

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & \times & \times & 0 \end{array}$$

From formula (66), we deduce the reflection of an arbitrary weight with respect to the i -th simple root, hence we see how to represent this operation in Dynkin diagram notation. Let a_i the coefficient on the i -th node of λ . The picture for $s_{\alpha_i}(\lambda)$ is obtained by λ changing the sign to a_i on the i -th node and by adding a_i to the coefficients on the adjacent nodes.

4. Completely reducible representations of \mathfrak{p}

Let \mathfrak{g} be a semisimple Lie algebra and let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subalgebra. Recall from Section 3 that $\mathfrak{p} \subset \mathfrak{g}$ corresponds to the choice of a grading on \mathfrak{g} such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is the direct sum of the nonnegative graded components. Every representation of \mathfrak{p} restricts to a representation of its subalgebra \mathfrak{g}_0 , which is reductive. A finite dimensional representation of a reductive Lie algebra \mathfrak{g}_0 is completely reducible if and only if $\mathfrak{z}(\mathfrak{g}_0)$ acts diagonalizable on it. Clearly, extending a completely reducible representation of \mathfrak{g}_0 trivially on \mathfrak{p}_+ , one gets a completely reducible representation of \mathfrak{p} . However, this is not an exceptional case:

THEOREM 13 ([10], p. 316). *Any completely reducible representation W of \mathfrak{p} is obtained by trivially extending a completely reducible representation of \mathfrak{g}_0 to \mathfrak{p} . Moreover, the grading element $Z \in \mathfrak{z}(\mathfrak{g}_0)$ acts by a scalar on each irreducible component of W .*

From now on, we will focus exclusively on representations of \mathfrak{p} which are completely reducible. We saw in (67) that $\Delta^0 \setminus \Sigma$ is a simple root system for a Cartan subalgebra $\mathfrak{h}'' \subset \mathfrak{g}_0^{ss}$. The irreducible representations of \mathfrak{g}_0^{ss} are therefore determined, in the sense of the Theorem of the highest weight, by weights $\sigma : \mathfrak{h}'' \rightarrow \mathbb{C}$ which are dominant and algebraically integral with respect to such simple system. We say that $\lambda \in \mathfrak{h}^*$ is \mathfrak{p} -dominant (respectively, \mathfrak{p} -algebraically integral) if $\langle \lambda, \alpha_i \rangle \geq 0$ for every $\alpha_i \in \Delta^0 \setminus \Sigma$ (respectively, if $\langle \lambda, \check{\alpha}_i \rangle$ is an integer for every $\alpha_i \in \Delta^0 \setminus \Sigma$). Let $\lambda \in \mathfrak{h}^*$ be a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight. Clearly, λ restricts to a linear functional $\lambda|_{\mathfrak{z}(\mathfrak{g}_0)} : \mathfrak{z}(\mathfrak{g}_0) \rightarrow \mathbb{C}$. Moreover, $\lambda|_{\mathfrak{h}''} : \mathfrak{h}'' \rightarrow \mathbb{C}$ is a linear functional dominant and algebraically integral with respect to $\Delta^0 \setminus \Sigma$. These two ingredients are equivalent to a unique irreducible representation of the reductive Lie algebra \mathfrak{g}_0 hence, by Theorem 13, to an irreducible representation of \mathfrak{p} . Since $\mathfrak{h} = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{h}''$, such a pair of linear functionals $\mathfrak{z}(\mathfrak{g}_0) \rightarrow \mathbb{C}$ and $\mathfrak{h}'' \rightarrow \mathbb{C}$ uniquely define a \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weight λ . We thus obtain the following

COROLLARY 7. *The isomorphism classes of finite-dimensional complex irreducible representations of \mathfrak{p} are in bijective correspondence with \mathfrak{p} -dominant and \mathfrak{p} -algebraically integral weights.*

Let $\lambda \in \mathfrak{h}^*$ a \mathfrak{p} -algebraically integral and \mathfrak{p} -dominant weight and let \mathbb{V} be the irreducible representation of \mathfrak{g}_0 of highest weight λ . Denote by $Z \in \mathfrak{z}(\mathfrak{g}_0)$ the grading element and by $Z(\lambda)$ the homogeneity of λ . Then, $Z(\lambda)$ is the factor scale describing the diagonalizable action of Z on \mathbb{V} . This can be computed by writing λ as linear combination of fundamental weights, then multiply the vector of coefficients for the inverse of the Cartan matrix and make the sum of all coefficients along the roots $\alpha_i \in \Sigma$, which correspond to the crossed nodes on the Dynkin diagram of λ .

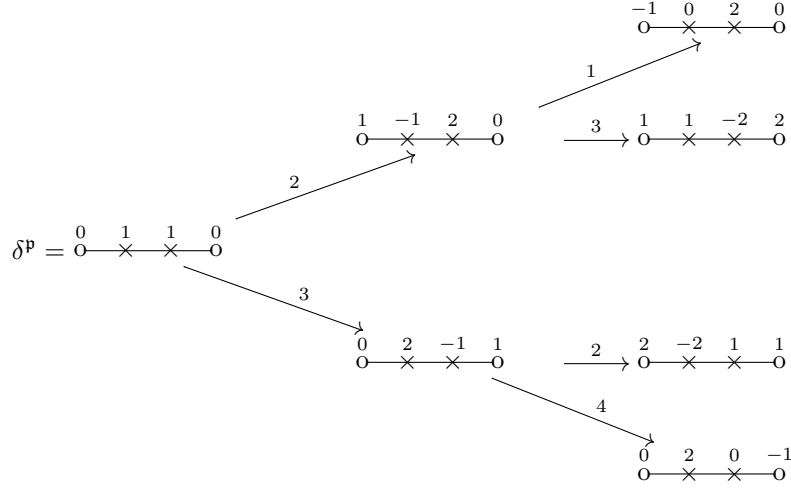
Recall that a subset $\Phi \subset \Delta^+$ is said to be *saturated* if, for every two elements $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Delta$, also $\alpha + \beta \in \Phi$. For $w \in W = W_{\mathfrak{g}}$, put $\Phi_w := w(\Delta^-) \cap \Delta^+$. Then $\Phi_w \subset \Delta^+$ is saturated with $\Delta^+ \setminus \Phi_w$ saturated and $w \mapsto \Phi_w$ is a bijection between W and the set of saturated subsets of Δ^+ having saturated complement in Δ^+ . The Weyl group $W_{\mathfrak{p}} := W(\mathfrak{g}_0^{ss}, \mathfrak{h}'')$ identifies with a subgroup of $W = W(\mathfrak{g}, \mathfrak{h})$, namely the subgroup generated by the simple reflections s_{α_i} for $\alpha_i \in \Delta^0 \setminus \Sigma$. It is possible to characterize $W_{\mathfrak{p}}$ in the picture of the bijection above. For this, recall the decomposition (68) of Δ^+ . In terms of the mentioned bijection, $W_{\mathfrak{p}}$ identifies with the subset of elements $w \in W$ such that $\Phi_w \subset \Delta^+(\mathfrak{g}_0)$. We define the *Hasse diagram* of \mathfrak{p} as the subset

$$W^{\mathfrak{p}} = \{ w \in W \mid \Phi_w \subset \Delta^+(\mathfrak{p}_+) \} \subset W$$

The Hasse diagram of \mathfrak{p} can be characterized as the subset $W^{\mathfrak{p}} \subset W$ of elements sending \mathfrak{g} -dominant weights to \mathfrak{p} -dominant weights. Indeed, since $w \in W$ acts orthogonally on \mathfrak{h}_0^* , for $\lambda \in \mathfrak{h}_0^*$ and $\alpha \in \Delta^+$ we have $\langle w(\lambda), \alpha \rangle = \langle \lambda, w^{-1}(\alpha) \rangle$. This identity shows that $w(\lambda)$ is a \mathfrak{p} -dominant weight for any \mathfrak{g} -dominant weight λ if and only if $w^{-1}(\alpha) \in \Delta^+$ for all $\alpha \in \Delta^+(\mathfrak{g}_0)$, hence if and only if $w \in W^{\mathfrak{p}}$. It thus follows from Corollary 7 that if λ is an algebraically integral and \mathfrak{g} -dominant weight and $w \in W^{\mathfrak{p}}$, there exists an irreducible representation of \mathfrak{p} of highest weight $w(\lambda)$. By Theorem 13, this can be also seen as an irreducible representation of \mathfrak{g}_0 on which $\mathfrak{z}(\mathfrak{g}_0)$ acts diagonalizable. We will denote by $W^{\mathfrak{p}}(r) \subset W^{\mathfrak{p}}$ the subset of elements of length r . We will see soon that $W^{\mathfrak{p}}(r)$ occurs in the description of the Lie algebra cohomology $H^r(\mathfrak{g}_-, \mathbb{V})$ given by Kostant's Theorem.

The Weyl group $W = W_{\mathfrak{g}}$ admits a structure of directed labeled graph. For $w, w' \in W$ we draw an arrow from w to w' if $l(w') = l(w) + 1$ and there exists a positive root α such that $w' = s_{\alpha}w$. Moreover, we put $w \leq w'$ if either $w = w'$ or there exists a finite sequence of arrows from w to w' . This defines the *Bruhat order*, a partial order on W with respect to which the comparable elements are those connected by a finite directed path in the graph defined above. This, in turn, induces a structure of directed graph on $W^{\mathfrak{p}}$. Let $\delta^{\mathfrak{p}} = \sum_{i \in \Sigma} \omega_i$ the lowest form adapted to the given standard parabolic. One can prove that mapping $w \in W$ to $w^{-1}(\delta^{\mathfrak{p}})$ restricts to a bijection between $W^{\mathfrak{p}}$ and the W -orbit of $\delta^{\mathfrak{p}}$. A graph of the orbit of $\delta^{\mathfrak{p}}$, hence of $W^{\mathfrak{p}}$, is thus obtained by applying simple reflections to $\delta^{\mathfrak{p}}$. To avoid redundant branches, we will not apply twice consequently the same simple reflection. From this graph, one can easily deduce $W^{\mathfrak{p}}$ from the graph of the orbit, namely as the composition of the simple reflections labelling the arrows in the opposite direction.

EXAMPLE 12. We perform the procedure to find $W^{\mathfrak{p}}$ in the case $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$ discussed in Example 11. Recall the properties of the standard simple root system $\Delta^0 = \{\alpha_1, \dots, \alpha_4\}$ from Example 8. The lowest form for the parabolic subalgebra is $\delta^{\mathfrak{p}} = \alpha_2 + \alpha_3$. The root reflection $s_{\alpha_i} : \mathfrak{h}_0^* \rightarrow \mathfrak{h}_0^*$ is given by $s_{\alpha_i}(\lambda) = \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i$ for all $\lambda \in \mathfrak{h}_0^*$, where $\check{\alpha}_i = 2 \frac{\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$. In Example 11, we explained how to represent weights and s_{α_i} in the Dynkin diagram notation. We will now represent the orbit of $\delta^{\mathfrak{p}}$ in the same notation. All elements of $W^{\mathfrak{p}}$ can be obtained as compositions of simple root reflections, whose action on the orbit of $\delta^{\mathfrak{p}}$ is free. Since $\langle \lambda, \check{\alpha}_i \rangle = 0$ if and only if $s_{\alpha_i}(\lambda) = \lambda$, at each step we only need to apply reflections with respect to simple roots which are not orthogonal to the weight λ from the previous step. Such roots correspond to those nodes in the Dynkin diagram of λ which are labelled with a nonzero coefficient. In order to avoid loops, it is enough to apply root reflections corresponding to nodes labelled with a strictly positive coefficient. We will not draw the whole graph for the orbit of $\delta^{\mathfrak{p}}$ below, performing only the first two steps will be enough for our purposes. Indeed, the reader will have enough informations to understand how the procedure works, and we will be able to deduce the element of $W^{\mathfrak{p}}$ of length one and two. Each arrow labelled by i represents to the action of s_{α_i} . Reading the labels on the arrows, we deduce that $W^{\mathfrak{p}}(1) = \{s_{\alpha_2}, s_{\alpha_3}\}$ and $W^{\mathfrak{p}}(2) = \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_2}s_{\alpha_3}, s_{\alpha_3}s_{\alpha_2}, s_{\alpha_3}s_{\alpha_4}\}$.



5. Lie algebra cohomology and Kostant's theorem

Let \mathfrak{g} be a finite dimensional Lie algebra and \mathbb{V} a finite dimensional representation of \mathfrak{g} . For $n \geq 0$ integer, define the space $C_n(\mathfrak{g}, \mathbb{V}) := \Lambda^n \mathfrak{g} \otimes \mathbb{V}$ of n -chains on \mathfrak{g} with coefficients in \mathbb{V} and the boundary operator $b : C_n(\mathfrak{g}, \mathbb{V}) \rightarrow C_{n-1}(\mathfrak{g}, \mathbb{V})$ by putting

$$\begin{aligned} b(X_1 \wedge \dots \wedge X_n \otimes v) &= \sum_{i=1}^n (-1)^i X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_n \otimes (X_i \cdot v) + \\ &+ \sum_{i < j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge \widehat{X}_j \wedge \dots \wedge X_n \otimes v \end{aligned}$$

A direct computation shows that $b^2 = 0$, hence that the differential defines a chain complex. One can thus define the quotient vector space

$$H_n(\mathfrak{g}, \mathbb{V}) := \frac{\text{Ker}(b : C_n(\mathfrak{g}, \mathbb{V}) \rightarrow C_{n-1}(\mathfrak{g}, \mathbb{V}))}{\text{Im}(b : C_{n+1}(\mathfrak{g}, \mathbb{V}) \rightarrow C_n(\mathfrak{g}, \mathbb{V}))}$$

known as the n -th homology group of the Lie algebra \mathfrak{g} with coefficients in \mathbb{V} . For $n \geq 0$ integer, define the space $C^n(\mathfrak{g}, \mathbb{V}) = \Lambda^n \mathfrak{g}^* \otimes \mathbb{V}$ of n -cochains on \mathfrak{g} with coefficients in \mathbb{V} and the coboundary operator $\partial : C^n(\mathfrak{g}, \mathbb{V}) \rightarrow C^{n+1}(\mathfrak{g}, \mathbb{V})$ via the formula

$$\begin{aligned} \partial \Phi(X_0, \dots, X_n) &= \sum_{i=0}^n (-1)^i X_i \cdot \Phi(X_0, \dots, \widehat{X}_i, \dots, X_n) + \\ (69) \quad &+ \sum_{i < j} (-1)^{i+j} \Phi([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_n) \end{aligned}$$

A direct computation shows that $\partial^2 = 0$, hence that the differential defines a cochain complex. One can thus define the quotient vector space

$$H^n(\mathfrak{g}, \mathbb{V}) := \frac{\text{Ker}(\partial : C^n(\mathfrak{g}, \mathbb{V}) \rightarrow C^{n+1}(\mathfrak{g}, \mathbb{V}))}{\text{Im}(\partial : C^{n-1}(\mathfrak{g}, \mathbb{V}) \rightarrow C^n(\mathfrak{g}, \mathbb{V}))}$$

known as the n -th cohomology group of the Lie algebra \mathfrak{g} with coefficients in \mathbb{V} .

5.1. Kostant's theorem. Suppose that $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is a $|k|$ -grading on a semisimple Lie algebra, corresponding to the choice of parabolic subalgebra $\mathfrak{p} = \bigoplus_{i=0}^k \mathfrak{g}_i = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ of \mathfrak{g} . As usual, denote by $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ the negative graded part of the grading. The Killing form of \mathfrak{g} gives an isomorphism $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$ of \mathfrak{p} -modules and an isomorphism $(\mathfrak{g}_-)^* \cong \mathfrak{p}_+$ of \mathfrak{g}_0 -modules. Clearly, $\mathfrak{g}/\mathfrak{p}$ and \mathfrak{g}_- identify as \mathfrak{g}_0 -modules, but only $\mathfrak{g}/\mathfrak{p}$ carries the structure of a \mathfrak{p} -module.

Let \mathbb{V} be a finite-dimensional representation of \mathfrak{g} . Both \mathfrak{g}_0 and \mathfrak{g}_- are subalgebras in \mathfrak{g} , so that \mathbb{V} can be also seen as a representation of \mathfrak{g}_0 and of \mathfrak{g}_- . It thus makes sense to consider the Lie algebra differential $\partial : C^n(\mathfrak{g}_-, \mathbb{V}) \rightarrow C^{n+1}(\mathfrak{g}_-, \mathbb{V})$ introduced in the previous section. The spaces of cochains carry a natural \mathfrak{g}_0 -module structure and ∂ can be easily seen to be a \mathfrak{g}_0 -homomorphism. Recall that ∂ defines the cohomology group $H^*(\mathfrak{g}_-, \mathbb{V})$ of \mathfrak{g}_- with coefficients in \mathbb{V} , which is naturally a \mathfrak{g}_0 -module. On the other hand, also $\mathfrak{p}_+ \subset \mathfrak{g}$ is a subalgebra and we can thus consider \mathbb{V}^* as a \mathfrak{p}_+ -module and the cohomology complex $(C^*(\mathfrak{p}_+, \mathbb{V}^*), \partial_{\mathfrak{p}_+})$ defining $H^*(\mathfrak{p}_+, \mathbb{V}^*)$. Now $\partial_{\mathfrak{p}_+}$ are \mathfrak{p} -homomorphisms and the maps ∂^* dual to $\partial_{\mathfrak{p}_+}$ write as

$$\partial^* : C_{n+1}(\mathfrak{p}_+, \mathbb{V}) \rightarrow C_n(\mathfrak{p}_+, \mathbb{V})$$

and they can be explicitly computed (see [10], p. 341). The explicit formula shows that $\partial^* = b$ is the boundary operator of the homology complex defining $H_*(\mathfrak{p}_+, \mathbb{V})$. Using the isomorphism $(\mathfrak{g}_-)^* \cong \mathfrak{p}_+$ given by the Killing form, we can look at the maps ∂^* as \mathfrak{g}_0 -homomorphisms $\partial^* : C^{n+1}(\mathfrak{g}_-, \mathbb{V}) \rightarrow C^n(\mathfrak{g}_-, \mathbb{V})$. One can put a positive definite scalar product on every space $C^n(\mathfrak{g}_-, \mathbb{V})$ such that ∂ and ∂^* are adjoint and define the *Kostant Laplacian*

$$\square_n := \partial \circ \partial^* + \partial^* \circ \partial : C^n(\mathfrak{g}_-, \mathbb{V}) \rightarrow C^n(\mathfrak{g}_-, \mathbb{V}).$$

This leads to the *Hodge decomposition*

$$C^n(\mathfrak{g}_-, \mathbb{V}) = \text{Im}(\partial^*) \oplus \text{Ker}(\square_n) \oplus \text{Im}(\partial)$$

as direct sum of \mathfrak{g}_0 -invariant submodules. Here, the sum of the first two summands coincides with $\text{Ker}(\partial^*)$, while the sum of the second and the third summands gives $\text{Ker}(\partial)$. This shows the existence of a natural identification of \mathfrak{g}_0 -modules

$$H^n(\mathfrak{g}_-, \mathbb{V}) = \text{Ker}(\partial)/\text{Im}(\partial) \cong \text{Ker}(\square_n) \cong \text{Ker}(\partial^*)/\text{Im}(\partial^*) = H_n(\mathfrak{p}_+, \mathbb{V}),$$

which endows each space $H^n(\mathfrak{g}_-, \mathbb{V})$ with the structure of a \mathfrak{p} -module on which \mathfrak{p}_+ acts trivially. Equivalently, $H^n(\mathfrak{g}_-, \mathbb{V})$ is a completely reducible \mathfrak{g}_0 -module. Clearly, the identification above also rewrites as an isomorphism $H^n(\mathfrak{g}_-, \mathbb{V}) \cong (H^n(\mathfrak{p}_+, \mathbb{V}^*))^*$ of \mathfrak{g}_0 -modules. The highest weight of a \mathfrak{g}_0 -irreducible component contained in $H^n(\mathfrak{p}_+, \mathbb{V}^*)$ is thus the lowest weight of a \mathfrak{g}_0 -irreducible component in $H^n(\mathfrak{g}_-, \mathbb{V})$.

The decomposition into \mathfrak{g}_0 -irreducible components for these cohomologies can be made explicit. Clearly, an analogous construction as above defines a \mathfrak{g}_0 -endomorphism \square_n of $C^n(\mathfrak{p}_+, \mathbb{V})$, hence an isomorphism $H^n(\mathfrak{p}_+, \mathbb{V}) \cong \text{Ker}(\square_n)$ saying how $H^n(\mathfrak{p}_+, \mathbb{V})$ sits inside $C^n(\mathfrak{p}_+, \mathbb{V})$. It is clear from the dualities above that the descriptions of $H^n(\mathfrak{g}_-, \mathbb{V})$ and $H^n(\mathfrak{p}_+, \mathbb{V})$ thus obtained are equivalent.

Let \mathbb{V} be a finite-dimensional representation of \mathfrak{g} . Consider the isomorphism of \mathfrak{g}_0 -modules $H^n(\mathfrak{p}_+, \mathbb{V}) \cong \text{Ker}(\square_n)$. The \mathfrak{g}_0 -actions on \mathfrak{p}_+ and on \mathbb{V} induce a \mathfrak{g}_0 -action on the set of alternating multilinear maps $\mathfrak{p}_+ \times \cdots \times \mathfrak{p}_+ \rightarrow \mathbb{V}$, hence on the tensor product $C^n(\mathfrak{p}_+, \mathbb{V}) = \Lambda^n \mathfrak{p}_+^* \otimes \mathbb{V}$. The weights of the induced representation on $C^n(\mathfrak{p}_+, \mathbb{V})$ write as difference between weights of \mathbb{V} and of \mathfrak{p}_+ . One can compute the action of the Laplacian \square_n on irreducible components of $C^n(\mathfrak{p}_+, \mathbb{V})$. The weights ν of the \mathfrak{g}_0 -module $H^n(\mathfrak{p}_+, \mathbb{V})$ correspond to the zero eigenvalue for the Laplacian \square_n , thus they can be expressed as solution of an equation involving the scalar product induced by the Killing form on the set of weights. Writing down the details leads to the following result, known as the Kostant's version of the Bott-Borel-Weil Theorem.

THEOREM 14 ([10], p. 351). *Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ a standard parabolic subalgebra, $W^{\mathfrak{p}}$ the Hasse diagram of the parabolic \mathfrak{p} and δ the lowest form of \mathfrak{g} . Suppose that \mathbb{V} is a finite-dimensional irreducible representation of \mathfrak{g} with highest weight λ and that ν is a \mathfrak{g}_0 -dominant weight. Denote by $H^*(\mathfrak{p}_+, \mathbb{V})^\nu$ the isotypical component of highest weight ν for the natural \mathfrak{g}_0 -representation on the cohomology. Then*

- (1) $H^*(\mathfrak{p}_+, \mathbb{V})^\nu \neq \{0\}$ if and only if there exists an element $w \in W^{\mathfrak{p}}$ such that $\nu = \nu_w := w \cdot \lambda$, where $w \cdot \lambda = w(\lambda + \delta) - \delta$ denotes the affine action on the set of weights of \mathbb{V} .
- (2) For any $w \in W^{\mathfrak{p}}$, $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ is irreducible and even the multiplicity of ν_w as a weight of $C^*(\mathfrak{p}_+, \mathbb{V})$ is one.
- (3) For $w \in W^{\mathfrak{p}}$, $H^*(\mathfrak{p}_+, \mathbb{V})^{\nu_w}$ is contained in the space $H^{l(w)}(\mathfrak{p}_+, \mathbb{V})$, where $l(w)$ denotes the length of w .

All informations about cohomologies can thus be deduced by analyzing the $W^{\mathfrak{p}}$ -orbits for the affine action on the set of weights. The computation of the cohomology $H^*(\mathfrak{g}_-, \mathfrak{g})$ can be always reduced to the case of \mathfrak{g} simple. At this stage, a finite number of cases should be taken in exam. In this way one gets a complete list, showing that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in nonpositive homogeneity, apart from few exceptional cases. The second cohomology groups $H^2(\mathfrak{g}_-, \mathfrak{g})$ can be similarly classified. [23] classifies the gradings such that $H^2(\mathfrak{g}_-, \mathfrak{g})$ is not concentrated in nonpositive homogeneity.

An important application of Kostant's theorem to parabolic geometries is the decomposition of $H^2(\mathfrak{g}_-, \mathfrak{g})$, since this encodes the admissible values for the harmonic curvature. If \mathfrak{g} is a simple Lie algebra, it acts irreducibly on itself under the adjoint representation. There is just one orbit for the affine action of the Weyl group, namely the orbit of the highest root $\lambda_{\mathfrak{g}}$ of the simple Lie algebra \mathfrak{g} . Let $Z \in \mathfrak{z}(\mathfrak{g}_0)$ be the grading element. For $\lambda \in \mathfrak{h}^*$, the *homogeneity* of λ is the real number $Z(\lambda)$. The homogeneity of λ is computed by writing λ as linear combination of simple roots and by taking the sum of the coefficients corresponding to roots in Σ . Consider the following subsets of $W^{\mathfrak{p}}(r)$:

$$W_t^{\mathfrak{p}}(r) := \{w \in W^{\mathfrak{p}}(r) \mid Z(-w \cdot \lambda_{\mathfrak{g}}) = t\}, \quad W_+^{\mathfrak{p}}(r) := \{w \in W^{\mathfrak{p}}(r) \mid Z(-w \cdot \lambda_{\mathfrak{g}}) \geq 1\}$$

The results of Kostant's Theorem write as follows:

$$H^r(\mathfrak{g}_-, \mathfrak{g})_t = \bigoplus_{w \in W_t^{\mathfrak{p}}(r)} V_{-w \cdot \lambda_{\mathfrak{g}}}, \quad H^r(\mathfrak{g}_-, \mathfrak{g})^1 = \bigoplus_{w \in W_+^{\mathfrak{p}}(r)} V_{-w \cdot \lambda_{\mathfrak{g}}}$$

where $w \cdot \lambda_{\mathfrak{g}}$ denotes the affine action of W on \mathfrak{h}^* . Otherwise put, the irreducible components of $H^r(\mathfrak{g}_-, \mathfrak{g})_t$ are indexed on $W_t^{\mathfrak{p}}(r)$. For regular parabolic geometries, the harmonic curvature takes values in the \mathfrak{g}_0 -submodule $H^2(\mathfrak{g}_-, \mathfrak{g})^1 \subset H^2(\mathfrak{g}_-, \mathfrak{g})$ of positive homogeneity.

EXAMPLE 13. In Example 12, we computed $W^{\mathfrak{p}}(1)$ and $W^{\mathfrak{p}}(2)$ for $\mathfrak{p} \subset \mathfrak{g} = \mathfrak{sl}(5, \mathbb{C})$ discussed in Example 11. The highest root $\lambda_{\mathfrak{g}} = \omega_1 + \omega_4$ and the lowest form $\delta = \omega_1 + \omega_2 + \omega_3 + \omega_4$ of \mathfrak{g} . Using formula (66) for $s_{\alpha_i}(\omega_i)$ and the fact that $s_{\alpha_i}(\omega_j) = \omega_j$ for $i \neq j$, we compute $w \cdot \lambda_{\mathfrak{g}}$ for $w \in W^{\mathfrak{p}}(1)$:

$$\begin{aligned} s_{\alpha_2} \cdot \lambda_{\mathfrak{g}} &= s_{\alpha_2}(\lambda_{\mathfrak{g}} + \delta) - \delta = +2\omega_1 + s_{\alpha_2}(\omega_2) + \omega_3 + 2\omega_4 - \delta = +2\omega_1 - 2\omega_2 + \omega_3 + \omega_4 \\ s_{\alpha_3} \cdot \lambda_{\mathfrak{g}} &= s_{\alpha_3}(\lambda_{\mathfrak{g}} + \delta) - \delta = +2\omega_1 + \omega_2 + s_{\alpha_3}(\omega_3) + 2\omega_4 - \delta = +\omega_1 + \omega_2 - 2\omega_3 + 2\omega_4 \end{aligned}$$

A direct computation shows that $W_+^{\mathfrak{p}}(1) = \{0\}$, hence $H^1(\mathfrak{g}_-, \mathfrak{g})^1 = \{0\}$. Similarly, we can compute the affine action $w \cdot \lambda_{\mathfrak{g}}$ for $w \in W^{\mathfrak{p}}(2) = \{s_{\alpha_2}s_{\alpha_1}, s_{\alpha_2}s_{\alpha_3}, s_{\alpha_3}s_{\alpha_2}, s_{\alpha_3}s_{\alpha_4}\}$, thus obtaining

the following weights:

$$\begin{array}{cccc}
 0 & -4 & 3 & 1 \\
 \circ & \times & \times & \circ \\
 1 & 3 & -4 & 0 \\
 \circ & \times & \times & \circ \\
 3 & -3 & 0 & 2 \\
 \circ & \times & \times & \circ \\
 2 & 0 & -3 & 3 \\
 \circ & \times & \times & \circ
 \end{array}$$

By Kostant's Theorem, these are the highest weights of the irreducible \mathfrak{g}_0 -representations, whose direct sum coincides with the \mathfrak{g}_0 -module $H^2(\mathfrak{p}_+, \mathfrak{g})$. Again, a direct computation shows that the opposite weights have all homogeneity one. More precisely, each weight can be represented as linear combination of simple roots, by multiplying the vector of coefficients above by the inverse of the Cartan matrix. The homogeneity of each weight, then, is the sum of the components along α_2 and α_3 . In this way one can verify that $W^{\mathfrak{p}}(2) = W_1^{\mathfrak{p}}(2)$, which implies that the second cohomology group is concentrated in homogeneity one. In formulas $H^2(\mathfrak{g}_-, \mathfrak{g}) = H^2(\mathfrak{g}_-, \mathfrak{g})_1$, which means that the harmonic curvature corresponds to torsion in homogeneity one.

Kostant's Theorem describes the cohomology for \mathfrak{g} complex graded semisimple. Relating the cohomologies associated to real graded semisimple and their complexifications leads to results for the real case. Let $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a real graded Lie algebra. The grading on \mathfrak{g} induces the grading $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_{-k}^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{g}_k^{\mathbb{C}}$ on the complexification of \mathfrak{g} . From the definition of the Lie algebra cohomology, one easily deduces the following

PROPOSITION 26. *Let \mathfrak{g} be a real graded semisimple Lie algebra with complexification $\mathfrak{g}^{\mathbb{C}}$.*

- (1) *Let V be a complex representation of \mathfrak{g} . Then, the real cohomology spaces $H_{\mathbb{R}}^*(\mathfrak{g}_-, V)$ are naturally complex vector spaces and $H_{\mathbb{R}}^*(\mathfrak{g}_-, V) \cong H_{\mathbb{C}}^*(\mathfrak{g}_-^{\mathbb{C}}, V)$ as a module over $\mathfrak{g}_0 \subset \mathfrak{g}_0^{\mathbb{C}}$.*
- (2) *If V is a real representation of \mathfrak{g} , then we have*

$$H_{\mathbb{C}}^*(\mathfrak{g}_-^{\mathbb{C}}, V \otimes \mathbb{C}) \cong H_{\mathbb{R}}^*(\mathfrak{g}_-, V) \otimes \mathbb{C}$$

EXAMPLE 14. Let $\mathfrak{g} = \oplus_{i=-2}^2 \mathfrak{g}_i$ be a real graded Lie algebra such that $\mathfrak{g}^{\mathbb{C}} = \oplus_{i=-2}^2 \mathfrak{g}_i^{\mathbb{C}}$ is the grading on $\mathfrak{sl}(5, \mathbb{C})$ described in Example 1. Then, Proposition 26 shows that $H_{\mathbb{C}}^*(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ is the complexification of $H_{\mathbb{R}}^*(\mathfrak{g}_-, \mathfrak{g})$. From the analysis of the weights and using Kostant's Theorem, one can verify that $H^1(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})^0 = \{0\}$ (see Example 13 of Appendix A). We thus also have that $H^1(\mathfrak{g}_-, \mathfrak{g})^0 = \{0\}$. Moreover, we have the isomorphism

$$H_{\mathbb{C}}^2(\mathfrak{g}_-^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}}) \cong H_{\mathbb{R}}^2(\mathfrak{g}_-, \mathfrak{g}) \otimes \mathbb{C}.$$

The condition $H^1(\mathfrak{g}_-, \mathfrak{g})_0 = \{0\}$ is equivalent to the fact that $\mathfrak{g}_0 \cong \mathfrak{der}_{gr}(\mathfrak{g}_-)$ via the adjoint action.

6. Cartan geometries

Let G be a Lie group and $H \subset G$ a closed subgroup. The projection $G \rightarrow G/H$ on the quotient is the simplest example of principal bundle with structure group H . The quotient space G/H is usually said to be an *homogeneous space*. Recall that the tangent space of a Lie group looks like the same in every point. The trivialization of TG via left-translations is encoded by a one-form $\omega_G \in \Omega^1(G, \mathfrak{g})$ on G , known as *Maurer–Cartan form*. In particular, ω_G satisfies the Maurer–Cartan equation:

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$$

The automorphisms of the homogeneous model are group automorphisms of G which lift automorphisms of G/H , namely

$$\text{Aut}(G, \omega_G) = \{\phi : G \rightarrow G \mid \phi(gh) = \phi(g)h \text{ } g \in G, h \in H, \phi^* \omega_G = \omega_G\}$$

One can show that $\text{Aut}(G, \omega_G) = G$. The notion of Cartan geometry is introduced to encode, at the same time, informations about a geometric space and about the diffeomorphisms preserving the structure of the space.

DEFINITION 21. *Let G be a Lie group with Lie algebra \mathfrak{g} and $H \subset G$ be a Lie subgroup in it. A Cartan geometry of type (G, H) on a manifold M is a H -principal fiber bundle $p : \mathcal{G} \rightarrow M$, endowed with a \mathfrak{g} -valued one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying the following conditions:*

- $(r^h)^*\omega = \text{Ad}(h^{-1}) \circ \omega$ for all $h \in H$;
- $\omega(\zeta_X(u)) = X$ for all $u \in \mathcal{G}$ and $X \in \mathfrak{g}$;
- $\omega_u : T_u\mathcal{G} \rightarrow \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

We will also say that ω is H -equivariant, it reproduces the vertical vector fields and, finally, that it trivializes the tangent bundle $T\mathcal{G}$ (or that it defines an absolute parallelism). In particular, $TM \cong \mathcal{G} \times_H (\mathfrak{g}/\mathfrak{h})$, where the representation $H \rightarrow GL(\mathfrak{g}/\mathfrak{h})$ is induced by the restriction to H of the adjoint representation.

The curvature form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ of a Cartan geometry $(p : \mathcal{G} \rightarrow M, \omega)$ is defined, for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$, by

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

In this language, saying that the homogeneous model $(G \rightarrow G/H, \omega_G)$ satisfies the Maurer–Cartan equation exactly means that has zero curvature. Through the isomorphism given by the Cartan connection, one can rather look at the curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2\mathfrak{g}^* \otimes \mathfrak{g}$, defined by $\kappa_u(X, Y) = K(\omega_u^{-1}(X), \omega_u^{-1}(Y))$ for all $X, Y \in \mathfrak{g}$. Since K annihilates by insertion of vertical vector fields, we can look at the curvature function as a map $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$. The restriction of the adjoint action $\text{Ad} : \mathfrak{h} \rightarrow GL(\mathfrak{g})$ induces an action on $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ in a natural way. One can verify that κ is equivariant with respect to this natural action.

A morphism between two Cartan geometries $(p : \mathcal{G} \rightarrow M, \omega)$, $(p' : \mathcal{G}' \rightarrow M', \omega')$ of same type (G, H) is a principal bundle morphism $\phi : \mathcal{G} \rightarrow \mathcal{G}'$ such that $\phi^*\omega' = \omega$. The set of automorphisms of a Cartan geometry, in particular, is a group and we denote it by $\text{Aut}(\mathcal{G}, \omega)$. It has dimension at most equal to the dimension of G and they are equal if and only if the Cartan geometry is locally isomorphic to the homogeneous model. The following result concerns the role of fundamental invariant played by the curvature in the description of Cartan geometries.

PROPOSITION 27 ([10], p. 74). *The curvature of a Cartan geometry $(p : \mathcal{G} \rightarrow M, \omega)$ is identically zero if and only if for every $x \in M$, there exists an open neighborhood U such that $(p^{-1}(U) \rightarrow U, \omega)$ is isomorphic to the restriction of the homogeneous model $(G \rightarrow G/H, \omega_G)$ to an open neighborhood of the identity.*

Zusammenfassung

Sei M eine endlich-dimensionale glatte Mannigfaltigkeit. Bezeichne mit $H \subset TM$ eine glatte Distribution und mit $[H, H] \subset TM$ jenes Teilbündel, das von Lie Klammern erster Ordnung von Schnitten von H erzeugt wird. Man nennt H *in einem Schritt klammererzeugend* wenn

$$TM = H + [H, H].$$

Die Hauptresultate in der Dissertation betreffen generische, klammererzeugende Distributionen vom Rang 4. Zunächst wird eine vollständige Klassifikation ihrer Typen bewiesen. Diese zeigt, dass solche Distributionen genau in den Dimensionen $5 \leq n \leq 10$ existieren, und wir betrachten die Fälle $n = 8$ und $n = 9$, da diese in der Literatur noch nicht behandelt wurden. Für $(4, 8)$ -Distributionen beweisen wir, dass es zwei verschiedene generische Typen gibt, die beide zu einer normalen parabolischen Geometrie äquivalent sind. Aufgrund dieser Äquivalenz können wir die lokalen Invarianten dieser Struktur aus der harmonischen Krümmung der korrespondierenden parabolischen Geometrie herauslesen, was eine vollständige Beschreibung bezüglich der lokalen Geometrie liefert. Dieselben Werkzeuge aus der Theorie der parabolischen Geometrie liefern eine Anwendung in der Beschreibung von submaximalen symmetrischen Modellen von generischen $(4, 8)$ -Distributionen vom hyperbolischen Typ.

Unter der Verwendung von Resultaten von Tanaka und Morimoto über die Verlängerungsprozedur, die eine bestimmte Konstruktion für klassische G -Strukturen auf Filtrierungen verallgemeinert, zeigen wir, dass der $(4, 9)$ -Fall der niedrigstdimensionale in einer abzählbaren Serie von in einem Schritt klammererzeugenden Distributionen ist, die eine kanonische lineare Konnexion auf dem Tangentialbündel TM bestimmt. Die Torsion und die Krümmung dieser Konnexion sind dann lokale Invarianten für diese Strukturen.

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