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Abstract

This master's thesis constitutes a study of *Linear Rational Term Structure Models* by Filipović, Larsson and Trolle (2016) [14] who present a term structure model under which an explicit formula for swaptions pricing exists. Additional important features of this term structure model are that with the right choice of parameters certain bounds, e.g. non-negativity of the interest rate can be ensured and it allows for unspanned stochastic volatility.

The beginning of this study provides an overview of interest rates and financial instruments such as bonds, swaps and options on swaps as well as the concept of term structure modelling and risk neutrality. We present the paper by Filipović et al. (2016) [14] in a self contained way. Eventually, the parameters of the explicit swaption pricing formula are calibrated with swaption prices derived with Black's formula from volatility data.

Diese Masterarbeit untersucht das Paper "Linear Rational Term Structure Models" von Filipović, Larsson and Trolle (2016) [14], in welchem ein Modell für die Zinsstruktur präsentiert wird und eine explizite Formel für Swaption Bepreisung bietet. Zusätzliche wichtige Eigenschaften dieses Modells sind, dass mit der richtigen Wahl von Parametern nicht-negative Zinsen garantiert werden können und "unspanned" stochastische Volatilität zulässt.

Der Anfang dieser Arbeit bietet einen Überblick über Zinsen und Finanzinstrumente wie Anleihen, Swaps und Optionen auf Swaps, sowie die Vorgehensweise beim Modellieren von Zinsstrukturen und das Konzept von Risikoneutralität. Weiters wird das Paper von Filipović et al. (2016) [14] zusammengefasst, sodass es nicht notwendigerweise herangezogen werden muss. Schlussendlich werden Swaption-Marktpreise mit Hilfe von Black's Formel aus Marktvolatilitäten berechnet und die Parameter der Swaption Bepreisungsformel damit kalibriert.

Contents

List of Tables

List of Figures

Introduction

In the financial market options on interest rate swaps have become an important tool to protect against fluctuations of interest rates. In an interest rate swap floating and fixed interest rates are exchanged such that one party of the swap pays floating interest rate to the other, whereas the other one pays a fixed interest rate.

Hull (2012) [17, p.660] provides a good example on how a swaption can be used: A company knowing that they will enter a loan agreement in 6 months with a floating rate for 5 years would want to enter into a payer swaption to ensure against high floating rates. That is, they enter a swaption for a 5-year period starting in 6 months giving the right but not the obligation to pay a certain fixed rate of interest and receive floating to pay the loan. If the floating rate turns out to be less than the fixed rate the company will not exercise. On the other hand, if the received floating rate is higher than the fixed rate of the swap the company will exercise and make use of good terms of the underlying swap. That is, the company is able to convert the floating-rate loan into a fixed-rate loan by entering a swaption at a predefined cost.

Swaptions are typically traded over-the-counter, that is, they are traded directly between two parties without the supervision of an exchange. The two parties, the buyer and the seller, agree on the price of the swaption, the length of the option period and the terms of the underlying swap. The swap then contains the notional amount, the fixed rate and the frequency of observation for the floating leg of the swap, usually LIBOR. Since a swap depends on future interest rates which are commonly unknown the pricing of a swaption can become very difficult. Interest rates are random and future changes are not observable which raises the issue to find a model which describes the term structure of interest rates in a appropriate way. Generally, for a fair pricing of financial instruments which depend on interest rates a suitable term structure is desired as well.

There are many different approaches in modelling the term structure of interest rate and not all are suitable for every problem. The popular exponential-affine framework (Cuchiero, Filipović and Teichmann, 2010 [9]) either only allows for unspanned stochastic volatility or can ensure non-negative interest rates but not both (Filipović et al., 2016, p.5 [14]). There have been a few studies about unspanned stochastic volatility (USV) suggesting that changes in swap rates can only partially describe changes in straddle returns (Collin-Dufresne and Goldstein, 2002, p.2 [6]) which rises the requirement of incorporating USV in term structure models. On the other hand, the assumption of non-negative interest rates has been challenged as the financial market continues to experience interest rates below zero. Nonetheless it is common to ensure non-negativity for interest rate in term structure models.

The main issue of this thesis is to find an appropriate term structure model for describing interest rates in which analytical swaption pricing is possible. The linear-rational term structure model by Filipović et al. (2016) [14] meets all of those requirements. Specifically, by choosing the parameters appropriately bounds for the interest rate can be defined. Additionally, USV can be incorporated and contrary to the exponential-affine framework the linear-rational term structure model also accommodates an analytical pricing formula for swaptions. These properties make the linear-rational term structure model particularly interesting and suitable for the topic of swaption pricing.

This thesis is organised as follows.

The first chapter of this thesis provides an introduction of the notion of different interest rates as well as the definition of zero-coupon bonds. This should help to keep this thesis consistent in itself and make the following calculations easier to understand. General mathematical procedures, e.g. the Itô Integral, are assumed to be known to the reader.

Further, the second chapter addresses the theory of arbitrage and risk-neutral pricing in financial markets. These are common assumptions since in an efficient market prices converge to arbitrage-free prices due to demand and supply effects. No arbitrage then leads to the notion of a state price density which can be understood as a stochastic discount factor. It takes over the role of the discounting term under the risk-neutral measure and its existence is sufficient to achieve an arbitrage-free model under the real probability measure. We also show how the price of a zero-coupon bond can be calculated when the state price density is known.

The next part of the theoretical overview constitutes the third chapter about swaps and swaptions which are important financial instruments. Since the pricing of swaptions is the main focus of the underlying paper by Filipović et al. (2016) [14], we draw the connection between zero-coupon bonds, the state price density and the swaption price. This connection is later used for the analytical swaption pricing formula. We also mention Black's formula for swaption prices which depends only on zero-coupon bond prices and implied volatilities. With Black's formula the swap rate is assumed lognormal which contradicts market observations. Nonetheless we will use this formula to compute swaption prices by using a data set of lognormal volatilities. From this we receive market prices for swaptions to calibrate relevant parameters of the swaption pricing formula.

The fourth chapter discusses how the term structure can be modelled and how unspanned stochastic volatility can be accommodated. Term structures can be defined directly through the dynamics of the interest rates or by using the prior introduced state price density. This lays the theoretical foundation on how Filipović et al. (2016) [14] defined the linear-rational term structure model while incorporating USV, no arbitrage and risk-neutrality.

The main part of this thesis constitutes the fifth chapter in which we study the linear-rational term structure model by Filipović et al. (2016) [14]. With this model bond prices become linear-rational functions of the underlying factors and an analytic formula for swaption prices is presented. Also USV factors can be accommodated and it can be ensured that the interest rate stays non-negative. This last feature is challenged in the current state of the financial market where we experience negative interest rates but it is still a common assumption for financial models. Within this chapter we also discuss a specification of the linear-rational term structure model in detail while incorporating a few assumptions. Eventually we will arrive at an explicit formula for pricing swaptions under the so-called linear-rational square-root model.

This formula is then used in the sixth and last chapter where an empirical analysis with swaption data is performed. Under a few simplifications we retrieve market swaption prices by using data of implied lognormal volatilities and Black's model for swaption prices. To do so we need corresponding zero-coupon bond prices which we calculate within in the linear-rational model by fixing a few parameters. Eventually we can calibrate the remaining parameters from the swaption formula of the model by using calculated market prices. All simplifications and restrictions as well as the used formulas and a short discussion on the outcome of the empirical analysis are shown is this chapter.

The calibration of the parameters is performed in R by the use of the function *nls* for non-linear least squares, and shows a very good fit to market observations. Even though we are using not real market data on swaption prices but calculated prices with volatility data we arrive at reasonable prices. This leads us to the belief that even with the implemented simplification and restrictions the model formula can provide adequate swaption prices. Nonetheless this study leaves a lot of starting points for further discussions, for example how the regression would look like without these restrictions or even when all parameters would be calibrated. However, even under restrictions this model allows for a calibration of parameters and retrieves useful values which is the aim of this study.

While Chapter 5 follows the underlying paper by Filipović, Larsson and Trolle (2016) [14], the theoretical chapters mainly use Björk (1996) [1], Filipović (2009) [13], as well as Collin-Dufresne and Goldstein (2002) [6] and Rogers (1997) [20].

1 Rates and Bonds

This first chapter will start with the main definitions of interest rates and bonds which will be used throughout this thesis. We are all acquainted with the notion of interest rates as everybody expects the amount of money at the bank to grow and to be rewarded when lending money. However, in current times when we experience negative interest rates this thought is challenged and lending becomes expensive. This should be kept in mind while we continue to assume non-negative interest rates for the rest of the thesis to not make the study more complicated.

Receiving an amount of money today is not equivalent to receiving the same amount tomorrow. Hence, money has a time value and a mathematical expression of such concepts is needed.

This chapter follows the structure of Filipović (2009) [13] and starts with the notion of a zero-coupon bond which is the basis for defining forward and short rates.

Definition 1. *A* ***zero-coupon bond*** *with maturity T is a security paying one unit of cash at a predetermined time T. Its price at time $t \leq T$ is denoted by $P(t,T)$ and it obviously must hold that $P(T,T) = 1$.*

A zero-coupon bond describes the time value of one Euro, that is, $P(t,T)$ denotes the value at time t of receiving one Euro at time T. This is, one Euro today is worth more than one Euro in the future under non-negative interest rates. In theory it is usually assumed that $P(t,T)$ is differentiable in T and there exist such zero-coupon bonds for each $T > 0$. For convenience we will also write T-bond for a zero-coupon bond with maturity T.

A *forward rate agreement* is an agreement to exchange interest rates in the future. At time t the interest rate for the period from the expiry date T to the maturity S, with $t < T < S$, is fixed. To visualize, consider this agreement as an investment, which

- starts with a zero net investment of selling one T-bond and buying $\frac{P(t,T)}{P(t,S)}$ S-bonds at time t,
- at time T pay one Euro,
- and at time S receive $\frac{P(t,T)}{P(t,S)}$ Euro.

This investment is effectively a forward investment of one dollar at time T which, with certainty, yields $\frac{P(t,T)}{P(t,S)}$ Euro at time S. Formally, this can be written as the *simply compounded forward rate* for $[T, S]$ at time t, which is

$$F(t; T, S) = \frac{1}{S - T}\left(\frac{P(t,T)}{P(t,S)} - 1\right). \tag{1.1}$$

The *simple spot rate* for $[t, T]$ is the constant rate for which an investment of $P(t, T)$ units at time t pays one unit of cash at maturity T. In formulas, that is

$$F(t, T) = F(t; t, T) = \frac{1}{T - t}\left(\frac{1}{P(t,T)} - 1\right). \tag{1.2}$$

The *continuously compounded forward rate* for $[T, S]$ at time t is defined as

$$R(t; T, S) = -\frac{\log P(t,S) - \log P(t,T)}{S - T}. \tag{1.3}$$

The *continuously compounded spot interest rate* for $[t, T]$, $R(t, T)$, is also called *yield on a zero-coupon bond* $P(t, T)$, which is the rate of which an investment $P(t, T)$ at t accrues continuously to yield one unit of cash at T. Therefore, this is

$$R(t, T) = R(t; t, T) = -\frac{\log P(t,T)}{T - t}. \tag{1.4}$$

Using this we get the *instantaneous forward rate* at time t with maturity T, which is

$$f(t, T) = \lim_{S \downarrow T} R(t; T, S) = -\frac{\partial \log P(t,T)}{\partial T}. \tag{1.5}$$

Following, the *short rate* at time t, the interest rate for short-term investment, is then

$$r(t) = f(t,t) = \lim_{T \downarrow t} R(t;t,T). \tag{1.6}$$

The requirement $P(T,T) = 1$ and the definition of the forward rate give the value of a T-bond at time t as

$$P(t,T) = e^{-\int_t^T f(t,u)\,\mathrm{d}u}. \tag{1.7}$$

For the following part of the thesis the bank account $B(t)$ is defined such that

$$\mathrm{d}B(t) = r(t)B(t)\,\mathrm{d}t, \tag{1.8}$$

for which, with $B(0) = 1$, it holds that

$$B(t) = e^{\int_0^t r(s)\,\mathrm{d}s}. \tag{1.9}$$

That is, after investing one unit at time 0 the bank account yields exactly the value of (1.9) at time t. This again under the assumption of non-negative interest rates, means that the investment grows for sure with the interest rate.

2 Arbitrage and Risk-Neutral Pricing

A common assumption when considering financial markets is that the market is arbitrage-free where no gains are possible without taking any risks. This demands for a characterization on how to ensure no-arbitrage in a model. The following approach of defining arbitrage and finding properties for an arbitrage-free model follows Björk (1996) [1].

Arbitrage is an investment strategy with no negative cash flows and a strictly positive cash flow in one future state. Formally, an arbitrage possibility exists if for some $T > 0$ and the value of a portfolio at time t, $V(t)$, it holds that

$$V(0) = 0 \quad \text{and} \quad V(T) \geq 0 \quad \text{and} \quad \mathbb{P}[V(T) > 0] > 0. \tag{2.1}$$

If no arbitrage portfolio exists for all $T > 0$ the model is called arbitrage-free. In an efficient market the assumption of no-arbitrage is justified by demand and supply effects where prices converge to arbitrage-free prices.

If there would exist a risk-free portfolio in an arbitrage-free model Y with dynamics

$$\mathrm{d}Y(t) = k(t)Y(t),$$

where k is some adapted process, it must hold that

$$k(t) = r(t).$$

This observation is clear, because if $k > r$ in some interval, arbitrage can be obtained by borrowing from the bank, investing in Y and making profit. If, on the other hand, it holds that $k < r$ one can sell Y short and invest in the bank account.

Throughout this thesis, let P denote the real probability measure and $\mathbb{E}_t[\cdot]$ be the $\mathcal{F}_t$-conditional expectation in the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Now we can investigate if a model is arbitrage-free by using the following definitions.

Definition 2. *Consider a financial market $S = (S_0, \dots, S_n)^\top$ and let, w.l.o.g., S_0 be a numéraire. Then the discounted price process vector $Z(t) = [Z_0, \dots, Z_n]$ is defined by*

$$Z(t) = \frac{S(t)}{S_0(t)}.$$

For the choice of numéraire usually the bank account B from (1.9) is taken which is of course not the only possible choice, and later on we will in fact also use the price process of a T-bond (for a fixed T) as numéraire.

Definition 3. *A probability measure $Q \sim P$ is called* ***equivalent martingale measure*** *EMM if the discounted price process Z is a Q martingale. The set of equivalent martingale measures is denoted by $\mathcal{P}$.*

This definition is a useful tool in arbitrage theory, where the first fundamental theorem of asset pricing connects the assumption of no-arbitrage and the existence of an EMM Q.

Theorem 2.1. ***(First Fundamental Theorem of Asset Pricing)*** *A model is arbitrage-free if there exists an EMM Q.*

Proof. The proof of this direction of the theorem is quite straight forward. Assume V is the discounted value process with $V(0) = 0$ and $V(T) \geq 0$ at some future date $T > 0$. Since V is a Q-martingale for some EMM it holds that

$$0 \leq \mathbb{E}^Q[V(T)] = V(0) = 0,$$

and we get that $V(T) = 0$ from which it follows due to the equivalence of P and Q that $\mathbb{P}[V(T) > 0] > 0$ does not hold. This then means that the model is arbitrage-free. $\square$

Thus, the existence of an EMM Q ensures no arbitrage and Q can be introduced into the bond market from the chapter before. Anticipating Definition 4, the EMM Q is called risk-neutral measure if the chosen numéraire is the bank account B.

Let S_0 be a numéraire and assume there exists an EMM Q for the price process denoted in units of S_o. When applying the definition of a martingale measure it holds for the price process $\Pi(t,T)$ of a claim C_T that

$$\frac{\Pi(t,T)}{S_0(t)} = \mathbb{E}_t^Q\left[\frac{\Pi(T,T)}{S_0(T)}\right] = \mathbb{E}_t^Q\left[\frac{C_T}{S_0(T)}\right]. \tag{2.2}$$

Recall that for the price of a cash flow it must hold that $\Pi(T,T) = C_T$. Therefore, the pricing formula of a cash flow C_T, with $t < T$, is

$$\Pi(t,T) = S_0(t)\mathbb{E}_t^Q\left[\frac{C_T}{S_0(T)}\right]. \tag{2.3}$$

These formulas hold for any numéraire and martingale measure couple (S_0, Q). Chosing bank account $B(t)$ from (1.9) as numéraire leads us to a specific name for the measure Q.

Definition 4. *A martingale measure Q is called a* ***risk neutral*** *measure if the bank account $B(t)$ is used as a numéraire. In the present context this means, for every fixed T the process*

$$Z(t,T) = \frac{P(t,T)}{B(t)}, \quad 0 \le t \le T,$$

is a Q-martingale.

The risk neutral measure Q is a theoretical probability which is used to price financial instruments. That is, under this risk neutral measure the price of a financial instrument equals its expected future pay-off discounted at the risk free rate, which is the bank account. Therefore, the basic pricing formula under the risk-neutral measure uses the representation of $B(t)$ from (1.9).

Proposition 2.1. *Assume that Q is a risk neutral martingale measure, and C_T is the pay-off of a claim at time T. Then the price $\Pi(t,T)$ of this pay-off is given by*

$$\Pi(t,T) = \mathbb{E}_t^Q\left[e^{-\int_t^T r(u)\,\mathrm{d}u} C_T\right], \tag{2.4}$$

where $\mathbb{E}_t^Q$ denotes the conditional expectation under the Q-measure. In partic-

ular, for the price of a T-bond at time t it follows that

$$P(t,T) = \mathbb{E}_t^Q \left[e^{-\int_t^T r(u)\,\mathrm{d}u} \right]. \tag{2.5}$$

This proposition formalizes the relation of the future payoff and the price of the underlying financial instrument. Using the risk neutral measure is one possible approach to price these instruments since the real probability measure is unknown in general. However, Filipović et al. (2016) [14] price bonds and swaptions under the real probability for which the notion of a state price density is required.

2.1 State Price Density

To guarantee the common hypothesis that models are free of arbitrage one usually assumes that there exists an EMM Q under which the price $\Pi(t,T)$ at time t of a contingent claim with pay-off C_T at $T > t$ is exactly given by (2.4). Thereof, we can define the state price density as the process ζ_t for which we receive the expected value under the real probability P. That is, we set the state price density equal to the discount process times the Radon-Nikodym derivative

$$\zeta_t = e^{-\int_0^t r(s)\,\mathrm{d}s} \frac{dQ}{dP}. \tag{2.6}$$

Plugging this into the basic pricing formula with $t = 0$, the price of a cash flow then equals

$$\Pi(0,T) = \mathbb{E}^Q \left[e^{-\int_0^T r(u)\,\mathrm{d}u} C_T \right] = \mathbb{E} \left[e^{-\int_0^T r(u)\,\mathrm{d}u} C_T \frac{dQ}{dP} \right] = \mathbb{E} \left[\zeta_T C_T \right].$$

Similarly, the price at time t of a cash flow C_T is then

$$\begin{aligned} \Pi(t,T) &= \mathbb{E}_t^Q \left[e^{-\int_t^T r(u)\,\mathrm{d}u} C_T \right] = \frac{dP}{dQ} \mathbb{E}_t \left[e^{-\int_t^T r(u)\,\mathrm{d}u} C_T \frac{dQ}{dP} \right] \\ &= \frac{dP}{dQ} e^{\int_0^t r(u)\,\mathrm{d}u} \mathbb{E}_t \left[e^{-\int_0^T r(u)\,\mathrm{d}u} C_T \frac{dQ}{dP} \right] = \zeta_t^{-1} \mathbb{E}_t \left[\zeta_T C_T \right] \end{aligned} \tag{2.7}$$

Hence, under the real probability measure P the price of a T-bond, that is, $C_T = 1$, becomes

$$P(t,T) = \zeta_t^{-1}\mathbb{E}_t\left[\zeta_T\right].$$

The state price density is also called pricing kernel or stochastic discount factor as it takes over the role of the discounting term under the risk-neutral measure.

Pricing a T-bond results in calculating the expected value of the state price density, therefore we need to know ζ_t and its expectation. Summarized, the existence of a state price density is sufficient to achieve an arbitrage-free model.

3 Swaps and Swaptions

An interest rate swap (IRS) is a financial derivative where two parties agree to exchange a payment stream based on a fixed rate of interest for a payment stream at a floating rate. The floating part is typically indexed to a reference rate, usually LIBOR (London Interbank Offered Rate). Specifically, it is called a payer interest rate swap when the holder of the swap pays the fixed rate and receives the floating leg.

This agreement is, for example, used to ensure against increasing interest rate. Imagine company A which has a loan of 1 million at a bank and its interest is regularly adjusted to the interest of the capital market. So to hedge against increasing interest rates company A enters a payer swap over 1 million with company B, to pay them a fixed, predetermined interest and receive floating. With the received floating rate company A can pay the variable interest on the loan of the bank.

Formally, a payer interest rate swap is specified by a number of payment dates, $T_0 < T_1 < \cdots < T_n$ with $\Delta = T_i - T_{i-1}$, a predetermined, fixed rate K and a nominal value N. This nominal value would be 1 million in our example before, but for convenience and without loss of generality we set $N = 1$. The cash flows take place at the dates $T_1, \dots, T_n$, where at T_i the holder of the payer swap contract pays the fixed rate ΔK and receives the floating rate of $F(T_{i-1}, T_i)\Delta$. In the floating leg $F(T_{i-1}, T_i)$ denotes the simple market interest rate, determined already at time T_{i-1}, which is defined as

$$F(T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{1}{P(T_{i-1}, T_i)} - 1 \right), \tag{3.1}$$

where $P(T_{i-1}, T_i)$ denotes the time-t price of a zero-coupon bond paying 1 at time T_i. Then the floating leg in the time interval $[T_{i-1}, T_i]$ is

$$\Delta F(T_{i-1}, T_i) = \frac{1}{P(T_{i-1}, T_i)} - 1, \tag{3.2}$$

which has the value at time t of

$$P(t, T_{i-1}) - P(t, T_i). \tag{3.3}$$

Combining the floating and the fixed leg the time-t value of a payer swap is

$$P(t, T_{i-1}) - P(t, T_i) - \Delta K P(t, T_i). \tag{3.4}$$

Hence, the price of a payer swap at $t \leq T_0$ is defined as the sum over (3.4) for each cash flow date,

$$\Pi_t^{swap} = P(t, T_0) - P(t, T_n) - \Delta K \sum_{i=1}^{n} P(t, T_i). \tag{3.5}$$

The time-t forward swap rate, $S_t^{T_0,T_n}$, is the rate K with what the value of the swap is equal to zero, that means the fixed and floating leg are worth the same. The forward swap rate is given by

$$S_t^{T_0,T_n} = \frac{P(t, T_0) - P(t, T_n)}{\Delta \sum_{i=1}^{n} P(t, T_i)}. \tag{3.6}$$

At time T_0 the forward swap rate becomes the spot swap rate.

A payer swaption is an option to enter into an IRS, which pays the fixed leg at a pre-determined rate and receives the floating leg. In particular, a European payer swaption on a swap described above has a value at expiration T_0 of

$$C_{T_0} = \left(\Pi_{T_0}^{swap}\right)^+ = \left(\sum_{i=0}^{n} c_i P(T_0, T_i)\right)^+ = \frac{1}{\zeta_{T_0}} \left(\sum_{i=0}^{n} c_i \mathbb{E}_{T_0}[\zeta_{T_i}]\right)^+. \tag{3.7}$$

The coefficients c_i are such that $c_0 = 1$, $c_n = -1 - \Delta K$ and $c_i = -\Delta K$ for $i = 1, \ldots, n-1$, and ζ_t denotes the state price density.

Following, the price of a swaption at time t is then defined as

$$\Pi_t^{swaption} = \frac{1}{\zeta_t}\mathbb{E}_t\left[\zeta_{T_0} C_{T_0}\right] = \frac{1}{\zeta_t}\mathbb{E}_t\left[\left(\sum_{i=0}^{n} c_i \mathbb{E}_{T_0}[\zeta_{T_i}]\right)^+\right]. \tag{3.8}$$

This formula will later be used in the specification of the linear-rational term structure model to define the analytical representation.

In the financial market swaption prices are presented in terms of implied volatilities, where the market standard is the normal implied volatility (NIV) $\sigma_{N,t}$ (Filipović et al., 2016, p.20 [14]). This volatility can then be plugged into the related pricing formula which assumes that the underlying forward swap rate has a normal distribution. For an at-the-money swaption, where the strike K equals the forward swap rate $S_t^{T_0,T_n}$, the relation between the swaption price and the NIV simplifies to

$$\Pi_t^{swaption} = \sqrt{T_0 - t}\frac{1}{\sqrt{2\pi}}\left(\sum_{i=1}^{n} \Delta P(t, T_i)\right)\sigma_{N,t}. \tag{3.9}$$

This so-called Normal model can be found in Corb (2012) [8], who also provides a closer look into the comparison with Black's formula. The latter assumes that the swap rate is lognormal, that is,

$$\Pi_t^{swaption} = \Delta\left(S_t^{T_0,T_n}\Phi(d_1(t)) - K\Phi(d_2(t))\right)\sum_{i=1}^{n} P(t, T_i), \tag{3.10}$$

where $\Phi(\cdot)$ is the standard Gaussian cumulative distribution function, and we have

$$d_{1,2}(t) = \frac{\log\left(\frac{S_t^{T_0,T_n}}{K}\right) \pm \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}},$$

with $\sigma(t)$ denoting the prevailing Black's swaption volatility. For at-the-money swaptions, i.e. $K = S_t^{T_0,T_n}$, it holds that

$$d_{1,2}(t) = \frac{\pm\frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}},$$

where Black's at-the-money swaption formula becomes

$$\Pi_t^{swaption} = S_t^{T_0,T_n} \left(\Phi(d_1(t)) - \Phi(d_2(t))\right) \Delta \sum_{i=1}^{n} P(t,T_i). \tag{3.11}$$

Zero-coupon bond prices can be observed in the market and the only unknown parameter in these formulas is the volatility. In the empirical analysis we use Black's formula to calculate market swaption prices for volatility data.

There have been a few studies about the differences and equivalences of the normal and the lognormal implied volatilites, as well as the Black-Merton Scholes and Bachelier option pricing formula. Bachelier's model assumes the swap rate to be normal distributed, that is, it uses the normal implied volatility, whereas in Black's model the swap rate is lognormally distributed. Two of these comparisons can be found in Schachermayer and Teichmann (2007) [21] as well as in Grunspan (2011) [15]. The discussion on which model provides a better fit will be left open for another study.

4 The Term Structure and Stochastic Volatility

In the financial market changes and movements of the interest rate constitute a major source of risk for market participants. Hence, the main motivation for modelling the term structure of interest rates is trying to find an explanatory model on how the interest rate evolves over time. A suitable model can then be used to solve real problems such as pricing an interest-rate-contingent claim. Therefore, the term structure model should be reasonable from an empirical perspective as well as not permitting arbitrage which is a common assumption in mathematical finance.

4.1 Term Structure Modelling

One possible and straight forward approach to model the term structure is to directly define the dynamics of the interest rate depending on the evolution of some unobserved factor which is also called state variable. The approach of using only a single-state variable to explain the random future movement of the interest rate is inadequate as the dynamics of the term strucutre are too complex to be summarized by a single source of uncertainty. An overview of different term structure models as well as an empirical study about their ability to capture the actual behavior of the interest rate in single- and multifactor models can be found in the paper by Chan, Karolyi, Longstaff and Sanders (1992) [4]. Following this evidence the linear-rational framework which is discussed in the next chapter of this thesis also considers a multifactor process.

Other than the approach of modelling the dynamics of the interest rate directly one possibility is to define a set of latent variables to serve as the state vector (Collin-Dufresne and Goldstein, 2002, p.2 [6]). The spot rate is then defined as a function of these state variables. Furthermore, to ensure that the model is arbitrage-free a state price density can be defined to price contingent claims as in Section 2.1. The approach of defining a state price density for term structure modelling appears in Constantinides (1992) [7]. Moreover, in Rogers (1997) [20] the underlying factor process is Markovian and the state price density is then modelled by a specific expression.

Most term structure models belong to the affine class in Duffie, Filipović and Schachermayer (2003) [10] in which all state variables show up in bond prices. Those models assume that bond markets are complete, that is, bonds are sufficient to perfectly replicate all fixed income derivatives (Collin-Dufresne and Goldstein, 2002, p.1 [6]). Therefore, in these models portfolios consisting solely of bonds are sufficient to hedge all sources of risk affecting fixed income derivatives (Collin-Dufresne and Goldstein, 2002, p.2 [6]). Furthermore, in the affine class bonds alone are sufficient to construct a complete fixed-income market. Additionally, under affine models swaption pricing can only be done approximately since the probability for their exercise region is difficult to compute as it is implicitly defined (Collin-Dufresne and Goldstein, 2001 [5]). Thus, the affine models are not suitable for an accurate pricing of swaptions.

4.2 Unspanned Stochastic Volatility

One of the first and most in depth papers about unspanned stochastic volatility is by Collin-Dufresne and Goldstein (2002) [6], who find that in most term structure models it is assumed that the market is complete, that is, bonds can be used to replicate all fixed income derivatives and hedge volatility risk (Collin-Dufresne and Goldstein, 2002, p.1-2 [6]).

Collin-Dufresne and Goldstein (2002) [6] investigate how many bonds are needed to hedge interest rate volatility-risk. In particular they examine how much of the variation in straddles can be explained by the variations in swap rates. Straddles are portfolios of at-the-money caps and floors which are highly sensitive to bond-price volatility risk. They find evidence that swap rates are only weakly correlated with straddles, in fact, swap rates are only able to explain a small percentage of the returns of straddles (Collin-Dufresne and Goldstein, 2002, p.2 [6]). These findings suggest the existence of one or more state variables which drive the volatility risk but do not affect swap rates, and thus, bond prices themselves. This feature is called "unspanned stochastic volatility" (USV). Subsequently, they also show via principal component analysis that a single USV factor explains most of the variation in straddles.

Following these findings, the aim is then to model the term structure such that the USV factors only drive the volatility, that is, we specify exogenous factors that feed into the martingale part of the term structure. One approach to capture the feature of unspanned stochastic volatility is to directly define the joint dynamics of forward rates and the state variable that drive forward rate volatility. Equivalently, one can specify the joint dynamics of a traded asset and its volatility. This way of proceeding entails one disadvantage which is that bond prices become inputs rather than predictions of the model (Collin-Dufresne and Goldstein, 2002, p.2 [6]).

For a more applicable approach one can follow the previously described procedure of defining a set of latent variables and a function of these variables for the short rate. Then the term structure process is modelled such that the USV factor process only drives its diffusion part but not the drift. Following, the joint factor process of these two processes can be defined in a way that bond prices become predictions and USV factors do not influence their dynamics.

In a model with unspanned stochastic volatility the existence of at least one state variable which drives innovations of interest rate derivatives leads to the implication that the bond market itself is incomplete. That is, a model which can generate an incomplete bond market can also inherit USV factors (Collin-Dufresne and Goldstein, 2002, p.2 [6]; Heidari and Wu, 2001 [16]).

The topic of pricing derivatives while considering stochastic volatility is part of Hull and White (1987) [18] and Casassus, Collin-Dufresne and Goldstein (2005) [3].

4.3 Mathematical Description of USV

We now raise the subject on how to describe USV factors mathematically, while a closer look into this topic for the specific term structure model is presented in the following chapter of this thesis. Filipović et al. (2016) [14] discuss these definitions in accordance with their proposed term structure model. Therefore, we will cover the general definitions now and go into more detail in the next chapter.

Unspanned factors can be understood as those factors which do not influence the term structure. If such factors change the term structure itself remains unchanged. These unspanned factors can then be described through directions

$\xi \in \mathbb{R}^d$ of the state vector along which the term structure remains unchanged. For this we first revise the notion of the *kernel* of a differentiable function f on $E \subset \mathbb{R}^d$, which is defined by

$$\mathrm{ker} f = \{\xi \in \mathbb{R}^d : \triangledown f(x)^\top \xi = 0 \text{ for all } x \in E\}.$$

Following, the definition of a kernel can be carried out further for the topic of term structures. One way to define the term structure would be through its functional form, which is the formulation of zero-coupon bond prices $P(t,T) = F(\tau, \cdot)$, with $\tau = T - t$. We can then define the term structure kernel analogously to the kernel of a function.

Definition 5. *The term structure kernel, denoted by* $\mathcal{U}$*, is given by*

$$\mathcal{U} = \bigcap_{\tau \geq 0} \mathrm{ker} F(\tau, \cdot). \tag{4.1}$$

That is, the term structure kernel defines those directions ξ along which the state of the term structure cannot be recovered only by the knowledge of time t bond prices $P(t, t+\tau)$, $\tau \geq 0$. Mathematically, the term structure kernel consists of all directions $\xi \in \mathbb{R}^d$ such that it holds for the gradient of $F(\cdot, x)$ that $\triangledown_x F(\tau, x)^\top \xi = 0$ for all $\tau \geq 0$ and all $x \in E$. In other words, the term structure kernel is unspanned by the term structure.

The definition of unspanned stochastic volatility using the specific term structure model by Filipović et al. (2016) [14] is shown in the next chapter.

5 The Linear-Rational Term Structure Model

In this chapter we review the Linear-Rational Term Structure Model of Filipović, Larsson and Trolle (2016) [14] and reproduce the relevant propositions. Eventually we describe the formula for swaption pricing and how it can be calculated under a certain specification of the model.

The main characteristic in Filipović, Larsson and Trolle (2016) [14] is that their term structure model comes with three important advantages. First, it can ensure non-negative interest rates, although this feature can be relaxed to be more realistic in view of the current market state. In general, one can define bounds in which the interest rate remains and this only depends on the choice of some parameters which is shown below. The second feature of the model is the easy accommodation of unspanned factors which affect volatility but do not influence the term structure itself. And lastly, but most important for the purpose of this thesis, this class of term structure admits semi-analytical solutions to swaptions.

Modelling the term structure starts with a multivariate factor process and a state price density, where bond prices and the short rate become linear rational functions.

5.1 The Linear-Rational Framework

Recalling the definition of a state price density we know its existence is sufficient to guarantee no-arbitrage in a model. The approach of the linear-rational term structure model is then to define this state price density together with a multivariate factor process defining the latent variables.

That is, the linear-rational term structure model consists of a multivariate factor process Z_t with a linear drift and state space $E \subset \mathbb{R}^m$. The process Z_t

is defined to have dynamics of the form

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\,\mathrm{d}t + \mathrm{d}M_t \tag{5.1}$$

for some $\kappa \in \mathbb{R}^{m\times m}$, $\theta \in \mathbb{R}^m$, and some m-dimensional martingale M_t. The state price density ζ_t is assumed to be a linear function of the process Z_t,

$$\zeta_t = e^{-\alpha t}(\phi + \psi^\top Z_t), \tag{5.2}$$

for some $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^m$ such that $\phi + \psi^\top x > 0$ for all $x \in E$, and some $\alpha \in \mathbb{R}$. The non-negativity of the short rate can be ensured by the parameter α. This feature is discussed below.

The drift of the process Z_t is linear which implies that the conditional expectations are of linear form, specifically it can be shown that

$$\mathbb{E}_t[Z_T] = \theta + e^{-\kappa(T-t)}(Z_t - \theta), \quad t \leq T. \tag{5.3}$$

The formal proof of this statement can be found in the Appendix.

Proceeding, using the basic pricing formula (2.7) and setting $C_T = 1$ the price of a zero-coupon bond becomes a linear-rational function of Z_t. Precisely, with $P(t,T) = F(T-t, Z_t)$ to display the dependence on Z_t, the definition of the state price density ζ_t and its expectation (5.3), we have

$$\begin{aligned}
F(T-t, Z_t) &= \frac{1}{\zeta}\mathbb{E}_t[\zeta_T] = \frac{1}{e^{-\alpha t}(\phi + \psi^\top Z_t)}\mathbb{E}_t\left[e^{-\alpha T}(\phi + \psi^\top Z_T)\right] \\
&= \frac{e^{-\alpha T}(\phi + \psi^\top \mathbb{E}_t[Z_T])}{e^{-\alpha t}(\phi + \psi^\top Z_t)} \\
&= \frac{e^{-\alpha T}(\phi + \psi^\top(\theta + e^{-\kappa(T-t)}(Z_t - \theta)))}{e^{-\alpha t}(\phi + \psi^\top Z_t)} \\
&= e^{-\alpha(T-t)}\frac{\phi + \psi^\top\theta + \psi^\top e^{-\kappa(T-t)}(Z_t - \theta)}{\phi + \psi^\top Z_t}.
\end{aligned}$$

Therefore, setting $T - t = \tau$, it follows that

$$F(\tau, z) = e^{-\alpha\tau}\frac{\phi + \psi^\top\theta + \psi^\top e^{-\tau\kappa}(z - \theta)}{\phi + \psi^\top z}. \tag{5.4}$$

Notice, that the T-bond is independent of the martingale part of the process

Z_t, this observation will later be used to accommodate USV factors.

The short rate is obtained via $r_t = -\partial_T \log P(t,T)|_{T=t}$, (1.6), where the partial derivation calculates as

$$
\begin{aligned}
-\frac{\partial}{\partial_T}\log P(t,T) &= -\frac{\partial}{\partial_T}\log\left(e^{-\alpha(T-t)}\frac{\phi+\psi^\top\theta+\psi^\top e^{-(T-t)\kappa}(Z_t-\theta)}{\phi+\psi^\top Z_t}\right)\\
&= \frac{\partial}{\partial_T}\left(\alpha(T-t)-\log\left(\frac{\phi+\psi^\top\theta+\psi^\top e^{-(T-t)\kappa}(Z_t-\theta)}{\phi+\psi^\top Z_t}\right)\right)\\
&= \frac{\partial}{\partial_T}\big(\alpha(T-t)-\log\big(\phi+\psi^\top\theta+\psi^\top e^{-(T-t)\kappa}(Z_t-\theta)\big)-\log(\phi+\psi^\top Z_t)\big)\\
&= \alpha-\frac{\phi+\psi^\top\theta+\psi^\top(-\kappa)e^{-(T-t)\kappa}(Z_t-\theta)}{(\phi+\psi^\top Z_t)}.
\end{aligned}
$$

Setting now $T=t$, we arrive at the formula for the short rate, which is

$$
r_t = \alpha - \frac{\psi^\top\kappa(\theta - Z_t)}{\phi+\psi^\top Z_t}. \tag{5.5}
$$

From this expression the role of α can be observed as it ensures that the short rate is bounded from below. Therefore the parameter α can be chosen large enough to guarantee the short rate stays non-negative, that is, set α as the smallest value for a non-negative short rate. Formulated, this is

$$
\alpha^* = \sup_{z\in E}\frac{\psi^\top\kappa(\theta-z)}{\phi+\psi^\top z} \quad \text{and} \quad \alpha_* = \inf_{z\in E}\frac{\psi^\top\kappa(\theta-z)}{\phi+\psi^\top z}, \tag{5.6}
$$

and set $\alpha=\alpha^*$, provided this is finite. The short rate satisfies $r_t\in[0,\alpha^*-\alpha_*]$. Notice that α^* and α_* depend on the drift parameters of Z_t, which are estimated from data. Whereas negative interest rates can easily be incorporated by setting $\alpha<\alpha^*$.

5.1.1 Term Structure Factors

The term structure kernel $\mathcal{U}$ in (4.1) can also be characterized within the linear-rational framework in terms of the parameters κ and ψ.

Theorem 5.1. *(Filipović et al., 2016, Theorem 1, p.8 [14]). Assume the term structure is not trivial, that is, the short rate r_t is not constant. Then the term structure kernel $\mathcal{U}$ is the largest subspace of* $\ker\psi^\top$ *that is invariant under κ,* *which is equivalent to*

$$\mathcal{U} = \mathrm{span}\{\psi, \kappa^\top \psi, \dots, \kappa^{(m-1)\top}\psi\}^\perp$$

The term structure Z_t exhibits no unspanned factors if the term structure kernel is zero. Under mild conditions it can then be reconstructed from a snapshot of the term structure at time t and the components of Z_t are called *term structure factors*. The case in which Z_t exhibits unspanned factors is described in the next section.

5.1.2 Unspanned Stochastic Volatility Factors

In the linear-rational term structure framework bond prices $P(t,T)$ in (5.4) only depend on the drift of Z_t. Therefore, we can specify factors which give rise to stochastic volatility, that is, they only feed into the martingale part of the process but the process Z_t itself does not exhibit unspanned components.

For the linear-rational model to inherit USV factors the framework (5.1)-(5.2) can be specialized such that the process Z_t has diffusive dynamics. The term structure becomes

$$\mathrm{d}Z_t = \kappa(\theta - Z_t)\,\mathrm{d}t + \sigma(Z_t, U_t)\,\mathrm{d}B_t, \tag{5.7}$$

where U_t is a l-dimensional USV factor process, B_t a d-dimensional Brownian motion and $\sigma(z,u)$ defines an $\mathbb{R}^{m\times d}$-valued dispersion function. The resulting joint factor process (Z_t, U_t) of the term structure and the unspanned factors has the state space $\mathcal{E} \subset \mathbb{R}^{m+l}$. The diffusion matrix of Z_t is defined by $a(z,u) = \sigma(z,u)\sigma(z,u)^\top$ and is assumed to be differentiable on $\mathcal{E}$.

The dynamics of the state price density can be calculated using Itô's formula and are the following

$$\begin{aligned}\frac{d\zeta_t}{\zeta_t} &= \frac{-\alpha e^{-\alpha t}\left(-\alpha(\phi + \psi^\top Z_t) + \psi^\top \kappa(\theta - Z_t)\right)dt + e^{-\alpha t}\psi^\top \sigma(Z_t, U_t)dB_t}{e^{-\alpha t}(\phi + \psi^\top Z_t)} \\ &= \left(-\alpha + \frac{\psi^\top \kappa(\theta - Z_t)}{\phi + \psi^\top Z_t}\right)dt + \frac{\psi^\top \sigma(Z_t, U_t)}{\phi + \psi^\top Z_t}dB_t.\end{aligned}$$

Using the definition of the short rate r_t, (5.5), and setting

$$\lambda_t = -\frac{\sigma(Z_t, U_t)^\top \psi}{\phi + \psi^\top Z_t}, \tag{5.8}$$

the dynamics of the state price density ζ_t become

$$\frac{d\zeta_t}{\zeta_t} = -r_t dt - \lambda_t^\top \, \mathrm{d}B_t. \tag{5.9}$$

For the dynamics of $P(t,T)$ it follows that

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = (r_t + \nu(t,T)^\top \lambda_t)\, \mathrm{d}t + \nu(t,T)^\top \, \mathrm{d}B_t, \tag{5.10}$$

where

$$\nu(t,T) = \frac{\sigma(Z_t, U_t)^\top \bigtriangledown_z F(T-t, Z_t)}{F(T-t, Z_t)}.$$

To see that the USV factors don't affect the term structure one can consider the time-t price of a zero-coupon bond with maturity T. Following Filipović et al. (2016) [14, p.9] and using the pricing formula (2.7) this price equals

$$\begin{aligned} P(t,T) &= \mathbb{E}_t[\frac{1}{\zeta_t} \cdot \zeta_T] = \mathbb{E}_t[\frac{1}{\zeta_t} \cdot \zeta_{t+\mathrm{d}t} \cdot \frac{1}{\zeta_{t+\mathrm{d}t}} \cdot \zeta_T] \\ &= \mathbb{E}_t\left[\frac{1}{\zeta_t}\zeta_{t+\mathrm{d}t}\right] \mathbb{E}_t\left[\frac{1}{\zeta_{t+\mathrm{d}t}}\zeta_T\right] + Cov_t\left[\frac{1}{\zeta_t}\zeta_{t+\mathrm{d}t}, \frac{1}{\zeta_{t+\mathrm{d}t}}\zeta_T\right] \\ &= P(t, t+\mathrm{d}t)\mathbb{E}_t[P(t+\mathrm{d}t, T)] + Cov_t\left[\frac{\zeta_{t+\mathrm{d}t}}{\zeta_t}, P(t+\mathrm{d}t, T)\right]. \end{aligned} \tag{5.11}$$

The price $P(t, t+dt) = 1 - r_t dt$ does not depend on U_t, but due to non-linear dependence on Z_{t+dt} the expected time-$(t+dt)$ price does,

$$\begin{aligned} \mathbb{E}_t[P(t+\mathrm{d}t, T)] = P(t,T) + \Bigg(&-\frac{\partial}{\partial \tau}F(T-t, Z_t) + \bigtriangledown_z F(T-t, Z_t)^\top \kappa(\theta - Z_t) \\ &+ \frac{1}{2}\mathrm{tr}(\bigtriangledown_z^2 F(T-t, Z_t) a(U_t, Z_t))\Bigg)\, \mathrm{d}t. \end{aligned}$$

But for the risk premium on the right hand side of (5.11) it holds that

$$\mathrm{Cov}_t\left[\frac{\zeta_{t+\mathrm{d}t}}{\zeta_t}, P(t+\mathrm{d}t, T)\right] = -P(t,T)\nu(t,T)^\top \lambda_t \, \mathrm{d}t.$$

Using market price of risk given by (5.8), those terms cancel each other out, we get

$$\frac{1}{2}\mathrm{tr}(\nabla_t^2 F(T-t, Z_t)a(U_t, Z_t)) + P(t,T)\nu(t,T)^\top \lambda_t = 0. \tag{5.12}$$

Therefore the USV process U_t cancels out and the dynamics of $P(t,T)$ are unaffected by it, which is the desired feature of USV.

Summarized, the factor process Z_t is now defined in a way that bond prices and the short rate are linear-rational in Z_t, as well as that it can exhibit USV factors. Using this construction of the process swaption prices can be calculated with a formula which will be provided in the next subsection.

We can now carry on the discussion of USV by identifying those USV factors that directly affect the instantaneous bond return covariances in the linear-rational term structure model, which can be written as

$$\mathrm{Cov}_t\left[\frac{\mathrm{d}P(t,T_1)}{P(t,T_1)}, \frac{\mathrm{d}P(t,T_2)}{P(t,T_2)}\right] = G(T_1-t, T_2-t, Z_t, U_t),$$

where

$$G(\tau_1, \tau_2, z, u) = \frac{\nabla_z F(\tau_1, z)^\top a(z,u) \nabla_z F(\tau_2, z)}{F(\tau_1, z)F(\tau_2, z)}, \tag{5.13}$$

(Filipović et al., 2016, p.10 [14]).

Analogously to the term structure kernel we can describe those directions $\xi \in \mathbb{R}^n$ of U_t along which the instantaneous bond return covariance matrix remains unchanged.

Definition 6. *The covariance kernel, denoted by $\mathcal{W}$, is given by*

$$\mathcal{W} = \bigcap_{\tau_1,\tau_2 \geq 0, (z,u)\in\mathcal{E}} \ker \nabla_u G(\tau_1, \tau_2, z, u). \tag{5.14}$$

With only the knowledge of the time-t instantaneous bond return covariances, with $T_1, T_2 \geq t$, the location of the process U_t cannot be recovered along the direction ξ. Nevertheless, expected future bond return covariances could be affected by the movements of U_t along these directions and the location of U_t, that is, the location of U_t can be recovered from time-t bond derivative prices.

The question then is to which extent the instantaneous bond return covariances are directly affected by USV factors. This depends on the u-gradient of the diffusion matrix $a(z,u)$ and how it transmits to the u-gradient of $G(\tau_1,\tau_2,z,u)$ through the defining relation.

Theorem 5.2. *The number of USV factors that directly affect the instantaneous bond return covariances is less than or equal to the dimension p of*

$$\operatorname{span}\{\nabla_u a_{ij}(z,u) : 1 \le i,j \le m, (z,u) \in \mathcal{E}\}.$$

Equality holds if the term structure kernel is zero, $\mathcal{U} = \{0\}$, κ is invertible, and $\phi + \psi^\top \theta \neq 0$.

Filipović et al. (2016) [14] also show a closer analysis of this topic including a contradiction to Collin-Dufresne and Goldstein (2002) [6]. The later state in Proposition 3 that in a two-factor term structure model no unspanned factors can be inherited. Whereas Filipović et al. present an example within their specification to show that the linear-rational term structure can exhibit USV factors in a two-factor case.

5.1.3 Swaptions

Continuing the review of swaption pricing the prior defined state price density of the linear-rational framework can be plugged into the formula (3.7). Therefore, the value of a payer swaption at expiration T_0

$$C_{T_0} = \frac{1}{\zeta_{T_0}} \left(\sum_{i=0}^{n} c_i \mathbb{E}_{T_0}[\zeta_{T_i}] \right)^+$$

can be calculated explicitly.

Since the expected value of Z_t is known, see (5.3), the sum can be calculated as

$$\begin{aligned} \sum_{i=0}^{n} c_i \mathbb{E}_{T_0}[\zeta_{T_i}] &= \sum_{i=0}^{n} c_i \mathbb{E}_{T_0}[e^{-\alpha T_i}(\phi + \psi^\top Z_{T_i})] = \sum_{i=0}^{n} c_i e^{-\alpha T_i}(\phi + \psi^\top \mathbb{E}_{T_0}[Z_{T_i})] \\ &= \sum_{i=0}^{n} c_i e^{-\alpha T_i}(\phi + \psi^\top(\theta + e^{-\kappa(T_i - T_0)}(Z_{T_0} - \theta))) = \\ &= \sum_{i=0}^{n} c_i e^{-\alpha T_i}(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)}(Z_{T_0} - \theta)) =: p_{swap}(Z_{T_0}). \end{aligned} \tag{5.15}$$

Hence, the value of a swaption at its maturity T_0 is $C_{T_0} = p_{swap}(Z_{T_0})^+/\zeta_{T_0}$.

With the fundamental pricing formula (2.7) the swaption price at time $t \leq T_0$ becomes

$$\Pi_t^{swaption} = \frac{1}{\zeta_t}\mathbb{E}_t[\zeta_{T_0} C_{T_0}] = \frac{1}{\zeta_t}\mathbb{E}_t[p_{swap}(Z_{T_0})^+]. \tag{5.16}$$

Computing the price results in an evaluation of the conditional expectation on the right side of (5.16), which can be done via direct numerical integration over $\mathbb{R}^m$ if the conditional distribution of Z_{T_0} is known. Since this is a challenging problem in general an alternative approach based on Fourier transform methods can be used (Theorem 4, Filipović, Larsson and Trolle, 2016, p.12 [14]).

Theorem 5.3. *Define $\hat{q}(x) = \mathbb{E}_t[\exp(x \cdot p_{swap}(Z_{T_0}))]$ for $x \in \mathbb{C}$ and let $\mu > 0$ be such that $\hat{q}(\mu) < \infty$. Then the swaption price is given by*

$$\Pi_t^{swaption} = \frac{1}{\zeta_t \pi}\int_0^\infty Re\left[\frac{\hat{q}(\mu + \mathrm{i}\lambda)}{(\mu + \mathrm{i}\lambda)^2}\right] \mathrm{d}\lambda \tag{5.17}$$

Proof. (Theorem 5.3) The proof uses the following identity from Fourier analysis, valid for any $\mu > 0$ and $s \in \mathbb{R}$:

$$s^+ = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(\mu+\mathrm{i}\lambda)s}\frac{1}{(\mu + \mathrm{i}\lambda)^2}\,\mathrm{d}\lambda. \tag{5.18}$$

Let $q(ds)$ denote the conditional distribution of the random variable $p_{swap}(Z_{T_0})$ so that $\hat{q}(x) = \int_{\mathbb{R}} e^{xs} q(ds)$ for $x \in \mathbb{C}$. Let $\mu > 0$ be such that $\hat{q}(\mu) < \infty$. Then, using Tonelli's theorem, it holds that

$$\begin{aligned}\int_{\mathbb{R}^2}\left|e^{(\mu+i\lambda)s}\frac{1}{(\mu+\mathrm{i}\lambda)^2}\right|\mathrm{d}\lambda q(\mathrm{d}s) &= \int_{\mathbb{R}^2}\frac{e^{\mu s}}{\mu^2+\lambda^2}\,\mathrm{d}\lambda q(\mathrm{d}s)\\ &= \int_{\mathbb{R}} e^{\mu s} q(\mathrm{d}s)\int_{\mathbb{R}}\frac{1}{\mu^2+\lambda^2}\,\mathrm{d}\lambda < \infty.\end{aligned}$$

This justifies applying Fubini's theorem in the following calculation, which uses the identity (5.18) on the second line:

$$
\begin{aligned}
\mathbb{E}_t[p_{swap}(Z_{T_0})^+] &= \int_{\mathbb{R}} s^+ q(ds) = \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{(\mu+\mathrm{i}\lambda)s} \frac{1}{(\mu+\mathrm{i}\lambda)^2} \,\mathrm{d}\lambda \right) q(\mathrm{d}s) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\hat{q}(\mu+\mathrm{i}\lambda)}{(\mu+\mathrm{i}\lambda)^2} \,\mathrm{d}\lambda = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{\hat{q}(\mu+\mathrm{i}\lambda)}{(\mu+\mathrm{i}\lambda)^2} \right] \mathrm{d}\lambda.
\end{aligned}
$$

The last equality uses the observation that the real part of $(\mu+\mathrm{i}\lambda)^{-2}\hat{q}(\mu+\mathrm{i}\lambda)$ is an even function of λ, therefore, it holds that

$$
\mathrm{Re}\left[\frac{\hat{q}(\mu+\mathrm{i}\lambda)}{(\mu+i\lambda)^2} \right] = \mathrm{Re}\left[\frac{\hat{q}(-(\mu+\mathrm{i}\lambda))}{(-(\mu+\mathrm{i}\lambda))^2} \right],
$$

together with the fact that the price of a swaption is real, and hence the right side is real. The resulting expression for the conditional expectation, together with (5.16), give the result. □

In this theorem the price of a swaption results in computing a simple line integral, for which $\hat{q}(\mu+i\lambda)$ has to be efficiently evaluated as λ varies through $\mathbb{R}$. In the empirical part we will focus on the square-root factor processes, which will be discussed in the next section, for which the computation of $\hat{q}(\mu+i\lambda)$ concludes in solving a system of ordinary differential equations.

5.2 The Linear-Rational Square-Root Model

Filipović, Larsson and Trolle (2016) [14] describe a specification of a linear-rational diffusion model (5.7) with term structure state space $\mathbb{E} = \mathbb{R}^m_+$ which they also use for their detailed empirical part. They call it the linear-rational square-root (LRSQ) model in which USV can easily be incorporated and swaptions can be priced efficiently. This specific model is also used for the empirical part of this thesis.

Starting point for the LRSQ model is a $(m+l)$-dimensional square-root diffusion process X_t which takes values in $\mathbb{R}^{m+l}_+$ and is of the form

$$
\mathrm{d}X_t = (b - \beta X_t)\,\mathrm{d}t + \mathrm{Diag}\left(\sigma_1\sqrt{X_{1t}}, \ldots, \sigma_{m+l}\sqrt{X_{m+l,t}}\right) \mathrm{d}B_t, \tag{5.19}
$$

where σ_i, $i = 1, \ldots, m+l$, denote the volatility parameters and $0 \leq l \leq k$ represents the desired number of USV factors. The joint factor process (Z_t, U_t) is defined as a linear transform of X_t, $(Z_t, U_t) = SX_t$, with state space $\mathcal{E} =$

$S(\mathbb{R}_+^{m+l})$. This needs an $(m+l)\times(m+l)$- matrix S such that the implied term structure state space is $E=\mathbb{R}_+^m$ and the USV process U_t feeds into the martingale part of Z_t, while the drift of Z_t does not depend on U_t.

Of course there are many possible constructions but we will restrict this study to the proposed example. The authors let the transform matrix S be of the form

$$S=\begin{pmatrix}\mathrm{Id}_m & A\\ 0 & \mathrm{Id}_m\end{pmatrix}\qquad \text{with } A=\begin{pmatrix}\mathrm{Id}_l\\ 0\end{pmatrix}.$$

In coordinates this is $Z_{it}=X_{it}+X_{m+i,t}$ and $U_{it}=X_{m+i,t}$ for $1\le i\le l$, and $Z_{it}=X_{it}$ for $l+1\le i\le m$. To ensure that Z_t has an autonomous linear drift, that is, it does not depend on U_t, the $(m+l)\times(m+l)$-matrix β in (5.19) is chosen of upper block-triangular of the form

$$\beta=S^{-1}\begin{pmatrix}\kappa & 0\\ 0 & A^T\kappa A\end{pmatrix}S=\begin{pmatrix}\kappa & \kappa A-AA^T\kappa A\\ 0 & A^T\kappa A\end{pmatrix}$$

for some $\kappa\in\mathbb{R}^{m\times m}$. While the constant drift term in (5.19) is specified as

$$b=\beta S^{-1}\begin{pmatrix}\theta\\ \theta_U\end{pmatrix}=\begin{pmatrix}\kappa\theta-AA^T\kappa A\theta_U\\ A^T\kappa A\theta_U\end{pmatrix}\tag{5.20}$$

for some $\theta\in\mathbb{R}^m$ and $\theta_U\in\mathbb{R}^l$. So the joint factor process (Z_t,U_t) is

$$\mathrm{d}Z_t=\kappa(\theta-Z_t)\,\mathrm{d}t+\sigma(Z_t,U_t)\,\mathrm{d}B_t\tag{5.21}$$

$$\mathrm{d}U_t=A^T\kappa A(\theta_U-U_t)\,\mathrm{d}t+\mathrm{Diag}\left(\sigma_{m+1}\sqrt{U_{1t}}\,\mathrm{d}B_{m+1,t},\ldots,\sigma_{m+l}\sqrt{U_{lt}}\,\mathrm{d}B_{m+l,t}\right),\tag{5.22}$$

with dispersion function of Z_t given by

$$\sigma(z,u)=(\mathrm{Id}_m,A)\mathrm{Diag}\left(\sigma_1\sqrt{z_1-u_1},\ldots,\sigma_{m+l}\sqrt{u_l}\right).$$

Filipović, Larsson and Trolle (2016) [14] provide a theorem that the parameters of the state price density ζ_t should be chosen such that $\phi=\mathbf{1}$ and $\psi=\mathbf{1}$, with $\mathbf{1}=(1,\ldots,1)^T$ to ensure that the short rate (5.5) is bounded from below. We will take this as given and use it in the empirical study. For a detailed proof consider Theorem 5, the appendix and internet appendix of their paper.

Accordingly, for the empirical part the state price density will be given by $\zeta_t = e^{-\alpha t}(1 + \mathbf{1}^T Z_t)$ and we also assume that it holds that

$$\{\mathbf{1}, \kappa^\top \mathbf{1}, \ldots, \kappa^{(m-1)\top}\mathbf{1}\} = \mathbb{R}^m. \tag{5.23}$$

The last equation follows from the assumption that Z_t itself does not exhibit unspanned components such that Z_t can be reconstructed from the term structure at time t.

The LRSQ(m,l) specification is obtained by choosing $\kappa \in \mathbb{R}^{m\times m}$ with non-positive off-diagonal elements and such (5.23) holds. The mean reversion levels θ and θ_U are chosen such that $b \in \mathbb{R}_+^{m+l}$ and the volatility parameters are $\sigma_1, \ldots, \sigma_{m+l} \geq 0$. This guarantees that a unique solution to the model in (5.19) and thus (5.21) exists.

5.2.1 Swaption Pricing with LRSQ

Under the LRSQ model the function $\hat{q}(x)$ of the swaption pricing formula in Theorem 5.3 can be explicitly solved. Since the process of X_t is affine the following exponential-affine transform formula holds.

Lemma 5.1. ***Exponential-Affine Transform Formula (Filipović et al., 2016, Internet Appendix, p.9 [14]).***
Let X_t be the square-root process (5.19). For any $0 \leq t \leq T$, and $u \in \mathbb{C}$, $v \in \mathbb{C}^{m+l}$ such that $\mathbb{E}\left[|\exp(v^T X_T)|\right] < \infty$ we have

$$\mathbb{E}_t\left[e^{u+v^\top X_T}\right] = e^{\Phi(T-t)+\Psi(T-t)^\top X_t},$$

where $\Phi : \mathbb{R}_+ \to \mathbb{C}$, $\Psi : \mathbb{R}_+ \to \mathbb{C}^{m+l}$ solve the system

$$\begin{aligned}
&\Phi'(\tau) = b^T \Psi(\tau) \\
&\Psi_i'(\tau) = -\beta_i^\top \Psi(\tau) + \frac{1}{2}\sigma_i^2 \Psi_i(\tau)^2, \quad i = 1, \ldots, m+l,
\end{aligned}$$

with initial condition $\Phi(0) = u$, $\Psi(0) = v$, where β_i denotes the ith column vector of β. The solution to this system is unique.

This theorem can be applied to $\hat{q}(x)$ because the joint factor process $(Z_t, U_t) = SX_t$ is a transform of the underlying affine process X_t.

For the rest of the thesis we will restrict the empirical study to the LRSQ(3,3) model, that is a three-factor process Z_t with three USV factors. In the empirical part of the underlying paper the study is done with one, two and three USV factors, whereof they favour the LRSQ(3,3) model specification (Filipović et al., 2016, Chapter IV [14]).

Considering the LRSQ(3,3) model we include a simplification for the vector β_i such that each vector only consists of one parameter β_i in the ith component. This together with its implications for other parameters is discussed in the next chapter. As a result, the system of Lemma 5.1 can be solved directly.

For the application of the exponential-affine transform formula we need to find the according representations for u and v. To do so, we specify the general characterizations from the section before. That is, in the LRSQ(3,3) model, while assuming β_i corresponds to the parameter β_i in the ith component,

$$\beta = \begin{pmatrix} \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_6 \end{pmatrix}, \tag{5.24}$$

the (6×6)-matrix S becomes

$$S = \begin{pmatrix} \mathrm{Id}_3 & A \\ 0 & \mathrm{Id}_3 \end{pmatrix} \quad \text{with } A = \begin{pmatrix} \mathrm{Id}_3 \\ 0 \end{pmatrix}.$$

From the definition of β, (5.20), and the property that it is diagonal, (5.24), we get

$$\beta = \begin{pmatrix} \kappa & \kappa A - AA^T \kappa A \\ 0 & A^T \kappa A \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} \Rightarrow \kappa = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix}. \tag{5.25}$$

With $(Z_t, U_t) = SX_t$ we get for the term structure coordinates that $Z_{it} = X_{it} + X_{i+3,t}$ for $i = 1, 2, 3$. Specifically, in LRSQ(3,3) with $(Z_t, U_t) = SX_t$ this is

$$Z_t = \begin{pmatrix} X_{1t} + X_{4t} \\ X_{2t} + X_{5t} \\ X_{3t} + X_{6t} \end{pmatrix}. \tag{5.26}$$

For the function $\hat{q}(\cdot)$ and Theorem 5.3 we can use the definition of p_{swap}, see (5.15), such that

$$\hat{q}(x) = \mathbb{E}_t \left[e^{x \cdot p_{swap}(Z_{T_0})} \right] = \mathbb{E}_t \left[e^{x \cdot \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)} (Z_{T_0} - \theta) \right)} \right].$$

To apply Lemma 5.1 we need the exponent to be of the form $u + v^\top X_{T_0}$ which we achieve by rearranging the parameters. The exponent can then be split up into a term that depends on Z_t and one that doesn't. Formally, the exponent can be simplified

$$\begin{aligned} x \cdot \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)} (Z_{T_0} - \theta) \right) = \\ = x \Bigg(\sum_{i=0}^{n} c_i e^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)} (-\theta) \right) \\ + \sum_{i=0}^{n} c_i e^{-\alpha T_i} (\psi^\top e^{-\kappa(T_i - T_0)} Z_{T_0}) \Bigg) \\ = \widetilde{u} + \widetilde{v}^\top Z_{T_0}. \end{aligned} \tag{5.27}$$

Since the exponential-affine transform formula is constructed for an affine process we need to express the exponent in terms of X_t, such that

$$\widetilde{u} + \widetilde{v}^\top Z_{T_0} = u + v^\top X_{T_0}.$$

We can then easily observe u from (5.27) and directly set it equal $\widetilde{u}$, specifically

$$u = \widetilde{u} = x \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)} (-\theta) \right).$$

For v we can use the relation of Z_t and X_t in (5.26) and define it such that the transition from Z_{T-0} to X_{T_0} is possible:

$$\begin{aligned}\widetilde{v}^\top Z_{T_0} &= \tilde{v_1}(X_{1T_0} + X_{4T_0}) + \tilde{v_2}(X_{2T_0} + X_{5T_0}) + \tilde{v_3}(X_{3T_0} + X_{6T_0}) \\ &= \tilde{v_1}X_{1T_0} + \tilde{v_2}X_{2T_0} + \tilde{v_3}X_{3T_0} + \tilde{v_1}X_{4T_0} + \tilde{v_2}X_{5T_0} + \tilde{v_3}X_{6T_0} \\ &= v^\top X_{T_0},\end{aligned}$$

with $v^\top = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3)$. The coefficients $\widetilde{v}_i$ can be read of (5.27), precisely,

$$\widetilde{v}^\top = \left(x \cdot \sum_{i=0}^{n} c_i e^{-\alpha T_i} (\psi^\top e^{-\kappa(T_i - T_0)}) \right).$$

Therefore, together with the assumption for the vector β_i, the system of ordinary differential equations in Lemma 5.1 becomes

$$\begin{aligned}\Psi(0) &= \begin{pmatrix} \widetilde{\Psi}(0) \\ \widetilde{\Psi}(0) \end{pmatrix}, \quad \text{with } \widetilde{\Psi}(0) = \widetilde{v} = \left(x \cdot \sum_{i=0}^{n} c_i e^{-\alpha T_i} (\psi^\top e^{-\kappa(T_i - T_0)}) \right)^\top \\ \Phi(0) &= x \cdot \sum_{i=0}^{n} c_i e^{-\alpha T_i} \left(\phi + \psi^\top \theta + \psi^\top e^{-\kappa(T_i - T_0)}(-\theta) \right) \\ \Psi_i'(\tau) &= -\beta_i \Psi_i(\tau) + \frac{1}{2}\sigma_i^2 \Psi_i(\tau)^2 \\ \Phi'(\tau) &= b^\top \Psi(\tau).\end{aligned}$$

Solving the first-order differential equation the components of Ψ are

$$\Psi_i(\tau) = \frac{2\beta_i \Psi_i(0) e^{-\beta_i \tau}}{2\beta_i + \sigma_i^2 \Psi_i(0) \left(e^{-\beta_i \tau} - 1\right)}. \tag{5.28}$$

Precisely, calculating the differential of $\Psi_i(0)$ we arrive at the differential equation,

$$\begin{aligned}\Psi_i'(\tau) &= \frac{-2\beta_i^2 \Psi_i(0) e^{-\beta_i \tau} \left(2\beta_i + \sigma_i^2 \Psi_i(0)(e^{-\beta\tau} - 1)\right) + 2\beta_i^2 \sigma_i^2 \Psi_i^2(0) e^{-2\beta_i \tau}}{\left(2\beta_i + \sigma_i^2 \Psi_i(0) \left(e^{-\beta_i \tau} - 1\right)\right)^2} \\ &= \frac{-2\beta_i^2 \Psi_i(0) e^{-\beta_i \tau}}{2\beta_i + \sigma_i^2 \Psi_i(0) \left(e^{-\beta_i \tau} - 1\right)} + \frac{2\beta_i^2 \sigma_i^2 \Psi_i^2(0) e^{-2\beta_i \tau}}{\left(2\beta_i + \sigma_i^2 \Psi_i(0) \left(e^{-\beta_i \tau} - 1\right)\right)^2} \\ &= -\beta_i \Psi_i(\tau) + \frac{1}{2}\sigma_i^2 \Psi_i(\tau)^2.\end{aligned}$$

Following, we get for Φ

$$\Phi'(\tau) = \sum_{i=1}^{6} b_i \left(\frac{2\beta_i \Psi_i(0) e^{-\beta_i \tau}}{2\beta_i + \sigma_i^2 \Psi_i(0) \left(e^{-\beta_i \tau} - 1\right)} \right)$$
$$\Rightarrow \Phi(\tau) = \sum_{i=1}^{6} b_i \frac{2\beta_i \tau - 2\log\left(2\beta_i e^{\beta_i \tau} - \Psi_i(0)\sigma_i^2 (e^{\beta_i \tau} - 1)\right)}{\sigma_i^2}. \tag{5.29}$$

With these functions we can now solve $\hat{q}(x)$ for the swaption pricing formula explicitly:

$$\hat{q}(x) = \mathbb{E}_t\left[e^{x \cdot p_{swap}(Z_{T_0})}\right] = \mathbb{E}_t\left[e^{u + v^\top X_{T_0}}\right] = e^{\Phi(T_0 - t) + \Psi(T_0 - t)^\top X_t}. \tag{5.30}$$

Together with Theorem 5.3 an analytical pricing of swaptions under the linear-rational square root model with three term structure factors and three USV factors is possible.

Obviously this is only one specification of the presented model and can be adjusted differently. For simplification we assumed a few restrictions to the model to make it easier and more applicable. In the next chapter we justify these constraints and show their implications within in the model for a parameter calibration with real swaption data.

6 Empirical Analysis

The empirical analysis of the linear-rational term structure model focuses on the LRSQ(3,3) specification and aims to calibrate the parameters influencing the swaption pricing formula. For that reason we first need market data on which we can regress our modelled prices on. Since swaption prices are usually stated through implied volatilities our market data contains lognormal volatilities of swaptions for different maturities and swap lengths. The maturities contain one to five years, seven and ten years, while we consider swap lengths from one to ten years. In total the dataset consists of 70 market implied volatilities for EUR ATM-swaptions of August 30, 2013.

At first market swaption prices need to be calculated by using the implied volatilities and since our data consists of lognormal volatilities we apply Black's model. The relevant formula for at-the-money swaptions, see (3.11), is

$$\Pi_t^{swaption} = S_t^{T_0,T_n} \left(\Phi(d_1(t)) - \Phi(d_2(t))\right) \Delta \sum_{i=1}^{n} P(t,T_i). \tag{6.1}$$

In our empirical analysis we only consider swaptions starting from today and we set $t = 0$. Using this Black's formula simplifies such that

$$\Pi_0^{swaption} = \Delta S_0^{T_0,T_n} \left(\Phi(d_1) - \Phi(d_2)\right) \sum_{i=1}^{n} P(0,T_i), \tag{6.2}$$

where $\Phi(\cdot)$ denotes the standard Gaussian cumulative distribution function,

$$d_{1,2} = d_{1,2}(0) = \pm\frac{1}{2}\sigma(0)\sqrt{T_0}, \quad S_0^{T_0,T_n} = \frac{P(0,T_0) - P(0,T_n)}{\Delta \sum_{i=1}^{n} P(0,T_i)}.$$

and $\sigma(0)$ is the prevailing Black's swaption volatility.

Since it holds that $d_1 = -d_2$ we can use the property of the normal distribution that $\Phi(-d_1) = 1 - \Phi(d_1)$ and Black's formula further reduces to

$$\Pi_0^{swaption} = S_0^{T_0,T_n} \left(2\Phi(d_1) - 1\right) \Delta \sum_{i=1}^{n} P(0, T_i). \tag{6.3}$$

Generally, this formula can be used to calculate prices of swaptions starting from today as usually zero-coupon bond prices as well as implied volatilities can be retrieved from the market.

6.1 Simplifications and Restrictions

The empirical analysis is performed within the LRSQ(3,3) model, that is $m = l = 3$.

Since our data set only consists of implied volatilities σ_i the remaining unknown inputs for Black's formula are the zero-coupon bond prices $P(0, T_i)$, for $i = 1, \ldots, n$. Thus, these bond prices need to be calculated too, for what we are using the model-implied formula for zero-coupon bond prices (5.4) where we fix the parameters. We ran some calculations with a few different values for α, θ and β, which implies κ, see (5.25), to find a combination which provides reasonable bond and swaption prices. The selection of values for those parameters is therefore arbitrary. If corresponding data for zero-coupon bond prices is available they could be used in order to calibrate these parameters. Lastly, we also fix values for the process X_0, to receive Z_0 through the dependency $Z_{i,0} = X_{i,0} + X_{m+i,0}$, for $i = 1, 2, 3$.

An additional specification regarding the state price density has been mentioned prior in this thesis and is also used in Filipović et al. (2016) [14, p.16]. Hence, we set $\phi = 1$ and $\psi = (1, 1, 1)^\top = \mathbf{1}^\top$, whereof it follows for the state price density that $\zeta_0 = 1 + \mathbf{1}^\top Z_0$. These steps have to be made to calculate $P(0, T_i)$ and eventually retrieve market swaption prices. For detailed explanations and justifications for this choice consult the underlying paper.

Next, we can use the swaption pricing formula of the linear-rational square-root model and calibrate the remaining parameters, σ and θ_U, with the calculated market prices. The pricing formula Theorem 5.3 is presented in Theorem

4 of Filipović et al. (2016) [14, p.12], which is, with $t = 0$,

$$\Pi_0^{swaption} = \frac{1}{\zeta_0 \pi} \int_0^{\infty} \mathrm{Re}\left[\frac{\hat{q}(\mu + i\lambda)}{(\mu + i\lambda)^2}\right] d\lambda,$$

where $\hat{q} = \mathbb{E}_0[\exp(x \cdot p_{swap}(Z_{T_0}))]$ for $x \in \mathbb{C}$ and $\mu > 0$ such that $\hat{q}(\mu) < \infty$. As already shown it holds that

$$\hat{q}(x) = \mathbb{E}_0\left[e^{x \cdot p_{swap}(Z_{T_0})}\right] = \mathbb{E}_0\left[e^{u + v^\top X_{T_0}}\right] = e^{\Phi(T_0) + \Psi(T_0)^\top X_0},$$

with Φ and Ψ as in (5.29) and (5.28) respectively.

We proceed the analysis within in the model set-up of Section 5.2 and 5.2.1 together with the specific simplifications for the functions $\Phi(\cdot)$ and $\Psi(\cdot)$ which we have mentioned at the beginning of this section. We assume that the column vector β_i contains zeros and the parameter β_i in ith component only. Without this restriction the solution to the equation system would result in a much more difficult calculation. As a consequence thereof other parameters which depend on β get restricted too.

First of all, with this simplification the matrix β is a 6×6 diagonal matrix with parameters $\beta_i = \beta_{ii}$, such that

$$\beta = \begin{pmatrix} \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_6 \end{pmatrix}.$$

This simplifies the equation system of Lemma 5.1 and the calculation of its solution, but might also influence the model in its fit for market data. We don't investigate the impact of this simplification explicitly, as it would open a discussion beyond the scope of this thesis.

Since we are within the LRSQ(3,3) model and $m = l = 3$ we have $A = \mathrm{Id}_3$ and for the matrix κ it follows from the fact that β is a diagonal matrix and from (5.25), that

$$\kappa = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

which implies $\beta_i = \beta_{i+3}$ for $i = 1, 2, 3$, since from (5.25) it has to hold that

$$\beta = \begin{pmatrix} \beta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_6 \end{pmatrix} = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix}.$$

This reduces the number of free parameters in the LRSQ model. Further, the vector b then is

$$b = \beta S^{-1} \begin{pmatrix} \theta \\ \theta_U \end{pmatrix} = \begin{pmatrix} \beta_1(\theta_1 - \theta_{U1}) \\ \beta_2(\theta_2 - \theta_{U2}) \\ \beta_3(\theta_3 - \theta_{U3}) \\ \beta_1\theta_{U1} \\ \beta_2\theta_{U2} \\ \beta_3\theta_{U3} \end{pmatrix}.$$

Additionally, we assume $\beta_i \neq 0$.

6.2 Results

Summarised, our market swaption prices are calculated using implied volatility data and modelled bond prices within the linear rational term structure model. A few parameters which are inputs for the zero coupon bond prices are taken as fixed, whereas the remaining nine parameters which only affect the swaption pricing formula, i. e. θ_U from (5.21) and the volatilities σ_i, are calibrated.

The calibration is performed in R using the function *nls*, for non-linear least squares, and starting values for θ_U and σ. The starting values are chosen randomly but adjusted such that the corresponding modelled swaption prices roughly fit the structure of the market prices.

We can see that even under the simplifications the LRSQ(3,3) model provides a good fit to market data after calibrating parameters. Figure 6.2 shows the calculated market swaption prices versus the modelled swaption prices with calibrated parameters. Both lines are very similar although the model, dotted line, overprices swaptions slightly.

However, even with altered starting parameters the calibration yields similar results which gives the belief that the model can be calibrated to market data very well and produces adequate prices.

Calibrated	**Parameters:**			
θ_{U1}	θ_{U2}	θ_{U3}	σ_1	σ_2
-3.106e-02	-1.264e+00	2.746e+00	-5.251e+01	8.991e-01
σ_3	σ_4	σ_5	σ_6	
5.872e+01	3.773e-01	1.203e+03	-8.666e+01	

Fixed	**Parameters:**			
α	θ_1	θ_2	θ_3	β_1
1.9	0.1	0.1	0.1	0.3
β_2	β_3	X_{01}	X_{02}	X_{03}
0.3	0.3	0.9	0.9	0.9
X_{04}	X_{05}	X_{06}		
0.9	0.9	0.9		

Table 6.1: Model parameters.

The first table shows the calibrated parameters from non-linear regression with a residual standard error of 0.009402. The sum of squared residuals of swaption prices is 0.005397. The second table shows the parameters which were fixed before regression because they were entered into (5.4) to calculate bond prices beforehand.

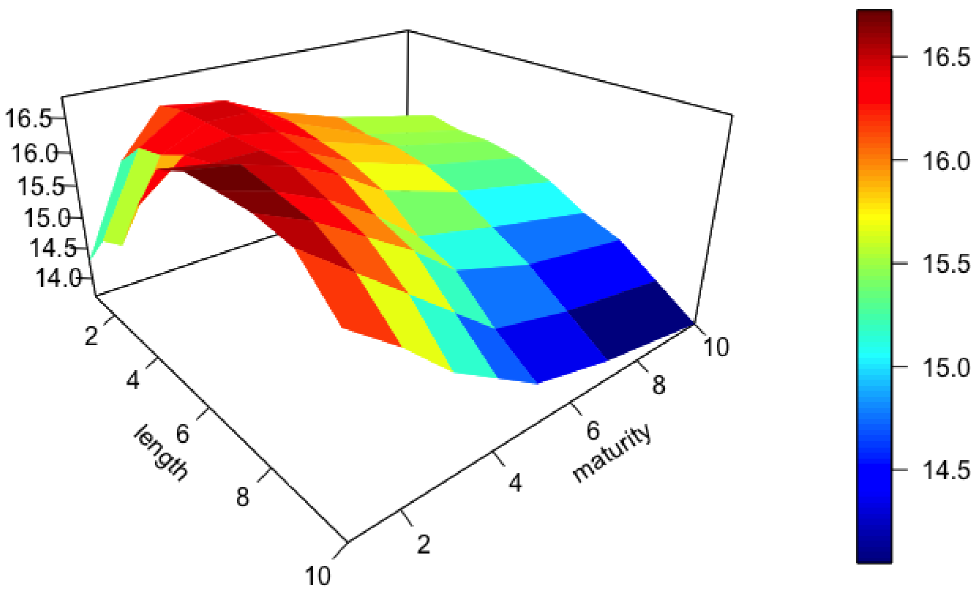

Figure 6.1: Market implied volatility surface.
This figures shows the sample data of market implied volatilities for at-the-money swaptions from August 30, 2013, with maturities 1, 2, 3, 4, 5, 7 and 10 years and swap lengths from 1 to 10 years.

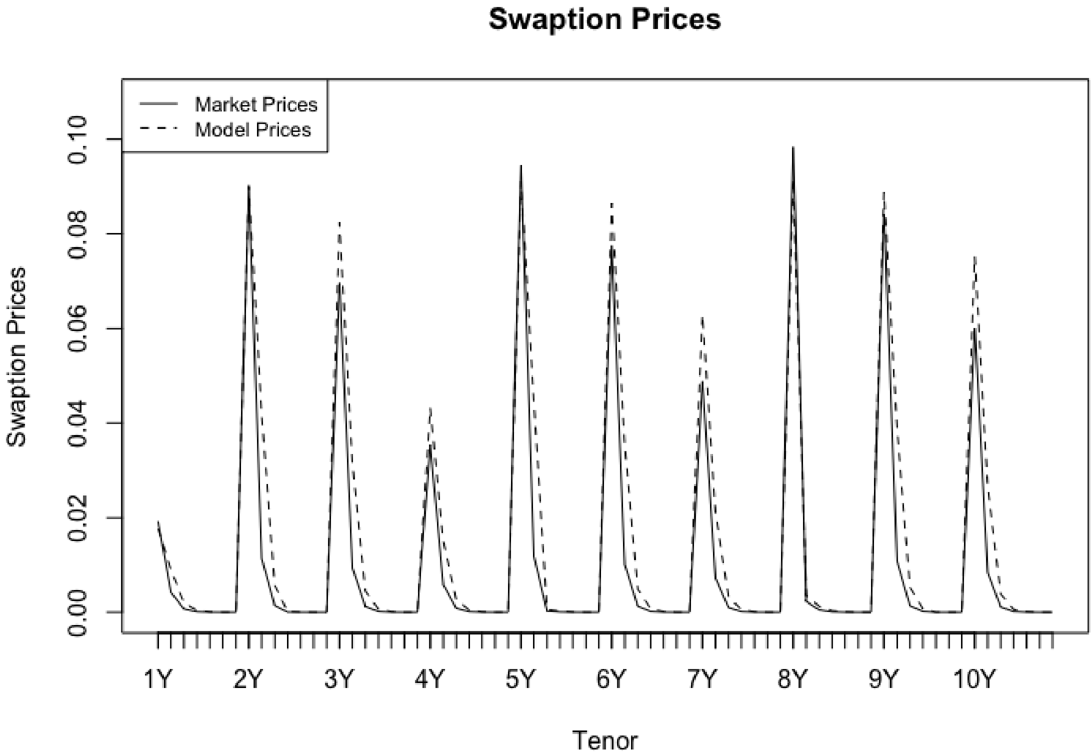

Figure 6.2: Market and model swaption prices.
The labels on the x-axis denote the length of the swap, which ranges from one to ten years. The first seven data points denote swaption prices for a one year swap with corresponding maturities of the swaptions increasing from one to five, seven and ten years. Further, the following seven data points denote prices for swaptions with swap length of 2 years and so on. The solid line shows swaption prices which are calculated using market implied volatilities and modelled bond prices whereas the dotted line displays the modelled swaption prices with calibrated parameters.

7 Conclusion

Filipović et al. (2016) [14] present the so-called linear-rational term structure models in which bond prices become linear-rational functions of the underlying factors. This class of term structure models inherits three important advantages, as i) one can define intervals within which the interest rate process remains by choosing the right parameters, ii) unspanned factors which affect volatility can be accommodated, as well as iii) it admits semi-analytical formulas for swaptions.

The linear-rational term structure model can be modified in various ways to find an appropriate process for observed market data. As we have seen in the empirical analysis of this thesis several parameters can be defined and chosen as desired. This leaves the freedom of adjusting the term structure model in various ways and may influence the level of difficulty of calculating the swaption pricing formula.

Under the selected restrictions and simplifications for the LRSQ(3,3) model the regression of the modelled prices to calculated market prices still provides a good fit. For the simple reason that the analysis would result in a much more difficult calculation we reduced the variable parameters to a minimum such that the calibration doesn't get overly difficult.

However, the model could be influenced in a way that the appropriateness of the term structure might be biased. Nonetheless, even with various restrictions this analysis is able to give an insight into the class of linear-rational term structure models and the advantages it accommodates.

A possible extension to the analysis could be to calibrate all parameters, including those which influence the zero-coupon bond prices. Since we fixed those parameters our calculated bond prices might not reflect the actual market very well which could influence the market swaption prices. That is, we compare partly modelled market swaption prices to fully modelled swaption prices such that the outcome could be biased. This extension to the empirical analysis could be part of an additional study as it would exceed the scope of this thesis.

Overall, the result of the empirical analysis shows that within the linear-rational term structure model an analytical pricing of swaptions even under a few simplifications is possible. Proper swaption prices can be generated and it is feasible to calibrate parameters to fit current market data. This makes the linear-rational term structure especially interesting when swaption pricing is the principal subject of interest.

Appendix

We now present the proof of formula (5.3).

Proof. For the expected value of the state price density in (5.3) we can use the process

$$Y_t = \theta + e^{-\kappa(\tau-t)}(Z_t - \theta),$$

which, using Itô's formula, satisfies $\mathrm{d}Y_t = e^{-\kappa(\tau-t)}\,\mathrm{d}M_t$. Integration by parts of Y_t yields

$$Y_t = Y_0 + e^{-\kappa(T-t)}M_t - \int_0^t M_s\kappa e^{-\kappa(T-s)}\,\mathrm{d}s.$$

Using Fubini's theorem we then get for any $0 \le t \le u$,

$$\begin{aligned}\mathbb{E}_t[Y_u] &= Y_0 + e^{-\kappa(T-u)}M_t - \int_0^t M_{s\wedge t}\kappa e^{-\kappa(T-s)}\,\mathrm{d}s \\ &= Y_t + M_t\left[e^{-\kappa(T-u)} - e^{-\kappa(T-t)} - \int_t^u \kappa e^{-\kappa(T-s)}\,\mathrm{d}s\right] = Y_t,\end{aligned}$$

which shows that Y_t is a true martingale. From the definition of Y_t it holds that $Z_\tau = Y_\tau$, such that $\mathbb{E}_0[Z_\tau] = Y_0 = \theta + e^{-\kappa(\tau-t)}(Z_t - \theta)$. Hence, we have shown the equality for $\tau = T - t$. □

Theorem 7.1. ***(Tonelli)*** *If (X, A, μ) and (Y, B, ν) are σ-finite measurable spaces, while $f : X \times Y \longrightarrow [0, \infty]$ is non-negative and measurable then*

$$\int_X \left(\int_Y f(x,y)dy\right)dx = \int_Y \left(\int_X f(x,y)dx\right)dy = \int_{X\times Y} f(x,y)d(x,y). \quad (7.1)$$

Theorem 7.2. ***(Fubini)*** *If $f(x,y)$ is $X \times Y$ integrable, meaning it is measurable and*
$\int_{X\times Y} |f(x,y)|d(x,y) < \infty$, then

$$\int_X \left(\int_Y f(x,y)dy\right)dx = \int_Y \left(\int_X f(x,y)dx\right)dy = \int_{X\times Y} f(x,y)d(x,y). \quad (7.2)$$

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