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Abstract

The elastic wave equation describes the evolution of a continuous body within the framework of classical linearized elasticity. We solve the Cauchy problem for this linear second-order system of partial differential equations via the variational approach. First we motivate the mathematical model and show how the elastic wave equation arises from the general equation of motion in continuum mechanics. Next we discuss the variational solution method which is based on the concept of weak solutions. In particular, we explain the abstract formulation of this method in Hilbert spaces. In the main part of the thesis, variational techniques are employed to establish existence, uniqueness, and regularity results for an abstract Cauchy problem associated to a general linear second-order evolution equation. Finally we apply these results to the elastic wave equation in a Sobolev space setting, assuming bounded and positive material parameters.

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Contents

Abstract	i
Acknowledgments	ii
1 The elastic wave equation	1
1.1 Modeling the motion of an elastic body	1
1.2 The Cauchy problem for linearized elasticity	4
1.3 Elastic waves	4
2 Variational methods	7
2.1 Weak solutions	7
2.2 Sobolev spaces	9
2.3 Abstract variational problems	11
2.3.1 Coercive bilinear forms in Hilbert spaces	11
2.3.2 Outline of the variational method for evolution equations	12
2.3.3 Technical tools: Weak convergence, compactness, and continuity	13
3 Variational solution of linear second-order evolution equations	17
3.1 The abstract Cauchy problem	17
3.1.1 Definition of the weak evolution equation	17
3.1.2 The existence and uniqueness result	19
3.1.3 Strong form	20
3.2 A priori estimates	23
3.2.1 Energy equality and integration by parts	24
3.2.2 Energy estimate	26
3.3 Uniqueness	29
3.3.1 Proof by the energy equality	29
3.3.2 Proof by an integrated energy equality	30
3.4 Existence	32
3.4.1 Proof by the Faedo-Galerkin method	32
3.4.2 Proof by parabolic regularization	37
3.5 Regularity	43
4 Application to the elastic wave equation	49
4.1 The weak form of the elastic wave equation	49
4.2 Positivity conditions for the elasticity tensor	51
4.3 Solution of displacement and traction problems	52
References	54
Zusammenfassung	57

Chapter 1

The elastic wave equation

The elastic wave equation describes the evolution of a continuous body in the setting of classical linearized elasticity. In Section 1.1 we discuss how the linear elastic wave equation arises from the general nonlinear governing equations in continuum mechanics. Then we introduce the Cauchy problem for the elastic wave equation (Section 1.2) and explain the connection to elastic waves (Section 1.3).

This chapter is mainly based on [MH83].

1.1 Modeling the motion of an elastic body

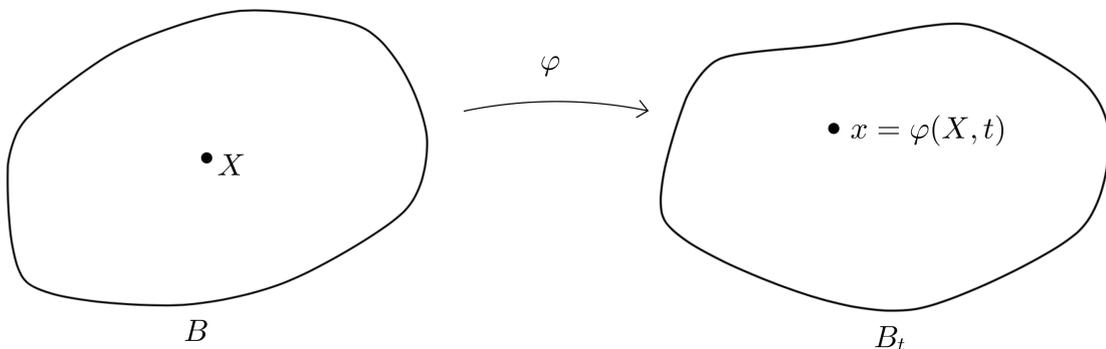
Elasticity is a subdiscipline of continuum mechanics, that describes the behavior of materials like solids, fluids, or gases. The underlying modeling hypothesis is that the exact molecular interior structure can be neglected.

Let $B \subseteq \mathbb{R}^n$ represent a continuous body that moves and deforms within the time interval $I = [t_0, t_1]$. The evolution is modeled by the **motion**

$$\varphi: B \times I \rightarrow \mathbb{R}^n. \quad (1.1)$$

As is illustrated in the figure below, for each $t \in I$, the motion maps a material point $X \in B$ to its new spatial position

$$x = \varphi(X, t) \in B_t := \varphi(B, t) \subseteq \mathbb{R}^n.$$



Remark 1.1 (Admissible motions). Since the function $\varphi: B \times I \rightarrow \mathbb{R}^n$ should model the evolution of a continuous body, several physically motivated admissibility conditions are imposed. For example, it is often required that the map $\varphi(\cdot, t)$, for fixed $t \in I$, is Lipschitz continuous, invertible, and orientation preserving, which locally means that $\det(\nabla\varphi(\cdot, t)) > 0$ a.e. on B . \lrcorner

1 The elastic wave equation

First- and second-order time derivatives of the motion φ are functions $B \times I \rightarrow \mathbb{R}^n$ that define the velocity and the acceleration field. We will denote them by $\dot{\varphi} = \partial_t \varphi = \frac{\partial \varphi}{\partial t}$ and $\ddot{\varphi} = \partial_t^2 \varphi = \frac{\partial^2 \varphi}{\partial t^2}$. The **mass density** of the body in its reference configuration at time t_0 is a function

$$\rho^0: B \rightarrow \mathbb{R}.$$

The motion is the solution of the **equation of motion**, which for $(X, t) \in B \times I$ reads

$$\rho^0(X) \ddot{\varphi}(X, t) - \operatorname{div} T^{\text{PK}}(X, t) = f(X, t). \quad (1.2)$$

This evolution equation is the continuum analog of Newton's Second Law, expressing balance of forces (conservation of linear momentum). The term $\rho^0 \ddot{\varphi}$ is the inertia and

$$f: B \times I \rightarrow \mathbb{R}^n$$

denotes the density of the external forces. These are the **body forces** describing long-range interactions. The term $\operatorname{div} T^{\text{PK}}$ models the **contact forces**, which represent the short-range forces that are assumed to exist in the interior of a continuous body. Here

$$T^{\text{PK}}: B \times I \rightarrow \mathbb{R}^{n \times n}$$

is the (first Piola-Kirchhoff) **stress tensor** and div stands for the row-wise divergence, in components (where the last equality explains the summation convention):

$$(\operatorname{div} T^{\text{PK}})_i = (\nabla \cdot T^{\text{PK}})_i = \partial_j T_{ij}^{\text{PK}} := \sum_{j=1}^n \partial_j T_{ij}^{\text{PK}}.$$

(Notation: Components are always defined with respect to Cartesian coordinates $(x_j)_{j=1}^n$ of \mathbb{R}^n , we write $\partial_j = \frac{\partial}{\partial x_j}$, and identify the derivative $\nabla \varphi$ with the matrix $(\partial_j \varphi_i)_{i,j=1}^n$.)

We briefly discuss the concept of stress: Contact forces are described via surface densities $\tau: B \times I \rightarrow \mathbb{R}^n$, the **traction**. By Cauchy's stress theorem, τ depends linearly on the unit normal vector $\nu: S \rightarrow \mathbb{R}^n$ of the surface $S \subseteq B$ on which it acts, that is,

$$\tau = T^{\text{PK}} \cdot \nu = (T_{ij}^{\text{PK}} \nu_j)_i.$$

Then, by the divergence theorem, the total surface force acting on a subbody $A \subseteq B$ reads

$$\int_{\partial A} \tau \, dS = \int_{\partial A} T^{\text{PK}} \cdot \nu \, dS = \int_A \operatorname{div} T^{\text{PK}} \, dV,$$

showing that contact forces have volume density $\operatorname{div} T^{\text{PK}}$. Thus the integral balance of forces

$$\frac{d^2}{dt^2} \int_A \rho^0 \varphi \, dV = \int_A f \, dV + \int_{\partial A} \tau \, dS$$

reduces to the local balance law $\rho^0 \ddot{\varphi} = f + \operatorname{div} T^{\text{PK}}$, which is (1.2).

If B is bounded, the following types of **boundary conditions** may be prescribed:

$$\varphi|_{\Gamma_D \times I} = \varphi_D \quad \text{and} \quad T^{\text{PK}} \cdot \nu|_{\Gamma_N \times I} = \tau_N. \quad (1.3)$$

The given functions φ_D and τ_N correspond to Dirichlet and Neumann data respectively, where one assumes that $\partial B = \Gamma_D \cup \Gamma_N$ is a disjoint union (e.g. one set is empty).

To determine the motion φ from Equations (1.4) and (1.3), its relation to T^{PK} has to be specified. This is the role of constitutive equations that define different classes of materials.

A body $B \subseteq \mathbb{R}^n$ is called **elastic** if there exists a response function $r: B \times GL(n) \rightarrow \mathbb{R}^{n \times n}$, such that for all $X \in B$,

$$T^{\text{PK}}(X, \cdot) = r(X, \nabla\varphi(X, \cdot)).$$

Here $GL(n) := \{F \in \mathbb{R}^{n \times n} : \det F \neq 0\}$, the linear group (invertible matrices, cf. Remark 1.1). We restrict ourselves to the hyperelastic theory, where $T^{\text{PK}} = r(\cdot, F) = \partial_F W(\cdot, F)$ for the elastic energy density

$$W: B \times GL(n) \rightarrow \mathbb{R}.$$

The specific form of W completely describes the elastic behavior of a material. The constitutive requirement of material frame-indifference implies that W actually depends on $\nabla\varphi$ only through the strain tensor $e(\varphi) := \frac{1}{2}(\nabla\varphi^T \cdot \nabla\varphi - 1_{n \times n})$.

With the constitutive relation for elasticity at hand, the equation of motion (1.2), together with the boundary conditions (1.3), then become the **governing equations of elasticity**:

$$\rho^0 \ddot{\varphi} - \nabla \cdot T^{\text{PK}} = f \quad \text{with} \quad T^{\text{PK}} = \partial_F W(\cdot, \nabla\varphi), \quad (1.4)$$

that is,

$$\rho^0 \ddot{\varphi} - \nabla \cdot (\partial_F W(\cdot, \nabla\varphi)) = f.$$

This is a system of nonlinear partial differential equations for φ , that has second order in space and time variables. More precisely, the system is quasilinear (in space variables):

$$\nabla \cdot T^{\text{PK}} = \nabla \cdot (\partial_F W(\cdot, \nabla\varphi)) = (\partial_F^2 W(\cdot, \nabla\varphi)) : \nabla\nabla\varphi + (\nabla \cdot \partial_F W)(\cdot, \nabla\varphi)$$

shows that the highest-order derivative (the Hessian $\nabla\nabla\varphi$) occurs linearly in the principal part but the coefficient $\partial_F^2 W(\cdot, \nabla\varphi)$, the generalized elasticity tensor, depends on $\nabla\varphi$.

In terms of the **displacement**

$$u: B \times I \rightarrow \mathbb{R}^n, \quad u(X, t) := \varphi(X, t) - X, \quad (1.5)$$

we have $\dot{\varphi} = \dot{u}$, $\ddot{\varphi} = \ddot{u}$, $\nabla\varphi = 1_{n \times n} + \nabla u$, and the governing equations (1.4) read

$$\rho^0 \ddot{u} - \nabla \cdot T^{\text{PK}} = f \quad \text{with} \quad T^{\text{PK}} = \partial_F W(\cdot, 1_{n \times n} + \nabla u).$$

Classical **linearized elasticity** results from a linearization of the governing equations around a stress-free reference state. This linear approximation is valid as long as displacements u , displacement gradients ∇u , and stresses are small. In most applications, especially in the case of seismic waves, a linear model (possibly generalized to nonzero prestress T^0) is sufficient.

Linearizing the elastic constitutive relation results in **Hooke's law**, the famous linear relation between stress and strain:

$$T = c : \nabla u, \quad \text{that is,} \quad T_{ij} = c_{ijkl} \partial_l u_k.$$

Here $T: B \times I \rightarrow \mathbb{R}^{n \times n}$ is the (linearized Cauchy) stress tensor and $c: B \rightarrow \mathbb{R}^{n \times n \times n \times n}$ is the classical (linearized) **elasticity tensor**, which has the symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \quad (1.6)$$

Then, denoting the density by $\rho: B \rightarrow \mathbb{R}$, the linearization of the equation of motion (1.4) reads

$$\rho \ddot{u} - \nabla \cdot T = f \quad \text{with} \quad T = c : \nabla u,$$

that is,

$$\rho \ddot{u} - \nabla \cdot (c : \nabla u) = f. \quad (1.7)$$

This linear second-order system of partial differential equations for the displacement field u is called the **elastic wave equation**. The name "wave equation" will be explained in Section 1.3.

1.2 The Cauchy problem for linearized elasticity

An important modeling problem in mechanics is to predict the evolution of a body starting from a known configuration. This is the **Cauchy problem** for the governing equations.

The Cauchy problem for linearized elasticity is to find the displacement

$$u: B \times I \rightarrow \mathbb{R}^n$$

that solves the elastic wave equation (1.7),

$$\rho \ddot{u} - \nabla \cdot T = f \quad \text{with} \quad T = c : \nabla u,$$

the **boundary conditions**

$$u|_{\Gamma_D \times I} = u_D \quad \text{and} \quad T \cdot \nu|_{\Gamma_N \times I} = \tau_N, \quad (1.8)$$

and the **initial conditions**

$$u(\cdot, t_0) = u^0 \quad \text{and} \quad \dot{u}(\cdot, t_0) = u^1. \quad (1.9)$$

In the Cauchy problem, the time interval $I = [t_0, t_1]$, the spatial domain $B \subseteq \mathbb{R}^n$ with boundary $\partial B = \Gamma_D \cup \Gamma_N$ (disjoint), the **material parameters** $\rho: B \rightarrow \mathbb{R}$ and $c: B \rightarrow \mathbb{R}^{n \times n \times n \times n}$, the **source** $f: B \times I \rightarrow \mathbb{R}^n$, the boundary data $u_D: \Gamma_D \times I \rightarrow \mathbb{R}^n$, $\tau_N: \Gamma_N \times I \rightarrow \mathbb{R}^n$, as well as the initial data $u^0: B \rightarrow \mathbb{R}^n$, $u^1: B \rightarrow \mathbb{R}^n$ are all assumed to be given.

1.3 Elastic waves

The relation of (1.7) to elastic waves is revealed if one considers a homogeneous and isotropic material. If the body B is part of the Earth, then the elastic waves are referred to as the seismic waves.

In isotropic media the elasticity tensor takes the form (with $i, j, k, l = 1, \dots, n$)

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

Here $\lambda, \mu: B \rightarrow \mathbb{R}$ are the Lamé constants, μ is the shear modulus (rigidity), λ is related to the bulk modulus (compressibility), and δ_{ij} is the Kronecker symbol. Hooke's law then reads

$$T_{ij} = \lambda (\partial_k u_k) \delta_{ij} + \mu (\partial_j u_i + \partial_i u_j) = (\lambda (\nabla \cdot u) 1_{n \times n} + \mu (\nabla u + \nabla u^T))_{ij}.$$

If the medium is homogeneous, all material parameters λ , μ , and ρ are constant. Then

$$(\nabla \cdot T)_i = \partial_j T_{ij} = \lambda \partial_i (\partial_k u_k) + \mu (\partial_j^2 u_i + \partial_i (\partial_j u_j)) = ((\lambda + \mu) \nabla (\nabla \cdot u) + \mu \Delta u)_i$$

where $\Delta = \nabla \cdot \nabla = \sum_{k=1}^n \partial_k^2$ is the Laplace operator. Then, with $f = 0$, (1.7) reduces to

$$\rho \ddot{u} = (\lambda + \mu) \nabla (\nabla \cdot u) + \mu \Delta u.$$

A Helmholtz decomposition

$$u = u_P + u_S \quad \text{where} \quad \nabla \times u_P = 0 \quad \text{and} \quad \nabla \cdot u_S = 0,$$

together with the identity $\Delta u = \nabla (\nabla \cdot u) - \nabla \times (\nabla \times u)$, shows that the irrotational part u_P and the divergence-free part u_S satisfy $\rho \ddot{u}_P = (\lambda + 2\mu) \Delta u_P$ and $\rho \ddot{u}_S = \mu \Delta u_S$ respectively. Consequently, assuming the positivity conditions

$$\rho > 0, \quad \mu > 0, \quad \lambda + 2\mu > 0, \quad (1.10)$$

the solution u of the elastic wave equation (1.7) splits into **P-waves** u_P and **S-waves** u_S , that fulfill the **classical wave equations** (however, note that u_P and u_S are vector-valued)

$$\ddot{u}_P = \alpha^2 \Delta u_P \quad \text{and} \quad \ddot{u}_S = \beta^2 \Delta u_S \quad (1.11)$$

with the real propagation speeds $\alpha := \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $\beta := \sqrt{\frac{\mu}{\rho}}$.

P- and S-waves are so-called body waves that can travel in all directions. Elastic waves can also propagate at surfaces, as is seen by solving the wave equations (1.11) in a half space. These surface waves cause the severe damage in earthquakes. Yet, seismology, which is the study of seismic waves, is the most accurate method to explore the deep interior of our planet.

We return to the general case of an anisotropic and inhomogeneous medium which is governed by the general linear elastic wave equation (1.7),

$$\rho \ddot{u} - \nabla \cdot (c : \nabla u) = f.$$

In components $i = 1, \dots, n$,

$$\rho \ddot{u}_i - \partial_j (c_{ijkl} \partial_l u_k) = f_i.$$

In more detailed notation at $(x, t) \in B \times I$ and without summation convention, this is

$$\rho(x) \frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l=1}^n \frac{\partial}{\partial x_j} \left(c_{ijkl}(x) \frac{\partial u_k}{\partial x_l} \right) = f_i(x, t)$$

or equivalently

$$\sum_{k=1}^n \left(\delta_{ik} \rho(x) \frac{\partial^2 u_k}{\partial t^2} - \sum_{j,l=1}^n \frac{\partial}{\partial x_j} \left(c_{ijkl}(x) \frac{\partial u_k}{\partial x_l} \right) \right) = f_i(x, t).$$

Following [Tré75, Chapter 47, p. 458], a second-order linear partial differential operator of the form

$$L := \frac{\partial^2}{\partial t^2} - \sum_{j,l=1}^n \frac{\partial}{\partial x_j} \left(a_{jl}(x) \frac{\partial}{\partial x_l} \right) \quad (1.12)$$

is **hyperbolic** (a **wave operator**), if the coefficients are bounded, that is $a_{jl} \in L^\infty(B)$, and the following positivity condition (uniform strong ellipticity, see Remark 4.5) holds:

There exist $\alpha > 0$ such that for a.a. $x \in B$,

$$\sum_{j,l=1}^n a_{jl}(x) \xi_j \xi_l \geq \alpha |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Note that this definition includes the classical wave operator $L = \partial_t^2 - \Delta$ (choose $a_{jl} = \delta_{jl}$).

A comparison with (1.12) suggests that the elastic wave equation (1.7) may indeed be classified as hyperbolic, provided that the density ρ (through which we have to divide) as well as the elasticity tensor c are bounded and positive, where positivity of the elasticities c_{ijkl} has to be defined in some appropriate sense. We will return to these issues in Chapter 4.

Chapter 2

Variational methods

Variational methods for partial differential equations are a versatile solution technique based on the weak formulation (Section 2.1). In Section 2.2 we discuss Sobolev spaces, which provide a favorable setting for the definition of weak solutions. The abstract formulation of variational methods in general Hilbert spaces is presented in Section 2.3. In particular, we outline the main steps of the variational approach to solve evolution equations.

The results of this chapter are based on [LM72, Tré75, DL88, DL92, RR04, Bre11].

2.1 Weak solutions

To study **partial differential equations (PDEs)** in a low regularity setting means that solutions, coefficients, initial, or boundary data are allowed to be non-smooth. In order to correctly state the equations and investigate solvability, the classical solution concept has to be generalized.

To illustrate the basic idea, consider a PDE $Pu = f$ on an open set $\Omega \subseteq \mathbb{R}^n$ with $f: \Omega \rightarrow \mathbb{R}$.

A solution $u: \Omega \rightarrow \mathbb{R}$ is a **classical solution**, if $(Pu)(x) = f(x)$ holds pointwise for all $x \in \Omega$. Yet, in order to discuss solvability in case of low regularity one has to relax the solution concept and consider **weak solutions (variational solutions)**. Their definition is based on reformulating the PDE in duality to suitable test functions and interpreting derivatives in the weak sense (see Definition 2.1 below).

Formally, a **weak form (variational form)** of a PDE $Pu = f$ can be obtained as follows: First one multiplies the equation with a test function φ and integrates over the domain,

$$\int_{\Omega} (Pu) \varphi \, dV = \int_{\Omega} f \varphi \, dV.$$

Then one tries to move the derivatives occurring in P to the test function side via integration by parts. By this procedure the regularity requirements on the solution u are gradually reduced. Moreover, it is guaranteed that classical solutions of the PDE are also weak solutions.

A suitable space of **test functions** φ are the smooth, compactly supported functions,

$$\mathcal{D}(\Omega) := \{\varphi \in \mathcal{C}^{\infty}(\Omega) : \text{supp}(\varphi) \text{ compact in } \Omega\},$$

endowed with the topology of uniform convergence of all derivatives with support in a fixed compact set, see e.g. [Tré67, Chapter 21]. The space of **distributions** $\mathcal{D}'(\Omega)$ consists of continuous linear functionals on $\mathcal{D}(\Omega)$. The action of a distribution $u \in \mathcal{D}'(\Omega)$ on a test function $\varphi \in \mathcal{D}(\Omega)$ is written in terms of a duality bracket: $u: \varphi \mapsto \langle u, \varphi \rangle \in \mathbb{R}$. Distributional duality of $u \in \mathcal{D}'(\Omega)^m$ and $\varphi \in \mathcal{D}(\Omega)^m$ is defined via scalar duality of the components, $\langle u, \varphi \rangle = \sum_{k=1}^m \langle u_k, \varphi_k \rangle$.

2 Variational methods

If $u \in L^1_{\text{loc}}(\Omega)$, then $\langle u, \varphi \rangle = \int_{\Omega} u \varphi \, dV$. Integration by parts,

$$\int_{\Omega} (\partial_j f) \varphi \, dV = - \int_{\Omega} f (\partial_j \varphi) \, dV,$$

which holds classically for $f \in \mathcal{C}^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, motivates the following definition:

Definition 2.1 (Distributional derivative, weak derivative). Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in \mathcal{D}'(\Omega)$, and $\varphi \in \mathcal{D}(\Omega)$. Then, for $1 \leq j \leq n$, the distributional derivative $\partial_j f \in \mathcal{D}'(\Omega)$ is defined by

$$\langle \partial_j f, \varphi \rangle := -\langle f, \partial_j \varphi \rangle. \quad (2.1)$$

Higher-order distributional derivatives are defined by iteration (written in multi-index notation: $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \dots + \alpha_n$, $D^\alpha := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, and $\partial_j^0 := 1$ for $j = 1, \dots, n$):

$$\langle D^\alpha f, \varphi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle.$$

The symbols ∂_i and D^α always denote distributional derivatives, which coincide with the classical derivatives for functions of sufficient regularity. \lrcorner

For linear PDEs of order $m \in \mathbb{N}_0$ and with constant or smooth coefficients a_α , that is, for PDEs of the form $\sum_{|\alpha| \leq m} a_\alpha D^\alpha u = f$, the concept of weak solutions naturally leads to functions in the Sobolev space H^k with $0 \leq k \leq m$. Whereas most linear PDEs will allow weak formulations in these Hilbert spaces, weak solutions of simple types of nonlinear PDEs are typically defined in the Banach spaces $W^{k,p}$ (see Section 2.2 for definitions).

Let us consider an example [DL88, Chapter IV, p. 142]:

Example 2.2 (Variational formulation of the Poisson equation). Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, and with smooth boundary. Then, according to regularity, the solution of

$$-\Delta u = f \quad \text{with} \quad u|_{\partial\Omega} = 0$$

can be interpreted as the solution of one of the following problems:

- (i) Find $u \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\int_{\Omega} (\Delta u + f) v \, dV = 0$ for all $v \in L^2(\Omega)$.
- (ii) Find $u \in H_0^1(\Omega)$ such that $\int_{\Omega} \nabla u \cdot \nabla v \, dV = \int_{\Omega} f v \, dV$ for all $v \in H_0^1(\Omega)$. (2.2)
- (iii) Find $u \in L^2(\Omega)$ such that $\int_{\Omega} (u \Delta v + f v) \, dV = 0$ for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$.

It is option (ii) that balances the regularity requirements on the solution u and on the test function v in an optimal way. Moreover, only in this case, $\partial\Omega$ can even be a Lipschitz domain. Therefore, a function $u \in H_0^1(\Omega)$ that satisfies (2.2) is called a **weak solution** of the Poisson equation. \lrcorner

Weak solutions may also solve a **variational problem**:

Example 2.3 (Euler-Lagrange equations). The weak solution $u \in H_0^1(\Omega)$ defined by (2.2) is a stationary point of the Dirichlet integral $J: H_0^1(\Omega) \rightarrow \mathbb{R}$,

$$J(u) := \int_{\Omega} F(u, \nabla u) \, dV \quad \text{with} \quad F(u, \nabla u) := \frac{1}{2} |\nabla u|^2 - fu.$$

Indeed, for test functions $\varphi \in \mathcal{D}(\Omega)$ (which is dense in $H_0^1(\Omega)$), the stationarity condition reads

$$\left. \frac{d}{d\varepsilon} J(u + \varepsilon\varphi) \right|_{\varepsilon=0} = \int_{\Omega} ((\partial_{\nabla u} F) \cdot \nabla \varphi + (\partial_u F)\varphi) \, dV = \int_{\Omega} (\nabla u \cdot \nabla \varphi - f\varphi) \, dV = 0.$$

Moreover, the classical Euler-Lagrange equations $\text{div}(\partial_{\nabla u} F) - \partial_u F = 0$ give $-\Delta u = f$. \lrcorner

In general, weak formulations that involve a symmetric bilinear form (like the Dirichlet integral) will also correspond to a variational problem (see also Lemma 2.10). This connection of weak solutions to calculus of variations explains the notion “variational solutions”.

2.2 Sobolev spaces

We present a collection of some basic facts about Sobolev spaces.

In this section, $\Omega \subseteq \mathbb{R}^n$ denotes an open set. A **Lipschitz domain** is an open, bounded, and connected set Ω , such that, locally, the boundary $\partial\Omega$ is the graph of a Lipschitz continuous function and Ω only lies on one side of the boundary. An exact definition is given in [Gri85, Def. 1.2.1.1, p. 5].

We recall that $L^p(\Omega)$ is the Banach space of (equivalence classes of) Lebesgue measurable functions f with norm $\|f\|_{L^p(\Omega)} < \infty$, where

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_{\Omega} |f(x)|^p \, dV \right)^{1/p} & 1 \leq p < \infty \\ \text{ess sup}_{x \in \Omega} |f(x)| & p = \infty. \end{cases} \quad (2.3)$$

Here, $\|f\|_{L^\infty(\Omega)}$ is the essential supremum of $|f(x)|$, that is, is the greatest lower bound (infimum) of all K such that $\sup_{x \in \Omega} |f(x)| \leq K$ a.e. on Ω .

Sobolev spaces consist of Lebesgue measurable and power integrable functions whose distributional derivatives also are elements of a Lebesgue space:

Definition 2.4 (Sobolev spaces). The L^p -based Sobolev space of order (exponent) $k \in \mathbb{N}_0$, is the set

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k : D^\alpha u \in L^p(\Omega) \right\}, \quad (2.4)$$

which is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty. \end{cases}$$

The L^2 -based Sobolev spaces $H^k(\Omega) := W^{k,2}(\Omega)$ are Hilbert spaces with inner product

$$\langle u|v \rangle_{H^k(\Omega)} := \sum_{|\alpha| \leq k} \langle D^\alpha u | D^\alpha v \rangle_{L^2(\Omega)} = \sum_{|\alpha| \leq k} \int_{\Omega} (D^\alpha u)(D^\alpha v) \, dV.$$

Moreover, $H_0^k(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{H^k(\Omega)}$. \lrcorner

By definition, $W^{0,p}(\Omega) = L^p(\Omega)$.

Remark 2.5 (Real exponents). For $\Omega = \mathbb{R}^n$, the Fourier transform allows us to define $H^s(\mathbb{R}^n)$ for arbitrary real Sobolev exponents $s \in \mathbb{R}$. Thereby, for Lipschitz domains Ω , one may also generalize $H^k(\Omega)$ to real exponents [DL88, Chapter V, §4, Prop. 3, p. 118]. \lrcorner

If $f \in W^{k,p}(\Omega)$ then $\partial_j f \in W^{k-1,p}(\Omega)$ for $j = 1, \dots, n$. In particular, partial derivatives are continuous maps $\partial_j : W^{k,p}(\Omega) \rightarrow W^{k-1,p}(\Omega)$,

$$\|\partial_j f\|_{W^{k-1,p}(\Omega)} \leq \|f\|_{W^{k,p}(\Omega)}. \quad (2.5)$$

Consequently, operators $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ with smooth coefficients map $H^s(\Omega)$ to $H^{s-m}(\Omega)$.

2 Variational methods

Dual spaces of Sobolev spaces may again be Sobolev spaces: We have $(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n)$. However, since test functions $\mathcal{D}(\Omega)$ are not dense in $H^k(\Omega)$ if $\Omega \subsetneq \mathbb{R}^n$ (and $k \in \mathbb{N}$; since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega) = H^0(\Omega)$), we only get

$$(H_0^k(\Omega))' = H^{-k}(\Omega).$$

By the standard (operator) norm for dual spaces, $\|f\|_{H^{-k}(\Omega)} := \sup_{\|u\|_{H_0^k(\Omega)} \leq 1, u \neq 0} \left(\frac{|\langle f, u \rangle|}{\|u\|_{H_0^k(\Omega)}} \right)$.

Similarly, duality for $W^{k,p}(\Omega)$ with $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ reads $(W_0^{k,p}(\Omega))' = W^{-k,p'}(\Omega)$, where the conjugate exponent p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$, including 1 and ∞ .

Remark 2.6 (The dual of H^1). The lack of density of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ implies that $(H^1(\Omega))'$ is not a space of distributions on Ω . However, the adjoint of the continuous projection $H^1(\Omega) \rightarrow H_0^1(\Omega)$ defines an embedding $H^{-1}(\Omega) \hookrightarrow (H^1(\Omega))'$, see [VP65, Section 3]. \lrcorner

The restriction of Sobolev spaces to surfaces reduces their exponent (e.g. [Tré75, Theorem 26.2], [Wlo87, Theorem 8.8]):

Lemma 2.7 (Trace operator). Let $\Omega \subseteq \mathbb{R}^n$ be a domain with smooth boundary and $k \in \mathbb{N}$. Then the restriction of $u \in \mathcal{C}^\infty(\bar{\Omega})$ to $\partial\Omega$ uniquely extends to a continuous and surjective linear map $T: H^k(\Omega) \rightarrow H^{k-\frac{1}{2}}(\partial\Omega)$ with $T(u) = u|_{\partial\Omega}$.

The space $H_0^k(\Omega)$ is the kernel of T . In particular, $H^1(\Omega)$ has traces in $H^{1/2}(\partial\Omega)$ and functions in $H_0^1(\Omega)$ vanish on the boundary $\partial\Omega$ (Lipschitz $\partial\Omega$ is enough for $k = 1$). Moreover, if $u \in H_0^1(\Omega)$ and $\Omega \subseteq \mathbb{R}^n$ is bounded, then **Poincaré's inequality** holds [DL88, Chapter IV, §7, Corollary 2, p. 126]:

$$\|u\|_{L^2(\Omega)}^2 \leq c_P \|\nabla u\|_{L^2(\Omega)}^2. \quad (2.6)$$

This implies $\|\nabla u\|_{L^2(\Omega)}^2 \leq \|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \leq (1 + c_P) \|\nabla u\|_{L^2(\Omega)}^2$, showing that $\|u\|_{H^1(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$ are equivalent norms on $H_0^1(\Omega)$.

Definition 2.8 (Continuous and compact embeddings). Let X, Y be Banach spaces with $X \subseteq Y$ as a subspace. Then $X \hookrightarrow Y$ (continuous embedding) if there exists an injective continuous map $X \rightarrow Y$ (there exists $c > 0$ with $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$). The embedding is compact, if $X \hookrightarrow Y$ and bounded sets in X are relatively compact in Y (sequences with $\|x_k\|_X \leq C$ have subsequences $(x_{k_l})_l$ that converge in Y to some $y \in Y$). \lrcorner

As is clear by definition, $H^k(\Omega) \hookrightarrow H^m(\Omega)$ for $m, k \in \mathbb{N}_0$ with $m \leq k$. Moreover, on bounded open sets $\Omega \subseteq \mathbb{R}^n$, we have $L^p(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p \leq \infty$. This translates to Sobolev spaces in the following way ([AF03, Theorem 4.12, p. 85] and [GT01, Theorem 7.26, p. 171]):

Lemma 2.9 (Sobolev embedding). Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain, $k \in \mathbb{N}_0$, $1 \leq p < \infty$.

- (i) If $k > \frac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow \mathcal{C}^0(\Omega) \cap L^\infty(\Omega)$ (and compactly in $\mathcal{C}^{0,\alpha}(\bar{\Omega})$ if $0 < \alpha < k - \frac{n}{p} \leq 1$).
- (ii) If $k = \frac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ (also compactly embedded).
- (iii) If $k < \frac{n}{p}$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \leq \frac{np}{n-kp}$ (and compactly for $q < \frac{np}{n-kp}$).

In particular, (i) shows that Sobolev regularity becomes classical regularity for sufficiently high exponents k, p or in sufficiently low dimension n .

2.3 Abstract variational problems

2.3.1 Coercive bilinear forms in Hilbert spaces

Weak formulations of linear PDEs may be defined in Sobolev spaces H^k via duality relations. This leads us to the following **abstract variational problem** on a general Hilbert space V :

Find $u \in V$, such that

$$a(u, v) = L(v) \quad \forall v \in V, \quad (2.7)$$

where $a: V \times V \rightarrow \mathbb{R}$ is bilinear and $L: V \rightarrow \mathbb{R}$ is linear and continuous, that is, $L \in V'$.

If a is an inner product (bilinear, symmetric, positive definite), then the Riesz representation theorem for Hilbert spaces yields solvability of the abstract variational problem: There exists a unique element $u \in V$ such that (2.7) holds.

This extends to the following result [Bre11, Corollary 5.8]:

Lemma 2.10 (Lax-Milgram for coercive bilinear forms). *Let V be a Hilbert space, $L \in V'$, and $a: V \times V \rightarrow \mathbb{R}$ be a continuous bilinear form that is coercive, that is, there exists $\alpha > 0$ such that*

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V. \quad (2.8)$$

Then there exists a unique solution $u \in V$ of the abstract variational problem (2.7).

Moreover, if a is symmetric, then the solution u minimizes the functional

$$J: V \rightarrow \mathbb{R}, \quad J(v) := \frac{1}{2}a(v, v) - L(v).$$

We recall that continuity of the bilinear map a means that there exists $c_a > 0$ such that

$$|a(u, v)| \leq c_a \|u\|_V \|v\|_V \quad \forall u, v \in V. \quad (2.9)$$

Thus, if the weak formulation of a linear PDE involves a continuous and coercive bilinear form, then existence and uniqueness of weak solutions are immediate.

Example 2.11 (Existence and uniqueness of weak solutions of the Poisson equation). The weak form (2.2) corresponds to an abstract variational problem (2.7) with

$$V = H_0^1(\Omega), \quad L = f \in V', \quad \text{and} \quad a: V \times V \rightarrow \mathbb{R}, \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dV = \langle \nabla u | \nabla v \rangle_{L^2(\Omega)}.$$

Bilinearity of a is clear and continuity follows from Cauchy-Schwarz inequality and (2.5):

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} = \|u\|_V \|v\|_V.$$

Coercivity of a follows from $\|u\|_{H^1(\Omega)}^2 \leq (1 + c_P) \|\nabla u\|_{L^2(\Omega)}^2$ by Poincaré's inequality (2.6):

$$a(u, u) = \|\nabla u\|_{L^2(\Omega)}^2 \geq \alpha \|u\|_{H^1(\Omega)}^2 = \alpha \|u\|_V^2 \quad \text{with} \quad \alpha = \frac{1}{1 + c_P}.$$

Consequently, by Lemma 2.10 there exists a unique weak solution $u \in V = H_0^1(\Omega)$ of the Poisson equation $-\Delta u = f$ for $f \in H^{-1}(\Omega)$. We note that the boundary condition $u|_{\partial\Omega} = 0$ is directly incorporated in the definition of the function space V . \lrcorner

Remark 2.12 (Classical solutions via elliptic regularity). If f lies in a better Sobolev space (or in a Hölder space $\mathcal{C}^{0,\alpha}$), then elliptic regularity implies that the weak solution of $-\Delta u = f$ is actually \mathcal{C}^2 , that is, we have found a classical solution, see e.g. [GT01]. \lrcorner

2.3.2 Outline of the variational method for evolution equations

The variational approach to the solution of partial differential equations consists of the following main parts (after [DL92, Chapter XVIII, p. 508] or [Bre11, Chapter 8, p. 201]):

I. Variational formulation:

Define the notion of a weak solution that generalizes the classical solution concept.

II. Existence and uniqueness:

Apply a “variational solution method” to prove that weak solutions exist and are unique.

III. Regularity:

Investigate the smoothness of weak solutions (weak solutions might possess enough regularity to be classical solutions). Moreover, questions of continuous dependence on data may be discussed.

For linear time-independent equations (equilibrium problems), a possible variational method in Part II is Lax-Milgram (Lemma 2.10) or the direct solution of the associated minimization problem. We have illustrated Parts I, II, and III in Examples 2.3, 2.11, and Remark 2.12 respectively.

A versatile variational solution method for **evolution equations** (dynamic problems) is the so-called **method of energy estimates**. In Part I we set up the **abstract Cauchy problem** P which involves a weak evolution equation. As in the static case, the boundary conditions of the PDE should be incorporated in the choice of function spaces. Then existence and uniqueness of weak solutions in Part II is obtained in the following way:

- **Step 0:** Prove uniqueness for P .
- **Step 1:** Define an approximate problem which has a solution u_m (e.g. by discretization).
Establish “a priori estimates” (the energy estimates).
- **Step 2:** Based on these estimates, infer boundedness of $(u_m)_m$ in some normed space.
By weak(-*) compactness, extract a weakly(-*) convergent subsequence.
- **Step 3:** Show that the limit satisfies the weak evolution equation.
- **Step 4:** Show that the limit satisfies the initial conditions.

Thereby, the limit obtained is the sought weak solution u of the abstract Cauchy problem P .

This variational solution method for second-order evolution equations will be discussed in detail in Chapter 3. The evolutionary counterpart of the abstract variational problem (2.7) will be (3.1),

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T),$$

which is coupled to the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$. If the bilinear forms a, b, c fulfill to certain hypotheses, then this abstract Cauchy problem will be **well-posed**: There exists a unique solution u that depends continuously on the data u^0, u^1, f (see Theorem 3.1 and Corollary 3.12).

In Chapter 4, we finally apply our findings to the elastic wave equation.

2.3.3 Technical tools: Weak convergence, compactness, and continuity

This section gathers various concepts and proves some statements that will be relevant for the variational solution of evolution equations (Chapter 3). A crucial ingredient is **compactness**, which allows us to infer the existence of a limit from mere boundedness of the sequence (see Part II, Step 2). Since compactness of closed balls with respect to the norm (strong topology) only holds in finite dimensional Banach spaces, we need to change the topology:

Definition 2.13 (Weak convergence, weak-* convergence). Let X be a Banach space with dual X' and write $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{X', X}$. A sequence $(x_n)_n \subseteq X$ converges weakly to $x \in X$,

$$x_n \rightharpoonup x \quad (n \rightarrow \infty), \quad \text{if} \quad \langle x'_n, x_n \rangle \rightarrow \langle x', x \rangle \quad \forall x' \in X' \quad (n \rightarrow \infty).$$

A sequence $(x'_n)_n \subseteq X'$ converges weakly-* to $x' \in X'$,

$$x'_n \rightharpoonup^* x' \quad (n \rightarrow \infty), \quad \text{if} \quad \langle x'_n, x \rangle \rightarrow \langle x', x \rangle \quad \forall x \in X \quad (n \rightarrow \infty). \quad \lrcorner$$

Lemma 2.14 (Weak compactness, weak-* compactness).

- (i) Let X be a reflexive Banach space. Then, from a bounded sequence in X , one can extract a weakly convergent subsequence (closed balls are weakly compact).
- (ii) Let X' be the dual of a separable Banach space. Then, from a bounded sequence in X' , one can extract a weakly-* convergent subsequence (closed balls are weakly-* compact).

These definitions and the proofs can be found e.g. in [DL88, Chapter VI, §5] and [Bre11, Theorems 3.17 & 3.16, p. 66].

Solutions of evolutionary PDEs are functions of space and time variables, say $(x, t) \mapsto u(x, t)$ with $x \in \Omega \subseteq \mathbb{R}^n$ and $t \in [0, T]$. However, in view of the variational formulation it is desirable to split the regularity requirements with respect to these variables. This leads us to the notion of functions of $t \in [0, T]$ with **values in a Banach space** X , that is, “vector-valued” functions. We list some examples to illustrate the notation:

Let X be a Banach space and $t \mapsto f(t) \in X$ be measurable on $(0, T)$. Then

$$\begin{aligned} f \in L^p((0, T); X) &: \iff t \mapsto \|f(t)\|_X \text{ is in } L^p(0, T); & \|f\|_{L^p((0, T); X)} &:= \|\|f(\cdot)\|_X\|_{L^p(0, T)} \\ f \in W^{k, p}((0, T); X) &: \iff t \mapsto \|f(t)\|_X \text{ is in } W^{k, p}(0, T); & \|f\|_{W^{k, p}((0, T); X)} &:= \|\|f(\cdot)\|_X\|_{W^{k, p}(0, T)} \\ f \in \mathcal{C}^0([0, T]; X) &: \iff t \mapsto \|f(t)\|_X \text{ is in } \mathcal{C}^0([0, T]); & \|f\|_{\mathcal{C}^0([0, T]; X)} &:= \max_{t \in [0, T]} \|f(t)\|_X. \end{aligned}$$

Spaces of vector-valued test functions $\mathcal{D}((0, T); X)$ and distributions $\mathcal{D}'((0, T); X)$ are defined in [Tré75, Chapter 39]. We will also need the test function space $\mathcal{D}([0, T]; X)$, which consists of the restrictions of $\mathcal{D}(\mathbb{R}; X)$ to $[0, T]$, see [DL92, Chapter XVIII, §1, Lemma 1, p. 473].

Lemma 2.15 (Weak-* convergence in L^∞ implies weak convergence in L^2). Let H be a Hilbert space. If $w_n \rightharpoonup^* w$ in $L^\infty((0, T); H)$, then $w_n \rightharpoonup w$ in $L^2((0, T); H)$ as $n \rightarrow \infty$.

Proof. Let $\varphi \in L^2((0, T); H) \subseteq L^1((0, T); H)$. If $w_n \rightharpoonup^* w$ in $L^\infty((0, T); H)$ as $n \rightarrow \infty$, then $w_n, w \in L^\infty((0, T); H) \subseteq L^2((0, T); H)$ and we get

$$\begin{aligned} \langle w_n, \varphi \rangle_{L^2((0, T); H), L^2((0, T); H)} &= \langle w_n, \varphi \rangle_{L^\infty((0, T); H), L^1((0, T); H)} \\ &\rightarrow \langle w, \varphi \rangle_{L^\infty((0, T); H), L^1((0, T); H)} = \langle w, \varphi \rangle_{L^2((0, T); H), L^2((0, T); H)} \end{aligned}$$

showing that $w_n \rightharpoonup w$ in $L^2((0, T); H)$ as $n \rightarrow \infty$. □

2 Variational methods

By Sobolev embedding (in the time variable), see Lemma 2.9,

$$H^1((0, T); V') \hookrightarrow \mathcal{C}^0([0, T]; V').$$

This result improves if we have additional spatial regularity, see also (3.59). We follow [RR04, Lemma 11.4, p. 383]:

Lemma 2.16 (Continuous embedding). *If V, H are Hilbert spaces such that $V \subseteq H$ is dense and with continuous embedding (that is we have $V \hookrightarrow H \hookrightarrow V'$), then*

$$L^2((0, T); V) \cap H^1((0, T); V') \hookrightarrow \mathcal{C}^0([0, T]; H). \quad (2.10)$$

Proof. Let $u \in \mathcal{D}([0, T]; V)$, which is dense in $L^2((0, T); V) \cap H^1((0, T); V')$. By the product rule, $\frac{d}{dt}\|u\|_H^2 = \frac{d}{dt}\langle u|u \rangle_H = 2\langle \dot{u}|u \rangle_H$ and integration on $(s, t) \subseteq [0, T]$ gives

$$\|u(t)\|_H^2 = \|u(s)\|_H^2 + 2 \int_s^t \langle \dot{u}(t')|u(t') \rangle_H dt'.$$

Continuity of $\|u(\cdot)\|_H^2$ allows us to choose s such that $\|u(s)\|_H^2$ equals the mean value of $\|u(\cdot)\|_H^2$ on $[0, T]$:

$$\|u(s)\|_H^2 = \frac{1}{T} \int_0^T \|u(t')\|_H^2 dt' = \frac{1}{T} \|u\|_{L^2((0, T); H)}^2.$$

With $\langle \dot{u}|u \rangle_H = \langle \dot{u}, u \rangle_{V, V'} \leq \|\dot{u}\|_{V'} \|u\|_V$ we arrive at the inequality

$$\|u(t)\|_H^2 \leq \frac{1}{T} \|u\|_{L^2((0, T); H)}^2 + 2 \int_0^T \|\dot{u}(t')\|_{V'} \|u(t')\|_V dt'.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} \int_0^T \|\dot{u}(t')\|_{V'} \|u(t')\|_V dt' &= \|\dot{u}\|_{V'} \|u\|_V \|L^1(0, T) \\ &\leq \|\dot{u}\|_{V'} \|L^2(0, T)\| \|u\|_V \|L^2(0, T)\| = \|\dot{u}\|_{L^2((0, T); V')} \|u\|_{L^2((0, T); V)} \end{aligned}$$

and with $H^1 \hookrightarrow L^2$ we obtain

$$\|u\|_{\mathcal{C}^0([0, T]; H)}^2 = \max_{t \in [0, T]} \|u(t)\|_H^2 \leq \frac{1}{T} \|u\|_{L^2((0, T); H)}^2 + 2 \|u\|_{H^1((0, T); V')} \|u\|_{L^2((0, T); V)}.$$

This estimate shows that $u \in L^2((0, T); V) \cap H^1((0, T); V')$ implies $u \in \mathcal{C}^0([0, T]; H)$ with continuous embedding (cf. Definition 2.8), which completes the proof. \square

Remark 2.17 (Aubin-Lions Lemma). Lemma 2.16 is similar to the following result on compact embeddings in $L^p((0, T); V)$ discussed in [CJL14]: If $V_1 \hookrightarrow V_2 \hookrightarrow V_3$ are Banach spaces such that $V_1 \hookrightarrow V_2$ is compact and $V_2 \hookrightarrow V_3$ is continuous (cf. Definition 2.8), then

$$\begin{aligned} L^p((0, T); V_1) \cap W^{1, q}((0, T); V_3) &\hookrightarrow L^p((0, T); V_2), \\ L^\infty((0, T); V_1) \cap W^{1, q}((0, T); V_3) &\hookrightarrow \mathcal{C}^0([0, T]; V_2) \end{aligned}$$

are both compact ($1 < p < \infty, 1 \leq q \leq \infty$). \lrcorner

Definition 2.18 (Weakly continuous functions $f: [0, T] \rightarrow X$). If X is a reflexive Banach space, then

$$\mathcal{C}_{\text{weak}}^0([0, T]; X) := \{f: [0, T] \rightarrow X : \forall x' \in X' : t \mapsto \langle x', f(t) \rangle_{X', X} \text{ is continuous on } [0, T]\}. \quad \lrcorner$$

Compared to (strongly) continuous functions

$$\mathcal{C}^0([0, T]; X) = \{f: [0, T] \rightarrow X : t \mapsto f(t) \in X \text{ is continuous on } [0, T]\},$$

weakly continuous functions are only continuous as scalar-valued functions $[0, T] \rightarrow \mathbb{R}$ obtained after taking the duality (the target space is equipped with the weak topology). Therefore, weak continuity is also referred to as “scalar continuity”, e.g. [DL92, Chapter XVIII, (5.129)]. Actually, in view of Remark 2.20 below, this name might be preferable.

From the estimate ($t_1, t_2 \in [0, T]$, $x' \in X'$)

$$|\langle x', f(t_1) \rangle_{X', X} - \langle x', f(t_2) \rangle_{X', X}| = |\langle x', f(t_1) - f(t_2) \rangle_{X', X}| \leq \|x'\|_{X'} \|f(t_1) - f(t_2)\|_X$$

it is clear that (strongly) continuous functions are weakly continuous (in the sense of Definition 2.18):

$$\mathcal{C}^0([0, T]; X) \subseteq \mathcal{C}_{\text{weak}}^0([0, T]; X).$$

Additional boundedness improves the continuity result (the statement is similar to Lemma 2.16, but with interchanged roles of space and time):

Lemma 2.19 (Weak continuity). *Let $V \hookrightarrow H$ be Hilbert spaces with V dense in H . Then*

$$\begin{aligned} \mathcal{C}^0([0, T]; H) \cap L^\infty((0, T); V) &\subseteq \mathcal{C}_{\text{weak}}^0([0, T]; V), \\ \mathcal{C}^0([0, T]; V') \cap L^\infty((0, T); H) &\subseteq \mathcal{C}_{\text{weak}}^0([0, T]; H). \end{aligned}$$

Proof. We follow [RR04, Section 11.2.4, p. 392]. Let $u \in \mathcal{C}^0([0, T]; H) \cap L^\infty((0, T); V)$ and $t \in (0, T)$. We first show $u(t) \in V$ for every $t \in (0, T)$, which will follow from the inequality

$$\|u(t)\|_V \leq \|u\|_{L^\infty((0, T); V)}$$

which we now prove (the nontrivial part is that the inequality holds for every t). By contradiction, suppose that there exists $t \in (0, T)$ with

$$\|u\|_{L^\infty((0, T); V)} < \|u(t)\|_V.$$

Since H is dense in V' there exists $h \in H$ with $\|h\|_{V'} \|u(t)\|_V = \langle h, u(t) \rangle$. Consequently, multiplication with $\|h\|_{V'}$ gives $\|h\|_{V'} \|u\|_{L^\infty((0, T); V)} < \langle h, u(t) \rangle$. By the assumption $u \in \mathcal{C}^0([0, T]; H)$, the inequality also holds for $s \in (0, T)$,

$$\|h\|_{V'} \|u\|_{L^\infty((0, T); V)} < \langle h, u(s) \rangle,$$

as long as $|s - t| < \varepsilon$ for some $\varepsilon > 0$ (note that it suffices to assume $u \in \mathcal{C}_{\text{weak}}^0([0, T]; H)$).

Then the function $g: (0, T) \rightarrow H$, $g(s) := \begin{cases} h, & |s - t| < \varepsilon \\ 0, & \text{otherwise} \end{cases}$ satisfies

$$\int_0^T \langle g(s), u(s) \rangle ds = \int_0^T \langle h, u(s) \rangle ds > \int_0^T \|h\|_{V'} \|u\|_{L^\infty((0, T); V)} ds = T \|h\|_{V'} \|u\|_{L^\infty((0, T); V)}.$$

But $\|g\|_{L^1((0, T); V')} = \int_0^T \|g(s)\|_{V'} ds = \int_0^T \|h\|_{V'} ds = T \|h\|_{V'}$. Thus we obtain

$$\int_0^T \langle g(s), u(s) \rangle ds > \|g\|_{L^1((0, T); V')} \|u\|_{L^\infty((0, T); V)}$$

which contradicts Hölder’s inequality. This shows that $u(t)$ is bounded in V for every $t \in (0, T)$.

2 Variational methods

To show weak continuity, let $f \in V$. By density, there exists a sequence $(f_n)_n$ in H such that

$$f_n \rightarrow f \quad \text{in } V' \quad (n \rightarrow \infty).$$

Then $\tilde{u}_n := \langle f_n, u(\cdot) \rangle$ converges uniformly to $\tilde{u} := \langle f, u(\cdot) \rangle$ on $[0, T]$, because

$$|\tilde{u}_n(t) - \tilde{u}(t)| = |\langle f_n, u(t) \rangle - \langle f, u(t) \rangle| = |\langle f_n - f, u(t) \rangle| \leq \|f_n - f\|_{V'} \|u(t)\|_V$$

holds for all $t \in [0, T]$. With $u \in \mathcal{C}^0([0, T]; H) \subseteq \mathcal{C}_{\text{weak}}^0([0, T]; H)$, the functions \tilde{u}_n are continuous on $[0, T]$. Hence uniform convergence of \tilde{u}_n implies continuity of the limit $\tilde{u} = \langle f, u(\cdot) \rangle$. This shows that u is weakly continuous, $u \in \mathcal{C}_{\text{weak}}^0([0, T]; V)$. The proof of the second statement is similar. \square

Remark 2.20 (Another concept of weak continuity for maps $F: X \rightarrow \mathbb{R}$). The notion of weak continuity according to Definition 2.18 must not be confused with the following weak continuity property for scalar-valued functions defined on a Banach space X [RR04, Definition 10.12, p. 347]: A map $F: X \rightarrow \mathbb{R}$ that satisfies

$$\lim_{n \rightarrow \infty} F(x_n) = F(x) \quad \text{whenever } x_n \rightharpoonup x \quad \text{in } X \quad (n \rightarrow \infty)$$

is called (sequentially) weakly continuous. Here the domain is equipped with the weak topology, which (in contrast to $\mathcal{C}_{\text{weak}}^0$) is more restrictive than (sequential, strong) continuity $F \in \mathcal{C}^0(X)$, which is formulated via strong convergence $x_n \rightarrow x$ (see also Definition 2.21). \lrcorner

Definition 2.21 (Lower semicontinuity, weak lower semicontinuity). Let X be a Banach space and $F: X \rightarrow \mathbb{R}$. If

$$\liminf_{n \rightarrow \infty} F(x_n) \geq F(x) \quad \text{whenever } x_n \rightarrow x \quad \text{in } X \quad (n \rightarrow \infty), \quad (2.11)$$

then F is called (sequentially) lower semicontinuous (**l.s.c.**).

If

$$\liminf_{n \rightarrow \infty} F(x_n) \geq F(x) \quad \text{whenever } x_n \rightharpoonup x \quad \text{in } X \quad (n \rightarrow \infty), \quad (2.12)$$

then F is called (sequentially) weakly lower semicontinuous (**w.l.s.c.**). \lrcorner

Lemma 2.22 (Composition with $\mathcal{C}_{\text{weak}}^0$). Let X be a Banach space.

(i) If $F: X \rightarrow \mathbb{R}$ is (strongly) continuous and convex, then F is w.l.s.c.

(ii) The norm $\|\cdot\|_X: X \rightarrow \mathbb{R}$ is w.l.s.c.

(iii) If $g \in \mathcal{C}_{\text{weak}}^0([0, T]; X)$ and $f: X \rightarrow \mathbb{R}$ is w.l.s.c, then $f \circ g: [0, T] \rightarrow \mathbb{R}$ is l.s.c.

In particular, if $g \in \mathcal{C}_{\text{weak}}^0([0, T]; X)$ then $t \mapsto \|g(t)\|_X$ is l.s.c. on $[0, T]$.

Proof. Claim (i) is established in [Bre11, Section 3.3, Remark 6, p. 61]. Claim (ii) then follows from (i) because of continuity and convexity of the norm: Let $x, y \in X$, $\lambda \in [0, 1]$, and consider $F = \|\cdot\|_X: X \rightarrow \mathbb{R}$. Then $|F(x) - F(y)| = |\|x\|_X - \|y\|_X| \leq \|x - y\|_X \rightarrow 0$ as $x \rightarrow y$; moreover

$$F(\lambda x + (1 - \lambda)y) = \|\lambda x + (1 - \lambda)y\|_X \leq \lambda \|x\|_X + (1 - \lambda)\|y\|_X = \lambda F(x) + (1 - \lambda)F(y).$$

We prove Claim (iii): If $t_n \rightarrow t$ ($n \rightarrow \infty$) in $[0, T]$, then $w_n := g(t_n) \rightharpoonup g(t) =: w$ in X from Definition 2.18. But since f is w.l.s.c, we obtain from (2.12) that

$$\liminf_{n \rightarrow \infty} (f \circ g)(t_n) = \liminf_{n \rightarrow \infty} f(g(t_n)) = \liminf_{n \rightarrow \infty} f(w_n) \geq f(w) = f(g(t)) = (f \circ g)(t). \quad \square$$

All these concepts of continuity will be relevant to establish the regularity of weak solutions.

Chapter 3

Variational solution of linear second-order evolution equations

We discuss the solution of linear evolution problems of second order with the variational approach, including detailed proofs. In view of the elastic wave equation, our focus lies on the solution of hyperbolic problems. In Section 3.1 the weak formulation of the evolution equation is introduced in the abstract Hilbert space setting and the solvability result is announced. Then we discuss a priori estimates based on the energy equality (Section 3.2). Next, uniqueness and existence of weak solutions are established in Sections 3.3 and 3.4 respectively. We present two different proofs for each of the statements. We conclude with the proof of continuity of weak solutions (Section 3.5).

The presentation closely follows [DL92, Chapter XVIII] and [RR04, Chapter 11], but we restrict to time-independent coefficients and real-valued functions.

3.1 The abstract Cauchy problem

3.1.1 Definition of the weak evolution equation

Within the context of linear evolutionary PDEs, the most natural Hilbert space framework for the dynamical variable u is $L^2((0, T); H)$ with a Hilbert space H and for finite time $0 < T < \infty$. At each time instant t , some sort of energy of the solution is required to be bounded, which is modeled by the condition $u(t) \in V$ for another Hilbert space $V \subseteq H$.

In this Hilbert space framework for space and time, the **Cauchy problem** is formulated as the following **abstract variational problem**:

Find $t \mapsto u(t) \in V$ that solves the **weak evolution equation**

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T) \quad (3.1)$$

and satisfies the initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1. \quad (3.2)$$

Here a, b, c are bilinear forms on V or on H , $\dot{u} = \frac{du}{dt}$ denotes the time derivative of u , and f represents the source.

We specify the assumptions on the **Hilbert spaces**: It is assumed that H and V are separable real Hilbert spaces where $V \subseteq H$ is a dense subspace with continuous embedding, that is,

3 Variational solution of linear second-order evolution equations

$V \hookrightarrow H$. Let V' denote the dual space of V . The Hilbert space H is identified with its dual H' . Therefore we obtain the following sequence of continuous and dense embeddings, a so-called **variational triple** with pivot space H :

$$V \hookrightarrow H \hookrightarrow V'.$$

We write $\langle \cdot, \cdot \rangle_H$ for the inner product in H and $\|\cdot\|_H$ for the associated norm; $\|\cdot\|_V$ and $\|\cdot\|_{V'}$ are the norms in V and V' respectively. The duality between V and V' is denoted by

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{V', V}.$$

The continuity of the embeddings implies the existence of $c_V, c'_V > 0$ such that

$$\frac{1}{c'_V} \|v\|_{V'} \leq \|v\|_H \leq c_V \|v\|_V \quad \forall v \in V. \quad (3.3)$$

Moreover, the duality $\langle \cdot, \cdot \rangle$ between V and V' is a continuous extension of the duality of $H' = H$ and H , that is,

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H \quad \text{on } H \times V. \quad (3.4)$$

Indeed, the inner product on H may be identified with the duality bracket and explicitly introducing the embedding $\iota: V \rightarrow H$ and its adjoint $\iota': H' = H \rightarrow V'$, where ι and ι' are given by $\iota v = v$ and $\iota' u = u$ for $v \in V$ and $u \in H$ respectively, we obtain

$$\langle u, v \rangle = \langle u, v \rangle_{V', V} = \langle \iota' u, v \rangle_{V', V} = \langle u, \iota v \rangle_{H', H} = \langle u, v \rangle_{H', H} = \langle u | v \rangle_H \quad \forall u \in H, v \in V.$$

Next we list the assumptions on the **bilinear forms**: We assume that a is a continuous bilinear form on V ,

$$a: V \times V \rightarrow \mathbb{R}, \quad a = a_0 + a_1. \quad (3.5)$$

By continuity there exists $c_a > 0$ such that

$$|a(u, v)| \leq c_a \|u\|_V \|v\|_V \quad \forall u, v \in V.$$

The principal part a_0 is symmetric and V -coercive with respect to H : There exist $\alpha, \lambda > 0$ such that

$$a_0(u, v) = a_0(v, u) \quad \text{and} \quad a_0(u, u) \geq \alpha \|u\|_V^2 - \lambda \|u\|_H^2 \quad \forall u, v \in V. \quad (3.6)$$

The bilinear form a_1 possesses additional regularity: There exist $c_{a_1} > 0$ and $c'_{a_1} > 0$ such that

$$|a_1(u, v)| \leq c_{a_1} \|u\|_V \|v\|_H \quad \forall u, v \in V \quad (3.7)$$

and

$$|a_1(u, v)| \leq c'_{a_1} \|u\|_H \|v\|_V \quad \forall u \in H, v \in V. \quad (3.8)$$

The bilinear forms b, c are continuous on H ,

$$b, c: H \times H \rightarrow \mathbb{R}. \quad (3.9)$$

By continuity there exist $c_b > 0$ and $c_c > 0$ such that

$$|b(w, v)| \leq c_b \|w\|_H \|v\|_H \quad \text{and} \quad |c(w, v)| \leq c_c \|w\|_H \|v\|_H \quad \forall w, v \in H.$$

In addition c is symmetric and H -coercive, i.e. there exists $\gamma > 0$ such that

$$c(w, v) = c(v, w) \quad \text{and} \quad c(w, w) \geq \gamma \|w\|_H^2 \quad \forall w, v \in H. \quad (3.10)$$

The assumptions on a, b, c aim at the modeling of hyperbolic equations. In the parabolic case, b is only defined on V and possesses a coercive principal part, which describes viscous damping (see Section 3.4.2).

3.1.2 The existence and uniqueness result

With the hypotheses on V , H , V' and a , b , c given above, we are ready to state the main result for the abstract Cauchy problem (3.1), (3.2):

Theorem 3.1 (Existence and uniqueness for the Cauchy problem). *Given the data*

$$u^0 \in V, \quad u^1 \in H, \quad \text{and} \quad f \in L^2((0, T); H), \quad (3.11)$$

there exists a unique solution

$$u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H) \quad (3.12)$$

of the weak evolution equation (3.1),

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T),$$

satisfying the initial conditions (3.2), that is, $u(0) = u^0$ and $\dot{u}(0) = u^1$.

This theorem coincides with the results of [DL92, Chapter XVIII, §5, Problem (P_2), p. 570; Theorems 3 and 4], restricting to time-independent operators and real-valued functions.

Let us check whether the weak formulation (3.1) makes sense as an equation in $\mathcal{D}'(0, T)$, if u has the asserted regularity (3.12): The validity of (3.1) as an equation for all $v \in V$ in the sense of $\mathcal{D}'(0, T)$ means that for all $\psi \in \mathcal{D}(0, T)$,

$$\left\langle \frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v), \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} = \left\langle \langle f, v \rangle, \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)},$$

which with the definition of the distributional derivative reads

$$- \left\langle c(\dot{u}, v), \dot{\psi} \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} + \left\langle b(\dot{u}, v) + a(u, v), \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} = \left\langle \langle f, v \rangle, \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)}.$$

Consequently, with (3.12), that is, $u \in \mathcal{C}^0([0, T]; V)$ and $\dot{u} \in \mathcal{C}^0([0, T]; H)$, all terms are defined.

We note that the equations above actually also make sense if we only assume $f \in L^2((0, T); V')$, $u \in L^2((0, T); V)$, and $\dot{u} \in L^2((0, T); H)$ (see also Lemma 3.3). Thus we call any function

$$u \in L^2((0, T); V) \cap H^1((0, T); H) \quad \text{or} \quad u \in L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H) \quad (3.13)$$

(which is a subset), that satisfies the weak evolution equation (3.1), a **weak solution**. However without any further assumption, the weak regularity (3.13) is not enough to make the initial conditions defined. By Sobolev embedding we only obtain

$$u \in H^1((0, T); H) \hookrightarrow \mathcal{C}^0([0, T]; H). \quad (3.14)$$

Further regularity will follow from the validity of the evolution equation, which eventually yields (3.12), see Section 3.5.

The principal part (with respect to time) in the abstract formulation (3.1) typically corresponds to the **acceleration term** (more precisely, inertia) in wave equations. In our setting c is time-independent. Therefore we can write $\frac{d}{dt}c(\dot{u}, \cdot) = \frac{d^2}{dt^2}c(u, \cdot)$. Yet, pulling the derivative inside the bilinear form, $\frac{d}{dt}c(\dot{u}, \cdot) = c(\ddot{u}, \cdot)$ requires more regularity, e.g. $H^2((0, T); H)$ (cf. Lemma 3.2 and Remarks 3.4, 3.5).

3.1.3 Strong form

The continuous bilinear forms a, b, c on V, H correspond to continuous linear operators A, B, C acting between V, H, V' . If X, Y are Banach spaces then

$$\langle Fx, y \rangle_{Y', Y} = f(x, y) \quad \forall x \in X, y \in Y$$

allows us to identify a continuous bilinear form $f: X \times Y \rightarrow \mathbb{R}$ with an operator $F \in \text{Lin}(X, Y')$. Specifically, $a = a_0 + a_1$ corresponds to the operator

$$A = A_0 + A_1 \in \text{Lin}(V, V')$$

defined by

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in V.$$

The additional regularity conditions (3.7) and (3.8) give

$$A_1 \in \text{Lin}(V, H) \cap \text{Lin}(H, V'). \quad (3.15)$$

Similarly, setting

$$\langle Bw|v \rangle_H = b(w, v) \quad \text{and} \quad \langle Cw|v \rangle_H = c(w, v) \quad \forall w, v \in H$$

yields the following operators corresponding to b and c :

$$B, C \in \text{Lin}(H, H).$$

Coercivity of C on H , that is $\langle Cw|w \rangle_H \geq \gamma \|w\|_H^2$, implies $\ker(C) = \{0\}$ and hence invertibility:

$$C^{-1} \in \text{Lin}(H, H).$$

In terms of the operators A, B, C the weak evolution equation (3.1),

$$\frac{d}{dt} c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle,$$

reads

$$\frac{d}{dt} \langle C\dot{u}|v \rangle_H + \langle B\dot{u}|v \rangle_H + \langle Au, v \rangle = \langle f, v \rangle,$$

or by (3.4),

$$\frac{d}{dt} \langle C\dot{u}, v \rangle + \langle B\dot{u}, v \rangle + \langle Au, v \rangle = \langle f, v \rangle.$$

A slightly better regularity than (3.13) is enough to pull the time derivative inside the first duality:

Lemma 3.2 (Acceleration term). *If $u \in L^2((0, T); V) \cap H^1((0, T); H)$ and $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$ then for all $v \in V$*

$$\frac{d}{dt} c(\dot{u}, v) = \left\langle \frac{d}{dt}(C\dot{u}), v \right\rangle \quad \text{in } \mathcal{D}'(0, T). \quad (3.16)$$

Formula (3.16) is [DL92, p. 559, Prop. 3], where it is formulated for $\dot{u} \in L^2((0, T); V)$ (but C time-dependent); here we establish the assertion in the case $\dot{u} \in L^2((0, T); H)$ by adapting the proof of [DL92, p. 477, Prop. 7].

Proof. By the assumptions $m := C\dot{u} \in L^2((0, T); H)$ with $\dot{m} = \frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$. Let $\psi \in \mathcal{D}(0, T)$. The duality $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V', V}$ satisfies the product rule:

$$\frac{d}{dt} \langle m(t), \psi(t)v \rangle = \langle \dot{m}(t), \psi(t)v \rangle + \langle m(t), \dot{\psi}(t)v \rangle.$$

Taking into account the conditions $\psi(0) = \psi(T) = 0$, integration yields

$$\int_0^T \langle \dot{m}, \psi v \rangle dt + \int_0^T \langle m, \dot{\psi} v \rangle dt = \langle m, \psi v \rangle \Big|_0^T = 0$$

which gives the integration by parts formula

$$\int_0^T \langle \dot{m}, \psi v \rangle dt = - \int_0^T \langle m, \dot{\psi} v \rangle dt.$$

The regularity of m guarantees that $\langle \dot{m}, v \rangle$ and $\langle m, v \rangle$ are elements of $L^1_{\text{loc}}(0, T)$ and thus

$$\begin{aligned} \langle \langle \dot{m}, v \rangle, \psi \rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} &= \int_0^T \langle \dot{m}, v \rangle \psi dt = \int_0^T \langle \dot{m}, \psi v \rangle dt = - \int_0^T \langle m, \dot{\psi} v \rangle dt = - \int_0^T \langle m, v \rangle \dot{\psi} dt \\ &= - \langle \langle m, v \rangle, \dot{\psi} \rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} = \left\langle \frac{d}{dt} \langle m, v \rangle, \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)}, \end{aligned}$$

where the last equality holds by definition of the derivative in $\mathcal{D}'(0, T)$. Consequently

$$\frac{d}{dt} c(\dot{u}, v) = \frac{d}{dt} \langle C\dot{u}, v \rangle = \frac{d}{dt} \langle m, v \rangle = \langle \dot{m}, v \rangle = \left\langle \frac{d}{dt} (C\dot{u}), v \right\rangle \quad \text{in } \mathcal{D}'(0, T),$$

which is (3.16). □

Lemma 3.2 enables us to write the evolution equation (3.1) in **(distributional) operator form**:

$$\left\langle \frac{d}{dt} (C\dot{u}) + B\dot{u} + Au, v \right\rangle = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T),$$

that is,

$$\frac{d}{dt} (C\dot{u}) + B\dot{u} + Au = f \quad \text{in } \mathcal{D}'((0, T); V').$$

Next we show that the additional regularity $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$ required in the Lemma already follows from the validity of the weak evolution equation alone. Moreover, the operator form holds in the strong L^2 sense:

Lemma 3.3 (Strong form of the evolution equation). *Solutions $u \in L^2((0, T); V) \cap H^1((0, T); H)$ of the weak evolution equation (3.1) satisfy*

$$\frac{d}{dt} (C\dot{u}) + B\dot{u} + Au = f \quad \text{a.e. in } L^2((0, T); V'). \quad (3.17)$$

Proof. Let $u \in L^2((0, T); V) \cap H^1((0, T); H)$ and $v \in V$. Then, by the assumptions,

$$C\dot{u} \in L^2((0, T); H) \subseteq L^2((0, T); V') \subseteq \mathcal{D}'((0, T); V')$$

On one hand, the second inclusion shows $\langle C\dot{u}, v \rangle \in \mathcal{D}'(0, T)$ and by construction of C we obtain $\frac{d}{dt} \langle C\dot{u}, v \rangle = \frac{d}{dt} c(\dot{u}, v) \in \mathcal{D}'(0, T)$. By the first inclusion $\langle C\dot{u}, v \rangle \in L^2(0, T)$ and therefore

3 Variational solution of linear second-order evolution equations

we get with $\psi \in \mathcal{D}(0, T)$,

$$\begin{aligned}
 \left\langle \frac{d}{dt}(C\dot{u}, v), \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} &= - \left\langle \underbrace{\langle C\dot{u}, v \rangle}_{\in L^2(0, T)}, \dot{\psi} \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} \\
 &= - \int_0^T \langle C\dot{u}, v \rangle \dot{\psi} \, dt = - \int_0^T \langle C\dot{u}, \dot{\psi} v \rangle \, dt = - \int_0^T \langle C\dot{u}, \frac{d}{dt}(\psi v) \rangle \, dt \\
 &= - \left\langle C\dot{u}, \frac{d}{dt}(\psi v) \right\rangle_{\mathcal{D}'((0, T); V'), \mathcal{D}((0, T); V)} \\
 &= \left\langle \frac{d}{dt}(C\dot{u}), \psi v \right\rangle_{\mathcal{D}'((0, T); V'), \mathcal{D}((0, T); V)}.
 \end{aligned}$$

On the other hand, the weak evolution equation (3.1) in terms of A, B, C yields

$$\begin{aligned}
 \left\langle \frac{d}{dt}(C\dot{u}, v), \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} &= \left\langle \underbrace{\langle f - B\dot{u} - Au, v \rangle}_{\in L^2(0, T) \subseteq L^1_{\text{loc}}(0, T)}, \psi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} \\
 &= \int_0^T \langle f - B\dot{u} - Au, v \rangle \psi \, dt \\
 &= \left\langle f - B\dot{u} - Au, \psi v \right\rangle_{\mathcal{D}'((0, T); V'), \mathcal{D}((0, T); V)}.
 \end{aligned}$$

Consequently, by the density of tensor products of test functions

$$\mathcal{D}(0, T) \otimes V := \{(t, x) \mapsto \psi(t)v(x) : \psi \in \mathcal{D}(0, T), v \in V\}$$

in $\mathcal{D}((0, T); V)$, we thus have derived the following representation of (3.1): For all $\varphi \in \mathcal{D}((0, T); V)$,

$$\left\langle \frac{d}{dt}(C\dot{u}) + B\dot{u} + Au, \varphi \right\rangle_{\mathcal{D}'((0, T); V'), \mathcal{D}((0, T); V)} = \left\langle f, \varphi \right\rangle_{\mathcal{D}'((0, T); V'), \mathcal{D}((0, T); V)},$$

that is, $\frac{d}{dt}(C\dot{u}) + B\dot{u} + Au = f$ in $\mathcal{D}'((0, T); V')$. Therefore, by the assumptions on u, A, B, f ,

$$\frac{d}{dt}(C\dot{u}) = f - B\dot{u} - Au \in L^2((0, T); V'),$$

which completes the proof. \square

Consequently, if u is a weak solution of (3.1), then $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$ holds by Lemma 3.3 and then Lemma 3.2 yields (3.16), that is, $\frac{d}{dt}c(\dot{u}, v) = \langle \frac{d}{dt}(C\dot{u}), v \rangle$.

Remark 3.4 (Strong and classical regularity). If $u \in H^2((0, T); V)$, which sometimes is called strong regularity, then (3.17) reads

$$C\ddot{u} + B\dot{u} + Au = f \quad \text{a.e. in } L^2((0, T); V'). \quad (3.18)$$

With classical regularity conditions, $u \in \mathcal{C}^2([0, T]; V)$ and $f \in \mathcal{C}^0([0, T]; V')$, Equation (3.18) holds in $\mathcal{C}^0([0, T]; V')$, i.e. everywhere on $[0, T]$. \lrcorner

Remark 3.5 (Better properties of the acceleration operator).

- (i) Assuming the higher regularity $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); H)$ compared to Lemma 3.2, (time-independence and) invertibility of C on H yields, with $u \in L^2((0, T); V) \cap H^1((0, T); H)$,

$$\frac{d}{dt}(C\dot{u}) = C\ddot{u} \in L^2((0, T); H) \implies \ddot{u} \in L^2((0, T); H),$$

and thus $u \in H^2((0, T); H)$. Then (3.16) gives

$$\frac{d}{dt}c(\dot{u}, v) = \left\langle \frac{d}{dt}(C\dot{u}), v \right\rangle = \left\langle \frac{d}{dt}(C\dot{u})|v \right\rangle_H = \langle C\ddot{u}|v \rangle_H = c(\ddot{u}, v) \quad \text{in } \mathcal{D}'(0, T).$$

Thereby the weak evolution equation (3.1) reads

$$c(\ddot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{in } \mathcal{D}'(0, T). \quad (3.19)$$

- (ii) However, as $A \in \text{Lin}(V, V')$, the condition $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); H)$ does not follow from the weak evolution equation (3.1) but we only get $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$, see Lemma 3.3. Yet, with C and $C^{-1} \in \text{Lin}(H, H)$, the implication

$$\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V') \implies \ddot{u} \in L^2((0, T); V')$$

generally does not hold. Its validity requires the existence of an extension of C to $\text{Lin}(V', V')$ which is invertible. By transposition (due to symmetry of c), this amounts to the hypothesis that

$$C \in \text{Lin}(V, V)$$

and is invertible. In this case the representation (3.19) is obtained as above. Moreover we may apply C^{-1} to the strong evolution equation (3.17) to get $\ddot{u} + C^{-1}B\dot{u} + C^{-1}Au = C^{-1}f$, that is

$$\ddot{u} + \tilde{B}\dot{u} + \tilde{A}u = \tilde{f} \quad \text{a.e. in } L^2((0, T); V') \quad (3.20)$$

where $\tilde{A} := C^{-1}A \in \text{Lin}(V, V')$, $\tilde{B} := C^{-1}B \in \text{Lin}(H, V')$, and $\tilde{f} := C^{-1}f \in L^2((0, T); V')$.

- (iii) In typical applications the variational triple $V \hookrightarrow H \hookrightarrow V'$ consists of L^2 -based Sobolev spaces. Within this framework operators $C \in \text{Lin}(H, H)$ that are elliptic pseudodifferential operators of order zero (e.g. nonzero multiplication or convolution operators) possess the desired properties. Thus the representations (3.19) and (3.20) are defined in this case. \lrcorner

Remark 3.6 (First-order system). If u is a weak solution and one defines

$$U := \begin{pmatrix} u \\ C\dot{u} \end{pmatrix}, \quad M := \begin{pmatrix} 0 & C^{-1} \\ -A & -BC^{-1} \end{pmatrix}, \quad F := \begin{pmatrix} 0 \\ f \end{pmatrix}$$

then $\frac{d}{dt}(C\dot{u}) + B\dot{u} + Au = f$ (3.17) can be written as the following system, [DL92, p. 559]:

$$\frac{d}{dt}U = MU + F \quad \text{a.e. in } L^2((0, T); H) \times L^2((0, T); V').$$

Indeed, $MU + F = \begin{pmatrix} C^{-1}C\dot{u} \\ -Au - BC^{-1}C\dot{u} \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} = \begin{pmatrix} \dot{u} \\ -Au - B\dot{u} + f \end{pmatrix} = \begin{pmatrix} \dot{u} \\ \frac{d}{dt}(C\dot{u}) \end{pmatrix} = \frac{d}{dt}U.$

Similarly, the first-order version of $\ddot{u} + \tilde{B}\dot{u} + \tilde{A}u = \tilde{f}$ (3.20) is

$$\frac{d}{dt}\tilde{U} = \tilde{M}\tilde{U} + \tilde{F} \quad \text{a.e. in } L^2((0, T); V')^2$$

with $\tilde{U} := \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$, $\tilde{M} := \begin{pmatrix} 0 & \text{Id}_H \\ -\tilde{A} & -\tilde{B} \end{pmatrix}$, and $\tilde{F} := \begin{pmatrix} 0 \\ \tilde{f} \end{pmatrix}$. \lrcorner

3.2 A priori estimates

A priori estimates are inequalities that a solution of an equation must necessarily satisfy, if it is assumed to exist. In case of the weak evolution equation (3.1), these estimates will be obtained as a consequence of the energy equality by employing the coercivity and continuity conditions.

3.2.1 Energy equality and integration by parts

The derivation of the energy equality relies on the generalization of the product rule (Leibniz rule) and integration by parts formula to continuous bilinear forms [DL92, Prop. 1 & 2, p. 558]:

Lemma 3.7 (Integration by parts for bilinear forms). *If $u, \varphi \in H^1((0, T); V)$, then*

$$\int_0^T \langle u, \dot{\varphi} \rangle dt = - \int_0^T \langle \dot{u}, \varphi \rangle dt + \langle u, \varphi \rangle \Big|_0^T \quad (3.21)$$

and

$$\int_0^T a_0(u, \dot{\varphi}) dt = - \int_0^T a_0(\dot{u}, \varphi) dt + a_0(u, \varphi) \Big|_0^T. \quad (3.22)$$

In particular,

$$\int_0^T a_0(u, \dot{u}) dt = \frac{1}{2} a_0(u, u) \Big|_0^T, \quad (3.23)$$

that is $a_0(u, \dot{u}) = \frac{d}{dt} \frac{1}{2} a_0(u, u)$.

If in addition $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$, then

$$\int_0^T \langle \frac{d}{dt}(C\dot{u}), \varphi \rangle dt = - \int_0^T c(\dot{u}, \dot{\varphi}) dt + c(\dot{u}, \varphi) \Big|_0^T. \quad (3.24)$$

In particular,

$$\int_0^T \langle \frac{d}{dt}(C\dot{u}), \dot{u} \rangle dt = \frac{1}{2} c(\dot{u}, \dot{u}) \Big|_0^T, \quad (3.25)$$

that is $\langle \frac{d}{dt}(C\dot{u}), \dot{u} \rangle = \frac{d}{dt} \frac{1}{2} c(\dot{u}, \dot{u})$.

Proof. The higher regularity $\dot{u}(t) \in V$ (instead of only $\dot{u}(t) \in H$) allows us to apply the product rule to the bilinear forms $\langle \cdot, \cdot \rangle_H = \langle \cdot, \cdot \rangle$, a_0 , and c : If $u, \varphi \in \mathcal{D}([0, T]; V)$, then

$$\frac{d}{dt} \langle u, \varphi \rangle = \langle u, \dot{\varphi} \rangle + \langle \dot{u}, \varphi \rangle \implies \int_0^T (\langle u, \dot{\varphi} \rangle + \langle \dot{u}, \varphi \rangle) dt = \langle u, \varphi \rangle \Big|_0^T$$

which results in (3.21). Moreover

$$\frac{d}{dt} a_0(u, \varphi) = a_0(u, \dot{\varphi}) + a_0(\dot{u}, \varphi) \implies \int_0^T (a_0(u, \dot{\varphi}) + a_0(\dot{u}, \varphi)) dt = a_0(u, \varphi) \Big|_0^T$$

which gives (3.22). Similarly,

$$\frac{d}{dt} c(\dot{u}, \varphi) = c(\ddot{u}, \varphi) + c(\dot{u}, \dot{\varphi}) \implies \int_0^T (c(\ddot{u}, \varphi) + c(\dot{u}, \dot{\varphi})) dt = c(\dot{u}, \varphi) \Big|_0^T$$

and (3.24) holds since

$$\langle \frac{d}{dt}(C\dot{u}), \cdot \rangle = c(\ddot{u}, \cdot)$$

for $u \in \mathcal{D}([0, T]; V)$. The identities (3.23) and (3.25) follow by symmetry and upon setting $u = \varphi$. Density of $\mathcal{D}([0, T]; V)$ in $H^1((0, T); V)$ concludes the proof. \square

By density of V in H , (3.21) and (3.24) also hold for $u \in H^1((0, T); H)$ and $\varphi \in H^1((0, T); V)$. Moreover, we note that if $u \in H^2((0, T); H)$ then (3.25) corresponds to

$$\left\langle \frac{d}{dt}(C\dot{u}), \dot{u} \right\rangle = \langle C\ddot{u} | \dot{u} \rangle_H = c(\ddot{u}, \dot{u}) = \frac{d}{dt} \frac{1}{2} c(\dot{u}, \dot{u}). \quad (3.26)$$

We are ready to discuss the following important necessary condition for a weak solution u of the abstract Cauchy problem [DL92, Lemma 7, p. 578]:

Proposition 3.8 (Energy equality). *Let $u \in L^\infty((0, T); V) \cap W^{1,\infty}((0, T); H)$ satisfy (3.1) for the data $u(0) = u^0 \in H$, $\dot{u}(0) = u^1 \in V$, and $f \in L^2((0, T); H)$. Then, for $t \in [0, T]$,*

$$\begin{aligned} & \frac{1}{2} (c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t))) + \int_0^t (b(\dot{u}(t'), \dot{u}(t')) + a_1(u(t'), \dot{u}(t'))) dt' \\ &= \frac{1}{2} (c(u^1, u^1) + a_0(u^0, u^0)) + \int_0^t \langle f(t') | \dot{u}(t') \rangle_H dt'. \end{aligned} \quad (3.27)$$

The pointwise validity on $[0, T]$ follows from property (3.12), $u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$.

Equation (3.27) is called energy equality, because in typical applications the function

$$E(u): [0, T] \rightarrow \mathbb{R}, \quad E(u)(t) := \frac{1}{2} (c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t))) \quad (3.28)$$

can be interpreted as the **stored energy** of the solution u as a function of time. In particular, the energy splits into the **kinetic energy** $\frac{1}{2} c(\dot{u}, \dot{u})$ plus the **potential energy** $\frac{1}{2} a_0(u, u)$.

In terms of E , (3.27) expresses **energy balance**:

$$E(u)(t) + \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) dt' = E(u)(0) + \int_0^t \langle f | \dot{u} \rangle_H dt'$$

or

$$\frac{d}{dt} E(u) + (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) = \langle f | \dot{u} \rangle_H,$$

where $b(\dot{u}, \dot{u}) + a_1(u, \dot{u})$ corresponds to the **dissipation rate** and $\langle f | \dot{u} \rangle_H$ is the **external power**.

We present a heuristic argument to prove the energy equality: First we write (3.1) via $\frac{d}{dt} c(\dot{u}, v) = \left\langle \frac{d}{dt}(C\dot{u}), v \right\rangle$ from (3.16):

$$\left\langle \frac{d}{dt}(C\dot{u}), v \right\rangle + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T).$$

Evaluation at a fixed time $t' \in (0, T]$ for the test function $v = \dot{u}(t')$ and applying integration by parts (formulas (3.25) and (3.23) of Lemma 3.7) yields

$$\begin{aligned} & \left\langle \frac{d}{dt}(C\dot{u}), \dot{u} \right\rangle + b(\dot{u}, \dot{u}) + a(u, \dot{u}) = \langle f, \dot{u} \rangle \\ \implies & \frac{1}{2} c(\dot{u}, \dot{u}) \Big|_0^t + \int_0^t (b(\dot{u}, \dot{u}) + a(u, \dot{u})) dt' = \int_0^t \langle f, \dot{u} \rangle dt' \\ \implies & \frac{1}{2} (c(\dot{u}, \dot{u}) + a_0(u, u)) \Big|_0^t + \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) dt' = \int_0^t \langle f, \dot{u} \rangle dt'. \end{aligned}$$

This coincides with (3.27), because by (3.4) $\langle f, \dot{u} \rangle = \langle f | \dot{u} \rangle_H$ if $f \in H$ and $\dot{u} \in V$. The result may alternatively be derived by taking \dot{u} in duality with the strong form (3.17):

$$\left\langle \frac{d}{dt}(C\dot{u}) + B\dot{u} + Au, \dot{u} \right\rangle_{L^2((0, T); V'), L^2((0, T); V)} = \left\langle f, \dot{u} \right\rangle_{L^2((0, T); V'), L^2((0, T); V)}.$$

3 Variational solution of linear second-order evolution equations

However, the calculations above are only formal since Theorem 3.1 only gives $\dot{u} \in \mathcal{C}^0([0, T]; H)$. Hence $v = \dot{u}(t') \in H$ for $t' \in (0, T)$, which might not be a valid test function $v \in V$. The problem is that $\dot{u} \in L^2((0, T); H)$ is not in duality with the equation, which only holds in $L^2((0, T); V')$. A rigorous proof of Proposition 3.8 based on double regularization techniques is provided in [LM72, Chapter 3, Section 8.4, Lemma 8.3, p. 276].

Nevertheless, what has been established by the calculation is the following result under the higher regularity $\dot{u} \in L^2((0, T); V)$:

Lemma 3.9 (Energy equality for higher regularity). *Let $u \in H^1((0, T); V)$ satisfy (3.1) for the data $u(0) = u^0 \in H$, $\dot{u}(0) = u^1 \in V$, and $f \in L^2((0, T); V')$. Then the energy equality (3.27) holds for a.a. $t \in (0, T)$.*

As will be seen in Section 3.4.2, Lemma 3.9 proves the energy equality for parabolic problems.

3.2.2 Energy estimate

The next statement is a key ingredient to obtain a priori estimates:

Lemma 3.10 (Gronwall's inequality). *Let $\phi \in L^\infty(0, T)$ and $\mu \in L^1(0, T)$ be such that $\phi, \mu \geq 0$ a.e. in $(0, T)$. Let $K \in \mathbb{R}$ and $t \in (0, T)$. If*

$$\phi(t) \leq K + \int_0^t \mu(t') \phi(t') dt',$$

then

$$\phi(t) \leq K e^{\int_0^t \mu(t') dt'}.$$

In particular, if $K = 0$, we deduce $\phi = 0$ a.e. in $(0, T)$.

Proof. We follow [DL92, Chapter XVIII, §5, Lemma 1, p. 559]. The assumptions $\phi \in L^\infty$ and $\mu \in L^1$ imply $\phi\mu \in L^1$. Consequently, the function $F: [0, T] \rightarrow \mathbb{R}$,

$$F(t) := K + \int_0^t \mu(t') \phi(t') dt'$$

is absolutely continuous, $F \in \mathcal{C}_{\text{abs}}$, with derivative $F' = \phi\mu$ a.e. in $(0, T)$. By the positivity assumptions on ϕ and μ we deduce

$$\phi \leq F \implies F' = \phi\mu \leq F\mu \implies \frac{F'}{F} \leq \mu.$$

Integration gives

$$\ln \left(\frac{F(t)}{F(0)} \right) \leq \int_0^t \mu(t') dt' \implies F(t) \leq F(0) e^{\int_0^t \mu(t') dt'}$$

for a.a. $t \in (0, T)$, which with $F(0) = K$ completes the proof. \square

The main result of this section is the a priori energy estimate. Specifically, it bounds the norm of a weak solution $u \in L^\infty((0, T); V) \cap W^{1,\infty}((0, T); H)$ in terms of the coefficients and data.

Proposition 3.11 (Energy estimate). *Let u be a solution of (3.1) for the data $u(0) = u^0 \in H$, $\dot{u}(0) = u^1 \in V$, and $f \in L^2((0, T); H)$. Then, for $t \in [0, T]$,*

$$\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \leq k_1 e^{k_2 t} \tag{3.29}$$

with

$$k_1 := \frac{(c_a + 2\lambda c_V^2)\|u^0\|_V^2 + c_c\|u^1\|_H^2 + \|f\|_{L^2((0,T);H)}^2}{\min(\alpha, \gamma)} \quad (3.30)$$

and

$$k_2 := \frac{1 + c_{a_1} + 2c_b + 2\lambda T}{\min(\alpha, \gamma)}. \quad (3.31)$$

In particular, equivalence of norms in \mathbb{R}^2 implies that there exists $k \geq 0$ such that for all $t \in [0, T]$

$$\|u(t)\|_V + \|\dot{u}(t)\|_H \leq k(\|u_0\|_V + \|u_1\|_H + \|f\|_{L^2((0,T);H)}). \quad (3.32)$$

The constant k is independent of the solution u or the data u^0, u^1, f .

Proof. Our starting point is twice the energy equality (3.27):

$$\begin{aligned} c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t)) &= c(u^1, u^1) + a_0(u^0, u^0) \\ &\quad + 2 \int_0^t \langle f | \dot{u} \rangle_H dt' - 2 \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) dt'. \end{aligned}$$

The left-hand side is bounded from below thanks to coercivity of c and a_0 ,

$$c(\dot{u}, \dot{u}) + a_0(u, u) \geq \gamma \|\dot{u}\|_H^2 + \alpha \|u\|_V^2 - \lambda \|u\|_H^2.$$

We estimate the right-hand side from above: Due to continuity of c and a_0 ,

$$c(u^1, u^1) + a_0(u^0, u^0) \leq c_c \|u^1\|_H^2 + c_a \|u^0\|_V^2,$$

and by continuity of b and a_1 , the Cauchy-Schwarz inequality

$$\langle f | \dot{u} \rangle_H \leq \|f\|_H \|\dot{u}\|_H,$$

as well as the estimate $2xy \leq x^2 + y^2$ for (the norms) $x, y \in \mathbb{R}$:

$$\begin{aligned} &2 \int_0^t \langle f | \dot{u} \rangle_H dt' - 2 \int_0^t (b(\dot{u}, \dot{u}) + a_1(u, \dot{u})) dt' \\ &\leq 2 \int_0^t (\|f\|_H \|\dot{u}\|_H + c_b \|\dot{u}\|_H^2 + c_{a_1} \|u\|_V \|\dot{u}\|_H) dt' \\ &\leq \int_0^t (\|f\|_H^2 + \|\dot{u}\|_H^2 + 2c_b \|\dot{u}\|_H^2 + c_{a_1} \|u\|_V^2 + c_{a_1} \|\dot{u}\|_H^2) dt' \\ &= \int_0^t (\|f\|_H^2 + (1 + c_{a_1} + 2c_b) \|\dot{u}\|_H^2 + c_{a_1} \|u\|_V^2) dt'. \end{aligned}$$

Combining both estimates yields the inequality

$$\begin{aligned} \gamma \|\dot{u}(t)\|_H^2 + \alpha \|u(t)\|_V^2 &\leq \lambda \|u\|_H^2 + c_c \|u^1\|_H^2 + c_a \|u^0\|_V^2 \\ &\quad + \int_0^t (\|f\|_H^2 + (1 + c_{a_1} + 2c_b) \|\dot{u}\|_H^2 + c_{a_1} \|u\|_V^2) dt'. \end{aligned}$$

The term $\lambda \|u(t)\|_H^2$ is estimated based on the fundamental theorem of calculus

$$u(t) = u^0 + \int_0^t \dot{u}(t') dt',$$

3 Variational solution of linear second-order evolution equations

which also holds for $u \in H^1((0, T); H) \subseteq \mathcal{C}_{\text{abs}}([0, T]; H)$ [DL92, Chapter XVIII, §5, p. 561]:

$$\begin{aligned} \|u(t)\|_H &= \|u^0 + \int_0^t \dot{u} \, dt'\|_H \leq \|u^0\|_H + \int_0^t \|\dot{u}\|_H \, dt' \\ &\leq \|u^0\|_H + \underbrace{\left(\int_0^t 1^2 \, dt'\right)}_{=t}^{1/2} \left(\int_0^t \|\dot{u}\|_H^2 \, dt'\right)^{1/2} \\ \implies \|u(t)\|_H^2 &\leq \left(\|u^0\|_H + \sqrt{t \int_0^t \|\dot{u}\|_H^2 \, dt'}\right)^2 \\ &\leq \|u^0\|_H^2 + 2 \underbrace{\|u^0\|_H \sqrt{t \int_0^t \|\dot{u}\|_H^2 \, dt'}}_{\leq \|u^0\|_H^2 + t \int_0^t \|\dot{u}\|_H^2 \, dt'} + t \int_0^t \|\dot{u}\|_H^2 \, dt' \\ \implies \|u(t)\|_H^2 &\leq 2\|u^0\|_H^2 + 2t \int_0^t \|\dot{u}\|_H^2 \, dt'. \end{aligned}$$

Thus, with $\|\cdot\|_H \leq c_V \|\cdot\|_V$ from (3.3) we obtain

$$\lambda \|u(t)\|_H^2 \leq 2\lambda c_V^2 \|u^0\|_V^2 + 2\lambda T \int_0^t \|\dot{u}\|_H^2 \, dt'$$

and the energy estimate takes the intermediate form

$$\begin{aligned} \gamma \|\dot{u}(t)\|_H^2 + \alpha \|u(t)\|_V^2 &\leq c_c \|u^1\|_H^2 + (c_a + 2\lambda c_V^2) \|u^0\|_V^2 + \int_0^t \|f\|_H^2 \, dt' \\ &\quad + \int_0^t ((1 + c_{a_1} + 2c_b + 2\lambda T) \|\dot{u}\|_H^2 + c_{a_1} \|u\|_V^2) \, dt', \end{aligned} \quad (3.33)$$

from which we deduce

$$\begin{aligned} \min(\alpha, \gamma) (\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2) &\leq c_c \|u^1\|_H^2 + (c_a + 2\lambda c_V^2) \|u^0\|_V^2 + \int_0^t \|f\|_H^2 \, dt' \\ &\quad + (1 + c_{a_1} + 2c_b + 2\lambda T) \int_0^t (\|u\|_V^2 + \|\dot{u}\|_H^2) \, dt'. \end{aligned}$$

In terms of the auxiliary function

$$\phi := \|u\|_V^2 + \|\dot{u}\|_H^2 \in L^\infty(0, T)$$

the estimate reads

$$\min(\alpha, \gamma) \phi(t) \leq c_c \|u^1\|_H^2 + (c_a + 2\lambda c_V^2) \|u^0\|_V^2 + \int_0^t \|f\|_H^2 \, dt' + (1 + c_{a_1} + 2c_b + 2\lambda T) \int_0^t \phi(t') \, dt'.$$

Consequently,

$$\phi(t) \leq k_1 + k_2 \int_0^t \phi(t') \, dt' \quad (3.34)$$

with constants (3.30) and (3.31):

$$k_1 = \frac{(c_a + 2\lambda c_V^2) \|u^0\|_V^2 + c_c \|u^1\|_H^2 + \int_0^T \|f\|_H^2 \, dt'}{\min(\alpha, \gamma)} \quad \text{and} \quad k_2 = \frac{1 + c_{a_1} + 2c_b + 2\lambda T}{\min(\alpha, \gamma)}.$$

Gronwall's inequality (Lemma 3.10) then gives $\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \leq k_1 e^{k_2 t}$, which is (3.29). \square

The energy estimate (3.29) quantifies how the total energy of the solution on the time interval $[0, T]$ depends on the initial data $\|u^0\|_V$ and $\|u^1\|_H$, the source $\|f\|_{L^2((0,T);H)}$, as well as on the operators and spaces, i.e. on the coercivity constants α, λ, γ , the continuity constants c_a, c_{a_1}, c_b, c_c , and the constant c_V for the embedding $V \hookrightarrow H$ from (3.3).

Moreover, due to linearity of the problem, the following statement is a direct consequence of the energy estimate (cf. [DL92, Theorem 2, p. 567]):

Corollary 3.12 (Continuous dependence on data). *Let u_1 be a solution of (3.1) for the data $u_1(0) = u_1^0 \in H, \dot{u}_1(0) = u_1^1 \in V, f_1 \in L^2((0, T); H)$. Let u_2 be another solution for the data $u_2(0) = u_2^0 \in H, \dot{u}_2(0) = u_2^1 \in V, f_2 \in L^2((0, T); H)$. Then there exists $k > 0$ such that*

$$\begin{aligned} & \|u_1 - u_2\|_{L^\infty((0,T);V)} + \|\dot{u}_1 - \dot{u}_2\|_{L^\infty((0,T);H)} \\ & \leq k(\|u_1^0 - u_2^0\|_V + \|u_1^1 - u_2^1\|_H + \|f_1 - f_2\|_{L^2((0,T);H)}). \end{aligned} \quad (3.35)$$

Proof. Since problem (3.1) is linear, the difference $u := u_1 - u_2$ is the solution of (3.1) with the initial data $u^0 := u_1^0 - u_2^0, u^1 := u_1^1 - u_2^1$ and the force given by $f := f_1 - f_2$. Consequently we have the estimate (3.32),

$$\|u(t)\|_V + \|\dot{u}(t)\|_H \leq k(\|u_0\|_V + \|u_1\|_H + \|f\|_{L^2((0,T);H)}).$$

with the same constant k . Taking $\sup_{t \in (0, T)}$ completes the proof. \square

The estimate (3.35) shows how the distance of the two solutions u_1 and u_2 is bounded in terms of the distances of their data. Therefore, it expresses continuity of the solution as a function of the data. Together with existence and uniqueness, which will be established in the two subsequent sections, the continuous dependence of the solution on the data is one of the essential features of a well-posed problem.

3.3 Uniqueness

3.3.1 Proof by the energy equality

Uniqueness in Theorem 3.1 immediately follows from Corollary (3.12), which is based on the energy estimate (3.29),

$$\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \leq k_1 e^{k_2 t},$$

which in turn was a consequence of the equality (3.27) (see Propositions 3.8 and 3.11).

We briefly repeat this quick approach to prove uniqueness:

Proof. Consider two different solutions of the weak evolution equation (3.1),

$$\tilde{u}, \tilde{\tilde{u}} \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H).$$

Then, by linearity, their difference $u := \tilde{u} - \tilde{\tilde{u}}$ solves the homogeneous equation

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = 0 \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T)$$

with zero initial values $u^0 = 0$ and $u^1 = 0$. But these conditions imply $k_1 = 0$, see (3.30), and the energy estimate (3.29) gives

$$\|u\|_V^2 + \|\dot{u}\|_H^2 = 0 \quad \text{a.e. on } (0, T).$$

3 Variational solution of linear second-order evolution equations

Consequently we deduce

$$u = 0, \quad \text{that is, } \tilde{u} = \tilde{\tilde{u}} \quad \text{in } L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H).$$

Since $\mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$ is contained in $L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H)$ this shows that solutions of (3.1) are unique. \square

3.3.2 Proof by an integrated energy equality

Uniqueness in Theorem 3.1 may be also established in an alternative way, which does not depend on the energy equality (3.27). The key idea is to choose a special integrated test function: In contrast to the derivative \dot{u} used in the heuristic proof of the energy equality, the antiderivative $\int u \, dt$ is a valid test function in the weak formulation. We follow [DL92, p. 572].

Proof. Assume that $u \in L^2((0, T); V) \cap H^1((0, T); H)$ is a solution of (3.1),

$$\frac{d}{dt}c(\dot{u}, v) + b(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T),$$

with $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$. Let $s \in (0, T)$ and define the **integrated test function**

$$\varphi(t) := \begin{cases} -\int_t^s u(t') \, dt', & t < s \\ 0, & t \geq s. \end{cases}$$

By construction, $\varphi \in L^2((0, T); V)$ with

$$\varphi(T) = 0, \quad \varphi(s) = 0, \quad \text{and} \quad \dot{\varphi}(t) = \begin{cases} u(t), & t < s \\ 0, & t \geq s. \end{cases}$$

In particular, the last property shows that $\dot{\varphi} \in L^\infty((0, T); V)$, implying

$$\varphi \in W^{1, \infty}((0, T); V) \subseteq H^1((0, T); V).$$

The validity of the weak evolution equation gives $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$ (Lemma 3.3) and therefore (3.16) holds: $\frac{d}{dt}c(\dot{u}, v) = \langle \frac{d}{dt}(C\dot{u}), v \rangle$. Evaluating the weak evolution equation (3.1) for $v = \varphi(t) \in V$ and integration thus leads to

$$\int_0^T \left(\langle \frac{d}{dt}(C\dot{u}), \varphi \rangle + b(\dot{u}, \varphi) + a(u, \varphi) \right) dt = \int_0^T \langle f, \varphi \rangle dt.$$

For $u \in H^1((0, T); H)$ and $\varphi \in H^1((0, T); V)$ the integration by parts formula (3.24) applies:

$$\int_0^T \langle \frac{d}{dt}(C\dot{u}), \varphi \rangle dt = - \int_0^T c(\dot{u}, \dot{\varphi}) dt - c(u, \varphi) \Big|_0^T.$$

Consequently, the integrated weak evolution equation reads

$$\int_0^T ((a_0 + a_1)(u, \varphi) + b(\dot{u}, \varphi) - c(\dot{u}, \dot{\varphi})) dt = \int_0^T \langle f, \varphi \rangle dt - c(u, \varphi) \Big|_0^T.$$

Upon changing T to s and by the properties of φ we obtain [DL92, Eq. (5.106)]

$$\int_0^s (a_0(\dot{\varphi}, \varphi) + a_1(u, \varphi) + b(\dot{u}, \varphi) - c(\dot{u}, \dot{\varphi})) dt = \int_0^s \langle f, \varphi \rangle dt + c(u^1, \varphi(0)).$$

Since the problem is linear, it suffices to show that the evolution equation (3.1) with zero data and source, i.e. with $u^0 = 0$, $u^1 = 0$, and $f = 0$, has the solution $u = 0$. With these assumptions the right-hand-side of the equation above vanishes and we get an **integrated energy equality**:

$$\begin{aligned} & \int_0^s \left(\frac{1}{2} \frac{d}{dt} a_0(\varphi, \varphi) + a_1(u, \varphi) + b(\dot{u}, \varphi) - \frac{1}{2} \frac{d}{dt} c(u, u) \right) dt = 0 \\ \implies & a_0(\varphi(0), \varphi(0)) + c(u(s), u(s)) = 2 \int_0^s (a_1(u, \varphi) + b(\dot{u}, \varphi)) dt \\ \implies & a_0(\varphi(0), \varphi(0)) + c(u(s), u(s)) = 2 \int_0^s (a_1(u, \varphi) - b(u, u)) dt. \end{aligned} \quad (3.36)$$

The last implication follows from bilinearity of b and the conditions $u^0 = 0$, $\varphi(s) = 0$:

$$0 = b(u, \varphi) \Big|_0^s = \int_0^s \frac{d}{dt} b(u, \varphi) dt = \int_0^s (b(\dot{u}, \varphi) + b(u, \dot{\varphi})) dt = \int_0^s (b(\dot{u}, \varphi) + b(u, u)) dt.$$

In the following we proceed as in the proof of the energy estimates (Proposition 3.11). Since $u(t) \in V \subseteq H$ and $\varphi(t) \in V$, the additional regularity (3.8), i.e. $A_1 \in \text{Lin}(H, V')$, gives $|a_1(u(t), \varphi(t))| \leq c'_{a_1} \|u(t)\|_H \|\varphi(t)\|_V$ and continuity of b on H yields $|b(u(t), u(t))| \leq c_b \|u(t)\|_H^2$. Thus we deduce an estimate for the right-hand-side of (3.36):

$$\begin{aligned} 2 \int_0^s (a_1(u, \varphi) - b(u, u)) dt & \leq 2 \int_0^s (|a_1(u, \varphi)| + |b(u, u)|) dt \\ & \leq 2 \int_0^s \left(c'_{a_1} \|u(t)\|_H \|\varphi(t)\|_V + c_b \|u(t)\|_H^2 \right) dt. \end{aligned}$$

By coercivity of a_0 and c , the left-hand side of (3.36) has the lower bound

$$a_0(\varphi(0), \varphi(0)) + c(u(s), u(s)) \geq \alpha \|\varphi(0)\|_V^2 - \lambda \|\varphi(0)\|_H^2 + \gamma \|u(s)\|_H^2.$$

In combination we get the inequality

$$\alpha \|\varphi(0)\|_V^2 + \gamma \|u(s)\|_H^2 \leq \int_0^s \left(\underbrace{c'_{a_1} 2 \|u(t)\|_H \|\varphi(t)\|_V}_{\leq \|u(t)\|_H^2 + \|\varphi(t)\|_V^2} + 2c_b \|u(t)\|_H^2 \right) dt + \underbrace{\lambda \|\varphi(0)\|_H^2}_{\leq \lambda \int_0^s \|u(t)\|_H^2 dt}.$$

Here, the estimate for the λ -term follows from $\varphi(0) = - \int_0^s u(t) dt$ and the triangle inequality for integrals:

$$\|\varphi(0)\|_H^2 = \left\| - \int_0^s u(t) dt \right\|_H^2 \leq \int_0^s \|u(t)\|_H^2 dt.$$

Thus

$$\alpha \|\varphi(0)\|_V^2 + \gamma \|u(s)\|_H^2 \leq \int_0^s \left((c'_{a_1} + 2c_b + \lambda) \|u(t)\|_H^2 + c'_{a_1} \|\varphi(t)\|_V^2 \right) dt$$

and we arrive at the estimate [DL92, (5.108)]

$$\|\varphi(0)\|_V^2 + \|u(s)\|_H^2 \leq M \int_0^s \left(\|\varphi(t)\|_V^2 + \|u(t)\|_H^2 \right) dt \quad \text{with} \quad M := \frac{c'_{a_1} + 2c_b + \lambda}{\min(\alpha, \gamma)}.$$

Next we consider the **antiderivative** v of the solution u :

$$v(t) := \int_0^t u(t') dt'.$$

By definition, $v \in H^1((0, T); V)$ and satisfies

$$v(s) = \int_0^s u = -\varphi(0) \quad \text{and} \quad v(s) - v(t) = \int_t^s u = -\varphi(t) \quad \text{for } t \leq s.$$

3 Variational solution of linear second-order evolution equations

These properties enable us to replace the integrated test function φ in the estimate by v :

$$\|v(s)\|_V^2 + \|u(s)\|_H^2 \leq M \int_0^s \left(\|v(s) - v(t)\|_V^2 + \|u(t)\|_H^2 \right) dt.$$

With

$$\int_0^s \|v(s) - v(t)\|_V^2 dt \leq 2 \int_0^s \left(\|v(s)\|_V^2 + \|v(t)\|_V^2 \right) dt = 2s\|v(s)\|_V^2 + 2 \int_0^s \|v(t)\|_V^2 dt$$

this yields

$$\|v(s)\|_V^2 + \|u(s)\|_H^2 \leq M \int_0^s \left(2\|v(t)\|_V^2 + \|u(t)\|_H^2 \right) dt + 2Ms\|v(s)\|_V^2$$

and hence

$$(1 - 2Ms)\|v(s)\|_V^2 + \|u(s)\|_H^2 \leq M \int_0^s \left(2\|v(t)\|_V^2 + \|u(t)\|_H^2 \right) dt.$$

Now we choose $s \leq s_0$ such that $1 - 2Ms_0 > 0$, i.e.

$$s_0 < \frac{1}{2M} = \frac{\min(\alpha, \gamma)}{2(c'_{a_1} + 2c_b + \lambda)}$$

(note that we can not simply assume $1 - 2Ms_0 \geq 1$, because this gives $s_0 \leq 0$). Then the estimate is of the form

$$\|v(s)\|_V^2 + \|u(s)\|_H^2 \leq M' \int_0^s \left(\|v(t)\|_V^2 + \|u(t)\|_H^2 \right) dt$$

with a constant $M' > 0$. Finally Gronwall's inequality (Lemma 3.10) yields

$$\|v\|_V^2 + \|u\|_H^2 = 0 \quad \text{a.e. in } (0, s_0).$$

In particular $u = 0$ a.e. on $(0, s_0)$. This implies uniqueness on the interval $(0, s_0)$. Then we consider the weak evolution problem (3.1) on (s_0, T) and by proceeding as above step-by-step we eventually get $u = 0$ a.e. in $(0, T)$, showing uniqueness of weak solutions (3.13). Starting from $u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$ (3.12) will give $u = 0$ everywhere in $[0, T]$. \square

3.4 Existence

3.4.1 Proof by the Faedo-Galerkin method

We discuss the Faedo-Galerkin method and will mainly follow [RR04, Section 11.2.2, p. 389].

Definition 3.13 (Galerkin approximation). Let $\{V_m\}_{m \in \mathbb{N}}$ be a family of finite dimensional subspaces of a separable Hilbert space V :

$$V_m \subseteq V \quad \text{with} \quad d_m := \dim V_m < \infty.$$

The space V_m is called the Galerkin approximation of V of order m , if $V_m \rightarrow V$ ($m \rightarrow \infty$) in the sense that there exists a dense subspace $W \subseteq V$ such that for all $v \in W$ there exists a sequence $(v_m)_{m \in \mathbb{N}}$ with $v_m \in V_m$ and $v_m \rightarrow v$ in V (see [DL92, p. 504]). \lrcorner

By definition, if $\{V_m\}_{m \in \mathbb{N}}$ is a Galerkin approximation of V , then every $u \in V$ can be obtained as a limit of some sequence $(v_m)_m$ with

$$v_m \in V_m \quad \text{and} \quad v_m \rightarrow u \quad \text{in} \quad V \quad (m \rightarrow \infty).$$

The Galerkin approximation of the abstract evolution equation leads to an approximated evolution problem, that corresponds to a space discretization of the PDE. This procedure is called the Faedo-Galerkin method (in contrast to the Galerkin method for time-independent problems).

Let $\{V_m\}_{m \in \mathbb{N}}$ be a Galerkin approximation of V (which exists thanks to separability of V). Density of V in H implies that $\{V_m\}_{m \in \mathbb{N}}$ also is a Galerkin approximation of H . Consequently, there exist two sequences $(u_m^0)_m, (u_m^1)_m$ with

$$\begin{aligned} u_m^0 &\in V_m, & u_m^0 &\rightarrow u^0 \quad \text{in} \quad V, \\ u_m^1 &\in V_m, & u_m^1 &\rightarrow u^1 \quad \text{in} \quad H. \end{aligned}$$

In particular, if $\Pi_m : V \rightarrow V_m$ and $P_m : H \rightarrow V_m$ are the orthogonal projections, we may choose

$$u_m^0 := \Pi_m u^0 \quad \text{and} \quad u_m^1 := P_m u^1. \quad (3.37)$$

Let

$$\{w_{m;j}\}_{j=1, \dots, d_m} \subseteq V_m$$

denote a basis of V_m . Then the original Cauchy problem (3.1), (3.2) is associated to the following **approximated Cauchy problem of order m** : Find the solution

$$u_m : t \mapsto u_m(t) = \sum_{k=1}^{d_m} g_k(t) w_{m;k} \in V_m \quad \text{with} \quad g_k : [0, T] \rightarrow \mathbb{R}$$

of the system

$$\begin{aligned} c(\ddot{u}_m, w_{m;j}) + b(\dot{u}_m, w_{m;j}) + a(u_m, w_{m;j}) &= \langle f, w_{m;j} \rangle \\ \text{for } j = 1, \dots, d_m \text{ and in the sense of } \mathcal{D}'(0, T), \end{aligned} \quad (3.38)$$

that satisfies

$$u_m(0) = u_m^0, \quad \dot{u}_m(0) = u_m^1. \quad (3.39)$$

Lemma 3.14 (Existence and uniqueness for the approximated Cauchy problem). *Problem (3.38), (3.39) possesses a unique solution*

$$u_m \in H^2((0, T); V_m). \quad (3.40)$$

In particular, $u_m \in \mathcal{C}^1([0, T]; V_m)$, $\dot{u}_m \in \mathcal{C}^0([0, T]; V_m)$, and $\ddot{u}_m \in L^2((0, T); V_m)$.

Proof. We observe that (3.38) is a linear second-order system of ordinary differential equations (ODEs) with constant coefficients for the components of the d_m -dimensional coefficient vector

$$g := (g_k)_{k=1}^{d_m} : [0, T] \rightarrow \mathbb{R}^{d_m}.$$

Indeed, due to bilinearity of a, b, c , the approximated evolution equation reads

$$\sum_{k=1}^{d_m} (c_{kj} \ddot{g}_k + b_{kj} \dot{g}_k + a_{kj} g_k) = f_j \quad \text{for } j = 1, \dots, d_m \quad (3.41)$$

with the constant (i.e. time-independent) scalar coefficients

$$a_{kj} := a(w_{m;k}, w_{m;j}), \quad b_{kj} := b(w_{m;k}, w_{m;j}), \quad c_{kj} := c(w_{m;k}, w_{m;j}),$$

3 Variational solution of linear second-order evolution equations

and the right-hand side

$$f_j := \langle f, w_{m;j} \rangle \in L^2(0, T)$$

for $k, j = 1, \dots, d_m$. The initial data are

$$g(0) = g^0 := (g_k^0)_{k=1}^{d_m} \quad \text{and} \quad \dot{g}(0) = g^1 := (g_k^1)_{k=1}^{d_m}$$

with components g_k^0 and g_k^1 given according to (3.39): $u_m^0 = \sum_{k=1}^{d_m} g_k^0 w_{m;k}$ and $u_m^1 = \sum_{k=1}^{d_m} g_k^1 w_{m;k}$.

Solutions of linear ODE-systems with constant coefficients always exist and are unique. In particular, in view of the invertibility of the leading coefficient matrix (c_{kj}) and since the source vector $(f_j)_{j=1}^{d_m}$ has components in $L^2(0, T)$, we deduce (e.g. by Fourier transform) the existence of a unique solution g of (3.41) that satisfies

$$g \in H^2(0, T)^{d_m}.$$

Consequently $\ddot{g} \in L^2(0, T)^{d_m}$ and Sobolev embedding yields $\dot{g} \in H^1 \subseteq \mathcal{C}^0$ and $g \in H^2 \subseteq \mathcal{C}^1$. Thus $u_m = \sum_{k=1}^{d_m} g_k w_{m;k} \in H^2((0, T); V_m)$, which concludes the proof. \square

The main result is the following:

Proposition 3.15 (Existence for the original Cauchy problem). *For $m \rightarrow \infty$ the sequence $(u_m)_m$ of solutions of the approximated Cauchy problem (3.38), (3.39) converges to a limit u in $\mathcal{D}'((0, T); H)$, which possesses the weak regularity (3.13):*

$$u \in L^\infty((0, T); V) \cap W^{1,\infty}((0, T); H).$$

Moreover u solves the original Cauchy problem, i.e. satisfies (3.1) with initial conditions (3.2).

Proof. Let $u_m \in H^2((0, T); V_m)$ be the approximate solution obtained in Lemma 3.14. We proceed in four steps (cf. Section 2.3.2): Convergence of u_m follows from boundedness results which are obtained via energy estimates (Steps 1 & 2). This also yields the asserted regularity of the limit u . In Step 3 we check whether this limit constructed from solutions u_m of the approximated problems solves the original evolution equation (3.1). In Step 4 we verify the initial conditions (3.2).

– **Step 1: Energy estimates.** Multiplication of (3.38) by \dot{g}_j and summation in j from 1 to d_m gives

$$c(\ddot{u}_m, \dot{u}_m) + b(\dot{u}_m, \dot{u}_m) + a(u_m, \dot{u}_m) = \langle f, \dot{u}_m \rangle.$$

Proceeding as in the heuristic proof of the energy equality is now rigorous as $\dot{u}_m \in V_m \subseteq V$ (see Lemma 3.9): We integrate twice the equation from 0 to $t \in (0, T)$, split $a = a_0 + a_1$, and employ the integration by parts formulas (3.23) and (3.25) to obtain

$$\begin{aligned} \frac{1}{2} c(\dot{u}_m, \dot{u}_m) \Big|_0^t + \int_0^t (b(\dot{u}_m, \dot{u}_m) + a(u_m, \dot{u}_m)) \, dt' &= \int_0^t \langle f, \dot{u}_m \rangle \, dt' \\ \implies \frac{1}{2} (c(\dot{u}_m, \dot{u}_m) + a_0(u_m, u_m)) \Big|_0^t + \int_0^t (b(\dot{u}_m, \dot{u}_m) + a_1(u_m, \dot{u}_m)) \, dt' &= \int_0^t \langle f, \dot{u}_m \rangle \, dt'. \end{aligned}$$

Insertion of the initial conditions $u_m^0 = u_m(0)$ and $u_m^1 = \dot{u}_m(0)$ from (3.39) yields the following **approximated energy equality**, which is similar to (3.27):

$$\begin{aligned} \frac{1}{2} (c(\dot{u}_m(t), \dot{u}_m(t)) + a_0(u_m(t), u_m(t))) + \int_0^t (b(\dot{u}_m(t'), \dot{u}_m(t')) + a_1(u_m(t'), \dot{u}_m(t'))) \, dt' \\ = \frac{1}{2} (c(u_m^1, u_m^1) + a_0(u_m^0, u_m^0)) + \int_0^t \langle f(t'), \dot{u}_m(t') \rangle \, dt'. \end{aligned} \quad (3.42)$$

In complete analogy to the proof of Proposition 3.11, coercivity and continuity of the bilinear forms then imply energy estimates for u_m , valid for $t \in [0, T]$:

$$\|u_m(t)\|_V^2 + \|\dot{u}_m(t)\|_H^2 \leq k_1 e^{k_2 t}. \quad (3.43)$$

– **Step 2: Boundedness and convergence to a limit.** The constants k_1, k_2 in the energy estimate (3.43) for u_m are the same as (3.30), (3.31) in the energy estimate for u :

$$k_1 = \frac{(c_a + 2\lambda c_V^2)\|u^0\|_V^2 + c_c\|u^1\|_H^2 + \|f\|_{L^2((0,T);H)}^2}{\min(\alpha, \gamma)} \quad \text{and} \quad k_2 = \frac{1 + c_{a_1} + 2c_b + 2\lambda T}{\min(\alpha, \gamma)},$$

where k_1 is obtained with the help of the projections to V_m , see (3.37):

$$\|u_m^0\|_V = \|\Pi_m u^0\|_V \leq \|u^0\|_V \quad \text{and} \quad \|u_m^1\|_H = \|P_m u^1\|_H \leq \|u^1\|_H.$$

Since the constants do not depend on the approximation order m , (3.43) implies that

$$\begin{aligned} \text{(i)} \quad & u_m \text{ lies in a bounded set of } L^\infty((0, T); V) \\ \text{(ii)} \quad & \dot{u}_m \text{ lies in a bounded set of } L^\infty((0, T); H), \end{aligned} \quad (3.44)$$

with bounds that are uniformly in m . Weak-* compactness allows us to extract corresponding subsequences which weakly-* converge as $m \rightarrow \infty$, written here without relabeling:

$$\begin{aligned} \text{(i)} \quad & u_m \rightharpoonup^* u \text{ in } L^\infty((0, T); V) \\ \text{(ii)} \quad & \dot{u}_m \rightharpoonup^* \dot{u} \text{ in } L^\infty((0, T); H). \end{aligned} \quad (3.45)$$

Actually, we first only get $\dot{u}_m \rightharpoonup^* w$ in $L^\infty((0, T); H)$ but $w = \dot{u}$ then follows from uniqueness of limits in $\mathcal{D}'((0, T); H)$. Thus there exists a limit function $u \in L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H)$ as was claimed in (3.13).

– **Step 3: The limit satisfies the evolution equation.** Let $h_j \in \mathcal{D}([0, T])$ and set

$$\varphi := \sum_{j=1}^n h_j w_{m,j} \in \mathcal{D}([0, T]) \otimes V_m$$

with $n \leq d_m$; functions of this form are dense in $\mathcal{D}([0, T]; V)$. Multiplication of the approximated equation (3.38) by h_j , summation in j from 1 to n , and time integration gives

$$\int_0^T (c(\ddot{u}_m, \varphi) + b(\dot{u}_m, \varphi) + a(u_m, \varphi)) dt = \int_0^T \langle f, \varphi \rangle dt,$$

from which by (3.24), $\int_0^T c(\ddot{u}_m, \varphi) dt = c(\dot{u}_m, \varphi)\Big|_0^T - \int_0^T c(\dot{u}_m, \dot{\varphi}) dt$, and in terms of operators,

$$\langle C\dot{u}_m, \varphi \rangle\Big|_0^T - \int_0^T \langle C\dot{u}_m, \dot{\varphi} \rangle dt + \int_0^T (\langle B\dot{u}_m, \varphi \rangle + \langle Au_m, \varphi \rangle) dt = \int_0^T \langle f, \varphi \rangle dt. \quad (3.46)$$

The assumptions $A \in \text{Lin}(V, V')$, $B, C \in \text{Lin}(H, H)$ together with the uniform boundedness of u_m in $L^\infty((0, T); V)$ and of \dot{u}_m in $L^\infty((0, T); H)$, see (3.44), imply that

$$\begin{aligned} \text{(i)} \quad & Au_m \text{ lies in a bounded set of } L^\infty((0, T); V') \\ \text{(ii)} \quad & B\dot{u}_m, C\dot{u}_m \text{ lie in bounded sets of } L^\infty((0, T); H), \end{aligned}$$

and weak-* compactness gives

$$\begin{aligned} \text{(i)} \quad & Au_m \rightharpoonup^* Au \text{ in } L^\infty((0, T); V') \\ \text{(ii)} \quad & B\dot{u}_m \rightharpoonup^* B\dot{u}, \quad C\dot{u}_m \rightharpoonup^* C\dot{u} \text{ in } L^\infty((0, T); H). \end{aligned} \quad (3.47)$$

3 Variational solution of linear second-order evolution equations

These convergence properties of Au_m , $B\dot{u}_m$, and $C\dot{u}_m$ are sufficient to take the limit $m \rightarrow \infty$ in (3.46).

In order to verify the **evolution equation** we further restrict to test functions

$$\varphi \in \mathcal{D}(0, T) \otimes V_m$$

making the first term in (3.46) disappear. The limit $m \rightarrow \infty$ then gives

$$-\int_0^T \langle C\dot{u}, \dot{\varphi} \rangle dt + \int_0^T (\langle B\dot{u}, \varphi \rangle + \langle Au, \varphi \rangle) dt = \int_0^T \langle f, \varphi \rangle dt$$

which coincides with

$$\left\langle \frac{d}{dt}(C\dot{u}) + B\dot{u} + Au, \varphi \right\rangle_{\mathcal{D}'((0,T);V'), \mathcal{D}((0,T);V)} = \langle f, \varphi \rangle_{\mathcal{D}'((0,T);V'), \mathcal{D}((0,T);V)}.$$

By density of functions φ in $\mathcal{D}((0, T); V)$, the limit u indeed satisfies the evolution equation (3.1).

– **Step 4: The limit satisfies the initial conditions.** We need to show $u(0) = u^0$ and $\dot{u}(0) = u^1$. For this purpose we consider test functions that vanish only at $t = T$:

$$\varphi \in \mathcal{D}([0, T]; V) \quad \text{with} \quad \varphi(T) = 0.$$

We first consider the equation for u . Since $u \in H^1((0, T); H)$, integration by parts (3.21) applies:

$$\int_0^T \langle \dot{u}, \varphi \rangle dt = -\langle u(0), \varphi(0) \rangle - \int_0^T \langle u, \dot{\varphi} \rangle dt.$$

With $u_m(0) = u_m^0$ the discrete analogue is

$$\int_0^T \langle \dot{u}_m, \varphi \rangle dt = -\langle u_m^0, \varphi(0) \rangle - \int_0^T \langle u_m, \dot{\varphi} \rangle dt.$$

The convergences (3.45), $u_m \rightharpoonup^* u$ in $L^\infty((0, T); V)$, $\dot{u}_m \rightharpoonup^* \dot{u}$ in $L^\infty((0, T); H)$, and $u_m^0 \rightarrow u^0$ in V allow us to take the limit $m \rightarrow \infty$:

$$\int_0^T \langle \dot{u}, \varphi \rangle dt = -\langle u^0, \varphi(0) \rangle - \int_0^T \langle u, \dot{\varphi} \rangle dt.$$

Comparison shows $\langle u(0), \varphi(0) \rangle = \langle u^0, \varphi(0) \rangle$, which with $\varphi(0) \in V$ gives $u(0) = u^0$ in V' .

Next we prove the equation for \dot{u} . On one hand, we already know from Step 3 that u is a solution, whence by Lemma 3.3, $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$. Consequently, taking the strong form (3.17) in duality with φ and integrating by parts via (3.24) gives

$$-\int_0^T \langle C\dot{u}, \dot{\varphi} \rangle dt + \int_0^T (\langle B\dot{u}, \varphi \rangle + \langle Au, \varphi \rangle) dt = \langle C\dot{u}(0), \varphi(0) \rangle + \int_0^T \langle f, \varphi \rangle dt.$$

On the other hand, with the approximated initial condition $\dot{u}_m(0) = u_m^1$, (3.46) reads

$$-\int_0^T \langle C\dot{u}_m, \dot{\varphi} \rangle dt + \int_0^T (\langle B\dot{u}_m, \varphi \rangle + \langle Au_m, \varphi \rangle) dt = \langle Cu_m^1, \varphi(0) \rangle + \int_0^T \langle f, \varphi \rangle dt$$

which with $u_m^1 \rightarrow u^1$ in H converges to

$$-\int_0^T \langle C\dot{u}, \dot{\varphi} \rangle dt + \int_0^T (\langle B\dot{u}, \varphi \rangle + \langle Au, \varphi \rangle) dt = \langle Cu^1, \varphi(0) \rangle + \int_0^T \langle f, \varphi \rangle dt.$$

By comparison, $\langle C\dot{u}(0), \varphi(0) \rangle = \langle Cu^1, \varphi(0) \rangle$, which thanks to invertibility of C shows $\dot{u}(0) = u^1$ in H and completes the proof. \square

We have established the existence of a weak solution $u \in L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H)$ of (3.1) that satisfies the initial conditions (3.2). Uniqueness has been proven in Section 3.3. The only missing assertion of Theorem 3.1 is the regularity $u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$ (3.12), which does not follow from the Faedo-Galerkin method.

Remark 3.16 (Existence for weaker assumptions on A_1). As the Faedo-Galerkin method suggests, existence can be established under the assumption $A_1 \in \text{Lin}(V, H)$ instead of $A_1 \in \text{Lin}(V, H) \cap \text{Lin}(H, V')$ (3.15). Yet, the condition $A_1 \in \text{Lin}(H, V')$ (3.8), seems to be indispensable in proving uniqueness [DL92, Chapter XVIII, §5, Remark 8, p. 574]. In both our uniqueness proofs, it is used to estimate the term involving a_1 in the energy equation (3.27) or its integrated counterpart (3.36). \lrcorner

Remark 3.17 (An alternative proof of $u(0) = u^0$). We recall that weak-* convergence in $L^\infty((0, T); H)$ implies weak convergence in $L^2((0, T); H)$ (Lemma 2.15). Consequently, properties (i) $u_m \rightharpoonup^* u$ in $L^\infty((0, T); V)$ and (ii) $\dot{u}_m \rightharpoonup^* \dot{u}$ in $L^\infty((0, T); H)$ of (3.45) imply

$$\begin{aligned} \text{(i)'} \quad u_m &\rightharpoonup u \quad \text{in } L^2((0, T); H) \\ \text{(ii)'} \quad \dot{u}_m &\rightharpoonup \dot{u} \quad \text{in } L^2((0, T); H) \end{aligned}$$

from which we further deduce

$$u_m \rightharpoonup u \quad \text{in } H^1((0, T); H).$$

By Sobolev embedding, $H^1((0, T); H) \hookrightarrow \mathcal{C}^0([0, T]; H)$ and thus the map $u \mapsto u(0)$ is continuous. This shows

$$u_m(0) \rightharpoonup u(0) \quad \text{in } H.$$

However, by definition of initial condition (3.39) in the approximated Cauchy problem,

$$u_m(0) \rightarrow u^0 \quad \text{in } V.$$

Therefore $u(0) = u^0$ holds, cf. [RR04, p. 390] but also [DL92, (5.127)]. \lrcorner

Remark 3.18 (Limit of the approximated energy equality). With strong convergence of u_m and \dot{u}_m , the energy equality (3.27) would directly follow from the approximated analog (3.42):

$$\begin{aligned} \frac{1}{2} (c(\dot{u}_m(t), \dot{u}_m(t)) + a_0(u_m(t), u_m(t))) + \int_0^t (b(\dot{u}_m, \dot{u}_m) + a_1(u_m, \dot{u}_m)) dt' \\ = \frac{1}{2} (c(u_m^1, u_m^1) + a_0(u_m^0, u_m^0)) + \int_0^t \langle f, \dot{u}_m \rangle dt'. \end{aligned}$$

However, at the moment we only have the weak or weak-* convergences of u_m and \dot{u}_m at hand. These are not sufficient, because products of weakly (-*) convergent sequences do not necessarily possess a limit. \lrcorner

3.4.2 Proof by parabolic regularization

An alternative existence proof is based on **parabolic regularization (vanishing viscosity method)**: The solution of the hyperbolic problem (3.1), (3.2) is constructed as the limit of solutions of associated parabolic problems as their viscosity vanishes (Proposition 3.23). The method exploits the better regularity of solutions in the parabolic case. We follow [DL92, Chapter XVIII, §5, Problem ($P_{2\varepsilon}$), p. 575].

3 Variational solution of linear second-order evolution equations

Parabolic regularization of the evolution equation relies on changing the operator $B \in \text{Lin}(H, H)$ to $B_\varepsilon \in \text{Lin}(V, V')$, which models a damping term with viscosity $\varepsilon > 0$. The definition is

$$B_\varepsilon := B + \varepsilon(A_0 + \lambda \text{Id}_V),$$

which splits in

$$B_\varepsilon = B_1 + B_{0,\varepsilon} \in \text{Lin}(V, V')$$

with the bounded contribution $B_1 := B \in \text{Lin}(H, H)$ and the principal part

$$B_{0,\varepsilon} := \varepsilon(A_0 + \lambda \text{Id}_V) \in \text{Lin}(V, V').$$

The artificial viscosity operator $B_{0,\varepsilon}$ disappears in the limit $\varepsilon \rightarrow 0$ (the exact meaning of “disappears” will be specified in the proof of Proposition 3.23).

On the level of bilinear forms, $b: H \times H \rightarrow \mathbb{R}$ is replaced by

$$b_\varepsilon: V \times V \rightarrow \mathbb{R}$$

defined by $b_\varepsilon(w, v) := \langle B_\varepsilon w, v \rangle$, that is

$$b_\varepsilon(w, v) = b(w, v) + \varepsilon(a_0(w, v) + \lambda \langle w, v \rangle) \quad \text{for } w, v \in V. \quad (3.48)$$

Again, the term $b_1 := b$ is a bounded perturbation of the principal viscous part

$$b_{0,\varepsilon} := \varepsilon(a_0(\cdot, \cdot) + \lambda \langle \cdot, \cdot \rangle).$$

Lemma 3.19 (Properties of the viscosity term). *The map b_ε is a continuous bilinear form on $V \times V$, where*

$$|b_\varepsilon(w, v)| \leq c_{b_\varepsilon} \|w\|_V \|v\|_V \quad \forall w, v \in V \quad (3.49)$$

holds with $c_{b_\varepsilon} := (c_b + \varepsilon(c_a + \lambda))c_V^2 > 0$. The principal part $b_{0,\varepsilon}$ is symmetric and V -coercive,

$$b_{0,\varepsilon}(w, w) \geq \beta_\varepsilon \|w\|_V^2 \quad \forall w \in V \quad (3.50)$$

with $\beta_\varepsilon := \varepsilon\alpha > 0$. The bounded component $b_1 = b$ has more regularity:

$$|b_1(w, v)| = |b(w, v)| \leq c_b \|w\|_H \|v\|_H.$$

Proof. By our assumptions, bilinearity of b_ε is immediate. Let $v, w \in V$. Then (3.4) allows us to replace $\langle w, v \rangle = \langle w|v \rangle_H$ in the definition of b_ε . The properties of b and a_0 , together with the continuous embedding (3.3), $\|\cdot\|_H \leq c_V \|\cdot\|_V$, yield

$$\begin{aligned} |b_\varepsilon(w, v)| &\leq |b(w, v)| + \varepsilon(|a_0(w, v)| + \lambda |\langle w|v \rangle_H|) \\ &\leq (c_b + \varepsilon(c_a + \lambda)) \|w\|_H \|v\|_H \\ &\leq (c_b + \varepsilon(c_a + \lambda)) c_V^2 \|w\|_V \|v\|_V = c_{b_\varepsilon} \|w\|_V \|v\|_V, \end{aligned}$$

showing continuity of b_ε on $V \times V$. Symmetry and coercivity of a_0 and $\langle \cdot | \cdot \rangle_H$ directly transfer to the principal part,

$$b_{0,\varepsilon}(v, w) = b_{0,\varepsilon}(w, v) \quad \text{and} \quad b_{0,\varepsilon}(w, w) = \varepsilon(a_0(w, w) + \lambda \|w\|_H^2) \geq \varepsilon\alpha \|w\|_V^2 = \beta_\varepsilon \|w\|_V^2.$$

Finally, continuity of $b_1 = b: H \times H \rightarrow \mathbb{R}$ is simply our hypothesis (3.9). \square

The proof shows that the desired V -coercivity of the artificial viscosity term $b_{0,\varepsilon}$ is constructed based on V -coercivity with respect to H of a_0 .

We have the following variational solution of the regularized Cauchy problem:

Theorem 3.20 (Existence and uniqueness for the regularized Cauchy problem). *With b_ε defined by (3.48), given the data*

$$u^0 \in V, \quad u^1 \in H, \quad \text{and} \quad f \in L^2((0, T); H), \quad (3.51)$$

there exists a unique solution

$$u_\varepsilon \in L^2((0, T); V), \quad \dot{u}_\varepsilon \in L^2((0, T); V), \quad \frac{d}{dt}(C\dot{u}_\varepsilon) \in L^2((0, T); V') \quad (3.52)$$

of the weak evolution equation

$$\frac{d}{dt}c(\dot{u}_\varepsilon, v) + b_\varepsilon(\dot{u}_\varepsilon, v) + a(u_\varepsilon, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T), \quad (3.53)$$

satisfying the initial conditions (3.2), i.e. $u_\varepsilon(0) = u^0$ and $\dot{u}_\varepsilon(0) = u^1$.

Due to V -coercivity of $b_{0,\varepsilon}$ the evolution equation (3.53) may be interpreted as a **parabolic** equation. In particular, comparing conditions (3.52) with (3.12) shows that weak solutions are more regular than in the hyperbolic case: We have $\dot{u}_\varepsilon(t) \in V$, which means that the initial condition $\dot{u}_\varepsilon(0) = u^1 \in H$ is smoothed by the evolution.

Proof. We first establish **a priori energy estimates**. Let u_ε be a weak solution of the parabolic problem (3.53) with regularity (3.52). Then $u_\varepsilon \in H^1((0, T); V)$ and the following calculation – in contrast to the hyperbolic case – is rigorous (see Lemma 3.9):

$$\begin{aligned} & \left\langle \frac{d}{dt}(C\dot{u}_\varepsilon), \dot{u}_\varepsilon \right\rangle + b_\varepsilon(\dot{u}_\varepsilon, \dot{u}_\varepsilon) + a(u_\varepsilon, \dot{u}_\varepsilon) = \langle f, \dot{u}_\varepsilon \rangle \\ \implies & \quad \frac{1}{2} c(\dot{u}_\varepsilon, \dot{u}_\varepsilon) \Big|_0^t + \int_0^t (b_\varepsilon(\dot{u}_\varepsilon, \dot{u}_\varepsilon) + a(u_\varepsilon, \dot{u}_\varepsilon)) dt' = \int_0^t \langle f, \dot{u}_\varepsilon \rangle dt' \\ \implies & \quad \frac{1}{2} (c(\dot{u}_\varepsilon, \dot{u}_\varepsilon) + a_0(u_\varepsilon, u_\varepsilon)) \Big|_0^t + \int_0^t (b_\varepsilon(\dot{u}_\varepsilon, \dot{u}_\varepsilon) + a_1(u_\varepsilon, \dot{u}_\varepsilon)) dt' = \int_0^t \langle f, \dot{u}_\varepsilon \rangle dt'. \end{aligned}$$

Thereby every weak solution u_ε satisfies the analog of the energy equality (3.27) for a.e. $t \in (0, T)$. Splitting of $b_\varepsilon = b_1 + b_{0,\varepsilon}$ and multiplication by two gives

$$\begin{aligned} & c(\dot{u}_\varepsilon(t), \dot{u}_\varepsilon(t)) + a_0(u_\varepsilon(t), u_\varepsilon(t)) + 2 \int_0^t b_{0,\varepsilon}(\dot{u}_\varepsilon, \dot{u}_\varepsilon) dt' \\ & = c(u^1, u^1) + a_0(u^0, u^0) + \int_0^t \langle f, \dot{u}_\varepsilon \rangle dt' - 2 \int_0^t (b_1(\dot{u}_\varepsilon, \dot{u}_\varepsilon) + a_1(u_\varepsilon, \dot{u}_\varepsilon)) dt', \end{aligned} \quad (3.54)$$

which is the starting point for estimates. In principle, the individual steps are the same as hyperbolic case (see the proof of Proposition 3.11). Upon replacing $b_1 = b$ and also employing the coercivity of $b_{0,\varepsilon}$ (Lemma 3.19), we arrive at the following counterpart of the intermediate estimate (3.33):

$$\begin{aligned} & \gamma \|\dot{u}_\varepsilon(t)\|_H^2 + \alpha \|u_\varepsilon(t)\|_V^2 + 2\beta_\varepsilon \int_0^t \|\dot{u}_\varepsilon\|_V^2 dt' \leq c_c \|u^1\|_H^2 + (c_a + 2\lambda c_V^2) \|u^0\|_V^2 \\ & \quad + \int_0^t \|f\|_H^2 dt' + \int_0^t ((1 + c_{a_1} + 2c_b + 2\lambda T) \|\dot{u}_\varepsilon\|_H^2 + c_{a_1} \|u_\varepsilon\|_V^2) dt'. \end{aligned} \quad (3.55)$$

With $\beta_\varepsilon = \varepsilon\alpha$ and estimating $2\alpha \geq \alpha \geq \min(\alpha, \gamma)$ this yields

$$\|u_\varepsilon(t)\|_V^2 + \|\dot{u}_\varepsilon(t)\|_H^2 + \varepsilon \int_0^t \|\dot{u}_\varepsilon\|_V^2 dt' \leq k_1 + k_2 \int_0^t (\|u_\varepsilon\|_V^2 + \|\dot{u}_\varepsilon\|_H^2) dt', \quad (3.56)$$

3 Variational solution of linear second-order evolution equations

where the constants are the same as in the hyperbolic case, (3.30) and (3.31):

$$k_1 = \frac{(c_a + 2\lambda c_V^2)\|u^0\|_V^2 + c_c\|u^1\|_H^2 + \int_0^T \|f\|_H^2 dt'}{\min(\alpha, \gamma)} \quad \text{and} \quad k_2 = \frac{1 + c_{a_1} + 2c_b + 2\lambda T}{\min(\alpha, \gamma)}.$$

In terms of $\phi_\varepsilon = \|u_\varepsilon\|_V^2 + \|\dot{u}_\varepsilon\|_H^2$ we thus obtained

$$\phi_\varepsilon(t) + \varepsilon \int_0^t \|\dot{u}_\varepsilon(t')\|_V^2 dt' \leq k_1 + k_2 \int_0^t \phi_\varepsilon(t') dt'.$$

In particular, omitting the viscosity term $\varepsilon \int_0^t \|\dot{u}_\varepsilon\|_V^2 dt' \geq 0$ also yields the estimate (3.34)

$$\phi_\varepsilon(t) \leq k_1 + k_2 \int_0^t \phi_\varepsilon(t') dt'.$$

Then Gronwall's inequality (Lemma 3.10) gives the energy estimate

$$\|u_\varepsilon(t)\|_V^2 + \|\dot{u}_\varepsilon(t)\|_H^2 \leq k_1 e^{k_2 t}.$$

Uniqueness then follows by the arguments given in Section 3.3.2. Existence may be deduced by the Faedo-Galerkin method similarly as in Section 3.4.1. Details and further results are provided in [DL92, Chapter XVIII, §5, Problem (P_1), p. 552]. \square

Lemma 3.21 (Continuity in the parabolic case). *If u satisfies (3.52), that is*

$$u \in L^2((0, T); V), \quad \dot{u} \in L^2((0, T); V), \quad \frac{d}{dt}(Cu) \in L^2((0, T); V'),$$

then

$$u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H).$$

In particular, the initial conditions (3.2), i.e. $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$, make sense.

Proof. If u has the regularity (3.52) then $\dot{u} \in W$ and $u \in \widetilde{W}$ for the Hilbert spaces

$$\begin{aligned} W &:= \{w \in L^2((0, T); V) : \frac{d}{dt}(Cw) \in L^2((0, T); V')\}, \\ \widetilde{W} &:= \{u \in L^2((0, T); V) : \dot{u} \in L^2((0, T); V), \frac{d}{dt}(Cu) \in L^2((0, T); V')\}. \end{aligned} \quad (3.57)$$

These are Hilbert spaces with norms $\|\cdot\|_W$ and $\|\cdot\|_{\widetilde{W}}$ given by

$$\begin{aligned} \|w\|_W^2 &= \|w\|_{L^2((0, T); V)}^2 + \left\| \frac{d}{dt}(Cw) \right\|_{L^2((0, T); V')}^2 = \int_0^T \|w(t)\|_V^2 dt + \int_0^T \left\| \frac{d}{dt}(Cw(t)) \right\|_{V'}^2 dt, \\ \|u\|_{\widetilde{W}}^2 &= \|u\|_{L^2((0, T); V)}^2 + \|\dot{u}\|_W^2 = \int_0^T \|u(t)\|_V^2 dt + \int_0^T \|\dot{u}(t)\|_V^2 dt + \int_0^T \left\| \frac{d}{dt}(Cu(t)) \right\|_{V'}^2 dt. \end{aligned}$$

Then the identities

$$u \in \widetilde{W} \subseteq \mathcal{C}^0([0, T]; V) \quad \text{and} \quad \dot{u} \in W \subseteq \mathcal{C}^0([0, T]; H)$$

hold by (3.58) and (3.59) below: The first result is true by Sobolev embedding,

$$\widetilde{W} \hookrightarrow H^1((0, T); V) \hookrightarrow \mathcal{C}^0([0, T]; V). \quad (3.58)$$

Secondly we have the Aubin-Lions type result [DL92, (5.28), Remark 1, p. 555]

$$W \hookrightarrow \mathcal{C}^0([0, T]; H). \quad (3.59)$$

If $C = \text{Id}_V$ (or if $C \in \text{Lin}(V, V)$ is symmetric and invertible) then W reduces to

$$W = \{w \in L^2((0, T); V) : \dot{w} \in L^2((0, T); V')\}.$$

In this case the embedding (3.59) coincides with

$$L^2((0, T); V) \cap H^1((0, T); V') \hookrightarrow \mathcal{C}^0([0, T]; H),$$

which is (2.10) of Lemma 2.16. In fact, by positivity and symmetry of $C \in \text{Lin}(H, H)$, one may obtain (3.59) by the same proof, with $\langle w|w \rangle_H$ replaced by $c(w, w) = \langle Cw|w \rangle_H$. \square

However, weak solutions of the hyperbolic evolution equations like (3.1) lie in

$$\widetilde{\widetilde{W}} := \{u \in L^2((0, T); V) : \dot{u} \in L^2((0, T); H), \frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')\}, \quad (3.60)$$

which is larger than \widetilde{W} (3.57). In particular, in contrast to the parabolic case, $u \in \widetilde{\widetilde{W}}$ does not directly imply $\dot{u} \in \mathcal{C}^0([0, T]; H)$; the proof of continuity requires more efforts (see Section 3.5).

Remark 3.22 (Perturbation of the viscosity term in parabolic problems). In order to prove existence and uniqueness of the parabolic problem (cf. Theorem 3.20), it suffices to assume that $b_1: V \times V \rightarrow \mathbb{R}$ enjoys additional regularity in only its second slot, that is, there exists $c_{b_1} > 0$ such that

$$|b_1(w, v)| \leq c_{b_1} \|w\|_V \|v\|_H \quad \forall w, v \in V.$$

From the algebraic inequality $2xy \leq (1/\delta)x^2 + \delta y^2$ applied to the norms and for $\delta = c_{b_1}/\beta \neq 0$ one then obtains the estimate

$$2|b_1(\dot{u}, \dot{u})| \leq 2c_{b_1} \|\dot{u}\|_V \|\dot{u}\|_H \leq \beta \|\dot{u}\|_V^2 + (c_{b_1}^2/\beta) \|\dot{u}\|_H^2.$$

The term $\beta \|\dot{u}\|_V^2$ is eventually absorbed in the coercivity estimate $2b_0(\dot{u}, \dot{u}) \geq 2\beta \|\dot{u}\|_V^2$ on the left-hand-side of (3.55). \lrcorner

We are ready to prove existence for the hyperbolic problem by letting the viscosity go to zero:

Proposition 3.23 (The vanishing viscosity limit). *If $\varepsilon \rightarrow 0$ then the solutions u_ε of the regularized problem (3.53) converge to u in $\mathcal{D}'((0, T); H)$ which has the weak regularity (3.13):*

$$u \in L^\infty((0, T); V) \cap W^{1, \infty}((0, T); H).$$

Moreover, u is a solution of the original Cauchy problem (3.1), (3.2).

Proof. Let $u_\varepsilon \in \widetilde{W} = \{u \in L^2((0, T); V) : \dot{u} \in L^2((0, T); V), \frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')\}$ be the solution of the regularized problem obtained in Theorem 3.20. As for the Faedo-Galerkin method, we take the steps of a variational method (cf. Section 2.3.2): Convergence of u_ε follows from boundedness, which comes from the estimates (Steps 1 & 2). Then we prove that the limit solves the original evolution equation (3.1) and check the initial conditions (3.2) (Steps 3 & 4).

– **Step 1: Energy estimates.** A priori energy estimates for u_ε have already been derived in the proof of Theorem 3.20. In particular, we have (3.56), where k_1, k_2 do not depend on ε :

$$\|u_\varepsilon(t)\|_V^2 + \|\dot{u}_\varepsilon(t)\|_H^2 + \varepsilon \int_0^t \|\dot{u}_\varepsilon\|_V^2 dt' \leq k_1 + k_2 \int_0^t (\|u_\varepsilon\|_V^2 + \|\dot{u}_\varepsilon\|_H^2) dt'.$$

3 Variational solution of linear second-order evolution equations

– **Step 2: Boundedness and convergence to a limit.** The energy estimate implies the following results for u_ε , with uniform bounds in ε :

$$\begin{aligned} \text{(i)} \quad & u_\varepsilon \text{ lies in a bounded set of } L^\infty((0, T); V), \\ \text{(ii)} \quad & \dot{u}_\varepsilon \text{ lies in a bounded set of } L^\infty((0, T); H), \\ \text{(iii)} \quad & \sqrt{\varepsilon} \dot{u}_\varepsilon \text{ lies in a bounded set of } L^2((0, T); V). \end{aligned} \quad (3.61)$$

By weak-* compactness, (i) & (ii) allow us to extract subsets $(u_{\varepsilon_k})_k \subseteq (u_\varepsilon)_\varepsilon$ & $(\dot{u}_{\varepsilon_k})_k \subseteq (\dot{u}_\varepsilon)_\varepsilon$ that are weakly-* convergent sequences as $k \rightarrow \infty$. To simplify the notation we again denote these sequences by u_ε & \dot{u}_ε and take the limit $\varepsilon \rightarrow 0$ (as before, convergence of \dot{u}_ε to \dot{u} holds because limits in $\mathcal{D}'((0, T); H)$ are unique):

$$\begin{aligned} \text{(i)} \quad & u_\varepsilon \rightharpoonup^* u \text{ in } L^\infty((0, T); V) \\ \text{(ii)} \quad & \dot{u}_\varepsilon \rightharpoonup^* \dot{u} \text{ in } L^\infty((0, T); H). \end{aligned}$$

Thereby, if $\varepsilon \rightarrow 0$ then u_ε converges a limit u , the **vanishing-viscosity solution** with the weak regularity (3.13):

$$u \in L^\infty((0, T); V) \quad \text{and} \quad \dot{u} \in L^\infty((0, T); H).$$

– **Step 3: The limit satisfies the evolution equation.** Our aim is to take the limit $\varepsilon \rightarrow 0$ in the evolution equation (3.53),

$$\frac{d}{dt} c(\dot{u}_\varepsilon, v) + b_\varepsilon(\dot{u}_\varepsilon, v) + a(u_\varepsilon, v) = \langle f, v \rangle,$$

that is,

$$\frac{d}{dt} (C\dot{u}_\varepsilon) + B_\varepsilon \dot{u}_\varepsilon + Au_\varepsilon = f \quad \text{a.e. in } L^2((0, T); V').$$

With $A = A_0 + A_1 \in \text{Lin}(V, V')$ and $B, C \in \text{Lin}(H, H)$, the uniform bounds (3.61) imply that

$$\begin{aligned} \text{(i)} \quad & Au_\varepsilon \text{ lies in a bounded set of } L^\infty((0, T); V'), \\ \text{(ii)} \quad & B\dot{u}_\varepsilon, C\dot{u}_\varepsilon \text{ lie in bounded sets of } L^\infty((0, T); H), \\ \text{(iii)} \quad & w_\varepsilon := \sqrt{\varepsilon}(A_0\dot{u}_\varepsilon + \lambda\dot{u}_\varepsilon) \text{ lies in a bounded set of } L^2((0, T); V') \end{aligned}$$

and weak-* compactness gives

$$\begin{aligned} \text{(i)} \quad & Au_\varepsilon \rightharpoonup^* Au \text{ in } L^\infty((0, T); V'), \\ \text{(ii)} \quad & B\dot{u}_\varepsilon \rightharpoonup^* B\dot{u}, \quad C\dot{u}_\varepsilon \rightharpoonup^* C\dot{u} \text{ in } L^\infty((0, T); H), \\ \text{(iii)} \quad & w_\varepsilon \rightharpoonup w \text{ in } L^2((0, T); V') \end{aligned} \quad (3.62)$$

as $\varepsilon \rightarrow 0$. By (iii), the artificial viscosity term disappears in the limit:

$$B_{0,\varepsilon}\dot{u}_\varepsilon = \varepsilon(A_0\dot{u}_\varepsilon + \lambda\dot{u}_\varepsilon) = \sqrt{\varepsilon} w_\varepsilon \rightharpoonup 0 \text{ in } L^2((0, T); V').$$

Indeed, if $\varphi \in L^2((0, T); V)$ then

$$\begin{aligned} \left\langle \sqrt{\varepsilon} w_\varepsilon, \varphi \right\rangle_{L^2((0, T); V'), L^2((0, T); V)} &= \sqrt{\varepsilon} \left\langle w_\varepsilon, \varphi \right\rangle_{L^2((0, T); V'), L^2((0, T); V)} \rightarrow 0 \quad (\varepsilon \rightarrow 0). \\ &\quad \downarrow \quad \downarrow \text{(iii)} \\ &0 \quad \left\langle w, \varphi \right\rangle_{L^2((0, T); V'), L^2((0, T); V)} \end{aligned}$$

Since weak-* convergence in L^∞ implies weak convergence in L^2 (see Lemma 2.15), properties (i), (ii) of (3.62) further imply

$$\begin{aligned} \text{(i)'} \quad & Au_\varepsilon \rightharpoonup Au \quad \text{in } L^2((0, T); V'), \\ \text{(ii)'} \quad & B\dot{u}_\varepsilon \rightharpoonup B\dot{u}, \quad C\dot{u}_\varepsilon \rightharpoonup C\dot{u} \quad \text{in } L^2((0, T); H), \quad \text{and thus also in } L^2((0, T); V'). \end{aligned}$$

In particular, we have obtained the following convergence result for $B_\varepsilon = B + B_{0,\varepsilon}$ as $\varepsilon \rightarrow 0$:

$$\text{(iii)'} \quad B_\varepsilon \dot{u}_\varepsilon \rightharpoonup B\dot{u} \quad \text{in } L^2((0, T); V'),$$

or, $b_\varepsilon(\dot{u}_\varepsilon, v) = \langle B_\varepsilon \dot{u}_\varepsilon, v \rangle \rightarrow \langle B\dot{u}, v \rangle = b(\dot{u}, v)$ for $v \in V$ and a.e. in $(0, T)$.

The regularized evolution equation and the convergence properties (i)' and (iii)' imply

$$\frac{d}{dt}(C\dot{u}_\varepsilon) = f - Au_\varepsilon - B_\varepsilon \dot{u}_\varepsilon \rightharpoonup f - Au - B\dot{u} \quad \text{in } L^2((0, T); V').$$

Moreover, since by (ii)' we already know $C\dot{u}_\varepsilon \rightharpoonup C\dot{u}$ in $L^2((0, T); H)$, uniqueness of distributional limits eventually gives equality

$$\frac{d}{dt}(C\dot{u}) = f - Au - B\dot{u}.$$

This shows that the limit u of the solution u_ε of the regularized problem satisfies the original hyperbolic equation in the vanishing viscosity limit $\varepsilon \rightarrow 0$.

– **Step 4: The limit satisfies the initial conditions.** One may argue similarly as in the proof of Proposition 3.15. An alternative way is suggested in [DL92, (5.127)], see Remark 3.17. \square

Finally we note that similar arguments as for the Faedo-Galerkin method (Remark 3.18) prevent us from deducing the energy equality (3.27) from its regularized counterpart (3.54) by $\varepsilon \rightarrow 0$.

3.5 Regularity

From Sections 3.3 and 3.4 it follows that there exists a unique solution u of the evolution equation (3.1) with weak regularity (3.13):

$$u \in L^\infty((0, T); V) \cap W^{1,\infty}((0, T); H).$$

In order to complete the proof of Theorem 3.1, we still need to show the continuity property (3.12):

$$u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H).$$

We have already seen that Sobolev embedding and the validity of the evolution equation yield the following additional regularity properties of weak solutions (see (3.14) and Lemma 3.3):

$$u \in \mathcal{C}^0([0, T]; H) \quad \text{and} \quad \frac{d}{dt}(C\dot{u}) \in L^2((0, T); V').$$

Let us summarize the results gathered so far:

Lemma 3.24 (An intermediate regularity result). *If u is a weak solution of (3.1), then the following continuity and boundedness properties hold:*

$$u \in \mathcal{C}^0([0, T]; H) \cap L^\infty((0, T); V), \tag{3.63}$$

$$C\dot{u} \in \mathcal{C}^0([0, T]; V') \cap L^\infty((0, T); H). \tag{3.64}$$

3 Variational solution of linear second-order evolution equations

Proof. Recall from assumption (3.13) that $u \in L^\infty((0, T); V)$ and $\dot{u} \in L^\infty((0, T); H)$. With the Sobolev embedding result (3.14), relation (3.63) is immediate. Boundedness in (3.64) follows from

$$\dot{u} \in L^\infty((0, T); H) \quad \text{and} \quad C \in \text{Lin}(H, H) \quad \implies \quad C\dot{u} \in L^\infty((0, T); H).$$

Thus $C\dot{u} \in L^\infty((0, T); H) \subseteq L^2((0, T); H) \subseteq L^2((0, T); V')$ and with $\frac{d}{dt}(C\dot{u}) \in L^2((0, T); V')$ from the evolution equation we obtain

$$C\dot{u} \in H^1((0, T); V'). \quad (3.65)$$

Sobolev embedding then gives continuity: $C\dot{u} \in \mathcal{C}^0([0, T]; V')$. \square

Remark 3.25 (Result for \dot{u}). If $C \in \text{Lin}(V', V')$ then (3.65) shows that weak solutions satisfy $\dot{u} \in H^1((0, T); V')$. Hence (3.64) holds with \dot{u} instead of $C\dot{u}$ (this is the result in [DL92, (5.128)]):

$$\dot{u} \in \mathcal{C}^0([0, T]; V') \cap L^\infty((0, T); H).$$

Moreover, $\ddot{u} \in L^2((0, T); V')$ and with $u \in L^2((0, T); V)$ we get $u \in H^2((0, T); V')$. \lrcorner

The Equations (3.63) and (3.64) are called “intermediate” because they assert continuity with the spaces H, V' that are too large compared to V, H in (3.12). However, we already have **weak continuity** for u, \dot{u} in the correct target spaces, because by Lemma 2.19,

$$\begin{aligned} \mathcal{C}^0([0, T]; H) \cap L^\infty((0, T); V) &\subseteq \mathcal{C}_{\text{weak}}^0([0, T]; V), \\ \mathcal{C}^0([0, T]; V') \cap L^\infty((0, T); H) &\subseteq \mathcal{C}_{\text{weak}}^0([0, T]; H). \end{aligned}$$

We briefly recall that $\mathcal{C}^0([0, T]; X) \subseteq \mathcal{C}_{\text{weak}}^0([0, T]; X)$ where (see Definition 2.18)

$$\begin{aligned} f \in \mathcal{C}^0([0, T]; X) &\iff t \mapsto \|f(t)\|_X \text{ is continuous on } [0, T], \\ f \in \mathcal{C}_{\text{weak}}^0([0, T]; X) &\iff \forall x' \in X' : t \mapsto \langle x', f(t) \rangle_{X', X} \text{ is continuous on } [0, T]. \end{aligned}$$

Thus Equations (3.63), (3.64) imply that u and \dot{u} are weakly continuous, if u is a weak solution of (3.1):

$$u \in \mathcal{C}_{\text{weak}}^0([0, T]; V) \quad \text{and} \quad \dot{u} \in \mathcal{C}_{\text{weak}}^0([0, T]; H). \quad (3.66)$$

Here the result for \dot{u} is deduced from $C\dot{u} \in \mathcal{C}_{\text{weak}}^0([0, T]; H)$ and from invertibility of C on H .

In order to show continuity, that is

$$u \in \mathcal{C}^0([0, T]; V) \quad \text{and} \quad \dot{u} \in \mathcal{C}^0([0, T]; H),$$

we first establish continuity of a variant of the stored energy:

Lemma 3.26 (Continuity of the energy). *Let u be a weak solution of (3.1). Then $t \mapsto E_\lambda(u)(t) := c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t)) + \lambda\|u(t)\|_H^2$ is continuous on $[0, T]$.*

With the energy $E(u)$ defined in (3.28), we thus have

$$E_\lambda(u) = c(\dot{u}, \dot{u}) + a_0(u, u) + \lambda\|u\|_H^2 = 2E(u) + \lambda\|u\|_H^2.$$

The crucial observation is that coercivity and continuity of c and a_0 yield the estimates

$$\gamma\|\dot{u}\|_H^2 + \alpha\|u\|_V^2 \leq E_\lambda(u) \leq c_c\|\dot{u}\|_H^2 + c_a\|u\|_V^2. \quad (3.67)$$

Next we present two proofs of Lemma 3.26. The first proof relies on the **energy equality** (Proposition 3.8), which is the approach taken in [DL92, Chapter XVIII, 5.3.3 (ii), p. 578]:

Proof. Let $s, t \in [0, T]$ such that $s < t$. Then, by the energy equality (3.27),

$$\begin{aligned} E_\lambda(u)(t) &= \lambda \|u(t)\|_H^2 + c(\dot{u}(t), \dot{u}(t)) + a_0(u(t), u(t)) \\ &= \lambda \|u(t)\|_H^2 + c(u^1, u^1) + a_0(u^0, u^0) + 2 \int_0^t (\langle f|\dot{u} \rangle_H - b(\dot{u}, \dot{u}) - a_1(u, \dot{u})) \, dt' \end{aligned}$$

and analogously for $E_\lambda(u)(s)$. Consequently,

$$\begin{aligned} |E_\lambda(u)(t) - E_\lambda(u)(s)| &= \left| \lambda \|u(t)\|_H^2 - \lambda \|u(s)\|_H^2 + 2 \int_s^t d(t') \, dt' \right| \\ &\leq \lambda \left| \|u(t)\|_H^2 - \|u(s)\|_H^2 \right| + 2 \int_s^t |d(t')| \, dt' \end{aligned}$$

with the dissipative term

$$d(t') := \langle f(t')|\dot{u}(t') \rangle_H - b(\dot{u}(t'), \dot{u}(t')) - a_1(u(t'), \dot{u}(t')).$$

Since by (3.13) weak solutions satisfy $u \in L^\infty((0, T); V)$, $\dot{u} \in L^\infty((0, T); H)$, and $f \in L^2((0, T); H)$, it follows that d is bounded:

$$d \in L^\infty(0, T) \subseteq L^1_{\text{loc}}(0, T) \quad \implies \quad \lim_{s \rightarrow t} \int_s^t |d(t')| \, dt' = 0.$$

With $u \in \mathcal{C}^0([0, T]; H)$ from (3.14) we get

$$\lim_{s \rightarrow t} \|u(s)\|_H^2 = \|u(t)\|_H^2.$$

Therefore, $E_\lambda(u)(s) \rightarrow E_\lambda(u)(t)$ as $s \rightarrow t$, which shows that $E_\lambda(u)$ is continuous. \square

In [RR04, Section 11.2.4, p. 393] another proof of Lemma 3.26 is presented. It is based on the **approximated energy equality** obtained from the Faedo-Galerkin method, but does not depend on the validity of the exact energy equality (Proposition 3.8):

Proof. We first show that $E_\lambda(u)$ is right-continuous at $t = 0$, that is,

$$\lim_{t \rightarrow 0^+} (E_\lambda(u)(t)) = E_\lambda(u)(0) = c(u^1, u^1) + a_0(u^0, u^0) + \lambda \|u^0\|_H^2. \quad (3.68)$$

If $E_\lambda(u)$ is considered as a function of u and \dot{u} separately, we obtain

$$\tilde{E}_\lambda(u, w) := c(w, w) + a_0(u, u) + \lambda \|u\|_H^2$$

for $u \in L^\infty((0, T); V)$ and $w \in L^\infty((0, T); H)$. By definition, $\tilde{E}_\lambda(u, \dot{u}) = E_\lambda(u)$. Moreover, by (3.67)

$$\gamma \|w\|_H^2 + \alpha \|u\|_V^2 \leq \tilde{E}_\lambda(u, w) \leq c_c \|w\|_H^2 + c_a \|u\|_V^2.$$

Therefore $\sup_{t \in [0, s]} (\tilde{E}_\lambda(\cdot, \cdot)(t))$ is equivalent to the squared norm in $L^\infty((0, s); V) \times L^\infty((0, s); H)$,

$$\sup_{t \in [0, s]} (\tilde{E}_\lambda(u, w)(t)) \cong \left\| \begin{pmatrix} u \\ w \end{pmatrix} \right\|_{L^\infty((0, s); V) \times L^\infty((0, s); H)}^2 := \sup_{t \in [0, s]} (\|u(t)\|_V^2 + \|w(t)\|_H^2),$$

for any $s \in (0, T]$. In particular, $t \mapsto \tilde{E}_\lambda(u, \dot{u})(t) = E_\lambda(u)(t)$ is (equivalent to) the composition $f \circ g$ of $g: t \mapsto (u(t), \dot{u}(t))$ and f the norm in the product space $V \times H$. But since g is weakly

3 Variational solution of linear second-order evolution equations

continuous by (3.66) and f is w.l.s.c. as a norm, their composition is l.s.c. (Lemma 2.22). Thereby we obtain the first inequality

$$\liminf_{t \rightarrow 0^+} (E_\lambda(u)(t)) = \liminf_{t \rightarrow 0^+} \left(\tilde{E}_\lambda(u, \dot{u})(t) \right) \geq \tilde{E}_\lambda(u, \dot{u})(0) = E_\lambda(u)(0).$$

Next, the converse inequality will be deduced with the help of the energy equality (3.42) for the solution $u_m \in H^2((0, T); V_m)$ of the approximated Cauchy problem (Lemma 3.14): For $t \in [0, T]$,

$$\begin{aligned} \underbrace{c(\dot{u}_m(t), \dot{u}_m(t)) + a_0(u_m(t), u_m(t))}_{= E_\lambda(u_m)(t) - \lambda \|u_m(t)\|_H^2} &= c(u_m^1, u_m^1) + a_0(u_m^0, u_m^0) + 2 \int_0^t \langle f | \dot{u}_m \rangle_H dt' \\ &\quad - 2 \int_0^t (b(\dot{u}_m, \dot{u}_m) + a_1(u_m, \dot{u}_m)) dt', \\ \implies \tilde{E}_\lambda(u_m, \dot{u}_m)(t) = E_\lambda(u_m)(t) &= c(u_m^1, u_m^1) + a_0(u_m^0, u_m^0) + \lambda \|u_m(t)\|_H^2 \\ &\quad + 2 \int_0^t \underbrace{(\langle f | \dot{u}_m \rangle_H - b(\dot{u}_m, \dot{u}_m) - a_1(u_m, \dot{u}_m))}_{=: d_m(t')} dt'. \end{aligned} \quad (3.69)$$

From the approximated energy estimate (3.43),

$$\sup_{t \in [0, s]} \left(\tilde{E}_\lambda(u_m, \dot{u}_m)(t) \right) < \infty$$

which by equivalence of norms means that the pair (u_m, \dot{u}_m) lies in a bounded subset of the product space $L^\infty((0, s); V) \times L^\infty((0, s); H)$. Bounded sets in this space are weakly-* compact and, in particular, weakly-* closed. Hence the limit $(u, \dot{u}) = \lim_{m \rightarrow \infty} (u_m, \dot{u}_m)$ exists and has the same bounds:

$$\sup_{t \in [0, s]} \left(\tilde{E}_\lambda(u, \dot{u})(t) \right) = \lim_{m \rightarrow \infty} \left(\sup_{t \in [0, s]} \left(\tilde{E}_\lambda(u_m, \dot{u}_m)(t) \right) \right) \leq \limsup_{m \rightarrow \infty} \left(\sup_{t \in [0, s]} \left(\tilde{E}_\lambda(u_m, \dot{u}_m)(t) \right) \right).$$

Actually, the weak-* convergence of (u_m, \dot{u}_m) has already been established in the course of the proof of Proposition 3.15, where we obtained (3.45):

- (i) $u_m \rightharpoonup^* u$ in $L^\infty((0, T); V)$
- (ii) $\dot{u}_m \rightharpoonup^* \dot{u}$ in $L^\infty((0, T); H)$.

Let us discuss the limits of the terms on the right hand side of (3.69) as $m \rightarrow \infty$. By construction of the Galerkin approximation (3.37), we have $u_m^0 \rightarrow u^0$ in V and $u_m^1 \rightarrow u^1$ in H , which by continuity of a_0 and c gives

$$c(u_m^1, u_m^1) + a_0(u_m^0, u_m^0) \rightarrow c(u^1, u^1) + a_0(u^0, u^0).$$

Moreover, $u_m \in H^2((0, T); V) \subseteq \mathcal{C}^0([0, T]; H)$ by Sobolev embedding and $u \in \mathcal{C}^0([0, T]; H)$ by (3.14), which with $u_m \rightharpoonup^* u$ in $L^\infty((0, T); H)$ from (i) implies that

$$\|u_m(t)\|_H^2 \rightarrow \|u(t)\|_H^2$$

holds for all $t \in [0, T]$. In particular we can write

$$\sup_{t \in [0, s]} \|u(t)\|_H = \max_{t \in [0, s]} \|u(t)\|_H = \|u\|_{\mathcal{C}^0([0, s]; H)}.$$

With

$$\sup_{t \in [0, s]} \int_0^t |d_m(t')| dt' \leq \int_0^s |d_m(t')| dt',$$

we then obtain in the limit superior $m \rightarrow \infty$, that

$$\sup_{t \in [0, s]} \left(\tilde{E}_\lambda(u, \dot{u})(t) \right) \leq c(u^1, u^1) + a_0(u^0, u^0) + \lambda \|u\|_{\mathcal{C}^0([0, s]; H)}^2 + \limsup_{m \rightarrow \infty} \int_0^s |d_m(t')| dt'.$$

Now we let s tend to zero. As in the previous proof, the dissipative term d_m as well as its limit d are bounded. Consequently

$$\lim_{s \rightarrow 0} \left(\limsup_{m \rightarrow \infty} \int_0^s |d_m(t')| dt' \right) = 0$$

and we arrive at

$$\limsup_{t \rightarrow 0+} (E_\lambda(u)(t)) = \limsup_{t \rightarrow 0+} \left(\tilde{E}_\lambda(u, \dot{u})(t) \right) \leq c(u^1, u^1) + a_0(u^0, u^0) + \lambda \|u^0\|_H^2 = E_\lambda(u)(0).$$

Together, the inequalities for $\liminf_{t \rightarrow 0+}$ and $\limsup_{t \rightarrow 0+}$ imply $\lim_{t \rightarrow 0+} E_\lambda(u)(t) = E_\lambda(u)(0)$, that is (3.68).

The same arguments with another arbitrary starting time instead of $t = 0$ allow us to deduce right-continuity of $t \mapsto E_\lambda(u)(t)$ at all $t \in [0, T]$. Finally, time-reversibility of the equations imply that we have actually established continuity of $E_\lambda(u)$ on $[0, T]$. \square

The following result eventually finishes the proof of Theorem 3.1:

Corollary 3.27 (Continuity). *If u is a weak solution of (3.1), then u possesses the regularity (3.12), that is $u \in \mathcal{C}^0([0, T]; V)$ and $\dot{u} \in \mathcal{C}^0([0, T]; H)$.*

Proof. Let $s, t \in [0, T]$. Coercivity, bilinearity, and symmetry give rise to the following long but simple calculation ([DL92, RR04] proceed along the same lines):

$$\begin{aligned} 0 &\leq \gamma \|\dot{u}(s) - \dot{u}(t)\|_H^2 + \alpha \|u(s) - u(t)\|_V^2 \\ &\leq c(\dot{u}(s) - \dot{u}(t), \dot{u}(s) - \dot{u}(t)) + a_0(u(s) - u(t), u(s) - u(t)) + \lambda \langle u(s) - u(t) | u(s) - u(t) \rangle_H \\ &= c(\dot{u}(s), \dot{u}(s)) - 2c(\dot{u}(s), \dot{u}(t)) + c(\dot{u}(t), \dot{u}(t)) \\ &\quad + a_0(u(s), u(s)) - 2a_0(u(s), u(t)) + a_0(u(t), u(t)) \\ &\quad + \lambda \langle u(s) | u(s) \rangle_H - 2\lambda \langle u(s) | u(t) \rangle_H + \lambda \langle u(t) | u(t) \rangle_H \\ &= E_\lambda(u)(s) - 2(c(\dot{u}(s), \dot{u}(t)) + a_0(u(s), u(t)) + \lambda \langle u(s) | u(t) \rangle_H) + E_\lambda(u)(t) =: \xi(s, t). \end{aligned}$$

In the limit $s \rightarrow t$,

$$\xi(s, t) = \underbrace{E_\lambda(u)(s)}_{\rightarrow E_\lambda(u)(t)} + E_\lambda(u)(t) - 2 \underbrace{(\langle C\dot{u}(t) | \dot{u}(s) \rangle_H + \langle A_0 u(t), u(s) \rangle + \lambda \langle u(t) | u(s) \rangle_H)}_{\rightarrow \langle C\dot{u}(t) | \dot{u}(t) \rangle_H + \langle A_0 u(t), u(t) \rangle + \lambda \langle u(t) | u(t) \rangle_H = E_\lambda(u)(t)} \rightarrow 0,$$

where the first convergence follows from continuity of $E_\lambda(u)$ (Lemma 3.26) and the second convergence holds by weak continuity (3.66) of u and \dot{u} . Consequently

$$0 \leq \gamma \|\dot{u}(s) - \dot{u}(t)\|_H^2 + \alpha \|u(s) - u(t)\|_V^2 \leq \xi(s, t) \rightarrow 0, \quad (s \rightarrow t).$$

This shows

$$\lim_{s \rightarrow t} \|u(s) - u(t)\|_V \rightarrow 0 \quad \text{and} \quad \lim_{s \rightarrow t} \|\dot{u}(s) - \dot{u}(t)\|_H = 0,$$

completing the proof. \square

Chapter 4

Application to the elastic wave equation

We solve the Cauchy problem for linearized elasticity by the variational method. In Section 4.1 the weak formulation of the Cauchy problem is introduced. Then we discuss the positivity conditions that will imply coercivity (Section 4.2). Finally, existence and uniqueness of weak solutions of displacement and traction problems is established (Section 4.3).

4.1 The weak form of the elastic wave equation

The Cauchy problem for the linear elastic wave equation was introduced in Section 1.2. We consider the problem on the open and bounded set $\Omega \subseteq \mathbb{R}^n$ and the time interval $I = [0, T]$. To avoid a clash of notation with the bilinear forms in the abstract setting, the classical elasticity tensor c will be denoted by Λ . First we simplify the boundary conditions (1.8) and restrict to conditions of homogeneous Dirichlet type ($\partial\Omega = \Gamma_D$ and $u_D = 0$).

The task is to find the displacement $u: \Omega \times [0, T] \rightarrow \mathbb{R}^n$ solving the elastic wave equation (1.7),

$$\rho \ddot{u} - \nabla \cdot (\Lambda : \nabla u) = f,$$

under the **homogeneous displacement conditions**

$$u|_{\partial\Omega \times [0, T]} = 0,$$

and the initial conditions (1.9), $u(\cdot, 0) = u^0$ and $\dot{u}(\cdot, 0) = u^1$.

Written at $(x, t) \in \Omega \times [0, T]$, (1.7) reads

$$\rho(x) \ddot{u}(x, t) - \nabla \cdot (\Lambda(x) : \nabla u(x, t)) = f(x, t).$$

We also recall the component version:

$$\rho(x) \ddot{u}_i(x, t) - \sum_{j,k,l=1}^n \partial_j (\Lambda_{ijkl}(x) \partial_l u_k(x, t)) = f_i(x, t) \quad \text{for } i = 1, \dots, n.$$

Classical solutions of the elastic wave equation are functions

$$u \in \mathcal{C}^2(\Omega \times [0, T])^n,$$

where it is assumed that density, elastic coefficients, force, and initial data have the regularity $\rho \in \mathcal{C}^0(\Omega)$, $\Lambda_{ijkl} \in \mathcal{C}^1(\Omega)$ for all $i, j, k, l = 1, \dots, n$, $f \in \mathcal{C}^0(\Omega \times [0, T])^n$, and $u^0, u^1 \in \mathcal{C}^0(\Omega)^n$.

4 Application to the elastic wave equation

Next we derive the variational formulation (Part I, see Section 2.3.2). If u is a classical solution, we can take the scalar product (in \mathbb{R}^n) with a **spatial test function** $v \in \mathcal{D}(\Omega)^n$, integrate over Ω , slightly rewrite the first term, and employ the divergence theorem in the second term:

$$\begin{aligned} & \int_{\Omega} (\rho \ddot{u} - \nabla \cdot (\Lambda : \nabla u)) \cdot v \, dV = \int_{\Omega} f \cdot v \, dV \\ \implies & \frac{d}{dt} \left(\int_{\Omega} \rho \dot{u} \cdot v \, dV \right) + \int_{\Omega} (\Lambda : \nabla u) : \nabla v \, dV = \int_{\Omega} f \cdot v \, dV. \end{aligned}$$

This expression makes sense as an equation in $\mathcal{D}'(0, T)$, if the coefficients ρ, Λ are in $L^\infty(\Omega)$,

$$u \in L^2((0, T); H_0^1(\Omega)^n) \quad \text{and} \quad \dot{u} \in L^2((0, T); L^2(\Omega)^n),$$

and for test functions $v \in H^1(\Omega)^n$. This motivates the following definition:

Definition 4.1 (Weak formulation). Let $\rho \in L^\infty(\Omega)$, $\Lambda_{ijkl} \in L^\infty(\Omega)$, and set

$$H := L^2(\Omega)^n \quad \text{and} \quad V := H_0^1(\Omega)^n. \quad (4.1)$$

Then a weak solution of the elastic wave equation is a function $u \in L^2((0, T); V) \cap H^1((0, T); H)$ that satisfies

$$\frac{d}{dt} c(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T), \quad (4.2)$$

where $f \in L^2((0, T); H)$ and

$$a: V \times V \rightarrow \mathbb{R}, \quad a(u, v) := \int_{\Omega} (\Lambda : \nabla u) : \nabla v \, dV, \quad (4.3)$$

$$c: H \times H \rightarrow \mathbb{R}, \quad c(w, v) := \int_{\Omega} \rho w \cdot v \, dV. \quad (4.4)$$

With the initial conditions (3.2), $u(0) = u^0 \in V$ and $\dot{u}(0) = u^1 \in H$, this defines a **weak solution of the Cauchy problem for the elastic wave equation**. \lrcorner

The spaces chosen in (4.1) indeed are separable real Hilbert spaces and satisfy $V \hookrightarrow H \hookrightarrow V'$ with continuous and dense embeddings (variational triple):

$$H_0^1(\Omega)^n \hookrightarrow L^2(\Omega)^n \hookrightarrow H^{-1}(\Omega)^n.$$

Equation (4.2) has the same form as the weak evolution equation (3.1), with $a = a_0$ and $b = 0$. Moreover, since f has values in H , the right hand side of (4.2) reads $\langle f, v \rangle = \langle f|_v \rangle_H$.

Remark 4.2 (Time-dependent coefficients). The methods developed in Chapter 3 can be extended to coefficients ρ, Λ (bilinear forms a, b, c) that are \mathcal{C}^1 functions of time, see [DL92, Chapter XVIII]. In this case the weak form (4.2) contains an additional term $b: H \times H \rightarrow \mathbb{R}$:

$$\rho \ddot{u} = \frac{d}{dt}(\rho \dot{u}) - \dot{\rho} \dot{u} \implies \int_{\Omega} \rho \ddot{u} \cdot v \, dV = \frac{d}{dt} \left(\int_{\Omega} \rho \dot{u} \cdot v \, dV \right) - \int_{\Omega} \dot{\rho} \dot{u} \cdot v \, dV = \frac{d}{dt} c(\dot{u}, v) + b(\dot{u}, v).$$

The form of the terms a (4.3) and c (4.4) is unchanged; they only involve t as an additional parameter via $\Lambda(\cdot, t)$ and $\rho(\cdot, t)$ respectively. \lrcorner

4.2 Positivity conditions for the elasticity tensor

In order to apply the theory developed in Chapter 3, coercivity of a and c has to be ensured. This is straightforward for c if the density ρ is strictly positive. In case of a , the elasticity tensor has to be positive definite in the sense given below:

Assumption 1 (Positivity of material parameters). Let $\rho \in L^\infty(\Omega)$ and $\Lambda_{ijkl} \in L^\infty(\Omega)$.

- (i) The density ρ is positive and uniformly bounded away from zero, that is, there exists $\mu > 0$ such that for a.a. $x \in \Omega$:

$$\mu < \rho(x).$$

- (ii) The elasticity tensor Λ is uniformly positive definite, in the sense that there exists $\lambda > 0$ such that for a.a. $x \in \Omega$:

$$X : \Lambda(x) : X \geq \lambda X : X \quad \text{for all } X \in \mathbb{R}^{n \times n}, \quad (4.5)$$

that is in components (and summation convention), $\Lambda_{ijkl}(x)X_{ij}X_{kl} \geq \lambda X_{ij}X_{ij}$. \lrcorner

The positivity condition (ii) in Assumption 1 is the **strict convexity** of the quadratic form associated to $\Lambda(x) \in \mathbb{R}^{n \times n \times n \times n}$ as a bilinear map on $\mathbb{R}^{n \times n}$ (uniformly in $x \in \Omega$).

Remark 4.3 (Convexity of the elastic energy density). Strict convexity of Λ is equivalent to strict convexity of the associated linearized elastic energy density $W : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$,

$$W(\varepsilon) = \frac{1}{2} \varepsilon : \Lambda : \varepsilon = \frac{1}{2} \nabla u : \Lambda : \nabla u$$

where $\varepsilon := \frac{1}{2}(\nabla u + \nabla u^T)$ is the (symmetric) linearized strain tensor. The second equality follows from the minor symmetries $\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk}$. In particular, $\Lambda = \partial_\varepsilon^2 W$, that is, $\Lambda_{ijkl} = \frac{\partial^2 W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$, which implies the major symmetry $\Lambda_{ijkl} = \Lambda_{klij}$. \lrcorner

Remark 4.4 (Rank-one convexity and reality of wave speeds). Strict convexity implies rank-one convexity (which is also known as the Legendre-Hadamard condition or strong ellipticity): There exists $\lambda > 0$, such that for a.a. $x \in \Omega$,

$$(\xi \otimes \eta) : \Lambda(x) : (\xi \otimes \eta) \geq \lambda |\xi|^2 |\eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^n, \quad (4.6)$$

that is in components (and summation convention), $\xi_i \eta_j \Lambda_{ijkl} \xi_k \eta_l \geq \lambda \xi_i^2 \eta_j^2$. This condition is equivalent to the positive definiteness of the acoustic tensor $A(\eta) \in \mathbb{R}^{n \times n}$, $A_{ik}(\eta) := \Lambda_{ijkl} \eta_j \eta_l$ for all propagation directions $\eta \in \mathbb{R}^n$ (indeed, $A_{ik}(\eta) \xi_i \xi_k \geq \alpha |\xi|^2$ with $\alpha := \lambda |\eta|^2 > 0$). In particular, rank-one convexity guarantees that plane wave solutions of $\rho \ddot{u} - \nabla \cdot (\Lambda : \nabla u) = 0$ in $\Omega = \mathbb{R}^n$ will propagate with real wave speeds [MH83, p. 240]. \lrcorner

Remark 4.5 (Ellipticity). Let $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a linear partial differential operator with smooth coefficients a_α on an open set $\Omega \subseteq \mathbb{R}^n$. Then P is called (uniformly) **elliptic** on Ω , if its principal symbol $p : \Omega \times \mathbb{R}^n \rightarrow \mathbb{C}$, $p(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x) (i\xi)^\alpha$ fulfills

$$p(x, \xi) \neq 0 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \xi \neq 0.$$

If P has second order, then $\xi \mapsto p(\cdot, \xi)$ is a quadratic form on \mathbb{R}^n , that is, $p(\cdot, \xi) = \xi \cdot A \cdot \xi$ with $A : \Omega \rightarrow \mathbb{R}^{n \times n}$. If A is uniformly positive definite, that is, if there exists $\alpha > 0$ such that

$$p(x, \xi) \geq \alpha |\xi|^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n, \quad (4.7)$$

i.e. for all $x \in \Omega$, $A(x)$ only has positive eigenvalues, then P is (uniformly) **strongly elliptic**. \lrcorner

4.3 Solution of displacement and traction problems

Application of Theorem 3.1 to the weak elastic wave equation yields the following result (which covers Parts II and III of the variational approach outlined in Section 2.3.2):

Theorem 4.6 (Existence and uniqueness for the homogeneous displacement problem). *Let $\rho \in L^\infty(\Omega)$ and $\Lambda_{ijkl} \in L^\infty(\Omega)$ both satisfy the positivity conditions of Assumption 1 and let a and c be defined by (4.3) and (4.4). With the choice (4.1) of spaces,*

$$H = L^2(\Omega)^n \quad \text{and} \quad V = H_0^1(\Omega)^n,$$

and given the data

$$u^0 \in V, \quad u^1 \in H, \quad \text{and} \quad f \in L^2((0, T); H),$$

there exists a unique solution $u \in \mathcal{C}^0([0, T]; V) \cap \mathcal{C}^1([0, T]; H)$ of the weak elastic wave equation (4.2),

$$\frac{d}{dt}c(\dot{u}, v) + a(u, v) = \langle f, v \rangle \quad \text{for all } v \in V \text{ in the sense of } \mathcal{D}'(0, T),$$

satisfying the initial conditions $u(0) = u^0$ and $\dot{u}(0) = u^1$.

Proof. It suffices to check the hypotheses (3.5) to (3.10) on $a: V \times V \rightarrow \mathbb{R}$ and $c: H \times H \rightarrow \mathbb{R}$ defined by (4.3) and (4.4):

$$a(w, v) = \int_{\Omega} (\Lambda : \nabla w) : \nabla v \, dV \quad \text{and} \quad c(w, v) = \int_{\Omega} \rho w \cdot v \, dV.$$

We use the abbreviations $L^2 := L^2(\Omega)$, $H^1 := H^1(\Omega)$, $L^\infty := L^\infty(\Omega)$ and will not indicate the dimension of spaces.

– **Properties of a .** Let $w, v \in V$. Continuity of a follows from boundedness of Λ :

$$|a(w, v)| = |\langle \Lambda : \nabla w | \nabla v \rangle_{L^2}| \leq \|\Lambda\|_{L^\infty} \|\nabla w\|_{L^2} \|\nabla v\|_{L^2} \leq \|\Lambda\|_{L^\infty} \|w\|_{H^1} \|v\|_{H^1} = \|\Lambda\|_{L^\infty} \|w\|_V \|v\|_V.$$

Symmetry of a is a consequence of the major symmetry $\Lambda_{ijkl} = \Lambda_{klij}$:

$$a(w, v) = \int_{\Omega} (\Lambda : \nabla w) : \nabla v \, dV = \int_{\Omega} \nabla v : \Lambda : \nabla w \, dV = \int_{\Omega} (\Lambda : \nabla v) : \nabla w \, dV = a(v, w).$$

Finally, positive definiteness of Λ from Assumption 1 (ii) yields coercivity of a :

$$\begin{aligned} a(w, w) &= \int_{\Omega} \underbrace{\nabla w(x) : \Lambda(x) : \nabla w(x)}_{\geq \lambda \nabla w(x) : \nabla w(x)} \, dV \geq \lambda \langle \nabla w | \nabla w \rangle_{L^2} = \lambda \|\nabla w\|_{L^2}^2 = \lambda \|w\|_{H^1}^2 - \lambda \|w\|_{L^2}^2 \\ &\implies a(w, w) \geq \lambda \|w\|_V^2 - \lambda \|w\|_H^2. \end{aligned}$$

– **Properties of c .** Let $w, v \in H$. Boundedness of ρ gives continuity of c :

$$|c(w, v)| = |\langle \rho w | v \rangle_{L^2}| \leq \|\rho\|_{L^\infty} \|w\|_{L^2} \|v\|_{L^2} = \|\rho\|_{L^\infty} \|w\|_H \|v\|_H.$$

Symmetry of c is clear and coercivity follows from the positivity of ρ , Assumption 1 (i):

$$c(w, w) = \int_{\Omega} \rho w \cdot w \, dV = \int_{\Omega} \underbrace{\rho(x)}_{> \mu} |w(x)|^2 \, dV \geq \mu \|w\|_{L^2}^2 = \mu \|w\|_H^2.$$

Since the bounded terms vanish identically ($a_1 = 0$, $b = 0$) the proof is complete. \square

An inspection of the proof shows that the **continuity** and **coercivity constants** of the bilinear forms are $c_a = \|\Lambda\|_{L^\infty}$, $c_c = \|\rho\|_{L^\infty}$ and $\alpha = \lambda$, $\gamma = \mu$. These material properties will appear in the energy estimates $\|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \leq k_1 e^{k_2 t}$ (3.29) through the constants k_1, k_2 .

Remark 4.7 (Weaker convexity assumptions). Existence and uniqueness of weak solutions of the Cauchy problem in classical linearized elasticity is still guaranteed, if the strict convexity (4.5) is replaced by rank-one convexity (4.6), see [HM78, MH83]. In the nonlinear setting also the intermediate conditions of polyconvexity and quasiconvexity play an important role, see [Bal02, Ant05]. \lrcorner

If Ω is a Lipschitz domain, then $u(t) \in V = H_0^1(\Omega)^n$ implies that the boundary conditions $u|_{\partial\Omega \times [0, T]} = 0$ hold in $H^{1/2}(\partial\Omega)$ in the sense of the Sobolev trace, (see Lemma 2.7).

More general boundary conditions can be incorporated by adapting the definition of the Hilbert spaces (4.1). The pivot space $H = L^2(\Omega)^n$ stays the same, but V will be defined as a closed subspace with

$$H_0^1(\Omega)^n \subseteq V \subseteq H^1(\Omega)^n.$$

In particular, if one chooses $V = H^1(\Omega)^n$, then the results of Theorem 4.6 will remain true for the **homogeneous traction conditions**

$$(\Lambda : \nabla u) \cdot \nu|_{\partial\Omega \times [0, T]} = 0.$$

On Lipschitz domains, these conditions can be interpreted to hold in $H^{-1/2}(\partial\Omega)$. We note that traction conditions arise as natural boundary conditions in the associated variational problem of stationary energy.

The **mixed conditions**

$$u|_{\Gamma_D \times [0, T]} = 0 \quad \text{and} \quad (\Lambda : \nabla u) \cdot \nu|_{\Gamma_N \times [0, T]} = 0$$

are covered by the choice $V = \{u \in H^1(\Omega)^n : u|_{\Gamma_D} = 0\}$. Finally, as is shown in [DL92, Example 10, p. 610], a modification of the source term in the weak formulation allows us to incorporate the general **inhomogeneous conditions** (1.8).

This flexibility is one of the main advantages of the variational approach to the solution of partial differential equations.

Remark 4.8 (Analysis for nonlinear elasticity). Problems concerning existence, uniqueness, and regularity of solutions of the nonlinear governing equations of elastodynamics (1.4),

$$\rho^0 \ddot{u} - \nabla \cdot T^{\text{PK}} = f \quad \text{with} \quad T^{\text{PK}} = \partial_F W(\cdot, 1_{n \times n} + \nabla u),$$

are challenging [MH83, Bal02, Ant05, Daf16]. The principal difficulty is quasilinearity, which makes the occurrence of shock waves possible; solvability results can typically only be established on short time intervals [HKM77, DH85]. Moreover, important requirements of frame-indifference, positive orientation, and global injectivity (Remark 1.1) are not easy to incorporate. In particular, they are partly incompatible with standard convexity assumptions and growth conditions. These difficulties already arise in elastostatics, where the equations reduce to $\nabla \cdot T^{\text{PK}} = 0$. This time-independent equation is typically formulated as an energy minimization problem and the direct method of calculus of variations can be invoked [Bal76, BM84, Cia88]. Yet, many questions, for instance concerning the regularity of minimizers and the validity of the Euler-Lagrange equations, are still open [Bal02]. \lrcorner

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Zusammenfassung

Die elastische Wellengleichung beschreibt das zeitliche Verhalten eines Kontinuums im Rahmen der klassischen linearen Elastizitätstheorie. In der vorliegenden Arbeit wird die Lösung des Cauchyproblems für dieses System linearer partieller Differentialgleichungen zweiter Ordnung mit Hilfe von Variationsmethoden präsentiert. Zunächst wird gezeigt, wie sich die elastische Wellengleichung aus der allgemeinen Bewegungsgleichung der Kontinuumsmechanik ergibt, wodurch das mathematische Modell motiviert wird. In weiterer Folge wird die variationelle Lösungsmethode diskutiert, welche auf dem Konzept von schwachen Lösungen beruht. Insbesondere wird die abstrakte Version dieser Methode in Hilberträumen erklärt. Die Beweise von Existenz, Eindeutigkeit und Regularität von Lösungen des abstrakten Cauchyproblems für eine allgemeine lineare Evolutionsgleichung zweiter Ordnung bilden den Hauptteil der Arbeit. Schließlich werden diese Ergebnisse auf die elastische Wellengleichung angewandt und somit die Lösbarkeit in Sobolevräumen für beschränkte und positive Materialparameter gezeigt.