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"The green paradox in competitive and monopoly markets"

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1 Introduction

In the past decades the extraction of non-renewable resources has been a highly discussed topic. Especially the implications on the environment are, at least after the Kyoto Protocol in 1997, considered of great importance. There have been several research papers written about how to reduce greenhouse gases and it is often argued that it is important to start taking measures as early as possible (e.g. Nordhaus 2008).

In 2008 Hans-Werner Sinn asked a new question, namely if it is possible that measures that are taken to decrease damage on the environment actually have the opposite effect, mostly because only the demand is targeted with said measures and the supply side is not considered. He called it the green paradox. An accurate definition for it is:

In the broad sense, any environmental policy that is formulated with the intention of improving environmental quality that turns out to have adverse consequences for the environment is called a green paradox outcome. In a more narrowly defined sense, the green paradox refers to environmental policy failure owing to the failure to recognize the intertemporal supply behavior of extractors of fossil fuel resources (Van Long (2014:2)).

After Sinn published his theory several papers were written about that issue. When Sinn talked about non-renewable resources, his main example was the oil industry. As there are only a few major oil exporting countries (OPEC-Countries), and they often meet to decide how much oil they want to produce, one can argue that it is best to model this problem using a monopoly approach. Although, like mentioned before, a lot of papers and different models were created about the green paradox, only few address this issue (e.g. Van der Ploeg and Withagen 2010).

My thesis is based upon two other papers concerning themselves with the green paradox, namely the ones by Gerlagh (2010) and Österle (2015). First I look at the implications of these models and then I extend them to the monopoly case. We will see that the results stay mostly the same, but not in all cases.

The models about the green paradox can be split up into two major groups. In chapter 2 we discuss models where a backstop, which is a clean alternative to the use of the non-renewable resource (e.g. wind energy), is assumed and the measure that should help the environment is e.g. a subsidy on the backstop. Important is that it is assumed that the backstop-price drops and a green paradox occurs if this leads to a higher environmental damage. In the models in chapter 3 there doesn't exist a backstop, but a revenue-tax can be implemented. In these models a green paradox arises if the introduction of such a tax has a negative implication on the environment. Chapter 4 will then conclude the thesis.

Following Gerlagh (2010) we differentiate between a weak and a strong green paradox. The weak green paradox occurs when the taken measure leads to an increase in present damage on the environment and the strong green paradox if it leads to an increase in overall damage.

2 The models with a backstop

This chapter is based on the models by Gerlagh (2010). All the models presented here are dynamic general equilibrium models. Firms own a fixed stock of the resource[1] and can extract and sell it, maximizing their profits. The extraction costs are either constant or linearly increasing.

There exists a backstop that has either a constant price (perfect backstop) or it increases with the amount used (imperfect backstop). When a perfect backstop is assumed, in the beginning only the resource and then solely the backstop is used. When assuming an imperfect backstop, there is a simultaneous use of both the resource and the backstop until the resource is depleted. After that only the backstop is used.

A linear demand function is assumed, although for the competitive equilibrium model with constant extraction costs and a perfect backstop this restriction is not needed. A green paradox arises, if a cheaper backstop has negative consequences for the environment. To be more precise, the weak green paradox occurs if a cheaper backstop implies more environmental damage at the initial period and a strong green paradox, if it implies a higher total damage.

We will see that the existence of the green paradoxes depends upon the assumptions about the extraction costs and the backstop.

2.1 The model with constant resource extraction costs and a perfect backstop

2.1.1 The competitive equilibrium

In this first model firms maximize the following discounted profit function Π, subject to a resource constraint, taking the price of the resource as given

$$\Pi = \int_0^\infty e^{-rt}(p_t - \zeta)q_t dt \tag{2.1.1}$$

$$s.t. \qquad \int_0^\infty q_t dt \leq S_0 \tag{2.1.2}$$

where p_t describes the price of the resource, ζ the (constant) extraction costs per unit, q_t

[1] When we talk about resources in this paper, we always mean non-renewable resources.

the amount of the resource that is extracted and sold, r the real interest rate and S_0 the initial resource stock.
Equation (2.1.2) states that the firms can't extract more of the resource than there is available in total.

We assume a strictly decreasing demand function $q_t = D(p_t)$, with a maximum demand α (D(0)=α) and a choke-price β (D(β)=0). Furthermore the price of the backstop is denoted by ψ, with $\zeta < \psi < \beta$. So the price of the backstop is higher than the extraction costs, but lower than the choke-price.
It is easy to imagine that under this assumption in the beginning solely the resource is used and a transition from the resource use to using the backstop after a finite time horizon occurs, as the resource either becomes too expensive for the consumers, or there are no resources left (or both).
Now we want to find out how the price-path in this model looks like. Therefore we redefine the problem a little and get

$$max\ \Pi = max \int_0^\infty e^{-rt}(p_t - \zeta)q_t dt \tag{2.1.3}$$

$$\begin{aligned} s.t. \quad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0\ \ given \end{aligned}$$

Note that this is really only a reformulation, in which the resource stock (S_t) is included, but the problem isn't changed at all. We will use this formulation in the following models as well. Trying to solve this "new" problem leads us to the following Hamiltonian

$$H = (p_t - \zeta)q_t - \lambda_t q_t + \nu_t S_t$$

Hence the optimality conditions are

$$\frac{\partial H}{q_t} = p_t - \zeta - \lambda_t = 0 \tag{2.1.4}$$

$$\begin{aligned} \dot{\lambda}_t &= r\lambda_t - \frac{\partial H}{S_t} \\ \Leftrightarrow \dot{\lambda}_t &= r\lambda_t - \nu_t \end{aligned} \tag{2.1.5}$$

$$\nu_t S_t = 0 \tag{2.1.6}$$

with λ_t being the shadow-price of the resource.
Looking at equation (2.1.6) we can now distinguish between two cases, $S_t = 0$ and $S_t > 0$ (implying $\nu_t = 0$). As we are only interested in the price when the resource stock is positive, we can restrict ourselves to the latter case and using (2.1.5) we get

$$\lambda_t = \lambda_0 e^{rt} \tag{2.1.7}$$

and combining this equation with (2.1.4)

$$p_t = \zeta + \lambda_0 e^{rt} \tag{2.1.8}$$

Using the price-path (equation (2.1.8)) we can write the profit function as

$$\Pi = \int_0^\infty e^{-rt}(p_t - \zeta) q_t dt = \lambda_0 S_0 \tag{2.1.9}$$

As p_t is strictly increasing, we can denote T as the time where p_t reaches the price of the backstop ψ and the resource is no longer used, as it gets more expensive than the backstop (note that therefore $p_T = \psi$). We will call this point in time the termination date.
In equilibrium it is optimal for the firm to use all of the available resource, as the output

in this case has to be higher than in the case where some part of the resource stays in the ground, so $\int_0^\infty q_t dt = S_0$. Using the demand function and the fact that after the termination date the resource is not used any more, we get

$$\int_0^T D(\zeta + e^{rt}\lambda_0)dt = S_0 \tag{2.1.10}$$

The price at time T is equal to $p_T = \zeta + e^{rT}\lambda_0$ (see equation (2.1.8)). As the price of the resource at the termination date has to equal the backstop-price ($p_T = \psi$), after some reformulation we get

$$Tr = ln(\psi - \zeta) - ln(\lambda_0) \tag{2.1.11}$$

Equations (2.1.10) and (2.1.11) fully characterize the equilibrium, as the only unknown variables in these two equations are λ_0 and T.

A weak green paradox occurs if and only if the extraction at time zero rises when the price of the backstop drops, as the damage on the environment at a certain point in time is higher the higher the extraction is.
Concerning the strong green paradox we need to define the total damage on the environment (Γ), being the present value of the overall influence of the resource extraction on the environment

$$\Gamma := \int_0^\infty e^{-rt}\theta_t q_t dt \tag{2.1.12}$$

Hereby we assume that the marginal damage of the resource extraction (θ_t) increases over time, but with a lower rate than the discount factor r, so that the term $e^{-rt}\theta_t$ is decreasing over time. This isn't a very strong assumption and rather common in relevant literature (see for example Hoel and Kverndokk 1996). Note that the marginal damage does not have to increase with a constant rate here, but we will need this additional assumption in chapter 3.
The initial damage on the environment is therefore equal to $\Gamma_0 = \theta_0 q_0$. We can now define that the weak green paradox holds, if $\frac{\partial \Gamma_0}{\partial \psi} < 0$ ($\Leftrightarrow \frac{\partial q_0}{\partial \psi} < 0$) and the strong green

paradox arises, if $\frac{\partial \Gamma}{\partial \psi} < 0$. Note that these definitions will be used in all three models in this chapter.

We can now show that, as the backstop gets cheaper, the resource is extracted faster. This has a negative influence on the environment.
To prove this, we first need to find out how the variables T and λ_0 change with a variation in ψ. We take total derivatives with respect to ψ in equation (2.1.10), which leads us to

$$\int_0^T e^{rt} D'(p_t) \cdot \frac{d\lambda_0}{d\psi} dt + D(p_T) \cdot \frac{dT}{d\psi} = 0 \tag{2.1.13}$$

Multiplying equation (2.1.13) with $\frac{d\psi}{d\lambda_0}$ and solving for $\frac{dT}{d\lambda_0}$ we get

$$\frac{dT}{d\lambda_0} = -\frac{\int_0^T e^{rt} D'(p_t) dt}{D(p_T)} \tag{2.1.14}$$

As the derivative of the demand function is always smaller than zero, the right hand side of the above equation is positive, so we get $\frac{dT}{d\lambda_0} > 0$.

Taking now total derivatives in equation (2.1.11) we get

$$r\frac{dT}{d\psi} = \frac{1}{\psi - \zeta} - \frac{1}{\lambda_0}\frac{d\lambda_0}{d\psi} \tag{2.1.15}$$

We multiply equation (2.1.15) again with $\frac{d\psi}{d\lambda_0}$, and end up with

$$r\frac{dT}{d\lambda_0} = \frac{1}{\psi - \zeta}\frac{d\psi}{d\lambda_0} - \frac{1}{\lambda_0} \tag{2.1.16}$$

As seen above, the left hand side of equation (2.1.16) is greater than zero. The second part on the right hand side is smaller than zero, so $\frac{d\psi}{d\lambda_0}$, and therefore also $\frac{d\lambda_0}{d\psi}$, has to be greater than zero.
From $\frac{dT}{d\lambda_0} > 0$ and $\frac{d\lambda_0}{d\psi} > 0$ we can immediately conclude that $\frac{dT}{d\psi}$ is also positive.
So the values of both the initial shadow-price and the termination date drop, when the backstop gets cheaper.

The last step is now to compare two cases of this model, where only the price for the backstop differs. We denote the values of the second case with a star and, assuming $\psi^* < \psi$, want to find out, if the crucial values corresponding to this cheaper backstop (Γ_0^* and Γ^*) are higher or lower than the original values corresponding to ψ.
From $\psi^* < \psi$ we can immediately conclude that $T^* < T$ and $\lambda_0^* < \lambda_0$.
To make it easier to compare Γ and Γ^*, we define $\tilde{\theta}_t = e^{-rt}\theta_t$. Note that, by assumption, $\tilde{\theta}_t$ is a strictly decreasing function.
As $p_t = \zeta + e^{rt}\lambda_0 \ \forall t : t < T$ we can conclude that $p_t^* < p_t \ \forall t : t < T^*$ and because of the strictly decreasing demand function we get $q_t < q_t^* \ \forall t : t < T^*$.
Additionally we know that $q_t > q_t^* \ \forall t : T^* < t < T$ as for these values of t $q_t^* = 0$ holds, and of course $q_t = q_t^* = 0 \ \forall t : t > T$. As explained previously in both cases the whole resource stock is used and thus we have $\int_0^T q_t dt = \int_0^{T^*} q_t^* dt = S_0$

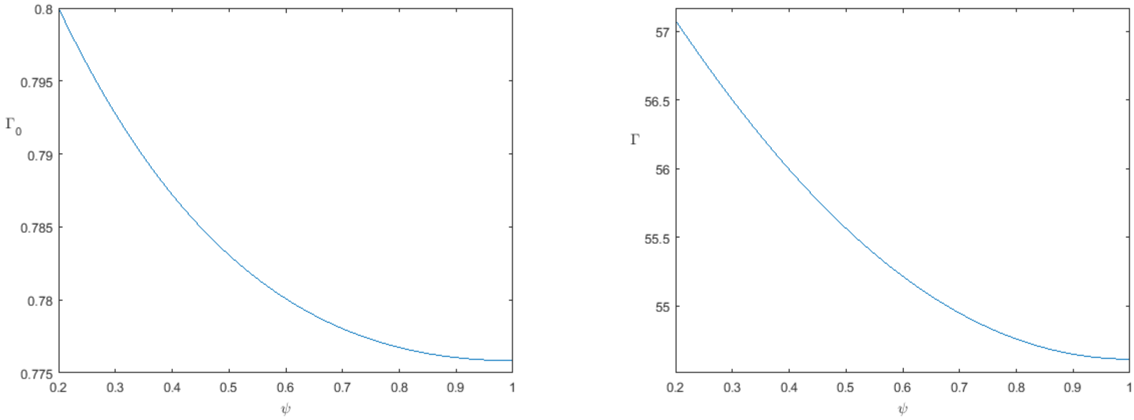
To prove that both green paradoxes occur, we formulate the following proposition using all the just derived facts about the paths that describe the extraction of the resource (q_t and q_t^*).

Proposition 1. *Consider two piecewise continuous paths $q_t, q_t^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $q_t = 0 \ \forall t : t > T$, $q_t^* = 0 \ \forall t : t > T^*$, $T > T^*$ and $\int_0^T q_t dt = \int_0^{T^*} q_t^* dt$. Moreover we know that $q_t < q_t^* \ \forall t : t < T^*$ and $q_t > q_t^* \ \forall t : T^* < t < T$. Additionally we have a strictly positive parameter θ_0 and a strictly decreasing weight function $\tilde{\theta}_t$.*
Then we can conclude that $\Gamma_0^* := \theta_0 q_0^*$ is strictly greater than $\Gamma_0 := \theta_0 q_0$ and $\Gamma^* := \int_0^{T^*} \tilde{\theta}_t q_t^* dt$ is strictly greater than $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$.

Proof. See appendix 5.1 □

The fact that Γ_0^* is strictly greater that Γ_0 is exactly the requirement for the weak green paradox to occur and from $\Gamma^* > \Gamma$ we can immediately conclude that the strong green paradox holds as well.
In figure 1 we see how the initial damage Γ_0 and the total damage Γ change with the backstop-price ψ. The used parameters and marginal damage function can be found below the figure.

It is not surprising to see that both the initial damage and the total damage are lower, the higher the price of the backstop is, as both green paradoxes occur. Moreover we see



(a) Initial damage Γ_0 in dependence upon ψ (b) Total damage Γ in dependence upon ψ

Figure 1: $\alpha = \beta = 1$, $\zeta = 0.2$, $r = 0.02$, $S_0 = 100$, $\theta_t = e^{0.01t}$

that both functions decline faster when the backstop-price is small. Keep in mind that we assumed $\zeta < \psi < \beta$, which are the borders we used for ψ in figure 1.

2.1.2 The monopoly allocation

Now we want to find out if the results are the same if we assume a monopoly. So now there is just one firm, which can then, of course, set the price of the resource maximizing its profit. So every combination of price and amount of the resource satisfying the demand function can be chosen. For the model this means that we are using a price-function depending on the amount of the resource extracted and sold $p_t = p(q_t)$. Therefore, when considering monopoly models, we are sometimes going to write $p(q_t)$ instead of p_t, when we want to highlight the fact that p_t depends on q_t, but there is, of course, no difference in the two expressions.

Again we write the maximization problem using the resource stock S_t

$$max \ \Pi = max \int_0^{\infty} e^{-rt}(p(q_t) - \zeta)q_t dt \tag{2.1.17}$$

$$\begin{aligned} s.t. \qquad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0 \ \ given \end{aligned}$$

Therefore we get the following Hamiltonian

$$H = (p(q_t) - \zeta)q_t - \lambda_t q_t + \nu_t S_t$$

The optimality conditions are

$$\frac{\partial H}{q_t} = \frac{\partial p_t}{\partial q_t} q_t + p(q_t) - \zeta - \lambda_t = 0 \tag{2.1.18}$$

$$\begin{aligned} \dot{\lambda}_t &= r\lambda_t - \frac{\partial H}{S_t} \\ \Leftrightarrow \dot{\lambda}_t &= r\lambda_t - \nu_t \end{aligned} \tag{2.1.19}$$

$$\nu_t S_t = 0 \tag{2.1.20}$$

We now assume that the demand function is linear

$$q_t = D(p_t) = \alpha(1 - \frac{p_t}{\beta}) \tag{2.1.21}$$

which immediately leads us to $p_t = p(q_t) = \beta(1 - \frac{q_t}{\alpha})$.
Note that, as in the competitive model, the maximum demand is equal to α and the choke-price equal to β.
Rewriting equation (2.1.18) using (2.1.19), (2.1.21) and because $\frac{\partial p_t}{\partial q_t} = -\frac{\beta}{\alpha}$ we get

$$p_t = \frac{\zeta + \beta}{2} + \frac{1}{2}\lambda_0 e^{rt} \tag{2.1.22}$$

as long as the resource stock is positive (because from $S_t > 0$ we can immediately conclude $\nu_t = 0$).
If the backstop-price is lower than $\frac{\zeta+\beta}{2}$ the price of the resource would be higher than the backstop-price and no resource would be sold. Therefore we have to distinguish between

the two cases $\psi > \frac{\zeta+\beta}{2}$ and $\psi \leq \frac{\zeta+\beta}{2}$.
In the first case we take the same approach as in the competitive case and get, as the whole resource stock is depleted until the termination date, similar to equation (2.1.10), just with a different price-path

$$\int_0^T D(\frac{\zeta+\beta}{2} + \frac{1}{2}e^{rt}\lambda_0)dt = S_0 \tag{2.1.23}$$

and the price-path at the termination date $p_T = \frac{\zeta+\beta}{2} + \frac{1}{2}\lambda_0 e^{rT}$ can be, using $p_T = \psi$, rewritten as

$$rT = ln(2\psi - \zeta - \beta) - ln(\lambda_0) \tag{2.1.24}$$

using the same arguments as in (2.1.11).
As in the competitive case we now take total derivatives of equations (2.1.23) and (2.1.24), multiply them with $\frac{d\psi}{d\lambda_0}$ and rearrange them. Equation (2.1.23) leads us to

$$\frac{dT}{d\lambda_0} = -\frac{\int_0^T \frac{e^{rt}}{2} D'(p_t)dt}{D(p_T)} \tag{2.1.25}$$

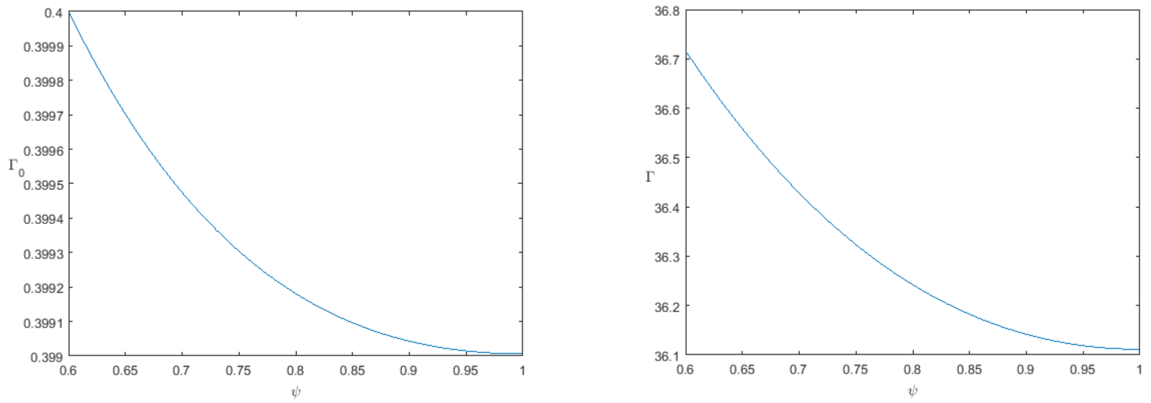
Thus we can conclude, like in the previous chapter that $\frac{dT}{d\lambda_0} > 0$, as the derivative of the demand function is smaller than zero.

Equation (2.1.24) gives us

$$r\frac{dT}{d\lambda_0} = \frac{2}{2\psi - \zeta - \beta}\frac{d\psi}{d\lambda_0} - \frac{1}{\lambda_0} \tag{2.1.26}$$

As the left hand side of this equation is positive and the second term of the right hand side is smaller than zero, the first term has to be positive, as we assumed $\psi > \frac{\zeta+\beta}{2}$. So we get $\frac{d\psi}{d\lambda_0} > 0$ and thus also $\frac{d\lambda_0}{d\psi} > 0$ and $\frac{dT}{d\psi} > 0$. As these are the same conclusions as we derived in the competitive case, the same conditions hold for the paths q_t and q_t^* and hence we can apply proposition 1 again to prove that Γ_0^* is strictly greater than Γ_0 and Γ^* is strictly greater than Γ and therefore, as in the competitive case, both green paradoxes occur.
Figure 2 shows how the damage in this model depends on the backstop-price ψ. The

same parameter values as in the competitive model are used. Note that now ψ has to be between $\frac{\zeta+\beta}{2}$ and β, as we have the additional assumption (compared to the competitive case) $\psi > \frac{\zeta+\beta}{2}$.



(a) Initial damage Γ_0 in dependence upon ψ (b) Total damage Γ in dependence upon ψ

Figure 2: $\alpha = \beta = 1$, $\zeta = 0.2$, $r = 0.02$, $S_0 = 100$, $\theta_t = e^{0.01t}$

As already mentioned, in the case that $\psi \leq \frac{\zeta+\beta}{2}$, the initial price would be higher than the price of the backstop and the firm would make zero profit. This can't be optimal, as, by assumption, the extraction costs are cheaper than the backstop-price ($\zeta < \psi$). So the best strategy for the firm is to set the price as high as possible, such that the resource still can be sold, which is exactly at the backstop-price, until the resource is depleted.
Assume now that the backstop-price drops ($\psi^* < \psi$). This would lead to a decrease in the price of the resource in each period ($p_t^* < p_t$). So the extraction in each period rises until the resource is depleted ($q_t^* > q_t \ \forall t : t < T^*$) and thus the termination date drops ($T^* < T$).
Therefore we have $T^* < T$, $q_t < q_t^* \ \forall t : t < T^*$, $q_t > q_t^* \ \forall t : T^* < t < T$ and $q_t = q_t^* = 0 \ \forall t : t > T$. This are the exact conditions we need to apply proposition 1 again and hence we get $\Gamma_0^* > \Gamma_0$ and $\Gamma^* > \Gamma$. So both green paradoxes arise in this case.

2.2 The model with increasing resource extraction costs and a perfect backstop

One can now argue that constant extraction costs, and hence a depletion of the resource is not likely, as it seems more realistic that the extraction costs rise when the resource

becomes more scarce. In the example of oil this argument can be supported by the consideration that, if a lot of oil is already extracted, the firms need to drill deeper to reach the still existing oil deposits, which is more expensive.
In this model we address this issue and assume that extraction costs rise with the amount of the resource that is already extracted. More precisely we assume a linear extraction cost function $\zeta_t = \eta s_t$, where s_t is the cumulative extraction up to point t.

$$s_t = \int_0^t q_{\hat{t}} d\hat{t} \tag{2.2.1}$$

Moreover we assume that $\eta S_0 > \psi$, so at some point in time, as the resource becomes more scarce, the extraction of it becomes more expensive than the backstop.

2.2.1 The competitive equilibrium

Again we want to formulate the model using the resource stock S_t. As s_t describes the cumulative extraction in period t, we get $S_t = S_0 - s_t$. Using this relationship we can conclude that $\zeta_t = \eta s_t = \eta(S_0 - S_t)$. We get the following optimization problem

$$max\ \Pi = max \int_0^\infty e^{-rt}(p_t - \eta(S_0 - S_t))q_t dt \tag{2.2.2}$$

$$\begin{aligned} s.t. \quad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0\ \ given \end{aligned}$$

Thus the Hamiltonian is

$$H = (p_t - \eta(S_0 - S_t))q_t - \lambda_t q_t + \nu_t S_t$$

which leads us to the following optimality conditions

$$\frac{\partial H}{q_t} = (p_t - \eta(S_0 - S_t)) - \lambda_t = 0$$
$$\Leftrightarrow p_t - \eta s_t - \lambda_t = 0 \tag{2.2.3}$$

$$\dot{\lambda}_t = r\lambda_t - \frac{\partial H}{\partial S_t}$$
$$\Leftrightarrow \dot{\lambda}_t = r\lambda_t - \eta q_t - \nu_t \tag{2.2.4}$$

$$\nu_t S_t = 0 \tag{2.2.5}$$

The goal is now to find λ_0 such that at the also unknown termination date T the resource-price equals the backstop-price ($p_T = \eta s_T + \lambda_T = \psi$) and the shadow-price of the resource equals zero ($\lambda_T = 0$).

At first we want to derive a system of two differential equations in s_t and λ_t. Therefore we again assume a linear demand function, as in (2.1.21). As in the previous model, we assume that the price of the backstop is smaller than the choke-price ($\psi < \beta$).
Using $\dot{s_t} = q_t$ (see (2.2.1)) and equations (2.1.21)and (2.2.3) we get

$$\dot{s_t} = -\frac{\alpha\eta}{\beta}s_t - \frac{\alpha}{\beta}\lambda_t + \alpha \tag{2.2.6}$$

Plugging equations (2.1.21) and (2.2.3) into (2.2.4), we get for $t < T$ (as then $\nu_t = 0$ holds)

$$\dot{\lambda}_t = \frac{\alpha\eta^2}{\beta}s_t + (r + \frac{\alpha\eta}{\beta})\lambda_t - \alpha\eta \tag{2.2.7}$$

In Figure 3 we see the phase plane of this system of differential equations. The used parameter values can be found right below the figure.
The red dot represents the steady state and the green lines the stable and the unstable

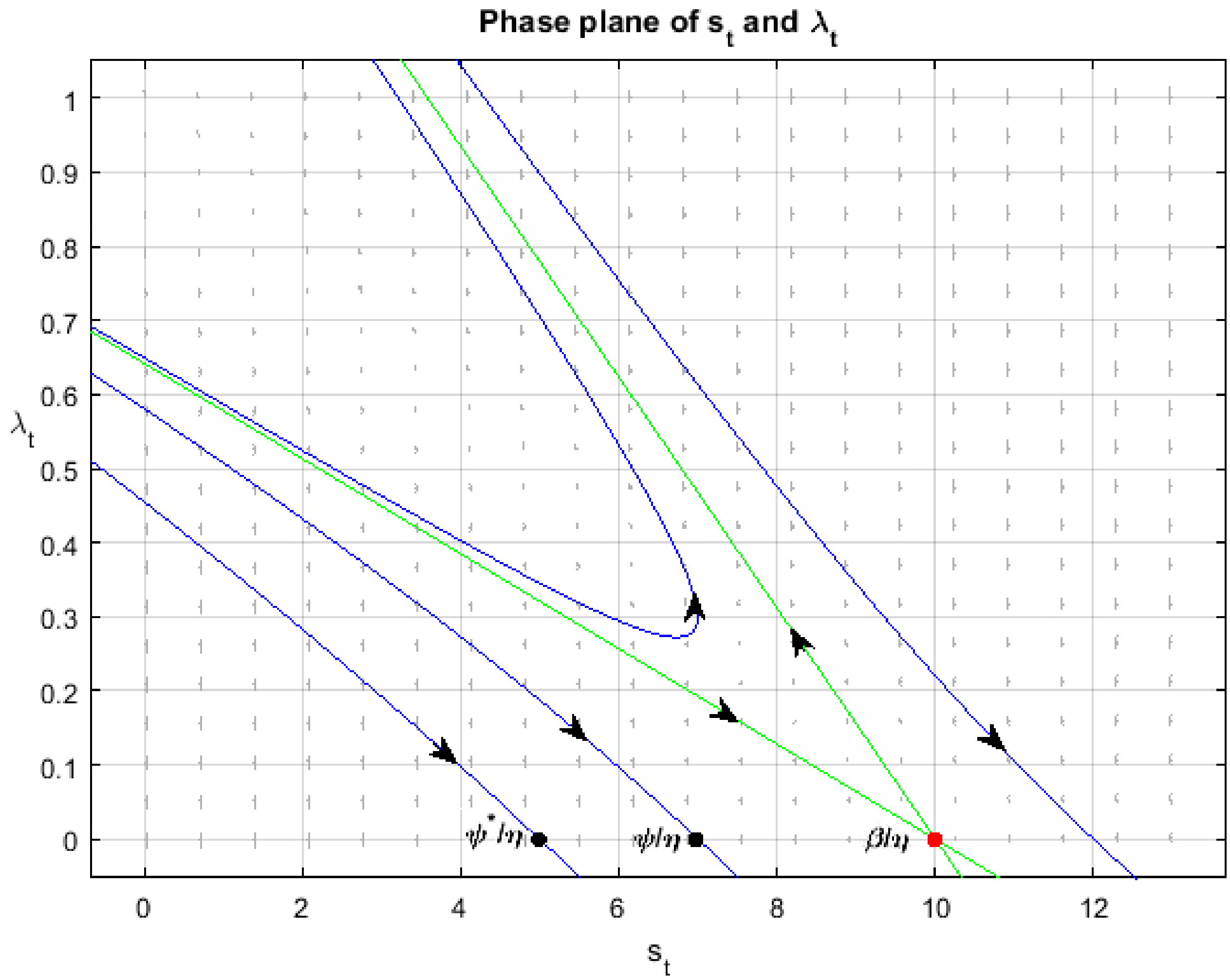


Figure 3: $\alpha = \beta = 1$, $\eta = 0.1$, $r = 0.02$

eigenvectors. We see that both eigenvectors are downwards sloping, where the one corresponding to the unstable eigenvalue has a greater absolute slope. Initially s_t is of course always zero. We know that, as $\psi < \beta$, $\zeta_t = \eta s_t$ and $\zeta_T = \psi$, at the termination date $s_T = \frac{\psi}{\eta}$ and $\lambda_T = 0$ hold, so the isocline has to end there, leaving some amount of the resource in the ground.

To get the equilibrium path, it is therefore necessary to find the right λ_0, such that the system ends in this particular point.

In figure 3 we have two different example values for the backstop $\psi^* < \psi(< \beta)$. We see that a cheaper backstop implies a lower λ_0 and therefore a lower p_0 which leads to a higher q_0. So we can conclude, without a formal proof yet, that the weak green paradox is going to hold. On the other hand the total amount of the resource extracted is lower, which, at this point, makes it hard to make a good guess if the strong green paradox arises or not. But we are going to show now that it, in fact, does not occur.

Proposition 2. *Given the system of differential equations, which is defined by equations (2.2.6) and (2.2.7), a strictly positive parameter θ_0 and a strictly decreasing weight function $\tilde{\theta}_t$ and knowing additionally that $s_0 = 0$, $s_T = \frac{\psi}{\eta}$, $\lambda_T = 0$, $\lambda_t = p_t - \eta s_t$ (see equation (2.2.3)) and $\dot{s}_t = q_t = \alpha(1 - \frac{p_t}{\beta})$ the function $\Gamma_0 := \theta_0 q_0$ depends negatively on the backstop-price ψ and the function $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$ depends positively on ψ.*

Proof. See appendix 5.1 □

As all the requirements for proposition 2 are fulfilled here, we know that if the backstop-price ψ drops, the initial damage Γ_0 is going to rise, but the total damage Γ drops. Thus the weak green paradox occurs, but the strong green paradox does not.

2.2.2 The monopoly allocation

As in the previous model, when considering the case of a monopoly, the only change in the model is that the price of the resource is a function of the amount of the resource that is extracted and sold ($p_t = p(q_t)$). Hence we get

$$max\ \Pi = max \int_0^\infty e^{-rt}(p(q_t) - \eta(S_0 - S_t))q_t dt \tag{2.2.8}$$

$$\begin{aligned} s.t. \quad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0 \ given \end{aligned}$$

Thus the Hamiltonian is

$$H = (p(q_t) - \eta(S_0 - S_t))\, q_t - \lambda_t q_t + \nu_t S_t$$

which leads us to the following optimality conditions

$$\begin{aligned}
&\frac{\partial H}{\partial q_t} = (p_t - \eta(S_0 - S_t)) + \frac{\partial p_t}{\partial q_t} q_t - \lambda_t = 0 \\
\Leftrightarrow & p_t - \eta s_t + \frac{\partial p_t}{\partial q_t} q_t - \lambda_t = 0
\end{aligned} \tag{2.2.9}$$

$$\begin{aligned}
&\dot{\lambda}_t = r\lambda_t - \frac{\partial H}{\partial S_t} \\
\Leftrightarrow & \dot{\lambda}_t = r\lambda_t - \eta q_t - \nu_t
\end{aligned} \tag{2.2.10}$$

$$\nu_t S_t = 0 \tag{2.2.11}$$

From (2.2.9) we can conclude, using the demand function (2.1.21) and the fact that $\frac{\partial p_t}{\partial q_t} = -\frac{\beta}{\alpha}$, that

$$p_t = \frac{1}{2}(\eta s_t + \lambda_t + \beta) \tag{2.2.12}$$

Like in the model with constant extraction costs, we have to distinguish between two cases.

At first we assume that the initial price of the resource has to be smaller than the backstop-price. This is the case when $\psi > \frac{\beta}{2}$ (see equation (2.2.12)), and can continue similar to the competitive case. Using the demand function (2.1.21), equations (2.2.10) and (2.2.12) as well as the fact that $\dot{s}_t = q_t$ we get

$$\dot{s}_t = \frac{1}{2}\left(-\frac{\alpha\eta}{\beta} s_t - \frac{\alpha}{\beta}\lambda_t + \alpha\right) \tag{2.2.13}$$

and

$$\dot{\lambda}_t = \frac{1}{2}\left(\frac{\alpha\eta^2}{\beta} s_t + (2r + \frac{\alpha\eta}{\beta})\lambda_t - \alpha\eta\right) \tag{2.2.14}$$

So we get a system of differential equations, defined by (2.2.13) and (2.2.14), which looks

similar as the system of differential equations in the competitive case (equations (2.2.6) and (2.2.7)) divided by 2. The only difference is that in equation (2.2.14) we have the term $2r$ instead of r.
Let us at first take a look at the phase plane of the monopoly case, using the same values for the parameters as in the competitive case.

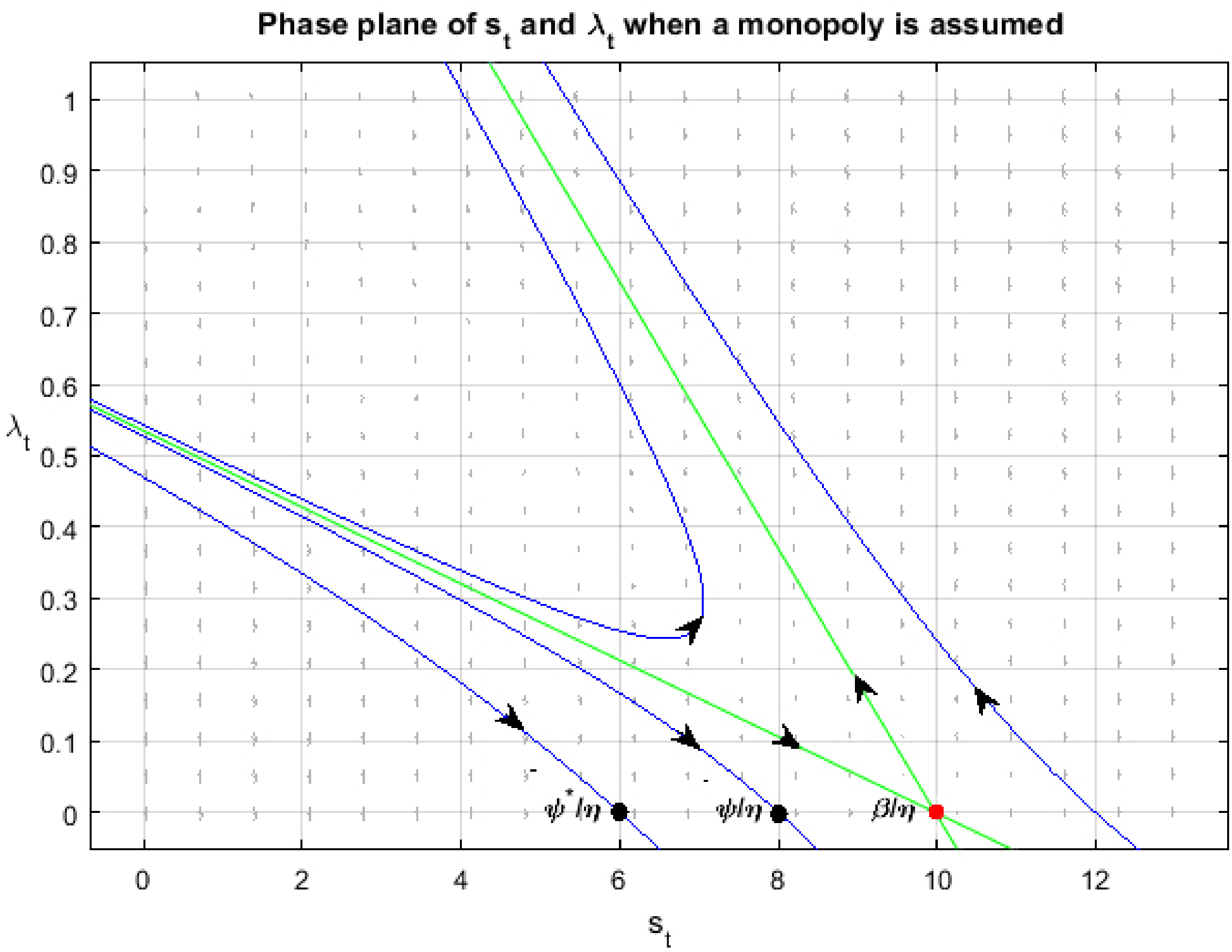


Figure 4: $\alpha = \beta = 1, \eta = 0.1, r = 0.02$

We see that the structure of the phase plane looks identical to the competitive case. Only the slope of the eigenvectors and therefore also the initial values of the shadow-price (λ_0) are different.

Now we want to prove that we in fact have the same implications as in the competitive case using the following proposition

Proposition 3. *Given the system of differential equations, which is defined by equations (2.2.13) and (2.2.14)), a strictly positive parameter* θ_0 *and a strictly decreasing weight function* $\tilde{\theta}_t$ *and knowing additionally that* $s_0 = 0$, $s_T = \frac{\psi}{\eta}$, $\lambda_T = 0$, $\lambda_t = 2p_t - \beta - \eta s_t$ *(see equation (2.2.12)) and* $\dot{s}_t = q_t = \alpha(1 - \frac{p_t}{\beta})$ *the function* $\Gamma_0 = \theta_0 q_0$ *depends negatively on the backstop-price* ψ *and the function* $\Gamma = \int_0^T \tilde{\theta}_t q_t dt$ *depends positively on* ψ.

Proof. See appendix 5.1 □

As the requirements for proposition 3 are fulfilled, we can immediately conclude that, as in the competitive case, the weak green paradox occurs, but the strong green paradox does not.

Now we are going to analyse the model when $\psi \leq \frac{\beta}{2}$. Then it is, like in the monopoly setup of the first model in the case where we assumed that $\psi \leq \frac{\zeta+\beta}{2}$, best to set the price of the resource equal to the backstop-price in all periods ($p_t = \psi \;\; \forall t$).
Thus we have

$$q_t = \alpha(1 - \frac{\psi}{\beta}) \tag{2.2.15}$$

Next we look at the cumulative extraction $s_t = \int_0^t q_{\hat{t}} d\hat{t}$ and using equation (2.2.15) we get

$$s_t = \alpha(1 - \frac{\psi}{\beta})t \tag{2.2.16}$$

Combining equation (2.2.16) with $s_T \eta = \psi$ and solving this for T, we get

$$T = \frac{\beta\psi}{\alpha\eta(\beta - \psi)} \tag{2.2.17}$$

We are going to show now that the occurrence of the strong green paradox depends on the values of the parameters used. To do this we will use a marginal damage function with a constant growth rate ($\theta_t = \theta_0 e^{\sigma t}$).

Proposition 4. *Assume that* $\dot{s}_t = q_t = \alpha(1 - \frac{\psi}{\beta})$, $T = \frac{\beta\psi}{\alpha\eta(\beta-\psi)}$, $\psi \leq \frac{\beta}{2}$, $\theta_0 > 0$ *and* $r, \sigma > 0$ *with* $r > \sigma$.
Then the function $\Gamma_0 = \theta_0 q_t$ *depends negatively upon* ψ *and the dependence of the function* $\Gamma = \int_0^\infty e^{-rt}\theta_0 e^{\sigma t} q_t$ *on* ψ *can't be uniquely determined, but depends on the parameters used. The chance of* $\frac{\partial\Gamma}{\partial\psi}$ *being lower than zero is high for a big difference between the real interest rate and the growth rate of the marginal damage function (*$r - \sigma$ *high) as well as for small values of* α *and* η.

Proof. See appendix 5.1 □

So we see that in this case the weak green paradox arises, but the occurrence of the strong green paradox depends on the parameters used. It is more likely to occur if there is a big difference between the real interest rate and the growth rate of the marginal damage function, as well as for small values of α and η.

2.3 The model with constant resource extraction costs and an imperfect backstop

In the third model we return to the assumption of a constant resource-price and deal with the issue of the perfect backstop.
In the previous models it was possible to get an infinitely large amount of the backstop at a constant price. It can be argued that this is not a very realistic assumption. Let's for example think about the backstop being wind power. There are spots that are more suited for a wind turbine and spots where turbines are less productive. At first obviously the best spots are used. As more and more turbines are put into operation the spots get worse and the marginal productivity of wind power drops.
Therefore the price of one unit of energy should rise with the amount of the backstop being used.
We assume in this model that the first unit of the backstop is cheaper than the extraction costs. This leads to a simultaneous use of both the resource and the backstop until the resource is depleted.

The initial price of the backstop is denoted by ψ_0. We assume that $\psi_0 < \zeta$ and that

the costs for the backstop rise by ψ' for every additional unit used. Therefore we get $p_t^e = \psi_0 + \psi' e_t$. Where e_t is the amount of the backstop used and p_t^e is the price of the backstop.
As the maximization problem for the firms is the same as in the first model, we get for the price-path (for $t < T$) again

$$p_t = \zeta + \lambda_t \tag{2.3.1}$$

and for the shadow-price

$$\lambda_t = \lambda_0 e^{rt} \tag{2.3.2}$$

Note that in equilibrium both resources are used simultaneously, therefore the prices have to coincide ($p_t^e = p_t$).

As the firm in this model is in competition with the supplier of the backstop in each period (because the resource and the backstop are used simultaneously) a monopoly model would not make any sense. Thus we will only discuss the competitive case.

2.3.1 The competitive equilibrium

The difference compared to the first model is that, when defining the demand function, one has to take the use of the backstop into account

$$D(p_t) = q_t + e_t = q_t + \frac{p_t - \psi_0}{\psi'} \tag{2.3.3}$$

Assuming again a linear demand function as in (2.1.21) we get, using (2.3.1) and (2.3.3), an expression for the resource extraction path

$$q_t = \alpha(1 - \frac{\zeta}{\beta}) - \frac{\zeta - \psi_0}{\psi'} - (\frac{\alpha}{\beta} + \frac{1}{\psi'})\lambda_t \tag{2.3.4}$$

To make further calculations easier, we want to write this function as $q_t = a(1 - \frac{\lambda_t}{b})$ with

$$a = \alpha(1 - \frac{\zeta}{\beta}) - \frac{\zeta - \psi_0}{\psi'} \tag{2.3.5}$$

and

$$b = \frac{a}{\frac{\alpha}{\beta} + \frac{1}{\psi'}} \tag{2.3.6}$$

It is easy to see that $\frac{\partial a}{\partial \psi_0} > 0$ and $\frac{\partial a}{\partial \psi'} > 0$, as $\psi_0 < \zeta$. We now want to find out how T and $\dot{q}_t$, and later on q_0 and Γ, depend upon the variable a and prove that neither T nor q_t depends on b.

We know that $q_T = 0$ has to hold, so we get $\lambda_T = b$ and thus because of (2.3.2) $\lambda_t = be^{r(t-T)}$.
This leads to

$$q_t = a(1 - e^{r(t-T)}) \tag{2.3.7}$$

and

$$S_0 = s_T = \int_0^T q_t dt = \int_0^T a(1 - e^{r(t-T)})dt = a(T - \frac{1 - e^{-rT}}{r}) \tag{2.3.8}$$

As q_t only depends on T and a, and, considering the above equation, we know that T only depends on a, we get $\frac{\partial q_t}{\partial b} = \frac{\partial T}{\partial b} = 0$.
Taking the derivative with respect to a from equation (2.3.8), we get

$$0 = T - \frac{1 - e^{-rT}}{r} + a \cdot \frac{\partial T}{\partial a} - ae^{-rT} \frac{\partial T}{\partial a} \tag{2.3.9}$$

And solving this for $\frac{\partial T}{\partial a}$ gives us

$$\frac{\partial T}{\partial a} = -\frac{T - \frac{1 - e^{-rT}}{r}}{a(1 - e^{-rT})} = -\frac{S_0}{a^2(1 - e^{-rT})} < 0 \tag{2.3.10}$$

The last inequality holds, because from (2.3.7) we know that $0 < q_0 = a(1 - e^{-rT})$.
For the change of the resource extraction we get $\dot{q}_t = -rae^{r(t-T)}$ (see equation (2.3.7))and therefore

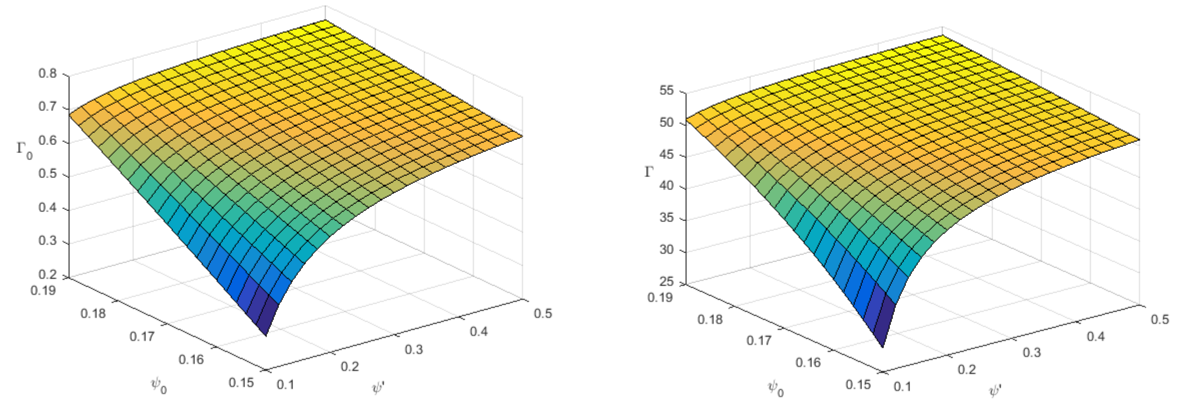
$$\frac{\partial \dot{q}_t}{\partial a} = -re^{r(t-T(a))} + r^2 ae^{r(t-T(a))} \cdot \frac{\partial T}{\partial a} < 0 \tag{2.3.11}$$

As neither T nor q_t depend upon b, we can solely focus our considerations on the changes in a when the price of the backstop drops.
Assume that there exists a cheaper backstop with $\psi_0^* < \psi_0$ and/or $\psi'^* < \psi'$ and label again all the variables corresponding to the cheaper backstop with a star. From the calculated derivatives we know that this leads to $a^* < a$ and thus to $T^* > T$ and $\dot{q_t}^* > \dot{q}_t \ \forall t : t < T$. Moreover as $\dot{q}_t = 0 \ \forall t : t > T$ and $\dot{q_{t^*}} < 0 \ \forall t : T < t < T^*$ we have $\dot{q_t}^* < \dot{q}_t \ \forall t : T < t < T^*$.
So we know that with a cheaper backstop the variable a falls, the resource becomes depleted later and the absolute change in the amount of resources used is smaller (as $\dot{q}_t < 0$).
To find out if the green paradoxes occur, we use the following proposition

Proposition 5. *Consider two continuous paths $q_t, q_t^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $q_t = 0 \ \forall t : t > T$, $q_t^* = 0 \ \forall t : t > T^*$, $T < T^*$ and $\int_0^T q_t dt = \int_0^{T^*} q_t^* dt$. Moreover we know that $\dot{q}_t < \dot{q_t^*} \ \forall t : t < T$ and $\dot{q}_t > \dot{q_t^*} \ \forall t : T < t < T^*$. Additionally we have a strictly positive parameter θ_0 and a strictly decreasing weight function $\tilde{\theta}_t$.*
Then we can conclude that $\Gamma_0^* := \theta_0 q_0^*$ is strictly smaller than $\Gamma_0 := \theta_0 q_t$ and $\Gamma^* := \int_0^{T^*} \tilde{\theta}_t q_t^* dt$ is strictly smaller than $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$.

Proof. See appendix 5.1 □

So the initial damage and the total damage corresponding to the cheaper backstop are smaller than the ones corresponding to the higher backstop, which means that neither of the green paradoxes does arise.
Figure 5 shows how the initial and the overall damage change with the initial backstop-price ψ_0 and the rise of the backstop-price per additionally unit used ψ'.
We see that both the initial and the total damage are higher the higher the initial backstop-price is and the faster it rises respectively. So if the price of the backstop



(a) Initial damage Γ_0 in dependence on ψ_0 and ψ' (b) Total damage Γ in dependence on ψ_0 and ψ'

Figure 5: $\alpha = \beta = 1$, $\zeta = 0.2$, $r = 0.02$, $S_0 = 100$, $\theta_t = e^{0.01t}$

gets lower and/or rises slower, the damage on the environment (both initial and total) declines. This goes in line with our findings that in this model the green paradoxes do not occur.

2.4 Discussion

The existence of the green paradox in the models with a backstop depends strongly on the assumptions made. In three different models, where always only one assumption is changed, we get three completely different outcomes concerning the green paradoxes.
If constant extraction costs and a perfect backstop are assumed both the weak and the strong green paradox occur. If we change the models by using increasing extraction costs, the strong green paradox doesn't arise any more, due to the fact that less of the resource is used and that outweighs the negative effect of the greater extraction in the beginning. If we assume constant extraction costs and an imperfect backstop (backstop-price rises the more units are used), both the initial backstop-price and how fast it rises has a positive correlation with both the initial and the total damage inflicted on the environment. Therefore in this case neither of the green paradoxes does arise, due to the fact that the cheaper the backstop-price, the more of the backstop and not the resource is used.
Note that it is not argued here that the models get more realistic from the first to the last in this chapter. There have already been presented arguments that support increasing extraction costs as well as ones supporting an imperfect backstop, but it can also be

argued that constant extraction costs are more realistic. One can't deny that it gets more difficult to extract resources when there are fewer left, but new technologies can be invented, which makes extraction cheaper again, so it may be true that it is closer to reality to assume constant extraction costs (see e.g. Stürmer and Schwerhoff 2012). Similarly it can be argued that maybe a perfect backstop is more realistic, as e.g. the spots where the wind turbines are put get less productive, but again the more wind turbines are installed, the better the technologies get and the costs for a unit of energy produced may stay the same. So if one should believe the green paradoxes occur or not is entirely based on which model she or he thinks reflects reality best.
When assuming a monopoly, what is, especially when we think about oil, a reasonable assumption, the structure of the solutions stay mostly the same. In the case of constant extraction costs and an additional parameter restriction ($\frac{\zeta+\beta}{2} > \psi$), the structure changes and the firm sets the price for the resource equal to the backstop-price in all periods. But what doesn't change is that still both green paradoxes arise.
The case of rising extraction costs and the additional parameter restriction $\frac{\beta}{2} > \psi$ is the only one where the conclusions about one of the green paradoxes may change, because there the strong green paradox may arises, depending on the chosen parameters, whereas in the competitive case we know that it does not. As already mentioned in the last model a monopoly case does not make any sense as the firm owning the resource is in competition with the backstop-owner.

3 The models with tax

This chapter is based on the model by Österle (2015). Here we study the second group of models that deal with the green paradox. The assumptions are a bit different compared to the models we looked at in the previous chapter. We drop the assumption of the existence of a backstop, but assume that the government can collect taxes on the resource. Therefore we have to change the definition of the green paradoxes. In this kind of models a weak green paradox arises if the introduction of a tax leads to a higher damage in the initial period and a strong green paradox if it leads to a higher overall damage.
The mentioned tax is a revenue tax, with tax rate τ_t, where $\tau_t = 1 - \kappa_t$. So κ_t denotes the fraction of the revenue that the firms can keep. We assume that $\kappa_t = \kappa_0 \cdot e^{-\hat{\kappa}t}$ with $\kappa_0 \in (0,1)$ and $\hat{\kappa} > 0$. So the tax increases with $\tau_t \to 1$ for $t \to \infty$. Note that if no tax is implemented, we have $\kappa_t = 1$ and therefore $\kappa_0 = 1$ and $\hat{\kappa} = 0$.
Another big difference is the structure of the demand function. Contrary to the previous chapter, we assume that the elasticity of the demand function is constant, which is equivalent to using the form $D(p_t) = p_t^{-\gamma}$, which is presented in Dasgupta and Heal (1979). They show that the property of this demand function leads to an optimal time horizon, in which the resource is extracted, that is equal to infinity.
As Österle omits extraction costs in her model (probably for simplicity), we will do the same in this chapter. Note that the cost-function in the model with an endogenous resource stock describes exploration and not extraction costs.

3.1 The model with an exogenous resource stock

In this first model with tax we assume a fixed resource stock, as in the models before. In the next section we will then relax this assumption.

3.1.1 The competitive equilibrium

The goal of the firms is again to maximize their discounted profits. This leads us to the following maximization problem

$$max\ \Pi = max \int_0^{\infty} p_t q_t \kappa_t e^{-rt} dt \tag{3.1.1}$$

$$\begin{aligned} s.t. \qquad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0\ \ given \end{aligned}$$

Therefore we get for the Hamiltonian

$$H = p_t q_t \kappa_t - \lambda_t q_t + \nu_t S_t$$

which leads to the optimality conditions

$$\frac{\partial H}{q_t} = p_t \kappa_t - \lambda_t = 0 \tag{3.1.2}$$

$$\begin{aligned} \dot{\lambda}_t =& r\lambda_t - \frac{\partial H}{\partial S_t} \\ \Leftrightarrow \dot{\lambda}_t =& r\lambda_t - \nu_t \end{aligned} \tag{3.1.3}$$

$$\nu_t S_t = 0 \tag{3.1.4}$$

As we know that the optimal time of extraction is equal to infinity, we get $S_t > 0\ \ \forall t$ and thus $\nu_t = 0\ \ \forall t$.
Hence we can conclude from (3.1.3) that $\lambda_t = \lambda_0 e^{rt}$ and therefore from (3.1.2)

$$p_t = \frac{\lambda_t}{\kappa_t} = \frac{\lambda_0}{\kappa_0} e^{(r+\hat{\kappa})t} = p_0 e^{(r+\hat{\kappa})t} \tag{3.1.5}$$

Thus we also know that

$$p_0 = \frac{\lambda_0}{\kappa_0} \tag{3.1.6}$$

As already mentioned, the demand function is defined as

$$q_t = D(p_t) = p_t^{-\gamma} \tag{3.1.7}$$

Using (3.1.5) and (3.1.7) we get for the resource extraction

$$q_t = p_t^{-\gamma} = p_0^{-\gamma} e^{-\gamma(r+\hat{\kappa})t} = q_0 e^{-\gamma(r+\hat{\kappa})t} \tag{3.1.8}$$

As there are no extraction costs, it is optimal to use all of the available resource, so the total extracted amount has to equal the initial resource stock $S_0 = \int_0^\infty q_t dt$. Using equation (3.1.8) we can rewrite this expression as

$$q_0 = S_0 \gamma (r + \hat{\kappa}) \tag{3.1.9}$$

Combining equations (3.1.8) and (3.1.9) we get

$$q_t = S_0 \gamma (r + \hat{\kappa}) e^{-\gamma(r+\hat{\kappa})t} \tag{3.1.10}$$

From equation (3.1.9) we can immediately conclude that the extraction at time zero is smallest if there is no tax ($\hat{\kappa} = 0$ compared to $\hat{\kappa} > 0$ in the case of a tax), therefore the weak green paradox arises. Moreover, as $\frac{\partial q_0}{\partial \hat{\kappa}} = S_0 \gamma > 0$, we know that the higher $\hat{\kappa}$ is, so the faster the tax rate grows, the higher is the initial damage. The initial tax rate τ_0 has no influence on the initial damage, as neither τ_0 nor κ_0 arise in equation (3.1.9).

To see if the strong green paradox arises, we look at the overall damage, restricting ourselves in this chapter to the case where $\theta_t = \theta_0 e^{\sigma t}$ and assuming, as in the previous chapter, $\sigma < r$. Thus we get $\Gamma = \int_0^\infty \theta_0 e^{-(r-\sigma)t} q_t dt$. Using equation (3.1.10) we can rewrite this expression as

$$\Gamma = \frac{S_0 \theta_0 \gamma (r + \hat{\kappa})}{r - \sigma + \gamma (r + \hat{\kappa})} \tag{3.1.11}$$

So Γ does not depend on the initial tax rate either.
Next we compute the derivative of Γ with respect to $\hat{\kappa}$

$$\frac{\partial \Gamma}{\partial \hat{\kappa}} = \frac{S_0 \theta_0 \gamma \left[r - \sigma + \gamma (r + \hat{\kappa}) \right] - S_0 \theta_0 \gamma^2 (r + \hat{\kappa})}{\left[r - \sigma - \gamma (r + \hat{\kappa}) \right]^2} =$$
$$= \frac{S_0 \theta_0 \gamma (r - \sigma)}{\left[r - \sigma + \gamma (r + \hat{\kappa}) \right]^2} > 0$$

So the overall damage rises if $\hat{\kappa}$ rises. Which means that the case without taxes is the best for the environment, as again the lowest possible value of $\hat{\kappa}$ is zero and this occurs in the case without taxes. Thus the strong green paradox occurs.
As before for the initial extraction, the faster the tax rate rises, the higher the damage. This results can be easily explained, as a faster growth of the tax rate makes an early extraction of the resource more profitable for the firm, which is exactly what increases the damage on the environment, when the total extraction stays constant.

3.1.2 The monopoly allocation

Again the firm maximizes the profit function with a price that depends on the amount of the resource extracted ($p_t = p(q_t)$)

$$max \ \Pi = max \int_0^\infty p(q_t) q_t \kappa_t e^{-rt} dt \tag{3.1.12}$$

$$\begin{aligned} s.t. \quad & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & S_0 \ \ given \end{aligned}$$

So the Hamiltonian is

$$H = p(q_t) q_t \kappa_t - \lambda_t q_t + \nu_t S_t$$

As mentioned before, because the optimal time horizon for the extraction equals infinity, ν_t equals zero. This leads to following optimality conditions

$$\begin{aligned} \frac{\partial H}{q_t} &= p(q_t)\kappa_t + \frac{\partial p_t}{\partial q_t} q_t \kappa_t - \lambda_t = 0 \\ \Leftrightarrow & p(q_t)\kappa_t - \frac{1}{\gamma}\kappa_t q_t^{-\frac{1}{\gamma}} - \lambda_t = 0 \end{aligned} \tag{3.1.13}$$

$$\begin{aligned} \dot{\lambda}_t &= r\lambda_t - \frac{\partial H}{\partial S_t} \\ \Leftrightarrow & \dot{\lambda}_t = r\lambda_t \end{aligned} \tag{3.1.14}$$

Rewriting (3.1.13), using the demand function (3.1.7), equation (3.1.14) and the definition of κ_t, we can get an expression for the price path

$$p_t = \frac{\lambda_0}{\kappa_0(1-\frac{1}{\gamma})} e^{(r+\hat{\kappa})t} = p_0 e^{(r+\hat{\kappa})t} \tag{3.1.15}$$

So we know that

$$p_0 = \frac{\lambda_0}{\kappa_0(1-\frac{1}{\gamma})} \tag{3.1.16}$$

To get the equation for the resource extraction path, we combine the demand function (3.1.7) with equation (3.1.15) and get

$$q_t = \left(\frac{\lambda_0}{\kappa_0(1-\frac{1}{\gamma})} \right)^{-\gamma} e^{-\gamma(r+\hat{\kappa})t} = q_0 e^{-\gamma(r+\hat{\kappa})t} \tag{3.1.17}$$

We see that q_t is only positive, if the elasticity of the demand function γ is greater than one. An elasticity smaller than or equal to one means that a higher price always implies a higher output. This leads to a price equal to infinity and an extraction of zero in each period. This is a common result in monopoly literature (see e.g. Varian 2010).

If the elasticity is greater than one, q_t is strictly positive and it is again optimal to use all of the available resource, so $S_0 = \int_0^\infty q_t dt$. Using this and equation (3.1.17) we get for the initial amount of the resource that is extracted the following expression

$$q_0 = S_0 \gamma (r + \hat{\kappa}) \tag{3.1.18}$$

and thus when combining equations (3.1.17) and (3.1.18) we end up with

$$q_t = S_0 \gamma (r + \hat{\kappa}) e^{-\gamma(r+\hat{\kappa})t} \tag{3.1.19}$$

which is exactly the same solution as in the competitive case. Hence, if we rule out the case where $\gamma \leq 1$, which would lead to no extraction at all, the results are the same as in the competitive case. So both the weak and the strong green paradox occur.

3.2 The model with endogenous resource exploration

The resource stock of oil and many other non-renewable resources has risen in the past decades (see e.g. Stürmer and Schwerhoff 2012). This is due to the fact that new technologies allow the extraction of resources that have not been extractable before.
In the following model we will account for that and assume that the initial resource stock is not fixed, but has to be explored. To do this, the firms have to pay exploration costs, which we will define as $C(S_0)$. The higher the initial resource stock should be, the more money has to be invested, so $C'(S_0) > 0$. Additionally we assume rising marginal costs ($C''(S_0) > 0$), because at first it is relatively easy to explore new resources, but as a lot of them are already explored, it gets more expensive to provide an additional unit of the resource. Thus the cost function has to be rising and convex, more precisely we assume $C(S_0) = \beta S_0^\alpha$ with $\beta > 0$ and $\alpha > 1$. A further assumption is that the exploration of the resource happens before the extraction. So the model can be divided into two phases, an exploration and an extraction phase.
Finally we assume that the reserves of the resource, being the maximal initial resource stock, are either infinity, or at least so high that it is economically not optimal to use all of the available reserves.

3.2.1 The competitive equilibrium

At first we take a look at the extraction phase and get a similar maximization problem as in the previous model.

$$max\ \Pi = max \int_0^\infty p_t q_t \kappa_t e^{-rt} dt - C(S_0) \tag{3.2.1}$$

$$\begin{aligned} s.t. \quad & \dot{S} = -q \\ & S_t \geq 0 \\ & S_0 \ \ endogenous \end{aligned}$$

As long as we don't use the fact that the initial resource stock is endogenous, the results are exactly the same as in the previous model, as $C(S_0)$ is a constant in the extraction phase and hence not relevant for maximisation. Thus we get for the initial price (see equation (3.1.6))

$$p_0 = \frac{\lambda_0}{\kappa_0} \tag{3.2.2}$$

For the initial resource extraction we get (see equation (3.1.9))

$$q_0 = S_0 \gamma (r + \hat{\kappa}) \tag{3.2.3}$$

and for the resource extraction path (see equations (3.1.8) and (3.1.10))

$$q_t = q_0 e^{-\gamma(r+\hat{\kappa})t} = S_0 \gamma (r + \hat{\kappa}) e^{-\gamma(r+\hat{\kappa})t} \tag{3.2.4}$$

At time zero the marginal utility of the resource, which is equal to the shadow-price λ_0, has to coincide with the marginal costs of the initial resource stock ($C'(S_0)$), otherwise it would be optimal to increase or decrease the amount of the resource made available.

Therefore we have $C'(S_0) = \lambda_0$ (see also Lasserre 1991). Using this equation we can get an expression for the initial extraction q_0

$$q_0 = \left(\frac{\kappa_0}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \tag{3.2.5}$$

For the derivation of equation (3.2.5) see appendix 5.2.1.

Now we compare the model where taxes have to be paid to the model without taxes. Remember that in the model without taxes we have $\kappa_0 = 1$ and $\hat{\kappa} = 0$.

The weak green paradox arises, if the initial damage in the case where taxes are imposed is higher than in the case without taxes ($\Gamma_0^{tax} > \Gamma_0^{notax} \Leftrightarrow q_0^{tax} > q_0^{notax}$). Thus, using equation (3.2.5) we can derive the condition for the weak green paradox to occur and get

$$\tau_0 < 1 - \left(1 + \frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \tag{3.2.6}$$

The derivation can again be found in appendix 5.2.1.

So the occurrence of the weak green paradox depends on the structure of the tax, that is implemented. A low initial tax rate τ_0 favours the weak green paradox to occur. The same is true if the tax rises fast ($\hat{\kappa}$ high). For which combination of κ_0 and $\hat{\kappa}$ the weak green paradox arises, can be seen in figure 6. The values of α and r are stated below the figure. What we can additionally conclude from (3.2.6) is that a high real interest rate r as well as high marginal exploration costs (α high) favour the weak green paradox to occur.

To see if the strong green paradox occurs we look at the total damage, again restricted to the case where $\theta_t = \theta_0 e^{\sigma t}$ and get

$$\Gamma = \frac{\theta_0}{\gamma(r+\hat{\kappa}) + r - \sigma} \left(\frac{\kappa_0}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \tag{3.2.7}$$

For the derivation see appendix 5.2.1.

As before we compare the case where a tax is collected to the case without a tax, to see

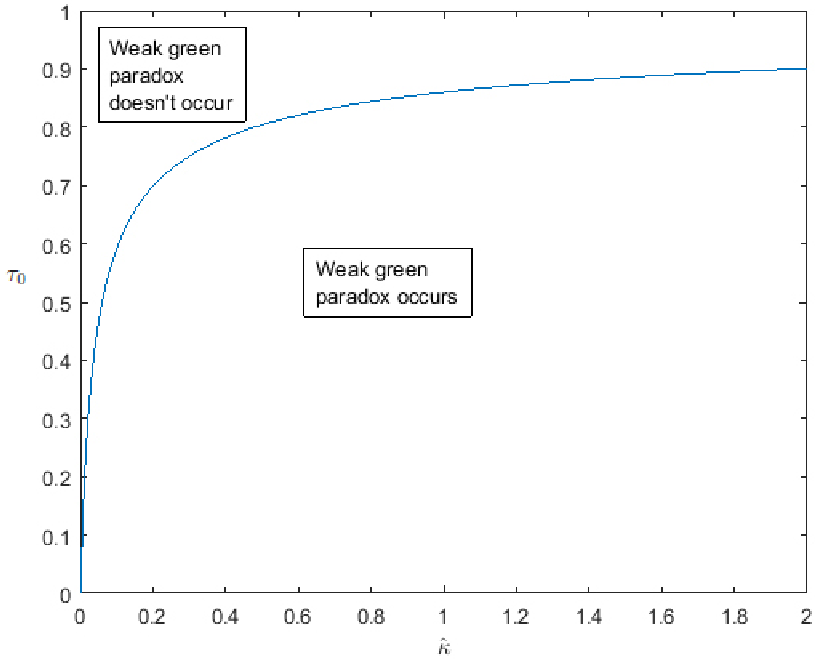


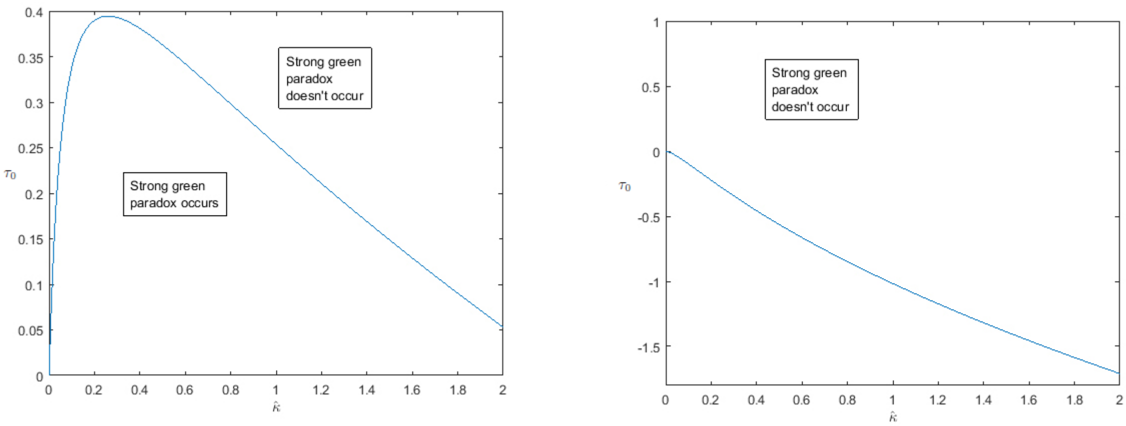
Figure 6: $\alpha = 1.5$, $r = 0.02$

when the strong green paradox occurs ($\Gamma^{tax} > \Gamma^{notax}$) and find out that it does if for the implemented tax the condition

$$\tau_0 < 1 - \left[\left(1 + \frac{\gamma\hat{\kappa}}{r(1+\gamma) - \sigma}\right)^{\alpha - 1 + \frac{1}{\gamma}} \left(1 + \frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \right] \tag{3.2.8}$$

holds. The derivations can again be found in appendix 5.2.1.

As it was before for the weak green paradox, a low initial tax rate makes it more likely for the strong green paradox to occur. What can be said additionally is that the strong green paradox is more likely if the growth rate of the marginal damage function σ is small. How fast the tax rises ($\hat{\kappa}$), as well as the effect of the real interest rate r and the parameter describing the elasticity of demand γ is ambiguous and depends on the values of the other parameters. But we know that for $\hat{\kappa} \to 0$ the right hand side of the inequality (3.2.8) goes to zero. Therefore for low values of $\hat{\kappa}$ the strong green paradox is not going to arise. On the other hand for $\hat{\kappa} \to \infty$ the right hand side goes to minus infinity, as the absolute value of the exponent of the first term in the square brackets is higher than the absolute value of the exponent of the second term in the square brackets. Thus there is no strong green paradox for high values of $\hat{\kappa}$ either. In between, depending on the other parameter values, it is either possible that there exist combinations of τ_0 and $\hat{\kappa}$ where the strong green paradox occurs (see figure 7a) or not (see figure 7b). This depends strongly, but not solely on how fast the marginal exploration costs rise.



(a) Fast rising marginal exploration costs $\alpha = 5$ (b) Slowly rising marginal exploration costs $\alpha = 2$

Figure 7: $r = 0.1$, $\sigma = 0.01$, $\gamma = 2$

3.2.2 The monopoly allocation

As there is only a solution in the monopoly case if $\gamma > 1$, we will take that assumption throughout this section. We already discussed in the previous section that equations (3.2.3) and (3.2.4) do not change, when a monopoly is assumed. What changes is the initial price p_0. We have

$$p_0 = \frac{\lambda_0}{\kappa_0(1 - \frac{1}{\gamma})} \tag{3.2.9}$$

instead of $p_0 = \frac{\lambda_0}{\kappa_0}$ as in the competitive case (see equation (3.1.16)). We will show that this doesn't lead to a different outcome concerning the green paradoxes.
Because the initial price is different, we also get a new expression for the initial extraction. Instead of (3.2.5) we get

$$q_0 = \left(\frac{\kappa_0(1 - \frac{1}{\gamma})}{\alpha\beta} \right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r + \hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \tag{3.2.10}$$

See appendix 5.2.2 for the derivation.

At first we again try to find out when the weak green paradox occurs. This is the case when the initial extraction in the tax-case is greater than the initial extraction in the case without a tax. After some calculations (see appendix 5.2.2) we find out that this is the case when

$$\tau_0 < 1 - \left(1 + \frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \tag{3.2.11}$$

We see that this is the exact same inequality as in the competitive case (see (3.2.6)). So the weak green paradox occurs for the same combinations of κ_0 and $\hat{\kappa}$ in the monopoly case as in the competitive case.

Next we calculate the overall damage (see appendix 5.2.2) and get

$$\Gamma = \frac{\theta_0}{\gamma(r+\hat{\kappa}) + r - \sigma} \left(\frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \tag{3.2.12}$$

Now it is just left to find out when the strong green paradox occurs by finding out when Γ^{tax} is greater than Γ^{notax}. This is the case when

$$\tau_0 < 1 - \left[\left(1 + \frac{\gamma\hat{\kappa}}{r(1+\gamma) - \sigma}\right)^{\alpha-1+\frac{1}{\gamma}} \left(1 + \frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}\right] \tag{3.2.13}$$

For the derivation see again appendix 5.2.2.

This is again the exact same condition as in the competitive case. Hence the strong green paradox arises also for the same combinations of κ_0 and $\hat{\kappa}$ in the monopoly case as in the competitive case.

3.3 Discussion

If we assume an exogenous resource stock when using the tax model, both green paradoxes occur. This is due to the fact that a rising revenue tax leads to more extraction earlier in time.

One can argue that it is more realistic that firms can invest money (e.g. in better extraction methods) to increase the size of the resource stock, as it grows with new technologies, which has happened for oil and a lot of other non-renewable resources in the past decades. When taking this into account we concluded that it depends on the structure of the implemented tax if the weak and the strong green paradox occur. The weak green paradox is more likely to occur when the initial tax rate is low and/or the tax rises fast (holding the other parameters constant). The strong green paradox is also more likely to arise with a low initial tax rate. Its occurrence is unlikely for a very slow or a fast rise of the tax rate (again holding the other parameters constant). For some parameter values, particularly for slowly rising marginal exploration costs (α small), the strong green paradox is not going to occur, no matter how the tax is structured.

When assuming a monopoly, the implications concerning the green paradoxes do not change at all in the tax models, when we assume that the elasticity of the demand is bigger than one, which guaranties that the monopoly models have a solution.

4 Conclusion

We showed that no matter if we look at the models with a backstop or the ones with tax, the occurrence of both the weak and the strong green paradox depends strongly upon the underlying assumptions. When changing an assumption such as e.g. the structure of the extraction costs, or the implemented tax, the results concerning the green paradoxes may change as well. As unfortunate as it is to not have an unambiguous result, it is nevertheless important (e.g. for policy makers) to know that they may be confronted with a green paradox. Moreover this coincides with the existing literature, where several papers conclude that a green paradox may occur, but it often depends on the assumptions (in most cases on the values of the parameters used).

In almost all cases the assumptions of a monopoly does not change the implications concerning the green paradoxes. The only exception is the backstop-model with increasing extraction costs where the strong green paradox may occur (for a low backstop-price ($\psi < \frac{\beta}{2}$) and some additional parameter restrictions), which it doesn't in the competitive case. So in the great majority of models looked at one need not to worry if the observed market is a competitive or a monopoly market, when the occurrence of the green paradoxes is analysed.

5 Appendix

5.1 Proofs of chapter 2

Proposition 1. *Consider two piecewise continuous paths* $q_t, q_t^* : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $q_t = 0 \ \forall t : t > T$, $q_t^* = 0 \ \forall t : t > T^*$, $T > T^*$ *and* $\int_0^T q_t dt = \int_0^{T^*} q_t^* dt$. *Moreover we know that* $q_t < q_t^* \ \forall t : t < T^*$ *and* $q_t > q_t^* \ \forall t : T^* < t < T$. *Additionally we have a strictly positive parameter* θ_0 *and a strictly decreasing weight function* $\tilde{\theta}_t$.
Then we can conclude that $\Gamma_0^* := \theta_0 q_0^*$ *is strictly greater than* $\Gamma_0 := \theta_0 q_0$ *and* $\Gamma^* := \int_0^{T^*} \tilde{\theta}_t q_t^* dt$ *is strictly greater than* $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$.

Proof. From $q_t < q_t^* \ \forall t : t < T$ we can immediately conclude that $q_0 < q_0^*$ and therefore $\Gamma_0^* > \Gamma_0$.
For the second part of the proof we take the difference between Γ^* and Γ and get

$$\begin{aligned}
\Gamma^* - \Gamma &= \int_0^{T^*} \tilde{\theta}_t q_t^* dt - \int_0^T \tilde{\theta}_t q_t dt = \\
&= \int_0^T \tilde{\theta}_t (q_t^* - q_t) dt \\
&= \int_0^{T^*} \tilde{\theta}_t (q_t^* - q_t) dt - \int_{T^*}^T \tilde{\theta}_t q_t dt > \\
&> \int_0^{T^*} \tilde{\theta}_{T^*} (q_t^* - q_t) dt - \int_{T^*}^T \tilde{\theta}_{T^*} q_t dt = \\
&= \tilde{\theta}_{T^*} \int_0^T (q_t^* - q_t) dt = \\
&= \tilde{\theta}_{T^*} \left(\int_0^T q_t^* dt - \int_0^T q_t dt \right) = \\
&= \tilde{\theta}_{T^*} \left(\int_0^{T^*} q_t^* dt - \int_0^T q_t dt \right) = 0
\end{aligned}$$

Thus we know that Γ^* has to be strictly greater than Γ. □

Proposition 2. *Given the system of differential equations, which is defined by equations (2.2.6) and (2.2.7), a strictly positive parameter* θ_0 *and a strictly decreasing weight function* $\tilde{\theta}_t$ *and knowing additionally that* $s_0 = 0$, $s_T = \frac{\psi}{\eta}$, $\lambda_T = 0$, $\lambda_t = p_t - \eta s_t$ *(see equation (2.2.3)) and* $\dot{s}_t = q_t = \alpha(1 - \frac{p_t}{\beta})$ *the function* $\Gamma_0 := \theta_0 q_0$ *depends negatively on the backstop-price* ψ *and the function* $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$ *depends positively on* ψ.

Proof. At first we need to find the stationary point of the system of differential equations (2.2.6) and (2.2.7)

$$0 = \dot{s^*} = -\frac{\alpha\eta}{\beta}s^* - \frac{\alpha}{\beta}\lambda^* + \alpha$$
$$0 = \dot{\lambda^*} = \frac{\alpha\eta^2}{\beta}s^* + (r + \frac{\alpha\eta}{\beta})\lambda^* - \alpha\eta$$

Solving this system of linear equations leads us to

$$s^* = \frac{\beta}{\eta} \tag{5.1.1}$$
$$\lambda^* = 0 \tag{5.1.2}$$

Now we derive the Jacobian matrix of the system

$$J = \begin{pmatrix} \frac{\delta \dot{s_t}}{\delta s_t} & \frac{\delta \dot{s_t}}{\delta \lambda_t} \\ \frac{\delta \dot{\lambda_t}}{\delta s_t} & \frac{\delta \dot{\lambda_t}}{\delta \lambda_t} \end{pmatrix} = \begin{pmatrix} -\frac{\alpha\eta}{\beta} & -\frac{\alpha}{\beta} \\ \frac{\eta^2\alpha}{\beta} & r + \frac{\alpha\eta}{\beta} \end{pmatrix}$$

To get the eigenvalues μ_i of this matrix we set the characteristic polynomial equal to zero

$$(-\frac{\alpha\eta}{\beta} - \mu)(r + \frac{\alpha\eta}{\beta} - \mu) + \frac{\eta^2\alpha^2}{\beta^2} = \mu^2 - r\mu - \frac{r\alpha\eta}{\beta} = 0 \tag{5.1.3}$$

Solving this equation, we get the two eigenvalues

$$\mu_1 = \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta}} \tag{5.1.4}$$
$$\mu_2 = \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta}} \tag{5.1.5}$$

Next we want to prove that the following (in)equalities hold

$$\mu_1 + \mu_2 = \frac{r}{2} > 0 \qquad -\frac{\alpha\eta}{\beta} < \mu_1 < 0 \qquad 0 < r < \mu_2 \tag{5.1.6}$$

$\mu_1 + \mu_2 = \frac{r}{2}$ we get immediately from (5.1.4) and (5.1.5).
To prove the chain of inequalities for μ_1 we have

$$\begin{aligned}\mu_1 &= \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta}} > \\ &> \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta} + (\frac{\alpha\eta}{\beta})^2} = \\ &= -\frac{\alpha\eta}{\beta}\end{aligned}$$

and

$$\begin{aligned}\mu_1 &= \frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta}} < \\ &< \frac{r}{2} - \sqrt{\frac{r^2}{4}} = 0\end{aligned}$$

For the last inequality of (5.1.6) the proof is similar

$$\begin{aligned}\mu_2 &= \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{r\alpha\eta}{\beta}} > \\ &> \frac{r}{2} + \sqrt{\frac{r^2}{4}} = r\end{aligned}$$

Knowing the eigenvalues μ_i and that in the steady state $s^* = \frac{\beta}{\eta}$, we can write the general solution of the cumulative extraction at time t as

$$s_t = \frac{\beta}{\eta} + Be^{\mu_1 t} + Ae^{\mu_2 t} \tag{5.1.7}$$

Using now $s_0 = 0$ gives us

$$\begin{aligned} 0 &= \frac{\beta}{\eta} + B + A \\ \Leftrightarrow B &= -(A + \frac{\beta}{\eta}) \end{aligned} \tag{5.1.8}$$

Plugging equation (5.1.8) into equation (5.1.7) we get

$$s_t = \frac{\beta}{\eta} - (A + \frac{\beta}{\eta})e^{\mu_1 t} + Ae^{\mu_2 t} \tag{5.1.9}$$

For the use of the resource we have

$$q_t = \dot{s}_t = -(A + \frac{\beta}{\eta})\mu_1 e^{\mu_1 t} + A\mu_2 e^{\mu_2 t} \tag{5.1.10}$$

The next step is to find out how A and T depend upon ψ.

At first we plug both s_t and q_t into the equation for λ_t and get using $p_t = \beta(1 - \frac{q_t}{\alpha})$

$$\begin{aligned} \lambda_t &= p_t - \eta s_t = \\ &= \beta(1 - \frac{q_t}{\alpha}) - \eta s_t = \\ &= \beta + \frac{\beta}{\alpha}(A + \frac{\beta}{\eta})\mu_1 e^{\mu_1 t} - \frac{\beta}{\alpha}A\mu_2 e^{\mu_2 t} - \beta + \eta(A + \frac{\beta}{\eta})e^{\mu_1 t} - \eta A e^{\mu_2 t} = \\ &= (A + \frac{\beta}{\eta})(\frac{\mu_1 \beta}{\alpha} + \eta)e^{\mu_1 t} - A(\frac{\mu_2 \beta}{\alpha} + \eta)e^{\mu_2 t} \end{aligned} \tag{5.1.11}$$

As we know that $\lambda_T = 0$ holds, we get

$$
\begin{aligned}
&(A+\frac{\beta}{\eta})(\frac{\mu_1\beta}{\alpha}+\eta)e^{\mu_1 T}-A(\frac{\mu_2\beta}{\alpha}+\eta)e^{\mu_2 T}=0\\
\Leftrightarrow & e^{(\mu_2-\mu_1)T}=\frac{(A+\frac{\beta}{\eta})(\frac{\mu_1\beta}{\alpha}+\eta)}{A(\frac{\mu_2\beta}{\alpha}+\eta)}\\
\Leftrightarrow & e^{T}=\left(\frac{(A+\frac{\beta}{\eta})(\mu_1\beta+\alpha\eta)}{A(\mu_2\beta+\alpha\eta)}\right)^{\frac{1}{\mu_2-\mu_1}}
\end{aligned}
\tag{5.1.12}
$$

To make calculations easier, we define $Z := \frac{\mu_1\beta+\alpha\eta}{\mu_2\beta+\alpha\eta}$. Using the inequalities in (5.1.6), we see that the numerator and the denominator are greater than zero. Moreover the numerator is smaller than the denominator, so we get $0 < Z < 1$.
We can rewrite equation (5.1.12) as

$$
e^{T}=\left(\frac{A+\frac{\beta}{\eta}}{A}Z\right)^{\frac{1}{\mu_2-\mu_1}}
\tag{5.1.13}
$$

The next step is finding out how big the total amount of the extracted resource is and how it depends upon A

$$
\begin{aligned}
s_T &= \frac{\beta}{\eta}-(A+\frac{\beta}{\eta})e^{\mu_1 T}+Ae^{\mu_2 T}=\\
&=\frac{\beta}{\eta}-(A+\frac{\beta}{\eta})\left(\frac{A+\frac{\beta}{\eta}}{A}Z\right)^{\frac{\mu_1}{\mu_2-\mu_1}}+A\left(\frac{A+\frac{\beta}{\eta}}{A}Z\right)^{\frac{\mu_2}{\mu_2-\mu_1}}=\\
&=\frac{\beta}{\eta}-(A+\frac{\beta}{\eta})^{\frac{\mu_2}{\mu_2-\mu_1}}A^{-\frac{\mu_1}{\mu_2-\mu_1}}Z^{\frac{\mu_1}{\mu_2-\mu_1}}+(A+\frac{\beta}{\eta})^{\frac{\mu_2}{\mu_2-\mu_1}}A^{-\frac{\mu_1}{\mu_2-\mu_1}}Z^{\frac{\mu_2}{\mu_2-\mu_1}}=\\
&=\frac{\beta}{\eta}-(A+\frac{\beta}{\eta})^{\frac{\mu_2}{\mu_2-\mu_1}}A^{-\frac{\mu_1}{\mu_2-\mu_1}}(Z^{\frac{\mu_1}{\mu_2-\mu_1}}-Z^{\frac{\mu_2}{\mu_2-\mu_1}})
\end{aligned}
\tag{5.1.14}
$$

Looking at equations (5.1.13) and (5.1.14), we see that for $A \to 0$, T goes to infinity and s_T to $\frac{\beta}{\eta}$. Moreover we have $\frac{\partial A}{\partial T} < 0$ and $\frac{\partial A}{\partial s_T} < 0$, as the exponents of $(A+\frac{\beta}{\eta})$ and A, as well as the last part in the brackets, are positive (because $0 < Z < 1$, $\mu_1 < 0$ and $\mu_2 > 0$). As we also know that $s_T = \frac{\psi}{\eta}$ we can easily conclude that $\frac{\partial A}{\partial \psi} < 0$ and $\frac{\partial T}{\partial \psi} > 0$.
Now we just need to find out how Γ_0 and Γ depend upon A and T.
For the initial extraction we have

$$q_0 = -\frac{\beta}{\eta}\mu_1 + (\mu_2 - \mu_1)A \tag{5.1.15}$$

We can easily see that $\frac{\partial q_0}{\partial A} > 0$ and therefore $\frac{\partial q_0}{\partial \psi} < 0$ and $\frac{\partial \Gamma_0}{\partial \psi} < 0$.

To see how Γ depends on ψ, we restrict ourselves to the case where $\tilde{\theta}_t = \theta_0 e^{(-r+\sigma)t}$ and generalize later. Note that in this case, as by assumption $\tilde{\theta}_t$ is strictly decreasing, $\sigma < r$ has to hold.

For the overall damage we get, using equation (5.1.13) for the last equality,

$$\begin{aligned}
\Gamma =& \int_0^T \theta_0 e^{(-r+\sigma)t} q_t dt = \\
=&\theta_0 \int_0^T -(A + \frac{\beta}{\eta})\mu_1 e^{(-r+\sigma+\mu_1)t} + A\mu_2 e^{(-r+\sigma+\mu_2)t} dt = \\
=&\theta_0 \left(-\frac{(A+\frac{\beta}{\eta})\mu_1}{-r+\sigma+\mu_1} e^{(-r+\sigma+\mu_1)T} + \frac{A\mu_2}{-r+\sigma+\mu_2} e^{(-r+\sigma+\mu_2)T} \right. \\
&+ \left. \frac{(A+\frac{\beta}{\eta})\mu_1}{-r+\sigma+\mu_1} - \frac{A\mu_2}{-r+\sigma+\mu_2} \right) = \\
=&\theta_0 \left(-\frac{(A+\frac{\beta}{\eta})\mu_1}{-r+\sigma+\mu_1} (e^{(-r+\sigma+\mu_1)T} - 1) + \frac{A\mu_2}{-r+\sigma+\mu_2} (e^{(-r+\sigma+\mu_2)T} - 1) \right) = \\
=&\theta_0 \left\{ -\frac{(A+\frac{\beta}{\eta})\mu_1}{-r+\sigma+\mu_1} \left[\left(\frac{(A+\frac{\beta}{\eta})}{A} Z \right)^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} - 1 \right] \right. \\
&+ \left. \frac{A\mu_2}{-r+\sigma+\mu_2} \left[\left(\frac{(A+\frac{\beta}{\eta})}{A} Z \right)^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} - 1 \right] \right\}
\end{aligned}$$

Now we define $X = \frac{\mu_1}{-r+\sigma+\mu_1}$ and $Y = \frac{\mu_2}{-r+\sigma+\mu_2}$ and get

$$
\begin{aligned}
\Gamma =& \theta_0 \left\{ -(A+\frac{\beta}{\eta})X \left[\left(\frac{(A+\frac{\beta}{\eta})}{A} Z \right)^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} - 1 \right] \right. \\
& \left. + AY \left[\left(\frac{(A+\frac{\beta}{\eta})}{A} Z \right)^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} - 1 \right] \right\} = \\
=& \theta_0 \left[AX + \frac{\beta}{\eta} X - X(A+\frac{\beta}{\eta})^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} A^{-\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} Z^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} \right. \\
& \left. - AY + Y(A+\frac{\beta}{\eta})^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} A^{-\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} Z^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} \right] = \\
=& \theta_0 \left[\frac{\beta}{\eta} X + A(X-Y) \right. \\
& \left. + (A+\frac{\beta}{\eta})^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} A^{-\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} \left(-XZ^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} + YZ^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} \right) \right] \qquad (5.1.16)
\end{aligned}
$$

As the damage at time t can only be positive for all t smaller than T, an increase in T can only lead to an increase in Γ, so $\frac{\partial \Gamma}{\partial T} > 0$.
The dependence upon A is a little more difficult.
It is easy to see that $X - Y < 0$ and that the exponents of $A + \frac{\beta}{\eta}$ and A are both positive.
If we can show that the last expression in the round brackets is lower than zero, we can conclude that $\frac{\partial \Gamma}{\partial A} < 0$.

$$
\begin{aligned}
& -XZ^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} + YZ^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} < 0 \\
\Leftrightarrow & XZ^{\frac{-r+\sigma+\mu_1}{\mu_2-\mu_1}} > YZ^{\frac{-r+\sigma+\mu_2}{\mu_2-\mu_1}} \\
\Leftrightarrow & \frac{X}{Y} > Z \qquad (5.1.17)
\end{aligned}
$$

Now we are going to prove that the inequality (5.1.17) holds, using the fact that the characteristic polynomial is equal to zero ($\mu_i^2 - r\mu_i - \frac{r\alpha\eta}{\beta} = 0$).

$$
\begin{aligned}
\frac{X}{Y} &= \frac{\mu_1(-r+\sigma+\mu_2)}{\mu_2(-r+\sigma+\mu_1)} = \\
&= \frac{-\mu_2 r + \mu_2\sigma + \mu_2^2}{-\mu_1 r + \mu_1\sigma + \mu_1^2} \cdot \frac{\mu_1^2}{\mu_2^2} = \\
&= \frac{\frac{r\alpha\eta}{\beta} + \mu_2\sigma}{\frac{r\alpha\eta}{\beta} + \mu_1\sigma} \cdot \frac{\mu_1^2}{\mu_2^2} > \\
&> \frac{\mu_1^2}{\mu_2^2} = \frac{r\mu_1 + \frac{r\alpha\eta}{\beta}}{r\mu_2 + \frac{r\alpha\eta}{\beta}} = \frac{\mu_1\beta + \alpha\eta}{\mu_2\beta + \alpha\eta} = Z
\end{aligned}
$$

So we know that $\frac{\partial\Gamma}{\partial A} < 0$ and $\frac{\partial\Gamma}{\partial T} > 0$. Thus, and because we know that $\frac{\partial A}{\partial\psi} < 0$ and $\frac{\partial T}{\partial\psi} > 0$, we get $\frac{\partial\Gamma}{\partial\psi} > 0$.

What is now left to prove is that this also holds in the case where $\tilde{\theta}_t$ is an arbitrary decreasing function. We assume again that the backstop drops from ψ to ψ^* ($\psi^* < \psi$). It is possible to find a σ, such that $\sigma < r$ holds and the growth rate of $\tilde{\theta}_t$ is greater than $-r+\sigma$ for all $t \in [0, T]$. Now we choose θ_0 such that $\theta_0 e^{(-r+\sigma)t}$ coincides with $\tilde{\theta}_t$ at T^* ($\theta_0 e^{(-r+\sigma)T^*} = \tilde{\theta}_{T^*}$). As we know that $T^* < T$, we get $\tilde{\theta}_t < \theta_0 e^{(-r+\sigma)t}$ $\forall t : t < T^*$ and $\tilde{\theta}_t > \theta_0 e^{(-r+\sigma)t}$ $\forall t : T^* < t < T$. Using $q_t^* > q_t$ $\forall t : t < T^*$ and $q_t^* < q_t$ $\forall t : T^* < t < T$ we get

$$
\begin{aligned}
\Gamma^* - \Gamma &= \int_0^{T^*} \tilde{\theta}_t q_t^* dt - \int_0^{T} \tilde{\theta}_t q_t dt = \\
&= \int_0^{T^*} \tilde{\theta}_t (q_t^* - q_t) dt - \int_{T^*}^{T} \tilde{\theta}_t q_t dt < \\
&< \int_0^{T^*} \theta_0 e^{(-r+\sigma)t} (q_t^* - q_t) dt - \int_{T^*}^{T} \theta_0 e^{(-r+\sigma)t} q_t dt = \\
&= \int_0^{T^*} \theta_0 e^{(-r+\sigma)t} q_t^* dt - \int_0^{T} \theta_0 e^{(-r+\sigma)t} q_t dt < 0
\end{aligned}
$$

As the expression in the last line is exactly the difference in the overall damage in the case where $\tilde{\theta}_t = \theta_0 e^{(-r+\sigma)t}$ and we know that in this case Γ depends positively on ψ, we can conclude that this expression is smaller than zero. Therefore also the difference in the overall damage using the generalized function $\tilde{\theta}_t$ is smaller than zero and thus the function Γ depends positively on ψ in this case as well. $\square$

Proposition 3. *Given the system of differential equations, which is defined by equations (2.2.13) and (2.2.14)), a strictly positive parameter* θ_0 *and a strictly decreasing weight function* $\tilde{\theta}_t$ *and knowing additionally that* $s_0 = 0$, $s_T = \frac{\psi}{\eta}$, $\lambda_T = 0$, $\lambda_t = 2p_t - \beta - \eta s_t$ *(see equation (2.2.12)) and* $\dot{s}_t = q_t = \alpha(1 - \frac{p_t}{\beta})$ *the function* $\Gamma_0 = \theta_0 q_0$ *depends negatively on the backstop-price* ψ *and the function* $\Gamma = \int_0^T \tilde{\theta}_t q_t dt$ *depends positively on* ψ.

Proof. The proof of this proposition is, unsurprisingly, very similar to the one of proposition 2. Some parts are in fact exactly the same. We will therefore refer to said proof for some implications.
At first we will show that the stationary point of the system of differential equations is the same as in the competitive case

$$
\begin{aligned}
0 = \dot{s^*} &= \frac{1}{2}\left(-\frac{\alpha\eta}{\beta}s^* - \frac{\alpha}{\beta}\lambda^* + \alpha\right) \\
0 = \dot{\lambda^*} &= \frac{1}{2}\left(\frac{\alpha\eta^2}{\beta}s^* + (2r + \frac{\alpha\eta}{\beta})\lambda^* - \alpha\eta\right)
\end{aligned}
$$

Solving this system of linear equations leads us again to

$$
s^* = \frac{\beta}{\eta} \tag{5.1.18}
$$

$$
\lambda^* = 0 \tag{5.1.19}
$$

Next we derive the Jacobian matrix of the system and get

$$
J = \begin{pmatrix} \frac{\delta \dot{s_t}}{\delta s_t} & \frac{\delta \dot{s_t}}{\delta \lambda_t} \\ \frac{\delta \dot{\lambda_t}}{\delta s_t} & \frac{\delta \dot{\lambda_t}}{\delta \lambda_t} \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} -\frac{\alpha\eta}{\beta} & -\frac{\alpha}{\beta} \\ \frac{\eta^2\alpha}{\beta} & 2r + \frac{\alpha\eta}{\beta} \end{pmatrix}
$$

Setting the characteristic polynomial equal to zero leads us to

$$
(-\frac{\alpha\eta}{\beta} - \mu)(2r + \frac{\alpha\eta}{\beta} - \mu) + \frac{\eta^2\alpha^2}{\beta^2} = \mu^2 - 2r\mu - \frac{2r\alpha\eta}{\beta} = 0 \tag{5.1.20}
$$

Solving this equation, we get the two eigenvalues

$$\mu_1 = r - \sqrt{r^2 + \frac{2r\alpha\eta}{\beta}} \tag{5.1.21}$$

$$\mu_2 = r + \sqrt{r^2 + \frac{2r\alpha\eta}{\beta}} \tag{5.1.22}$$

Now we are going to show that the following (in)equalities hold

$$\mu_1 + \mu_2 = r > 0 \qquad -\frac{\alpha\eta}{\beta} < \mu_1 < 0 \qquad 0 < 2r < \mu_2 \tag{5.1.23}$$

$\mu_1 + \mu_2 = r$ we get immediately from (5.1.21) and (5.1.22).
Moreover we know

$$\begin{aligned} \mu_1 &= r - \sqrt{r^2 + \frac{2r\alpha\eta}{\beta}} > \\ &> r - \sqrt{r^2 + \frac{2r\alpha\eta}{\beta} + (\frac{\alpha\eta}{\beta})^2} = \\ &= -\frac{\alpha\eta}{\beta} \end{aligned} \tag{5.1.24}$$

and

$$\begin{aligned} \mu_1 &= r - \sqrt{r^2 + \frac{2r\alpha\eta}{\beta}} < \\ &< r - \sqrt{r^2} = 0 \end{aligned}$$

For the last inequality in (5.1.23) concerning μ_2 we have

$$\begin{aligned} \mu_2 &= r + \sqrt{r^2 + \frac{2r\alpha\eta}{\beta}} > \\ &> r + \sqrt{r^2} = 2r \end{aligned}$$

As the steady state is the same as in the competitive case, we can also write s_t as

$$s_t = \frac{\beta}{\eta} + Be^{\mu_1 t} + Ae^{\mu_2 t} \tag{5.1.25}$$

which leads us to

$$q_t = \dot{s}_t = -(A + \frac{\beta}{\eta})\mu_1 e^{\mu_1 t} + A\mu_2 e^{\mu_2 t} \tag{5.1.26}$$

Look at the proof of proposition 2 (5.1.7)-(5.1.10) for confirmation.
Now we plug s_t and q_t into the equation for λ_t and get using $p_t = \beta(1 - \frac{q_t}{\alpha})$

$$\begin{aligned}
\lambda_t &= 2p_t - \beta - \eta s_t = \\
&= 2\beta(1 - \frac{q_t}{\alpha}) - \beta - \eta s_t = \\
&= 2\beta + \frac{2\beta}{\alpha}(A + \frac{\beta}{\eta})\mu_1 e^{\mu_1 t} - \frac{2\beta}{\alpha}A\mu_2 e^{\mu_2 t} - \beta - \beta + \eta(A + \frac{\beta}{\eta})e^{\mu_1 t} - \eta A e^{\mu_2 t} = \\
&= (A + \frac{\beta}{\eta})(\frac{2\mu_1\beta}{\alpha} + \eta)e^{\mu_1 t} - A(\frac{2\mu_2\beta}{\alpha} + \eta)e^{\mu_2 t} \qquad (5.1.27)
\end{aligned}$$

Using $\lambda_T = 0$ we get

$$\begin{aligned}
&(A + \frac{\beta}{\eta})(\frac{2\mu_1\beta}{\alpha} + \eta)e^{\mu_1 T} - A(\frac{2\mu_2\beta}{\alpha} + \eta)e^{\mu_2 T} = 0 \\
\Leftrightarrow & e^{(\mu_2 - \mu_1)T} = \frac{(A + \frac{\beta}{\eta})(\frac{2\mu_1\beta}{\alpha} + \eta)}{A(\frac{2\mu_2\beta}{\alpha} + \eta)} \\
\Leftrightarrow & e^T = \left(\frac{(A + \frac{\beta}{\eta})(2\mu_1\beta + \alpha\eta)}{A(2\mu_2\beta + \alpha\eta)}\right)^{\frac{1}{\mu_2 - \mu_1}} \qquad (5.1.28)
\end{aligned}$$

Now we define $Z := \frac{2\mu_1\beta + \alpha\eta}{2\mu_2\beta + \alpha\eta}$ and end up with the same equation as in the proof of proposition 2. So as long as we don't need the definition of Z, which we do not between equations (5.1.13) and (5.1.17), we can follow the same steps as there. From equation (5.1.15) we know that Γ_0 depends negatively on ψ. To show that Γ depends positively upon ψ, we just need to prove that $\frac{X}{Y} > Z$ holds for $X = \frac{\mu_1}{-r+\sigma+\mu_1}$, $Y = \frac{\mu_2}{-r+\sigma+\mu_2}$ and

$Z = \frac{2\mu_1\beta+\alpha\eta}{2\mu_2\beta+\alpha\eta}$ (see equations (5.1.16) and (5.1.17)). We will do that using the characteristic polynomial (5.1.20)

$$\begin{aligned}
\frac{X}{Y} &= \frac{\mu_1(-r+\sigma+\mu_2)}{\mu_2(-r+\sigma+\mu_1)} = \\
&= \frac{-\mu_2 r+\mu_2\sigma+\mu_2^2}{-\mu_1 r+\mu_1\sigma+\mu_1^2} \cdot \frac{\mu_1^2}{\mu_2^2} = \\
&= \frac{\frac{2r\alpha\eta}{\beta}+\mu_2\sigma+\mu_2 r}{\frac{2r\alpha\eta}{\beta}+\mu_1\sigma+\mu_1 r} \cdot \frac{\mu_1^2}{\mu_2^2} > \\
&> \frac{\mu_1^2}{\mu_2^2} = \frac{2r\mu_1+\frac{2r\alpha\eta}{\beta}}{2r\mu_2+\frac{2r\alpha\eta}{\beta}} = \frac{\mu_1\beta+\alpha\eta}{\mu_2\beta+\alpha\eta} > \frac{2\mu_1\beta+\alpha\eta}{2\mu_2\beta+\alpha\eta} = Z
\end{aligned}$$

So we can conclude that Γ depends positively upon ψ.
For the generalization to $\tilde{\theta}_t$ being an arbitrary decreasing function we again refer to the proof of proposition 2 (last part). $\square$

Proposition 4. *Assume that* $\dot{s}_t = q_t = \alpha(1-\frac{\psi}{\beta})$, $T = \frac{\beta\psi}{\alpha\eta(\beta-\psi)}$, $\psi \leq \frac{\beta}{2}$, $\theta_0 > 0$ *and* $r, \sigma > 0$ *with* $r > \sigma$.
Then the function $\Gamma_0 = \theta_0 q_t$ *depends negatively upon* ψ *and the dependence of the function* $\Gamma = \int_0^\infty e^{-rt}\theta_0 e^{\sigma t} q_t$ *on* ψ *can't be uniquely determined, but depends on the parameters used. The chance of* $\frac{\partial\Gamma}{\partial\psi}$ *being lower than zero is high for a big difference between the real interest rate and the growth rate of the marginal damage function* ($r-\sigma$ *high*) *as well as for small values of* α *and* η.

Proof. It is relatively easy to prove that Γ_0 depends negatively on the ψ, as the initially extracted amount of the resource is $q_0 = \alpha(1-\frac{\psi}{\beta})$ and if the backstop-price drops, q_0 rises and therefore Γ_0 rises as well.
For the overall damage we get

$$\begin{aligned}
\Gamma &= \int_0^T \tilde{\theta}_t q_t dt = \\
&= \alpha(1-\frac{\psi}{\beta})\int_0^T \theta_0 e^{(-r+\sigma)t} dt = \\
&= \alpha(1-\frac{\psi}{\beta})\frac{\theta_0}{-r+\sigma}(e^{(-r+\sigma)T}-1) = \\
&= \alpha(1-\frac{\psi}{\beta})\frac{\theta_0}{r-\sigma}(1-e^{-\frac{\beta\psi(r-\sigma)}{\alpha\eta(\beta-\psi)}})
\end{aligned}$$

Now we need to calculate the derivative of Γ with respect to ψ and get

$$\begin{aligned}\frac{\partial\Gamma}{\partial\psi} =& \alpha(1-\frac{\psi}{\beta})\frac{\theta_0}{r-\sigma}\frac{(r-\sigma)\beta\alpha\eta(\beta-\psi)+(r-\sigma)\beta\psi(\alpha\eta)}{\alpha^2\eta^2(\beta-\psi)^2}e^{-\frac{\beta\psi(r-\sigma)}{\alpha\eta(\beta-\psi)}} \\ & -\frac{\alpha\theta_0}{\beta(r-\sigma)}(1-e^{-\frac{\beta\psi(r-\sigma)}{\alpha\eta(\beta-\psi)}})\end{aligned}$$

Factoring out the term $\frac{\alpha\theta_0}{\beta(r-\sigma)}$, defining $x := \frac{\beta\psi(r-\sigma)}{\alpha\eta(\beta-\psi)}$ and simplifying the expression gives us

$$\frac{\partial\Gamma}{\partial\psi} = \frac{\alpha\theta_0}{\beta(r-\sigma)}\left[(\frac{\beta}{\psi}x+1)e^{-x}-1\right]$$

Now we want to show that this term can be positive or negative, depending on the values of the parameters.
To do this the first factor is irrelevant here, as it is always positive. So we can drop it. Moreover we define $a := \frac{\beta}{\psi}$ and get

$$\frac{\partial\Gamma^*}{\partial\psi} = (ax+1)e^{-x}-1 \tag{5.1.29}$$

From $\beta > \psi$ we can conclude that $a > 1$. Moreover we know that $x > 0$ holds, but x can get very small (when σ is close to r) or very large (when the term $\alpha\eta$ is small), holding ψ and β, and therefore a constant. So it is enough now to prove that there exist some x greater than zero for which equation (5.1.29) is positive and there also exist some x greater than zero for which it is negative.
To do that we are going to prove that equation (5.1.29) has two roots for non-negative x (one at $x = 0$) and the only extremum is a maximum turning point in between. Note that it would be enough to find two example values to prove the statement, but we want to get an understanding of how the solution is structured.
It is easy to see that equation (5.1.29) has a root at $x = 0$. To find the turning point we calculate the first derivative of (5.1.29) with respect to x and get

$$\frac{\partial[(ax+1)e^{-x}-1]}{\partial x} = (a-ax-1)e^{-x} \tag{5.1.30}$$

Equation (5.1.30) is equal to zero if $x = \frac{a-1}{a} (> 0)$. Plugging this into the second derivative of (5.1.29) with respect to x gives us

$$\frac{\partial^2 \left[(ax+1)e^{-x} - 1\right]}{\partial^2 x^2} = (ax - 2a + 1)e^{-x} = -ae^{-\frac{a-1}{a}} \tag{5.1.31}$$

As this expression is smaller than zero we have a single extremum, which is a maximum turning point at $x = \frac{a-1}{a}$. Now we only need to show that if x gets high enough the value of (5.1.29) gets negative.
We set $x = a$ and get

$$(ax+1)e^{-x} - 1 = (a^2+1)e^{-a} - 1 \tag{5.1.32}$$

This expression is smaller than zero if and only if $a^2 + 1 < e^a \ \forall a : a > 1$. This is the case because the equation is fulfilled for $a = 1$ and the derivation of the left hand side is smaller than the derivation of the right hand side ($2a < e^a$) if a is greater than one.
So if x is high enough we have $\frac{\partial \Gamma}{\partial \psi} < 0$ and vice versa. The variable x is higher the higher $(r - \sigma)$ is and the lower α and η are respectively.

$\square$

Proposition 5. *Consider two continuous paths $q_t, q_t^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $q_t = 0 \ \forall t : t > T$, $q_t^* = 0 \ \forall t : t > T^*$, $T < T^*$ and $\int_0^T q_t dt = \int_0^{T^*} q_t^* dt$. Moreover we know that $\dot{q}_t < \dot{q}_t^* \ \forall t : t < T$ and $\dot{q}_t > \dot{q}_t^* \ \forall t : T < t < T^*$. Additionally we have a strictly positive parameter θ_0 and a strictly decreasing weight function $\tilde{\theta}_t$.*
Then we can conclude that $\Gamma_0^* := \theta_0 q_0^*$ is strictly smaller than $\Gamma_0 := \theta_0 q_t$ and $\Gamma^* := \int_0^{T^*} \tilde{\theta}_t q_t^* dt$ is strictly smaller than $\Gamma := \int_0^T \tilde{\theta}_t q_t dt$.

Proof. First we define $\Delta_t = q_t^* - q_t$.
From the above implications we get $\dot{\Delta}_t = \dot{q_t}^* - \dot{q}_t > 0 \ \forall t : t < T$ and $\dot{\Delta}_t = \dot{q_t}^* < 0 \ \forall t : T < t < T^*$.
Moreover we have $\Delta_T = q_T^* - q_T = q_T^* > 0$ and thus

$$\int_0^T \Delta_t dt = \int_0^T q_t^* dt - \int_0^T q_t dt = \int_0^T q_t^* dt - \int_0^{T^*} q_t^* dt = -\int_T^{T^*} q_t^* dt < 0$$

So Δ_t is a strictly increasing, continuous function with $\Delta_T > 0$ and a negative integral from zero to T. This implies that $\Delta_0 < 0$ and the existence of a $t^* \in (0, T)$ with $\Delta_{t^*} = 0$. From $\Delta_0 < 0$ we can conclude $q_0^* < q_0$ and $\Gamma_0^* < \Gamma_0$.
To see if $\Gamma^* < \Gamma$ holds, we look at the difference in the total damage

$$\begin{aligned}
\Gamma^* - \Gamma &= \int_0^{T^*} \tilde{\theta}_t q_t^* dt - \int_0^T \tilde{\theta}_t q_t dt = \\
&= \int_0^{T^*} \tilde{\theta}_t \Delta_t dt = \\
&= \int_0^{t^*} \tilde{\theta}_t \Delta_t dt + \int_{t^*}^{T^*} \tilde{\theta}_t \Delta_t dt < \\
&< \int_0^{t^*} \tilde{\theta}_{t^*} \Delta_t dt + \int_{t^*}^{T^*} \tilde{\theta}_{t^*} \Delta_t dt = \\
&= \tilde{\theta}_{t^*} \int_0^{T^*} \Delta_t dt = \\
&= \tilde{\theta}_{t^*} \left(\int_0^{T^*} q_t^* dt - \int_0^{T^*} q_t dt \right) = \\
&= \tilde{\theta}_{t^*} \left(\int_0^{T^*} q_t^* dt - \int_0^{T} q_t dt \right) = 0
\end{aligned}$$

$\square$

5.2 Derivations in chapter 3

5.2.1 The competitive case

For the derivation of equation (3.2.5) we use the definition of $C(S_0)$, the fact that $C'(S_0) = \lambda_0$, as well as equations (3.2.2), (3.2.3) and the demand function (3.1.7)

$$\begin{aligned}
& C'(S_0) = \lambda_0 \\
\Leftrightarrow & \alpha\beta S_0^{\alpha-1} = p_0 \kappa_0 \\
\Leftrightarrow & \alpha\beta \left(\frac{q_0}{\gamma(r+\hat{\kappa})} \right)^{\alpha-1} = q_0^{-\frac{1}{\gamma}} \kappa_0 \\
\Leftrightarrow & q_0^{\alpha-1+\frac{1}{\gamma}} = (\alpha\beta)^{-1} [\gamma(r+\hat{\kappa})]^{\alpha-1} \kappa_0 \\
\Leftrightarrow & q_0 = \left(\frac{\kappa_0}{\alpha\beta} \right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}
\end{aligned}$$

To get the inequality (3.2.6) we use $\tau_0 = 1 - \kappa_0$, equation (3.2.5) as well as the fact that in the case with no tax $\kappa_0 = 1$ and $\hat{\kappa} = 0$ holds

$$
\begin{aligned}
& q_0^{tax} > q_0^{notax} \\
\Leftrightarrow & \frac{\kappa_0}{\alpha\beta}^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \left(\frac{1}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} (\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > (\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} > \left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0 > \left(\frac{r}{r+\hat{\kappa}}\right)^{\alpha-1} \\
\Leftrightarrow & \kappa_0 > \left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \\
\Leftrightarrow & \tau_0 < 1 - \left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}
\end{aligned}
$$

Equation (3.2.7) can be derived using the fact that $\theta_t = \theta_0 e^{\sigma t}$, as well as equations (3.2.4)and (3.2.5)

$$
\begin{aligned}
\Gamma = & \int_0^\infty e^{-rt}\theta_t q_t dt = \\
= & \int_0^\infty \theta_0 e^{(-r+\sigma)t} q_t dt = \\
= & \int_0^\infty \theta_0 e^{(-r+\sigma)t} q_0 e^{-\gamma(r+\hat{\kappa})t} dt = \\
= & \int_0^\infty \theta_0 \left(\frac{\kappa_0}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} e^{-[\gamma(r+\hat{\kappa})+r-\sigma]t} dt = \\
= & \frac{\theta_0}{\gamma(r+\hat{\kappa})+r-\sigma} \left(\frac{\kappa_0}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}
\end{aligned}
$$

For the derivation of (3.2.8) we use equation (3.2.7) and again $\tau_0 = 1 - \kappa_0$ and the fact that in the case with no tax $\kappa_0 = 1$ and $\hat{\kappa} = 0$ holds

$$
\begin{aligned}
&\Gamma^{tax} > \Gamma^{notax}\\
\Leftrightarrow&\frac{\theta_0}{\gamma(r+\hat{\kappa})+r-\sigma}\left(\frac{\kappa_0}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}}[\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \frac{\theta_0}{\gamma r+r-\sigma}\left(\frac{1}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}}(\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}\\
\Leftrightarrow&\frac{1}{\gamma(r+\hat{\kappa})+r-\sigma}\kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}}[\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \frac{1}{\gamma r+r-\sigma}(\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}\\
\Leftrightarrow&\kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} > \frac{\gamma(r+\hat{\kappa})+r-\sigma}{\gamma r+r-\sigma}\left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}\\
\Leftrightarrow&\kappa_0 > \left(\frac{\gamma(r+\hat{\kappa})+r-\sigma}{\gamma r+r-\sigma}\right)^{\frac{(\alpha-1)\gamma+1}{\gamma}}\left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\alpha-1}\\
\Leftrightarrow&\kappa_0 > \left(1+\frac{\gamma\hat{\kappa}}{r(1+\gamma)-\sigma}\right)^{\alpha-1+\frac{1}{\gamma}}\left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}\\
\Leftrightarrow&\tau_0 < 1-\left[\left(1+\frac{\gamma\hat{\kappa}}{r(1+\gamma)-\sigma}\right)^{\alpha-1+\frac{1}{\gamma}}\left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}\right]
\end{aligned}
\tag{5.2.1}
$$

5.2.2 The monopoly model

For the derivation of equation (3.2.10) we use again the definition of $C(S_0)$, the fact that $C'(S_0) = \lambda_0$, as well as equations (3.2.9), (3.2.3) and the demand function (3.1.7)

$$
\begin{aligned}
&C'(S_0) = \lambda_0\\
\Leftrightarrow&\alpha\beta S_0^{\alpha-1} = p_0\kappa_0(1-\frac{1}{\gamma})\\
\Leftrightarrow&\alpha\beta\left(\frac{q_0}{\gamma(r+\hat{\kappa})}\right)^{\alpha-1} = q_0^{-\frac{1}{\gamma}}\kappa_0(1-\frac{1}{\gamma})\\
\Leftrightarrow&q_0^{\alpha-1+\frac{1}{\gamma}} = (\alpha\beta)^{-1}[\gamma(r+\hat{\kappa})]^{\alpha-1}\kappa_0(1-\frac{1}{\gamma})\\
\Leftrightarrow&q_0 = \left(\frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}}[\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}
\end{aligned}
$$

To get the inequality (3.2.11) we use equation (3.2.10) as well as $\tau_0 = 1-\kappa_0$ and the fact that in the case with no tax $\kappa_0 = 1$ and $\hat{\kappa} = 0$ holds

$$
\begin{aligned}
& q_0^{tax} > q_0^{notax} \\
\Leftrightarrow & \frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \left(\frac{1-\frac{1}{\gamma}}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} (\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > (\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} > \left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow & \kappa_0 > \left(\frac{r}{r+\hat{\kappa}}\right)^{\alpha-1} \\
\Leftrightarrow & \kappa_0 > \left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \\
\Leftrightarrow & \tau_0 < 1-\left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}
\end{aligned}
$$

Equation (3.2.12) can be derived using the fact that $\theta_t = \theta_0 e^{\sigma t}$, as well as equations (3.2.4) and (3.2.10)

$$
\begin{aligned}
\Gamma &= \int_0^\infty e^{-rt}\theta_t q_t dt = \\
&= \int_0^\infty \theta_0 e^{(-r+\sigma)t} q_t dt = \\
&= \int_0^\infty \theta_0 e^{(-r+\sigma)t} q_0 e^{-\gamma(r+\hat{\kappa})t} dt = \\
&= \int_0^\infty \theta_0 \left(\frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} e^{-[\gamma(r+\hat{\kappa})+r-\sigma]t} dt = \\
&= \frac{\theta_0}{\gamma(r+\hat{\kappa})+r-\sigma} \left(\frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}} [\gamma(r+\hat{\kappa})]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}}
\end{aligned}
$$

For the derivation of (3.2.13) we use equation (3.2.12) and again $\tau_0 = 1-\kappa_0$, as well as the fact that in the case with no tax $\kappa_0 = 1$ and $\hat{\kappa} = 0$ holds

$$\begin{aligned}
&\Gamma^{tax} > \Gamma^{notax} \\
\Leftrightarrow &\frac{\theta_0}{\gamma(r+\hat{\kappa})+r-\sigma}\left(\frac{\kappa_0(1-\frac{1}{\gamma})}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}}\left[\gamma(r+\hat{\kappa})\right]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \frac{\theta_0}{\gamma r+r-\sigma}\left(\frac{1-\frac{1}{\gamma}}{\alpha\beta}\right)^{\frac{\gamma}{(\alpha-1)\gamma+1}}(\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow &\frac{1}{\gamma(r+\hat{\kappa})+r-\sigma}\kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}}\left[\gamma(r+\hat{\kappa})\right]^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} > \frac{1}{\gamma r+r-\sigma}(\gamma r)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow &\kappa_0^{\frac{\gamma}{(\alpha-1)\gamma+1}} > \frac{\gamma(r+\hat{\kappa})+r-\sigma}{\gamma r+r-\sigma}\left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\frac{(\alpha-1)\gamma}{(\alpha-1)\gamma+1}} \\
\Leftrightarrow &\kappa_0 > \left(\frac{\gamma(r+\hat{\kappa})+r-\sigma}{\gamma r+r-\sigma}\right)^{\frac{(\alpha-1)\gamma+1}{\gamma}}\left(\frac{\gamma r}{\gamma(r+\hat{\kappa})}\right)^{\alpha-1} \\
\Leftrightarrow &\kappa_0 > \left(1+\frac{\gamma\hat{\kappa}}{r(1+\gamma)-\sigma}\right)^{\alpha-1+\frac{1}{\gamma}}\left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)} \\
\Leftrightarrow &\tau_0 < 1-\left[\left(1+\frac{\gamma\hat{\kappa}}{r(1+\gamma)-\sigma}\right)^{\alpha-1+\frac{1}{\gamma}}\left(1+\frac{\hat{\kappa}}{r}\right)^{-(\alpha-1)}\right] \qquad (5.2.2)
\end{aligned}$$

5.3 Abstract

A current question in environmental economics is, if measures that are taken to protect the environment (e.g. subsidies for renewable energy or taxes on non-renewable resources) can have the opposite effect, because the supply side is not considered and non-renewable resources are then extracted faster.
This thesis looks at the implications concerning this so called green paradox using existing competitive equilibrium models and finds out that changes in only one of the assumptions of such a model can change the occurrence of the green paradox.
These models are then expanded by a monopoly case, as for example in the oil industry it can be argued that this is closer to reality. It is then shown that in nearly all cases the assumption of a monopoly instead of a competitive setup doesn't change the occurrence of the green paradox.

Eine aktuelle Frage in der Umweltökonomie ist, ob Maßnahmen, die eigentlich die Umwelt schützen sollen (z.B. Förderungen für erneuerbare Energien oder Steuern auf nichterneuerbare Ressourcen), negative Auswirkungen auf diese haben können, weil die Angebotsseite nicht beachtet wird und nichterneuerbare Ressourcen dadurch früher abgebaut werden.
Diese Arbeit betrachtet dieses sogenannte grüne Paradoxon unter Verwendung von existierenden Modellen, die einen vollständigen Wettbewerb annehmen. Hierbei wird gezeigt, dass eine Veränderung in nur einer Annahme die Resultate bezüglich des Auftretens eines grünen Paradoxons verändern können.
Danach werden die verwendeten Modelle unter der Annahme eines Monopols betrachtet, da argumentiert werden kann, dass dies zum Beispiel im Fall von Erdöl besser der Realität entspricht. Es kann gezeigt werden, dass diese Veränderung in den Modellannahmen in fast allen Fällen keinen Einfluss auf ein Auftreten des grünen Paradoxons hat.

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