MASTERARBEIT/ MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

ROBUST OPTIMIZATION

verfasst von / submitted by

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angestrebter akademischer Grad/in partial fulfilment of the requirements for the degree of

Master of Science (M.Sc.)

Wien, 2017 / Vienna, 2017

Studienkennzahl lt. Studienblatt/ degree programme code as it appears on the student record sheet:	A 066 920
Studienrichtung lt. Studienblatt/ degree programme as it appears on the student record sheet:	Quantitative Economics, Management and Finance
Betreut von / Supervisor:	Univ.-Prof. Mag. Dr. Immanuel Bomze

Acknowledgement

I would first like to thank my supervisor Dr. Immanuel Bomze for his effort and help. In the previous semester I had the opportunity to participate in one of his lectures called "Applied Optimization" that motivated me writing my master's thesis in this field. I highly appreciate his valuable insights as well as his quick feedbacks that helped me working through the underlying topics of this thesis. This work is an important component to fulfil the requirements of my degree and I am glad that Dr. Immanuel Bomze accepted me to write this thesis under his supervision.

Furthermore, I am very grateful to my parents for the wonderful, exciting and informative possibility of studying in Vienna. Their generosity and patience were an important and highly valued support on my educational path. Studying in a city with some distance to my home brought some difficulties along but their understanding, even when I missed some family events, let me enjoy my time.

In addition, I want to thank my colleagues for the collaborations, discussions and cheerfulness during the time of my studies. They made it possible to enjoy the time in university even when exercises, deadlines and exams kept us busy. Among those, I especially want to thank Stefan M. Haas and Jasmin Riegler for their interest, suggestions of improvements and support during this writing process.

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Abstracts

This master's thesis studies optimization problems handling data influenced by uncertainties as they appear in various 'real life' applications. The transformation of a general optimization problem into the according robust optimization problem by developing the robust counterpart is of special interest. This optimization approach can ensure feasibility of solutions without significantly decreasing the optimal value of the objective function for particular problems. Furthermore, specified restrictions on the constraints, the cone in use as well as on the uncertainty set itself influences the structure of the robust counterpart. After giving a short overview of the topic, this thesis is structured by analyzing linear optimization problems with uncertainties, which are well researched and in real life applications the most commonly used ones, and the process of relaxing some restrictions, that lead to more general optimization problems. The quadratic optimization problems and especially the quadratically constrained quadratic optimization problems are in the focus of the last sections. An essence on researching these problems are the tractability properties, which are investigated by trying to reformulate the problems into explicit solvable forms.

Diese Masterarbeit behandelt das Thema Robuste Optimierung. Dies sind Optimierungsprobleme die von Unsicherheiten in den Daten beeinflusst werden, wie dies auch häufig in praktischen Anwendungen der Fall ist. Die Umformulierung eines gewöhlichen Optimierungsproblems in ein robustes Optimierungsproblem steht hier im Vordergrund. Mit dieser Methode der Optimierung kann für Probleme von speziellen Strukturen die Lösbarkeit von Anfang an sichergestellt werden. Ebenso kann garantiert werden, dass eine Lösung alle gewünschten Bedingungen erfüllt. Gezielte Einschränkungen der Bedingungen, des verwendeten Kegels und der Menge, welche die Unsicherheiten der Daten beschreibt, verändert die Struktur des Optimierungsproblems maßgeblich. Beginnend mit einer kurzen Einführung in das Thema ergibt sich die weitere Struktur der Arbeit durch das Analysieren der in den Anwendungen am häufigsten vorkommenden, linearen Optimierungsprobleme und durch das weitere Lockern einzelner Einschränkungen, durch das sich Probleme übergeordneter Strukturen ergeben. Im Mittelpunkt dieses Dokuments ist die Eigenschaft der Lösbarkeit solcher Probleme, welche durch das Umformulieren des ursprünglich Problems in explizit lösbaren Formen untersucht wird.

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1 Motivation

The Robust Optimization approach, which is also called Robust Counterpart approach, can be applied to all generic optimization problems with uncertain or partly uncertain data. These uncertain informations are often separated into an numerical data part, with uncertain data belonging to a uncertainty set, and the structure part, that is known for all possible realizations of the uncertainties. The main structure of the optimization problem is not only defined by the latter one of those but also depends on the choice of the uncertainty set.
For many years data uncertainty has already been an important aspect in the areas of operations research and mathematical programming.
It is quite common that optimization problems which describe real world processes have to work with uncertain data. Nevertheless, some informations are often available that make it possible to work with approximations or mean values. While there are many different sources for data uncertainties in such problems the most frequent ones are:

1. The data values are unknown at the exact moment when the problem is solved. This is the case for problems incorporating future values like returns, temperatures or other state conditions. Nevertheless, some prior values or approximations do exist that help making a forecast. The deviation of those values from the real value are called forecast/prediction errors.

2. Other data components such as parameters of technological processes or certain conditions on remote places (space, deep sea) exist but metering them exactly is not possible. Those values are mostly given by nominal values of rough measurements such that they are often distorted by measurement errors.

3. The third possible explanation for the use of uncertainties is given when the determined solution cannot be realized as calculated. Even if all informations are given and the optimal solution for a process can be detected it might happen that the machines or processes in use are not working precisely enough. These deflections from the optimal values are called implementation errors.

All of the uncertainties from the above sources can be modelled in the same way. Contrary to most of the usual approaches on problems with data uncertainties, the Robust Optimization approach is a practical tool that allows to ensure feasibility of problems. It has been shown in the past that even small changes in the values and parameters can lead to suboptimal, unpractical, or even (highly) infeasible solutions. While there are various ways to deal with data uncertainties we recall the two most popular approaches. Beginning with the most common one of dealing with those kind of problems leads to the Stochastic Optimization approach. Another well known approach is given by the ignorance of the uncertainties in combination with a sensitivity analysis that can be interpreted as a special form of the Stochastic Optimization approach. The most important characteristics and the main differences to the Robust Counterpart approach are discussed in detail in the following paragraph, since the Robust Counterpart approach offers some beneficial characteristics that are not, or only partly given by the above mentioned ones.

1.1 Stochastic vs. Robust Optimization Approach

This approach considers the data to be of random kind that follows a distributional structure. In the best and utopic case this distribution is known, which only happens in a rare number of events. In most practical applications the information on the distribution is sparse. The robust optimization approach seems much more conservative since all the worst-case scenarios are considered. While on the other hand, the Stochastic Optimization approach works with constraints where it usually uses nominal values with particular confidence intervals depending only on the previous observations and simulations. It is possible that in these events no rare outcomes or extreme values were attained. Also the specification or approximations of the underlying distributions are a complicated routine for most processes. Particularly in real-life applications it is difficult to find good and reliable approximations for the probability distribution. Even if the distribution can be considered to belong to a certain family of distributions, a large enough sample size is required to estimate the relevant parameters accurately. Especially for multidimensional probability distributions, an enormous size of observations is required to fit the relevant parameters.

All of the above leads to the suspicion that Stochastic Optimization approaches have to work with strong and crucial simplifications and it is difficult to determine the influence of these simplifications on the quality of the results. Determining the tractability of Stochastic Optimization problems and especially of the ones that

incorporate information via chance-constraints let us observe that the tractability of those is the exception, since the feasible sets are mostly non-convex. For optimization problems with sparse distribution information this approach might become unstable.
On the other side of the coin, the fact that no distribution information is incorporated in the Robust Optimization approach is often held against it. Nevertheless, the consideration of the worst-case scenarios and the accompanied conservatism should be viewed as a huge plus point. Specifically in construction calculations for bridges and truss designs the engineers usually work with thicker (up to 50% more) materials to compensate for the possibility of modelling deviations. This could simpler be taken care of by the Robust Optimization approach through expanding the uncertainty set. For the Stochastic Optimization approach this is more complicated due to the probabilistic structure. If the probability for a scenario gets increased the likelihood of another (or multiple) scenario(s) get reduced.

1.2 'Ignorance' and Sensitivity Analysis vs. Robust Optimization

Besides the Stochastic Optimization approach and the Robust Optimization approach another typical strategy to handle data uncertainties is the Sensitivity Analysis. This approach can be interpreted as a special case of the Stochastic Optimization approach since the values with the largest probabilities or the mean of the uncertain data variables are taken to solve the problem. These nominal data values are taken to be certain such that the problem can be solved without considering the uncertainties. Only after calculating the optimal solution by ignoring the uncertainties, the influence of small changes in the data components are observed and analyzed. This process is applied in the hope that the real values are very close to the used ones and the optimality and the feasibility remains, or at least does not change drastically. But as shown in numerous examples, solutions might become suboptimal or even infeasible and therefore unpractical for small changes.

Overall, this approach is a practical option to simplify a problem and determine a solution in almost no time, but it should not be put to use for problems where the violation of a constraint has a potentionally critical impact as it might be the case for construction problems or in portfolio theory (see [4]).

One of the major characteristics and motivation for the use of the Robust Optimization approach is that the feasibility of a solution can be ensured, while with the Stochastic Optimization approach outcomes with small probabilities might be overlooked. In a general Robust Mathematical Optimization approach, various possible data realizations are taken into account but constraint violations are possible since those violations are only added via a cost term. With this focus on the stability of the results, the feasibility of the results might not always be guaranteed. The Robust Mathematical approach is similar to the Robust Optimization approach if the constraints are taken to be binding and no constraint violations are allowed. This can be accomplished by a significant increase of the cost terms.

To conclude, one of the major characteristics of the Robust Counterpart method in comparison to the other approaches mentioned above is the set-based and deterministic structure instead of the usual stochastic one. A so constructed solution takes every possible realization of the uncertain parameters in the uncertainty set into account. This set-based uncertainty fits the parameter uncertainty well in numerous applications (as seen in [7] & [8]). In consideration of all possible uncertainty realizations the question of tractability and solvability occurs. It is shown in [3] and various other papers that the robust counterpart of a tractable optimization problem does not need to be tractable itself, as there could be infinitely many constraints and the robust counterpart might become a semi-infinite optimization problem. Therefore a focus on tractable results depending on the characteristic of the nominal problem is of special interest. This leads to the structure of this work as particular classes of problems, defined by the objective function and the constraint functions in combination with an specified uncertainty set, are of explicit forms that are efficiently solvable. The structure of an uncertainty set has a significant impact on the tractability of the problem.

Sometimes it might be the case that the Robust Counterpart approach seems too pessimistic to be practical. Since this approach looks mostly on worst-case scenarios, it might be impractical in some situations, as it could be the case in the financial sector (portfolio theory). Nevertheless this approach finds numerous applications in various fields that handle optimization problems with different uncertainties. It is remarkable that such a 'reliable solution' does not decrease the optimal value or the functionality of the original problem significantly. For further information on this property consult [4] and/or [1].

Remark 1. Not only medium-sized problems, as they occur in fields like management, combinatorial optimization, engineering and controlling, benefit from this approach but especially the large-scale problems, like they often arise in signal and image processing problems or quantum mechanics, highly appreciate the 'a priori'-tractability characteristics of this approach.

Like already stated, the missing incorporation of distribution information is sometimes held against the Robust Counterpart approach. However, it is possible to embody a probabilistic guarantee via the design and the dimension of the uncertainty set. This and further benefits of the Robust Optimization approach that go beyond the scope of this work can be reviewed in [9] and other works listed in the bibliography. In this work we concentrate on robust linear optimization problems as well as on more general robust conic optimization problems where the robust counterpart is of a particular structure that lead to interesting insights and feasibility of the problems.

In the following chapter the basic framework and most common notational conventions are introduced. A lot of the definitions and notations are compatible with the ones in [4] and [3]. After introducing the uncertainty set and some practical and commonly used simplification tricks in Chapter 2, we begin analyzing optimization problems with particular structures. Starting with the well-researched linear optimization problems and allowing for uncertainties in the data component, let us develop the related robust counterpart in Chapter 3. After considering the robust counterpart for different (ellipsoidal) uncertainty set structures we continue with 'Robust Conic Optimization' in Chapter 4 by weakening the restriction on the constraint functions and allow for more complex ones. This lets us immediately look on robust quadratic programs and robust quadratically constrained quadratic programs with uncertain data components. By analogy to the 'Robust Linear Optimization' chapter we look at the properties of those problems as well as on those of their robust counterparts.

2 Introduction to Robust Optimization

The definition of an *Optimization Problem* is given by

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x,\zeta) \\ &\text{subject to } \widetilde{F}(x,\zeta) \in \widetilde{\mathbf{K}} \subset \mathbb{R}^m, \end{aligned} \tag{2.1}$$

with $x \in \mathbb{R}^n$ being the vector of decision variables, $\zeta \in \mathbb{R}^K$ being the parameter elements, while $f(\cdot,\cdot)$ and $\widetilde{F}(\cdot,\cdot)$ represent the objective and the constraint functions respectively. In most literature the cone $\widetilde{\mathbf{K}}$ is taken to be convex and also in this and the following chapters we stick to that assumption. With the prerequisite that the data of the optimization problem is only partly known or entirely unknown, an uncertainty set $\mathcal{U} \subset \mathbb{R}^m$ is introduced. This set contains all given and relevant informations about the data ζ. To ensure robustness and feasibility of the solutions the constraints $\widetilde{F}(x,\zeta) \in \widetilde{\mathbf{K}}$ must be fulfilled for every possible realization of $\zeta \in \mathcal{U}$ and an appropriate candidate solution x.
As stated in the previous chapter, this approach considers the worst-case scenarios. Therefore, the following analysis is done with the largest possible value of the objective function.
For an uncertain optimization problem (2.1) with the additional assumption that the objective function is certain (see Section 2.2), the robust counterpart is now defined by

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &\text{subject to } F(x,\zeta) \in \mathbf{K} \subset \mathbb{R}^{m+1}, \forall \zeta \in \mathcal{U}. \end{aligned} \tag{2.2}$$

Remark 2. An objective function that depends on the uncertain parameters can be avoided by introducing a new variable and adapting the constraint function. This procedure is shown in detail in (2.5). The constraint function in use is given by $F(x,\zeta) = \begin{pmatrix} \widetilde{F}(x,\zeta) \\ s - f(x,\zeta) \end{pmatrix} \in \mathbf{K}$ with $\mathbf{K} = \widetilde{\mathbf{K}} \times \mathbb{R}_+$.

The solution to this problem is now called robust feasible solution of (2.1). In the following a solution value to (2.2) is called *robust optimal value* to the uncertain optimization problem (2.1) .

For the Robust Counterpart approach to be considered in practice at all, it must be numerically solvable. In general the robust counterpart (2.2) of an uncertain problem as given by (2.1) cannot be solved since it might be a semi-infinite program depending on the constraints. The necessary and sufficient restrictions that imply solvability of the optimization problems and the according robust counterparts are therefore of special interest. Moreover, for the robust optimization approach to be really useful for applications the problem must not only be theoretical numerically solvable, but we must also be able to find such a solution in an efficient manner. Therefore, the robust counterpart of an uncertain convex problem must be transformed into a solvable convex optimization problem. This transformation does not only depend on the characteristics of the original problem itself but also on the used class of the uncertainty set $\mathcal{U}$.

A convenient type of uncertainty sets is given by ellipsoidal uncertainty sets as explained in the following Section 2.1. These ellipsoidal uncertainty sets have a convenient representation and the necessary flexibility to incorporate various restrictions and structures on the unknown data part.

2.1 Uncertainty Sets

As stated in [2] ellipsoids and intersections of finitely many ellipsoids are reasonable types of uncertainty sets. Ellipsoids have an easy parametric representation, and of equal importance is that various cases of stochastic uncertainties can be expressed as ellipsoidal deterministic uncertainties. One of the practical benefits with this set-based optimization approach is that no complicated assumptions or guesses on distributions have to be made. Ellipsoidal uncertainty sets are practical in various scenarios since partial uncertainties as well as boundaries on certain values can be incorporated. In the following sections we work with ellipsoids in $\mathbb{R}^K$ that are given by the definition

$$\mathcal{E}_1\left(\Pi, \Lambda\right) = \Pi(B) + \Lambda \tag{2.3}$$

with a certain affine embedding from $\mathbb{R}^L$ into $\mathbb{R}^K$ given by Π, the L-dimensional euclidean unit ball $B = \left\{u \in \mathbb{R}^L : \|u\|_2 \leq 1\right\}$ and a linear subspace in $\mathbb{R}^K$ represented by Λ.

For problems with partly certain and partly uncertain data, this structure can be incorporated with an uncertainty set in the form of a *flat ellipsoid*. In the case of restricted data values that are bounded from above and/or from below, they can be represented by the use of *ellipsoidal cylinders* as uncertainty sets. The *standard* $K-$*dimensional ellipsoids,* as well as the just mentioned flat ellipsoids and the ellipsoidal cylinders can be represented by the definition (2.3) above.

A standard K-dimensional ellipsoid is stated by the definition (2.3) if $L = K$ and $\Lambda = \{0\}$. The following figure 2.1 shows a possible function $\Pi(B) + \Lambda$ for the two dimensional case.

Standard Ellipsoid function from $\mathbb{R}^2 \to \mathbb{R}^2$

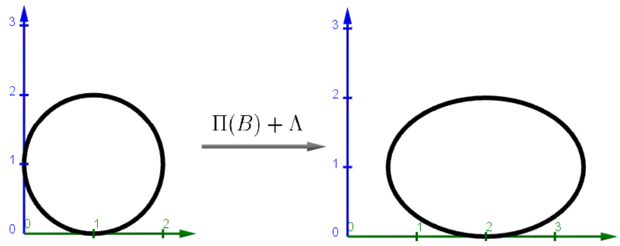

Figure 2.1: Standard Ellipsoid

This figure shows an affine embedding Π *from* $\mathbb{R}^2$ *into* $\mathbb{R}^2$ *with the addition of a subspace* $\Lambda = \{0\}$. *Important to mention is that the dimension of the domain equals the dimension of the codomain from* $\Pi(B) + \Lambda$.

The case of partial uncertainty and the corresponding flat ellipsoid is covered by $\Lambda = \{0\}$ while the dimensions of the affine embedding are of the order $L < K$. A few possible examples of such a projection $\Pi(B) + \Lambda$ are illustrated below.

Flat Ellipsoid function from $\mathbb{R}^2 \to \mathbb{R}^3$

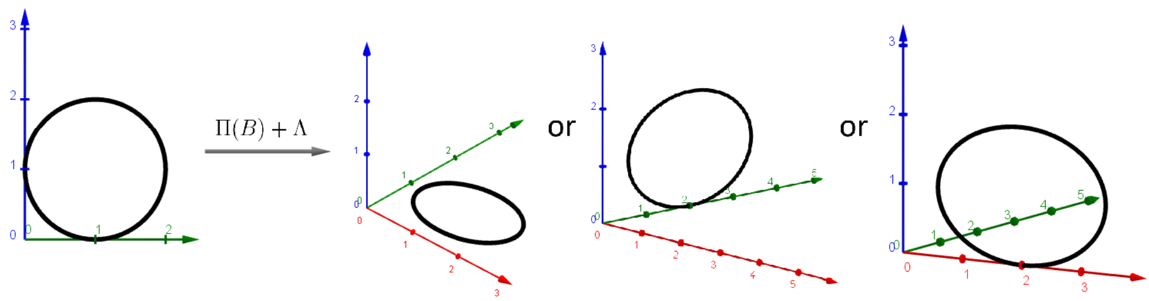

Figure 2.2: Flat Ellipsoid

The crucial point of this projection is that the codomain of the function $\Pi(B) + \Lambda$ *is larger than the domain itself and the additional subspace is again given by* $\Lambda = \{0\}$.

The case of ellipsoidal cylinders can be constructed with a non-trivial Λ and an affine embedding Π. Three possible images of such a function are illustrated by the following figure.

Ellipsoidal Cylinder function from $\mathbb{R}^2 \to \mathbb{R}^3$

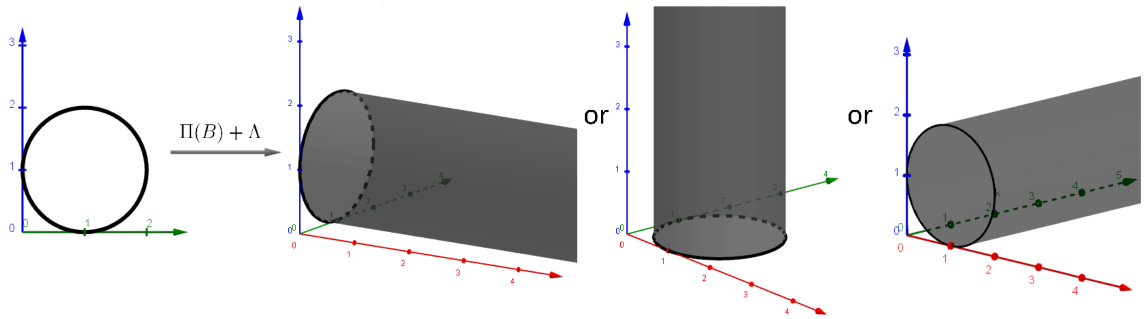

Figure 2.3: Ellipsoidal Cylinder

As in the previous two cases the function Π is still given as an affine embedding while the non-trivial subspace Λ adds the relevant characteristics.

The figures above are simply added for a better understanding of the uncertainty set $\mathcal{U}$, although the images of the function $\Pi(B) + \Lambda$ could look strongly different, e.g. the axis of the ellipsoidal cylinders need not have to be orthogonal to the standard axis/planes. The following definition considers the intersection of such sets.

Definition 1. *We talk about an* <u>*$\cap-$ellipsoidal*</u> *uncertainty set $\mathcal{U}$ if the following three assumptions hold:*

1. *$\mathcal{U}$ can be written as finitely many intersections of ellipsoids with explicitly given affine transformations Π_k and linear subspaces Λ_k such as $\mathcal{U} = \bigcap_{k=1}^{L} \mathcal{E}(\Pi_k, \Lambda_k)$.*

2. *The uncertainty set $\mathcal{U}$ is bounded.*

3. *All ellipsoids $\mathcal{E}(\Pi_k, \Lambda_k)$ for $k \leq L$ have at least one data representative in their relative interior in common:*
   $\exists \zeta \in \mathbb{R}^K : \forall k \leq L \; \exists u_k$ *with* $\|u_k\|_2 < 1, \lambda_k \in \Lambda_k$ *s.t.* $\Pi_k(u_k) + \lambda_k = \zeta \; \forall k \leq L$.

Remark 3. An uncertainty set $\mathcal{U}$ given by only one ellipsoid is simply called <u>*ellipsoidal uncertainty set*</u>.

Remark 4. An ellipsoidal uncertainty set $\mathcal{E}_1(\Pi_1, \Lambda)$ from above, can equivalently be written as $\mathcal{E}_2(\Pi_2, Q) = \{\Pi_2(v) : \|Qv\|_2 \leq 1\}$, with $\Pi_2(\cdot)$ being a certain affine embedding. This property is shown in the following paragraph.

Let an ellipsoid $\mathcal{E}_1(\Pi_1, \Lambda) = \Pi_1(\mathbf{B}) + \Lambda = \{Au + a : \|u\|_2 \leq 1\} + \Lambda$ be given for a particular matrix $A \in \mathbb{R}^{K\times L}$, a vector $a \in \mathbb{R}^K$ and a linear subspace $\Lambda \subset \mathbb{R}^K$. The existence of such an equivalent representation is shown by a case distinction:

Case $\Lambda = \{0\}$: Let the matrix $Q \in \mathbb{R}^{L\times L}$ be given by the $L-$dimensional identity matrix I_l. Furthermore define $C := A$ and $c := a$ to enable the following reformulations

$$\begin{aligned}\Pi_1(\mathbf{B}) + \Lambda &= \{Au + a : \|u\|_2 \leq 1\} + \Lambda \\ &= \left\{Au + a + \lambda t : u^\top u \leq 1, \lambda \in \Lambda, t \in \mathbb{R}\right\} \\ &= \left\{ Cv + c : v^\top \underbrace{Q^\top Q}_{I_L} v \leq 1 \right\} = \mathcal{E}_2(\Pi_2, Q).\end{aligned}$$

Case $\Lambda \neq \{0\}$: The reformulations below are fulfilled for matrix $C := (A, \lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^{K\times(L+m)}$ with the basis vectors $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^L$ from the linear subspace Λ, the matrix $Q := (I_L, 0, \ldots, 0) \in \mathbb{R}^{L\times(L+m)}$, with the L-dimensional identity matrix I_L and the vectors $c := a \in \mathbb{R}^K, t \in \mathbb{R}^m$ and $v^\top = \left[u^\top, t^\top\right]$:

$$\begin{aligned}\Pi_1(\mathbf{B}) + \Lambda &= \left\{Au + a : u^\top u \leq 1\right\} + \Lambda \\ &= \left\{Au + a + \lambda_1 t_1 + \ldots + \lambda_m t_m : u^\top u \leq 1, t_1, \ldots, t_m \in \mathbb{R}\right\} \\ &= \left\{ Cv + c : v^\top \underbrace{Q^\top Q}_{I_L} v \leq 1 \right\} = \mathcal{E}_2(\Pi_2, Q).\end{aligned}$$

After showing that $\mathcal{E}_1(\Pi_1, \Lambda)$ can be restated by $\mathcal{E}_2(\Pi_2, Q)$ we have to show the other direction as well to prove the equality of both formulations.

Let $\mathcal{E}_2(\Pi_2, Q) = \left\{Cv + c : v^\top Q^\top Q v \leq 1\right\}$ be given for explicit matrices $C \in \mathbb{R}^{K\times L}, Q \in \mathbb{R}^{M\times L}$ and a vector $c \in \mathbb{R}^K$. We recall that $Q^\top Q$ is positive semi-definite. Also this direction is shown by the use of a case distinction.

Case 1: $Q^\top Q \in \mathbb{R}^{L\times L}$ is positive definite. Therefore the condition of the ellipsoid can be transformed by using the Cholesky-factorization as shown below:

$$v^\top Q^\top Qv \leq 1 \Leftrightarrow v^\top U^\top \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_L \end{pmatrix} Uv \leq 1$$
$$\Leftrightarrow v^\top U^\top \begin{pmatrix} \sqrt{q_1} & & \\ & \ddots & \\ & & \sqrt{q_L} \end{pmatrix}^\top \begin{pmatrix} \sqrt{q_1} & & \\ & \ddots & \\ & & \sqrt{q_L} \end{pmatrix} Uv \leq 1.$$

The $q_1, q_2, \ldots, q_L > 0$ state the eigenvalues of $Q^\top Q$. Now we obtain the required formulation by defining $u := \begin{pmatrix} \sqrt{q_1} & & \\ & \ddots & \\ & & \sqrt{q_L} \end{pmatrix} Uv$ that leads to the corresponding $v = U^\top \begin{pmatrix} \frac{1}{\sqrt{q_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{q_L}} \end{pmatrix} u$. In this way the ellipsoid $\{Cv + c : \|Qv\|_2 \leq 1\}$ can be given by the formulation

$$\left\{Au + a : u^\top u \leq 1\right\} + \Lambda,$$

for the matrix $A = CU^\top \begin{pmatrix} \frac{1}{\sqrt{q_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{q_L}} \end{pmatrix}$, the vector $a = c$ and the linear subspace given by $\Lambda = \ker Q^\top Q = \{0\}$.

The second case considers a positive semi-definite matrix $Q^\top Q$. We can use the notation from the first case with the modification that $q_1, \ldots, q_n > 0$, while $q_{n+1} = \ldots = q_L = 0$. With this observation, the formulation is given by choosing

$$u := \begin{pmatrix} \sqrt{q_1} & & \\ & \ddots & \\ & & \sqrt{q_n} \end{pmatrix} Uv \in \mathbb{R}^n, A = CU^\top \begin{pmatrix} \frac{1}{\sqrt{q_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{q_n}} \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} \in \mathbb{R}^{k\times L},$$

$a = c$ and a nontrivial subspace $\Lambda = C\left(\ker Q^\top Q\right)$.

It is demonstrated below that such ellipsoidal uncertainty sets are able to represent polytopes. Since polytopes are defined by finitely many intersections of half spaces it is not trivial that they can be designed by an uncertainty set of the above structure. Let the uncertainty set be given in the form of a polytope. The commonly used definition of a polytope is a bounded set defined by finitely many linear inequalities, so the uncertainty set is given (for $d_i \neq 0$) by $\mathcal{U} = \left\{ u \in \mathbb{R}^k \mid d_i^T u \leq r_i, i = 1, 2, \dots M \right\}$. Due to the boundedness assumption of the uncertainty set follows that $d_i^\top u$ attains its minimum. Let this minimum be given by s_i such that the uncertainty set $\mathcal{U}$ can equivalently be given by $\left\{ u \in \mathbb{R}^k \mid s_i \leq d_i^T u \leq r_i, i = 1, 2, \dots M \right\}$.
Let the ellipsoidal set be given as in the previous mentioned form $\mathcal{E}_2 \left(\Pi_2, Q_i \right) = \{ \Pi_2(v) : \|Q_i v\|_2 \leq 1 \}$ with the matrix $Q_i = \frac{2}{r_i - s_i} d_i^\top \in \mathbb{R}^{1 \times K}$. Furthermore, the condition of the ellipsoidal set $v^\top Q_i^\top Q_i v = \left[\frac{2}{(r_i - s_i)} d_i^\top v \right]^2 \leq 1$ implies that $\mid d_i^\top v \mid \leq \frac{r_i - s_i}{2}$. On the other hand, the vector u is given by $u = \Pi_2(v) = I_k v + p_i$, for the $k-$dimensional identity matrix I_k and an arbitrary vector $p_i \in \mathbb{R}^k$. As the condition of the polytope is tested, we obtain that $d_i^\top u = d_i^\top v + d_i^\top p_i \leq \mid d_i^\top v \mid + d_i^\top p_i \leq \frac{r_i - s_i}{2} + d_i^\top p_i \leq r_i$ holds for p_i that fulfil $d_i^\top p_i = \frac{r_i + s_i}{2}$.
To show that the polytopic set is a subset of the ellipsoidal set we suppose that a point x belongs to $\mathcal{U}$ and therefore fulfils $d_i^\top x \leq r_i$. If the condition of the ellipsoidal set is now tested we observe:

$$
\begin{aligned}
v^\top Q_i^\top Q_i v = \left(\frac{2}{r_i - s_i} d_i^\top (x - p) \right)^2 \leq 1 \Leftrightarrow \\
d_i^\top x - d_i^\top p \leq r_i - d_i^\top p = r_i - \frac{r_i + s_i}{2} = \frac{r_i - s_i}{2}.
\end{aligned}
$$

Therefore it is shown that polytopes can be described by ellipsoidal sets as defined in this chapter.

Not only the easy representation and the possibility to express polytopes but also the reason that more complicated sets can be well approximated by intersections of ellipsoidal uncertainty sets justifies their use.
An ellipsoidal uncertainty set in applications is often given in the form

$$
\mathcal{U} = \left\{ D_o + \sum_{i=1}^{L} p_i D_i : p \in \mathcal{P} \right\}, \tag{2.4}
$$

with D_0 denoting the nominal values and p belonging to a perturbation set $\mathcal{P}$.

The robust feasible solution set is given by

$$\mathcal{X} = \{x : F(x,\zeta) \in \mathbf{K}\ \forall \zeta \in \mathcal{U}\} = \bigcap_{\zeta \in \mathcal{U}} \{x : F(x,\zeta) \in \mathbf{K}\}.$$

For the main part of this work we focus on this specified uncertainty sets. In order to keep the notation for the uncertainty sets relatively easy and short a few simplifications are presented in the following Section 2.2.

2.2 Simplification

This section introduces a few very useful tools for the following chapters that can be used to handle problems with uncertain data values. Since the tractability of a general robust optimization problem is not given in most of the cases, some simplification steps are introduced to keep the search for tractable problems as simple as possible.
To catch up the last point of the previous section we start this chapter with a standardization trick for the perturbation sets.

Remark 5. Whenever a perturbation set $\mathcal{P}$ can be given as an image of a different perturbation set $\bar{\mathcal{P}}$ by an affine mapping $p \mapsto \bar{p} = \alpha + Pp$, it can be switched between those perturbations without changing the relevant structure of the problem. Because of this statement more complicated forms can be represented by standard and normalized geometries. In particular, some perturbation sets with 'box-like' geometries such as parallelotopes or rectangles can be reformulated by an affine mapping to perturbation sets given by the unit box $\left\{u \in \mathbb{R}^L : -1 \leq u_l \leq 1, l = 1, 2 \ldots, L\right\}$. In the same way, can we represent ellipsoidal and 'circle-like' perturbation sets with the help of the standard euclidean ball $\left\{b \in \mathbb{R}^L : \|b\|_2 \leq 1, l = 1, 2, \ldots, L\right\}$. Those reformulations for the sets work as follows:

$$\begin{aligned}
\mathcal{U} &= \left\{ D_0 + \sum_{j=1}^{L} p_j D_j : p \in \mathcal{P} \right\} \\
&= \left\{ D_0 + \sum_{j=1}^{L} \left[\alpha_j + \sum_{k=1}^{K} P_{j,k} \bar{p}_k \right] D_j : \bar{p} \in \bar{\mathcal{P}} \right\} \\
&= \left\{ \underbrace{\left[D_0 + \sum_{j=1}^{L} \alpha_j D_j \right]}_{\bar{D}_0} + \sum_{k=1}^{K} \bar{p}_k \underbrace{\left[\sum_{j=1}^{L} P_{j,k} D_j \right]}_{\bar{D}_k} : \bar{p} \in \bar{\mathcal{P}} \right\}.
\end{aligned}$$

The first step in developing the robust counterpart of a general optimization problem given by (2.1) is to simplify the objective function. By adding a new variable s and attaching an additional constraint $s - f(x, \zeta) \geq 0$ to the optimization problem, the objective function can be written as a linear function without dependency on the data uncertainties

$$\min_{s, x \in \mathbb{R}^n} \left\{ s : s - f(x, \zeta) \geq 0, \widetilde{F}(x, \zeta) \in \widetilde{\mathbf{K}} \subset \mathbb{R}^{m-1} \right\}. \tag{2.5}$$

By assuming that this was already done we continue with an uncertain conic optimization problem of the form

$$\min_{x\in\mathbb{R}^n} \left\{ c^T x + d : F(x,\zeta) \in \mathbf{K} \subset \mathbb{R}^m \right\}, \tag{2.6}$$

with the new $F(\cdot,\cdot)$ already incorporating the additional constraint $s - f(x,\zeta)$.

The assumption of a certain objective function and in awareness that a shift term d only influences the optimal value and not the solution itself, this term can be left aside and be taken care of after finding an optimal solution for the problem without this part. With an optimal solution x^* for the reduced objective function $c^\top x$ the term $c^\top x^* + \sup\{d\}$ solves the robust counterpart if d belongs to the projection of $\mathcal{U}$.

So we finally end up with an even simpler form of the optimization problem given by

$$\min_{x\in\mathbb{R}^n} \left\{ c^T x : F(x,\zeta) \in \mathbf{K}\ \forall \zeta \in \mathcal{U} \right\}, \mathbf{K} \subset \mathbb{R}^m, \mathcal{U} \subset \mathbb{R}^k. \tag{2.7}$$

This simplifications are used in the following chapters and give us the possibility to look at linear and uncertainty independent objective functions. In a broad spectrum of practical applications linear optimization problems are in use. While data uncertainties can source from various different reasons they can be treated in similar ways. Due to the high usability of linear optimization problems in numerous areas such as statistics, engineering and finance they are put on a closer look in the following chapter.

3 Robust Linear Optimization

3.1 Uncertain Linear Optimization Problems

We start this chapter with explaining a standard linear optimization problem and continue by looking at the robust counterpart of the original problem arising from allowing uncertainties in the data.
For most applications of Linear Optimization problems, the data components are not known exactly. In these cases of data uncertainties the real values are often estimated or approximated. While some data are simply not known exactly, other data uncertainties arise from the fact that they cannot be measured or implemented exactly enough to describe the problem sufficiently. In various applications of linear optimization problems with very small uncertainties around 1% the deviation from the real values are mostly ignored and the problems are solved as in a case with no uncertainty. This method is used with the belief that small changes in the data do not influence the feasibility or the optimality properties of the obtained optimal solution. But this "good hope" does not always hold true as it can easily be shown (Example 1.1.1. in [6]).

Definition 2. *A general* Linear Optimization *problem is given by*

$$\min_x \left\{ c^T x + d : Ax \leq b \right\}, \tag{3.1}$$

where $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ denote the objective while $x \in \mathbb{R}^n$ represents the vector of decision variables.

These decision variables can describe various different processes from actions up to some conditions like temperature or pressure. The matrix $A \in \mathbb{R}^{m \times n}$ and the vector $b \in \mathbb{R}^m$ define the constraints former introduced in (2.1) as $\widetilde{F}(x, \zeta)$.

Remark 6. For $f(x) = c^\top x + d, \zeta = (A, b), F(x, \zeta) = b - Ax$ and the cone $\mathbf{K}$ being the $m-$dimensional positive orthant $\mathbb{R}^m_+$ in the definition (2.2), we obtain the form (3.1).

The variables c, d, A and b might be affected by uncertainties. Therefore an uncertainty set $\mathcal{U}$ was introduced in Section 2.1 that describes and incorporates the uncertain parameters. The problem does not have a predefined method to be solved if the uncertainties influence the variables c and/or d of the objective function. Instead of handling the linear optimization problem with data uncertainties in c, d, A and b we usually use the before stated simplification steps (2.5) to get a linear and uncertainty independent objective function.

Further to mention is, that the constraints of linear optimization problems are usually written in various forms of equalities and inequalities. Nevertheless, those can be rewritten equivalently as constraints of the form $a^T x \leq b$ as it is shown for some cases in the following Section 3.1.1. After this is shown it is sufficient to assume that all constraints of the original optimization problem are given in the form of $Ax \leq b$.

3.1.1 Equivalent Constraint Representations

An important point to mention is, that the constraints of the uncertain linear optimization problem can be given in numerous forms of equalities and inequalities. Without trouble, these forms can be reformulated to end up with only linear inequalities of the same kind. However not all reformulations of the constraints lead to the same robust counterpart of the nominal problem. Sometimes, the robust counterpart can become more or less conservative [6] and this might lead to huge differences in solving the problem or even could induce infeasibility of the robust counterpart.
Additional to mention is, that slack variables might contradict the first of the three assumptions (3.1.1) in this section and should therefore be eliminated and avoided when reformulations are made. In the same sense should state variables, as they often appear in supply chain problems, be avoided since they might cause a different robust counterpart.

The legitimate transformations of constraints that lead to equivalent robust counterparts of the nominal optimization problem are called *equivalent constraint representations.*

Definition 3. *We call a set $\bar{X} \subset \mathbb{R}_x^n \times \mathbb{R}_u^m$ an equivalent representation of another set $X \subset \mathbb{R}_x^n$ if the projection of $\bar{X}$ onto the space of x-variables is X.*

This can be clarified in the following example: A vector x belongs exactly to a set X when there exists a another vector $u \in \mathbb{R}_u^k$ such that $(x, u) \in \bar{X}$. The set X can be formally stated as $X = \left\{x : \exists u : (x, u) \in \bar{X}\right\}$. With the help of this definition nonlinear inequalities like

$$|x| + |y| \leq 1$$

can equivalently be represented by the five linear inequality constraints

$$-u_1 \leq x_1 \leq u_1, -u_2 \leq y \leq u_2, u_1 + u_2 \leq 1.$$

In the same manner the single constraint $\sum_{j=1}^{n} |x_j| \leq 1$ can be represented by the $2n + 1$ inequality constraints

$$-u_i \leq x_i \leq u_i,\ i = 1, \ldots, n,\ \sum_{i=1}^{n} u_i \leq 1.$$

For an arbitrary optimization problem in the general form of

$$\min_{x} \left\{f\left(x\right) \mid x \text{ fulfills } \mathcal{S}_i, i = 1, 2, \ldots, m\right\}, \tag{3.2}$$

with a constraint system $\mathcal{S}_i$, we can now find an equivalent representation of the problem with an amended constraint system $\bar{\mathcal{S}}_i$ by

$$\min_{x, y_1, \ldots, y_m} \left\{f\left(x\right) \mid (x, y_i) \text{ fulfills } \bar{\mathcal{S}}_i, i = 1, 2, \ldots, m\right\}, \tag{3.3}$$

with constraints variables (x, y_i). If a solution (x, y_i) is feasible for (3.3) then the first component x is also feasible for (3.2) and since the objective is identical they also have the same value. This implies that also the optimal values are identical. The main motivation of this equivalent representation is that the new representation might have wanted additional characteristics. The goal of this paragraph is to find a representation with finitely many convex constraints instead of a semi-infinite problem, which finally leads to an explicit convex and computationally tractable problem. For further information on this refer to [6].

If we look at a single row of an inequality equation $a^\top x \leq \beta$ from the system of linear constraints $Ax \leq b$ with a and β belonging to an uncertainty set $\mathcal{U}$ given by

$$\mathcal{U} = \left\{ [a, \beta] = \left[a^0, \beta^0\right] + \sum_{k=1}^{L} p_k \left[a^k, \beta^k\right] : p \in \mathcal{P} \right\},$$

then the *equivalent representation* for the robust counterpart is given by

$$\left\{ a^\top x \leq \beta \; \forall \, [a, \beta] \in \mathcal{U} \right\}.$$

Before analyzing the structures of robust counterparts for general ellipsoidal uncertainty sets, two examples with specified uncertainty sets are presented. The application of the just mentioned steps for two particular cases of perturbation sets $\mathcal{P}$, are shown in the following examples.

Example 1. With an interval uncertain perturbation set in the form of the unit box given by $\mathcal{P} = \left\{p \in \mathbb{R}^L : \|p\|_\infty \leq 1\right\}$, the robust counterpart can equivalently be represented by inequality constraints:

$$
\begin{aligned}
& a^\top x \leq \beta,\ \forall [a, \beta] \in \mathcal{U} \\
\Leftrightarrow & \left[a^0\right]^\top x + \sum_{l=1}^{L} p_l \left[a^l\right]^\top x \leq \beta^0 + \sum_{l=1}^{L} p_l \beta^l,\ \forall p : \|p\|_\infty \leq 1 \\
\Leftrightarrow & \sum_{l=1}^{L} p_l \left[a^l\right]^\top x - \sum_{l=1}^{L} p_l \beta^l \leq - \left[a^0\right]^\top x + \beta^0,\ \forall p : |p_l| \leq 1,\ l = 1, \ldots, L \\
\Leftrightarrow & \sum_{l=1}^{L} p_l (\left[a^l\right]^\top x - \beta^l) \leq - \left[a^0\right]^\top x + \beta^0,\ \forall p : |p_l| \leq 1,\ l = 1, \ldots, L \\
\Leftrightarrow & \max_{-1 \leq p_l \leq 1} \sum_{l=1}^{L} p_l (\left[a^l\right]^\top x - \beta^l) \leq - \left[a^0\right]^\top x + \beta^0 \\
\Leftrightarrow & \sum_{l=1}^{L} | \left[a^l\right]^\top x - \beta^l | \leq \beta^0 - \left[a^0\right]^\top x
\end{aligned}
$$

The last line can equivalently be represented by the $2L + 1$ inequality constraints

$$
-u_l \leq \left[a^l\right]^\top x - \beta^l \leq u_l \ \textit{for}\ l = 1, 2, \ldots, L, \quad \text{and} \ \sum_{l=1}^{L} u_l + \left[a^0\right]^\top x \leq \beta^0,
$$

as shown previously in this section.

Example 2. Suppose in the following example that the perturbation has the form of a centred ball with radius r and is represented by the set $\mathcal{P} = \left\{p \in \mathbb{R}^L : \|p\|_2 \leq r\right\}$. For this structure the reformulation to a problem with convex constraints looks as follows:

$$
\begin{aligned}
& a^\top x \leq b, \; \forall [a,b] \in \mathcal{U} \\
\Leftrightarrow\ & \left[a^0\right]^\top x + \sum_{l=1}^{L} p_l \left[a^l\right]^\top x \leq b^0 + \sum_{l=1}^{L} p_l b^l, \; \forall p : \|p\|_2 \leq r \\
\Leftrightarrow\ & \sum_{l=1}^{L} p_l \left[a^l\right]^\top x - \sum_{l=1}^{L} p_l b^l \leq -\left[a^0\right]^\top x + b^0, \; \forall p : \|p\|_2 \leq r \\
\Leftrightarrow\ & \sum_{l=1}^{L} p_l (\left[a^l\right]^\top x - b^l) \leq -\left[a^0\right]^\top x + b^0, \; \forall p : \|p\|_2 \leq r \\
\Leftrightarrow\ & \max_{\|p\|_2 \leq r} \sum_{l=1}^{L} p_l (\left[a^l\right]^\top x - b^l) \leq -\left[a^0\right]^\top x + b^0 \\
\Leftrightarrow\ & r \sqrt{\sum_{l=1}^{L} \left([a^l]^\top x - b^l\right)^2} \leq b^0 - \left[a^0\right]^\top x \\
\Leftrightarrow\ & \left[a^0\right]^\top x - b^0 + r \sqrt{\sum_{l=1}^{L} \left([a^l]^\top x - b^l\right)^2} \leq 0.
\end{aligned}
$$

The last line states a conic quadratic inequality which is also a tractable convex constraint.

Definition 4. *The definition of an uncertain linear optimization problem in general is given by the set*

$$\left\{ \min_x \left\{ c^T x + d : Ax \leq b \right\} \right\}_{(c,d,A,b) \in \mathcal{U}} \tag{3.4}$$

of linear optimization problems where the data variables take values in the uncertainty set $\mathcal{U} \subset \mathbb{R}^{(m+1)\times(n+1)}$.

Without applying the simplification steps we combine the data from the linear programming problem (3.1) into a matrix $D \in \mathbb{R}^{m+1 \times n+1}$ such as

$$D = \begin{bmatrix} c^T & d \\ A & b \end{bmatrix}.$$

The uncertainty set is defined as in Section 2.1 and therefore given by

$$\mathcal{U} = \left\{ \underbrace{\begin{bmatrix} c^T & d \\ A & b \end{bmatrix}}_{D} = \underbrace{\begin{bmatrix} c_0^T & d_0 \\ A_0 & b_0 \end{bmatrix}}_{D_0} + \sum_{l=1}^{L} p_l \underbrace{\begin{bmatrix} c_l^T & d_l \\ A_l & b_l \end{bmatrix}}_{D_l} : p \in \mathbf{P} \subset \mathbb{R}^L \right\}. \tag{3.5}$$

The uncertain data D is separated as explained in (2.4) into a nominal part D_0 and a shift part D_l with a factor p_l from a perturbation set $\mathbf{P}$. Without considering the kind of uncertainties that we are dealing with for different problems, the nominal part does not change for different realizations of the problem and therefore defines the major characteristic of the program in question. The shift-part on the other hand only depends on the possible realizations of the values belonging to the uncertainty set.

To develop the robust counterpart of an uncertain linear problem like (3.4) three assumptions are defined to set the basic structure of the framework (for more details see [6]).

3.1.1. Assumptions:

1. First, all values in the decision vector x should be specific numerical values assigned according to the knowledge of the data given at that exact moment. These values are determined before the actual values are finally known.
2. Second, the choice of the uncertainty set is crucial because the person in charge is accountable for the obtained results only if the revealed data belongs to the predefined uncertainty set $\mathcal{U}$.
3. The third assumption is that all constraints of the uncertain linear optimization problem must be fulfilled for data realizations belonging to the specified uncertainty set.

Remark 7. The combination of the second and the third assumption assures a solution $x \in \mathbb{R}^n$ of the uncertain linear optimization problem to be robust feasible. Explicitly, a vector $x \in \mathbb{R}^n$ is called *robust feasible* if $Ax \leq b$ is fulfilled for all uncertain realizations of c, d, A, b belonging to a specified uncertainty set $\mathcal{U}$.

Remark 8. These assumptions above do not guarantee the uniqueness of a solution. This instance is taken care of by taking the supremum of the objective function which represents a "worst-case" scenario for all relevant data from the uncertainty set. These assumptions give us some important characteristics, concerning the feasibility and the boundedness of the nominal problem, that are required later in Section 3.5.

Now all prerequesites are introduced to define the robust counterpart of the uncertain linear optimization problem.

3.2 Robust Counterpart of Uncertain Linear Optimization Problems

Definition 5. *The definition of a* Robust Counterpart *to an uncertain linear problem is given by*

$$\min_{x \in \mathbb{R}^n} \left\{ \sup_{(c,d,A,b) \in \mathcal{U}} \left[c^T x + d \right] : Ax \leq b, \ \forall (c, d, A, b) \in \mathcal{U} \right\}. \tag{3.6}$$

As already mentioned in (2.5) we can consider an objective function independent of the data uncertainties and without a shift term d. Therefore, also the robust counterpart simplifies to

$$\min_{x\in\mathbb{R}^n}\left\{\sup_{(A,b)\in\mathcal{U}}\left[c^Tx\right] : Ax\leq b\ \forall(A,b)\in\mathcal{U}\right\}. \tag{3.7}$$

In order to develop a consistent uncertainty set that is convex and closed, it is necessary to show that the constraints of an uncertain linear optimization problem can be interpreted constraint-wise. Since the objective function can be taken as certain, the uncertainty set is solely defined by the single rows of the constraints. Already works in 1973 [19] considered the column-wise uncertainty dependency which might be even more conservative since values can take the worst scenario realizations in all instances. The row constraint-wise version incorporates the more realistic assumption that not all worst case scenarios can be realized at the same time. Let a_i denote the i-th row of the matrix A while b_i represents the i-th entry of the vector b, it follows:

$$(Ax)_i\leq b_i\Rightarrow a_ix\leq b_i.$$

This yields for the robust counterpart

$$a_ix\leq b_i\ \forall[a_i;b_i]\in\mathcal{U}_i,$$

with $\mathcal{U}_i := \{[a_i;b_i]:[A;b]\in\mathcal{U}\}$ being a projection of $\mathcal{U}$. Let now $\bar{x}$ be a robust feasible solution to the i-th constraint $a_ix\leq b_i$, then $\bar{x}$ is also robust feasible if the uncertainty set $\mathcal{U}_i$ is replaced by its convex hull $conv(\mathcal{U}_i)$. If we have $[\hat{a}_i;\hat{b}_i]\in conv(\mathcal{U}_i)$ then there exists a linear combination $\sum_{j=1}^{J}\lambda_j[a_{j,i};b_{j,i}]$ with $[a_{j,i};b_{j,i}]\in\mathcal{U}_i$, $\lambda_j\geq 0$ and $\sum_{j=1}^{J}\lambda_j=1$ such that $[\hat{a}_i;\hat{b}_i]=\sum_{j=1}^{J}\lambda_j[a_{j,i};b_{j,i}]$. With this structure we can easily prove that the i-th constraint also holds for the convex uncertainty set:

$$\hat{a}_i\bar{x}=\sum_{j=1}^{J}\lambda_ja_{j,i}\bar{x}\leq\sum_{j=1}^{J}\lambda_jb_{j,i}=\hat{b}_i.$$

Therefore we can assume that the given uncertainty set of an uncertain linear optimization problem is convex. With a similar argumentation it can be shown that the set of robust feasible solutions remains unchanged if the uncertainty set is replaced by its closure. These properties let us rewrite the uncertainty set for a robust counterpart of an uncertain linear optimization problem from (3.5) as the direct product of convex and closed uncertainty sets defined by the row-wise constraints:

$$\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \dots \mathcal{U}_m \text{ where } \mathcal{U}_i = \left\{ [a_i^0; b_i^0] + \sum_{l=1}^{L_i} p_l [a_i^l; b_i^l] : p \in \mathcal{P} \right\} \tag{3.8}$$

for a convex and closed perturbation set $\mathcal{P}$ and $1 \leq i \leq m$. The individual uncertainty sets are solely defined by the single row-constraints and have no dependencies with each other.

Now the solvability question arises: 'Are general robust counterparts tractable?' It is not clear from the beginning that the robust counterpart of an uncertain linear optimization problem is solvable. In general the robust counterpart might have infinitely many constraints and can be NP-hard to solve as shown below (Example 1.2.7 in [6]):

Let the constraints of an uncertain linear program be given by $\{\|Ax - a\|_1 \leq 1\}_{[A,a]\in\mathcal{U}}$ where the uncertainty only influences the vector a while the matrix A is certain.[1] The vector a is characterized by the perturbations in a: $\{a = B\zeta : \|\zeta\|_\infty \leq 1\}$ for a positive semidefinite matrix B.[2] This restricts the perturbations to be in the unit box. Verifying if $x = 0$ is robust feasible is equivalent to analyzing if $\|B\zeta\|_1 \leq 1$ holds whenever $\|\zeta\|_\infty \leq 1$. With the fact that $\|z\|_1 = \max\limits_{\|\alpha\|_\infty \leq 1} \alpha^\top z$ it follows that the problem can be rewritten to

$$\max_{\alpha,\zeta} \left\{ \alpha^\top B\zeta : \|\alpha\|_\infty \leq 1, \|\zeta\|_\infty \leq 1 \right\} \leq 1.$$

By using the Cauchy-Schwarz inequality

$$\begin{aligned} \alpha^\top B\zeta &= \left(\sqrt{B}\alpha\right)^\top \left(\sqrt{B}\zeta\right) \leq \left\|\sqrt{B}\alpha\right\|_2 \left\|\sqrt{B}\zeta\right\|_2 \\ &\leq \max_{\|\alpha\|_2 \leq 1} \sqrt{\alpha^\top B\alpha} \max_{\|\zeta\|_2 \leq 1} \sqrt{\zeta^\top B\zeta} = \max_{\|\zeta\|_2 \leq 1} \zeta^\top B\zeta \end{aligned}$$

and α equals ζ without loss of generality for a positive semidefinite matrix B, with its symmetric square-roots matrix.

To sum up the above, analyzing if $x = 0$ is a robust feasible solution for the program with the linear inequality is analog to checking if the maximum, with ζ belonging to the unit box, of the nonnegative quadratic form $\zeta^\top B\zeta$ is smaller than

[1] Where $\|\cdot\|_1$ states the 'Taxicab/Manhattan' norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$.
[2] The $\|\cdot\|_\infty$ states the 'infinity/maximum' norm: $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$.

one. This problem is well known as 'max-cut problem' and is NP-hard to solve [15]. Therefore, also our problem of interest is NP-hard to solve.
So the answer to the above question is 'No' in most cases, but some restrictions on the uncertainty sets lead to tractable robust counterparts for linear optimization problems with uncertainties. This specifications are in the focus of the next section.

3.3 Tractability of Robust Counterparts

The general robust counterpart of a linear optimization problem with uncertainties can be reformulated to

$$\min \left\{c^T x : x \in G_{\mathcal{U}}\right\}, \quad \text{for } G_{\mathcal{U}} = \{x : Ax \leq b,\ \forall [A;b] \in \mathcal{U}\}$$

with $G_{\mathcal{U}}$ being closed and convex as shown in the previous section. Without restricting the uncertainty set to be of ellipsoidal structure we look at some properties. Various works (e.g. [14]) show that an efficient separation oracle is sufficient for a linear objective function to be efficiently minimized over a closed and convex set. An efficient separation oracle for a given set $G \subset \mathbb{R}^n$ states for an input vector $y \in \mathbb{R}^n$ if the vector y belongs to the set G or not. In the case that y does not lie in G the separation oracle returns a separator $s_y^\top \in \mathbb{R}^n$ for G and y like e.g. $s_y y > \underset{x \in G}{sup}\{s_y x\}$. With this observation the tractability question for the robust counterpart of a linear optimization problem changes to the possible geometrical forms of uncertainty sets $\mathcal{U}$ that have an efficient separation oracle. To simplify the notation we reformulate the uncertainty set and the notation of the problem from $\mathcal{U}(x) = \{Ax \leq b : [A;b] \in \mathcal{U}\}$ to

$$\mathcal{U}(x) = \left\{Ax \geq 0 : A \in \mathcal{U}, f^\top x = 1\right\}.$$

Remark 9. For reasons of simplicity the constraints are often reformulated as $Ax \geq 0$. This formulation is obtained by defining $c := (c, 0)^\top, A := (-A, b)^\top$ and $x := (x, 1)^\top$ in the above definition (3.1) of a linear optimization problem.

This notational convention is solely introduced for a simpler representation and makes reformulations more understandable and convenient as seen in Section 3.5. Notable for the following is that the earlier stated (in 2.1) cone in this particular case of linear optimization problems is the positive orthant $\mathbb{R}^m_+$.

It is shown in the following that instead of looking for an efficient separation oracle we can look for an efficient inclusion oracle. The latter one should report for an input vector $y \in \mathbb{R}^n$ if the convex set $\mathcal{U}(y) = \{Ay : A \in \mathcal{U}\}$ is a subset of the nonnegative orthant $\mathbb{R}^m_+$. If it is not a subset of the nonnegative orthant $\mathbb{R}^m_+$ a matrix $A_y \in \mathcal{U}$ is returned as separator like in the case before, such that $A_y y$ does not belong to $\mathbb{R}^m_+$.
Now, for any given vector $x \in \mathbb{R}^n$ combined with an efficient inclusion oracle we can construct an efficient separation oracle as follows:
The simple condition $f^\top x = 1$ is tested first. If this constraint does not hold then obviously x does not belong to the also adapted $G_\mathcal{U} = \left\{x : Ax \geq 0,\ \forall [A] \in \mathcal{U}, f^\top x = 1\right\}$ and a separator for x and $G_\mathcal{U}$ is given by f in case $f * x - 1 < 0$ (or respectively $-f$ in case of , $f^\top x - 1 > 0$).
For the case when $f^\top x = 1$ holds true we have to evaluate if $\mathcal{U}(x)$ is a subset of $\mathbb{R}^m_+$. In the case of being a subset it follows immediately that x belongs to $G_\mathcal{U}$, but otherwise a separator matrix $A_x \in \mathcal{U}$ is required. In the case of $\mathcal{U}(x)$ not being a subset of $\mathbb{R}^m_+$ there exists a matrix $A_x \in \mathcal{U}$ with $A_x x$ not belonging to $\mathbb{R}^m_+$. This means in particular that at least one value is non-positive. Without loss of generality let this be the j-th value ($(A_x x)_j < 0$). Then a separator is given by $s_x^\top := -e_j^\top A_x$ for an input value $y \in \mathbb{R}^n$, the matrix $A_x \in \mathcal{U}$ from above and $e_j^\top$ denoting the j-th row of the m-dimensional unit matrix. For $y \in G_\mathcal{U}$ it follows that $-e_j^\top \underbrace{A_x y}_{\geq 0} \leq 0$ holds because $Ay \in \mathbb{R}^m_+$, for all $A \in \mathcal{U}$ and in particular for A_x. Contrary, for the point x it holds that $s_x x = -e_j^\top \underbrace{A_x x}_{<0} > 0$, and therefore s_x is a valid separator for x and $G_\mathcal{U}$.

It can be concluded from [3] and [14] that the theoretical tractability of an uncertainty set $\mathcal{U}$ is equivalent to finding an efficient separation/inclusion oracle for this set. This oracle can always be found for an uncertainty set in the form of a convex hull for a finite set of possible realizations $A_1, A_2, \ldots, A_L$. Any given vector $x \in \mathbb{R}^n$ can be tested for $A_i x$ being greater or equal to zero for $i = 1, \ldots, L$ and therefore also the convex hull $\operatorname{conv}(A_1 x, A_2 x, \ldots, A_L x)$ is a subset of $\mathbb{R}^m_+$. In case $A_j x < 0$ for $j \in \{1, 2, \ldots, L\}$ then A_j can be used as separator and an efficient separation oracle is given. Additionally in this particular case the "Tractability Principle" from [3] implies an efficient separation oracle for the formerly defined set $G_\mathcal{U}$ and this further implies the computational tractability of the robust counterpart of a linear optimization problem with an uncertainty set of this kind.

To sketch this statement for a convex uncertainty set, like it is the case for an ellipsoidal uncertainty set, we have to analyse if $\mathcal{U}(x) \subset \mathbb{R}^m_+$ holds true for a given vector $x \in \mathbb{R}^n$. This is equivalent to solving the convex programs

$$\min\{e_i^\top Ax : A \in \mathcal{U}\}, i = 1, 2, \ldots, m. \tag{3.9}$$

Now similar to the "finite realization set-case" from before, it follows that if all optimal values are greater or equal to zero, then the set $\mathcal{U}(x)$ is a subset of $\mathbb{R}^m_+$. If this is not the case then at least one optimal value is negative. Without loss of generality let this again be the j-th optimal value of the convex program (3.9) that is smaller than zero. So a feasible solution for A_j that has a negative objective value works as a separator and implies an efficient separation oracle.
While the "Tractability Principle" in [3] states the computational tractability for all robust counterparts of linear optimization problems with "reasonable closed convex uncertainty sets" there might be a crucial difference to the practical solvability. Especially the enormous dimensions of linear programs conducted from applications require a justified simple structure on the uncertainty sets to achieve practical solvability of the robust counterparts.

In this and the following chapters we stick to ellipsoidal uncertainty sets. These sets have various benefits such as a simple geometrical and structural representation as well as the necessary flexibility to model different characteristics on the data. The term "ellipsoidal uncertainty" already appeared in [19] in 1973 and was revisited thereafter in many works dealing with Robust Optimization.

Motivated by various works (e.g. [3]) it is shown in the following section that the robust counterpart of a linear program with ellipsoidal uncertainties is a conic quadratic program. These conic quadratic programs can be efficiently solved in practice for problems incorporating enormous amounts of data with the developments on interior point algorithms and other methods.

3.4 Uncertainty Sets in Linear Optimization Problems

In the case of linear problems the following definition of an ellipsoid in $\mathbb{R}^K$ from Section 2.1 is used:

$$\mathcal{E} = \mathcal{E}\left(\Pi, Q\right) = \{\Pi(b) : \|Qb\|_2 \leq 1\}$$

with $\Pi : b \mapsto \Pi(b)$ being the certain affine embedding of $\mathbb{R}^L$ into $\mathbb{R}^K$ and $Q \in \mathbb{R}^{M\times L}$ being a matrix replacing the additional subspace in the previous Definition (2.3). To recall, we can cover the three required cases of normal ellipsoids, flat ellipsoids and ellipsoidal cylinders with properly chosen dimensions of the affine embedding and the right matrix Q.
A regular K-dimensional ellipsoid is represented by a nonsingular matrix Q and the matching dimensions $L = K = M$.
If a part of the data is certain or known the relevant characteristics can be incorporated by the usage of flat ellipsoids as uncertainty sets. Flat ellipsoids can be portrayed by the above definition if Q is nonsingular and the corresponding dimensions of the affine embedding are of the order $L = M$ and $M \leq K$.
And finally, ellipsoidal cylinders as uncertainty sets that connect ellipsoidal or interval restrictions on the data with the optimization problem. These are taken care of by the above definition if a singular matrix Q is given.
The three conditions in (2.1) for an ellipsoidal uncertainty set can be slightly restated for the case of linear problems with uncertainties as:

1. $\mathcal{U}$ can be written as finitely many intersections of ellipsoids with explicitly given affine transformations Π_k and matrices Q_k such as $\mathcal{U} = \bigcap_{k=0}^{L} \mathcal{E}(\Pi_k, Q_k)$.
2. The uncertainty set $\mathcal{U}$ is bounded.
3. All ellipsoids $\mathcal{E}(\Pi_k, Q_k)$ for $k \leq L$ share at least one data representative.

The importance of these assumptions becomes clear in the next sections.

3.5 Explicit Form of Robust Counterpart

To derive the representation of the robust counterpart for an uncertain linear problem we assume that the uncertainty set is given as stated above (3.4). While problems might be given in various forms for numerous applications they can be reformulated to determine the tractability of a problem.

Remark 10. Not all different formulations of a problem lead to the same robust counterpart as shown in the 'equivalent constraint representations' Section 3.1.1.

In the following section, we take a look at the structure of the robust counterpart when the uncertainty set $\mathcal{U}$ is given as a simple ellipsoid or as direct product of ellipsoids.

3.5.1 Ellipsoidal and Constraint-wise Uncertainty Set

First we consider the case that the uncertainty $\mathcal{U}$ is a standard ellipsoid of the form $\mathcal{U} = \left\{A = D^0 + \sum_{j=1}^{k} b_j D^j \mid \|b\|_2 \leq 1\right\}$ with $D^j \in \mathbb{R}^{m\times n}, j = 1,\ldots,k$. Here $\Pi(B)$ is given by $\Pi(B) = D^0 + \sum_{j=1}^{k} b_j D^j$ and is an affine embedding of a k-dimensional standard ball $B = \{b : \|b\|_2 \leq 1\}$. We define $\left(r_i^j\right)^\top$ to be the i-th row of D^j for $j = 1, 2, \ldots, k$ and further set the matrix R_i of the dimension $n \times k$ to have these r_i^j as columns. In this way the matrix is given by $R_i = \left(r_i^1, r_i^2, \ldots, r_i^k\right)$. With the above notation from [3] the i-th row of the affine embedding $\Pi(B)$ can be simplified to $(r_i^0 + R_i b)^\top$ as shown below.
The data matrix that defines the uncertainty set is notated by

$$D^0 = \begin{pmatrix} d_{11}^0 & \cdots & d_{1n}^0 \\ \vdots & \ddots & \vdots \\ d_{m1}^0 & \cdots & d_{mn}^0 \end{pmatrix} \text{and } D^j = \begin{pmatrix} d_{11}^j & \cdots & d_{1n}^j \\ \vdots & \ddots & \vdots \\ d_{m1}^j & \cdots & d_{mn}^j \end{pmatrix} \in \mathbb{R}^{m\times n} \text{ for } j = 1, 2, \ldots, k.$$

The affine embedding $\Pi(B)$ and the i-th row thereof $\Pi(B)_i$ are given by

$$\begin{aligned} \Pi(B) &= \Pi(b : \|b\|_2 \leq 1) = \{D^0 + b_1 D^1 + b_2 D^2 + \cdots + b_k D^k\} \text{ with } \|b\|_2 \leq 1, \text{ and} \\ \Pi(B)_i &= \left\{\left(d_{i1}^0, \cdots, d_{in}^0\right) + b_1\left(d_{i1}^1, \cdots, d_{in}^1\right) + \ldots + \left(d_{i1}^k, \cdots, p_{in}^k\right)\right. \\ &= \left.\left(d_{i1}^0 + b_1 d_{i1}^1 + \ldots + b_k d_{i1}^k, \cdots, d_{in}^0 + b_1 d_{i1}^1 + \ldots + b_k d_{in}^k\right)\right\}, \text{ with } \|b\|_2 \leq 1, \\ &\text{for } i \in \{1, 2, \ldots, m\}. \end{aligned}$$

Now from the definition of $r_i^j := \left(d_{i1}^j, \cdots, d_{in}^j\right)$ for $j = 1, 2, \ldots, k$ and R_i given by

$$R_i := \left(r_i^1, r_i^2, \ldots, r_i^k\right) = \begin{pmatrix} d_{i1}^1 & d_{i1}^2 & \cdots & d_{i1}^k \\ \vdots & \vdots & \ddots & \vdots, \\ d_{in}^1 & d_{in}^2 & \cdots & d_{in}^k \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, m,$$

follows that an element of $\Pi(B)_i$ can be written as $(r_i^0 + R_i b)^\top$ for $i = 1, \ldots n$:

$$r_i^0 + R_i b = \begin{pmatrix} d_{i1}^0 \\ \vdots \\ d_{in}^0 \end{pmatrix} + \begin{pmatrix} d_{i1}^1 & d_{i1}^2 & \cdots & d_{i1}^k \\ \vdots & \vdots & \ddots & \vdots \\ d_{in}^1 & d_{in}^2 & \cdots & d_{in}^k \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} = (\Pi(B)_i)^\top$$

Towards getting the explicit form of the robust counterpart a definition for a *robust feasible* vector with the above stated notations and structures is required.

Definition 6. *A vector $x \in \mathbb{R}^n$ is called robust feasible iff $f^\top x = 1$ holds true and*

$$\forall i \in \{1, 2, \ldots, m\} \quad \left(r_i^0\right)^\top x + (R_i b)^\top x \geq 0, \ \forall\, b : \|b\|_2 \leq 1.$$

This is exactly the case when $f^\top x = 1$ and

$$\begin{aligned} \min_{b:\|b\|_2 \leq 1} \left\{ \left(r_i^0\right)^\top x + (R_i b)^\top x \right\} &= \min_{b:\|b\|_2 \leq 1} \left\{ \left(r_i^0\right)^\top x + b^T R_i^\top x \right\} \\ &= \left(r_i^0\right)^\top x - \left\|R_i^\top x\right\|_2 \geq 0, \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

The above statement is true iff $\min\limits_{\|b\|_2 \leq 1} \{b^\top R_i^\top x\} = -\left\|R^\top x\right\|_2$ holds and this can easily be shown with the help of the Cauchy-Schwarz inequality:

$$b^\top \underbrace{R_i^\top x}_{=:r} \geq -\left\|b^\top\right\|_2 \|r\|_2 \geq -1 \|r\|_2 = -\|r\|_2, \text{ for } \|b\|_2 \leq 1.$$

The validity of the equality sign is shown by the two following case distinctions.
(1) The first case for $r = 0$ is trivial because both sides become zero.
(2) For the $r \neq 0$ case, we define $b := -\frac{r}{\|r\|_2}$ and therefore $\|b\|_2 = 1$ holds always true with

$$b^T r = -\frac{r^\top}{\|r\|_2} r = -\frac{\|r\|_2^2}{\|r\|_2} = -\|r\|_2.$$

And finally, the robust counterpart of a linear program with a simple ellipsoid as uncertainty set is given as a conic quadratic program of the form

$$\min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid r_i^0 x \geq \left\|R_i^\top x\right\|_2; i = 1, 2, \ldots, m; f^\top x = 1 \right\}. \tag{3.10}$$

or again stated in terms of D^0 and D^j from $\mathcal{U} = \left\{ A = D^0 + \sum_{j=1}^k b_j D^j \mid b^T b \leq 1 \right\}$ as

$$\min_{x \in \mathbb{R}^n} \left\{ c^\top x \mid \begin{pmatrix} d_{i1}^0 \\ \vdots \\ d_{in}^0 \end{pmatrix}^\top \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \geq \sqrt{\sum_{j=1}^k \left(\begin{pmatrix} d_{i,1}^1 & d_{i,2}^1 & \cdots & d_{i,n}^1 \\ d_{i,1}^2 & d_{i,2}^2 & \cdots & d_{i,n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{i,1}^m & d_{i,2}^m & \cdots & d_{i,n}^m \end{pmatrix} x \right)^2} \right\},$$
$$\text{,for } i = 1, 2, \ldots, m; \text{ and } f^\top x = 1.$$

If the uncertainty set $\mathcal{U}$ is given as the direct product $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$, where the single uncertainty sets $\mathcal{U}_i$ for $i = 1, 2, \ldots, m$ represent the uncertainty set of each row and they all are of ellipsoidal structure as above, the robust counterpart turns out to be again a conic quadratic program.

Let the separated uncertainty sets be given by

$$\mathcal{U}_i = \left\{ A \mid A^\top \hat{e}_i = r_i^0 + R_i b^i \text{ with } b^i \in \mathbb{R}^{m_i} \text{ and } \left\| b^i \right\|_2 \leq 1; i = 1, 2, \ldots, m \right\}$$

with $\hat{e}_i$ denoting the i-th unit vector, $r_i^\top \in \mathbb{R}^n$ and $R_i \in \mathbb{R}^{n \times m_i}$.

Analogue to the simple ellipsoid case we get as robust counterpart for the uncertain linear problem

$$\min_{x \in \mathbb{R}^n} \left\{ c^T x \mid \left[r_i^0 \right]^\top x \geq \left\| R_i^T x \right\|_2 ; i = 1, 2, \ldots, m; f^\top x = 1 \right\}.$$

3.5.2 General ellipsoidal Uncertainty set

Even if the uncertainty set has a more general ellipsoidal form, the robust counterpart turns out to be a conic quadratic program (see [3] (Theorem 3.1)). The importance of this finding is given by the 'a priori-tractability' since such problems can be solved efficiently up to a huge data size. Due to the wide variety in applications of such problems and the therefore accompanied interest, the theorem and the proof is shown in detail below.

Theorem 3.1. *The robust counterpart of a linear optimization problem*

$$\left\{ \min_x \left\{ c^T x : Ax \geq 0 \right\} \right\}_{A \in \mathcal{U}} \tag{3.11}$$

for a given uncertainty set of ellipsoidal or $\cap$-ellipsoidal structure is a conic quadratic program of the form

$$\min_{x \in \mathbb{R}^n} \left\{ c^T x \mid a_i^T x + \alpha_i \geq \left\| B_i x + b_i \right\|_2 , i = 1, 2, \ldots, M \right\} \tag{3.12}$$

with a_i, b_i being certain vectors, α_i being a fixed value and B_i representing a certain matrix.

Remark 11. The terms of this conic quadratic program are defined by the structural parameters of the nominal linear problem as well as the characteristics of the uncertainty set. As a result it follows that the conic quadratic program is given by the affine embeddings Π_k and the attendant matrices Q_k.

Proof: A general ellipsoidal uncertainty set is given by

$$\mathcal{U} = \bigcap_{k=1}^{L} \mathcal{E}(\Pi_k, Q_k) = \bigcap_{k=1}^{L} \left\{ \Pi_k \left(b^k\right) \mid \left\|Q_k b^k\right\|_2 \leq 1 \right\}.$$

Similar to the simple ellipsoid case and the constraint wise ellipsoid case earlier, we look at a robust feasible point $x \in \mathbb{R}^n$. A solution x with $f^\top x = 1$ is robust feasible iff, forall $i \in \{1, 2, \ldots\}$ the ideal/minimum value of the program, which is defined by the inner product of the i-th row of $\Pi_0\left(b^0\right)$ and x, conditioned on $\Pi_k\left(b^k\right) = \Pi_0\left(b^0\right)$ and $\left\|Q_k b^k\right\|_2 \leq 1$ for $k = 1, 2, \ldots, L$, is nonnegative in every coordinate. The terms b^k simply state the design variables for $k \in \{1, 2, \ldots, L\}$. The feasible solution set of this program is the definition of the uncertainty set $\mathcal{U}$ with $\Pi_0(b^0) = \Pi_k(b^k)$ for $k = 1, 2, \ldots, L$. Let 'OP_0' be the above optimization problem given by

$$\begin{gathered} \text{minimize the } i\text{-th entry of } \Pi_0(b^0)x \\ \text{s.t. } \Pi_k(b^k) = \Pi_0(b^0) \text{ for } k = 1, 2, \ldots, L, \text{ and } \|Q_k b_k\|_2 \leq 1 \text{ for } k = 0, 1, \ldots, L. \end{gathered} \tag{3.13}$$

As stated earlier in this section, OP_0 is a quadratic optimization program and therefore can be written in the form

$$\begin{gathered} \min\ cy + d \text{ s.t.} \\ Ry = r \text{ and } \|A_k y - b_k\|_2 \leq c_k y - d_k \text{ for } k = 0, 1, \ldots, L, \end{gathered} \tag{3.14}$$

with the decision vector y and the restrictions defined by the matrices A_k, the vectors b_k, c, c_k and the variables d_k and d for $k \in \{0, 1, \ldots L\}$. Nestrov, Yu and Nemirovski show in their work [16] in 1994 (Thm.4.2.1) that the dual of a conic quadratic program like (3.14) is also a conic quadratic problem and in the particular case of (3.14) the dual program is then represented by

$$\begin{gathered} \max\ r^\top \lambda + \sum_{k=0}^{L} \left[d_k \nu_k + b_k^\top \mu_k\right] + d \text{ s.t.} \\ R^\top \lambda + \sum_{k=1}^{L} \left[c_k \nu_k + A_k^\top \mu_k\right] = c^\top \text{ and } \|\mu_k\|_2 \leq \nu_k \text{ for } k = 0, 1, \ldots, L. \end{gathered} \tag{3.15}$$

where λ, μ_k and ν_k represent the design variables for $k = 0, 1 \ldots L$. Due to the boundedness of (3.14) from below in addition to the feasibility of the problem, the dual problem is also solvable and shares the optimal values with the primal problem. Since the nominal problem (3.13) only depends affinely on the decision variable x in the objective and the conic quadratic problem (3.14) is the same as (3.13) for some row i, the dual conic quadratic problem (3.15) can be written as

$$\begin{aligned}
\max \left[r^{(i)}\right]^\top \lambda^{(i)} + \sum_{k=0}^{L} \left[d_k^{(i)}\nu_k^{(i)} + \left[b_k^{(i)}\right]^\top \mu_k^{(i)}\right] + d^{(i)}(x) \,\text{s.t.}& \\
\left[R^{(i)}\right]^\top \lambda^{(i)} + \sum_{k=1}^{L} \left[c_k^{(i)}\nu_k^{(i)} + \left[A_k^{(i)}\right]^\top \mu_k^{(i)}\right] = \left[c^{(i)}\right]^\top (x) \text{ and}& \\
\left\|\mu_k^{(i)}\right\|_2 \leq \nu_k^{(i)} \text{ for } k = 0, 1, \ldots, L.&
\end{aligned} \tag{3.16}$$

As before, $\lambda^{(i)}, \mu_k^{(i)}$ and $\nu_k^{(i)}$ are the design variables while $d^{(i)}(x)$ and $c^{(i)}(x)$ are affine functions depending on x. The terms $r^{(i)}$ as well as $b_k^{(i)}, c_k^{(i)}, d_k^{(i)}$ for $k = 1, 2, \ldots, L$ and the matrices $A_l^{(i)}$for $k = 1, 2, \ldots, L$ and $R^{(i)}$ are independent of the variable x. These independent terms are solely defined by the affine mappings $\Pi_k(\cdot)$ and $Q_k(\cdot)$. The earlier mentioned (3.1.1) characteristics of the assumptions play an important role here, since they ensure the feasibility and the boundedness from below of the problem (3.13) which imply that the problem (3.13) has the same optimal value as the problem (3.16).

Bringing all this together states that a vector $x \in \mathbb{R}^n$ can be called robust feasible for a given uncertainty set $\mathcal{U}$ iff the following constraints are fulfilled $\forall i \in \{1, 2, \ldots, m\}$:

$$\begin{aligned}
f^\top x = 1& \\
\left[r^{(i)}\right]^\top \lambda^{(i)} + \sum_{k=0}^{L} \left[d_k^{(i)}\nu_k^{(i)} + \left[b_k^{(i)}\right]^\top \mu_k^{(i)}\right] + d^{(i)}(x) \geq 0& \\
\left[R^{(i)}\right]^\top \lambda^{(i)} + \sum_{k=1}^{L} \left[c_k^{(i)}\nu_k^{(i)} + \left[A_k^{(i)}\right]^\top \mu_k^{(i)}\right] = \left[c^{(i)}\right]^\top (x)& \\
\left\|\mu_k^{(i)}\right\|_2 \leq \nu_k^{(i)} \text{ for } k = 0, 1, \ldots, L&
\end{aligned} \tag{3.17}$$

and given design variables $\lambda^{(i)}, \mu_k^{(i)}$ and $\nu_k^{(i)}$ for $k = 0, 1, \ldots, L$. Now we have that the robust counterpart of the nominal problem is equivalent to the program minimizing $\bar{c}^\top x$ such that $x, \lambda^{(i)}, \mu_k^{(i)}$ and $\nu_k^{(i)}$ for $k = 0, 1, \ldots, L$ satisfy the conditions (3.17). This problem is of conic quadratic form where the variables are defined by the affine functions $\Pi_k(\cdot)$ and $Q_k(\cdot)$. $\square$

4 Robust Conic Optimization

Like in the previous chapter, we are dealing now with conic optimization problems that are influenced by data uncertainties. The main difference to Chapter 3 is that we are not restricting the cone to be the positive orthant anymore and allow the constraint functions, that are influenced by the choice of the uncertainty set, to be given by a more complex (non-linear) structure. To recall, the initial problem of interest is

$$\left\{ \min_x \left\{ c^T x + d : F(x, \zeta) \in \mathbf{K} \right\} \right\}, \tag{4.1}$$

with $\mathbf{K} \in \mathbb{R}^N$ being a nonempty pointed and closed convex cone. The data uncertainties might influence the constraint functions represented by $F(\cdot, \cdot)$ as well as the variables of the objective function $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.
<u>Recall:</u> With the simplification steps from 2.2 we can again assume that the objective function is linear and certain without a shift part. Now the uncertain conic problem can be stated similar to the linear case by

$$\left\{ \min_x \left\{ c^T x : F(x, \zeta) \in \mathbf{K} : \zeta \in \mathcal{U} \right\} \right\}, \tag{4.2}$$

with $\mathcal{U}$ denoting the uncertainty set. Also in this case the robust conic optimization problem can be represented by the robust counterpart as

$$\left\{ \min_x \left\{ c^T x : F(x, \zeta) \in \mathbf{K} : \forall \zeta \in \mathcal{U} \right\} \right\}. \tag{4.3}$$

A general problem in the form of such a robust counterpart seems again not practical at all since the resulting problem states now a semi-infinite conic problem. The required computational effort looks overwhelming to be useful for practical applications. Unlike as in the previous chapter the cone $\mathbf{K}$ as well as the constraint function $F(\cdot, \cdot)$ are allowed to be of a more intricate structure than just the positive orthant and linear functions. With a similar approach as in the linear case, we are able to obtain some information when putting restrictions on the cone $\mathbf{K}$, the uncertainty set $\mathcal{U}$ and look at particular (non-linear) functions $F(\cdot, \cdot)$.

In order to work with such kind of problems and create a convenient framework some assumptions and restrictions are introduced in the following, before the focus is on optimization problems with specified constraint functions and perturbation sets.
The following assumptions imply convexity for our problem in focus.

Assumptions:

1. The cone $\mathbf{K}$ must be closed, convex and nonempty. Remark: This was obviously fulfilled in the case of the uncertain linear optimization problems since the cone was the positive orthant $\mathbb{R}^m_+$.

2. The 'decision vector' x belongs to a set X that is also closed, convex and nonempty.

3. Also the uncertainty set $\mathcal{U}$ is closed, convex and nonempty.

4. The constraint function $F(\cdot,\cdot)$ is a continuously differentiable function on its domain $(X \times \mathcal{U})$ of definition and is $\mathbf{K}$-concave in the first component $(x \in X)$.

Definition 7. *A function $F(\cdot,\cdot)$ is called $\mathbf{K}$-concave in the first component if*

$$\begin{aligned} &\forall\,(x_1, x_2 \in X, \zeta \in \mathcal{U})\ \&\ \forall\,(\lambda \in [0,1]):\\ &F(\lambda x_1 + (1-\lambda)\, x_2, \zeta) \geq_{\mathbf{K}} \lambda F(x_1,\zeta) + (1-\lambda)\, F(x_2,\zeta)\,, \end{aligned}$$

with $a \geq_{\mathbf{K}} b$ meaning that $a - b$ belongs to $\mathbf{K}$.

Remark 12. The definition of a function being $\mathbf{K}$-convave in the second component is by analogy.

These assumptions ensure the problem of interest to be a convex problem for ζ belonging to the specified uncertainty set $\mathcal{U}$. As in the previous chapter we want the uncertainty set to be closed and convex. As we see next, the robust counterpart of a convex problem with uncertainties does not change when the uncertainty set is replaced by its closure or its convex hull, iff the constraint function is $\mathbf{K}$-concave in the second component.

Remark 13. If the constraint function is $\mathbf{K}$-concave in the second component then due to the last of the above assumptions it follows immediately that we talk about a function that is $\mathbf{K}$-concave in both components.

Obviously the robust counterpart does not change when we replace $\mathcal{U}$ by its closure since this is already taken care of by the third assumption. For the replacement of $\mathcal{U}$ by the convex hull, suppose a robust feasible vector x for a data vector $\zeta \in \mathrm{conv}(\mathcal{U})$ is given. This vector ζ can also be written as a convex combination by $\sum_{i=1}^{k} \lambda_i \zeta_i$ with $\zeta_i \in \mathcal{U}$ for $i = 1, 2, \ldots, k$. So $\sum_{i=1}^{k} \lambda_i F(x, \zeta_i) \geq_{\mathbf{K}} 0$ holds true and since $F(\cdot, \cdot)$ is given as a concave function it follows that also $F(x, \zeta) \geq_{\mathbf{K}} \sum_{i=1}^{k} \lambda_i F(x, \zeta_i)$ is true. This in extension implies that $F(x, \zeta) \geq_{\mathbf{K}} 0$ and therefore also the robust counterpart remains equivalent if the uncertainty set is replaced by its convex hull.

Furthermore we say that the uncertainty is an *affine uncertainty* when the mapping $F(\cdot, \cdot)$ is affine in the second component whenever the values of x belong to the definition set X.

Remark 14. Not only the linear problems in Chapter 3 but also the quadratic (and semidefinite problems) in this chapter are given by constraint functions that are affine in the data component.

Analyzing the structure for the robust counterpart of a convex program with uncertainties in the data under the above assumptions exhibits that its feasible set is again a convex and closed set. These assumptions also imply from the feasibility of the robust counterpart that the nominal uncertainty affected problem of interest is feasible for all instances. Furthermore, it is clear that the optimal value of the feasible robust counterpart is at least as great as the optimal value of the convex uncertain problem for all possible ζ realizations in $\mathcal{U}$. Not only might the optimal value of the robust counterpart be significantly worse than for the nominal problem but it might even happen that the initial uncertainty affected optimization problem is feasible while this is not the case for the attendant robust counterpart. It turns out in the following that this difference in the optimal values does not occur if the uncertainty is considered to be 'constraint-wise' and affine.

A uncertainty is called to be *contstraint-wise* in this chapter when the cone is given by the positive orthant $\mathbf{K} = \mathbf{R}_{+}^{m}$ and the constraints are given by independent functions as follows

$$F(x, \zeta) = \begin{pmatrix} f_1(x, \zeta) \\ f_2(x, \zeta) \\ \vdots \\ f_m(x, \zeta) \end{pmatrix} \text{ with } f_i(x, \zeta) \geq 0 \text{ for } i = 1, 2, \ldots, m.$$

Accordingly, the robust counterpart can be stated by the individual functions as

$$\min_x \left\{c^\top x : x \in X, f_i\left(x, \zeta\right) \geq 0,\ \forall\, i = 1, 2, \ldots, m, \forall\, \zeta \in \mathcal{U}\right\}.$$

Instead of looking on one single data vector ζ the robust counterpart stays the same when we assume that there are disconnected data vectors for every constraint function. This leads to a reinterpretation of the uncertainty set $\mathcal{U}$ by simply being the direct product of the individual uncertainty sets $\mathcal{U}_i$ for $i = 1, 2, \ldots, m$. The separated uncertainty set $\mathcal{U}_i$ is again closed and convex as well as defined on the specified area for its particular ζ_i for each $i = 1, 2, \ldots, m$.

This chapter often considers positive definite and positive semi-definite matrices via the *Schur-complement* and is therefore shortly recalled.

Definition 8. *The Schur-complement goes back to* Issai Schur *and gives the following statement about block-matrices (also see [12]). Let a matrix* $M \in \mathbb{R}^{(m+n)\times(m+n)}$ *be given by four sub-matrices* $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{m\times n}, C \in \mathbb{R}^{n\times m}$ *and* $D \in \mathbb{R}^{n\times n}$ *in the following way:*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

If A is invertible then the Schur-complement of M with respect to A is given by

$$M/A := D - CA^{-1}B.$$

Respectively the Schur-complement of the (invertible) matrix D in M is given by

$$M/D := A - CD^{-1}B.$$

Remark 15. Furthermore, a symmetric matrix M with the above structure is positive (semi-)definite if and only if the matrix A and the Schur-complement M/A is positive (semi-)definite, provided that A is invertible.

We continue this chapter by looking at specified cases of robust conic problems and start with quadratic optimization problems that incorporate not entirely certain data.

4.1 Robust Quadratic Optimization

These problems are of special interest since they often occur when a distance function is minimized in real world applications. An optimization problem of the form

$$\min_x \frac{1}{2} x^\top Q x + c^\top x + d \text{ such that} \tag{4.4}$$

$$Ax \leq a \text{ and } Bx = b \tag{4.5}$$

for a symmetric matrix $Q \in \mathcal{S} \subset \mathbb{R}^{n \times n}$, real matrices $A, B \in \mathbb{R}^{m \times n}$,vectors $a, b \in \mathbb{R}^m$ and $x, c \in \mathbb{R}^n$, and the real variable d, is called *Quadratic Optimization Problem.*

Remark 16. Note that if the symmetric matrix Q is the zero matrix we get back to a linear problem case. Thus, the statements and observations in this chapter also stay true for the special case of linear optimization problems.

Remark 17. Every quadratic problem of the above form (4.4) with Q positive-definite can be reduced to a second order cone program. With the usual simplification trick (2.5) we get a linear objective function as well as an additional constraint of the above quadratic form $t \geq x^\top Q x$. Since every positive definite matrix Q can be written as product of two matrices $Q = A^\top A$ with the Cholesky-factorization we can reformulate the term as

$$x^\top Q x = x^\top A^\top A x = \|Ax\|_2^2 .$$

Now lets say a vector $[Ax; t; t+1]$ belongs to the second order cone L^{n+2}. Explicitly written is this the case when

$$t + 1 \geq \sqrt{\|Ax\|_2^2 + t^2}.$$

Due to the positivity of both sides, squaring them is a monotone transformation and by further rearrangements of the terms we obtain the second order cone problem

$$t \geq \frac{\|Ax\|_2^2}{2} - \frac{1}{2}.$$

The fact that we are looking at a minimization problem let us ignore the addition/subtraction-term of a constant in this inequality, since it does not influence the solution but only the optimal value of the objective.

We have seen now that every convex quadratic optimization problem can be stated as a second order cone problem. On the other hand, not every second order cone problem is a quadratic optimization problem, e.g. the following quadratically constrained quadratic optimization problems.

Definition 9. *An optimization problem is called* Quadratically Constrained Quadratic Optimization Problem *if it can be stated as*

$$\begin{aligned} \min_x \frac{1}{2} x^\top Q x + c^\top x + d \text{ such that} \\ \frac{1}{2} x^\top Q_i x + c_i^\top x + d_i \le 0, i = 1, 2, \dots, m, \\ Bx = b, \end{aligned} \tag{4.6}$$

with Q_i being symmetric matrices, c_i, d_i being vectors for $i = 1, 2, \dots, m$.

Now we assume that the data elements are partly or entirely uncertain. As mentioned in the previous chapters, equality constraints should be avoided or reformulated to inequalities. Suppose that by equivalent constraint representations and the use of the simplification steps, we end up with the following representation.
In the following, let a *Quadratically Constrained Convex Optimization Problem* with uncertainties in the data be given by the second order cone program

$$\min_x \left\{ c^T x : -x^T [A^i]^T A^i x + 2[b^i]^T x + \gamma^i \ge 0 \right\} \tag{4.7}$$

with $i = 1, 2, \dots, m$ and $(A^1, b^1, \gamma^1; A^2, b^2, \gamma^2; \dots; A^m, b^m, \gamma^m) \in \mathcal{U}$. The matrices A^i are of the dimension $l_i \times n$. This notation is used to allow comparisions and extensional readings in [3] without complicated formulation adjustments.

Remark 18. This can be obtained through equation (2.2) by restricting $\mathbf{K}$ to be $\mathbb{R}^m_+$, defining ζ to be $(A^1, b^1, c^1; A^2, b^2, c^1; \dots; A^m, b^m, c^m)$ with $A^i \in \mathbb{R}^{l_i \times n}, b^i \in \mathbb{R}^n$ and $\gamma^i \in \mathbb{R}$, and setting $F(x, \zeta) = \begin{pmatrix} -x^T [A^1] A^1 x + 2 [b^1]^T x + \gamma^1 \\ -x^T [A^2] A^2 x + 2 [b^2]^T x + \gamma^2 \\ \dots \\ -x^T [A^m] A^m x + 2 [b^m]^T x + \gamma^m \end{pmatrix}$.
With this notation we are again able to work with the positive orthant of the according dimension as cone $\mathbf{K}$.

In the following section we are looking at the robust counterparts of such quadratically constrained quadratic optimization problems or alternatively on the robust counterpart of their second order cone representations.

4.2 Robust Counterpart of Uncertain Quadratic Optimization Problems

With the same order of steps as in the linear case chapter we first consider the case of the uncertainty set to be an ellipsoid and further look at an finite intersection of ellipsoids as a uncertainty set.

4.2.1 Ellipsoid and Constraint-wise Uncertainty Set

By assuming that the uncertainty set $\mathcal{U}$ is a bounded ellipsoid we can define $\mathcal{U}_i$ as a projection of the uncertainty set on the data described by the $i-th$ constraint as:

$$\mathcal{U}_i = \left\{ \left(A^i, b^i, \gamma^i\right) = \left(A^{i0}, b^{i0}, \gamma^{io}\right) + \sum_{j=1}^{k} u_j \left(A^{ij}, b^{ij}, \gamma^{ij}\right) \mid \|u\|_2 \leq 1 \right\}, i = 1, 2, \ldots, m.$$

(For further details recall the beginning of this chapter and/or [2].)
The robust counterpart for (4.7) does not change if the uncertainty set is given by the combination of its constraint-wise ellipsoidal set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$.
In the following we interpret the inequality sign '$\geq$' for two matrices A and B as:

$A \geq B$ *implies that* A *and* B *are symmetric and* $A - B$ *is positive semidefinite.*

The robust counterpart of (4.7) is given by

$$\min_x \left\{ c^T x \text{ s.t. } - x^T A^T A x + 2b^T x + \gamma \geq 0 \;\; \forall \;\; (A, b, \gamma) \in \mathcal{U}_i, i = 1, 2, \ldots, m \right\}.$$

For a fixed i and a separation of the uncertain data into the structure part, that is certain for all possible data realizations, and the numerical part, that incorporates the characteristics for the different realizations of the uncertain data, the robust counterpart is given by

$$\min_x c^T x$$

$$\begin{aligned} \text{s.t. } & - x^T \left[A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right]^T \left[A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right] x \\ & + 2\left[b^{i0} + \sum_{j=1}^{k} u_j b^{ij}\right]^T x + \left[\gamma^{i0} + \sum_{j=1}^{k} u_j \gamma^{ij}\right] \geq 0 \; \forall \; \left(u : \|u\|_2 \leq 1\right). \end{aligned} \tag{4.8}$$

The constraint term (4.8) remains the same when it is multiplicated by a positive term τ^2:

$$\begin{aligned} s.t. \quad & -\tau^2 x^T \left[A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right]^T \left[A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right] x \\ & + \tau^2 2 \left[b^{i0} + \sum_{j=1}^{k} u_j b^{ij}\right]^T x + \tau^2 \left[\gamma^{i0} + \sum_{j=1}^{k} u_j \gamma^{ij}\right] \geq 0 \; \forall \; (u : \|u\|_2 \leq 1) \,. \end{aligned} \tag{4.9}$$

The following equation (4.10) is obtained by replacing u with $\hat{u}$ defined by

$$\hat{u} = \begin{cases} \dfrac{u}{\tau}, & \text{if } \tau > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{s.t.} \quad & -x^T \left[\tau A^{i0} + \sum_{j=1}^{k} \underbrace{\tau \hat{u}_j}_{u_j} A^{ij}\right]^T \left[\tau A^{i0} + \sum_{j=1}^{k} \underbrace{\tau \hat{u}_j}_{u_j} A^{ij}\right] x \\ & + 2\tau \left[\tau b^{i0} + \sum_{j=1}^{k} \underbrace{\tau \hat{u}_j}_{u_j} b^{ij}\right]^T x + \tau \left[\tau \gamma^{i0} + \sum_{j=1}^{k} \underbrace{\tau \hat{u}_j}_{u_j} \gamma^{ij}\right] \geq 0 \, \forall \left((u, \tau) : \|\hat{u}\|_2 \leq 1\right) . \end{aligned} \tag{4.10}$$

Remark, that the last term $\|\hat{u}\|_2 \leq 1$ can be written as $\|u\|_2 \leq \tau$. It is shown in [2] that (4.10) is an even function in (τ, u) and can therefore be restated as

$$\begin{aligned} \text{s.t.} \quad & -x^T \left[\tau A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right]^T \left[\tau A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right] x \\ & + 2\tau \left[\tau b^{i0} + \sum_{j=1}^{k} u_j b^{ij}\right]^T x + \tau \left[\tau \gamma^{i0} + \sum_{j=1}^{k} u_j \gamma^{ij}\right] \geq 0 \, \forall \left((u, \tau) : \|u\|_2^2 \leq \tau^2\right) . \end{aligned} \tag{4.11}$$

With the following Lemma 4.1 (for additional information consult [2]) the line to an existing factor $\lambda_i \geq 0$ can be drawn. This factor is a required tool to construct the robust counterpart as an explicit semidefinite problem.

Lemma 4.1 (S-Lemma). *For two symmetric matrices P, Q and with z_0 satisfying $z_0^\top P z_0 > 0$ the implication*

$$z^\top P z \geq 0 \Rightarrow z^\top Q z \geq 0$$

holds true if and only if there exists a $\lambda \geq 0$ such that $Q \geq \lambda P$.

From (4.8) - (4.11), it can be concluded as shown in [2] that the implication

$$P(\tau, u) \geq 0 \Rightarrow Q_i^x(\tau, u) \geq 0 \tag{4.12}$$

holds for

$$\begin{aligned} P(\tau, u) =& \tau^2 - u^\top u \geq 0, \\ Q_i^x(\tau, u) =& - x^T \left[\tau A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right]^T \left[\tau A^{i0} + \sum_{j=1}^{k} u_j A^{ij}\right] x \\ &+ 2\tau \left[\tau b^{i0} + \sum_{j=1}^{k} u_j b^{ij}\right]^T x + \tau \left[\tau \gamma^{i0} + \sum_{j=1}^{k} u_j \gamma^{ij}\right]. \end{aligned}$$

We now use for this observation the S-Lemma 4.1 that states for the implication (4.12) the existence of a factor $\lambda^i \geq 0$ with

$$Q_i^x(\tau, u) - \lambda^i \left(\tau^2 - u^\top u\right) \geq 0. \tag{4.13}$$

Remark 19. The term $Q_i^x(\tau, u) - \lambda^i \left(\tau^2 - u^\top u\right)$ is positive definite. For every matrix $A \in \mathbb{R}^{m \times n}$ that is invertible, it follows that $A^\top A$ is positive definite since $x^\top A^\top A x = \|Ax\|_2^2 \geq 0 \; \forall x \in \mathbb{R}^n$ with $x \neq 0$ holds. Furthermore, the term $A^\top A$ is symmetric due to the fact that $\left[A^\top A\right]^\top = A^\top \left[A^\top\right]^\top = A^\top A$ is true.

The quadratic term (4.13) can be rewritten as follows

$$Q_i^x(\tau, u) - \lambda^i \left(\tau^2 - u^\top u\right) = \begin{pmatrix} \tau \\ u \end{pmatrix}^\top \left[S^i(x) - \left[R^i(x)\right]^\top R^i(x)\right] \begin{pmatrix} \tau \\ u \end{pmatrix} - \lambda^i \left(\tau^2 - u^\top u\right)$$

where $S^i(x) \in \mathbb{R}^{(k+1)\times(k+1)}$ is a symmetric matrix and $R^i(x) \in \mathbb{R}^{l_i \times (k+1)}$ is a rectangular matrix. Note that the matrices $R^i(x)$ depend affinely on x (see Remark 20) and the product $\left[R^i(x)\right]^\top R^i(x)$ is not only positive definite but also symmetric. Furthermore, is the sum of symmetric matrices again a symmetric matrix and so is the term given by $S^i(x) - \left[R^i(x)\right]^\top R^i(x)$. We observe that the Schur-complement (see [11]) for the quadratic form of $Q_i^x(\tau,u)-\lambda^i\left(\tau^2-u^\top u\right)$ is positive semidefinite for some λ^i exactly when

$$\begin{pmatrix} S^i+\begin{pmatrix} -\lambda^i & \\ & \lambda^i I_k \end{pmatrix} & \left[R^i(x)\right]^\top \\ R^i(x) & I_{l_i} \end{pmatrix}$$

is positive semidefinite. The statements of this section brought together form the following theorem.

Theorem 4.1. *The robust counterpart of a quadratically constrained quadratic problem, as given in* (4.7)*, with a simple ellipsoid as uncertainty set has the structure of a semidefinite problem.*

Remark 20. Furthermore, if the uncertainty set is of the structure given by the following two lines

$$\mathcal{U}_i=\left\{\left(A^i,b^i,\gamma^i\right)=\left(A^{i0},b^{i0},\gamma^{io}\right)+\sum_{j=1}^{k}u_j\left(A^{ij},b^{ij},\gamma^{ij}\right)|u^Tu\leq 1\right\},i=1,2,\dots,m$$
$$\mathcal{U}=\mathcal{U}_1\times\mathcal{U}_2\times\ldots\times\mathcal{U}_m,$$

the semidefinite problem representing the robust counterpart can explicitly be written as:

$$\min_x\left\{c^\top x\right\}$$

with respect to $\lambda^i\in\mathbb{R}$ for $i=1,2,\dots,m$ s.t.

$$\left(\begin{array}{c|cccc|c}
\gamma^{i0}+2x^\top b^{i0}-\lambda^i & \frac{\gamma^{i1}}{2}+x^\top b^{i1} & \frac{\gamma^{i2}}{2}+x^\top b^{i2} & \cdots & \frac{\gamma^{ik}}{2}+x^\top b^{ik} & \left[A^{i0}x\right]^\top \\
\hline
\frac{\gamma^{i1}}{2}+x^\top b^{i1} & \lambda^i & & & & \left[A^{i1}x\right]^\top \\
\frac{\gamma^{i2}}{2}+x^\top b^{i2} & & \lambda^i & & & \left[A^{i2}x\right]^\top \\
\vdots & & & \ddots & & \vdots \\
\frac{\gamma^{i(k-1)}}{2}+x^\top b^{i(k-1)} & & & & & \left[A^{i(k-1)}x\right]^\top \\
\frac{\gamma^{ik}}{2}+x^\top b^{ik} & & & & \lambda^i & \left[A^{ik}x\right]^\top \\
\hline
A^{i0}x & A^{i1}x & A^{i2}x & \cdots & A^{ik}x & I_{l_i}
\end{array}\right)\succeq 0,$$
$$i=1,2,\dots,m,$$

where I_{l_i} is the $(l_i\times l_i)$ identity matrix. Important to mention is the non-negativity of λ^i for $i=1,2,\dots,m$. This statement becomes clearer in the Section 4.3 as a step by step derivation for a similar case is shown.

4.2.2 General Ellipsoidal Uncertainty Set

This section shows the difficulties in finding and solving the explicit robust counterpart when the uncertainty set is given as intersection of ellipsoidal uncertainty sets with the help of a short example. To show the intractability for the robust counterpart of a quadratically constrained problem of the form

$$\min_x \left\{ c^\top x \ s.t. \ -x^T A^T A x + 2b^T x + \gamma \geq 0 \ \forall \ (A, b, \gamma) \in \mathcal{U} \right\}$$

if the uncertainty set $\mathcal{U}$ is given by intersections of ellipsoids, it is sufficient to consider a simple uncertainty set of the form

$$\mathcal{U} = \left\{ \xi = (A, b, \gamma) \,|\, A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{pmatrix}, b = 0, \gamma = 1 \right\}_{a \in \mathbf{B}},$$

with $\mathbf{B}$ being a $n-$dimensional polytope. More specifically, let $\mathbf{B} \subset \mathbb{R}^n$ be a centralized parallelotope. As mentioned in Section 2.1 polytopes can be expressed by intersections of ellipsoidal sets. Now the intractability is shown when the feasibility for a specific solution $x = (1, 1, \ldots, 1)^\top \in \mathbb{R}^n$ is tested. This task is equivalent to the proof that the euclidean norm of the variable a is smaller than one for all possible $a \in \mathbf{B}$. In other words, it must be shown that the polytope $\mathbf{B}$ is included in the unit euclidean ball. This latter problem is known to be NP-hard (for further information on this example see [18] and/or [17]). In the following section we consider more general conic optimization problems.

4.3 Robust Conic Quadratic Optimization

Suppose the uncertain optimization problem is given in the form of a *Conic Quadratic Problem* by the set

$$\left\{\min_x \left\{c^T x + d : F(x,\zeta) \in \mathbf{K}\right\}\right\}_{\zeta \in \mathcal{U}}.$$

Let the function $F(\cdot,\cdot)$ for the uncertain data $\zeta = (A^1, b^1, d^1, \gamma^1, \ldots, A^m, b^m, d^m, \gamma^m)$ be constraint-wise functions of the form

$$F(x,\zeta) = \begin{pmatrix} f_1(x,\zeta) \\ f_2(x,\zeta) \\ \vdots \\ f_m(x,\zeta) \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} A^1 x + b^1 \\ [d^1]^\top x + \gamma^1 \end{bmatrix} \\ \begin{bmatrix} A^2 x + b^2 \\ [d^2]^\top x + \gamma^2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A^m x + b^m \\ [d^m]^\top x + \gamma^m \end{bmatrix} \end{pmatrix} \in \mathbf{K}.$$

In addition, suppose that the cone $\mathbf{K}$ is given as direct product of cones: $\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2 \times \cdots \times \mathbf{K}_m$. If these cones $\mathbf{K}_i$ are *Lorentz-cones* (also known as second order-, Minkowski-, light- or ice cream-cones) for $i = 1, 2, \ldots, m$ then the *robust conic quadratic optimization problem* is given by

$$\left\{\min_x \left\{c^T x + d : \left\|A^i x + b^i\right\|_2 \leq \left[d^i\right]^\top x + \gamma^i \text{ for } i = 1, 2, \ldots, m\right\}\right\}_{\zeta \in \mathcal{U}}, \tag{4.14}$$

for matrices $A^i \in \mathbb{R}^{k_i \times n}$, vectors $b^i \in \mathbb{R}^{k_i}, d^i \in \mathbb{R}^n$ and the scalars $\gamma^i \in \mathbb{R}$.

Definition 10. *A* Lorentz-cone $L^k \subset \mathbb{R}^k$ *is defined as*

$$L^k = \left\{x \in \mathbb{R}^k : x_k \geq \sqrt{\sum_{j=1}^{k-1} x_j^2}\right\}.$$

Remark 21. The one dimensional Lorentz-cone L^1 is given by the nonnegative orthant $\mathbb{R}_+$ since the sum over an empty set is defined as zero. A Lorentz-cone L^k is regular and self dual.

Remark 22. The robust conic program (4.14) is just another formulation for the vector $\begin{bmatrix} A^i x + b^i \\ c^\top x + d^i \end{bmatrix}$ belonging to the Lorentz-cone L^{k_i+1} for $i = 1, 2, \ldots, m$.

Since the robust counterpart of a quadratically constrained quadratic problem, where the uncertainty set is given by an intersection of ellipsoids, leads to an intractable problem, this is also the case for those problems of a more general structure.

Therefore, we continue by checking the properties for the case of a simple ellipsoidal uncertainty set.

4.3.1 Ellipsoidal Case

Let the uncertainty be constraint wise such that the uncertainty set $\mathcal{U}$ is given by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$. Each $\mathcal{U}_i$ depends on its constraint function $f_i(\cdot,\cdot) \in \mathbf{K}_i$ for $i = 1, 2, \ldots, m$. The ellipsoidal sets $\mathcal{U}_i$ are given as direct product of both ellipsoids deducted from components of the according constraint functions. This specifically means that every single uncertainty set $\mathcal{U}_i$ for $i = 1, 2, \ldots, m$ can be written as

$$
\begin{aligned}
\mathcal{U}_i &= \bar{\mathcal{U}_i} \times \hat{\mathcal{U}}_i, \text{with} \\
\bar{\mathcal{U}_i} &= \left\{ \left[A^i; b^i\right] = \left[A^{i0}; b^{i0}\right] + \sum_{j=1}^{\bar{k}_i} u_j \left[A^{ij}; b^{ij}\right] \mid \|u\|_2^2 \leq 1 \right\}, \\
\hat{\mathcal{U}_i} &= \left\{ \left[c^i; \gamma^i\right] = \left[c^{i0}; \gamma^{i0}\right] + \sum_{j=1}^{\hat{k}_i} u_j \left[c^{ij}; \gamma^{ij}\right] \mid \|u\|_2^2 \leq 1 \right\}.
\end{aligned}
$$

Analogous to the previous cases, the robust counterpart of the uncertain quadratic conic problem (4.14) is given by

$$
\min_x \left\{ c^T x + d : \left\|A^i x + b^i\right\|_2 \leq \left[d^i\right]^\top x + \gamma^i \text{ for } i = 1, 2, \ldots, m,\ \forall \zeta \in \mathcal{U} \right\}. \tag{4.15}
$$

In order to research the robust counterpart of this problem it is sufficient to observe the robust formulation of one single constraint. Let one robust constraint be given by

$$
\|Ax + b\|_2 \leq d^\top x + \gamma \ \ \forall (A, b, d, \gamma) \in \mathcal{U}, \tag{4.16}
$$

with $(A, b) \in \bar{\mathcal{U}}$ and $(d, \gamma) \in \hat{\mathcal{U}}$.

More precisely, this constraint is given by

$$
\begin{gathered}
\forall (u, v) : \|u\|_2^2 \leq 1, \|v\|_2^2 \leq 1 \\
x^\top d^0 + \gamma^0 + \sum_{j=1}^{\hat{k}} u_j \left(x^\top d^j + \gamma^j\right) \geq \left\| A^0 x + b^0 + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i\right) \right\|_2 .
\end{gathered}
$$

Obviously there exists a $\lambda \geq 0$ which lets us rewrite the constraint by the following two lines

$$x^\top d^0 + \gamma^0 + \sum_{j=1}^{\hat{k}} u_j \left(x^\top d^j + \gamma^j \right) \geq \lambda \;\; \forall\, u : \|u\|_2^2 \leq 1$$

$$\lambda \geq \left\| A^0 x + b^0 + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i \right) \right\|_2 \quad \forall\, v : \|v\|_2^2 \leq 1.$$

Furthermore a pair (x, λ) fulfills the constraints above exactly when the matrix

$$\begin{pmatrix} [d^0]^\top x + \gamma^0 - \lambda & [d^1]^\top x + \gamma^1 & [d^2]^\top x + \gamma^2 & \cdots & \left[d^{\hat{k}}\right]^\top x + \gamma^{\hat{k}} \\ [d^1]^\top x + \gamma^1 & [d^0]^\top x + \gamma^0 - \lambda & & & \\ [d^2]^\top x + \gamma^2 & & [d^0]^\top x + \gamma^0 - \lambda & & \\ \vdots & & & \ddots & \\ \left[d^{\hat{k}}\right]^\top x + \gamma^{\hat{k}} & & & & [d^0]^\top x + \gamma^0 - \lambda \end{pmatrix} \tag{4.17}$$

is positive definite (and symmetric) for a $\lambda \geq 0$ and

$$\lambda^2 \geq \left\| A^0 x + b^0 + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i \right) \right\|_2^2 \quad \forall\, v : \|v\|_2^2 \leq 1 \tag{4.18}$$

holds. Due to the positivity of the terms, the constraint (4.18) can be written as

$$\forall\, (t, v), \|v\|_2^2 \leq t^2 : \lambda^2 t^2 \geq \left\| t(A^0 x + b) + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i \right) \right\|_2^2.$$

In this sense, a pair (x, λ) with a positive λ fulfills (4.18) exactly when the nonnegativity of the quadratic term $(t^2 - v^\top v)$ implies that the other quadratic term $\lambda^2 t^2 - \left\| t(A^0 x + b) + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i \right) \right\|_2^2$ is nonnegative.
This indication statement can as well be given (see [2] for further information) according to the S-Lemma by the existence of an $\alpha \geq 0$ that states

$$\lambda^2 t^2 - \left\| t(A^0 x + b) + \sum_{i=1}^{\bar{k}} v_i \left(A^i x + b^i \right) \right\|_2^2 - \alpha \left(t^2 - v^\top v \right) \geq 0.$$

Since λ is nonnegative, let α be given as $\alpha = \mu \lambda$ for a $\mu \geq 0$. Note, if λ is zero then also μ is considered to be zero.

To sum up the last few lines, for a nonnegative λ to a given x which fulfills (4.18) there exists another variable μ that satisfies the following statement for the quadratic term:

$$\begin{aligned}
&\text{For } \lambda, \mu \geq 0 \,(\text{ if } \lambda = 0 \Rightarrow \mu = 0) \\
&\left(\lambda^2 - \lambda\mu\right) t^2 + \lambda\mu v^\top v - \begin{pmatrix} t & v^\top \end{pmatrix} [R(x)]^\top R(x) \begin{pmatrix} t \\ v \end{pmatrix} \geq 0 \\
&\text{for the matrix } R = \begin{pmatrix} A^0 x + b^0, & A^1 x + b^1, & \cdots & , A^{\hat{k}} x + b^{\hat{k}} \end{pmatrix}.
\end{aligned} \tag{4.19}$$

Similar, as the above constraints could be expressed by the positive semi-definiteness of the matrix (4.17), we want to find a symmetric matrix S depending on the three just introduced variables x, λ and μ. Suppose that an analogous way to check the statement (4.19) is to determine if the matrix

$$S(x, \lambda, \mu) = \begin{pmatrix} \lambda - \mu & & & & \left(A^0 x + b^0\right)^\top \\ & \mu & & & \left(A^1 x + b^1\right)^\top \\ & & \ddots & & \vdots \\ & & & \mu & \left(A^{\bar{k}} x + b^{\bar{k}}\right)^\top \\ \left(A^0 x + b^0\right) & \left(A^1 x + b^1\right) & \cdots & \left(A^{\bar{k}} x + b^{\bar{k}}\right) & \lambda I_l \end{pmatrix} \tag{4.20}$$

is positive semi-definite. To show that this statement is equivalent a case distinction is used.

Case $\lambda > 0$ (and therefore $\mu > 0$): The positiveness of λ lets us rewrite the quadratic term of (4.19) by a division of λ to

$$(\lambda - \mu) t^2 + \mu v^\top v - \begin{pmatrix} t & v^\top \end{pmatrix} R^\top(x) \left(\lambda \hat{I}_l\right)^{-1} R(x) \begin{pmatrix} t \\ v \end{pmatrix} \geq 0.$$

This on the other hand, is exactly the Schur-complement of the matrix $S(x, \lambda, \mu)$. We recall that the symmetric matrix $S(x, \lambda, \mu)$ with an invertible upper left block is positive-semidefinite if this upper left block matrix and its Schur-complement is positive semi-definite.

Case $\lambda = 0 \Rightarrow \mu = 0$: For this case the quadratic term of the statement (4.19) is true exactly when $R(x) = 0$. From that follows that the matrix $S(x, \lambda, \mu)(= S(x, 0, 0))$ obviously is positive semi-definite and (4.18) is also satisfied with $\lambda = 0$ if and only if $R(x) = 0$.

Bringing together the statements of this section, for an uncertain conic quadratic problem where the uncertainty set is given as the direct product of the individual uncertainty sets and each of them is the product of two ellipsoidal spaces corresponding to their data parts of the form

$$
\begin{aligned}
\mathcal{U} &= \mathcal{U}_1 \times \cdots \times \mathcal{U}_m \text{ with } \mathcal{U}_i = \bar{\mathcal{U}_i} \times \hat{\mathcal{U}_i} \\
\bar{\mathcal{U}_i} &= \left\{ \left[A^i; b^i\right] = \left[A^{i0}; b^{i0}\right] + \sum_{j=1}^{\bar{k_i}} u_j \left[A^{ij}; b^{ij}\right] \mid \|u\|_2^2 \leq 1 \right\}, \\
\hat{\mathcal{U}_i} &= \left\{ \left[c^i; \gamma^i\right] = \left[c^{i0}; \gamma^{i0}\right] + \sum_{j=1}^{\hat{k_i}} u_j \left[c^{ij}; \gamma^{ij}\right] \mid \|u\|_2^2 \leq 1 \right\}.
\end{aligned}
$$

The corresponding robust counterpart can equivalently be written as the semidefinite problem by

$$
\min_x c^\top x \text{ w.r.t. } x \in \mathbb{R}^n, \lambda^1, \ldots, \lambda^m \in \mathbb{R}, \mu^1, \ldots, \mu^m \in \mathbb{R} \text{ s.t.}
$$

$$
\begin{pmatrix}
[d^0]^\top x + \gamma^0 - \lambda^i & [d^1]^\top x + \gamma^1 & [d^2]^\top x + \gamma^2 & \cdots & \left[d^{\hat{k}}\right]^\top x + \gamma^{\hat{k}} \\
[d^1]^\top x + \gamma^1 & [d^0]^\top x + \gamma^0 - \lambda^i & & & \\
[d^2]^\top x + \gamma^2 & & [d^0]^\top x + \gamma^0 - \lambda^i & & \\
\vdots & & & \ddots & \\
\left[d^{\hat{k}}\right]^\top x + \gamma^{\hat{k}} & & & & [d^0]^\top x + \gamma^0 - \lambda^i
\end{pmatrix} \geq 0,
$$

$$
\text{and} \quad
\begin{pmatrix}
\lambda^i - \mu & & & & (A^0 x + b^0)^\top \\
& \mu & & & (A^1 x + b^1)^\top \\
& & \ddots & & \vdots \\
& & & \mu & \left(A^{\bar{k}} x + b^{\bar{k}}\right)^\top \\
(A^0 x + b^0) & (A^1 x + b^1) & \cdots & \left(A^{\bar{k}} x + b^{\bar{k}}\right) & \lambda^i \hat{I}_l
\end{pmatrix} \geq 0,
$$

forall $i = 1, 2, \ldots, m$.

5 Summary

Nearly all optimization models that describe practical processes have to deal with uncertain variables or at least partly uncertain data. In contrast to the most common approaches for these kind of problems, that try to find or estimate a fitting family of distributions for the uncertain variables, the approach of this work operates with a set based structure. This has the simple benefit that various error sources are ruled out. Although it might not be necessary for every problem to consider the strict observance of all constraints, it is important for some, since small perturbations in the uncertain data of the problem could lead to a practically worthless solution [4].
All information on the uncertain data are used to construct a so called *uncertainty set*, such that the constraints are fulfilled for all possible data realizations belonging to this set. Especially, if a constraint violation has the potential to implement a crucial incident, as it might be the case in construction- or chemical-processes, the use of the robust counterpart approach could offer some important benefits. Another perspective is that even if parameters are not known, they simply cannot violate their constraints, e.g. the positivity of supply.
Even though in some cases the constraint violations can be compensated for e.g. by a purchase from a different company, the approach in this thesis only considers hard constraints. Extensions like allowing soft constraints in the robust optimization approach can be found in [6].
Not only the set based approach itself is interesting on its own but also the fact that numerous problems turn out to be tractable if the underlying uncertainty set is specified accordingly.
A wide range of real life processes and procedures, like shipping and distribution problems, can be modelled by linear programs that incorporate uncertainties. These kind of problems are tractable by the use of the robust optimization approach for ellipsoidal uncertainty sets, since the corresponding robust counterpart is a conic quadratic program. Even if the uncertainty set is given by the intersection of finitely many ellipsoids this is still the case. This conic quadratic program can explicitly be given in terms of the initial problem and the choice/characteristics of the uncertainty set.

A more general and also heavily used family of problems is given by the conic quadratic problems with uncertainties, as they often appear in applications when a distance function is optimized. Like in the uncertain linear optimization case the problems turn out to be safe tractable for an ellipsoidal uncertainty set, due to the fact that the robust counterpart can be stated as an explicit semi-definite program. The same holds true for a quadratically constrained quadratic optimization problem that incorporates partial uncertain data via an ellipsoidal uncertainty set. More complicated to solve is the robust counterpart of these problems, if the uncertainty set is given by an intersection of finitely many ellipsoids. For this cases the corresponding robust counterpart states an NP-hard problem.
Another wide spread and well researched kind of problems is given by the semi-definite programs with uncertainties. Although their robust counterpart for an ellipsoidal uncertainty set is NP-hard to solve, the problems turn out to be safe tractable for particular structured ellipsoidal uncertainty sets. For more on this consult the work [2].

Other possible extensions of this thesis are different ways to find safe tractable approximations for the cases when the robust counterpart turns out to be NP-hard. Another interesting area is that links between different models of multi-stage decision-making problems and the robust optimization approach can be found and researched [7]. Also the link between stochastic optimization problems with choice-constraints and their safe tractable approximations with the help of robust optimization are possible extensions that can be looked up in [6].
Overall, this approach presents a reliable way to obtain a feasible solution for particular problems that have to deal with uncertainties. Due to its sparse use of distribution information and other valuable characteristics, this way of problem solving is widely used in many areas like engineering (see for instance [1] and references therein), finance and statistics. Other cases of practical use of this approach are TV-tubes manufacturings as well as processes in inventory- and supply-chain management [9].

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