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DISSERTATION / DOCTORAL THESIS

Titel der Dissertation / Title of the Doctoral Thesis

“Extremal Bounds of Gaussian Gabor Frames and Properties of Jacobi’s Theta Functions”

verfasst von / submitted by

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angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of
Doktor der Naturwissenschaften (Dr. rer. nat.)

Wien, 2016 / Vienna, 2016

Studienkennzahl lt. Studienblatt /

degree programme code as it appears on the student
record sheet:

A 796 605 405

Dissertationsgebiet lt. Studienblatt /

field of study as it appears on the student record
sheet:

Mathematik

Betreut von / Supervisor:

Univ.-Prof. Dr. Karlheinz Gröchenig

Acknowledgements

I gratefully acknowledge the support of Karlheinz Gröchenig during the time of my PhD studies. The goals I wanted to reach for my thesis were ambitious and the outcome was unclear. Nevertheless, he let me take on this challenge and provided resources for me to attend international conferences and doctoral schools to broaden my network and my knowledge in the field of time-frequency analysis as well as in related fields. I very much appreciate the fact that I could work in an autonomous way and his honesty in the many discussions we had about mathematical as well as non-mathematical topics.

I am very happy to have found an enthusiastic collaborator in Stefan Steinerberger. I really enjoyed working with him and hope for more joint work to come. His way of handling mathematical problems is very refreshing and the cooperation with him cannot be overvalued from my side.

Moreover, I want to thank Hans Georg Feichtinger and Maurice de Gosson for their support since my master studies and I am very pleased that I was already integrated into NuHAG by that time. Also, I want to thank all my colleagues here at NuHAG for many valuable and enjoyable discussions and for the great working atmosphere.

A big thank you has to go to my parents Gabi and Walter for their lifelong and unconditional support and of course to my sister Katrin, who is always ready to lend an ear to me.

I also want to thank my friends, with some of them I unfortunately lost touch, and at the same time I apologize that I did not always find enough time for you.

Lastly, I want to thank the person without whom I would not have reached the goal of finishing this thesis. I am grateful for your kind and loving words when I was struggling with myself and that you put faith in me when I did not. You have decorated my life.

Thank you, Barbara!

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The following work contains results obtained by the author and his collaborators. The results were achieved during the author's activity as research assistant affiliated with the "Numerical Harmonic Analysis Group" (NuHAG) at the Faculty of Mathematics at the University of Vienna. Therefore, parts of this thesis can be found verbatim in original research papers published by the author and his collaborators, in particular in the articles by Faulhuber [22, 23] and Faulhuber & Steinerberger [24]. The plots and figures in this work were created either with MATLAB [63] or Mathematica [76]. The author was supported by the Austrian Science Fund (FWF): P26273-N25.

Abstract

This work deals with a special topic in the area of Gabor frames belonging to the field of time-frequency analysis. The focus of this thesis is the investigation of sharp frame bounds of Gabor frames with Gaussian window and how they behave under a deformation of the lattice. The study of the deformation of Gabor frames itself is not new and at the same time it is not yet fully understood.

When it comes to distorting Gabor frames one quickly stumbles across the frame set of a window function. The frame set of a function is the set of all lattices which yield a frame for this particular window. Under certain decay and smoothness assumptions it is known that the frame set is open (which is not true in general). However, it is not clear how to determine the frame set of a class of functions or even a single function.

The first and hitherto only window for which the entire frame is known, is the Gaussian window. In this particular case the necessary conditions imposed by the Balian-Low Theorem and by the density theorem are already sufficient and therefore the frame set is the largest possible. In contrast to the result for the Gaussian window, conjectures about a simple structure of the frame set for the Hermite functions of higher order were disproved by counterexamples.

Gabor frames with Gaussian windows are well examined, but still, there are open problems and conjectures which will be tackled in this work and we will present solutions for particular cases. One problem is to understand the behavior of the frame bounds within the frame set. This leads to the question whether there exists a unique lattice within the frame set which leads to extremal frame bounds. We mention that the question posed is not entirely meaningful for the whole frame set, but rather for a subset which consists of lattices of the same volume.

Considering only rectangular (separable) lattices of even redundancy, we will prove that the square lattice maximizes the lower frame bound and minimizes the upper frame bound. For general lattices of even redundancy we will prove that the hexagonal lattice minimizes the upper frame bound. The results need the notion of theta functions on a lattice which in the separable case can be split into products of the classical Jacobi theta functions. In order to prove these results, new properties of Jacobi's theta functions are established.

Introduction

This thesis deals with the concept of Gabor frames, which describes intermediate cases between pure time analysis and pure frequency analysis (Fourier analysis). If we are given a signal (function), described by its temporal behavior, we may use the Fourier transform to learn the distribution of the Fourier coefficients. With these coefficients we may approximate or reconstruct our signal (function) by trigonometric functions. The drawback is that we do not get any information about the temporal distribution of the signal (function) from its Fourier coefficients. Therefore, Gabor proposed to have a two-dimensional representation of a one-dimensional signal (function) which simultaneously uses information about the distribution of the signal in time and the behavior of its frequencies, in particular the description of the distribution of the Fourier coefficients.

This leads to interesting questions. One of the very first questions in this context is the following. Is it possible to exactly determine the appearance of a certain frequency at a certain point in time? This is not possible because of uncertainty principles and, therefore, the next question is whether we can at least learn the frequency distribution in a neighborhood of a certain point in time. This is done by multiplying the signal f with a window g which is localized around some point in time. By shifting the window g in the time domain, we successively get an idea of how the frequency distribution is changing over time. This is the basic idea of Gabor frames. In the spirit of Fourier analysis we would like to be able to write f as a convergent series of some simple atoms, which in the Fourier case are the trigonometric monomials. This gives rise to the question of which functions should serve as a window g and how to spread the window in the time-frequency plane to gain a stable frame. The quality of the Gabor system is measured by two constants called frame bounds, which are obtained from the frame inequality.

These topics are briefly touched on in Section 1. We describe some key ingredients and basic concepts of Gabor analysis and have a brief look at the state of the art in time-frequency analysis. Although much of the theory will be set up in a quite general way, the Gaussian will be chosen as a window when it comes to explicit examples and calculations. The Gaussian is a popular choice due to its good decay and smoothness properties, its invariance under the Fourier transform and its property of uniquely minimizing the classical uncertainty principle.

In Section 2 we will encounter the symplectic group (a matrix group) and the metaplectic group (a group of unitary operators). We will study how the action of the symplectic group on the spreading of the atoms can be compensated by allowing the metaplectic group to act on the window, hence the atoms.

In Section 3 we will use these concepts for the Gaussian and we will see that in this case Hamiltonian mechanics paints a nice picture of the action of the aforementioned groups. We will learn how the frame property is preserved without affecting the frame bounds. Also, we will see that certain geometric characteristics are preserved.

In Section 4 we will explicitly compute frame bounds for Gaussian Gabor frames. We will encounter the Zak transform and the pre-Gramian as tools to obtain explicit formulas for the frame bounds.

In Section 5 we will study a conjecture about optimal frame bounds for Gabor frames with Gaussian windows and separable lattices. The conjecture is as follows. Given the standard Gaussian window, which lattice of prescribed fixed density minimizes the frame condition number of the resulting Gabor system? Formulated as a problem for Gabor frames, the question appeared in the literature at the latest in 2003 in the work of Strohmer & Beaver [72]. In another context an analogous problem already arose in 1995 [33]. The expected solution in [33] was the square lattice. By numerical observations it was shown in [72] that a hexagonal lattice outperforms the square lattice in the sense of the frame condition number. At least since the publication of [72] it seems that anybody who ever dealt with this problem is convinced that the solution provided by Strohmer & Beaver is optimal. However, if only separable lattices are considered, the solution is expected to be the square lattice. The mentioned lattices are shown in Figure 1. In the separable case, we

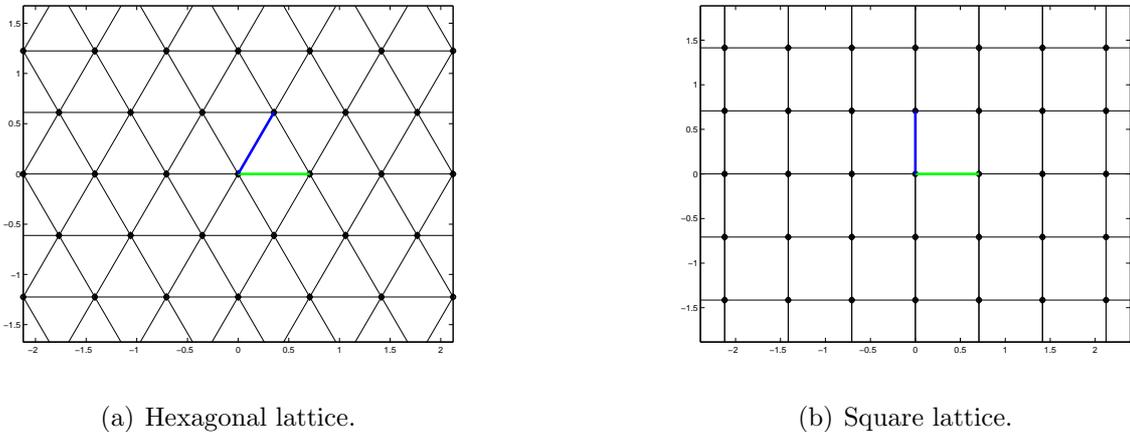


Figure 1: A hexagonal lattice and the square lattice of redundancy 2. The coloured vectors are a basis of the corresponding lattice.

will prove the conjecture to be true for special cases. For the proofs it is necessary to leave the field of Gabor analysis. It turns out that the optimization of the frame bounds turns out to be equivalent to an optimization problem for theta functions over a lattice. In the separable case, these theta functions split into products involving either Jacobi's theta-3 or Jacobi's theta-4 function. Although the problems of finding extremal points for the lower and the upper frame bound look alike, they are very different to prove. A common theme is an algebraic simplification which allows us to ignore the parameter which describes the redundancy of the Gabor system. This simplification cannot be overestimated, since the functions under consideration tend to a constant function in the limit case of the parameter. Therefore, a direct analysis of the critical points of the functions seems to be impossible. The desired extremal results about the square lattice follow from new monotonicity results about the logarithmic derivatives of Jacobi's theta functions on a logarithmic scale. We will not only prove the monotonicity results, but also some identities which are needed to establish the results as well as some consequences. Also, we would like to mention

that some of the newly established properties we prove for Jacobi's theta-3 function have already found their way into the work of other researchers.

The case of a general lattice is treated in Section 6. We will not prove the correctness of the Strohmer & Beaver conjecture, but we will prove that, under certain assumptions on the density, the hexagonal lattice minimizes the upper frame bound of a Gabor frame with standard Gaussian window. This will follow from a result by Montgomery on minimal theta functions from 1988 [65]. We will also give an analytic proof that, for redundancy 2, the frame condition number of a standard Gaussian Gabor frame with a hexagonal lattice is smaller than the condition number using the square lattice.

In Section 7 we will study a packing problem for holomorphic functions which at first seems to be unrelated to the problems studied in the previous sections. A theorem by Landau states that there exists a positive constant $\mathcal{L} > 0$ such that each holomorphic mapping f from the complex unit disc \mathbb{D} into \mathbb{C} , with $|f'(0)| = 1$ contains an open disc of minimal radius $r(f) \geq \mathcal{L}$. The smallest upper bound $\mathcal{L}_+ \geq \mathcal{L}$ was given by Rademacher by constructing a function which maps \mathbb{D} to \mathbb{C}/Λ_h where Λ_h is a hexagonal lattice with covering radius \mathcal{L}_+ . Therefore, the largest disc contained in $f(\mathbb{D})$ has exactly radius \mathcal{L}_+ and it is conjectured that $\mathcal{L} = \mathcal{L}_+$. Curiously, numerical inspections yield that $\frac{1}{\mathcal{L}_+}$ might give the value of the lower frame bound for a Gaussian Gabor frame with hexagonal lattice of redundancy 2. However, we will see a proof that the value of the lower frame bound of a Gaussian Gabor frame with a square lattice of redundancy 2 is given by the reciprocal of the radius of the largest disc of the holomorphic function, fulfilling the conditions of Landau's theorem, which maps \mathbb{D} onto $\mathbb{C}/\Lambda_\square$ where Λ_\square is a square lattice with the described covering radius.

1 Time-Frequency Analysis

A Gabor system (or Weyl-Heisenberg system) for $L^2(\mathbb{R}^d)$ is generated by a (fixed, non-zero) window function $g \in L^2(\mathbb{R}^d)$ and an index set $\Lambda \subset \mathbb{R}^{2d}$. It consists of time-frequency shifted versions of g which are called atoms. We say $\lambda = (x, \omega)^T \in \mathbb{R}^d \times \mathbb{R}^d$ is a point in the time-frequency plane and use the following notation for a time-frequency shift by λ

$$\pi(\lambda)g(t) = M_\omega T_x g(t) = e^{2\pi i \omega \cdot t} g(t - x), \quad x, \omega, t \in \mathbb{R}^d.$$

Hence, for a window function g and an index set Λ the Gabor system is

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g \mid \lambda \in \Lambda\}.$$

In order to be a frame, $\mathcal{G}(g, \Lambda)$ has to fulfil the frame inequality

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d) \quad (1.1)$$

for some positive constants $A, B > 0$ called frame bounds. If the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame it is called a Gabor frame. Usually, in this work when we speak about frame bounds we mean the tightest possible bounds which we will also call optimal frame bounds.

Throughout this work, the index set will usually be a lattice. A lattice is a discrete subgroup of the time-frequency plane, i.e. $\Lambda \subset \mathbb{R}^{2d}$ is generated by a non-unique, invertible $2d \times 2d$ matrix S , in the sense that $\Lambda = SZ^{2d}$. Whereas the generating matrix is non-unique, in fact there are countably many generators for one and the same lattice as we will see, the volume of the lattice which is defined as

$$vol(\Lambda) = |\det(S)|$$

is unique and therefore is a characteristic number for a lattice. We call the reciprocal of the volume the density or redundancy of the system

$$\delta(\Lambda) = \frac{1}{vol(\Lambda)}.$$

In the subsequent paragraphs we will see that it is meaningful to define Gabor frames for other function spaces than just $L^2(\mathbb{R}^d)$.

The first Gabor system was studied in 1932 by von Neumann [67] and by Gabor [35] in 1946 using a Gaussian window function, namely $e^{-\pi t^2}$, and the integer lattice $\mathbb{Z} \times \mathbb{Z}$. Later on, we will see that this Gabor system just fails to be a frame, due to the Balian-Low theorem. Since their introduction a lot of research has been done on the subject of Gabor frames and they have become an integral part of wireless communications [13, 51, 72], of signal processing [30, 31] and of speech processing and the analysis of acoustic signals [19]. Gabor frames are also used as tools in several mathematical fields in order to characterize smoothness properties and phase space concentration [27, 31, 41] as well as in the study of pseudodifferential operators [42]. Thus, it is of interest to further investigate and understand the structure of Gabor frames for different window functions and different types of lattices.

1.1 Basic Concepts in Time-Frequency Analysis

What we understand up to now is that the frame property of a Gabor family $\mathcal{G}(g, \Lambda)$ defined in (1.1) crucially depends on the window function g and on the lattice Λ . Although frame theory is often built upon a Hilbert space, in our case $L^2(\mathbb{R}^d)$, we know that the right setting for time-frequency analysis are the modulation spaces $M^p(\mathbb{R}^d)$ and, among many others of course, the work by Feichtinger has to be mentioned at this point [25, 26]. But before introducing modulation spaces, we start with more classical concepts. First, we fix our notation for the Fourier transform and, related to it, the short-time Fourier transform (STFT).

Definition 1.1 (Fourier Transform). For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ we define the Fourier transform of f by

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi i\omega \cdot t} dt.$$

We will use both notations $\mathcal{F}f$ and \widehat{f} in this work. The Fourier transform satisfies Plancherel's formula.

Theorem 1.2 (Plancherel). For $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ the following identity holds true

$$\|f\|_2 = \|\widehat{f}\|_2.$$

Hence, by a density argument the Fourier transform extends to a unitary operator on the Hilbert space $L^2(\mathbb{R}^d)$. Therefore, we get the following inversion formula.

Theorem 1.3 (Inversion formula). For $f \in L^2(\mathbb{R}^d)$ we have

$$f(x) = \int_{\mathbb{R}^d} \widehat{f}(\omega)e^{2\pi i x \cdot \omega} d\omega.$$

Definition 1.4 (Inverse Fourier transform). For $f \in L^2(\mathbb{R}^d)$ we define the inverse Fourier transform by

$$\mathcal{F}^{-1}f(\omega) = \mathcal{F}f(-\omega) = \int_{\mathbb{R}^d} f(t)e^{2\pi i\omega \cdot t} dt.$$

A useful tool which comes along with the Fourier transform and which we shall use frequently in this work is the Poisson summation formula.

Proposition 1.5 (Poisson summation formula). Let f be a function with the properties

$$\sum_{n \in \mathbb{Z}^d} f(x+n) \in L^2(\mathbb{T}^d)$$

and

$$\left(\widehat{f}(k)\right)_{k \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d).$$

The Poisson summation formula is then given by

$$\sum_{n \in \mathbb{Z}^d} f(n+x) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(k) e^{-2\pi i k \cdot x}$$

with equality almost everywhere.

Definition 1.6 (Short-Time Fourier Transform). For a fixed, non-zero window function $g \in L^2(\mathbb{R}^d)$ the short-time Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ with respect to the window g is defined via

$$\mathcal{V}_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \langle f, \pi(\lambda) g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega \cdot t} dt.$$

where $\lambda = (x, \omega) \in \mathbb{R}^d \times \mathbb{R}^d$.

At this point we want to introduce another concept for a quadratic representation of a function, which is closely related to the STFT.

Definition 1.7 (Ambiguity Function). The ambiguity function of a function $f \in L^2(\mathbb{R}^d)$ is given by

$$\mathcal{A}f(x, \omega) = \int_{\mathbb{R}^d} f\left(t + \frac{x}{2}\right) \overline{f\left(t - \frac{x}{2}\right)} e^{-2\pi i \omega \cdot t} dt.$$

In a similar way we define the cross-ambiguity function of two functions $f, g \in L^2(\mathbb{R}^d)$

$$\mathcal{A}_g f(x, \omega) = \int_{\mathbb{R}^d} f\left(t + \frac{x}{2}\right) \overline{g\left(t - \frac{x}{2}\right)} e^{-2\pi i \omega \cdot t} dt.$$

The cross-ambiguity function and the ambiguity function are closely related to the STFT. In fact, they only differ by a phase factor, which is a complex number of modulus 1, i.e. $c \in \mathbb{C}$, $|c| = 1$. We have

$$\mathcal{A}_g f(x, \omega) = e^{\pi i x \cdot \omega} \mathcal{V}_g f(x, \omega).$$

The appearance of the phase factor is due to the fact that the translation and modulation operators do not commute.

$$M_\omega T_x = e^{2\pi i x \cdot \omega} T_x M_\omega \tag{1.2}$$

Equation (1.2) is called the commutation relation for time-frequency shifts. The ambiguity function is somehow a more symmetric time-frequency representation of a signal than the short-time Fourier transform and in absolute values they are the same. The usual interpretation of the ambiguity function is that it tells us how much a function is spread out in time and frequency, similar to the interpretation of the Wigner distribution in physics.

Definition 1.8 (Wigner Distribution). The Wigner distribution of a function $f \in L^2(\mathbb{R}^d)$ is given by

$$\mathcal{W}f(x, \omega) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{f\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t} dt.$$

For $f, g \in L^2(\mathbb{R}^d)$ the cross-Wigner distribution is defined as

$$\mathcal{W}_g f(x, \omega) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t} dt.$$

The Wigner distribution is related to the ambiguity function by the symplectic Fourier transform which is given by

$$\mathcal{F}f(J\lambda) = \mathcal{F}f(\omega, -x), \quad \lambda = (x, \omega), \quad x, \omega \in \mathbb{R}^d.$$

$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ denotes the standard symplectic matrix which we will encounter several more times in this work.

$$\mathcal{W}f(\lambda) = \mathcal{W}f(x, \omega) = \mathcal{F}(\mathcal{A}f)(\omega, -x) = \mathcal{F}(\mathcal{A}f)(J\lambda), \quad x, \omega \in \mathbb{R}^d,$$

Similarly, we find the following relation between the cross-Wigner distribution and the cross-ambiguity function

$$\mathcal{W}_g f(\lambda) = \mathcal{F}(\mathcal{A}_g f)(J\lambda), \quad \lambda = (x, \omega), \quad x, \omega \in \mathbb{R}^d$$

We introduced 3 different kinds of quadratic representations of a function $f \in L^2(\mathbb{R}^d)$ which are all alike and have similar interpretations. We will now introduce the inversion formula for the STFT (see e.g. [41]).

Proposition 1.9 (Inversion of the STFT). For $g \in L^2(\mathbb{R}^d)$ we have

$$f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} \mathcal{V}_g f(\lambda) \pi(\lambda) g d\lambda$$

for all $f \in L^2(\mathbb{R}^d)$.

In a similar way we can reconstruct f , given g , from the cross-ambiguity function or from the cross-Wigner distribution. We note the fine difference, that given the ambiguity function or the Wigner distribution, we can only reconstruct f up to a phase factor $|c| = 1$, since $\mathcal{A}(f) = \mathcal{A}(cf)$.

Proposition 1.10. For $f \in \mathcal{S}(\mathbb{R}^d)$ with $f(0) \neq 0$ we have

$$\overline{f(0)} f(x) = \int_{\mathbb{R}^d} \mathcal{A}f(x, \omega) e^{\pi i x \cdot \omega} d\omega.$$

If we want to reconstruct g from the STFT, the cross-ambiguity function or the cross-Wigner distribution this can be done since $\mathcal{V}_g(cg) = c\mathcal{V}_g(g)$, $\mathcal{A}_g(cg) = c\mathcal{A}_g(g)$ and $\mathcal{W}_g(cg) = c\mathcal{W}_g(g)$ for $c \in \mathbb{C}$. The difference between Proposition 1.9 and Proposition 1.10 is that in one case we assume that g is known whereas in the other case we assume that we are given the auto-correlations and have to find g . Speaking in terms of the STFT, given $\mathcal{V}_g g$ without prior knowledge of the window, the task of finding g is also only solvable up to a phase factor since we cannot distinguish between $\mathcal{V}_g g$ and $\mathcal{V}_{(cg)}(cg)$.

In this work we will mostly use the ambiguity function to state properties of a function f in the time-frequency plane.

So far, we have worked in the Hilbert space $L^2(\mathbb{R}^d)$. We introduce some more function spaces which play essential roles in time-frequency analysis.

Definition 1.11 (Modulation Space). For $1 \leq p \leq \infty$ and any (non-zero) window function $g \in \mathcal{S}(\mathbb{R}^d)$ in the Schwartz space, the modulation space $M^p(\mathbb{R}^d)$ consists of all elements $f \in \mathcal{S}'(\mathbb{R}^d)$ in the space of tempered distributions such that the norm

$$\|f\|_{M^p}^p = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle f, M_\omega T_x g \rangle|^p dx d\omega$$

is finite with the usual adjustment for the ∞ -norm.

The definition of the short-time Fourier transform can be extended by duality principles or the use of Banach-Gelfand triples [29, 43]. Therefore, the short-time Fourier transform is also defined for $(f, g) \in (\mathcal{S}', \mathcal{S})$ or $(f, g) \in (\mathcal{S}'_0, \mathcal{S}_0) = (M^\infty, M^1)$.

The modulation spaces are independent of the choice of $g \in \mathcal{S}(\mathbb{R}^d)$ and also, $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. The modulation space $M^1(\mathbb{R}^d)$, called Feichtinger's algebra and often denoted by \mathcal{S}_0 , is the smallest function space invariant under time-frequency shifts and the Fourier transform which also contains the Schwartz space. It was first introduced by Feichtinger in 1981 [25]. It is also a Banach space, embedded in $L^1(\mathbb{R}^d) \cap \mathcal{C}_0(\mathbb{R}^d)$. Consequently, in time-frequency analysis $M^1(\mathbb{R}^d)$ is a natural choice for the window functions g . Its dual space $M^\infty(\mathbb{R}^d)$, often denoted by $\mathcal{S}'_0(\mathbb{R}^d)$, is the canonical space of distributions in time-frequency analysis. Thus, in time-frequency analysis the pair (M^1, M^∞) is the appropriate substitute for the usual pair $(\mathcal{S}, \mathcal{S}')$ in analysis [43]. With the modulation spaces as instruments in our hands, we revisit the Poisson summation formula.

Proposition 1.12 (Poisson Summation Formula). *Let $f \in M^1(\mathbb{R}^d)$, then the Poisson summation formula*

$$\sum_{n \in \mathbb{Z}^d} f(x + n) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{-2\pi i k \cdot x}$$

holds for all $x \in \mathbb{R}^d$.

The condition $f \in M^1(\mathbb{R}^d)$ implies that also $\hat{f} \in M^1(\mathbb{R}^d)$. Therefore both sums converge absolutely. In contrast to Proposition 1.5 the formula now holds pointwise since in addition both f and \hat{f} are continuous.

We also want to mention a second class of function spaces frequently used in time-frequency analysis, the Wiener amalgam spaces $W(L^p(\mathbb{R}^d), \ell^q(\mathbb{Z}^d))$.

Definition 1.13 (Wiener Amalgam Space). A function f is an element of the Wiener amalgam space $W(L^p(\mathbb{R}^d), \ell^q(\mathbb{Z}^d))$ if the norm defined via

$$\|f\|_W = \left(\sum_{k \in \mathbb{Z}^d} \|f \cdot T_k \chi_{[0,1]^d}\|_p^q \right)^{1/q}$$

is finite, with the usual adjustment for the ∞ -norm.

The Wiener amalgam spaces allow us to describe a function in terms of its local as well as its global behavior. At this point we mention that the modulation space $M^1(\mathbb{R}^d)$ is a subspace of the intersection of the Wiener space $W(L^\infty(\mathbb{R}^d), \ell^1(\mathbb{Z}^d))$ and its image under the Fourier transform

$$M^1(\mathbb{R}^d) \subset W(L^\infty(\mathbb{R}^d), \ell^1(\mathbb{Z}^d)) \cap \mathcal{F}W(L^\infty(\mathbb{R}^d), \ell^1(\mathbb{Z}^d)).$$

For more properties about Wiener amalgam spaces we refer to the textbooks [30, 41] and the references therein.

As can already be seen from the frame property in equation (1.1), there is a canonical operator, the coefficient operator, which can be associated to the Gabor family $\mathcal{G}(g, \Lambda)$.

Definition 1.14 (Coefficient Operator). Let $f, g \in L^2(\mathbb{R}^d)$ and Λ be a lattice in \mathbb{R}^{2d} . The coefficient or analysis operator is then

$$C_{g, \Lambda} f = (\langle f, \pi(\lambda)g \rangle)_{\lambda \in \Lambda} = V_g f|_{\Lambda}.$$

The adjoint operator of $C_{g, \Lambda}$ is called synthesis operator.

Definition 1.15 (Synthesis Operator). Let $g \in L^2(\mathbb{R}^d)$, Λ be a lattice in \mathbb{R}^{2d} , $\lambda \in \Lambda$ and let $c = (c_\lambda)_{\lambda \in \Lambda} \in \ell^2(\Lambda)$. The synthesis operator is then

$$D_{g, \Lambda} c = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g.$$

We can now define the frame operator, which maps functions from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Definition 1.16 (Frame Operator). The frame operator is given by

$$D_{g, \Lambda} C_{g, \Lambda} f = S_{g, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.$$

If the frame operator $S_{g, \Lambda}$ is invertible and bounded on $L^2(\mathbb{R}^d)$, which is equivalent to $C_{g, \Lambda}$ being bounded from above and below, which is equivalent to (1.1), then $\mathcal{G}(g, \Lambda)$ is a Gabor frame. The invertibility of $S_{g, \Lambda}$ implies the existence of a dual window $\gamma = S_{g, \Lambda}^{-1}g \in L^2(\mathbb{R}^d)$ and we get a Gabor expansion for an arbitrary function $f \in L^2(\mathbb{R}^d)$.

$$\begin{aligned} f &= S_{g, \Lambda}^{-1} S_{g, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma \\ &= S_{g, \Lambda} S_{g, \Lambda}^{-1} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda)g. \end{aligned} \tag{1.3}$$

The convergence of the series in (1.3) is unconditional in $L^2(\mathbb{R}^d)$. If $g \in M^1(\mathbb{R}^d)$ and $1 \leq p \leq \infty$, then the coefficient operator $C_{g,\Lambda}$ is bounded from $M^p(\mathbb{R}^d)$ into $\ell^p(\Lambda)$, $D_{g,\Lambda}$ is bounded from $\ell^p(\Lambda)$ into $M^p(\mathbb{R}^d)$ and the frame operator $S_{g,\Lambda}$ is bounded on $M^p(\mathbb{R}^d)$. In the case that g is in the Wiener amalgam space $\mathcal{W}(L^\infty(\mathbb{R}^d), \ell^1(\mathbb{Z}^d))$ the frame operator is bounded on $L^2(\mathbb{R}^d)$ [43].

We would also like to mention the following connection between the Gabor frame bounds and the frame operator

$$\begin{aligned} A^{-1} &= \|(S_{g,\Lambda})^{-1}\|_{Op} \\ B &= \|S_{g,\Lambda}\|_{Op}. \end{aligned}$$

1.2 The Fine and the Coarse Structure of Gabor Frames

Although we know that the modulation space $M^1(\mathbb{R}^d)$ provides a nice setting for the window functions, this does not guarantee that we get a Gabor frame for an arbitrary window $g \in M^1(\mathbb{R}^d)$ and an arbitrary lattice $\Lambda \subset \mathbb{R}^{2d}$. We will now deal with the fine structure of Gabor frames which describes connections between a window g and lattices which together with g give a frame. Therefore we introduce the full and the reduced frame set as described in [43].

Definition 1.17 (Frame Set). For given window g , the full frame set is defined as the set of all 2d-dimensional lattices Λ , which together with g generate a Gabor frame.

$$\mathcal{F}_{full}(g) = \{\Lambda \subset \mathbb{R}^{2d} : \mathcal{G}(g, \Lambda) \text{ is a frame}\}$$

The reduced frame set is defined as the set of all lattice parameters of separable 2d-dimensional lattices Λ , which together with g generate a Gabor frame.

$$\mathcal{F}_{(\alpha,\beta)}(g) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{G}(g, \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d) \text{ is a frame}\}$$

Clearly, $(\alpha, \beta) \in \mathcal{F}_{(\alpha,\beta)}(g)$ implies $\alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d \in \mathcal{F}_{full}(g)$. Sometimes we will identify a separable lattice $\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with its lattice parameters and write $\Lambda_{(\alpha,\beta)} \in \mathcal{F}_{(\alpha,\beta)}$.

With this definition in our hands, we may rephrase the question about when a Gabor system forms a frame in the following way. For any given g what is its (full or reduced) frame set? At this point we want to emphasize that there is no general idea of how to determine the frame set of a class of functions or even a single function. The case of the 1-dimensional standard Gaussian window

$$g_0(t) = 2^{1/4} e^{-\pi t^2},$$

which we will encounter several more times in this work, has been fully analyzed: results of Lyubarskii [61] and Seip [70] give the full frame set for Gabor frames with Gaussian window g_0 as

$$\mathcal{F}_{full}(g_0) = \{\Lambda \subset \mathbb{R}^2 \mid vol(\Lambda) < 1\}.$$

This implies that the reduced frame set is given by

$$\mathcal{F}_{(\alpha,\beta)}(g_0) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1\}.$$

Due to classical results from harmonic analysis, the result that the frame set is the largest possible holds true for generalized Gaussians of the form

$$\tilde{g}_0(t) = K e^{-L\pi(1+i\gamma)t^2}, \quad K \in \mathbb{C}, \gamma, L \in \mathbb{R}_+.$$

which we will investigate in the upcoming sections. At the time of this work, there is no other function in a modulation space for which we know the full frame set. Still there are some functions for which the reduced frame set is known and is the largest possible, i.e. $\mathcal{F}_{(\alpha,\beta)}(g) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1\}$. The list of functions for which this is known is manageable and consists of

- the Gaussian $e^{-\pi t^2}$ [61, 70] (Lyubarskii, Seip both 1992)
- the hyperbolic secant $(e^{\pi t} + e^{-\pi t})^{-1}$ [58] (Janssen & Strohmer 2002)
- the two-sided exponential function $e^{-|t|}$ [56] (Janssen 2003)
- and its Fourier transform $\frac{2}{1+4\pi^2 t^2}$

In Section 2 we will see that it is easy to determine the frame set of \hat{g} once the frame set of g is known. Since the Gaussian and the hyperbolic secant are invariant under the Fourier transform the list does not get longer, besides taking dilates of the mentioned functions or multiplication with a constant $K \in \mathbb{C}$. We can add another function to the list, if we admit functions from $L^2(\mathbb{R})$ instead of only $M^1(\mathbb{R})$

- the one-sided exponential $e^{-t}\chi_{\mathbb{R}_+}$ [54] (Janssen 1996)
- and its Fourier transform $\frac{1}{1-2\pi it}$.

In 2013 the list was extended by a whole class of functions, namely totally positive functions of finite type. A totally positive function of finite type M can easily be described via its Fourier transform and satisfies

$$\hat{g}(\omega) = \prod_{k=1}^M (1 + 2\pi i \delta_k \omega)^{-1},$$

with non-zero parameters $\delta_k \in \mathbb{R}$ and $M \in \mathbb{N}$. M is called the type of the function. Whenever $M \geq 2$ the function is already in the modulation space $M^1(\mathbb{R})$ [43]. We also see that for $M = 1$ we will have a scaled version of the one-sided exponential function which is not in Feichtinger's algebra. Due to the results by Gröchenig & Stöckler [47] we know that for $M \geq 2$ the reduced frame set of a totally positive window of finite type is given by $\mathcal{F}_{(\alpha,\beta)}(g) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1\}$. We do not go into the details, but we mention that all functions in the above lists belong to the class of totally positive functions. More

results on totally positive functions followed for discrete Gabor frames in 2014 [7]. We refer to the mentioned literature and the references therein for more information about totally positive functions.

From the above results, one might get the impression that the frame set is always of such simple nature, but this is not true. Already for the characteristic function of an interval the reduced frame set is very complicated and was not fully known until the work of Dai & Sun in 2015 [17]. Janssen already described parts of the frame set in 2003, but left some white spots. The pattern is now known as Janssen's tie [57]. Another class of well examined functions are the Hermite functions

$$h_n(t) = \frac{2^{1/4}}{(n!(2\pi)^n 2^n)^{1/2}} e^{\pi t^2} \left(\frac{d^n}{dt^n} e^{-2\pi t^2} \right), \quad n \in \mathbb{N}_0.$$

We note that $h_0(t) = g_0(t)$ is the standard Gaussian which is the only function for which the full frame set is known, as we already mentioned earlier. It is known that at least for $\alpha\beta < \frac{1}{n+1}$ the system $\mathcal{G}(h_n, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Gabor frame [44, 45]. In 2014 Gröchenig formulated some conjectures on the frame set of the Hermite functions in his survey [43] based on known results to that time. For even Hermite functions the conjecture was that the reduced frame set is given by $\mathcal{F}_{(\alpha,\beta)}(h_{2n}) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1\}$. Due to results of Lyubarskii & Nes in 2013 [62], showing that for odd functions the parameters (α, β) never give a frame when $\alpha\beta = \frac{n-1}{n}$, $2 \leq n \in \mathbb{N}$, the conjecture for odd Hermite functions was that $\mathcal{F}_{(\alpha,\beta)}(h_{2n+1}) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \alpha\beta < 1, \alpha\beta \neq \frac{n-1}{n}, n = 2, 3, \dots\}$. Just recently Lemvig found counterexamples to these conjectures, leaving little hope that the frame sets of the Hermite functions might have a simple structure [60].

We will now turn to the coarse structure of Gabor frames, which describes general properties of the frame set.

Theorem 1.18 (Coarse Structure). *Let $g \in M^1(\mathbb{R}^d)$, then the full frame set $\mathcal{F}_{\text{full}}(g)$ is an open subset of $\{\Lambda : \text{vol}(\Lambda) < 1\}$ and it contains a neighbourhood of 0. Hence, the reduced frame set $\mathcal{F}_{(\alpha,\beta)}(g)$ is open in $\{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ : \alpha\beta < 1\}$ and contains a neighbourhood of $(0, 0)$ in $\mathbb{R}_+ \times \mathbb{R}_+$.*

Theorem 1.18 combines several fundamental results from the field of time-frequency analysis, namely the density theorem, the Balian-Low theorem, the theorem on the existence of Gabor frames as well as perturbation results. We will now state these results and start with the density theorem, which gives a necessary condition on a lattice to generate a Gabor frame.

Theorem 1.19 (Density Theorem). *Let $g \in L^2(\mathbb{R}^d)$ be a window generating a Gabor frame $\mathcal{G}(g, \Lambda)$, then*

$$\text{vol}(\Lambda) \leq 1. \tag{1.4}$$

The density theorem can be seen as one of many uncertainty principles in time-frequency analysis, as it states that the time-frequency shifts of the window must cover the time-frequency plane densely enough. It has been studied extensively and there exists a variety

of proofs and formulations [43, 49]. Equality in (1.4) is possible if $g \in L^2(\mathbb{R}^d)$ and in that case we say that we generate a frame at critical density. For $g = \chi_{[0,1]^d}$ and $\Lambda = \mathbb{Z}^{2d}$ we even have an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^d)$, i.e.

$$\|f\|_2^2 = \sum_{\lambda \in \mathbb{Z}^{2d}} |\langle f, \pi(\lambda)\chi_{[0,1]^d} \rangle|^2.$$

In this particular case, $\mathcal{G}(\chi_{[0,1]^d}, \mathbb{Z}^{2d})$ consists of the translated Fourier basis functions of the d -dimensional torus.

However, the indicator function $\chi_{[0,1]^d}$ is not an element of the modulation space $M^1(\mathbb{R}^d)$, which we described to be a natural choice to pick our window functions from. The next theorem can be seen as a “no-go result” and has been stated in many different versions. We will give a statement for the Hilbert space case and discuss some consequences of the result.

Theorem 1.20 (Balian-Low Theorem). *Let $g \in L^2(\mathbb{R}^d)$ be a window generating a Gabor frame $\mathcal{G}(g, \Lambda)$ with $\text{vol}(\Lambda) = 1$. Then, either $xg \notin L^2(\mathbb{R}^d)$ or $\omega \hat{g} \notin L^2(\mathbb{R}^d)$, with both $x, \omega \in \mathbb{R}^d$.*

This result means that the existence of an orthonormal basis which allows good concentration in time and frequency at the same time is not possible. We can also formulate a version of the Balian-Low theorem for Feichtinger’s algebra. If $\mathcal{G}(g, \Lambda)$ is a Gabor frame with $\text{vol}(\Lambda) = 1$, then $g \notin M^1(\mathbb{R}^d)$. In comparison to Theorem 1.19, where we have a window from $L^2(\mathbb{R}^d)$, the inequality in equation (1.4) becomes strict by taking a window from $M^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. At this point, we want to remark that an amalgam version of the Balian-Low theorem can as well be formulated, meaning that under the milder assumption $g \in \mathcal{W}(L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d), \ell^1(\mathbb{Z}^d))$ the reduced frame set still does not contain lattice parameters with $\alpha\beta = 1$. Hence,

$$\mathcal{F}_{(\alpha,\beta)}(g) \subset \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ : \alpha\beta < 1\}$$

is a necessary condition for windows from the Wiener amalgam space to constitute a frame. This impossibility to produce an orthonormal basis with nice decay properties in the time as well as in the frequency domain was the first strong argument to use frames in analysis. Furthermore, there is quite a subtle point in Theorem 1.18, namely the statement that the frame set $\mathcal{F}_{full}(g)$ contains a neighborhood of 0 which asserts the existence of Gabor frames [43].

The fact that for $g \in M^1(\mathbb{R}^d)$ the (full or reduced) frame set is open, is not true if g is not in Feichtinger’s algebra, e.g. for $g = \chi_{[0,1]} \notin M^1(\mathbb{R})$ the frame set is not open. The openness of the frame set may as well be interpreted as a strong perturbation result. A first result in this direction was presented by Feichtinger & Kaiblinger in 2004 [28] and states that a Gabor frame over a rectangular lattice remains a frame under small perturbations on the generating matrix S_1 . If $g \in M^1(\mathbb{R}^d)$ and $\mathcal{G}(g, S_1\mathbb{Z}^{2d})$ is a frame, then $\mathcal{G}(g, S_2\mathbb{Z}^{2d})$ is a frame as well, if $\|S_2 - S_1\| < \varepsilon$ for $\varepsilon > 0$ sufficiently small in some matrix norm. This

result is remarkable, since the perturbation in the generating matrix may be small, but the perturbations on the lattice points accumulate and, hence, might be quite large.

We want to close the section with recent results on deformations of Gabor Frames.

Theorem 1.21. *Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^{2d}$. Let $T_n : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$, $n \in \mathbb{N}$ be a sequence of differentiable maps with Jacobian DT_n . Assume*

$$\sup_{\lambda \in \mathbb{Z}^{2d}} |DT_n(\lambda) - Id| \rightarrow 0, \quad n \rightarrow \infty,$$

then the following holds. If $\mathcal{G}(g, \Lambda)$ is a frame, then $\mathcal{G}(g, T_n\Lambda)$ is a frame for sufficiently large n .

This result is a special case of what is called Lipschitz deformation [46]. It also holds for Riesz sequences which we do not discuss in this work. Also, Theorem 1.21 is already quite general in the sense that it holds for non-uniform Gabor frames, which do not have a lattice structure, under non-linear deformations and therefore generalizes the result from [28]. It is still an open question how to reasonably formulate a general concept of the deformation of Gabor systems [38, 43]. Usually deformation results only decide whether the frame property is kept at all, but say little about the quality of the resulting Gabor frame, meaning that in general they do not give information on the sharp frame bounds.

In the next section we will derive some deformation results where the sharp frame bounds are kept under a deformation of the Gabor system. We will see that under certain aspects the deformation of the lattice can be compensated by a suitable deformation of the window function and vice versa.

2 The Symplectic and the Metaplectic Group

There are two obvious ways of changing a Gabor system $\mathcal{G}(g, \Lambda)$. One option is to deform the lattice Λ the other option is to change the window function g . Since a lattice can be described by a matrix the first idea is to change the matrix or to multiply it with another matrix. A perturbation of the window can be performed by letting a unitary operator act on the window.

For the lattice part we first note that we only need to consider matrices of even dimension. Also, we want to deform our lattices in a continuous way and we want to keep the subgroup property of our lattice as well as the volume of the lattice. Therefore we focus on matrices with determinant 1 and we will study a subgroup of the special linear group $SL(2d, \mathbb{R})$.

For the part concerning the deformation of the window we will focus on a subgroup of the group of unitary operators $\mathcal{U}(L^2(\mathbb{R}^d))$. This subgroup, called the metaplectic group, and its elements, the metaplectic operators, are widely used in quantum mechanics and in time-frequency analysis. There is a close connection between the symplectic and the metaplectic group and this interplay might be used to solve problems in quantum mechanics once the solution for the corresponding classical problem is known [36]. In time-frequency analysis this property can be used to deform Gabor frames without destroying their frame property and even keeping the optimal frame bounds [38, 34, 41].

2.1 Symplectic Matrices

Definition 2.1 (Symplectic Matrix). A matrix $S \in GL(2d, \mathbb{R})$ is called symplectic if and only if

$$SJS^T = S^TJS = J, \quad (2.1)$$

where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here, 0 denotes the $d \times d$ zero matrix and I is the $d \times d$ identity matrix. J is called the standard symplectic matrix.

We note that condition (2.1) is redundant. Actually, we have

$$SJS^T = J \iff S^TJS = J.$$

From (2.1) we conclude that all symplectic matrices $S \in Sp(2d, \mathbb{R})$ must have determinant equal to ± 1 . In fact, if $S \in Sp(2d, \mathbb{R})$ then $\det(S) = 1$, see [36, 37, 41]. Also, $Sp(2d, \mathbb{R})$ is a subgroup of $SL(2d, \mathbb{R})$ and in the case $d = 1$ we have $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$. In all other cases where $d > 1$, $Sp(2d, \mathbb{R})$ is a proper subgroup of $SL(2d, \mathbb{R})$.

Lemma 2.2 (Symplectic Group). *The set of all symplectic matrices forms a group denoted by $Sp(2d, \mathbb{R})$ and is called the symplectic group.*

Proof. Let $S_1, S_2 \in Sp(2d, \mathbb{R})$. It follows from equation (2.1) that the product $S_1 S_2 \in Sp(2d, \mathbb{R})$. Taking the inverse of the double equality in (2.1) and using the fact that $J^{-1} = -J$ we see that $S^{-1} \in Sp(2d, \mathbb{R})$ if $S \in Sp(2d, \mathbb{R})$. \square

It is convenient to write symplectic matrices as block matrices in the following form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are $d \times d$ matrices. With this notation we have the following formula for the inverse of a symplectic matrix

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}.$$

In the case $d = 1$ this reduces to the well-known inversion formula for a matrix S belonging to $SL(2, \mathbb{R})$, as $A, B, C, D \in \mathbb{R}$ are scalars. For general dimension d , condition (2.1) already implies $d(2d + 1)$ constraints on the block matrices A, B, C, D [37]. We want to state the 3 universal constraints which hold for arbitrary dimension d .

$$A^T C = C^T A \tag{2.2}$$

$$B^T D = D^T B \tag{2.3}$$

$$A^T D - C^T B = I \tag{2.4}$$

or equivalently

$$AB^T = BA^T \tag{2.5}$$

$$CD^T = DC^T \tag{2.6}$$

$$AD^T - BC^T = I \tag{2.7}$$

We see that the products in (2.2), (2.3), (2.5) and (2.6) are symmetric. Furthermore, for the special case $d = 1$ these conditions in fact collapse to equation (2.4) or (2.7) and state that S has determinant $\det(S) = 1$.

2.2 Free Symplectic Matrices

We will now introduce the building blocks of the symplectic group and the free symplectic matrices which factor into these building blocks. At the end of this section we will see that any symplectic matrix is the product of the mentioned building blocks [36, 40].

The motivation comes from Hamiltonian mechanics. We want to describe the motion of a particle depending on two variables usually called position x and momentum p which depend on time t and are coupled by Hamilton's equations

$$\begin{aligned} x'(t) &= \frac{\partial}{\partial p} H(x(t), p(t)) \\ p'(t) &= -\frac{\partial}{\partial x} H(x(t), p(t)). \end{aligned}$$

or in a more compact notation

$$\begin{pmatrix} x'(t) \\ p'(t) \end{pmatrix} = J \begin{pmatrix} \frac{\partial}{\partial x} H \\ \frac{\partial}{\partial p} H \end{pmatrix}.$$

Here, $H(x(t), p(t))$ is the Hamiltonian or Hamilton function and J is the already known standard symplectic matrix. The coupled pair $(x, p) \in \mathbb{R}^d \times \mathbb{R}^d$ describes a point in phase space. For more details on Hamiltonian mechanics see e.g. Arnold's textbook [4].

Given now initial position x and final position \tilde{x} of a particle we want to know the initial and final momentum p and \tilde{p} , assuming that the motion is linear. This means, we are given a linear system $(x, p) = S(\tilde{x}, \tilde{p})$ and knowing the pair (x, \tilde{x}) we are trying to find (p, \tilde{p}) . This is equivalent to

$$\begin{aligned} x &= A\tilde{x} + B\tilde{p} \\ p &= C\tilde{x} + D\tilde{p}. \end{aligned}$$

In order to solve this system of equations for (p, \tilde{p}) , clearly B has to be invertible. We could also ask the reverse question, given (p, \tilde{p}) how can we determine the pair (x, \tilde{x}) . In this case, of course, C has to be invertible. Therefore, we want to put on record that the following definition could have been done with a condition on C and hence, all follow up results can be adjusted and reformulated with a condition on C .

Definition 2.3 (Free Symplectic Matrix). We call a symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2d, \mathbb{R})$ a free symplectic matrix if B is invertible.

Definition 2.4 (Generator Matrices). We define the following $2d \times 2d$ matrices.

- The standard symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

- The shearing matrices

$$V_P = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}$$

with P being a symmetric matrix.

- The dilation matrices

$$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}$$

with L being an invertible matrix.

We call these matrices the generator matrices for the free symplectic matrices.

We note that the matrices V_P and M_L are not free, still the name generator matrix is justified by the following propositions.

Proposition 2.5. *With the notation of Definition 2.4 we get that any free symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ can be factored as*

$$S = V_{DB^{-1}} M_{B^{-1}} J V_{B^{-1}A}. \quad (2.8)$$

Proof. Since S is a free symplectic matrix, it follows that B is invertible. Therefore, we have the factorization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ DB^{-1} & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & DB^{-1}A - C \end{pmatrix} \begin{pmatrix} B^{-1}A & I \\ -I & 0 \end{pmatrix}.$$

The rest of the proof follows by conditions (2.2) – (2.7). From the mentioned conditions we conclude that $DB^{-1}A - C = (B^{-1})^T$ since $AD^T - BC^T = I$. Therefore, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ DB^{-1} & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & (B^{-1})^T \end{pmatrix} \begin{pmatrix} B^{-1}A & I \\ -I & 0 \end{pmatrix}$$

and the last matrix can be factored as

$$\begin{pmatrix} B^{-1}A & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ B^{-1}A & I \end{pmatrix}.$$

□

A proof is also given in [36] or [34].

2.3 Generating Functions

In the case of time-frequency analysis the proper way to use and interpret Hamiltonian mechanics is by replacing position by time, momentum by frequency and phase space by time-frequency plane. We will now return to the notation of time-frequency analysis and point out connections between free symplectic matrices and quadratic forms.

Proposition 2.6. *Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2d, \mathbb{R})$ be a free symplectic matrix. Let P, Q be $d \times d$ symmetric matrices and let L be a $d \times d$ invertible matrix.*

(i) *Then we have*

$$(x, \omega) = S(\tilde{x}, \tilde{\omega}) \iff \begin{cases} \omega = \partial_x W(x, \tilde{x}), \\ \tilde{\omega} = -\partial_{\tilde{x}} W(x, \tilde{x}) \end{cases}$$

where W is the quadratic form

$$W(x, \tilde{x}) = \frac{1}{2} DB^{-1}x^2 - B^{-1}x \cdot \tilde{x} + \frac{1}{2} B^{-1}A\tilde{x}^2 \quad (2.9)$$

where DB^{-1} and $B^{-1}A$ are symmetric.

(ii) To every quadratic form

$$W(x, \tilde{x}) = \frac{1}{2}Px^2 - Lx \cdot \tilde{x} + \frac{1}{2}Q\tilde{x}^2 \quad (2.10)$$

we can associate the free symplectic matrix

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}. \quad (2.11)$$

We call the quadratic form in (2.10) the generating function of S_W in (2.11) and S_W factors as

$$S_W = V_P M_L J V_Q.$$

We note the connection between the generating function in equation (2.9) and the factorization of a free symplectic matrix given in equation (2.8) in Proposition 2.5. The next theorem and the resulting corollary, whose proofs can be found in [36], show the importance of free symplectic matrices and their factorization.

Theorem 2.7. *For every $S \in Sp(2d, \mathbb{R})$ there exist two (non-unique) free symplectic matrices S_{W_1} and S_{W_2} such that $S = S_{W_1} S_{W_2}$.*

Corollary 2.8. *The set of all matrices*

$$\{V_P, M_L, J\}$$

generates the symplectic group $Sp(2d, \mathbb{R})$.

2.4 Metaplectic Operators and the Quadratic Fourier Transform

After the study of the symplectic group we will now investigate the metaplectic group. We start with a rather abstract characterization of the metaplectic group which allows a very quick identification with the symplectic group. A sequence of group homomorphisms

$$G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_n \rightarrow G_{n+1}$$

is called exact, if the image of each homomorphism equals the kernel of the next homomorphism

$$im(G_{k-1} \rightarrow G_k) = ker(G_k \rightarrow G_{k+1}), \quad k = 1, \dots, n.$$

A short exact sequence is of the form

$$0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow 0$$

and in this case we have an identification rule via the following isomorphism

$$G_3 \cong G_2 / im(G_1 \rightarrow G_2).$$

Definition 2.9 (Metaplectic Group). The metaplectic group $Mp(2d, \mathbb{R})$ is the connected two-fold cover of the symplectic group $Sp(2d, \mathbb{R})$. Equivalently, we can define $Mp(2d, \mathbb{R})$ by saying that the sequence of homomorphisms

$$0 \rightarrow \mathbb{Z}_2 \rightarrow Mp(2d, \mathbb{R}) \rightarrow Sp(2d, \mathbb{R}) \rightarrow 0$$

is exact.

Since, the sequence in Definition 2.9 is actually a short exact sequence, we can identify the symplectic group with the metaplectic group modulo $im(\mathbb{Z}^2 \rightarrow Mp(2d, \mathbb{R})) = \{\pm I\}$

$$Sp(2d, \mathbb{R}) \cong Mp(2d, \mathbb{R}) / \{\pm I\},$$

where I now denotes the identity element of the group $(Mp(2d, \mathbb{R}), \circ)$.

We would also like to present a more constructive approach to define the metaplectic group. The metaplectic group is a group of unitary operators on $L^2(\mathbb{R}^d)$, well described e.g. in [36, 34, 41, 69]. We define the following operators.

Definition 2.10. For a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ in the Schwartz space we define the following operators.

- The modified Fourier transform \mathcal{J} defined by

$$\mathcal{J}\psi(\omega) = i^{-d/2} \int_{\mathbb{R}^d} \psi(t) e^{-2\pi i \omega \cdot t} dt = i^{d/2} \mathcal{F}\psi(\omega). \quad (2.12)$$

- The linear chirps

$$\mathcal{V}_P\psi(t) = e^{\pi i P t \cdot t} \psi(t) \quad (2.13)$$

with P being a real, symmetric $d \times d$ matrix.

- The dilation operator

$$\mathcal{M}_{L,n}\psi(t) = i^n \sqrt{|\det(L)|} \psi(Lt), \quad (2.14)$$

where L is invertible and n is an integer corresponding to a choice of $arg(\det(L))$, to be more precise

$$n\pi \equiv arg(\det(L)) \pmod{2\pi}. \quad (2.15)$$

The class modulo 4 of the integer n appearing in the definition of the dilation operator (2.14) is called Maslov index [36, 40]. At this point we remark that we have chosen the phase factors in equation (2.12) and equation (2.14) according to the existing literature [36, 34, 41] and that the choice is not clear from the context so far. However, as the main motivation for this section was to gain tools for a deformation of Gabor frames without changing the frame bounds, we do not have to care about phase factors. The reason why they do not influence the frame bounds can be directly seen from the frame inequality (1.1). Therefore we will not discuss their choice any further and might as well just ignore them.

We will now associate quadratic forms to metaplectic operators and we will also see parallels between the symplectic and the metaplectic group.

Definition 2.11 (Quadratic Fourier Transform). Let S_W be the free symplectic matrix

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}$$

associated to the quadratic form $W(\tilde{t}, t) = \frac{1}{2}P\tilde{t}^2 - L\tilde{t} \cdot t + \frac{1}{2}Qt^2$ (compare Proposition 2.6 equations (2.10) and (2.11)). Let the operators \mathcal{J} , \mathcal{V}_P and $\mathcal{M}_{L,n}$ be defined as in (2.12), (2.13) and (2.14) respectively. We call the operator

$$\mathcal{S}_{W,n} = \mathcal{V}_P \mathcal{M}_{L,n} \mathcal{J} \mathcal{V}_Q \quad (2.16)$$

the quadratic Fourier transform associated to the free symplectic matrix S_W .

For $\psi \in \mathcal{S}(\mathbb{R}^d)$ we have the explicit formula

$$\mathcal{S}_{W,n}\psi(\tilde{t}) = i^{n-\frac{d}{2}} \sqrt{|\det(L)|} \int_{\mathbb{R}^d} \psi(t) e^{2\pi i W(\tilde{t}, t)} dt, \quad (2.17)$$

where $W(\tilde{t}, t)$ is again the quadratic form as defined in (2.10) and Definition 2.11. We note that to each quadratic form $W(\tilde{t}, t)$ we can actually associate not one but two metaplectic operators $\mathcal{S}_{W,n}$ and $\mathcal{S}_{W,n+2} = -\mathcal{S}_{W,n}$. Due to (2.15), both are equally good choices. This reflects the fact that the metaplectic operators are elements of the two-fold cover of the symplectic group.

Although, all statements in this section were formulated for the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, they also hold for the modulation spaces $M^p(\mathbb{R}^d)$ including the Hilbert space $L^2(\mathbb{R}^d)$. Also, we will frequently drop one or both of the indices W and n and will write S instead of S_W and \mathcal{S} or \mathcal{S}_W instead of $\mathcal{S}_{W,n}$. When the context allows, we will also use other indices than the ones mentioned.

As can be seen by formula (2.16) a quadratic Fourier transform is a manipulation of a (suitable) function by a chirp, a modified Fourier transform, a dilation and another chirp. This is the exact same way in which the fractional Fourier transform is described in [3] with an additional dilation in between the modified Fourier transform and the second chirp. Hence, the quadratic Fourier transform is an extension of the fractional Fourier transform in the sense that the directions in the time-frequency plane are scaled by some factor depending on the angle. For more details on the fractional Fourier transform see also [40].

Proposition 2.12. *Every operator $\mathcal{S}_{W,n}$ extends to a unitary operator on $L^2(\mathbb{R}^d)$ and the inverse is given by*

$$\mathcal{S}_{W,n}^{-1} = \mathcal{S}_{W^*, n^*},$$

with $W^*(\tilde{t}, t) = -W(t, \tilde{t})$ and $n^* = d - n$.

Proof. The fact that $\mathcal{S}_{W,n}$ is a unitary operator is clear since, \mathcal{V}_P , $\mathcal{M}_{L,n}$ and \mathcal{J} are unitary. Obviously, we have

$$\mathcal{V}_P^{-1} = \mathcal{V}_{-P}, \quad \mathcal{M}_{L,n}^{-1} = \mathcal{M}_{L^{-1}, -n}$$

and the inverse of the modified Fourier transform is given by

$$\mathcal{J}^{-1}\psi(t) = i^{d/2} \int_{\mathbb{R}^d} \psi(\omega) e^{2\pi i t \cdot \omega} d\omega = i^{d/2} \mathcal{F}^{-1}\psi(t)$$

We note that

$$\mathcal{J}^{-1}\mathcal{M}_{L^{-1}, -n} = \mathcal{M}_{-L^T, d-n}\mathcal{J}$$

and hence,

$$\mathcal{S}_{W,n}^{-1} = \mathcal{V}_{-Q}\mathcal{J}^{-1}\mathcal{M}_{L^{-1}, -n}\mathcal{V}_{-P} = \mathcal{S}_{W^*, n^*}.$$

□

From Definition 2.11 it follows that the metaplectic operators are a subset of the group $\mathcal{U}(L^2(\mathbb{R}^d))$. In fact the metaplectic operators form a subgroup of $\mathcal{U}(L^2(\mathbb{R}^d))$ [36, 41].

Definition 2.13 (Metaplectic Group). The group generated by the quadratic Fourier transforms $\mathcal{S}_{W,n}$ is called the metaplectic group and is denoted by $Mp(2d, \mathbb{R})$. Its elements are called metaplectic operators.

Theorem 2.14. *For every $\mathcal{S} \in Mp(2d, \mathbb{R})$ there exist two quadratic Fourier transforms \mathcal{S}_{W_1, n_1} and \mathcal{S}_{W_2, n_2} such that $\mathcal{S} = \mathcal{S}_{W_1, n_1}\mathcal{S}_{W_2, n_2}$.*

The factorization in Theorem 2.14 is not unique as the identity operator can always be written as $\mathcal{S}_{W,n}\mathcal{S}_{W^*, n^*}$.

Corollary 2.15. *The set of all operators*

$$\{\mathcal{V}_P, \mathcal{M}_{L,n}, \mathcal{J}\}$$

generates the metaplectic group.

We close this section by introducing the natural projection of the metaplectic group $Mp(2d, \mathbb{R})$ onto the symplectic group $Sp(2d, \mathbb{R})$, which we will denote by π^{Mp} . For the details we refer to [36].

Theorem 2.16. *The mapping*

$$\begin{aligned} \pi^{Mp} : Mp(2d, \mathbb{R}) &\longrightarrow Sp(2d, \mathbb{R}) \\ \mathcal{S}_{W,n} &\longmapsto S_W \end{aligned}$$

which associates a free symplectic matrix with generating function W to a quadratic Fourier transform, is a surjective group homomorphism. Hence,

$$\pi^{Mp}(\mathcal{S}_1\mathcal{S}_2) = \pi^{Mp}(\mathcal{S}_1)\pi^{Mp}(\mathcal{S}_2).$$

and the kernel of π^{Mp} is given by

$$\ker(\pi^{Mp}) = \{\pm I\}.$$

Therefore, $\pi^{Mp} : Mp(2d, \mathbb{R}) \mapsto Sp(2d, \mathbb{R})$ is a two-fold covering of the symplectic group.

Definition 2.17. The mapping π^{Mp} in Theorem 2.16 is called the natural projection of $Mp(2d, \mathbb{R})$ onto $Sp(2d, \mathbb{R})$.

The natural projections of the metaplectic generator elements are the symplectic generator elements.

$$\pi^{Mp}(\pm\mathcal{V}_P) = V_P, \quad \pi^{Mp}(\pm\mathcal{M}_{L,n}) = M_L, \quad \pi^{Mp}(\pm\mathcal{J}) = J.$$

For more information on the interplay of the symplectic and the metaplectic group we refer to [34, 36].

3 Gabor Frame Sets of Invariance

In this section we will study examples of generalized Gaussian Gabor frames where the frame bounds stay invariant under a change of the lattice or a change of the Gaussian. The machinery working in the background is the interplay between symplectic and the metaplectic group as described in Section 2. We will have a look at the geometric aspects of the window and the lattice in the time-frequency plane and we will use the geometric intuition to quickly derive some classical as well as some non-obvious results.

We introduce the following notation. Assume $\mathcal{G}(g_1, \Lambda_1)$ and $\mathcal{G}(g_2, \Lambda_2)$ are Gabor frames with the same optimal frame bounds then we write

$$\mathcal{G}(g_1, \Lambda_1) \cong \mathcal{G}(g_2, \Lambda_2).$$

We note that a priori we cannot say anything more about the relation between the two frames from the fact $\mathcal{G}(g_1, \Lambda_1) \cong \mathcal{G}(g_2, \Lambda_2)$. The windows as well as the lattices might be totally unrelated to each other, but we are particularly interested in cases where the windows g_1 and g_2 can be derived from each other by the action of an element of the metaplectic group.

Theorem 3.1. *Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame. Let $\mathcal{S} \in Mp(2d, \mathbb{R})$ and let $S \in Sp(2d, \mathbb{R})$ be the natural projection of the metaplectic operator $\pi^{Mp}(\mathcal{S}) = S$. Then*

$$\mathcal{S}\mathcal{G}(g, \Lambda) = \mathcal{G}(\mathcal{S}g, S\Lambda)$$

and therefore $\mathcal{G}(\mathcal{S}g, S\Lambda) \cong \mathcal{G}(g, \Lambda)$.

Proof. The key ingredient in the proof is the following relation between metaplectic operators and the symmetric time-frequency shifts defined by

$$\rho(\lambda) = \rho(x, \omega) = M_{\omega/2} T_x M_{\omega/2}.$$

We have the following covariance principle

$$\mathcal{S}^{-1} \rho(S\lambda) \mathcal{S} = \rho(\lambda),$$

with $\pi^{Mp}(\mathcal{S}) = S$. From this we conclude that

$$\sum_{\lambda \in S\Lambda} |\langle f, \rho(\lambda) \mathcal{S}g \rangle|^2 = \sum_{\lambda \in S\Lambda} |\langle f, \mathcal{S} \rho(S^{-1}\lambda) g \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle \mathcal{S}^{-1} f, \rho(\lambda) g \rangle|^2.$$

From the frame inequality

$$A \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda) g \rangle|^2 \leq B \|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d)$$

we finally conclude that $\mathcal{G}(\mathcal{S}g, S\Lambda) \cong \mathcal{G}(g, \Lambda)$ since $\|f\|_2^2 = \|\mathcal{S}^{-1} f\|_2^2$ for all $f \in L^2(\mathbb{R}^d)$. \square

The theorem can be found in [38] and follows from classical results in harmonic analysis. Similar results and the covariance principle can also be found in the textbooks of de Gosson [36], Folland [34] or Gröchenig [41].

The first observation we make in the direction of frame bounds is that a phase factor $c \in \mathbb{C}$ with $|c| = 1$ is negligible since $\mathcal{G}(g, \Lambda) \cong \mathcal{G}(cg, \Lambda)$ as can also directly be seen from the frame inequality. Actually we already used that fact in the argumentation above as the covariance principle holds for the symmetric time-frequency shifts whereas the frame inequality is formulated for usual time-frequency shifts. Therefore, in what follows, we will not care too much about appearing phase factors.

Theorem 3.1 is a particular case of the notion of a “Hamiltonian deformation of Gabor frames” as described in [38]. It tells us under which conditions the frame property as well as the optimal frame bounds are kept when a Gabor frame suffers some deformations. This is a very special case, as in general neither the optimal frame bounds nor the frame property might be kept under some general deformation of the frame. However, as already presented in Section 1.1, there are cases when the frame property might be kept without keeping the optimal frame bounds [39, 28, 46]. This is usually done by either deforming the window and fixing the lattice or the other way round. By Theorem 3.1, we know that these approaches are equivalent as long as we stick to symplectic and metaplectic deformations.

What we will see in the following sections is that it is possible to keep both, the frame property and the optimal frame bounds under certain lattice deformations, without changing the window. This is due to the fact that generalized Hermite functions, including the generalized Gaussians, are eigenfunctions of certain metaplectic operators with eigenvalues of modulus 1. Hence, the corresponding symplectic matrix will deform the lattice, while the window remains unchanged up to a phase factor.

3.1 Lattice Rotations and the Standard Gaussian

From this point on, we will only consider the 1-dimensional case. The most popular 1-dimensional window function is probably the standard Gaussian

$$g_0(t) = 2^{1/4} e^{-\pi t^2}.$$

Although Gabor frames with Gaussian window have been studied intensively, we still want to explore and exploit the Gabor family $\mathcal{G}(g_0, \Lambda)$ with $vol(\Lambda) < 1$. We recall that by the work of Lyubarskii [61] and Seip [70] the frame set of the Gaussian window is the largest possible

$$\mathcal{F}_{full}(g_0) = \{\Lambda \subset \mathbb{R}^2 \mid vol(\Lambda) < 1\}.$$

One of the simplest manipulations of our Gabor frame is to rotate the lattice and calculate the corresponding window. This means that our lattice is deformed by the rotation matrix

$$S_\tau = \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix}$$

and the corresponding deformation of the window is given by the action of the quadratic Fourier transform on the window g_0 . To derive an explicit formula for the resulting window we use Proposition 2.6 and equation (2.17).

$$\mathcal{S}_\tau g_0(\tilde{t}) = i^{n(\tau)-\frac{1}{2}} \sqrt{\frac{1}{|\sin \tau|}} \int_{\mathbb{R}} e^{2\pi i W_\tau(t, \tilde{t})} g_0(t) dt,$$

where $n(\tau) \in \{0, 1, 2, 3\}$ depends on τ and the choice of $\arg\left(\sqrt{\sin(\tau)}\right)$ and where

$$W_\tau(t, \tilde{t}) = \frac{1}{2 \sin \tau} ((t^2 + \tilde{t}^2) \cos \tau - 2t\tilde{t}).$$

This manipulation is meaningful whenever $\tau \neq k\pi$, $k \in \mathbb{Z}$. The case $\tau = k\pi$ is obvious, since the matrix $S_{k\pi}$ equals $\pm I$. Hence, by the factorization $I = -J^2$, we find that the corresponding metaplectic operator can be assigned and is (up to the sign) the identity operator. In the case that $\tau = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, we simply recover, up to a phase factor, the (modified) Fourier transform as the resulting metaplectic operator, which reflects the fact that changing from the time domain to the frequency domain is equivalent to a rotation of the time-frequency plane by 90 degrees. We would like to know the resulting window for general τ . Performing the calculations, we get

$$\mathcal{S}_\tau g_0(t) = 2^{1/4} i^{n(\tau)} e^{-i\frac{\tau}{2}} e^{-\pi t^2} = c g_0(t),$$

with $|c| = 1$. The calculations above need a change of variables and the Fourier invariance of the standard Gaussian, $\mathcal{F}g_0 = g_0$. For a proof of the Fourier invariance of g_0 see [34, 41]. Hence, we have the result

$$\mathcal{G}(g_0, \Lambda) \cong \mathcal{G}(g_0, S_\tau \Lambda),$$

which means that the frame bounds of a Gabor frame with window g_0 stay invariant under a rotation of the lattice. A heuristic explanation is given by looking at the ambiguity function of the standard Gaussian which is given by

$$\mathcal{A}g_0(x, \omega) = e^{-\frac{\pi}{2}(x^2 + \omega^2)}.$$

It is rotation symmetric in the time-frequency plane. Therefore a rotation of the time-frequency plane will yield a rotated lattice and the same ambiguity function. We also want to mention the other quadratic representations of g_0 we introduced. The STFT is given by

$$\mathcal{V}_{g_0} g_0(x, \omega) = e^{\pi i x \omega} e^{-\frac{\pi}{2}(x^2 + \omega^2)}$$

whereas the Wigner distribution is given by

$$\mathcal{W}g_0(x, \omega) = 2e^{-2\pi(x^2 + \omega^2)}.$$

We note the relations

$$|\mathcal{V}_{g_0} g_0(x, \omega)| = \mathcal{A}g_0(x, \omega) = \mathcal{F}^{-1}(\mathcal{W}g_0)(-\omega, x).$$

3.2 Elliptic Deformations and Dilated Gaussians

In Section 3.1 we saw that, using the standard Gaussian window, the Gabor frame bounds stay invariant under a rotation of the lattice. We will now extend this result using ideas from Hamiltonian mechanics. For an introduction to Hamiltonian mechanics we refer to Arnold's textbook [4].

We introduce the harmonic oscillator with mass $m = 1$ via its Hamiltonian given by

$$H(x, \omega; \tau) = \frac{x(\tau)^2}{2} + \frac{\omega(\tau)^2}{2}. \quad (3.1)$$

Assuming a conservative system, the Hamiltonian gives the full energy of the system and is therefore constant. Hamilton's equations are given by

$$\frac{d}{d\tau}\lambda = J \begin{pmatrix} \frac{\partial}{\partial x} H \\ \frac{\partial}{\partial \omega} H \end{pmatrix} = J\lambda, \quad (3.2)$$

where $\lambda = (x, \omega)$ and both, x and ω depend on τ . The rotation matrix

$$S_\tau = \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix}$$

determines the phase flow of the harmonic oscillator with mass $m = 1$. This means that if the pair (x_0, ω_0) is an initial state satisfying (3.1) and (3.2), then the pair $(x_\tau, \omega_\tau) = S_\tau(x_0, \omega_0)$ is a solution to (3.1) and (3.2).

Allowing arbitrary mass m , the Hamiltonian of the harmonic oscillator is given by

$$H_m(x, \omega; \tau) = \frac{mx^2}{2} + \frac{\omega^2}{2m}.$$

The trajectories of the initial value problem induced by Hamilton's equations

$$\frac{d}{d\tau}\lambda = J \begin{pmatrix} \frac{\partial}{\partial x} H_m \\ \frac{\partial}{\partial \omega} H_m \end{pmatrix}, \quad \lambda(0) = \lambda_0 \quad (3.3)$$

will be ellipses in standard position with semi-axis ratio m .

Assume now we are given the Gabor frame $\mathcal{G}(g_0, \Lambda)$ with standard Gaussian window and arbitrary lattice Λ with $\text{vol}(\Lambda) < 1$. Any dilation of the lattice by a symplectic matrix $M_{\sqrt{m}}$ can be compensated by a metaplectic dilation of the window such that the frame bounds remain unchanged, so

$$\mathcal{G}(g_0, \Lambda) \cong \mathcal{G}(\mathcal{M}_{\sqrt{m}} g_0, M_{\sqrt{m}} \Lambda).$$

The dilated standard Gaussian is

$$g_{0,m}(t) = \mathcal{M}_{\sqrt{m}} g_0(t) = c(2m)^{1/4} e^{-\pi m t^2},$$

where $|c| = 1$. Next, we compute the ambiguity function $\mathcal{A}g_{0,m}$.

$$\begin{aligned}\mathcal{A}g_{0,m}(x, \omega) &= \sqrt{2m} \int_{\mathbb{R}} e^{-\pi m(t+x/2)^2} e^{-\pi m(t-x/2)^2} e^{-2\pi i \omega t} dt \\ &= \sqrt{2m} e^{-\pi m x^2/2} \int_{\mathbb{R}} e^{-\pi m 2t^2} e^{-2\pi i \omega t} dt \\ &= e^{-\frac{\pi}{2} \left(m x^2 + \frac{\omega^2}{m} \right)}.\end{aligned}$$

Hence, any level set of $\mathcal{A}g_{0,m}$ will be an ellipse in standard position with semi-axis ratio m .

As already mentioned, the trajectories of System (3.3) are the ellipses

$$m x^2 + \frac{\omega^2}{m} = H_m = \text{const.}$$

In general a set X in phase space will change its shape under the action of the phase flow φ_τ associated to a Hamiltonian system, but it follows from the general theory on Hamiltonian mechanics that the volume is preserved $\text{vol}(X) = \text{vol}(\varphi_\tau X)$. Also, if $x \in X$ then $\varphi_\tau x \in \varphi_\tau X$ and if $X_1 \cap X_2 = \{\}$ then $\varphi_\tau X_1 \cap \varphi_\tau X_2 = \{\}$. In the particular case of System (3.3) there exist sets which even keep their shape, regardless of their size and position in phase space. These sets are the possibly translated ellipses in standard position. Therefore the symplectic matrices

$$S_{\tau,m} = \begin{pmatrix} \cos \tau & \frac{1}{m} \sin \tau \\ -m \sin \tau & \cos \tau \end{pmatrix}.$$

with m fixed are the right candidates for deforming the lattice of the Gabor system $\mathcal{G}(g_{0,\sqrt{m}}, \Lambda)$ without changing the frame bounds.

Theorem 3.2. *Let $g_{0,m}(t) = (2m)^{1/4} e^{-\pi m t^2}$ be the dilated standard Gaussian and let $\Lambda \subset \mathbb{R}^2$ be a lattice with $\text{vol}(\Lambda) < 1$. Let*

$$S_{\tau,m} = \begin{pmatrix} \cos \tau & \frac{1}{m} \sin \tau \\ -m \sin \tau & \cos \tau \end{pmatrix}.$$

be the deformation matrix acting on the lattice. Then

$$\mathcal{G}(g_{0,m}, \Lambda) \cong \mathcal{G}(g_{0,m}, S_{\tau,m} \Lambda).$$

Proof. It is sufficient to show that

$$\mathcal{S}_{\tau,m} g_{0,m}(t) = c g_{0,m}(t),$$

with $\pi^{Mp}(\mathcal{S}_{\tau,m}) = S_{\tau,m}$. Since the ambiguity function determines a function up to a phase factor we will show

$$\mathcal{A}(\mathcal{S}_{\tau,m} g_{0,m})(x, \omega) = \mathcal{A}g_{0,m}(x, \omega).$$

First, we note that

$$S_{\tau,m} M_{\sqrt{m}} = \begin{pmatrix} \frac{\cos \tau}{\sqrt{m}} & \frac{\sin \tau}{\sqrt{m}} \\ -\sqrt{m} \sin \tau & \sqrt{m} \cos \tau \end{pmatrix} = M_{\sqrt{m}} S_{\tau},$$

where $S_{\tau} = S_{\tau,1}$ is a rotation by $-\tau$. This means, that imposing the elliptic flow $S_{\tau,m}$ on the dilated lattice is the same as rotating the lattice by the corresponding angle followed by the same dilation. As a next step we recall the covariance principle

$$\rho(\lambda) \mathcal{S} = \mathcal{S} \rho(S^{-1} \lambda)$$

for symmetric time-frequency shifts. Since $\mathcal{A}f(\lambda) = \langle f, \rho(\lambda) f \rangle$, the covariance principle implies that

$$\mathcal{A}(\mathcal{S}f)(\lambda) = \mathcal{A}f(S^{-1} \lambda).$$

Using the fact that

$$\mathcal{A}g_0(\lambda) = \mathcal{A}g_0(x, \omega) = e^{-\frac{\pi}{2}(x^2 + \omega^2)} = e^{-\frac{\pi}{2} \langle \lambda, \lambda \rangle}$$

we compute

$$\begin{aligned} \mathcal{A}(\mathcal{S}_{\tau,m} g_{0,m})(x, \omega) &= \mathcal{A}(\mathcal{S}_{\tau,m} \mathcal{M}_{\sqrt{m}} g_0)(x, \omega) \\ &= \mathcal{A}g_0\left((S_{\tau,m} M_{\sqrt{m}})^{-1} \lambda\right) \\ &= e^{-\frac{\pi}{2} \langle (S_{\tau,m} M_{\sqrt{m}})^{-1} \lambda, (S_{\tau,m} M_{\sqrt{m}})^{-1} \lambda \rangle} \\ &= e^{-\frac{\pi}{2} \langle (M_{\sqrt{m}} S_{\tau})^{-1} \lambda, (M_{\sqrt{m}} S_{\tau})^{-1} \lambda \rangle} \\ &= e^{-\frac{\pi}{2} \langle S_{\tau}^{-1} M_{\sqrt{m}}^{-1} \lambda, S_{\tau}^{-1} M_{\sqrt{m}}^{-1} \lambda \rangle} \\ &= e^{-\frac{\pi}{2} \langle M_{\sqrt{m}}^{-1} \lambda, M_{\sqrt{m}}^{-1} \lambda \rangle} \\ &= e^{-\frac{\pi}{2} \left(m x^2 + \frac{\omega^2}{m}\right)} \\ &= \mathcal{A}g_{0,m}(x, \omega). \end{aligned}$$

Therefore, we find that the ambiguity function $\mathcal{A}g_{0,m}$ stays invariant under the action of $\mathcal{S}_{\tau,m}$ on the window $g_{0,m}$. This implies that the dilated standard Gaussian $g_{0,m}$ is an eigenfunction of $\mathcal{S}_{\tau,m}$ with an eigenvalue of modulus 1.

$$\mathcal{S}_{\tau,m} g_{0,m}(t) = c g_{0,m}(t), \tag{3.4}$$

with $|c| = 1$. Hence, equation (3.4) implies that

$$\mathcal{G}(g_{0,m}, \Lambda) \cong \mathcal{G}(g_{0,m}, S_{\tau,m} \Lambda),$$

□

We note that according to Theorem 3.2 for each Gaussian there is an uncountable family of lattices, all with different Euclidean geometry, such that the resulting frame bounds are always the same. The only case where the Euclidean geometry remains unchanged is for $m = 1$, meaning that we only rotate the lattice. Hence, taking only the geometry into account, there is only 1 lattice for the standard Gaussian for which the frame bounds are kept.

In order to derive Theorem 3.2 we were motivated by a geometric approach and a clear picture in mind about the flow induced by the harmonic oscillator. The crucial ingredient for Theorem 3.2 to work is that we could explicitly calculate the eigenfunctions of the metaplectic operator via the ambiguity function. We note that similar approaches have already been made by Daubechies in 1988 [18], characterizing the (dilated) Hermite functions as eigenfunctions of certain localization operators. A geometric approach in the time-frequency plane has also been used in [20] to construct frames consisting of eigenfunctions of localization operators. We only stated Theorem 3.2 for the dilated Gaussian window, but the result holds for all dilated Hermite functions since they are eigenfunctions of the quadratic Fourier transform defined in (2.17) and have eigenvalues of modulus 1. We recall that we defined the n -th Hermite function as

$$h_n(t) = \frac{2^{1/4}}{(n!(2\pi)^n 2^n)^{1/2}} e^{\pi t^2} \left(\frac{d^n}{dt^n} e^{-2\pi t^2} \right), \quad n \in \mathbb{N}_0.$$

In fact we find the following characterization of the Hermite functions in Folland's textbook [34].

Proposition 3.3. *For $f \in L^2(\mathbb{R})$ the ambiguity function $\mathcal{A}f(x, \omega)$ is rotation-invariant if and only if f is a Hermite function $f = c h_n$, $c \in \mathbb{C}$, $n \in \mathbb{N}_0$.*

As a consequence we get the following conjecture for the Hermite functions.

Conjecture 3.4. *Let $g \in L^2(\mathbb{R})$, $\Lambda \subset \mathbb{R}^2$ such that the system $\mathcal{G}(g, \Lambda)$ is a frame and let*

$$S_{\tau, m} = \begin{pmatrix} \cos \tau & \frac{1}{m} \sin \tau \\ -m \sin \tau & \cos \tau \end{pmatrix}.$$

Then the following are equivalent.

- (i) *For all $\tau \in \mathbb{R}$ we have $\mathcal{G}(g, \Lambda) \cong \mathcal{G}(g, S_{\tau, m}\Lambda)$.*
- (ii) *For all $\tau \in \mathbb{R}$ we have $\mathcal{A}(S_{\tau, m}g)(x, \omega) = \mathcal{A}g(x, \omega)$ with $\pi^{Mp}(S_{\tau, m}) = S_{\tau, m}$*
- (iii) *$g(t) = c \mathcal{M}_{\sqrt{m}} h_n(t)$, $c \in \mathbb{C}$.*

The part needing verification is (i) \Rightarrow (ii). (ii) \Leftrightarrow (iii) is Proposition 3.3 and (ii) \Rightarrow (i) follows from Theorem 3.2.

As a next step, we will present results where the frame bounds are kept under the action of a discrete, non-compact deformation group leaving the lattice invariant.

3.3 Modular Deformations of Gabor Frames

In this section, we will investigate discrete deformations of Gabor frames for dimension $d = 1$. In particular, the objects of interest are taken from the modular group which we define as follows.

Definition 3.5 (Modular Group). The modular group $SL(2, \mathbb{Z})$ consists of all 2×2 matrices with integer entries and determinant 1.

The modular group is therefore a discrete subgroup of $SL(2, \mathbb{R}) = Sp(2, \mathbb{R})$. In the literature, the modular group is also defined as the group of linear fractional transformations on the complex upper half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

which have the form

$$z \mapsto \frac{az + b}{cz + d},$$

with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. It is obvious how to switch between these definitions and how to identify elements of the mentioned groups. For more details on the modular group see, e.g., the textbook of Stein & Shakarchi [71].

Consider the integer lattice \mathbb{Z}^2 . The action of the modular group leaves \mathbb{Z}^2 invariant, i.e. $\mathcal{B}\mathbb{Z}^2 = \mathbb{Z}^2$ for $\mathcal{B} \in Sp(2, \mathbb{Z})$. In other words, \mathcal{B} is just another choice for a basis of \mathbb{Z}^2 . In particular, any $\mathcal{B} \in Sp(2, \mathbb{Z})$ provides a basis for \mathbb{Z}^2 . Taking any symplectic matrix $S \in Sp(2, \mathbb{R})$ and any basis $\mathcal{B} \in Sp(2, \mathbb{Z})$ for \mathbb{Z}^2 this implies that

$$S\mathbb{Z}^2 = S\mathcal{B}\mathbb{Z}^2.$$

We stay with the square lattice for the beginning. Let

$$\Lambda_{\square}^{\delta} = \frac{1}{\sqrt{\delta}}\mathbb{Z} \times \frac{1}{\sqrt{\delta}}\mathbb{Z}$$

be the square lattice of density $\delta > 1$ such that the system $\mathcal{G}(g, \Lambda_{\square}^{\delta})$ is a Gabor frame. For

$$\mathcal{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{Z})$$

the corresponding metaplectic operator is given by

$$\mathfrak{B}g(\tilde{t}) = i^{-\frac{1}{2}} \sqrt{\frac{1}{|b|}} \int_{\mathbb{R}} e^{2\pi i W(\tilde{t}, t)} g(t) dt,$$

where $W(\tilde{t}, t) = \frac{1}{2} \frac{d}{b} \tilde{t}^2 - \frac{1}{b} \tilde{t}t + \frac{1}{2} \frac{a}{b} t^2$ and $b \neq 0$. In general $\mathfrak{B}g$ will differ from g by more than just a phase factor as we apply a chirp, a modified Fourier transform, a dilation and again a chirp, but the lattice remains invariant under a modular deformation. Hence,

$$\mathcal{G}(g, \Lambda_{\square}^{\delta}) \cong \mathcal{G}(\mathfrak{B}g, \Lambda_{\square}^{\delta}).$$

This result can be extended in an obvious way. Let $S \in Sp(2, \mathbb{R})$ and let $\mathcal{S} \in Mp(2, \mathbb{R})$ be the corresponding metaplectic operator, then

$$\mathcal{G}(Sg, S\Lambda_{\square}^{\delta}) \cong \mathcal{G}(\mathcal{S}\mathfrak{B}g, S\Lambda_{\square}^{\delta}).$$

Therefore, given any lattice $\Lambda = S\Lambda_{\square}^{\delta}$ there are countably many possible windows, resulting from one and the same window, which lead to the same Gabor frame bounds. We sum up the results in the following theorem.

Theorem 3.6. *Let $S \in Sp(2, \mathbb{R})$, $\mathcal{B} \in Sp(2, \mathbb{Z})$ and let \mathcal{S} and \mathfrak{B} be the corresponding metaplectic operators. Let $\Lambda_{\square}^{\delta} = \frac{1}{\sqrt{\delta}}\mathbb{Z} \times \frac{1}{\sqrt{\delta}}\mathbb{Z}$ with $\delta > 1$ and let $g \in L^2(\mathbb{R})$ be a window function. Then*

$$\mathcal{G}(Sg, S\Lambda_{\square}^{\delta}) \cong \mathcal{G}(\mathcal{S}\mathfrak{B}g, S\Lambda_{\square}^{\delta}) \cong \mathcal{G}(g, \Lambda_{\square}^{\delta}).$$

Whereas the deformations in the previous section had been derived from a continuous, compact group, the deformations in the current section were derived from a discrete, non-compact group. Continuous deformation groups will in general change the lattice, whereas the window might stay invariant under the corresponding deformation. Discrete deformation groups will in general change the window, whereas the lattice might stay invariant under the corresponding deformation.

We also note the importance of the order of the operators in Theorem 3.6. The window is chosen according to a choice of basis for the window before performing the deformation of the frame. Once the window is chosen we apply Theorem 3.1 in order to derive Theorem 3.6. What we learn from Theorem 3.6 is that the intuition that we need nicely concentrated windows in order to derive good frame bounds if we choose a ‘nice’ lattice such as the quadratic or the hexagonal lattice, is misleading. This results from the fact that the basis for the integer lattice \mathbb{Z}^2 might be far from the standard orthonormal basis in \mathbb{R}^2 . We will discuss this property in more detail in the next section.

3.4 Examples for Generalized Gaussians

We will now illustrate our geometric approach to keep the frame bounds when deforming a Gabor frame by example, using different Gaussians and different lattices.

Example 3.7. We start with an example inspired by the article of Strohmer & Beaver [72]. For this purpose, let

$$\Lambda_h^{\delta} = \frac{1}{\sqrt{\delta}} \sqrt{\frac{2}{\sqrt{3}}} \begin{pmatrix} \cos(\pi/6) & \cos(\pi/6) \\ -\sin(\pi/6) & \sin(\pi/6) \end{pmatrix} \mathbb{Z}^2$$

be a version of the hexagonal lattice of density $\delta > 1$. We choose the standard Gaussian g_0 as window function. The resulting Gabor system $\mathcal{G}(g_0, \Lambda_h^{\delta})$ is then of course a Gabor frame. We apply the dilation matrix $M_{3^{-1/4}}$ on the lattice and the rescaling operator $\mathcal{M}_{3^{-1/4}}$ on the window. Theorem 3.2 tells us that

$$\mathcal{G}(g_0, \Lambda_h^{\delta}) \cong \mathcal{G}(\mathcal{M}_{3^{-1/4}}g_0, M_{3^{-1/4}}\Lambda_h^{\delta}).$$

We compute

$$M_{3^{-1/4}}\Lambda_h^\delta = \frac{1}{\sqrt{\delta}} \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \mathbb{Z}^2 = S_{\pi/4}\Lambda_\square^\delta,$$

which is a 45 degrees rotated version of the square lattice of density $\delta > 1$. Recall, that the ambiguity function of g_0 is given by

$$\mathcal{A}g_0(x, \omega) = e^{-\frac{\pi}{2}(x^2 + \omega^2)}$$

and the ambiguity function of $\mathcal{M}_{3^{-1/4}}g_0 = g_{0, \frac{1}{\sqrt{3}}}$ is given by

$$\mathcal{A}g_{0, \frac{1}{\sqrt{3}}}(x, \omega) = e^{-\frac{\pi}{2}\left(\frac{x^2}{\sqrt{3}} + \sqrt{3}\omega^2\right)}.$$

So far, we derived the observation from [72] that $\mathcal{G}(g_{0,1}, \Lambda_h^\delta) \cong \mathcal{G}(g_{0, \frac{1}{\sqrt{3}}}, S_{\pi/4}\Lambda_\square^\delta)$. Basically this means that for a Gaussian whose ambiguity function has ellipses with axis ratio $\sqrt{3}$ and as level lines, we can choose a 45° rotated version of the square lattice and have the same frame bounds as for the standard Gaussian and the hexagonal lattice of same density. Applying the matrix

$$S_{\tau, \frac{1}{\sqrt{3}}} = \begin{pmatrix} \cos \tau & \sqrt{3} \sin \tau \\ -\frac{\sin \tau}{\sqrt{3}} & \cos \tau \end{pmatrix},$$

derived from the flow of the harmonic oscillator with mass $m = \frac{1}{\sqrt{3}}$ on the lattice will leave the frame bounds unchanged and we have

$$\mathcal{G}(g_0, \Lambda_h^\delta) \cong \mathcal{G}\left(g_{0, \frac{1}{\sqrt{3}}}, S_{\pi/4}\Lambda_\square^\delta\right) \cong \mathcal{G}\left(g_{0, \frac{1}{\sqrt{3}}}, S_{\tau, \frac{1}{\sqrt{3}}} S_{\pi/4}\Lambda_\square^\delta\right).$$

The deformation process is illustrated in Figure 2. Unless $\tau = k\frac{\pi}{3}$, $k \in \mathbb{Z}$, the lattice will not be a rotated version of the square lattice. The property which stays invariant under the deformation is the shape of the Gaussian. If we consider the level line of the Gaussian which gives the densest packing in the time-frequency plane, we observe that the packing density and the shape of the level line are the features which stay invariant under the chosen deformation. As already mentioned, we have chosen the operator according to the shape of the Gaussian and in general shapes are not kept under Hamiltonian deformations. The only property which stays invariant in general is the packing property. We extended the observations from [72] a bit, by providing an uncountable family of lattices which give the same packing property and the same frame bounds for a dilated Gaussian. We note again that for the standard Gaussian we can only rotate the lattice in order to keep these properties.

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Example 3.8. Let $\Lambda_\square^\delta = \frac{1}{\sqrt{\delta}}\mathbb{Z}^2$ be the square lattice of density $\delta > 1$ and let $g_0(t) = 2^{1/4}e^{-\pi t^2}$ be the standard Gaussian. The standard symplectic form J belongs to the modular group $Sp(2, \mathbb{Z})$. Hence, $J\Lambda_\square^\delta = \Lambda_\square^\delta$ and $\mathcal{J}g_0 = c g_0$ with $|c| = 1$. In this case neither

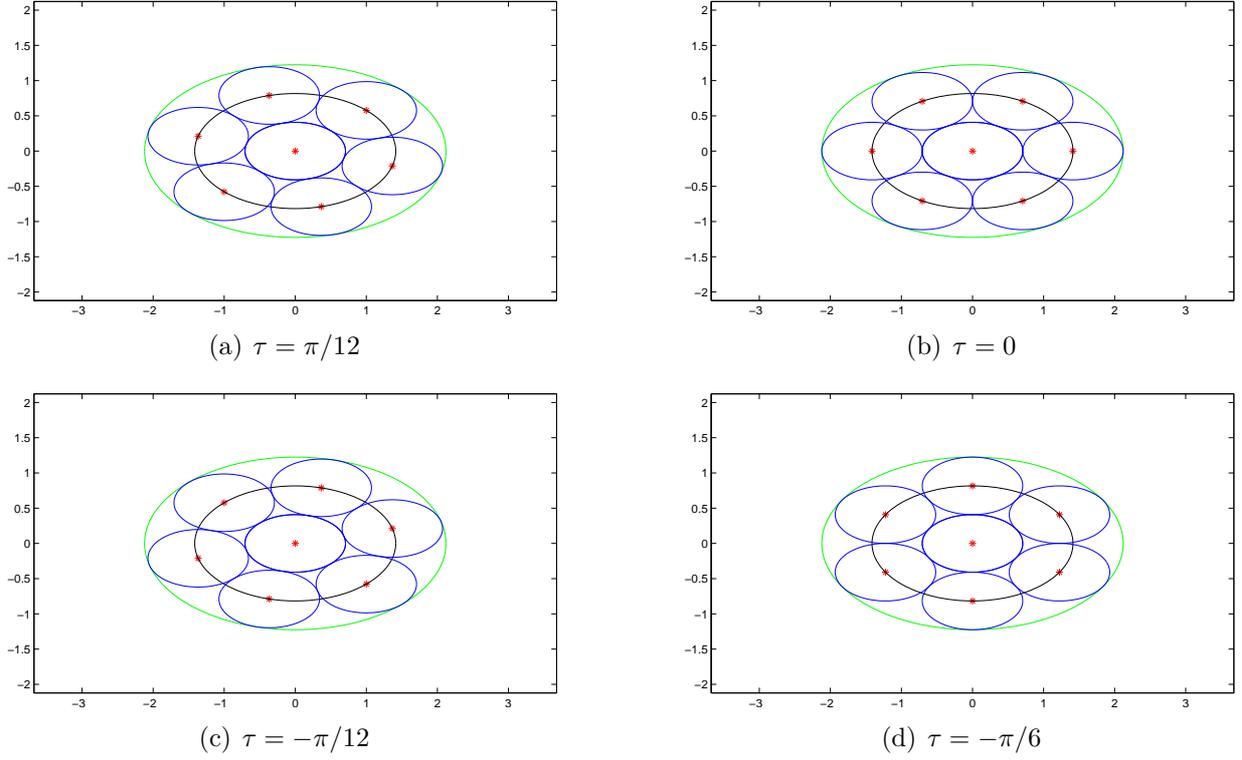


Figure 2: Illustration of the action of $S_{\tau, m}$ on the lattice and of $\mathcal{S}_{\tau, m}$ on the ambiguity function. The small ellipses illustrate the ambiguity functions centered at lattice points. The ellipses centered at the origin indicate flow lines of the harmonic oscillator.

the change of basis, nor the metaplectic operation have an effect on the Gabor frame. This is due to the fact that we could interpret the change of basis as a rotation of the time-frequency plane. Using a quadratic representation of g_0 , such as the ambiguity function, we see that a rotation does not have an effect as $\mathcal{A}g_0(x, \omega) = e^{-\frac{\pi}{2}(x^2 + \omega^2)}$.

Let us now consider the case of $g_{0, \sqrt{3}}$ and $S_{\frac{\pi}{4}} \Lambda_{\square}^{\delta}$ which is a 90 degrees rotated version of the window in Example 3.7. It can also be seen as a deformation of the window under a change of basis. Starting from the ambiguity function of

$$\mathcal{A}g_{0, \sqrt{3}}(x, \omega) = e^{-\frac{\pi}{2} \left(\sqrt{3}x^2 + \frac{\omega^2}{\sqrt{3}} \right)},$$

we rotate our lattice by the matrix $S_{-\frac{\pi}{4}}$ and apply the corresponding operator $\mathcal{S}_{-\frac{\pi}{4}}$ on the

window. Hence, the ambiguity function of the new window becomes

$$\begin{aligned}
\mathcal{A} \left(\mathcal{S}_{-\frac{\pi}{4}} g_{0,\sqrt{3}} \right) (x, \omega) &= \mathcal{A} g_{0,\sqrt{3}} \left(S_{-\frac{\pi}{4}}^{-1} \lambda \right) \\
&= \mathcal{A} g_0 \left(M_{3^{1/4}}^{-1} S_{-\frac{\pi}{4}}^{-1} \lambda \right) \\
&= e^{-\frac{\pi}{2} \left\langle \left(S_{-\frac{\pi}{4}} M_{3^{1/4}} \right)^{-1} \lambda, \left(S_{-\frac{\pi}{4}} M_{3^{1/4}} \right)^{-1} \lambda \right\rangle} \\
&= e^{-\frac{\pi}{2} \left(M_{3^{-1/4}} S_{\frac{\pi}{4}} \lambda \right)^T \cdot \left(M_{3^{-1/4}} S_{\frac{\pi}{4}} \lambda \right)} \\
&= e^{-\frac{\pi}{2} \lambda^T S_{\frac{\pi}{4}}^T M_{3^{-1/4}}^T M_{3^{-1/4}} S_{\frac{\pi}{4}} \lambda} \\
&= e^{-\frac{\pi}{2} \lambda^T S_{-\frac{\pi}{4}} M_{3^{-1/2}} S_{\frac{\pi}{4}} \lambda} \\
&= e^{-\frac{\pi}{2} \frac{2}{\sqrt{3}} (x^2 + x\omega + \omega^2)}.
\end{aligned}$$

Therefore, the level lines of the ambiguity function will be ellipses rotated by 45 degrees with axis ratio equal to $\sqrt{3}$ and the lattice will be a square lattice with density $\delta > 1$. The action of the metaplectic operator can be interpreted in a very natural and geometric way as can be seen by the calculations above. Every action imposed on the lattice effects the shape of the ambiguity function in exactly the same way as the geometry of the lattice is changed. We could also have rotated the lattice $\Lambda = \frac{1}{\sqrt{\delta}} S_{\frac{\pi}{4}} \mathbb{Z}^2$ in the opposite direction by $S_{\frac{\pi}{4}}$. Then the lattice would have been $\frac{1}{\sqrt{\delta}} J \mathbb{Z}^2$, which leads to the same lattice, but with another choice of basis. Hence, the window as well as the ambiguity function would have changed.

$$\mathcal{A} \left(\mathcal{S}_{-\frac{\pi}{4}}^{-1} g_{0,\sqrt{3}} \right) (x, \omega) = e^{-\frac{\pi}{2} \frac{2}{\sqrt{3}} (x^2 - x\omega + \omega^2)} = \mathcal{A} \left(\mathcal{S}_{-\frac{\pi}{4}} g_{0,1/\sqrt{3}} \right) (x, \omega)$$

Switching from $\mathcal{A} \left(\mathcal{S}_{-\frac{\pi}{4}}^{-1} g_{0,\sqrt{3}} \right)$ to $\mathcal{A} \left(\mathcal{S}_{-\frac{\pi}{4}}^{-1} g_{0,\frac{1}{\sqrt{3}}} \right)$, illustrated in Figure 3, can be interpreted as a rotation, a shearing or as choosing a different basis.

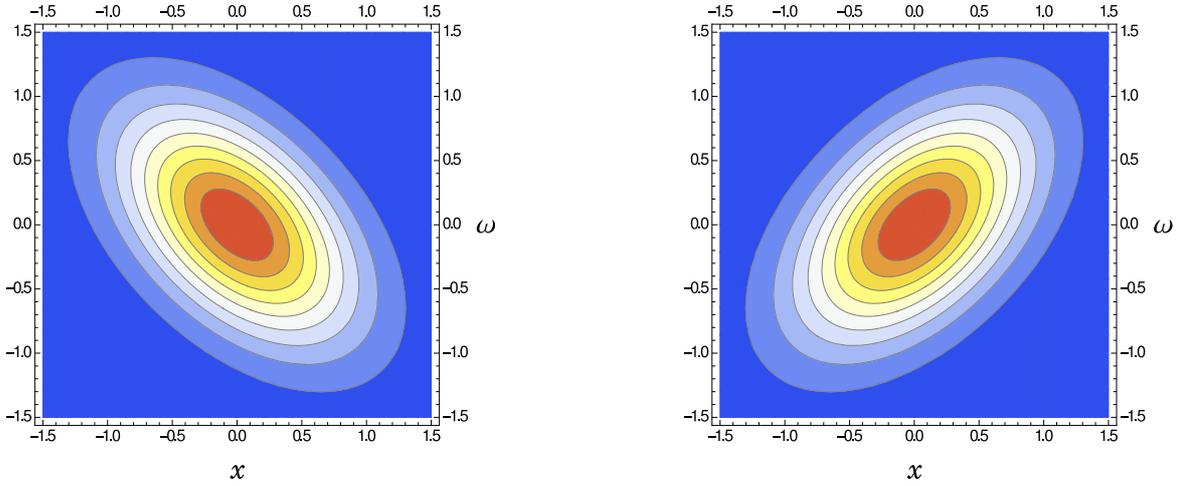
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Example 3.9. In this example we discuss Theorem 3.6 by considering the ambiguity function of the window. We choose the window $g_{0,\frac{1}{3}}$ which has ambiguity function

$$\mathcal{A} g_{0,\frac{1}{3}}(x, \omega) = e^{-\frac{\pi}{2} \left(\frac{x^2}{3} + 3\omega^2 \right)}.$$

This window is quite concentrated in the frequency direction whereas it is pretty spread in the time direction. Nonetheless we choose again the hexagonal lattice

$$\Lambda_h^\delta = S_h^\delta \mathbb{Z}^2 = \frac{1}{\sqrt{\delta}} \sqrt{\frac{2}{\sqrt{3}}} \begin{pmatrix} \cos(\pi/6) & \cos(\pi/6) \\ -\sin(\pi/6) & \sin(\pi/6) \end{pmatrix} \mathbb{Z}^2.$$



(a) $\mathcal{A}\left(\mathcal{S}_{-\frac{\pi}{4}} g_{0,\sqrt{3}}\right)$

(b) $\mathcal{A}\left(\mathcal{S}_{-\frac{\pi}{4}} g_{0,\frac{1}{\sqrt{3}}}\right)$

Figure 3: Contour plots of the ambiguity functions of two generalized Gaussians which lead to the same frame bounds for the scaled integer lattice of density $\delta > 1$.

Then we have

$$\mathcal{G}\left(g_0, \Lambda_h^\delta\right) \cong \mathcal{G}\left(g_{0,\frac{1}{3}}, \Lambda_h^\delta\right).$$

The basis chosen for \mathbb{Z}^2 in the example are the Euclidean standard Basis and

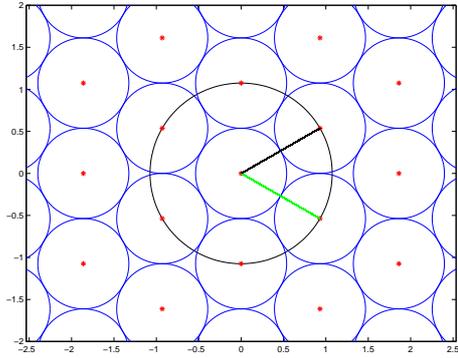
$$\tilde{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In particular there exist a generalized Gaussian window \tilde{g} and metaplectic operators \mathcal{S}_h^δ with $\pi^{Mp}(\mathcal{S}_h^\delta) = S_h^\delta$ and $\tilde{\mathfrak{B}}$ with $\pi^{Mp}(\tilde{\mathfrak{B}}) = \tilde{\mathcal{B}}$ such that

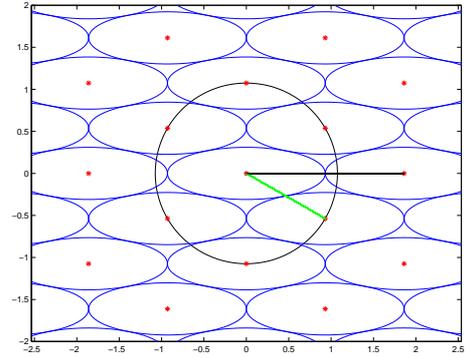
$$\mathcal{G}(\tilde{g}, \mathbb{Z}^2) = \mathcal{G}\left((\mathcal{S}_h^\delta)^{-1} g_0, \mathbb{Z}^2\right) = \mathcal{G}\left(\tilde{\mathfrak{B}}^{-1} (\mathcal{S}_h^\delta)^{-1} g_{0,\frac{1}{3}}, \mathbb{Z}^2\right)$$

By choosing another basis we can create even less concentrated (generalized) Gaussian windows which still lead to Gabor frames which have the same frame bounds as $\mathcal{G}(g_0, \Lambda_h^\delta)$. We also note that in contrast to Example 3.8 the systems $\mathcal{G}\left(g_{0,\frac{1}{3}}, \Lambda_h^\delta\right)$ and $\mathcal{G}\left(g_{0,3}, \Lambda_h^\delta\right)$ do not possess the same optimal frame bounds since Λ_h^δ is not invariant under a rotation of 90° .

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(a) Hexagonal arrangement for the standard Gaussian.



(b) Hexagonal arrangement for the dilated standard Gaussian

Figure 4: Illustrated are the shapes of the ambiguity functions of the time-frequency shifted standard Gaussian g_0 and of the time-frequency shifted dilated Gaussian $g_{0, \frac{1}{3}}$. In both cases the time-frequency shifts are carried out in the same hexagonal pattern. The straight lines indicate the chosen basis \mathcal{B} for the lattice, the large circle is for visual aid to identify the hexagonal pattern more easily.

We have seen that the geometric intuition we have about packing problems is helpful for deforming Gaussian Gabor frames without changing the frame bounds, but we have to be aware that we might encounter some surprising results if we do not choose the standard basis for \mathbb{Z}^2 . These heuristic arguments lead to the conjectures that for the standard Gaussian among all separable lattice the square lattice yields the best frame bounds and that among general lattices the hexagonal lattice gives the best frame bounds. We will prove parts of these conjectures in the upcoming sections.

4 Optimal Frame Bounds

In this section we will outline methods to compute sharp frame bounds of a Gabor system $\mathcal{G}(g, \Lambda)$. These methods will also serve as tools to investigate how the frame bounds change in dependence of the lattice parameters.

4.1 The Zak Transform and the Pre-Gramian

So far, we found conditions under which two Gabor frames possess the same optimal frame bounds. As a next step, we will study how to explicitly compute those optimal bounds. A tool heavily exploited when it comes to calculating frame bounds is the Zak transform which we will define right away.

Definition 4.1 (Zak Transform). The Zak transform of the function $f \in L^2(\mathbb{R})$ with respect to the parameter $T > 0$ is given by

$$(\mathcal{Z}_T f)(x, \omega) = \sqrt{T} \sum_{k \in \mathbb{Z}} f(x - Tk) e^{2\pi i Tk \omega}, \quad x, \omega \in \mathbb{R}.$$

For an extensive study on the Zak transform we refer to [41, 52]. The origin of this transform goes at least back to Weil [74], according to Janssen [52] it was already studied by Gauss. It became more popular after Zak's work on finding eigenfunctions for a complete set of commuting operators in quantum mechanics in 1967 [78]. Since then, the Zak transform has often been used as tool in time-frequency analysis to compute the frame bounds of Gabor frames [48, 52, 54, 62].

Lemma 4.2. *Let $1 < \frac{1}{\alpha\beta} = n \in \mathbb{N}$ be fixed and let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and $f \in L^2(\mathbb{R})$. Then the optimal lower and upper frame bound of $G(f, \Lambda)$ are given by*

$$A = \operatorname{ess\,inf}_{(x, \omega)} \sum_{k=0}^{n-1} \left| (\mathcal{Z}_\alpha f) \left(x + \frac{k}{n}, \omega \right) \right|^2 \quad (4.1)$$

$$B = \operatorname{ess\,sup}_{(x, \omega)} \sum_{k=0}^{n-1} \left| (\mathcal{Z}_\alpha f) \left(x + \frac{k}{n}, \omega \right) \right|^2 \quad (4.2)$$

respectively, where the essential infimum and supremum may be taken over any unit square in \mathbb{R}^2 .

Janssen uses an equivalent approach to compute frame bounds for windows in $L^2(\mathbb{R})$ on separable lattices [53, 54], which we will follow. First, we define the operator G which maps elements of $L^2(\mathbb{R})$ onto double sequences and is often called the pre-Gramian of the Gabor frame $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$

$$Gf = \left(\langle f, M_{k/\beta} T_{l/\alpha} g \rangle \right)_{k, l \in \mathbb{Z}}, \quad f, g \in L^2(\mathbb{R}).$$

Then the operator GG^* , called the Gramian of $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$, maps double sequences onto double sequences and has matrix elements

$$(GG^*)_{(k,l);(k',l')} = \langle M_{l'/\alpha} T_{k'/\beta} g, M_{l/\alpha} T_{k/\beta} g \rangle,$$

with $k, l, k', l' \in \mathbb{Z}$, $g \in L^2(\mathbb{R})$. In the case that $1 \leq \frac{1}{\alpha\beta} = n \in \mathbb{N}$, by using the commutation relations (1.2) it can easily be shown that

$$(GG^*)_{(k,l);(k',l')} = \langle g, M_{(l-l')/\alpha} T_{(k-k')/\beta} g \rangle,$$

with $k, l, k', l' \in \mathbb{Z}$, $g \in L^2(\mathbb{R})$. Also, in this case the Gramian GG^* has Toeplitz structure and the spectrum of GG^* is therefore contained in the interval $[\tilde{A}, \tilde{B}]$, where

$$\tilde{A} = \text{ess inf}_{(x,\omega)} F_g(x, \omega; \alpha, \beta) \quad (4.3)$$

$$\tilde{B} = \text{ess sup}_{(x,\omega)} F_g(x, \omega; \alpha, \beta) \quad (4.4)$$

where F_g is the Fourier series associated to the Toeplitz operator GG^* [14].

$$F_g(x, \omega; \alpha, \beta) = \sum_{(k,l) \in \mathbb{Z}^2} \langle g, M_{l/\alpha} T_{k/\beta} g \rangle e^{2\pi i k x} e^{2\pi i l \omega} \quad (4.5)$$

and the essential infimum and supremum may be taken over any unit square. The reader familiar with Riesz sequences will have noticed that these are the Riesz bounds of the adjoint Gabor system. We find that

$$\frac{1}{\alpha\beta} F_g(x, \omega; \alpha, \beta) = \sum_{k=0}^{n-1} \left| (\mathcal{Z}_\alpha f) \left(x + \frac{k}{n}, \omega \right) \right|^2 \quad (4.6)$$

(see e.g. [54]) and so from (4.1), (4.2), (4.3), (4.4) and (4.6) we conclude that the spectral bounds of GG^* coincide with the Gabor frame bounds of $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ up to the constant factor $n = \frac{1}{\alpha\beta}$.

The operator G already appeared in the work by Wexler & Raz [75] where they study the construction of dual windows. The method became known as the Wexler-Raz biorthogonality relations.

We will now compute the frame bounds for a chirped Gaussian window on a rectangular lattice with (fixed) integer redundancy. This is equivalent to computing the frame Bounds for a Gabor frame with the standard Gaussian g_0 on a general lattice of (fixed) integer redundancy.

The chirped standard Gaussian, with chirping parameter γ , is given by

$$\mathcal{V}_\gamma g_0(t) = g_\gamma(t) = e^{\pi i \gamma t^2} 2^{1/4} e^{-\pi t^2} = 2^{1/4} e^{-\pi t^2(1-i\gamma)}.$$

The frame bounds are then given by the minimum and maximum of the Fourier series

$$F_{g_\gamma}(t, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \langle g_\gamma, M_{l/\alpha} T_{k/\beta} g_\gamma \rangle e^{2\pi i k x} e^{2\pi i l \omega}.$$

Furthermore, we compute

$$\begin{aligned} \left\langle g_\gamma, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g_\gamma \right\rangle &= 2^{1/2} \int_{\mathbb{R}} e^{\pi i \gamma t^2} e^{-\pi t^2} e^{-\pi i \gamma (t-k/\beta)^2} e^{-\pi (t-k/\beta)^2} e^{-2\pi i l/\alpha t} dt \\ &= 2^{1/2} \int_{\mathbb{R}} e^{\pi i \gamma (t^2 - (t-k/\beta)^2)} e^{-\pi (t^2 + (t-k/\beta)^2)} e^{-2\pi i l/\alpha t} dt \\ &= 2^{1/2} e^{-\pi \frac{k^2}{\beta^2} (1-i\gamma)} \frac{1}{\sqrt{2}} \int_{\mathbb{R}} e^{-\pi t^2} e^{-2\pi i (\frac{l}{\alpha} + \frac{k}{\beta}(-\gamma+i)) / \sqrt{2} t} dt \\ &= e^{-\pi \frac{k^2}{\beta^2} (1+i\gamma)} e^{-\frac{\pi}{2} (\frac{l}{\alpha} + \frac{k}{\beta}(-\gamma+i))^2} \\ &= e^{-\pi i \frac{kl}{\alpha\beta}} e^{-\frac{\pi}{2} (\frac{k^2}{\beta^2} + (\frac{l}{\alpha} - \frac{k\gamma}{\beta})^2)}. \end{aligned}$$

Since we have $(\alpha\beta)^{-1} = n \in \mathbb{N}$, the complex exponential will give an alternating sign. Therefore, the lower and upper frame bound are given by

$$A(\alpha, \beta) = \min_{(x, \omega) \in [0, 1]^2} n \sum_{k, l \in \mathbb{Z}} (-1)^{(kln)} e^{-\frac{\pi}{2} (\frac{k^2}{\beta^2} + (\frac{l}{\alpha} - \frac{k\gamma}{\beta})^2)} e^{2\pi i k x} e^{2\pi i l \omega}, \quad (4.7)$$

$$B(\alpha, \beta) = \max_{(x, \omega) \in [0, 1]^2} n \sum_{k, l \in \mathbb{Z}} (-1)^{(kln)} e^{-\frac{\pi}{2} (\frac{k^2}{\beta^2} + (\frac{l}{\alpha} - \frac{k\gamma}{\beta})^2)} e^{2\pi i k x} e^{2\pi i l \omega} \quad (4.8)$$

respectively. We will study these series intensively in the Section 5.

4.2 Janssen's Representation

We want to mention a certain representation of the frame operator which also goes back to Janssen [53] and is now widely known as Janssen's representation. Before doing so, we formulate the definition of the adjoint lattice.

Definition 4.3. For a lattice $\Lambda \subset \mathbb{R}^2$ the adjoint lattice is given by

$$\Lambda^\circ = \text{vol}(\Lambda)^{-1} \Lambda.$$

Proposition 4.4 (Janssen's representation). *Let $g \in L^2(\mathbb{R}^d)$ with the property that*

$$\sum_{\lambda^\circ \in \Lambda^\circ} |\langle g, \pi(\lambda^\circ)g \rangle| < \infty. \quad (4.9)$$

Then, the frame operator can be written in the form

$$S_{g, \Lambda} = \text{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ).$$

We note that the coefficients, given by the inner products of g and its time-frequency shifted version $\pi(\lambda^\circ)g$, appearing in Janssen's representation are the same as the coefficients appearing in the Fourier series associated to the Toeplitz operator GG^* in equation (4.5). We will make use of this representation in Section 6. Note that if the lattice Λ is very dense, then the adjoint lattice Λ° is very coarse which usually gives computational advantages.

5 The Separable Case and Jacobi's Theta Functions

In this section we will focus on the case where Λ is a separable lattice, i.e. it can be written in the form $\alpha\mathbb{Z} \times \beta\mathbb{Z}$, $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$. The section is motivated by the following question. Given a Gaussian window function, which lattice $\Lambda \subset \mathbb{R}^2$ minimizes B/A ? Due to a conjecture in the work of Floch, Alard & Berrou in 1995 [33] the square lattice was a promising candidate to provide a solution to this problem. This was disproved in 2003 by Strohmer & Beaver [72] by providing evidence that the hexagonal lattice is preferable over the square lattice in terms of the frame condition. The new conjecture was therefore that B/A is minimized by the hexagonal lattice among all lattices of any fixed redundancy (this conjecture was also stated by Abreu & Dörfler in 2012 [1]). However, Strohmer & Beaver also claim that it is 'plausible' to assume that the square lattice minimizes the frame condition among all rectangular lattices. We believe that an even stronger statement is true, which we formulate in the following conjecture.

Conjecture 5.1. ¹. *Let $g_0(t) = 2^{1/4}e^{-\pi t^2}$ be the standard Gaussian window. Among all pairs (α, β) in the set*

$$\mathcal{F}_{(\alpha, \beta)}^\delta(g_0) = \{(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid (\alpha\beta)^{-1} = \delta > 1, \delta \text{ fixed}\}.$$

the pair $(\frac{1}{\sqrt{\delta}}, \frac{1}{\sqrt{\delta}})$ is the unique maximizers for the lower frame bound and the unique minimizer for the upper frame bound.

For even redundancy, we will prove the correctness of the conjecture. Given a standard Gaussian Gabor frame with a separable lattice of integer redundancy $1 < (\alpha\beta)^{-1} = n \in 2\mathbb{N}$, we will show that the problems of finding lattice parameters maximizing the lower frame bound and of finding lattice parameters minimizing the upper frame bound are both uniquely solved for the parameters $\alpha = \beta = \frac{1}{\sqrt{n}}$. Hence, the resulting lattice is the square lattice. These results were first proven by Faulhuber & Steinerberger [24]. For odd redundancy, we will also see a proof that the upper frame bound is again minimized if $\alpha = \beta = \frac{1}{\sqrt{n}}$.

Unfortunately, several details of [24, 7. Proof of Theorem 2.6], which implies the proof of the maximality of the lower frame bound for the square lattice of odd redundancy, were omitted and the author was not able to reproduce all the details necessary before completion of the thesis. We will therefore not state this maximality result as a theorem, but only try to make reasoning of [24, 7. Proof of Theorem 2.6] more plausible.

We will see that the proofs do not require any special knowledge on Gabor frames, but rather need identities for Jacobi's theta functions. The results and techniques presented in this section have already been adapted and reused by other researchers. In particular, some results from [24] were already used in [10] and [11].

¹In a private discussion with Peter Søndergaard at the Strobl conference in 2014 we light-heartedly called our intuition, that among all separable lattice the square lattice should minimize the frame condition number, 'The Obvious Conjecture'. At that time we only had numerical inspections and no idea how to prove it.

5.1 Even Redundancy.

Theorem 5.2. Consider the window function $g_0(t) = 2^{1/4}e^{-\pi t^2}$. For the Gabor frame

$$\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$$

with $1 < (\alpha\beta)^{-1} = n \in 2\mathbb{N}$ we denote the lower and upper frame bound by

$$A = A_{g_0, n}(\alpha, \beta)$$

and

$$B = B_{g_0, n}(\alpha, \beta)$$

respectively. Then we have that

$$A_{g_0, n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \geq A_{g_0, n}(\alpha, \beta) \quad (5.1)$$

$$B_{g_0, n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \leq B_{g_0, n}(\alpha, \beta) \quad (5.2)$$

for all $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $1 < (\alpha\beta)^{-1} = n \in \mathbb{N}$ with equality if and only if $(\alpha, \beta) = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right)$.

Corollary 5.3. For $g_0(t) = 2^{1/4}e^{-\pi t^2}$, the condition number of the frame operator for the Gabor frame $\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is minimized for the square lattice. This means that

$$\frac{B_{g_0, n}}{A_{g_0, n}} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \leq \frac{B_{g_0, n}}{A_{g_0, n}}(\alpha, \beta)$$

for all $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $1 < (\alpha\beta)^{-1} = n \in 2\mathbb{N}$ with equality if and only if $(\alpha, \beta) = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right)$.

The proof of Theorem 5.2 will follow from statements about products of Jacobi's theta-3 and theta-4 function. Recall equation (4.7) and equation (4.8) which tell us that for a chirped Gaussian the lower and upper frame bound can be computed, up to the factor $(\alpha\beta)^{-1} = n$, via the minimum and the maximum of the Fourier series

$$F_{g_\gamma}(x, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{(kln)} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \left(\frac{l}{\alpha} - \frac{k\gamma}{\beta} \right)^2 \right)} e^{2\pi i k x} e^{2\pi i l \omega}.$$

For $\gamma = 0$ this reduces to

$$F_{g_0}(x, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{kln} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{2\pi i k x} e^{2\pi i l \omega}. \quad (5.3)$$

We will need to prove many properties about Jacobi's theta-3 and theta-4 function in order to derive the results. This section will naturally split into two separate parts again,

one proving results for θ_3 and the result for the upper bound and one proving results of θ_4 and the result for the lower bound.

If n is even the alternating sign in the Fourier series (5.3) vanishes and therefore we treat this case first. We also see that we can split the double sum in the following way

$$F_{g_0}(x, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{k^2}{\beta^2}} e^{2\pi i k x} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{l^2}{\alpha^2}} e^{2\pi i l \omega}.$$

With the following definition we can rewrite F_{g_γ} as a product of theta functions.

Definition 5.4 (Theta Function). For $z \in \mathbb{C}$ and $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$ (complex upper half plane) we define the function

$$\Theta(z, \tau) = \sum_{k=-\infty}^{\infty} e^{\pi i k^2 \tau} e^{2\pi i k z}$$

called theta function.

From this definition we see that

$$\begin{aligned} \Theta(z, \tau) &= \Theta(z + 1, \tau), \\ \Theta(z, \tau) &= \Theta(z, \tau + 2). \end{aligned}$$

Furthermore, Θ is an entire function in the variable $z \in \mathbb{C}$ and holomorphic in the variable $\tau \in \mathbb{H}$. Also, for τ fixed we have the property that

$$\Theta(1/2 + \tau/2 + k + l\tau, \tau) = 0, \quad k, l \in \mathbb{Z},$$

in particular $\Theta(1/2, 0) = 0$ and $\Theta(1/2 + \tau/2, \tau) = 0$. There is also a product representation of the theta function, known as Jacobi's triple product representation, which will be useful later on.

Proposition 5.5 (Jacobi triple product). For $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ we have

$$\Theta(z, \tau) = \prod_{k \geq 1} (1 - e^{2k\pi i \tau}) (1 + e^{(2k-1)\pi i \tau} e^{2\pi i z}) (1 + e^{(2k-1)\pi i \tau} e^{-2\pi i z}). \quad (5.4)$$

For more details on theta functions and the product representation we refer to the textbook of Stein & Shakarchi [71].

As Janssen already stated [54], it follows from the product representation that for real z and purely imaginary $\tau = is$, $s \in \mathbb{R}_+$ the function $\Theta(z, is)$ is maximal for $z \in \mathbb{Z}$ and minimal for $z \in \mathbb{Z} + \frac{1}{2}$. Therefore, the Fourier series

$$\begin{aligned} F_{g_0}(x, \omega; \alpha, \beta) &= \sum_{k \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{k^2}{\beta^2}} e^{2\pi i k x} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{l^2}{\alpha^2}} e^{2\pi i l \omega} \\ &= \Theta\left(x, \frac{1}{2\beta^2}\right) \Theta\left(\omega, \frac{1}{2\alpha^2}\right) \\ &= \Theta\left(x, \frac{1}{2\beta^2}\right) \Theta\left(\omega, \frac{n^2\beta^2}{2}\right) \end{aligned}$$

assumes its minimum at $(x, \omega) = (k + 1/2, l + 1/2)$ and its maximum at $(x, \omega) = (k, l)$, $k, l \in \mathbb{Z}$. For the special values $z = 0$ and $z = 1/2$ the theta function is known as Jacobi's theta-3 and theta-4 function respectively. We have

$$\theta_3(s) = \sum_{k=-\infty}^{\infty} e^{-\pi k^2 s} = \Theta(0, is)$$

$$\theta_4(s) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-\pi k^2 s} = \Theta(1/2, is)$$

with $s \in \mathbb{R}_+$. They appear in many different areas: θ_4 , for example, is a rescaling of the cumulative distribution function of the Kolmogorov-Smirnov distribution in probability theory whereas θ_3 is the heat kernel on the torus evaluated at 0.

For $(\alpha\beta)^{-1} = n \in 2\mathbb{N}$ fixed, we can now write the lower and upper frame bound of $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ in dependence of one lattice parameter and in terms of Jacobi's theta functions. We have

$$A = A(\beta) = n \theta_4 \left(\frac{1}{2\beta^2} \right) \theta_4 \left(\frac{n^2 \beta^2}{2} \right) \quad (5.5)$$

$$B = B(\beta) = n \theta_3 \left(\frac{1}{2\beta^2} \right) \theta_3 \left(\frac{n^2 \beta^2}{2} \right). \quad (5.6)$$

From these formulas it is easy to see that $A(\beta)$ as well as $B(\beta)$ have critical points at $\beta = 1/\sqrt{n}$. The corresponding separable lattice is the quadratic lattice $\frac{1}{\sqrt{n}}\mathbb{Z} \times \frac{1}{\sqrt{n}}\mathbb{Z}$. The work lies in showing that there are no more critical points and that the resulting extrema are global and unique.

By making the substitution $\beta = \sqrt{s/n}$ we get

$$A \left(\sqrt{s/n} \right) = \tilde{A}(s) = n \theta_4 \left(\frac{n}{2} s \right) \theta_4 \left(\frac{n}{2} \frac{1}{s} \right) \quad (5.7)$$

$$B \left(\sqrt{s/n} \right) = \tilde{B}(s) = n \theta_3 \left(\frac{n}{2} s \right) \theta_3 \left(\frac{n}{2} \frac{1}{s} \right) \quad (5.8)$$

which is slightly easier to handle.

As a next step we take the algebraic structure into account. A straight-forward analysis of the above products seems difficult because they have extremely small derivatives around $s = 1$ (see Figures 5 and 6). The following lemma will be quite useful.

Lemma 5.6. *Let $F_r(s) = f(rs)f(r/s)$ with $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ differentiable and $r \in \mathbb{R}_+$ fixed. If*

$$s \frac{f'(s)}{f(s)} \quad \text{is strictly increasing (decreasing) for } s > 0,$$

then the only critical point of $F_r(s)$ is at $s = 1$ and the this point is the global minimum (maximum).

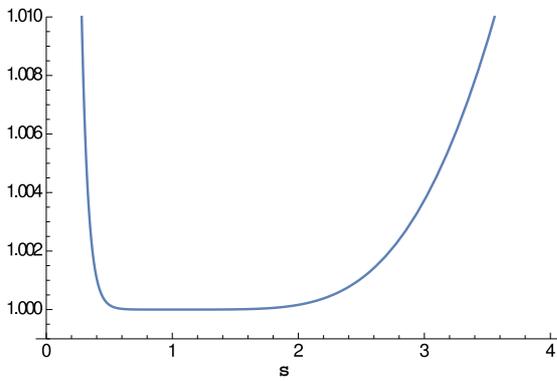
Proof. The algebraic structure implies that $F_r(s) = F_r(1/s)$ and therefore there exists either a local minimum or a local maximum. Each critical point of F_r satisfies

$$0 = \frac{d}{ds}F_r(s) = rf'(rs)f(r/s) - \frac{r}{s^2}f(rs)f'(r/s),$$

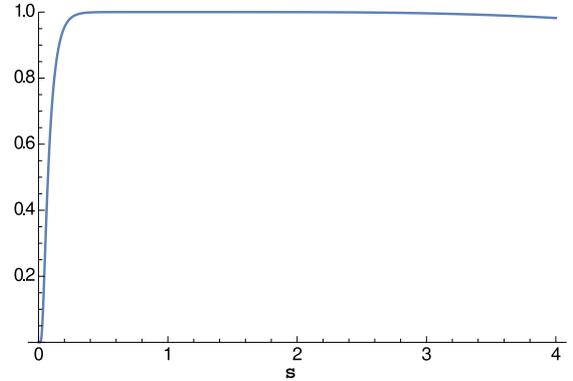
which is equivalent to

$$rs \frac{f'(rs)}{f(rs)} = \frac{r}{s} \frac{f'(r/s)}{f(r/s)}.$$

The monotonicity assumption implies $rs = r/s$ and thus $s = 1$ is the only solution. Also, due to the monotonicity assumption, we find that the global minimum (maximum) is achieved only for $s = 1$. \square

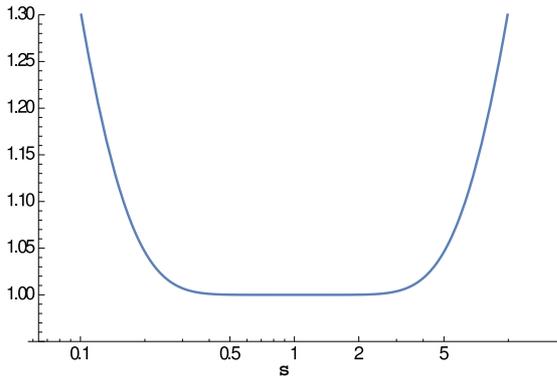


(a) The function $\theta_3(6s)\theta_3(6/s)$.

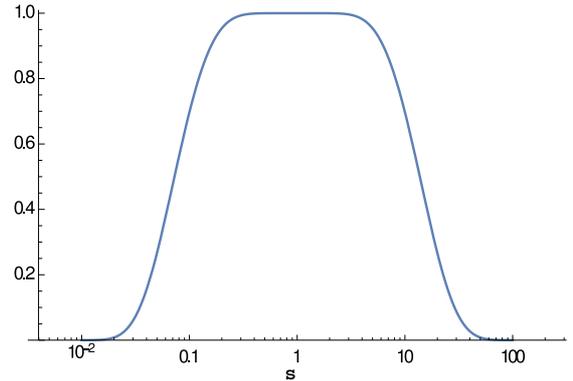


(b) The function $\theta_4(6s)\theta_4(6/s)$.

Figure 5: Both functions take their global extremum for $s = 1$.



(a) The function $\theta_3(6s)\theta_3(6/s)$ with logarithmic scale on the abscissa.



(b) The function $\theta_4(6s)\theta_4(6/s)$ with logarithmic scale on the abscissa.

Figure 6: On a logarithmic scale the symmetry around $s = 1$ becomes more visible.

We will now proof some properties of the Jacobi theta-3 and theta-4 functions, which we could not find in the literature prior to the work of Faulhuber & Steinerberger [24] and in the end they will imply Theorem 5.2 for even redundancy.

5.1.1 Properties of Theta-3 and a Statement for the Upper Frame Bound

This section is devoted to prove equation (5.2)

$$B_{g_0,n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \leq B_{g_0,n}(\alpha, \beta)$$

of Theorem 5.2 for even redundancy.

We start with two identities for the Jacobi theta-3 function. The first identity is well-known and follows from the Poisson summation formula.

Lemma 5.7 (Jacobi Identity).

$$\sqrt{s}\theta_3(s) = \theta_3\left(\frac{1}{s}\right). \quad (5.9)$$

To our knowledge the next identity was not found in the literature before the publication of [24].

Lemma 5.8. *For all $s > 0$ we have*

$$s \frac{\theta_3'(s)}{\theta_3(s)} + \frac{1}{s} \frac{\theta_3'(1/s)}{\theta_3(1/s)} = -\frac{1}{2}.$$

Proof. We use the Jacobi identity given in equation (5.9). Differentiating the identity on both sides gives

$$\frac{1}{2\sqrt{s}}\theta_3(s) + \sqrt{s}\theta_3'(s) = -\frac{1}{s^2}\theta_3'\left(\frac{1}{s}\right).$$

Multiplying both sides with $\frac{s^{1/2}}{\theta_3(s)}$ yields

$$\frac{1}{2} + s \frac{\theta_3'(s)}{\theta_3(s)} = -\frac{1}{s} \frac{\theta_3'\left(\frac{1}{s}\right)}{s^{1/2}\theta_3\left(\frac{1}{s}\right)}.$$

Using the Jacobi identity once more we get

$$\frac{1}{2} + s \frac{\theta_3'(s)}{\theta_3(s)} = -\frac{1}{s} \frac{\theta_3'\left(\frac{1}{s}\right)}{\theta_3\left(\frac{1}{s}\right)}$$

and the proof is complete. □

At this point we remark that Riemann apparently used the following identities, which he attributed to Jacobi, to establish the functional equation for his zeta function (see e.g. the textbook [21]). The identities are

$$\frac{1 + 2\psi(s)}{1 + 2\psi\left(\frac{1}{s}\right)} = \frac{1}{\sqrt{s}} \quad (5.10)$$

and

$$\frac{1}{2} + \psi(1) + 4\psi'(1) = 0, \quad (5.11)$$

where $\psi(s) = \sum_{k \geq 1} e^{-\pi k^2 s}$. We find out that equation (5.10) is the Jacobi identity for the theta-3 function in Lemma 5.7 and equation (5.11) follows by differentiating $1 + 2\psi(s) = \frac{1}{\sqrt{s}} (1 + 2\psi(\frac{1}{s}))$ on both sides and evaluating for $s = 1$. We note that equation (5.11) can also be written as $\frac{1}{2}\theta_3(1) + 2\theta_3'(1) = 0$ which is equivalent to $\frac{\theta_3'(1)}{\theta_3(1)} = -\frac{1}{4}$. This is a special case of the identity in Lemma 5.8 for $s = 1$.

The next result follows from estimates on θ_3 and its derivatives by using geometric series.

Lemma 5.9. *The function $s \frac{\theta_3'(s)}{\theta_3(s)}$ is strictly increasing for $s \geq 1$.*

Proof. We will now show that the derivative $\frac{d}{ds} (s\theta_3'(s)/\theta_3(s))$ is positive. This is equivalent to showing that

$$s\theta_3''(s)\theta_3(s) + \theta_3'(s)\theta_3(s) - s\theta_3'(s)^2 > 0,$$

where the first term is positive and the last two terms are negative. Since $\theta_3(s) \geq 1$ the statement above is implied by the stronger inequality

$$s\theta_3''(s) + \theta_3'(s)\theta_3(s) - s\theta_3'(s)^2 > 0,$$

which can be equivalently written as

$$0 < s \underbrace{\left(2 \sum_{k \geq 1} \pi^2 k^4 e^{-\pi k^2 s}\right)}_{(I)} - \underbrace{\left(2 \sum_{k \geq 1} \pi k^2 e^{-\pi k^2 s}\right)}_{(II)} \underbrace{\left(1 + 2 \sum_{k \geq 1} e^{-\pi k^2 s}\right)}_{\theta_3(s)} - s \underbrace{\left(2 \sum_{k \geq 1} \pi k^2 e^{-\pi k^2 s}\right)^2}_{(III)}. \quad (5.12)$$

We will now establish this inequality using term-by-term estimates. The first term (I) can be easily bounded from below by

$$(I) = s \left(2 \sum_{k \geq 1} \pi^2 k^4 e^{-\pi k^2 s}\right) \geq 2s\pi^2 e^{-\pi s}.$$

In order to control (II) and (III) we bound the expressions by geometric series. The geometric series starting at index m is given by

$$\sum_{k \geq m} q^k = q^m \sum_{k \geq 0} q^k = q^m \frac{1}{1-q}.$$

Differentiating the classical geometric series $\sum_{k \geq 0} q^k = \frac{1}{1-q}$ with a subsequent multiplication by q gives

$$\sum_{k \geq 0} kq^k = \frac{q}{(1-q)^2}.$$

By combining these results we show that

$$\begin{aligned} \sum_{k \geq m} kq^k &= q^m \left(\sum_{k \geq 0} kq^k + m \sum_{k \geq 0} q^k \right) \\ &= q^m \left(\frac{q}{(1-q)^2} + m \frac{1}{1-q} \right) \\ &= q^m \frac{m - (m-1)q}{(1-q)^2}, \end{aligned}$$

which allows us to estimate (II) and (III).

$$\begin{aligned} (II) &= 2 \sum_{k \geq 1} \pi k^2 e^{-\pi k^2 s} \\ &\leq 2\pi e^{-\pi s} + 2\pi \sum_{k \geq 4} k e^{-\pi k s} \\ &= 2\pi \left(e^{-\pi s} + e^{-4\pi s} \frac{(4 - 3e^{-\pi s})}{(1 - e^{\pi s})^2} \right) \\ (III) &= (II)^2 \leq \left(2\pi \left(e^{-\pi s} + e^{-4\pi s} \frac{(4 - 3e^{-\pi s})}{(1 - e^{\pi s})^2} \right) \right)^2 \end{aligned}$$

For $s \geq 1$ it is quickly checked that $\frac{(4-3e^{-\pi s})}{(1-e^{\pi s})^2} \leq \frac{(4-3e^{-\pi})}{(1-e^{\pi})^2} \leq 1$ which gives us the even rougher estimates

$$\begin{aligned} (II) &\leq 2\pi (e^{-\pi s} + e^{-4\pi s}) \\ (III) &\leq (2\pi (e^{-\pi s} + e^{-4\pi s}))^2 \leq (2\pi (2e^{-\pi s}))^2. \end{aligned}$$

Since θ_3 is strictly decreasing on \mathbb{R}_+ we estimate it in the following way

$$\theta_3(s) < \theta_3(1) < \frac{\pi}{2}$$

for all $s > 1$. The last estimate is very rough as the concrete value is $\theta_3(1) = \frac{\pi^{1/4}}{\Gamma(3/4)} \approx 1.08643$ [64] but it is helpful since it allows us to pull out the factor π^2 when we want to

show equation (5.12). Combining all these estimates we claim that

$$\begin{aligned} & s \left(2 \sum_{k \geq 1} \pi^2 k^4 e^{-\pi k^2 s} \right) - \left(2 \sum_{k \geq 1} \pi k^2 e^{-\pi k^2 s} \right) \left(1 + 2 \sum_{k \geq 1} e^{-\pi k^2 s} \right) - s \left(2 \sum_{k \geq 1} \pi k^2 e^{-\pi k^2 s} \right)^2 \\ & \geq 2s\pi^2 e^{-\pi s} - 2\pi (e^{-\pi s} + e^{-4\pi s}) \frac{\pi}{2} - s (2\pi (2e^{-\pi s}))^2 > 0. \end{aligned}$$

This means that we have to check that

$$\pi^2 e^{-\pi s} (2s - 1 - e^{-3\pi s} - 16s e^{-\pi s}) > 0$$

which follows from the following chain of inequalities. For $s \geq 1$ we have

$$s (2 - 16e^{-\pi s}) \geq \underbrace{2 - 16e^{-\pi}}_{\approx 1.30858} > \underbrace{1 + e^{-3\pi}}_{\approx 1.00008} \geq 1 + e^{-3\pi s}$$

We see that the desired Inequality (5.12) holds at least for $s \geq 1$. □

As a consequence we get the following result which is illustrated in Figure 7.

Theorem 5.10. *The function $s \frac{\theta'_3(s)}{\theta_3(s)}$ is strictly increasing on \mathbb{R}_+ .*

Proof. From Lemma 5.9 we know that $s \frac{\theta'_3(s)}{\theta_3(s)}$ is strictly increasing for $s \geq 1$. From Lemma 5.8 we know that the identity

$$s \frac{\theta'_3(s)}{\theta_3(s)} + \frac{1}{s} \frac{\theta'_3(1/s)}{\theta_3(1/s)} = -\frac{1}{2}.$$

holds for $s \in \mathbb{R}_+$. Consequently the function $\frac{1}{s} \frac{\theta'_3(1/s)}{\theta_3(1/s)}$ must be decreasing for $s \geq 1$. This implies that $s \frac{\theta'_3(s)}{\theta_3(s)}$ is strictly increasing for $s \in (0, 1]$. Therefore, the statement follows immediately. □

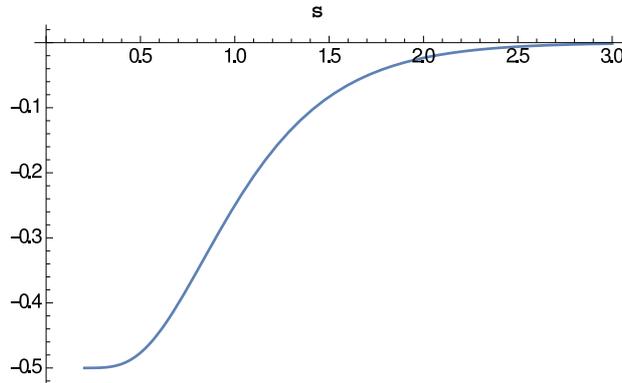


Figure 7: The function $s \theta'_3(s) / \theta_3(s)$. The figure indicates that Mathematica 10 has some troubles with evaluating the function near $s = 0$.

We saw that the proof of Theorem 5.10 is quite curious. This is because $\theta_3(s)$ and its derivatives are complicated to evaluate if s is close to the origin (less than 1), since many different terms start to contribute to the sum, but $\theta_3(s)$ and its derivatives are quite simple to handle for $s \geq 1$ because then the series are essentially characterized by one term only. The identity from Lemma 5.8 helped us to backpropagate the behavior to the origin. An approach using the product representation instead of the series representation leads to the same problems and arguments. We are not aware of a method avoiding the use of the identity from Lemma 5.8.

As a consequence we get the following result.

Theorem 5.11. *For all $r > 0$ fixed and $s \in \mathbb{R}_+$ we have*

$$\theta_3(rs)\theta_3\left(\frac{r}{s}\right) \geq \theta_3(r)^2$$

with equality only for $s = 1$.

Proof. The result is a combination of Theorem 5.10 and Lemma 5.6. From Theorem 5.10 we know that $s\theta_3'(s)/\theta_3(s)$ is strictly increasing and therefore, as a consequence of the algebraic structure, the result follows. \square

Theorem 5.11 in combination with equation (5.8) implies particularly that

$$B(1/\sqrt{n}) \leq n\theta_3\left(\frac{n}{2}s\right)\theta_3\left(\frac{n}{2s}\right).$$

Using again the substitution $\beta = \sqrt{s/n}$ and equation (5.6) we get

$$B(1/\sqrt{n}) \leq n\theta_3\left(\frac{1}{2\beta^2}\right)\theta_3\left(\frac{n^2\beta^2}{2}\right).$$

The fact that we fixed the product $(\alpha\beta)^{-1} = n \in 2\mathbb{N}$ finally gives

$$B_{g_0,n}\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \leq B_{g_0,n}(\alpha, \beta)$$

for even redundancy.

In Appendix A we give an alternative proof of Theorem 5.11 for the case that $r = 1$ which corresponds to the minimization of the upper frame bound for redundancy 2. We will now state more properties of Jacobi's theta-3 function and some of these properties appeared for the first time in [24]. But first we start with an elementary proof which implies the log-convexity of θ_3 .

Lemma 5.12. *Let $s \in \mathbb{R}_+$ and let $a_k, b_k \geq 0$ be sequences of positive real numbers such that $\sum_k a_k < \infty$. Then*

$$f(s) = \sum_{k=1}^{\infty} a_k e^{-b_k s} \quad \text{satisfies} \quad (\log(f(s)))'' \geq 0.$$

Proof. We prove the statement in the form $f''(s)f(s) - f'(s)^2 \geq 0$. Let

$$f(s) = \sum_{k=1}^{\infty} a_k e^{-b_k s}.$$

Then, by direct computation, our statement can be written as

$$f''(s)f(s) - f'(s)^2 = \left(\sum_{k=1}^{\infty} b_k^2 a_k e^{-b_k s} \right) \left(\sum_{k=1}^{\infty} a_k e^{-b_k s} \right) - \left(\sum_{k=1}^{\infty} b_k a_k e^{-b_k s} \right)^2 \geq 0.$$

Using the fact that

$$\sum_{k=1}^{\infty} b_k a_k e^{-b_k s} = \sum_{k=1}^{\infty} (b_k \sqrt{a_k} e^{-b_k s/2}) (\sqrt{a_k} e^{-b_k s/2}),$$

the proof follows by applying the Cauchy-Schwarz inequality. \square

As a direct application of Theorem 5.10 we have the following result (see also [24]).

Proposition 5.13 (Refined Logarithmic Convexity for θ_3). *For $s \in \mathbb{R}_+$ we have*

$$\theta_3''(s)\theta_3(s) - \theta_3'(s)^2 > -\frac{\theta_3'(s)\theta_3(s)}{s} > 0.$$

Proof. Theorem 5.10 states that $\frac{d}{ds} \left(s \frac{\theta_3'(s)}{\theta_3(s)} \right) > 0$ for $s \in \mathbb{R}_+$. By a direct calculation we get

$$0 < \frac{d}{ds} \left(s \frac{\theta_3'(s)}{\theta_3(s)} \right) = \frac{\theta_3'(s)}{\theta_3(s)} + s \frac{\theta_3''(s)\theta_3(s) - \theta_3'(s)^2}{\theta_3(s)^2}$$

which is equivalent to

$$\theta_3''(s)\theta_3(s) - \theta_3'(s)^2 > -\frac{\theta_3'(s)\theta_3(s)}{s}.$$

We finally show that the right-hand side of the last inequality is positive. Since,

$$\theta_3(s) = 1 + 2 \sum_{k \in \mathbb{N}} e^{-\pi k^2 s} > 1$$

and

$$\theta_3'(s) = -2\pi \sum_{k \in \mathbb{N}} k^2 e^{-\pi k^2 s} < 0,$$

we see that

$$\frac{\theta_3'(s)\theta_3(s)}{s} < 0$$

and the proof is complete. \square

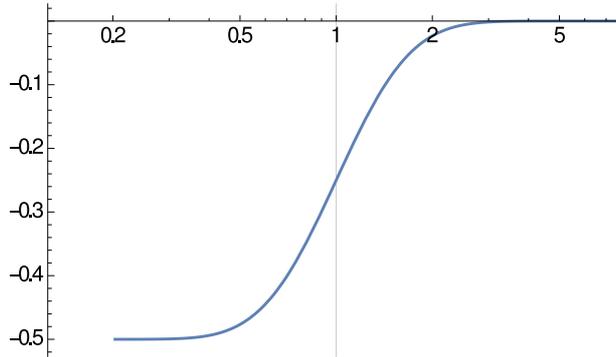


Figure 8: The function $s \frac{\theta_3'(s)}{\theta_3(s)}$ plotted on a logarithmic scale. On this scale, the (only) point of inflection is at $s = 1$.

The following proposition states that on a logarithmic scale the logarithmic derivative of Jacobi's theta-3 function has exactly one point of inflection and that it is antisymmetric with respect to this point. (see Figure 8). The proof is quite lengthy and we need careful estimates for Jacobi's theta-3 functions and its logarithmic derivatives.

Proposition 5.14. *For $s \in \mathbb{R}_+$, the function*

$$s \frac{d}{ds} \left(s \frac{d}{ds} \left(s \frac{d}{ds} \log(\theta_3(s)) \right) \right) \quad (5.13)$$

is positive for $s \in (0, 1)$ and negative for $s > 1$. Also, the function is antisymmetric in the following sense

$$s \frac{d}{ds} \left(s \frac{d}{ds} \left(s \frac{d}{ds} \log(\theta_3(s)) \right) \right) = -s \frac{d}{ds} \left(s \frac{d}{ds} \left(s \frac{d}{ds} \log \left(\theta_3 \left(\frac{1}{s} \right) \right) \right) \right).$$

Before we prove this proposition we have some remarks. The computer algebra system Mathematica [76] was used at some points in the proof in order to calculate explicit values or closed expressions for geometric series, but in principle all computations can also be checked by hand.

We also want to explain the occurrence of the differential operator $s \frac{d}{ds}$. We actually establish symmetry results for $\log(\theta_3(s))$ on a logarithmic scale, which means that we use the variable transformation $s \mapsto e^s$. Therefore, we get an extra exponential factor each time we take a derivative. By reversing the transformation of variables, we come back to an ordinary scale, but the factor s stays.

Proof of Proposition 5.14. To simplify notation we set

$$\psi(s) = (\log \circ \theta_3)'(s) = \frac{\theta_3'(s)}{\theta_3(s)}.$$

and start with the following calculation

$$s \frac{d}{ds} \left(s \frac{d}{ds} (s \psi(s)) \right) = s \psi(s) + 3s^2 \psi'(s) + s^3 \psi''(s). \quad (5.14)$$

From Lemma 5.8 we already know that

$$s \psi(s) + \frac{1}{s} \psi \left(\frac{1}{s} \right) = -\frac{1}{2}.$$

which means that $s \psi(s)$ is antisymmetric with respect to the point $(1, -\frac{1}{4})$ on a logarithmic scale. Differentiating the last identity on both sides with respect to s and a subsequent multiplication by s gives

$$s \psi(s) + s^2 \psi'(s) - \frac{1}{s} \psi \left(\frac{1}{s} \right) - \frac{1}{s^2} \psi' \left(\frac{1}{s} \right) = 0.$$

This means that the first derivative of $s\psi(s)$ is symmetric with respect to the line $s = 1$ on a logarithmic scale. By repeating the process once more, we find out that on a logarithmic scale the second derivative of $s\psi(s)$ is antisymmetric with respect to the point $(1, 0)$ which is equivalent to the statement

$$s \psi(s) + 3s^2 \psi'(s) + s^3 \psi''(s) + \frac{1}{s} \psi \left(\frac{1}{s} \right) + 3\frac{1}{s^2} \psi' \left(\frac{1}{s} \right) + \frac{1}{s^3} \psi'' \left(\frac{1}{s} \right) = 0.$$

In particular the last equality implies that

$$\psi(1) + 3\psi'(1) + \psi''(1) = 0.$$

By repeating this process, we can actually show that on a logarithmic scale all higher order derivatives of $\log(\theta_3(s))$ are either symmetric with respect to the line $s = 1$ or antisymmetric with respect to the point $(1, 0)$.

Since ψ is the logarithmic derivative of θ_3 , we use Jacobi's triple product formula for θ_3 , given by equation (5.4), to obtain a series representation for ψ . In order to control the expression in equation (5.14) we compute the derivatives of ψ up to order 2.

$$\begin{aligned} \psi(s) &= \sum_{k \geq 1} \left(\frac{2k\pi e^{-2k\pi s}}{1 - e^{-2k\pi s}} - 2 \frac{(2k-1)\pi e^{-(2k-1)\pi s}}{1 + e^{-(2k-1)\pi s}} \right) \\ \psi'(s) &= \sum_{k \geq 1} \left(-\frac{(2k\pi)^2 e^{-2k\pi s}}{(1 - e^{-2k\pi s})^2} + 2 \frac{((2k-1)\pi)^2 e^{-(2k-1)\pi s}}{(1 + e^{-(2k-1)\pi s})^2} \right) \\ \psi''(s) &= \sum_{k \geq 1} \left(\frac{(2k\pi)^3 e^{-2k\pi s} (1 + e^{-2k\pi s})}{(1 - e^{-2k\pi s})^3} \right. \\ &\quad \left. - 2 \frac{((2k-1)\pi)^3 e^{-(2k-1)\pi s} (1 - e^{-(2k-1)\pi s})}{(1 + e^{-(2k-1)\pi s})^3} \right) \end{aligned}$$

We start with an estimation for ψ from above. For $s > 0$, we have

$$\psi(s) = \frac{\theta'_3(s)}{\theta_3(s)} < 0$$

since $\theta'_3(s) < 0$ and $\theta_3(s) > 0$. This bound was not hard to establish. In what follows we will estimate parts of the series by the leading term, by using geometric series and by their values at $s = 1$. We proceed with an upper bound for ψ' for $s > 1$.

$$\begin{aligned} \psi'(s) &= \sum_{k \geq 1} \left(-\frac{(2k\pi)^2 e^{-2k\pi s}}{(1 - e^{-2k\pi s})^2} + 2 \frac{((2k-1)\pi)^2 e^{-(2k-1)\pi s}}{(1 + e^{-(2k-1)\pi s})^2} \right) \\ &< \sum_{k \geq 1} \left(2 \frac{((2k-1)\pi)^2 e^{-(2k-1)\pi s}}{(1 + e^{-(2k-1)\pi s})^2} \right) \\ &< \sum_{k \geq 1} (2\pi^2 k^2 e^{-k\pi s}) \\ &= 2\pi^2 e^{-\pi s} \underbrace{(e^{-\pi s} + 1)}_{< 1.05} \underbrace{(1 - e^{-\pi s})^{-3}}_{< 1.15} \\ &< 24 e^{-\pi s}. \end{aligned}$$

With the same techniques we bound ψ'' from above for $s > 1$.

$$\begin{aligned} \psi''(s) &= \sum_{k \geq 1} \left(\frac{(2k\pi)^3 e^{-2k\pi s} (1 + e^{-2k\pi s})}{(1 - e^{-2k\pi s})^3} \right. \\ &\quad \left. - 2 \frac{((2k-1)\pi)^3 e^{-(2k-1)\pi s} (1 - e^{-(2k-1)\pi s})}{(1 + e^{-(2k-1)\pi s})^3} \right) \\ &< \sum_{k \geq 1} \left((2k\pi)^3 e^{-2k\pi s} \underbrace{(1 + e^{-2\pi})}_{< 1.002} \underbrace{(1 - e^{-2\pi})^{-3}}_{< 1.006} \right) - 2\pi^3 e^{-\pi s} \\ &< 1.01 \pi^3 e^{-\pi s} \underbrace{(e^{-5\pi s} + 4e^{-3\pi s} + e^{-\pi s})}_{< 0.045} \underbrace{(1 - e^{-2\pi s})^{-4}}_{< 1.008} - 2\pi^3 e^{-\pi s} \\ &< 0.05 \pi^3 e^{-\pi s} - 2\pi^3 e^{-\pi s} \\ &< -60 e^{-\pi s}. \end{aligned}$$

Therefore we have for $s > 1$

$$\begin{aligned} s \frac{d}{ds} \left(s \frac{d}{ds} (s \psi(s)) \right) &= s \psi(s) + 3s^2 \psi'(s) + s^3 \psi''(s) \\ &< 72 s^2 e^{-\pi s} - 60 s^3 e^{-\pi s}. \end{aligned}$$

It is quickly verified that $72s^2e^{-\pi s} - 60s^3e^{-\pi s} < 0$ for $s > \frac{6}{5}$. In particular this shows that

$$s\psi(s) + 3s^2\psi'(s) + s^3\psi''(s) < 0 \quad \text{for } s > \frac{6}{5}.$$

As a next step we compute that

$$s\frac{d}{ds}(s\psi(s) + 3s^2\psi'(s) + s^3\psi''(s)) = s\psi(s) + 7s^2\psi'(s) + 6s^3\psi''(s) + s^4\psi'''(s).$$

The last expression is again symmetric on a logarithmic scale with respect to $s = 1$. We continue with an estimation on ψ''' for $s > 1$.

$$\begin{aligned} \psi'''(s) &= \sum_{k \geq 1} \left(-\frac{(2k\pi)^4 e^{-\pi ks} (e^{-5\pi ks} + 4e^{-3\pi ks} + e^{-\pi ks})}{(1 - e^{-2\pi ks})^4} \right. \\ &\quad \left. + \frac{2((2k-1)\pi)^4 e^{-(2k-1)\pi s} (e^{-2(2k-1)\pi s} - 4e^{-(2k-1)\pi s} + 1)}{(1 + e^{-(2k-1)\pi s})^4} \right) \\ &< -16\pi^4 e^{-\pi s} \underbrace{(e^{-5\pi} + 4e^{-3\pi} + e^{-\pi})}_{>0.04} \\ &\quad + 2 \sum_{k \geq 1} ((2k-1)\pi)^4 e^{-(2k-1)\pi s} \underbrace{(1 + e^{-(2k-1)\pi s})^{-2}}_{<1} \\ &< -0.64\pi^4 e^{-\pi s} + 2\pi^4 e^{-\pi s} + 2 \sum_{k \geq 3} (k\pi)^4 e^{-k\pi s} \\ &= -0.64\pi^4 e^{-\pi s} + 2\pi^4 e^{-\pi s} \underbrace{(1 - e^{-\pi s})^{-5}}_{<1.25} \\ &\quad \times \left(1 + \underbrace{(16e^{-7\pi s} - 79e^{-6\pi s} + 155e^{-5\pi s} - 149e^{-4\pi s} + 81e^{-3\pi s})}_{<0.0065} \right) \\ &< (2 \cdot 1.259 - 0.64)\pi^4 e^{-\pi s} \\ &< 183e^{-\pi s} \end{aligned}$$

It follows that

$$s\psi(s) + 7s^2\psi'(s) + 6s^3\psi''(s) + \psi'''(s) < (168s^2 - 360s^3 + 183s^4) e^{-\pi s}.$$

It is quickly verified that $(168s^2 - 360s^3 + 183s^4) e^{-\pi s} < 0$ for $s \in \left(1, \frac{60+2\sqrt{46}}{61}\right)$. We note that $\frac{60+2\sqrt{46}}{61} > 1.205$. Therefore, the function $s\psi(s) + 3s^2\psi'(s) + s^3\psi''(s)$ is strictly decreasing at least on the interval $(1, 1.205)$. Since we already proved that the function is negative for $s > 1.2$ and that the value at $s = 1$ is zero, we can finally conclude that the expression given in equation(5.13) is negative for $s > 1$. Due to the already mentioned antisymmetry with respect to the point $(1, 0)$ the function has to be positive for $0 < s < 1$. \square

It seems as if we were quite lucky regarding the last proof. This impression is due to the fact that our estimates were quite rough, but at the same time fine enough for the proof to work. Numerical inspections beforehand suggested that

$$s\psi(s) + 3s^2\psi'(s) + s^3\psi''(s) < 0 \quad \text{for } s > 1.05.$$

and that

$$s\psi(s) + 7s^2\psi'(s) + 6s^3\psi''(s) + \psi'''(s) < 0 \quad \text{for } 1 < s < 1.5.$$

We adjusted the proof to a level which should be quite accessible with no or only little help of a computer algebra software. We remark that the proof does not work if $\psi'''(s)$ is estimated by $184e^{-\pi s}$ instead of $183e^{-\pi s}$.

5.1.2 Properties of Theta-4 and a Statement for the Lower Frame Bound

This section is devoted to prove equation (5.1)

$$A_{g_0,n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \geq A_{g_0,n}(\alpha, \beta)$$

of Theorem 5.2 for even redundancy.

We will proceed in the same manner as for the upper frame bound, meaning that we will use the algebraic lemma (Lemma 5.6) to establish that $s \frac{\theta'_4(s)}{\theta_4(s)}$ is strictly decreasing (see Figure 9) for $s \in \mathbb{R}_+$. Since θ_4 has alternating signs, it is difficult to handle and the proof contains a certain ‘magic’ element of algebraic simplification. Interestingly, this argument does not work for θ_3 even though there exists an analogous representation of θ_3 as an infinite product, as already mentioned.

Theorem 5.15. *The function $s \frac{\theta'_4(s)}{\theta_4(s)}$ is strictly decreasing on \mathbb{R}_+ .*

Proof. We use the Jacobi triple product representation from Proposition 5.5 to rewrite θ_4 as

$$\theta_4(s) = \prod_{k \geq 1} (1 - e^{-2k\pi s}) (1 - e^{-(2k-1)\pi s})^2 = \prod_{k \geq 1} \theta_{4,k}(s).$$

Using the product rule we will show that for every $k \in \mathbb{N}$ and $s > 0$

$$\frac{d}{ds} \left(s \frac{\theta'_{4,k}(s)}{\theta_{4,k}(s)} \right) < 0$$

which then immediately implies

$$\frac{d}{ds} \left(s \frac{\theta'_4(s)}{\theta_4(s)} \right) = \frac{d}{ds} \left(s \frac{(\prod_{k \geq 1} \theta_{4,k}(s))'}{\prod_{k \geq 1} \theta_{4,k}(s)} \right) = \frac{d}{ds} \left(s \sum_{k \geq 1} \frac{\theta'_{4,k}(s)}{\theta_{4,k}(s)} \right) = \sum_{k \geq 1} \frac{d}{ds} \left(s \frac{\theta'_{4,k}(s)}{\theta_{4,k}(s)} \right) < 0.$$

A simple calculation yields

$$\begin{aligned} s \frac{\theta'_{4,k}(s)}{\theta_{4,k}(s)} &= s \left(\frac{2k\pi e^{-2k\pi s}}{1 - e^{-2k\pi s}} + 2 \frac{(2k-1)\pi e^{-(2k-1)\pi s}}{1 - e^{-(2k-1)\pi s}} \right) \\ &= \frac{2k\pi s}{e^{2k\pi s} - 1} + 2 \frac{(2k-1)\pi s}{e^{(2k-1)\pi s} - 1}. \end{aligned}$$

Both terms in the last sum are of the form $ms/(e^{ms} - 1)$ for some $m \in \mathbb{R}_+$. Note that

$$\frac{d}{ds} \left(\frac{ms}{e^{ms} - 1} \right) = m \frac{e^{ms} - (1 + ms e^{ms})}{(e^{ms} - 1)^2} < 0,$$

which is quickly checked using the elementary inequality

$$e^y < 1 + y e^y \quad \Leftrightarrow \quad 1 - y < e^{-y} \quad \forall y > 0.$$

Therefore, the statement follows since all terms involved are negative. \square

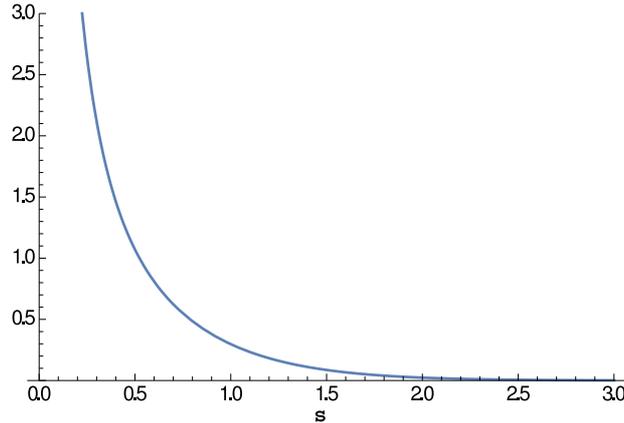


Figure 9: The function $s\theta'_4(s)/\theta_4(s)$.

In comparison to the proof of Theorem 5.10 the proof of Theorem 5.15 was rather straight forward. At this point we want to mention that the use of the product representation is crucial for the proof. We were not able to establish the result with the series representation of θ_4 .

As a consequence we get the following result.

Theorem 5.16. *For all $r > 0$ fixed and $s \in \mathbb{R}_+$ we have*

$$\theta_4(rs)\theta_4\left(\frac{r}{s}\right) \leq \theta_4(r)^2$$

with equality only for $s = 1$.

Proof. The result is a combination from Theorem 5.15 and Lemma 5.6. From Theorem 5.15 we know that $s\theta'_4(s)/\theta_4(s)$ is strictly decreasing and therefore, as a consequence of the algebraic structure, the result follows. \square

Theorem 5.16 in combination with equation (5.7) implies particularly that

$$A(1/\sqrt{n}) \geq n\theta_4\left(\frac{n}{2}s\right)\theta_4\left(\frac{n}{2}\frac{1}{s}\right).$$

Using again the substitution $\beta = \sqrt{s/n}$ and equation (5.5) we get

$$A(1/\sqrt{n}) \geq n\theta_4\left(\frac{1}{2\beta^2}\right)\theta_4\left(\frac{n^2\beta^2}{2}\right).$$

The fact that we fixed the product $(\alpha\beta)^{-1} = n \in 2\mathbb{N}$ finally gives

$$A_{g_0,n}\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\right) \geq A_{g_0,n}(\alpha, \beta)$$

for even redundancy.

In Appendix A we find an alternative proof of Theorem 5.16 for the case that $r = 1$ using the series representation. However, this proof is not suitable for generalization to arbitrary $r > 0$. We will now state some more properties of Jacobi's theta-4 function. As a consequence of Theorem 5.15 we get the following statement (see also [24]).

Proposition 5.17 (Refined Logarithmic Concavity for θ_4). *We have, for $s > 0$,*

$$\theta_4''(s)\theta_4(s) - \theta_4'(s)^2 < -\frac{\theta_4'(s)\theta_4(s)}{s} < 0.$$

Proof. Due to Theorem 5.15 we know that the function $s\frac{\theta_4'(s)}{\theta_4(s)}$ is strictly decreasing. This means

$$0 > \frac{d}{ds} \left(s \frac{\theta_4'(s)}{\theta_4(s)} \right) = \frac{\theta_4'(s)}{\theta_4(s)} + s \frac{\theta_4''(s)\theta_4(s) - \theta_4'(s)^2}{\theta_4(s)^2}$$

which is equivalent to

$$\theta_4''(s)\theta_4(s) - \theta_4'(s)^2 < -\frac{\theta_4'(s)\theta_4(s)}{s}.$$

Next we show that the right-hand side of the last inequality is negative. Since

$$\theta_4(s) = 1 + 2 \sum_{k \in \mathbb{N}} (-1)^k e^{-\pi k^2 s}$$

we have

$$\theta_4'(s) = -2\pi \sum_{k \in \mathbb{N}} (-1)^k k^2 e^{-\pi k^2 s}.$$

We see that $\lim_{s \rightarrow \infty} \theta_4(s) = 1$ and that $\lim_{s \rightarrow \infty} s \theta_4'(s) = 0$ therefore

$$\lim_{s \rightarrow \infty} s \frac{\theta_4'(s)}{\theta_4(s)} = 0.$$

Theorem 5.15 states that $s \frac{\theta_4'(s)}{\theta_4(s)}$ is strictly decreasing, which implies that

$$s \frac{\theta_4'(s)}{\theta_4(s)} > 0 \quad \text{for all } s > 0.$$

Thus, dividing by s^2 and multiplying by $\theta_4(s)^2$ gives

$$\frac{\theta_4'(s)\theta_4(s)}{s} > 0$$

which completes Theorem 5.17. □

Next, we will prove an identity involving θ_2 and θ_4 which is analogous to the identity of θ_3 stated in Lemma 5.8.

Lemma 5.18. *For $s > 0$, we have*

$$s \frac{\theta_4'(s)}{\theta_4(s)} + \frac{1}{s} \frac{\theta_2'(\frac{1}{s})}{\theta_2(\frac{1}{s})} = -\frac{1}{2}$$

and

$$s \frac{\theta_2'(s)}{\theta_2(s)} + \frac{1}{s} \frac{\theta_4'(\frac{1}{s})}{\theta_4(\frac{1}{s})} = -\frac{1}{2}$$

Proof. We start with the identity

$$\sqrt{s} \theta_4(s) = \theta_2\left(\frac{1}{s}\right)$$

which follows from the Poisson summation formula and is actually a classical result for Jacobi's theta functions. By differentiating this identity on both sides we get

$$\frac{1}{2s^{1/2}} \theta_4(s) + s^{1/2} \theta_4'(s) = -\frac{1}{s^2} \theta_2'\left(\frac{1}{s}\right).$$

Multiplying both sides with $\frac{s^{1/2}}{\theta_4(s)}$ and using the identity $\sqrt{s} \theta_4(s) = \theta_2(\frac{1}{s})$ once more gives

$$\frac{1}{2} + s \frac{\theta_4'(s)}{\theta_4(s)} = -\frac{1}{s} \frac{\theta_2'(\frac{1}{s})}{s^{1/2} \theta_4(s)} = -\frac{1}{s} \frac{\theta_2'(\frac{1}{s})}{\theta_2(\frac{1}{s})}$$

and the proof is complete. □

Corollary 5.19. *The function $s \frac{\theta'_2(s)}{\theta_2(s)}$ is strictly decreasing on \mathbb{R}_+ .*

Proof. Theorem 5.15 tells us that $s \frac{\theta'_4(s)}{\theta_4(s)}$ is strictly decreasing on \mathbb{R}_+ . Using Lemma 5.18 we get

$$s \frac{\theta'_2(s)}{\theta_2(s)} = -\frac{1}{s} \frac{\theta'_4\left(\frac{1}{s}\right)}{\theta_4\left(\frac{1}{s}\right)} - \frac{1}{2}$$

and the statement follows. \square

In [24] the statement in Corollary 5.19 was proved for $s > \frac{1}{4}$. We note that the identities in Lemma 5.18 hold for arbitrary differentiable functions f and g (not necessarily different from each other) which fulfil the identity $\sqrt{s}f(s) = g\left(\frac{1}{s}\right)$.

5.2 Odd Redundancy

In the spirit of Theorem 5.2 we would like to prove

$$A_{g_0,n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \geq A_{g_0,n}(\alpha, \beta) \quad (5.15)$$

$$B_{g_0,n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \leq B_{g_0,n}(\alpha, \beta) \quad (5.16)$$

for the case $n \in 2\mathbb{N} + 1$. The proof of (5.16) will easily follow from the already achieved results. Regrettably, the proof of (5.15) turns out to be very hard to establish and we will not see a full proof of for this case.

First of all, we recall the Fourier series from equation (5.3)

$$F_{g_0}(x, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{kl} n e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{2\pi i k x} e^{2\pi i l \omega} \quad (5.17)$$

and that up to the factor n the frame bounds are given by the minimum and the maximum of these series. The difference to the case of even redundancy is the alternating sign. We note that the sign is only negative when both indices, k and l , are odd and that n does not change the sign, because it is odd, and therefore we can ignore it.

We use the absolute convergence of the above series to rewrite in different ways. First, we observe that

$$\begin{aligned} F_{g_0}(x, \omega; \alpha, \beta) &= 2 \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{(2k)^2}{\beta^2} + \frac{(2l)^2}{\alpha^2} \right)} e^{2\pi i (2k)x} e^{2\pi i (2l)\omega} \\ &\quad - \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{k+l} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{2\pi i k x} e^{2\pi i l \omega}. \end{aligned}$$

By rewriting the alternating sign as a complex exponential the series takes the following form

$$F_{g_0}(x, \omega; \alpha, \beta) = 2 \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{(2k)^2}{\beta^2} + \frac{(2l)^2}{\alpha^2} \right)} e^{2\pi i(2k)x} e^{2\pi i(2l)\omega} \\ - \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{2\pi i k(x + \frac{1}{2})} e^{2\pi i l(\omega + \frac{1}{2})}.$$

It is easy to see that the first double sum is maximal whenever $(x, \omega) \in \mathbb{Z} \times \mathbb{Z}$ and from the product representation in equation (5.4) we already know that the second double sum is minimal whenever $(x, \omega) \in \mathbb{Z} \times \mathbb{Z}$. It follows that the upper frame bound is therefore given by

$$B_{g_0}(\alpha, \beta) = n F_{g_0}(0, 0; \alpha, \beta).$$

Now rewrite F_{g_0} in a another way.

$$F_{g_0}(x, \omega; \alpha, \beta) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{2\pi i k x} e^{2\pi i l \omega} \\ - 2 \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{(2k+1)^2}{\beta^2} + \frac{(2l+1)^2}{\alpha^2} \right)} e^{2\pi i(2k+1)x} e^{2\pi i(2l+1)\omega}$$

We already know from the product representation in equation (5.4) that the first double sum is minimal whenever $(x + \frac{1}{2}, \omega + \frac{1}{2}) \in \mathbb{Z} \times \mathbb{Z}$ and it is easy to see that the second double sum is maximal whenever $(x, \omega) \in \mathbb{Z} \times \mathbb{Z}$, because then the complex exponentials take the value 1, or if $(x + \frac{1}{2}, \omega + \frac{1}{2}) \in \mathbb{Z} \times \mathbb{Z}$, because then both complex exponentials take the value -1 and hence their product is 1. Therefore the Fourier series in equation (5.17) takes its minimum whenever $(x + \frac{1}{2}, \omega + \frac{1}{2}) \in \mathbb{Z} \times \mathbb{Z}$. This means that the lower frame bound is now given by

$$A_{g_0}(\alpha, \beta) = n F_{g_0}\left(\frac{1}{2}, \frac{1}{2}; \alpha, \beta\right).$$

To simplify notation we introduce the function

$$\theta_o(s) = \sum_{k \in \mathbb{Z}} e^{-\pi(2k+1)^2 s},$$

which can be understood as θ_3 with summation restricted to the odd integers. It is closely related to Jacobi's theta-2 function which is defined as

$$\theta_2(s) = \sum_{k \in \mathbb{Z}} e^{-\pi(k + \frac{1}{2})^2 s}.$$

Now we can write the lower and upper frame bound as

$$A(\beta) = n \left(\theta_4\left(\frac{1}{2\beta^2}\right) \theta_4\left(\frac{n^2\beta^2}{2}\right) - 2\theta_o\left(\frac{1}{2\beta^2}\right) \theta_o\left(\frac{n^2\beta^2}{2}\right) \right) \quad (5.18)$$

$$B(\beta) = n \left(\theta_3\left(\frac{1}{2\beta^2}\right) \theta_3\left(\frac{n^2\beta^2}{2}\right) - 2\theta_o\left(\frac{1}{2\beta^2}\right) \theta_o\left(\frac{n^2\beta^2}{2}\right) \right). \quad (5.19)$$

The following result will already imply that the upper frame bound is minimized for $\alpha = \beta = \frac{1}{\sqrt{n}}$.

Proposition 5.20. *Let $r > 0$ be fixed, $s \in \mathbb{R}_+$. Then we have that*

$$\theta_o(r)^2 \geq \theta_o(rs)\theta_o\left(\frac{r}{s}\right)$$

with equality only for $s = 1$.

Proof. Using the Poisson summation formula the result follows from Theorem 5.16. We have

$$\begin{aligned} \theta_o(rs) &= \sum_{k \in \mathbb{Z}} e^{-\pi(2k-1)^2 rs} = \sum_{k \in \mathbb{Z}} e^{-4\pi(k-1/2)^2 rs} \\ &= \frac{1}{2\sqrt{rs}} \sum_{l \in \mathbb{Z}} e^{-\pi il} e^{-\frac{\pi l^2}{4rs}} = \frac{1}{2\sqrt{rs}} \theta_4\left(\frac{1}{4rs}\right). \end{aligned}$$

In the same manner we derive

$$\theta_o(r/s) = \frac{\sqrt{s}}{2\sqrt{r}} \theta_4\left(\frac{s}{4r}\right).$$

Therefore, we have

$$\theta_o(rs)\theta_o(r/s) = \frac{1}{4r} \theta_4\left(\frac{1}{4rs}\right) \theta_4\left(\frac{s}{4r}\right)$$

and the statement follows from the results about the θ_4 -function. \square

We remark that this result can also be derived by a proper rescaling of θ_2 . It is quickly checked that

$$\theta_4(s) = \sum_{k \in \mathbb{Z}} e^{-\pi k^2 s} e^{\pi ik} = \frac{1}{\sqrt{s}} \sum_{l \in \mathbb{Z}} e^{-\pi(l-\frac{1}{2})^2 \frac{1}{s}} = \frac{1}{\sqrt{s}} \theta_2\left(\frac{1}{s}\right)$$

by using the Poisson summation formula. We have the following result.

Corollary 5.21. *For $r > 0$ fixed, $s \in \mathbb{R}_+$ we have*

$$\theta_3(rs)\theta_3(r/s) - 2\theta_o(rs)\theta_o(r/s) > 0$$

and the function is minimal if and only if $s = 1$.

Proof. From Theorem 5.11 we know that $\theta_3(rs)\theta_3(r/s)$ assumes its global minimum only for $s = 1$. Proposition 5.20 tells us that $\theta_o(rs)\theta_o(r/s)$ takes its global maximum only for $s = 1$. Therefore the function assumes its minimum only for $s = 1$. It is quickly checked that

$$\theta_3(1)^2 > 2\theta_o(1)^2.$$

First, we observe that $\theta_3(1)^2 > 1$. Next, we see that

$$\theta_o(1) = \sum_{k \in \mathbb{Z}} e^{-\pi(2k+1)^2} < 2 \sum_{k \geq 1} e^{-\pi k} = 2 \frac{1}{e^\pi - 1} < 1.$$

which follows easily by using the identity for the geometric series. Therefore the function $\theta_3(rs)\theta_3(r/s) - 2\theta_o(rs)\theta_o(r/s)$ is positive. \square

By using the substitution $\beta = \sqrt{s/n}$ again, we find that for odd redundancy equation (5.2)

$$B_{g_0, n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \leq B_{g_0, n}(\alpha, \beta)$$

holds for all $(\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+$ with $(\alpha\beta)^{-1} = n \in 2\mathbb{N} + 1$.

We will now turn to an analogue statement of Corollary 5.21 involving Jacobi's theta-4 function. With the following identity, we can rewrite the problem in many different ways.

Lemma 5.22. *Let $r_1, r_2, s \in \mathbb{R}^+$. Then we have the following identity.*

$$\begin{aligned} & \theta_3 \left(\frac{r_1 s}{2} \right) \theta_3 \left(\frac{r_2}{2s} \right) + \theta_4 \left(\frac{r_1 s}{2} \right) \theta_4 \left(\frac{r_2}{2s} \right) \\ &= 2 \left(\theta_3(2r_1 s) \theta_3 \left(\frac{2r_2}{s} \right) + \theta_2(2r_1 s) \theta_2 \left(\frac{2r_2}{s} \right) \right) \end{aligned}$$

Proof. The proof only needs the unconditional convergence of the series of the Jacobi theta function so the rearrangements of the terms are justified.

$$\begin{aligned} & \theta_3 \left(\frac{r_1 s}{2} \right) \theta_3 \left(\frac{r_2}{2s} \right) + \theta_4 \left(\frac{r_1 s}{2} \right) \theta_4 \left(\frac{r_2}{2s} \right) \\ &= \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(r_1 k^2 s + \frac{r_2 l^2}{s} \right)} + \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{k+l} e^{-\frac{\pi}{2} \left(r_1 k^2 s + \frac{r_2 l^2}{s} \right)} \\ &= 2 \left(\sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(r_1 (2k)^2 s + \frac{r_2 (2l)^2}{s} \right)} + \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(r_1 (2k+1)^2 s + \frac{r_2 (2l+1)^2}{s} \right)} \right) \\ &= 2 \left(\theta_3(2r_1 s) \theta_3 \left(\frac{2r_2}{s} \right) + \theta_2(2r_1 s) \theta_2 \left(\frac{2r_2}{s} \right) \right). \end{aligned}$$

\square

Similar identities can be found in [16, Chap. 4, p. 104] and the identity in Lemma 5.22 could as well be established from other known identities involving Jacobi's theta-2, theta-3 and theta-4 functions.

Proposition 5.23. *Let $r \geq \frac{1}{2}$, $s \in \mathbb{R}_+$. Then*

$$\begin{aligned}
& \theta_4(rs)\theta_4\left(\frac{r}{s}\right) - 2\theta_o(rs)\theta_o\left(\frac{r}{s}\right) \\
&= \theta_4(rs)\theta_4\left(\frac{r}{s}\right) - \frac{1}{2r}\theta_4\left(\frac{s}{4r}\right)\theta_4\left(\frac{1}{4rs}\right) \\
&= 2\theta_3(4rs)\theta_3\left(\frac{4r}{s}\right) - \theta_3(rs)\theta_3\left(\frac{r}{s}\right) \\
&> 0.
\end{aligned}$$

Proof. The identities follow from Lemma 5.22 and the Poisson summation formula.

For the part concerning the positivity of the expressions we show a stronger argument, namely the positivity of

$$f_r(s) = \theta_4(rs)\theta_4\left(\frac{r}{s}\right) - \theta_4\left(\frac{s}{4r}\right)\theta_4\left(\frac{1}{4rs}\right).$$

It follows from Proposition 5.17 that θ_4 is strictly increasing. Therefore, we have for $r > \frac{1}{2}$ that

$$\theta_4(rs) > \theta_4\left(\frac{s}{4r}\right)$$

and

$$\theta_4\left(\frac{r}{s}\right) > \theta_4\left(\frac{1}{4rs}\right).$$

In particular this implies that

$$\theta_4(rs)\theta_4\left(\frac{r}{s}\right) > \frac{1}{2r}\theta_4\left(\frac{s}{4r}\right)\theta_4\left(\frac{1}{4rs}\right), \quad \text{for } r > \frac{1}{2}$$

and the positivity statement is proved. □

We could add more identities by using Poisson summation on single terms. We use the first notation since it seems natural to write the lower frame bound in this way as we derive a formula for the upper bound by substituting theta-4 by theta-3 as can be seen from equation (5.18) and equation (5.19). The second notation was used to proof the proposition. The third notation was introduced because it seems curious that it enjoys the same properties as the first or the second equation.

From numerical inspections it seems true that for $r > \frac{1}{2}$ the the expressions in Proposition 5.23 are maximal if and only if $s = 1$. We will give some arguments that

$$\theta_4(rs)\theta_4\left(\frac{r}{s}\right) - \frac{1}{2r}\theta_4\left(\frac{s}{4r}\right)\theta_4\left(\frac{1}{4rs}\right)$$

should be maximal if and only if $s = 1$. In [24] it is argued that the approximation

$$\begin{aligned} \theta_4(rs)\theta_4\left(\frac{r}{s}\right) - 2\theta_o(rs)\theta_o\left(\frac{r}{s}\right) &\sim (1 - 2e^{-\pi rs})(1 - 2e^{-\pi\frac{r}{s}}) - 2(2e^{-\pi rs}2e^{-\pi\frac{r}{s}}) \\ &= 1 - 2e^{-\pi rs} - 2e^{-\pi\frac{r}{s}} - 4e^{-\pi r(s+\frac{1}{s})} \end{aligned}$$

is very accurate (see Figure 10). The second derivative of the approximation is given by

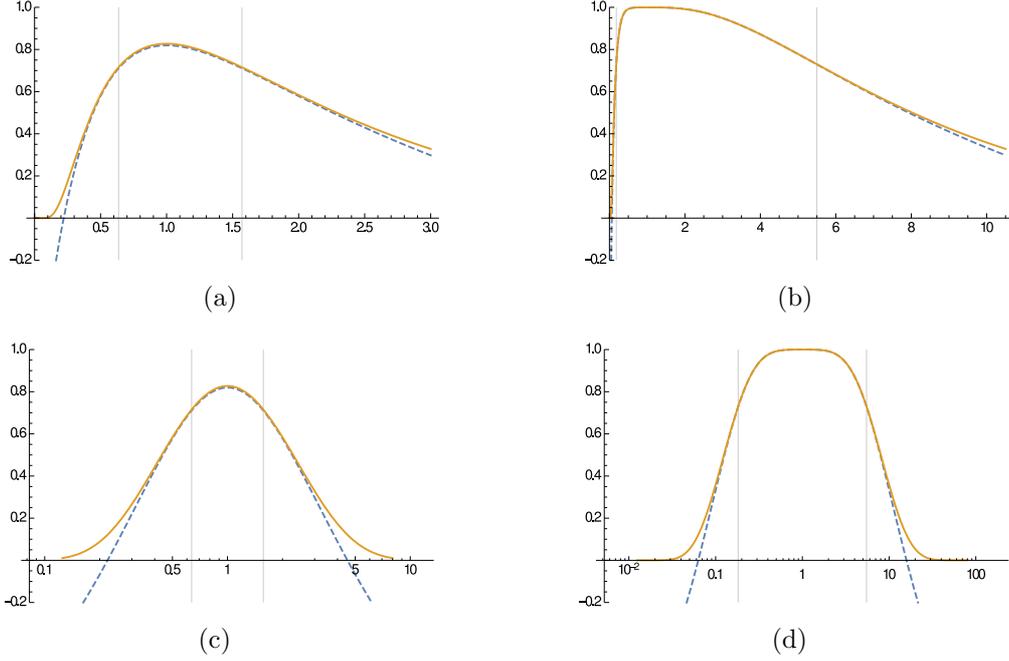


Figure 10: The function $\theta_4(rs)\theta_4\left(\frac{r}{s}\right) - 2\theta_o(rs)\theta_o\left(\frac{r}{s}\right)$ (solid graph) and its approximation for (dashed graph) for different parameters r on a normal scale ((a) and (b)) and on a logarithmic scale ((c) and (d)). The parameters are $r = 1$ ((a) and (c)) and $r = \frac{7}{2}$ ((b) and (d)). The borders of the interval $\left(\frac{2}{\pi r}, \frac{\pi r}{2}\right)$ are marked by the vertical lines.

$$s^{-4}e^{-\pi r(s+\frac{1}{s})} \left((4\pi rs - 2\pi^2 r^2) e^{\pi rs} + 8\pi rs - 4\pi^2 r^2 - 4\pi^2 r^2 s^4 + 8\pi^2 r^2 s^2 - 2\pi^2 r^2 s^4 e^{\frac{\pi r}{s}} \right)$$

For $s > 0$, we will now show that this expression is negative at least for $s < \frac{\pi r}{2}$. We see that

$$4\pi rs - 2\pi^2 r^2 < 0, \quad \text{for } s < \frac{\pi r}{2}.$$

Also

$$8\pi rs - 4\pi^2 r^2 < 0 \quad \text{for } s < \frac{\pi r}{2}.$$

Since

$$-4\pi^2 r^2 s^4 + 8\pi^2 r^2 s^2 < 0 \quad \text{for } s^2 > 2$$

we have to check that

$$-4\pi^2 r^2 s^4 + 8\pi^2 r^2 s^2 - 2\pi^2 r^2 s^4 e^{\frac{\pi r}{s}} < 0 \quad \text{for } s > 1.$$

The statement is certainly true for $s = 1$ as we have $-4\pi^2 r^2 + 8\pi^2 r^2 < 40r^2$ and $19 \cdot 23r^2 < 2\pi^2 r^2 s^4 e^{\pi r}$. Assuming $r > 1$ this gives

$$-4\pi^2 r^2 + 8\pi^2 r^2 < 2\pi^2 r^2 s^4 e^{\pi r}.$$

For $s > 1$, it is quickly checked that $-4\pi^2 r^2 s^4 + 8\pi^2 r^2 s^2$ is strictly decreasing and that $2\pi^2 r^2 s^4 e^{\frac{\pi r}{s}}$ is strictly increasing.

This implies that the one-term approximation

$$1 - 2e^{-\pi r s} - 2e^{-\pi \frac{r}{s}} - 4e^{-\pi r \left(s + \frac{1}{s}\right)}$$

has a local maximum at $s = 1$ for $s \in \left[1, \frac{\pi r}{2}\right)$. By a symmetry argument we conclude that for $s \in \left(\frac{2}{\pi r}, \frac{\pi r}{2}\right)$ we have a local maximum for $s = 1$. For $s \notin \left(\frac{2}{\pi r}, \frac{\pi r}{2}\right)$ and $r \geq 1$ the function cannot assume a global maximum since for $s \geq \frac{\pi r}{2}$

$$\begin{aligned} 1 - 2e^{-\pi r s} - 2e^{-\pi \frac{r}{s}} - 4e^{-\pi r \left(s + \frac{1}{s}\right)} &\leq \underbrace{1 - 2e^{-2}}_{<0.72} < \underbrace{1 - 4e^{-\pi} - 4e^{-2\pi}}_{>0.8} \\ &\leq 1 - 2e^{-\pi r s} - 2e^{-\pi \frac{r}{s}} - 4e^{-\pi r \left(s + \frac{1}{s}\right)} \Big|_{s=1}. \end{aligned}$$

The same argumentation works for $s \leq \frac{2}{\pi r}$ and therefore the unique global maximum is attained at $s = 1$. Although the approximation is very accurate, it only suggests that the expression

$$\theta_4(rs)\theta_4\left(\frac{r}{s}\right) - 2\theta_o(rs)\theta_o\left(\frac{r}{s}\right)$$

is maximal if and only if $s = 1$. From the algebraic structure it is clear that the expression has a critical point at $s = 1$ and by adding an additional term in the approximation it is possible to show that the expression cannot assume a global maximum outside the interval $\left(\frac{2}{\pi r}, \frac{\pi r}{2}\right)$. In order to gain a proper proof for the statement, it would be sufficient to show that there are no more critical points inside the interval $\left(\frac{2}{\pi r}, \frac{\pi r}{2}\right)$. A direct analysis seems hard, since the function is almost constant on a large part of this interval if r is big.

By using the substitution $\beta = \sqrt{s/n}$, the above arguments suggest particularly that

$$A_{g_0, n} \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \right) \geq A_{g_0, n}(\alpha, \beta)$$

for all $(\alpha\beta)^{-1} \in 2\mathbb{N} + 1$.

5.3 Critical Density

Although there exist already lots of results which tell us that $\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ cannot be a frame for $\alpha\beta = 1$, we will give a proof for this fact using results about Jacobi's theta functions.

We investigate the lower frame bound which we expect to cause troubles. We recall the following fact for Jacobi's theta functions from Lemma 5.22. For $r_1, r_2, s \in \mathbb{R}^+$, we have

$$\begin{aligned} & \theta_3\left(\frac{r_1 s}{2}\right) \theta_3\left(\frac{r_2}{2s}\right) + \theta_4\left(\frac{r_1 s}{2}\right) \theta_4\left(\frac{r_2}{2s}\right) \\ &= 2 \left(\theta_3(2r_1 s) \theta_3\left(\frac{2r_2}{s}\right) + \theta_2(2r_1 s) \theta_2\left(\frac{2r_2}{s}\right) \right). \end{aligned}$$

If we now take a look the lower frame bound we find, by using Lemma 5.22, that we can write it in the following way

$$\begin{aligned} A_{g_0,1}(\alpha, \beta) &= \theta_4\left(\frac{1}{2\beta^2}\right) \theta_4\left(\frac{\beta^2}{2}\right) - 2 \theta_2\left(\frac{2}{\beta^2}\right) \theta_2(2\beta^2) \\ &= 2 \theta_3(2\beta^2) \theta_3\left(\frac{2}{\beta^2}\right) - \theta_3\left(\frac{\beta^2}{2}\right) \theta_3\left(\frac{1}{2\beta^2}\right). \end{aligned}$$

From the Jacobi identity we conclude that $\theta_3\left(\frac{2}{\beta^2}\right) = \frac{\beta}{\sqrt{2}} \theta_3\left(\frac{\beta^2}{2}\right)$ and that $\theta_3\left(\frac{1}{2\beta^2}\right) = \sqrt{2} \beta \theta_3(2\beta^2)$. Therefore we can rewrite the lower frame bound as

$$A_{g_0,1}(\alpha, \beta) = \sqrt{2} \beta \left(\theta_3(2\beta^2) \theta_3\left(\frac{\beta^2}{2}\right) - \theta_3\left(\frac{\beta^2}{2}\right) \theta_3(2\beta^2) \right) = 0.$$

From this we see that $\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ never is a frame if $\alpha\beta = 1$. We remark that setting $r = \frac{1}{2}$ in Proposition 5.23 corresponds exactly to the case of the lower bound for a Gaussian Gabor frame at critical density. Therefore the condition $r > \frac{1}{2}$ cannot be weakened any more in Proposition 5.23. Also, the sign changes for $r < \frac{1}{2}$.

6 The General Case and Theta Functions on a Lattice

This section deals with the already mentioned conjecture formulated by Strohmer & Beaver in 2003 [72] which we briefly recall. They claim that the condition number of the Gabor frame operator for the standard Gaussian window and a hexagonal lattice of fixed density $\delta > 1$ is minimal among all lattice of same fixed density δ . In their work Strohmer & Beaver show that the hexagonal lattice is preferable over the quadratic lattice, which, until then, was the candidate to give the optimal condition number for a standard Gaussian Gabor frame due to the conjecture by Floch, Alard & Berrou in 1995 [33]. We state the following, stronger conjecture.

Conjecture 6.1. *Let $g_0(t) = 2^{1/4}e^{-\pi t^2}$ be the standard Gaussian window. Among all lattices in the set*

$$\mathcal{F}_{full}^\delta(g_0) = \{\Lambda \subset \mathbb{R}^2 \mid \text{vol}^{-1}(\Lambda) = \delta > 1, \delta \text{ fixed}\}.$$

the hexagonal lattice is the unique maximizer for the lower frame bound and the unique minimizer for the upper frame bound.

For even redundancy we will see that the hexagonal lattice minimizes the upper frame bound. We conjecture that it also maximizes the lower frame bound, unfortunately we do not yet have a proof for that. It turns out that optimizing the frame bounds in the case of even redundancy is equivalent to finding the maximum and the minimum of the heat kernel of the flat Laplacian on the torus \mathbb{R}^2/Λ and then optimizing among all lattices of fixed area. Investigations in this direction have been carried out by Montgomery in 1988 [65]. Using Montgomery's theorem about minimal theta functions [65, Theorem 1], we will prove Theorem 6.2. Besides Montgomery's theorem we will use the Poisson summation formula which holds pointwise for the Gaussian window.

Theorem 6.2. *Let S be the generating matrix for the lattice $\Lambda = S\mathbb{Z}^2$ of density $2n$, $n \in \mathbb{N}$ and let $g_0(t) = 2^{1/4}e^{-\pi t^2}$ be the standard Gaussian. Let*

$$\Lambda_h = S_h\mathbb{Z}^2 = \frac{1}{\sqrt{2n}} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \mathbb{Z}^2$$

be the hexagonal lattice. We denote the upper frame bound of $\mathcal{G}(g_0, \Lambda)$ by

$$B = B_{g_0, n}(\Lambda).$$

Then

$$B_{g_0, n}(\Lambda_h) \leq B_{g_0, n}(\Lambda)$$

with equality only for $\Lambda = \tilde{S}_h\mathbb{Z}^2$ with

$$\tilde{S}_h = QS_h\mathcal{B}$$

where Q is an orthogonal matrix and $\mathcal{B} \in SL(2, \mathbb{Z})$.

The theorem tells us that the upper frame bound is minimized by a hexagonal lattice. The matrix \mathcal{B} is an element of the modular group and \mathbb{Z}^2 is invariant under the action of this group. In fact, it is just another choice for a basis of our lattice. Furthermore, the action of the matrix Q does not change the geometry of the lattice. Therefore, those matrices will be negligible in our proofs and we will focus on lattices generated by lower triangular matrices which is justified by the results of Section 3.

In Section 6.1 we will recall Janssen's formulas [54] for sharp frame bounds of a Gaussian Gabor frame. We will show that, starting from a rectangular lattice, the upper frame bound is always improved by shearing the lattice, or, equivalently, by chirping the window. In Section 6.2 we will show that among a certain class of lattices, which we will call quincunx lattices, the hexagonal lattice leads to the smallest possible upper frame bound. In Section 6.3 we will prove Theorem 6.2 and finally, in Section 6.4 we will state some open problems about frame bounds and also some conjectures about concrete values for redundancy 2 in the spirit of Strohmer & Beaver [72]. Also, we will make the observation that Landau's constant, Baxter's 4-colouring constant and the frame bounds of a Gabor frame with standard Gaussian window and hexagonal lattice of redundancy 2 might be related to each other.

6.1 Chirped Gaussians and Sheared Lattices

As already mentioned, due to the results in Section 3 we only need to consider lattices where the generating matrix takes the form

$$S_\gamma = \begin{pmatrix} \alpha & 0 \\ \gamma\alpha & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

with $\alpha, \beta > 0$ and $\alpha\beta = \frac{1}{2^n}$. It is enough to look at lattices of the type $\Lambda = S_\gamma\mathbb{Z}^2$ because any lattice $\Lambda \subset \mathbb{R}^2$ can be represented by $\Lambda = QS_\gamma\mathbb{Z}^2$, where Q is an orthogonal matrix (QR-decomposition). The rotation imposed by Q does not affect the frame bounds as the Gaussian is an eigenfunction with eigenvalue 1 of the corresponding metaplectic operator, which is the fractional Fourier transform described already in [3] or [18]. Instead of looking at lattices of the mentioned type and the standard Gaussian, we can also look at rectangular lattices, i.e. $\gamma = 0$, paired with chirped Gaussians. A chirped (standard) Gaussian is of the form

$$g_\gamma(t) = 2^{1/4} e^{\pi i \gamma t^2} e^{-\pi t^2}.$$

As we also already studied in detail in Section 3, the two systems

$$\mathcal{G}(g_0, S_\gamma\mathbb{Z}^2) \quad \text{and} \quad \mathcal{G}(g_{-\gamma}, S_0\mathbb{Z}^2)$$

possess the same sharp frame bounds.

We choose the Gabor system $\mathcal{G}(g_{-\gamma}, S_0\mathbb{Z}^2)$ as object of investigation. As already stated before, due to the work of Janssen [54], we know that for $(\alpha\beta)^{-1} \in \mathbb{N}$ the lower and upper

frame bound are given by the minimum and maximum, respectively, of the Fourier series

$$F_{g_{-\gamma}}(x, \omega) = \frac{1}{\alpha\beta} \sum_{k, l \in \mathbb{Z}} \left\langle g_{-\gamma}, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g_{-\gamma} \right\rangle e^{2\pi i k x} e^{2\pi i l \omega}.$$

From its definition it is clear that we only need to know $F_{g_{\gamma}}$ on the unit square, i.e. $(x, \omega) \in [0, 1] \times [0, 1]$. We already computed the inner product in Section 4 to be

$$\left\langle g_{-\gamma}, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g_{-\gamma} \right\rangle = e^{-\pi i \frac{kl}{\alpha\beta}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \left(\frac{l}{\alpha} + \frac{k\gamma}{\beta} \right)^2 \right)} = (-1)^{\frac{kl}{\alpha\beta}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \left(\frac{l}{\alpha} + \frac{k\gamma}{\beta} \right)^2 \right)}.$$

Since we assume $(\alpha\beta)^{-1} \in 2\mathbb{N}$ we have

$$F_{g_{-\gamma}}(x, \omega) = 2n \sum_{k, l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \left(\frac{l}{\alpha} + \frac{k\gamma}{\beta} \right)^2 \right)} e^{2\pi i k x} e^{2\pi i l \omega},$$

which, as Montgomery did [65], can also be identified as the heat kernel of the flat Laplacian on the torus \mathbb{R}^2/Λ where $\Lambda = S_{\gamma}\mathbb{Z}^2$. We note that

$$F_{g_{-\gamma}}(x, \omega) = F_{g_{\gamma}}(-x, \omega) = F_{g_{-\gamma}}(-x, -\omega) = F_{g_{\gamma}}(x, -\omega).$$

It is easy to see that the functions $F_{g_{-\gamma}}$ and $F_{g_{\gamma}}$ take their maximum whenever $(x, \omega) \in \mathbb{Z} \times \mathbb{Z}$. This implies that in the case of even redundancy the optimal upper frame bound for a Gabor frame with standard Gaussian window is given by the formula

$$\begin{aligned} B &= B(\alpha, \beta, \gamma) = B(\alpha, \beta, -\gamma) \\ &= 2n \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{-\frac{\pi}{2} \left(\frac{k^2 \gamma^2}{\beta^2} + 2 \frac{kl\gamma}{\alpha\beta} \right)} \\ &= 2n \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{-\frac{\pi}{2} \left(\frac{k^2 \gamma^2}{\beta^2} - 2 \frac{kl\gamma}{\alpha\beta} \right)}. \end{aligned} \tag{6.1}$$

Our goal is to find the global minimum of this function with respect to the parameters α , β and γ . Since the product $\alpha\beta = \frac{1}{2n}$ is fixed, this is a minimization problem in 2 variables. It is similar to the problem considered in Section 5. Unfortunately, the techniques used there cannot be applied directly, as the double sum does not factor into a product of two sums for $\gamma \neq 0$. Still, the double sum converges very nicely, in particular absolutely. We will now show some properties of B .

Proposition 6.3. *For α, β fixed with $\alpha\beta = \frac{1}{2n}$, $n \in \mathbb{N}$, B is periodic in γ with period $\frac{\beta}{\alpha}$ and symmetric with respect to the points $\frac{\beta}{2\alpha}\mathbb{Z}$. Furthermore, B takes its global maximum only for $\gamma \in \frac{\beta}{\alpha}\mathbb{Z}$, i.e., for rectangular lattices.*

Proof. The only property that needs verification is that B assumes its global maximum for $\gamma \in \frac{\beta}{\alpha}\mathbb{Z}$. The periodicity and symmetry follow from the according properties of the lattice.

We split the double sum in the following way.

$$\begin{aligned} B(\gamma) &= 2n \sum_{k \in \mathbb{Z}} \left(e^{-\frac{\pi}{2} \frac{k^2}{\beta^2} (1+\gamma^2)} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{l^2}{\alpha^2} - 2 \frac{kl\gamma}{\alpha\beta} \right)} \right) \\ &= 2n \sum_{k \in \mathbb{Z}} \left(e^{-\frac{\pi}{2} \frac{k^2}{\beta^2} (1+\gamma^2)} \sum_{l \in \mathbb{Z}} e^{-\pi n \left(\frac{\beta}{\alpha} l^2 - 2kl\gamma \right)} \right). \end{aligned}$$

We will now use Poisson summation to rewrite the inner sum of the expression. For k fixed we have that

$$\sum_{l \in \mathbb{Z}} e^{-\pi n \left(\frac{\beta}{\alpha} l^2 - 2kl\gamma \right)} = \sum_{l \in \mathbb{Z}} e^{-\pi n \frac{\beta}{\alpha} \left(l - \frac{\alpha}{\beta} \gamma k \right)^2} e^{\pi n \frac{\alpha}{\beta} \gamma^2 k^2} = \sqrt{\frac{\alpha}{n\beta}} \sum_{m \in \mathbb{Z}} e^{-\frac{\pi}{n} \frac{\alpha}{\beta} (m^2 - n^2 \gamma^2 k^2)} e^{-2\pi i m \frac{\alpha}{\beta} \gamma k}.$$

Using the fact that $(\alpha\beta)^{-1} = 2n$ and due to the convergence properties of the double series we can now rewrite it as

$$\begin{aligned} B(\gamma) &= 2n \sqrt{\frac{\alpha}{n\beta}} \sum_{k \in \mathbb{Z}} \left(e^{-\frac{\pi}{2} \frac{k^2}{\beta^2} (1+\gamma^2)} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{n} \frac{\alpha}{\beta} (l^2 - n^2 \gamma^2 k^2)} e^{-2\pi i l \frac{\alpha}{\beta} \gamma k} \right) \\ &= 2n \sqrt{2} \alpha \sum_{k \in \mathbb{Z}} \left(e^{-2\pi \alpha^2 n^2 k^2 (1+\gamma^2)} \sum_{l \in \mathbb{Z}} e^{-2\pi \alpha^2 (l^2 - n^2 k^2 \gamma^2)} e^{-2\pi i k l \frac{\alpha}{\beta} \gamma} \right) \\ &= 2n \sqrt{2} \alpha \sum_{k, l \in \mathbb{Z}} e^{-2\pi \alpha^2 (l^2 + k^2 n^2)} e^{-2\pi i k l \frac{\alpha}{\beta} \gamma} \\ &= 2n \sqrt{2} \alpha \sum_{k, l \in \mathbb{Z}} e^{-2\pi \alpha^2 (l^2 + k^2 n^2)} \cos \left(2\pi k l \frac{\alpha}{\beta} \gamma \right). \end{aligned}$$

By using the identity $\cos(2x) = \cos(x)^2 - \sin(x)^2 = 1 - 2\sin(x)^2$ we get

$$B = 2n \sqrt{2} \alpha \sum_{k, l \in \mathbb{Z}} e^{-2\pi \alpha^2 (l^2 + k^2 n^2)} \left(1 - 2 \sin \left(\pi k l \frac{\alpha}{\beta} \gamma \right)^2 \right)$$

and we see that $B(\gamma)$ is maximal for $\gamma \in \frac{\beta}{\alpha} \mathbb{Z}$. \square

Proposition 6.3 shows that for even redundancy the upper frame bound of any Gabor frame with rectangular lattice and standard Gaussian window will become smaller by either shearing the lattice or by chirping the window ($\gamma \notin \frac{\beta}{\alpha} \mathbb{Z}$). This gives analytic evidence that the quadratic lattice cannot be optimal for the upper frame bound among general lattices. We will now state a lemma by Montgomery from which we will be able to conclude when B assumes its minimum. The proof of the upcoming result needs a lot of cumbersome computations and estimates. Therefore, we refer to the original work where this lemma has been proved [65, Lemma 4].

Lemma 6.4 (Montgomery). *Let $c > 0$ be fixed, $r \in (0, \frac{1}{2})$ and $s \geq \frac{1}{2}$. We define*

$$\vartheta(r, s; c) = \sum_k \left(e^{-c\pi s k^2} \sum_{l \in \mathbb{Z}} e^{-\frac{c\pi}{s}(l+kr)^2} \right).$$

Then

$$\frac{\partial}{\partial r} \vartheta(r, s; c) < 0.$$

The parameter r in Montgomery's lemma in principle represents the shearing parameter of the lattice whereas the parameter s corresponds to the lattice parameter α (or β as one prefers). Fixing α and β , B only depends on the shearing parameter γ . By rewriting equation (6.1) in the following way

$$\begin{aligned} B(\gamma) &= 2n \sum_{k, l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{l-k\gamma}{\alpha} \right)^2} e^{-\frac{\pi}{2} \frac{k^2}{\beta^2}} \\ &= 2n \sum_{k \in \mathbb{Z}} \left(e^{-2\pi\alpha^2 n^2 k^2} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2\alpha^2} (l-\gamma\frac{\alpha}{\beta}k)^2} \right) \\ &= 2n \vartheta \left(-\gamma \frac{\alpha}{\beta}, 2\alpha^2 n; n \right), \end{aligned}$$

Montgomery's lemma implies the following proposition.

Proposition 6.5. *For α, β fixed with $\alpha \geq \frac{1}{\sqrt{2}\sqrt{2n}}$ and $\alpha\beta = \frac{1}{2n}$, $n \in \mathbb{N}$, B assumes its global minimum only for $\gamma \in \frac{\beta}{\alpha} \left(\frac{1}{2} + \mathbb{Z} \right)$.*

We want to remark that Montgomery has already noticed that the condition $s \geq \frac{1}{2}$ in Lemma 6.4 is sufficient for the statement to hold. In his work [65], Montgomery states that it seems likely that the result is still true for $s \geq \frac{1}{\sqrt{12}}$. Making the substitutions $s = 2\alpha^2 n = \frac{1}{2}$ and $r = -\gamma\frac{\alpha}{\beta} = \frac{1}{2}$ in Lemma 6.4 we get the value of the upper frame bound for $\mathcal{G}(g_0, \Lambda_\diamond)$ where

$$\Lambda_\diamond = \frac{1}{\sqrt{2n}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix} \mathbb{Z}^2$$

is a 45° rotated version of the square lattice of redundancy $2n$. Using the parameters $s = 2\alpha^2 n = \frac{1}{\sqrt{12}}$ and $r = -\gamma\frac{\alpha}{\beta} = \frac{1}{2}$ we find the upper frame bound of the Gabor frame $\mathcal{G}(g_0, \Lambda_h)$ where

$$\Lambda_h = \frac{1}{\sqrt{2n}} \begin{pmatrix} \frac{1}{\sqrt{2}\sqrt[4]{3}} & 0 \\ -\frac{\sqrt{3}}{\sqrt{2}} & \sqrt{2}\sqrt[4]{3} \end{pmatrix} \mathbb{Z}^2$$

is the hexagonal lattice from Theorem 6.2 with another choice of basis and rotated by 30°. Testing for some values $s < \frac{1}{\sqrt{12}}$ suggests that Lemma 6.4 might hold for $s \geq \frac{1}{4}$. We find a local maximum for $r = \frac{1}{2}$ by testing numerically for some values $s < \frac{1}{4}$. Numerically the minimum was found by symmetrically truncating the theta function and finding critical points for the truncated series using 'NSolve' in Mathematica 10 [76].

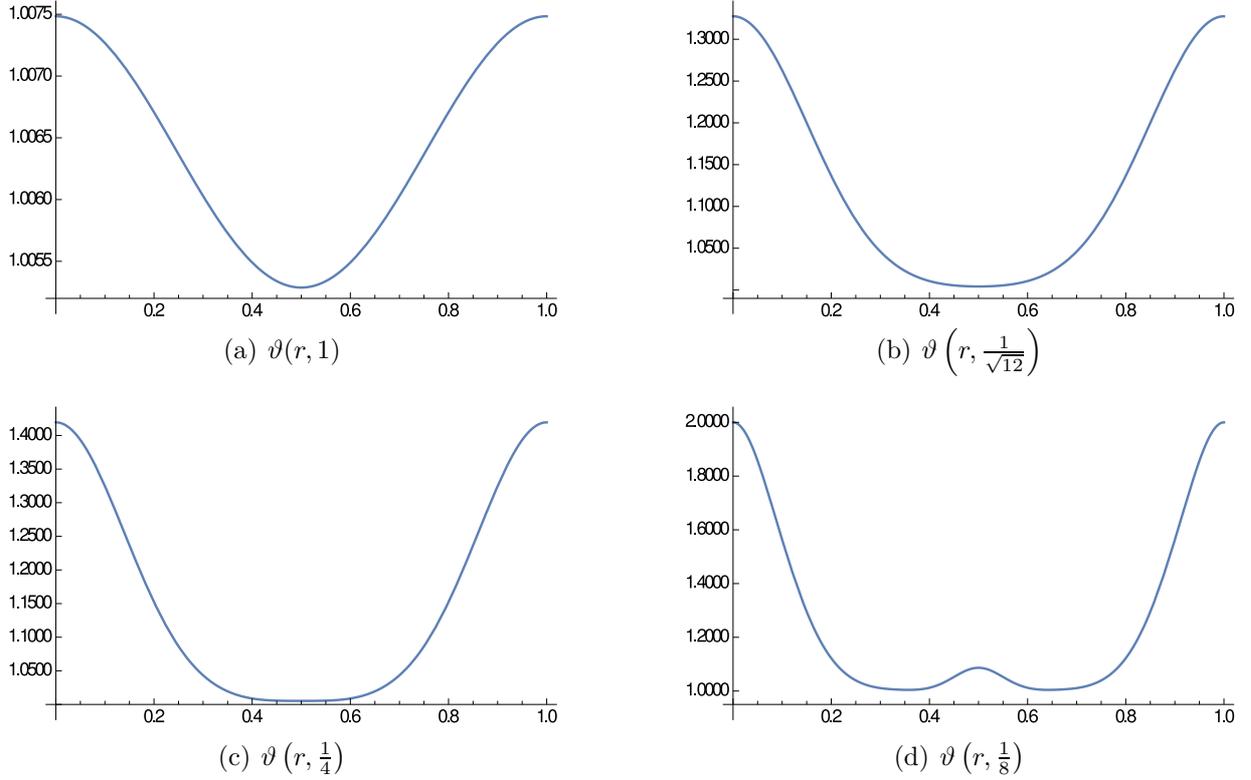


Figure 11: The function $\vartheta(r, s)$ from Lemma 6.4 with $c = 2$ and different fixed values for s . The ordinates in (a)-(d) are shown in different scales. The plots were created in Mathematica 10 [76] by symmetrically truncating the theta function.

6.2 Dilated Quincunx Lattices

We will now consider the case $\gamma = \frac{\beta}{2\alpha}$. This leads us to dilated versions of the quincunx lattice which can be described by a matrix of the form

$$S = \begin{pmatrix} 1 & 0 \\ \frac{\beta}{2\alpha} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \beta/2 & \beta \end{pmatrix}.$$

Often the quincunx lattice is the 45° rotated version of the square lattice. A generating matrix has the form $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix}$.

From Proposition 6.5 we already know that for $\alpha\beta = \frac{1}{2n}$ and $\alpha \geq \frac{1}{\sqrt{2}\sqrt{2n}}$ the upper frame bound assumes its global minimum for $\gamma \in \frac{\beta}{\alpha} (\frac{1}{2} + \mathbb{Z})$. For the standard Gaussian window, we want to find out whether there exists a unique triple of lattice parameters (α, β, γ) such that $\Lambda = \begin{pmatrix} \alpha & 0 \\ \gamma\alpha & \beta \end{pmatrix} \mathbb{Z}^2$ with $(\alpha\beta)^{-1} \in 2\mathbb{N}$ and $\gamma = \frac{\beta}{2\alpha}$ which minimizes the upper frame bound.

We can now write the upper frame bound as

$$\begin{aligned}
B &= 2n \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)} e^{-\frac{\pi}{2} \left(\frac{k^2 \gamma^2}{\beta^2} - 2 \frac{kl\gamma}{\alpha\beta} \right)} \\
&= 2n \sum_{k, l \in \mathbb{Z}} e^{-\frac{\pi}{2\alpha^2} \left((4n^2\alpha^4 + \frac{1}{4})k^2 + kl + l^2 \right)}.
\end{aligned} \tag{6.2}$$

We will now give some definitions in order to state a theorem formulated by Montgomery [65]. For $\rho > 0$ and a positive definite quadratic form $q(u_1, u_2) = au_1^2 + bu_1u_2 + cu_2^2$ of discriminant $D = -1$ we define the theta function

$$\theta_q(\rho) = \sum_{k, l \in \mathbb{Z}} e^{-2\pi\rho q(k, l)}. \tag{6.3}$$

The discriminant D of the quadratic form is given by

$$D = b^2 - 4ac$$

and for positive definite quadratic forms we have $D < 0$. Also, we can associate the symmetric matrix

$$S_q^T S_q = G_q = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

to the quadratic form $q(u_1, u_2) = (u_1, u_2) \cdot G_q \cdot (u_1, u_2)^T = \|S_q u\|_2^2$. We observe that

$$4 \det(G_q) = -D.$$

We say the quadratic form \tilde{q} is (integrally) equivalent to the quadratic form q if

$$\tilde{q}(u_1, u_2) = (u_1, u_2) \cdot \underbrace{\mathcal{B}^T G_q \mathcal{B}}_{G_{\tilde{q}}} \cdot (u_1, u_2)^T$$

where $\mathcal{B} \in SL(2, \mathbb{Z})$. This means that the Gram matrices $(S_q \mathcal{B})^T (S_q \mathcal{B})$ and $S_q^T S_q$ associated to the quadratic forms \tilde{q} and q respectively, are associated to the same lattice $S_q \mathbb{Z}^2$. At this point, we remark that we used the index q to emphasize that the Gram matrix G_q is associated to the quadratic form. In what follows we will also use the index Λ to emphasize the association to a lattice. For more details on quadratic forms associated to a lattice we refer to the textbook of Conway & Sloane [16].

Theorem 6.6 (Montgomery). *Let $h(u_1, u_2) = \frac{1}{\sqrt{3}}(u_1^2 + u_1u_2 + u_2^2)$. For any $\rho > 0$ and any positive definite quadratic form $q(u_1, u_2)$ with discriminant $D = -1$ we have*

$$\theta_q(\rho) \geq \theta_h(\rho).$$

If we have equality for some $\rho > 0$, then q and h are equivalent forms and $\theta_q \equiv \theta_h$.

We will see that the quadratic form $h(u_1, u_2)$ in Montgomery's theorem is associated to the hexagonal lattice, which was certainly known to Montgomery, but not mentioned in his work [65]. This will imply, that among all theta functions associated to a quadratic form with fixed discriminant, the form associated to a hexagonal lattice minimizes the theta function.

The quadratic form associated to the upper frame bound in equation (6.2) is given by

$$\rho q_\alpha(k, l) = \frac{1}{4\alpha^2} \left(\left(4n^2\alpha^4 + \frac{1}{4} \right) k^2 + kl + l^2 \right)$$

which is not yet normalized. In order to minimize B we have to find values for $\alpha > 0$ such that

$$4n^2\alpha^4 + \frac{1}{4} = 1.$$

The unique solution to this problem is given by

$$\alpha = \frac{\sqrt[4]{3}}{\sqrt{2}\sqrt{2n}}.$$

Therefore,

$$\rho q_\alpha(k, l) = \frac{n}{\sqrt{3}} (k^2 + kl + l^2).$$

Since, we want to have that the discriminant $D = -1$, we get that $\rho = n$. We compute the other parameters β and γ . As the product $\alpha\beta = \frac{1}{2n}$ is fixed we have

$$\beta = \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{2n}}$$

and γ was set to $\frac{\beta}{2\alpha}$ which gives

$$\gamma = \frac{1}{\sqrt{3}}.$$

On the other hand, the resulting lattice is generated by the matrix

$$S_h = \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{3}} & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt[4]{3}}{\sqrt{2}\sqrt{2n}} & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{2n}} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt[4]{3}}{\sqrt{2}\sqrt{2n}} & 0 \\ \frac{\sqrt{2}}{2\sqrt[4]{3}\sqrt{2n}} & \frac{\sqrt{2}}{\sqrt[4]{3}\sqrt{2n}} \end{pmatrix} = \frac{1}{\sqrt{2n}} \begin{pmatrix} \frac{\sqrt[4]{3}}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{2\sqrt[4]{3}} & \frac{\sqrt{2}}{\sqrt[4]{3}} \end{pmatrix}.$$

Its Gram matrix is given by

$$G_h = S_h^T S_h = \frac{1}{2n} \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

The determinant of the Gram matrix is $\det(G_h) = \frac{1}{4n^2} = -\frac{1}{4}D$ which is equivalent to the fact that $D = \frac{-1}{n^2}$. Therefore, the resulting (non-normalized) quadratic form is given by

$$\rho h(u_1, u_2) = \frac{1}{n\sqrt{3}} (u_1^2 + u_1u_2 + u_2^2)$$

and we find out that $\rho = \frac{1}{n}$. The associated theta function is

$$\theta_h \left(\frac{1}{n} \right) = \sum_{k,l \in \mathbb{Z}} e^{-2\pi \frac{1}{n\sqrt{3}}(k^2+kl+l^2)}$$

The discrepancy about the appearing factors in the quadratic form is explained by the following identity

$$\rho \theta_q(\rho) = \theta_q \left(\frac{1}{\rho} \right)$$

which is true for quadratic forms with discriminant $D = -1$ and can be verified by using Poisson summation. The identity can also be looked up in [65]. So, we find out that

$$B_h = 2n \sum_{k,l \in \mathbb{Z}^2} e^{-2\pi \frac{n}{\sqrt{3}}(k^2+kl+l^2)} = 2 \sum_{k,l \in \mathbb{Z}^2} e^{-2\pi \frac{1}{n\sqrt{3}}(k^2+kl+l^2)}.$$

We will discuss this property in more detail in the upcoming section.

For even redundancy $\alpha\beta = \frac{1}{2n}$, the results show that for fixed $\alpha \geq \frac{1}{\sqrt{2}\sqrt{2n}}$ the combination of the standard Gaussian window and a quincunx lattice minimizes the upper frame bound. Among these quincunx lattices we find that the hexagonal lattice uniquely minimizes the upper frame bound which implies that it uniquely minimizes the upper frame bound among all lattices of redundancy $2n$ with $\alpha \geq \frac{1}{\sqrt{2}\sqrt{2n}}$.

6.3 General Lattices

As a next step we want to show that the hexagonal lattice uniquely minimizes the upper frame bound among all lattices of (fixed) even redundancy. Therefore, the only assumptions on the triple (α, β, γ) defining the lattice will be that $(\alpha\beta)^{-1} = 2n$, $n \in \mathbb{N}$. The statement will again follow from Montgomery's theorem.

We recall Janssen's representation that for $g \in L^2(\mathbb{R}^d)$ with

$$\sum_{\lambda^\circ \in \Lambda^\circ} |\langle g, \pi(\lambda^\circ)g \rangle| = \sum_{\lambda^\circ \in \Lambda^\circ} |\mathcal{A}g(\lambda^\circ)| < \infty$$

the frame operator can be represented as

$$S_{g,\Lambda} = \text{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ).$$

We also recall that for the chirped standard Gaussian and a separable lattice of even redundancy, the Gabor frame bounds are given by the minimum and the maximum of the Fourier series

$$F_{g-\gamma}(x, \omega) = \frac{1}{\alpha\beta} \sum_{k,l \in \mathbb{Z}} \left\langle g_{-\gamma}, M_{\frac{l}{\alpha}} T_{\frac{k}{\beta}} g_{-\gamma} \right\rangle e^{2\pi i k x} e^{2\pi i l \omega}.$$

which we already encountered earlier. This shows that for a Gabor frame with separable lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with $\alpha\beta = \frac{1}{2n}$ and window g_γ , the chirped standard Gaussian, the $\ell^1(\Lambda^\circ)$ -norm of the coefficients in Janssen's representation of the frame operator (including the factor $\text{vol}(\Lambda)^{-1}$) gives us precisely the upper frame bound by evaluating the Fourier series $F_{g_{-\gamma}}$ at $(x, \omega) = (0, 0)$. We will now show that for Gabor frames with standard Gaussian window g_0 and general lattices the same is true for the $\ell^1(2\Lambda)$ -norm of the samples of the ambiguity function $\mathcal{A}g_0$ of the standard Gaussian.

For $\Lambda = S_\gamma\mathbb{Z}^2 = \begin{pmatrix} \alpha & 0 \\ \alpha\gamma & \beta \end{pmatrix} \mathbb{Z}^2$, the associated Gram matrix is given by

$$G_\Lambda = \begin{pmatrix} (1 + \gamma^2)\alpha^2 & \alpha\beta\gamma \\ \alpha\beta\gamma & \beta^2 \end{pmatrix}$$

with determinant $\det(G_\Lambda) = \frac{1}{\alpha^2\beta^2} = \frac{1}{4n^2} = -\frac{1}{4}D$. The adjoint lattice, its associated matrix and the Gram matrix are given by

$$\Lambda^\circ = S^\circ\mathbb{Z}^2 = \begin{pmatrix} \frac{1}{\beta} & 0 \\ \frac{\gamma}{\beta} & \frac{1}{\alpha} \end{pmatrix} \mathbb{Z}^2 = \text{vol}(\Lambda)^{-1}\Lambda \quad \text{and} \quad G_{\Lambda^\circ} = \begin{pmatrix} \frac{1+\gamma^2}{\beta^2} & \frac{\gamma}{\alpha\beta} \\ \frac{\gamma}{\alpha\beta} & \frac{1}{\alpha^2} \end{pmatrix}.$$

We recall the identity

$$\theta_q(\rho) = \frac{1}{\rho} \theta_q\left(\frac{1}{\rho}\right)$$

where $\theta_q(\rho)$ is defined as in equation (6.3) and q is a quadratic form of discriminant -1 . Using the last identity for theta functions, we compute the upper frame bound of $\mathcal{G}(g_0, \Lambda)$ as

$$\begin{aligned} B(\Lambda) &= 2n \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{A}g_0(\lambda^\circ) \\ &= 2n \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \left(\frac{1+\gamma^2}{\beta^2} k^2 + 2\frac{\gamma k l}{\alpha\beta} + \frac{1}{\alpha^2} l^2 \right)} \\ &= \det(G_{\Lambda^\circ})^{1/2} \sum_{k, l \in \mathbb{Z}} e^{-\frac{\pi}{2} \langle S^\circ(k, l), S^\circ(k, l) \rangle} \\ &= 2(-D)^{-1/2} \sum_{k, l \in \mathbb{Z}} e^{-2\pi(-D)^{-1/2} q_\Lambda(k, l)} \\ &= 2 \sum_{k, l \in \mathbb{Z}} e^{-\frac{\pi}{2} q_\Lambda(2k, 2l)} \\ &= 2 \sum_{\lambda \in 2\Lambda} \mathcal{A}g_0(\lambda). \end{aligned}$$

Here, $q_\Lambda(k, l) = (-D)^{-1/2} \langle S(k, l), S(k, l) \rangle$ is a quadratic form with discriminant -1 . We find that

$$B(\Lambda) = 2\theta_{q_\Lambda}\left(\frac{1}{n}\right) = 2n \theta_{q_\Lambda}(n). \quad (6.4)$$

Therefore, Montgomery's theorem (Theorem 6.6) implies that for a Gabor frame with Gaussian window and any lattice of even redundancy, only a lattice with associated Gram matrix

$$G_h = \frac{1}{2n} \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

yields the minimal norm for the Gabor frame operator $S_{g_0, \Lambda}$ or equivalently the smallest possible upper frame bound. Therefore, only rotated versions of the hexagonal lattice give the optimal upper frame bound. Any other quadratic form which also minimizes the theta function $\theta_q(\rho)$ is as well associated to a hexagonal lattice. In particular the quadratic form associated to the lattice will be of the form

$$\tilde{G}_h = \mathcal{B}^T G_h \mathcal{B}, \quad \mathcal{B} \in SL(2, \mathbb{Z})$$

where $\mathcal{B} \in SL(2, \mathbb{Z})$ is just another choice for a basis for Λ_h . This proves Theorem 6.2. For more details on the connections between lattices and quadratic forms we refer again to the textbook by Conway & Sloane [16].

As a final remark we note that the case of redundancy 2 is special as we have $2\Lambda = \Lambda^\circ$. Also, this means that $n = 1$ in equation (6.4) and in this case and we have

$$B(\Lambda) = 2\theta_{q_\Lambda}(1).$$

6.4 Open Problems and Observations

In this section we will have a look at some properties of the often encountered Fourier series associated to the Gabor system $\mathcal{G}(g, \Lambda)$. First, we mention the problem for the lower frame bound. Whereas it is easy to see that the maximum of the Fourier series

$$F_q(x, \omega; \rho) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-2\pi\rho q(k,l)} e^{2\pi i(kx+l\omega)}$$

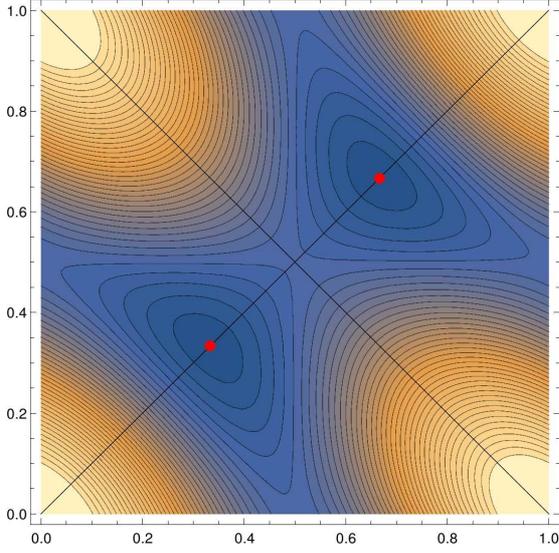
is achieved if $(x, \omega) \in \mathbb{Z} \times \mathbb{Z}$ for any $\rho > 0$ and any positive definite quadratic form q with discriminant -1 , it is not as easy to locate its minimum (see Figure 12). In the separable case, i.e. $\gamma = 0$ or, equivalently, when the quadratic form q has no mixed term, we already saw in Section 5 that the minimum is taken for $(x, \omega) \in (\mathbb{Z} + \frac{1}{2}) \times (\mathbb{Z} + \frac{1}{2})$ as was also described by Janssen [54]. Also, in the separable case the maximum and the minimum of the Fourier series associated to a Gabor frame of integer redundancy $(\alpha\beta)^{-1} = n > 1$

$$F_{(\alpha,\beta)}(x, \omega) = 2n \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} (-1)^{kln} e^{-\frac{\pi}{2} \left(\frac{k^2}{\alpha^2} + \frac{l^2}{\beta^2} \right)} e^{2\pi i(kx+l\omega)}$$

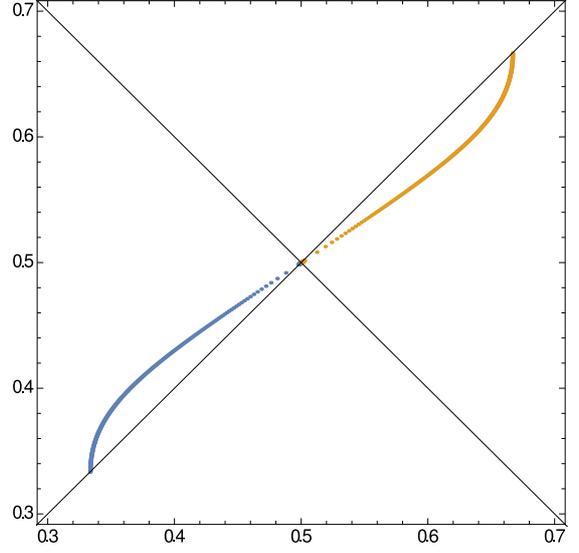
are taken at integer and half integer points respectively. Note the alternating sign in the last formula which vanishes for $n \in 2\mathbb{N}$.

Taking $h(u_1, u_2) = \frac{1}{\sqrt{3}}(u_1^2 - u_1u_2 + u_2^2)$, which is equivalent to the quadratic form from Theorem 6.6, and an adaption of a result by Baernstein [6] shows that the series

$$F_h(x, \omega; \rho) = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e^{-2\pi\rho h(k,l)} e^{2\pi i(kx+l\omega)}$$



(a) The hexagonal heat kernel $F_h(x, \omega; 1)$. The maximum is taken at the corners of the unit square, the minima are marked and taken at the points $(\frac{1}{3}, \frac{1}{3})$ and $(\frac{2}{3}, \frac{2}{3})$.



(b) The *argmin* of a family of sheared heat kernels for a shearing factor $0 \leq \gamma \leq \frac{1}{\sqrt{3}}$. The parameters α and β are chosen such that for $\gamma = \frac{1}{\sqrt{3}}$ we obtain the hexagonal heat kernel.

Figure 12: In the separable case, i.e. $\gamma = 0$, the minimum of any heat kernel is taken at the point $(\frac{1}{2}, \frac{1}{2})$. By varying γ , we will obtain two points, located symmetrically to the point $(\frac{1}{2}, \frac{1}{2})$, which yield the minimum.

assumes its minimum at $(x, \omega) = (\frac{1}{3}, \frac{1}{3})$ and, due to symmetry, also at $(x, \omega) = (\frac{2}{3}, \frac{2}{3})$ and of course at all integer shifts of these points (see Figure 12).

The sharp results about frame bounds in this section are restricted to the case of even redundancy. However, due to the results by Janssen [53] and Tolimieri & Orr [73] the norm of the Gabor frame operator is always bounded from above by the $\ell_1(\Lambda^\circ)$ -norm of the coefficients in Janssen's representation as given by equation (4.9) in Proposition 4.4, i.e.

$$B(\Lambda) \leq \text{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} |\langle g, \pi(\lambda^\circ)g \rangle|.$$

This result holds for any Gabor system with $g \in M^1(\mathbb{R}^d)$. In the case of a Gabor frame with standard Gaussian window $\mathcal{G}(g_0, \Lambda)$ the result is sharp whenever $\text{vol}(\Lambda) = \frac{1}{2n}$, $n \in \mathbb{N}$. Therefore, the upper frame bound of the system $\mathcal{G}(g_0, \Lambda)$ can always be estimated by

$$B(\Lambda) \leq 2 \sum_{\lambda \in \Lambda} \mathcal{A}g_0(2\lambda) = 2n \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{A}g_0(\lambda^\circ).$$

The first sum, involving 2Λ , has computational advantages whenever $\frac{1}{2} \leq \text{vol}(\Lambda)$ because in this case the adjoint lattice Λ° is denser than 2Λ . If $\frac{1}{2} \geq \text{vol}(\Lambda)$ summing over Λ° is preferable for computations. From the frame inequality given in equation (1.1) and the

last estimate for the upper frame bound, we see that

$$\sum_{\lambda \in \Lambda} |\langle g_0, \pi(\lambda)g_0 \rangle|^2 = \sum_{\lambda \in \Lambda} \mathcal{A}g_0(\lambda)^2 \leq B \leq 2 \sum_{\lambda \in \Lambda} \mathcal{A}g_0(2\lambda). \quad (6.5)$$

Although we do not have a sharp bound, the inequalities in Formula (6.5) give indication that Theorem 6.2 might hold for lattices of arbitrary fixed density greater than 1. In fact, as the redundancy increases we find out that the estimates in (6.5) become sharper (see Figure 13).

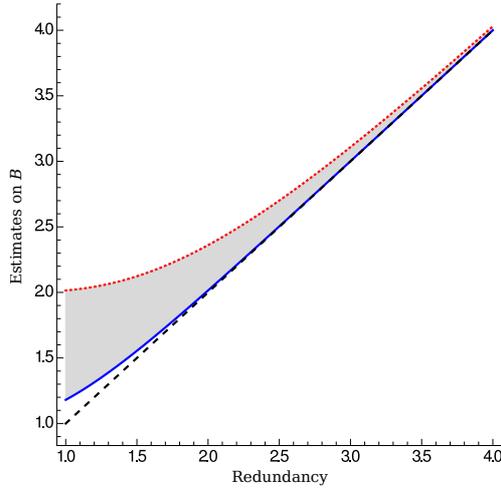


Figure 13: The lower estimate (solid line) and the upper estimate (dotted line) on B from equation (6.5) for the hexagonal lattice for different redundancies greater than 1. As the redundancy increases, B tends to take the value of the redundancy.

This argument can be made rigorous by observing that for large redundancy we can sample and sum the ambiguity function over the adjoint lattice. Therefore, we see that only the term $\mathcal{A}g_0(0)$ contributes a lot to the sums in equation (6.5) since $\mathcal{A}g_0$ decays like the standard Gaussian in all directions. As a consequence we get

$$\lim_{vol(\Lambda) \rightarrow 0} vol(\Lambda) \sum_{\lambda \in \Lambda} \mathcal{A}g_0(\lambda)^2 = \lim_{vol(\Lambda) \rightarrow 0} vol(\Lambda) B(\Lambda) = \lim_{vol(\Lambda) \rightarrow 0} vol(\Lambda) 2 \sum_{\lambda \in \Lambda} \mathcal{A}g_0(2\lambda) = 1.$$

This is true for any lattice which indicates that as the redundancy increases the geometry of the lattice is negligible for the frame bounds. It is also interesting to observe, that the mean of $F_q(x, \omega; \rho)$ is equal to 1, independent of the quadratic form q and the parameter ρ

$$\iint_{[0,1] \times [0,1]} F_q(x, \omega; \rho) dx d\omega = 1.$$

This implies that the upper frame bound is always greater than or equal to the redundancy of the Gabor system $\mathcal{G}(g_0, \Lambda)$. Of course this is a well-known fact which follows e.g. from

Janssen's proof of the density theorem [55]. Furthermore, since the hexagonal lattice yields the smallest maximum for all heat kernels on the torus \mathbb{R}^2/Λ with $\text{vol}(\Lambda)$ fixed, the fact that all heat kernels possess the same mean, leads to the conjecture that the heat kernel associated to the hexagonal lattice might also give the largest minimum, because since the maximum is close to the mean we would expect the minimum to be close to the mean as well.

We want to make a final remark on frame bounds of the Gabor frame $\mathcal{G}(g_0, \Lambda)$ for redundancy 2. If we take the standard Gaussian and the square lattice $\frac{1}{\sqrt{2}}\mathbb{Z} \times \frac{1}{\sqrt{2}}\mathbb{Z}$ we find out that the sharp lower and upper frame bound are given by

$$A_{\square} = 2 \left(\sum_{k \in \mathbb{Z}} (-1)^k e^{-\pi k^2} \right)^2 = 2 \left(\frac{\pi^{1/4}}{2^{1/4} \Gamma\left(\frac{3}{4}\right)} \right)^2$$

$$B_{\square} = 2 \left(\sum_{k \in \mathbb{Z}} e^{-\pi k^2} \right)^2 = 2 \left(\frac{\pi^{1/4}}{\Gamma\left(\frac{3}{4}\right)} \right)^2,$$

where Γ is the usual gamma function defined as

$$\Gamma(t) = \int_{\mathbb{R}_+} x^{t-1} e^{-x} dx$$

The two formulas for A_{\square} and B_{\square} are actually classical results about Jacobi's theta-3 and theta-4 function. They can also be found in [64]. It is easy to see that the condition number is $B_{\square}/A_{\square} = \sqrt{2}$ which was also observed by Strohmer & Beaver in [72]. Furthermore, they mentioned that in the case of the hexagonal lattice of redundancy 2, the condition number is approximately $B_h/A_h \approx 1.2599$ which, as they say, 'is suspiciously close to $\sqrt[3]{2}$ '. We will now prove that it is exactly the expected value.

Proposition 6.7. *For redundancy 2, the ratio of the frame bounds for the standard Gaussian window and a hexagonal lattice is given by*

$$\frac{B_h}{A_h} = \sqrt[3]{2}.$$

Proof. We already observed that the frame bounds are given by

$$A_h = 2 \sum_{k, l \in \mathbb{Z}} e^{-2\pi \frac{1}{\sqrt{3}}(k^2 - kl + l^2)} e^{2\pi i(k+l)/3} = 2 \sum_{k, l \in \mathbb{Z}} e^{-2\pi \frac{1}{\sqrt{3}}(k^2 + kl + l^2)} e^{2\pi i(k-l)/3}$$

$$B_h = 2 \sum_{k, l \in \mathbb{Z}} e^{-2\pi \frac{1}{\sqrt{3}}(k^2 + kl + l^2)}.$$

We will now use a result on cubic theta functions by Hirschhorn, Garvan & Borwein derived in 1993 [50]². To stick close to their notation we introduce the following functions

$$\begin{aligned} a(q) &= \sum_{k,l \in \mathbb{Z}} q^{k^2+kl+l^2} \\ b(q) &= \sum_{k,l \in \mathbb{Z}} q^{k^2+kl+l^2} \zeta_3^{k-l} \\ c(q) &= \sum_{k,l \in \mathbb{Z}} q^{\left(k+\frac{1}{3}\right)^2 + \left(k+\frac{1}{3}\right)\left(l+\frac{1}{3}\right) + \left(l+\frac{1}{3}\right)^2} \end{aligned}$$

where $\zeta_3^3 = 1$ and $\zeta_3 \neq 1$ and $|q| < 1$. These functions fulfil the identity

$$a(q)^3 = b(q)^3 + c(q)^3.$$

Setting $q = e^{-2\pi/\sqrt{3}}$ we will now prove that actually $b\left(e^{-2\pi/\sqrt{3}}\right) = c\left(e^{-2\pi/\sqrt{3}}\right)$ by using Poisson summation. We start with the observation that $e^{-\frac{2\pi}{\sqrt{3}}(k^2+kl+l^2)}$ and $e^{-\frac{2\pi}{\sqrt{3}}(k^2-kl+l^2)}$ are the 2-dimensional Fourier transforms of each other. Therefore we have

$$\begin{aligned} \sum_{k,l \in \mathbb{Z}} e^{-\frac{2\pi}{\sqrt{3}}(k^2+kl+l^2)} e^{2\pi i \frac{(k-l)}{3}} &= \sum_{k,l \in \mathbb{Z}} e^{-\frac{2\pi}{\sqrt{3}}(k^2-kl+l^2)} e^{-2\pi i \frac{(k+l)}{3}} \\ &= \sum_{k,l \in \mathbb{Z}} e^{-\frac{2\pi}{\sqrt{3}}\left(\left(k+\frac{1}{3}\right)^2 + \left(k+\frac{1}{3}\right)\left(l+\frac{1}{3}\right) + \left(l+\frac{1}{3}\right)^2\right)}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} A_h &= 2b\left(e^{-2\pi/\sqrt{3}}\right) = 2c\left(e^{-2\pi/\sqrt{3}}\right) \\ B_h &= 2a\left(e^{-2\pi/\sqrt{3}}\right) \end{aligned}$$

which gives

$$B_h^3 = 2A_h^3$$

and the proof is finished. □

We remark that in the case of the square lattice of redundancy 2 we already know

$$B_{\square}^2 = 2A_{\square}^2.$$

This is equivalent to the identity

$$\theta_3(1)^4 = 2\theta_4(1)^4 = \theta_2(1)^4 + \theta_4(1)^4,$$

which actually holds for arbitrary $s > 0$ and not only for the presented case $s = 1$. This is one reason why the functions $a(q)$, $b(q)$ and $c(q)$ introduced in the last proof are called

²The author wishes to thank Thomas Strohmmer for pointing out reference [50]

cubic analogues of the Jacobi theta functions. Also, Proposition 6.7 gives an analytic proof that for a Gabor frame with standard Gaussian window and a lattice of redundancy 2 the hexagonal lattice yields a better frame condition number than the square lattice.

We close this section with some numerical observations. Trying to link A_h and B_h to known constants gives two promising candidates. Computations³ show that (at least) the first 2048 digits of the following numbers match

$$A_h \approx \frac{1}{\mathcal{L}_+} = \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} = \sqrt[3]{2}\mathcal{C}^2, \quad (6.6)$$

where \mathcal{L}_+ is the best known upper bound for Landau's constant \mathcal{L} first described by Landau in 1929 [59] for which the inequality

$$\frac{1}{2} < \mathcal{L} \leq \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)} = \mathcal{L}_+ \quad (6.7)$$

is known and the upper bound is conjectured to be sharp [32, Section 7.1]. This goes back to the work of Rademacher in 1943 [68] in which he computed the upper value \mathcal{L}_+ given in (6.7) and conjectured that this might be the true value of \mathcal{L} . Landau himself had a slightly larger value for the upper bound on this constant. The constant \mathcal{C} is Baxter's 4-colouring constant for the hexagonal (triangular) lattice given by

$$\mathcal{C}^2 = \prod_{k=1}^{\infty} \frac{(3k-1)^2}{(3k-2)(3k)} = \frac{3}{4\pi^2} \Gamma\left(\frac{1}{3}\right)^3$$

which can be found in [8], [9], [32, Section 5.24.1]. We will now prove the last equality in equation (6.6). We could not find this identity in the literature.

Proposition 6.8.

$$\frac{1}{\mathcal{L}_+} = \sqrt[3]{2}\mathcal{C}^2.$$

Proof. We establish the result by using two identities for the Gamma function. We have the Legendre duplication formula which says

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

Setting $z = \frac{1}{6}$ gives us, after some rearrangements, that

$$\Gamma\left(\frac{1}{6}\right) = 2^{2/3} \sqrt{\pi} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}. \quad (6.8)$$

³The computations were performed in Mathematica 10.0 by truncating the series representation of A_h for indices $|k|, |l| > 41$ and using the built-in function 'Gamma' to evaluate Γ .

The second identity we use, is Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}.$$

For $z = \frac{1}{3}$ and $z = \frac{1}{6}$ this gives us the identities

$$\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} \quad (6.9)$$

$$\Gamma\left(\frac{5}{6}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{6}\right)}. \quad (6.10)$$

By using all these identities we will be able to complete the proof.

$$\begin{aligned} \frac{1}{\mathcal{L}_+} &= \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} \stackrel{(6.10)}{=} \frac{\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(\frac{1}{3}\right)2\pi} \stackrel{(6.8)}{=} \frac{\Gamma\left(\frac{1}{3}\right)^2 2^{4/3}\pi}{\Gamma\left(\frac{2}{3}\right)^2 \Gamma\left(\frac{1}{3}\right) 2\pi} \\ &\stackrel{(6.9)}{=} \frac{\Gamma\left(\frac{1}{3}\right)^3 2^{4/3}\pi}{\frac{(2\pi)^2}{3} 2\pi} = \frac{3\Gamma\left(\frac{1}{3}\right)^3 2^{1/3}}{4\pi^2} = 2^{1/3}\mathcal{C}^2. \end{aligned}$$

□

As a deduction of equation (6.6) and Proposition 6.7 which shows that $B_h/A_h = \sqrt[3]{2}$ we get the following approximation

$$B_h \approx \frac{\sqrt[3]{2}}{\mathcal{L}_+} = \frac{\sqrt[3]{2}\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)} = 2^{2/3}\mathcal{C}^2. \quad (6.11)$$

Again, the approximation is very accurate and at least the first 2048 digits are correct. In the upcoming section we will have a closer look at Landau's constant and a possible connection to Gabor frames.

7 Gabor Frame Bounds and “Weltkonstanten”

In the first part of this section we will study a packing problem for holomorphic functions. In 1929 Landau [59] gave some estimates on two constants, which he called “Weltkonstanten” (universal constants). These constants go back to an article by Bloch in 1925 [12]. In this article Bloch showed the existence of one of the former mentioned “Weltkonstanten”. In 1937 Ahlfors & Grunsky gave an upper bound on Bloch’s constant which they already believed to be its true value and in 1943 Rademacher [68] gave an estimate on the related Landau constant which he also conjectured to be the true value.

In the second part we have a look at lower frame bounds for two particular Gabor frames. For the standard Gaussian g_0 we investigate the Gabor frames $\mathcal{G}(g_0, \Lambda_{\square}^2)$, where Λ_{\square}^2 is the square lattice of redundancy 2, and the Gabor frame $\mathcal{G}(g_0, \Lambda_h^2)$, where Λ_h^2 is the hexagonal lattice of redundancy 2.

7.1 A Packing Problem for Holomorphic Functions

In what follows \mathbb{D} will be the open unit disc in \mathbb{C} . As Landau already wrote in his article [59]: The following theorem is due to Bloch [12].

Theorem 7.1 (Bloch 1925). *Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'(0)| = 1$. Then there exist an open subset $U \subseteq \mathbb{D}$ and an open disc $D_f(r)$ of radius $r > 0$ in $f(\mathbb{D})$ such that f is biholomorphic from U to $D_f(r)$.*

Bloch’s constant \mathcal{B} is then defined in the following way. First, we define for each $f \in \mathcal{H}(\mathbb{D})$ a constant b depending on f by

$$b(f) = \sup\{r \in \mathbb{R}_+ \mid \exists U \subseteq \mathbb{D}: f(U) = D_f(r) \text{ and } f|_U \text{ is biholomorphic}\}$$

where $D_f(r)$ is a disc of radius r contained in $f(\mathbb{D})$. Bloch’s constant is now

$$\mathcal{B} = \inf_{\substack{f \in \mathcal{H}(\mathbb{D}), \\ |f'(0)|=1}} \{b(f)\}.$$

The smallest upper bound for \mathcal{B} was found by Ahlfors & Grunsky in 1937 [2] and is conjectured to be the true value of \mathcal{B} . We have the following estimates

$$\frac{\sqrt{3}}{4} + 2 \cdot 10^{-4} < \mathcal{B} \leq \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right) (1 + \sqrt{3})^{1/2}}.$$

We are interested in the related constant \mathcal{L} which is called Landau’s constant. It is defined in a way similar to Bloch’s constant by dropping the property that f should be biholomorphic on $D_f(r)$. We have following theorem.

Theorem 7.2 (Landau 1929). *Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'(0)| = 1$. Then there exists a disc $D_f(r)$ of radius $r > 0$ in $f(\mathbb{D})$.*

Landau's constant is now defined as follows. Again, for $f \in \mathcal{H}(\mathbb{D})$ we define a constant ℓ depending on f by

$$\ell(f) = \sup\{r \in \mathbb{R}_+ \mid D_f(r) \subseteq f(\mathbb{D})\}.$$

Landau's constant is now

$$\mathcal{L} = \mathcal{L}_+ = \inf_{\substack{f \in \mathcal{H}(\mathbb{D}), \\ |f'(0)|=1}} \{\ell(f)\}.$$

We have the following estimates

$$\frac{1}{2} + 10^{-335} < \mathcal{L} \leq \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)} = \mathcal{L}_+.$$

The best upper estimate, which is conjectured to be the true value of \mathcal{L} , was achieved by Rademacher in 1943 [68] using analogue techniques to the ones of Ahlfors & Grunsky in their article [2].

Conjecture 7.3 (Rademacher 1943).

$$\mathcal{L} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}.$$

The work by Baernstein & Vinson [5] gives a collection of techniques and results concerning Bloch's constant as well as Landau's constant. The textbooks [15] and [71] contain information about Schwarz-Christoffel mappings, which map the unit disc onto a regular polygon. For mappings from the unit disc onto a hyperbolic polygon we refer to [66, V.7]. These mappings have been the main tools in Rademacher's work and we will also use them later on.

It is beneficial for us to repeat Rademacher's process of finding the upper bound \mathcal{L}_+ . The procedure is as follows [68]:

- Map the interior of a hyperbolic equilateral triangle in \mathbb{D} conformally onto the interior of a Euclidean equilateral triangle
- By successive reflections create a universal covering map from \mathbb{D} onto \mathbb{C}/Λ_h , where Λ_h is the (shifted) hexagonal lattice generated by the Euclidean triangle with optimal covering radius 1.

Consider the following function as already described by Rademacher in [68]

$$\begin{aligned} \phi_\alpha: \mathbb{D} &\rightarrow \mathbb{C} \\ \phi_\alpha(z) &= z \cdot \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6} + \frac{\alpha}{2}\right)}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{6} + \frac{\alpha}{2}\right)} \frac{{}_2F_1\left(\frac{5}{6} - \frac{\alpha}{2}, \frac{1}{2} - \frac{\alpha}{2}, \frac{4}{3}; z^3\right)}{{}_2F_1\left(\frac{1}{6} - \frac{\alpha}{2}, \frac{1}{2} - \frac{\alpha}{2}, \frac{2}{3}; z^3\right)}. \end{aligned}$$

Here, ${}_2F_1(a, b, c; z)$ denotes Gauss' hypergeometric function. It is given by

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(q)_n$ denotes the rising Pochhammer symbol which is

$$(q)_n = \begin{cases} 1 & n = 0 \\ q(q+1) \cdots (q+n-1) & n > 0. \end{cases}$$

We note that an analytic continuation of ${}_2F_1$ is possible along any path in the complex plane, as long as the branch points 0 and 1 are avoided. For $z = 1$ we have the value

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}(c) > \operatorname{Re}(a+b)$$

The function ϕ_α maps the interior of the unit disc \mathbb{D} onto an equilateral, equiangular circular triangle of angles $\alpha\pi$, $0 \leq \alpha \leq 1$ (see Figure 14). We note the following properties of ϕ_α

$$\begin{aligned} \phi_\alpha(0) &= 0 \\ \phi_\alpha(1) &= 1 \\ \phi_\alpha(\zeta_3) &= \zeta_3 \\ \phi_\alpha(\zeta_3^2) &= \zeta_3^2 \end{aligned}$$

which hold for all $\alpha \in [0, 1]$. Here, $\zeta_3 = e^{2\pi i/3}$ and the formulas for $\alpha = 0$ hold in the limit case $\lim_{\alpha \rightarrow 0^+} \phi_\alpha(z) = \phi_0(z)$. Also, the points on the unit circle are mapped to the edges of the triangle. By using the inverse function for $\alpha = 0$

$$\phi_0^{-1}(z)$$

we introduce the following conformal mapping

$$f(z) = \phi_{\frac{1}{3}}(\phi_0^{-1}(z)).$$

This function maps a hyperbolic triangle, to be more precise the zero-angled, equilateral, circular triangle, onto the Euclidean equilateral triangle. By successive reflections (see Figure 15⁴) f extends to a universal covering map from \mathbb{D} onto \mathbb{C}/Λ_h . The largest disc contained in $f(\mathbb{D})$ has radius 1. We compute the derivative of f

$$f'(z) = \frac{\phi'_{\frac{1}{3}}(\phi_0^{-1}(z))}{\phi'_0(\phi_0^{-1}(z))}$$

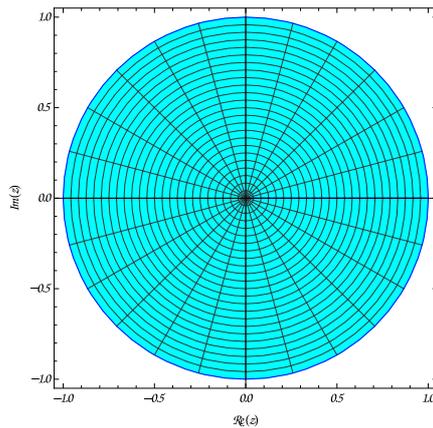
and evaluate at zero

$$f'(0) = \frac{\phi'_{\frac{1}{3}}(0)}{\phi'_0(0)} = \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{6})\Gamma(\frac{1}{3})}.$$

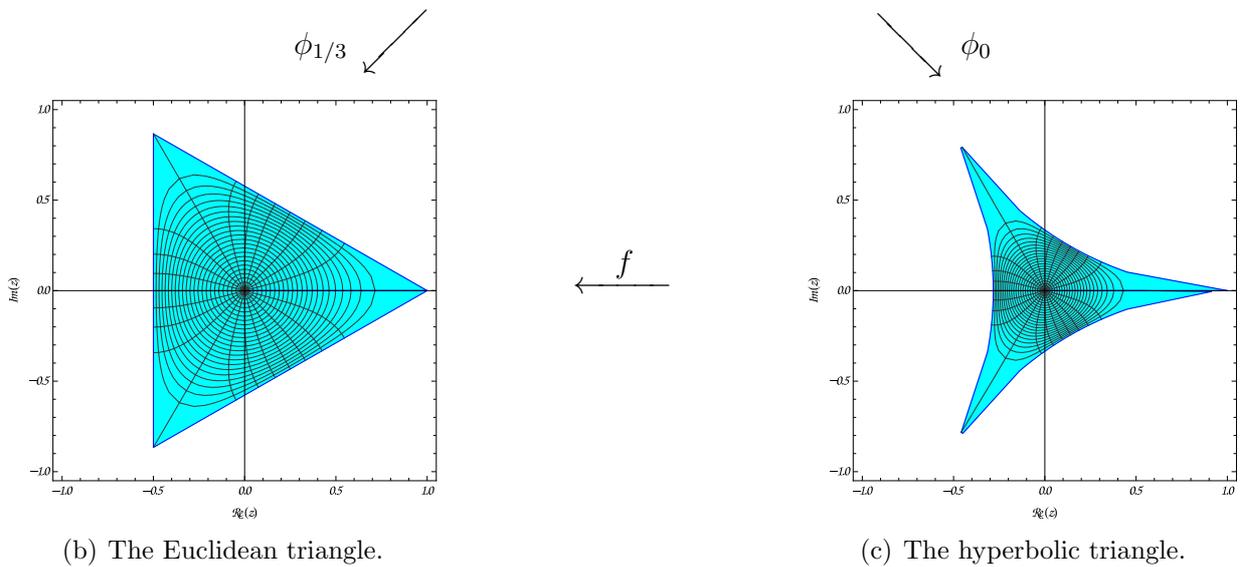
⁴The Mathematica code to produce the pictures of the hyperbolic triangles in Figure 15 was downloaded from the Wolfram Demonstrations Project [77].
demonstrations.wolfram.com/ReflectingARegularPolygonAcrossItsSidesInTheHyperbolicPlane

Since, we want to have $|f'(0)| = 1$, we scale f by the factor $\frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})}$ which now means that the largest disc contained in $f(\mathbb{D})$ has exactly radius $\frac{\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{6})}$.

We remark that Λ_h , obtained from f in the described way, is actually not a lattice since it does not contain the origin. However, a simple shift of Λ_h does not affect any of the results, since this would mean that we add a constant to the function $f(z)$ which does not change its derivative.



(a) The unit disc \mathbb{D} .

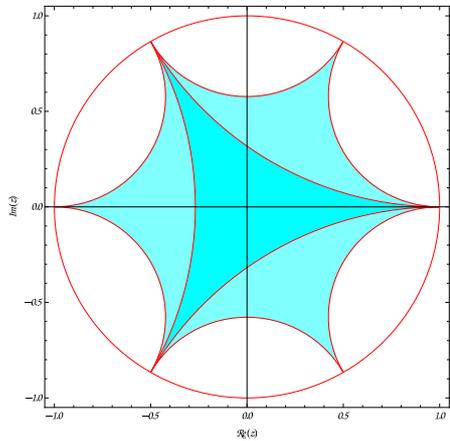


(b) The Euclidean triangle.

(c) The hyperbolic triangle.

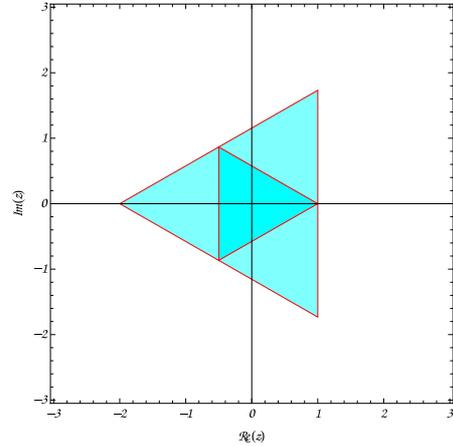
Figure 14:

Illustration of the conformal mappings ϕ_0 and $\phi_{1/3}$. The unit disc is mapped to the hyperbolic equilateral zero-angled triangle and to the Euclidean equilateral triangle. f maps the hyperbolic triangle onto the Euclidean triangle.

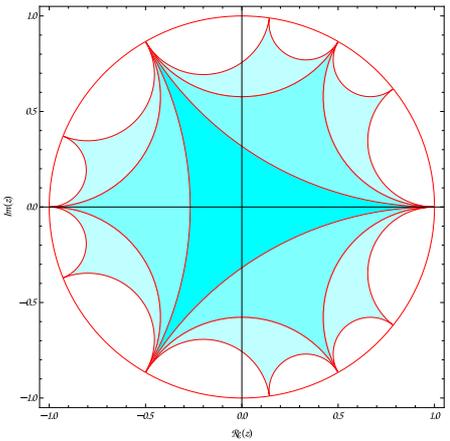


(a) First step of hyperbolic reflections in the unit disc.

f →

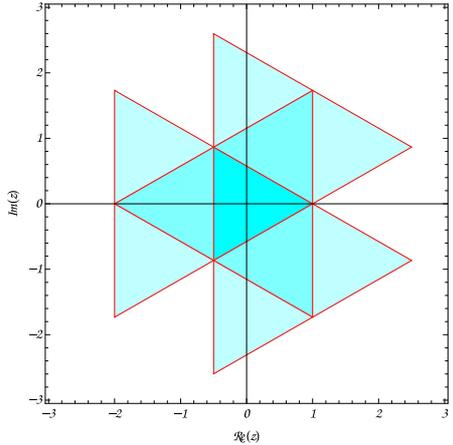


(b) First step of Euclidean reflections in the plane.



(c) Second step of hyperbolic reflections in the unit disc.

f →



(d) Second step of Euclidean reflections in the plane.

Figure 15: Successive reflections of the hyperbolic triangles along their edges provide a tessellation of the unit disc. We note that all hyperbolic triangles in the disc are disjoint from each other whereas in the plane two different triangles might be reflected to one and the same triangle after the second step of reflections. Therefore the map f is not biholomorphic.

We will now consider a mapping similar to ϕ_α above, namely

$$\Phi_\alpha : \mathbb{D} \rightarrow \mathbb{C}$$

$$\Phi_\alpha(z) = z \cdot \frac{\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{a}{2} + \frac{3}{4}\right) {}_2F_1\left(\frac{1}{2} - \frac{a}{2}, \frac{3}{4} - \frac{a}{2}; \frac{5}{4}; z^4\right)}{\Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{a}{2} + \frac{1}{4}\right) {}_2F_1\left(\frac{1}{4} - \frac{a}{2}, \frac{1}{2} - \frac{a}{2}; \frac{3}{4}; z^4\right)}$$

which maps the disc \mathbb{D} onto a circular quadrangle with angles $\alpha\pi$, $0 \leq \alpha \leq 1$ (see Figure 16).

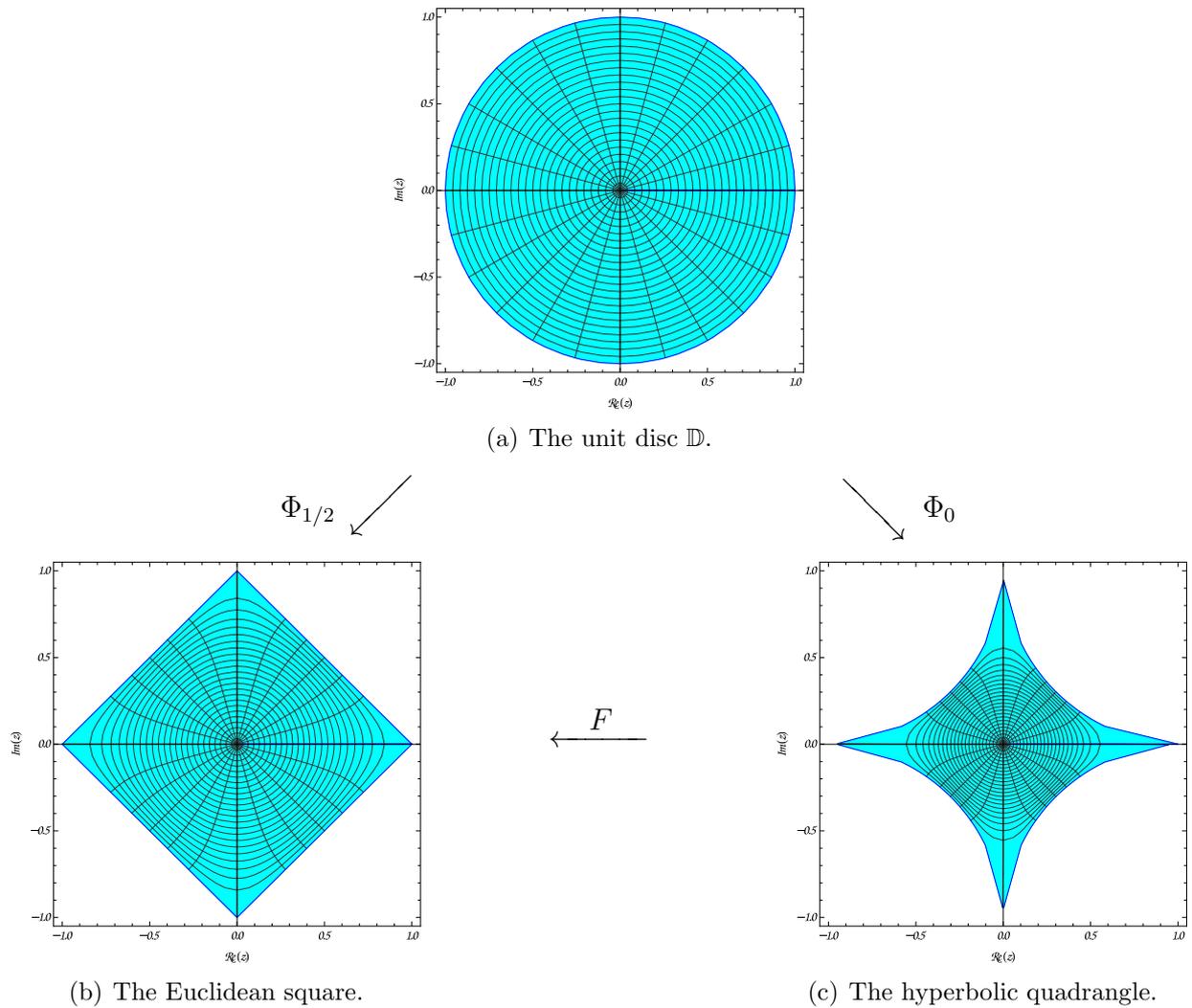
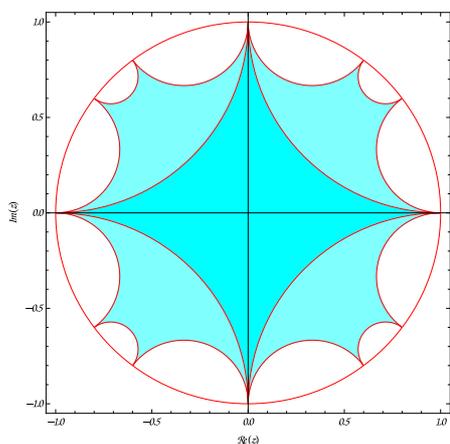


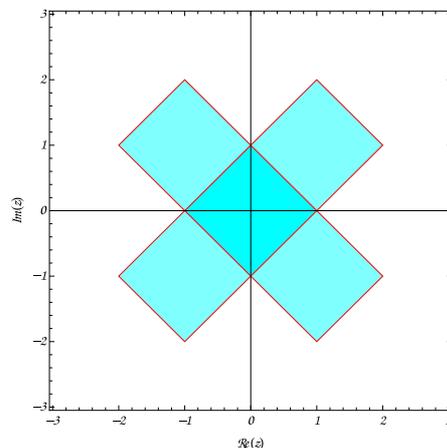
Figure 16:

Illustration of the conformal mappings Φ_0 and $\Phi_{1/2}$. The unit disc is mapped to the hyperbolic equilateral zero-angled quadrangle and to the Euclidean square. F maps the hyperbolic quadrangle onto the Euclidean square.

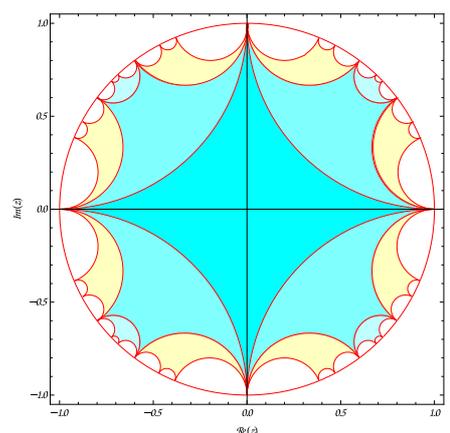


(a) First step of hyperbolic reflections in the unit disc.

f \rightarrow

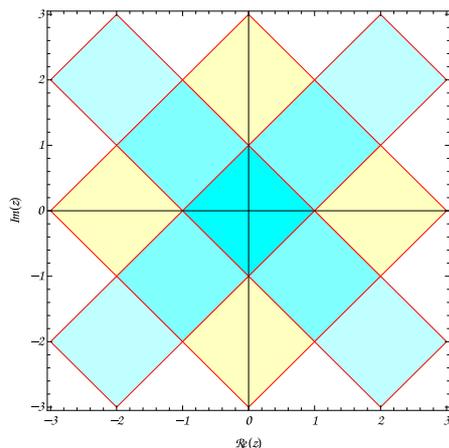


(b) First step of Euclidean reflections in the plane.



(c) Second step of hyperbolic reflections in the unit disc.

f \rightarrow



(d) Second step of Euclidean reflections in the plane.

Figure 17: Successive reflections of the circular quadrangles along their edges provide a tessellation of the unit disc. We note that all hyperbolic quadrangles in the disc are disjoint from each other whereas in the plane two different squares might be reflected to one and the same square after the first step of reflections (plotted yellow in step 2). Therefore the map F is not biholomorphic.

We note the following properties of Φ_α

$$\begin{aligned}\Phi_\alpha(0) &= 0 \\ \Phi_\alpha(1) &= 1 \\ \Phi_\alpha(i) &= i \\ \Phi_\alpha(i^2) &= i^2 \\ \Phi_\alpha(i^3) &= i^3\end{aligned}$$

hold for all $\alpha \in [0, 1]$. The formulas for $\alpha = 0$ hold in the limit case $\lim_{\alpha \rightarrow 0^+} \phi_\alpha(z) = \phi_0(z)$. Also, the points on the unit circle are mapped to border sides of the quadrangle. Hence, by successive reflections (see Figure 17 ⁵) the function

$$F(z) = \Phi_{\frac{1}{2}}(\Phi_0^{-1}(z))$$

extends to a universal covering map which maps \mathbb{D} onto $\mathbb{C}/\Lambda_\diamond$, where Λ_\diamond is a 45° rotated square lattice with covering radius 1, i.e. the largest disc contained in $F(\mathbb{D})$ has radius 1. The points Λ_\diamond are branch points of F with infinite order. Computing the derivative in zero gives

$$F'(0) = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}.$$

Again, we remark that Λ_\diamond is actually not a lattice, but, as before, adding the right constant (e.g. -1) to the function $F(z)$ shifts the point set such that one point is the origin. Multiplying $F(z)$ with $e^{\frac{\pi i}{4}}$ rotates the image $F(\mathbb{D})$ by 45° such that the branch points of F create the square lattice with covering radius 1.

7.2 Gabor Frame Bounds Revisited

We will investigate the lower frame bounds of the Gabor frames

$$\mathcal{G}(g_0, \Lambda_h^2) \quad \text{and} \quad \mathcal{G}(g_0, \Lambda_\square^2)$$

where Λ_h^2 is the hexagonal lattice of redundancy 2 and Λ_\square^2 is the square lattice of redundancy 2. From here on we will sometimes drop the superscript 2 which indicates that the lattice has redundancy 2, since we do not consider other redundancies. We already know that finding the lower frame bound of $\mathcal{G}(g_0, \Lambda)$, $\Lambda \in \mathcal{F}_{full}^2(g_0)$ is equivalent to the following optimization problem

$$A(\Lambda) = \inf_{(x,\omega) \in [0,1]^2} F_{g_0}(x, \omega) = 2 \sum_{\lambda \in \Lambda} \langle g_0, \pi(\lambda)g_0 \rangle e^{2\pi i(kx+l\omega)}.$$

⁵The Mathematica code to produce the pictures of the hyperbolic quadrangles in Figure 17 was downloaded from the Wolfram Demonstrations Project [77].

demonstrations.wolfram.com/ReflectingARegularPolygonAcrossItsSidesInTheHyperbolicPlane

Now, we define the following number

$$\mathcal{A} = \sup_{\Lambda \in \mathcal{F}_{full}^2(g_0)} A(\Lambda).$$

Having a look at the problem when only separable lattices are allowed, we define the following constant

$$\mathfrak{A} = \sup_{\Lambda \in \mathcal{F}_{(\alpha,\beta)}^2(g_0)} A(\Lambda).$$

As we already know from Section 5 (see also [24]), this value is given by

$$\mathfrak{A} = 2 \sum_{k,l \in \mathbb{Z}} e^{-2\pi \frac{1}{2}(k^2+l^2)} e^{2\pi i \frac{1}{2}(k+l)} = 2 \vartheta_4(1)^2 = 2 \left(\frac{\pi^{1/4}}{2^{1/4} \Gamma(\frac{3}{4})} \right)^2.$$

The last equality actually goes back to a classical result about Jacobi's theta-4 function and can also be found in [64]. We note the following curious property.

Proposition 7.4.

$$\mathfrak{A} = F'(0).$$

Proof. We already know that $\mathfrak{A} = 2 \left(\frac{\pi^{1/4}}{2^{1/4} \Gamma(\frac{3}{4})} \right)^2$ and that $F'(0) = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{4})}$. Therefore we have to show that

$$2 \frac{\pi^{1/2}}{2^{1/2} \Gamma(\frac{3}{4})^2} = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{4})}. \quad (7.1)$$

We use Euler's reflection formula

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}$$

for $z = \frac{1}{4}$. This gives

$$\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \sqrt{2} \pi.$$

Therefore, equation (7.1) reads

$$\frac{\sqrt{2\pi}}{\Gamma(\frac{3}{4})^2} = \frac{\sqrt{2} \pi}{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{4})^2}$$

and, since, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ the proof is complete. \square

For the general case, we conjecture that Λ being a hexagonal lattice always maximizes the lower frame bound for a Gaussian Gabor frame. Therefore, we conjecture that the value of \mathcal{A} defined above, is given by

$$\mathcal{A} = 2 \sum_{k,l \in \mathbb{Z}} e^{-2\pi \frac{1}{\sqrt{3}}(k^2 - kl + l^2)} e^{2\pi i \frac{1}{3}(k+l)}.$$

Furthermore, the numerical results from Section 6 suggest that the following might be true.

$$\mathcal{A} = \frac{1}{\mathcal{L}_+} = \frac{\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}.$$

Therefore, we end this work with the following conjecture.

Conjecture 7.5. *\mathcal{A} is the “Weltkonstante”*

$$\mathcal{A} = \frac{1}{\mathcal{L}}.$$

A On Two Products of Jacobi's Theta Functions

We start with recalling Jacobi's theta functions. Also, for the special arguments $z = 0$ and $z = \frac{1}{2}$ we present alternative proofs to Theorem 5.11 and Theorem 5.16 in Section 5 but without the additional parameter r (i.e. $r = 1$). For Gabor frames with standard Gaussian window and rectangular lattices of redundancy 2, the results imply the optimality results for the frame bounds using the square lattice. The proofs are very different in style compared to the proofs in Section 5.

Recall that

$$\begin{aligned}\theta_3(s) &= \sum_{k \in \mathbb{Z}} e^{-\pi k^2 s} \\ \theta_4(s) &= \sum_{k \in \mathbb{Z}} (-1)^k e^{-\pi k^2 s}\end{aligned}$$

with $s \in \mathbb{R}_+$. Since the proof of the upcoming lemma relies heavily on it, we recall the Jacobi identity

$$\theta_3(s) = s^{-1/2} \theta_3(1/s), \quad (\text{A.1})$$

which follows from the Poisson summation formula.

Lemma A.1. *For $s \in \mathbb{R}_+$ we have*

$$\theta_3(s)\theta_3(1/s) \geq \theta_3(1)^2$$

with equality only for $s = 1$.

Proof. We want to find the critical points of $\theta_3(s) \cdot \theta_3(1/s)$. Hence, we need to find $s \in \mathbb{R}_+$ such that

$$(\theta_3(s) \cdot \theta_3(1/s))' = 0.$$

This is equivalent to

$$\theta_3'(s) \cdot \theta_3(1/s) - \theta_3(s) \cdot \theta_3'(1/s) s^{-2} = 0. \quad (\text{A.2})$$

From (A.2) we see that $\theta_3(s) \cdot \theta_3(1/s)$ has a critical point at $s = 1$. Also, equation (A.2) is equivalent to

$$\frac{\theta_3(s)}{\theta_3(1/s)} \frac{\theta_3'(1/s)}{\theta_3'(s)} = s^2,$$

which by (A.1) is equivalent to

$$\frac{\theta_3'(1/s)}{\theta_3'(s)} = s^{5/2}. \quad (\text{A.3})$$

We will now use (A.1) again to compute

$$\begin{aligned}\theta_3'(s) &= (s^{-1/2} \theta_3(1/s))' \\ &= -\frac{1}{2} s^{-3/2} \theta_3(1/s) - s^{-5/2} \theta_3'(1/s).\end{aligned}$$

This can be reformulated as

$$\theta_3'(1/s) = \left(-\frac{1}{2}s^{-3/2} \theta_3(1/s) - \theta_3'(s) \right) s^{5/2}. \quad (\text{A.4})$$

Plugging (A.4) into (A.3) and using (A.1) again yields

$$\frac{-\frac{1}{2}s^{-1} \theta_3(s) - \theta_3'(s)}{\theta_3'(s)} = 1,$$

which leads to the differential equation

$$\theta_3'(s) = -\frac{1}{4}s^{-1} \theta_3(s). \quad (\text{A.5})$$

The solution to (A.5) is given by

$$\theta_3(s) = c s^{-1/4}, \quad (\text{A.6})$$

where $c = \theta_3(1)$. So, we have to check that $\theta_3(s) \neq \theta_3(1) s^{-1/4}$ except for $s = 1$. For this purpose we define the function $f(s) = \theta_3(1) s^{-1/4}$ and want to find $s \in \mathbb{R}_+$ such that $\theta_3(s) = f(s)$. To make the analysis easier, we define the functions

$$\tilde{\theta}_3(s) = \theta_3(\exp(s))$$

and

$$\begin{aligned} \tilde{f}(s) &= f(\exp(s)) \\ &= \theta_3(1) e^{-s/4} \\ &= \tilde{\theta}_3(0) e^{-s/4}. \end{aligned}$$

Instead of comparing $\theta_3(s) \cdot \theta_3(1/s)$ with $f(s) \cdot f(1/s) \equiv \theta_3(1)^2$ on \mathbb{R}_+ we compare the symmetric products $\tilde{\theta}_3(s) \cdot \tilde{\theta}_3(-s)$ and $\tilde{f}(s) \cdot \tilde{f}(-s) \equiv \tilde{\theta}_3(0)^2$ on the whole real line. The function $\tilde{\theta}_3(s)$ is positive, strictly decreasing and

$$\lim_{s \rightarrow \infty} \tilde{\theta}_3(s) = 1.$$

Also, the function $\tilde{f}(s)$ is positive, strictly decreasing and

$$\lim_{s \rightarrow \infty} \tilde{f}(s) = 0.$$

We will make use of the mentioned properties of $\tilde{\theta}_3$ and \tilde{f} as well as of the properties of their derivatives. We compute

$$\tilde{\theta}_3'(s) = -\pi e^s \sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2 e^s} \quad (\text{A.7})$$

and

$$\tilde{f}'(s) = -\frac{1}{4}\tilde{\theta}_3(0)e^{-s/4}. \quad (\text{A.8})$$

From equations (A.7) and (A.8) and setting $s = 0$ we conclude that

$$4\pi = \frac{\tilde{\theta}_3(0)}{\sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2}}. \quad (\text{A.9})$$

We will now compare $\tilde{\theta}_3'$ with \tilde{f}' .

$$\tilde{\theta}_3'(s) = \tilde{f}'(s) \quad (\text{A.10})$$

$$\iff -\pi e^s \sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2 e^s} = -\frac{1}{4}\tilde{\theta}_3(0)e^{-s/4} \quad (\text{A.11})$$

$$\iff 4\pi \sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2 (e^s + 5s/4)} = \tilde{\theta}_3(0) \quad (\text{A.12})$$

$$\stackrel{(\text{A.9})}{\iff} \sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2 (e^s + 5s/4)} = \sum_{k \in \mathbb{Z}} k^2 e^{-\pi k^2} \quad (\text{A.13})$$

We consider 3 cases.

Case 1. Let $s = 0$. From the equivalence of (A.10) and (A.13) we see immediately that $\tilde{\theta}_3'(0) = \tilde{f}'(0)$. Also, we have $\tilde{\theta}_3(0) \cdot \tilde{\theta}_3(0) = \tilde{f}(0) \cdot \tilde{f}(0)$, which is certainly true by the definitions of the functions.

Case 2. Let $s < 0$. Then, equation (A.13) becomes an inequality and the left-hand side of (A.13) is bigger than the right-hand side. This is due to the fact that

$$e^s + 5s/4 < 1 \quad \text{for } s < 0.$$

By the equivalence of (A.10) and (A.13) this is equivalent to

$$\tilde{\theta}_3'(s) < \tilde{f}'(s) \quad \text{for } s < 0.$$

Case 3. Let $s > 0$. With similar arguments as for the case $s < 0$, we conclude that

$$\tilde{\theta}_3'(s) > \tilde{f}'(s) \quad \text{for } s > 0.$$

Recall that both functions, $\tilde{\theta}_3$ and \tilde{f} , are strictly decreasing and by equation (A.5) or (A.13) they touch at $s = 0$. Using the results from the case analysis, we are able to conclude that $\tilde{\theta}_3(s) \geq \tilde{f}(s)$ with equality if and only if $s = 0$. In the same way $\tilde{\theta}_3(-s) \geq \tilde{f}(-s)$ holds with equality if and only if $s = 0$. Hence, we finally get

$$\tilde{\theta}_3(s) \cdot \tilde{\theta}_3(-s) \geq \tilde{f}(s) \cdot \tilde{f}(-s) \equiv \tilde{\theta}_3(0)^2$$

for all $s \in \mathbb{R}$ with equality if and only if $s = 0$. Therefore, $\tilde{\theta}_3(s) \cdot \tilde{\theta}_3(-s)$ takes its global minimum at $s = 0$ and consequently, $\theta_3(s) \cdot \theta_3(1/s)$ takes its global minimum only for $s = 1$. \square

Lemma A.2. For $s \in \mathbb{R}_+$ we have

$$\theta_4(s)\theta_4(1/s) \leq \theta_4(1)^2$$

with equality only for $s = 1$.

Proof. First we rewrite the sum over the natural numbers

$$\theta_4(s) = 1 + 2 \sum_{k \geq 1} (-1)^k e^{-\pi k^2 s}. \quad (\text{A.14})$$

We will show that outside a neighborhood of 1 the product $\phi(s) = \theta_4(s)\theta_4(1/s)$ is smaller than some threshold and that inside this neighborhood we only find one critical point, which is a local maximum and hence, the global maximum. For this purpose we compute the derivatives up to order 3.

$$\phi'(s) = \theta_4\left(\frac{1}{s}\right)\theta_4'(s) - \frac{\theta_4(s)\theta_4'\left(\frac{1}{s}\right)}{s^2} \quad (\text{A.15})$$

$$\begin{aligned} \phi''(s) &= \frac{\theta_4(s)\theta_4''\left(\frac{1}{s}\right)}{s^4} + \frac{2\theta_4(s)\theta_4'\left(\frac{1}{s}\right)}{s^3} \\ &\quad - \frac{2\theta_4'(s)\theta_4'\left(\frac{1}{s}\right)}{s^2} + \theta_4\left(\frac{1}{s}\right)\theta_4''(s) \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} \phi'''(s) &= -\frac{\theta_4(s)\theta_4'''\left(\frac{1}{s}\right)}{s^6} - \frac{6\theta_4(s)\theta_4''\left(\frac{1}{s}\right)}{s^5} - \frac{6\theta_4(s)\theta_4'\left(\frac{1}{s}\right)}{s^4} \\ &\quad + \frac{3\theta_4'(s)\theta_4''\left(\frac{1}{s}\right)}{s^4} + \frac{6\theta_4'(s)\theta_4'\left(\frac{1}{s}\right)}{s^3} - \frac{3\theta_4'\left(\frac{1}{s}\right)\theta_4''(s)}{s^2} \\ &\quad + \theta_4\left(\frac{1}{s}\right)\theta_4'''(s) \end{aligned} \quad (\text{A.17})$$

In order to get estimates for $\phi^{(n)}$ in a neighborhood of 1 it is sufficient to have estimates for $\theta_4^{(m)}$ where $m = 0, \dots, n$ in this neighborhood. We compute

$$\theta_4'(s) = +2\pi \sum_{k \geq 1} (-1)^{k+1} k^2 e^{-\pi k^2 s} \quad (\text{A.18})$$

$$\theta_4''(s) = -2\pi^2 \sum_{k \geq 1} (-1)^{k+1} k^4 e^{-\pi k^2 s} \quad (\text{A.19})$$

$$\theta_4'''(s) = +2\pi^3 \sum_{k \geq 1} (-1)^{k+1} k^6 e^{-\pi k^2 s}. \quad (\text{A.20})$$

Next, we want to estimate $\theta_4^{(m)}(s)$, $m = 1, 2, 3$ in a neighborhood of 1. As the sums in (A.18), (A.19) and (A.20) are absolutely convergent we compare two successive terms and show that they are monotonically decreasing for s large enough.

Let $k > 1$, $r > 0$, then there exists a value s_r such that

$$k^r e^{-\pi k^2 s} \leq (k-1)^r e^{-\pi(k-1)^2 s} \quad (\text{A.21})$$

for all $s \geq s_r = \frac{r}{3\pi}$. We start with the observation that the Inequality (A.21) is equivalent to

$$\left(1 + \frac{1}{k-1}\right)^r \leq e^{\pi(k^2 - (k-1)^2)s}$$

which is equivalent to

$$\underbrace{\left(\left(1 + \frac{1}{k-1}\right)^{(k-1)}\right)^{\frac{r}{(k-1)}}}_{\leq e^{\frac{r}{k-1}}} \leq e^{\pi(2k-1)s}.$$

Therefore, we compare the exponents

$$\frac{r}{k-1} \leq (2k-1)\pi s$$

which is equivalent to

$$\frac{r}{(k-1)(2k-1)} \leq \pi s.$$

The last statement is certainly true for $k > 1$, $r > 0$ and $s > \frac{r}{3\pi}$. Using equation (A.21) and the fact that

$$0 < \theta_{4,2}'''(s) = +2\pi^3 \sum_{k=1}^2 (-1)^{k+1} k^6 e^{-\pi k^2 s} \leq \theta_4'''(s), \quad s \geq \frac{2}{\pi}$$

we conclude that $\theta_4'''(s) > 0$ and decreasing on the interval $[\frac{2}{\pi}, \infty)$. Using (A.21) and the fact that the sum is alternating we conclude that

$$|\theta_4'''(s)| \leq \frac{2\pi^3}{e^2} = 8.39249\dots \quad (\text{A.22})$$

on the interval $[\frac{2}{\pi}, \infty)$. As we want to estimate $\phi'''(s)$ via equation (A.20) we also need estimates on the lower order derivatives of $\theta_4(s)$ and on $\theta_4(s)$ itself. By using the Inequality A.21 on $\theta_4''(s)$ we conclude that $\theta_4''(s) < 0$ and increasing for $s \in [\frac{4}{3\pi}, \infty) \supset [\frac{2}{\pi}, \infty)$. Also, we conclude that

$$|\theta_4''(s)| \leq \frac{2\pi^2}{e^2} = 2.67141\dots \quad (\text{A.23})$$

on the interval $[\frac{2}{\pi}, \infty)$. In the same manner, we get that $\theta_4'(s) > 0$ and decreasing for $s \in [\frac{2}{3\pi}, \infty) \supset [\frac{2}{\pi}, \infty)$. Hence, we have

$$|\theta_4'(s)| \leq \frac{2\pi}{e^2} = 0.850337\dots \quad (\text{A.24})$$

on the interval $[\frac{2}{\pi}, \infty)$. At last, we conclude that $0 \leq \theta_4(s) \leq 1$ and increasing, since it is a cumulative distribution function on \mathbb{R}_+ , in particular

$$|\theta_4(s)| \leq 1 \quad (\text{A.25})$$

on \mathbb{R}_+ . Since, estimates on $|\phi^{(k)}(s)|$ involve estimates on the derivatives of our function θ_4 with argument s as well as with $\frac{1}{s}$, we cannot estimate $|\phi^{(k)}|$ on the the interval $[\frac{2}{\pi}, \infty)$, but only on the interval $[\frac{2}{\pi}, \frac{\pi}{2}]$. We use Equations (A.22), (A.23), (A.24) and (A.25) to get an estimate on $\phi'''(s)$ on the interval $[\frac{2}{\pi}, \frac{\pi}{2}]$. We plug in all those estimates into equation (A.17) and estimate all negative terms by 0 so we get an estimate from above.

$$\begin{aligned} \phi'''(s) &\leq 0 + 6 \left(\frac{\pi}{2}\right)^5 \frac{2\pi^2}{e^2} + 0 \\ &+ 0 + 6 \left(\frac{\pi}{2}\right)^3 \frac{2\pi}{e^2} \frac{2\pi}{e^2} + 3 \left(\frac{\pi}{2}\right)^2 \frac{2\pi}{e^2} \frac{2\pi^2}{e^2} \\ &+ \frac{2\pi^3}{e^2} \\ &= \frac{2\pi^3}{e^2} + \frac{6\pi^5}{e^4} + \frac{3\pi^7}{8e^2} \\ &= K_+ = 195.304\dots \end{aligned} \quad (\text{A.26})$$

In the same manner we estimate positive terms by 0 to get an estimate from below.

$$\begin{aligned} \phi'''(s) &\geq - \left(\frac{\pi}{2}\right)^6 \frac{2\pi^3}{e^2} + 0 - 6 \left(\frac{\pi}{2}\right)^4 \frac{2\pi}{e^2} \\ &- 3 \left(\frac{\pi}{2}\right)^4 \frac{2\pi}{e^2} \frac{2\pi^2}{e^2} + 0 + 0 \\ &+ 0 \\ &= - \frac{3\pi^5}{4e^2} - \frac{3\pi^7}{4e^4} - \frac{\pi^9}{32e^2} \\ &= K_- = -198.61986\dots \end{aligned} \quad (\text{A.27})$$

Since $|K_+| < |K_-|$ we get the following estimate for $|\phi'''(s)|$

$$|\phi'''(s)| \leq |K_-| = K = 198.61986\dots \quad (\text{A.28})$$

As a next step we want to show that $\phi''(1) < 0$ by using the information we have about $\phi'''(s)$. First, we write

$$\phi''(s) - \phi''(1) = \int_1^s \phi'''(u) du.$$

This implies that

$$\phi''(s) = \int_1^s \phi'''(u) du + \phi''(1).$$

We will now estimate how large $\phi''(1)$ can be at most. Therefore, we observe that $\theta_4''(s)$ is always less or equal to a truncation of the series after any even term, in particular

$$\theta_4''(s) \leq 2\pi^2(-e^{-\pi s} + 16e^{-4\pi s}) = \theta_{4,2}''(s)$$

and

$$\theta_4''(1) \leq 2\pi^2(-e^{-\pi} + 16e^{-4\pi}) = \theta_{4,2}''(1) = -0.851907\dots$$

In the same manner we can estimate $\theta_4'(s)$ by a truncation of the series after any odd term, in particular

$$\theta_4'(s) \leq 2\pi e^{-\pi s} = \theta_{4,1}'(s)$$

and

$$\theta_4'(1) \leq 2\pi e^{-\pi} = \theta_{4,1}'(1) = 0.271521\dots$$

According to equation (A.16) we know that

$$\begin{aligned} \phi''(1) &= 2(\theta_4(1)\theta_4''(1) + \theta_4(1)\theta_4'(1) - \theta_4'(1)^2) \\ &\leq 2\theta_4(1)(\theta_{4,2}''(1) + \theta_{4,1}'(1)) \\ &= -1.16077\theta_4(1). \end{aligned}$$

The value of $\theta_4(1) = \frac{\pi^{1/4}}{2^{1/4}\Gamma(\frac{3}{4})} = 0.913579\dots$ where $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s} dt$ is the Euler gamma function. Collecting all the information we have, we see that

$$\phi''(1) \leq \tilde{K} = -1.060457\dots \quad (\text{A.29})$$

and consequently

$$\phi''(s) = \int_1^s \phi'''(u) du + \phi''(1) \leq K|s-1| + \tilde{K}$$

where K and \tilde{K} are defined as in (A.28) and (A.29) respectively. Finally, we found the desired neighborhood of 1

$$|s-1| < -\frac{\tilde{K}}{K} = 0.00533913\dots$$

In particular, this means that in the smaller interval $I_0 = [1/s_0, s_0] = [\frac{1}{1.005}, 1.005]$ the function $\phi(s)$ has a local maximum in $s = 1$.

Further on, $\phi(1.005) = \phi(\frac{1}{1.005})$ and outside this region we can dominate $\phi(s)$ by the product $\theta_{4,2}(s)\theta_{4,2}(1/s)$ where $\theta_{4,2}(s) = 1 - 2e^{-\pi s} + 2e^{-4\pi s}$ is the series from equation (A.14) truncated after $k = 2$. On the interval $[1, s_0]$ we know that $\phi(s)$ takes its maximum at $s = 1$ and is decreasing for $s \geq 1$. Also, we calculate that $\theta_{4,2}(s_0)\theta_{4,2}(1/s_0) = 0.8346118\dots < 0.834627\dots = \theta_4(1)^2$ and hence

$$\phi(1) = \theta_4(1)^2 > \theta_{4,2}(s_0)\theta_{4,2}(1/s_0) > \phi(s_0).$$

By the mean value theorem there exists some $\xi \in (1, s_0)$ such that $\phi(\xi) = \theta_{4,2}(s_0)\theta_{4,2}(1/s_0)$. For $s \geq 1$, we define the function

$$g(s) = \begin{cases} \theta_4(s)\theta_4(1/s), & 1 \leq s \leq \xi \\ \theta_{4,2}(s_0)\theta_{4,2}(1/s_0) & \xi < s \leq s_0 \\ \theta_{4,2}(s)\theta_{4,2}(1/s), & s_0 < s \leq \frac{3\pi}{\log(4)} \\ \theta_4\left(\frac{\log(4)}{3\pi}\right), & \frac{3\pi}{\log(4)} < s \end{cases}$$

On the interval $\left[1, \frac{3\pi}{\log(4)}\right]$ we have $g(s) \geq \phi(s)$ and g is decreasing. And at $s = \frac{3\pi}{\log(4)}$ we have $1 - \frac{3}{2^{5/3}} = \theta_{4,2}\left(\frac{\log(4)}{3\pi}\right) = 0.0550592\dots$. Since, $\theta_{4,2}(s) \geq \theta_4(s)$ this shows that $g \geq \theta_4$ on $[1, \infty)$. By the symmetry $\phi(s) = \phi(1/s)$ we finally get that $\phi(s)$ takes its global maximum at $s = 1$. \square

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Deutsche Zusammenfassung

Die vorliegende Arbeit behandelt ein Thema im Bereich von Gabor Frames welche zum Gebiet der Zeit-Frequenz Analyse gehören. Der Fokus wird auf die Untersuchung von scharfen Schranken von Gabor Frames mit Gauß Fenster gerichtet und wie sich diese Schranken unter Deformation des zugehörigen Gitters verhalten. Die Forschung bezüglich der Deformation von Gabor Systemen ist weithin bekannt, gleichzeitig gibt es noch kein allgemeines Verständnis dafür, wann ein Gabor System auch tatsächlich einen Frame bildet.

Möchte man das Verhalten eines Gabor Frames mit einem bestimmten Fenster untersuchen, stößt man schnell auf den Begriff des Frame Sets. Das Frame Set eines bestimmten Fensters beschreibt welche Gitter einen Frame liefern. Fordert man eine bestimmte Glattheit und ein bestimmtes Abklingverhalten von einem Fenster, so weiß man, dass das Frame Set eine offene Menge ist (was im Allgemeinen nicht wahr ist). Die Bestimmung des Frame Sets gestaltet sich im Allgemeinen als sehr schwierig und man kennt kein allgemeines Konzept mit welchem man das Frame Set für eine Klasse von Funktionen oder selbst einer einzelnen Funktion bestimmen kann.

Die erste und bislang einzige Funktion für welche man das gesamte Frame Set kennt, ist die Gauß Funktion. In diesem Fall sind die notwendigen Bedingungen welche sich aus dem Balian-Low Theorem beziehungsweise dem Dichtheitsatz ergeben schon hinreichend. Im Gegensatz zu dem Resultat für die Gauß Funktion wurden erst kürzlich Vermutungen über eine ähnlich einfache Struktur des Frame Sets von Hermite Funktionen höherer Ordnung durch Gegenbeispiele widerlegt.

Es mag den Anschein erwecken, dass wir viel über Gabor Frames mit einem Gauß Fenster wissen, dennoch gibt es auch in diesem Fall noch genügend offene Fragen, welche es wert sind, untersucht zu werden. Ein offenes Problem ist es, das Verhalten der Frame Schranken in Abhängigkeit des Frame Sets zu beschreiben. Konkret stellt sich die Frage, ob es ein einziges optimales Gitter gibt, welches die Schranken eines Gabor Frames mit Gauß Fenster optimiert. An dieser Stelle weisen wir darauf hin, dass diese Fragestellung erst unter der Einschränkung sinnvoll wird, dass die betrachteten Gitter alle dieselbe Dichte haben.

Beschränkt man sich auf den Fall von Rechtecksgittern (welche auch separabel genannt werden) mit geradzahlig Dichte, so werden wir Beweise sehen, dass das quadratische Gitter sowohl die untere als auch die obere Frame Schranke optimiert. Für allgemeine Gitter geradzahlig Dichte, werden wir einen Beweis führen welcher zeigt, dass das hexagonale Gitter die obere Frame Schranke minimiert. Für den Beweis ist es notwendig Theta Funktionen über einem Gitter zu betrachten. Im Fall von Rechtecksgittern zerfallen diese Theta Funktionen in Produkte der klassischen Jacobi Theta Funktionen und die erwähnten Resultate für die Frame Schranken folgen durch in dieser Arbeit gezeigte, neue Eigenschaften der Jacobi Theta Funktionen.

