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„One-Loop Corrections to the Seesaw Mechanism and Models of Neutrino Masses and Lepton Mixing“

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Abstract

The observations of neutrino oscillations verify the existence of small but nonzero neutrino masses. However, in the Standard Model of particle physics neutrinos are assumed massless, since only left-handed neutrinos and right-handed antineutrinos can be observed in weak decays and cross sections. The subject of this master thesis is the implementation of neutrino masses in the Standard Model by means of the seesaw mechanism of type I. The particle content of the Standard Model is being extended by heavy right-handed neutrinos and additional scalar doublets. After a general discussion on Dirac, Majorana and hybrid mass terms, the seesaw mechanism type I is formulated for an arbitrary number of left and right-handed neutrinos and for an arbitrary number of scalar doublets. Since even after this procedure some left-handed neutrinos might remain massless, radiative corrections to the seesaw mechanism, in particular one-loop corrections, are considered. After a general discussion of dominant one-loop corrections to the seesaw mechanism, this knowledge is applied to two different models. In the first model, which is based on the standard gauge group $SU(2) \times U(1)$, the mass correction for the left-handed neutrinos is derived explicitly. Afterwards, the special case of a minimal extension within this model is considered. In the second model, the so-called scotogenic model, an additional exact $\mathbb{Z}_2$-symmetry is introduced, which causes the left-handed neutrinos to acquire mass only at one-loop level. After the discussion of the minimal extension, this model is generalized for arbitrary numbers of left and right-handed neutrinos, as well as a scalar doublet. In particular, the correspondence of non-vanishing neutrino masses and terms in the scalar potential is outlined.

Zusammenfassung

Die Beobachtungen von Neutrinooszillationen belegen, dass Neutrinos sehr kleine, aber von Null verschiene Massen haben. Im Standardmodell der Teilchenphysik werden Neutrinos jedoch als masselos angenommen, da in schwachen Zerfällen und Wirkungsquerschnitten nur linkshändige Neutrinos und rechtshändige Antineutrinos beobachtet werden können. In dieser Masterarbeit wird die Implementierung von Neutrinomassen im Standardmodell mit Hilfe des Seesaw-Mechanismus vom Typ I behandelt. Hierbei werden die Teilchen im Standardmodell durch schwere rechtshändige Neutrinos sowie durch zusätzliche skalare Doubletts erweitert. Nach einer einführenden Diskussion über Dirac-, Majorana-, sowie Hybridmassenterme wird der Seesaw-mechanismus für beliebige Anzahlen von linkshändigen und rechtshändigen Neutrinos, sowie für eine beliebige Anzahl von skalaren Doubletts formuliert. Da im Allgemeinen durch diese Methode nicht alle linkshändigen Neutrinos Masse erhalten, werden anschließend radiative Korrekturen im Rahmen von Einschleifen-Korrekturen zum diskutiert. Im Anschluß an die allgemeine Diskussion von dominanten Einschleifenkorrekturen zum Seesaw-mechanismus werden diese Resultate in zwei verschiedenen Modellen angewendet. Im ersten Modell, welches auf der Standardeichgruppe $SU(2) \times U(1)$ basiert, wird die Massenkorretur der linkshändigen Neutrions explizit berechnet, sowie der Spezialfall der minimalen Erweiterung des Standardmodells betrachtet. Im zweiten Modell wird eine zusätzliche exakte $\mathbb{Z}_2$-symmetrie eingeführt, wodurch die Neutrinos ausschließlich durch die Einschleifen-Korrekturen Masse erhalten. In Anschluß an den Spezialfall der minimalen Erweiterung wird auch für dieses Modell eine Verallgemeinerung für beliebige links- und rechtshändige Neutrions sowie für beliebig viele skalare Doubletts formuliert. Insbesondere wird der Zusammenhang zwischen nichtverschwindenden Neutrinomassen und Termen im skalaren Potential hergestellt.

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Contents

Introduction

In the Standard Model of particle physics neutrinos are assumed massless, since in electroweak processes only left-handed neutrinos and right-handed antineutrinos occur. Nevertheless, in the last two decades numerous neutrino experiments confirmed the theory of neutrino oscillations [1, 2], like for example the Super-Kamiokande [3], the Homestake [4], the SAGE [5], and SNO [6] experiments and many others [7, 8, 9, 10, 11, 12, 13, 14, 15]. Therefore, neutrinos cannot be massless and lepton mixing must exist. Hence, the Standard Model has to be extended in order to include the phenomena of massive neutrinos. Throughout the years many theoretical suggestions have been made and one of those shall be the topic of the master thesis at hand. The seesaw mechanism of type I [16, 17, 18] is a very promising framework not only for implementing non zero neutrino masses into the Standard Model, but also for reproducing the smallness of these masses. For applying this mechanism, the Standard Model is extended by right-handed neutrino fields, which are massive but sterile. This means they do not take part in any weak interactions. Furthermore, in this extension of the Standard Model additional scalar doublets are allowed. This model exhibits three mass scales; the small mass scale below the electroweak scale, the electroweak scale itself (~ 100 GeV) and the even larger seesaw scale.

The extension with right-handed neutrino gauge singlets gives rise to neutrino mass terms and the magnitude of the light neutrino masses will be reciprocal to the heavy seesaw mass scale introduced by the right-handed neutrinos. In addition radiative corrections to the seesaw mechanism are considered, where one-loop corrections lead to additional light neutrino masses.

This master thesis starts with an introductory section, where a brief overview of the Standard Model of particle physics is presented. In particular the electroweak unification and the Higgs mechanism are discussed. In the second section the principle of different possible neutrino mass terms is considered, i.e. the Dirac, the Majorana and the hybrid Dirac-Majorana mass term. For each of these mass terms consequences of lepton (neutrino) mixing and lepton number conservation are investigated. The main part is treated in the third section, where the seesaw mechanism type I is employed for arbitrary numbers of left and right-handed neutrinos and afterwards also for an arbitrary number of Higgs doublets [19]. In the last section one-loop corrections to the seesaw masses are calculated for the most general model [20]. Furthermore, two special models are considered. The first one [21, 22] is a minimal extension of the Standard Model with one right-handed neutrino and one additional scalar doublet, whereas the second one [23] exhibits an additional discrete symmetry, which leads to massless neutrinos on tree level. Finally this second model will be generalized for arbitrary numbers of additional scalar doublets.

1 The Standard Model of Particle Physics

1.1 Fundamentals

The Standard Model (SM) of particle physics is a theory to describe fundamental particles and all their interactions, except gravity. The particle content of the SM can be divided into three groups - leptons, quarks, and mediators, which are listed in figure 1 below.

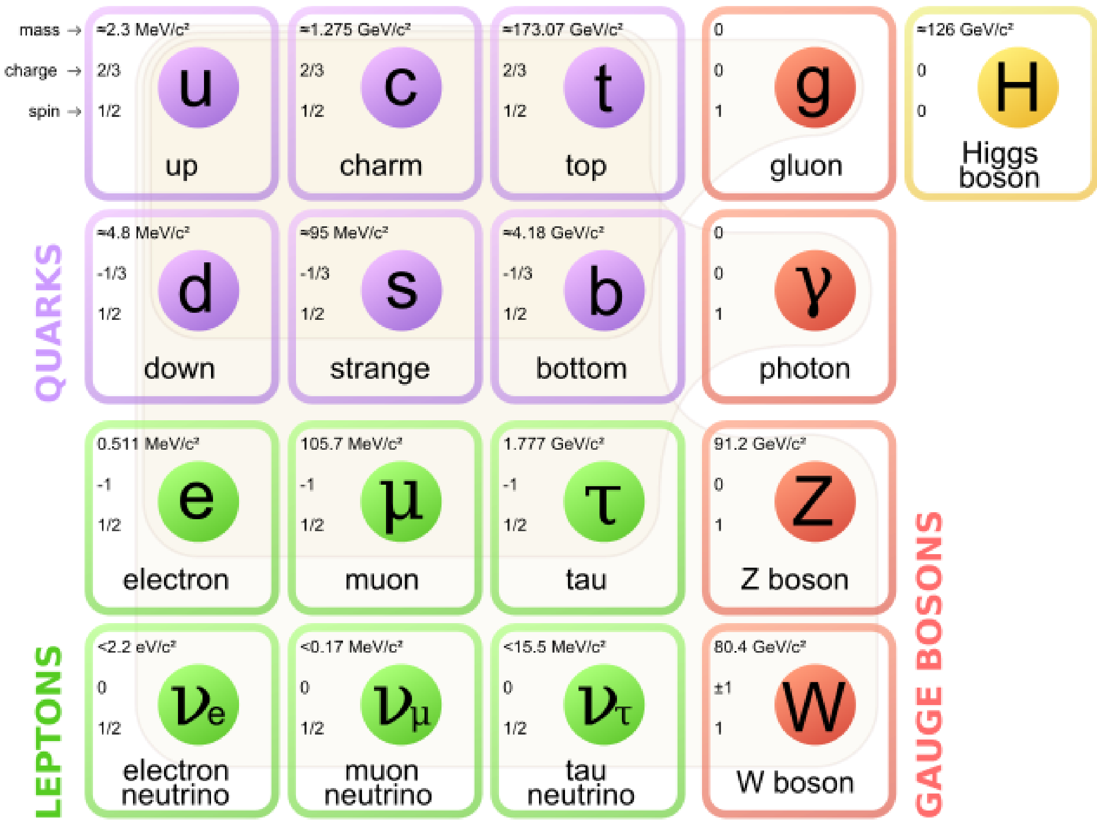


Figure 1: Fundamental particles of the Standard Model. Reprinted from [24].

There are three generations or families of quarks and leptons (first three columns), where one generation is consisting of an up-type and a down-type quark, a charged lepton and its corresponding neutrino as well as one antiparticle for each of them. In addition each quark and antiquark comes in three different colors. Therefore, the particle content per generation sums up to sixteen fundamental particles. Equal particle types of different generations differ by their flavor quantum numbers and their masses, which increase going to higher generations.

Furthermore, the four types of gauge bosons are listed in the forth column which mediate the electromagnetic (γ), weak ($W^{\pm}, Z^0$) and strong interaction ($g^a, a = 1, \ldots, 8$). In addition there is the Higgs boson, which gives mass to the fundamental particles through the Higgs mechanism. So finally we end up with 12 leptons, 36 quarks, 12 mediators and at least one Higgs boson, which gives a minimum of 61 fundamental particles [25, p.48].

The SM is a relativistic quantum field theory (QFT) or, more precisely, it is a collection of related theories. It contains a description of the electromagnetic forces in terms of quantum electrodynamics (QED), the Glashow-Weinberg-Salem (GWS) theory of electroweak processes, and the theory of quantum chromodynamics (QCD) to describe the strong nuclear forces. All those fundamental interactions are derived from the requirement of local gauge invariance [25, p.3].

In QFT one achieves a uniform description of particles and interaction forces via fields. Quarks and leptons can be seen as field quanta, i.e. excited states of corresponding particle fields. Forces which are already described as vector fields in classical field theories can be represented in terms of field quanta, called gauge bosons. The Lagrangians and equations of motions of the SM can be formulated in terms of these fields.

1.2 Fields and Field Equations

As mentioned above in QFT all particles and interaction forces are described as fields. In this section all three types of fields are introduced as well as their field equations. Since QFT is a relativistic theory, all equations of motion need to be Lorentz invariant[1], i.e. the corresponding Lagrangians need to be Lorentz scalars [26, p.35].

1.2.1 Klein-Gordon Equation

This equation describes the kinematics of a free spin 0 particle, i.e. a scalar field ϕ like for example the Higgs field. Scalar fields behave under Lorentz transformations Λ like

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x), \tag{1.1}$$

and their gradient $\partial_\mu \phi(x)$ transforms as a covariant vector field

$$\partial_\mu \phi(x) \to \partial_\mu \left(\phi(\Lambda^{-1}x) \right) = (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x). \tag{1.2}$$

Thus, the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 \tag{1.3}$$

transforms like a Lorentz scalar

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x), \tag{1.4}$$

which is shown in [26, p.36]. Hence, the action is Lorentz invariant, since it is obtained by integration of $\mathcal{L}$ over spacetime. As an immediate consequence of the principle of least action the equation of motion is Lorentz invariant. This field equation

$$\boxed{(\Box + m^2)\phi(x) = 0} \tag{1.5}$$

is received via the Euler-Lagrange equation[2] and it is called *Klein-Gordon equation*.

[1]For more detailed information on Lorentz invariance see appendix C.
[2]Remember the Euler-Lagrange equation is $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$.

1.2.2 Proca Equation

For a description of gauge bosons, i.e. spin 1 particles, we need vector fields V^μ which transform, according to [26, p.37], as

$$V^\mu(x) \to \Lambda^\mu_\nu V^\nu(\Lambda^{-1}x). \tag{1.6}$$

This type of field also carries an orientation, which must be transformed as well when the point of evaluation of the field is changed. The Lagrangian for vector fields is given in [27, p.5/1] or in [25, p.246] by

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu V^\nu - \partial^\nu V^\mu)(\partial_\mu V_\nu - \partial_\nu V_\mu) + \frac{1}{2}m^2 V_\nu V^\nu. \tag{1.7}$$

It is useful to introduce the shorthand notation

$$V^{\mu\nu} \equiv \partial^\mu V^\nu - \partial^\nu V^\mu, \tag{1.8}$$

and so the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{4}V^{\mu\nu}V_{\mu\nu} + \frac{1}{2}m^2 V^\nu V_\nu. \tag{1.9}$$

Again via Euler-Lagrange equation the equation of motion is received as

$$\partial_\mu V^{\mu\nu} + m^2 V^\nu = 0, \tag{1.10}$$

and it is called *Proca equation*. Alternatively it can be also written in a way similar to the Klein-Grodon equation,

$$\boxed{(\Box + m^2)V^\mu = 0}\,, \tag{1.11}$$

with vanishing divergence of the vector field $\partial_\mu V^\mu = 0$.

1.2.3 Dirac Equation

Finally we need a way to describe fermions, i.e. spin $1/2$ particles. We have to find an equation consistent with the relativistic energy-momentum formula

$$p^\mu p_\mu - m^2 = 0, \tag{1.12}$$

where p^μ denotes the four-momentum and m the mass of the free particle. Dirac's strategy was to factor the energy-momentum relation (1.12) as shown in [25, p.214ff]. In order to do so the necessity of some anticommuting coefficients γ^μ arises, which have to fulfill the relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{1.13}$$

with the Minkowski metric $g^{\mu\nu}$. So the coefficients have to be at least 4×4 matrices and they are called *Dirac* or *gamma matrices*[3]. Since they satisfy relation (1.13), the set of gamma matrices γ^μ forms a Clifford algebra, as noted in [28, p.89f].

[3]Properties and representations of the Dirac matrices are treated in appendix A.

Another crucial point is a proper description of the fermion fields ψ. Since they carry spin $1/2$, the spin vector has to be changed when the field is undergoing a Lorentz transformation. Hence, fermion fields do not transform like four vectors even though they carry four components. Fermion fields transform according to a different representation of the Lorentz group called *spinorial representation*[4]:

$$\psi(x)^a \to \psi^{a\prime}(x') = (\Lambda_D)^a_b \psi^b(\Lambda^{-1}x'), \tag{1.14}$$

$$\text{with} \quad \Lambda_D = \exp\left(\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{2}\right), \tag{1.15}$$

$$\text{and} \quad \sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{1.16}$$

Since fermion fields ψ are spinor fields, they are also referred to as Dirac spinors or bi-spinors, as noted in [25, p.216], and their four complex components ψ^a are labeled with Dirac indices $a = 1, \ldots, 4$.

Now we are finally able to write down the Lagrangian properly as

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi, \tag{1.17}$$

where the Dirac adjoint spinor is denoted as

$$\psi = \psi^\dagger\gamma^0. \tag{1.18}$$

As usual we can receive the corresponding field equation for fermions, called the *Dirac equation*, using the Euler-Lagrange formula, i.e.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}\,. \tag{1.19}$$

Employing the Feynman slash notation $\gamma^\mu a_\mu = \not{a}$ we can rewrite the Dirac Lagrangian and the Dirac equation as

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi \tag{1.20}$$

$$(i\not{\partial} - m)\psi = 0. \tag{1.21}$$

1.2.4 Weyl Equation

Before we discuss Weyl spinors, we should discuss handedness of fermions. In order to do this, two different properties have to be distinguished as done in [29, p.10ff]. A fermion obeying the Dirac equation can be classified in left or right-handed (LH resp. RH) due to the relative orientation of its spin[5] $\mathbf{\Sigma}$ and its momentum $\mathbf{p}$. Therefore, the *helicity* of a fermion

$$h = \frac{\mathbf{\Sigma}\cdot\mathbf{p}}{p} \tag{1.22}$$

is defined as the projection of the spin vector onto the direction of the momentum. As a fact h has eigenvalues ± 1, we call an eigenstate with eigenvalue $+1$ RH and one with

[4] A more comprehensive discussion can be found in appendix C.3.
[5] These spin matrices are given by $\Sigma^i = \varepsilon^{ijk}\frac{1}{2}\sigma_{jk}$, compare to equation (C.46).

eigenvalue -1 LH. Even though helicity is invariant under rotations, it is not under boosts for massive particles which cannot move at the speed of light. For these fermions it is possible to transform into a frame of reference where the direction of the momentum is changed, whereas the spin vector stays unaffected. But helicity is a conserved quantity for a free particle, since h commutes with the Dirac Lagrangian.

The second property which can be defined is called *chirality*. It is a more abstract concept where fermions are distinguished according to their behaviour under Lorentz transformations. The spinorial representation of the Lorentz group for a Dirac spinor is reducible and decomposes, when using the gamma matrices in the Weyl basis[6], into two irreducible representations acting on two-spinors, i.e. the transformation matrix becomes block digaonal[7]. Hence, the two upper and lower components of a Dirac spinor are decoupled under Lorentz transformations. The upper part transforms according to the LH spinorial representation, whereas the lower part transforms according to the RH spinorial representation. Therefore in general, a Dirac spinor can carry both, LH and RH, chirality components and it is possible to project the fermion field onto either its LH or RH component. Formally chirality can be defined by the fifth gamma matrix[8] γ^5, which has eigenvalues ± 1. Using γ^5 we can define two projection operators[9]

$$P_L = \frac{1}{2}(\mathbb{1}_4 - \gamma^5) \quad \text{and} \quad P_R = \frac{1}{2}(\mathbb{1}_4 + \gamma^5), \tag{1.23}$$

with properties

$$\begin{aligned} (P_L)^2 &= P_L = (P_L)^\dagger, \\ (P_R)^2 &= P_R = (P_R)^\dagger, \\ P_L P_R &= 0, \\ P_L + P_R &= \mathbb{1}_4. \end{aligned} \tag{1.24}$$

These projection operators act on the spinors in such a way that the Dirac spinor can be formally decomposed into two components of different chirality

$$\begin{aligned} \psi &= \psi_L + \psi_R, \\ \text{with} \quad \psi_L &\equiv P_L\psi \quad \text{and} \quad \psi_R \equiv P_R\psi, \end{aligned} \tag{1.25}$$

or alternatively we can say

$$\begin{aligned} P_L\psi_L &= \psi_L, \qquad P_R\psi_R = \psi_R, \\ P_L\psi_R &= 0 \ \ , \qquad P_R\psi_L = 0. \end{aligned} \tag{1.26}$$

Those components ψ_L and ψ_R are LH and RH respectively and transform as indicated above according to the LH and RH Weyl representation[10] Λ_L and Λ_R. Since the transfor-

[6] See appendix A.3.3.
[7] For details see appendix C.3.3 .
[8] We are using $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. For a proper definition and properties see appendix A.1.2.
[9] Note that in some literature different conventions could be used.
[10] They transform the same way under rotations, but oppositely under boosts. For details see again appendix C.3.3.

mation behaviour of the upper two components is decoupled from the one of the two lower components it is possible and also useful to denote the upper resp. lower components of ψ_L and ψ_R as two two-component spinors[11] χ_L and ξ_R. So we can write the Dirac spinor[12] as

$$\psi = \begin{pmatrix} \chi_L \\ \xi_R \end{pmatrix}, \tag{1.27}$$

$$\text{with} \quad \psi_L \equiv P_L \psi = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R \equiv P_R \psi = \begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \tag{1.28}$$

and those two independent two-spinors χ_L and ξ_R are called *Weyl or chiral spinors*.

Unlike helicity the property of chirality is Lorentz invariant but not conserved, because γ^5 does not commute with the mass term of the Dirac Lagrangian. Nevertheless, in the massless limit the problem of Lorentz invariance of h disappears as well as the problem with the conserved value of γ^5, since γ^5 does indeed commute with the mass-independent term of the Dirac-Lagrangian. It can be shown[13] that helicity and chirality become equivalent[14] for massless fermions. Therefore Weyl spinors are used to describe massless fermions. Following now [30, p.51f], we insert (1.27) into the Dirac Lagrangian (1.20) we get

$$\mathcal{L} = i(\chi_L)^\dagger \overline{\sigma}^\mu \partial_\mu \chi_L + i(\xi_R)^\dagger \sigma^\mu \partial_\mu \xi_R - m[(\chi_L)^\dagger \xi_R + (\xi_R)^\dagger \chi_L], \tag{1.29}$$

where we have introduced a new notation for the Pauli matrices

$$\sigma^\mu = (\mathbb{1}_2, \sigma^i) \quad \text{and} \quad \overline{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i). \tag{1.30}$$

Having a closer look at equation (1.29), as done in [31, p.91f], we can see that both Weyl spinors are coupled in the mass term, which means a massive fermion requires both χ_L and ξ_R. For a massless fermion this equation decouples into two similar equations one for each Weyl spinor:

$$\boxed{i\overline{\sigma}^\mu \partial_\mu \chi_L = 0 \quad \text{and} \quad i\sigma^\mu \partial_\mu \xi_R = 0}\,. \tag{1.31}$$

Hence, a massless fermion can be described by one single Weyl spinor with the corresponding one of the equations of motion above, which are called *Weyl equations*.

1.2.5 Majorana Equation

In contrast to the previous subsection we now want to return to massive fermions. We want to know, if there are real solutions of the Dirac equation (1.19). Putting it differently we want to investigate, if there are real spinor fields which fulfil the Dirac equation. To discuss this, we might follow [29, p.4ff] and start with the observation that the answer

[11] We denote them χ_L and ξ_R to emphasize their independence from each other and to distinguish them from the Majorana spinor, which will be discussed in the next section 1.2.5.

[12] This is the reason why Dirac spinors are also called bi-spinors.

[13] For this proof see [29, p.13].

[14] This equivalence can be used to define chirality on two-component Weyl spinors. Even though chirality can not be defined on these objects, because we can not use γ^5, it is possible to define helicity eigenstate vectors in the two-component notation using Pauli matrices instead, as done in [29, p.18].

depends on the used representation of the gamma matrices $\{\gamma^\mu\}$. If every gamma matrix has only purely imaginary non-zero elements, then the Dirac equation is real and therefore we are able to find a real solution, i.e. a spinor field $\tilde{\psi}$ for which[15]

$$\tilde{\psi}^* = \tilde{\psi}. \tag{1.32}$$

Fortunately, it is possible to define gamma matrices in such a way that the condition claimed above can be satisfied, namely if

$$\tilde{\gamma}^{\mu *} = -\tilde{\gamma}^\mu. \tag{1.33}$$

This can be achieved by using the Majorana basis[16] for the gamma matrices. But even if we are using another basis we would still be able to obtain a similar condition like (1.32) for the spinor field. Due to Pauli's fundamental theorem[17] we are able to change to another basis of gamma matrices γ^μ using a unitary matrix U by

$$\gamma^\mu = U\tilde{\gamma}^\mu U^\dagger. \tag{1.34}$$

If a spinor field $\tilde{\psi}$ is a solution of the Dirac equation in the Majorana basis $\{\tilde{\gamma}^\mu\}$, then

$$\psi = U\tilde{\psi} \tag{1.35}$$

is a solution likewise in another basis $\{\gamma^\mu\}$. Thus, the Majorana condition (1.32) for an arbitrary basis is given by

$$\psi = UU^T\psi^*, \tag{1.36}$$

which is easily derived from equation (1.32) and (1.35)[18]. Of course the matrix product UU^T is also unitary, because U is unitary. It is customary to introduce another unitary matrix C defined[19] as

$$UU^T \equiv C{\gamma^0}^T, \tag{1.37}$$

which we use to rewrite the right hand side of equation (1.36) and denote it in a more compact notation, as

$$\psi^C \equiv C{\gamma^0}^T\psi^* = C\overline{\psi}^T. \tag{1.38}$$

Hence, we can achieve the *Majorana condition*[20]

$$\boxed{\psi = \psi^C}, \tag{1.39}$$

which defines a Majorana field in any basis[21].

[15]To prevent confusion of notation, here quantities with tilde refer to the Majorana basis.
[16]For an explicit display of the $\tilde{\gamma}^\mu$ in this basis see appendix A.3.4.
[17]For a complete statement of the theorem and a reference to its proof see appendix A.3.1.
[18]From equation (1.35) we get $\tilde{\psi} = U^\dagger\psi$ and inserting this into (1.32) we obtain $U^\dagger\psi = (U^\dagger\psi)^* = U^T\psi^*$ which leads to (1.36), where we used the unitarity of U.
[19]The reason why we use an extra factor of ${\gamma^0}^T$ in the definition is discussed in appendix B.2.2.
[20]It can be shown that this condition is Lorentz invariant [29, p.8f].
[21]For the case of a Majorana basis of gamma matrices we have simply $C_{\text{Maj}} = \mathbb{1}_4$ (see appendix B.2.2 equation (B.48)) and the Majorana condition has the form (1.32).

The matrix C is also known as *charge conjugation matrix* and its defining relation[22] is given by

$$\boxed{(\gamma^\mu)^T = -C^{-1}\gamma^\mu C}\,. \tag{1.40}$$

The equation (1.38) defines the charge conjugated Dirac spinor.

So far, a Majorana fermion field in the Majorana representation has four real components. Now we want to represent it in terms of two-component Weyl spinors analogously to the Dirac field in equation (1.27), but now we have to take care of the Majorana condition. It is interesting to see how the Majorana condition (1.39) looks in terms of the decomposition into LH and RH Weyl spinors. Using relation (1.27) and the form of C in the Weyl basis[23] we get

$$\begin{aligned}
\begin{pmatrix}\chi_L\\ \xi_R\end{pmatrix} = \psi \overset{!}{=} \psi^C & \\
&= C_{\text{Weyl}}\gamma^{0^T}\psi^* \\
&= -i\gamma^2\gamma^{0^2}\psi^* \\
&= \begin{pmatrix}0 & -i\sigma^2\\ i\sigma^2 & 0\end{pmatrix}\begin{pmatrix}(\chi_L)^*\\ (\xi_R)^*\end{pmatrix} \\
&= i\sigma^2\begin{pmatrix}-(\xi_R)^*\\ (\chi_L)^*\end{pmatrix}.
\end{aligned} \tag{1.41}$$

Thus, the Majorana condition in the Weyl basis reads as

$$\chi_L = -i\sigma^2(\xi_R)^* \tag{1.42}$$

$$\text{and} \quad \xi_R = i\sigma^2(\chi_L)^* := \chi_R. \tag{1.43}$$

Since the two two-spinor components are correlated by this relation, we denote the lower one as χ_R to emphasize this correlation, and hence we can write a Majorana field in the chiral representation in the form

$$\psi(x) = \begin{pmatrix}\chi_L(x)\\ \chi_R(x)\end{pmatrix} = \begin{pmatrix}\chi_L(x)\\ i\sigma^2(\chi_L)^*(x)\end{pmatrix} \tag{1.44}$$

or, since we only need one two-spinor for describing the Majorana spinor, we might use $\chi_L =: \chi$ for a shorter notation. Analogously to (1.25) and (1.28) for the Dirac field a Majorana field can be defined by the sum of a spinor with distinct chirality and its charge conjugated spinor, i.e.

$$\begin{aligned}
\psi &= \psi_L + (\psi_L)^C\,, \\
\text{with} \quad \psi_L &= \begin{pmatrix}\chi\\ 0\end{pmatrix}, \\
\text{and} \quad (\psi_L)^C &= \begin{pmatrix}0\\ i\sigma^2\chi^*\end{pmatrix}.
\end{aligned} \tag{1.45}$$

[22] This relation follows from (1.34) and (1.37). In formula (B.20) in appendix B.2 it is derived also in an alternative way.

[23] See appendix B.2.2 equation (B.47).

Finally, we want to obtain the Majorana equation, as done in [30, p.207f]. First, we establish the Dirac Lagrangian of a Majorana field in the four-component notation[24]

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left(\overline{\psi} i\partial\!\!\!/ \psi - m\overline{\psi}\psi\right) \\
=& \frac{1}{2}\left[\overline{\psi}_L i\partial\!\!\!/ \psi_L + \overline{(\psi_L)^C} i\partial\!\!\!/ (\psi_L)^C - m\left(\overline{\psi}_L (\psi_L)^C + \overline{(\psi_L)^C}\psi_L\right)\right].
\end{aligned}
\tag{1.46}
$$

Using the Weyl decomposition and the chiral basis of gamma matrices[25] we get analogously as for the Weyl Lagrangian (1.29) the following Lagrangian for the Majorana fields:

$$
\mathcal{L} = \frac{i}{2}\left[\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi + \chi^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \chi^* - m\left(\chi^\dagger \sigma^2 \chi^* - \chi^T \sigma^2 \chi\right)\right], \tag{1.47}
$$

where we used $(\sigma^2)^\dagger = \sigma^2$.

Furthermore, it can be shown[26], as done in [29, p.22] that $(\sigma^2\sigma^\mu\sigma^2)^T = \overline{\sigma}^\mu$ and therefore

$$
\chi^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \chi^* = -\partial_\mu \chi^\dagger \overline{\sigma}^\mu \chi. \tag{1.48}
$$

For further simplification it should be remembered that the physical important quantity is the action $S = \int d^4x\mathcal{L}$, which has been done in [32, p.4]. Using partial integration the derivative of the second term can be transferred form $\chi^\dagger$ to the field χ and hence

$$
S = \int d^4x 2i\left[\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi + \chi^T \sigma^2 \sigma^\mu \sigma^2 \partial_\mu \chi^*\right] - \frac{i}{2}m\left(\chi^\dagger \sigma^2 \chi^* - \chi^T \sigma^2 \chi\right). \tag{1.49}
$$

Thus, we obtained the Majorana Lagrangian

$$
\mathcal{L} = i\chi^\dagger \overline{\sigma}^\mu \partial_\mu \chi + \frac{i}{2}m\left(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^*\right). \tag{1.50}
$$

The equation of motion, i.e. the *Majorana equation* that follows from this Lagrangian[27] is given by

$$
\boxed{\overline{\sigma}^\mu \partial_\mu \chi - m\sigma^2 \chi^* = 0}. \tag{1.51}
$$

1.3 Electroweak Theory

For the purpose of this thesis it is sufficient to restrict ourselves to electroweak interaction since only quarks participate in strong interaction. In the Glashow-Weinberg-Salem (GWS) model electromagnetic and weak interactions are unified to the so-called electroweak (EW) interaction that means the two elementary interactions can be seen as two aspects of the same force.

[24] "The overall factor of 1/2 compared to the general Dirac Lagrangian (1.20) is usual for self-conjugate fields, introduced to ensure a consistent normalization of the field operators in QFT" [29, p.21]. For details of the derivation of this Lagrangian see appendix F.1.

[25] See appendix A.3 and note that in the Weyl basis we can use notation (1.30) and write $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$.

[26] It should be emphasized, as done in [29, p.22] that commutation of two fermion fields produces a minus sign.

[27] The calculation can be found in [29, p.22] or in [30, p.208].

1.3.1 V − A Structure of Weak Currents

As mentioned in [25, p.331], the crucial structural difference between the electromagnetic (EM) and weak currents is that the EM current (see equation (B.65)) is purely vectorial, whereas the weak currents exhibit vector and axial vector contributions.

It is noted in [33, p.236] that the experimental data is in accordance with the assumption that interactions of leptons only appear in the following forms:

$$J_\mu(x) = \sum_\ell \overline{\psi}_\ell \gamma_\mu (1-\gamma_5)\psi_{\nu_\ell}(x), \tag{1.52}$$

$$J_\mu^\dagger(x) = \sum_\ell \overline{\psi}_{\nu_\ell} \gamma_\mu (1-\gamma_5)\psi_\ell(x), \tag{1.53}$$

where ℓ labels the charged leptons, i.e. $\ell = e, \mu, \tau$, and ν_ℓ the corresponding neutrinos. Then, in analogy to the interaction Lagrangian of QED, we can write for the weak leptonic interaction

$$\mathcal{L}^I_{\text{weak}}(x) = g_W J^{\mu\dagger}(x) W_\mu(x) + g_W J^\mu(x) W_\mu^\dagger(x), \tag{1.54}$$

where g_W is a dimensionless coupling constant and the field $W_\mu(x)$ describes the mediating particles of the (charged) weak interaction[28]. In [33, p.241] the $V - A$ structure of the current is emphasized by writing the current as the actual difference

$$J^\mu(x) = J^\mu_V(x) - J^\mu_A(x) \tag{1.55}$$

of the vector current and the axial vector current given by

$$J^\mu_V = \sum_\ell \overline{\psi}_\ell(x)\gamma^\mu \psi_{\nu_\ell}(x), \tag{1.56}$$

$$J^\mu_A(x) = \sum_\ell \overline{\psi}_\ell(x)\gamma^\mu \gamma^5 \psi_{\nu_\ell}. \tag{1.57}$$

It is easy to see that the vector current J^μ_V changes sign under the parity transformation[29], whereas the axial vector current $J^\mu_A(x)$ does not. Therefore, parity is not conserved, because the interaction Lagrangian (1.54) is not invariant under spatial inversions.

This specific structure has striking consequences for massless particles, e.g. neutrinos in the SM, or for particles in the high energy limit. In these two cases, chirality and helicity become equivalent and the LH chirality projector given in (1.23) appears in the V-A current (1.55). In analogy to the property (1.26) we define the LH neutrino field operators as

$$\psi^L_{\nu_\ell}(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi_{\nu_\ell}(x). \tag{1.58}$$

In the currents (1.52) - (1.53) only those LH operators appear, which can annihilate only LH neutrinos and create only RH antineutrinos. Thus, RH neutrinos and LH antineutrinos do not enter in weak interactions and only LH neutrinos and RH antineutrinos take part in weak interactions. For the LH charged lepton fields the LH field operator $\psi^L_{\nu_\ell}$ can be

[28] This will be discussed in detail in subsection 1.3.2.
[29] See appendix B.1.

defined analogously and the leptonic current (1.52) can be written as

$$J_\mu(x) = 2\sum_\ell \overline{\psi}^L_\ell(x)\gamma_\mu\psi^L_{\nu_\ell}(x), \tag{1.59}$$

where only the LH fields are involved for the neutrinos and also for the charged leptons.

1.3.2 Electroweak Gauge Theory

In appendix B.4.4 global gauge transformations for the weak interactions are discussed, as well as the appearance of LH isodoublets $\Psi^L_\ell = \left(\psi^L_{\nu_\ell}, \psi^L_\ell\right)^T$ and RH isosinglets[30] ψ^R_ℓ and $\psi^R_{\nu_\ell}$. In this section we follow [33, p.268ff], while using the notation in [34]. We generalize the $SU(2)_L$ and $U(1)_Y$ transformations discussed in appendix B.4.4 form global to local phase transformations, like in the section B.4.1 on QED. First we discuss the local $SU(2)_L$ transformations

$$U = \exp\left(igT_j\omega_j(x)\right) \tag{1.60}$$

under which the lepton (fermion) fields transform as

$$\Psi^L_\ell(x) \to \exp\left(igT_j\omega_j(x)\right)\Psi^L_\ell(x), \tag{1.61}$$

$$\overline{\Psi}^L_\ell(x) \to \overline{\Psi}^L_\ell(x)\exp\left(-igT_j\omega_j(x)\right), \tag{1.62}$$

$$\psi^R_\ell(x) \to \psi^R_\ell(x), \quad \psi^R_{\nu_\ell}(x) \to \psi^R_{\nu_\ell}(x), \tag{1.63}$$

$$\overline{\psi}^R_\ell(x) \to \overline{\psi}^R_\ell(x), \quad \overline{\psi}^R_{\nu_\ell}(x) \to \overline{\psi}^R_{\nu_\ell}(x). \tag{1.64}$$

The $\omega_j(x)$ $(j = 1, 2, 3)$ denote arbitrary real differentiable functions of x, and g is a real constant[31]. $T_j = \frac{1}{2}\tau_j$ are the generators of the algebra, with τ_j denoting the Pauli matrices defined in (A.35). To make the free Lagrangian

$$\mathcal{L}^0_{\text{EW}} = i\left[\overline{\Psi}^L_\ell(x)\partial\!\!\!/\Psi^L_\ell(x) + \overline{\psi}^R_\ell(x)\partial\!\!\!/\psi^R_\ell(x) + \overline{\psi}^R_{\nu_\ell}(x)\partial\!\!\!/\psi^R_{\nu_\ell}(x)\right] \tag{1.65}$$

invariant under these local phase transformations, we introduce the covariant derivative

$$\partial^\mu \to D^\mu\Psi^L_\ell(x) = \partial^\mu + igW^\mu_j(x)T_j, \tag{1.66}$$

where $W^\mu_j(x)$ are three real gauge fields[32]. Thus, the Lagrangian is modified to

$$\mathcal{L}^0_{\text{EW}} \to i\left[\overline{\Psi}^L_\ell(x)D\!\!\!\!/\,\Psi^L_\ell(x) + \overline{\psi}^R_\ell(x)\partial\!\!\!/\psi^R_\ell(x) + \overline{\psi}^R_{\nu_\ell}(x)\partial\!\!\!/\psi^R_{\nu_\ell}(x)\right]. \tag{1.67}$$

Since $SU(2)_L$ is in contrast to $U(1)$ a non-abelian gauge group the transformation of the the gauge field is more complicated, because it has to be taken into account that the matrices U do not commute. Hence, we get for the gauge fields

$$W^\mu_jT_j \to UW^\mu_jT_jU^{-1} + \frac{i}{g}\left(\partial_\mu U\right)U^{-1}, \tag{1.68}$$

[30] Even though RH neutrinos are not included in the SM interactions, we already introduce the RH neutrino field operator and its transformation properties, because we will need this later on.

[31] This constant will be identified later in the EW unification as coupling constant.

[32] In contrast to QED we now need three gauge fields, since we have three generators of the group or also three conserved charges (see appendix B.4.4).

and the field strength tensor is given by

$$F^j_{\mu\nu} T_j \equiv F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig\left[W_\mu, W_\nu\right], \tag{1.69}$$

and therefore it transforms like

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1}. \tag{1.70}$$

Analogously to QED a free part for the gauge fields has to be added to the Lagrangian, but first we want to investigate the local $U(1)_Y$ transformations.

The local phase transformations for all the LH and RH fermion fields are

$$\psi(x) \rightarrow \exp\left[ig'\frac{Y}{2}f(x)\right]\psi(x), \tag{1.71}$$

$$\overline{\psi}(x) \rightarrow \overline{\psi}(x)\exp\left[-ig'\frac{Y}{2}f(x)\right], \tag{1.72}$$

where g' is a real number[33] and $f(x)$ is an arbitrary real differentiable function. The weak hypercharge Y takes different values for the LH doublets and RH singlets, which are listed in appendix B.4.4 in table 23. Since the gauge transformations associated with the weak hypercharge are $U(1)$ transformations, the following procedure is analogous to the QED case. We introduce the covariant derivative

$$\partial_\mu \rightarrow D_\mu \equiv \partial_\mu + ig'\frac{Y}{2}B_\mu(x), \tag{1.73}$$

with the real gauge field $B_\mu(x)$ transforming like

$$B_\mu(x) \rightarrow B_\mu(x) + \frac{1}{g'}\partial_\mu f(x). \tag{1.74}$$

The corresponding field strength tensor is simply

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{1.75}$$

If we combine both covariant derivatives (1.66) and (1.73), i.e.

$$\boxed{D^\mu \equiv \partial^\mu + ig\vec{T}\vec{W}^\mu(x) + ig'\frac{Y}{2}B^\mu(x)}, \tag{1.76}$$

to ensure invariance under the whole local gauge group $SU(2) \times U(1)$, the Lagrangian for the leptons becomes

$$\mathcal{L}^0_{\mathrm{EW}} \rightarrow i\overline{\Psi}^L_\ell(x)\not{D}\Psi^L_\ell(x) + \overline{\psi}^R_\ell(x)\not{D}\psi^R_\ell(x) + \overline{\psi}^R_{\nu_\ell}(x)\not{D}\psi^R_{\nu_\ell}(x), \tag{1.77}$$

where $\vec{T} = \frac{1}{2}\vec{\tau}$ for the LH doublets and $\vec{T} = 0$ for the RH singlets and the weak hypercharge $Y = -1, -2, 0$ for LH doublets, RH singlets ψ_ℓ and ψ_{ν_ℓ} respectively. This Lagrangian now contains the free lepton Lagrangian $\mathcal{L}^0_{\mathrm{EW}}$ as well as the interaction part $\mathcal{L}^I_{\mathrm{EW}}$ of the fermions with the four gauge fields, which come from the covariant derivative term, i.e.

$$\mathcal{L}^I_{\mathrm{EW}} = -gJ^\mu_i(x)W_{i\mu}(x) - g'J^\mu_Y(x)B_\mu(x), \tag{1.78}$$

[33]The meaning of g' will be determined in section 1.3.4.

where $J_i^\mu(x)$ are the weak isospin currents given in equation (B.101) and $J_Y^\mu(x)$ the weak hypercharge current given in equation (B.107).

Furthermore, like in the case of QED, we have to add a free part to the Lagrangian for the gauge bosons W_i and B. Again we assume them to be massless and we use the Proca Lagrangian given in (1.9) to write the gauge boson Lagrangian as

$$\boxed{\mathcal{L}_{\mathrm{EW}}^{\mathrm{G}} = -\frac{1}{4} W_{\mu\nu i} W_i^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}}\,. \tag{1.79}$$

So the complete Lagrangian[34] for fermions and gauge bosons and their interaction is given by

$$\boxed{\mathcal{L}_{\mathrm{EW}} = \mathcal{L}_{\mathrm{EW}}^{\mathrm{G}} + \mathcal{L}_{\mathrm{EW}}^{0} + \mathcal{L}_{\mathrm{EW}}^{I}}\,, \tag{1.80}$$

where now also quark fields are included to describe the full theory, i.e.

$$\begin{aligned}\mathcal{L}_{\mathrm{EW}}^{0} + \mathcal{L}_{\mathrm{EW}}^{I} = \sum_{\ell=e,\mu,\tau} i \left[\overline{\Psi}_\ell^L(x) \not{D} \Psi_\ell^L(x) + \overline{\psi}_\ell^R(x) \not{D} \psi_\ell^R(x) + \overline{\psi}_{\nu_\ell}^R(x) \not{D} \psi_{\nu_\ell}^R(x) \right] + \\ \sum_{q=d,s,b} i \left[\overline{\Psi}_q^L(x) \not{D} \Psi_q^L(x) + \overline{\psi}_q^R(x) \not{D} \psi_q^R(x) \right] + \sum_{q=u,c,t} i \left[\overline{\psi}_q^R(x) \not{D} \psi_q^R(x) \right].\end{aligned} \tag{1.81}$$

1.3.3 The Higgs Field

So far we have obtained a $SU(2)_L \times U(1)_Y$ invariant description of EW interactions of fermions and gauge bosons, but unfortunately the particles are still massless in this model. As it is mentioned in [25, p.360], "the principle of local gauge invariance works beautifully for the EM interactions" discussed in section B.4.1, since the photon is massless and the mass term of the fermions does not destroy the $U(1)_{\mathrm{EM}}$ gauge invariance of the Lagrangian. In the EW theory the necessity arises for massive gauge bosons. This can be achieved by the procedure of spontaneous symmetry breaking (SSB) and the Higgs mechanism[35].

Spontaneous symmetry breaking appears in a theory if the state of lowest energy of a Lagrangian is degenerate. According to [33, p.280], this means "there is no unique eigenstate to represent the ground state. If we arbitrarily select one of the degenerate states as the ground state, then the ground state no longer shares the full symmetry with the Lagrangian". In the GSW model a complex scalar field ϕ is introduced, which provides this property. In [25, p.362] it is emphasized that therefore SSB is no consequence of an external agency, but rather that "the true symmetry of the system is hidden by the arbitrary selection of a particular (asymmetrical) ground state".

The spontaneous breaking of a continuous symmetry gives rise to massless particles. In [26, p.351] it is explained that in an $O(N)$ symmetric theory a rotation in N dimensions can be performed in any one of $N(N-1)/2$ planes, which is the number of continuous

[34] Note that the Lagrangian is subscripted EW because it already contains the electromagnetic interaction as well. It will be more obvious in the section on gauge bosons 1.3.4. Furthermore, we will have to include a scalar field (Higgs field) in section 1.3.3. After doing all this we will be finally able to really write down the full Lagrangian of the GSW model of the EW interaction.

[35] For a more elaborated discussion see appendix B.4.2 and B.4.3.

symmetries in this theory. After SSB there remain $(N-1)(N-2)/2$ symmetries, which correspond to rotations of $(N-1)$ massless fields. The number of broken symmetries is $N(N-1)/2-(N-1)(N-2)/2 = N-1$, which is the difference of unbroken symmetries before and after SSB. The massless fields are called *Goldstone bosons* and the general statement that for every spontaneously broken continuous symmetry, the theory must contain such a massless particle, is called *Goldstone theorem*.

Finally, to obtain massive gauge bosons we have to apply the *Higgs mechanism*. As indicated in [26, p.692] a massless vector boson has only two physical polarization states, whereas a massive vector boson must have three. In the Higgs mechanism the gauge bosons are said to be *eating* the Goldstone boson and therefore they get their third degree of freedom. Thus, the Goldstone boson itself does not appear any longer in the Lagrangian of the theory, which would be problematic, since they do not represent physical fields.

Now we want to apply the procedure discussed above to the case of EW theory. In order to break the $SU(2)$ symmetry in EW theory, as mentioned in [33, p.290], a weak isospin doublet (Higgs doublet) of complex scalar fields has to be introduced, i.e.

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{1.82}$$

where $\phi_1(x)$ and $\phi_2(x)$ are complex scalar fields invariant under Lorentz transformations. This Higgs doublet transforms under the $SU(2) \times U(1)$ gauge group like the isospin doublets $\Psi_\ell^L(x)$ in (1.61) and (1.71):

$$SU(2)_L : \begin{cases} \Phi(x) \to & \exp\left[\frac{ig\tau_j\omega_j(x)}{2}\right]\Phi(x), \\ \Phi^\dagger(x) \to & \Phi^\dagger(x)\exp\left[\frac{-ig\tau_j\omega_j(x)}{2}\right], \end{cases} \tag{1.83}$$

$$U(1)_Y : \begin{cases} \Phi(x) \to & \exp\left[ig'\frac{Y}{2}f(x)\right]\Phi(x), \\ \Phi^\dagger(x) \to & \Phi^\dagger(x)\exp\left[-ig'\frac{Y}{2}f(x)\right]. \end{cases} \tag{1.84}$$

The Lagrangian[36] of this field is given by

$$\boxed{\mathcal{L}_{\text{EW}}^{\Phi} = (D_\mu\Phi)^\dagger\,(D^\mu\Phi) + V(\Phi)}, \tag{1.85}$$

with the covariant derivative D_μ given in (1.76) and the potential

$$V(\Phi) = -\mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2, \tag{1.86}$$

where λ and μ are real constants.

We assume $\lambda > 0$, otherwise the potential would not have any lowest energy state. Further, for our purpose we assume $\mu^2 < 0$, because in the other case the ground state would not be degenerate. It can be shown, as in [33, p.291] that $V(\Phi)$ possesses a minimum at

$$\Phi^\dagger\Phi = |\phi_1|^2 + |\phi_2|^2 = -\frac{\mu^2}{2\lambda}. \tag{1.87}$$

[36] Compare to the Klein-Gordon Lagrangian (1.3) without the explicit mass term.

Hence, without loss of generality, we can choose one particular vacuum state as

$$\Phi_0 = \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, \tag{1.88}$$

where the chosen vacuum expectation value (VEV) is

$$v = \sqrt{-\frac{\mu^2}{\lambda}} \quad (> 0). \tag{1.89}$$

Therefore, the $SU(2)_L \times U(1)_Y$ symmetry is spontaneously broken to $U(1)_{\mathrm{EM}}$, because after this particular choice the vacuum ground state (1.88) is, in general, not invariant under the $SU(2)_L$ symmetry. Nevertheless, it must be invariant under $U(1)_{\mathrm{EM}}$, to ensure charge conservation and zero mass for the photon. Assigning a weak hypercharge of $Y_\Phi = {}^1\!/_2$ to the Higgs field, the lower component $\phi_2(x)$ of the Higgs field is electrically neutral due to the Gell-Mann-Nishijima formula (B.109). Hence, in case of our choice (1.88) we find that "spontaneous symmetry breaking only occurs in the electrically neutral component of the vacuum field (1.88), and charge conservation holds exactly" [33, p.291]. Thus, we might denote the components of the Higgs doublet in a different way to emphasize their EM charge as

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{1.90}$$

Following [33, p.291] we parametrize the Higgs doublet in terms of its deviations from the vacuum field Φ_0 by

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \eta_1(x) + i\eta_2(x) \\ v + h(x) + i\eta_3(x) \end{pmatrix}, \tag{1.91}$$

where $h(x)$ and $\eta_i(x),\ i = 1, 2, 3$ are four real fields. Only $h(x)$ is a massive field, whereas the three fields η_i are massless as predicted[37], since all three generators of $SU(2)$ are broken equally. These Goldston bosons are unphysical fields and can be removed by choosing a particular gauge for the Higgs doublet, called the *unitary gauge*, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tag{1.92}$$

which no longer contains the unphysical fields $\eta_i(x)$. Hence, in this particular gauge the Lagrangian (1.85) becomes after SSB:

$$\mathcal{L}_{\mathrm{EW}}^{\Phi} \overset{\mathrm{SSB}}{\rightarrow} \frac{1}{2} \left(D_\mu h\right) \left(D^\mu h\right) - \lambda v^2 h^2 \left(1 + \frac{1}{2v} h\right)^2 + \frac{\mu^4}{4\lambda^2}. \tag{1.93}$$

We can now easily read off the mass term of the field $h(x)$, which we call the *Higgs field*,

$$\boxed{m_h^2 = 2\lambda v^2}. \tag{1.94}$$

[37] See appendix B.4.2.

1.3.4 Gauge Boson Masses

We now want to apply the Higgs mechanism to obtain mass terms for the gauge bosons. In doing so, we will find the real physical gauge boson fields, which will be the actual mass eigenstates. In section 1.3.2 we found four gauge fields W_i^μ and B^μ for the $SU(2)_L \times U(1)_Y$ invariant EW theory. According to [34, p.IX.5], B^μ transforms as a $SU(2)$ singlet and does not couple to other gauge bosons, thus $T_B^3 = 0$ and $Y_B = 0$. Therefore using the Gell-Mann-Nishijima formula (B.109) we find B^μ is neutrally charged, i.e. $Q_B = 0$. The three W_i^μ bosons transform according to the adjoint representation of $SU(2)$. However, each of them does not couple to other gauge bosons, thus $Y_W = 0$. Using the Gell-Mann-Nishijima formula (B.109) again, we obtain the charge matrix of the triplet

$$Q_W = T_3 = (-i\varepsilon_{3ab}) = I \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.95}$$

This has eigenvalues ± 1 and 0 for the eigenvectors

$$\vec{e}_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 1, & \pm i, & 0 \end{pmatrix}^T, \quad \vec{e}_3 = \begin{pmatrix} 0, & 0, & 1 \end{pmatrix}^T, \tag{1.96}$$

respectively. Hence, $\vec{W}^\mu$ can be decomposed into

$$\vec{W}_\mu = W_\mu^3 \vec{e}_3 + W_\mu^+ \vec{e}_+ + W_\mu^- \vec{e}_-, \tag{1.97}$$

with the charge eigenstates of $\vec{W}_\mu$

$$W_\mu^+ = \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}}, \qquad W_\mu^- = \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}. \tag{1.98}$$

We can achieve the mass terms of the gauge bosons by investigating the first term of the Lagrangian $\mathcal{L}_{\text{EW}}^\Phi$ (1.85) after SSB and using the unitary gauge of Φ given in (1.92)

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) \overset{SBB}{\to} \frac{g^2 v^2}{4} W_\mu^- W^{+\mu} + \frac{v^2}{8} \left(-gW_\mu^3 + g' B_\mu\right) \left(-gW^{3\mu} + g' B^\mu\right), \tag{1.99}$$

where the last term can be rewritten while introducing the Weinberg angle θ_W

$$c_W = \cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad s_W = \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{1.100}$$

When doing this, we find the neutrally charged gauge bosons W^3 and B are mixed to the physical mass eigenstates Z^0 and A

$$\begin{pmatrix} Z^0 \\ A \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} W^3 \\ B \end{pmatrix}. \tag{1.101}$$

The part of the Lagrangian that gives the desired mass terms for the gauge bosons becomes

$$(D_\mu \Phi)^\dagger (D^\mu \Phi) \overset{SBB}{\to} m_W^2 W_\mu^+ W^{-\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu, \tag{1.102}$$

with the masses

$$\boxed{m_W^2 = \frac{g^2 v^2}{4}, \quad m_Z^2 = \frac{(g^2 + g'^2) v^2}{4}, \quad m_A = 0}\,, \tag{1.103}$$

where the following relation is valid

$$\frac{m_W}{m_Z} = \cos\theta_W. \tag{1.104}$$

Hence, we have obtained three massive gauge bosons for the weak interaction and one massless gauge boson for the EM interaction, i.e. the photon. Now we are able to rewrite the covariant derivative D_μ, defined in equation (1.76), in terms of the physical gauge boson fields

$$D_\mu = \partial_\mu + \frac{ig}{\sqrt{2}}\left(T_+ W_\mu^+ + T_- W_\mu^-\right) + \frac{ig}{c_W}\left(T_3 - s_W^2 Q\right) Z_\mu + ieQA_\mu, \tag{1.105}$$

where we defined

$$T_\pm = T_1 \pm iT_2 = \begin{cases} 0 & \text{for singlets,} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for doublets,} \end{cases} \tag{1.106}$$

and

$$g s_W = g' c_W := e. \tag{1.107}$$

The first term gives of the covariant derivative above will give the kinetic term for the free fermions, the second term the charge weak current, the third the neutral weak current, and the fourth term the EM current.

1.3.5 Fermion Masses

So far the fermions of the theory are still massless, since the Lagrangian $\mathcal{L}_{\text{EW}}$ in equation (1.80) does not contain any fermion mass terms. Citing [33, p.292], "to obtain non-vanishing fermion masses, we must augment the Lagrangian by adding a suitable term", where the fermions are coupled to the Higgs field through so called Yukawa interactions. The Yukawa Lagrangian according to [35, p.13] is given by

$$-\mathcal{L}_{\text{EW}}^{\text{Yuk}} = \sum_{i,j=1}^{3}\left(\overline{q}_{iL}\Phi d_{jR}\Gamma_{ij}^{(q)} + \overline{q}_{iL}\tilde{\Phi} u_{iR}\Delta_{ij}^{(q)} + \overline{D}_{iL}\Phi \ell_{jR}\Gamma_{ij}^{(\ell)}\right) + \text{H.c.}\,, \tag{1.108}$$

where we introduced a different notation for the LH quark and lepton doublets, q_{iL} and D_{iL} respectively, and RH quark and lepton singlets[38], d_{jR}, u_{jR} and ℓ_{jR} for convenience. $\Gamma_{ij}^{(q)}$, $\Delta_{ij}^{(q)}$ and $\Gamma_{ij}^{(\ell)}$ are the 3×3 Yukawa coupling matrices of the fermion fields to the

[38]Note that in this theory RH neutrinos are not included.

Higgs doublet. The *charge conjugated Higgs doublet* $\tilde{\Phi}$ is defined as

$$\tilde{\Phi} \equiv \epsilon \Phi^* \quad \text{with } \epsilon = i\tau_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.109}$$

Its transformation property follows from the one of Φ given in (1.83) and from the fact that for any $U \in SU(2)$, $\epsilon U^* = U\epsilon$ is valid. Thus, $\tilde{\Phi}$ transforms under $SU(2)$ as $\tilde{\Phi} \to U\tilde{\Phi}$, and its weak hypercharge is of opposite sign compared to Y_Φ, i.e.$Y_{\tilde{\Phi}} = -Y_\Phi$.

We called this the charge conjugated Higgs doublet since we derive from (1.109):

$$\tilde{\Phi} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \overset{SSB}{\to} \frac{1}{\sqrt{2}} \begin{pmatrix} v^* + h(x) \\ 0 \end{pmatrix}, \tag{1.110}$$

where we used $\phi^{+*} = \phi^-$ and also displayed its form under SSB and in unitary gauge like Φ before in (1.92). Hence, we can see that Φ will give mass to the lower components of the LH doublets, whereas $\tilde{\Phi}$ will give mass to the upper components.

Following [34, p.IX.7], we achieve fermion mass terms by SSB and inserting the Higgs doublets Φ and $\tilde{\Phi}$ in unitary gauge in the Yukawa Lagrangian. Thus, we obtain

$$-\mathcal{L}_{\text{Yuk}} \overset{\text{SSB}}{\to} \overline{d}_L M_d d_R + \overline{u}_L M_u u_R + \overline{\ell}_L M_\ell \ell_R + \text{H.c.}\,, \tag{1.111}$$

with fermion mass matrices

$$M_d = \frac{v}{\sqrt{2}} \Gamma^{(q)}, \qquad M_u = \frac{v}{\sqrt{2}} \Delta^{(q)}, \qquad M_\ell = \frac{v}{\sqrt{2}} \Gamma^{(\ell)}, \tag{1.112}$$

where $d_{L/R}$, $u_{L/R}$ and $\ell_{L/R}$ denote the vector of the down-type and up-type quark spinor fields, as well as the vector of the charged lepton spinor fields respectively, which are still no mass eigenfields. For obtaining these we diagonalize the mass matrices by a biunitary transformation (theorem E.2.1):

$$U_L^{d\dagger} M_d U_R^d \equiv \hat{M}_d = \text{diag}(m_d, m_s, m_b), \tag{1.113}$$
$$U_L^{u\dagger} M_u U_R^u \equiv \hat{M}_u = \text{diag}(m_u, m_c, m_t), \tag{1.114}$$
$$U_L^{\ell\dagger} M_\ell U_R^\ell \equiv \hat{M}_\ell = \text{diag}(m_e, m_\mu, m_\tau). \tag{1.115}$$

We can now define the chiral mass eigenfields[39] $d'_{L/R}, u'_{L/R}, \ell'_{L/R}$ as

$$d_{L/R} = U^d_{L/R} d'_{L/R}, \quad u_{L/R} = U^u_{L/R} u'_{L/R}, \tag{1.116}$$
$$\ell_{L/R} = U^\ell_{L/R} \ell'_{L/R}, \qquad \nu_{\ell L} = U^\ell_L \nu'_{\ell L}. \tag{1.117}$$

Inserting all this into the Lagrangian (1.111) and using the explicit form of the physical mass eigenfields defined according to (1.25)

$$d' = d'_L + d'_R = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad u' = u'_L + u'_R = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad \ell' = \ell'_L + \ell'_R = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \tag{1.118}$$

[39] Note that in the minimal SM neutrinos are only LH and massless and no lepton mixing occurs, when ν'_L is defined in this way.

we achieve

$$\boxed{-\mathcal{L}_{\mathrm{EW}}^{\mathrm{Yuk}} = \sum_f m_f \overline{f} f \left(1 + \frac{h}{v}\right)}, \tag{1.119}$$

where $f = u, c, t, d, s, b, e, \mu, \tau$ are Dirac fields, even though we started with chiral fields.

1.3.6 Electroweak Currents and Quark Mixing

Finally, we discussed all necessary Lagrangians to write down the complete Lagrangian of the EW theory, which is given by

$$\boxed{\mathcal{L}_{\mathrm{EW}} = \sum_\psi \mathcal{L}_{\mathrm{EW}}^{\psi} + \mathcal{L}_{\mathrm{EW}}^{G} + \mathcal{L}_{\mathrm{EW}}^{\Phi} - \mathcal{L}_{\mathrm{EW}}^{\mathrm{Yuk}}}, \tag{1.120}$$

where we combined the Lagrangians given in (1.81), (1.79), (1.85) and (1.108). So far we have also already calculated partially the Lagrangians $\mathcal{L}_{\mathrm{EW}}^{\Phi}$ and $\mathcal{L}_{\mathrm{EW}}^{\mathrm{Yuk}}$ after SSB in unitary gauge, where we obtained the mass terms (1.93), (1.102) and (1.119). Furthermore, we can rewrite $\mathcal{L}_{\mathrm{EW}}^{G}$ in terms of the physical gauge fields found in (1.98) and (1.101), which has been done in [33, p.299], as well as in chapter two in [36].

At last we should investigate the fermion Lagrangian $\sum_\psi \mathcal{L}_{\mathrm{EW}}^{\psi}$, given in equation (1.81), when we insert the covariant derivative (1.105). Beside the kinetic terms of the fermion fields, the covariant derivative also yields the coupling of the fermion fields to the gauge fields, i.e. the charged weak, the neutral weak and the EM currents. Citing the results in [27, p.15/19ff] we find

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} \sum_\psi \overline{\psi} \gamma^\mu \left(T_+ W_\mu^+ + T_- W_\mu^-\right) \psi, \tag{1.121}$$

$$\mathcal{L}_{NC} = -\frac{g}{\cos\theta_W} \sum_\psi \overline{\psi} \gamma^\mu \left(T_3 - \sin^2\theta_W Q\right) \psi Z_\mu, \tag{1.122}$$

$$\mathcal{L}_{EM} = -e \sum_\psi Q_\psi \overline{\psi} \gamma^\mu \psi A_\mu, \tag{1.123}$$

where ψ denote all chiral quark and lepton flavor fields, which contribute to the charged, neutral and electromagnetic interactions.

Inserting the physical columns field d', u', ℓ', ν_ℓ' into $\mathcal{L}_{CC}$ we achieve

$$\mathcal{L}_{CC} = -\frac{g}{2\sqrt{2}} \left[\overline{u'} \gamma^\mu (\mathbb{1} - \gamma_5) V_{\mathrm{CKM}} d' + \overline{\nu'} \gamma^\mu (\mathbb{1} - \gamma_5) \ell'\right] W_\mu^+ + \mathrm{H.c.}\,, \tag{1.124}$$

with a 3×3 unitary matrix V_{CKM} called the *Cabibbo-Kobayashi-Maskawa (CKM) matrix*[40]:

$$\boxed{V_{\mathrm{CKM}} = U_L^{u\dagger} U_L^d}. \tag{1.125}$$

This matrix describes the mixing of quark flavors in charge weak interactions. It gives the relation between the flavor eigenfields d, s, b and the physical mass eigenfields d', s', b'

[40]Sometimes it is also called just Kobayashi-Maskawa (KM) matrix.

by

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix} = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}. \tag{1.126}$$

According to [25, p.321], "there are nine entries in the CKM matrix, but they are not all independent. V_{CKM} can be reduced to a kind of canonical form, in which there remain just three *generalized Cabibbo angles* , $(\theta_1, \theta_2, \theta_3)$ and one phase factor (δ):"

$$V_{\text{CMK}} = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix},$$

where c_i denotes $\cos\theta_i$ and s_i for $\sin\theta_i$. In [25, p.67] it is noted that although at weak vertices only members of the same generation are coupled, there is no flavor conservation in weak interactions. Hence, the EW theory is sometimes called *flavordynamics*.

It is mentioned in [36, p.79] that "there are several different ways to parametrize the CKM matrix. The parametrization advocated by the Particle Data Group is:

$$V_{\text{CKM}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix} \tag{1.127}$$

in which c_{ij} and s_{ij} are shorthand for $\cos\theta_{ij}$ and $\sin\theta_{ij}$ respectively, and the mixing angles, θ_{ij}, are experimentally known to satisfy $\theta_{13} \ll \theta_{23} \ll \theta_{12} \ll 1$."

Finally we also insert the physical fields into $\mathcal{L}_{\text{NC}}$ and $\mathcal{L}_{EM}$ and obtain

$$\mathcal{L}_{NC} = -\frac{g}{2\cos\theta_W} \sum_f \overline{f}\gamma^\mu \left(a_f - b_f\gamma_5\right) f Z_\mu \tag{1.128}$$

$$\mathcal{L}_{EM} = -e\sum_f Q_f \overline{f}\gamma^\mu f A_\mu. \tag{1.129}$$

Here $f = f_L + f_R$ denotes the physical fields u', d', ℓ', ν'_ℓ with charge Q_f and the coefficients in the neutral current interactions are given by

$$a_f = t^L_{3\,f} - 2\sin^2\theta_W Q_f, \quad b_f = t^L_{3\,f}. \tag{1.130}$$

The parameter $t^L_{3\,f}$ denotes the eigenvalue of T_3 in the SU(2) doublets, which are given in table 2 below.

f	u'	d'	ℓ'	ν'_ℓ
a_f	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$\frac{1}{2}$	$-\frac{1}{2} + 2\sin^2\theta_W$
b_f	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$

Table 2: Coefficients for neutral current weak interactions of different fermion types. Reprinted from [27, p.15/21].

2 Neutrino Mass Terms and Mixing Schemes

Now we want to discuss how neutrino mass terms can be included in the SM and for some introductory remarks we will refer to the discussions done in [36, p.397f], [37, p.4], [38, p.2ff,17], [39, p.6,28], and [2, p.678].

In the last section we have introduced an appropriate field description of the particle content of the SM and discussed the electroweak unification. We have seen that elementary fermions can be described by LH and RH Weyl spinors[41]. The crucial difference between neutrinos and other fermions is that they enter the SM Lagrangian with just one chirality, whereas charged leptons and quarks all come in pairs of LH and RH fields, which can be combined to Dirac spinors $\psi = \psi_L + \psi_R$. Since neutrinos are electrically and color neutral they only participate in weak interactions and therefore only one definite chirality state of neutrinos is required in the SM. In the Goldhaber experiment [40] it has been confirmed that only LH neutrinos take part in the weak interaction, which is in accordance with the fact that the EW theory is a chiral theory, where parity is maximally violated due to the V-A structure of the charged EW currents. Thus, the existence of only LH neutrinos is strongly connected to parity violation, since the EW Lagrangian[42] is not invariant under parity transformations, which has been confirmed in the famous Wu experiment [41].

A consequence of the absence of RH neutrinos in the SM is that no Yukawa interactions with the Higgs doublet exist for neutrinos as they do for charged leptons and quarks. Thus, neutrino masses can not be generated by the Higgs mechanism like the other fermion masses and they remain massless after SSB in the minimal version of the SM. The only chance to obtain neutrino mass terms is to extend the SM and use a *beyond the SM* mechanism for mass generation. Whereas gauge symmetry and Lorentz invariance need to be maintained, there are two other properties of the SM, which might be rethought, namely the particle content and renormalizabilty[43].

The simplest approach, where renormalizability of the theory is maintained, is to work in low energy regime (below the EW scale) and to extend the particle content of the SM by assuming there are RH neutrinos. Those have not been observed yet, because their interaction with other matter is too weak, as noted in [37, p.4]. So we simply introduce so-called *sterile RH neutrino fields* ν_R, which do not enter any (EW) interaction since they are

[41] We will describe in the following all fields by 4-component spinors. So if we are talking about LH resp. RH Weyl spinors we think of LH resp. RH 4-component spinors ψ_L, ψ_R like defined in (1.28).

[42] More precisely the charged currents cause parity violation.

[43] According to [42], "in the SM without RH singlet neutrino fields there are no renormalizable interactions that give masses to the neutrinos after the SSB of the EW gauge symmetry with the Higgs mechanism. However, there is a general belief that the SM is the low-energy manifestation of a more complete theory [43, 44] (for reviews see [39, p.28ff] or [45, 46]). The effect of this new theory is to induce in the Lagrangian of the SM non-renormalizable interactions which preserve the EW gauge symmetry above the EW symmetry breaking scale, but violate the conservation of lepton and baryon numbers (see [47] and references therein). These non-renormalizable interactions are operators of dimension $d > 5$ and must be multiplied by coupling constants that have dimension $\mathcal{M}^{4-d}$, where $\mathcal{M}$ is a mass scale characteristic of the new theory".

assumed to be total singlets under the entire SM gauge group, i.e. $Q = T = T_3 = Y = 0$. In this way both helicity states for neutrinos and antineutrinos are available to form a mass term. The number of RH neutrino fields $\nu_{\ell' R}$ can be arbitrary and can exceed the number of LH neutrinos $\nu_{\ell L}$. We will denote the number of RH neutrinos by n_R, counting by the flavor index $\ell' = e, \mu, \tau, \dots$ and analogously we will denote the number of LH neutrinos by n_L and count by the flavor index $\ell = e, \mu, \tau, \dots$. For simplicity we might assume in this section $n_R = n_L = n$, so the number of RH neutrinos coincides with the number of LH neutrinos[44].

Another open question is, whether neutrinos are of Dirac or Majorana nature. As mentioned in [42, p.5f], if we impose "total lepton number conservation[45], neutrinos with definite masses are four-component Dirac particles, since L is an adequate quantum number to distinguish between particle and antiparticle. But if the total lepton number is not conserved, massive neutrinos are truly neutral two-component Majorana particles". These possibilities correspond to the existence of different neutrino mass terms[46], as found in the Dirac Lagrangian (1.17) and Majorana Lagrangian (1.47). These terms are Lorentz invariant bilinears of a LH and a RH field and hence of the form $m_D\overline{\psi}\psi$ resp. $m_M\psi^T C\psi$, where m_D and m_M denotes the masses in a Dirac or Majorana mass term respectively . We will discuss those mass terms and their consequences on lepton number conservation and lepton mixing in detail in the following subsections.

2.1 Dirac Mass Term

In section 1.3.5, where we discussed mass generation for fermions, we have obtained the desired Dirac mass terms for all charged fermions by applying the Higgs mechanism to the Yukawa Lagrangian (1.108). Since we have added RH neutrinos to the theory we are now able to achieve Dirac mass terms for the neutrinos in a similar way. Therefore, like done in [2, p.679f], [37, p.4f] and [35, p.30], we introduce Yukawa interactions, which couple RH neutrinos to the LH ones via the Higgs field. As explained in [36, p.421], these Yukawa interaction terms are allowed, since they are gauge invariant, because the introduced RH neutrinos are singlets under the entire gauge group. So Yukawa couplings are the only possibility for those fields to couple to the ordinary SM particles. Thus, the new terms added to the SM Lagrangian are Lorentz and gauge invariant and also renormalizable[47]. So, besides a kinetic term for the RH neutrinos $\sum_{\ell'} i\overline{\nu_{\ell R}}\partial\!\!\!/\nu_{\ell' R}$, we add the mentioned Yukawa interactions to the leptonic part of Yukawa Lagrangian given in

[44]Of course in the SM $n_L = 3$, but we want to discuss the following in a more general way, where more than three lepton generations are possible.

[45]For details see appendix B.3.2.

[46]It should be emphasized like in [37, p.49] that even if the following mass terms are referred to as *Dirac* and *Majorana* they have not much to do with the appearance of Dirac or Majorana particles, apart from two special cases, which will be discussed in subsection 2.3.

[47]All terms of the Lagrangian have to be of dimension ≤ 4. For $\hbar = c = 1$ we have dim(mass) = 1, dim(length) $=-1$, dim(action) = 0, dim(vector field) = 1 = dim(scalar field) and dim(spinor field) = $^3/_2$. Thus, a Dirac mass term $m\overline{\psi}\psi$ has dimension $1 + {}^3/_2 + {}^3/_2 = 4$.

(1.108). Hence, we obtain

$$-\mathcal{L}_{\text{Yuk}}^{(\ell)} = \sum_{\ell,\ell'=e,\mu,\tau,\dots} \left(\overline{D}_{\ell L} \Phi \ell_{\ell' R} \Gamma_{\ell\ell'}^{(\ell)} + \overline{D}_{\ell L} \tilde{\Phi} \nu_{\ell' R} \Delta_{\ell\ell'}^{(\ell)} \right) + \text{H.c.}\,, \tag{2.1}$$

where $D_L = (\nu_L, \ell_L)^T$ denote again the LH lepton doublets and $\Gamma^{(\ell)}, \Delta^{(\ell)}$ are now $n \times n$ Yukawa coupling matrices.

2.1.1 Mass Term

Now in analogy to section 1.3.5 we apply the Higgs mechanism by SSB of the extended Yukawa Lagrangian (2.1) and inserting the unitary gauge for the Higgs doublet Φ and its complex conjugate $\tilde{\Phi}$ given in (1.92) and (1.110) respectively.

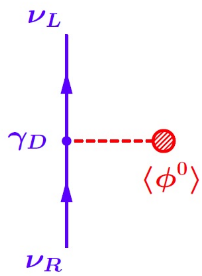


Figure 3: Neutrino Dirac Mass term generation via SSB and Higgs mechanism by coupling to the standard Higgs doublet Φ with Yukawa coupling γ_D.
Adapted from [48, p.5].

This procedure, visualized in figure 3, leads to

$$-\mathcal{L}_{\text{Yuk}}^{(\ell)} \overset{SSB}{\to} \mathcal{L}_D = - \sum_{\ell,\ell'=e,\mu,\tau,\dots} \overline{\ell}_{\ell L} (M_\ell)_{\ell\ell'} \ell_{\ell' R} + \overline{\nu}_{\ell L} (M_D)_{\ell\ell'} \nu_{\ell' R} + \text{H.c.}\,, \tag{2.2}$$

where the first term for charged lepton masses is already known, but the second term is the new Dirac mass term for neutrinos with a $n \times n$ mass matrix[48], M_D given by[49]

$$M_D = \frac{v}{\sqrt{2}} \Delta^{(\ell)\dagger}. \tag{2.3}$$

So we obtain the desired Dirac mass term for neutrinos

$$\mathcal{L}_D = - \sum_{\ell,\ell'=e,\mu,\tau,\dots} \overline{\nu_{\ell' R}} (M_D)_{\ell\ell'} \nu_{\ell L} + \text{H.c.} = -\overline{\nu_R} M_D \nu_L + \text{H.c.}\,, \tag{2.4}$$

where we introduced the $n-$component column vectors of neutrino flavor fields

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ \vdots \end{pmatrix}, \quad \nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \vdots \end{pmatrix}. \tag{2.5}$$

[48] The index D indicates the the mass matrix corresponds to a Dirac mass.
[49] We follow the convention in [35] by defining this matrix by the Hermitian conjugate of the Yukawa coupling $\Delta^{(\ell)}$.

Following [2, p.680] and using the same procedure as before in section 1.3.5, we diagonalize the mass matrix, which is assumed to be non-degenerate, by a biunitary transformation (theorem E.2.1). For doing so we introduce two $n \times n$ unitary matrices U_D^L and U_D^R, such that

$$U_D^{L\dagger} M_D U_D^R \equiv m_D, \tag{2.6}$$

with $(m_D)_{ik} = (m_D)_k \delta_{ik}$, $(m_D)_k \geq 0$. Again we introduce chiral mass eigenfields ν'_L and ν'_R, which are combined to physical mass eigenfields $\nu' = (\nu_k)_k$ with $k = 1, \dots, n$, i.e.

$$\nu_L = U_D^L \nu'_L, \quad \nu_R = U_D^R \nu'_R, \quad \nu' = \nu'_L + \nu'_R = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_n \end{pmatrix}, \tag{2.7}$$

where $\nu_k(x)$ is the field of a neutrino with mass $(m_D)_k$. Inserting this into (2.4) we get

$$\boxed{\mathcal{L}_D = -\overline{\nu'}_R m_D \nu'_L + \text{H.c.} = -\overline{\nu'} m_D \nu' = -\sum_{k=1}^{n} (m_D)_k \overline{\nu_k} \nu_k.}, \tag{2.8}$$

We have seen that the Dirac mass matrix M_D is proportional to the vacuum expectation value (VEV)[50] of the Higgs field, which is $\frac{v}{\sqrt{2}} = 174$ GeV. Following the arguments of [48, p.4], we would need Yukawa couplings that are very small ($\Delta^\nu \sim 10^{-12}$) compared to the Yukawa couplings of the electron ($\Gamma^e \sim 3 \times 10^{-6}$) for example, in order to accommodate the observed small neutrino masses $m_D \sim 0.1$ eV. This fact seems to be quite unnatural, and so it is noted in [38, p.18] that, "it is very unlikely that neutrino masses are of the same SM origin as masses of quarks and charged leptons". Furthermore, it is explained there, that these "extremely small values of the neutrino Yukawa couplings are commonly considered as a strong argument against this origin of neutrino masses" and hence we have to consider models of neutrino mass generation beyond the SM.

2.1.2 Mixing Matrix

In equation (2.7), we have introduced the physical mass eigenfields. We might rewrite these equations by carrying out the matrix multiplication more explicitly

$$\nu_{\ell L} = \sum_{k=1}^{n} (U_D^L)_{\ell k} \nu_{kL} \qquad (\ell = e, \mu, \tau, \dots), \tag{2.9}$$

$$\nu_{\ell' R} = \sum_{k=1}^{n} (U_D^R)_{\ell' k} \nu_{kR} \qquad (\ell' = e, \mu, \tau, \dots). \tag{2.10}$$

[50] In [37, p.5] it is explained that "in the early universe, when the temperature was high enough that Higgs particles were present in the primordial plasma ($T > T_{\text{EW}} \sim 140$ GeV for a Higgs mass $m_H \sim 125$ GeV), this interaction allowed ν_R-particles to participate in various different scattering processes. At energies much below the mass of the W-boson one can in good approximation replace the Higgs field ϕ by its VEV $\frac{v}{\sqrt{2}} = 174$ GeV [49, 50]. [...] Thus, at $T \ll T_{\text{EW}}$ the only effect of the Yukawa interaction is the generation of the Dirac mass term M_D, and the only way how the fields ν_R interact with the SM is via their mixing with ν_L due to M_D."

Hence, the n flavor fields $\nu_{\ell L}$ ($\ell = e, \mu, \tau, \dots$) are linear unitary combinations of the LH components ν_{kL} of n mass eigenfields of neutrinos with masses $(m_D)_k$ ($k = 1, \dots, n$) in the Dirac case, as noted in [42, p.6]. Even though we equivalently have (2.10), there is no mixing of sterile and active neutrinos in this scheme, since the sterile RH fields do not occur in the standard EW interaction Lagrangian.

Now we want to consider the lepton mixing matrix in analogy to the CKM quark mixing matrix (1.125) discussed in section 1.3.6. As done there before we now insert the physical neutrino fields (1.124) in the charged current Lagrangian, and obtain for the leptonic part

$$\mathcal{L}_{\text{CC}}^{(\ell)} = -\frac{g}{2\sqrt{2}} \left[\overline{\ell'} \gamma^\mu (\mathbb{1} - \gamma_5) V_{\text{PMNS}} \nu' W_\mu^- + \overline{\nu'} \gamma^\mu (\mathbb{1} - \gamma_5) V_{\text{PMNS}}^\dagger \ell' W_\mu^+ \right], \tag{2.11}$$

where we introduced in analogy to the CKM matrix the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix[51], which is given by

$$\boxed{V_{\text{PMNS}} = U_L^{\ell\,\dagger} U_D^L}\,. \tag{2.12}$$

In [36, p.412] in the case of $n = 3$ it is discussed that that "the PMNS matrix describes 3-flavor oscillations because lepton number conservations ensures that there are only three distinct mass eigenvalues amongst which oscillations can take place. In particular there are no new observable oscillation effects between the neutrinos and their antiparticles, since these are guaranteed to have exactly equal masses by the lepton symmetry".

According to [36, p.400f], [37, p.8] and also [42, p.7,10], in the case of three neutrino flavors $n = 3$, the mixing matrix may be generally parametrized in terms of three mixing angles and six phases. A convenient parametrization of the PMNS matrix is the one proposed in [51], as $V_{\text{PMNS}} = e^{i\hat{\varphi}} V K$ with

$$e^{i\hat{\varphi}} = \operatorname{diag}(e^{i\varphi_e}, e^{i\varphi_\mu}, e^{i\varphi_\tau}), \tag{2.13}$$

$$K = \operatorname{diag}\left(e^{i\alpha_1/2}, e^{i\alpha_2/2}, e^{i\alpha_3/2}\right), \tag{2.14}$$

$$\begin{aligned} V &= V_{23} V_{13} V_{12} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}, \end{aligned} \tag{2.15}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$.

The three unphysical phases φ_i can be absorbed into the LH charged lepton fields in the basis where M_ℓ is diagonal. By re-phasing these fields with $\ell''_L = e^{-i\hat{\varphi}} \ell'_L$ the phase factor $e^{i\hat{\varphi}}$ is removed from the charged current Lagrangian (2.11).

[51] Note the difference in the defintion to the CKM matrix: $V_{\text{CKM}} \longleftrightarrow V_{\text{PMNS}}^\dagger$ as emphasised in [35, p.30].

Comparing the matrix V with the CKM matrix in the parametrization (1.127), we see that it is CKM-like, whereas the full PMNS matrix is different, because of the appearance of the extra phases in the matrix K. As mentioned in [36, p.400f], "such phases are conventionally removed from the CKM matrix by performing phase rotations on the first- and second-generation quarks, and this is possible within the SM because the rest of the Lagrangian conserves flavor and so is unchanged by this re-phasing. The same rotations also remove the phases in the PMNS matrix in our case where neutrinos are Dirac particles", because the extra phases can be absorbed by re-phasing the neutrino fields. However, only one phase is measurable [52] and the PMNS-matrix is analogous to the CKM-matrix and depends on four parameters„ i.e. three angles and one CP-violating phase[52] δ_{13}.

2.1.3 Lepton Number Conservation

As outlined in [30] the Lagrangian has only one remaining global $U(1)$ symmetry, since a phase convention has been defined for the PMNS matrix, which fixed the parameters. According to [2, p.680] resp. [42, p.6] it is not difficult to convince oneself that not only the Dirac mass term but even the total Lagrangian is invariant under $U(1)$ transformations

$$\begin{aligned} &\nu_{\ell L}(x) \to e^{i\Lambda}\nu_{\ell L}(x), \quad \nu_{\ell R}(x) \to e^{i\Lambda}\nu_{\ell R}(x), \\ &\ell(x) \to e^{i\Lambda}\ell(x), \end{aligned} \tag{2.16}$$

where Λ is a constant parameter, since it is the same for all neutrino and charged lepton fields. Then, using Noether's theorem, on can see that the invariance of the Lagrangian under the transformation (2.16) implies that the total lepton charge or number[53] L is conserved and therefore, L is the quantum number that distinguishes a neutrino from an antineutrino.

This implies that the neutrino mass eigenfields ν_i are Dirac particles with $L(\nu_i) = 1$ and $L(\overline{\nu}_i) = -1$ for the antineutrinos $\overline{\nu}_i$. The Lagrangian of the theory containing the Dirac[54] mass term (2.4) is invariant with respect to the global gauge transformations

$$\begin{aligned} &\nu_i(x) \to e^{i\Lambda}\nu_i(x), \\ &\ell(x) \to e^{i\Lambda}\ell(x) \qquad \ell = e, \mu, \tau, \dots , \end{aligned} \tag{2.17}$$

where Λ is a constant parameter and we used (2.9).

Another important fact is that the family lepton numbers L_ℓ $(\ell = e, \mu, \tau, \dots)$ will not be conserved in a theory with a neutrino mass term given in (2.4) if the matrix M_D is

[52] It is mentioned in [42, p.7] that "in this parametrization the CP-violating δ_{13} phase is associated with s_{13} and hence it is clear that CP violation is negligible in the lepton sector if the mixing angle θ_{13} is small. More generally, it is possible to show that if any of the elements of the mixing matrix is zero, the CP-violating phase can be rotated away by a suitable re-phasing of the charged lepton and neutrino fields".

[53] See appendix B.3.2 for definition and discussion.

[54] So in this case the name *Dirac* mass term is really justified.

not diagonal. However, in case of this neutrino mass term, the total lepton charge is conserved and hence decays like

$$\begin{aligned} \mu^+ &\to e^+ + \gamma, & \mu &\to e^+ + e^- + e^+, \\ K^+ &\to \pi^+ + \mu^\pm + e^\mp, & \mu^- + (A,Z) &\to e^- + (A,Z), \end{aligned} \tag{2.18}$$

etc. are possible in a model, where neutrino mixing is given by (2.9). In the neutrino beams oscillations of the type $\overset{(-)}{\nu_\ell} \leftrightarrows \overset{(-)}{\nu_{\ell'}} \quad (\ell \neq \ell')$ should be observed. But beside this, as a consequence of the conservation of the total lepton charge L, neutrinoless double-β decay

$$(A,Z) \to (A, Z+2) + e^- + e^-, \tag{2.19}$$

and decays like

$$\begin{aligned} \mu^- + (A,Z) &\to e^+ + (A, Z-2), \\ K^+ &\to \pi^- + e^+ + \mu^+, \end{aligned} \tag{2.20}$$

etc. are forbidden.

2.2 Majorana Mass Term

As discussed before mass terms are Lorentz invariant bilinears of a LH and a RH field. Besides the Dirac mass term discussed above, we can also obtain another type of mass term for the LH neutrinos, called *Majorana mass term*, without the use of the introduced RH neutrino singlets. In fact, it is easy to see that the charge conjugated field $(\nu_{\ell L})^C$ is a RH field[55]. Hence, as noted by [48, p.4], a Majorana mass term requires only one helicity type of Weyl spinor.

As discussed in section 1.3 on the unified EW theory, Majorana masses are forbidden for quarks and charged leptons due to color and EM gauge invariance. However in the case of neutrinos, which are electrically and color neutral anyhow, Majorana mass terms are possible for both active LH neutrinos and sterile RH neutrinos, which will be carried out in detail in this section.

2.2.1 Mass Term for Sterile Neutrinos

Let us consider the easier case of the Majorana mass term for the sterile neutrinos first. The mass term is formed by the RH fields and their LH charge conjugated fields given in the column notation as defined in (2.5)

$$\nu_R = \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \\ \vdots \end{pmatrix}, \quad (\nu_R)^C = \begin{pmatrix} (\nu_{eR})^C \\ (\nu_{\mu R})^C \\ \nu_{(\tau R)}^C \\ \vdots \end{pmatrix}. \tag{2.21}$$

[55] For a proof see appendix B.2.4 calculation (**??**).

Thus, as noted in [39, p.8] and [37, p.5], the mass term is given by:

$$\mathcal{L}_M^{(R)} = -\frac{1}{2}\overline{(\nu_R)^C} M_R \nu_R + \text{H.c} = -\frac{1}{2} \sum_{\ell_1', \ell_2' = e, \mu, \tau, \dots} \overline{(\nu_{\ell_1' R})^C} (M_R)_{\ell_1' \ell_2'} \nu_{\ell_2' R} + \text{H.c}, \tag{2.22}$$

where M_R is a complex $n_R \times n_R$ matrix. It should be remembered, as in [30, p.207] that ν_R are RH Majorana spinor fields and the factor $^1/_2$ is introduced to compensate for double counting and ensuring a proper normalization of the fields.

It is explained in [48, p.5] that this mass term does not break EW gauge symmetry, because the RH neutrinos were introduced as sterile and hence, they are isospin singlets. Thus, this bare mass term can in principle[56] occur in the theory and, as we will see when discussing the seesaw mechanism, the mass of the sterile neutrinos plays a crucial role for mass generation of light LH neutrinos. Here we should note that the mass matrix M_R has to be symmetric. This can be easily shown by using the relation (B.49) and the properties of the charge conjugation matrix (B.20), (B.31), and (B.32) (as well as the fact that a minus sign appears when interchanging two fermionic field operators). Quoting the arguments in [2, p.681], we have

$$\overline{(\nu_R)^C} M_R \nu_R = -(\nu_R^T C^{-1} M_R \nu_R)^T = \nu_R^T (C^{-1})^T M_R^T \nu_R = \overline{(\nu_R)^C} M_R^T \nu_R. \tag{2.23}$$

So this implies $M_R^T = M_R$ and hence M_R is symmetric. Furthermore, it M_R can always be assumed to be non-degenerate and also already diagonal, such that ν_R are already mass eigenfields.

2.2.2 Mass Term for Active Neutrinos

Let us now proceed with the Majorana mass term for the LH neutrinos as discussed in [2, p.681]. The Majorana mass term is build in the same way as for RH neutrinos:

$$\mathcal{L}_M^{(L)} = -\frac{1}{2}\overline{(\nu_L)^C} M_L \nu_L + \text{H.c} = -\frac{1}{2} \sum_{\ell_1, \ell_2 = e, \mu, \tau, \dots} \overline{(\nu_{\ell_1 L})^C} (M_L)_{\ell_1 \ell_2} \nu_{\ell_2 L} + \text{H.c.}\,, \tag{2.24}$$

where we used the column notation for the fields given in (2.5) as before

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ \vdots \end{pmatrix}, \quad (\nu_L)^C = \begin{pmatrix} (\nu_{eL})^C \\ (\nu_{\mu L})^C \\ \nu_{(\tau L)}^C \\ \vdots \end{pmatrix}, \tag{2.25}$$

and where M_L is a complex $n_L \times n_L$ matrix, which is again symmetric by the same arguments as above in (2.23), so we also have $M_L^T = M_L$.

However, things are more complicated in this case, since the LH neutrino fields are

[56] It is mentioned in [48, p.5] that "in some models such a bare mass term for the RH neutrinos might be forbidden by new physics, and M_R is instead generated by the VEV of a SM Higgs triplet field S, or by a higher-dimension operator". The necessity of such a procedure will be also discussed shortly for the Majorana mass term of LH neutrinos.

no singlets under EW gauge transformations. Therefore, this kind of mass term is, like in the Dirac case, a priori forbidden since it would break gauge invariance and it can be only generated in the framework of a *beyond the SM* physics, as noted in [38, p.20]. As explained in [42, p.10], a Majorana mass term is only possible, if the scalar sector of the minimal SM is extended [53], in particular at tree-level a Higgs triplet is required [54, 55, 56]. According to [48, p.5] a Majorana mass M_L of the LH neutrinos can be achieved by a coupling as illustrated in figure 4. To ensure appropriately small masses, the Higgs triplet has to have a small Yukawa coupling γ_T and/or a small VEV $\langle\phi_T^0\rangle$. Besides, the possibility of "radiative mass generation at the one-loop level with a single charged Higgs singlet (plus an additional Higgs doublet [45, 57, 58, 59, 60]) or even at the 2-loop level with a doubly charged Higgs singlet (plus an additional singly charged scalar [61])" is mentioned in [42, p.9].

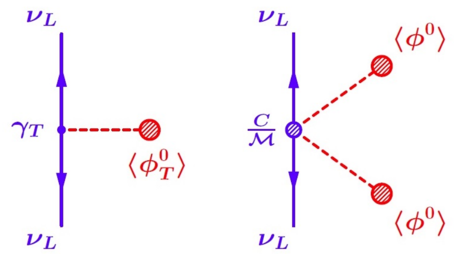


Figure 4: Left: Majorana mass term generated by a Higgs triplet. Right: Majorana mass term generated by a higher-dimension operator. Adapted from [48, p.5].

In [48, p.5] and [37, p.7] also a different approach concerning the repeal of renormalizability is considered[57]. The Majorana mass term for the active neutrinos can be constructed without adding any new degrees of freedom to the SM from a higher-dimensional operator - the so-called Weinberg operator [62]

$$\frac{1}{2}\overline{\ell_L}\tilde{\phi} f \tilde{\phi}^T \ell_L^C + \text{H.c.}\,, \tag{2.26}$$

where f is some flavor matrix of dimension mass^{-1}.

This dimension-5 operator is not renormalizable and is constructed of two Higgs doublets and a coefficient $f \sim {}^{C}/_{\mathcal{M}}$. After SSB this operator generates the mass term as indicated in figure 4 above. In an effective field theory approach it can be understood as the low energy limit of renormalizable operators, which are obtained after integrating out heavier degrees of freedom. These can be very heavy Majorana sterile neutrinos (the type I seesaw), a heavy scalar triplet (the type II seesaw [63]), a fermion triplet (the type III seesaw [64]), or new degrees of freedom in a string theory [65].

In this thesis we will deal with low energy scales, where renormalizability is maintained and the mass terms will be generated by heavy RH neutrino singlets, i.e. we will consider the seesaw mechanism type I. In order to proceed analogously to the previous case of a Dirac mass term, we diagonalize the mass matrix M_L and rewrite the Lagrangian in terms of mass eigenfields.

[57] A more elaborate discussion can be found e.g. in [39, p.28ff].

It is noted in [36, p.399f] that since M_L is a symmetric matrix, it has fewer independent entries than the quark or charged lepton mass matrix or the Dirac mass matrix and we shall be interested in the case of non-degenerate eigenvalues. To diagonalize the mass matrix M_L we follow [2, p.681f] and apply Schur's theorem (theorem E.2.2), which tells us that a complex symmetric matrix can always be diagonalized with a unitary matrix U_M:

$$M_L = (U_M^\dagger)^T m_L U_M^\dagger, \tag{2.27}$$

where U_M is a $n_L \times n_L$ matrix and $(m_L)_{ik} = (m_L)_k \delta_{ik},\ (m_L)_k \geq 0$ for $k = 1, \dots, n_L$ We now insert (2.27) into (2.24) and obtain[58]

$$\mathcal{L}_M^{(L)} = -\frac{1}{2}\overline{(N_L)^C} m_L N_L - \frac{1}{2}\overline{N}_L m_L (N_L)^C, \tag{2.28}$$

where we defined chiral physical fields in analogy to (2.7) by

$$N_L = U_M^\dagger \nu_L, \quad (N_L)^C = C\overline{N_L}^T. \tag{2.29}$$

Here we combined similarly to (2.7) both chiral fields to

$$\xi = N_L + (N_L)^C = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n_L} \end{pmatrix}, \tag{2.30}$$

and finally obtain[59] the neutrino mass term in the form

$$\boxed{\mathcal{L}_M^{(L)} = -\frac{1}{2}\overline{\xi} m_L \xi = -\frac{1}{2}\sum_{k=1}^{n_L} (m_L)_k \overline{\xi_k}\xi_k}, \tag{2.31}$$

where ξ_k is the field of a neutrino with mass $(m_L)_k$. These fields ξ_k satisfy the condition

$$\xi_k(x) = C\overline{\xi_k}^T(x), \qquad k = 1, 2, \dots, n_L \tag{2.32}$$

which means that the $\xi_k(x)$ are Majorana fields, since it is simply the Majorana condition given in (1.39).

2.2.3 Mixing Matrix

Again following [2, p.682], we obtain from (2.29) and (2.30) the mixing which connects the flavor fields to the mass eigenfields in the following from

$$\nu_L = U_M \xi_L, \tag{2.33}$$

[58] For a detailed derivation see appendix F.2.
[59] For a detailed derivation see appendix F.3.

i.e. in component notation

$$\nu_{\ell L} = \sum_{k=1}^{n_L} (U_M)_{\ell k} \xi_{kL} \qquad \ell = e, \mu, \tau, \ldots . \tag{2.34}$$

Hence, the LH flavor fields are linear combinations of the LH component of the fields of Majorana neutrinos with definite masses and there are as many Majorana neutrinos as flavor neutrinos (i.e. the number of charged leptons). There are $2n_L$ states with different chirality of the n_L massive Majorana neutrinos, which correspond to the $2n_L$ neutrinos and antineutrinos $(\nu_e, \nu_\mu, \nu_\tau, \overline{\nu_e}, \overline{\nu_\mu}, \overline{\nu_\tau}, \ldots)$.

As done in the section before we now insert the physical neutrino fields (1.124) in the charged current Lagrangian, and obtain for the leptonic part

$$\mathcal{L}_{\mathrm{CC}}^{(\ell)} = -\frac{g}{2\sqrt{2}} \left[\overline{\ell'} \gamma^\mu (\mathbb{1} - \gamma_5) V_{\mathrm{PMNS}} \xi W_\mu^- + \overline{\xi} \gamma^\mu (\mathbb{1} - \gamma_5) V_{\mathrm{PMNS}}^\dagger \ell' W_\mu^+ \right], \tag{2.35}$$

where we have introduced again in analogy to the CKM matrix the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix[60], which is given by

$$\boxed{V_{\mathrm{PMNS}} = U_L^{\ell\,\dagger} U_M} . \tag{2.36}$$

Now following the arguments in [36, p.400f], we note that the neutrino mass matrix was diagonalized by redefining the neutrino fields. But there are not enough fields, which can be independently rotated to achieve the diagonalization, because the neutrino mass matrix does not connect different LH and RH fields. Therefore we get less independent physical parameters from these rotations, which leads to more CP-violating phases for leptons than for quarks.

As in the Dirac case, we consider three neutrino flavors $n_L = 3$, where the mixing matrix may be parametrized in terms of three mixing angles and six phases. Three of the phases can be absorbed into the LH charged lepton fields as discussed in the Dirac case. We want to restate the convenient parametrization of the PMNS matrix we used before in the form[61] of $V_{\mathrm{PMNS}} = VK$ with $K = \mathrm{diag}\left(e^{i\alpha_1/2}, e^{i\alpha_2/2}, e^{i\alpha_3/2}\right)$ and

$$\begin{aligned} V &= V_{23} V_{13} V_{12} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta_{13}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta_{13}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta_{13}} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta_{13}} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta_{13}} & c_{23} c_{13} \end{pmatrix}, \end{aligned} \tag{2.37}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$.

[60] Note again the difference in the defintion to the CKM matrix: $V_{\mathrm{CKM}} \longleftrightarrow V_{\mathrm{PMNS}}^\dagger$ as emphasised in [35, p.30]

[61] We assume the three phases are already absorbed in the LH charged lepton fields.

As already discussed in the Dirac case in section 2.1.2 the PMNS matrix differs form the CKM matrix by the extra phases in the matrix K. But now in the Majorana case we are not able to perform an appropriate re-phasing to absorb these phases. As we will see in the next section the Majorana neutrino mass term is not invariant under lepton number transformations and so it is not conserved by re-phasing of the LH neutrino states as in the Dirac case before.

In [36, p.401] it is noted that "the phases δ_{13} and α_i can have physical implications because they introduce CP-violation into neutrino physics. Since the phase δ_{13} is the direct analogue of the CP-violation phase in the CKM matrix, its effects disappear in the limit $\theta_{13} \to 0$. One of the phases α_i can be rotated away by making a common phase rotation of the charged leptons (conventionally α_3 is removed), and the other two are not observable in processes which conserve total lepton number L".

2.2.4 Lepton Number Conservation

Finally we want to discuss lepton number conservation in the case of a Majorana mass term as we did before in the case of a Dirac mass term in section 2.1.3 and we are following again the discussion in [2, p.682f].

The mass term (2.24) can not be invariant under any global gauge transformation and hence, in this case lepton charges are not conserved. Therefore, it is not possible to distinguish neutrino from antineutrino and the mass eigenfields in the mass term (2.24) are Majorana neutrinos.

The difference between the Dirac and Majorana cases can be emphasized in the following way. It follows from (B.27) and (2.32) that the LH and the RH components of the Majorana field $\xi_k(x)$ are related by

$$\xi_{kR}(x) = C\overline{\xi_{kL}}^T(x), \tag{2.38}$$

whereas in the case of a Dirac field $\nu_k(x)$ the LH and the RH components are independent.

In the Majorana case the charged lepton mass term and the weak currents are invariant under global transformation of the fields $\xi_k(x)$ and $\ell(x)$ of the form

$$\begin{aligned} \xi_{kL}(x) \to \xi'_{kL}(x) = e^{i\Lambda}\xi_{kL}(x), \\ \ell(x) \to \ell'(x) = e^{i\Lambda}\ell(x). \end{aligned} \tag{2.39}$$

Nevertheless, the neutrino massterm breaks this symmetry because due to (2.38) we have[62]

$$\xi'_{kR}(x) = e^{-i\Lambda}\xi_{kR}(x). \tag{2.40}$$

Thus, in a theory with Majorana mass term (2.24) neither any of the lepton charges L_ℓ, $\ell = e, \mu, \tau, \ldots$ is conserved nor is the total lepton charge $L = \sum_\ell L_\ell$. Consequently, in this theory also $(\beta\beta)_{0\nu}$ decay and other similar processes are not forbidden. Furthermore,

[62] This is the reason why we cannot absorb the extra CP-violating phases into the neutrino fields, as discussed in the previous section 2.2.3.

in this case of massive Majorana neutrinos the mixing scheme also leads to oscillations in the neutrino beams like in the Dirac case.

2.3 Hybrid Dirac-Majorana Mass Term

In the previous two sections we have discussed possible Dirac and Majorana mass terms for neutrinos separately. However, in a theory where both, active LH neutrino fields as well as sterile RH neutrino singlets are present, the most general mass term includes the Dirac mass term as well as Majorana mass terms for LH and RH neutrinos. This hybrid mass term is called Dirac-Majorana mass term and will be discussed in this section. We consider n_L active neutrinos and n_R sterile neutrinos, where we take for simplicity[63] $n_L = n_R = n$ as before.

2.3.1 Mass Term

Again we start by writing down the possible mass term as it is done in [42, 7.f], [39, p.8f], or [2, p.683]. For a first approach we simply sum up the Lagrangians $\mathcal{L}_D, \mathcal{L}_M^{(L)}$ and $\mathcal{L}_M^{(R)}$, we have already encountered in (2.4), (2.22) and (2.24):

$$\begin{aligned}\mathcal{L}_{D+M} =&\mathcal{L}_D + \mathcal{L}_M^{(L)} + \mathcal{L}_M^{(R)}\\ =&-\sum_{\ell,\ell'=e,\mu,\tau,\dots} \overline{\nu}_{\ell' R}(M_D)_{\ell\ell'}\nu_{\ell L} + \text{H.c.}\\ &-\frac{1}{2}\sum_{\ell_1,\ell_2=e,\mu,\tau,\dots} \overline{(\nu_{\ell_1 L})^C} M_{L\ell_1\ell_2}\nu_{\ell_2 L} + \text{H.c}\\ &-\frac{1}{2}\sum_{\ell_1',\ell_2'=e,\mu,\tau,\dots} \overline{(\nu_{\ell_1' R})^C} M_{R\ell_1'\ell_2'}\nu_{\ell_2' R} + \text{H.c.}\,.\end{aligned} \tag{2.41}$$

The matrices M_D, M_L and M_R are complex $n \times n$ matrices.

It is convenient and useful to rewrite this Lagrangian in a more compact form. For doing this we introduce the LH column fields with $2n$ components

$$\omega_L = \begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix}, \tag{2.42}$$

where the $n-$component column fields are defined like in (2.5) by

$$\nu_L = \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \\ \vdots \end{pmatrix}, \quad (\nu_R)^C = \begin{pmatrix} (\nu_{eR})^C \\ (\nu_{\mu R})^C \\ (\nu_{\tau R})^C \\ \vdots \end{pmatrix}. \tag{2.43}$$

Applying this knowledge we can rewrite[64] (2.41) in terms of this LH column vector ω_L

[63]In the next section on the seesaw mechanism we will consider also the general case of $n_L \neq n_R$.

[64]In [35, p.43] this derivation is done more explicit. The calculus is a bit lengthy and therefore can be found in the appendix F.4.

as

$$\boxed{\mathcal{L}_{D+M} = -\frac{1}{2}\overline{(\omega_L)^C} M \omega_L + \text{H.c.}}. \tag{2.44}$$

The mass matrix M is a symmetric[65] $2n \times 2n$ matrix, which is built of blocks of the mass matrices M_D, M_L and M_R such that

$$M = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{2.45}$$

Now we diagonalize the matrix M like the matrix M_L using Schur's theorem (theorem E.2.2) and assuming M to be not degenerate, which states the existence of a unitary $2n \times 2n$ matrix U such that

$$M = (U^\dagger)^T m U^\dagger, \tag{2.46}$$

where m is the diagonal matrix

$$m_{ik} = m_k \delta_{ik}, \quad m_k \geq 0, \quad i,k = 1,2,\ldots,2n. \tag{2.47}$$

We repeat the procedure done before and by defining the chiral physical fields

$$\omega_L' = U^\dagger \omega_L, \tag{2.48}$$

and inserting ω_L' and (2.46) into the Lagrangian (2.44), we achieve

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{(\omega_L')^C} m \omega_L' + \text{H.c.}. \tag{2.49}$$

In the last step we introduce the physical mass eigenfields χ as

$$\chi = \omega_L' + (\omega_L')^C = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{2n} \end{pmatrix}, \tag{2.50}$$

and rewrite the Lagrangian (2.49) in terms of this field to obtain

$$\mathcal{L}^{D+M} = -\frac{1}{2}\overline{\chi} m \chi = -\frac{1}{2}\sum_{k=1}^{2n} m_k \overline{\chi}_k \chi_k. \tag{2.51}$$

Finally, we should note that the general Dirac-Majorana mass term (2.44) contains some special cases and limits as discussed in [48, p.6], [37, p.9f], [36, p.414f], and [66, p.298]. We will distinguish different scenarios due to the relative size between M_R and M_D (in particular between their eigenvalues). For simplicity we assume only one generation (n=1) of neutrinos[66] for this discussion, where we have $M_D = m_D$, $M_L = m_L$ and $M_R =$

[65]This is quite obvious, because $M^T = \begin{pmatrix} M_L^T & M_D^T \\ (M_D^T)^T & M_R^T \end{pmatrix} = M$, since M_L and M_R are symmetric.

[66]For this case an elaborate discussion can be found in [42, p.11f].

m_R. According to [66, p.298] we can find the following eigenvalues for M:

$$m_{1,2} = \frac{1}{2}\left[(m_R + m_L) \pm \sqrt{(m_L - m_R)^2 + 4m_D^2}\right]. \tag{2.52}$$

We can now consider six different cases or limits of the general Dirac-Majorana mass term (2.44):

(i) $\mathbf{m_L = m_R = 0}$: Here we have $m_{1,2} = m_D$ and (2.44) reduces to the Dirac mass term given in (2.4) and the result a Dirac neutrinos and conserved lepton numbers. We should, as in [37, p.49], note that in this *Dirac case* we are able to combine the chiral Weyl fields into Dirac spinors, because the unbroken subgroup $U(1)_{\mathrm{EM}}$ with its associated conserved current leads to a mass matrix that allows this combination.

(ii) $\mathbf{m_D = 0}$: Similarly in this case we have $m_{1,2} = m_L, m_R$ and this limit of (2.44) leads to the sum of the Majorana mass terms for LH and RH neutrinos we considered in (2.24) and (2.22). In [37, p.49] it is emphasized that in this case we obtain $n_L + n_R$ (in the simplification considered here it is 2) Majorana spinors. In the simplest case, where the mass matrices M_L and M_R are diagonal, we can find the mass terms in the form of $\mathcal{L}_M^{(L)}$ and $\mathcal{L}_M^{(R)}$. As we have seen in this *Majorana case* we have no mixing between active LH and sterile RH neutrino fields, which are both Majorana neutrinos and therefore there is no conservation of lepton numbers.

(iii) $\mathbf{m_D \approx m_L}$ and/or $\mathbf{m_D \approx m_R}$: If all mass matrices are approximately the same size this leads in general to $n_L + n_R$ Majorana neutrinos with comparable masses. This means that the sterile RH neutrinos are quite as light as the active LHn neutrinos. In this particular case there occurs quite strong mixing between the active LH and sterile RH fields and we call this the *active-sterile mixed case.*

(iv) $\mathbf{m_D \gg m_L, m_R}$: We call this limit the *pseudo-Dirac limit*, since $m_{1,2} \approx m_D$, because the smallness of the Majorana masses m_L and m_R leads just to a small shift from m_D in the eigenvalues. According to [36, p.415], for $n_L < n_R$ there will be $2n_L$ massive neutrino states participating in the weak interaction. In the general case, where it can be $n_L \geq n_R$ or $n_L < n_R$, we have $\min(n_L, n_R)$ almost-degenerate pairs of massive neutrinos plus $|n_L - n_R|$ light states.

(v) $\mathbf{m_D \ll m_L, m_R}$: This case might be called the *Pseudo-Majorana case*, since the eigenvalues are $m_{1,2} \approx m_L, m_R$. The LH and RH neutrinos do not mix significantly, and the observed neutrino oscillations are completely described by the Majorana mass matrix m_L.

(vi) $\mathbf{m_R \gg m_D, m_L}$: The last case considered here is the one we will consider in the rest of this thesis and it is called the *seesaw limit.* As noted in [48, p.6] for $n_L = 1 = n_R$ we have one mainly sterile state, with heavy mass $m_2 \simeq m_R$ and one very light active state. We will show in section 3 how $m_1 \simeq m_L - {m_D^2}/{m_R}$ is achieved. For $m_L = 0$, this yields an elegant explanation for the smallness of the (active) neutrino masses $|m_1| \ll m_D$. This specific case when $m_L = 0$ is referred to as seesaw mechanism type I.

2.3.2 Mixing Matrix

Following [2, p.683f] we find the relation between the flavor fields $\nu_{\ell L}(x)$ and $(\nu_{\ell R}(x))^C$ and the LH components of the $2n$ Majorana fields $\chi_k(x)$ by using the unitarity of the matrix U. Then we find

$$\omega_L = U\omega'_L = UP_L\chi, \tag{2.53}$$

and might denote $P_L\chi \equiv \chi_L$. Hence, we obtain

$$\nu_{\ell L} = \sum_{k=1}^{2n} U_{\ell k}\chi_{kL}, \quad (\nu_{\ell' R})^C = \sum_{k=1}^{2n} U_{\ell' k}\chi_{kL}, \tag{2.54}$$

where $\ell = e, \mu, \tau, \ldots$ (altogether n values) and the index ℓ' takes the n lower rows of the mixing matrix U.

Thus, the n flavor fields $\nu_{\ell L}(x)$ are linear combinations of the LH components of $2n$ Majorana mass eigenfields in the hybrid mass term case. The crucial point is that the sterile fields $(\nu_{\ell R}(x))^C$ are also linear combinations of the LH components of the same $2n$ Majorana fields. From (2.53) we find for the RH components χ_R

$$(\omega_L)^C = U^* P_R\chi, \tag{2.55}$$

where we denote analogously $P_R\chi = \chi_R$ and hence we achieve

$$(\nu_{\ell L})^C = \sum_{k=1}^{2n} U^*_{\ell k}\chi_{kR}, \quad \nu_{\ell' R} = \sum_{k=1}^{2n} U^*_{\ell' k}\chi_{kR}. \tag{2.56}$$

Finally, we should investigate the charged current Lagrangian (1.124) in terms of the mass eigenfields. As it is discussed in [36, p.414], it should be emphasized that in the case of a Dirac-Majorana mass term the PMNS-matrix is no longer a square matrix[67], even in the case for $n_L = n_R = n$. The matrix $U_L^{(\ell)}$ transforming charged lepton flavor fields into mass eigenfields is a $n \times n$ matrix, whereas the neutrino mass eigenfields are obtained by mixing of $2n$ Majorana fields via the $2n \times 2n$ matrix U, as derived above in (2.54). If we write down the charged current Lagrangian explicitly, we find

$$\begin{aligned}\mathcal{L}_{\mathrm{CC}}^{(\ell)} &= -\frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau,\ldots} \overline{\ell}_{\ell L}\gamma^\mu \nu_{\ell L} W_\mu^- + \text{H.c.} \\ &= -\frac{g}{\sqrt{2}} \sum_{\ell=e,\mu,\tau,\ldots} \sum_{i=1}^{n}\sum_{k=1}^{2n} \overline{\ell'}_{iL} (U_L^{(\ell)})^*_{i\ell}\gamma^\mu U_{\ell k}\chi_{kL} W_\mu^- + \text{H.c.}\,.\end{aligned} \tag{2.57}$$

The matrix multiplication can be carried out since the transformation into neutrino eigenfields given in (2.54) takes only the first n rows of U and hence we achieve for the

[67] We will discuss this topic for $n_L \neq n_R$ in section 3.1.4.

PMNS mixing matrix

$$K := \sum_{\ell=e,\mu,\tau,\dots} (U_L^{(\ell)})^*_{i\ell} U_{\ell k}, \tag{2.58}$$

which is now a $n \times 2n$ rectangular matrix[68]. The crucial point in this is that "the existence of these new mixing elements implies that the introduces RH neutrinos can contribute due to mixing to the charged current weak interactions" [36, p.414].

Furthermore, this particular mixing leading to mass eigenfields (2.54) also has consequences on the neutral current Lagrangian (1.122). In the previous cases of purely Dirac or Majorana mass terms, where the full summation over all rows of the unitary matrices U_D resp. U_M is performed, no mixing effects occur in the neutral currents in terms of mass eigenfields. Now we find for the neutrino part of the neutral current Lagrangian using $Q_\nu = 0$ and $t^L_{3\,f} = {}^1\!/_2$ in equation (1.128):

$$\begin{aligned}
\mathcal{L}_{\text{NC}}^{(\nu)} &= -\frac{g}{\cos\theta_W} \sum_{\ell=e,\mu,\tau,\dots} \overline{\nu}_{\ell L} \gamma^\mu \left(T_3 - \sin^2\theta_W Q\right) \nu_{\ell L} Z_\mu + \text{H.c.} \\
&= -\frac{g}{2\cos\theta_W} \sum_{\ell=e,\mu,\tau,\dots} \overline{\nu}_{\ell L} \gamma^\mu \nu_{\ell L} Z_\mu + \text{H.c.} \\
&= -\frac{g}{2\cos\theta_W} \sum_{\ell=e,\mu,\tau,\dots} \sum_{k=1}^{2n} \sum_{j=1}^{2n} \overline{\chi}_{jL} U^*_{j\ell} \gamma^\mu U_{\ell k} \chi_{kL} Z_\mu + \text{H.c.}\,.
\end{aligned} \tag{2.59}$$

Here the summation over ℓ runs only over the the first n columns of $U^\dagger$ and the first n rows of U, and thus the unitarity property of U cannot be applied. The matrix multiplication is just performed for the $2n \times n$ submatrix of U^* and the $n \times 2n$ submatrix of U and we obtain

$$P_{jk} := \sum_{\ell=e,\mu,\tau,\dots} U^*_{j\ell} U_{\ell k}, \tag{2.60}$$

where P is a $2n \times 2n$ matrix[68].

2.3.3 Lepton Number Conservation

As noted in [2, p.683f], the mass eigenfields χ defined in (2.50) obviously satisfy the Majorana condition

$$\chi_k(x) = C\overline{\chi}_k^T(x). \tag{2.61}$$

Therefore, neutrinos are Majorana particles in this case and hence, as already discussed in section 2.2, it is evident that the Dirac-Majorana mass term is not invariant in the general case under any global gauge transformation of the neutrino fields. This implies that in the theory under discussion no lepton charges are conserved and in a theory with a Dirac-Majorana mass term decays like

$$\begin{aligned}
&\mu^+ \to e^+ + \gamma, \\
&\mu^+ \to e^+ + e^- + e^+, \quad \text{etc.,}
\end{aligned} \tag{2.62}$$

[68] We use the notation of [67].

as well as such as $(\beta\beta)_{0\nu}$ decay are allowed.

Besides these analogies a crucial difference can be found concerning neutrino oscillations. Whereas in the Dirac and Majorana case oscillations only occur between active neutrinos, in the hybrid case oscillations between active and sterile neutrinos are possible.

3 Seesaw Mechanism

We have already mentioned the *seesaw limit* in section 2.3.1, where we discussed several possible cases and limits for the Dirac-Majorana mass term. We indicated there how RH neutrino singlets added to the SM can lead to very light active neutrinos. As noted in [37, p.11], increasing the scale m_R of the eigenvalues of the Majorana mass matrix M_R gives rise to heavy[69] sterile neutrinos whereas the masses of the active ones decrease. This mechanism is therefore called *seesaw mechanism.* According to [19], the mass scale m_R should be more precisely[70] understood as the order of magnitude of the eigenvalues of $\sqrt{M_R^\dagger M_R}$ and the seesaw limit $m_R \gg m_L, m_D$ should be understood in that sense that the eigenvalues of $\sqrt{M_R^\dagger M_R}$ are all much larger than the matrix elements of M_L and M_D.

3.1 Seesaw Mechanism Type I

We will concentrate in this master thesis on the seesaw mechanism of type I, which was first considered in [68, 17, 16, 69, 18], where the SM is augmented by RH neutrinos singlets. The addition of these RH heavy neutrino singlets leads to mass generation of neutrinos at tree-level (see figure 5 below). We have already discussed in section 2.2.2 that a Majorana mass term for active LH neutrinos can only be generated by extension of the scalar sector by a Higgs triplet or by a higher dimensional operator in a non-renormalizable theory, where a heavy virtual fermion singlet field is exchanged[71]. In this thesis we want to focus on a model, as done in [19], where renormalizability is maintained and the presence of a Higgs triplet is avoided. Hence, in the seesaw model considered here no Majorana mass term for active neutrinos at tree-level occurs, i.e. $M_L = 0$.

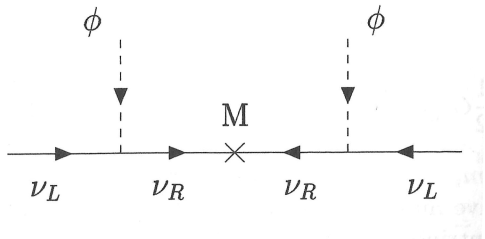


Figure 5: LH neutrino aquires small Majorana mass through seesaw mechanism via exchange of a virutal heavy RH neutrino. Reprinted from [70, p.280].

[69]Since "the Majorana mass terms need not to vanish in the limit where the EW symmetry is unbroken, the sterile neutrinos are naturally expected to be at the weak SSB scale or larger", as it is noted in [36, p.422]. We will discuss different mass scales in more detail in subsection 3.1.3.

[70]We will denote the scale of the Dirac mass matrix M_D, or also here more precisely the scale of eigenvalues of $\sqrt{M_D^\dagger M_D}$ by m_D. The scale m_L for the Majorana masses of LH neutrinos is understood analogously.

[71]For a discussion of this see [38], [39] or [36].

3.1.1 General Mass Term

For model building we use what we have already encountered in the previous section 2 on neutrino mass terms, especially what we have discussed in the case of the Dirac-Majorana mass term in section 2.3. However, here we do not longer restrict ourselves to a number of sterile neutrinos coinciding with the number of active neutrinos. We will examine the mass term in the most general case for $n_L \neq n_R$ and discuss the consequences on mass generation and mixing in the cases $n_L \leq n_R$ or $n_L > n_R$. We might introduce a more useful and illustrative index notation. The flavor indices $\ell = 1, \dots, n_L$ will denote the n_L LH active neutrino fields, whereas $\ell' = 1, \dots, n_R$ will denote the n_R RH sterile ones. The most general Dirac-Majorana mass term is given in analogy to equation (2.41), but as indicated before we will stick to a model where the symmetric $n_L \times n_L$ Majorana matrix for the active neutrinos $M_L = 0$. Hence, the actual mass term for the seesaw mechanism we like to concentrate on is

$$\begin{aligned}\mathcal{L}_{D+M} =& \mathcal{L}_D + \mathcal{L}_M^{(R)} \\ =& -\sum_{\ell=1}^{n_L}\sum_{\ell'=1}^{n_R} \overline{\nu}_{\ell' R} (M_D)_{\ell\ell'} \nu_{\ell L} + \text{H.c.} \\ & -\frac{1}{2}\sum_{\ell_1', \ell_2'=1}^{n_R} \overline{(\nu_{\ell_1' R})^C} M_{R\ell_1'\ell_2'} \nu_{\ell_2' R} + \text{H.c.}\ .\end{aligned} \tag{3.1}$$

The mass matrices M_D and M_R are complex $n_L \times n_R$, $n_R \times n_R$ matrices, respectively. As before in section 2.3, the Majorana mass matrix M_R is symmetric, whereas it should be noted that in the most general case for $n_L \neq n_R$ the Dirac mass matrix M_D will be a rectangular and not a square matrix.

Following the procedure done before, we are able to write this mass term in a more compact way. Using the LH column vectors ω_L defined in (2.42):

$$\omega_L = \begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix}\ , \tag{3.2}$$

we can rewrite again (3.1) as

$$\boxed{\mathcal{L}_{D+M} = -\frac{1}{2}\overline{(\omega_L)^C} M_{\text{D+M}} \omega_L + \text{H.c.}}\ . \tag{3.3}$$

The mass matrix M_{D+M} is now a symmetric $(n_L + n_R) \times (n_L + n_R)$ matrix, which is built of blocks of the Dirac mass matrix M_D and the Majorana mass matrix M_R for RH neutrino singlets[72]:

$$M_{\text{D+M}} = \begin{pmatrix} 0 & M_D^T \\ \underbrace{M_D}_{n_L} & \underbrace{M_R}_{n_R} \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix}\ . \tag{3.4}$$

[72]These are the matrices introduced in (2.3) and (2.22) discussed in the sections 2.1.1 and 2.2.1.

Now we proceed by diagonalizing the mass matrix and introduce mass eigenfields like in the section before. We will apply the notation used in [20, 21, 71]. The symmetric mass matrix $M_{\text{D+M}}$ can be diagonalized by a unitary $(n_L + n_R) \times (n_L + n_R)$ matrix U according to Schur's theorem (theorem E.2.2) as

$$U^T M_{\text{D+M}} U = \hat{m} = \text{diag}(m_1, m_2, \dots, m_{n_L+n_R}), \tag{3.5}$$

where the diagonal entries m_i for $i = 1, \dots, n_L + n_R$ are real and non-negative. The unitary matrix U can be decomposed into two submatrices

$$U = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix}, \tag{3.6}$$

where U_L is a $n_L \times (n_L + n_R)$ and U_R a $n_R \times (n_L + n_R)$ matrix. Since U is unitary, i.e. $UU^\dagger = U^\dagger U = \mathbb{1}_{n_L+n_R}$, the submatrices have to fulfil the following properties[73], as noted in [20]:

$$U_L U_L^\dagger = \mathbb{1}_{n_L}, \tag{3.7}$$
$$U_R U_R^\dagger = \mathbb{1}_{n_R}, \tag{3.8}$$
$$U_L U_R^T = 0_{n_L \times n_R}, \tag{3.9}$$
$$U_L^\dagger U_L + U_R^T U_R^* = \mathbb{1}_{n_L+n_R}. \tag{3.10}$$

Applying this decomposition to equation (3.5) we get

$$\begin{aligned} U_L^* \hat{m} U_L^\dagger &= 0_{n_L \times n_L} (= M_L), \\ U_R \hat{m} U_R^T &= M_R, \\ U_R \hat{m} U_L^\dagger &= M_D. \end{aligned} \tag{3.11}$$

Using the last two results we further obtain the following useful relation:

$$U_R^\dagger M_D = \hat{m} U_L^\dagger. \tag{3.12}$$

Further we can introduce again chiral mass eigenfields ω'_L

$$\omega'_L = U^\dagger \omega_L, \tag{3.13}$$
$$\overline{(\omega'_L)^C} = \overline{(\omega_L)^C} U^{\dagger T}, \tag{3.14}$$

and define Majorana mass eigenfields χ by

$$\chi = \omega'_L + (\omega'_L)^C = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{n_L+n_R} \end{pmatrix}, \tag{3.15}$$

$$P_R \chi = (\omega'_L)^C, \; P_L \chi = \omega'_L. \tag{3.16}$$

[73] For a proof of this see Appendix F.5.

Hence, we can obtain the mass term Lagrangian in mass eigenfields:

$$\begin{aligned}\mathcal{L}_{D+M} &= -\frac{1}{2}\overline{(\omega_L)^C}{U^\dagger}^T\hat{m}U^\dagger\omega_L + \text{H.c.}\\ &= -\frac{1}{2}\overline{(\omega'_L)^C}\hat{m}\omega'_L + \text{H.c.}\\ &= -\frac{1}{2}\overline{\chi}\hat{m}\chi.\end{aligned} \tag{3.17}$$

The crucial point here is that in the case where less RH neutrinos than LH neutrinos are present in the model not all LH neutrinos will acquire mass by the seesaw mechanism. This has been shown by G.C. Branco, W. Grimus and L. Lavoura in [72, p.500]. The statement given there says, if $n_R \geq n_L$ all $n_L + n_R$ Majorana neutrinos of the model are in general massive at the tree-level and the seesaw mechanism delivers n_L light and n_R heavy neutrinos. But in the case where $n_R < n_L$ we will end up with $(n_L - n_R)$ Majorana neutrinos remaining massless at tree-level apart from n_R heavy and also just n_R light neutrinos. But it can be shown, e.g. in [73, 74] that the massless fields acquire mass in radiative corrections. This procedure, especially concerning one-loop corrections, will be discussed in more detail in section 4.

3.1.2 Mass Matrix and Disentangling of Light and Heavy Masses

In this section we should investigate the neutrino mass matrix M_{D+M} given in (3.4) in a bit more in detail. We show how M_{D+M} can be transformed in a block diagonal form, where mass matrices for light and heavy neutrinos become decoupled and an effective mass matrix for each of them is derived. This procedure[74] considered by W. Grimus and L. Lavoura [19] is done performing a unitary transformation of the neutrino fields via an unitary $(n_L + n_R) \times (n_L + n_R)$ matrix W, which is defined by

$$\omega_L = \begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix} = W \begin{pmatrix} \nu_{\text{light}} \\ \nu_{\text{heavy}} \end{pmatrix}_L, \tag{3.18}$$

such that the transformation[75] acting on the neutrino mass matrix M_{D+M} leads to the following block diagonal form:

$$W^T \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} W = \begin{array}{c} \begin{pmatrix} M_{\text{light}} & 0 \\ 0 & M_{\text{heavy}} \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix} \\ \begin{matrix} \underbrace{\phantom{M_{li}}}_{n_L} & \underbrace{\phantom{M_{he}}}_{n_R} \end{matrix} \end{array}, \tag{3.19}$$

where M_{light} and M_{heavy} are symmetric $n_L \times n_L$ and $n_R \times n_R$ matrices, respectively. Hence, we obtained disentangled light and heavy mass matrices and two $n_L \times n_R$ zero submatrices as off-diagonal elements.

[74] In [19] $M_L \neq 0$ has been assumed, but we will adapt the calculations for our assumption $M_L = 0$.

[75] It should be noted that this transformation has a similar form like the diagonalizing transformation done in (3.5). However, the matrices M_{light} and M_{heavy} need not to be diagonal after this transformation.

In [19] the following ansatz[76] for the unitary $(n_L+n_R)\times(n_L+n_R)$ matrix W is suggested

$$W = \begin{pmatrix} \sqrt{\mathbb{1} - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbb{1} - B^\dagger B} \end{pmatrix}, \tag{3.20}$$

where B is a $n_L \times n_R$ matrix and the square roots in this ansatz should be understood as a power series.

To fix B as a function of the mass matrices M_D and M_R we use the vanishing off-diagonal submatrices. For doing this we are inserting the ansatz into (3.19):

$$\begin{aligned}
&\begin{pmatrix} M_{\text{light}} & 0 \\ 0 & M_{\text{heavy}} \end{pmatrix} = W^T M_{\text{D+M}} W \qquad (3.21)\\
&= \begin{pmatrix} \sqrt{\mathbb{1} - B^* B^T} & -B^* \\ B^T & \sqrt{\mathbb{1} - B^T B^*} \end{pmatrix} \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \begin{pmatrix} \sqrt{\mathbb{1} - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbb{1} - B^\dagger B} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\mathbb{1} - B^* B^T} & -B^* \\ B^T & \sqrt{\mathbb{1} - B^T B^*} \end{pmatrix} \begin{pmatrix} -M_D^T B^\dagger & M_D^T \sqrt{\mathbb{1} - B^\dagger B} \\ M_D \sqrt{\mathbb{1} - BB^\dagger} - M_R B^\dagger & M_D B + M_R \sqrt{\mathbb{1} - B^\dagger B} \end{pmatrix}.
\end{aligned}$$

We might perform the last step of matrix multiplication for each of the four submatrices separately, starting with the upper right (UR) and lower left (LL), which are obliged to be zero:

$$\begin{aligned}
(\text{UR}) : 0 \overset{!}{=} & \sqrt{\mathbb{1} - B^* B^T} M_D^T \sqrt{\mathbb{1} - B^\dagger B} - B^* \left(M_D B + M_R \sqrt{\mathbb{1} - B^\dagger B} \right) \qquad (3.22)\\
= & \sqrt{\mathbb{1} - B^* B^T} M_D^T \sqrt{\mathbb{1} - B^\dagger B} - B^* M_D B + B^* M_R \sqrt{\mathbb{1} - B^\dagger B}, \\
(\text{LL}) : 0 \overset{!}{=} & - B^T M_D^T B^\dagger + \sqrt{\mathbb{1} - B^T B^*} \left(M_D \sqrt{\mathbb{1} - BB^\dagger} - M_R B^\dagger \right) \qquad (3.23)\\
= & - B^T M_D^T B^\dagger + \sqrt{\mathbb{1} - B^T B^*} M_D \sqrt{\mathbb{1} - BB^\dagger} - \sqrt{\mathbb{1} - B^T B^*} M_R B^\dagger.
\end{aligned}$$

We assume B is a power series in $(m_R)^{-1}$ with coefficients B_j proportional to $(m_R)^{-j}$ and expand the square root:

$$B = B_1 + B_2 + B_3 + \dots\,, \tag{3.24}$$

$$\sqrt{\mathbb{1} - BB^\dagger} = \mathbb{1} - \frac{1}{2} B_1 B_1^\dagger - \frac{1}{2}(B_1 B_2^\dagger + B_2 B_1^\dagger) - \dots\,. \tag{3.25}$$

All these coefficients B_j can be determined recursively and it can be shown that all even coefficients of B are equal to zero[77]. However, in this thesis we are only interested in the lowest order of this expansion and therefore, we use

$$\sqrt{\mathbb{1} - BB^\dagger} \approx \mathbb{1} - \frac{1}{2} B_1 B_1^\dagger, \tag{3.26}$$

[76] It should be noted that an equivalent ansatz has been proposed in [75]. The ansatz (3.20) is also used in [76], but in a more restrictive way.

[77] For a sketch of the proof see [19].

such that

$$W \approx \begin{pmatrix} 1 - \frac{1}{2}B_1 B_1^\dagger & B_1 \\ -B_1^\dagger & 1 - \frac{1}{2}B_1^\dagger B_1 \end{pmatrix} \tag{3.27}$$

for further calculations.

As a first step we need to determine the first coefficient B_1 as a function of the mass matrices M_R and M_D. For this we insert this approximative expansion for B into the (LL) submatrix[78] (3.23), which has to be zero, and rearrange the occurring terms by their order of $m_D(m_R)^{-1}$:

$$\begin{aligned} 0 &\stackrel{!}{=} -B_1^T M_D^T B_1^\dagger + \left(\mathbb{1} - \frac{1}{2}B_1^T B_1^*\right) M_D \left(\mathbb{1} - \frac{1}{2}B_1 B_1^\dagger\right) - \left(\mathbb{1} - \frac{1}{2}B_1^T B_1^*\right) M_R B_1^\dagger \qquad (3.28) \\ &= \underbrace{-B_1^T M_D^T B_1^\dagger}_{\frac{m_D^3}{m_R^2}} + \underbrace{M_D}_{\frac{m_D^1}{m_R^0}} - \underbrace{\frac{1}{2}M_D B_1 B_1^\dagger}_{\frac{m_D^3}{m_R^2}} - \underbrace{\frac{1}{2}B_1^T B_1^* M_D}_{\frac{m_D^3}{m_R^2}} + \underbrace{\frac{1}{4}B_1^T B_1^* M_D B_1 B_1^\dagger}_{\frac{m_D^5}{m_R^4}} - \underbrace{M_R B_1^\dagger}_{\frac{m_D^1}{m_R^0}} + \underbrace{\frac{1}{2}B_1^T B_1^* M_R B_1^T}_{\frac{m_D^3}{m_R^1}} \\ &= \underbrace{M_D - M_R B_1^\dagger}_{\frac{m_D^1}{m_R^0}} + \underbrace{\frac{1}{2}B_1^T B_1^* M_R B_1^T - B_1^T M_D^T B_1^\dagger - \frac{1}{2}M_D B_1 B_1^\dagger - \frac{1}{2}B_1^T B_1^* M_D}_{\frac{m_D^3}{m_R^2}} + \underbrace{\frac{1}{4}B_1^T B_1^* M_D B_1 B_1^\dagger}_{\frac{m_D^5}{m_R^4}}. \end{aligned}$$

Since also the lowest order must fulfil the equation, we easily achieve an expression for B_1 from the condition obtained in this way $0 = M_D - M_R B_1^\dagger$:

$$\begin{aligned} &\boxed{B_1 = \left(M_R^{-1} M_D\right)^\dagger}, \quad B_1^\dagger = M_R^{-1} M_D, \\ &B_1^* = M_D^T M_R^{-1}, \qquad B_1^T = M_R^{-1*} M_D^*. \end{aligned} \tag{3.29}$$

With B_1 fixed we can simply calculate the other submatrices which correspond to M_{light} and M_{heavy}. First we start with the upper left (UL) submatrix which gives us the mass matrix for the light neutrinos:

$$\begin{aligned} (\text{UL}) : M_{\text{light}} &= -\sqrt{\mathbb{1} - B^* B^T} M_D^T B^\dagger - B^* \left(M_D \sqrt{\mathbb{1} - BB^\dagger} - M_R B^\dagger\right) \\ &= -\sqrt{\mathbb{1} - B^* B^T} M_D^T B^\dagger - B^* M_D \sqrt{\mathbb{1} - BB^\dagger} + B^* M_R B^\dagger \\ &\approx -\left(\mathbb{1} - \frac{1}{2}B_1^* B_1^T\right) M_D^T B_1^\dagger - B_1^* M_D \left(\mathbb{1} - \frac{1}{2}B_1 B_1^\dagger\right) + B_1^* M_R B_1^\dagger \\ &= -M_D^T B_1^\dagger + \frac{1}{2}B_1^* B_1^T M_D^T B_1^\dagger - B_1^* M_D + \frac{1}{2}B_1^* M_D B_1 B_1^\dagger + B_1^* M_R B_1^\dagger \\ &= \underbrace{-M_D^T B_1^\dagger - B_1^* M_D + B_1^* M_R B_1^\dagger}_{\frac{m_D^2}{m_R^1}} + \underbrace{\frac{1}{2}B_1^* B_1^T M_D^T B_1^\dagger + \frac{1}{2}B_1^* M_D B_1 B_1^\dagger}_{\frac{m_D^4}{m_R^1}} \\ &\stackrel{(3.29)}{\approx} -M_D^T M_R^{-1} M_D \underbrace{- M_D^T M_R^{-1} M_D + M_D^T M_R^{-1} \overbrace{M_R M_R^{-1}}^{\mathbb{1}} M_D}_{=0}. \qquad (3.30) \end{aligned}$$

In this calculation we restricted ourselves to the lowest order of the square root expansion,

[78] We could have also used the UR submatrix, which would be equivalent.

to the lowest order in m_R^{-1} for fixing B_1as well as to the lowest order in m_R^{-1} of the occurring terms in (UL) and hence we achieved the famous so-called *seesaw formula* for the mass matrix of the light neutrinos:

$$\boxed{M_{\text{light}} \approx -M_D^T M_R^{-1} M_D}\,. \tag{3.31}$$

Finally we repeat this procedure for the lower right (LR) submatrix to achieve a formula for M_{heavy}:

$$\begin{aligned}
(\text{LR}): M_{\text{heavy}} &= B^T M_D^T \sqrt{\mathbb{1} - B^\dagger B} + \sqrt{\mathbb{1} - B^T B^*}\left(M_D B + M_R \sqrt{\mathbb{1} - B^\dagger B}\right) \\
&= B^T M_D^T \sqrt{\mathbb{1} - B^\dagger B} + \sqrt{\mathbb{1} - B^T B^*} M_D B + \sqrt{\mathbb{1} - B^T B^*} M_R \sqrt{\mathbb{1} - B^\dagger B} \\
&\approx B_1^T M_D^T \left(\mathbb{1} - \frac{1}{2} B_1^\dagger B_1\right) + \left(\mathbb{1} - \frac{1}{2} B_1^T B_1^*\right) M_D B_1 \\
&\quad + \left(\mathbb{1} - \frac{1}{2} B_1^T B_1^*\right) M_R \left(\mathbb{1} - \frac{1}{2} B_1^\dagger B_1\right) \\
&= B_1^T M_D^T - \frac{1}{2} B_1^T M_D^T B_1^\dagger B_1 + M_D B_1 - \frac{1}{2} B_1^T B_1^* M_D B_1 \\
&\quad + M_R - \frac{1}{2} B_1^T B_1^* M_R - \frac{1}{2} M_R B_1^\dagger B_1 + \frac{1}{4} B_1^T B_1^* M_R B_1^\dagger B_1 \\
&= \underbrace{M_R}_{\frac{m_D^0}{m_R^{-1}}} + \underbrace{B_1^T M_D^T + M_D B_1 - \frac{1}{2} B_1^T B_1^* M_R - \frac{1}{2} M_R B_1^\dagger B_1}_{\frac{m_D^2}{m_R^1}} \\
&\quad \underbrace{- \frac{1}{2} B_1^T M_D^T B_1^\dagger B_1 - \frac{1}{2} B_1^T B_1^* M_D B_1 + \frac{1}{4} B_1^T B_1^* M_R B_1^\dagger B_1}_{\frac{m_D^4}{m_R^3}}.
\end{aligned} \tag{3.32}$$

Again we only consider the lowest order and therefore we can skip the last step we have done before for M_{light}, i.e. inserting B_1 given in (3.29), and hence we find:

$$\boxed{M_{\text{heavy}} \approx M_R}\,. \tag{3.33}$$

Before discussing this results concerning the different possible mass scales in the next section, we should emphasise the connection between the unitary diagonalizing transformation done by U in section 3.1.1 and the unitary transformation W introduced in this section, which leads to block diagonalization of the mass matrix and mass scale disentanglement.

First we should also use the result we obtained for B_1 in (3.29) and find by expanding the square root into a power series and taking only the lowest order[79] the following

[79]This means simply approximation the square root by $\sqrt{\mathbb{1} - BB^\dagger} \approx \mathbb{1}$.

approximation for W:

$$\begin{aligned} W &\approx \begin{pmatrix} 1-\frac{1}{2}B_1B_1^\dagger & B_1 \\ -B_1^\dagger & 1-\frac{1}{2}B_1^\dagger B_1 \end{pmatrix} \\ &\approx \begin{pmatrix} \mathbb{1} & M_D^\dagger M_R^{*-1} \\ -M_R^{-1}M_D & \mathbb{1} \end{pmatrix}. \end{aligned} \tag{3.34}$$

Then we remember how we decomposed U in equation (3.6)

$$U = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix}, \tag{3.35}$$

and in [77] we found the following correlation, based on results of J. Schechter and J.W.F Valle in [75]:

$$U = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix} \simeq \begin{pmatrix} \mathbb{1} & M_D^\dagger M_R^{*-1} \\ -M_R^{-1}M_D & \mathbb{1} \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2^* \end{pmatrix}, \tag{3.36}$$

which is U given in the leading order of the inverse of the high scale m_R. At that order the matrices $n_L \times n_L$ resp. $n_R \times n_R$ matrices V_1 and V_2 are defined due to diagonalizing M_{light} resp. $M_{\text{heavy}} \approx M_R$:

$$V_1^T M_{\text{light}} V_1 = \hat{M}_{\text{light}}, \tag{3.37}$$

$$V_2^\dagger M_{\text{heavy}} V_2^* = \hat{M}_{\text{heavy}}. \tag{3.38}$$

Furthermore, it is emphasized in [77] that the matrix U is the lepton mixing matrix and the part of the mixing matrix relevant for the light neutrinos is given by V_1. But before we consider the mixing schemes in section 3.1.4 we should discuss the seesaw formula and mass scales for M_R briefly in the next section.

3.1.3 Mass Scales

These first order approximations obtained above in equation (3.31) and (3.33) reveal quite unambiguously the advantages of the seesaw mechanism. On the one hand we found that of course the masses of the heavy neutrinos are mainly determined by the masses of the introduced RH neutrino singlets. On the other hand we showed how the masses of the light neutrinos are affected by the inverse heavy mass scale. Equation (3.31) indicates quite illustratively how the light neutrino masses decrease when we increase the heavy mass scale while the Dirac mass scale is fixed by a fixed Yukawa coupling, since we derived in equation (2.3) $M_D \sim \Delta^{(\ell)\dagger}$.

Since all parameters of the sterile neutrino masses are free, "the choice of M_R is a matter of theoretical prejudice", as mentioned in [48, p.17]. Thus there are various combinations for choosing M_R and the Yukawa coupling, which are displayed in the diagram 6 below.

However, according to [48, p.17], in standard approaches Yukawa couplings are taken to be ~ 1 and $m_R \sim 10^{10} - 10^{15}$ GeV, because such Majorana masses of the RH neutrino

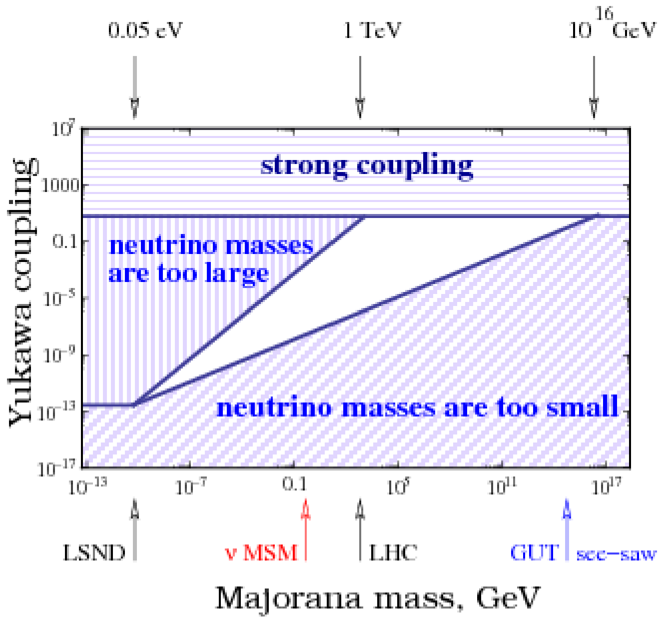


Figure 6: Possible value of the Yukawa couplings and Majorana masses of the sterile neutrinos in seesaw models. Adapted from [48, p.17].

singlets are much larger than the energy scale of the EW SSB, i.e. larger than the masses $W^{\pm}$ and Z^0 bosons as well as the Higgs boson. In [42, p.12] it is mentioned that this scale could be also as low as the TeV scale, but even as high as the GUT[80] scale for $m_R \sim 10^{15} - 10^{16}$ GeV or even the Plank scale $m_R \sim 10^{19}$ GeV.

3.1.4 Mixing Matrix

Remembering (2.54) and (2.56) we are able to express the flavor eigenfields as a linear superposition of the $n_L + n_R$ physical mass eigenfields:

$$\nu_{\ell L} = \sum_{k=1}^{n_L+n_R} U_{\ell k} P_L \chi_k, \quad (\nu_{\ell L})^C = \sum_{k=1}^{n_L+n_R} U^*_{\ell k} P_R \chi_k, \tag{3.39}$$

$$\nu_{\ell R} = \sum_{k=1}^{n_L+n_R} U^*_{\ell' k} P_R \chi_k, \quad (\nu_{\ell R})^C = \sum_{k=1}^{n_L+n_R} U_{\ell' k} P_L \chi_k, \tag{3.40}$$

where the flavor indices take different values, in particular $\ell = 1, \dots, n_L$ and $\ell' = n_L + 1, \dots, n_L + n_R$. Hence, we can apply the decomposition[81] of U introduced in equation (3.6):

$$\nu_{\ell L} = \sum_{k=1}^{n_L+n_R} (U_L)_{\ell k} P_L \chi_k, \quad \nu_{\ell R} = \sum_{k=1}^{n_L+n_R} (U_R)_{\ell' k} P_R \chi_k. \tag{3.41}$$

So we obtained the following mixing correlation between flavor and mass eigenfields:

$$\nu_L = U_L P_L \chi = U_L \omega'_L, \quad \nu_R = U_R P_R \chi = U_R (\omega'_L)^C. \tag{3.42}$$

[80]GUT=Grand Unified Theory

[81]Here it becomes clear, why we used the complex conjugate of U_R in the definition of the decomposition of U.

Like in the case of the Dirac-Majorana mass term this mixing patterns have consequences for the charged and neutral current Lagrangians (1.124) and (1.122). We follow the discussion done by [67] and [78] and adapt what we have already obtained in section 2.3.2 for the Dirac-Majorana mass term with $n_L = n_R = n$. The charged current Lagrangian in terms of mass eigenfields reads as

$$\begin{aligned}\mathcal{L}_{\text{CC}}^{(\ell)} &= -\frac{g}{\sqrt{2}} \sum_{\ell=1}^{n_L} \overline{\ell}_{\ell L} \gamma^\mu \nu_{\ell L} W_\mu^- + \text{H.c.} \\ &= -\frac{g}{\sqrt{2}} \sum_{\ell,i=1}^{n_L} \sum_{k=1}^{n_L+n_R} \overline{\ell'}_{iL} (U_L^{(\ell)})^*_{i\ell} \gamma^\mu U_{\ell k} \chi_{kL} W_\mu^- + \text{H.c.}\,,\end{aligned} \tag{3.43}$$

and we find the rectangular $n_L \times (n_L + n_R)$ matrix

$$K = \sum_{\ell=1}^{n_L} (U_L^{(\ell)})^*_{i\ell} U_{\ell k}, \tag{3.44}$$

which can be also expressed using the decomposition introduced for U in (3.6) as

$$K = (U_L^{(\ell)})^\dagger U_L. \tag{3.45}$$

As shown e.g. in [79], K satisfies

$$KK^\dagger = \mathbb{1}, \tag{3.46}$$

but $K^\dagger K$ does not equal the identity matrix. It is mentioned in [78, p.107] that this matrix K may be also decomposed as

$$K = (K_L, K_H), \tag{3.47}$$

where K_L denotes an $n_L \times n_L$ submatrix corresponding to light neutrinos and K_H denotes an $n_L \times n_R$ submatrix. The light mass eigenstates are effectively described by a mixing matrix K_L, which is non-unitary, according to [78, p.107]. Finally the charged current interaction can be written in matrix notation as

$$\mathcal{L}_{\text{CC}}^{(\ell)} = -\frac{g}{2\sqrt{2}} \left[\overline{\ell'} \gamma^\mu (\mathbb{1} - \gamma_5) K \chi W_\mu^- + \overline{\chi} \gamma^\mu (\mathbb{1} - \gamma_5) K^\dagger \ell' W_\mu^+ \right]. \tag{3.48}$$

At last we should also discuss the neutral current interaction Lagrangian. As discussed before for the Dirac-Majorana mass term with $n_L = n_R = n$ in section 2.3.2 and according to [78, p.107], the neutral current couplings of mass eigenstate neutrinos are no longer diagonal, like in theories where no sterile singlet neutrinos are present. Following [67] we find again

$$\begin{aligned}\mathcal{L}_{\text{NC}}^{(\nu)} &= -\frac{g}{\cos\theta_W} \sum_{\ell=1}^{n_L} \overline{\nu}_{\ell L} \gamma^\mu \left(T_3 - \sin^2\theta_W Q \right) \nu_{\ell L} Z_\mu + \text{H.c.} \\ &= -\frac{g}{2\cos\theta_W} \sum_{\ell=1}^{n_L} \overline{\nu}_{\ell L} \gamma^\mu \nu_{\ell L} Z_\mu + \text{H.c.} \\ &= -\frac{g}{2\cos\theta_W} \sum_{\ell=1}^{n_L} \sum_{k=1}^{n_L+n_R} \sum_{j=1}^{n_L+n_R} \overline{\chi}_{jL} U^*_{\ell j} \gamma^\mu U_{\ell k} \chi_{kL} Z_\mu + \text{H.c.}\end{aligned} \tag{3.49}$$

for the neutral current interactions in terms of mass eigenfields and also again the $(n_L + n_R) \times (n_L + n_R)$ matrix

$$P_{jk} := \sum_{\ell=1}^{n_L} U_{j\ell}^* U_{\ell k}, \tag{3.50}$$

which may be also expressed in terms of the decomposition of U given in (3.6)

$$P = U_L^\dagger U_L. \tag{3.51}$$

Alternatively we may use a reformulation the definition of K in (3.44) in terms of U

$$\sum_{\ell=1}^{n_L} (U_L^\ell)_{i\ell} K_{\ell k} = U_{i\ell}, \tag{3.52}$$

we express P by

$$P = K^\dagger K, \tag{3.53}$$

or using the decomposition for K we might write as done in [78, p.108]

$$P = \begin{pmatrix} K_L^\dagger K_L & K_L^\dagger K_H \\ K_H^\dagger K_L & K_H^\dagger K_H . \end{pmatrix} \tag{3.54}$$

This matrix P is Hermitian,

$$P = P^\dagger, \tag{3.55}$$

and also a projection operator, since

$$P^2 = P. \tag{3.56}$$

Again using matrix notation we can write the neutral current Lagrangian as

$$\mathcal{L}_{\mathrm{NC}}^{(\nu)} = -\frac{g}{4\cos\theta_W} \overline{\chi} \gamma^\mu (\mathbb{1} - \gamma_5) P \chi Z_\mu + \text{H.c.} \tag{3.57}$$

$$= -\frac{g}{4\cos\theta_W} Z_\mu \overline{\chi} \gamma^\mu \left[P_L (U_L^\dagger U_L) - P_R (U_L^T U_L^*) \right] \chi + \text{H.c.}\,, \tag{3.58}$$

where the last line is the result given in [20, 71], which utilizes the property (3.10) of the decomposition matrices U_L and U_R.

Finally we may cite what has been noted in [67], namely "the fact that $P \neq \mathbb{1}$ is a statement that the GIM mechanism[82] is unnatural for lepton theories with massive neutrinos". Furthermore, it is said there that as a physical consequence the n_R "heavier neutrinos can decay in n_L lighter ones and the neutral current interactions should also show oscillation effects as a neutrino beam evolves".

3.2 Seesaw Mechanism in a Multi-Higgs Model

In the previous section we discussed the seesaw mechanism of type I at tree-level for an arbitrary number n_L of LH neutrinos and n_R RH neutrinos. The Dirac mass term is obtained via SSB of the single standard Higgs doublet. In this section the scalar sector

[82] Glashow-Iliopolus-Maiani mechanism

shall be extended too. We introduce the multi-Higgs model, which is used in [20] and [21]. In this model the SM is not only extended by an arbitrary number of LH and RH neutrinos, but also by an arbitrary number n_H of scalar doublets. In this section we adapt the results for Yukawa and weak interactions for the arbitrary number of Higgs doublets. We start in the following section by investigating the scalar mass eigenfields.

3.2.1 Scalar Mass Eigenfields

So let n_L denote the number of LH neutrinos, n_R the number of RH neutrinos and n_H the number of scalar doublets ϕ_k ($k = 1, 2, \dots, n_H$) in this theoretical framework. We follow the considerations done [20] and in [71, Appendix A] and define the doublets analogously to the case $n_H = 1$, where the lower component is neutral and the upper charged

$$\phi_k = \begin{pmatrix} \varphi_k^+ \\ \varphi_k^0 \end{pmatrix}, \tag{3.59}$$

and we define the conjugated doublet by

$$\tilde{\phi}_k = i\tau_2 \phi_k^*. \tag{3.60}$$

We assume the vacuum expectation value (VEV) of the neutral component[83] to be

$$\langle 0|\varphi_k^0|0\rangle = \frac{v_k}{\sqrt{2}}. \tag{3.61}$$

Then we can write

$$\varphi_k^0 = \frac{v_k}{\sqrt{2}} + \varphi_k^{0\prime}, \quad \text{with } \langle 0|\varphi_k^{0\prime}|0\rangle = 0. \tag{3.62}$$

We want to express ϕ_k in their physical mass eigenstates, hence we have to investigate the quadratic terms in the scalar potential

$$V = \sum_{i,j} \mu_{ij}^2 \phi_i^\dagger \phi_j + \sum_{i,j,k,l} \lambda_{ijkl} \left(\phi_i^\dagger \phi_j\right) \left(\phi_k^\dagger \phi_l\right), \tag{3.63}$$

with

$$\mu_{ij}^2 = \left(\mu_{ji}^2\right)^*, \quad \lambda_{ijkl} = \lambda_{klij}, \quad \lambda_{ijkl} = \lambda_{jilk}^*. \tag{3.64}$$

In order to find all quadratic terms we insert (3.62) in a different form, where we split $\varphi_k^{0\prime}$ into its real and imaginary part[84]

$$\varphi_k^0 = \frac{v_k + \rho_k + i\sigma_k}{\sqrt{2}}. \tag{3.65}$$

Thus, we get

$$\langle 0|\rho_k|0\rangle = 0 = \langle 0|\sigma_k|0\rangle, \quad \rho_k^\dagger = \rho_k, \quad \sigma_k^\dagger = \sigma_k. \tag{3.66}$$

[83] Note that it is assumed that the VEV is only acquired by the neutral components and the symmetry of the model shall be broken again to a $U(1)_{\mathrm{EM}}$.

[84] Great thanks to Prof. Grimus for giving a detailed explanation of the following calculations.

We start by investigating the μ_{ij}^2 terms, which are

$$\begin{aligned}\sum_{i,j}\mu_{ij}^2\phi_i^\dagger\phi_j &= \sum_{i,j}\mu_{ij}^2(\varphi_i^-,\varphi_i^{0*})\begin{pmatrix}\varphi_j^+\\ \varphi_j^0\end{pmatrix} \\ &= \sum_{i,j}\mu_{ij}^2[\varphi_i^-\varphi_i^+ + \frac{1}{2}(v_i^*+\rho_i-i\sigma_i)(v_j+\rho_j+i\sigma_j)] \\ &= \sum_{i,j}\mu_{ij}^2[\underline{\varphi_i^-\varphi_j^+} + \frac{1}{2}(v_i^*v_j+v_i^*\rho_j+iv_i^*\sigma_j+\rho_iv_j\underline{+\rho_i\rho_j+i\rho_i\sigma_j}-i\sigma_iv_j\underline{-i\sigma_i\rho_j+\sigma_i\sigma_j})],\end{aligned} \tag{3.67}$$

where the terms contributing to the mass terms, i.e. quadratic in the fields, are underlined. Similarly we find the quadratic terms in the λ_{ijkl} part

$$\begin{aligned}\sum_{i,j,k,l}\lambda_{ijkl}\left(\phi_i^\dagger\phi_j\right)\left(\phi_k^\dagger\phi_l\right) &= \sum_{i,j,k,l}\lambda_{ijkl}\left(\varphi_i^-\varphi_j^+ + \varphi_i^{0*}\varphi_j^0\right)\left(\varphi_k^-\varphi_l^+ + \varphi_k^{0*}\varphi_l^0\right) \\ = \sum_{i,j,k,l}\lambda_{ijkl}[&\underbrace{\left(\varphi_i^-\varphi_j^+\right)\left(\varphi_k^-\varphi_l^+\right)}_{=:1} + \underbrace{\left(\varphi_i^-\varphi_j^+\right)\left(\varphi_k^{0*}\varphi_l^0\right)}_{=:2} + \underbrace{\left(\varphi_i^{0*}\varphi_j^0\right)\left(\varphi_k^-\varphi_l^+\right)}_{=:3} + \underbrace{\left(\varphi_i^{0*}\varphi_j^0\right)\left(\varphi_k^{0*}\varphi_l^0\right)}_{=:4}].\end{aligned} \tag{3.68}$$

The first term 1 is of course not quadratic in the scalar fields, but we can find quadratic terms for the charged scalar fields when we insert (3.65) in terms 2 and 3:

$$2 = \frac{1}{2}\sum_{i,j,k,l}\lambda_{ijkl}\underline{(\varphi_i^-\varphi_j^+)(v_k^*v_l} + v_k^*\rho_l + iv_k^*\sigma_l + \rho_kv_l + \rho_k\rho_l + i\rho_k\sigma_l - i\sigma_kv_l - i\sigma_k\rho_l + \sigma_k\sigma_l), \tag{3.69}$$

$$3 = \frac{1}{2}\sum_{i,j,k,l}\underbrace{\lambda_{ijkl}}_{=\lambda_{klij}}\underline{(\varphi_k^-\varphi_l^+)(v_i^*v_j} + v_i^*\rho_j + iv_i^*\sigma_j + \rho_iv_j + \rho_i\rho_j + i\rho_i\sigma_j - i\sigma_iv_j - i\sigma_i\rho_j + \sigma_i\sigma_j). \tag{3.70}$$

If we use the symmetry property (3.64) of λ_{ijkl} in 3 and rename the indices $(k,l \leftrightarrow i,j)$ we obtain the following quadratic term of the charged scalar fields

$$\sum_{i,j,k,l}\lambda_{ijkl}\varphi_i^-\varphi_j^+v_k^*v_l. \tag{3.71}$$

In the last term 4 we can find quadratic terms of the neutral scalar fields. We insert again (3.65) and obtain

$$\begin{aligned}4 =& \frac{1}{4}\sum_{i,j,k,l}\lambda_{ijkl}(\underline{\underline{v_i^*v_j}} + v_i^*\rho_j + iv_i^*\sigma_j + \rho_iv_j + \underline{\rho_i\rho_j + i\rho_i\sigma_j} - i\sigma_iv_j\underline{-i\sigma_i\rho_j+\sigma_i\sigma_j}) \\ &\cdot(\underline{v_k^*v_l} + v_k^*\rho_l + iv_k^*\sigma_l + \rho_kv_l + \underline{\underline{\rho_k\rho_l + i\rho_k\sigma_l}} - i\sigma_kv_l\underline{\underline{-i\sigma_k\rho_l+\sigma_k\sigma_l}}) \\ =& \frac{1}{4}\sum_{i,j,k,l}\lambda_{ijkl}\underline{\underline{v_i^*v_j}}\left[(\rho_k\rho_l+\sigma_k\sigma_l) + i\left(\rho_k\sigma_l-\sigma_k\rho_l\right)\right] \\ &+ \frac{1}{4}\sum_{i,j,k,l}\underbrace{\lambda_{ijkl}}_{=\lambda_{klij}}\underline{v_k^*v_l}\left[(\rho_i\rho_j+\sigma_i\sigma_j) + i\left(\rho_i\sigma_j-\sigma_i\rho_j\right)\right] \\ &+ \frac{1}{4}\sum_{i,j,k,l}\lambda_{ijkl}(v_i^*\rho_j + iv_i^*\sigma_j + \rho_iv_j - i\sigma_iv_j)(v_k^*\rho_l + iv_k^*\sigma_l + \rho_kv_l - i\sigma_kv_l),\end{aligned} \tag{3.72}$$

where we used the symmetry property (3.64) of λ_{ijkl} in the second sum and hence, again by renaming the indices in the second sum ($k, l \leftrightarrow i, j$), we find the following quadratic terms for the neutral scalar fields in the term 4:

$$\begin{aligned}&\frac{1}{2}\sum_{i,j,k,l}\lambda_{ijkl}v_i^*v_j\left[(\rho_k\rho_l+\sigma_k\sigma_l)+i\left(\rho_k\sigma_l-\sigma_k\rho_l\right)\right]\\&+\frac{1}{4}\sum_{i,j,k,l}\lambda_{ijkl}(v_i^*\rho_j+iv_i^*\sigma_j+\rho_iv_j-i\sigma_iv_j)(v_k^*\rho_l+iv_k^*\sigma_l+\rho_kv_l-i\sigma_kv_l).\end{aligned} \tag{3.73}$$

Finally we collect all quadratic terms we have found in the scalar potential V:

$$\begin{aligned}&\sum_{i,j}\mu_{ij}^2\varphi_i^-\varphi_j^+ + \sum_{i,j,k,l}\lambda_{ijkl}\varphi_i^-\varphi_j^+v_k^*v_l+\\&+\sum_{i,j}\mu_{ij}^2[(\rho_i\rho_j+\sigma_i\sigma_j)+i(\rho_i\sigma_j-\sigma_i\rho_j)]+\\&+\frac{1}{2}\sum_{i,j,k,l}\lambda_{ijkl}v_i^*v_j\left[(\rho_k\rho_l+\sigma_k\sigma_l)+i\left(\rho_k\sigma_l-\sigma_k\rho_l\right)\right]\\&+\frac{1}{4}\sum_{i,j,k,l}\lambda_{ijkl}(v_i^*\rho_j+iv_i^*\sigma_j+\rho_iv_j-i\sigma_iv_j)(v_k^*\rho_l+iv_k^*\sigma_l+\rho_kv_l-i\sigma_kv_l).\end{aligned} \tag{3.74}$$

From the first line we can read off the mass matrix of the charged scalars, which is a complex and Hermitian $n_H \times n_H$ matrix

$$\mathcal{M}_+^2 = \mu^2 + \Lambda, \tag{3.75}$$

where the Hermitian matrix Λ is defined as:

$$\Lambda_{ij} = \sum_{k,l}\lambda_{ijkl}v_k^*v_l, \tag{3.76}$$

and $\mu^2 = (\mu_{ij}^2)$ is also Hermitian by definition (3.64). It follows from the Hermiticity of μ^2 that $\text{Re}\mu^2$ is symmetric and $\text{Im}\mu^2$ is antisymmetric, i.e.

$$\text{Re}\mu_{ij}^2 = \text{Re}\mu_{ji}^2, \quad \text{Im}\mu_{ij}^2 = -\text{Im}\mu_{ji}^2. \tag{3.77}$$

To obtain a mass matrix for the neutral scalars we now apply this knowledge to the second line of (3.74) some terms will vanish since the first term $(\rho_i\rho_j + \sigma_i\sigma_j)$ is symmetric in i, j and the second term $i(\rho_i\sigma_j - \sigma_i\rho_j)$ is antisymmetric .

So the only contributing terms come from the purely symmetric and purely antisymmetric combinations, which are

$$\frac{1}{2}\sum_{i,j}\left[\text{Re}\mu_{ij}^2(\rho_i\rho_j+\sigma_i\sigma_j)-\text{Im}\mu_{ij}^2(\rho_i\sigma_j-\sigma_i\rho_j)\right], \tag{3.78}$$

or in matrix form we may write

$$\frac{1}{2}(\rho^T,\sigma^T)\begin{pmatrix}\text{Re}\mu^2 & -\text{Im}\mu^2\\(-\text{Im}\mu^2)^T & \text{Re}\mu^2\end{pmatrix}\begin{pmatrix}\rho\\\sigma\end{pmatrix}. \tag{3.79}$$

It is then quite obvious that the third line in (3.74) leads to an analogous term, since

Λ is Hermitian too and hence its real part is symmetric, whereas its imaginary part is antisymmetric. Thus, we obtain

$$\frac{1}{2}\sum_{i,j}\left[\mathrm{Re}\Lambda_{ij}(\rho_i\rho_j+\sigma_i\sigma_j)-\mathrm{Im}\Lambda_{ij}(\rho_i\sigma_j-\sigma_i\rho_j)\right], \tag{3.80}$$

or again in matrix form

$$\frac{1}{2}(\rho^T,\sigma^T)\begin{pmatrix}\mathrm{Re}\Lambda & -\mathrm{Im}\Lambda\\ (-\mathrm{Im}\Lambda)^T & \mathrm{Re}\Lambda\end{pmatrix}\begin{pmatrix}\rho\\ \sigma\end{pmatrix}. \tag{3.81}$$

Finally, the forth line on (3.74) has to be discussed. Therefore, we rewrite it in the following way

$$\frac{1}{4}\sum_{i,j,k,l}[\underbrace{(v_i^*\rho_j+iv_i^*\sigma_j)}_{=:a}+\underbrace{(\rho_iv_j-i\sigma_iv_j)}_{=:b}]\cdot[\underbrace{(v_k^*\rho_l+iv_k^*\sigma_l)}_{=:c}\underbrace{(\rho_kv_l-i\sigma_kv_l)}_{=:d}], \tag{3.82}$$

and we rewrite the terms from $ac+bd$ by defining the symmetric[85] matrix

$$K_{ik}=\sum_{j,l}\lambda_{ijkl}v_jv_l, \tag{3.83}$$

and by suitable renaming of indices as well as using the symmetry property of the matrix, as

$$\frac{1}{2}\sum_{i,k}\left[\mathrm{Re}K_{ik}(\rho_i\rho_k-\sigma_i\sigma_k)+\mathrm{Im}K_{ik}(\rho_i\sigma_k+\sigma_i\rho_k)\right]. \tag{3.84}$$

In a similar way we treat the terms from $ad+bc$ by using the third relation of equation (3.64) and by defining the Hermitian[86] matrix

$$K'_{il}=\sum_{j,k}\lambda_{ijkl}v_jv_k^*, \tag{3.85}$$

and by suitable renaming of indices and taking the symmetry properties of K' into account we get

$$\frac{1}{2}\sum_{i,l}\left[\mathrm{Re}K'_{il}(\rho_i\rho_l+\sigma_i\sigma_l)-\mathrm{Im}K'_{il}(\rho_i\sigma_l-\sigma_i\rho_l)\right]. \tag{3.86}$$

Now we collect all reformulated results from (3.78), (3.80), (3.84) and (3.86) and rearrange the sum of them by terms of $\rho_i\rho_j$, $\sigma_i\sigma_j$ and mixed terms. Hence, we achieve

$$\begin{aligned}\frac{1}{2}\sum_{ij}\Big\{&\mathrm{Re}\mu^2_{ij}+\mathrm{Re}\Lambda_{ij}+\mathrm{Re}K_{ij}+\mathrm{Re}K'_{ij}]\rho_i\rho_j\\ &+[\mathrm{Re}\mu^2_{ij}+\mathrm{Re}\Lambda_{ij}-\mathrm{Re}K_{ij}+\mathrm{Re}K'_{ij}]\sigma_i\sigma_j\\ &+[-\mathrm{Im}\mu^2_{ij}-\mathrm{Im}\Lambda_{ij}+\mathrm{Im}K_{ij}-\mathrm{Im}K'_{ij}]\rho_i\sigma_j\\ &+[\mathrm{Im}\mu^2_{ij}+\mathrm{Im}\Lambda_{ij}+\mathrm{Im}K_{ij}+\mathrm{Im}K'_{ij}]\sigma_i\rho_j\Big\}.\end{aligned} \tag{3.87}$$

The last line can be rewritten by using again the symmetry properties of the imaginary

[85]This means in contrast to the matrices μ^2, Λ and K' that $\mathrm{Re}K_{ij}=\mathrm{Re}K_{ji}$ and also $\mathrm{Im}K_{ij}=\mathrm{Im}K_{ji}$.
[86]Hence, $\mathrm{Re}K'_{ij}=\mathrm{Re}K'_{ji}$ and $\mathrm{Im}K'_{ij}=-\mathrm{Im}K'_{ji}$.

parts of the matrices, which leads to

$$[-\mathrm{Im}\mu^2_{ji} - \mathrm{Im}\Lambda_{ji} + \mathrm{Im}K_{ij} - \mathrm{Im}K'_{ji}]\rho_j\sigma_i, \tag{3.88}$$

and by renaming the indices in this line we can sum up the last two lines. Hence, we achieve

$$\begin{aligned}\frac{1}{2}\sum_{ij}\Big\{&\mathrm{Re}\mu^2_{ij} + \mathrm{Re}\Lambda_{ij} + \mathrm{Re}K_{ij} + \mathrm{Re}K'_{ij}]\rho_i\rho_j\\ &+[\mathrm{Re}\mu^2_{ij} + \mathrm{Re}\Lambda_{ij} - \mathrm{Re}K_{ij} + \mathrm{Re}K'_{ij}]\sigma_i\sigma_j\\ &+2[-\mathrm{Im}\mu^2_{ij} - \mathrm{Im}\Lambda_{ij} + \mathrm{Im}K_{ij} - \mathrm{Im}K'_{ij}]\rho_i\sigma_j\Big\}.\end{aligned} \tag{3.89}$$

We introduce the real $n_H \times n_H$ matrices[87]

$$A = \mathrm{Re}(\mu^2 + \Lambda + K') + \mathrm{Re}K, \tag{3.90}$$

$$B = \mathrm{Re}(\mu^2 + \Lambda + K') - \mathrm{Re}K, \tag{3.91}$$

$$C = -\mathrm{Im}(\mu^2 + \Lambda + K') + \mathrm{Im}K, \tag{3.92}$$

which determine the $2n_H \times 2n_H$ mass matrix $\mathcal{M}_0^2$ of the neutral scalars. Together with the mass term for the charged scalars obtained in (3.75) we can write the mass terms of the scalar potential in the form

$$V_{\mathrm{mass}} = \sum_{i,j}\varphi_i^-(\mathcal{M}_+^2)_{ij}\varphi_j^+ + \frac{1}{2}[A_{ij}\rho_i\rho_j + B_{ij}\sigma_i\sigma_j + 2C_{ij}\rho_i\sigma_j]. \tag{3.93}$$

To find the mass eigenfields we look at the eigenvalue equations

$$\mathcal{M}_+^2 a = m_a^2 a, \tag{3.94}$$

$$\mathcal{M}_0^2\begin{pmatrix}\mathrm{Re}\,b\\ \mathrm{Im}\,b\end{pmatrix} = \begin{pmatrix}A & C\\ C^T & B\end{pmatrix}\begin{pmatrix}\mathrm{Re}\,b\\ \mathrm{Im}\,b\end{pmatrix} = m_b^2\begin{pmatrix}\mathrm{Re}\,b\\ \mathrm{Im}\,b\end{pmatrix}, \tag{3.95}$$

or equivalently for the neutral scalars

$$\left(\mu^2 + \Lambda + K'\right)b + Kb^* = m_b^2 b. \tag{3.96}$$

The eigenvectors a and $b = \mathrm{Re}\,b + i\,\mathrm{Im}\,b$ are complex $n_H \times 1$ vectors, which fulfil the following orthonormality equations

$$\sum_k(\mathrm{Re}\,b_k\mathrm{Re}\,b'_k + \mathrm{Im}\,b_k\mathrm{Im}\,b'_k) = \mathrm{Re}(b^\dagger b') = \delta_{bb'}, \quad \sum_k a_k^* a'_k = a^\dagger a' = \delta_{aa'}, \tag{3.97}$$

$$\sum_a = a_k^* a'_k = \sum_b \mathrm{Re}\,b_k\mathrm{Re}\,b_{k'} = \sum_b \mathrm{Im}\,b_k\mathrm{Im}\,b_{k'} = \delta_{kk'}, \quad \sum_b \mathrm{Re}\,b_k\mathrm{Im}\,b_{k'} = 0. \tag{3.98}$$

The physical charged scalar mass eigenstate S_a^+ and the physical neutral scalar mass eigenstate S_b^0 are given by

$$S_a^+ = \sum_k a_k^*\varphi_k^+, \tag{3.99}$$

[87] A and B are symmetric.

$$S_b^0 = \sum_k \mathrm{Re}\left(b_k^*(\rho_k + i\sigma_k)\right), \tag{3.100}$$

or equivalently

$$\phi_k = \begin{pmatrix} \sum_a a_k S_a^+ \\ (v_k + \sum_b b_k S_b^0)/\sqrt{2} \end{pmatrix}, \tag{3.101}$$

and indeed one has

$$V_{\text{mass}} = \sum_a m_a^2 S_a^- S_a^+ + \frac{1}{2}\sum_b m_b^2 (S_b^0)^2. \tag{3.102}$$

The Goldstone bosons of this multi-Higgs model correspond to zero eigenvalues of the mass matrices of the charged and neutral scalars $\mathcal{M}_+^2$ and $\mathcal{M}_0^2$. The corresponding eigenvectors linked to the longitudinal modes of the W and Z vector bosons are given by

$$a_W = \frac{1}{v}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_H} \end{pmatrix}, \quad b_Z = \frac{i}{v}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_H} \end{pmatrix}, \tag{3.103}$$

where

$$v = \sqrt{|v_1|^2 + |v_2|^2 + \ldots + |v_{n_H}|^2} = \frac{2m_W}{g}. \tag{3.104}$$

Indeed, as it is shown in [71, p.38], $m_{a_W} = 0$ and $m_{b_Z} = 0$ and we denote the Goldstone bosons by $G^0 = S_{b_Z}^0$ and $G^\pm = S_{a_W}^\pm$.

3.2.2 Yukawa Interactions and Mass Terms

In the multi-Higgs model the Yukawa Lagrangian reads as

$$\begin{aligned} \mathcal{L}_{\text{Yuk}} &= -\sum_{k=1}^{n_H}\sum_{i_1,i_2=1}^{n_L}\sum_{j=1}^{n_R}\left[\phi_k^\dagger \bar{\ell}_{i_1 R}(\Gamma_k)_{i_1 i_2} + \tilde{\phi}_k^\dagger \bar{\nu}_{jR}(\Delta_k)_{j i_2}\right] D_{i_2 L} + \text{H.c.} \\ &= -\sum_{k=1}^{n_H}(\phi_k^\dagger \bar{\ell}_R \Gamma_k + \tilde{\phi}_k^\dagger \bar{\nu}_R \Delta_k) D_L + \text{H.c.}, \end{aligned} \tag{3.105}$$

where $D_L = \binom{\nu_L}{\ell_L}$ denote the LH lepton doublets as before. Γ_k is a $n_L \times n_L$ Yukawa coupling matrix and Δ_k is one with $n_R \times n_L$. The $n_L \times n_L$ charged lepton mass matrix M_ℓ is obtained as before in (1.112) as well as the $n_R \times n_L$ Dirac neutrino mass matrix M_D analogously[88] to (2.3) via SSB

$$M_\ell = \frac{1}{\sqrt{2}}\sum_k v_k^* \Gamma_k, \qquad M_D = \frac{1}{\sqrt{2}}\sum_k v_k \Delta_k. \tag{3.106}$$

We assume M_ℓ to be already diagonal with real positive diagonal elements without loss of generality, i.e.

$$M_\ell = \mathrm{diag}(m_e, m_\mu, m_\tau, \ldots). \tag{3.107}$$

[88] Note that in (2.3) M_D was defined with $\Delta^\dagger$. Here we want to stuck to the convention used in [20].

The total neutrino mass term includes also a Majorana mass term for the RH neutrino singlets ν_R as in (3.1) or we can write as in [20]

$$\mathcal{L}_{\text{D+M}} = -\overline{\nu}_R M_D \nu_L - \frac{1}{2}\overline{\nu}_R C M_R \overline{\nu}_R^T + \text{H.c.}\,, \tag{3.108}$$

where M_R is symmetric and non-singular as in the previous section 3.1 and C is the charge conjugation matrix. This Lagrangian can be be written in a more compact form as in (3.17) by introducing the $(n_L + n_R) \times (n_L + n_R)$ symmetric neutrino mass matrix M_{D+M} given in (3.4) by

$$M_{\text{D+M}} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{3.109}$$

We simply follow the procedure done in 3.1, where M_{D+M} is diagonalized by the unitary matrix $U = \binom{U_L}{U_R^*}$ defined in (3.6) such that

$$U^T M_{D+M} U = \hat{m} = \text{diag}(m_1, m_2, \ldots, m_{n_L+n_R}). \tag{3.110}$$

Further we remember that we can write the flavor neutrino fields ν_L and ν_R as linear superpositions of the physical Majorana neutrino fields χ_i as done in (3.42) and in the following,

$$\nu_L = U_L P_L \chi, \quad \nu_R = U_R P_R \chi. \tag{3.111}$$

In the next step we like to discuss the lepton Yukawa couplings of the neutral and charged scalar mass eigenfields S_a^+ and S_b^0. Therefore it is useful to rewrite the Lagrangian (3.105) in terms of the components of the scalar doublets given in (3.59):

$$\begin{aligned}
\mathcal{L}_{\text{Yuk}} = &- \sum_k \Bigg[\overline{\ell}_R \Gamma_k (\varphi_k^-, \varphi_k^{0*}) \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} + (\overline{\nu}_L, \overline{\ell}_L) \begin{pmatrix} \varphi_k^+ \\ \varphi_k^0 \end{pmatrix} \Gamma_k^\dagger \ell_R \\
&+ \overline{\nu}_R \Delta_k (\varphi_k^0, -\varphi_k^+) \begin{pmatrix} \nu_L \\ \ell_L \end{pmatrix} + (\overline{\nu}_L, \overline{\ell}_L) \begin{pmatrix} \varphi_k^{0*} \\ -\varphi_k^- \end{pmatrix} \Delta_k^\dagger \nu_R \Bigg] \\
= &- \sum_k \Big[\overline{\ell}_R \Gamma_k \varphi_k^- \nu_L + \overline{\ell}_R \Gamma_k \varphi_k^{0*} \ell_L + \overline{\nu}_L \varphi_k^+ \Gamma_k^\dagger \ell_R + \overline{\ell}_L \varphi_k^0 \Gamma_k^\dagger \ell_R \\
&+ \underline{\overline{\nu}_R \Delta_k \varphi_k^0 \nu_L} - \overline{\nu}_R \Delta_k \varphi_k^+ \ell_L + \underline{\overline{\nu}_L \varphi_k^{0*} \Delta_k^\dagger \nu_R} - \overline{\ell}_L \varphi_k^- \Delta_k^\dagger \nu_R \Big].
\end{aligned} \tag{3.112}$$

First we want to concentrate on the couplings of the neutral scalars to the neutrinos and extract the underlined terms from above and insert the neutral scalar mass eigenfields[89] according to (3.101).

This part of the Yukawa Lagrangian gives

$$\begin{aligned}
\mathcal{L}_{\text{Yuk}}^\nu(\varphi^0) &= - \sum_k \left[\overline{\nu}_R \Delta_k \varphi_k^0 \nu_L + \overline{\nu}_L \varphi_k^{0*} \Delta_k^\dagger \nu_R \right] \\
&= - \frac{1}{\sqrt{2}} \sum_{k,b} \left[\overline{\nu}_R \Delta_k \left(v_k + b_k S_b^0 \right) \nu_L + \overline{\nu}_L \left(v_k^* + b_k^* S_b^{0*} \right) \Delta_k^\dagger \nu_R \right] \\
&= - \frac{1}{\sqrt{2}} \sum_{k,b} \left[\overline{\nu} v_k \Delta_k \nu_L + \overline{\nu}_L v_k^* \Delta_k^\dagger \nu_R \right] - \frac{1}{\sqrt{2}} \sum_{k,b} \left[\overline{\nu}_R \Delta_k b_k S_b^0 \nu_L + \overline{\nu}_L b_k^* S_b^0 \Delta_k^\dagger \nu_R \right].
\end{aligned} \tag{3.113}$$

[89]Note that S_b^0 is real per definition (3.100).

The first line simply gives the mass term with M_D defined as before, but the second line describes the Yukawa interactions of the neutral scalar mass eigenfields S_b^0 with the neutrinos. We introduce the notation[90]

$$\Delta_b = \sum_k b_k \Delta_k, \tag{3.114}$$

and insert the neutrino mass eigenfields χ given in (3.111). Thus, we obtain

$$\begin{aligned}\mathcal{L}^{\nu}_{\text{Yuk}}(S_b^0) =& -\frac{1}{\sqrt{2}}\sum_b \left[(\overline{\nu}_R \Delta_b S_b^0 \nu_L)\Delta_b S_b^0 + \overline{\nu}_L S_b^0 \Delta_b^\dagger \nu_R\right] \\ =& -\frac{1}{\sqrt{2}}\sum_b S_b^0 \left[(\overline{\chi} P_L \underbrace{U_R^\dagger)\Delta_b(U_L}_{=:A_b} P_L \chi) + (\overline{\chi} P_R \underbrace{U_L^\dagger)\Delta_b^\dagger(U_R}_{=A_b^\dagger} P_R \chi)\right].\end{aligned} \tag{3.115}$$

We like to rewrite this expression by using the property $P_L^2 = P_L$ of the projection operator and the Majorana condition of the neutrino fields from which we can derive[91]

$$\overline{\chi} A_b P_L \chi = \overline{\chi} P_L A_b^T \chi \Rightarrow \overline{\chi} A_b P_L \chi = \frac{1}{2}\left(\overline{\chi} A_b P_L \chi + \overline{\chi} P_L A_b^T \chi\right). \tag{3.116}$$

Hence, we achieve

$$\begin{aligned}\mathcal{L}^{\nu}_{\text{Yuk}}(S_b^0) =& -\frac{1}{2\sqrt{2}}\sum_b S_b^0 \overline{\chi}\left[\left(U_R^\dagger \Delta_b U_L + U_L^T \Delta_b^T U_R^*\right) P_L \right.\\ &\left. + \left(U_L^\dagger \Delta_b^\dagger U_R + U_R^T \Delta_b^* U_L^*\right) P_R\right]\chi.\end{aligned} \tag{3.117}$$

After we have calculated the Yukawa interaction of the neutral scalar mass eigenfields with the Majorana neutrino mass eigenfields we want to obtain the Yukawa Lagrangian for the charged scalar mass eigenfields too. Therefore, we extract the following terms of (3.112): in the third line the first and third term and in the forth line the two remaining terms. Thus, we get

$$\begin{aligned}\mathcal{L}^{\nu}_{\text{Yuk}}(\varphi^\pm) =& -\sum_k \left[\varphi_k^-(\overline{\ell}_R \Gamma_k \nu_L - \overline{\ell}_L \Delta_k^\dagger \nu_R) + \varphi_k^+(\overline{\nu}_L \Gamma_k^\dagger \ell_R - \overline{\nu}_R \Delta_k \ell_L)\right] \\ =& -\sum_k \left[\varphi_k^-(\overline{\ell}_R \Gamma_k \nu_L - \overline{\ell}_L \Delta_k^\dagger \nu_R) + \text{H.c.}\right].\end{aligned} \tag{3.118}$$

Now we proceed similar to the neutral case by inserting the charged scalar mass eigenfields $S_a^\pm$ according to (3.101) and the neutrino mass eigenfields χ given in (3.111) and we achieve[92]

$$\mathcal{L}_{\text{Yuk}}(S^\pm) = \sum_a S_a^- \overline{\ell}\left[P_R(\Delta_a^\dagger U_R - P_L(\Gamma_a U_L)\right]\chi + \text{H.c.}\,, \tag{3.119}$$

where we defined analogously[93]

$$\Delta_a = \sum_k a_k \Delta_k, \qquad \Gamma_a = \sum_k a_k^* \Gamma_k. \tag{3.120}$$

[90]For the neutral Goldstone boson G^0 we have $\Delta_{b_Z} = \frac{ig}{\sqrt{2}m_W} M_D$.

[91]For a proper demonstration of this see Appendix F.6.

[92]We also use the properties of the projection operator P_L and P_R in (1.26) and the assumption that M_ℓ is already diagonal, i.e. $\ell = \ell_L + \ell_R$ are already mass eigenfields.

[93]For the charged Goldstone bosons $G^\pm = S_{a_W}^\pm$ we therefore have $\Delta_{a_W} = \frac{g}{\sqrt{2}m_W} M_D$ and $\Gamma_{a_W} = \frac{g}{\sqrt{2}m_W} M_\ell$.

3.2.3 Weak Interactions

Of course the extension of the model to arbitrary Higgs doublets affects also the masses of the weak gauge bosons, but the procedure of SSB and therefore mass generation of the interaction bosons is straightforward analogous to the procedure in the SM done in section 1.3. The charged and neutral current Lagrangians also have been already derived in the previous section in (3.43) and (3.57).

4 One-Loop Corrections to the Seesaw Mechanism

In the previous section we discussed the seesaw mechanism of type I on tree level. In this case we got a vanishing upper left submatrix of the mass matrix $M_{\mathrm{D+M}}$ given in (3.4) corresponding to a zero Majorana Mass term for LH neutrinos. We obtained the seesaw formula (3.31), which is[94]

$$M_\nu^{\mathrm{tree}} = -M_D^T M_R^{-1} M_D. \tag{4.1}$$

Now we will show that one-loop corrections will lead to a nonvanishing contribution to the upper left submatrix in M_{D+M}, i.e.

$$M_{D+M} = \begin{pmatrix} \delta M_L & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{4.2}$$

4.1 One-Loop Corrected Multi-Higgs Model

First we want to follow the considerations done in [20] or [21]. In the introduction of the latter it is indicated that at tree-level $n_L - n_R$ neutrinos stay massless, what have been already mentioned in the end of section 3.1.1. At one-loop level the remaining massless neutrinos split up in two parts[95]. $n_L - n_R - n_0$ of them will acquire mass and n_0 neutrinos will still stay massless[96], where $n_0 = \max(0, n_L - n_H n_R)$.

4.1.1 Preliminary Considerations

According to [20], the following corrections[97] at one-loop level to the light neutrino mass matrix occur:

$$M_\nu = M_\nu^{\mathrm{tree}} + \delta M_L. \tag{4.3}$$

These corrections are obtained by taking contributions from the neutrino self-energy $\Sigma(p)$ into account. It is possible to decompose the self-energy, as in (D.85), by

$$\Sigma(p) = A_L(p^2)\not{p} P_L + A_R(p^2)\not{p} P_R + B_L(p^2) P_L + B_R(p^2) P_R, \tag{4.4}$$

where p is the neutrino momentum and we split the coefficients into LH and RH non-absortive parts[98], which obey the relations

$$A_{L,R} = A_{L,R}^\dagger, \qquad \text{and} \qquad B_L = B_R^\dagger. \tag{4.5}$$

[94] From now on we will use the notation M_ν^{tree} instead of M_{light}.
[95] For an elaborated proof of this see appendix F.7.
[96] At two-loop level even those neutrinos become massive. See [74] and [80, 61].
[97] Note that the full correction is given by
$M_\nu = M_\nu^{\mathrm{tree}} + \delta M_L - \delta M_D^T M_R^{-1} M_D - M_D^T M_R^{-1} \delta M_D + M_D^T M_R^{-1} \delta M_R M_R^{-1} M_D$.
But we want to concentrate in this master thesis on the correction δM_L. The other corrections are discussed in more detail in [20].
[98] As done in [20], absorptive parts have been neglected.

The self-energy itself must fulfil

$$\Sigma(p) = C\left[\Sigma(p)\right]^T C^{-1}, \tag{4.6}$$

since it has to be consistent with the Majorana condition of the neutrino fields. Hence, the coefficients fulfil

$$A_L = A_R^T, \quad B_L = B_L^T, \quad B_R = B_R^T. \tag{4.7}$$

The self-energy leads to a correction of the neutrino propagator like in (D.88) in QED. Hence, the neutrino propagator has the following structure

$$\frac{i}{\not{p} - m - A\not{p} - B} = \frac{i}{(1-A)\not{p} - (m+B)} = \frac{i}{\left[\not{p} - \frac{m+B}{1-A}\right](1-A)}, \tag{4.8}$$

with $A = A_L P_L + A_R P_R$ and $B = B_L P_L + B_R P_R$. For one loop corrections it is only necessary to take corrections from B to the mass into account[99]. If we would also allow corrections from A this would lead to higher order corrections which would be necessary for two-loop or higher loop calculations. Hence, we assume $A_L = 0 = A_R$ for our considerations of mass corrections. Furthermore, we only need to take the part B_L into account, because we know from (4.5) and (4.7) $B_L = B_R^*$. It can be easily shown[100] that both parts in Σ will lead to the same mass corrections. We also recognize that there will be no counterterm for δM_L in renormalization, i.e. $\delta^c M_L = 0$, since at tree-level we had $M_L = 0$. By rewriting equation (3.5) we find

$$\delta M_L = \delta^{1-\text{loop}} M_L + \delta^c M_L = \delta^{1-\text{loop}} M_L = U_L^* B_L(0) U_L^\dagger, \tag{4.9}$$

where we assume the neutrino momentum to be $p = 0$, because its mass is very small in comparison to the masses of the Z^0 boson and the neutral scalars. According to [20] only the self-energies of the neutrino caused by Z^0 boson, S_b^0 and G^0 interactions contribute significantly to the mass corrections. Hence, we just consider the Feynman diagrams displayed in figure 7 below.

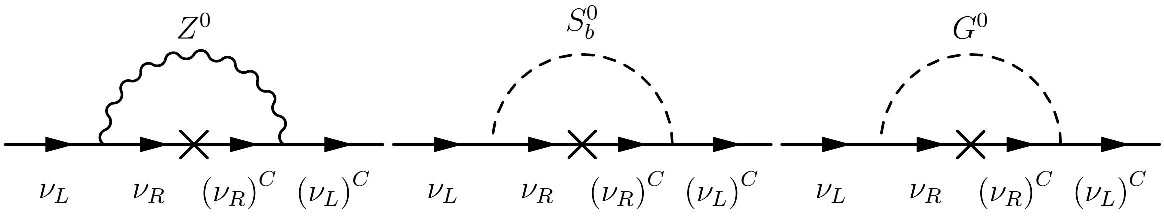


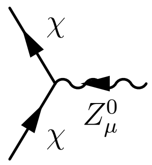
Figure 7: These three Feynman diagrams for neutrino self-energies contribute to the corrections of the neutrino mass matrix M_{D+M}. They illustrate the interaction of the chiral flavor eigenfields ν_L and ν_R. The vertex represents the Yukawa interaction $\mathcal{L}_{\text{Yuk}}$ given in (3.113) and the internal neutrino line represents the Majorana mass term of the RH neutrinos.

[99] The shifted mass can be written as $\hat{m} + B$.
[100] See Appendix F.8.

In the next sections we will calculate each of these contributions to $\Sigma(p)$ separately in the basis where the tree-level neutrino mass matrix is diagonal and using a procedure similar to the one showed in appendix D.4 used for the photon contribution to the fermion self-energy in QED .

4.1.2 Neutrino Self-energy from Z^0

We like to start with the contribution of the Z^0 boson. In equation (3.57) we have seen how it interacts with the mass eigenfield Majorana neutrinos. Combining this with our knowledge on the Feynman rule for this kind of vertex given in (D.62) we can reformulate the Feynman rule for the vertex



$$= i\frac{g}{2c_W}\gamma_\mu\left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right). \tag{4.10}$$

The propagators of the boson and the neutrino can be simply taken from the list of Feynman rules given in appendix D.3. Hence we can write for the neutrino self-energy:

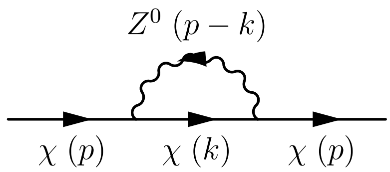


Figure 8: Feynman diagram of the Z^0 boson contribution to the neutrino self-energy, given in neutrino mass eigenfields.

$$\begin{aligned}
-i\Sigma^Z_{ij}(p) = &\sum_\iota \int \frac{d^dk}{(2\pi)^d}\frac{ig}{2c_W}\gamma^\mu\left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{i\iota}\frac{i(\not k + m_\iota)}{k^2 - m_\iota^2 + i\varepsilon} \\
&\cdot \left\{-\frac{ig_{\mu\nu}}{(k-p)^2 - m_Z^2} + \frac{i(k-p)_\mu (k-p)_\nu}{m_Z^2}\right. \\
&\cdot \left.\left(\frac{1}{(k-p)^2 - m_Z^2} - \frac{1}{(k-p)^2 - \xi_Z m_Z^2}\right)\right\} \\
&\cdot \frac{ig}{2c_W}\gamma^\nu\left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\iota j},
\end{aligned} \tag{4.11}$$

where the summation index ι runs over all $n_L + n_R$ neutrino mass eigenstates. As discussed before, we are only interested in the part of $\Sigma(p)$ not proportional to $\not k$ and therefore, we only take the part m_ι of the nominator of the fermion propagator.

Hence, we get

$$\begin{aligned}
-i\Sigma^Z_{ij}(p)|_{m_\ell} = &- \frac{i^4 g^2}{4c_W^2} \sum_\iota \int \frac{d^dk}{(2\pi)^d} \frac{m_\iota}{k^2 - m_\iota^2 + i\varepsilon} \\
&\left\{ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\iota} \gamma^\mu \frac{g_{\mu\nu}}{(k-p)^2 - m_Z^2} \gamma^\nu \left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\iota j} \right. \\
&+ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\iota} \gamma^\mu \frac{i(k-p)_\mu (k-p)_\nu}{m_Z^2} \\
&\left. \left(-\frac{1}{(k-p)^2 - m_Z^2} + \frac{1}{(k-p)^2 - \xi_Z m_Z^2}\right) \gamma^\nu \left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\iota j} \right\},
\end{aligned} \tag{4.12}$$

where we pulled out all constant factors, split the Z^0 boson propagator and used

$$\gamma^\mu \gamma_5 = -\gamma_5 \gamma^\mu \quad \Rightarrow \quad \gamma^\mu P_L = P_R \gamma^\mu. \tag{4.13}$$

In the next step we contract all quantities with four-indices μ and ν. We use in particular

$$\gamma^\mu g_{\mu\nu} \gamma^\nu = \gamma^\mu \gamma_\mu = d\mathbb{1}_d, \tag{4.14}$$

$$\gamma^\mu (k-p)_\mu (k-p)_\nu \gamma^\nu = (\not{k} - \not{p})^2 = (k-p)^2. \tag{4.15}$$

So we obtain

$$\begin{aligned}
-i\Sigma^Z_{ij}(p)|_{m_\iota} = &- \frac{g^2}{4c_W^2} \sum_\iota \int \frac{d^dk}{(2\pi)^d} \frac{m_\iota}{k^2 - m_\iota^2 + i\varepsilon} \\
&\left\{ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\iota} \frac{d}{(k-p)^2 - m_Z^2} \left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\iota j} \right. \\
&+ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\iota} \frac{(k-p)^2}{m_Z^2} \\
&\left. \left(-\frac{1}{(k-p)^2 - m_Z^2} + \frac{1}{(k-p)^2 - \xi_Z m_Z^2}\right) \left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\iota j} \right\}.
\end{aligned} \tag{4.16}$$

Since we already know (4.9) we only need the part $B_L(p)$ from $\Sigma^Z_{ij}(p)|_{m_\iota}$, which means we only take the terms proportional to P_L. We know from (4.4) $-i\Sigma \propto B_L P_L$ and hence we get

$$\begin{aligned}
(B_L)^Z_{ij}(p) = &\frac{ig^2}{4c_W^2} \sum_\ell \int \frac{d^dk}{(2\pi)^d} \frac{m_\iota}{k^2 - m_\iota^2 + i\varepsilon} (U_L^T U_L^*)_{i\iota} (U_L^\dagger U_L)_{\iota j} \left\{ \frac{d}{(k-p)^2 - m_Z^2} \right. \\
&\left. + \frac{(k-p)^2}{m_Z^2} \left(-\frac{1}{(k-p)^2 - m_Z^2} + \frac{1}{(k-p)^2 - \xi_Z m_Z^2}\right) \right\}.
\end{aligned} \tag{4.17}$$

Using (4.9) together with the unitarity of U_L (3.7) and using matrix notation we achieve

$$\begin{aligned}
\delta M_L(Z) = &i \frac{g^2}{4c_W^2} \int \frac{d^dk}{(2\pi)^d} \left\{ \underbrace{\frac{d}{k^2 - m_Z^2}}_{(Z,1)} + \frac{1}{m_Z^2} \left(\underbrace{\frac{k^2}{k^2 - \xi_Z m_Z^2}}_{(Z,2)} - \underbrace{\frac{k^2}{k^2 - m_Z^2}}_{(Z,3)} \right) \right\} \\
&\cdot U_L^* \frac{\hat{m}}{k^2 - \hat{m}^2 + i\varepsilon} U_L^\dagger,
\end{aligned} \tag{4.18}$$

and we split this formula in three parts for the following calculations

$$\delta M_L(Z,1) = i\frac{g^2}{4c_W^2}\int\frac{d^dk}{(2\pi)^d}\frac{d}{k^2-m_Z^2}U_L^*\frac{\hat{m}}{k^2-\hat{m}^2+i\varepsilon}U_L^\dagger, \tag{4.19}$$

$$\delta M_L(Z,2) = i\frac{g^2}{4c_W^2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{m_Z^2}\frac{k^2}{k^2-\xi_Z m_Z^2}U_L^*\frac{\hat{m}}{k^2-\hat{m}^2+i\varepsilon}U_L^\dagger, \tag{4.20}$$

$$\delta M_L(Z,3) = -i\frac{g^2}{4c_W^2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{m_Z^2}\frac{k^2}{k^2-m_Z^2}U_L^*\frac{\hat{m}}{k^2-\hat{m}^2+i\varepsilon}U_L^\dagger. \tag{4.21}$$

We will now calculate $\delta M_L(Z,1)$ and $\delta M_L(Z,3)$ separately and in section 4.1.4 we will discuss the term $\delta M_L(Z,2)$, because this part will vanish together with the contribution from the Goldstone boson. So we start with the first part and repeat the procedure for evaluating a loop integral as it is shown in the appendix D.4. First we use the Feynman parametrization (E.45) for $A = k^2-\hat{m}^2+i\varepsilon$ and $B = k^2-m_Z^2$

$$\begin{aligned}\delta M_L(Z,1) &= i\frac{g^2}{4c_W^2}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx U_L^*\frac{d\hat{m}}{(xA-(1-x)B+i\varepsilon)^2}U_L^\dagger\\ &= i\frac{g^2}{4c_W^2}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx U_L^*\frac{d\hat{m}}{(k^2-x\hat{m}^2-(1-x)m_Z^2+i\varepsilon)^2}U_L^\dagger,\end{aligned} \tag{4.22}$$

and then we can employ the formula for Wick rotation (E.46) and get

$$\begin{aligned}\delta M_L(Z,1) &= i^2\frac{g^2}{4(4\pi)^{\frac{d}{2}}c_W^2}\int_0^1 dx U_L^*\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\frac{d\hat{m}}{(k^2-x\hat{m}^2-(1-x)m_Z^2)^{2-\frac{d}{2}}}U_L^\dagger\\ &= -\frac{g^2}{4\cdot 16\pi^2 c_W^2}\int_0^1 dx\left(\frac{1}{\varepsilon}-\gamma_E+\ldots\right)U_L^*(4-2\varepsilon)\hat{m}\underbrace{\frac{(4\pi)^\varepsilon}{(x\hat{m}^2+(1-x)m_Z^2)^\varepsilon}}_{(*)}U_L^\dagger,\end{aligned} \tag{4.23}$$

where we inserted a dimension $d = 4-2\varepsilon$ and the expansion of $\Gamma(\varepsilon)$ mentioned in (D.81). For further calculations we rewrite again the term

$$(*) = \varepsilon^{\varepsilon\ln(\frac{4\pi}{(x\hat{m}^2+(1-x)m_Z^2)^\varepsilon})} = 1+\varepsilon\ln\left(\frac{4\pi}{(x\hat{m}^2+(1-x)m_Z^2)}\right)+\ldots\,, \tag{4.24}$$

which gives together with the factors $(4\hat{m}-2\varepsilon\hat{m})(\frac{1}{\varepsilon}-\gamma_E+\ldots)$ the following expression $(4\hat{m}-2\varepsilon\hat{m})(\frac{1}{\varepsilon}-\gamma_E+\ldots)$ in the lowest orders in ε

$$\begin{aligned}&(4\hat{m}-2\varepsilon\hat{m})(\frac{1}{\varepsilon}-\gamma_E+\ldots)(1+\varepsilon\ln\left(\frac{4\pi}{(x\hat{m}^2+(1-x)m_Z^2)}\right)+\ldots)\\ &= -2\hat{m}+4\hat{m}\left(\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi)-\ln(x\hat{m}^2+(1-x)m_Z^2)\right)+\mathcal{O}(\varepsilon).\end{aligned} \tag{4.25}$$

Therefore, we have to evaluate the following expression

$$\begin{aligned}\delta M_L(Z,1) = &-\frac{g^2}{4\cdot 16\pi^2 c_W^2}\int_0^1 dx\,U_L^*\\ &\cdot\left\{4\hat{m}\left[\frac{1}{\varepsilon}-\gamma_E+\ln(4\pi)-\ln\left(x\hat{m}^2+(1-x)m_Z^2\right)\right]-2\hat{m}\right\}U_L^\dagger,\end{aligned} \tag{4.26}$$

where we can simply pull out the terms independent of x, such that we obtain

$$\begin{aligned}\delta M_L(Z,1) =& -\frac{g^2}{4\cdot 16\pi^2 c_W^2} U_L^* \left\{4\hat{m}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right] - 2\hat{m}\right\} U_L^\dagger \\ &+ \frac{g^2}{4\cdot 16\pi^2 c_W^2} \int_0^1 dx U_L^* 4\hat{m} \ln\left(x\hat{m}^2 + (1-x)m_Z^2\right) U_L^\dagger.\end{aligned} \tag{4.27}$$

The remaining integration can be done by the integration formula for this type of logarithmic integral given in (E.48), which gives

$$\begin{aligned}\int_0^1 dx \ln\left(x\hat{m}^2 + (1-x)m_Z^2\right) &= \ln(m_Z^2 - \hat{m}^2 - m_Z^2) - 1 + \frac{m_Z^2}{\hat{m}^2 - m_Z^2}\left(\ln(\hat{m}^2) - \ln(m_Z^2)\right) \\ &= \ln(m_Z^2) - 1 + \frac{1}{\frac{\hat{m}^2}{m_Z^2} - 1} \ln\left(\frac{\hat{m}^2}{m_Z^2}\right).\end{aligned} \tag{4.28}$$

Furthermore, we introduce the following shorthand notations

$$r_Z := \frac{\hat{m}^2}{m_Z^2}, \qquad k := -\frac{1}{\varepsilon} + \gamma_E - \ln(4\pi) - 1. \tag{4.29}$$

Thus, we get the following expression:

$$\begin{aligned}\delta M_L(Z,1) &= -\frac{g^2}{4\cdot 16\pi^2 c_W^2} U_L^* \left[4\hat{m}\left(-k - \ln(\hat{m}^2) - \frac{\ln r_Z}{r_Z - 1}\right) - 2\hat{m}\right] U_L^\dagger \\ &= -\frac{g^2}{4\cdot 16\pi^2 c_W^2} U_L^* (-4\hat{m}) \left(k + \ln(\hat{m}^2) + \frac{\ln r_Z}{r_Z - 1} + \frac{1}{2}\right) U_L^\dagger,\end{aligned} \tag{4.30}$$

and we achieve the final result for the first part of the Z contribution to δM_L, given by

$$\boxed{\delta M_L(Z,1) = \frac{g^2}{16\pi^2 c_W^2} U_L^* \hat{m} \left(k + \frac{1}{2} + \ln(\hat{m}^2) + \frac{\ln r_Z}{r_Z - 1}\right) U_L^\dagger}. \tag{4.31}$$

As mentioned before we postpone the discussion on $\delta M_L(Z,2)$ until we investigate the contribution of the Goldstone boson in section 4.1.4. Therefore, we proceed with the calculation of $\delta M_L(Z,3)$ given in (4.21) as

$$\delta M_L(Z,3) = -i\frac{g^2}{4c_W^2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{m_Z^2} \frac{k^2}{k^2 - m_Z^2} U_L^* \frac{\hat{m}}{k^2 - \hat{m}^2 + i\varepsilon} U_L^\dagger. \tag{4.32}$$

We use the same procedure, which we applied for $\delta M_L(Z,1)$. Feynman parametrization for $A = k^2 - \hat{m}^2$ and $B = k^2 - m_Z^2$ leads to the denominator

$$(Ax + (1-x)B - i\varepsilon)^2 = (k^2 - x\hat{m}^2 - (1-x)m_Z^2 + i\varepsilon)^2. \tag{4.33}$$

Hence, we get

$$\begin{aligned}\delta M_L(Z,3) &= -i\frac{g^2}{4c_W^2}\frac{1}{m_Z^2} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx U_L^* \frac{k^2\hat{m}}{(k^2 - x\hat{m}^2 - (1-x)m_Z^2 + i\varepsilon)^2} U_L^\dagger \\ &= -i\frac{g^2}{4c_W^2}\frac{1}{m_Z^2} \int_0^1 dx U_L^* \frac{i\hat{m}^3}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} (x\hat{m}^2 + (1-x)m_Z^2)^{\frac{d}{2}-2} U_L^\dagger,\end{aligned} \tag{4.34}$$

where we applied again Wick rotation formula (E.46) in the second step. Now we set $d = 4 - 2\varepsilon$ and use again the expansion of $\Gamma(\varepsilon)$, which gives

$$\begin{aligned}\delta M_L(Z,3) &= \frac{g^2}{4(4\pi)^2 c_W^2}\frac{1}{m_Z^2}\int_0^1 dx U_L^* \hat{m}^3 \left(\frac{1}{\varepsilon} - \gamma_E + \ldots\right)\frac{(4\pi)^\varepsilon}{(x\hat{m}^2 + (1-x)m_Z^2)^\varepsilon} U_L^\dagger \\ &= \frac{g^2}{4(4\pi)^2 c_W^2}\frac{1}{m_Z^2}\int_0^1 dx U_L^* \hat{m}^3 \left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(x\hat{m}^2 + (1-x)m_Z^2)\right] U_L^\dagger,\end{aligned} \tag{4.35}$$

and we used again the same trick (4.24) for rewriting and expanding the $(\ldots)^\varepsilon$ term. Finally, we integrate over dx using (E.48) and use the relation $c_W \cdot m_Z = m_W$ to obtain

$$\begin{aligned}\delta M_L(Z,3) =& \frac{g^2}{4(4\pi)^2 m_W^2} U_L^* \hat{m}^3 \left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(\hat{m}^2) + 1\right. \\ & \left. - \frac{m_Z^2}{\hat{m}^2 - m_Z^2}\left(\ln(\hat{m}^2) - \ln(m_Z^2)\right)\right] U_L^\dagger.\end{aligned} \tag{4.36}$$

We introduce again the shorthand notations (4.29) and achieve the following final result for the third part of the Z boson contribution:

$$\boxed{\delta M_L(Z,3) = -\frac{g^2}{64\pi^2 m_W^2} U_L^* \hat{m}^3 \left(k + \ln(\hat{m}^2) + \frac{\ln(r_Z)}{r_Z - 1}\right) U_L^\dagger}. \tag{4.37}$$

4.1.3 Neutrino Self-energy from S_b^0

In this section we want to calculate now the contribution of the neutral scalar mass eigenfields to the neutrino self-energy, which corresponds to the Feynman diagram drawn in figure 9 below.

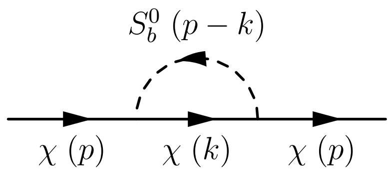


Figure 9: Feynman diagram of the S_b^0 scalar fields contribution to the neutrino self-energy, given in neutrino mass eigenfields.

The self-energy formula for this contribution is derived by the Feynman rules for the fermion propagator (D.53), the propagator of the neutral scalar (D.57) and the one for the interaction vertex (D.64), where the neutral scalar couples to the neutrino via Yukawa interaction given in (3.117).

Hence, we can write

$$\begin{aligned}
-i\Sigma_{ij}^{S_b^0}(p) =&\sum_\iota \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k-p)^2 - m_b^2 + i\varepsilon} \\
&\cdot \frac{i}{\sqrt{2}} \left(U_b P_L + U_b^\dagger P_L\right)_{i\iota} \frac{i}{\not{k} - m_\iota + i\varepsilon} \frac{i}{\sqrt{2}} \left(U_b P_L + U_b^\dagger P_L\right)_{\iota j} \\
=&\frac{1}{2} \sum_\iota \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k-p)^2 - m_b^2 + i\varepsilon} \\
&\cdot \left(U_b P_L + U_b^\dagger P_L\right)_{i\iota} \frac{\not{k} + m_\ell}{k^2 - m_\iota^2 + i\varepsilon} \left(U_b P_L + U_b^\dagger P_L\right)_{\iota j},
\end{aligned} \tag{4.38}$$

where we introduced the shorthand notation $U_b = \left(U_R^\dagger \Delta_b U_L + U_L^T \Delta_b^T U_R^*\right)$ and pulled out all constant factors in the second step. Again we are only interested in the part for $B_L^{S^0}$ which is proportional to P_L and independent of $\not{k}$. Further we assume again $\not{p} = 0$ and so we obtain[101] by

$$(B_L)_{ij}^{S_b^0}(0) = \frac{i}{2} \sum_\iota \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_b^2 + i\varepsilon} (U_b)_{i\iota} \frac{m_\iota}{k^2 - m_\iota^2 + i\varepsilon} (U_b)_{\iota j}. \tag{4.39}$$

We change to matrix notation and use relation (4.9) to achieve

$$\delta M_L(S_b^0) = \frac{i}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_b^2 + i\varepsilon} U_L^* U_b \frac{\hat{m}}{k^2 - \hat{m}^2 + i\varepsilon} U_b U_L^\dagger. \tag{4.40}$$

Applying the properties for U_L and U_R given in (3.7)-(3.9) the terms remaining are

$$U_L^* U_b = U_L^* \left(U_R^\dagger \Delta_b U_L + U_L^T \Delta_b^T U_R^*\right) = \Delta_b^T U_R^*, \tag{4.41}$$

$$U_b U_L^\dagger = \left(U_R^\dagger \Delta_b U_L + U_L^T \Delta_b^T U_R^*\right) U_L^\dagger = U_R^\dagger \Delta_b, \tag{4.42}$$

and therefore we get

$$\delta M_L(S_b^0) = \frac{i}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_b^2 + i\varepsilon} \Delta_b^T U_R^* \frac{\hat{m}}{k^2 - \hat{m}^2 + i\varepsilon} U_R^\dagger \Delta_b. \tag{4.43}$$

We proceed like in the previous section and introduce Feynman parametrization (E.45) with $A = k^2 - \hat{m}^2$ and $B = k^2 - m_b^2$ which gives a similar denominator to the previous case

$$(xA - (1-x)B)^2 = (k^2 - x\hat{m}^2 - (1-x)m_b^2 + i\varepsilon)^2. \tag{4.44}$$

Thus, we obtain

$$\begin{aligned}
\delta M_L(S_b^0) =&\frac{i}{2} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \Delta_b^T U_R^* \frac{\hat{m}}{(k^2 - x\hat{m}^2 - (1-x)m_b^2 + i\varepsilon)^2} U_R^\dagger \Delta_b \\
=&\frac{i}{2} \int_0^1 dx \Delta_b^T U_R^* \hat{m} \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)} \frac{1}{(k^2 - x\hat{m}^2 - (1-x)m_b^2 + i\varepsilon)^{2-\frac{d}{2}}} U_R^\dagger \Delta_b,
\end{aligned} \tag{4.45}$$

where we applied again the Wick rotation formula (E.46) in the second line. We follow

[101] Note that the self-energy function is defined with a factor of $-i$ which has to be absorbed by multiplying by i to obtain B_L.

previous considerations setting $d = 4 - 2\varepsilon$ and expanding the gamma function again according to (D.81) and using the trick (4.24) to rewrite the terms $(\dots)^\varepsilon$. Doing so we obtain

$$\begin{aligned}\delta M_L(S_b^0) &= -\frac{1}{2(4\pi)^2}\int_0^1 dx \Delta_b^T U_R^* \hat{m}\left(\frac{1}{\varepsilon} - \gamma_E + \dots\right)\frac{(4\pi)^\varepsilon}{(k^2 - x\hat{m}^2 - (1-x)m_b^2)^\varepsilon} U_R^\dagger \Delta_b \qquad (4.46)\\ &= -\frac{1}{32\pi^2}\int_0^1 dx \Delta_b^T U_R^* \hat{m}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(k^2 - x\hat{m}^2 - (1-x)m_b^2)\right] U_R^\dagger \Delta_b\\ &= -\frac{1}{32\pi^2}\Delta_b^T U_R^* \hat{m}\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \ln(\hat{m}^2) + 1 - \frac{1}{\frac{\hat{m}^2}{m_b^2}}\left(\ln(\hat{m}^2) - \ln(m_b^2)\right)\right] U_R^\dagger \Delta_b.\end{aligned}$$

In the last step we performed the integration over the Feynman parameter like in the formula (E.48). Finally we rewrite the result by using some shorthand notations as the divergent variable k defined in (4.29) and analogously to r_Z we define

$$r_b = \frac{\hat{m}^2}{m_b^2} \qquad (4.47)$$

and hence we achieve

$$\boxed{\delta M_L(S_b^0) = -\frac{1}{32\pi^2}\Delta_b^T U_R^* \hat{m}\left[k + \ln(\hat{m}^2) + \frac{\ln(r_b)}{r_b - 1}\right] U_R^\dagger \Delta_b}. \qquad (4.48)$$

4.1.4 Neutrino Self-energy from G^0

Finally we want to investigate the contribution of the Goldstone boson. The corresponding Feynman diagram is shown below in figure 10.

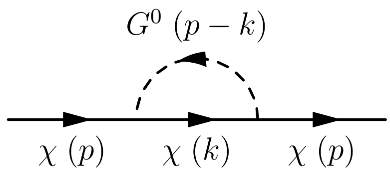


Figure 10: Feynman diagram of the G^0 scalar fields contribution to the neutrino self-energy, given in neutrino mass eigenfields.

This can be obtained rather simple since we have already calculated $\delta M_L(S_b^0)$ in (4.43) and we know the Goldstone boson is the scalar $G^0 = S_{b_Z}^0$ corresponding to the eigenvector b_Z given in (3.103). Its propagator is given in (D.59) and we noted in section 3.2.2 that

$$\Delta_{b_Z} = \frac{ig}{\sqrt{2}m_W} M_D. \qquad (4.49)$$

Inserting all this in (4.43) and using matrix notation leads to

$$\begin{aligned}\delta M_L(G^0) =& \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2}\Delta_{bz}^T U_R^* \frac{\hat{m}}{k^2-\hat{m}^2}U_R^\dagger \Delta_{bz} \\ =& -\frac{ig^2}{4m_W^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2}M_D^T U_R^* \frac{\hat{m}}{k^2-\hat{m}^2}U_R^\dagger M_D \\ =& -\frac{ig^2}{4m_W^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2}U_L^* \underbrace{\hat{m}\frac{\hat{m}}{k^2-\hat{m}^2}\hat{m}}_{\frac{\hat{m}^3}{k^2-\hat{m}^2}}U_L^\dagger,\end{aligned} \tag{4.50}$$

where we used the relation (3.12) in the last step. This gives exactly the same contribution up to the minus sign as the second term of the Z^0 boson

$$\delta M_L(Z,2) = i\frac{g^2}{4c_W^2 m_Z^2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2}U_L^* \frac{k^2\hat{m}}{k^2-\hat{m}^2+i\varepsilon}U_L^\dagger, \tag{4.51}$$

when we insert $m_W^2 = c_W^2 m_Z^2$ and since we can show that

$$\begin{aligned}U_L^*\frac{k^2\hat{m}^2}{k^2-\hat{m}^2}U_L^\dagger &= U_L^*\frac{(k^2-\hat{m}^2)\hat{m}+\hat{m}^3}{k^2-\hat{m}^2}U_L^\dagger \\ &= U_L^*\hat{m}U_L^\dagger + U_L^*\frac{\hat{m}^3}{k^2-\hat{m}^2}U_L^\dagger = U_L^*\frac{\hat{m}^3}{k^2-\hat{m}^2}U_L^\dagger,\end{aligned} \tag{4.52}$$

because the first term vanishes according to the first relation in (3.11).

4.1.5 Correction $\delta \mathrm{M_L}$ to the Neutrino Mass

In this section we want to achieve the final equation for the one loop correction δM_L to the LH neutrino masses. In the sections before we have already calculated all contributing parts. In [20] is has been pointed out that the charged bosons $W^\pm$ does only contribute to the $\not{p}$ dependent part A_L and A_R of the neutrino self-energy $\Sigma(p)$ shown in figure 11 since it gives

$$\begin{aligned}\Sigma_{ij}^{(W)}(p) =& i\sum_\ell \frac{g^2}{2}\int\frac{d^dk}{(2\pi)^2}S_{\mu\nu}^W(k-p)\frac{1}{k^2-m_\ell} \\ &\cdot\left[(U_L^\dagger)_{i\ell}(U_L)_{\ell j}\gamma^\mu\not{k}\gamma^\nu P_L + (U_L^T)_{i\ell}(U_L^*)_{\ell j}\gamma^\mu\not{k}\gamma^\nu P_R\right].\end{aligned} \tag{4.53}$$

where $S_{\mu\nu}^W(k-p)$ is the propagator of $W^\pm$ with momentum $k-p$ given in (D.56) and we used as usual the Feynman rule (D.53) for the fermion propagator and the appropriate vertex is given in (D.63). Because of it only contains terms dependent on $\not{k}$ it only contributes to $A_L(p^2)$ and $A_R(p^2)$.

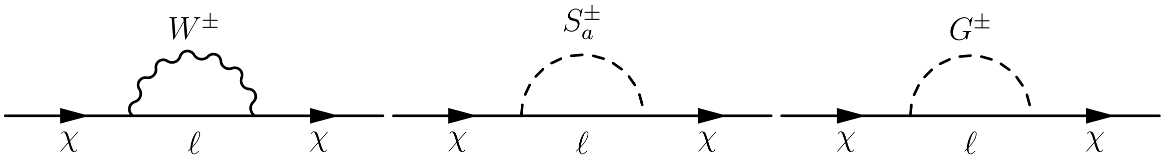


Figure 11: Feynman diagrams for neutrino self-energies not contributing to neutrino mass corrections δM_L.

In [20] it is also indicated that the exchange of the charged scalar mass eigenfields $S_a^\pm$ and the charged Goldstone bosons $G^\pm$ do not contribute either, since one obtains

$$\begin{aligned}(B_L)_{ij}^{S_a^\pm}(p^2) = &- i \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k-p)^2 - m_a^2} \frac{m_\ell}{k^2 - m_\ell^2} \\ &\cdot \left[(U_R^\dagger \Delta_a)_{i\ell} (\Gamma_a U_L)_{\ell j} + (U_L^T \Gamma_a^T)_{i\ell} (\Delta_a^T U_R^*)_{\ell j} \right] ,\end{aligned} \tag{4.54}$$

where we used again the Feynman rules for the fermion propagator (D.53), for the charged scalar (D.58) and the rule for the interaction vertex (D.66) resp. (D.68) as well as the notations Δ_a and Γ_a from (3.120). From this we get the part for the charged Goldstone boson analogously to the case of the neutral one, by inserting the eigenvector a_w corresponding to $G^\pm$ given in (3.103) and using the Feynman rule for the charged Goldstone boson propagator (D.60):

$$\begin{aligned}(B_L)_{ij}^{G^\pm}(p^2) = &- \frac{ig^2}{2m_W^2} \sum_\ell \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k-p)^2 - \xi_W m_W^2} \frac{m_\ell^2}{k^2 - m_\ell^2} \\ &\cdot \left[m_i (U_L^\dagger)_{i\ell} (U_L)_{\ell j} + (U_L^T)_{i\ell} (U_L^*)_{\ell j} m_j \right] .\end{aligned} \tag{4.55}$$

Clearly these two parts do not contribute since one has to apply (4.9) which causes together with (3.9) these contributions to vanish.

Hence, all contributing parts have been derived in the previous sections and we sum up all results as

$$\boxed{\delta M_L = \delta M_L(Z) + \delta M_L(G^0) + \sum_{b \neq b_Z} \delta M_L(S_b^0)} . \tag{4.56}$$

Because the correction δM_L has no counterterm since M_L vanishes at tree level, δM_L itself must be already gauge-independent and finite. We want to show this in the current section and start with some observations we have already made. In section (4.1.2) we split up the Z^0 contribution in three parts due to the form of the boson propagator in R_ξ gauge and hence we might write

$$\delta M_L = \delta M_L(Z,1) + \delta M_L(Z,2) + \delta M_L(Z,3) + \delta M_L(G^0) + \sum_{b \neq b_Z} \delta M_L(S_b^0). \tag{4.57}$$

In section 4.1.4 we have already shown that

$$\delta M_L(Z,2) + \delta M_L(G^0) = 0, \tag{4.58}$$

which means the parts containing the unphysical parameter ξ_Z cancel each other and δM_L is indeed gauge-independent.

In the next step we should deal with the divergences which occur in the remaining contributions we derived in (4.31), (4.37) and (4.48), which all contain the divergent quantity k defined in (4.29). First we show that the infinities drop out in the sum

$$\sum_{b \neq b_Z} \delta M_L(S_b^0) + \delta M_L(Z,3), \tag{4.59}$$

because all terms independent of the boson masses m_b and m_Z cancel each other. We might split those contributions into two parts and indicate the term dependent on m_b resp. m_Z with one prime and the part independent of those masses with a double prime, such that

$$\delta M_L(S_b^0) = \delta M_L''(S_b^0) + \delta M_L'(S_b^0), \tag{4.60}$$

$$\delta M_L(Z,3) = \delta M_L''(Z,3) + \delta M_L'(Z,3). \tag{4.61}$$

To show

$$\sum_{b\neq b_Z} \delta M_L''(S_b^0) + \delta M_L''(Z,3) = 0, \tag{4.62}$$

we need the orthogonality relation for the eigenvectors b we had in equation (3.98). For our purpose we write

$$\sum_{b\neq b_Z} b_j b_k + (b_Z)_j (b_Z)_k = 0, \tag{4.63}$$

with b_Z defined in (3.103) and we remember the definition of Δ_b given in (3.114). Thus, we can rewrite the part of $\sum_{b\neq b_Z} \delta M_L(S_b^0)$ independent of m_b in the following way:

$$\begin{aligned}
\sum_{b\neq b_Z} \delta M_L''(S_b^0) &= \frac{1}{32\pi^2} \sum_{b\neq b_Z} \Delta_b^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger \Delta_b \\
&\overset{(3.114)}{=} \frac{1}{32\pi^2} \sum_{b\neq b_Z} \sum_{k,j} b_k \Delta_k^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger b_j \Delta_j \\
&\overset{(4.63)}{=} -\frac{1}{32\pi^2} \sum_{k,j} (b_Z)_k \Delta_k^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger (b_Z)_j \Delta_j \\
&\overset{(3.103)}{=} \frac{1}{32\pi^2 v^2} \sum_{k,j} v_k \Delta_k^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger v_j \Delta_j \\
&\overset{(3.106)}{=} \frac{1}{32\pi^2 v^2} \sqrt{2} M_D^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger \sqrt{2} M_D \\
&= \frac{1}{16\pi^2 v^2} M_D^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger M_D.
\end{aligned} \tag{4.64}$$

To compare this result with the part of $\delta M_L(Z,3)$ independent of the boson mass m_Z we use the third relation in equation (3.11) in the rewritten forms

$$U_R^\dagger M_D = \hat{m} U_L^\dagger \Leftrightarrow M_D^T U_R^* = U_L^* \hat{m}, \tag{4.65}$$

and the definition of v given in (3.104) . Hence, we get

$$\begin{aligned}
\delta M_L''(Z,3) &= -\frac{g^2}{64\pi^2 m_W^2} U_L^* \hat{m}^3 (k + \ln \hat{m}^2) U_L^\dagger \\
&= -\frac{g^2}{64\pi^2 m_W^2} U_L^* \hat{m}\hat{m}(k + \ln \hat{m}^2) \hat{m} U_L^\dagger \\
&= -\frac{4}{64\pi^2 v^2} M_D^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger M_D \\
&= -\frac{1}{16\pi^2 v^2} M_D^T U_R^* \hat{m}(k + \ln \hat{m}^2) U_R^\dagger M_D,
\end{aligned} \tag{4.66}$$

which is exactly the same up to the minus sign we obtained before from $\sum_{b\neq b_Z} \delta M_L''(S_b^0)$.

Therefore, we showed (4.62). Thus, in the sum only $\sum_{b\neq b_Z}\delta M'_L(S^0_b)+\delta M'_L(Z,3)$ remain, i.e.

$$\begin{aligned}\sum_{b\neq b_Z}\delta M_L(S^0_b)+\delta M_L(Z,3) &= \sum_{b\neq b_Z}\delta M'_L(S^0_b)+\delta M'_L(Z,3) &(4.67)\\ =&\frac{1}{32\pi^2}\sum_{b\neq b_Z}\Delta_b^T U_R^*\hat{m}\frac{\ln r_b}{r_b-1}U_R^\dagger\Delta_b-\frac{g^2}{64\pi^2 m_W^2}U_L^*\hat{m}^3\frac{\ln r_z}{r_z-1}U_L^\dagger. &(4.68)\end{aligned}$$

Finally, we have to discuss the divergences in $\delta M_L(Z,1)$ given in (4.19). We find all terms proportional to $\hat{m}$, but not to $\hat{m}\ln\hat{m}$, vanish, since we had

$$U_L^*\hat{m}U_L^\dagger=0 \tag{4.69}$$

in the first relation of equation (3.11). Again we split $\delta M_L(Z,1)$ into two parts analogous to (4.60), where the double primed part denotes the terms proportional to $\hat{m}$ and obviously get

$$\delta M''_L(Z,1)=\frac{g^2}{16\pi^2 c_W^2}U_L^*\hat{m}(k+\frac{1}{2})U_L^\dagger=0. \tag{4.70}$$

The remaining part of $\delta M_L(Z,1)$ is then

$$\delta M'_L(Z,1)=\frac{g^2}{16\pi^2 c_W^2}U_L^*\underbrace{\hat{m}\left(\ln\hat{m}^2+\frac{\ln r_Z}{r_Z-1}\right)}_{=:(**)}U_L^\dagger=\frac{g^2}{16\pi^2 c_W^2}U_L^*\hat{m}\frac{r_Z\ln r_Z}{r_Z-1}U_L^\dagger. \tag{4.71}$$

The second equality can be easily shown by rewriting the factor $(**)$ in the following way

$$\begin{aligned}(**)=&\hat{m}\left(\ln\hat{m}^2+\frac{\ln r_Z}{r_Z-1}\right)\\ =&\hat{m}\ln\hat{m}^2+\frac{\hat{m}\,m_Z^2}{\hat{m}^2-m_Z^2}(\ln\hat{m}^2-\ln m_Z^2)\\ =&\frac{\hat{m}(\hat{m}^2-m_Z^2)\ln\hat{m}^2+\hat{m}\,m_Z^2\ln\hat{m}^2}{\hat{m}^2-m_Z^2}-\frac{\hat{m}\,m_Z^2}{\hat{m}^2-m_Z^2}\ln m_Z^2\\ =&\frac{\hat{m}^3}{\hat{m}^2-m_Z^2}\ln\hat{m}-\frac{\hat{m}\,m_Z^2}{\hat{m}^2-m_Z^2}\ln m_Z^2\\ =&\frac{\hat{m}\,m_Z^2}{\hat{m}^2-m_Z^2}\left[\frac{\hat{m}^2}{m_Z^2}\ln\hat{m}^2-\ln m_Z^2\right]\\ =&\hat{m}\frac{r_Z}{r_Z-1}\left[\ln\hat{m}^2-\frac{1}{r_Z}\ln m_Z^2\right].\end{aligned} \tag{4.72}$$

Since we know $U_L^*\hat{m}U_L^\dagger=0$, we can add and subtract terms as long as they are proportional to $\hat{m}$ and vanish due to this property. To achieve the wanted result we simply add a suitable term

$$\begin{aligned}(**)=&\hat{m}\frac{r_Z}{r_Z-1}\left[\ln\hat{m}^2-\frac{1}{r_Z}\ln m_Z^2+\frac{1}{r_Z}\ln m_Z^2-\ln m_Z^2\right]\\ =&\hat{m}\frac{r_Z}{r_Z-1}\left[\ln\hat{m}^2-\ln m_Z^2\right]=\hat{m}\frac{r_Z}{r_Z-1}\ln r_Z,\end{aligned} \tag{4.73}$$

and this leads to the second equality in (4.71).

Now it is quite obvious to see that

$$\delta M_L'(Z,1) = -4\delta M_L'(Z,3) \tag{4.74}$$

using $c_W^2 m_Z^2 = m_W^2$, since we have found

$$\begin{aligned}\delta M_L'(Z,1) &= \frac{g^2}{16\pi^2 c_W^2} U_L^* \hat{m} \frac{r_Z \ln r_Z}{r_Z - 1} U_L^\dagger \\ &= \frac{g^2}{16\pi^2 m_W^2} U_L^* \hat{m}^3 \frac{\ln r_Z}{r_Z - 1} U_L^\dagger,\end{aligned} \tag{4.75}$$

$$\delta M_L'(Z,3) = -\frac{g^2}{64\pi^2 m_W^2} U_L^* \hat{m}^3 \frac{\ln r_z}{r_z - 1} U_L^\dagger. \tag{4.76}$$

So we achieved a final result for δM_L, which does not contain any divergences and is gauge independent. We can write it in the final form

$$\begin{aligned}\delta M_L &= \sum_{b\neq b_Z} \delta M_L'(S_b^0) + \delta M_L'(Z,1) + \delta M_L'(Z,3) \\ &= \sum_{b\neq b_Z} \delta M_L'(S_b^0) - 3\delta M_L'(Z,1) \\ &= \sum_{b\neq b_Z} \frac{1}{32\pi^2} \sum_{b\neq b_Z} \Delta_b^T U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b + \frac{3g^2}{64\pi^2 m_W^2} U_L^* \hat{m}^3 \frac{\ln r_z}{r_z - 1} U_L^\dagger.\end{aligned} \tag{4.77}$$

We could also rewrite the second part using equation (3.12) so we obtain

$$\boxed{\delta M_L = \sum_{b\neq b_Z} \frac{1}{32\pi^2} \Delta_b^T U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b + \frac{3g^2}{64\pi^2 m_W^2} M_D^T U_R^* \hat{m} \frac{\ln r_z}{r_z - 1} U_R^\dagger M_D}. \tag{4.78}$$

In [20] some arguments for the dominance of δM_L in the one-loop corrections to M_ν^{tree} are stated. It has been argued that the factor $(16\pi^2)^{-1}$ is the main reason for the terms of δM_L to be smaller than the tree-level masses of the light neutrinos, since in the considered multi-Higgs model the scalar masses are assumed to be of the order of the electroweak scale. While corrections to M_R are irrelevant, because the heavy neutrino masses are free parameters of the theory, corrections δM_D to M_D are negligible compared to the correction δM_L, since one can find[102]

$$\delta^{1-\text{loop}} M_D(S_a^\pm) \sim Y^2 m_D, \qquad \delta^{1-\text{loop}} M_D(S_b^0) \sim Y^2 m_D, \tag{4.79}$$

where Y is a typical Yukawa coupling and m_D resp. m_R denote the scales of M_D resp. M_R.

Talking about mass scales we should also notice that in δM_L only terms of first order in $(m_R)^{-1}$ are relevant, if we assume m_R to be much larger than the EW scale. To emphasize this we follow the considerations in [20] and introduce the approximation

$$U_R \simeq (0, W), \qquad \text{with} \quad W^\dagger M_R W^* \simeq \tilde{m} \equiv \text{diag}(m_{n_L+1}, \ldots, m_{n_L+n_R}), \tag{4.80}$$

where W is a unitary $n_R \times n_R$ matrix whose elements are not suppressed by $m_D(m_R)^{-1}$.

[102] In [20] it is mentioned that there is no Z^0 contribution for the corrections to M_D.

Furthermore, we assume the mass scale of the RH neutrinos to be much larger then the masses of the scalar mass eigenfields, i.e. $m_R \gg m_b$. Hence, we can write for our result (4.78) of δM_L

$$\boxed{\begin{aligned}\delta M_L = &\sum_{b \neq b_Z} \frac{m_b^2}{32\pi^2} \Delta_b^T W^* \left(\frac{1}{\tilde{m}} \ln \frac{\tilde{m}^2}{m_b^2} \right) W^\dagger \Delta_b \\ &+ \frac{3g^2}{64\pi^2 c_W^2} M_D^T W^* \left(\frac{1}{\tilde{m}} \ln \frac{\tilde{m}^2}{m_Z^2} \right) W^\dagger M_D,\end{aligned}} \tag{4.81}$$

where we used again $c_W m_Z = m_W$ and omitted all terms not proportional to $(\tilde{m})^{-1}$, i.e. of the scale $(m_R)^{-1}$. In this form the dependence of the dominant correction δM_L on the different mass scales m_b, m_Z and m_R, m_D becomes more evident and we might illustrate those dependence in a even more simplified way in terms of mass scales as

$$\delta m_L \sim \frac{1}{16\pi^2} \left(\sum_b \frac{m_b^2}{m_R} \ln \frac{m_R^2}{m_b^2} + \frac{m_D^2}{m_R} \ln \frac{m_R^2}{m_Z^2} \right), \tag{4.82}$$

where δm_L should indicate the mass scale of δM_L.

4.2 A Special One-Loop Corrected Seesaw Models

In this section a special model will be discussed, where neutrino masses are generated by one loop corrections. This model represents a minimal extensions of the SM with one additional Higgs doublet ($n_H = 2$) and one RH neutrino singlet ($n_R = 1$). We also consider only three generations LH neutrinos ($n_L = 3$), which is in accordance with reality and the experimental data.

This model has been discussed by W. Grimus and H. Neufeld in [22] and also in their previous paper [21]. Here it will be explicitly shown that one light LH neutrino gets massive via seesaw mechanism and a second one acquires mass due to one loop corrections, whereas one neutrino still remains massless at one-loop level, as it has been indicated before in the beginning of section 4.1.

4.2.1 Tree-level Neutrino Mass Matrix

The neutrino mass matrix on tree level[103] given in (3.4) reduces in this set up to a complex symmetric 4×4 matrix

$$M_{D+M}^{(0)} = \begin{pmatrix} 0 & M_D^T \\ \underbrace{M_D}_{3} & \underbrace{M_R}_{1} \end{pmatrix} \begin{matrix} \}3 \\ \}1 \end{matrix}, \tag{4.83}$$

where the Dirac mass matrix M_D becomes a 1×3 matrix, i.e. M_D^T is a 3-vector and the

[103] To indicate tree-level quantities resp. one loop level quantities we will introduce superscripts (0) resp. (1).

mass matrix of the RH neutrinos M_R reduces to a 1×1 matrix, i.e. a scalar number which we will assume to be positive and real and we denote it simply by m_R .

According to Schur's theorem (theorem E.2.2) this complex symmetric 4×4 tree-level matrix can be diagonalized by a unitary 4×4 matrix $U^{(0)}$ such that

$$U^{(0)^T} M^{(0)}_{D+M} U^{(0)} = \hat{m}^{(0)} = \mathrm{diag}(m_1^{(0)}, m_2^{(0)}, m_3^{(0)}, m_4^{(0)}), \tag{4.84}$$

where the diagonal elements $m_i^{(0)}$ are nonnegative and real. In order to construct the diagonalizing matrix $U^{(0)}$ we are regarding M_D as linear mapping

$$M_D : \mathbb{C}^3 \to \mathbb{C}. \tag{4.85}$$

The rank-nullity theorem (theorem E.3) leads to

$$\dim \ker M_D = \dim \mathbb{C}^3 - \dim \mathrm{im}\, M_D \geq 3 - 1 = 2. \tag{4.86}$$

Hence, there are in general two orthonormal vectors $u_1', u_2' \in \mathbb{C}^3$, which are elements of the kernel of M_D, i.e.

$$M_D u_i' = 0 \quad \text{for } i = 1, 2. \tag{4.87}$$

We know[104] that the columns of the diagonalizing matrix must be orthogonal and since we also know $u_1' \perp u_2'$ and

$$M_D u_1' = 0 = M_D u_2', \tag{4.88}$$

there must exist a third vector $u_3' \in \mathbb{C}^3$ orthonormal to u_1', u_2', which must be of the form

$$u_3' = \frac{M_D^\dagger}{||M_D^\dagger||}, \tag{4.89}$$

where we denote $||M_D^\dagger|| = \sqrt{M_D M_D^\dagger} =: m_D$. Now we can make a first attempt to construct the diagonalizing matrix, which we call $V^{(0)}$. Its columns must consist of four orthogonal 4×1 vectors which can be constructed from the u_i', for $i = 1, 2, 3$ by attaching a zero in the forth component. For the vector of the forth column being orthogonal to the first three ones we can simply take $(0, 0, 0, 1)^T$, since M_R is already a diagonal matrix because it is just a 1×1 matrix and we might denote it as m_R. Thus, we write

$$\begin{aligned} V^{(0)^T} M^{(0)}_{D+M} V^{(0)} &= \begin{pmatrix} u_1'^T & 0 \\ u_2'^T & 0 \\ u_3'^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & M_{D1} \\ 0 & 0 & 0 & M_{D2} \\ 0 & 0 & 0 & M_{D3} \\ M_{D1} & M_{D2} & M_{D3} & M_R \end{pmatrix} \begin{pmatrix} u_1' & u_2' & u_3' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_D \\ 0 & 0 & m_D & m_R \end{pmatrix}, \end{aligned} \tag{4.90}$$

[104]See Appendix E.2.

where we were able to eliminate some off-diagonal elements. In the next step we rotate the lower right block matrix by a unitary matrix

$$W^{(0)} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{pmatrix}, \tag{4.91}$$

where we introduced the shorthand notation $c := \cos\theta$ and $s := \sin\theta$. We calculate

$$\begin{aligned} &W^{(0)T} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_D \\ 0 & 0 & m_D & m_R \end{pmatrix} W^{(0)} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -m_D cs - (m_D c - m_R s)s & c(m_D c - m_R s) - m_D s^2 \\ 0 & 0 & m_D c^2 - s(m_R c + m_D s) & m_D cs + c(m_R c + m_D s) \end{pmatrix}, \end{aligned} \tag{4.92}$$

and by enforcing vanishing off-diagonal elements we obtain a defining relation for the rotation angle θ. If we demand the upper right entry off-diagonal entry to be zero

$$\begin{aligned} 0 &\overset{!}{=} \cos(\theta)(m_D\cos(\theta) - m_R\sin(\theta)) - m_D\sin(\theta)^2 \\ &= m_D(\sin(\theta)^2 + \cos(\theta)^2) - m_R\cos(\theta)\sin(\theta) \\ \Leftrightarrow \frac{m_D}{m_R} &= \frac{\sin(\theta)\cos(\theta)}{\cos(\theta)^2 - \sin(\theta)^2} \overset{(E.43)}{=} \frac{\sin(\theta)\cos(\theta)}{\cos(2\theta)} \\ &\overset{(E.44)}{=} \frac{1}{2}\frac{\sin(2\theta)}{\cos(2\theta)} = \frac{1}{2}\tan(2\theta), \end{aligned} \tag{4.93}$$

and hence we achieve the following condition

$$\boxed{\tan(2\theta) = 2\frac{m_D}{m_R}}. \tag{4.94}$$

Using this result we can show that the diagonal entries are the eigenvalues of $M^{(0)}_{\mathrm{D+M}}$, which can be calculated by

$$\det\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_D \\ 0 & 0 & m_D & m_R \end{pmatrix} - \lambda\mathbb{1}_4\right] = 0. \tag{4.95}$$

This gives us solutions

$$\lambda_{1,2} = 0, \tag{4.96}$$

$$\lambda_{3,4} = \frac{m_R}{2} \pm \sqrt{\frac{m_R^2}{4} + m_D^2}. \tag{4.97}$$

Inserting (4.94) into the lower two diagonal elements (3,3) and (4,4) in (4.93) and using the relations (E.41)-(E.44) we find

$$\begin{aligned}
(4,4) &= m_D\cos(\theta)\sin(\theta) + \cos(\theta)(m_R\cos(\theta) + m_D\sin(\theta)\\
&= 2m_D\cos(\theta)\sin(\theta) + m_R\cos(\theta)^2\\
&= m_D\sin(2\theta) + \frac{1}{2}m_R + \frac{1}{2}m_R\cos(2\theta)\\
&= m_D\frac{\tan(2\theta)}{\sqrt{1+\tan(2\theta)^2}} + \frac{1}{2}m_R + \frac{1}{2}m_R\frac{1}{\sqrt{1+\tan(2\theta)^2}}\\
&= m_D\frac{2\frac{m_D}{m_R}}{\sqrt{1+\left(\frac{2m_D}{m_R}\right)^2}} + \frac{1}{2}m_R + \frac{1}{2}m_R\frac{1}{\sqrt{1+\left(\frac{2m_D}{m_R}\right)^2}}\\
&= \frac{1}{2}m_R + \frac{\sqrt{1+4\frac{m_D^2}{m_R^2}}}{1+4\frac{m_D^2}{m_R^2}}\left(2\frac{m_D^2}{m_R} + \frac{1}{2}m_R\right)\\
&= \frac{1}{2}m_R + m_R\frac{\sqrt{m_R^2+4m_D^2}}{m_R^2+4m_D^2}\left(2\frac{m_D^2}{m_R} + \frac{1}{2}m_R\right)\\
&= \frac{1}{2}m_R + \frac{\sqrt{m_R^2+4m_D^2}}{m_R^2+4m_D^2}\frac{1}{2}\left(4m_D^2+m_R^2\right)\\
&= \frac{1}{2}m_R + \frac{1}{2}\sqrt{m_R^2+4m_D^2}\\
&= \frac{1}{2}m_R + \sqrt{\frac{1}{4}m_R^2+4m_D^2} = \lambda_4,
\end{aligned} \tag{4.98}$$

$$\begin{aligned}
(3,3) &= -m_D\cos(\theta)\sin(\theta) - (m_D\cos(\theta) - m_R\sin(\theta))\sin(\theta)\\
&= -2m_D\cos(\theta)\sin(\theta) + m_R\sin(\theta)^2\\
&= -m_D\frac{2\frac{m_D}{m_R}}{\sqrt{1+\left(\frac{2m_D}{m_R}\right)^2}} + \frac{1}{2}m_R - \frac{1}{2}m_R\frac{1}{\sqrt{1+\tan(2\theta)^2}}\\
&= \frac{1}{2}m_R + \frac{\sqrt{m_R^2+4m_D^2}}{m_R^2+4m_D^2}\frac{m_R}{2}\left(-4\frac{m_D^2}{m_R} - m_R\right)\\
&= \frac{1}{2}m_R - \frac{1}{2}\sqrt{m_R^2+4m_D^2}\\
&= \frac{1}{2}m_R - \sqrt{\frac{1}{4}m_R^2+4m_D^2} = \lambda_3.
\end{aligned} \tag{4.99}$$

Finally we have to ensure non negative real diagonal entries, since we are interested in mass eigenvalues. Because the eigenvalue λ_3 is not positive we have to insert a factor i by

$$K^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.100}$$

such that we obtain the final result

$$\begin{aligned}
&K^{(0)T}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}m_R - \sqrt{\frac{1}{4}m_R^2 + 4m_D^2} & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_R + \sqrt{\frac{1}{4}m_R^2 + 4m_D^2} \end{pmatrix} K^{(0)} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1}{4}m_R^2 + 4m_D^2} - \frac{1}{2}m_R & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{4}m_R^2 + 4m_D^2} + \frac{1}{2}m_R \end{pmatrix},
\end{aligned} \tag{4.101}$$

where all diagonal entries are non-negative. Hence, we constructed the diagonalizing matrix U as

$$U^{(0)} = V^{(0)}.W^{(0)}.K^{(0)} = \begin{pmatrix} u_1 & u_2 & i\cos(\theta)u_3' & \sin(\theta)v_3 \\ 0 & 0 & -i\sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{4.102}$$

such that

$$U^{(0)T} M_{D+M}^{(0)} U^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_3^{(0)} & 0 \\ 0 & 0 & 0 & m_4^{(0)} \end{pmatrix}, \tag{4.103}$$

with mass eigenvalues

$$m_3^{(0)} = \sqrt{\frac{1}{4}m_R^2 + m_D^2} - \frac{1}{2}m_R, \tag{4.104}$$

$$m_4^{(0)} = \sqrt{\frac{1}{4}m_R^2 + m_D^2} + \frac{1}{2}m_R. \tag{4.105}$$

In the seesaw limit for $m_R \gg m_D$ we obtain by expanding the square-root the following approximation

$$m_3^{(0)} \simeq \frac{m_R}{2}\left(1 + \frac{1}{2}\cdot\frac{4m_D^2}{m_R^2}\right) - \frac{m_R}{2} = \frac{m_D^2}{m_R}, \tag{4.106}$$

$$m_4^{(0)} \simeq \frac{m_R}{2}\left(1 + \frac{1}{2}\cdot\frac{4m_D^2}{m_R^2}\right) + \frac{m_R}{2} = m_R, \tag{4.107}$$

which reproduces exactly what we have already found in (3.31) and (3.33). But here we have shown also that only one light LH neutrino acquires mass via seesaw mechanism, whereas the other two LH neutrinos remain massless on tree level.

4.2.2 One-loop Neutrino Mass Matrix

In the next step we will show that after one-loop corrections a second LH neutrino becomes massive and only one remains massless. To show this explicitly we have to diagonalize the corrected neutrino mass matrix given in (4.2), which reduces in our set up to the symmetric[105] complex 4×4 matrix

$$M^{(1)}_{\mathrm{D+M}} = \begin{array}{c} \begin{pmatrix} \delta M_L & M_D^T \\ M_D & m_R \end{pmatrix} \begin{array}{l} \}3 \\ \}1 \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{\delta M_L}}_{3} & \underbrace{\hphantom{m_R}}_{1} \end{array} \end{array}, \tag{4.108}$$

where the correction δM_L has been derived in the previous section 4.1 and is given in equation (4.78). For this purpose we decompose the matrix $U^{(0)}$ given in (4.102) in the following way:

$$U^{(0)} = \begin{array}{c} \begin{pmatrix} U'_L & U''_L \\ 0 & U''^*_R \end{pmatrix} \begin{array}{l} \}n_L \\ \}n_R \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{U'_L}}_{n_L - n_R} & \underbrace{\hphantom{U''_L}}_{2n_R} \end{array} \end{array}, \tag{4.109}$$

where U'_L is in our special case the 3×2 matrix (u'_1, u'_2), U''_L is a 3×2 matrix and U''^*_R is 1×2. A straightforward calculation[106] leads to

$$\begin{aligned} U^{(0)^T} M^{(1)}_{\mathrm{D+M}} U^{(0)} =& U^{(0)^T} M^{(0)}_{\mathrm{D+M}} U^{(0)} + U^{(0)^T} \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix} U^{(0)} \\ =& \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix} + \begin{array}{c} \begin{pmatrix} U'^T_L \delta M_L U'_L & U'^T_L \delta M_L U''_L \\ U''^T_L \delta M_L U'_L & U''^T_L \delta M_L U''_L \end{pmatrix} \begin{array}{l} \}2 \\ \}2 \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{U'^T_L \delta M_L U'_L}}_{2} & \underbrace{\hphantom{U''^T_L \delta M_L U''_L}}_{2} \end{array} \end{array}. \end{aligned} \tag{4.110}$$

This second matrix, which can be considered as a perturbation of the tree-level mass matrix, can be diagonalized by a unitary transformation $V^{(1)}$ as mentioned in [21], where $V^{(1)} - \mathbb{1}$ is of one-loop order

$$V^{(1)} \simeq \mathbb{1} + i\Omega, \qquad \Omega = \Omega^\dagger. \tag{4.111}$$

The unitary matrix actually diagonalizing $M^{(1)}_{\mathrm{D+M}}$ is constructed as the product

$$U = U^{(0)} V^{(1)}, \tag{4.112}$$

and the additional corrective terms to $\hat{m}^{(0)}$, more exactly to $m_3^{(0)}$ and $m_4^{(0)}$, due to matrix multiplication with $V^{(1)}$ will be really small and hence negligible. Thus, for an appropriate choice[107] of $V^{(1)}$ we can achieve

$$\begin{aligned} U^T M^{(1)}_{\mathrm{D+M}} U \simeq& \hat{m}^{(0)} + \begin{pmatrix} \mathrm{Re}\left(U'^T_L \delta M_L U'_L\right) & 0 \\ 0 & 0 \end{pmatrix} \\ =& \begin{pmatrix} \mathrm{Re}\left(u'^T_1 \delta M_L u'_1\right) & \mathrm{Re}\left(u'^T_1 \delta M_L u'_2\right) & 0 & 0 \\ \mathrm{Re}\left(u'^T_2 \delta M_L u'_1\right) & \mathrm{Re}\left(u'^T_2 \delta M_L u'_2\right) & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}. \end{aligned} \tag{4.113}$$

[105] This is quite obvious since δM_L is symmetric.
[106] The full calculation can be found in appendix F.9.
[107] This procedure is discussed in general in appendix E.2.2 and more explicitly for this case in F.9.

The remaining off-diagonal entries in the upper left submatrix $U_L'^T \delta M_L U_L'$ can be removed by an appropriate choice of u_1' and u_2'. We recognize that due to its form

$$\delta M_L = \sum_{b \neq b_Z} \frac{1}{32\pi^2} \Delta_b^T U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b + \frac{3g^2}{64\pi^2 m_W^2} M_D^T U_R^* \hat{m} \frac{\ln r_z}{r_z - 1} U_R^\dagger M_D, \tag{4.114}$$

obtained in (4.78), the Z^0 contribution to δM_L, i.e. the second term, will not contribute at all since we had $M_D u_i' = 0$ for $i = 1, 2$. So if we choose $u_1' \perp \Delta_1$ and $u_1' \perp \Delta_2$ the remaining off-diagonal entries in (4.113) are removed since $\Delta_b = b_1 \Delta_1 + b_2 \Delta_2$ and we obtain

$$\begin{aligned} {U^{(0)}}^T M_{\mathrm{D+M}}^{(0)} U^{(0)} &= \hat{m}^{(0)} + \begin{pmatrix} U_L'^T \delta M_L U_L' & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{Re}\left(u_2'^T \delta M_L u_2'\right) & 0 & 0 \\ 0 & 0 & m_3^{(0)} & 0 \\ 0 & 0 & 0 & m_4^{(0)} \end{pmatrix}. \end{aligned} \tag{4.115}$$

Furthermore, the overall phase of u_2' can be chosen in a way such that $u_2'^T \delta M_L u_2'$ is positive. This procedure also shows that even after one-loop corrections one LH neutrino remains massless[108].

Finally, we would like to give an estimation on the order of magnitude of $m_2^{(1)}$ as it is done in [22]. An upper boundary for $m_2^{(1)}$ can be found in the following way. We know from construction of $U^{(0)}$ that u_2' is orthonormal to u_1' and u_3' and its phase is fixed by the positivity of $m_2^{(1)}$. We use u_2' to define the following quantity

$$c_\alpha = \frac{v}{\sqrt{2} m_D} u_2'^T \Delta_\alpha^T \in \mathbb{C}, \tag{4.116}$$

for $\alpha = 1, 2$ and with the Yukawa coupling constants Δ_α and $v = \sqrt{|v_1|^2 + |v_2|^2}$. This quantities fulfil the relation

$$v_1^* c_1 + v_2^* c_2 = 0, \tag{4.117}$$

since we had M_D given in equation (3.106) and $m_D = ||M_D||$. v_1 and v_2 denote the VEV of the two Higgs doublets as defined in (3.61). Nevertheless, $|c_1|^2 + |c_2|^2$ remains an independent parameter in this model, because we have no restrictions on u_2' and Δ_α apart from *naturalness*, which requires $|c_1|^2 + |c_2|^2$ being $\mathcal{O}(1)$. Above, we have found

$$\begin{aligned} m_2 &= u_2'^T \delta M_L u_2' \\ &= \sum_{b \neq b_Z} \frac{1}{32\pi^2} u_2'^T \Delta_b^T U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b u_2' \\ &= \sum_{b \neq b_Z} \frac{1}{32\pi^2} u_2'^T \left(\sum_\alpha b_\alpha \Delta_\alpha^T \right) U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b - 1} U_R^\dagger \left(\sum_\beta b_\beta \Delta_\beta \right) u_2', \end{aligned} \tag{4.118}$$

where we used $\Delta_b = b_1 \Delta_1 + b_2 \Delta_2$.

[108] However, it will become massive at two loop level as shown in [81, 82, 74].

We can now rewrite this in terms of our newly defined quantity c_α and use the Cauchy-Schwarz inequality (E.4):

$$\begin{aligned} m_2 &= \sum_{b\neq b_Z} \frac{2m_D^2}{32\pi^2 v^2}\left(\sum_\alpha b_\alpha c_\alpha\right) U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger \left(\sum_\beta b_\beta c_\beta\right) \\ &\leq \sum_{b\neq b_Z} \frac{2m_D^2}{32\pi^2 v^2}\left| U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger\right| \left|\sum_\alpha b_\alpha c_\alpha\right|^2 \\ &\leq \sum_{b\neq b_Z} \frac{2m_D^2}{32\pi^2 v^2}\left| U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger\right| \left(\sum_\alpha |b_\alpha|^2\right)\left(\sum_\alpha |c_\alpha|^2\right). \end{aligned} \tag{4.119}$$

In (3.97) we had $||b|| = \sum_\alpha b_\alpha = 1$ and hence we get

$$m_2 \leq \sum_{b\neq b_Z} \frac{2m_D^2}{32\pi^2 v^2}\left| U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger\right| \left(|c_1|^2 + |c_2|^2\right). \tag{4.120}$$

For a more meaningful result we should simplify the factor $U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger$. This can be done in a similar way to the procedure used for (4.81) where we assumed the 1×4 matrix to be

$$(\underbrace{0}_{2}, \underbrace{U_R''}_{2}) = U_R \simeq (\underbrace{0}_{3}, \underbrace{W}_{1}). \tag{4.121}$$

In our case we have $W = \mathbb{1}$ and hence $\tilde{m} = W^\dagger m_R W^* = m_R \gg m_b$. Thus, we obtain

$$m_2 \leq \sum_{b\neq b_Z} \frac{m_b^2}{16\pi^2 v^2} W^* \left(\frac{1}{m_R} \ln \frac{{m_R}^2}{m_b^2}\right) W^\dagger \left(|c_1|^2 + |c_2|^2\right). \tag{4.122}$$

More explicitly we might calculate this factor by inserting $U_R = (0, 0, -i\sin\theta, \cos\theta)$ and $\hat{m}^{(0)} = \mathrm{diag}(0, 0, m_3^{(0)}, m_4^{(0)})$. Doing so we get

$$U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger = \frac{m_b^2}{2m_R}\left(-\sin(\theta)^2 \frac{m_R^2}{m_D^2}\ln\left(\frac{m_D^4}{m_b^2 m_R^2}\right) + \cos(\theta)^2 \ln\left(\frac{m_R^2}{m_b^2}\right)\right). \tag{4.123}$$

Furthermore, we can achieve for $m_D \ll m_R$

$$\cos(\theta)^2 = \frac{1}{2}(1+\cos(2\theta)) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1+\left(\frac{2m_D}{m_R}\right)^2}}\right) \approx 1, \tag{4.124}$$

$$\sin(\theta)^2 = \frac{1}{2}(\cos(2\theta)-1) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\left(\frac{2m_D}{m_R}\right)^2}} - 1\right) \approx 0, \tag{4.125}$$

which leads to

$$U_R^* \hat{m}^{(0)} \frac{\ln r_b}{r_b-1} U_R^\dagger \simeq \frac{m_b^2}{2m_R} \ln\left(\frac{m_R^2}{m_b^2}\right). \tag{4.126}$$

Hence, we have achieved the following result for an upper boundary of m_2 as

$$m_2 \leq \sum_{b\neq b_Z} \frac{m_D^2}{32\pi^2 v^2}\frac{m_b^2}{m_R} \ln\left(\frac{m_R^2}{m_b^2}\right)\left(|c_1|^2 + |c_2|^2\right), \tag{4.127}$$

which can be used for an estimation on the order of magnitude

$$\boxed{m_2^{(1)} \sim \frac{1}{16\pi^2} m_3 \frac{M_0^2}{v^2} \ln \frac{M_R}{M_0}}, \tag{4.128}$$

where M_0 is a generic physical neutral scalar mass.

It is also noted that for $M_0 \sim v$ the relation $m_2^{(1)} \ll m_3^{(0)}$ comes solely from the numerical factor $(16\pi^2)^{-1}$ appearing in the loop integration. In [22] it has been also stated that in general cancellations in the summation over the vectors b do not appear, because those vectors are connected to the diagonalizing matrix of the neutral scalar mass matrix, but these matrix elements are independent of the masses m_b^2.

5 Scotogenic Model

5.1 Original Model by E. Ma

This model, proposed by E. Ma in [23], is an extension of the SM with three RH neutrino singlets ($n_R = 3 = n_L$) and one additional scalar doublet ($n_H = 2$). Moreover, the theory is supposed to exhibit an additional exact symmetry and the symmetry group is given by $SU(2)_L \times U(1)_Y \times \mathbb{Z}_2$. All SM particles ($D_L$, ℓ_R, ϕ) transform evenly under $\mathbb{Z}_2$, whereas all new particles (ν_R and η) transform oddly, i.e.

$$\begin{gathered} D_L \to D_L, \qquad \ell_R \to \ell_R, \qquad \phi \to \phi, \\ \nu_R \to -\nu_R, \qquad \eta \to -\eta. \end{gathered} \tag{5.1}$$

This symmetry is assumed to remain unbroken after SSB, i.e.

$$SU(2)_L \times U(1)_Y \times \mathbb{Z}_2 \overset{\text{SSB}}{\longrightarrow} U(1)_{\text{EM}} \times \mathbb{Z}_2. \tag{5.2}$$

Hence, some terms are forbidden in the Lagrangian, which has to be invariant under $\mathbb{Z}_2$ transformations. In particular the Majorana mass term for the RH neutrinos given in (2.22) is invariant. Moreover, the mass matrix of the RH neutrinos can be chosen diagonal without loss of generality, i.e. $M_R = \text{diag}(m_{R1}, m_{R2}, m_{R3})$. The 6×6 neutrino mass matrix $M^{(0)}_{\text{D+M}}$ at tree-level is a priori given according to (3.4).

5.1.1 Scalar Mass Eigenfields

For investigating the scalar sector we use the general results of section 3.2.1 and start by adapting the scalar potential (3.63) for $n_H = 2$, i.e.

$$V = \sum_{i,j=1}^{2} \mu^2_{ij} \phi^\dagger_i \phi_j + \sum_{i,j,k,l=1}^{2} \lambda_{ijkl} \left(\phi^\dagger_i \phi_j\right) \left(\phi^\dagger_k \phi_l\right). \tag{5.3}$$

Because of the additional symmetry terms linear and cubic in η are forbidden, since V has to be also invariant under $\mathbb{Z}_2$ transformations. Therefore, we have

$$\begin{aligned} V =& \mu^2_{11}\phi^\dagger_1\phi_1 + \mu^2_{22}\phi^\dagger_2\phi_2 + \lambda_{1111}\left(\phi^\dagger_1\phi_1\right)^2 + \lambda_{2222}\left(\phi^\dagger_2\phi_2\right)^2 + 2\lambda_{1122}\left(\phi^\dagger_1\phi_1\right)\left(\phi^\dagger_2\phi_2\right) \\ &+ 2\lambda_{1221}\left(\phi^\dagger_1\phi_2\right)\left(\phi^\dagger_2\phi_1\right) + \lambda_{1212}\left(\phi^\dagger_1\phi_2\right)^2 + \lambda^*_{1212}\left(\phi^\dagger_2\phi_1\right)^2, \end{aligned} \tag{5.4}$$

where we used the properties of λ_{ijkl} given in (3.64). The inadmissibility of linear and cubic terms in η means

$$\mu^2_{12} = \mu^2_{21} = 0, \tag{5.5}$$

$$\lambda_{1112} = \lambda_{1222} = 0, \tag{5.6}$$

and hence all $\lambda_{ijkl} = 0$, with $(ijkl)$ being permutations of (1112) and (1222), because of (3.64).

Introducing the notation used in [23], we define

$$\phi_1 = \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad \phi_2 = \eta = \begin{pmatrix} \eta^+ \\ \eta^0 \end{pmatrix}, \tag{5.7}$$

$$\mu_{11}^2 = m_1^2, \qquad \mu_{22}^2 = m_2^2, \tag{5.8}$$

$$\lambda_{1111} = \frac{1}{2}\lambda_1, \quad \lambda_{2222} = \frac{1}{2}\lambda_2, \quad \lambda_{1122} = \frac{1}{2}\lambda_3, \quad \lambda_{1221} = \frac{1}{2}\lambda_4, \quad \lambda_{1212} = \frac{1}{2}\lambda_5. \tag{5.9}$$

Due to Hermiticity of all terms (except the last one) all coefficients are real[109], i.e.

$$m_1^2, m_2^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}. \tag{5.10}$$

Moreover $\lambda_5 \in \mathbb{R}$ can be assumed, since a possible phase can be absorbed into the field η, as it has been noted in [84]. Thus, the scalar potential V reads as

$$\begin{aligned} V =& m_1^2\phi^\dagger\phi + m_2^2\eta^\dagger\eta + \frac{1}{2}\lambda_1\left(\phi^\dagger\phi\right)^2 + \frac{1}{2}\lambda_2\left(\eta^\dagger\eta\right)^2 \\ &+ \lambda_3\left(\phi^\dagger\phi\right)\left(\eta^\dagger\eta\right) + \lambda_4\left(\phi^\dagger\eta\right)\left(\eta^\dagger\phi\right) + \frac{1}{2}\lambda_5\left(\left(\phi^\dagger\eta\right)^2 + \text{H.c.}\right). \end{aligned} \tag{5.11}$$

Finding VEV's in this potential is only possible if V is bounded from below, because only if V does not tend to minus infinity anywhere, the potential has stable minima, which has been discussed in [84] or [85, p.90f]. This leads to some constraints for the coupling coefficients λ_i, which have been investigated in [85, 84, 86, 87]:

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \lambda_3 > -\sqrt{\lambda_1\lambda_2}, \qquad \lambda_3 + \lambda_4 - |\lambda_5| > -\sqrt{\lambda_1\lambda_2}. \tag{5.12}$$

Without loss of generality it can be assumed that $\lambda_5 \geq 0$, as done in [23]. Furthermore in [84] the requirements of the parameters m_i^2 for a stable $\mathbb{Z}_2$ symmetric vacuum can be divided in two classes:

$$\text{(a)} \qquad m_2^2 \geq 0 \qquad\qquad \text{if } m_1^2 \geq 0 \Leftrightarrow \langle\phi^0\rangle^2 = 0 = \langle\eta^0\rangle^2, \tag{5.13}$$

$$\text{(b)} \qquad m_2^2 \geq \frac{\lambda_3 + \lambda_4 + \lambda_5}{\lambda_1} m_1^2 \qquad \text{if } m_1^2 < 0 \Leftrightarrow \langle\phi^0\rangle^2 = -\frac{m_1^2}{\lambda_1}, \quad \langle\eta^0\rangle^2 = 0. \tag{5.14}$$

In Ma's scotogenic model [23] the case $m_1^2 < 0$ and $m_2^2 > 0$ has been considered, which is included in case (b) above. Therefore, the new scalar doublet η has vanishing VEV and for the standard Higgs doublet we can choose its usual VEV, c.f. equation (1.89),

$$v^2 \equiv \langle\phi^0\rangle^2 = \left(\frac{v_1}{\sqrt{2}}\right)^2 = -\frac{m_1^2}{\lambda_1}. \tag{5.15}$$

Because of only one nonvanishing VEV, v can be chosen real, according to [26, 84, 85].

A simplified plot of the potential with this constraints is shown in figure 12 below, where we only considered the real parts of the neutral component of the scalar doublets.

[109]This has been also mentioned in [83, p.242]

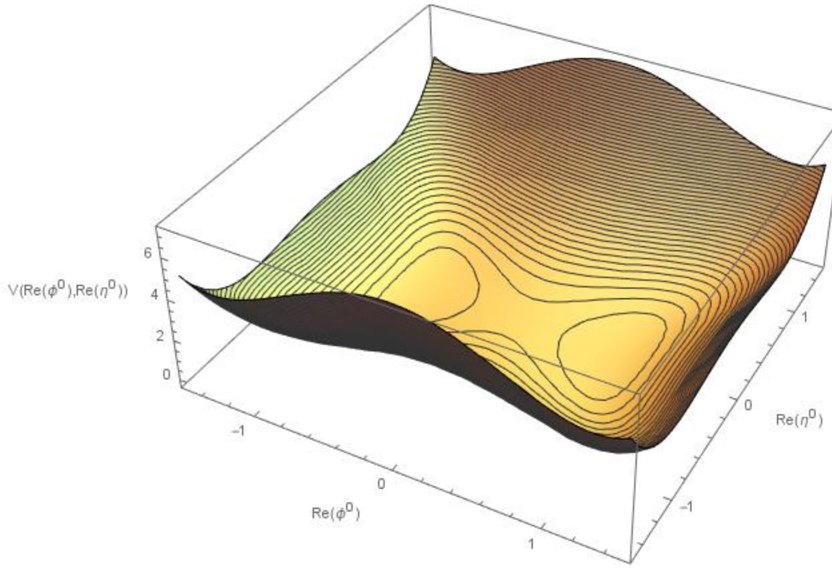


Figure 12: This is a simplified plot, done with Mathematica 10, of the scalar potential as a function of the real part of the neutral components of the scalar fields, i.e. $V = -\left(\mathrm{Re}(\phi^0)\right)^2 + \left(\mathrm{Re}(\eta^0)\right)^2 + \left(\mathrm{Re}(\phi^0)\right)^4 + \left(\mathrm{Re}(\eta^0)\right)^4 - \left(\mathrm{Re}(\phi^0)\right)^2 \left(\mathrm{Re}(\eta^0)\right)^2$.

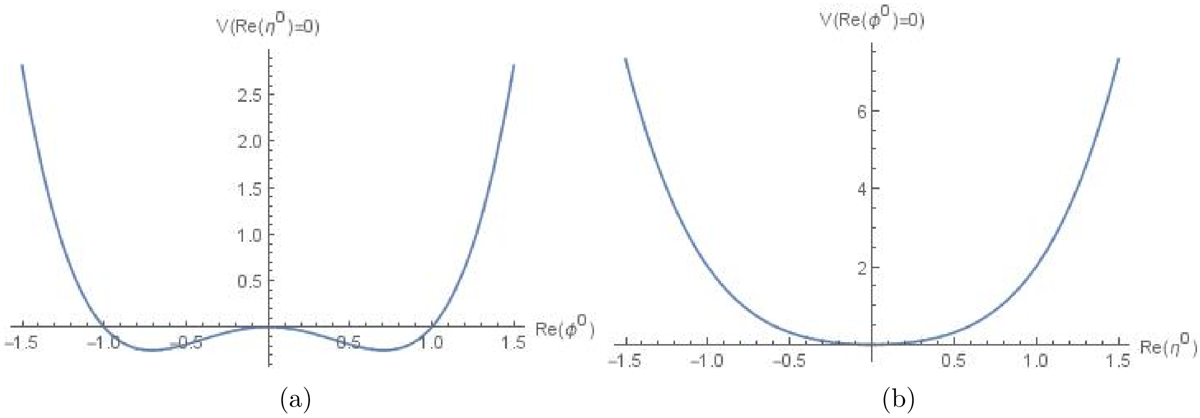
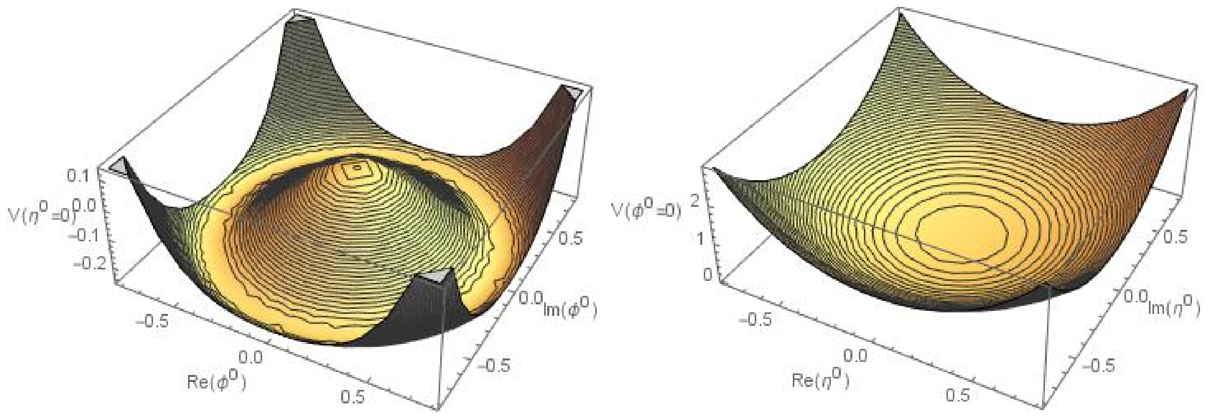


Figure 13: Here two different cross sections of the potential $V(\mathrm{Re}(\phi^0), \mathrm{Re}(\eta^0))$ are illustrated using Mathematica 10. In (a) the cross section of the $\mathrm{Re}(\eta^0) = 0$ plane and in (b) the one of $\mathrm{Re}(\phi^0) = 0$ is shown respectively. Apparently the VEV of $\mathrm{Re}(\phi^0)$ is degenerate whereas the VEV for $\mathrm{Re}(\eta^0)$ is zero.

In order to find the masses of the scalar mass eigenfields we can use again the results of section 3.2.1. To apply the equations found there we need the matrices μ^2 as well as Λ, K and K' given in equation (3.76),(3.83) and (3.85) respectively in terms of our model here. We find

$$\begin{aligned} \mu^2 &= \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \qquad \Lambda = v^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \end{pmatrix}, \\ K &= v^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_5 \end{pmatrix}, \quad K' = v^2 \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_4 \end{pmatrix}. \end{aligned} \tag{5.16}$$



(a) (b)

Figure 14: Now the complex valued scalar potential $V(\phi, \eta)$ has been plotted in Mathematica 10 for (a) $\eta^0 = 0$ and (b) $\phi^0 = 0$ respectively. The shape of (a) clearly shows a whole circle of minima, i.e. the VEV is infinitely degenerate, whereas in (b) one global minimum appears at zero; c.f. figure 22 in appendix B.4.2.

Hence, we can apply equation (3.75) to find the masses of the charged scalar mass eigenfields $\phi^\pm$ and $\eta^\pm$

$$\mathcal{M}_+^2 = \begin{pmatrix} m_1^2 + \lambda_1 v^2 & 0 \\ 0 & m_2^2 + \lambda_3 v^2 \end{pmatrix}. \tag{5.17}$$

Taking (5.15) into account we obtain

$$m^2(\phi^+) = 0, \qquad m^2(\eta^+) = m_2^2 + \lambda_3 v^2. \tag{5.18}$$

For the masses of the neutral scalar mass eigenfields we need the matrices A, B and C given in (3.90), (3.91) and (3.92) respectively. Since all coefficients are real we get $C = 0_{2\times 2}$ and

$$A = \begin{pmatrix} m_1^2 + 3\lambda_1 v^2 & 0 \\ 0 & m_2^2 + (\lambda_3 + \lambda_4 + \lambda_5) v^2 \end{pmatrix}, \tag{5.19}$$

$$B = \begin{pmatrix} m_1^2 + \lambda_1 v^2 & 0 \\ 0 & m_2^2 + (\lambda_3 + \lambda_4 - \lambda_5) v^2 \end{pmatrix}. \tag{5.20}$$

The formula for the neutral scalar masses is given in (3.95) resp. (3.96) and hence, using (5.15), we achieve

$$m^2(\sqrt{2}\mathrm{Re}\phi^0) = 2\lambda_1 v^2, \quad m^2(\sqrt{2}\mathrm{Re}\eta^0) = m_2^2 + (\lambda_3 + \lambda_4 + \lambda_5) v^2, \tag{5.21}$$

$$m^2(\sqrt{2}\mathrm{Im}\phi^0) = 0, \qquad m^2(\sqrt{2}\mathrm{Im}\eta^0) = m_2^2 + (\lambda_3 + \lambda_4 - \lambda_5) v^2. \tag{5.22}$$

The scalar doublets can be parametrized by their mass eigenfields analogously to (3.101) as follows

$$\phi = \begin{pmatrix} 0 \\ \frac{v_1 + h}{\sqrt{2}} \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta^+ \\ \frac{\rho + i\sigma}{\sqrt{2}} \end{pmatrix}, \tag{5.23}$$

where we used the notation $h = \sqrt{2}\mathrm{Re}(\phi^0)$, $\rho = \sqrt{2}\mathrm{Re}(\eta^0)$ and $\sigma = \sqrt{2}\mathrm{Im}(\eta^0)$. This is the parametrization in the unitary gauge, since we have chosen the VEV of ϕ^0 in the appropriate form given in (5.15). Therefore $\phi^\pm$ and $g = \mathrm{Im}(\phi^0)$ are massless, i.e. the Goldstone bosons of the theory. Furthermore $\mathcal{M}_0^2$ in (3.95) is already diagonal, as shown in (5.19) above, which mean the fields $\phi^\pm, \eta^\pm$ as well as h, g, ρ and σ are already the scalar mass eigenfields since no mixing occurs. Thus, the 2×1 eigenvectors b for the scalar mass eigenfields h, g, ρ, σ have to be the complex unit vectors

$$b^{(h)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad b^{(g)} = \begin{pmatrix} i \\ 0 \end{pmatrix}, \qquad b^{(\rho)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad b^{(\sigma)} = \begin{pmatrix} 0 \\ i \end{pmatrix}. \tag{5.24}$$

Since ϕ is the SM Higgs doublet, the real part of its neutral component becomes the Higgs field after SSB and its imaginary part remains massless and represents the Goldstone boson, which will give mass to the gauge boson Z^0 via the Higgs mechanism discussed in section 1.3.4. The charged component of ϕ remains also massless due to the particular choice of the VEV, i.e. the unitary gauge, and it will give mass to the gauge bosons $W^\pm$ via the Higgs mechanism. So we end up with one massive real neutral scalar mass eigenfield h from the Higgs doublet as usual.

However, the second scalar doublet η leads to three additional mass eigenfields $\eta^\pm, \mathrm{Re}\eta^0$ and $\mathrm{Im}\eta^0$. In particular we recognize from (5.21) and (5.22) above that the masses of $\mathrm{Re}\eta^0$ and $\mathrm{Im}\eta^0$ differ by $2\lambda_5 v^2$. If the coupling coefficient $\lambda_5 = 0$, the real and imaginary part of η^0 have the same mass, i.e. $m^2(\mathrm{Re}(\sqrt{2}\eta^0)) = m^2(\mathrm{Im}(\sqrt{2}\eta^0))$ which means the mass eigenvalue is two-fold degenerate.

5.1.2 Yukawa Interactions

Now considering the Yukawa Lagrangian we find that some terms there are also forbidden due to the additional symmetry, since $\mathcal{L}_{\mathrm{Yuk}}$ has to be also invariant under $\mathbb{Z}_2$ transformations. Applying the result (3.105) for the Yukawa Lagrangian in a multi-Higgs model in section 3.2.2 to our special case $n_H = 2, n_R = 3$ we find

$$\begin{aligned} -\mathcal{L}_{\mathrm{Yuk}} &= \sum_{k=1}^{2} \sum_{i_1,i_2,j=1}^{3} \left[\phi_k^\dagger \overline{\ell}_{i_1 R} (\Gamma^{(k)})_{i_1 i_2} + \tilde{\phi}_k^\dagger \overline{\nu}_{jR} (\Delta^{(k)})_{j i_2} \right] D_{i_2 L} + \text{H.c.} \\ &= \sum_{k=1}^{2} (\phi_k^\dagger \overline{\ell}_R \Gamma^{(k)} + \tilde{\phi}_k^\dagger \overline{\nu}_R \Delta^{(k)}) D_L + \text{H.c.}\,. \end{aligned} \tag{5.25}$$

Here ϕ_1 is identified with the standard Higgs doublet ϕ and the additional scalar doublet ϕ_2 will be called η. Hence, we obtain

$$\begin{aligned} -\mathcal{L}_{\mathrm{Yuk}} &= (\phi^\dagger \overline{\ell}_R \Gamma^{(\phi)} + \tilde{\phi}^\dagger \overline{\nu}_R \Delta^{(\phi)}) D_L + (\eta^\dagger \overline{\ell}_R \Gamma^{(\eta)} + \tilde{\eta}^\dagger \overline{\nu}_R \Delta^{(\eta)}) D_L + \text{H.c.} \\ &= \left((\phi^-, \phi^{0*}) \overline{\ell}_R \Gamma^{(\phi)} + (\phi^0, -\phi^+) \overline{\nu}_R \Delta^{(\phi)} \right) \begin{pmatrix} \nu_{\ell L} \\ \ell_L \end{pmatrix} \\ &\quad + \left((\eta^-, \eta^{0*}) \overline{\ell}_R \Gamma^{(\eta)} + (\eta^0, -\eta^+) \overline{\nu}_R \Delta^{(\eta)} \right) \begin{pmatrix} \nu_{\ell L} \\ \ell_L \end{pmatrix} + \text{H.c.}\,. \end{aligned} \tag{5.26}$$

Because of the $\mathbb{Z}_2$ symmetry all terms linear in the new fields ν_R and η are forbidden. Thus, we get

$$
\begin{aligned}
-\mathcal{L}_{\text{Yuk}} =& \overline{\ell}_R \Gamma^{(\phi)} \left(\phi^- \nu_L + \phi^{0*} \ell_L \right) + \overline{\nu}_R \Delta^{(\eta)} \left(\eta^0 \nu_L - \eta^+ \ell_L \right) + \text{H.c.} \\
=& \Gamma^{(\phi)}_{ij} \overline{\ell}_{iR} \phi^- \nu_{jL} + \Gamma^{(\phi)}_{ij} \overline{\ell}_{iR} \phi^- \phi^{0*} \ell_{jL} \\
&+ \Delta^{(\eta)}_{ij} \overline{\nu}_{iR} \eta^0 \nu_{jL} - \Delta^{(\eta)}_{ij} \overline{\nu}_{iR} \eta^+ \ell_{jL} + \text{H.c.}\,,
\end{aligned} \tag{5.27}
$$

for $\Delta^{(\phi)} = 0$ and $\Gamma^{(\eta)} = 0$. This restrictions on the 3×3 Yukawa coupling matrices take into account the $\mathbb{Z}_2$ symmetry. The second term in the second line will give the charge lepton masses as in the SM, since the standard Higgs doublet acquires VEV $v \neq 0$.

But there is no Dirac mass term for the neutrinos, since we have $M_D = 0$ according to equation (2.3) and $\langle \eta^0 \rangle = 0$. Hence, in this model no neutrino masses are generated at tree-level via the seesaw mechanism since the seesaw formula, we derived in (3.31), gives $M^{(0)}_{\text{light}} = 0$. Moreover, no mixing of the RH and LH neutrinos occurs since $M^{(0)}_{D+M}$ is already diagonal, because the only nonzero submatrix is M_R, which has been assumed to be already diagonal. Thus, ν_R and ν_L are already chiral mass eigenfields, which means $\omega_L = \omega'_L$ in equation (2.48) and the diagonalizing matrix is simply

$$
U^{(0)} = \mathbb{1}_6. \tag{5.28}
$$

Nevertheless, neutrino masses will be generated at one-loop analogously to section 4.1. In the following section we will use the results we achieved before and therefore, we have to rewrite the Yukawa Lagrangian in terms of scalar mass eigenfields, which are of course all real. Hence the part of the Yukawa Lagrangian concerning neutrino-neutral scalar couplings is given by

$$
\begin{aligned}
-\mathcal{L}^{(\nu)}_{\text{Yuk}} &= \overline{\nu}_R \Delta^{(\eta)} \eta^0 \nu_L + \overline{\nu}_L \eta^{0*} \Delta^{(\eta)\dagger} \nu_R \\
&= \frac{1}{\sqrt{2}} \left[\overline{\nu}_R \Delta^{(\eta)} \left(b_2^{(\rho)} \rho + b_2^{(\sigma)} \sigma \right) \nu_L + \overline{\nu}_L \left(b_2^{(\rho)*} \rho + b_2^{(\sigma)*} \sigma \right) \Delta^{(\eta)\dagger} \nu_R \right] \\
&= \frac{1}{\sqrt{2}} \left[\overline{\nu}_R \Delta^{(\eta)} \left(\rho + i\sigma \right) \nu_L + \overline{\nu}_L \left(\rho - i\sigma \right) \Delta^{(\eta)\dagger} \nu_R \right].
\end{aligned} \tag{5.29}
$$

We used that the parametrization for ϕ^0_k given in (3.101) reduces in this model to what was given in (5.23), i.e.

$$
\sum_b \frac{1}{\sqrt{2}} \left(v_2 + b_2 S^0_b \right) = \frac{1}{\sqrt{2}} \left[(v_2 + b_2^{(\rho)} \rho) + (v_2 + b_2^{(\sigma)} \sigma) \right] = \frac{1}{\sqrt{2}} (\rho + i\sigma), \tag{5.30}
$$

since $v_2 = 0$ and the eigenvectors b are given above in (5.24).

5.1.3 One-Loop Neutrino Masses

The light neutrino masses will be generated at one-loop in this model by exchange of the neutral scalar field η^0 and the heavy sterile RH neutrinos ν_R according to the Feynman diagram in figure 15 below.

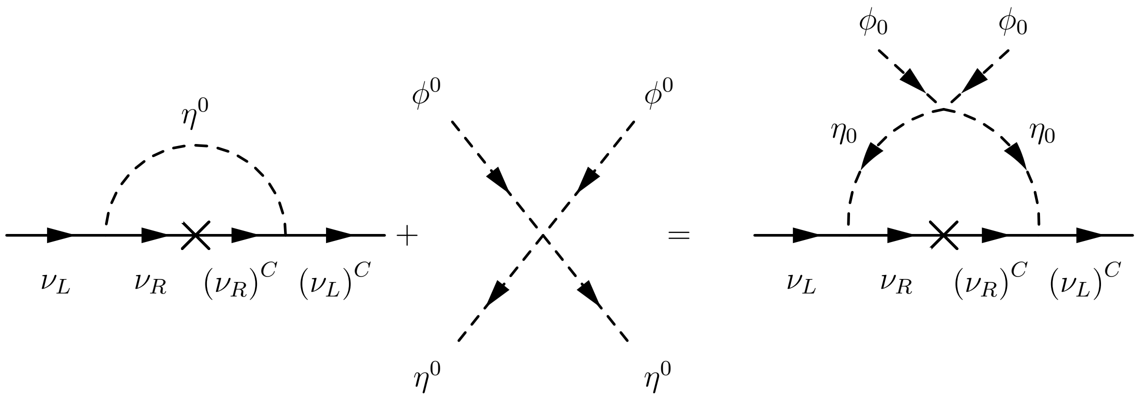


Figure 15: The one-loop Feynman diagram for the scotogenic model is a composition of the one-loop correction of the neutrino propagator via a η^0 selfexchange and the crossinteraction of the neutral scalar fields ϕ^0 and η^0. The mass eigenfields σ and ρ contribute to the first diagram with opposite sign, whereas the crossinteraction (λ^5-term) leads to the splitting of the masses of ρ and σ.

The one-loop corrected 6×6 neutrino mass matrix according to (4.2) will be

$$M^{(1)}_{D+M} = \begin{pmatrix} \delta M_L & 0 \\ 0 & M_R \end{pmatrix}, \tag{5.31}$$

since $M_D = 0$ in Ma's scotogentic model. We simply apply the result for δM_L found in (4.78), again noticing that the second term vanishes since $M_D = 0$. Hence, we get

$$\begin{aligned} \delta M_L &= \sum_{b \neq b_Z} \frac{1}{32\pi^2} \Delta_b^T U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b \\ &= \frac{1}{32\pi^2} \left[\Delta^T_{b(\rho)} U_R^* \hat{m} \frac{\ln\left(\frac{\hat{m}^2}{m_\rho^2}\right)}{\frac{\hat{m}^2}{m_\rho^2} - 1} U_R^\dagger \Delta_{b(\rho)} + \Delta^T_{b(\sigma)} U_R^* \hat{m} \frac{\ln\left(\frac{\hat{m}^2}{m_\sigma^2}\right)}{\frac{\hat{m}^2}{m_\sigma^2} - 1} U_R^\dagger \Delta_{b(\sigma)} \right]. \end{aligned} \tag{5.32}$$

The matrices Δ_b were defined in (3.114) and reduce in this model to

$$\Delta_b = \sum_k b_k \Delta_k = b_2 \Delta^{(\eta)}. \tag{5.33}$$

Since we know $U^{(0)} = \mathbb{1}_6$ from equation (5.28), the submatrix of the decomposition given in (3.6) is

$$U_R^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.34}$$

Thus, we obtain

$$U_R^* \hat{m} \ln\left(\frac{\hat{m}^2}{m_\rho^2}\right) \frac{1}{\frac{\hat{m}^2}{m_\rho^2} - 1} U_R^\dagger = \tilde{m}^{(\rho)} = \delta_{ij} m_{ij}^{(\rho)} = \mathrm{diag}(m_1^{(\rho)}, m_2^{(\rho)}, m_3^{(\rho)}), \tag{5.35}$$

$$U_R^* \hat{m} \ln\left(\frac{\hat{m}^2}{m_\sigma^2}\right) \frac{1}{\frac{\hat{m}^2}{m_\sigma^2} - 1} U_R^\dagger = \tilde{m}^{(\sigma)} = \delta_{ij} m_{ij}^{(\rho)} = \mathrm{diag}(m_1^{(\sigma)}, m_2^{(\sigma)}, m_3^{(\sigma)}), \tag{5.36}$$

with diagonal entries m_i for $i = 1, 2, 3$ given by

$$m_i^{(\rho)} = m_{Ri} \ln\left(\frac{m_{R_i}^2}{m_\rho^2}\right) \frac{m_\rho^2}{m_{Ri}^2 - m_\rho^2}, \qquad m_i^{(\sigma)} = m_{Ri} \ln\left(\frac{m_{R_i}^2}{m_\sigma^2}\right) \frac{m_\sigma^2}{m_{Ri}^2 - m_\sigma^2}. \tag{5.37}$$

Inserting all this in to δM_L above becomes

$$\delta M_L = \frac{1}{32\pi^2}\left[b_2^{(\rho)}\Delta^{(\eta)^T}\tilde{m}^{(\rho)}b_2^{(\rho)}\Delta^{(\eta)} + b_2^{(\sigma)}\Delta^{(\eta)^T}\tilde{m}^{(\sigma)}b_2^{(\sigma)}\Delta^{(\eta)}\right], \tag{5.38}$$

or alternatively in matrix component notation and using (5.35) and (5.36) we obtain

$$\begin{aligned}(\delta M_L)_{il} =& \frac{1}{32\pi^2}\sum_{j,k=1}^{3}\left[\underbrace{\left(b_2^{(\rho)}\right)^2}_{1^2}\left(\Delta^{(\eta)^T}\right)_{ij}\delta_{jk}\tilde{m}_{jk}^{(\rho)}\left(\Delta^{(\eta)}\right)_{kl} + \underbrace{\left(b_2^{(\sigma)}\right)^2}_{(-i)^2}\left(\Delta^{(\eta)^T}\right)_{ij}\delta_{jk}\tilde{m}_{jk}^{(\sigma)}\left(\Delta^{(\eta)}\right)_{kl}\right] \\ =& \frac{1}{32\pi^2}\sum_{j=1}^{3}\left[\left(\Delta^{(\eta)}\right)_{ji}\tilde{m}_j^{(\rho)}\left(\Delta^{(\eta)}\right)_{jl} - \left(\Delta^{(\eta)}\right)_{ji}\tilde{m}_j^{(\sigma)}\left(\Delta^{(\eta)}\right)_{jl}\right].\end{aligned} \tag{5.39}$$

Therefore, the final result for the correction to the neutrino mass matrix at one-loop in the scotogenic model is

$$\boxed{\begin{aligned}(\delta M_L)_{il} =& \frac{1}{32\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl} m_{Rj} \\ &\cdot\left[\ln\left(\frac{m_{R_j}^2}{m_\rho^2}\right)\frac{m_\rho^2}{m_{Rj}^2 - m_\rho^2} - \ln\left(\frac{m_{R_j}^2}{m_\sigma^2}\right)\frac{m_\sigma^2}{m_{Rj}^2 - m_\sigma^2}\right],\end{aligned}} \tag{5.40}$$

where m_ρ^2 and m_σ^2 are the masses of $\sqrt{2}\mathrm{Re}(\eta^0)$ and $\sqrt{2}\mathrm{Im}(\eta^0)$ respectively, hence using (5.21) and (5.22) we know

$$m_\rho^2 = m_2^2 + (\lambda_3 + \lambda_4 + \lambda_5)v^2 \tag{5.41}$$

$$m_\sigma^2 = m_2^2 + (\lambda_3 + \lambda_4 - \lambda_5)v^2. \tag{5.42}$$

Apparently, if $\lambda_5 = 0$ and therefore $m_\rho^2 = m_\sigma^2$, the neutrinos will remain massless also at one-loop level, since the two terms in (5.40) will cancel. Hence, only non-degenerate mass eigenvalues of the neutral scalar field give rise to massive LH neutrinos at one-loop.

The effects of this mass shift between $\mathrm{Re}(\eta^0)$ and $\mathrm{Im}(\eta^0)$ will be now investigated. First, we define the mean value and shift as

$$m_0^2 = \frac{m_\rho^2 + m_\sigma^2}{2}, \qquad \delta m^2 = \frac{m_\rho^2 - m_\sigma^2}{2} = \lambda_5 v^2, \tag{5.43}$$

and clearly

$$m_\rho^2 = m_0^2 + \delta m^2, \qquad m_\sigma^2 = m_0^2 - \delta m^2. \tag{5.44}$$

Furthermore, the terms[110] in (5.40) can be considered as a function

$$f(x) = \frac{x}{x-a}\ln\left(\frac{x}{a}\right), \tag{5.45}$$

with $x = m_\rho^2$, m_σ^2 and $a = m_{Rj}^2$, such that

$$(\delta M_L)_{il} = \frac{1}{32\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl} m_{Rj}\left[f(m_\rho^2) - f(m_\sigma^2)\right]. \tag{5.46}$$

[110]Hence, we like to follow the notation in [23], we took the reciprocal value of the fraction in the logarithm and compensate the appearing minus sign in the nominator of the factor.

For writing δM_L in terms of m_0^2 we expand the function $f(x)$ around $x_0 = m_0^2$ as

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} (x - x_0) + \dots . \tag{5.47}$$

Since an expansion up to first order will be sufficient, we calculate the first derivative

$$\frac{df}{dx} = \frac{x-a-x}{(x-a)^2} \ln\left(\frac{x}{a}\right) + \frac{x}{x-a} = \frac{1}{x-a}\left(1 - \frac{a \ln\left(\frac{x}{a}\right)}{(x-a)}\right) . \tag{5.48}$$

Applying all this to (5.40) and inserting our proper variables for a, x and x_0 gives

$$\begin{aligned}(\delta M_L)_{il} =& \frac{1}{32\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} m_{Rj} \\ &\cdot \left[f(m_0^2) + \left.\frac{df}{dm_\rho^2}\right|_{m_0^2} (m_\rho^2 - m_0^2) - f(m_0^2) - \left.\frac{df}{dm_\sigma^2}\right|_{m_0^2} (m_\sigma^2 - m_0^2) \right] \\ =& \frac{1}{32\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} m_{Rj} \left[\left.\frac{df}{dm_\rho^2}\right|_{m_0^2} \delta m^2 - \left.\frac{df}{dm_\sigma^2}\right|_{m_0^2} (-\delta m^2) \right] \\ =& \frac{2\delta m^2}{32\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} m_{Rj} \left.\frac{df}{dm_\rho^2}\right|_{m_0^2} .\end{aligned} \tag{5.49}$$

By inserting the derivative (5.48) in terms of our proper variables we achieve our final result:

$$\boxed{(\delta M_L)_{il} = \frac{\lambda_5 v^2}{16\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} \frac{m_{Rj}}{m_0^2 - m_{Rj}^2} \left[1 - \frac{m_{Rj}^2 \ln\left(\frac{m_0^2}{m_{Rj}^2}\right)}{(m_0^2 - m_{Rj}^2)} \right]} . \tag{5.50}$$

5.1.4 Special Limits

Now we like to discuss some possible limits comparing the mass ranges of the mean value of the neutral scalar masses m_0^2 and the masses of the RH neutrinos m_{Rj}.

(i) $\mathbf{m_{Rj} \gg m_0^2}$:
For the limit $\frac{m_0^2}{m_{Rj}^2} \to 0$, where the mass of the RH neutrino is much heavier than the mean value of the masses of the LH neutrinos. Thus, we rewrite (5.50) in the following way:

$$\begin{aligned}(\delta M_L)_{il} &= \frac{\lambda_5 v^2}{16\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} \frac{1}{m_{Rj}} \frac{1}{\frac{m_0^2}{m_{Rj}^2} - 1} \left[1 - \frac{1}{\frac{m_0^2}{m_{Rj}^2} - 1} \ln\left(\frac{m_0^2}{m_{Rj}^2}\right) \right] \\ &\to \frac{\lambda_5 v^2}{16\pi^2} \sum_{j=1}^{3} \left(\Delta^{(\eta)}\right)_{ji} \left(\Delta^{(\eta)}\right)_{jl} \frac{1}{m_{Rj}} \left[-1 - \ln\left(\frac{m_0^2}{m_{Rj}^2}\right) \right] .\end{aligned} \tag{5.51}$$

(ii) $\mathbf{m_{Rj} \ll m_0^2}$:
Here, where the mass of the RH neutrino is much lighter than the mean value of the masses of the LH neutrinos, i.e. the limit $\frac{m_{Rj}^2}{m_0^2} \to 0$, we rewrite (5.50) analogously.

Hence, we obtain

$$
\begin{aligned}
(\delta M_L)_{il} &= \frac{\lambda_5 v^2}{16\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl}\frac{m_{Rj}}{m_0^2}\frac{1}{1-\frac{m_{Rj}^2}{m_0^2}}\left[1-\frac{m_{Rj}^2}{m_0^2}\frac{1}{\frac{m_{Rj}^2}{m_0^2}-1}\ln\left(\frac{m_0^2}{m_{Rj}^2}\right)\right] \\
&\to \frac{\lambda_5 v^2}{16\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl}\frac{m_{Rj}}{m_0^2}.
\end{aligned}
$$

(iii) $\mathbf{m_{Rj} \simeq m_0^2}$:
To calculate δM_L in the limit $(m_{Rj} - m_0^2) \to 0$ we will expand the logarithm according to $\ln(1+x) = x - \frac{x^2}{2} + \ldots$, since the second order will be sufficient. For a clearer calculation we display the structure of the summands of (5.50) as

$$
\frac{1}{x-y}\left[1-\frac{y}{x-y}\ln\left(\frac{x}{y}\right)\right] = \frac{1}{\delta}\left[1-\frac{y}{\delta}\ln\left(\frac{y+\delta}{y}\right)\right], \tag{5.52}
$$

where $x = m_0^2$ and $m_{Rj}^2 = y$ and for $\delta := x - y$ we will take the limit $\delta \to 0$:

$$
\begin{aligned}
\lim_{\delta\to 0}\frac{1}{\delta}\left[1-\frac{y}{\delta}\ln\left(\frac{y+\delta}{y}\right)\right] &= \lim_{\delta\to 0}\frac{1}{\delta}\left[1-\frac{y}{\delta}\ln\left(1+\frac{\delta}{y}\right)\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left[1-\frac{y}{\delta}\left(\frac{\delta}{y}-\frac{1}{2}\frac{\delta^2}{y^2}+\ldots\right)\right] \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left[1-1+\frac{1}{2}\frac{\delta}{y}+\ldots\right] = \frac{1}{2y}.
\end{aligned} \tag{5.53}
$$

Thus, we get

$$
\begin{aligned}
(\delta M_L)_{il} &= \frac{\lambda_5 v^2}{16\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl}\frac{m_{Rj}}{m_0^2}\frac{1}{1-\frac{m_{Rj}^2}{m_0^2}}\left[1-\frac{m_{Rj}^2}{m_0^2}\frac{1}{\frac{m_{Rj}^2}{m_0^2}-1}\ln\left(\frac{m_0^2}{m_{Rj}^2}\right)\right] \\
&\to \frac{\lambda_5 v^2}{32\pi^2}\sum_{j=1}^{3}\left(\Delta^{(\eta)}\right)_{ji}\left(\Delta^{(\eta)}\right)_{jl}\frac{1}{m_{Rj}}.
\end{aligned} \tag{5.54}
$$

It should be noted that the results obtained in this thesis differ by a factor of $^1/_2$ from the original results in [23]. As discussed in [88] this additional factor is necessary due to rescaling the real scalar fields by a factor of $^1/_{\sqrt{2}}$.

A general feature of this model with three massive RH neutrinos is that all three LH neutrinos become massive at one-loop level, if all Yukawa couplings are nonzero. Moreover, the masses of the LH neutrinos are appropriately small (~ 1 eV) due to

(i) the loop factor $\frac{1}{16\pi^2}$,

(ii) the small Yukawa couplings,

(iii) small coupling λ_5 ($\sim 10^{-4}$),

(iv) and the large seesaw scale $m_R \sim 10^9$ GeV.

Of course the one-loop corrected mass matrix $M^{(1)}_{\mathrm{D+M}}$ has to be diagonalized to write down the mass eigenfields of the neutrinos. In the case of the scotogenic model this is a rather simple task, since M_R has been assumed diagonal and $M_D = 0$. Hence, $U^{(0)} = \mathbb{1}_6$ and the procedure done in section 4.2 simplifies to finding a matrix Ω of one-loop order, according to

$$U = U^{(0)}V^{(1)} = (\mathbb{1} + i\Omega), \tag{5.55}$$

such that

$$U^T M^{(1)}_{\mathrm{D+M}} U = \hat{m}^{(1)} = \mathrm{diag}(m_{L1}, m_{L2}, m_{L3}, m_{R1}, m_{R2}, m_{R3}). \tag{5.56}$$

Besides, this model also gives rise to dark matter candidates[111]. These can be fermionic or bosonic, since the RH neutrinos or the real or imaginary part of the neutral component of the scalar doublet η might be the lightest, but still very heavy, stable particle (LSP), as pointed out in [23]. There, Ma briefly discussed the two possible scenarios, whether if one of the RH neutrinos or one of the neutral scalars is the LSP.

(a) Let $m_{R1} < m_{R2} < m_{R3}$ and $m_{R1} < m_\rho, m_\sigma$, then ν_{R1} is LSP and

$$\eta^\pm \to \ell^\pm \nu_{R1,2,3}, \qquad \nu_{R2} \to \ell^\pm \ell^\mp \nu_{R1}, \qquad \nu_{R3} \to \ell^\pm \ell^\mp \nu_{R1,2},$$

will be observable decays.

(b) If otherwise $m_\rho, m_\sigma < m_{R1,2,3}$, observable decays will be

$$\nu_{R1,2,3} \to \ell^\pm \eta^\mp, \qquad \eta^\mp \to \eta^0 + W^\mp.$$

Since ρ must be slightly heavier ($\delta m^2 = \lambda_5 v^2$), this case would explain their coannihilation in the early universe, according to [23].

5.2 Generalization of the Scotogenic Model

Now, in this last section the scotogenic model by E. Ma [23] will be generalized for arbitrary numbers of scalar doublets

$$\phi_k = \begin{pmatrix} \phi_k^+ \\ \phi_k^0 \end{pmatrix} \qquad \text{for } k = 1, \ldots, n_H, \tag{5.57}$$

$$\eta_{k'} = \begin{pmatrix} \eta_{k'}^+ \\ \eta_{k'}^0 \end{pmatrix} \qquad \text{for } k' = 1, \ldots, n_\eta, \tag{5.58}$$

where n_H and n_η denote the number of scalar doublets ϕ_k and $\eta_{k'}$ respectively. Analogously to the special case discussed in the previous section, they transform evenly resp. oddly under $\mathbb{Z}_2$, i.e.

$$\phi_k \to \phi_k, \qquad \eta_{k'} \to -\eta_{k'} \tag{5.59}$$

for all $k = 1, \ldots, n_H$ and all $k' = 1, \ldots, n_\eta$.

[111]Hence, Ma proposed that neutrino masses are due to the existence of dark matter. Therefore the name *scotogenic* means caused by darkness.

5.2.1 Generalized Scalar Sector and Yukawa Interactions

Again analogously, we assume the VEV's to be

$$\langle 0|\phi_k^0|0\rangle = \frac{v_k}{\sqrt{2}} \qquad \text{for } k = 1, \dots, n_H, \tag{5.60}$$

$$\langle 0|\eta_{k'}^0|0\rangle = 0 \qquad \text{for } k' = 1, \dots, n_\eta, \tag{5.61}$$

and hence, we find the following parametrization for the neutral components:

$$\phi_k^0 = \frac{v_k + h_k + i g_k}{\sqrt{2}} \qquad \text{for } k = 1, \dots, n_H, \tag{5.62}$$

$$\eta_{k'}^0 = \frac{\rho_{k'} + i\sigma_{k'}}{\sqrt{2}} \qquad \text{for } k' = 1, \dots, n_\eta. \tag{5.63}$$

For the full scalar potential we generalize the one given in (5.11) according to the general form we derived in (3.63). Thus, we obtain

$$\begin{aligned}
V = &\sum_{i,j=1}^{n_H} m^{2(1)}_{ij} \phi_i^\dagger \phi_j + \sum_{i,j,k,l=1}^{n_H} \lambda^{(1)}_{ijkl} \left(\phi_i^\dagger \phi_j\right)\left(\phi_k^\dagger \phi_l\right) \\
&+ \sum_{i',j'=1}^{n_\eta} m^{2(2)}_{i'j'} \eta_{i'}^\dagger \eta_{j'} + \sum_{i',j',k',l'=1}^{n_\eta} \lambda^{(2)}_{i'j'k'l'} \left(\eta_{i'}^\dagger \eta_{j'}\right)\left(\eta_{k'}^\dagger \eta_{l'}\right) \\
&+2 \sum_{i,j=1}^{n_H} \sum_{k',l'=1}^{n_\eta} \lambda^{(3)}_{ijk'l'} \left(\phi_i^\dagger \phi_j\right)\left(\eta_{k'}^\dagger \eta_{l'}\right) + 2 \sum_{i,l=1}^{n_H} \sum_{j',k'=1}^{n_\eta} \lambda^{(4)}_{ij'k'l} \left(\phi_i^\dagger \eta_{j'}\right)\left(\eta_{k'}^\dagger \phi_l\right) \\
&+ \left[\sum_{i,k=1}^{n_H} \sum_{j',l'=1}^{n_\eta} \lambda^{(5)}_{ij'kl'} \left(\phi_i^\dagger \eta_{j'}\right)\left(\phi_k^\dagger \eta_{l'}\right) + \text{H.c.}\right],
\end{aligned} \tag{5.64}$$

where all coefficients fulfil conditions similar to (3.64). In particular, to ensure Hermiticity of the terms in the potential, they obey the following relations[112]:

$$m^{2(1)}_{ij} = \left(m^{2(1)}_{ji}\right)^*, \qquad \lambda^{(1)}_{ijkl} = \lambda^{(1)}_{klij}, \qquad \lambda^{(1)}_{ijkl} = \left(\lambda^{(1)}_{jilk}\right)^*, \tag{5.65}$$

$$m^{2(2)}_{i'j'} = \left(m^{2(2)}_{j'i'}\right)^*, \qquad \lambda^{(2)}_{i'j'k'l'} = \lambda^{(2)}_{k'l'i'j'}, \qquad \lambda^{(2)}_{i'j'k'l'} = \left(\lambda^{(2)}_{j'i'l'k'}\right)^*, \tag{5.66}$$

$$\lambda^{(3)}_{ijk'l'} = \left(\lambda^{(3)}_{jil'k'}\right)^*, \qquad \lambda^{(4)}_{ij'k'l} = \left(\lambda^{(4)}_{lk'j'i}\right)^*, \qquad \lambda^{(5)}_{ij'kl'} = \lambda^{(5)}_{kl'ij'}. \tag{5.67}$$

All terms in this potential have to be invariant under $\mathbb{Z}_2$ transformations, therefore all terms linear or cubic in $\eta_{k'}$ have to vanish, which means the corresponding coupling constants have to be zero. The superscripts of the couplings indicate their correspondence to the constants used in (5.11). Note that there is no mixing of ϕ_k and $\eta_{k'}$ doublets to scalar mass eigenfields, because of the $\mathbb{Z}_2$ symmetry. Therefore, the only mass terms for the ϕ_k doublets come from the first line of the potential and the procedure is analogous to section 3.2.1. The mass terms for the $\eta_{k'}$ doublets originate from the remaining lines of the potential. Only the second term in the second line will not contribute since $\langle 0|\eta_{k'}^0|0\rangle = 0$.

[112] Note that especially $\lambda^{(3)}_{ijk'l'}$, $\lambda^{(4)}_{ij'k'l}$ and $\lambda^{(5)}_{ij'kl'}$ fulfil not exactly the same conditions given in (3.64), since interchanging of primed and unprimed indices is not possible, because they indicate the different fields ϕ_k and $\eta_{k'}$ respectively.

The mass terms for the $\eta^0_{k'}$ are

$$
\begin{aligned}
V^{(\eta^0)}_{\text{mass}} &= \frac{1}{2}\sum_{i',j'=1}^{n_\eta} m^{2\,(2)}_{i'j'}\left[(\rho_{i'}\rho_{j'}+\sigma_{i'}\sigma_{j'})+i\left(\rho_{i'}\sigma_{j'}-\sigma_{i'}\rho_{j'}\right)\right] \\
&+\frac{1}{2}\sum_{i,j=1}^{n_H}\sum_{k',l'=1}^{n_\eta}\lambda^{(3)}_{ijk'l'}v_i^*v_j\left[(\rho_{k'}\rho_{l'}+\sigma_{k'}\sigma_{l'})+i\left(\rho_{k'}\sigma_{l'}-\sigma_{k'}\rho_{l'}\right)\right] \\
&+\frac{1}{2}\sum_{i,l=1}^{n_H}\sum_{j',k'=1}^{n_\eta}\lambda^{(4)}_{ij'k'l}v_i^*v_l\left[(\rho_{j'}\rho_{k'}+\sigma_{j'}\sigma_{k'})+i\left(\rho_{j'}\sigma_{k'}-\sigma_{j'}\rho_{k'}\right)\right] \\
&+\frac{1}{4}\left\{\sum_{i,k=1}^{n_H}\sum_{j',l'=1}^{n_\eta}\lambda^{(5)}_{ij'kl'}v_i^*v_k^*\left[(\rho_{j'}\rho_{l'}-\sigma_{j'}\sigma_{l'})+i\left(\rho_{j'}\sigma_{l'}+\sigma_{j'}\rho_{l'}\right)\right]+\text{H.c.}\right\}.
\end{aligned}
\tag{5.68}
$$

In the next step we rename the indices of the last three lines to match the notation in the first one to collect terms quadratic in the fields $\rho_{i'}$, $\rho_{j'}$ and $\sigma_{i'}$, $\sigma_{j'}$ as well as mixed terms properly. Hence, we find

$$
\begin{aligned}
V^{(\eta^0)}_{\text{mass}} &= \frac{1}{2}\sum_{i',j'=1}^{n_\eta} m^{2\,(2)}_{i'j'}\left[(\rho_{i'}\rho_{j'}+\sigma_{i'}\sigma_{j'})+i\left(\rho_{i'}\sigma_{j'}-\sigma_{i'}\rho_{j'}\right)\right] \\
&+\frac{1}{2}\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\lambda^{(3)}_{kli'j'}v_k^*v_l\left[(\rho_{i'}\rho_{j'}+\sigma_{i'}\sigma_{j'})+i\left(\rho_{i'}\sigma_{j'}-\sigma_{i'}\rho_{j'}\right)\right] \\
&+\frac{1}{2}\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\lambda^{(4)}_{kj'i'l}v_k^*v_l\left[(\rho_{i'}\rho_{j'}+\sigma_{i'}\sigma_{j'})+i\left(\rho_{i'}\sigma_{j'}-\sigma_{i'}\rho_{j'}\right)\right] \\
&+\frac{1}{4}\left\{\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\left[(\rho_{i'}\rho_{j'}-\sigma_{i'}\sigma_{j'})+i\left(\rho_{i'}\sigma_{j'}+\sigma_{i'}\rho_{j'}\right)\right]\right. \\
&\left.+\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\left(\lambda^{(5)}_{ki'lj'}\right)^*v_kv_l\left[(\rho_{i'}\rho_{j'}-\sigma_{i'}\sigma_{j'})-i\left(\rho_{i'}\sigma_{j'}+\sigma_{i'}\rho_{j'}\right)\right]\right\},
\end{aligned}
\tag{5.69}
$$

and rearranging the terms gives

$$
\begin{aligned}
V^{(\eta^0)}_{\text{mass}} &= \frac{1}{2}\sum_{i',j'=1}^{n_\eta}\left[m^{2\,(2)}_{i'j'}+v_k^*v_l\left(\lambda^{(3)}_{kli'j'}+\lambda^{(4)}_{kj'i'l}\right)+\frac{1}{2}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*+\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right)\right]\rho_{i'}\rho_{j'} \\
&+\frac{1}{2}\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\left[m^{2\,(2)}_{i'j'}+v_k^*v_l\left(\lambda^{(3)}_{kli'j'}+\lambda^{(4)}_{kj'i'l}\right)-\frac{1}{2}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*+\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right)\right]\sigma_{i'}\sigma_{j'} \\
&+\frac{i}{2}\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\left[m^{2\,(2)}_{i'j'}+v_k^*v_l\left(\lambda^{(3)}_{kli'j'}+\lambda^{(4)}_{kj'i'l}\right)+\frac{1}{2}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*-\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right)\right]\rho_{i'}\sigma_{j'} \\
&+\frac{i}{2}\sum_{k,l=1}^{n_H}\sum_{i',j'=1}^{n_\eta}\left[-m^{2\,(2)}_{i'j'}-v_k^*v_l\left(\lambda^{(3)}_{kli'j'}+\lambda^{(4)}_{kj'i'l}\right)+\frac{1}{2}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*-\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right)\right]\sigma_{i'}\rho_{j'}.
\end{aligned}
$$

We introduce, similarly to (5.16) in the special case before, the matrices

$$
\begin{aligned}
M^{2\,(2)} &= \left(m^{2\,(2)}\right)_{i'j'}, & \Lambda^{(\eta)}_{i'j'} &= \sum_{k,l=1}^{n_H}\lambda^{(3)}_{kli'j'}v_k^*v_l, \\
K'^{(\eta)}_{i'j'} &= \sum_{k,l=1}^{n_H}\lambda^{(4)}_{kj'i'l}v_kv_l^*, & K^{(\eta)}_{i'j'} &= \sum_{k,l=1}^{n_H}\lambda^{(5)}_{ki'lj'}v_k^*v_l^*,
\end{aligned}
\tag{5.70}
$$

with $M^{2(2)}$, $\Lambda^{(\eta)}$ and $K'^{(\eta)}$ being Hermitian and $K^{(\eta)}$ being symmetric due to the relations (5.65) of the coefficients. Thus, we know

$$\begin{aligned}
\mathrm{Re}M^{2(2)}_{i'j'} &= \mathrm{Re}M^{2(2)}_{j'i'}, &\quad \mathrm{Im}M^{2(2)}_{i'j'} &= -\,\mathrm{Im}M^{2(2)}_{j'i'}, &\quad (5.71)\\
\mathrm{Re}\Lambda^{(\eta)}_{i'j'} &= \mathrm{Re}\Lambda^{(\eta)}_{j'i'}, & \mathrm{Im}\Lambda^{(\eta)}_{i'j'} &= -\,\mathrm{Im}\Lambda^{(\eta)}_{j'i'}, & (5.72)\\
\mathrm{Re}K'^{(\eta)}_{i'j'} &= \mathrm{Re}K'^{(\eta)}_{j'i'}, & \mathrm{Im}K'^{(\eta)}_{i'j'} &= -\,\mathrm{Im}K'^{(\eta)}_{j'i'}, & (5.73)\\
\mathrm{Re}K^{(\eta)}_{i'j'} &= \mathrm{Re}K^{(\eta)}_{j'i'}, & \mathrm{Im}K^{(\eta)}_{i'j'} &= \quad \mathrm{Im}K^{(\eta)}_{j'i'}. & (5.74)
\end{aligned}$$

Furthermore, we can simplify the terms with $\lambda^{(5)}_{ki'lj'}$, because

$$\sum_{k,l=1}^{n_H}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^* + \left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right) = K^{(\eta)}_{i'j'} + K^{(\eta)*}_{i'j'} = 2\mathrm{Re}K^{(\eta)}_{i'j'} \tag{5.75}$$

$$\sum_{k,l=1}^{n_H}\left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^* - \left(\lambda^{(5)}_{ki'lj'}v_k^*v_l^*\right)^*\right) = K^{(\eta)}_{i'j'} - K^{(\eta)*}_{i'j'} = 2i\mathrm{Im}K^{(\eta)}_{i'j'}. \tag{5.76}$$

We use this knowledge on symmetry behaviour in the next step and recognize that the field part of the first two lines in $V^{(\eta^0)}_{\text{mass}}$ above is symmetric in $i' \leftrightarrow j'$, whereas the one of the last two lines is antisymmetric. So the contributing parts come from purely symmetric or antisymmetric combinations and the potential reduces to:

$$\begin{aligned}
V^{(\eta^0)}_{\text{mass}} &= \frac{1}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Re}M^{2(2)}_{i'j'} + \mathrm{Re}\Lambda^{(\eta)}_{i'j'} + \mathrm{Re}K'^{(\eta)}_{i'j'} + \mathrm{Re}K^{(\eta)}_{i'j'}\right]\rho_{i'}\rho_{j'}\\
&+ \frac{1}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Re}M^{2(2)}_{i'j'} + v_k^*v_l\mathrm{Re}\Lambda^{(\eta)}_{i'j'} + \mathrm{Re}K'^{(\eta)}_{i'j'} - \mathrm{Re}K^{(\eta)}_{i'j'}\right]\sigma_{i'}\sigma_{j'}\\
&+ \frac{i^2}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Im}M^{2(2)}_{i'j'} + \mathrm{Im}\Lambda^{(\eta)}_{i'j'} + \mathrm{Im}K'^{(\eta)}_{i'j'} + \mathrm{Im}K^{(\eta)}_{i'j'}\right]\rho_{i'}\sigma_{j'}\\
&+ \frac{i^2}{2}\sum_{i',j'=1}^{n_\eta}\left[-\mathrm{Im}M^{2(2)}_{i'j'} - \mathrm{Im}\Lambda^{(\eta)}_{i'j'} - \mathrm{Im}K'^{(\eta)}_{i'j'} + \mathrm{Im}K^{(\eta)}_{i'j'}\right]\sigma_{i'}\rho_{j'}.
\end{aligned} \tag{5.77}$$

In the next step we can use the symmetry properties of the real and imaginary parts of the matrices (5.71) and rewrite the last line as

$$+\frac{i^2}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Im}M^{2(2)}_{j'i'} + \mathrm{Im}\Lambda^{(\eta)}_{j'i'} + \mathrm{Im}K'^{(\eta)}_{j'i'} + \mathrm{Im}K^{(\eta)}_{j'i'}\right]\sigma_{i'}\rho_{j'}. \tag{5.78}$$

If we rename the indices $i' \leftrightarrow j'$ in this line, then we can sum up the last two lines and obtain

$$\begin{aligned}
V^{(\eta^0)}_{\text{mass}} &= \frac{1}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Re}M^{2(2)}_{i'j'} + \mathrm{Re}\Lambda^{(\eta)}_{i'j'} + \mathrm{Re}K'^{(\eta)}_{i'j'} + \mathrm{Re}K^{(\eta)}_{i'j'}\right]\rho_{i'}\rho_{j'}\\
&+ \frac{1}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Re}M^{2(2)}_{i'j'} + v_k^*v_l\mathrm{Re}\Lambda^{(\eta)}_{i'j'} + \mathrm{Re}K'^{(\eta)}_{i'j'} - \mathrm{Re}K^{(\eta)}_{i'j'}\right]\sigma_{i'}\sigma_{j'}\\
&+ \frac{2i^2}{2}\sum_{i',j'=1}^{n_\eta}\left[\mathrm{Im}M^{2(2)}_{i'j'} + \mathrm{Im}\Lambda^{(\eta)}_{i'j'} + \mathrm{Im}K'^{(\eta)}_{i'j'} + \mathrm{Im}K^{(\eta)}_{i'j'}\right]\rho_{i'}\sigma_{j'}.
\end{aligned} \tag{5.79}$$

Finally, we introduce again matrices A, B and C as

$$A = \mathrm{Re}(M^{2(2)} + \Lambda^{(\eta)} + K'^{(\eta)}) + \mathrm{Re}K^{(\eta)}, \tag{5.80}$$
$$B = \mathrm{Re}(M^{2(2)} + \Lambda^{(\eta)} + K'^{(\eta)}) - \mathrm{Re}K^{(\eta)}, \tag{5.81}$$
$$C = \mathrm{Im}(M^{2(2)} + \Lambda^{(\eta)} + K'^{(\eta)}) + \mathrm{Im}K^{(\eta)} \tag{5.82}$$

in analogy to (3.90)-(3.92) in section 3.2.1. Note that the matrices A and B are real and symmetric per definition, whereas C is just real. Using this notation, the mass term of the potential can be written in a very compact form as

$$V_{\mathrm{mass}}^{(\eta^0)} = \frac{1}{2} \sum_{i',j'=1}^{n_\eta} A_{i'j'}\rho_{i'}\rho_{j'} + B_{i'j'}\sigma_{i'}\sigma_{j'} + 2C_{i'j'}\rho_{i'}\sigma_{j'}, \tag{5.83}$$

and the mass matrix for the neutral scalar fields of $\eta-$type is given by

$$M_\eta^2 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{5.84}$$

The mass eigenvalue equation is also given analogously to (3.95) by

$$\mathcal{M}_0^2 \begin{pmatrix} \mathrm{Re}\, b \\ \mathrm{Im}\, b \end{pmatrix} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, b \\ \mathrm{Im}\, b \end{pmatrix} = m_b^2 \begin{pmatrix} \mathrm{Re}\, b \\ \mathrm{Im}\, b \end{pmatrix}, \tag{5.85}$$

where m_b^2 denote the masses of the neutral real scalar mass eigenfields generated by mixing of the $\eta_{k'}^0$ fields. The eigenvectors b are $n_\eta \times 1$ complex vectors and if none of the $2n_\eta$ mass eigenvalues m_b^2 is degenerate, i.e. there are $2n_\eta$ neutral scalar mass eigenfields, there are of course also $2n_\eta$ eigenvectors b.

The generalization of the Yukawa Lagrangian in (5.25) is then given by

$$\begin{aligned} -\mathcal{L}_{\mathrm{Yuk}} =& \sum_{k=1}^{n_H} (\phi_k^\dagger \overline{\ell}_R \Gamma^{(\phi_k)} + \tilde{\phi}_k^\dagger \overline{\nu}_R \Delta^{(\phi_k)}) D_L + \mathrm{H.c.} \\ &+ \sum_{k'=1}^{n_\eta} (\eta_{k'}^\dagger \overline{\ell}_R \Gamma^{(\eta_{k'})} + \tilde{\phi}_k^\dagger \overline{\nu}_R \Delta^{(\eta_{k'})}) D_L + \mathrm{H.c.}\,. \end{aligned} \tag{5.86}$$

Now only considering the neutrino part of this Lagrangian the only terms remaining are analogously to (5.29)

$$-\mathcal{L}_{\mathrm{Yuk}}^{(\nu)} = \sum_{k'=1}^{n_\eta} \overline{\nu}_R \Delta^{(\eta_{k'})} \eta_{k'}^0 \nu_L + \overline{\nu}_L {\eta_{k'}^0}^* {\Delta^{(\eta_{k'})}}^\dagger \nu_R, \tag{5.87}$$

where the $\mathbb{Z}_2$ symmetry again leads to vanishing couplings $\Delta^{(\phi_k)}$ and hence $M_D = 0$.

5.2.2 One-Loop Neutrino Masses

Therefore, the one-loop correction to the neutrino masses is given by the formula (4.78), but since $M_D = 0$ the second term does not contribute. Moreover, only the neutral scalar mass eigenfields from the $\eta_{k'}$ doublets contribute to the first part, because the ϕ_k doublets do not contribute since $\Delta^{(\phi_k)} = 0$.

Hence, the final result in the generalized scotogenic model for the one-loop corrections to the neutrino masses is given by

$$\boxed{\delta M_L = \sum_b \frac{1}{32\pi^2} {\Delta^{(\eta)}}^T_b U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b^{(\eta)}}, \tag{5.88}$$

with $r_b = \frac{\hat{m}^2}{m_b^2}$ and $\Delta_b^{(\eta)} = \sum_{k'} b_{k'} \Delta^{(\eta_{k'})}$ defined analogously to (4.47) and (3.114). As in the previous section, it can be assumed that the diagonalized tree-level neutrino mass matrix $\hat{m}$ equals the undiagonalized mass matrix $M^{(0)}_{\mathrm{D+M}}$, if M_R is assumed diagonal and $\hat{m} = \mathrm{diag}(m_{1R}, m_{2R}, m_{3R})$.

Finally it should be investigated what happens, if the couplings $\lambda^{(5)}_{ij'kl'}$ are zero. It has been discussed before in the special case $n_H = 1 = n_\eta$, a vanishing Yukawa coupling λ_5 causes a degeneracy of the mass eigenvalues, i.e. there is only one mass eigenvalue. Therefore, no one-loop corrections to the neutrino masses appear and the neutrinos remain massless. It will be shown in the following that this is also true in the generalized model.

If $\lambda^{(5)}_{ij'kl'} = 0$, then $K = 0$ and hence

$$A = \mathrm{Re}(\mu^2 + \Lambda + K') = B. \tag{5.89}$$

Per definition we know that A and B are real and symmetric $n_\eta \times n_\eta$ matrices, whereas C is a real and in this case also an antisymmetric $n_\eta \times n_\eta$ matrix. The mass eigenvalue equation then can be written as

$$\mathcal{M}_0^2 \begin{pmatrix} \mathrm{Re} b \\ \mathrm{Im} b \end{pmatrix} = \begin{pmatrix} A & C \\ -C & A \end{pmatrix} \begin{pmatrix} \mathrm{Re} b \\ \mathrm{Im} b \end{pmatrix} = m_b^2 \begin{pmatrix} \mathrm{Re} b \\ \mathrm{Im} b \end{pmatrix}. \tag{5.90}$$

Thus, $\mathcal{M}_0^2$ is a real symmetric $2n_\eta \times 2n_\eta$ matrix and fulfils the following equation[113]

$$S^T \mathcal{M}_0^2 S = \mathcal{M}_0^2, \tag{5.91}$$

for the anti-symmetric matrix

$$S = \begin{pmatrix} 0 & \mathbb{1}_{n_\eta} \\ -\mathbb{1}_{n_\eta} & 0 \end{pmatrix}, \tag{5.92}$$

$$S S^T = \mathbb{1}_{2n_\eta}. \tag{5.93}$$

If $b = x + iy$ is eigenvector of $\mathcal{M}_0^2$ to the eigenvalue m_b^2 then we find

$$\begin{aligned} \mathcal{M}_0^2 \begin{pmatrix} -y \\ x \end{pmatrix} &= \mathcal{M}_0^2 S \begin{pmatrix} x \\ y \end{pmatrix} \overset{(5.93)}{=} S S^T \mathcal{M}_0^2 S \begin{pmatrix} x \\ y \end{pmatrix} \\ &\overset{(5.91)}{=} S \mathcal{M}_0^2 \begin{pmatrix} x \\ y \end{pmatrix} \overset{(5.90)}{=} S m_b^2 \begin{pmatrix} x \\ y \end{pmatrix} \qquad = m_b^2 \begin{pmatrix} -y \\ x \end{pmatrix}. \end{aligned} \tag{5.94}$$

Thus, if $b = x + iy$ is eigenvector to m_b^2 then $b' = -y + ix$ is also eigenvector to the

[113] Great thanks to Prof. Grimus for these crucial observations.

same eigenvalue and they are linear independent, since they are orthogonal

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -y \\ x \end{pmatrix} = 0. \tag{5.95}$$

Hence, the eigenvalue m_b^2 is at least two-fold degenerate and the eigenspace E_b has at least dimension 2. The pairs of eigenvectors $b = x + iy$ and $b' = -y + ix$ are related via

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}, \qquad b' = ib. \tag{5.96}$$

They form a complete set of orthonormal vectors spanning the eigenspaces and hence, all eigenspaces have even dimension. As we will see in the following those pairings will cause vanishing one-loop contributions to the neutrino masses.

Now we can express the Yukawa Lagrangian (5.87) in terms of scalar mass eigenfields. Adapting the result in (3.101) we obtain

$$\eta_k^0 = \frac{1}{\sqrt{2}}\left(\sum_b b_k S_b^0 + \sum_{b'} b'_k S_{b'}^0\right) = \sum_{b,b'}\left(b_k S_b^0 + i b_k S_{b'}^0\right), \tag{5.97}$$

where we used the relation (5.96) above. Then we achieve in the Yukawa Lagrangian

$$\begin{aligned} -\mathcal{L}_{\text{Yuk}}^{(\nu)} &= \frac{1}{\sqrt{2}}\sum_{k=1}^{n_\eta}\sum_{b,b'}\overline{\nu}_R \Delta^{(\eta_k)}\left(b_k S_b^0 + i b_k S_{b'}^0\right)\nu_L + \text{H.c.} \\ &= \frac{1}{\sqrt{2}}\sum_{b,b'}\overline{\nu}_R\left(\Delta_b^{(\eta)} S_b^0 + i\Delta_b^{(\eta)} S_{b'}^0\right)\nu_L + \text{H.c.}\,, \end{aligned} \tag{5.98}$$

where we used again the definition of Δ_b given in (3.114).

If we now apply all this to the formula for the one-loop mass correction (4.78) resp. to the result we achieved before in (5.88) and use $m_b^2 = m_{b'}^2$, we get

$$\begin{aligned} \delta M_L =& \sum_b \frac{1}{32\pi^2}(\Delta_b^{(\eta)})^T U_R^* \hat{m}\frac{\ln r_b}{r_b - 1}U_R^\dagger \Delta_b^{(\eta)} \\ &+ \sum_{b'} \frac{1}{32\pi^2}(i\Delta_b^{(\eta)})^T U_R^* \hat{m}\frac{\ln r_{b'}}{r_{b'} - 1}U_R^\dagger i\Delta_b^{(\eta)} \\ =& \sum_b \frac{1}{32\pi^2}(\Delta_b^{(\eta)})^T U_R^* \hat{m}\frac{\ln r_b}{r_b - 1}U_R^\dagger \Delta_b^{(\eta)} \\ &+ \sum_{b'} \frac{1}{32\pi^2}(i\Delta_b^{(\eta)})^T U_R^* \hat{m}\frac{\ln r_b}{r_b - 1}U_R^\dagger i\Delta_b^{(\eta)} \\ =& \sum_b \frac{1}{32\pi^2}(\Delta_b^{(\eta)} + i\Delta_b^{(\eta)})^T U_R^* \hat{m}\frac{\ln r_b}{r_b - 1}U_R^\dagger (\Delta_b^{(\eta)} + i\Delta_b^{(\eta)}) \\ =& \sum_b \frac{1}{32\pi^2}\underbrace{(1+i)^2}_{=0}\Delta_b^{(\eta)^T} U_R^* \hat{m}\frac{\ln r_b}{r_b - 1}U_R^\dagger \Delta_b^{(\eta)} = 0. \end{aligned} \tag{5.99}$$

Hence, it has been shown that for vanishing Yukawa couplings $\lambda_{ijkl}^{(5)}$ in the generalized model there are no one-loop mass corrections and the neutrinos remain massless even at one-loop level.

5.2.3 Generalized Fermion Sector

In the last step this model will be generalized for arbitrary numbers of LH neutrinos n_L and RH neutrinos n_R. The Yukawa Lagrangian (5.87) for the neutrinos can be adapted in the following way;

$$-\mathcal{L}_{\text{Yuk}}^{(\nu)} = \sum_{k'=1}^{n_\eta} \sum_{i=1}^{n_L} \sum_{j=1}^{n_R} \overline{\nu}_{Rj} \Delta^{(\eta_{k'})}{}_{ji} \eta_{k'}^0 \nu_{Li} + \text{H.c.}\,, \tag{5.100}$$

where the Yukawa couplings[114] $\Delta^{(\eta_{k'})}$ are now $n_R \times n_L$ matrices as in section 3.2.2.

In the case of arbitrary n_L and n_R we like to find a formula to describe the numbers of neutrinos remaining massless at one-loop level, analogously to (F.50). First we recognize of course, due to the $\mathbb{Z}_2$ symmetry the $n_R \times n_L$ Dirac mass matrix M_D is zero and hence, the number of neutrinos remaining massless at tree-level is n_L in contrary to the model without this symmetry, where this number is $n_L - n_R$.

In the generalized scotogenic model the tree-level neutrino mass matrix $M_{D+M}^{(0)} = \hat{m}$ is diagonal, since we can assume M_R to be diagonal. Hence, no diagonalization at tree-level is necessary and the diagonalizing matrix is simply $U^{(0)} = \mathbb{1}_{n_L+n_R}$. In this case we can not apply the approximation (F.70) for calculating one-loop masses, but we can easily diagonalize the one-loop corrected neutrino mass matrix

$$M_{D+M}^{(1)} = \begin{array}{c} \begin{pmatrix} \delta M_L & 0 \\ 0 & M_R \end{pmatrix} \begin{array}{l} \}n_L \\ \}n_R \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n_L} & \underbrace{\quad}_{n_R} \end{array} \end{array}\,, \tag{5.101}$$

via Schur's theorem (theorem E.2.2) by a $(n_L + n_R) \times (n_L + n_R)$ unitary matrix

$$U^{(1)} = \begin{pmatrix} U_L^{(1)} & 0 \\ 0 & \mathbb{1}_{n_R} \end{pmatrix}, \tag{5.102}$$

with $U_L^{(1)}$ being a unitary $n_L \times n_L$ matrix per construction, such that

$$U^{(1)T} M_{D+M}^{(1)} U^{(1)} = \text{diag}(m_{L1}, \ldots, m_{Ln_L}, m_{R1}, \ldots, m_{Rn_R}). \tag{5.103}$$

Thus, the interesting part is the upper left $n_L \times n_L$ block matrix, where

$$U_L^{(1)T} \delta M_L U_L^{(1)} = \text{diag}(m_{L1}, \ldots, m_{Ln_L}). \tag{5.104}$$

To find the number of neutrinos remaining massless at one-loop level we investigate the column vectors u_{Li} for $i = 1, \ldots, n_L$ of $U_L^{(1)}$, which give

$$u_{Li}^T \delta M_L u_{Li} = m_{Li}. \tag{5.105}$$

The $n_L \times n_L$ one-loop correction matrix δM_L in the most general scotogenic model is

[114] We only considered here the neutrino part of the Yukawa Lagrangian, but in the general Yukawa Lagrangian $\Delta^{(\phi_k)} = 0_{n_R \times n_L}$ and the couplings of the charged leptons $\Gamma^{(\phi_k)}$ and $\Gamma^{(\eta_{k'})}$ are $n_L \times n_L$ matrices.

analogous to (5.88)

$$\delta M_L = \sum_b \frac{1}{32\pi^2} \Delta^{(\eta)}{}_b^T U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b^{(\eta)}, \tag{5.106}$$

but now $\Delta_b^{(\eta)}$ are $n_R \times n_L$ matrices, U_R^* is $n_R \times (n_L + n_R)$ and $\hat{m}$ is of course $(n_L + n_R) \times (n_L + n_R)$. Hence, the matrix structure of δM_L is analogously to (F.46)

$$\delta M_L = \sum_b \sum_{k=1}^{n_\eta} \Delta^{(\eta_k)}{}^T \tilde{m}_b \Delta^{(\eta_k)}, \tag{5.107}$$

and therefore

$$m_{Li} = u_{Li}{}^T \delta M_L u_{Li} = \sum_b \sum_{k=1}^{n_\eta} u_{Li}{}^T \Delta^{(\eta_k)}{}^T \tilde{m}_b \Delta^{(\eta_k)} u_{Li}. \tag{5.108}$$

The LH neutrino masses m_{Li} remain zero at one-loop level, if the column vectors $u_{Li} \in \ker(\Delta^{\eta_k})$ for all $k = 1, \ldots, n_\eta$ and if all Yukawa couplings $\Delta^{(\eta_k)}$ are assumed linearly independent. Therefore, we can apply the same procedure as done in appendix F.7 in equations (F.51)-(F.55) and we find analogously

$$\boxed{n_0 = \max(0, n_L - n_R n_\eta)}, \tag{5.109}$$

as the number of neutrinos remaining massless after one-loop correction in the most general scotogenic model.

It should be noted that in this model the number of scalar doublets ϕ_k does not have any effect on the number of massive neutrinos. Furthermore, we can also see that for example in the case $n_L = 3$, $n_R = 2$ and $n_\eta = 2$ all LH neutrinos will acquire mass after one-loop corrections.

6 Summary

In this thesis we discussed extensions of the SM, to explain neutrino masses. In the most general set up we considered n_L active LH neutrinos and extended the particles content of the SM by n_R sterile RH neutrinos and n_H Higgs doublets. In section 2 it has been discussed that in such a model a Majorana mass term for the RH neutrinos is allowed. Furthermore, since RH neutrinos have been included, the Yukawa Lagrangian leads to a Dirac neutrino mass term analogously to the mass term for the charged leptons in the SM. In this thesis we focused on the seesaw mechanism of type I, and hence we excluded a Majorana mass term for the LH neutrinos. To be able to employ this mechanism, the masses of the RH neutrinos are assumed to be much larger than the EW scale. The general mass term is then given in (3.3) and the neutrino fields are Majorana fields. The neutrino mass matrix M_{D+M} is the symmetric $(n_L + n_R) \times (n_L + n_R)$ matrix

$$M_{\mathrm{D+M}} = \begin{array}{c} \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \begin{array}{l} \}n_L \\ \}n_R \end{array} \\ \begin{array}{cc} \underbrace{}_{n_L} & \underbrace{}_{n_R} \end{array} \end{array} . \tag{6.1}$$

where the block matrices are the Dirac mass matrix M_D and the Majorana mass matrix M_R for RH neutrino singlets.

It has been shown in section 3.1.2 that light (LH) and heavy (RH) neutrino masses can be disentangled such that the neutrino mass matrix appears as a block diagonal matrix

$$\begin{pmatrix} M_{\mathrm{light}} & 0 \\ 0 & M_{\mathrm{heavy}} \end{pmatrix}, \tag{6.2}$$

with M_{light} being the $n_L \times n_L$ mass matrix of the light neutrinos and M_{heavy} the $n_R \times n_R$ mass matrix of the heavy neutrinos. It has been shown that in the lowest order of the inverse seesaw scale, i.e. m_R^{-1}, the light neutrino mass matrix is given by the seesaw formula

$$M_{\mathrm{light}} \approx -M_D^T M_R^{-1} M_D, \tag{6.3}$$

whereas M_{heavy} is just the RH neutrino mass matrix M_R in this approximation. Since the RH neutrinos are assumed to be very heavy, which means the eigenvalues of M_R are of the order of the very large seesaw scale, the seesaw formula shows that the masses of the light neutrinos will be very light indeed.

Nevertheless, if $n_L > n_R$, then there are at least $n_L - n_R$ neutrinos remaining massless at tree level, but including radiative corrections will lead to more massive LH neutrinos. Therefore, one-loop corrections to the seesaw mechanism have been the focus of this thesis. It has been shown that the corrected neutrino mass matrix is given by

$$M_{D+M} = \begin{pmatrix} \delta M_L & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{6.4}$$

The correction δM_L has been derived in equation (4.78) as

$$\delta M_L = \sum_{b \neq b_Z} \frac{1}{32\pi^2} \Delta_b^T U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger \Delta_b + \frac{3g^2}{64\pi^2 m_W^2} M_D^T U_R^* \hat{m} \frac{\ln r_z}{r_z - 1} U_R^\dagger M_D. \tag{6.5}$$

The correction has contributions from the scalar mass eigenfields as well as from the Z^0 boson. In terms of mass scales this result means

$$\delta m_L \sim \frac{1}{16\pi^2} \left(\sum_b \frac{m_b^2}{m_R} \ln \frac{m_R^2}{m_b^2} + \frac{m_D^2}{m_R} \ln \frac{m_R^2}{m_Z^2} \right), \tag{6.6}$$

with neutral scalar masses m_b and the mass of the Z^0 boson m_Z. Besides the factor $(16\pi^2)^{-1}$ and the logarithmic structure, which come from the one-loop calculations, the m_R^{-1} dependence should be noticed. The factors $\frac{m_b^2}{m_R}$ and $\frac{m_D^2}{m_R}$ are both of the same order as the scale of the light neutrino masses given by the seesaw formula. In general, including one-loop corrections leads to $n_0 = \max(0, n_L - n_R n_H)$ massless neutrinos, whereas $(n_L - n_R - n_0)$ are massive at one-loop level, which has been shown in appendix F.7. These results have been examined by employing them to a special model with $n_L = 3$ and a minimal extension $n_R = 1$ and $n_H = 2$. In this special case one neutrino acquires mass at tree-level, a second one at one-loop level and the third one will remain massless.

Finally, in section 5, all general results have been used to investigate a special type of SM extension, i.e. the scotogenic model by E. Ma. An additional exact $\mathbb{Z}_2$ symmetry does not allow Dirac mass terms and all LH neutrinos remain massless at tree-level. Nevertheless, at one-loop level light neutrino masses are generated. In the original model of Ma, the SM with $n_L = 3$ is extended by three RH neutrinos $n_R = 3$ and one scalar doublet η. The added particles transform oddly under $\mathbb{Z}_2$, whereas the SM particles transform evenly. Because of the $\mathbb{Z}_2$ symmetry, only the two scalar mass eigenfields of η contribute to the one-loop neutrino mass correction. The scalar mass eigenfields are exactly the real and imaginary part of η and their masses only differ by their λ_5 couplings. This means the mass split is given by $\lambda_5 v$, where v is the VEV of the Higgs doublet. Therefore, only if $\lambda_5 \neq 0$ the neutrinos become massive at one-loop level.

This model has been generalized for an arbitrary numbers of RH neutrinos n_R and of scalar doublets of the SM Higgs doublet-type n_H, which transform evenly under $\mathbb{Z}_2$, as well as for an arbitrary number n_η of scalar doublets of η-type, which transform oddly under $\mathbb{Z}_2$. It has been shown that if the λ^5-*type* couplings are zero, there will be no massive neutrinos at one-loop level, analogous to the special case. This happens because the scalar mass eigenfields split exactly into pairs, such that their contributions to the one-loop mass correction cancel. Besides, for the generally extended fermion sector it has been found in (5.109) that the number of neutrinos remaining massless at one-loop level is given by $n_0 = \max(0, n_L - n_R n_\eta)$, analogously to the result obtained before.

A Appendix - Gamma Matrices

This section is a summary of the most relevant facts on gamma matrices found in [25, p.215f, 224, 239], [26, p.40ff, 50, 133ff], [28, p.89ff] and [33, p.327ff]. For a better understanding of the more mathematical aspects of the Clifford algebra, which is formed by the Dirac matrices, we like to refer to some literature by D. Hestenes [89],[90] and [91].

A.1 Definition

A.1.1 The Set of Dirac Matrices

When deriving the Dirac equation the necessity of some anticommuting objects occurs, which satisfy

$$\gamma^0\gamma^1 = -\gamma^1\gamma^0, \quad \gamma^0\gamma^2 = -\gamma^2\gamma^0, \quad \gamma^0\gamma^3 = -\gamma^3\gamma^0, \tag{A.1}$$

$$\gamma^1\gamma^2 = -\gamma^2\gamma^1, \quad \gamma^1\gamma^3 = -\gamma^3\gamma^1, \quad \gamma^2\gamma^3 = -\gamma^3\gamma^2, \tag{A.2}$$

and also

$$(\gamma^0)^2 = +\mathbb{1}_4, \tag{A.3}$$

$$(\gamma^i)^2 = -\mathbb{1}_4. \tag{A.4}$$

This can be written in a more compact form

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{A.5}$$

where we used the notation $\{A, B\} = AB+BA$ for the anticommutator and the Minkowski metric[115] $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. "A set of objects γ^μ (clearly d of them in d-dimensional spacetime, in our case $d = 4$) satisfying this anticommutation relation is said to form a Clifford algebra" [28, p.90].

Furthermore, the Dirac matrices can be normalized such that γ^0 is Hermitian while γ^i for $i = 1, 2, 3$ is anti-Hermitian[116], i.e.

$$(\gamma^0)^\dagger = +\gamma^0, \tag{A.6}$$

$$(\gamma^i)^\dagger = -\gamma^i, \tag{A.7}$$

which can be also expressed by the Hermiticity condition

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \quad \forall\mu. \tag{A.8}$$

[115] Note that a different convention for the Minkowski metric requires a difference in normalization for the Dirac matrices. As mentioned in [36, p.519], in the chiral basis the relationship is $\gamma^\mu_{\text{Bjorken Drell}} = i\gamma^\mu_{\text{Pauli}}$. The Bjorken Drell convention is the *mostly minus* convention, which is used in this thesis. The Pauli convention is also called *mostly plus* since the metric tensor is then defined as $g^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ [36, p.518].

[116] Note that when Pauli convention for the Minkowski metric is used, γ^0 is anti-Hermitian, whereas γ^i are Hermitian [36, p.519].

So far the γ-matrices have been defined in their contravariant form. We now define the corresponding covariant matrices by the relation

$$\gamma_\mu = g_{\mu\nu}\gamma^\nu = \{+\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3\}. \tag{A.9}$$

A.1.2 The Fifth Gamma Matrix

A fifth anticommuting γ-matrix is defined[117] by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.10}$$

and γ^5 has the properties

$$\{\gamma^\mu, \gamma^5\} = 0 \quad \forall\mu, \tag{A.11}$$

$$(\gamma^5)^2 = \mathbb{1}, \tag{A.12}$$

$$\gamma^{5\dagger} = \gamma^5. \tag{A.13}$$

We can also define the covariant form of the fifth matrix γ_5 through

$$\gamma_5 \equiv \frac{i}{4!}\varepsilon_{\lambda\mu\nu\pi}\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\pi = \gamma^5, \tag{A.14}$$

"where the completely antisymmetric alternating symbol $\varepsilon_{\lambda\mu\nu\pi}$ is equal to $+1$ for (λ, μ, ν, π) an even permutation of $(0, 1, 2, 3)$, is equal to -1 for an odd permutation, and vanishes if two or more indices are the same" [33, p.329]. It satisfies the following contraction identities:

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\tau} = -2(\delta^\mu_\nu\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma), \tag{A.15}$$

$$\varepsilon^{\alpha\beta\gamma\nu}\varepsilon_{\alpha\beta\gamma\tau} = -6\delta^\nu_\tau, \tag{A.16}$$

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -24. \tag{A.17}$$

A.2 Properties

In calculations, where γ-matrices are involved it is very useful to know the algebraic identities below, which can be easily derived from the anticommutation relation (A.5):

A.2.1 Contraction Theorems

$$\gamma_\mu\gamma^\mu = 4, \tag{A.18}$$

$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu, \tag{A.19}$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\mu = 4g^{\nu\lambda}, \tag{A.20}$$

$$\gamma_\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu = -2\gamma^\sigma\gamma^\lambda\gamma^\nu, \tag{A.21}$$

$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\delta\gamma^\mu = 2(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\nu + \gamma^\nu\gamma^\beta\gamma^\alpha\gamma^\delta). \tag{A.22}$$

[117] As noted in [36, p.520], the sign of this matrix is convention dependent, but not a consequence of a particular choice of Minkowski metric .

A.2.2 Trace Theorems

$$\begin{aligned}
\mathrm{Tr}(\gamma^\alpha\gamma^\beta\ldots\gamma^\mu\gamma^\nu) &= 0 \quad \text{if } (\ldots) \text{ contains an odd number of } \gamma\text{−matrices}, && \text{(A.23)}\\
\mathrm{Tr}(1) &= 4, && \text{(A.24)}\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu}, && \text{(A.25)}\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) &= 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}), && \text{(A.26)}\\
\mathrm{Tr}(\gamma^5\gamma^\mu) &= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) \overset{\text{(A.23)}}{=} 0, && \text{(A.27)}\\
\mathrm{Tr}(\gamma^5) &= 0, && \text{(A.28)}\\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) &= 0, && \text{(A.29)}\\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) &= 4i\varepsilon^{\mu\nu\lambda\sigma}. && \text{(A.30)}
\end{aligned}$$

A.3 Representations

So far we have discussed the γ-matrices in a representation-free way, relying only on the anticommutation relation (A.5) and the Hermiticity condition (A.8) of the γ-matrices. There are infinitely many ways of writing γ^μ ($\mu = 0, \ldots, 3$) as 4×4 matrices such that equations (A.5) and (A.8) hold. Nevertheless, there exists, up to an equivalence given by a similarity transformation, only one unique complex matrix representation , which is irreducible.

A.3.1 Pauli's Fundamental Theorem

The similarity transformation for this equivalence of representations is stated in Pauli's fundamental theorem for gamma matrices, which can be found e.g. in [92, p.54f]

Theorem A.3.1: If γ^μ ($\mu = 0, \ldots, 3$) and $\tilde{\gamma}^\mu$ ($\mu = 0, \ldots, 3$) are two sets of matrices, which both satisfy the anticommutation relation(A.5), i.e. $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = \{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\}$, then the there exists a nonsingular matrix S (called the similarity transformation) such that

$$\tilde{\gamma}^\mu = S\gamma^\mu S^{-1} \;, \tag{A.31}$$

and S is unique up to a multiplicative constant[118].

If the two sets of matrices fulfil also the Hermiticity condition (A.8), then the similarity transformation S can be chosen to be unitary and the transformation can be written as

$$\tilde{\gamma}^\mu = S\gamma^\mu S^\dagger. \tag{A.32}$$

Proof. For a proof see e.g. [93, p.17f]. □

Thus, since $-\gamma^{\mu T}$ and $\gamma^{\mu\dagger}$ also satisfy the condition (A.5), there exist matrices C and

[118] And hence we can always choose $\det S = +1$.

A such that

$$-\gamma^{\mu T} = C^{-1}\gamma^{\mu}C, \tag{A.33}$$
$$\gamma^{\mu\dagger} = A^{-1}\gamma^{\mu}A, \tag{A.34}$$

where C appears as charge conjugation matrix in the discussion on Majorana spinors in section 1.2.5. Finally we shall introduce three particular representations for Dirac matrices which are useful in practice[119]. All of them are written in terms of the Pauli 2×2 spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.35}$$

Note that in these three representation γ^0 is not only Hermitian but also symmetric, i.e.

$$\gamma^0 = \gamma^{0T}. \tag{A.36}$$

A.3.2 Dirac-Pauli Representation

In this representation the Dirac γ-matrices can be written as

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{A.37}$$

A.3.3 Weyl or Chiral Representation

This is another common and very convenient choice where γ^5 is diagonal

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{A.38}$$

A.3.4 Majorana Representation

There is also a third kind of basis in which the Dirac matrices are purely imaginary, whereas the spinors in the Majorana representation are real:

$$\begin{array}{ll} \gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, & \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \\ \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, & \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}, \end{array} \quad \gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}. \tag{A.39}$$

[119] A elaborated discussion can be found in [94, p.85ff]

B Appendix - Symmetries in Particle Physics

B.1 Parity Transformation

B.1.1 Motivation and Definition

A discussion of this topic can be found in [25, p.125f] and [26, p.65f]. First, one should distinguish between reflections and inversions. Through reflections we produce mirror images, where the plane of reflection, i.e. the mirror, can be chosen arbitrarily, whereas through inversions "every point is carried through the origin to the diametrically opposite location" [25, p.125]. This difference is visualized in figure 16 below. An inversion can be also understood as a reflection followed by a rotation by 180°. For our purpose we will use

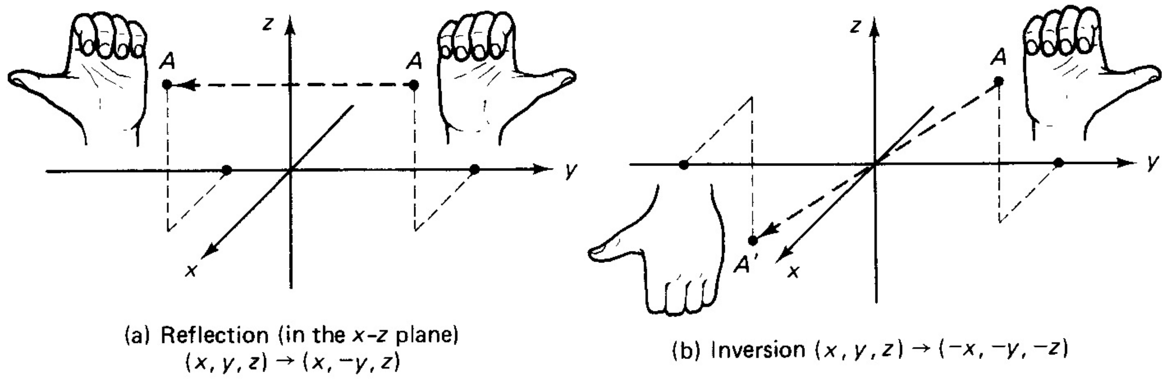


Figure 16: Reflections and inversions. Reprinted from [25, p.126].

inversions to be able to discuss parity of objects like spinors, since inversion of a spinor fields means reversing the momentum of a particle without flipping its spin. This causes a change of chirality of the spinor field, i.e. under inversions a LH spinor is transformed into a RH spinor and vice versa[120].

Mathematically inversions can be described by a unitary linear operator $\mathcal{P}$, called the *parity operator*, which can be represented by a 3×3 or 4×4 matrix P acting on objects in the three dimensional euclidean space $\mathbb{E}^3$ or in the four dimensional spacetime $\mathcal{M}_4$ respectively. For those three and four dimensional representations P the following holds:

$$PP^\dagger = P^\dagger P = \mathbb{1}, \tag{B.1}$$

$$P = \mathrm{diag}(-1,-1,-1) \qquad \text{for } \mathbb{E}^3, \tag{B.2}$$

$$P = \mathrm{diag}(1,-1,-1,-1) \quad \text{for } \mathcal{M}_4. \tag{B.3}$$

[120]Hence, since weak interaction is chiral, it is not invariant under parity, whereas strong and electromagnetic exhibit such an invariance.

The transformation of an object Θ in terms of the operator $\mathcal{P}$ is written as $\mathcal{P}\Theta\mathcal{P}^{-1}$, whereas for the matrix P acting on an object we simply write $P\Theta$.

Furthermore, it is clear that applying the operator resp. the matrix twice leads to the initial object

$$\mathcal{P}^2 = \mathbb{1}, \text{ resp. } P^2 = \mathbb{1}, \tag{B.4}$$

and hence it follows that the eigenvalues of $\mathcal{P}$ resp. P are ± 1.

B.1.2 Parity in $\mathbb{E}^3$ of Scalars and Vectors

Now we should consider the behaviour of some customary objects like scalars and vectors in three dimensions. In doing so, we find that we have to distinguish two types of each of them. One kind is transforming in the *ordinary* way under parity, whereas the other one is not. Therefore we will call the latter pseudoscalars or pseudovectors. Mathematically we can distinguish them whether they have eigenvalue $+1$ (scalars and pseudovectors) or eigenvalue -1 (vectors and pseudoscalars) and their transformation behaviour is shown below in table 17.

Scalar	:	$P(s)$	$=$		s
Pseudoscalar	:	$P(p)$	$=$	$-$	p
Vector (or polar vector)	:	$P(\vec{v})$	$=$	$-$	$\vec{v}$
Pseudovector (or axial vector)	:	$P(\vec{a})$	$=$		$\vec{a}$

Table 17: Scalars and vectors under P-transformation. Reprinted from [25, p.127].

We should note that the cross product of two vectors is a pseudovector, and the dot product of two vectors does not change sign under P whereas the dot product of a vector and a pseudovector does. In a theory with parity invariance, we must never add a vector to a pseudovector[121]. We might want to classify some well-known quantities according to this property in table B.1.2 below.

Scalar	**Pseudoscalar**	**Vector**	**Pseudovector**
time	helicity	position	angular momentum
mass	magnetic charge	momentum	magnetic field
energy	magnetic flux	electric field	magnetization

Table 18: Examples of some physical quantities and their behaviour under P.

[121] This is exactly the reason why the theory of weak interactions exhibits no invariance under parity transformations.

B.1.3 Parity in $\mathcal{M}_4$ of Four-Vectors and Spinors

Finally, we should discuss the behaviour of four-vectors and spinors under parity transformations. In order to ensure Lorentz invariance of the transformation we are obliged to use the Lorentz covariant conjugate[122] $\mathcal{P}\ldots\mathcal{P}^{-1}$ of the operator $\mathcal{P}$.

The behaviour under $\mathcal{P}$ of four-vectors can be written down straight forward, whereas the parity transformation of a spinor field ψ and its Dirac adjoint $\overline{\psi}$ can be obtained through claiming that they have to satisfy the parity transformed Dirac equation. So in their transformation behaviour [123] are shown in table 19 below.

Four-vector	:	$\mathcal{P}v^\mu\mathcal{P}^{-1}$	$=$	$(v^0,-\vec{v})^T$	
Pseudo-four-vector	:	$\mathcal{P}w^\mu\mathcal{P}^{-1}$	$=$	$(w^0,\vec{w})^T$	
Dirac spinor	:	$\mathcal{P}\psi(t,\vec{x})\mathcal{P}^{-1}$	$=$	$\eta\gamma^0\psi(t,-\vec{x})$	$=: \psi^P$
Adjoint Dirac spinor	:	$\mathcal{P}\overline{\psi}(t,\vec{x})\mathcal{P}^{-1}$	$=$	$\eta^*\gamma^0\overline{\psi}(t,-\vec{x})$	$=: \overline{\psi}^P$

Table 19: Four-vectors and spinors under $\mathcal{P}$-transformation.

In various calculations the transformation behaviour of the various Dirac field bilinears under parity transformations will be important. Those bilinears are[124]:

$$\overline{\psi}\psi, \quad \overline{\psi}\gamma^\mu\psi, \quad i\overline{\psi}[\gamma^\mu,\gamma^\nu]\psi, \quad \overline{\psi}\gamma^5\gamma^\mu\psi, \quad i\overline{\psi}\gamma^5\psi. \tag{B.5}$$

For the first two bilinears we obtain the behaviour of a scalar and vector respectively under parity transformation, i.e.

$$\mathcal{P}\overline{\psi}\psi\mathcal{P}^{-1} = |\eta|^2\overline{\psi}\gamma^0\gamma^0\psi(t,-\vec{x}) = \quad +\overline{\psi}\psi(t,-\vec{x}), \tag{B.6}$$

$$\mathcal{P}\overline{\psi}\gamma^\mu\psi\mathcal{P}^{-1} = \overline{\psi}\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x}) = \begin{cases} +\overline{\psi}\gamma^\mu\psi(t,-\vec{x}) & \text{for } \mu=0, \\ -\overline{\psi}\gamma^\mu\psi(t,-\vec{x}) & \text{for } \mu=1,2,3. \end{cases} \tag{B.7}$$

Similarly we find for the last two bilinears a pseudoscalar and a pseudovector behaviour respectively, i.e.

$$\mathcal{P}i\overline{\psi}\gamma^5\psi\mathcal{P}^{-1} = i\overline{\psi}\gamma^0\gamma^5\gamma^0\psi(t,-\vec{x}) = \quad -i\overline{\psi}\gamma^5\psi(t,-\vec{x}), \tag{B.8}$$

$$\mathcal{P}\overline{\psi}\gamma^\mu\gamma^5\psi\mathcal{P}^{-1} = \overline{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(t,-\vec{x}) = \begin{cases} -\overline{\psi}\gamma^\mu\gamma^5\psi(t,-\vec{x}) & \text{for } \mu=0, \\ +\overline{\psi}\gamma^\mu\gamma^5\psi(t,-\vec{x}) & \text{for } \mu=1,2,3. \end{cases} \tag{B.9}$$

B.2 Charge Conjugation

Another important discrete symmetry is the particle-antiparticle symmetry, which is discussed e.g. in [25, p.128f], [26, 70f], [28, 97f] or [95, p.26ff,299ff].

[122] As it is used e.g. in [26, p.65]
[123] η are possible phases, which can be set to 1.
[124] The factors of i have been chosen to make all these quantities Hermitian.

B.2.1 Charge Conjugation Operator

We should mention here briefly as it is done in [25, p.128f] that such a charge conjugation operator $\mathcal{C}$ acting on the Fock space of particle states is a generalization of the notion of changing the sign of the charge, under which classical electrodynamics is invariant and can be implemented as a unitary linear operator like $\mathcal{P}$. Charge conjugation not only changes the sign of the electric charges but it rather flips the sign of all *internal* quantum numbers (charge Q, baryon number B, lepton number L, leptonic family numbers L_ℓ, strangeness s, charm c, beauty b, truth t) while leaving mass m, energy E, momentum p, angular momentum J, spin Σ and helicity h untouched.

Most particles in nature are not eigenstates of $\mathcal{C}$, in contrast to the parity operator $\mathcal{P}$. Nevertheless, there are particle states which are identical to their antiparticle state. This is only possible for neutral particles and so these are the only possible eigenstates of the charge conjugation operator $\mathcal{C}$.

B.2.2 Charge Conjugation Matrix

After this discussion in terms of Hilbert space and state vectors we should investigate the meaning of charge conjugation in terms of Dirac spinor theory. We want to discuss how the operation of charge conjugation, $\mathcal{C}\psi\mathcal{C}^{-1}$, can be performed on Dirac spinor fields and we are searching for an appropriate definition of a charge conjugated Dirac spinor, which we will denote by ψ^C. There are some authors like [28, p.97f], [30, p.71], [94, p.101ff], [95, p.299f] and [96, p.108f], who have chosen a similar and quite accessible approach, which we will follow here.

Quoting [94, p.101ff] we start by the fact that "the Dirac theory implies the existence of particles and antiparticles", and hence the Dirac equation (1.19) must be invariant under a symmetry, which interchanges particles and antiparticles, i.e. the transformation called charge conjugation

$$\psi \to \psi^C, \tag{B.10}$$

which flips the sign of the charge q of the Dirac spinor field ψ. The fermion field ψ satisfies the following equation[125]

$$\left[\gamma^\mu \left(i\partial_\mu - qA_\mu\right) - m\right]\psi = 0, \tag{B.11}$$

where A_μ denotes the electromagnetic vector potential. Hence, the charge conjugated field ψ^C has to obey the same equation, but with the opposite sign of charge q

$$\left[\gamma^\mu \left(i\partial_\mu + qA_\mu\right) - m\right]\psi^C = 0. \tag{B.12}$$

Now we want to rewrite equation (B.11) for ψ to achieve such a form as in (B.12). If

[125] This equation for a Dirac field coupled to a EM vector potential is derived in appendix B.4.1 equation (B.61). It is simply the Dirac equation (1.19) plus the interaction of the fermion and the EM vector potential, i.e. the current.

we take the complex conjugate of equation (B.11), we get

$$\left[-\gamma^{\mu *}\left(i\partial_\mu + qA_\mu\right) - m\right]\psi^* = 0. \tag{B.13}$$

This equation has already the right sign of charge, but differs from the typical form of the Dirac equation because we have $-\gamma^{\mu *}$.

This actually means we are in a different basis of Dirac matrices and we already know from Pauli's fundamental theorem (theorem A.3.1) that we are able to find a unitary matrix S to switch into another Dirac matrix representation. For reasons, which will get clearer later, we like to denote the matrix S as $(C\gamma^{0^T})$ and so we achieve the change of the basis by

$$-\gamma^{\mu *} = \left(C\gamma^{0^T}\right)^{-1}\gamma^\mu\left(C\gamma^{0^T}\right). \tag{B.14}$$

Now we proceed by multiplying equation (B.13) by this matrix $(C\gamma^{0^T})$ from the left and use the relation (B.14) above to achieve

$$\begin{aligned}\left(C\gamma^{0^T}\right)\left[-\gamma^{\mu *}\left(i\partial_\mu + qA_\mu\right) - m\right]\psi^* &= 0,\\ \left(C\gamma^{0^T}\right)\left[\left(C\gamma^{0^T}\right)^{-1}\gamma^\mu\left(C\gamma^{0^T}\right)\left(i\partial_\mu + qA_\mu\right) - m\right]\psi^* &= 0,\\ \left[\gamma^\mu\left(i\partial_\mu + qA_\mu\right) - m\right]\left(C\gamma^{0^T}\right)\psi^* &= 0,\end{aligned} \tag{B.15}$$

which has the form we were looking for. Comparing (B.15) with the equation (B.12) for the charge conjugated spinor we find

$$\psi^C = \left(C\gamma^{0^T}\right)\psi^*, \tag{B.16}$$

which tells us how the charge conjugated field is connected to the initial field. To emphasise this we may rewrite this result in terms of the adjoint spinor field (1.18)

$$\boxed{\mathcal{C}\psi\mathcal{C}^{-1} = \psi^C = \left(C\gamma^{0^T}\right)\psi^* = C\overline{\psi}^T}, \tag{B.17}$$

and we call C the *charge conjugation matrix*.

Its defining property can be derived from equation (B.14), where we have introduced the matrix C by rewriting it like

$$C\underbrace{\gamma^{0^T}\gamma^{\mu *}\gamma^{0^T}}_{\left(\gamma^0\gamma^{\mu\dagger}\gamma^0\right)^T}C^{-1} = -\gamma^\mu. \tag{B.18}$$

Now we use the Hermiticity condition of the Dirac matrices (A.8) in a rewritten form:

$$\gamma^\mu = \gamma^0\left(\gamma^\mu\right)^\dagger\gamma^0. \tag{B.19}$$

Thus, inserting this in relation (B.18), we obtain the defining relation for C in any representation of Dirac matrices as

$$\boxed{\left(\gamma^\mu\right)^T = -C^{-1}\gamma^\mu C}. \tag{B.20}$$

The calculation above, in order to obtain this defining relation in such a concise form, explains why we have introduced the matrix C with a γ^{0^T} attached before.

Besides this derivation of the charge conjugation matrix done above, we should also reconsider how we introduced it in the section on the Majorana equation 1.2.5. In relation (1.37), i.e.

$$UU^T \equiv C\gamma^{0^T} , \tag{B.21}$$

we introduced the matrix C also with a γ^{0^T} attached, where U was the unitary matrix introduced to perform the change of basis from the Majorana representation to an arbitrary representation of Dirac matrices where γ^μ need not to be purely imaginary. We had in equation (1.34)

$$\gamma^\mu = U\tilde{\gamma^\mu}U^\dagger, \tag{B.22}$$

where $\tilde{\gamma^\mu}$ denote the Dirac matrices in the Majorana representation and therefore they fulfil the relation

$$\tilde{\gamma}^{\mu *} = -\tilde{\gamma}^\mu. \tag{B.23}$$

We should show that the matrix C defined in (1.37) in section 1.2.5 is the same one we have introduced here above. Thus we prove C defined via U must fulfil the same defining relation (B.20). We start by expressing $\tilde{\gamma^\mu}$ and taking the complex conjugate of equation (1.34)

$$\begin{aligned} \tilde{\gamma^\mu} &= U^\dagger\gamma^\mu U, \\ (\tilde{\gamma^\mu})^* &= U^T(\gamma^\mu)^* U^*, \\ -\tilde{\gamma^\mu} &= U^T(\gamma^\mu)^* U^*, \end{aligned} \tag{B.24}$$

and in the last step we used relation (B.23). Now we can insert again (B.24) and use the unitarity of U, i.e. $U^T U^* = \mathbb{1} = U^* U^T$ to get

$$\begin{aligned} -U^\dagger\gamma^\mu U &= U^T(\gamma^\mu)^* U^*, \\ U^* U^\dagger\gamma^\mu U U^T &= -(\gamma^\mu)^*, \\ (UU^T)^\dagger\gamma^\mu UU^T &= -(\gamma^\mu)^*, \\ (UU^T)^{-1}\gamma^\mu UU^T &= -(\gamma^\mu)^*, \end{aligned} \tag{B.25}$$

where we used the fact that if U is unitary UU^T is also unitary. Finally we insert the relation (1.37) where we have introduced C and express $-\gamma^\mu$ by using the unitarity of $(C\gamma_0^T)$:

$$\begin{aligned} (C\gamma_0^T)^{-1}\gamma^\mu(C\gamma_0^T) &= -(\gamma^\mu)^* \\ -\gamma^\mu &= (C\gamma_0^T)(\gamma^\mu)^*(C\gamma_0^T)^{-1}. \end{aligned} \tag{B.26}$$

This equation is actually the same as (B.18), if we remember the properties of γ^0 given in (A.3) and (A.6). Hence, the last steps to obtain the defining relation for C are the same as above and we showed that the C matrix defined in (1.37) in section 1.2.5 ist the charge conjugation matrix defined in this appendix.

Furthermore, we can also obtain from the defining relation (B.20)

$$C^{-1}\gamma_5 C = \gamma_5^T. \tag{B.27}$$

Proof. We rewrite the defining relation (B.20) by

$$\gamma^\mu C = -C(\gamma^\mu)^T \tag{B.28}$$

to see explicitly how to interchange the charge conjugation with a Dirac matrix.

Thus, for interchanging γ^5 with C we obtain by using equation (B.28) four times:

$$\begin{aligned}\gamma^5 C &= i\gamma^0\gamma^1\gamma^2\gamma^3 C = -i\gamma^0\gamma^1\gamma^2 C{\gamma^3}^T \\ &= i\gamma^0\gamma^1 C{\gamma^2}^T{\gamma^3}^T = -i\gamma^0 C{\gamma^1}^T{\gamma^2}^T{\gamma^3}^T \\ &= +iC{\gamma^0}^T{\gamma^1}^T{\gamma^2}^T{\gamma^3}^T = C{\gamma^5}^T.\end{aligned} \tag{B.29}$$

□

It might be also useful to note that consequently we get

$$C^{-1}\gamma^\mu\gamma_5 C = C^{-1}\gamma^\mu \underbrace{CC^{-1}}\gamma_5 C = -{\gamma^\mu}^T{\gamma_5}^T = -\left(\gamma_5\gamma^\mu\right)^T = \left(\gamma^\mu\gamma_5\right)^T, \tag{B.30}$$

where we used the anticommutator relation (A.11) in the last step.

Finally, we want to discuss some matrix properties of the charge conjugation matrix C, which are also valid in any basis of Dirac matrices. The C is per definition a unitary 4×4 matrix

$$C^{-1} = C^\dagger, \tag{B.31}$$

and it can be shown from (B.20) that C has to be also antisymmetric

$$C^T = -C. \tag{B.32}$$

Proof. First we transpose the defining relation for C (B.20) and get

$$C^T{\gamma^\mu}^T {C^{-1}}^T = -\gamma^\mu, \tag{B.33}$$

and we express $-{\gamma^\mu}^T$, which is also expressed in (B.20). Thus, we get

$$C^T\gamma^\mu {C^T}^{-1} = -{\gamma^\mu}^T = C^{-1}\gamma^\mu C. \tag{B.34}$$

Hence, we can rewrite this as

$$C{C^T}^{-1}\gamma^\mu = \gamma^\mu C{C^T}^{-1}, \tag{B.35}$$

and find that the matrix $C{C^T}^{-1}$ commutes with all Dirac matrices. Therefore it also commutes with all 16 linear independent matrices $\Gamma = \{\mathbb{1}_4, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma_{\mu\nu}\}$, since these matrices are all products of Dirac matrices (see (C.44)). The set of these matrices spans the space of 4×4 matrices and thus, if $C{C^T}^{-1}$ commutes with all all of them, it has to commute with all 4×4 matrices. This is only possible if $C{C^T}^{-1}$ is proportional to the identity matrix, i.e.

$$C{C^T}^{-1} = \lambda\mathbb{1}_4, \tag{B.36}$$

for $\lambda\in\mathbb{C}$ a priori. But we can find by multiplying the equation above with C^T from the right

$$C = \lambda C^T, \tag{B.37}$$

and by transposing and multiplying with C from the left we get

$$C^T = \lambda C. \tag{B.38}$$

Thus, we obtain by inserting (B.38) into (B.37)

$$C = \lambda^2 C \quad \Rightarrow \quad \lambda^2 = 1 \quad \Leftrightarrow \quad \lambda = \pm 1. \tag{B.39}$$

To find the right sign for λ we multiply all nontrivial matrices from the spanning set with C from the right and take the transposed:

$$(\gamma^\mu C)^T = C^T \gamma^{\mu T} \overset{(B.38)}{=} \lambda C \gamma^{\mu T} \underbrace{C^{-1} C}\overset{(B.20)}{=} -\lambda \gamma^\mu C, \tag{B.40}$$

$$(\gamma_5 C)^T = C^T \gamma_5^T \overset{(B.38)}{=} \lambda C \gamma_5^T \underbrace{C^{-1} C} \overset{(B.27)}{=} \lambda \gamma_5 C. \tag{B.41}$$

For the other two more complicated matrices we apply the same procedure:

$$(\gamma^\mu \gamma_5 C)^T \overset{(B.38)}{=} \lambda \underbrace{C \gamma_5^T C^{-1}} \underbrace{C \gamma^{\mu T} C^{-1}} C \underset{(B.20)}{\overset{(B.38)}{=}} -\lambda \gamma_5 \gamma^\mu C \overset{(A.11)}{=} \lambda \gamma^\mu \gamma_5 C, \tag{B.42}$$

$$\begin{aligned}
(\sigma_{\mu\nu} C)^T &= C^T \sigma_{\mu\nu}^T \underset{(B.38)}{\overset{(C.44)}{=}} \frac{i}{2} \lambda C \left[\gamma_\nu^T, \gamma_\mu^T\right] \\
&= \frac{i}{2} \lambda \left(\underbrace{C \gamma_\nu^T C^{-1}} \underbrace{C \gamma_\mu^T C^{-1} C} - \underbrace{C \gamma_\mu^T C^{-1}} \underbrace{C \gamma_\nu^T C^{-1}} C \right) \\
&= \frac{i}{2} \lambda \left(\gamma_\nu \gamma_\mu C - \gamma_\mu \gamma_\nu C\right) = -\lambda \frac{i}{2} \left[\gamma_\mu, \gamma_\nu\right] C = -\lambda \sigma_{\mu\nu} C.
\end{aligned} \tag{B.43}$$

If we now assume $\lambda = 1$, we end up with ten linear independent antisymmetric 4×4 matrices form (B.40) and (B.43), which is a contradiction since there can be only six linear independent antisymmetric 4×4 matrices. Thus we must have $\lambda = -1$ which leads to just six linear independent antisymmetric 4×4 matrices given in (B.41) and (B.42). So finally with $\lambda = -1$ we obtained $C = -C^T$ from (B.37) and hence, we showed that the charge conjugation matrix is antisymmetric in any basis of Dirac matrices. □

B.2.3 Charge Conjugation Matrix in Different Representations

Now we want to investigate the actual appearance of the matrix C in the different bases and hence we shall follow [94, p.102ff] in this subsection, since an elaborated discussion is given there. Initially in [94, p.102] it is noted that "it suffices to construct the charge conjugation matrix in some particular representation of the gamma matrices; the unitary transformation which transforms to another representation the gives the matrix C in this new representation". It is noted in [28, p.89] that in the Weyl basis as well as in the Dirac basis, only γ^2 is imaginary. Hence, it follows from (B.18) that $C\gamma^{0^T}$ commutes with γ^2 but anticommutes with the other three Dirac matrices. So we might choose[126]

$$\boxed{C = -i\gamma^2\gamma^0}. \tag{B.44}$$

[126] Note that other authors might use different conventions, also depending on the Minkowski metric they use . For example when the Pauli metric is used the charge conjugation matrix is defined as $C = i\gamma^2\gamma^0$ [36, p.520] or even definitions like $C = \gamma^2\gamma^0$ can be found.

In this definition C is real and thus it fulfils

$$C^{-1} = C^{\dagger} = C^{T} = -C, \tag{B.45}$$

according to the relations (B.31) and (B.32).

(i) ***Dirac Representation***
Using the Dirac representation of gamma matrices we obtain the charge conjugation matrix C explicitly in the form

$$C_{\text{Dirac}} = -i\gamma^2\gamma^0 = -i\begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}. \tag{B.46}$$

A proof for C_{Dirac} fulfilling all properties of a charge conjugation matrix (B.20), (B.31), and (B.32), can be found in [94, p.102-107].

(ii) ***Weyl Representation***
Using now the Weyl representation of gamma matrices (or alternatively performing a unitary transformation on C to change the basis) we obtain the charge conjugation matrix C explicitly in the form

$$C_{\text{Weyl}} = -i\gamma^2\gamma^0 = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \tag{B.47}$$

Again a proof for C_{Weyl} fulfilling all properties of a charge conjugation matrix (B.20), (B.31), and (B.32), can be found in [94, p.107-108].

(iii) ***Majorana Representation***
According to [31, p.97] in the Majorana basis, where the gamma matrices are purely imaginary, we have simply

$$C_{\text{Majorana}} = \mathbb{1}_4 \,. \tag{B.48}$$

B.2.4 Charge Conjugated Dirac Bilinears

Finally we should investigate how the charge conjugation operator acts on Lorentz bilinears of spinor fields. In section B.2.2 we found how a charge conjugation is performed on a spinor. For the following we need to know how charge conjugation on the Dirac adjoint spinor performed. Using its definition (1.18) and how charge conjugation acts on a spinor $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ according to (B.17) we derive

$$\begin{aligned}
\mathcal{C}\overline{\psi}\mathcal{C}^{-1} &= \mathcal{C}\psi^{\dagger}\gamma^0\mathcal{C}^{-1} = \mathcal{C}(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)\gamma^0\mathcal{C}^{-1} \\
&= (\mathcal{C}\psi_1^*\mathcal{C}^{-1}, \mathcal{C}\psi_2^*\mathcal{C}^{-1}, \mathcal{C}\psi_3^*\mathcal{C}^{-1}, \mathcal{C}\psi_4^*\mathcal{C}^{-1})\gamma^0 = \left(C{\gamma^0}^T\psi^*\right)^{\dagger}\gamma^0 \\
&= \psi^T\gamma^{0*}C^{\dagger}\gamma^0 = \psi^T\gamma^{0*}\underbrace{C^{\dagger}\gamma^0 C}_{-{\gamma^0}^T}C^{-1} = \boxed{-\psi^T C^{-1} = \overline{(\psi)}^C},
\end{aligned} \tag{B.49}$$

where we used in the last step the defining relation for C (B.20) together with the unitarity of C (B.31) as well as the Hermiticity of γ^0 (A.6).

So now we are able to investigate how the various Lorentz bilinears of Dirac spinors transform under charge conjugation. We may start with a general discussion given in [97, p.259ff] of the procedure we will need to use for that. We use the transformation behaviour of the Dirac spinor and the adjoint Dirac spinor obtained in (B.17) resp. (B.49) and have a look at an arbitrary Dirac bilinear $\overline{\psi}A\psi$, where A stands for a product of Dirac matrices. This bilinear transforms under charge conjugation as

$$\mathcal{C}\left(\overline{\psi}A\psi\right)\mathcal{C}^{-1} = -\psi^T C^{-1} A C \overline{\psi}^T. \tag{B.50}$$

If we take the transposed of the right hand side and take into account the exchange of two fermion fields by an extra minus sign we get

$$\mathcal{C}\left(\overline{\psi}A\psi\right)\mathcal{C}^{-1} = \overline{\psi}\left(C^{-1}AC\right)^T\psi. \tag{B.51}$$

Hence, we know of course all possible cases for $(C^{-1}AC)$, where A can be $\gamma^\mu, \gamma_5, \gamma^\mu\gamma_5$ from the relations (B.20), (B.29) and (B.30).

Investigating the scalar $\overline{\psi}\psi$, where $A = \mathbb{1}$ we find that middle part $(C^{-1}AC)$ is trivial, if we remember that the charge conjugation matrix is unitary. Hence, we have

$$\mathcal{C}\overline{\psi}\psi\mathcal{C}^{-1} = \overline{\psi}\left(C^{-1}\mathbb{1}C\right)^T\psi = \overline{\psi}\psi. \tag{B.52}$$

For the vector $\overline{\psi}\gamma^\mu\psi$, where $A = \gamma^\mu$, we use (B.20) and therefore we get

$$\mathcal{C}\overline{\psi}\gamma^\mu\psi\mathcal{C}^{-1} = \overline{\psi}\left(C^{-1}\gamma^\mu C\right)^T\psi = -\overline{\psi}\gamma^\mu\psi. \tag{B.53}$$

In case of the pseudoscalar $\overline{\psi}\gamma_5\psi$ we have $A = \gamma_5$ and use relation (B.29) to obtain

$$\mathcal{C}\overline{\psi}\gamma_5\psi\mathcal{C}^{-1} = \overline{\psi}\left(C^{-1}\gamma_5 C\right)^T\psi = \overline{\psi}\gamma_5\psi. \tag{B.54}$$

And finally we apply this procedure for the pseudovector $\overline{\psi}\gamma^\mu\gamma_5\psi$ where $A = \gamma^\mu\gamma_5$ and we can apply relation (B.30) to get

$$\mathcal{C}\overline{\psi}\gamma^\mu\gamma_5\psi\mathcal{C}^{-1} = \overline{\psi}\left(C^{-1}\gamma^\mu\gamma_5 C\right)^T\psi = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{B.55}$$

Therefore, we have found that just the vector is odd under charge conjugation whereas the scalar, pseudoscalar and pseudovector transform even under $\mathcal{C}$.

B.2.5 Charge Conjugation and Chirality

Finally we want to show how charge conjugation changes the chirality or handedness of a Weyl spinor and thus of a Majorana particle. Hence we like to show that the RH projector acting on a charge conjugated LH spinor gives $P_R(\psi_L)^C = (\psi_L)^C$, which shows that $(\psi_L)^C$ is indeed a RH spinor.

Proof.

$$
\begin{aligned}
P_R(\psi_L)^C \overset{(B.17)}{=} P_R\left(C\gamma^{0T}\psi_L^*\right) &= \frac{1}{2}\left(\mathbb{1}_4+\gamma^5\right)C\gamma^{0T}\psi_L^* \\
&= \frac{1}{2}\left(C\gamma^{0T}\psi_L^*\right)+\frac{1}{2}\left(\gamma^5 C\gamma^{0T}\psi_L^*\right) \overset{(B.29)}{=} \frac{1}{2}C\left(\mathbb{1}_4+\gamma^{5T}\right)\gamma^{0T}\psi_L^* \\
&\overset{(A.11)}{=} \frac{1}{2}C\gamma^{0T}\left(\mathbb{1}_4-\gamma^{5T}\right)\psi_L^* \overset{(A.13)}{=} C\gamma^{0T}\left(\frac{1}{2}\left(\mathbb{1}_4-\gamma^5\right)\psi_L\right)^* \\
&= C\gamma^{0T}P_L\psi_L^* \overset{(1.26)}{=} C\gamma^{0T}\psi_L^*
\end{aligned}
\tag{B.56}
$$

□

B.3 Flavor Symmetry

In section 1.1 all fundamental particles of the Standard Model have been presented. The different types of quarks and leptons are said to have a specific flavor. This means the different species of fermions possess specific flavor quantum numbers, which are used to describe and distinguish them. As indicated in section 1.1 quarks and leptons come in six flavors each. In this section we want to discuss some flavor quantum numbers and their conservation in interactions.

B.3.1 Quark Flavor

In six flavors of quarks are introduced, which are classified according to EM charge (Q), strangeness (s), charm (c), beauty or bottomness[127] (b), and truth or topness (t). For consistency we might include upness (u) and downness (u). The values of these quantum numbers and the baryon number for particular quarks are shown in table 20. The Baryon number B is defined as

$$\mathrm{B} = \frac{1}{3}\left(n_q - n_{\overline{q}}\right), \tag{B.57}$$

where n_q denotes the number of quarks and $n_{\overline{q}}$ the number of antiquarks. Hence Baryons, which consist of three quarks, have a baryon number B = 1, and antibaryons B = −1. Mesons, which consist of a quark and a antiquark, have baryon number B = 0. All these quantum numbers are displayed in table B.3.1 below for the different quarks.

q	Q	B	u	d	c	s	t	b
u	$^2/_3$	$^1/_3$	1	0	0	0	0	0
d	-$^1/_3$	$^1/_3$	0	-1	0	0	0	0
c	$^2/_3$	$^1/_3$	0	0	1	0	0	0
s	-$^1/_3$	$^1/_3$	0	0	0	-1	0	0
t	$^3/_3$	$^1/_3$	0	0	0	0	1	0
b	$-^1/_3$	$^1/_3$	0	0	0	0	0	-1

Table 20: Quark flavor classification. Adapted from [25, p.47] and [98, p.8].

[127] Sometimes quark flavors are all denoted by capital letters, but then bottomness should be denoted by B' to distinguish it from the baryon number B. Here we are using the notation from [98, p.8].

As mentioned in [25, p.74], "flavor is conserved at strong or EM vertices, but not at weak vertices". But the flavors are said to be conserved approximately, since the weak forces are so weak.

B.3.2 Lepton Number

As noted in [25, p.47], similarly to the classification of quarks, the six flavors of leptons can be classified according to their EM charge (Q) and their family lepton number, i.e. the electronic number (L_e), the muonic number (L_μ), and the tauonic number (L_τ). And again we want to display the value for this quantum numbers for each elementary lepton in the following table 21.

ℓ	Q	L_e	L_μ	L_τ
e	-1	1	0	0
ν_e	0	1	0	0
μ	-1	0	1	0
ν_μ	0	0	1	0
τ	-1	0	0	1
ν_τ	0	0	0	1

Table 21: Lepton classification. Reprinted from [25, p.47].

In addition a total lepton number is defined as

$$L = L_e + L_\mu + L_\tau. \tag{B.58}$$

As discussed in [25, p.74], "the strong forces do not touch leptons at all and in EM interaction the same particle comes out as went in. The weak interaction in the SM (without massive and RH neutrinos) only mix together leptons form the same generation, so the electron, muon and tau numbers are all conserved". There is no similar conservation of generation types for quarks, because due to CKM mixing quark generations are mixed in the weak interaction.

B.4 Gauge Symmetry

B.4.1 Quantum Electrodynamics

First we should consider the simplest case and part of the SM gauge theory, thus we follow [33, p.77f, 262f] in this subsection. In quantum electrodynamics (QED) we want the free Dirac fields ψ, describing all fermions q and ℓ, to couple to the electromagnetic (EM) field in order to describe electromagnetic interaction. This can be formally achieved by *minimal coupling or substitution* . In QED the interaction of the Dirac fields and the four-vector EM potential $A^\mu(x) = (\phi, \vec{A})$ is obtained by introducing the covariant derivative as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \to D_\mu = [\partial_\mu + ieQA_\mu(x)] \, , \tag{B.59}$$

where e is the elementary charge and Q the charge of the fermion field in units of e. Furthermore, we have to add also a kinetic part for the free four-vector potential A_μ to achieve the full QED Lagrangian. This self coupling is described by the Proca Lagrangian (1.9), but without the mass term, which would break the gauge invariance of the Lagrangian. Hence, the Lagrangian of QED for a single fermion[128] can be written as

$$\boxed{\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \left(i\gamma^\mu D_\mu - m \right) \psi}, \tag{B.60}$$

with the electromagnetic field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. And we note that the equation of motion for the single fermion field coupled to the EM potential is given by

$$(i\partial\!\!\!/ - eQA\!\!\!/ - m)\psi = 0. \tag{B.61}$$

Now we want to discuss the gauge invariance of the Lagrangian. According to [25, p.348], the free Dirac Lagrangian (1.17) is invariant under the abelian gauge group[129] $U(1)_{\text{EM}}$. The Dirac fields transform under this global gauge transformations as

$$\begin{aligned} \psi &\to \exp(i\theta)\psi, \\ \overline{\psi} &\to \exp(-i\theta)\overline{\psi}. \end{aligned} \tag{B.62}$$

But if the phase factor $\theta = \theta(x)$ is a function of spacetime x^μ, the Lagrangian will be not invariant under such a local gauge transformation since we pick up an extra term from the derivative of θ. This local gauge transformation can be written as

$$\begin{aligned} \psi &\to \exp(-iQ\lambda(x))\psi, \\ \overline{\psi} &\to \exp(iQ\lambda(x))\overline{\psi}, \end{aligned} \tag{B.63}$$

where it is convenient to introduce the notation $\theta(x) = -Q\lambda(x)$. Hence, to obtain local gauge invariance of the complete Lagrangian, it is necessary to add a coupling to a gauge field A_μ, which transforms under local gauge transformation according to

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \lambda(x). \tag{B.64}$$

This gauge field A_μ is identified with the electromagnetic four-vector potential. As mentioned above, for the full Lagrangian we have to add a free term of this gauge field. This gauge field must be massless ($m_A = 0$), because even though the field strength tensor $F^{\mu\nu}$ is invariant under (B.64), the mass term $A^\nu A_\nu$ will break the gauge symmetry.

Due to Noether's theorem the invariance under the $U(1)_{\text{EM}}$ gauge group leads to a conserved EM current

$$J^\mu_{\text{EM}}(x) = Q\overline{\psi}\gamma^\mu\psi, \tag{B.65}$$

[128] For a QED Lagrangian describing the full particle content of all quarks and leptons we can simply sum over all possible flavors q, ℓ, noticing different charges Q_q and Q_ℓ for different fermion fields.

[129] The subscript EM is just to indicate the correspondence to EM interaction and thus to the EM charge, which generates this group, as we well see in this section.

where Q denotes the charge of the fermion field and the current satisfies indeed

$$\partial_\mu J^\mu_{\text{EM}}(x) = 0. \tag{B.66}$$

This current coupled to the A_μ is exactly the minimal coupling interaction introduced by the covariant derivative. Hence, the interaction part of the Lagrangian can be written as

$$\mathcal{L}^I_{\text{QED}} = Q\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x). \tag{B.67}$$

Thus, the complete Lagrangian is given by

$$\mathcal{L}_{\text{QED}} = \mathcal{L}^0_{\text{QED}} + \mathcal{L}^I_{\text{QED}}, \tag{B.68}$$

where the free part of the Lagrangian is

$$L^0_{\text{QED}} \equiv \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Proca}}, \tag{B.69}$$

with $m_A = 0$. Hence, we achieve the conservation law for the EM charge (operator) $\hat{Q}$,

$$\hat{Q} = \int d^3x J^0_{\text{EM}} = Q\int d^3x\psi^\dagger(x)\psi(x), \tag{B.70}$$

which is the generator of the Lie group $U(1)_{\text{EM}}$ of gauge transformations.

B.4.2 Spontaneous Symmetry Breaking

In this section we want to discuss the procedure of spontaneous symmetry breaking (SSB) following [33, p.281]. The simplest example of a field theory exhibiting spontaneous symmetry breaking is the Goldstone model. Its Lagrangian is given by

$$\mathcal{L}(x) = (\partial^\mu\phi^*)(\partial_\mu\phi) - \mu^2(\phi^*\phi) - \lambda(\phi^*\phi)^2. \tag{B.71}$$

We assume $\lambda > 0$, otherwise the potential would not have any lowest energy state. Furthermore, for our purpose we assume $\mu^2 < 0$, because otherwise the ground state would not be degenerate as shown in figure 22.

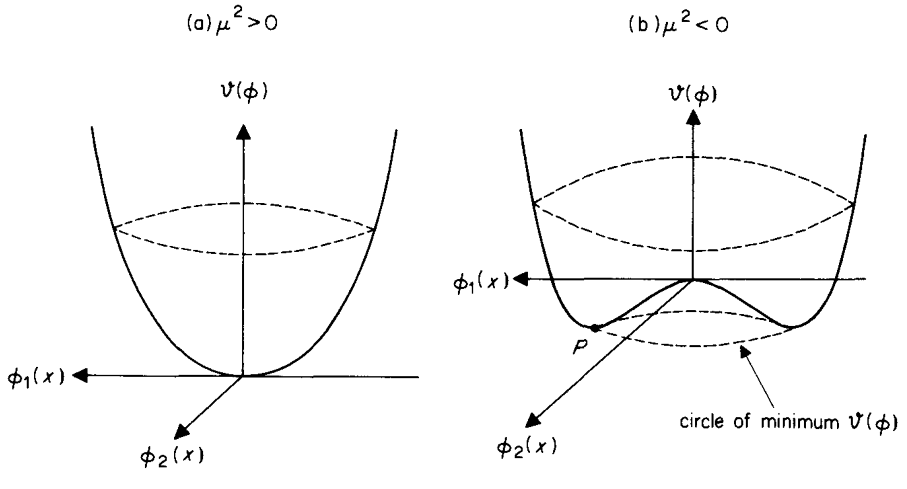


Figure 22: The potential of the scalar field $V(|\Phi|)$. Reprinted from [33, p.282].

Since ϕ is a complex scalar field we can decompose it into its real and imaginary part

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i\phi_2\right), \tag{B.72}$$

and therefore, the Lagrangian can be written as

$$\mathcal{L}_\phi^0 = \frac{1}{2} \left(\partial_\mu \phi_1\right) \left(\partial^\mu \phi_1\right) + \frac{1}{2} \left(\partial_\mu \phi_2\right) \left(\partial^\mu \phi_2\right) - \frac{1}{2}\mu^2 \left(\phi_1^2 + \phi_2^2\right) - \frac{1}{4}\lambda \left(\phi_1^2 + \phi_2^2\right)^2. \tag{B.73}$$

It can be easily seen that the potential has a whole circle of absolute minima at

$$\phi_{\text{min}}(x) = -\frac{\mu^2}{2\lambda} e^{i\theta}, \qquad 0 \leq \theta < 2\pi, \tag{B.74}$$

with a phase angle θ determining the direction in the complex (ϕ_1, ϕ_2)-plane. Obviously the ground state is not unique and the Lagrangian exhibits a $U(1)$ symmetry. Thus, we are free to choose one particular direction, e.g. $\theta = 0$ such that

$$\phi_0 = -\frac{\mu^2}{2\lambda} = \frac{1}{\sqrt{2}} v \quad (> 0). \tag{B.75}$$

is real. Now we introduce two real fields $\sigma(x)$ and $\eta(x)$, which measure the deviations of the field $\phi(x)$ from the chosen vacuum state ϕ_0. Inserting the scalar field in term of these real fields, i.e.

$$\phi(x) = \frac{1}{\sqrt{2}} \left[v + \sigma(x) + i\eta(x)\right], \tag{B.76}$$

into the Lagrangian (1.85), we obtain

$$\mathcal{L}_\phi = \frac{1}{2} \left(\partial_\mu \sigma\right) \left(\partial^\mu \sigma\right) - \frac{1}{2}(2\lambda v^2)\sigma^2 + \frac{1}{2} \left(\partial_\mu \eta\right) \left(\partial^\mu \eta\right) \tag{B.77}$$

$$- \lambda v \sigma \left[\sigma^2 - \eta^2\right] - \frac{1}{2}\lambda \left[\sigma^2 + \eta^2\right]^2 + \frac{\mu^4}{4\lambda^2}. \tag{B.78}$$

Having a closer look at the Lagrangian we find a mass term for the σ field and see that the η field is massless since no term quadratic in η appears. Thus we get

$$m_\sigma = \sqrt{2\lambda v^2}, \tag{B.79}$$

$$m_\eta = 0, \tag{B.80}$$

and η is the predicted Goldstone boson.

B.4.3 Higgs Mechanism

For an example of the procedure of the Higgs mechanism we follow again [33, p.284f]. The Lagrangian (B.71) can be made invariant under $U(1)$ gauge transformations in a similar way as done before for the free Dirac Lagrangian (B.60). A gauge field $A_\mu(x)$ is introduced by inserting the covariant derivative

$$D_\mu = \phi \left[\partial_\mu + iqA_\mu(x)\right] \phi(x). \tag{B.81}$$

Again we have to add the term for the free gauge field to the Lagrangian

$$-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x), \tag{B.82}$$

where

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x). \tag{B.83}$$

Thus, we achieve the Lagrangian

$$\mathcal{L} = (D^\mu\phi)^* (D_\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{B.84}$$

which defines the Higgs model. This Lagrangian is invariant under the $U(1)$ gauge transformations

$$\phi(x) \to \phi(x)e^{-iqf(x)}, \tag{B.85}$$

$$\phi^*(x) \to \phi^*(x)e^{iqf(x)}, \tag{B.86}$$

$$A_\mu(x) \to A_\mu(x) + \partial_\mu f(x). \tag{B.87}$$

We follow the procedure of SSB shown in the previous chapter and obtain a circle of minima and choose ϕ_0 as in (B.75) for the vacuum state[130]. We define the real fields σ and η like in (B.76). Hence, the Lagrangian is written as

$$\begin{aligned}\mathcal{L}_\Phi =& \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}(2\lambda v^2)\sigma^2 \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(qv)^2 A_\mu A^\mu \\ &+ \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) \\ &+ qvA^\mu\partial_\mu\eta + \text{interaction terms.}\end{aligned} \tag{B.88}$$

Here the interaction terms, which are cubic and quartic in the fields, and an insignificant constant term has been skipped. Nevertheless, it is not obvious that the second and third lines describes a massive vector boson and a massless scalar boson, because the product term of A_μ and η indicates that they are not independent normal coordinates. An analysis of the degrees of freedom[131] of equation (B.84) and (B.88) leads to the insight that the second Lagrangian must contain an unphysical field, which does not represent real particles. It can be eliminated by a particular choice of gauge - called *unitary gauge,* where ϕ is transformed into a real field of the form

$$\boxed{\phi(x) = \frac{1}{\sqrt{2}}(v + \sigma(x))}. \tag{B.89}$$

[130] Note that the vector field A_μ must vanish for the vacuum.

[131] (B.84) has 4 degrees of freedom; two from the complex scalar field and two from the real massless vector field. Whereas (B.88) has five, the two real scalar fields give one each and the real massive vector field gives three.

Inserting this in the Lagrangian (B.88) we obtain

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}\left(\partial_\mu \sigma\right)\left(\partial^\mu \sigma\right) - \frac{1}{2}(2\lambda v^2)\sigma^2 \\ &- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(qv)^2 A_\mu A^\mu \\ &- \lambda v \sigma^3 - \frac{1}{4}\lambda\sigma^4 + \frac{1}{2}q^2 A_\mu A^\mu \left(2v\sigma + \sigma^2\right).\end{aligned} \tag{B.90}$$

The first line of this equation describes a real Klein-Gordon field with mass $\sqrt{2\lambda v^2}$ and the second a neutral vector boson with mass $|qv|$. Now the number of degrees of freedom is four as in equation (B.84) and one of the two degrees of freedom of the complex scalar field has been *eaten up* by the vector field A_μ which has become massive in this process. The remaining degree of freedom form ϕ shows up as the real scalar field σ, which is called Higgs boson or Higgs scalar.

B.4.4 Weak Isospin and Hypercharge

To develop a gauge theory of weak interactions it is necessary and helpful to introduce two important quantities, called *weak isospin* and *weak hypercharge*. In doing so we might follow [33, p.264ff] and [96, p.292ff] and use the same systematics as for QED in subsection B.4.1. We want to find a set of gauge transformations which leaves the free-lepton Lagrangian invariant and leads to conservation of the weak currents $J_\mu(x)$ and $J^\dagger_\mu(x)$ in equation (1.52) and (1.53). We start at the free-lepton Lagrangian for massless leptons given by

$$\mathcal{L}^0_{\mathrm{EW}} = i\left[\overline{\psi}_\ell(x)\partial\!\!\!/\psi_\ell(x) + \overline{\psi}_{\nu_\ell}(x)\partial\!\!\!/\psi_{\nu_\ell}(x)\right], \tag{B.91}$$

where $\ell = e, \mu, \tau$ is understood as summation over all lepton flavors.

We rewrite this Lagrangian taking into account that only LH leptons enter the interaction given in (1.54). Hence, we use the chiral projection operators (1.23) to split the Dirac spinors in its RH and LH components like in (1.26). Thus, we obtain

$$\mathcal{L}^0_{\mathrm{EW}} = i\left[\overline{\Psi}^L_\ell(x)\partial\!\!\!/\Psi^L_\ell(x) + \overline{\psi}^R_\ell(x)\partial\!\!\!/\psi^R_\ell(x) + \overline{\psi}^R_{\nu_\ell}(x)\partial\!\!\!/\psi^R_{\nu_\ell}(x)\right], \tag{B.92}$$

where we combined the LH fields ψ^L_ℓ and $\psi^L_{\nu_\ell}$ into a two-component field [132]

$$\Psi^L_\ell = \begin{pmatrix}\psi^L_{\nu_\ell} \\ \psi^L_\ell\end{pmatrix}, \tag{B.93}$$

and correspondingly

$$\overline{\Psi}^L_\ell = \left(\overline{\psi}^L_{\nu_\ell}, \overline{\psi}^L_\ell\right). \tag{B.94}$$

We find that the bilinear terms formed by the two-component spinor field are invariant under $SU(2)$ transformations. This gauge group is generated by three 2×2 Hermitian matrices, i.e. $^1/_2$ the Pauli spin matrices τ_k given in (A.35).

[132] We shall see shortly, why it is useful to write the Lagrangian in such a unsymmetrical way.

They satisfy the commutation relation

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k, \tag{B.95}$$

where ε_{ijk} is the usual completely antisymmetric tensor. Any transformation of $SU(2)$ can be represented by a unitary matrix

$$U(\boldsymbol{\alpha}) \equiv \exp\left(\frac{i\alpha_j\tau_j}{2}\right), \tag{B.96}$$

for any three real numbers $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. Hence, for the transformation of the two-component field we get

$$\Psi_\ell^L(x) \to U(\boldsymbol{\alpha})\Psi_\ell^L(x) \equiv \exp\left(\frac{i\alpha_j\tau_j}{2}\right)\Psi_\ell^L(x), \tag{B.97}$$

$$\overline{\Psi}_\ell^L(x) \to \overline{\Psi}_\ell^L(x)U^\dagger(\boldsymbol{\alpha}) \equiv \overline{\Psi}_\ell^L(x)\exp\left(\frac{-i\alpha_j\tau_j}{2}\right), \tag{B.98}$$

which leaves the bilinear term of Ψ_ℓ^L in (B.92) invariant. This two-component field is also called *weak isospinor*, due to its behaviour under the $SU(2)$ transformation. Thus, we might classify quantities according to their transformation property under $SU(2)$. Furthermore, we find that the RH lepton fields are weak isoscalars, i.e. they are invariant under $SU(2)$ transformations:

$$\psi_\ell^R(x) \to \psi_\ell^R(x), \qquad \psi_{\nu_\ell}^R(x) \to \psi_{\nu_\ell}^R(x), \tag{B.99}$$

$$\overline{\psi}_\ell^R(x) \to \overline{\psi}_\ell^R(x), \qquad \overline{\psi}_{\nu_\ell}^R(x) \to \overline{\psi}_{\nu_\ell}^R(x). \tag{B.100}$$

The full Lagrangian (B.92) remains unchanged, i.e. it is invariant, under these $SU(2)$ transformations. Hence, we obtain three conserved currents

$$J_i^\mu(x) = \frac{1}{2}\overline{\Psi}_\ell^L(x)\gamma^\mu\tau_i\Psi_\ell^L(x), \qquad i = 1, 2, 3, \tag{B.101}$$

which are called weak isospin currents and we also have three conserved quantities

$$T_i^W = \int d^3x J_i^0(x) = \frac{1}{2}d^3s\overline{\Psi}_\ell^{L^\dagger}(x)\tau_i\Psi_\ell^L(x), \qquad i = 1, 2, 3, \tag{B.102}$$

which are called the weak isospin charges and generate an $SU(2)_L$ algebra[133]

$$[T_i, T_j] = i\varepsilon^{ijk}T_k. \tag{B.103}$$

According to [33, p.266], "the leptonic currents $J^\mu(x)$ and $J^{\mu\dagger}(x)$ in (1.52) and (1.53) can be written as linear combinations of the conserved weak isospin currents $J_1^\mu(x)$ and $J_2^\mu(x)$" as

$$J^\mu(x) = 2\left[J_1^\mu - iJ_2^\mu(x)\right] \tag{B.104}$$

$$J^{\mu\dagger}(x) = 2\left[J_1^\mu + iJ_2^\mu(x)\right]. \tag{B.105}$$

[133] The subscript L is to remind us that the weak isospin current couples only LH fermions.

But besides, there is still a third conserved current, given by

$$J_3^\mu(x) = \frac{1}{2}\left[\overline{\psi}_{\nu_\ell}^L(x)\gamma^\mu\overline{\psi}_{\nu_\ell}^L(x) - \overline{\psi}_\ell^L(x)\gamma^\mu\overline{\psi}_\ell^L(x)\right]. \tag{B.106}$$

Since this current only couples electrically neutral leptons, or electrically charged leptons, this current is called a *neutral current*. The currents $J^\mu(x)$ and $J^{\mu\dagger}$ are charged currents, since they couple electrically neutral with electrically charged leptons. Hence, we can define the weak hypercharge current $J_Y^\mu(x)$ by

$$\frac{1}{2}J_Y^\mu(x) = \frac{1}{e}J_{\mathrm{EM}}^\mu(x) - J_3^\mu(x). \tag{B.107}$$

Following [33, p.267], "the corresponding charge

$$Y = \int d^3x J_Y^0(x) \tag{B.108}$$

is called the *weak hypercharge*. We see from (B.107) that Y is related to the electric charge Q_{EM} and the weak isocharge T_3^W by the Gell-Mann-Nishijima formula[134]

$$\boxed{\frac{Y}{2} = Q - T_3^W}. \tag{B.109}$$

The conservation of the electric charge Q_{EM} and of the weak isocharge T_3^W implies conservation of the weak hypercharge Y". But its conservation also follows from the invariance of the Lagrangian under $U(1)_Y$ transformations

$$U(\beta) = \exp\left(i\beta\frac{Y}{2}\right), \tag{B.110}$$

since the lepton fields transform as

$$\Psi_\ell^L(x) \to e^{\left(i\beta\frac{Y}{2}\right)}\Psi_\ell^L(x), \qquad \overline{\Psi}_\ell^L(x) \to \overline{\Psi}_\ell^L(x)e^{\left(-i\beta\frac{Y}{2}\right)}, \tag{B.111}$$

$$\psi_\ell^R(x) \to e^{\left(i\beta\frac{Y}{2}\right)}\psi_\ell^R(x), \qquad \overline{\psi}_\ell^R(x) \to \overline{\psi}_\ell^R(x)e^{\left(-i\beta\frac{Y}{2}\right)}, \tag{B.112}$$

$$\psi_{\nu_\ell}^R(x) \to e^{\left(i\beta\frac{Y}{2}\right)}\psi_{\nu_\ell}^R(x), \qquad \overline{\psi}_{\nu_\ell}^R(x) \to \overline{\psi}_{\nu_\ell}^R(x)e^{\left(-i\beta\frac{Y}{2}\right)}. \tag{B.113}$$

In order to determine the weak hypercharge, we need the values of the third component of the weak isospin T_3 for each field. As we have seen so far and as it is mentioned in [99, p.151], in the theory of weak interactions the leptons (and quarks) are divided into isospinors and isoscalars. The isospinors are formed by LH isodoublets, which are assigned with values $T = {}^1\!/_2$ and $T_3 = \pm{}^1\!/_2$ of the isospin and its third component, whereas the isoscalars are formed by RH isosinglets, which are assigned with values $T = 0$ and $T_3 = 0$. Now we can calculate the values for the weak hypercharge easily via the Gell-Mann-Nishijima formula (B.109). In the following table 23 all values of Q, T, T_3, Y for all fermion fields are summarized. There, u and d refer to up-type quarks (u, c, t) and down-type quarks (d, s, b).

[134]Note that some authors use a different scale for Y like [33], where $Y = Q - T_3^W$.

Fermion	Q	T	T_3	Y
ν_ℓ^L	0	$^1/_2$	$^1/_2$	-1
ℓ^L	-1	$^1/_2$	$-^1/_2$	-1
ℓ^R	-1	0	0	-2
u^L	$^2/_3$	$^1/_2$	$^1/_2$	$^1/_3$
d^L	$-^1/_3$	$^1/_2$	$-^1/_2$	$^1/_3$
u^R	$^2/_3$	0	0	$^4/_3$
d^R	$-^1/_3$	0	0	$-^2/_3$

Table 23: Weak isospin and hypercharge quantum numbers of fermions. Reprinted from [96, p.295].

C Appendix - Lorentz Transformations

According to [26, p.16], "the Lagrangian formulation of field theory is particularly suited to relativistic dynamics because all expressions are explicitly Lorentz invariant". Hence, it is useful to remember some facts on Lorentz transformations and four-vector notation, which will be done in this appendix.

C.1 Lorentz Transformation and Four-Vectors

The following can be found for example in [100, p.293, 312] and [101, p.12ff].

A four-vector x with $ct = x^0$ is given by

$$x \equiv (x^\mu) = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}. \tag{C.1}$$

The Minkowski metric tensor[135] g for the Minkowski spacetime $\mathcal{M}_4$

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad g^{-1} = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{C.2}$$

with $g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$, where $\delta^\mu{}_\rho$ denotes the Kronecker delta[136]. Using the metric we can achieve covariant four-vectors from contravariant four-vectors and express the scalar product of four-vectors as

$$x_\mu = g_{\mu\nu} x^\nu = \begin{cases} x^0 & \text{for } \mu = 0, \\ -x^\mu & \text{for } \mu = 1, 2, 3, \end{cases} \tag{C.3}$$

$$(x, y) = x \cdot y = g_{\mu\nu} x^\mu y^\nu = x^\mu y_\mu = x_\mu y^\mu. \tag{C.4}$$

In order to transform a four-vector from one inertial reference frame to another, we can make the following ansatz without loss of generality:

$$x'^\mu = a^\mu f^\mu(x), \tag{C.5}$$

$$f^\mu(x = 0) = 0, \tag{C.6}$$

where the $f^\mu(x)$ are yet undetermined functions of x and $a = (a^\mu)$ is a constant four-vector. It is easy to see that because of homogeneity of time and a Euclidean structure of space, $f^\mu(x)$ must be homogeneous, linear functions. Hence, we achieve

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{C.7}$$

[135] Note that other authors might use a different convention. In this thesis the *Bjorken and Drell* or *mostly minus* convention is used. The other common convention is the *Pauli* or *mostly plus* metric, where $g_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ [36, p.518].

[136] This is valid, however only for a Cartesian basis and not for the general case.

where the constants Λ^{μ}_{ν} can be written in the compact form of a 4×4 matrix $\Lambda = (\Lambda^{\mu}{}_{\nu})$. In [93, p.13f] it is noted that

$$x'^{\mu} x'_{\mu} = \Lambda^{\mu}{}_{\nu} \Lambda^{\alpha}{}_{\mu} x^{\nu} x_{\alpha} := x^{\mu} x_{\mu}, \tag{C.8}$$

which implies

$$\Lambda^{\mu}{}_{\nu} \Lambda_{\mu}{}^{\alpha} = \Lambda^{T\alpha}{}_{\mu} \Lambda^{\mu}{}_{\nu} = \delta^{\alpha}{}_{\nu} = g^{\alpha}{}_{\nu}. \tag{C.9}$$

This shows that

$$\Lambda^{T\alpha}{}_{\mu} = \Lambda^{-1\alpha}{}_{\mu}, \tag{C.10}$$

and hence, Λ is an orthogonal transformation, i.e.

$$(\Lambda^{-1})_{\alpha\mu} = (\Lambda^{T})_{\alpha\mu} = \Lambda_{\alpha\mu}. \tag{C.11}$$

Therefore, we can rewrite (C.9) as

$$(\Lambda^{T})_{\alpha\mu} \Lambda^{\mu}_{\nu} = (\Lambda^{T})_{\alpha\mu} g^{\mu\beta} \Lambda_{\beta\nu} = g_{\alpha\nu}, \tag{C.12}$$

or in matrix notation as

$$\boxed{\Lambda^{T} g \Lambda = g}. \tag{C.13}$$

This relation (C.13) is often used as the definition of a Lorentz transformation Λ. Furthermore, we know

$$\Lambda^{T} = \Lambda^{-1}, \tag{C.14}$$

and hence, we can also write $\Lambda g \Lambda^{T} = g$ and the metric can be written as

$$g_{\alpha\beta} = g_{\mu\nu} \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} = \Lambda_{\nu\alpha} \Lambda^{\nu}_{\beta} = \Lambda^{\nu}_{\alpha} \Lambda_{\beta\nu}. \tag{C.15}$$

C.2 Lorentz Group

In this subsection we will follow [101, p.15ff] and [102, p.49-52, 143ff].

The transformations that correspond to a transition from one inertial frame to another, as seen above, form a group known as the *inhomogeneous Lorentz group* or *Poincaré group*[137]. We will call these group elements (Λ, a) and they are defined by

$$(\Lambda, a) : x^{\mu} \longmapsto x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}, \tag{C.16}$$

with $a = (a^{\mu})$ being an arbitrary constant four-vector and Λ satisfying (C.13). So the group is given as the following set:

$$\mathscr{L} = \{\Lambda \in \mathcal{M}_4 | \Lambda^{T} g \Lambda = g\} \approx \mathcal{O}(1,3). \tag{C.17}$$

As mentioned in [101, p.16], "the transformations, which leave the coordinate origin unal-

[137] It is indeed a group, since it fulfils all four group conditions.

tered likewise form a subgroup of the Poincaré group, which is called the *(homogeneous) Lorentz group*. A general element of this group has the form $(\Lambda, 0)$", i.e.

$$(\Lambda, 0) : x^\mu \longmapsto x'^\mu = \Lambda^\mu_\nu x^\nu. \tag{C.18}$$

From (C.13) it follows that Lorentz transformations fulfil

$$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1, \tag{C.19}$$

$$(\Lambda^0_0)^2 \geq 1 \Rightarrow \Lambda^0_0 \geq 1 \text{ or } \Lambda^0_0 \leq -1. \tag{C.20}$$

Hence we can introduce the following notation for the special subgroups

$$\mathscr{L}_+ = \{\Lambda \in \mathscr{L} | \det \Lambda = 1\}, \tag{C.21}$$

$$\mathscr{L}^\uparrow = \{\Lambda \in \mathscr{L} | \Lambda^0_0 \geq 1\}, \tag{C.22}$$

where $\mathscr{L}_+$ is called *proper Lorentz group* and $\mathscr{L}^\uparrow$ *orthochronous Lorentz group*. As we can see the group is not connected, and decomposes into four separated connected components:

$$\mathscr{L}^\uparrow_+ = \{\Lambda \in \mathscr{L} | \det \Lambda = 1,\ \Lambda^0_0 \geq 1\} \approx \mathrm{SO}(1,3), \tag{C.23}$$

$$\mathscr{L}^\uparrow_- = \{\Lambda \in \mathscr{L} | \det \Lambda = -1,\ \Lambda^0_0 \geq 1\} \approx \mathcal{P}\mathscr{L}^\uparrow_+, \tag{C.24}$$

$$\mathscr{L}^\downarrow_+ = \{\Lambda \in \mathscr{L} | \det \Lambda = 1,\ \Lambda^0_0 \leq -1\} \approx \mathcal{T}\mathscr{L}^\uparrow_+, \tag{C.25}$$

$$\mathscr{L}^\downarrow_- = \{\Lambda \in \mathscr{L} | \det \Lambda = -1,\ \Lambda^0_0 \leq -1\} \approx \mathcal{P}\mathcal{T}\mathscr{L}^\uparrow_+, \tag{C.26}$$

where $\mathscr{L}^\uparrow_+ = \mathscr{L}_+ \cap \mathscr{L}^\uparrow$ is called *proper orthochronous Lorentz group.* The symmetry operators are defined as

$$\mathcal{T} = \mathrm{diag}(-1, 1, 1, 1) \in \mathscr{L} \quad (\text{time reversal}), \tag{C.27}$$

$$\mathcal{P} = \mathrm{diag}(1, -1, -1, -1) \in \mathscr{L} \quad (\text{spatial parity}). \tag{C.28}$$

C.3 Representations of the Lorentz Group

As mentioned in [102, p.134], "for a systematic investigation of all possible representations it is, however, necessary to study the Lorentz group itself more closely, since the structure of the group has implications on the structure of the set of all its representations". In the previous section we learned that Lorentz transformations are homogeneous transformations satisfying the (pseudo)orthogonality relations (C.13). Since we know $g_{\mu\nu} = g_{\nu\mu}$, we have ten relations, which restrict the sixteen matrix elements of Λ. Furthermore, we can only choose six matrix elements independently, because those relations are independent from each other. It can be shown that the Lorentz Group is also a Lie group [102, p.136].

Now we will follow the discussion on this done by [103]. Let us denote the infinitesimal generators of the Lorentz group $O(1,3)$ by $J^{\mu\nu}$. Their commutation relation

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}) \tag{C.29}$$

defines the associated algebra. Hence, an arbitrary element Λ of the Lorentz group can be represented by

$$\Lambda = \exp\left(-\frac{i\omega_{\mu\nu}J^{\mu\nu}}{2}\right), \tag{C.30}$$

where $\omega_{\mu\nu}$ denote the parameters of the transformation. According to [103], "the Lorentz group has both finite-dimensional and infinite-dimensional representations. However, it is non-compact, therefore its finite-dimensional representations are not unitary (the generators are not Hermitian). The generators of the finite-dimensional representation can be chosen to be Hermitian".

It is sufficient for our purpose to restrict ourselves to the finite-dimensional representations of the restricted Lorentz group $SO(1,3)$. "These representations act on finite-dimensional vector spaces (the base space). Elements of these vector spaces are said to transform according to the given representation", as noted in [103].

C.3.1 Trivial Representation

The one-component objects, which transform according to this representation, denoted by (0,0), are called *Lorentz scalars*. The transformation is given by

$$\phi \xrightarrow{\Lambda} \Lambda_S \cdot \phi = \phi, \tag{C.31}$$

with $\Lambda_S = \exp\left(\frac{-i\omega_{\mu\nu}J^{\mu\nu}}{2}\right) = 1$ where $J^{\mu\nu} = 0$. Hence, the matrices representing the Lorentz transformation acting on this one-dimensional vector space $\mathbb{R}$ are simply 1×1 matrices. For scalar fields $\phi(x)$ the transformation is analogous, apart from the necessity for a transformation of the spacetime variable x, i.e.

$$\phi(x) \rightarrow \phi'(x') = \phi(\Lambda^{-1}x). \tag{C.32}$$

C.3.2 Vector Representation

This representation is denoted by $(\frac{1}{2}, \frac{1}{2})$ and the generators $\mathcal{J}^{\mu\nu}$ are 4×4 matrices given by

$$(\mathcal{J}^{\mu\nu})^{\rho}_{\sigma} = i(g^{\mu\rho}\delta^{\nu}_{\sigma} - g^{\nu\rho}\delta^{\mu}_{\sigma}). \tag{C.33}$$

Hence, this representation is the fundamental representation of the Lorentz group, because it can be shown that the elements of $SO(1,3)$ are exactly those matrices $\mathcal{J}^{\mu\nu}$. Therefore, the transformation of the 4-component objects in $\mathbb{R}^4$, called *Lorentz four-vectors*, is given by

$$V^{\rho} \xrightarrow{\Lambda} (\Lambda_V)^{\rho}_{\sigma} V^{\rho}, \tag{C.34}$$

where $(\Lambda_V)^{\rho}_{\sigma} = \exp\left(\frac{-i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}{2}\right)^{\rho}_{\sigma}$ is the usual Lorentz transformation matrix.

Again one can express the transformation for vector fields analogous, but like in the scalar case a transformation of the spacetime variable is necessary. Thus, we get

$$V^{\rho}(x) \xrightarrow{\Lambda} (\Lambda_V)^{\rho}_{\sigma} V^{\rho}(\Lambda^{-1}x). \tag{C.35}$$

C.3.3 Spinoraial Representation

It can be shown that the proper orthochronous Lorentz group $SO(1,3)$ is the double cover of the $SL(2,\mathbb{C})$, i.e.

$$\mathcal{L}_+^\uparrow \cong SL(2,\mathbb{C})/_{\{\mathbb{1}_2,-\mathbb{1}_2\}}. \tag{C.36}$$

Hence we can construct two two-dimensional representations of the $SO(1,3)$. On the one hand we have the fundamental representation of $SL(2,\mathbb{C})$, which we will denote by $(\frac{1}{2},0)$. The Lorentz transformations act on the complex two-component objects ψ, called *LH Weyl spinors*, according to

$$\psi^\alpha \stackrel{\Lambda}{\to} (\Lambda_L)_{\alpha\beta}\psi_\beta, \tag{C.37}$$

where $\Lambda_L = \exp\left(\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{2}\right)$. The generators $\sigma^{\mu\nu} = J^{\mu\nu}$ are 2×2 matrices such that in this representation rotation and boost generators (in the Weyl basis) appear as

$$\sigma^{ij} \equiv \frac{1}{2}\varepsilon^{ijk}\sigma^k, \tag{C.38}$$

$$\text{and} \quad \sigma^{0i} \equiv -\frac{i}{2}\sigma^i. \tag{C.39}$$

On the other hand we can construct a second spinorial representation, i.e. the contragradient reperesenation of $SL(2,\mathbb{C})$, denoted by $(0,\frac{1}{2})$. This representation acts on so-called *RH Weyl spinors*, according to

$$\psi^\alpha \stackrel{\Lambda}{\to} (\Lambda_R)_{\alpha\beta}\psi_\beta, \tag{C.40}$$

where $\Lambda_R = \exp\left(\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{2}\right)$. The rotation and boost generators (in the Weyl basis) are now

$$\sigma^{ij} \equiv \frac{1}{2}\varepsilon^{ijk}\sigma^k \quad \text{again}, \tag{C.41}$$

$$\text{but now} \quad \sigma^{0i} \equiv \frac{i}{2}\sigma^i. \tag{C.42}$$

According to [103], "by taking the direct sum $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ of the two representations, we obtain a 4-dimensional (reducible) representation of the Lorentz group which acts upon complex four-component objects called *Dirac spinors*". Those Dirac spinors transform according to

$$\Psi_a \stackrel{\Lambda}{\to} (\Lambda_D)_{ab}\Psi_b\ , \tag{C.43}$$

where $\Lambda_D = \exp\left(\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{4}\right)$. Now the generators $\sigma^{\mu\nu}$ are 4×4 matrices[138,139] given by

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu], \tag{C.44}$$

and satisfy

$$\sigma^{\mu\nu\dagger} = \gamma^0\sigma^{\mu\nu}\gamma^0. \tag{C.45}$$

[138] Note that in other literature a different convention might be used, where for Λ a factor of $^1/_4$ is introduced, because $\sigma^{\mu\nu}$ is defined with a factor of $^1/_2$

[139] Those are the remaining six basis elements of the four-dimensional Clifford algebra. All together, the basis has dimension 16, $\Gamma = \{\mathbb{1}_4, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$ [28, p.91f].

Here the generators for rotations and boots[140] are actually the direct sum of the 2×2 matrices for the left-handed and right-handed Weyl spinors:

$$\sigma^{ij} = \frac{1}{2}\varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \varepsilon^{ijk}\Sigma^k, \tag{C.46}$$

$$\text{and} \quad \sigma^{0i} = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \quad \text{for gamma matrices in Weyl basis,} \tag{C.47}$$

$$\text{resp.} \quad \sigma^{0i} = \frac{i}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{for gamma matrices in Dirac basis.} \tag{C.48}$$

Thus, as noted in [34, p.VI.5], a Dirac spinor (also called bispinor) transforms like $\binom{\psi_L}{\psi_R}$ where ψ_L and. ψ_R are LH or RH Weyl spinors respectively for gamma matrices in the Weyl basis.

In [34, p.VI.5] a more detailed discussion is given, where the the coefficients $\omega^{\mu\nu}$ are given explicitly. As before it is recognized that the spinors in proper orthochronous Lorentz group transform according to double cover of the $SL(2,\mathbb{C})$. Let be $A \in SL(2,\mathbb{C})$, then a Dirac spinor in the Weyl basis transforms according to

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} A\psi_L \\ A^{-1\dagger}\psi_R \end{pmatrix} = \Lambda_D \psi, \tag{C.49}$$

where Λ_D is given by

$$\Lambda_D = \begin{pmatrix} A & 0 \\ 0 & A^{-1\dagger} \end{pmatrix}, \tag{C.50}$$

and A denotes a matrix of the fundamental representation and $A^{-1\dagger}$ one of the contragradient representation of the $SL(2,\mathbb{C})$ resp. of the Lorentz group[141].

Hence, we see that the spinorial representation is reducible and that the two blocks correspond to the different transformation behaviour of the LH and RH Weyl spinors, where the LH ones transform according to the fundamental representation of the $SL(2,\mathbb{C})$ and the RH according to the contragradient representation.

They can be written as

$$A = \exp\left(-i\frac{(\vec{\alpha} - i\vec{u}) \cdot \vec{\sigma}}{2}\right), \qquad A^{-1\dagger} = \exp\left(i\frac{(\vec{\alpha} - i\vec{u}) \cdot \vec{\sigma}^T}{2}\right), \tag{C.51}$$

where $\vec{u}, \vec{\alpha} \in \mathbb{R}^3$ denoting the parameters for boosts and rotations of the Lorentz transformation induced by the matrices A resp. $A^{-1\dagger}$. Comparing this to the general results for Λ_L and Λ_R given before, we now found the coefficients in an explicit form.

[140] The rotation generator σ^{ij} is just the spinor transformation matrix in the two-dimensional representation of the rotation group replicated twice. The boost generators σ^{0i} are not Hermitian, and thus our implementation of boots ist not unitary (this was also true for the vector representation). In fact the Lorentz group, being noncompact , has no faithful, finite-dimensional representations that are unitary [26, p.41] .

[141] These two representations are equivalent, while the fundamental and the complex conjugated representation are not (see appendix E.1.2).

For an arbitrary basis of Dirac matrices Λ_D is given as before by

$$\Lambda_D = \exp\left(-i\frac{\sigma_{\mu\nu}\omega^{\mu\nu}}{4}\right), \tag{C.52}$$

with

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \tag{C.53}$$

The generators $\omega^{\mu\nu}$ can now be written down explicitly by

$$\omega^{\mu\nu} = \begin{pmatrix} 0 & -u_1 & -u_2 & -u_3 \\ u_1 & 0 & \alpha_3 & -\alpha_2 \\ u_2 & -\alpha_3 & 0 & \alpha_1 \\ u_3 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}, \tag{C.54}$$

with $\vec{u}, \vec{\alpha} \in \mathbb{R}^3$ again denoting the parameters for boosts and rotations.

Finally we recognize, according to [33, p.335] that the nonsingular 4×4 matrix Λ_D fulfils

$$\gamma^\nu = \Lambda^\nu_\mu \Lambda_D \gamma^\mu \Lambda_D^{-1} \tag{C.55}$$

and

$$\Lambda_D^{-1} = \gamma^0 \Lambda_D^\dagger \gamma^0. \tag{C.56}$$

The covariance of the Dirac equation (1.19) is established by (C.43). Furthermore, we can obtain the transformation behaviour of the adjoint spinor via (C.56) and (C.43) and it is given by

$$\overline{\Psi}(x) \to \overline{\Psi}'(x') = \overline{\Psi}(x)\Lambda_D^{-1}. \tag{C.57}$$

C.4 Bilinear Lorentz Covariants - Dirac Bilinears

According to [33, p.335f] we can obtain from (C.43), (C.55) and (C.57) five basic bilinear covariants of the Dirac theory. Under a Lorentz transformation

$$\left.\begin{matrix} \overline{\psi}\psi \\ \overline{\psi}\gamma^\mu\psi \\ \overline{\psi}\sigma^{\mu\nu}\psi \\ \overline{\psi}\gamma^5\gamma^\mu\psi \\ \overline{\psi}\gamma^5\psi \end{matrix}\right\} \text{transforms as a} \begin{cases} \text{scalar,} \\ \text{vector,} \\ \text{antisymmetric second} - \text{rank tensor,} \\ \text{pseudo (or axial)} - \text{vector,} \\ \text{pseudo} - \text{scalar.} \end{cases} \tag{C.58}$$

"The terms pseudo-vector and pseudo-scalar arise from the fact that these quantities transform as a vector and scalar respectively under continuous Lorentz transformations, but with an additional sign change under parity transformation" [26, p.49].

Now we want to prove what is claimed above. For this purpose we will follow [31, p.88f,95] for the rest of this section and quote what has been shown there.[31]

Claim C.4.1: $\overline{\psi}\psi$ is a Lorentz scalar, i.e. it transforms under a Lorentz transformation like a scalar field (C.32).

Proof. In order to investigate how this term behaves under a Lorentz transformation, we use the spinorial representation of the Lorentz transformations derived in section C.3.3 as well as the fact that

$$\Lambda_D^\dagger = \exp\left(\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu\dagger}\right) = \gamma^0\Lambda_D^{-1}\gamma^0, \tag{C.59}$$

because of (C.45). So we can conclude the following:

$$\begin{aligned}\overline{\psi}(x)\psi(x) &= \psi^\dagger(x)\gamma^0\psi(x) \\ &\to \psi^\dagger(\Lambda^{-1}x)\Lambda_D^\dagger\gamma^0\Lambda_D\psi(\Lambda^{-1}x) \\ &\overset{(C.59)}{=} \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x)\psi \\ &= \overline{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x).\end{aligned} \tag{C.60}$$

Hence, the term is a Lorentz scalar, because it transforms like (C.32). □

Claim C.4.2: $\overline{\psi}\gamma^\mu\psi$ is a Lorentz vector, which means that[142]

$$\overline{\psi}(x)\gamma^\mu\psi(x) \to \Lambda^\mu_\nu\overline{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x). \tag{C.61}$$

Proof. Suppressing the x argument, under a Lorentz transformation we have,

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}\Lambda_D^{-1}\gamma^\mu\Lambda_D\psi. \tag{C.62}$$

If $\overline{\psi}\gamma^\mu\psi$ is to transform as a vector, we must have

$$\Lambda_D^{-1}\gamma^\mu\Lambda_D = \Lambda^\mu_\nu\gamma^\nu. \tag{C.63}$$

To show this we will work infinitesimally, so that[143]

$$\Lambda = \exp\left(\frac{1}{2}\omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \approx 1 + \frac{1}{2}\omega_{\rho\sigma}\mathcal{M}^{\rho\sigma} + \dots, \tag{C.64}$$

$$\Lambda_D = \exp\left(\frac{1}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma}\right) \approx 1 + \frac{1}{2}\omega_{\rho\sigma}\sigma^{\rho\sigma} + \dots. \tag{C.65}$$

So the requirement (C.63) becomes

$$-[\sigma^{\rho\sigma}, \gamma^\mu] = (\mathcal{M}^{\rho\sigma})^\mu_\nu\gamma^\nu, \tag{C.66}$$

where we have suppressed the α, β indices on γ^μ and $\sigma^{\rho\sigma}$, but otherwise left all other indices explicit. In fact equation (C.66) follows from

$$[\sigma^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \tag{C.67}$$

which is has been proven in [31, p.84]. To see this, we write the right-hand side of (C.66)

[142] This equation means that we can treat the $\mu = 0, 1, 2, 3$ index on the γ^μ as a true vector index. In particular we can form Lorentz scalars by contracting it with other Lorentz indices.

[143] $\mathcal{M}^{\rho\sigma\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta\sigma\mu\eta^{\rho\nu} = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu$ denote the six antisymmetric 4×4 matrices which are called the generators of the Lorentz transformations [31, p.82].

by expanding $\mathcal{M}$,

$$(\mathcal{M}^{\rho\sigma})^{\mu}_{\nu}\gamma^{\nu} = (\eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu})\gamma^{\nu} = \eta^{\rho\mu}\gamma^{\mu} - \eta^{\sigma\mu}\gamma^{\rho}, \tag{C.68}$$

which means that the proof follows, if we can show

$$-[\sigma^{\rho\sigma}, \gamma^{\mu}] = \eta^{\rho\mu}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho}, \tag{C.69}$$

which is exactly what was claimed in (C.67). □

Claim C.4.3: $\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi$ transforms as a Lorentz tensor. More precisely, the symmetric part is a Lorentz scalar, proportional to $\eta^{\mu\nu}\overline{\psi}\psi$, while the antisymmetric part is a Lorentz tensor, proportional to $\overline{\psi}\sigma^{\mu\nu}\psi$.

Proof. This is analogous to proof of claim C.4.2 □

Claim C.4.4: $\overline{\psi}\gamma^{5}\psi$ transforms as a pseudo Lorentz scalar and $\overline{\psi}\gamma^{5}\gamma^{\mu}\psi$ transforms as a pseudo Lorentz vector or axial Lorentz vector.

Proof. This is again analogous to proof of claim C.4.2 and taking in account their behaviour under parity transformation (B.8) and (B.9). □

D Appendix - One-Loop Correction in QED

When discussing one-loop corrections to the seesaw mechanism ourselves to one-loop corrections of the fermion propagator, i.e. fermion self-energies.

In the first three sections we introduced the basics of the SM and different possible mass terms as well as the seesaw mechanism just on tree level. This means we have only taken into account the lowest order of perturbation theory. As it is indicated in [33, p.175], "on taking higher orders into account, one expects corrections of the order of the fine structure constant α to the lowest-order results, known as radiative corrections" and in particular one-loop corrections.

First, we should introduce the path integral formalism and correlation functions, which will be used in the following sections. Then we briefly discuss perturbation theory in this framework to calculate the fermion self-energy using dimensional regularization. Finally, we will discuss renormalization of the QED and give a more or less complete list of Feynman rules, which will be useful in section 4. In doing so we will follow [27] throughout the following sections.

D.1 Correlation Functions and Path Integral Formalism

D.1.1 Scalar Field

First of all we should define the correlation function or n-point function of a real scalar field ϕ, as done in [27, p.1/8ff].

Definition D.1.1: A *n-point function or correlation function* is a vacuum expectation value of time-ordered products of scalar field operators

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle, \tag{D.1}$$

where T denotes the time ordering operator given by

$$T\phi(x_1)\dots\phi(x_n) = \phi(x_{i_1})\dots\phi(x_{i_n}), \tag{D.2}$$

for permutations $i_1,\dots,i_n$ of $1,\dots,n$, such that the operators are rearranged descending in times $x^o_{i_1} > \dots > x^o_{i_1}$.

One very important case of the correlation function is the the *two-point function*

$$\langle 0|T\phi(x)\phi(y)|0\rangle, \tag{D.3}$$

which plays a central role in perturbation expansion of interacting theories. The two-point function is translation invariant, i.e.

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \langle 0|T\phi(x-y)\phi(0)|0\rangle, \tag{D.4}$$

and defines the *free propagator of the boson*[144] by

$$\triangle(x) := i\langle 0|T\phi(x)\phi(0)|0\rangle, \tag{D.5}$$

which can be interpreted as the probability amplitude of a particle propagating from 0 to x in spacetime. This propagator is a Green's function of the field equation, which in case of a scalar field the Klein-Gordon equation (1.5), i.e.

$$(\Box + m^2)\triangle(x) = \delta^{(4)}(x). \tag{D.6}$$

Thus, the Green's function can be written as the following Fourier integral:

$$\triangle(x) = \int \frac{d^4x}{(2\pi)^4} \frac{e^{-ipx}}{m^2 - p^2 - i\varepsilon}. \tag{D.7}$$

Here, we have taken into account the Feynman boundary conditions by $m^2 \to m^2 - i\varepsilon$, which ensures heaving only positive frequency solutions for $x^0 > 0$ and only negative frequency solutions for $x^0 < 0$.

When introducing path integral formalism, we find that n-point functions are given by such a basic path integral formula:

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{1}{\mathcal{N}}\int [d\varphi]e^{iS[\varphi]}\varphi(x_1)\dots\varphi(x_n), \tag{D.8}$$

with commuting c-number fields φ and a normalization factor $\mathcal{N}$ given by

$$\mathcal{N} = \int [d\varphi]e^{iS[\varphi]}, \tag{D.9}$$

and $S[\varphi]$ should be the classical action of the model for the free scalar field, i.e.

$$S[\varphi] = \int d^4x[\partial_\mu\varphi(x)\partial^\mu\varphi(x) - m^2\varphi(x)^2]. \tag{D.10}$$

The integration measure $[d\varphi]$ can be understood in the following way; The metric allows us to define a volume element

$$ds^2 = \int d^4x[d\varphi], \tag{D.11}$$

which can be seen as the distance between two neighbouring field configurations[145]. It can be easily shown that the integration measure is translation invariant, i.e.

$$[d\varphi] = [d\varphi'] \qquad \text{for} \quad \varphi(x) = \varphi'(x) + k(x), \tag{D.12}$$

for an arbitrary function $k(x)$.

[144] Similarly we find the propagator for a complex scalar field as $\triangle(x) := i\langle 0|T\phi(x)\phi^\dagger(y)|0\rangle$.

[145] In QM the fields would be various paths of particle trajectories. The volume element thus would be a distance between two paths and that's why this formalism is called path integral method. An introductory discussion can be found in [28, p.7ff] .

The generating functional is given by

$$\begin{aligned} Z[f] &= \frac{1}{\mathcal{N}} \int [d\varphi] e^{iS[\varphi]} e^{i\int d^4x f(x)\varphi(x)} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n f(x_1)\dots f(x_n) \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle \\ &=: \langle 0|Te^{i\int d^4x f(x)\phi(x)}|0\rangle, \end{aligned} \tag{D.13}$$

and so the n-point functions can be obtained as functional derivations of $Z[f]$ as

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{1}{i^n} \left. \frac{\delta^n Z[f]}{\delta f(x_1)\dots f(x_n)} \right|_{f=0}. \tag{D.14}$$

We can compute the generating functional for the scalar field by recognizing a Fresnel type integral and find

$$\begin{aligned} Z[f] =& \{0|Te^{i\int d^4x f(x)\phi(x)}|0\rangle \\ =& \exp\left(\frac{i}{2}\int d^4x d^4y f(x) f(y) \triangle(x-y)\right). \end{aligned} \tag{D.15}$$

With this and applying Wick's theorem (theorem D.3.1) it can be shown that one can decompose every n-point function in a product of two-point functions, i.e.

$$\begin{aligned} \langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle &= \sum_{\text{pairings}} \frac{1}{i}\triangle(x_{i_1}-x_{i_2})\dots\frac{1}{i}\triangle(x_{i_{n-1}}-x_{i_n}) \\ &= \sum_{\text{pairings}} \frac{1}{i}\langle 0|T\phi(x_{i_1})\phi(x_{i_2})|0\rangle \dots \langle 0|T\phi(x_{i_{n-1}})\phi(x_{i_n})|0\rangle. \end{aligned} \tag{D.16}$$

D.1.2 Dirac Field

All considerations above are similar for fermion operators. The two-point function of Dirac fields is given by

$$\langle 0|\psi_a(x)\overline{\psi}_b(y)|0\rangle = \frac{1}{i}(i\not{\partial}_x - m)_{ab}\triangle(x-y)) =: \frac{1}{i}S(x-y), \tag{D.17}$$

where $S(x-y)$ is called the *Dirac propagato*r. It is a Green's function of the Dirac equation (1.19), since

$$(m - i\not{\partial}_x)S(x-y) = \delta^{(4)}(x-y), \tag{D.18}$$

with Feynman boundary conditions. Its Fourier representation is written as

$$S(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{\not{k}+m}{m^2-k^2-i\varepsilon} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{m-\not{k}-i\varepsilon}. \tag{D.19}$$

It can be shown that the two following types of two-point functions vanish:

$$\langle 0|\psi_a(x)\psi_b(y)|0\rangle \; = 0, \tag{D.20}$$

$$\langle 0|\overline{\psi}_a(x)\overline{\psi}_b(y)|0\rangle \; = 0. \tag{D.21}$$

Therefore, we only get nonvanishing results for the general case of a n-point function only if the number of ψ's equals the number of $\overline{\psi}$'s, since it can be decomposed as a product of two-point functions, similarly to the bosonic case by

$$\begin{aligned}&\langle 0|T\psi_{a_1}(x_1)\overline{\psi}_{b_1}(y_1)\dots\psi_{a_n}(x_n)\overline{\psi}_{b_n}(y_n)|0\rangle\\&=\sum_{\sigma\in\mathcal{S}_n}(-1)^\sigma\frac{1}{i}S_{a_1b_{\sigma(1)}}(x_1-y_{\sigma(1)}\dots\frac{1}{i}S_{a_nb_{\sigma(n)}}(x_n-y_{\sigma(n)}),\end{aligned}\tag{D.22}$$

with

$$(-1)^\sigma=\begin{cases}+1 & \text{if } \sigma \text{ is an even permutation of } 1,\dots,n,\\ -1 & \text{if } \sigma \text{ is an odd permutation of } 1,\dots,n.\end{cases}\tag{D.23}$$

The fermionic path integral is a bit more complicated since fermionic variables are elements of a Grassmann algebra, i.e. they are anticommuting objects. A discussion on this can be also found in [27, p.4/1ff] and it is also discussed in [28, p.121ff]. But after some considerations we obtain the generating functional as a functional of two Grassmann variables η and $\overline{\eta}$ given by

$$\begin{aligned}Z[\eta,\overline{\eta}]=&\langle 0|T\exp\left(i\int d^4x[\overline{\eta}(x)\psi(x)-\overline{\psi}(x)\eta(x)]\right)|0\rangle\\=&\frac{1}{\mathcal{N}}\int[d\psi,d\overline{\psi}]\exp\left(i\int d^4x[S[\psi]+\overline{\eta}(x)\psi(x)-\overline{\psi}(x)\eta(x)]\right),\end{aligned}\tag{D.24}$$

with a translation invariant integration measure $[d\psi,d\overline{\psi}]$ of Grassmann variables. We can compute the generating functional for the free Dirac field similarly as before and get

$$\begin{aligned}Z[\eta,\overline{\eta}]=&\langle 0|T\exp\left(i\int d^4xd^4y[\overline{\eta}(x)S(x-y)\eta(y)]\right)|0\rangle\\=&\frac{1}{\mathcal{N}}\int[d\psi,d\overline{\psi}]\exp\left(i\int d^4xd^4y[\overline{\eta}S(x-y)\eta(x)]\right),\end{aligned}\tag{D.25}$$

which leads to the pairing rules stated above.

D.1.3 Vector Field

In the case of a (real) massive vector field the two-point function can be also obtained from the propagator of the vector boson given by

$$\langle 0|TV^\mu(x)V^\nu(y)|0\rangle=i\triangle^{\mu\nu}(x-y),\tag{D.26}$$

where the propagator is a Green's function of the Proca equation (1.10). Its Fourier representation is written as

$$\triangle^{\mu\nu}(x)=\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{g^{\mu\nu}-\frac{k^\mu k^\nu}{M^2}}{M^2-k^2-i\varepsilon}.\tag{D.27}$$

For a complex vector field we get analogously

$$\langle 0|TV_\mu(x)V_\nu^\dagger(y)|0\rangle=i\triangle^{\mu\nu}(x-y),\tag{D.28}$$

but it should be noted that other two-point functions vanish

$$\langle 0|TV_\mu(x)V_\nu(y)|0\rangle = 0, \tag{D.29}$$

$$\langle 0|TV_\mu^\dagger(x)V_\nu^\dagger(y)|0\rangle = 0. \tag{D.30}$$

Again, we obtain the generating functional, which is in this case given by

$$\begin{aligned} Z[J] =& \{0|Te^{-i\int d^4x V^\mu(x)J_\mu(x)}|0\rangle \\ =& \frac{1}{\mathcal{N}}\int [dV^\mu]\exp\left(i\int d^4x[S[V^\mu]-V^\mu J_\mu]\right), \end{aligned} \tag{D.31}$$

where $J_\mu(x)$ is an external current and the integration $[dV^\mu]$ is translation invariant. Computing the generating functional for the real vector field leads to

$$Z[J] = \exp\left(-\frac{i}{2}\int d^4x d^4y J_\mu(x)\triangle^{\mu\nu}(x-y)J_\nu(y)\right), \tag{D.32}$$

and similarly we get for the complex vector field

$$Z[J,J^*] = \exp\left(-i\int d^4x d^4y J *_\mu (x)\triangle^{\mu\nu}(x-y)J_\nu(y)\right). \tag{D.33}$$

D.1.4 Gauge Field

Finally we should also discuss the propagator of an abelian gauge field, i.e. the photon field, which is a massless vector field, since a mass term is forbidden by the gauge invariance of the Lagrangian. In order to derive the Fourier representation of the propagator, we can find the Green's function only by adding a gauge fixing term containing a gauge parameter ξ. In [26, p.732ff] an elaborated discussion can be found. We utilize the conventions there and state the result for the gauge fixed Lagrangian of the interacting photon

$$\mathcal{L}_{\text{eff}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{free photon}}\underbrace{-\frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{gauge fixing}}\underbrace{-J_\mu A^\mu}_{\text{interaction}}. \tag{D.34}$$

The derivation can be found in [27, p.6/1ff] and here we just want to state the result for the photon propagator:

$$D^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{-i}{k^2+i\varepsilon}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2}\right]. \tag{D.35}$$

As it is mentioned in [83, p.150], the gauge fixing parameter is an arbitrary real non-negative number. In the Landau gauge we have $\xi = 0$, in the Feynman-'t Hooft gauge $\xi = 1$ and in the unitary gauge $\xi = \infty$. It is important to be aware of the fact that physically meaningful quantities have to be independent of the gauge parameter. In any gauge where the propagator falls off with $\frac{1}{k^2}$ "the perturbation theory will be renormalizable, in the sense that the divergences are removed by a finite set of counterterms", as is mentioned in [26, p.738]. Thus, we call these gauges R_ξ *gauges* to indicate the corresponding parameter ξ and their renormalizability. As it is noted in [70, p.35ff], this more general framework of gauges is more convenient to use if we would like to calculate Feynman diagrams beyond tree level. Even though in unitary gauge the physical structure becomes more obvious, renormalizability becomes obscure, since propagators behave as $\mathcal{O}(1)$ rather than $\mathcal{O}(k^{-2})$ for large k.

At last we state the result for the generating functional of the photon field

$$Z[J] = \exp\left(-\frac{i}{2}\int d^4x d^4y D^{\mu\nu}(x-y)J_\mu(x)J_\nu(y)\right), \tag{D.36}$$

which is indeed independent of the gauge parameter ξ for a conserved current J, as it is shown in [27, p.6/4ff].

It should be only noted here that the propagators for the other gauge bosons can be found similarly, as done e.g. in [83, p.148ff]. Here we will simply state the results obtained there in the section D.3 on Feynman rules.

D.2 Perturbation Theory and the S-Operator

In this section we will briefly state some important results and considerations found in [27, p.7/1ff]. If we deal with interactions, which are small in some sense, perturbation theory can be employed. We might decompose the Hamiltonian[146] of the theory in a free and interacting part, i.e.

$$H = H_0 + H_{\text{int}}. \tag{D.37}$$

We are working in the interaction picture (IP), where time evolution of operators is defined as

$$A_{\text{IP}}(t) := e^{iH_0t}A_S e^{-iH_0}, \tag{D.38}$$

where the index S indicates an operator in the Schrödinger picture. If we want to calculate a S-matrix element in a scattering process, we use

$$\langle\chi \text{ out}|\phi \text{ in}\rangle = \langle\chi|S|\phi\rangle, \tag{D.39}$$

where S denotes the S-operator defined as

$$S := Te^{-i\int d^4x\mathcal{H}_{\text{int}}^{\text{IP}}}. \tag{D.40}$$

This is the starting point for perturbative expansion as we will see.

If one does some explicit calculations of S-matrix elements, a specific structure becomes apparent . All S-matrix elements exhibit the form

$$(2\pi)^4\delta^{(4)}\left(\sum_i p_i^{\text{in}} - \sum_j q_j^{\text{out}}\right) i\mathcal{M}(p_1^{\text{in}}, p_2^{\text{in}}, \ldots \to q_1^{\text{out}}, q_2^{\text{out}}, \ldots), \tag{D.41}$$

and we call $i\mathcal{M}(p_1^{\text{in}}, p_2^{\text{in}}, \ldots \to q_1^{\text{out}}, q_2^{\text{out}}, \ldots)$ the *invariant matrix element*, which contains all relevant physical information of the scattering process. This invariant matrix element can be easily written down, if one uses Feynman rules, which we will discuss in the next section.

[146]or respectively the Lagrangian

But before, we should consider two-point functions in an interacting theory as it is done in [27, p.16/1ff]. Let $|\Omega\rangle$ denote the vacuum of the interacting theory and we assume that

$$\langle\Omega|\phi(0)|\Omega\rangle = 0.$$

Then we achieve for the two-point function of a scalar field

$$\begin{aligned}\langle\Omega|\phi(x)\phi(0)|\Omega\rangle =& \frac{Z}{i}\triangle(x;m_{\text{ph}}) + \dots \\ =& \frac{Z}{i}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{m_{\text{ph}}^2 - k^2 - i\varepsilon} + \dots,\end{aligned} \tag{D.42}$$

where we focused on one particle states with momentum p and the dots indicate continuum contributions for particles $n \geq 2$. The pole of this two-point function determines the physical mass m_{ph} and the *field renormalization constant* Z is given by

$$\sqrt{Z} = \langle\Omega|\phi(0)|p\rangle. \tag{D.43}$$

Analogously, we can obtain the two-point function of fermions in an interacting theory as

$$\begin{aligned}\langle\Omega|T\psi(x)\overline{\psi}(0)|\Omega\rangle =& \frac{Z}{i}S(x;m_{\text{ph}}) + \dots \\ =& \frac{Z}{i}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{m_{\text{ph}} - \not{k} - i\varepsilon} + \dots,\end{aligned} \tag{D.44}$$

with the renormalization constant Z given by

$$\langle\Omega|\psi(0)|p,s\rangle = \sqrt{Z}u(p,s;m_{\text{ph}}), \tag{D.45}$$
$$\langle p,s|\overline{\psi}(0)|\Omega\rangle = \sqrt{Z}\overline{u}(p,s;m_{\text{ph}}), \tag{D.46}$$
$$\langle\overline{p,s}|\psi(0)|\Omega\rangle = \sqrt{Z}v(p,s;m_{\text{ph}}), \tag{D.47}$$
$$\langle\Omega|\overline{\psi}(0)|\overline{p,s}\rangle = \sqrt{Z}\overline{v}(p,s;m_{\text{ph}}). \tag{D.48}$$

In order to calculate the perturbative expansion in first order, we note what is mentioned in [27, p.18/1ff]. The action can be decomposed in a free and interaction part

$$S[\varphi] = S_0[\varphi] + S_{\text{int}}[\varphi], \tag{D.49}$$

and φ denotes the fields present in the theory (e.g. $\varphi = \psi, \overline{\psi}, A^\mu, \dots$). We like to rewrite the n-point function of the interacting theory using path integral formalism. Thus, when splitting S into these two parts, we obtain

$$\begin{aligned}\langle\Omega|T\phi(x_1)\dots\phi(x_n)|\Omega\rangle =& \frac{\int[d\varphi]e^{iS[\varphi]}\varphi(x_1)\dots\varphi(x_n)}{\int(d\varphi]e^{iS[\varphi]}} \\ =& \frac{\langle\langle e^{iS_{\text{int}}[\varphi]}\varphi(x_1)\dots\varphi(x_n)\rangle\rangle}{\langle\langle e^{iS_{\text{int}}[\varphi]}\rangle\rangle},\end{aligned} \tag{D.50}$$

where $\langle\langle\dots\rangle\rangle$ denotes the Gaussian mean value, which is defined as

$$\langle\langle\mathcal{F}[\varphi]\rangle\rangle = \frac{\int[d\varphi]e^{iS_0[\varphi]}\mathcal{F}[\varphi]}{\int[d\varphi]e^{iS_0[\varphi]}}, \tag{D.51}$$

for a functional $\mathcal{F}$. To perform perturbative expansion we insert the power series expansion of the exponential function, and obtain

$$e^{iS_{\text{int}}[\varphi]} = \sum_{n=0}^{\infty} \frac{i^n}{n!} (S_{\text{int}}[\varphi])^n. \tag{D.52}$$

D.3 Feynman Rules

Theorem D.3.1: Let $A_1, \ldots, A_n$ be a set of n creation and annihilation operators, then we can evaluate their product in a systematic way:

$$\begin{aligned} A_1 \ldots A_n = &: A_1 \ldots A_n : + \\ &+ : \underbracket{A_1 A_2} A_3 A_4 \ldots A_n : + \ldots \qquad \text{one contraction} \\ &+ : \underbracket{A_1 A_2}\, \underbracket{A_3 A_4} \ldots A_n : + \ldots \qquad \text{two contractions} \\ &+ \ldots , \end{aligned}$$

where a *Wick contraction* means $\underbracket{A_1 A_2} := \langle 0 | A_1 A_2 | 0 \rangle$ and : ... : denotes the normal ordering of the creation and annihilation operator, which means rearranging the product in a way such that all creation operators stand on the left side of all annihilation operators.

Proof. A proof of this theorem is done in [27, p.10/2ff]. □

This *Wick contraction theorem* can be now used to give a list of all Feynman rules we need.

D.3.1 Propagators

Here we like to give a comprehensive list of Feynman rules of the propagator of all particles considered in this thesis. We start with the fermion propagator derived in section D.1.2 and the photon propagator we already mentioned in section D.1.4 in R_ξ gauge.

Similarly, the propagators of the weak interaction bosons can be found and will just be stated here. Finally, for sake of completeness, we present also the propagators of the scalars (including the Goldstone bosons), which will be useful in calculations for the loop corrections to the seesaw mechanism in section 4. Those propagators can be found in [83, p.150f, 158], where ξ_A, ξ_Z and ξ_W denote the different gauge parameters corresponding to the gauge bosons A, Z^0 and $W^\pm$ of the EW theory. The Feynman rules for the propagators are listed on the next page.

$$(f)\colon\ \underset{k}{\longrightarrow} = i\frac{\not k + m_f}{k^2 - m_f^2}, \tag{D.53}$$

$$(\gamma)\colon\ \underset{k}{\rightsquigarrow} = \frac{-ig_{\mu\nu}}{k^2} + (1-\xi_A)\frac{ik_\mu k_\nu}{k^4}, \tag{D.54}$$

$$(Z^0)\colon\ \underset{k}{\rightsquigarrow} = \frac{-ig_{\mu\nu}}{k^2 - m_Z^2} + \frac{ik_\mu k_\nu}{m_Z^2}\left(\frac{1}{k^2 - m_Z^2} - \frac{1}{k^2 - \xi_Z m_Z^2}\right), \tag{D.55}$$

$$(W^\pm)\colon\ \underset{k}{\rightsquigarrow} = \frac{-ig_{\mu\nu}}{k^2 - m_W^2} + \frac{ik_\mu k_\nu}{m_W^2}\left(\frac{1}{k^2 - m_W^2} - \frac{1}{k^2 - \xi_W m_W^2}\right), \tag{D.56}$$

$$(S_a^0)\colon\ \underset{k}{\dashrightarrow} = \frac{i}{k^2 - m_a^2}, \tag{D.57}$$

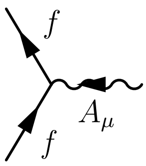
$$(S_b^\pm)\colon\ \underset{k}{\dashrightarrow} = \frac{i}{k^2 - m_b^2}, \tag{D.58}$$

$$(G^0)\colon\ \underset{k}{\dashrightarrow} = \frac{i}{k^2 - \xi_Z m_Z^2}, \tag{D.59}$$

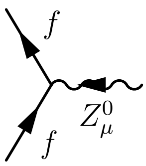
$$(G^\pm)\colon\ \underset{k}{\dashrightarrow} = \frac{i}{k^2 - \xi_W m_W^2}. \tag{D.60}$$

D.3.2 Vertices

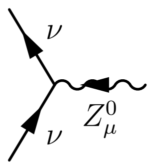
Furthermore, we want to give a list of all interaction vertices needed in this master thesis, i.e. EW interactions of leptons with the gauge bosons, which can be found in [83, p.159]:



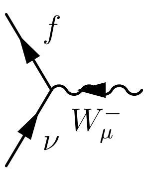
$$= -ieq_f\gamma_\mu, \tag{D.61}$$



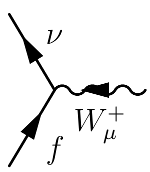
$$= i\frac{g}{c_W}\gamma_\mu\left(-\frac{1}{2}P_L + s_w^2\right),$$



$$= i\frac{g}{2c_W}\gamma_\mu P_L, \tag{D.62}$$

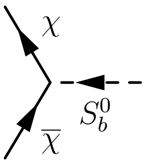


$$= i\frac{g}{\sqrt{2}}\gamma_\mu P_L,$$

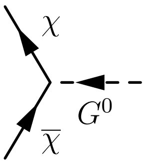


$$= i\frac{g}{\sqrt{2}}\gamma_\mu P_L. \tag{D.63}$$

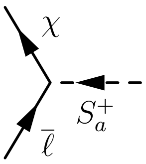
In addition, the vertices for the Yukawa interactions of the neutral and charged scalar mass eigenfields with the neutrino mass eigenfields will be needed. They are given in [83, p.160], but we already note them in the notation, which is needed in section 4.1 and has been introduced there:



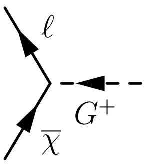
$$= -\frac{i}{\sqrt{2}}\left(\left[U_R^\dagger \Delta_b U_L + U_L^T \Delta_b^T U_R^*\right] P_L + \left[U_L^\dagger \Delta_b^\dagger U_R + U_R^T \Delta_b^* U_L^*\right] P_R\right), \tag{D.64}$$



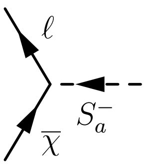
$$= -\frac{i}{\sqrt{2}}\left(\left[U_R^\dagger \Delta_{bZ} U_L + U_L^T \Delta_{bZ}^T U_R^*\right] P_L + \left[U_L^\dagger \Delta_{bZ}^\dagger U_R + U_R^T \Delta_{bZ}^* U_L^*\right] P_R\right), \tag{D.65}$$



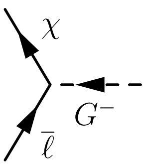
$$= i\left[(U_R^\dagger \Delta_a) P_R - (U_L^\dagger \Gamma_a^\dagger) P_L\right], \tag{D.66}$$



$$= i\left[(U_R^\dagger \Delta_{aW}) P_R - (U_L^\dagger \Gamma_{aW}^\dagger) P_L\right], \tag{D.67}$$



$$= i\left[P_R\left(\Delta_a^\dagger U_R\right) - P_L\left(\Gamma_a U_L\right)\right], \tag{D.68}$$



$$= i\left[P_R\left(\Delta_{aW}^\dagger U_R\right) - P_L\left(\Gamma_{aW} U_L\right)\right]. \tag{D.69}$$

D.4 Fermion Self-Energy

D.4.1 Fermion Two-Point Function at One-Loop

A discussion on this can be also found in [26, p.217], but we follow again [27, p.18/1ff] and use the path integral formalism for the perturbative expansion of the Green's function to compute the one-loop correction to the fermion propagator. The two-point function is given by

$$\langle \Omega | T \psi_a(x) \overline{\psi}_b(y) | \Omega \rangle = \frac{\langle\langle e^{iS_{\text{int}}} \psi_a(x) \overline{\psi}_b(y) \rangle\rangle}{\langle\langle e^{iS_{\text{int}}} \rangle\rangle}, \tag{D.70}$$

where we insert the QED interaction part of the action ($q = -e$ for e^-)

$$S_{\text{int}} = -q \int d^4x \overline{\psi}(x) \not{A}(x) \psi(x). \tag{D.71}$$

Inserting the perturbative expansion into the denominator , we obtain

$$\langle\langle e^{iS_{\text{int}}} \psi_a(x) \overline{\psi}_b(y) \rangle\rangle = \langle\langle (1 + iS_{\text{int}} + \frac{i^2}{2!} S_{\text{int}}^2 + \ldots) \psi_a(x) \overline{\psi}_b(y) \rangle\rangle. \tag{D.72}$$

We recognize that the first term (0th order) simply gives the fermion propagator at tree-level and the second (1st order) term vanishes since gauge condition leads to $\langle\langle A_\mu \rangle\rangle = 0$. The third terms (2nd order) yield the additional one-loop corrections and we use Wick's theorem (theorem D.3.1) to find all possible contractions. The only one which contributes to the one loop corrections is the fermion self-energy shown in figure 24 below.

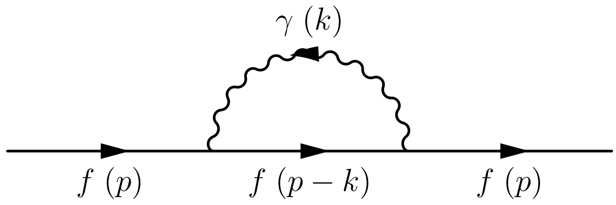
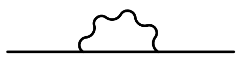


Figure 24: Fermion self-energy - photon loop

After some calculations we achieve the corrected two-point function in the following form:

$\langle \Omega | T \psi(x) \overline{\psi}(y) | \Omega \rangle \Big|_{1-\text{loop}} =$  $+$ 

(D.73)

$$= \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \left\{ \frac{i}{\not{p} - m + i\varepsilon} + \frac{i}{\not{p} - m + i\varepsilon} \left(-i\Sigma(p)\right) \frac{i}{\not{p} - m + i\varepsilon} \right\},$$

with the self-energy function Σ of the fermion given by the loop-integral

$$-i\Sigma(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2 + i\varepsilon} \gamma^\mu \frac{1}{k^2 + i\varepsilon}. \tag{D.74}$$

D.4.2 Regularization

This loop integral (D.74) exhibits not only an IR-divergence but also an UV-divergence. The first on can be easily fixed by adding a small photon mass m_γ to the photon propagator. The latter could be treated in different ways, but here we choose dimensional regularization. This means we introduce a (higher) space-time dimension d such that the loop integral is convergent.

Thus, we make the following substitutions:

$$\int d^4x \to \int d^dx, \quad \int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}, \quad g^\mu_\mu = 4, \quad \gamma_\mu \not{a} \gamma^\mu = (2-d)\not{a}. \tag{D.75}$$

Hence, we obtain the following form for fermion self-energy:

$$-i\Sigma(p) = -e^2 \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\not{p}-\not{k})+dm}{(p-k)^2 - m^2 + i\varepsilon} \frac{1}{k^2 - m_\gamma^2 + i\varepsilon}. \tag{D.76}$$

In the next step we employ Feynman parametrization (E.45) to contract the two denominators and we obtain

$$-i\Sigma(p) = -e^2 \int_0^1 \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\not{p}-\not{k})+dm}{\{x[(k-p)^2 - m^2 + i\varepsilon] + (1-x)[k^2 - m_\gamma^2 + i\varepsilon]\}}. \tag{D.77}$$

After completing the square in the polynomial denominator, we perform a formal shift of the integration variable k by introducing $k = k - xp$, and hence we end up with an integral

$$-i\Sigma(p) = -e^2 \int_0^1 \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(1-x)\not{p}+dm}{k^2 + x(1-x)p^2 - xm^2 - (1-x)m_\gamma^2 + i\varepsilon}. \tag{D.78}$$

Now, the integration over k can be performed by Wick rotation and using formula (E.46) with $\alpha = 2$ and thus we obtain

$$\begin{aligned}-i\Sigma(p) = &- e^2 \int_0^1 [(2-d)(1-x)\not{p} + dm] \frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(2)} \\ &[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon]^{\frac{d}{2}-2}.\end{aligned} \tag{D.79}$$

Since $\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$ we see that $\Sigma(p)$ has poles at $d = 4, 6, 8, \ldots$. To investigate the behaviour of Σ in the vicinity of the physical relevant spacetime dimension $d = 4$ we set $d = 4 - 2\varepsilon$ for $\varepsilon \to 0$ such that $d \to 4$. Inserting d of this form into the equation above we get

$$\begin{aligned}-i\Sigma(p) \underset{d\to 4}{\to} &- e^2 \int_0^1 [(-2+2\varepsilon)(1-x)\not{p} + (4-2\varepsilon)m] \\ &\frac{i\Gamma(\varepsilon)}{(4\pi)^2(4\pi)^{-\varepsilon}} [xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon]^{-\varepsilon}.\end{aligned} \tag{D.80}$$

Finally, we use the following expansion of the gamma function Γ:

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} \underbrace{+\Gamma'(\varepsilon)}_{-\gamma_E} + \mathcal{O}(\varepsilon), \tag{D.81}$$

where γ_E denotes the Euler-Mascheroni constant. Furthermore, we rewrite and expand the term of the power $-\varepsilon$ in the following way:

$$a^{-\varepsilon} = \varepsilon^{-\varepsilon \ln(a)} = 1 - \varepsilon \ln(a) + \mathcal{O}(\varepsilon^2). \tag{D.82}$$

Applying all this to formula (D.80) and only taking into account the two lowest orders in ε of the expansion we achieve

$$\begin{aligned} -i\Sigma(p) = &- \frac{ie^2}{(4\pi)^2}\left\{\left[\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right](-\not{p} + 4m) + \not{p} - 2m \right. \\ &\left. - \int_0^1 [-2(1-x)\not{p} + 4m]\ln[xm^2 + (1-x)m_\gamma^2 - x(1-x)p^2 - i\varepsilon]\right\}. \end{aligned} \tag{D.83}$$

Introducing an arbitrary mass scale μ to reintroduce d instead of ε quite nicely[147] we can finally achieve

$$\Sigma|_{\not{p}=m} = \frac{e^2 m}{(4\pi)^2}\mu^{d-4}\left\{-6\underbrace{\left[\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln(4\pi)\right)\right]}_{=:\Lambda_d} - 3\ln\left(\frac{m^2}{\mu^2}\right) + 4\right\}. \tag{D.84}$$

D.4.3 Structure and Interpretation

As it is mentioned in [27, p.18/20], the fermion self energy can be decomposed into two parts

$$\Sigma(\not{p}) = A(p^2)\not{p} + B(p^2)m. \tag{D.85}$$

When we take a closer look at the final result in (D.84) we see that of course we always have a part proportional to $\not{p}$ and one proportional to m, if we take into account[148] that

$$\not{p}^n = \begin{cases} p^{n-1}\not{p} & \text{for odd } n, \\ p^n & \text{for even } n, \end{cases} \tag{D.86}$$

where $n \geq 1$, because Σ can only depend on powers of $\not{p}$.

Besides this structure, we like to discuss the structure of the propagator in more detail, as it is done in [27, p.18/19ff] and also in [26, p.220f]. We find that the corrected fermion propagator in momentum space, i.e. the Fourier transformed two-point function, can be written as

$$\int d^4x \langle\Omega|T\psi(x)\overline{\psi}(0)|\rangle e^{ipx} = \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}(-i\Sigma)\frac{i}{\not{p}-m} + \mathcal{O}\left(\frac{\Sigma}{\not{p}-m}\right)^2 \tag{D.87}$$

$$= \frac{i}{\not{p}-m-\Sigma}, \tag{D.88}$$

where we noticed the expansion to be a geometric series[149]. We recognize that the pole of the two-point function (in momentum space) is no longer at $\not{p} = m$, but shifted to the

[147] Note that $\mu^{-2\varepsilon}\frac{1}{\varepsilon} = \frac{1}{\varepsilon} - \ln\mu^2$ and $[e^2] = 4 - d = 2\varepsilon$.

[148] Great thanks to my college Maximilian Löschner for his very helpful note.

[149] We are able to write the series in this compact form because the the right sign appears in the denominator. This is the reason for defining the self-energy with a factor of $-i$. Thanks to my college Maximilian Löschner for pointing this out.

physical mass $\not{p} = m_{\text{ph}}$, which is given by corrections to the bare mass m as

$$m_{\text{ph}} = m + \Sigma|_{\not{p}=m_{\text{ph}}} + \text{corrections of higher order.} \tag{D.89}$$

The behaviour of the one-loop propagator in the vicinity of the physical mass $p^2 = m_{\text{ph}}^2$ can be obtained by the expansion of

$$\not{p} - m - \Sigma(p) = \not{p} - m - \Sigma(p)|_{\not{p}=m_{\text{ph}}} - \left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m_{\text{ph}}} (\not{p} - m_{\text{ph}}). \tag{D.90}$$

Inserting this result in (D.88), we get

$$\frac{i}{\not{p} - m - \Sigma(p)} \underset{p^2\to m_{\text{ph}}^2}{\rightarrow} \frac{iZ_2}{\not{p} - m_{\text{ph}}} \tag{D.91}$$

, where we introduced the notation

$$Z_2 = 1 + \left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m_{\text{ph}}}, \tag{D.92}$$

which is called the *wave function renormalization constant of the electron field.* Inserting the result for Σ we obtained in (D.84), we get

$$Z_2|_{1-\text{loop}} = 1 + \left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m} = 1 + \frac{e^2}{(4\pi)^2}\mu^{d-4}\left(2\Lambda_d + \ln\left(\frac{m^2}{\mu^2}\right) - 4 - 2\ln\left(\frac{m_\gamma^2}{m^2}\right)\right), \tag{D.93}$$

for the renormalization constant at one-loop[150].

D.5 Renormalization of QED

We have seen in the last section that the one-loop correction to the fermion propagator[151] leads to a shift in the pole, i.e. a shift in mass. When renormalizing QED, we want so establish renormalized quantities (ψ_r, A_r^μ) and physical parameters $(m_{\text{phys}}, e_{\text{phys}})$ in the Lagrangian (B.60) instead of the bare quantities and parameters (ψ, A^μ, m, e). Those quantities are connected by

$$\psi = \sqrt{Z_2}\psi_r, \quad A^\mu = \sqrt{Z_3}A_r^\mu, \quad eZ_2\sqrt{Z_3} = e_{\text{phys}}\sqrt{Z_1}, \tag{D.94}$$

and we introduce the notations

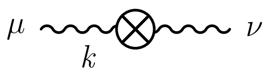
$$\delta_m = (Z_2 m - m_{\text{phys}}), \quad \delta_3 = (Z_3 - 1), \quad \delta_2 = (Z_2 - 1). \tag{D.95}$$

Therefore, we obtain the Lagrangian in the following form

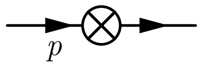
$$\begin{aligned}\mathcal{L}_{\text{QED}} = &-\frac{1}{4}F_{r\mu\nu}F_r^{\mu\nu} + \overline{\psi}_r(i\partial - m_{\text{phys}})\psi_r + e_{\text{phys}}\overline{\psi}_r\gamma_\mu\psi_r A_r^\mu \\ &-\frac{1}{4}\delta_3 F_{r\mu\nu}F_r^{\mu\nu} + \overline{\psi}_r(i\delta_2\partial - \delta_m)\psi_r - e_{\text{phys}}\delta_1\overline{\psi}_r\gamma_\mu\psi_r A_r^\mu.\end{aligned} \tag{D.96}$$

[150] Note that in the last term the IR-divergence shows up, since the logarithm would explode for $m_\gamma = 0$.
[151] In case of the photon propagator and the interaction vertex one can employ a similar procedure and obtain corrections and renormalization constants Z_3 resp. Z_2.

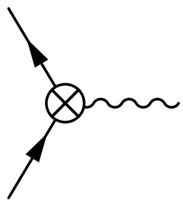
The first two terms in the first line represent the free Lagrangian $\mathcal{L}_0$, the third term in the first line is the EM interaction and the terms in the second line are called *counterterms*. We can also define Feynman rules for these conterterms by



$$= -i\left(g^{\mu\nu}k^2 - k^\mu k^\nu\right)\delta_3, \tag{D.97}$$



$$= i\left(\not{p}\delta_2 - \delta_m\right), \tag{D.98}$$



$$= ie_{\text{phys}}\gamma^\mu\delta_1. \tag{D.99}$$

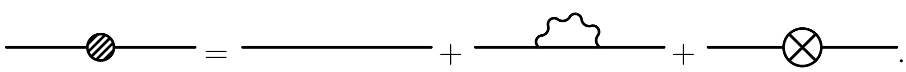
Their coefficients are fixed by renormalization conditions, e.g. the coefficients δ_2 and δ_m are fixed by

$$\tilde{\Sigma}(\not{p} = m_{\text{phys}}) = 0, \quad \frac{\partial}{\partial\not{p}}\tilde{\Sigma}(\not{p})\Big|_{\not{p}=m_{\text{phys}}} = 0, \tag{D.100}$$

with the renormalized self-energy $\tilde{\Sigma}$ given by

$$-i\tilde{\Sigma}(p) = -i\Sigma(p) + i(\not{p}\delta_2 - \delta_m). \tag{D.101}$$

The renormalized fermion propagator is composed of the tree-level contribution, the one-loop contribution (i.e. self-energy) as well as the counterterm. This can be visualized by



(D.102)

E Appendix - Mathmatical Tools

E.1 Algebraic Structures

In this section some useful definitions and properties of groups, Lie groups, and Lie algebras are summarized using [102], [104], [105], [106], and [107].

E.1.1 Defnitions

Definition E.1.1: A ***group*** $\mathcal{G}$ is a set of elements $\{g_1, g_2, \ldots\}$ together with a binary operation $\cdot$ (group law or operation) which satisfies the following group properties

(G1) *Closure*: If $g_1, g_2 \in \mathcal{G} \Rightarrow g_1 \cdot g_2 \in \mathcal{G}$

(G2) *Associativity*: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3) \quad \forall g_1, g_2, g_3 \in \mathcal{G}$

(G3) *Identity*: $\exists e \in \mathcal{G}$ such that $e \cdot g = g \cdot e = g \; \forall g \in \mathcal{G}$

(G4) *Inverse*: $\forall g \in \mathcal{G} \; \exists g^{-1} \in \mathcal{G}$ such that $g^{-1} \cdot g = g \cdot g^{-1} = e$

A group is called an ***abelian group***, if their group elements satisfy also

(G5) *Commutativity*: $g_1 \cdot g_2 = g_2 \cdot g_1 \quad \forall g_1, g_2 \in \mathcal{G}$

Definition E.1.2: A subset $\mathcal{H}$ of a group $\mathcal{G}$ ($\mathcal{H} \subseteq \mathcal{G}$) is called a ***subgroup*** if it satisfies the following conditions

(H1) *Closure*: If $h_1, h_2 \in \mathcal{H} \Rightarrow h_1 \cdot h_2 \in \mathcal{H}$

(H2) *Identity*: $\exists e \in \mathcal{H}$

(H3) *Inverse*: $\forall h \in \mathcal{H} \Rightarrow h^{-1} \in \mathcal{H}$

and we denote this by $\mathcal{H} \leq \mathcal{G}$. The so-called *induced operation* on $\mathcal{H}$ is defined via the group operation on $\mathcal{G}$ and hence it satisfies (G2). Furthermore, every group $\mathcal{G}$ obviously has two trivial subgroups, namely $\mathcal{G}$ itself and $\{e\}$.

Definition E.1.3: Let be $(\mathcal{G}, \cdot)$ and $(\mathcal{G}', *)$ two groups then a mapping $\varphi : \mathcal{G} \to \mathcal{G}'$ satisfying the condition

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2) \tag{E.1}$$

for all $g_1, g_2 \in \mathcal{G}$ is called a ***group homomorphism***. If this mapping is bijective, it is called a ***group isomorphism*** and we write $\mathcal{G} \cong \mathcal{G}'$.

Definition E.1.4: A set $\mathcal{K}$ assigned with two binary operations $(\mathcal{K}, +, \cdot)$ is called a ***field***, if the following conditions are satisfied:

(F1) $(\mathcal{K}, \cdot)$ forms an *abelian group* with neutral element 0

(F2) $(\mathcal{K}, +)$ forms an *abelian group* with neutral element 1

(F3) *Distributivity*: $a \cdot (b + c) = a \cdot b + a \cdot c$. and $(a + b) \cdot c = a \cdot c + b \cdot c \quad \forall a, b, c \in \mathcal{K}$

Definition E.1.5: A set $\mathcal{V}$ is called a ***vector space*** over a field $(\mathcal{K}, +, \cdot)$, if the two binary operations vector addition $\oplus : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and scalar multiplication $\odot : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ satisfy the following properties $\forall u, v, w \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{K}$

(V1) *Associativity:* $u \oplus (v \oplus w) = (u \oplus v) \oplus w$

(V2) *Identity:* $\exists 0_{\mathcal{V}} \in \mathcal{V} \quad \forall v \in \mathcal{V}$ such that $u \oplus 0_{\mathcal{V}} = 0_{\mathcal{V}} \oplus v = v$

(V3) *Inverse:* $\forall v \in \mathcal{V} \quad \exists - v \in \mathcal{V}$ such that $v \oplus (-v) = (-v) \oplus v = 0_{\mathcal{V}}$

(V4) *Commutativity:* $v \oplus u = u \oplus v \; \forall u, v \in \mathcal{V}$

(S1) *Left distributivity:* $\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$

(S2) *Right distributivity:* $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$

(S3) *Associativity:* $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$

Definition E.1.6: Let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathcal{K}$-vector spaces. A map $f : \mathcal{V} \to \mathcal{W}$ is called ***linear map*** or ***vector space homomorphism***, if it is homogeneous and additive, i.e. it satisfies the following conditions for all $x, y \in \mathcal{V}$ and $a \in \mathcal{K}$.

$$f(a \odot_{\mathcal{V}} x) = a \odot_{\mathcal{W}} f(x) \tag{E.2}$$

$$f(x \oplus_{\mathcal{V}} y) = f(x) \oplus_{\mathcal{W}} f(y) \tag{E.3}$$

Definition E.1.7: A $\mathcal{K}$-vector space $\mathcal{A}$ equipped with an additional binary operation $* : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a ***algebra*** over a field $\mathcal{K}$ (or $\mathcal{K}$-***algebra***), if the binary operation is bilinear, i.e. it satisfies the following properties $\forall x, y, z \in \mathcal{A}$ and $\forall \alpha, \beta \in \mathcal{K}$

(A1) *Left distributivity:* $(x \oplus y) * z = x * z \oplus y * z$

(A2) *Right distributivity:* $x * (y \oplus z) = x * y \oplus x * z$

(A3) *Compatibility with scalars:* $(\alpha \cdot x) * (\beta \cdot y) = (\alpha \cdot \beta)(x * y)$

If the binary operation $*$ also fulfils

(A4) *Associativity:* $x * (y * z) = (x * y) * z$

$\mathcal{A}$ is called a ***associative algebra*** over a field $\mathcal{K}$.

Definition E.1.8: Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over a field $\mathcal{K}$. A map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a ***algebra homomorphism***, if it fulfils for all $k \in \mathcal{K}$ and $x, y \in \mathcal{A}$

$$\phi(k *_{\mathcal{A}} x) = k *_{\mathcal{B}} \phi(x) \tag{E.4}$$

$$\phi(x \oplus_{\mathcal{A}} y) = \phi(x) \oplus_{\mathcal{B}} \phi(y) \tag{E.5}$$

$$\phi(x \odot_{\mathcal{A}} y) = \phi(x) \odot_{\mathcal{B}} \phi(y) \tag{E.6}$$

If in addition ϕ is bijective it is called an ***algebra isomorphism***.

Definition E.1.9: A ***Lie group*** is a smooth manifold $\mathcal{G}$ endowed with a group structure with smooth[152] operation. This means that we have a smooth operation $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, an inversion $\nu : \mathcal{G} \to \mathcal{G}$ and a unit element $e \in \mathcal{G}$ such that the group axioms (G1)-(G4) are satisfied. We will write $g_1 \cdot g_2$ for $\mu(g_1, g_2)$ and g^{-1} for $\nu(g)$ for $g, g_1, g_2 \in \mathcal{G}$.

Definition E.1.10: If $\mathcal{H}$ is a subgroup and also a submanifold of the Lie group $\mathcal{G}$, then $\mathcal{H} \subset \mathcal{G}$ is called a ***Lie subgroup*** of $\mathcal{G}$ and $\mathcal{H}$ itself is also a Lie group.

Definition E.1.11: A ***homomorphism of Lie groups*** $\mathcal{G}$ and $\mathcal{G}'$ is a smooth map $\varphi : \mathcal{G} \to \mathcal{G}'$ which is a group homomorphism.

Definition E.1.12: A ***Lie algebra*** over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a $\mathbb{K}-$vector space $\mathfrak{g}$ together with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* of $\mathfrak{g}$, which is skew symmetric, i.e.

$$[Y, X] = -[X, Y] \quad \forall X, Y \in \mathfrak{g} \tag{E.7}$$

and satisfies the Jacobi identity, i.e.

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \quad \forall X, Y, Z \in \mathfrak{g} \tag{E.8}$$

The Lie algebra of a Lie group $\mathcal{G}$ ist the tangent space $\mathfrak{g} := T_e\mathcal{G}$.

Definition E.1.13: Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. A ***Lie subalgebra*** of $\mathfrak{g}$ is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ which is closed under the Lie bracket, i.e. such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. We write $\mathfrak{h} \leq \mathfrak{g}$ if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Of course, in this case $(\mathfrak{h}, [\,,\,])$ is a Lie algebra, too.

Definition E.1.14: If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a ***homomorphism of Lie algebras*** $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a linear mapping which is compatible with the Lie brackets, i.e. such that for all $X, Y \in \mathfrak{g}$

$$[\varphi(X), \varphi(Y)] = \varphi([X, Y]). \tag{E.9}$$

If such a homomorphism is bijective, we call the mapping an ***isomorphism of Lie algebras*** and we write $\mathfrak{g} \cong \mathfrak{h}$.

[152]Smooth means infinitely differentiable ($\mathcal{C}^\infty$).

Proposition E.1.15: Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\varphi : \mathcal{G} \to \mathcal{H}$ is a smooth homomorphism, then $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras, i.e. $\varphi'([X,Y]))[\varphi'(X), \varphi'(Y)]$ for all $X, Y \in \mathfrak{g}$.

Proof. See [105, p.5] □

E.1.2 Representations

Definition E.1.16: A ***representation of a group*** $\mathcal{G}$ on a vector space V is a homomorphism $\rho : \mathcal{G} \to GL(V)$ and $GL(V)$ is the general linear group of all automorphism of V.

Definition E.1.17: A n-dimensional ***matrix representation*** of a group $\mathcal{G}$ is a homomorphism $R : \mathcal{G} \to GL_n(K)$, where K is a field and $GL_n(K)$ is the general linear group of all invertible $n \times n$ matrices with coefficients in K. All representations of a group on a finite dimensional vector space can be reduced to matrix representations.

Definition E.1.18: Two representations of a group $\mathcal{G}$ $\rho : \mathcal{G} \to GL(V)$ on the K-vector space V and $\pi : \mathcal{G} \to GL(W)$ on the K-vector space W are called ***equivalent or isomorphic***, if there exists a vector space isomorphism $\alpha : V \to W$ so that for all $g \in \mathcal{G}$,

$$\alpha \circ \rho(g) \circ \alpha^{-1} = \pi(g) \tag{E.10}$$

Definition E.1.19: Let $\mathcal{K}$ be a field and $\mathcal{A}$ a $\mathcal{K}$-algebra. An algebra homomorphism $\pi : \mathcal{A} \to L(\mathcal{V})$ is said to be a ***representation of an algebra*** $\mathcal{A}$, where $\mathcal{V}$ is a $\mathcal{K}$-vector space and $L(\mathcal{V})$ denotes the algebra of linear operators on $\mathcal{V}$.

Definition E.1.20: Let $\mathcal{A}$ be a $\mathcal{K}$-algebra and $\mathcal{V}_1$ and $\mathcal{V}_2$ two $\mathcal{K}$-vector spaces. Two algebra representations $\pi_1 : \mathcal{A} \to \mathcal{V}_1$ and $\pi_2 : \mathcal{A} \to \mathcal{V}_2$ are said to be ***equivalent*** ($\pi_1 \sim \pi_2$), if there exists a vector space isomorphism $T : \mathcal{V}_1 \to \mathcal{V}_2$ such that for all $a \in \mathcal{A}$

$$\pi_1(a) = T^{-1} \circ \pi_2 \circ T. \tag{E.11}$$

Definition E.1.21: A ***representation of a Lie group*** $\mathcal{G}$ on a finite dimensional vector space V is a smooth homomorphism $\varphi : \mathcal{G} \to GL(V)$. A ***representation of a Lie algebra*** $\mathfrak{g}$ on V is a Lie algebra homomorphism $\varphi' : \mathfrak{g} \to L(V, V)$, where $L(V, V)$ is the space of all linear maps form V to V. In particular, for any representation φ of a Lie group, one obtains a representation φ'. Now any Lie group $\mathcal{G}$ has a canonical representation on its Lie algebra, called the ***adjoint representation***.

Definition E.1.22: Given representations of $\mathcal{G}$ on V and W, a linear map $f : V \to W$ is called a ***morphism or $\mathcal{G}$-equivalent*** if $f(g \cdot v) = g \cdot f(v)$ for all $g \in \mathcal{G}$. An isomorphism of representations is a ***$\mathcal{G}$-equivalent linear isomorphism*** $f : V \to W$. If such an isomorphism exists, then we say that V and W are isomorphic and write $V \cong W$.

Definition E.1.23: Given representations of a Lie group $\mathcal{G}$ on V_1 and V_2 then there is an representation $V_1 \oplus V_2$ defined by

$$g \cdot (v_1, v_2) := (g \cdot v_1, g \cdot v_2). \tag{E.12}$$

This construction is referred to as the ***direct sum of representations***. Of course, the natural inclusion of the two summands into $V_1 \oplus V_2$ are $\mathcal{G}$-equivalent.

Definition E.1.24: A representation V of $\mathcal{G}$ is called ***decomposable*** if it is isomorphic to a direct sum $V_1 \oplus V_2$ with $\dim(V_1), \dim(V_2) > 0$. If this ist not the case, then the representation is called ***indecomposable***.

Definition E.1.25: Quite analogously, given two representations of a Lie algebra $\mathfrak{g}$ on vector spaces V and W, there is an obvious representation on the direct sum $V \oplus W$, defined by

$$X \cdot (v, w) := (X \cdot v, X \cdot w). \tag{E.13}$$

This is called the ***direct sum*** of the representations V and W. Similarly, we can construct a natural representation on the space $L(V, W)$. Namely, for $\varphi : V \to W$ and $X \in \mathfrak{g}$ we define

$$(X \cdot \varphi)(v) := X \cdot (\varphi(v)) - \varphi(X \cdot v). \tag{E.14}$$

Definition E.1.26: Let V be a representation of $\mathcal{G}$. A linear subspace $W \subset V$ is called ***$\mathcal{G}$-invariant or a subrepresentation*** if $g \cdot w \in W$ for all $g \in \mathcal{G}$ and $w \in W$. In that case, we obtain representations of $\mathcal{G}$ on W and on V/W, defined by restriction respectively by

$$g \cdot (v + W) := (g \cdot v) + W \tag{E.15}$$

Definition E.1.27: for any representation of $\mathcal{G}$ on V, the subspaces $\{0\}$ and V of V are evidently invariant. If these are the only invariant subspaces the the representation is called ***irreducible***.

Definition E.1.28: To obtain decomposability of a representation, one dose not only need a nontrivial invariant subspace but also a complementary subspace, which is invariant, too. A representation of $\mathcal{G}$ is called ***completely reducible*** if any $\mathcal{G}$-invariant subspace $W \subset V$ admits a $\mathcal{G}$-invariant complement.

Proposition E.1.29: If a representation is completely reducible, then V is a direct sum of irreducible subrepresentations.

Proof. See [105, p.38] □

Definition E.1.30: A representation of $\mathcal{G}$ on V is called ***unitary***, if there is a positive definite inner product $\langle\,,\,\rangle$ on V (Hermitian if V is complex) which is $\mathcal{G}$-invariant in the sense that

$$\langle g \cdot v, g \cdot w\rangle = \langle v, w\rangle \tag{E.16}$$

for all $g \in \mathcal{G}$ and $v, w \in V$.

Proposition E.1.31: Any unitary representation is completely reducible.

Proof. See [105, p.39] □

Definition E.1.32: A ***trivial representation*** of a (Lie) group $\mathcal{G}$ (or even a Lie algebra $\mathfrak{g}$ on V is a representation on which all elements of $\mathcal{G}$ (or $\mathfrak{g}$) act as the identity mapping of V.

Definition E.1.33: Let be ρ a representation of $\mathcal{G}$ on a vector space V, then the ***dual or contragradient representation*** $\tilde{\rho}$ is defined over the dual vector space $\tilde{V}$ as $\tilde{\rho}(g) = \rho(g^{-1})^T$ for all $g \in \mathcal{G}$.

Definition E.1.34: For a representation ρ of group $\mathcal{G}$ on a complex vector space V the ***complex conjugate representation*** $\overline{\rho}$ is defined over the complex conjugate[153] vector space $\overline{V}$ such that $\overline{\rho}(g)$ is the complex conjugate of $\rho(g)$ for all $g \in \mathcal{G}$.

Definition E.1.35: A *complex conjugate contragradient representation* of a group on a vector space is simply the combination of the two definitions above.

E.2 Matrix Diagonalization

E.2.1 General Theorems

In this section two theorems on matrix diagonalization should be stated, which are found in [108, p.1056] and are of great use for diagonalization of fermion mass matrices. First we state a theorem for the most general case of an arbitrary complex square matrix. This transformation is called ***biunitary transformation***.

Theorem E.2.1: Let M be an arbitrary complex $n \times n$ matrix. Then there exist unitary $n \times n$ matrices U_L and U_R, such that

$$\hat{M} = U_L^{\dagger} M U_R \tag{E.17}$$

is diagonal, real and non-negative.

Proof. See [108, p.1056f] proof of theorem III. □

Note that this proof uses another theorem on matrix diagonalization, which tells us the obviously Hermitian matrix $M^{\dagger}M$ can be diagonalized by an unitary matrix V. Thus, the eigenvalues of $\hat{M}$ in the theorem above correspond to the eigenvalues of the matrix $M^{\dagger}M$.

[153] Note that in mathematics complex conjugations is mostly denoted by $\overline{x}$ whereas in physics complex conjugation is mostly denoted by V^*, which may be confused with the V^* indicating a dual space in mathematics

The second theorem, which applies for example for Majorana mass matrices, is a bit more specific, since it applies for symmetric matrices.

Theorem E.2.2: Let M be a complex symmetric $n \times n$ matrix. Then there exists a unitary $n \times n$ matrix U, such that

$$\hat{M} = U^T M U \tag{E.18}$$

is diagonal, real and non-negative.

Proof. See [108, p.1056] proof of theorem II. □

This theorem was first stated implicitly by I. Schur in his paper on quadratic forms [109], therefore we will refer to this theorem in this thesis also as *Schur's theorem* (on matrix diagonalization), although this type of factorization is also known as *Takagi's factorization* (see e.g. [110, p.204f]). Besides, it should be emphasized that the diagonal elements of $\hat{M}$ are the non-negative square roots of the eigenvalues of the Hermitian matrix $MM^\dagger$ and the columns of U are the corresponding eigenvectors[154]. Finally, it should be noted that this kind of factorization is no eigendecomposition of M.

E.2.2 Corrective Diagonalization

Let M_0 be a $n \times n$ matrix, which has been diagonalized as above. Hence we have

$$U^{(0)^T} M_0 U^{(0)} = \hat{m}^{(0)} = \mathrm{diag}(m_1^{(0)}, \ldots, m_n^{(0)}), \tag{E.19}$$

with real $m_i \geq 0$ for $i = 1, \ldots, n$. Let $U^{(0)}$ consist of n column vectors $u_i^{(0)}$ with n entries, i.e.

$$U^{(0)} = (u_1^{(0)}, \ldots, u_n^{(0)}), \tag{E.20}$$

and since $U^{(0)}$ is unitary its column vectors form an orthonormal system (ONS) which means

$$u_i^{(0)^\dagger} u_j^{(0)} = \delta_{ij} \quad \text{or} \quad \left| u_i^{(0)} \right|^2 = 1. \tag{E.21}$$

We can rewrite (E.19) as

$$M_0 u_i^{(0)} = m_i u_i^{(0)^*}, \tag{E.22}$$

which is the analogue to an eigenvalue equation in the case of diagonalization via Schur's theorem. Now we want to consider a small correction[155] to M_0, such as

$$M = M_0 + \lambda M_1, \tag{E.23}$$

[154] A proof of this is given in [110, p.205] and it should be noted that it can be shown that the eigenvectors to different eigenvalues of a Hermitian matrix are pairwise orthogonal.

[155] Great thanks to Prof. Grimus for his helpful advice concerning the following considerations.

with small $\lambda \in \mathbb{R}$. The corrected matrix M then can be diagonalized again according to Schur's theorem by a unitary matrix $U = (u_1, \ldots, u_n)$. The columns u_i will be are a linear combination of the $u_i^{(0)}$ since they form an ONS. Furthermore, the u_i will be close to $u_i^{(0)}$ since we consider a very small correction. Hence we get approximately

$$u_i \simeq c_i u_i^{(0)} + u_i^{(1)}, \tag{E.24}$$

with a complex factors c_i and a small correction $u_i^{(1)}$ being of the order of λ. Those $u_i^{(1)}$ are also linear combinations of the ONS $\{u_i^{(0)}\}$ and we assume $u_i^{(1)} \perp u_i^{(0)}$. To ensure the u_i being normalized we find

$$1 \stackrel{!}{=} |u_i|^2 = \left|c_i u_i^{(0)}\right|^2 + \left|u_i^{(1)}\right|^2 = |c_i|^2 \left|u_i^{(0)}\right|^2 + \left|u_i^{(1)}\right|^2 \propto |c_i|^2 + \lambda^2 \simeq |c_i|^2 . \tag{E.25}$$

Thus, c_i must be a phase factor $e^{i\alpha_i} \simeq (1 + i\alpha_i)$ with small[156] phase α_i.

The small correction to M_0 will cause a small correction to the mass eigenvalues such that we can write approximately

$$m_i \simeq m_i^{(0)} + m_i^{(1)}. \tag{E.26}$$

Under this considerations the eigenvalue equation analogue for the corrected matrix M can be written as

$$Mu_i = (M_0 + \lambda M_1)(c_i u_i^{(0)} + u_i^{(1)}) = (m_i^{(0)} + \lambda m_i^{(1)})(c_i u_i^{(0)} + u_i^{(1)})^*. \tag{E.27}$$

Multiplication with $\left(u_j^{(0)}\right)^T$ from the left leads to

$$\left(u_j^{(0)}\right)^T (M_0 + \lambda M_1)(c_i u_i^{(0)} + u_i^{(1)}) = \left(u_j^{(0)}\right)^T (m_i^{(0)} + \lambda m_i^{(1)})(c_i u_i^{(0)} + u_i^{(1)})^*. \tag{E.28}$$

Using the condition (E.21) as well as relation (E.22) and its transposed, the left hand side gives

$$\begin{aligned}
&\left(u_j^{(0)}\right)^T (M_0 + \lambda M_1)((1 + i\alpha_i)u_i^{(0)} + u_j^{(1)}) \\
=&(1 + i\alpha_i)\left(u_j^{(0)}\right)^T M_0 u_i^{(0)} + \left(u_j^{(0)}\right)^T M_0 u_i^{(1)} \\
&+ \lambda(1 + i\alpha_i)\left(u_i^{(0)}\right)^T M_1 u_i^{(0)} + \lambda\left(u_j^{(0)}\right)^T M_0 u_i^{(1)} \\
\simeq&(1 + i\alpha_i)m_j^{(0)}\delta_{ji} + m_j^{(0)}\left(u_j^{(0)}\right)^\dagger u_i^{(1)} + \lambda\left(u_j^{(0)}\right)^T M_1 u_i^{(0)}
\end{aligned} \tag{E.29}$$

where we only considered terms of maximal order λ. Analogously we get for the right hand side

$$\begin{aligned}
&\left(u_j^{(0)}\right)^T (m_i^{(0)} + \lambda m_i^{(1)})((1 + i\alpha_i)u_i^{(0)} + u_i^{(1)})^* \\
=&(1 - i\alpha_i)\left(u_j^{(0)}\right)^T m_i^{(0)}\left(u_i^{(0)}\right)^* + \left(u_j^{(0)}\right)^T m_i^{(0)}\left(u_i^{(1)}\right)^* \\
&+ \lambda(1 - i\alpha_i)\left(u_j^{(0)}\right)^T\left(u_i^{(0)}\right)^* + \lambda\left(u_j^{(0)}\right)^T m_i^{(1)}\left(u_j^{(1)}\right)^* \\
\simeq&(1 - i\alpha_i)m_i^{(0)}\delta_{ji} + m_i^{(0)}\left(u_j^{(0)}\right)^T\left(u_i^{(1)}\right)^* + \lambda m_i^{(1)}\delta_{ji}.
\end{aligned} \tag{E.30}$$

[156] We have $1 = \left|e^{i\alpha_i}\right|^2 + \left|u_i^{(1)}\right|^2 \simeq |(1 + i\alpha_i|^2 + \lambda^2) = 1 - |\alpha_i|^2 + \lambda^2$. Hence, α_i is of the order of λ.

Therefore we have

$$\begin{aligned}(1+i\alpha_i)m_j^{(0)}\delta_{ji} + m_j^{(0)}\left(u_j^{(0)}\right)^\dagger u_i^{(1)} + \lambda\left(u_j^{(0)}\right)^T M_1 u_i^{(0)} \\ = (1-i\alpha_i)m_i^{(0)}\delta_{ji} + m_i^{(0)}\left(u_j^{(0)}\right)^T \left(u_i^{(1)}\right)^* + \lambda m_i^{(1)}\delta_{ji}.\end{aligned} \tag{E.31}$$

In the case $i = j$ this leads to

$$(1+i\alpha_i)m_i^{(0)} + \lambda\left(u_i^{(0)}\right)^T M_1 u_i^{(0)} = (1-i\alpha_i)m_i^{(0)} + \lambda m_i^{(1)} \tag{E.32}$$

where we used the assumption $u_i^{(1)} \perp u_i^{(0)}$. We obtain the following expression for the correction to the diagonal matrix $\hat{m}$

$$\begin{aligned}\lambda m_i^{(1)} =& 2i\alpha_i m_i^{(0)} + \lambda\left(u_i^{(0)}\right)^T M_1 u_i^{(0)} \\ =& \lambda\mathrm{Re}\left(\left(u_i^{(0)}\right)^T M_1 u_i^{(0)}\right) + i\left(\mathrm{Im}\left(\left(u_i^{(0)}\right)^T M_1 u_i^{(0)}\right) + 2\alpha_i m_i^{(0)}\right)\end{aligned} \tag{E.33}$$

Since $m_i^{(1)}$ has to be real the phase α_i has to be such that the two last imaginary terms cancels, namely

$$\alpha_i = \frac{\mathrm{Im}\left(\left(u_i^{(0)}\right)^T M_1 u_i^{(0)}\right)}{2m_i^{(0)}}. \tag{E.34}$$

Therefore we find as final result

$$\boxed{m_i = m_i^{(0)} + \lambda m_i^{(1)} = m_i^{(0)} + \lambda\mathrm{Re}\left(\left(u_i^{(0)}\right)^T M_1 u_i^{(0)}\right)}. \tag{E.35}$$

In the case of $i \neq j$ we get

$$m_j^{(0)}\left(u_j^{(0)}\right)^\dagger u_i^{(1)} + \lambda\left(u_j^{(0)}\right)^T M_1 u_i^{(0)} = m_i^{(0)}\left(\left(u_j^{(0)}\right)^\dagger u_i^{(1)}\right)^*. \tag{E.36}$$

Introducing the notation $A_{ji} = \left(u_j^{(0)}\right)^\dagger u_i^{(1)}$ and $B_{ji} = \left(u_j^{(0)}\right)^T M_1 u_i^{(0)}$ we obtain

$$\left(m_j^{(0)} - m_i^{(0)}\right)\mathrm{Re}(A_{ji}) + i\left(m_j^{(0)} + m_i^{(0)}\right)\mathrm{Im}(A_{ji}) = -\mathrm{Re}(B_{ji}) - i\mathrm{Im}(B_{ji}) \tag{E.37}$$

and therefore we get

$$\left(u_j^{(0)}\right)^\dagger u_i^{(1)} = \mathrm{Re}(A_{ji}) + i\mathrm{Im}(A_{ji}) = -\frac{\mathrm{Re}(B_{ji})}{\left(m_j^{(0)} - m_i^{(0)}\right)} - i\frac{\mathrm{Im}(B_{ji})}{\left(m_j^{(0)} + m_i^{(0)}\right)}. \tag{E.38}$$

This gives us also a relation for the correction of the columns $u_i^{(0)}$.

E.3 Rank-Nullity Theorem

Theorem E.3.1: Let V and W be vector spaces over a field and A a linear mapping $A: V \to W$, then the following holds:

$$\dim V = \dim(\operatorname{im} A) - \dim(\ker A)$$

where the image and the kernel of the mapping A are defined as

$$\begin{aligned} \operatorname{im} A &= \{w \in W | Av = w \,\text{for some}\, v \in V\} \\ \ker A &= \{v \in V | Av = 0\} \end{aligned}$$

E.4 Cauchy-Schwarz Inequality

Let be V a n-dimensional $\mathbb{C}$ vector space with inner product $\langle .,. \rangle$ and norm $||.||$. For two vectors $x = (x_i)_i$, $y = (y_i)_i \in V$ the Cauchy-Schwarz inequality tells us

$$|\langle x, y \rangle| \leq ||x|| \cdot ||y||. \tag{E.39}$$

In component notation we may write

$$\left| \sum_{i=1}^{n} x_i y_i^* \right|^2 \leq \sum_{j=1}^{n} |x_i|^2 \cdot \sum_{k=1}^{n} |y_i|^2 \tag{E.40}$$

where * denotes complex conjugation. Note that for $z \in \mathbb{C}$ the absolute value is $|z| = zz^*$.

E.5 Trigonometric Functions

In this section we will briefly state some properties and relations of trigonometric functions, especially sine and cosine. The following relations are valid:

$$\cos(\theta)^2 = \frac{1}{2}(1 + \cos(2\theta)), \tag{E.41}$$

$$\sin(\theta)^2 = \frac{1}{2}(1 - \cos(2\theta)), \tag{E.42}$$

$$\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2, \tag{E.43}$$

$$\frac{1}{2}\sin(2\theta) = \sin(\theta)\cos(\theta). \tag{E.44}$$

E.6 Integration Tricks for Loop-Integrals

E.6.1 Feynman Parametrization

According to [26, p.189f], this useful trick enables us to squeeze different denominator factors into a single quadratic polynomial. After performing a shift of the integration variable, we can complete the square in this polynomial and evaluate the remaining spherically symmetric integral. In the simplest case we consider two different denominators,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2}, \tag{E.45}$$

but this can be extended straight forward for n denominators[157]. Nevertheless, this version will be sufficient for our purpose. The parameter x is called Feynman parameter.

E.6.2 Wick Rotation

Another very useful trick is Wick Rotation. It can be used to evaluate integrals of a specific form, which appear in loop calculations. For those integrals we can use

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - a + i\varepsilon)^\alpha} = \frac{i(-1)^\alpha \Gamma(\alpha - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(\alpha)} (a - i\varepsilon)^{\frac{d}{2} - \alpha}, \tag{E.46}$$

where the Gamma function is defined as

$$\Gamma(n) = (n-1)!. \tag{E.47}$$

E.6.3 Logarithmic Integrals

In our one-loop calculations we have to solve the following specific integral:

$$\begin{aligned} \int_0^1 dx \ln(ax+b) &= \frac{a+b}{a} \ln(a+b) - 1 - \frac{b}{a} \ln b \\ &= \ln(a+b) - 1 + \frac{a}{b} \left(\ln(a+b) - \ln a \right). \end{aligned} \tag{E.48}$$

[157] This formula can be found in [26, p.190] .

F Appendix - Calculations and Derivations

For the sake of completeness all more or less lengthy computations and derivations, which have been skipped in the main part of this thesis for a more fluent legibility, have been summarized in this appendix.

F.1 Majorana Lagrangian for Chiral Fields

We like to rewrite the Majorana field in the form defined in (1.45) using the properties (1.25) and (1.26), hence we can write

$$\psi = P_L\psi_L + P_R\left(\psi_L\right)^C . \tag{F.1}$$

Inserting this in the Dirac Lagrangian (1.46) gives

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left[\overline{\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)} i\partial\!\!\!/ \left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right. \\
&\left. - m\overline{\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)}\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right] \\
=& \frac{1}{2}\left[\left(\psi_L P_R + \left(\psi_L\right)^C P_L\right) i\partial\!\!\!/ \left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right. \\
&\left. - m\left(\psi_L P_R + \left(\psi_L\right)^C P_L\right)\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right] \\
=& \frac{1}{2}\left[\left(\psi_L\gamma^\mu P_L + \left(\psi_L\right)^C \gamma^\mu P_R\right) i\partial_\mu \left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right. \\
&\left. - m\left(\psi_L P_R + \left(\psi_L\right)^C P_L\right)\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)\right] .
\end{aligned} \tag{F.2}$$

For achieving this we used the anticommutation relation (A.11) for γ^5 as well as the fact that

$$\overline{P_L\psi_L} = \psi_L^\dagger P_L^\dagger\gamma^0 = \psi_L^\dagger P_L\gamma^0 = \psi_L^\dagger\gamma^0 P_R = \overline{\psi}_L P_R, \tag{F.3}$$

and hence

$$\overline{\psi} = \overline{\left(P_L\psi_L + P_R\left(\psi_L\right)^C\right)} = \psi_L P_R + \left(\psi_L\right)^C P_L. \tag{F.4}$$

This can be achieved by using the properties of the chiral projection operators given in (1.24), which also cause terms of the type $P_L P_R$ to vanish in the Lagrangian. Noticing this and reabsorbing the projection operators in the fields according to (1.25) leads to

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2}\left[\overline{\psi}_L i\partial\!\!\!/ P_L^2\psi_L + \overline{\left(\psi_L\right)^C} i\partial\!\!\!/ P_R^2\left(\psi_L\right)^C - m\left(\overline{\psi}_L P_R^2\left(\psi_L\right)^C + \overline{\left(\psi_L\right)^C} P_L^2\psi_L\right)\right] \\
=& \frac{1}{2}\left[\overline{\psi}_L i\partial\!\!\!/ \psi_L + \overline{\left(\psi_L\right)^C} i\partial\!\!\!/ \left(\psi_L\right)^C - m\left(\overline{\psi}_L\left(\psi_L\right)^C + \overline{\left(\psi_L\right)^C}\psi_L\right)\right] .
\end{aligned} \tag{F.5}$$

F.2 Majorana Mass Lagrangian for Chiral Physical Fields

From (2.29) and the unitarity of U_M (i.e. $U_M^* U_M^T = \mathbb{1} = U_M^T U_M^*$) we get $\nu_L = U_M n_L$. Hence, we obtain

$$\begin{aligned}
\overline{\nu}_L &= \overline{(U_M N_L)} = \overline{N}_L U_M^\dagger & &\Rightarrow \overline{N}_L = \overline{\nu}_L U_M, \\
(\nu_L)^C &= C\overline{\nu}_L^T = C(U_M^\dagger)^T \overline{N}_L^T = U_M^*(N_L)^C & &\Rightarrow (N_L)^C = U_M^T(\nu_L)^C, \\
\overline{(\nu_L)^C} &= \overline{(U_M^*(N_L)^C)} = \overline{(N_L)^C} U_M^T & &\Rightarrow \overline{(N_L)^C} = \overline{(\nu_L)^C}(U_M^\dagger)^T.
\end{aligned}$$

Now being aware of this, the computation is done easily as follows:

$$\begin{aligned}
\mathcal{L}_M^{(L)} &= -\frac{1}{2}\left[\overline{(\nu_L)^C} M_L \nu_L + \overline{\nu}_L M_L^\dagger (\nu_L)^C\right] = -\frac{1}{2}\left[\overline{(\nu_L)^C}(U_M^\dagger)^T m_L U_M^\dagger \nu_L + \overline{\nu}_L U_M m_L^\dagger U_M^T (\nu_L)^C\right] \\
&= -\frac{1}{2}\overline{(N_L)^C} m_L N_L - \frac{1}{2}\overline{N}_L m_L (N_L)^C
\end{aligned} \tag{F.6}$$

F.3 Majroana Mass Lagrangian for Physical Fields

To achieve the Majorana mass term for the LH fields in terms of the physical fields, we simply use $\xi = P_L\xi + P_R\xi$ and $P_L\xi = N_L$ resp. $P_R\xi = (N_L)^C$. Thus, we get

$$\begin{aligned}
\mathcal{L}_M^{(L)} &= -\frac{1}{2}\overline{(N_L)^C} m_L N_L - \frac{1}{2}\overline{N}_L m_L (N_L)^C = -\frac{1}{2}\overline{(P_R\xi)} m_L P_L\xi - \frac{1}{2}\overline{(P_L\xi)} m_L P_R\xi \\
&= -\frac{1}{2}\overline{(P_L\xi + P_R\xi)} m_L (P_R\xi + P_L\xi) \quad = -\frac{1}{2}\overline{\xi} m_L \xi.
\end{aligned} \tag{F.7}$$

F.4 General Dirac-Majorana Lagrangian in Compact Notation

In this section we want to do the calculations explicitly as done in [35, p.43] for obtaining the Dirac-Majorana mass Lagrangian in terms of the LH $2n$ column field ω_L, which has been defined in (2.42) as

$$\omega_L = \begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix}. \tag{F.8}$$

We want to show how the Lagrangian (2.44) is obtained form (2.41), which is given as the sum of the Dirac mass term and the Majorana mass term for LH and RH neutrinos as

$$\begin{aligned}
\mathcal{L}_{\text{D+M}} &= \mathcal{L}_D + \mathcal{L}_M^{(L)} + \mathcal{L}_M^{(R)} \\
&= \left[-\overline{\nu}_R M_D \nu_L - \frac{1}{2}\overline{(\nu_L)^C} M_L \nu_L - \frac{1}{2}\overline{(\nu_R)^C} M_R \nu_R\right] + \text{H.c.}\,.
\end{aligned} \tag{F.9}$$

We will need the following relations, which can be obtained from relations of the charge conjugated spinors as well as from the defining relation for the charge conjugation matrix C, which we have already discussed in section B.2.4 and B.2.2.

These relations are

$$(\nu_{L/R})^C \equiv C\overline{(\nu_{L/R})}^T, \tag{F.10}$$

$$\nu^*_{L/R} = -C^{-1}\gamma^0(\nu_{L/R})^*, \tag{F.11}$$

$$(\nu_{L/R})^{CC} = \nu_{L/R}, \tag{F.12}$$

$$\overline{\nu_{L/R}} = \overline{\nu^{CC}_{L/R}} = -\left[(\nu_{L/R})^C\right]^T C^{-1}, \tag{F.13}$$

$$\gamma^{0^T} C^{-1}\gamma^0 = -C^{-1}. \tag{F.14}$$

Furthermore, we should be aware of

$$\overline{\nu}_R M_D \nu_L = \nu_L^T M_D \overline{\nu}_R^T = \overline{(\nu_L)^C} M_D{}^T (\nu_R)^C, \tag{F.15}$$

where we used according to (B.49)

$$\overline{(\nu_{L/R})^C} = -\nu^T_{L/R} C^{-1}. \tag{F.16}$$

Thus, we can rewrite the Dirac mass term in the form

$$\begin{aligned}
\mathcal{L}_D &= -\overline{\nu}_R M_D \nu_L + \text{H.c.} \\
&\overset{(F.13)}{=} \left[(\nu_R)^C\right]^T C^{-1} M_D \nu_L + \text{H.c.} \\
&= \frac{1}{2}\left\{\left[(\nu_R)^C\right]^T C^{-1} M_D \nu_L + \nu_L^T C^{-1} M_D^T (\nu_R)^C\right\} + \text{H.c.} \\
&= -\frac{1}{2}\left[\overline{\nu}_R M_D \nu_L + \overline{(\nu_L)^C} M_D^T (\nu_R)^C\right] + \text{H.c.}\,,
\end{aligned} \tag{F.17}$$

and also the Majorana mass term for RH neutrino singlets as

$$\begin{aligned}
\mathcal{L}_M^{(R)} &= \frac{1}{2}\left\{\nu_R^T C^{-1} M_R^* \nu_R + \left(\nu_R^T C^{-1} M_R^* \nu_R\right)^\dagger\right\} \\
&= \left(\frac{1}{2}\nu_R^T C^{-1} M_R^* \nu_R\right)^\dagger + \text{H.c.} \\
&= \frac{1}{2}\nu_R^\dagger C M_R \nu_R^* + \text{H.c.} \\
&\overset{(F.11)}{=} \frac{1}{2}\left[-C^{-1}\gamma^0(\nu_R)^C\right]^T C M_R \left[-C^{-1}\gamma^0(\nu_R)^C\right] + \text{H.c.} \\
&= \frac{1}{2}\left(\nu_R^C\right)^T \left(\gamma^0\right)^T \left(-C^{-1}\right)^T M_R \underbrace{C\left(-C^{-1}\right)}_{-\mathbb{1}}\gamma^0 \nu_R^C + \text{H.c.} \\
&= -\frac{1}{2}\left(\nu_R^C\right)^T \left(\gamma^0\right)^T C^{-1} M_R \gamma^0 \nu_R^C + \text{H.c.} \\
&= \frac{1}{2}\left[(\nu_R)^C\right]^T \left(-\gamma^{0^T} C^{-1}\gamma^0\right) M_R (\nu_R)^C + \text{H.c.} \\
&\overset{(F.14)}{=} \frac{1}{2}\left[(\nu_R)^C\right]^T C^{-1} M_R (\nu_R)^C \\
&= -\frac{1}{2}\left[\overline{(\nu_R)^C}\right]^C M_R (\nu_R)^C + \text{H.c.} \\
&= -\frac{1}{2}\overline{\nu}_R M_R (\nu_R)^C + \text{H.c.}\,.
\end{aligned} \tag{F.18}$$

Of course we can rewrite the Majorana mass term for LH neutrinos in a complete analogous way by

$$\begin{aligned}\mathcal{L}_M^{(L)} &= \left(\frac{1}{2}\nu_L^T C^{-1} M_L^* \nu_R\right)^\dagger + \text{H.c.} \\ &= \frac{1}{2}\left[(\nu_L)^C\right]^T C^{-1} M_L (\nu_L)^C + \text{H.c.} \\ &= -\frac{1}{2}\overline{\nu}_L M_L (\nu_L)^C + \text{H.c.}\,.\end{aligned} \tag{F.19}$$

Now we like to compare this results with (2.44) inserting the adjoint and charge conjugated LH column vector ω_L, which can be written as

$$\overline{(\omega_L)^C} = -\omega_L^T C^{-1} = -\left(\nu_L^T C^{-1}, \left[(\nu_R)^C\right]^T C^{-1}\right) = \left(\overline{(\nu_L)^C}.\overline{\nu_R}\right), \tag{F.20}$$

Applying this as well as (F.10) and (F.16) we obtain,

$$\begin{aligned}\mathcal{L}_{D+M} &= -\frac{1}{2}\overline{(\omega_L)^C} M \omega_L + \text{H.c.} \\ &= -\frac{1}{2}\left(\overline{(\nu_L)^C}, \overline{\nu_R}\right)\begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix}\begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix} + H.c. \\ &= -\frac{1}{2}\left(\overline{(\nu_L)^C} M_L + \overline{\nu_R} M_D, \overline{(\nu_L)^C} M_D^T + \overline{\nu_R} M_R\right)\begin{pmatrix} \nu_L \\ (\nu_R)^C \end{pmatrix} + H.c. \\ &= -\frac{1}{2}\left[\overline{(\nu_L)^C} M_L \nu_L + \overline{\nu_R} M_D \nu_L + \overline{(\nu_L)^C} M_D^T (\nu_R)^C + \overline{\nu_R} M_R (\nu_R)^C\right] \\ &= \underbrace{-\frac{1}{2}\left[\overline{\nu}_R M_D \nu_L + \overline{(\nu_L)^C} M_D^T (\nu_R)^C\right]}_{\mathcal{L}_D} \underbrace{-\frac{1}{2}\overline{\nu}_R M_R (\nu_R)^C}_{\mathcal{L}_M^{(R)}} \underbrace{-\frac{1}{2}\overline{(\nu_L)^C} M_L \nu_L}_{\mathcal{L}_M^{(L)}} + \text{H.c.}\,,\end{aligned} \tag{F.21}$$

and these terms are exactly those obtained in (F.17), (F.18) and (F.19).

F.5 Properties of the Diagonalization Matrix U

In section 3.1 we introduced the unitary matrix U to diagonalize the neutrino mass matrix M_{D+M}, i.e.

$$U^T M_{\mathrm{D+M}} U = \hat{m} = \mathrm{diag}(m_1, m_2, \ldots, m_{n_L+n_R}), \tag{F.22}$$

where m_i are real and non-negative. We decomposed U into two submatrices

$$U = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix}, \tag{F.23}$$

and hence

$$U^\dagger = (U_L^\dagger, U_R^T). \tag{F.24}$$

Of course unitarity means

$$U^\dagger U = \mathbb{1}_{(n_L+n_R)\times(n_L+n_R)} = UU^\dagger. \tag{F.25}$$

Thus, the first equation gives

$$\mathbb{1}_{(n_L+n_R)\times(n_L+n_R)} = U^\dagger U = (U_L^\dagger, U_R^T)\begin{pmatrix} U_L \\ U_R^* \end{pmatrix} = (U_L^\dagger U_L + U_R^T U_R^*), \tag{F.26}$$

and the second one

$$\begin{aligned} \mathbb{1}_{(n_L+n_R)\times(n_L+n_R)} &= UU^\dagger = \begin{pmatrix} U_L \\ U_R^* \end{pmatrix}(U_L^\dagger, U_R^T) = \begin{pmatrix} U_L U_L^\dagger & U_L U_R^T \\ U_R^* U_L^\dagger & U_R^* U_R^T \end{pmatrix}, \\ \begin{pmatrix} \mathbb{1}_{n_L\times n_L} & 0_{n_L\times n_R} \\ 0_{n_R\times n_L} & \mathbb{1}_{n_R\times n_R} \end{pmatrix} &= \begin{pmatrix} U_L U_L^\dagger & U_L U_R^T \\ U_R^* U_L^\dagger & U_R^* U_R^T \end{pmatrix}. \end{aligned} \tag{F.27}$$

So we can simply extract the relations for the submatrices and get

$$\mathbb{1}_{n_L\times n_L} = U_L U_L^\dagger, \quad \mathbb{1}_{n_R\times n_R} = U_R U_R^\dagger, \quad 0_{n_R\times n_L} = U_R^* U_L^\dagger. \tag{F.28}$$

Furthermore, we can show that applying this decomposition to equation (F.22) leads to

$$\begin{aligned} M_{D+M} &= U^* \hat{m} U^\dagger, \\ \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} &= \begin{pmatrix} U_L^* \\ U_R \end{pmatrix} \hat{m} (U_L^\dagger, U_R^T), \end{aligned} \tag{F.29}$$

where we can after performing the matrix multiplication on the right hand side simply read off the submatrices:

$$U_L^* \hat{m} U_L^\dagger = 0_{n_L\times n_L} (= M_L), \tag{F.30}$$

$$U_R \hat{m} U_R^T = M_R, \tag{F.31}$$

$$U_R \hat{m} U_L^\dagger = M_D. \tag{F.32}$$

Using the last two results we also obtain the following useful relation:

$$U_R^\dagger M_D = U_R^\dagger U_R \hat{m} U_L^\dagger = (\mathbb{1}_{n_L+n_R} - U_L^T U_L^*) \hat{m} U_L^\dagger = \hat{m} U_L^\dagger. \tag{F.33}$$

F.6 Yukawa Interactions of Neutral Scalar Mass Fields

In equation (3.116) of section 3.2.2 we indicated that we used the Majorana condition $\chi^C = \chi$ to obtain

$$\overline{\chi} A P_L \chi = \overline{\chi} P_L A^T \chi. \tag{F.34}$$

Here we want to give an elaborated derivation of this property. We start with the left hand side of the relation above and employ the Majorana condition:

$$\begin{aligned} \overline{\chi} A P_L \chi &= (C\gamma_0^T \chi^*)^\dagger \gamma_0 A P_L (C\gamma_0^T \chi^*) \\ &= \chi^T \gamma_0^T C^\dagger \gamma_0 A P_L C \gamma_0^T \chi^* \\ &= \chi^T \gamma_0^T C^\dagger \gamma_0 \overbrace{C C^{-1}} A P_L C \gamma_0^T \chi^* \\ &= \chi^T \gamma_0^T \underbrace{C^\dagger \gamma_0 C}_{-\gamma_0^T} A \underbrace{C^{-1} P_L C}_{P_L^T} \gamma_0^T \chi^* \end{aligned}$$

$$
\begin{aligned}
&= -\chi^T \underbrace{\gamma_0^T \gamma_0^T}_{\mathbb{1}} A P_L^T \gamma_0^T \chi^* \\
&= -\chi^T A P_L^T \gamma_0^T \chi^* \\
&= -\chi_{ia} P_{L\,ab}^T A_{ij} \gamma_{0\,bc}^T \chi_{ij}^* \\
&= \chi_{jc}^* \gamma_{0cb} A_{ji}^T P_{Lba} \chi_{ia} \\
&= \overline{\chi} P_L A^T \chi,
\end{aligned}
\tag{F.35}
$$

where we switched in the third last step to index notation with Dirac indices a, b, c and flavor indices i, j. We commute the numbers to obtain the second last line, but have to take into account that the fermion fields χ are Grassmann variables, i.e. they anticommute. Hence, we get an additional minus sign when interchanging them.

F.7 Number of Neutrinos Remaining Massless on Tree and One-Loop Level

It has been shown in [72] that the number of neutrinos remaining massless at tree-level is $n_L - n_R$ if $n_L > n_R$. More generally one can say the number of neutrinos remaining massless at tree-level is $\max(0, n_L - n_R)$. To prove this we investigate the entries of the diagonalized neutrino mass matrix given in (3.110). For this purpose we construct the diagonalizing matrix U by the following observations as done in [21].

The $n_R \times n_L$ matrix M_D can be regarded as linear mapping $M_D : \mathbb{C}^{n_L} \to \mathbb{C}^{n_R}$. The rank-nullity theorem (theorem E.3) tells us

$$\dim(\ker M_D) = \dim(\mathbb{C}^{n_L}) - \dim(\operatorname{im} M_D) = n_L - n_R, \tag{F.36}$$

and therefore we can find $n_L - n_R$ orthonormal vectors $u_i' \in \mathbb{C}^{n_L}$ spanning the kernel of M_D, i.e. the subspace $\ker M_D = \{u \in \mathbb{C}^{n_L} | M_D u = 0\} \subseteq \mathbb{C}^{n_L}$. Hence, for $1 \leq i \leq n_L - n_R$ we have

$$M_D u_i' = 0. \tag{F.37}$$

If we append n_R zeros to these vectors, we obtain $n_L - n_R$ orthonormal vectors $u_i = \binom{u_i'}{0} \in \mathbb{C}^{n_L + n_R}$ and we can find $2n_R$ vectors v_j such that the set $\{u_1, \dots u_{n_L - n_R}, v_1, \dots v_{2n_R}\}$ forms an orthonormal system of $\mathbb{C}^{n_L+n_R}$. Since U is unitary its column vectors have to form an ONS and we choose[158] the system constructed above as those columns.

Now we decompose U in four block-matrices, as

$$U = \begin{pmatrix} U_L' & U_L'' \\ \underbrace{0}_{n_L - n_R} & \underbrace{U_R''^*}_{2n_R} \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix} , \tag{F.38}$$

where

$$U_L' = (u_1', \dots, u_{n_L - n_R}') \tag{F.39}$$

[158]We can choose any appropriate U since the Takagi- or Schur-factorization is not unique in general.

$$\begin{pmatrix} U''_L \\ U''^*_R \end{pmatrix} = (v_1, \ldots, v_{2n_R}). \tag{F.40}$$

Therefore, equation (3.110) gives

$$\hat{m} = U^T M_{D+M} U = \begin{pmatrix} 0 & {U'_L}^T M_D^T U''^*_R \\ {U''_R}^\dagger M_D U'_L & {U''_L}^T M_D^T U''^*_R + {U''}^\dagger_R M_D U'_L + {U''_R}^\dagger M_R U''^*_R \end{pmatrix}. \tag{F.41}$$

The off-diagonal submatrices vanish since per construction we have $M_D U'_L = 0$. For the lower right submatrix being diagonal, U''^*_R and U''_L have to be chosen appropriately. This means we find a suitable choice for the column vectors v_i, such that

$${U''_R}^\dagger M_D U''_L + {U''_L}^T M_D^T U''^*_R + {U''_R}^\dagger M_R U''^*_R = \hat{M}'. \tag{F.42}$$

In this way we can achieve

$$\mathrm{diag}(m_{L1}, \ldots, m_{Ln_L}, m_{R1}, \ldots, m_{Rn_R}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{M}' \end{pmatrix} \begin{matrix} \}n_L \\ \}n_R \end{matrix} . \tag{F.43}$$

$$\underbrace{}_{n_L - n_R}\underbrace{}_{2n_R}$$

From this equation above it is obvious that at tree-level $n_L - n_R$ neutrinos remain massless, since $m_{Li} = 0$ for all $i = 1, \ldots n_L - n_R$. This number of massless neutrinos coincides per construction with the dimension of the kernel of the linear mapping M_D given in (F.36).

This is valid if $n_L \geq n_R$, but otherwise if $n_L < n_R$ we have $\dim(\ker M_D) \leq 0$ and since the dimension of a subspace must be non-negative we get $\dim(\ker M_D) = 0$. This means the kernel is the subspace $\{0\} \subseteq \mathbb{C}^{n_L}$ and hence there exists no such nontrivial $u'_i \in \mathbb{C}^{n_l}$ such that $M_D u'_i = 0$ and therefore also no orthonormal vectors u'_i spanning the kernel. This leads to all LH neutrinos becoming massive on tree level.

In the next step we derive the number of neutrinos remaining massless even at one-loop level. In [21] this number has been denoted by

$$n_0 = \max(0, n_L - n_R n_H), \tag{F.44}$$

which is dependent on the number of scalar doublets (n_H) and RH neutrino singlets (n_R). To show this, as done in [21], we use the result for the one-loop corrected masses, which will be shown in (F.70), i.e.

$$m_i^{(1)} = \mathrm{Re}\left({u'}_i^T \delta M_L {u'}_i\right), \tag{F.45}$$

for $i = 1, \ldots, n_L - n_R$. Hence, we want show how many $m_i^{(1)} = 0$. Therefore, we investigate the matrix structure of δM_L given in (4.78), which is of the form

$$\delta M_L = \sum_b \sum_{k=1}^{n_H} \Delta_k^T \tilde{m}_b \Delta_k + M_D^T \tilde{m}_Z M_D, \tag{F.46}$$

where we introduced the shorthand notations

$$\tilde{m}_b = \frac{1}{32\pi^2} U_R^* \hat{m} \frac{\ln r_b}{r_b - 1} U_R^\dagger, \tag{F.47}$$

$$\tilde{m}_Z = \frac{3g^2}{64\pi^2 m_W^2} U_R^* \hat{m} \frac{\ln r_z}{r_z - 1} U_R^\dagger \tag{F.48}$$

with $U_R^* = (0, U_R''^*)$. Then we find

$$\begin{aligned} {u'_i}^T \delta M_L u'_i &= \sum_b \sum_{k=1}^{n_H} {u'_i}^T \Delta_k^T \tilde{m}_b \Delta_k u'_i + {u'_i}^T M_D^T \tilde{m}_Z M_D u'_i \\ &= \sum_b \sum_{k=1}^{n_H} {u'_i}^T \Delta_k^T \tilde{m}_b \Delta_k u'_i, \end{aligned} \tag{F.49}$$

because $M_D u'_i = 0$ per construction of the column vectors u'_i.

Finding the number of neutrinos remaining massless at one-loop level, i.e. $m_i^{(1)} = 0$ means searching for the number of vectors u'_i such that

$$0 = \sum_b \sum_{k=1}^{n_H} {u'_i}^T \Delta_k^T \tilde{m}_b \Delta_k u'_i. \tag{F.50}$$

Assuming no relations among the Yukawa couplings Δ_k or excluding them, leads to the following condition to ensure the equation above. For all $i = 1, \ldots, n_L - n_R$ it has to be

$$\Delta_k u'_i = 0, \tag{F.51}$$

for all $k = 1, \ldots, n_H$. This means for $m_i^{(1)} = 0$ the vector u'_i is an element of the kernels of all linear maps Δ_k, i.e.

$$u'_i \in \bigcap_{k=1}^{n_H} \ker \Delta_k = \ker \Delta'_{n_H}, \tag{F.52}$$

where the map Δ'_{n_H} is constructed in the following way:

Let us consider the simple case of $n_H = 2$. Then, we can write

$$\begin{aligned} \ker \Delta_1 \cap \ker \Delta_2 &= \{v \in \mathbb{C}^{n_L} | \Delta_1 v = 0 \wedge \Delta_2 v = 0\} \\ &= \{v \in \ker \Delta_1 | \Delta_2 v = 0\} = \ker \left(\Delta_2 |_{\ker \Delta_1}\right). \end{aligned} \tag{F.53}$$

The dimension of this subspace can be derived as before by the rank-nullity theorem (theorem E.3) by

$$\dim(\ker\left(\Delta_2 |_{\ker \Delta_1}\right)) = \dim(\ker \Delta_1) - \dim(\operatorname{im} \Delta_2) = n_L - 2n_R. \tag{F.54}$$

Repeating this procedure until we reach Δ_{n_H} restricted to the kernel of Δ_{n_H-1} restricted to the kernel of Δ_{n_H-2} and so on until Δ_1. The dimension of the intersection of all kernels is then analogously derived as

$$\dim\left(\bigcap_{k=1}^{n_H} \ker \Delta_k\right) = n_L - n_H n_R, \tag{F.55}$$

which is the number of orthonormal vectors spanning this subspace. Since we want these vectors to give us $m_i^{(1)} = 0$, those vectors have to be the column vectors u'_i of the diagonalizing matrix U.

To achieve this we have to ensure $u'_i \in \ker \Delta_1$, which is the starting point of our procedure above. According to [21], this can be obtained by performing a basis transformation on the scalar doublets such that $v_1 \neq 0$ and $v_k = 0$ for $2 \leq k \leq n_H$. In this basis the

Dirac mass matrix reduces to

$$M_D = \frac{1}{\sqrt{2}} v_1 \Delta_1, \tag{F.56}$$

and since $u'_i \in \ker M_D$ per construction it follows $u'_i \in \ker \Delta_1$. Since the u'_i form an ONS they span the kernel $\ker \Delta'_{n_H}$. Hence, the dimension of this kernel is the number of the orthonormal column vectors u'_i for which (F.50) holds.

So the number of neutrinos remaining massless at one-loop level is given by the number $n_0 = n_L - n_H n_R$ for the case $n_L > n_H n_R$. In the case $n_L \leq n_H n_R$ the intersection of the kernel will be the zero space $\{0\}$ which has dimension zero, hence $n_0 = 0$ in this case. Thus the formula (F.44) is valid.

F.8 One-Loop Mass Corrections from $\mathrm{B_R}$

In section 4.1.2 equation (4.16) we obtained the following result for the Z^0-Boson contribution to neutrino self-energy:

$$\begin{aligned}
-i\Sigma^Z_{ij}(p)|_{m_\ell} = &-\frac{g^2}{4c_W^2} \sum_\ell \int \frac{d^dk}{(2\pi)^d} \frac{m_\ell}{k^2 - m_\ell^2 + i\varepsilon} \\
&\left\{ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\ell} \frac{4}{(k-p)^2 - m_Z^2} \left(P_L U_L^\dagger U_L - P_R U_L^T U_L^*\right)_{\ell j} \right. \\
&+ \left(P_R U_L^\dagger U_L - P_L U_L^T U_L^*\right)_{i\ell} \frac{(k-p)^2}{m_Z^2} \\
&\left. \left(-\frac{1}{(k-p)^2 - m_Z^2} + \frac{1}{(k-p)^2 - \xi_Z m_Z^2}\right) \left(P_L U_L^T U_L^* - P_R U_L^\dagger U_L\right)_{\ell j} \right\}.
\end{aligned} \tag{F.57}$$

We also know from (4.4) $-i\Sigma \propto B_R P_R$ and hence we might take the P_R terms instead of the ones proportional to P_L, which gives us

$$\begin{aligned}
(B_R)^Z_{ij}(p) = &\frac{ig^2}{4c_W^2} \sum_\ell \int \frac{d^dk}{(2\pi)^d} \frac{m_\ell}{k^2 - m_\ell^2 + i\varepsilon} (U_L^\dagger U_L)_{j\ell} (U_L^T U_L^*)_{\ell i} \\
&\cdot \left\{ \frac{4}{(k-p)^2 - m_Z^2} + \frac{(k-p)^2}{m_Z^2} \left(-\frac{1}{(k-p)^2 - m_Z^2} + \frac{1}{(k-p)^2 - \xi_Z m_Z^2} \right) \right\}.
\end{aligned} \tag{F.58}$$

From (4.5) and (4.7) we have the relation $B_L = B_R^*$, which is obviously true, if we compare the result for B_R above with the form of B_L given in (4.17). This is also the case for all other boson contributions to the neutrino self-energy. The correction to the mass matrix δM_L is computed analogously but with an additional complex conjugation

$$\delta M_L = U_L^* B_R(0)^* U_L^\dagger \tag{F.59}$$

which gives exactly the same mass correction.

F.9 Diagonalization of the Tree-Level and One-Loop Corrected Mass Matrix

In this appendix detailed calculations will be given, which have been skipped in section 4.2. There we have constructed the unitary 4×4 matrix $U^{(0)}$, which diagonalizes the tree-level neutrino mass matrix $M^{(0)}_{\mathrm{D+M}}$. We showed

$$U^{(0)^T} M^{(0)}_{D+M} U^{(0)} = \hat{m}^{(0)} = \mathrm{diag}(0, 0, m^{(0)}_3, m^{(0)}_4), \tag{F.60}$$

for

$$M^{(0)}_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix}, \tag{F.61}$$

and

$$U^{(0)} = \begin{pmatrix} U'_L & U''_L \\ 0 & U''^*_R \end{pmatrix} = \begin{pmatrix} u'_1 & u'_2 & icu'_3 & su'_3 \\ 0 & 0 & -is & c \end{pmatrix}, \tag{F.62}$$

where u'_i, $i = 1, 2, 3$ are orthogonal 3-vectors with $u'_1, u'_2 \perp M_D$ and $u'_3 = \frac{M_D}{m_D}$ and $s = \sin(\theta)$ and $c = \cos(\theta)$. The rotation angle θ is defined via $\tan(2\theta) = \frac{2m_D}{m_R}$. Furthermore, we showed in section 4.1 that one-loop corrections enter into the neutrino mass matrix in the form

$$M^{(1)}_{\mathrm{D+M}} = M^{(0)}_{D+M} + \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix}, \tag{F.63}$$

and hence we have

$$\begin{aligned} U^{(0)^T} M^{(0)}_{\mathrm{D+M}} U^{(0)} &= U^{(0)^T} M^{(0)}_{\mathrm{D+M}} U^{(0)} + U^{(0)^T} \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix} U^{(0)} \\ &= \hat{m}^{(0)} + \begin{pmatrix} U'^T_L \delta M_L U'_L & U'^T_L \delta M_L U''_L \\ U''^T_L \delta M_L U'_L & U''^T_L \delta M_L U''_L \end{pmatrix}. \end{aligned} \tag{F.64}$$

The second matrix of the second line can be easily calculated as

$$\begin{aligned} U^{(0)^T} \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix} U^{(0)} &= \begin{pmatrix} U'^T_L & 0 \\ U''^T_L & U''^\dagger_R \end{pmatrix} \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U'_L & U''_L \\ 0 & U''^*_R \end{pmatrix} \\ &= \begin{pmatrix} U'^T_L & U''^T_L \\ 0 & U''^\dagger_R \end{pmatrix} \begin{pmatrix} \delta M_L U'_L & \delta M_L U''_L \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} U'^T_L \delta M_L U'_L & U'^T_L \delta M_L U''_L \\ U''^T_L \delta M_L U'_L & U''^T_L \delta M_L U''_L \end{pmatrix} \\ &= \begin{pmatrix} {u'_1}^T \delta M_L u'_1 & {u'_1}^T \delta M_L u'_2 & ic.{u'_1}^T \delta M_L u'_3 & s.{u'_1}^T \delta M_L u'_3 \\ {u'_2}^T \delta M_L u'_1 & {u'_2}^T \delta M_L u'_2 & ic.{u'_2}^T \delta M_L u'_3 & s.{u'_2}^T \delta M_L u'_3 \\ ic.{u'_3}^T \delta M_L u'_1 & ic.{u'_3}^T \delta M_L u'_2 & -c^2.{u'_3}^T \delta M_L u'_3 & ics.{u'_3}^T \delta M_L u'_3 \\ s.{u'_3}^T \delta M_L u'_1 & s.{u'_3}^T \delta M_L u'_2 & isc.{u'_3}^T \delta M_L u'_3 & s^2.{u'_3}^T \delta M_L u'_3 \end{pmatrix}. \end{aligned} \tag{F.65}$$

If we choose $u'_1 \perp \Delta_1, \Delta_2$, we get vanishing entries in the first line and first column, i.e.

$$U^{(0)T} \begin{pmatrix} \delta M_L & 0 \\ 0 & 0 \end{pmatrix} U^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & {u'_2}^T \delta M_L u'_2 & ic.{u'_2}^T \delta M_L u'_3 & s.{u'_2}^T \delta M_L u'_3 \\ 0 & ic.{u'_3}^T \delta M_L u'_2 & -c^2.{u'_3}^T \delta M_L u'_3 & ics.{u'_3}^T \delta M_L u'_3 \\ 0 & s.{u'_3}^T \delta M_L u'_2 & isc.{u'_3}^T \delta M_L u'_3 & s^2.{u'_3}^T \delta M_L u'_3 \end{pmatrix}. \tag{F.66}$$

This is the matrix, which has to be diagonalized with a suitable matrix $V^{(1)}$. This matrix must be unitary and hence can be represented by

$$V^{(1)} = \mathrm{e}^{i\Omega} \simeq \mathbb{1} + i\Omega, \tag{F.67}$$

with Ω Hermitian and of one-loop order. To find a suitable choice of Ω, we can use the considerations done in appendix E.2.2. The diagonalizing matrix U of the corrected mass matrix $M^{(1)}_{D+M} = M^{(0)}_{D+M} + \delta M$ is according to (4.112) given as

$$U = U^{(0)} V^{(1)} = U^{(0)} + iU^{(0)}\Omega, \tag{F.68}$$

and hence

$$\begin{aligned} \hat{m}^{(1)} &= U^T M^{(1)}_{D+M} U \\ &= (U^{(0)} + iU^{(0)}\Omega)^T (M^{(0)}_{D+M} + \delta M)(U^{(0)} + iU^{(0)}\Omega) \\ &\simeq \hat{m}^{(0)} + i\hat{m}^{(0)}\Omega + i(\hat{m}^{(0)}\Omega)^T + U^{(0)T} \delta M U^{(0)}. \end{aligned} \tag{F.69}$$

Now applying the results of appendix E.2.2, we obtain for the corrected diagonalized mass matrix

$$m^{(1)}_i = m^{(0)}_i + \mathrm{Re}\left({u'}^{(0)T}_i \delta M_L {u'}^{(0)}_i \right), \tag{F.70}$$

or more explicitly

$$m^{(1)}_1 = 0 + \mathrm{Re}\left({u'_1}^{(0)T} \delta M_L {u'_1}^{(0)} \right) = 0 \tag{F.71}$$

$$m^{(1)}_2 = 0 + \mathrm{Re}\left({u'_2}^{(0)T} \delta M_L {u'_2}^{(0)} \right) = \mathrm{Re}\left({u'_2}^{(0)} \delta M_L {u'_2}^{(0)} \right) \tag{F.72}$$

$$m^{(1)}_3 = m^{(0)}_3 + \mathrm{Re}\left(-c^2.{u'_3}^{(0)T} \delta M_L {u'_3}^{(0)} \right) \simeq \frac{m_D^2}{m_R} \tag{F.73}$$

$$m^{(1)}_4 = m^{(0)}_4 + \mathrm{Re}\left(s^2.{u'_3}^{(0)T} \delta M_L {u'_3}^{(0)} \right) \simeq m_R. \tag{F.74}$$

We used the results $m^{(0)}_3$ and $m^{(0)}_4$ given in (4.106) and (4.107), which are both very large in comparison to the corrections, which are therefore negligible. The lightest neutrino mass $m^{(1)}_1$ remains zero, since we have chosen $u'_1 \perp \Delta_1, \Delta_2$. Thus, we achieved two massive LH neutrinos at one-loop level.

The matrix $\Omega = -iU^{(0)\dagger} U$ can be determined by (E.38), which fixes all off-diagonal elements. The matrix B_{ji}, which has been introduced for this result, is in this case exactly given in (F.65) above. The diagonal elements of Ω are all zero, since we assumed $u^{(0)}_i \perp u^{(1)}_i$, i.e. $u^{(0)\dagger}_i u^{(1)}_i = 0$.

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