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„Fusion of defects in Landau-Ginzburg models in a functorial approach"

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Abstract

In thesis we consider two-dimensional supersymmetric Landau-Ginzburg models with defects in it. When we impose B-type gluing conditions, a defect is described by a factorisation of the corresponding superpotential W. We represent this factorisation in a module homomorphism framework of $\mathbb{Z}_2$-graded modules over a polynomial ring R, they are called matrix factorisations. By considering functors $U : R\text{-}mod_f \to R'\text{-}mod_f$, where $R\text{-}mod_f$ denotes a R-module category with the morphisms given by the module homomorphisms, our aim is to construct a category of these functors and to find relations between the morphisms in the functor category and morphisms between matrix factorisations which are induced by the functors between the module categories. For this purpose we define a functor Ξ. It turns out that this functor is surjective on the morphism spaces in the case $R = \mathbb{C}[x]$ but fails to be injective. There are similar results in the case $R = \mathbb{C}[x, y]$ but we can show surjectivity only in special cases. We also managed to determine the kernel of Ξ in one special case.

Zusammenfassung

In der vorliegenden Arbeit betrachten wir zweidimensionale supersymmetrische Landau-Ginzburg Modelle mit Defekten darin. Diese Defekte können durch Faktorisierungen des zugehörigen Superpotentiales W beschrieben werden, wenn wir sog. B-Typ Bedinguingen von den Defekten fordern. Wir beschreiben diese Faktorisierung des Superpotentiales in Termen von Modulhomomorphismen über einem $\mathbb{Z}_2$ graduierten Modul eines Polynomringes R, diese speziellen Homomorphismsen werden im Folgenden Matrixfaktorisierungen genannt. Betrachten wir einen Funktor $U : R\text{-}mod_f \to R'\text{-}mod_f$, wobei $R\text{-}mod_f$ eine R-Modulkategorie bezeichnet in der die zugehörigen Morphismen durch die Modulhomomorphismen gegeben sind, ist es unser Ziel eine Kategorie aus diesen Funktoren zu bilden. Genauer wollen wir Relationen zwischen den Morphismen der Funktoren und der Morphismen zwischen Matrixfaktorisierungen, welche durch diese Funktoren induziert werden, untersuchen. Darum definieren wir einen Funktor Ξ zwischen der Kategorie der Fusionsfunktoren und der Kategorie Matrixfaktorisierungen. Konkret stellen wir uns die Frage ist Ξ surjektiv und oder injektiv auf den Morphismen? Es stellt sich heraus dass Ξ im Fall $R = \mathbb{C}[x]$ surjektiv aber nicht injektiv ist. Für den Fall $R = \mathbb{C}[x, y]$ gibt es ähnliche Resultate, wobei wir hier aber einschränkende Bedingungen fordern müssen. Es gelang für beide Fälle die zugehörigen Kerne in einem wichtigen Spezialfall zu bestimmen.

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Contents

Chapter 1

Introduction

The physical model we use in this thesis is similar to the well known model in solid state physics, the Landau-Ginzburg model. The only difference is that we are supposing our model is supersymmetric. These models are often used in quantum field theories and especially in String Theory.

Roughly speaking supersymmetry means that every bosonic particle has a supersymmetric fermionic partner particle. The idea of this uncommon symmetry in nature was born in the 1970's. Supersymmetry is a "hot" candidate for a grand unifying theory because it would extend the standard model in a way that possibly all forces are of equal strength at high energies, as in the young universe. The theory also provides an explanation for dark matter.

In particular supersymmetry says that physics has to be invariant under a bigger class of symmetry operations than only translations, rotations and boosts. We call these symmetries internal symmetries, where the symmetries which are described by the Poincaré-group are called spatial symmetries. In particular this means that we have to extend the Poincaré-algebra in a non trivial way. When we try to do this using extensions of ordinary Lie-algebras there is a very general result from Coleman and Mandula in 1967, which states that all interactions in a field theory would vanish if the symmetry group of the theory mixes spatial and internal symmetries.

But when we extend the Poincaré-algebra in the framework of graded Lie algebras it turns out there are nontrivial possible extensions of the Poincaré-algebra in $3+1$ or higher dimensions. The super-Poincaré-algebras. A system which is invariant under these transformations is called supersymmetric.

The supersymmetry transformation that changes fermionic particles into their bosonic superpartners can be viewed as a translation in a generalized space called superspace. The infinitesimal generators of these transformations we call supercharges. Superspace consists of the ordinary spatial coordinates and anti-commuting coordinates, so called Grassmann variables. This translation can be represented by a differential operator involving a mixture of ordinary derivatives and Grassmann derivatives. One important thing about supersymmetry transformations is that one can have more than one supercharge which generates the supersymmetry transformations. In this work we consider the case $N = 2$.

In the first part we follow the discourse of [1]. Suppose we have several $N = (2, 2)$ supersymmetric theories, each located at a special space-time region. If these space-time regions have a common boundary or include defects, the natural question that arises is: What happens at the boundary or at the defects? Our aim is to construct a theory which takes care of the boundaries and defects and is still supersymmetric (in a certain sense). Therefore we impose several conditions to our supercharges, how they should behave at the boundaries or defects. Common conditions are the so called A- and B -Type conditions. In general the variation under supersymmetry, taking these conditions into account, does not vanish. To achieve this one needs to introduce certain non-chiral superfields such that they compensate the supersymmetric variation of the full space theory. We will formulate a condition when the introduced fields compensates the variation. It turns out that these conditions include the factorisation of the superpotential (or the difference of the superpotentials in the case of a shared boundary) of the corresponding theory called matrix factorisation of the corresponding superpotential. We consider the situation where a Landau-Ginzburg model with superpotential W_1 is separated by a defect from a Landau-Ginzburg model with superpotential W_2. The B-type defects between the models are described by matrix factorisation of the difference $W_1 - W_2$ of the superpotentials. By modelling the superpotentials as polynomials we can understand these matrix factorisations as homomorphisms between modules over a unital polynomial ring R. The aim of this thesis is to work out a relation between the module homomorphisms and a category of functors between R-module categories.

Next we introduce the theory of functors in a general framework following the ideas in [2]. We consider $\mathbb{C}$-linear functors on free R-modules. Our aim here is to construct a category of functors. To form such a category we first need to define morphisms $\phi^{(n)}_{U,V}$ between two functors U and V. The defined morphisms are graded in that sense that they act on a sequence of R-module homomorphisms of a certain length n. By defining a differential d we can consider the cohomology classes of the graded morphisms and discuss the structure of morphism spaces.

In the following we have a closer look at functor categories. The objects of these categories are called fusion functors. These fusion functors satisfy a certain homogeneity condition for a specific polynomial R-module homomorphism W, the superpotential. The morphisms $\phi^{(n)}_{U,V}$ between the fusion functors U and V remain the same as in the section before.

One important thing about matrix factorisations is that they form a category too, where the morphisms between the objects are realised by the zeroth cohomology class of the differential δ. By defining a suitable differential δ_{Q_1,Q_2} on the space of morphisms between matrix factorisations Q_1, Q_2 we obtain a graded structure there too. In fact we have a $\mathbb{Z}_2$-grading on this space where morphisms of an odd degree are called fermionic and morphisms of an even degree are called bosonic, e.g. if $\psi^{(i)}$, with i odd, is a fermionic morphism, δ_{Q_1,Q_2} maps it to a bosonic morphism $\delta_{Q_1,Q_2}\psi^{(i)}$ between Q_1 and Q_2.

The authors of [2] defined a map Ξ from the space of morphisms of degree n to the morphism space of matrix factorisations, with $\Xi^Q\left(\phi^{(n)}\right) = \phi^{(n)}(Q, ..., Q)$, where Q is a matrix factorisation. It turns out that Ξ preserves the differential structures of the morphism spaces, i.e. we map d-closed or exact morphisms to δ-closed or exact morphisms. In this thesis we analyse the properties of this map Ξ^{I_W}, where I_W is called the identity defect. Since matrix factorisations are R module homomorphisms, they could be mapped to R' module homomorphisms under a fusion functor U. One can show that for a fusion functor U, a matrix factorisation Q and the identity defect I_W the following relation holds: $U\left(I_W\right) \otimes Q \cong U(Q)$.

The first question concerns the surjectivity of Ξ^{I_W} for modules over R. With the relation above we consider a given closed morphism between matrix factorisations $U(I_W)$ and $V(W_W)$, we want to construct the pre-image up to d exact terms. The authors in [2] showed that this map is indeed surjective in the case $R = \mathbb{C}[x]$.

By analysing the methods [2] we can construct a counter example which shows that Ξ fails to be injective. Here injectivity means that a morphism $\phi^{(n)}_{U,V}$ of degree n between the fusion functors U and V that is mapped to an exact morphism $\delta\psi^{(i)}$ where $\psi^{(i)}$ is bosonic for $i = 0$ or fermionic for $i = 1$ is exact itself. This means there is a morphism $\chi^{(n-1)}_{U,V}$ of degree $n-1$ such that $d\chi^{(n-1)}_{U,V} = \phi^{(n)}_{U,V}$, which we will see is not the case.
We then analyse the kernel of the map Ξ^{I_W} and find that it is given by the so-called Jacobi ideal $\langle \partial_x W \rangle$.

We also analysed the surjectivity in the case $R = \mathbb{C}[x_1, x_2]$. It turns out that we were not able to proof that the map Ξ^{I_W} is surjective because of a strange mixture of terms which are of order zero in the matrix factorisation and quadratic terms of the corresponding induced morphism. This mixture occurs because of the defining property of matrix factorisations. But it turns out that for $U = V = id$ the kernel is again given by the Jacobi ideal $\langle \partial_{x_1} W, \partial_{x_2} W \rangle$.

Chapter 2

Defects in Landau-Ginzburg models

To analyse the fusion of defects in Landau-Ginzburg models we need to describe defects in such models. The defects which preserve the so-called B-type supersymmetry can be represented by matrix factorisations of the difference of the superpotentials. The composition or "fusion" of defects preserving the B-type supersymmetry as well as their action on B-type boundary conditions is described in this framework. Many concepts of the underlying ideas of this chapther are rather involved, such that we will only focus on the most important parts. For further details there is a lot of literature introducing the reader to the theory of topological QFT. In the following we are presenting a summary of the introduction part in [1], covering everything we need to understand the physical background of this thesis.

2.1 Defects in $N = 2$ theories

In this thesis we consider only two-dimensional supersymmetric field theories with $N = (2, 2)$ supersymmetries. This means there are four odd elements of this super algebra which we denote by $Q_\pm$ and the conjugated operators by $\bar{Q}_\pm$. They satisfy the following anti commutation relation

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P \tag{2.1}$$

where P is the momentum and H the Hamiltonian. All other combinations of anti commutation relations between the odd elements are vanishing. By $\pm$ we distinguish between so-called left and right movers.

The first question that we have to ask ourselves is what happens if we consider a physical model which involves two supersymmetric theories on not necessarily disjoint regions of space. To be more precise suppose we have two theories which are supersymmetric with $N = (2, 2)$ defined on a certain region in $\mathbb{R}^2 \cong \mathbb{C}$ which share a common one-dimensional subset, called *defect* or *interface*. But what happens with supersymmetry in these *glued* theories? In the current chapter we want to give a answer to this question.

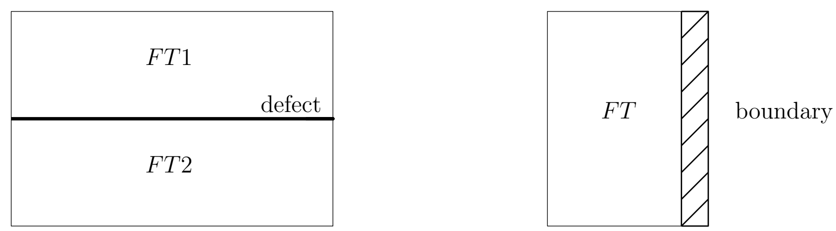

Figure 2.1: Left: Two field theories separated by a defect. Right: A theory with a boundary.

In the following we investigate two cases of possible bulk theories, therefore see figure 2.1 . The first case covers two such theories that are glued together at a defect, which is a common region of space-time with codimension 1 in both theories. The second case concerns a field theory which has a boundary. In the case of $N = 2$ theories there are two possible ways to implement such defects or boundaries. These are called *A*- and *B-type* defects or boundaries respectively. These *A*- and *B-type* conditions do not preserve the whole supersymmetry of the theories which is not possible in general, but they preserve supersymmetry with a modified variation. In the case of two theories glued together one can suppose that one preserves a kind of "diagonal supersymmetry" which will be discussed in detail below.

Note that boundaries break symmetries in every case and defects in general break some symmetries too, to see this consider the following counterexample. Suppose two two-dimensional theories in the complex plane $\mathbb{C}$ are glued together at the real line $\mathbb{R} \subset \mathbb{C}$. It is clear that this glued theory is not symmetric under translations, even if the two originating theories are invariant under translations. When we consider a boundary instead, the translational invariance is always broken.

For the following we need to understand the case of two theories glued together along a defect. Here we consider situations where two such theories have a common one-dimensional interface. We investigate a setup of supersymmetry preserving defects, i.e. those defects whose presence still allow the total theory to be supersymmetric under a constraint supersymmetric variation. Modelling the defect on the real line $\mathbb{R} \subset \mathbb{C}$ separating two possibly different theories on the upper and lower half plane we demand that:

- **B-type defect:** For B-type defects the following combinations of the supercharges are conserved

$$\begin{aligned} Q_B &= Q_+ + Q_-, \\ \bar{Q}_B &= \bar{Q}_+ + \bar{Q}_-. \end{aligned} \tag{2.2}$$

This immediately implies that along the defect line the supercharges have to fullfill the following "gluing conditions":

$$\begin{aligned} Q_+^{(1)} + Q_-^{(1)} &= Q_+^{(2)} + Q_-^{(2)}, \\ \bar{Q}_+^{(1)} + \bar{Q}_-^{(1)} &= \bar{Q}_+^{(2)} + \bar{Q}_-^{(2)}. \end{aligned} \tag{2.3}$$

The subscripts $^{(1)}$ and $^{(2)}$ refer to the two theories on upper and lower half plane respectively.

- **A-type defects:** On the other hand, the gluing conditions along the defect can be twisted by an automorphism of the supersymmetry algebra which exchanges $Q_\pm$ with $\bar{Q}_\pm$:

$$\begin{aligned} Q_+^{(1)} + \bar{Q}_-^{(1)} &= Q_+^{(2)} + \bar{Q}_-^{(2)}, \\ \bar{Q}_+^{(1)} + Q_-^{(1)} &= \bar{Q}_+^{(2)} + Q_-^{(2)}. \end{aligned} \tag{2.4}$$

They ensure that the combinations

$$\begin{aligned} Q_A &= Q_+ + \bar{Q}_-, \\ \bar{Q}_A &= \bar{Q}_+ + Q_-. \end{aligned} \tag{2.5}$$

are conserved.

If we deal with theories which have a boundary instead of a defect, we replace the phrase "defect" by "boundary" and apply the same conditions as above where the superscript $^{(2)}$ now stands for the boundary. In situations where defects as well as boundary conditions are present, A- or B-type supersymmetry can be preserved, in case all defects and boundaries are of$A$- and *B-type* respectively.

As mentioned above we cannot preserve the whole $N = (2,2)$ supersymmetry algebra in general when we glue two arbitrary theories together. But there are two special classes of defects which preserve the whole $N = (2,2)$ supersymmetry algebra. The first class is given by

$$Q_\pm^{(1)} = Q_\pm^{(2)}, \quad \bar{Q}_\pm^{(1)} = \bar{Q}_\pm^{(2)} \text{ on } \mathbb{R}, \tag{2.6}$$

which satisfies the conditions (2.4) and (2.3). One particular defect of this kind is the trivial defect, which implements the separation of one and the same theory. Defects of the second kind are related to the one of the first kind by mirror symmetry. They obey the following gluing conditions

$$\begin{aligned} Q_+^{(1)} &= Q_+^{(2)}, \quad \bar{Q}_+^{(1)} = \bar{Q}_+^{(2)} \\ Q_-^{(1)} &= \bar{Q}_-^{(2)}, \quad \bar{Q}_-^{(1)} = Q_-^{(2)} \text{ on } \mathbb{R}, \end{aligned} \tag{2.7}$$

The supersymmetry algebra immediately implies that defects of these two classes preserve translational invariance since they are defined on the same space-time region and at most differ by a sign. From the gluing conditions of the supercharges it directly follows

$$P^{(1)} = P^{(2)} \text{ and } H^{(1)} = H^{(2)} \text{ on } \mathbb{R}. \tag{2.8}$$

But note that this is not possible for the two dimensional boundary conditions which automatically break one half of the local translation symmetries and therefore can at most preserve the half of the bulk symmetry.

In order to study defects in Landau-Ginzburg theories one can apply many techniques obtained from the study of theories with boundaries, which were developed in [3] and [4].

2.2 Defects in Landau-Ginzburg models

Now we investigate defects in supersymmetric Landau-Ginzburg models in two dimensions. Therefore we repeat a few definitions of the basics of supersymmetry. Then we show how boundary and defect conditions can be satisfied in supersymmetric Landau-Ginzburg models by introducing new fields to our theory which factorize the superpotential following the discourse [1]. Such a factorisation of the superpotential leads to the definition of matrix factorisations. Whenever we can obtain such a matrix factorisation it is equivalent to say that we can satisfy the supersymmetry conditions on boundaries or defects. A fundamental property of matrix factorisations is that they form a category with morphisms defined below. We are also going to discuss the case where one and the same theory is separated by several defects and what happens when these defects come closer and closer together.

2.2.1 Bulk action

Consider the two-dimensional $N = (2,2)$ superspace, the space is spanned by two spatial coordinates (bosonic coordinates) $x^{\pm} = x_0 \pm x_1$ and four Grassmann variables (fermionic coordinates) $\theta^{\pm}, \bar{\theta}^{\pm}$.The supercharges can be represented as differential operators acting on this superspace, they are given by

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm}\,, \quad \bar{Q}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm}\,. \tag{2.9}$$

Note that we can view supersymmetry transformations as translation in the superspace. To derive the group of the supersymmetry algebra we use the exponential map. We can choose right or left action of the group on fields on this superspace. Supercharges are given by a left action and the operators corresponding to the right action are called covariant derivatives. They are given by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\partial_\pm\,, \quad \bar{D}_\pm = -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\,. \tag{2.10}$$

Definition 2.2.1. *Chiral superfields are fields on the $N = (2,2)$ superspace, $X = X(x, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$ which satisfy*

$$\bar{D}_\pm X = 0. \tag{2.11}$$

In a similar way we define:

Definition 2.2.2. *Antichiral superfields are fields on the $N = (2,2)$ superspace, $X = X(x, \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$ which satisfy*

$$D_\pm \bar{X} = 0. \tag{2.12}$$

Remark 2.2.1 (Chiral superfields). *When we perform a coordinate transformation $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$ the covariant derivative $\bar{D}_\pm$ is represented by the differential operator $\bar{D}_\pm = \frac{\partial}{\partial y^\pm}$ such that a chiral superfield X can generally be represented by*

$$X = \phi\left(y^\pm\right) + \theta^\alpha\psi_\alpha\left(y^\pm\right) + \theta^+\theta^- F\left(y^\pm\right), \tag{2.13}$$

where $\alpha \in \{\pm\}$ and ϕ, ψ, F are arbitrary functions of $y^\pm$.

The underlying model of this thesis has the following action

$$S = S_D + S_F, \tag{2.14}$$

We demand that this theory have only a finite amount of superfields X_i. The D-term is given by

$$S_D = \int K\left(X_i, \bar{X}_i\right) d^4\theta d^2x, \tag{2.15}$$

where K is to so called Kähler potential which we assume to be flat and diagonal, i.e. $K = \sum_i \bar{X}_i X_i$. The F-term

$$S_F = \int W\left(X_i\right)\big|_{\bar{\theta}^\pm=0} d\theta^+ d\theta^- d^2x + \int \bar{W}\left(\bar{X}_i\right)\big|_{\bar{\theta}^\pm=0} d\theta^+ d\theta^- d^2x \tag{2.16}$$

is determined by the superpotential W, which is a holomorphic function of the chiral superfields X_i. At this point we make an very important definition for the rest of thesis, which holds for the later chapters if nothing other is mentioned.

Definition 2.2.3. *The superpotential W is a polynomial in the chiral superfields X_i.*

Note that polynomials are holomorphic.

In a two-dimensional theory without boundaries and defects which is $N = (2,2)$ supersymmetric, the variation of the action is given by

$$\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+, \tag{2.17}$$

which vanishes for all $\epsilon_\pm, \bar{\epsilon}_\pm$. The corresponding conserved supercharges can then be represented as

$$\begin{aligned} Q_\pm &= \int \left((\partial_0 \pm \partial_1)\bar{\phi}_{\bar{j}} \psi^j_\pm \mp i \bar{\psi}^{\bar{i}}_\mp \partial_{\bar{i}} \bar{W} \right) dx^1, \\ Q_\pm &= \int \left(\bar{\psi}^{\bar{j}}_\pm (\partial_0 \pm \partial_1)\phi_j \pm i \psi^i_\mp \partial_i W \right) dx^1. \end{aligned} \tag{2.18}$$

2.2.2 B-type boundary conditions and matrix factorisations

Let us now discuss the formulation of a Landau-Ginzburg model on the upper half plane (UHP) with a boundary at the real line $\mathbb{R}$. As mentioned above this problem is very similar to that one with a theory with defects. So we begin our discussion with the easier case of a boundary. The coordinates of our problem are defined by

$$z = \bar{z} = t, \quad \theta^+ = \theta^- = \theta, \quad \bar{\theta}^+ = \bar{\theta}^- = \bar{\theta}. \tag{2.19}$$

The presence of the boundary at $\mathbb{R} \subset \mathbb{C}$ then reduces the number of supersymmetries

$$\delta_B = \epsilon Q - \bar{\epsilon} \bar{Q}, \tag{2.20}$$

due to the B-type conditions, only the supersymmetry generators

$$Q = Q_+ + Q_-, \quad \bar{Q} = \bar{Q}_+ + \bar{Q}_-, \tag{2.21}$$

are compatible with the B-type boundary conditions because of the fact that supersymmetry transformations of the form (2.17) only preserve the boundary if $\epsilon_+ = -\epsilon_- =: \epsilon$ and $\bar{\epsilon}_+ = -\bar{\epsilon}_- =: \epsilon$.

A not very surprising result is that the restriction of the bulk Landau-Ginzburg action defined on $\mathbb{C}$ with B-type boundary conditions is not invariant under B-type supersymmetry (2.20), the variation produces in general non vanishing terms which can be split as

$$\delta_B S = \delta_B S_D + \delta_B S_F. \tag{2.22}$$

One can show by straightforward calculation that the variation of the F-term is given by

$$\delta_B S_F = i \int_{\partial\Sigma} \bar{\epsilon} W d\bar{\theta} dt - i \int_{\partial\Sigma} \epsilon \bar{W} d\bar{\theta} dt. \tag{2.23}$$

To "repair" the B-type supersymmetry one has two possibilities to achieve that the variation of the action $\delta_B S$ vanishes. Firstly one can introduce additional boundary terms whose variation compensates the terms occurring in $\delta_B S$. Or second one can introduce boundary conditions on the fields which ensure that the variation is trivial.

The authors in [5] and [6] showed that the introduction a suitable boundary term to our action (2.22) can always compensate the variation of the D-term.

The more interesting result is that the F-term (2.23) can be cancelled out by adding extra non-chiral fermionic boundary superfields $\pi_1, ..., \pi_r$ to the theory, which obey

$$\bar{D}\pi_i = E_i. \tag{2.24}$$

The new fields produce a variation term at the boundary which has the following form

$$\delta_B S_\pi = i \int_{\partial\Sigma} J_i \pi_i d\theta dt + c.c, \tag{2.25}$$

this exactly cancels with the remaining non-zero term of our original action, if

$$\sum_i J_i E_i = W \tag{2.26}$$

is satisfied. So we have shown that we can preserve B-type supersymmetry for Landau-Ginzburg theories with boundaries in it. The procedure explained above is the underlying key idea of the concepts that will be developed in the later chapters. We saw that we can achieve B-type supersymmetry by finding a resolution of W into products. In this discussion we neglect the D-term since it can always be compensated by introducing a boundary term which does not change the action away from the boundary.

This suggests that to compensate the supersymmetric variation of the bulk theory is equivalent to find a homomorphism Q (not the supercharge) of the following form

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix}, \tag{2.27}$$

which has to satisfy $Q^2 = W \cdot Id_{M_Q}$ where M_Q is a module of the polynomial ring $R = \mathbb{C}[x]$. This homomorphism is called matrix factorisation. Let us summarize this in the following definition.

Definition 2.2.4. *Let W be the superpotential, which is a polynomial in the chiral superfields X_i of the theory. We define a matrix factorisation as a homomorphism $Q : M_Q \to M_Q$, where M_Q is a free $\mathbb{Z}_2$ graded module over $R = \mathbb{C}[x]$ of the form*

$$M_Q = M_{Q,1} \oplus M_{Q,2}, \tag{2.28}$$

with Q satisfying $Q^2 = W \cdot \mathbb{1}_{M_Q}$.

A very important feature of these matrix factorisations is that they have a certain algebraic structure. In fact they form a category (e.g A.2.1), where the objects of the category are given by the matrix factorisations themselves and the morphism space between the matrix factorisations is realised as the zeroth cohomology of $\mathbb{Z}_2$-graded module homomorphisms with respect to a certain differential.

2.2.3 B-type defects and matrix factorisations

As mentioned above the situation of a theory with a boundary is very similar to the problem of a theory with a defect in it. For this we will indeed follow the same strategy used for the characterisation of B-type boundary conditions in Landau-Ginzburg models reviewed in 2.2.2 above. We follow the discourse in [1] and consider two two-dimensional supersymmetric Landau-Ginzburg models which are separated by a defect on the real line $\mathbb{R} \subset \mathbb{C}$. Let the theory on the upper half plane (UHP) have a finite number of chiral superfields X_i and a superpotential $W_1(X_1, ..., X_n)$. Analogously the theory on the lower half plane (LHP) have a finite number of corresponding chiral superfields Y_i and a superpotential W_2 depending on them, too. Since we want to describe defects which preserve the B-type supersymmetry, we therefore impose B-type conditions. Similar to the case of a theory with a boundary, the B-type supersymmetric variation does not vanish in general. The only slightly difference is that the non-vanishing terms on the UHP and LHP differ by a sign which comes from the different orientation of the boundary on the real line $\mathbb{R}$. Therefore, the total B-type supersymmetry variation of the action of the first Landau-Ginzburg model on the UHP and the second one on the LHP is given by

$$\begin{aligned} \delta_B S &= \delta_B S_D + \delta_B S_F \\ \delta_B S_F &= \pm i \int \left(\bar{\epsilon}(W_1 - W_2) - \epsilon(\bar{W}_1 - \bar{W}_2) \right). \end{aligned} \tag{2.29}$$

Again $\delta_B S_D$ can be compensated by introducing an appropriate boundary term and $\delta_B S_F$ can be cancelled by introducing additional fermionic fields $\pi_1, ..., \pi_r$ which satisfy

$$\bar{D}\pi_i = E_i. \tag{2.30}$$

The same reasoning as outlined in 2.2.2 for the case of boundary conditions leads to the conclusion that B-type defects between the two Landau-Ginzburg models are characterised by matrix factorisations of the difference $W = W_1 - W_2$ of the respective superpotentials, which squares to $W_1 - W_2$, i.e.

$$\sum_i J_i E_i = (W_1 - W_2) \cdot \mathbb{1}_M. \tag{2.31}$$

2.2.4 Fusion of B-type defects

The question we consider in this section is what happens when there are several defects separating the theory one theory and what happens if the area of separation become infinitesimal small, this process is called fusion. This question was answered by [7] in the case of two defects.

We consider $\mathbb{C}$ with two defects and corresponding matrix factorisations Q_1 and Q_2 which separate the complex plane into three theories, see figure 2.2.

The superpotentials for each theory depend on a set of variables $\{x_{1,i}\}_{i \in I_1}$, $\{x_{2,i}\}_{i \in I_2}$ and $\{x_{3,i}\}_{i \in I_3}$ respectively. The matrix factorisations Q_1 and Q_2 satisfy the relations $Q_1^2 = W_1(\{x_{1,i}\}_{i \in I_1}) - W_2(\{x_{2,i}\}_{i \in I_2})$ and $Q_2^2 = W_2(\{x_{2,i}\}_{i \in I_2}) -$

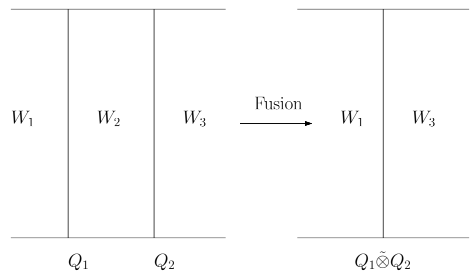

Figure 2.2: Fusion of defects

$W_3(\{x_{3,i}\}_{i\in I_3})$, derived above. We introduce the shorthand notation $x_j = \{x_{j,i}\}_{i\in I_j}$ for the variables corresponding to the theory of W_i. There is a theorem which states that we can always fuse the defects to a new defect with the corresponding matrix factorisation $Q_1 \otimes Q_2$. The only smack of this theorem is that $Q_1 \otimes Q_2$ still depends on the variables x_2. Another statement that is a basic result, shows that one can find an equivalent matrix factorisation Q_3 which only depends on x_1 and x_3.

This is one possible motivation by investigating the morphisms between matrix factorisations we hope to understand the process of fusing, i.e. a way to express Q_3 without difficult calculations. For this purpose we will define so called fusion functors (which form a category too) later on, which implement such fusions. At this point it is not clear if one can model any fusion process by such fusion functors but we hope to find the answer by understanding the structure of the morphism space of matrix factorisations.

Chapter 3

Functors

In this chapter we will give an introduction to functor categories following the ideas in [2] from where we also take over definitions and proofs of this section. Therefore we require basic knowledge in category theory. The underlying definitions can be found in the appendix A. In this section we will define graded morphisms between functors over R-module categories such that they form a differential graded category and discuss the structure of the morphism spaces.

3.1 Linear functors on free modules

In the following we consider categories of free finite rank R-modules, where R is a unital polynomial ring over $\mathbb{C}$, which we denote with $R\text{-}mod_f$. We are looking for functors $U : R\text{-}mod_f \to R'\text{-}mod_f$ between two such module categories $R\text{-}mod_f$ and $R'\text{-}mod_f$ which are linear in the module-homomorphisms, i.e.

$$\forall M, N \in R\text{-}mod_f, \forall f, g \in Hom(M, N), \forall \alpha, \beta \in \mathbb{C} : U(\alpha f + \beta g) = \alpha U(f) + \beta U(g). \tag{3.1}$$

In this thesis it is of major importance to investigate maps between those functors, which fullfill certain properties like linearity or associativity. Therefore our aim is to regard these functors themselves as objects of a category and define morphisms between the functors $U, V : R\text{-}mod_f \to R'\text{-}mod_f$, which can be done in the following way.

Definition 3.1.1. *Let $n \in \mathbb{N}$ and $M_{n+1} \xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1$ be any sequence of n module-homomorphisms. We call $\phi^{(n)}_{UV} \in Hom_n(U, V)$ a morphism of degree n between the functors U and V, when it maps any sequence of n module-homomorphisms to a morphism from $U(M_1)$ to $V(M_{n+1})$ which is $\mathbb{C}$-linear in each morphism entry f_i with $1 \leq i \leq n$. Here $Hom_n(U, V)$ denotes* *the set of all morphisms of degree n between U and V. Thus we can write:*

$$\begin{aligned}\phi_{UV}^{(n)} : & \left(M_{n+1} \xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1 \right) \\ & \mapsto \phi_{UV}^{(n)} \left(M_{n+1} \xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1 \right) \in Hom(U(M_1), V(M_{n+1})).\end{aligned} \tag{3.2}$$

The next ingredient we need to build a category of functors of module categories is the composition of the above defined morphisms between these functors to gain a notion of associativity.

Definition 3.1.2. *The composition of two given morphisms $\phi_{UV}^{(n)}$ and $\phi_{VW}^{(n')}$ of the degree n and n' respectively, is given by*

$$\begin{aligned}& \left(\phi_{VW}^{(n')} \circ \phi_{UV}^{(n)} \right) \left(M_{n+n'+1} \xleftarrow{f_{n+n'}} ... \xleftarrow{f_1} M_1 \right) \\ & = \phi_{VW}^{(n')} \left(M_{n+n'+1} \xleftarrow{f_{n+n'}} ... \xleftarrow{f_{n+1}} M_{n+1} \right) \circ \phi_{UV}^{(n)} \left(M_{n+1} \xleftarrow{f_n} ... \xleftarrow{f_1} M_1 \right).\end{aligned} \tag{3.3}$$

which is a morphism of degree $n + n'$.

Note that the compositions on the left-hand side and on the right hand-side are not in the same space. On the left-hand side we mean the composition of a morphisms of degree n and n' in the functor category and on the right-hand side we have a composition of morphisms in the R'-mod_f category. Also note that by this definition the composition of morphisms in the functor category is associative.

Finally we need to define an identity morphism in our functor category, this can be done as below.

Definition 3.1.3. *The identity morphism is of degree 0 and is denoted by $id_U \in Hom_0(U, U)$ and it is defined by*

$$id_U(M_1) = U(\mathbf{1}_{M_1}) = \mathbf{1}_{U(M_1)}. \tag{3.4}$$

Now we are able to state the definition of the functor category between two module categories:

Definition 3.1.4. *The functor category $Fun_{R,R'}$ has $\mathbb{C}$-linear functors as objects and the set of morphisms between two functors $U, V \in Fun_{R,R'}$ from R-mod_f to R'-mod_f is given by*

$$Hom(U, V) = \bigoplus_{n=0}^{\infty} Hom_n(U, V). \tag{3.5}$$

3.2 Differential graded functor category

We want to equip our functor category $Fun_{R,R'}$ with an additional structure to obtain a differential graded category, therefore we need to define a suitable differential d on the cochain complex (c.l. A.3.2) of the graded morphism spaces, which maps a morphism $\phi_{UV}^{(n)}$ of degree n to a morphism $\phi_{UV}^{(n+1)}$ of degree $n+1$. The important thing about this differential below is that this differential is exactly the same as in the Hochschild cohomolgy. We will make use of this fact to state our result on the structure of morphism spaces in this category.

Definition 3.2.1. *For two functors $U, V \in Fun_{R,R'}$ the differential maps a morphism $\phi_{UV}^{(n)}$ to a morphism of degree $n+1$ which is defined by*

$$\begin{aligned}
&\left(d\phi_{UV}^{(n)}\right)\left(M_{n+2} \xleftarrow{f_{n+1}} M_{n+1} \xleftarrow{f_{n-1}} M_n ... \xleftarrow{f_1} M_1\right) \\
&\quad = V(f_{n+1}) \circ \phi_{UV}^{(n)}\left(M_{n+1} \xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1\right) \\
&\quad + (-1)\phi_{UV}^{(n)}\left(M_{n+2} \xleftarrow{f_{n+1}\circ f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1\right) \\
&\quad + ... \\
&\quad + (-1)^n \phi_{UV}^{(n)}\left(M_{n+2} \xleftarrow{f_{n+1}} M_{n+1} \xleftarrow{f_n} M_n ... \xleftarrow{f_2\circ f_1} M_1\right) \\
&\quad + (-1)^{n+1} \phi_{UV}^{(n)}\left(M_{n+2} \xleftarrow{f_{n+1}} M_{n+1} \xleftarrow{f_n} M_n ... \xleftarrow{f_2} M_2\right) \circ U(f_1)
\end{aligned} \tag{3.6}$$

This indeed leads to a differential graded category (c.l. A.2.2). The fact that $d \circ d = 0$ will be proved in the following.

Lemma 3.2.1. *The differential d satisfies $d \circ d = 0$ and the following Leibniz property*

$$d\left(\phi_{VW}^{(n)} \circ \phi_{UV}^{(n')}\right) = \left(d\phi_{VW}^{(n)}\right) \circ \phi_{UV}^{(n')} + (-1)^n \phi_{VW}^{(n)} \circ \left(d\phi_{UV}^{(n')}\right). \tag{3.7}$$

Proof. See in Appendix B. "Proof of lemma 3.2.1". □

Due to the lemma and the definition of d we can now have a closer look at the cohomology groups, denoted by $H^n(Hom(U,V))$, of the morphism space in $Fun_{R,R'}$. This leads to the following definition:

Definition 3.2.2. *$\widetilde{Fun}_{R,R'}$ is a category with the same objects as $Fun_{R,R'}$ and with the morphisms between two functors U, V given by*

$$H^*(Hom(U,V)) = \bigoplus_{n=0}^{\infty} H^n(Hom(U,V)). \tag{3.8}$$

By the definition of d we immediately see that the degree 0 morphisms of the category $\widetilde{Fun}_{R,R'}$ are natural transformations (c.l. definition A.2.5) between any two functors $U, V \in \widetilde{Fun}_{R,R'}$ since $\phi^{(0)}_{UV}$ is closed (note that all morphisms with representative in the corresponding cohomology group are closed), i.e.

$$0 = \left(d\phi^{(0)}_{UV}\right)\left(M_2 \xleftarrow{f_1} M_1\right) = V(f_1) \circ \phi^{(0)}_{UV}(M_1) - \phi^{(0)}_{UV}(M_2) \circ U(f_1). \tag{3.9}$$

To continue with the main part of the thesis we first have to show that we can reduce the spaces of morphisms in $Fun_{R,R'}$ without losing information on the structure. It is enough for any two linear functors $U, V \in Fun_{R,R'}$ to consider the subspace of morphisms of degree n that vanish when one of their entries is the identity, i.e.

$$Hom^{red}_n(U, V) = \{\phi^{(n)}_{UV} \in Hom_n(U, V), \phi^{(n)}_{UV}(f_n, ..., 1, ..., f_1) = 0\}. \tag{3.10}$$

The space of the reduced morphisms is then given by

$$Hom^{red}(U, V) = \bigoplus_{n=0}^{\infty} Hom^{red}_n(U, V). \tag{3.11}$$

Our first important result is the following proposition.

Proposition 3.2.1. *Let $n \in \mathbb{N}$ then*

$$H^n\left(Hom^{red}(U, V)\right) \cong H^n(Hom(U, V)), \tag{3.12}$$

for all n.

The strategy of proving this result is to show that for an arbitrary closed morphism $\phi^{(n)}_{UV} \in Hom_n(U, V)$ there is a morphism $\tilde{\phi}^{(n)}_{UV} \in Hom^{red}_n(U, V)$ such that their difference is exact, i.e.

$$\phi^{(n)}_{UV} - \tilde{\phi}^{(n)}_{UV} = d\psi^{(n-1)}_{UV}, \tag{3.13}$$

where $\psi^{(n-1)}_{UV} \in Hom_{n-1}(U, V)$. Indeed this follows by induction from the following lemma.

Lemma 3.2.2. *Let $S \in \mathbb{N}$ and $\phi_S \in Hom_n(U, V)$ be a closed morphism of degree n that vanishes if any of its last S arguments is the identity map. Then there is a closed morphism $\phi_{S+1} \in Hom_n(U, V)$ that vanishes if any of its last $S+1$ arguments is the identity, such that $\phi_S - \phi_{S+1}$ is exact.*

Proof. This proof is taken over from [2]. The first step is to write out what does it mean that ϕ_S is closed

$$d\phi_S = 0. \tag{3.14}$$

With the definition of our differential, this in turn implies

$$\begin{aligned} &V(f_{n+1}) \circ \phi_S(f_n, ..., f_1) - \phi_S(f_{n+1} \circ f_n, ..., f_1) + ... \\ &+ (-1)^n \phi_S(f_{n+1}, ..., f_3, f_2 \circ f_1) + (-1)^{n+1} \phi_S(f_{n+1}, ..., f_2) \circ U(f_1) = 0. \end{aligned} \tag{3.15}$$

Now set $f_{S+1} = f_{S+2} = \mathbf{1}$, where $\mathbf{1}$ is the identity map for R-modules and insert this into (3.15) and make use of the property that ϕ_S is zero when one of its last S entries is the identity,

$$\begin{aligned}
&V(f_{n+1}) \circ \phi_S(f_n, ..., \mathbf{1}, \mathbf{1}, ..., f_1) \\
&- \phi_S(f_{n+1} \circ f_n, ..., f_{S+3}, \mathbf{1}, \mathbf{1}, ..., f_1) + ... \\
&(-1)^{n-S-2}\phi_S(f_{n+1}, ..., f_{S+4} \circ f_{S+3}, \mathbf{1}, \mathbf{1}, f_S, ..., f_1) \\
&+ (-1)^{n-S-1}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_S, ..., f_1) = 0.
\end{aligned} \tag{3.16}$$

Next define a morphism of degree $n-1$ in the following way,

$$\psi_S(f_{n-1}, ..., f_1) := \phi_S(f_{n-1}, ..., f_{S+1}, \mathbf{1}, f_S, ..., f_1). \tag{3.17}$$

Note that ϕ_S is a morphism of degree n and we are setting its $S+1$ argument to the identity map. In the final step we show that

$$\phi_{S+1} := \phi_S + (-1)^{n-S-1} d\psi_S, \tag{3.18}$$

satisfies the claim of the lemma. First we compute the differential of ψ_S,

$$\begin{aligned}
d\psi_S(f_{n+1}, ..., f_{S+3}, f_{S+1}, ..., f_1) &= V(f_{n+1}) \circ \phi_S(f_n, ..., f_{S+3}, f_{S+1}, \mathbf{1}, f_S, ..., f_1) \\
&- \phi_S(f_{n+1} \circ f_n, ..., f_{S+3}, f_{S+1}, \mathbf{1}, f_S, ..., f_1) + ... \\
&+ (-1)^{n-S-1}\phi_S(f_{n+1}, ..., f_{S+4}, f_{S+3} \circ f_{S+1}, \mathbf{1}, f_S, ..., f_1) \\
&+ (-1)^{n-S}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_{S+1} \circ f_S, f_{S-1}, ..., f_1) + ... \\
&+ (-1)^{n-1}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_{S+1}, ..., f_3, f_2 \circ f_1) \\
&+ (-1)^{n}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_{S+1}, ..., f_2) \circ U(f_1).
\end{aligned} \tag{3.19}$$

It is easy to see that $d\psi_S$ vanishes if any of its last S arguments is the identity map, so that ϕ_{S+1} has this property too. We now evaluate $d\psi_S$ in the case where $f_{S+1} = \mathbf{1}$,

$$\begin{aligned}
d\psi_S(f_{n+1}, ..., f_{S+3}, f_{S+1}, ..., f_1) &= V(f_{n+1}) \circ \phi_S(f_n, ..., f_{S+3}, \mathbf{1}, \mathbf{1}, f_S, ..., f_1) \\
&- \phi_S(f_{n+1} \circ f_n, ..., f_{S+3}, \mathbf{1}, \mathbf{1}, f_S, ..., f_1) + ... \\
&+ (-1)^{n-S-2}\phi_S(f_{n+1}, ..., f_{S+5}, f_{S+4} \circ f_{S+4}, \mathbf{1}, \mathbf{1}, f_S, ..., f_1) \\
&+ (-1)^{n-S}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_{S+1} \circ f_S, f_{S-1}, ..., f_1) + ... \\
&= -(-1)^{n-S-1}\phi_S(f_{n+1}, ..., f_{S+3}, \mathbf{1}, f_S, ..., f_1).
\end{aligned} \tag{3.20}$$

In the last step we used (3.16). From our definition in (3.18) we conclude that ϕ_{S+1} vanishes if the entry at position $s+1$ is the identity. In addition with the property of ϕ_{S+1} mentioned before ϕ_{S+1} vanishes if any of its last $S+1$ arguments is the identity. □

3.3 Linearity

The question we are asking us now is what is the best way to deduce if two morphism of degree n between the functors U and$V$ are in the same cohomology class? It would be great to find "special" sequences of module homomorphisms which determine the behaviour of the morphisms for all other sequences.

Above we identified the degree zero morphism between the functors U and V with the natural transformations (c.l. definition A.2.5) in the categorical sense. This directly implies that for R-modules M and N we have

$$\phi_{UV}^{(0)}(N) = V(g) \circ \phi_{UV}^{(0)}(M) \circ U(g^{-1}), \tag{3.21}$$

provided $g : N \to M$ is invertible, in other words g is an isomorphism. So this means that we only have to check the behaviour of $\phi_{UV}^{(0)}$ on the class of isomorphic modules.
What is about the higher degree morphisms? This consideration clearly carries over for higher degree morphisms on sequences of isomorphisms. But the various possibilities of combinations make things complicated to work with in an efficient way. One needs to define a system of representatives $\{R_i\}$ with $i \in \mathcal{I}$ of isomorphic modules. Then one is able to show that when two morphisms $\phi_{UV}^{(n)}$ and $\psi_{UV}^{(n)}$ between the functors U and V coincide on every sequence of the representative modules R_i, they belong to the same cohomology class.

But we have not taken into account the linearity yet. It turns out that this reduces the sequences where we have to check the equivalence of two morphisms ϕ_{UV} and ψ_{UV} in an enormous way. Let us consider two modules M_1 and M_2 and $M_1 \oplus M_2$ their direct sum. Denote with

$$\pi_1 : M_1 \oplus M_2 \to M_1 \quad , \quad \pi_2 : M_1 \oplus M_2 \to M_2, \tag{3.22}$$

the projections of $M_1 \oplus M_2$ on M_1 and M_2 respectively. Also denote with

$$\iota_1 : M_1 \to M_1 \oplus M_2 \quad , \quad \iota_2 : M_2 \to M_1 \oplus M_2, \tag{3.23}$$

the inclusion of M_1 and M_2 into $M_1 \oplus M_2$ respectively. From the linearity of the functor U one can conclude

$$U(\pi_1) \oplus U(\pi_2) : U(M_1 \oplus M_2) \xrightarrow{\cong} U(M_1) \oplus U(M_2) \tag{3.24}$$

is an isomorphism with inverse

$$U(\iota_1) \oplus U(\iota_2) : U(M_1) \oplus U(M_2) \xrightarrow{\cong} U(M_1 \oplus M_2). \tag{3.25}$$

For a closed morphism of degree zero $\phi^{(0)}$ linearity implies that

$$\phi^{(0)}(M_1 \oplus M_2) = V(\iota_1) \circ \phi^{(0)}(M_1) \circ U(\pi_1) + V(\iota_2) \circ \phi^{(0)}(M_2) \circ U(\pi_2). \tag{3.26}$$

In our case all objects of consideration are isomorphic to R^n, and the result above implies that natural transformations are completely fixed as soon as we have defined $\phi^{(0)}(R)$. Again the procedure takes over for higher degree morphisms. Indeed the next result tells us when we are analysing higher degree morphisms, we can reduce our considerations on sequences of endomorphisms of R i.e. $R \xleftarrow{f_n} R \xleftarrow{f_{n-1}} ... \xleftarrow{f_1} R$.

To prove the claim above we need to introduce the following notation. For a module $R^m = R \oplus ... \oplus R$ we denote the projection on the k^{th} summand by π_k and analogously the embedding of R as the k^{th} summand in R^m as ι_k, such that

$$\sum_{k=1}^{m} \iota_k \circ \pi_k = \mathbf{1}_{R^m}. \tag{3.27}$$

To proceed we need to prove the following lemma, but this proof is very technical and takes several pages, therefore we shift it to the appendix.

Lemma 3.3.1. *Let $\phi^{(n)}_{UV} \in Hom_n(U, V)$ be a closed morphism of degree n between the functors U and V. Then there exists a morphism $\phi^{(n-1)}_{UV}$ of degree $n-1$ such that on sequences of homomorphisms $R^{m_{n+1}} \xleftarrow{f_n} R^{m_n} \xleftarrow{f_{n-1}} ... \xleftarrow{f_1} R^1$ between modules R^{m_i} we have*

$$\begin{aligned} \phi^{(n)}_{UV}(f_n, ..., f_1) = \\ &\sum_{k_1=1}^{m_1} ... \sum_{k_{n+1}=1}^{m_{n+1}} V(\iota_{k_{n+1}}) \circ \phi^{(n)}_{UV}(\pi_{k_{n+1}} \circ f_n \circ \iota_{k_n}, ..., \pi_{k_2} \circ f_1 \circ \iota_{k_1}) \circ U(\pi_{k_1}) \\ &+ d\phi^{(n-1)}_{UV}(f_{n-1}, ..., f_1). \end{aligned} \tag{3.28}$$

Proof. See in Appendix B. "Proof of lemma 3.3.1". This proof is taken over from [2]. □

The following result follows directly from the lemma above:

Proposition 3.3.1. *Let $\phi, \psi \in Hom_n(U, V)$ be two closed morphisms of degree n between the functors U and V. Assume that $\phi^{(n)}$ and $\psi^{(n)}$ coincide on all sequences of endomorphisms of the rank one module R, $R \xleftarrow{f_n} R \xleftarrow{f_{n-1}} ... \xleftarrow{f_1} R$. Then $\phi^{(n)}_{UV}$ and $\psi^{(n)}_{UV}$ are in the same cohomology class.*

Since we can reduce our considerations on chain complexes of the form $R^{m_{n+1}} \xleftarrow{f_n} R^{m_n} \xleftarrow{f_{n-1}} ... \xleftarrow{f_1} R^1$ to sequences $R \xleftarrow{f_n} R \xleftarrow{f_{n-1}} ... \xleftarrow{f_1} R$, the morphisms between U and V acting on them map them to an element of $Hom(U(R), V(R))$ which is a bimodule over R, where the left and right action is given by $r \cdot \phi = V(r)\phi$ and $\phi \cdot r = \phi U(r)$. When we have a closer look at A.3.5 we recognize the structure of the cohomolgies. We may identify them with Hochschild cohomologies over a unital ring, note that the polynomial ring R is an associative R algebra, i.e. $H^n(Hom(U,V))$ is isomorphic to the Hochschild cohomology of the polynomial ring R with values in the R-bimodule $Hom(U(R),V(R)) =: M$. Note that everything discussed so far also holds for the reduced morphisms $H^n(Hom^{red}(U,V))$ between the functors U and V.

3.4 Structure of the morphism spaces

Now we want to analyse the structure of the cohomology groups $H^*(Hom(U,V))$ in detail. We have already seen that $H^0(Hom(U,V))$ consists of all natural transformations between U and V in the categorical sense (c.l. definition A.2.5). Due to proposition 3.3.1 we can restrict our considerations to the case of the elementary module R, where the closure condition reads

$$\phi^{(0)}(R) \circ U(p) = V(p) \circ \phi^{(0)}(R). \tag{3.29}$$

Here $p \in R$ is any polynomial in R and is viewed as an endomorphism on R itself. Since the functors U and V are $\mathbb{C}$-linear it is enough to require the closure condition above for monomials, $p \in \{x_1, ..., x_n\}$, thus the zeroth cohomology group is given by

$$H^0(Hom(U,V)) \cong \{f \in Hom(U(R),V(R)) : \forall i \in \{1,...,n\}, f \circ U(x_i) = V(x_i) \circ f\}. \tag{3.30}$$

What are about the higher cohomolgy groups? The first thing we should ask us is, if there are only finitely many non trivial cohomology groups? To answer this question for a polynomial ring in n variables our notation becomes very cumbersome and such a proof would take a few pages of straightforward calculation which is unreadable by the pedestrian approach. Instead of doing that we use some nice results from homological algebra following the discourse in [8]. The statements can be found in appendix A.

Therefore let $U, V : R\text{-}mod_f \to R'\text{-}mod_f$ be functors of module categories, where $R = \mathbb{C}[x_1, ..., x_n]$ and $R' = \mathbb{C}[y_1, ..., y_m]$ with m and n not necessarily equal. From section 3.3 we know that the n-th cohomology group is isomorphic to the Hochschild cohomology of bimodules over a unital polynomial ring R, i.e.

$$H^n(Hom^{red}(U,V)) \cong H^n(R, \underbrace{Hom(U(R),V(R))}_{=:M}). \tag{3.31}$$

Here M has a is a bimodule structure over R with the left multiplication $r \cdot \phi = V(r)\phi$ and the corresponding right multiplication $\phi \cdot r = \phi U(r)$. This leads to the following theorem.

Theorem 3.4.1. *Let $R = \mathbb{C}[x_1, ..., x_n]$, then $H^p(R,M) = 0$ for any $p > n$ and for every bimodule M.*

Proof. First we need to express the enveloping algebra R^e as a polynomial ring over R. The enveloping algebra $R^e = R \otimes_{\mathbb{C}} R^{op}$ is simply $\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$ since R is commutative and therefore we have $R \cong R^{op}$.
Looking at the module homomorphism $R^e \to R$ which acts via $u \otimes v \to uv$ we see that the kernel consists of those polynomials $f(x_1, ..., x_n, y_1, ..., y_n)$ which vanish if all $x_i = y_i$, $1 \leq i \leq n$. In other words the kernel is generated by the regular sequence $(x_1 - y_1, ..., x_n - y_n)$. By the first isomorphism theorem we obtain that

$$R^e \big/ (x_1 - y_1, ..., x_n - y_n)R^e \cong R. \tag{3.32}$$

This makes available the Koszul resolution (c.f. A.3.2) of R, i.e.

$$0 \to \Lambda^n((R^e)^n) \to ... \to \Lambda^2((R^e)^n) \to (R^e)^n \to R^e \to R^e \big/ (x_1 - y_1, ..., x_n - y_n)R^e \to 0 \tag{3.33}$$

is exact. By A.3.1 we have also that

$$Ext^p_{R^e}\left(R^e \big/ (x_1 - y_1, ..., x_n - y_n)R^e, M\right) = H^p(x, M) = H^p(Hom(K(x), M)). \tag{3.34}$$

Here x denotes our regular sequence $(x_1 - y_1, ..., x_n - y_n)$, $K(x)$ the corresponding Koszul complex of the regular sequence x and I the ideal $(x_1 - y_1, ..., x_n - y_n)R^e$ spanned by x. Acting with the contravariant functor $Hom(\cdot, M)$ on the exact sequence (3.34) we obtain

$$0 \leftarrow Hom\left(\Lambda^n((R^e)^n), M\right) \leftarrow ... \leftarrow Hom((R^e)^n, M) \leftarrow Hom(R^e, M) \leftarrow Hom(R, M) \leftarrow 0. \tag{3.35}$$

When we look at the left hand side of the upper expression, we see that the cohomology groups for $p > n$ are trivial and this completes the proof. □

We already identified the zeroth cohomology group as the natural transformations betweem the functors U and V which is consistent with our analysis above. To see this we have a closer look on the the contravariant functor $Hom(\cdot, M)$, and we note that this functor is right exact, i.e. the sequence

$$Hom(\Lambda^2 (R^e)^n, M) \leftarrow Hom((R^e)^n, M) \leftarrow Hom(R^e, M) \leftarrow Hom(R, M) \leftarrow 0, \tag{3.36}$$

is exact in the first two entries. Therefore we obtain

$$\begin{aligned} H^0(x, M) &= ker(Hom(Hom(R^e, M) \to Hom((R^e)^n, M)) \\ &\cong Hom(R, M) = Hom(R, Hom(U(R), V(R))). \end{aligned} \tag{3.37}$$

Unfortunately the functor $Hom(\cdot, M)$ does not map the whole exact sequence to a exact sequence so that we have to investigate the higher cohomolgy groups separately from the zeroth group. When we for example look at the first cohomology group

$$H^1(x, M) = \frac{ker(Hom((R^e)^n, M) \to Hom(\Lambda^2 (R^e)^n, M)}{im(Hom(R^e, M) \to Hom((R^e)^n, M)} \tag{3.38}$$

it becomes clear that we could not simplify this expression in general without knowing M. One can show that in the case $M = R = \mathbb{C}[x_1, ..., x_n]$ that $H^1(x, M) = R^n$ and for the higher cohomologies one obtains $H^p(x, M) = \Lambda^p(R^n)$

Let us have a closer look on degree one morphisms. The closure condition of degree one morphisms reads

$$d\phi^{(1)}(g, f) = V(g) \circ \phi^{(1)}(f) - \phi^{(1)}(g \circ f) + \phi^{(1)}(g) \circ U(f) = 0, \tag{3.39}$$

for all module homomorphisms $f : M \to N$, $g : N \to O$. Now let us consider the case $R = \mathbb{C}[x]$. If we consider an exact morphism, this implies

$$\tilde{\phi}^{(1)}(f) = d\phi^{(0)}(f) = V(f) \circ \phi^{(0)}(M) + \phi^{(0)}(N) \circ U(f), \tag{3.40}$$

where f is again a module homomorphism $f : M \to N$. Following the reasoning in proposition 3.3.1 we can restrict our consideration to the case, where all modules involved are isomorphic to R, i.e. $M \cong N \cong O \cong R$. The homomorphisms can be represented by polynomials $p, q \in R$ and therefore the closure condition then reads

$$d\phi^{(1)}(p, q) = V(p) \circ \phi^{(1)}(q) - \phi^{(1)}(pq) + \phi^{(1)}(p) \circ U(q) = 0. \tag{3.41}$$

We now have that $\phi^{(1)}(f)$ is completely determined in terms of $\phi^{(1)}(x)$, where $x \in R$, which is a homomorphism from $U(R)$ to $V(R)$. Evaluating the exact homomorphisms $\tilde{\phi}^{(1)}$ on $x \in R$ one finds

$$\tilde{\phi}^{(1)}(x) = V(x) \circ \phi^{(0)}(R) - \phi^{(0)}(R) \circ U(x), \tag{3.42}$$

where $\phi^{(0)}(R) : U(R) \to V(R)$ is an arbitrary homomorphism. So we conclude that

$$H^1(Hom(U, V)) \cong \frac{Hom(U(R), V(R))}{\{V(x)f - fU(x), f \in Hom(U(R), V(R))\}}. \tag{3.43}$$

Chapter 4

Fusion functors

In this chapter we discuss a certain species of functor categories, the fusion functor categories. The basic constuction is again taken over from [2]. We want to analyse the structure of its morphism space and therefore we define a map Ξ which relates the morphisms between fusion functors with the morphisms between matrix factorisations. To understand the structure of the morphisms of matrix factorisations it is required to investigate the properties of the map Ξ. We do this by presenting an introduction into fusion functor categories and proving the surjectivity of Ξ in the one variable case following the results of [2]. We can show that the map Ξ is not injective. Roughly speaking, this tells us that the space of morphisms between fusion functors is larger then the space of morphisms between matrix factorisations. So it would be nice if we can write out the kernel of Ξ explicitly to introduce a suitable equivalence relation on the space of morphisms between fusion functors, such that Ξ becomes injective. We find that when we choose $U = V = id$ and $M = R$, the kernel is given by the Jacobi ideal.

We will also turn to the case of two variable module categories and try to investigate the property of surjectivity in this case. Indeed one can show that every odd morphism between matrix factorisations has a suitable preimage but for even morphisms there is a strange occurrence of quadratic terms which complicates the process of finding a preimage under Ξ. We also showed that the kernel of Ξ for $U = V = id$ and $M = R$ is the Jacobi ideal.

4.1 Fusion functors and matrix factorisations

Let R and R' be polynomial rings over $\mathbb{C}$, and $W \in R$ and $W' \in R'$ specific polynomials. We define:

Definition 4.1.1. *A (W,W')-fusion functor U is a $\mathbb{C}$-linear functor from $R-mod_f$ to $R'-mod_f$ with the property*

$$U(W \cdot f) = W' \cdot U(f) \tag{4.1}$$

for any module homomorphism f.

In order to build a category out of these fusion functors we need to define morphisms between the functors, therefore we use the definition of morphisms from the previous chapter and additionally demand a certain homogeneity condition.

Definition 4.1.2. *A morphism of degree n between fusion functors U and V is a morphism $\phi_{UV}^{(n)} \in Hom_n^{red}(U,V)$ with the property that*

$$\phi_{UV}^{(n)}(f_n, ..., W \cdot f_i, ..., f_1) = W' \phi_{UV}^{(n)}(f_n, ..., f_1) \tag{4.2}$$

for all $i \in \{1, ..., n\}$.

In an analogous way as in the chapter before we define the fusion functor category. Here we denote the reduced morphism spaces between fusion functors in the same way as we did it before for general functors.

Definition 4.1.3. *The fusion functor category $FFun_{W,W'}$ has $\mathbb{C}$-linear fusion functors as objects and the set of morphisms between two functors $U, V \in Fun_{W,W'}$ is given by*

$$Hom(U,V) = \bigoplus_{n=0}^{\infty} Hom_n^{red}(U,V). \tag{4.3}$$

The differential d from (3.2.1) applies to the morphisms of the fusion functor category too and we can consider the zeroth cohomology which defines a category $\widetilde{FFun}_{W,W'}$.

4.2 Action on matrix factorisation

Fusion functors have an important relation to matrix factorisations Q. Let $M = M_0 \oplus M_1$ be a $\mathbb{Z}_2$-graded free finite rank R-module, and Q a module homomorphism which satisfies $Q^2 = W \cdot \mathbf{1}_M$.

Proposition 4.2.1. *Let U be a (W,W')-fusion functor, and $Q : M \to M$ a matrix factorisation of W. Then $U(Q) : U(M) \to U(M)$ is a matrix factorisation of W'.*

Proof. This proof is taken over from [2]. The proof follows by straightforward calculation.

$$U(Q) \circ U(Q) = U\left(Q^2\right) = U\left(W \cdot \mathbf{1}_M\right) = W' \cdot U\left(\mathbf{1}_M\right) = W' \cdot \mathbf{1}_{U(M)}. \tag{4.4}$$

□

To proceed note that a morphism between two fusion functor U, V evaluated on a matrix factorisation Q induces a morphism between $U(Q)$ and $V(Q)$. This motivates the following definition of the space of morphisms in a matrix factorisation category.

Definition 4.2.1. *The space of morphisms between matrix factorisations $Q_1 : M_1 \to M_1$ and $Q_2 : M_2 \to M_2$ is defined as the zeroth cohomology of $\mathbb{Z}_2$-graded module homomorphisms $\phi^{(n)} \in Hom(M_1, M_2)$ ($n = 0$ describing even (bosonic) homomorphisms, and $n = 1$ odd (fermionic) homomorphisms) with respect to the differential*

$$\delta_{Q_1,Q_2}\phi^{(n)} = Q_2 \circ \phi^{(n)} + (-1)^{n+1}\phi^{(n)} \circ Q_1. \tag{4.5}$$

Now we can define our map Ξ.

Definition 4.2.2. *Let U, V be (W, W')-fusion functors, and $Q : M \to M$ a W-matrix factorisation. Then we define the map*

$$\Xi^Q_{U,V} : Hom^{red}_{U,V} \to Hom(U(M), V(M)), \tag{4.6}$$

$$\Xi^Q_{U,V}\left(\phi^{(n)}\right) = \phi^{(n)}_{UV}(Q, ..., Q). \tag{4.7}$$

The most important property of the map $\Xi^Q_{U,V}$ is, that it is compatible with the differentials d and $\delta_{U(Q),V(Q)}$. This leads us to our next proposition.

Proposition 4.2.2. *We have*

$$\Xi^Q_{U,V}\left(d\phi^{(n)}\right) = \delta_{U(Q),V(Q)}\Xi^Q_{U,V}\left(\phi^{(n)}\right). \tag{4.8}$$

In particular closed morphisms are mapped to closed homomorphisms, and exact morphisms are mapped to exact homomorphisms, so that $\Xi^Q_{U,V}$ induces a map on the cohomologies

$$\tilde{\Xi}^Q_{U,V} : H^0_d\left(Hom^{red}(U, V)\right) \to H^0_{\delta_{U(Q),V(Q)}}\left(Hom(U(M), V(M))\right). \tag{4.9}$$

Proof. This proof is taken over from [2]. Start with the left hand side of (4.8):

$$\begin{aligned}\Xi^Q_{U,V}\left(d\phi^{(n)}\right) =& d\phi^{(n)}_{UV}(Q, ..., Q)\\ =& V(Q) \circ \phi^{(n)}_{UV}(Q, ..., Q)\\ &- \phi^{(n)}_{UV}(Q, ..., Q, Q^2) + ... + (-1)^n\phi^{(n)}_{UV}(Q^2, Q, ..., Q)\\ &+ (-1)^{n+1}\phi^{(n)}_{UV}(Q, ..., Q) \circ U(Q).\end{aligned} \tag{4.10}$$

Now we have a closer look on the second term of the left hand side of (4.8). We have

$$\phi^{(n)}_{UV}(Q, ..., Q, Q^2) = \phi^{(n)}_{UV}(Q, ..., Q, W \cdot \mathbf{1}) = W' \cdot \phi^{(n)}_{UV}(Q, ..., Q, \mathbf{1}) = 0, \tag{4.11}$$

where in the last step we used the fact that $\phi^{(n)}_{UV} \in Hom^{red}(U, V)$, i.e. it is a morphism that vanishes when any of its entries is the identity map. The same argument applies for all other terms of (4.10) as well. The only remaining terms are

$$\begin{aligned}\Xi^{Q}_{U,V}\left(d\phi^{(n)}\right) =& V(Q)\circ\phi^{(n)}_{UV}(Q,...,Q)+(-1)^{n+1}\phi^{(n)}_{UV}(Q,...,Q)\circ U(Q)\\ =&\delta_{U(Q),V(Q)}\Xi^{Q}_{U,V}\left(\phi^{(n)}\right).\end{aligned}\tag{4.12}$$

Thus (4.8) is proved. Also note that $\phi^{(n)}$ is mapped to an even (bosonic) homomorphism for n even and to an odd (fermionic) homomorphism for n odd. □

4.3 Fusion functors describe fusion: operator-like interfaces

Fusion functors have a very useful property which we want to investigate now. Let M_{I_W} be a (R,R)-bimodule (c.l. A.1.2). That means that M_{I_W} is both a left R-module and a right R-module. Then consider one specific bimodule homomorphism $I_W : M_{I_W} \to M_{I_W}$ called the identity defect, it is a (W,W') matrix bifactorisation which satisfies the relation

$$I_W^2 = W\otimes\mathbf{1}-\mathbf{1}\otimes W. \tag{4.13}$$

By adapting the definition of U slightly, extending it to a functor of a (R,R)-bimodule, acting trivial on the second factor, we see that $U(M_{I_W})$ is a (R',R)-bimodule and therefore $U(I_W)$ is a (W',W) matrix bifactorisation.

The key ingredient is to note that for our factorisation I_W, which separates one and the same theory and therefore it is called the identity defect, there exist closed homomorphisms

$$\lambda_Q : M_{I_W}\otimes M_Q\to M_Q \quad , \quad \lambda_Q^{-1} : M_Q\to M_{I_W}\otimes M_Q, \tag{4.14}$$

such that

$$\lambda_Q\lambda_Q^{-1} = \mathbf{1}_{M_Q}+(\delta_{Q,Q}\text{-exact terms}), \tag{4.15}$$

$$\lambda_Q^{-1}\lambda_Q = \mathbf{1}_{M_{I_W}\otimes M_Q}+(\delta_{I_W\otimes Q,I_W\otimes Q}\text{-exact terms}). \tag{4.16}$$

Thus $I_W\otimes Q$[1] and Q are isomorphic matrix factorisations[2], i.e.

$$U(I_W)\otimes Q\cong U(Q). \tag{4.17}$$

Recall the map $\Xi^{I_W}_{U,V}$ that maps any morphism between fusion functors U,V to a morphism between the factorisations $U(I_W)$ and $V(I_W)$. The question we are considering in this thesis is how these concepts are related. The relation

[1] Here one has to be careful using $\otimes$. This tensor product has to respect the $\mathbb{Z}_2$-grading of the modules.

[2] This construction can be easily generalised to (W,W'')-matrix bifactorisations Q for some $W''\in R''$

(4.17) helps to simplify things when we are want to investigate the properties of the maps $\Xi^{Q}_{U,V}$ for an arbitrary matrix factorisation because it is enough to consider it for $\Xi^{I_W}_{U,V}$. In fact we only have to consider the properties of Ξ on I_W. From now on we will call matrix bifactorisations which are induced from fusion functors operator-like interfaces.

4.4 The case of one variable

By the one variable case we mean that we restrict ourselves to a R-module category where $R = \mathbb{C}[x]$. Let W be our superpotential, the authors in [9] showed that the identity defect for this case can be represented by

$$I_W = \begin{pmatrix} 0 & x - \tilde{x} \\ \frac{W(x)-W(\tilde{x}}{x-\tilde{x}} & 0 \end{pmatrix}, \tag{4.18}$$

acting on a free rank two (R,R)-bimodule which we can decompose as

$$M_{I_W} = M_{I_W,0} \oplus M_{I_W,1} = R \otimes_{\mathbb{C}} R \oplus R \otimes_{\mathbb{C}} R, \tag{4.19}$$

where we denote the variable of the second factor R by $\tilde{x}$. We also can decompose the modules $M_{I_W} \otimes_R M_Q$ and M_Q as

$$M_{I_W} \otimes_R M_Q = R \otimes_{\mathbb{C}} M_{Q,0} \oplus R \otimes_{\mathbb{C}} M_{Q,1} \oplus R \otimes_{\mathbb{C}} M_{Q,0} \oplus R \otimes_{\mathbb{C}} M_{Q,1}, \tag{4.20}$$

where

$$M_Q = M_{Q,0} \oplus M_{Q,1}. \tag{4.21}$$

The isomorphism (4.17) can be implemented in the one variable case by the following maps ([9])

$$\lambda_Q = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} : M_{I_W} \otimes_R M_Q \to M_Q, \tag{4.22}$$

$$\lambda_Q^{-1} = \begin{pmatrix} \iota_{M_{Q,0}} & 0 \\ \frac{Q^{(0)}(x)-Q^{(0)}(\tilde{x})}{x-\tilde{x}} & 0 \\ 0 & \frac{Q^{(1)}(x)-Q^{(1)}(\tilde{x})}{x-\tilde{x}} \\ 0 & \iota_{M_{Q,1}} \end{pmatrix} : M_Q \to M_{I_W} \otimes_R M_Q. \tag{4.23}$$

Here ι and μ are defined in the following way:
Suppose $M_{Q,i}$ is a free R-module and isomorphic to $R \otimes_{\mathbb{C}} E$, with some complex vector space E (which is finite dimensional for finite-rank module). Therefore

$$R \otimes_{\mathbb{C}} M_{Q,i} \cong R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} E. \tag{4.24}$$

Then we define

$$\begin{aligned} \iota : R \otimes_{\mathbb{C}} E &\to R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} E \\ r \otimes e &\to r \otimes 1 \otimes e, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \mu : R \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} E &\to R \otimes_{\mathbb{C}} E \\ r \otimes s \otimes e &\to (rs) \otimes e. \end{aligned} \tag{4.26}$$

This means that we can view μ as a kind of projection to the first factor. The problem we want to consider in the following is if the map Ξ is surjective applied to the identiy defect. Therefore let U and V be two given fusion functors and ϕ a given morphism between the matrix factorisations $U(I_W)$ and $V(I_W)$, surjectivity would mean that there is a corresponding morphism Φ between the functors U and V which is mapped to ϕ under Ξ^{I_W}.

To prove this we will compute the induced morphism $\phi_Q : U(Q) \to V(Q)$ for a given $\phi : U(I_W) \to V(I_W)$ where we make use of our isomorphism (4.17). Now we try to guess the morphism Φ by knowing how it has to act on $\Phi(Q, ..., Q) = \phi_Q$.

4.4.1 Even morphisms

In the following chapters we construct morphisms between two fusion functors U and V such that for a given $\phi \in Hom(U(I_W), V(I_W))$ with $\delta\phi = 0$ there exists a morphism Φ such that $\Xi^{I_W}_{U,V}(\Phi) = \phi + \delta$exact-terms.

Let ϕ be a closed bosonic morphism ϕ between the matrix factorisations $U(I_W)$ and $V(I_W)$. We want to construct a degree zero d-closed morphism Φ between the functors U and V. To do this we look at the induced morphism ϕ_Q between $U(Q)$ and $V(Q)$, using the relation (4.17) and the homomorphisms λ_Q and λ_Q^{-1} we have

$$\phi_Q = V(\lambda_Q) \circ (\phi \otimes \mathbf{1}) \circ U\left(\lambda_Q^{-1}\right). \tag{4.27}$$

Here we implemented the morphism between $U(I_W) \otimes Q$ and $V(I_W)$ as we did before by $\phi \otimes \mathbf{1}$. We write the closed bosonic morphism ϕ as

$$\phi = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \phi^{(1)} \end{pmatrix}, \tag{4.28}$$

where $\phi^{(i)}$ is a map $\phi^{(i)}: U(R) \otimes_{\mathbb{C}} R \to V(R) \otimes_{\mathbb{C}} R$. We obtain

$$\begin{aligned}
\phi_Q =& V \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix} \\
&\circ \begin{pmatrix} \phi^{(0)} \otimes \mathbf{1}_{M_{Q,0}} & 0 & 0 & 0 \\ 0 & \phi^{(1)} \otimes \mathbf{1}_{M_{Q,1}} & 0 & 0 \\ 0 & 0 & \phi^{(1)} \otimes \mathbf{1}_{M_{Q,0}} & 0 \\ 0 & 0 & 0 & \phi^{(0)} \otimes \mathbf{1}_{M_{Q,1}} \end{pmatrix} \\
&\circ U \begin{pmatrix} \iota_{M_{Q,0}} & 0 \\ \frac{Q^{(0)}(x) - Q^{(0)}(\tilde{x})}{x - \tilde{x}} & 0 \\ 0 & \frac{Q^{(1)}(x) - Q^{(1)}(\tilde{x})}{x - \tilde{x}} \\ 0 & \iota_{M_{Q,1}} \end{pmatrix} \\
=& \begin{pmatrix} V(\mu) \circ \left(\phi^{(0)} \otimes \mathbf{1}_{M_{Q,0}}\right) \circ U\left(\iota_{M_{Q,0}}\right) & 0 \\ 0 & V(\mu) \circ \left(\phi^{(0)} \otimes \mathbf{1}_{M_{Q,1}}\right) \circ U\left(\iota_{M_{Q,1}}\right) \end{pmatrix}.
\end{aligned} \tag{4.29}$$

In the above expression we note that ϕ_Q only depends on $M_Q = M_{Q,0} \oplus M_{Q,1}$. This suggests to define the degree zero morphism

$$\Phi(M) = V(\mu) \circ \left(\phi^{(0)} \otimes \mathbf{1}_M\right) \circ U(\iota_M), \tag{4.30}$$

between the functors U and V which produces ϕ_Q when we apply it to M_Q. The only thing which remains to show is that Φ is a d-closed morphism. In order to do that we consider the maps that enter in the above expression. First we have the map $V(\mu)$ which acts as

$$\begin{aligned} V(\mu) : V(R) \otimes_{\mathbb{C}} R \otimes_{\mathbb{C}} E &\to V(R) \otimes_{\mathbb{C}} E \\ v \otimes r \otimes e &\to (V(r)v) \otimes e. \end{aligned} \tag{4.31}$$

Where $R \otimes_{\mathbb{C}} E \cong M_{Q,i}$. The map $\phi^{(0)} : U(R) \otimes_{\mathbb{C}} R \to V(R) \otimes_{\mathbb{C}} R$ can be decomposed as

$$\phi^{(0)} = \sum_m \phi_m^{(0)} \otimes r_m \text{ with } r_m \in R \text{ and } \phi_m^{(0)} : U(R) \to V(R), \tag{4.32}$$

where the sum is finite. For example one can choose the monomials $r_m(x) = x^m$. This in turn implies that a module homomorphism $f : R \otimes_{\mathbb{C}} E \to R \otimes_{\mathbb{C}} F$ can be written as

$$f = \sum_i r_i \otimes f_i, \tag{4.33}$$

where $f_i : E \to F$ are vector space homomorphisms, and the sum over i is finite.

Since $d\Phi$ is a morphism of degree one, it acts on such a module homomorphism defined above. In other words we need to show that $d\Phi(f) = 0$, i.e.

$$d\Phi(f) = V(f) \circ \Phi(R \otimes_{\mathbb{C}} F) - \Phi(R \otimes_{\mathbb{C}} E) \circ U(F) = 0. \tag{4.34}$$

$d\Phi(f)$ defines a map from $U(R) \otimes_{\mathbb{C}} E$ to $V(R) \otimes_{\mathbb{C}} F$. Evaluating it on $u \otimes e \in U(R) \otimes_{\mathbb{C}} E$ we obtain

$$
\begin{aligned}
d\Phi(f)(u \otimes e) = & \sum_i V(r_i) \circ V(\mu) \circ \phi^{(0)}(u \otimes 1) \otimes (f_i e) \\
& - \sum_i V(\mu) \circ \phi^{(0)} \left(U(r_i) u \otimes 1\right) \otimes (f_i e).
\end{aligned} \tag{4.35}
$$

When the morphism $\phi^{(0)}$ is arbitrary, $d\Phi(f)$ does not vanish, but since $\phi^{(0)}$ is part of an $\delta_{U(I_W),V(I_W)}$-closed morphism ϕ we have

$$
V(I_W) \circ \phi - \phi \circ U(I_W) = 0. \tag{4.36}
$$

When we look at the upper right entry we obtain the following relation

$$
\phi^{(0)} \circ U(x - \tilde{x}) = V(x - \tilde{x}) \circ \phi^{(1)}. \tag{4.37}
$$

As before we denote by $x - \tilde{x}$ a homomorphism from $R \otimes_{\mathbb{C}} R$ to itself, and x and $\tilde{x}$ denote the variables of the first and second factor of R respectively.

The functors U and V act trivially on the $\tilde{x}$-variables, so we can write e.g. $V(x - \tilde{x}) = V(x) - \tilde{x} \cdot \mathbf{1}_{V(R) \otimes_{\mathbb{C}} R}$. If we now apply from the left the map $V(\mu) : V(R) \otimes_{\mathbb{C}} R \to V(R)$, it acts trivially on the x-variable, but replaces $\tilde{x}$ by $V(x)$. So we obtain

$$
V(\mu) \circ \phi^{(0)} \circ U(x - \tilde{x}) = 0, \tag{4.38}
$$

$$
\Longrightarrow \; V(\mu) \circ \phi^{(0)} \circ U(x) = V(\mu) \circ \phi^{(0)} \circ \tilde{x} = V(x) \circ V(\mu) \circ \phi^{(0)}. \tag{4.39}
$$

Because of the functorial property and the linearity of U and V, one can replace x in the above equation by an arbitrary polynomial $p(x)$,

$$
V(\mu) \circ \phi^{(0)} \circ U(p(x)) = V(p(x)) \circ V(\mu) \circ \phi^{(0)}. \tag{4.40}
$$

It follows that $d\Phi$ in (4.35) indeed vanishes on homomorphisms $f : R \to R$, i.e. when $E \cong F \cong \mathbb{C}$. By linearity this is also true for homomorphisms $f : R \otimes_{\mathbb{C}} E \to R \otimes_{\mathbb{C}} F$ with arbitrary finite-dimensional vector spaces E and F. So we have shown that – for the case of one variable – any bosonic morphism ϕ between $U(I_W)$ and $V(I_W)$ is induced by a closed degree 0 morphism between the functors U and V.

4.4.2 Odd morphisms

Now we turn to closed fermionic morphisms between the matrix factorisations, therefore let ψ be a fermionic morphism between $U(I_W)$ and $V(I_W)$, i.e. it is a homomorphism

$$
\psi = \begin{pmatrix} 0 & \psi^{(1)} \\ \psi^{(0)} & 0 \end{pmatrix} : U(R) \otimes_{\mathbb{C}} R \oplus U(R) \otimes_{\mathbb{C}} R \to V(R) \otimes_{\mathbb{C}} R \oplus V(R) \otimes_{\mathbb{C}} R, \tag{4.41}
$$

satisfying

$$V\left(I_W\right)\circ\psi+\psi\circ U\left(I_W\right)=0. \tag{4.42}$$

Analogously to the bosonic case above we calculate the induced morphism $\psi_Q : U(Q)\to V(Q)$ using again (4.17). We obtain

$$\begin{aligned}
\psi_Q =&V\begin{pmatrix}\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu\end{pmatrix}\\
&\circ\begin{pmatrix}0 & 0 & \phi^{(1)}\otimes\mathbf{1}_{M_{Q,0}} & 0\\ 0 & 0 & 0 & \phi^{(0)}\otimes\mathbf{1}_{M_{Q,1}}\\ \phi^{(0)}\otimes\mathbf{1}_{M_{Q,0}} & 0 & 0 & 0\\ 0 & \phi^{(1)}\otimes\mathbf{1}_{M_{Q,1}} & 0 & 0\end{pmatrix}\\
&\circ U\begin{pmatrix}\iota_{M_{Q,0}} & 0\\ \frac{Q^{(0)}(x)-Q^{(0)}(\tilde{x})}{x-\tilde{x}} & 0\\ 0 & \frac{Q^{(1)}(x)-Q^{(1)}(\tilde{x})}{x-\tilde{x}}\\ 0 & \iota_{M_{Q,1}}\end{pmatrix}\\
=&V\left(\mu\right)\circ\left(\phi^{(1)}\otimes\mathbf{1}_{M_Q}\right)\circ U\begin{pmatrix}0 & \frac{Q^{(1)}(x)-Q^{(1)}(\tilde{x})}{x-\tilde{x}}\\ \frac{Q^{(0)}(x)-Q^{(0)}(\tilde{x})}{x-\tilde{x}} & 0\end{pmatrix}.
\end{aligned} \tag{4.43}$$

This suggests to guess the morphism Ψ, which is a degree one morphism between the fusion functors U and V acting on a sequence of length one as

$$\Psi(f)=V\left(\mu\right)\circ\left(\phi^{(1)}\otimes\mathbf{1}_{M_N}\right)\circ U\left(\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right) : U(M)\to V(N). \tag{4.44}$$

It remains to check two things. First we need to show that Ψ is closed under d, which can be done by straightforward calculation

$$\begin{aligned}
d\Psi(g,f) =& V(g)\circ\Psi(f)-\Psi(g\circ f)+\Psi(g)\circ U(f)\\
=&V(g)\circ V(\mu)\circ\psi^{(1)}\circ U\left(\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right)-V(\mu)\circ\psi^{(1)}\circ U\left(\frac{g\circ f(x)-g\circ f(\tilde{x})}{x-\tilde{x}}\right)\\
&+V(\mu)\circ\psi^{(1)}\circ U\left(\frac{g(x)-g(\tilde{x})}{x-\tilde{x}}\right)\circ U(f)\\
=&V(g)\circ V(\mu)\circ\psi^{(1)}\circ U\left(\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right)\\
&-V(\mu)\psi^{(1)}\circ U\left(g(\tilde{x})\circ\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right)\\
=&0.
\end{aligned} \tag{4.45}$$

The remaining second property to show is that Ψ respects the homogeneous condition for the superpotential W.

$$\Psi(W\cdot f)=W'\cdot\Psi(f). \tag{4.46}$$

Note that we are do not need to show that on the degree zero morphism in the bosonic case above, since a degree zero morphism only act on sequences of length zero. To show this we need to take into account that the corresponding fermionic morphism ψ between the matrix factorisations $U(I_W)$ and $V(I_W)$ is closed with respect to $\delta_{U(I_W),V(I_W)}$. Using this fact we obtain the following relation

$$V(x-\tilde{x}) \circ \psi^{(0)} + \psi^{(1)} \circ U\left(\frac{W(x)-W(\tilde{x})}{x-\tilde{x}}\right) = 0. \tag{4.47}$$

By applying $V(\mu)$ from the left hand side, the first term vanishes so that we end up with

$$V(\mu) \circ \psi^{(1)} \circ U\left(\frac{W(x)-W(\tilde{x})}{x-\tilde{x}}\right) = 0. \tag{4.48}$$

Using this result we can show (4.46):

$$\begin{aligned}
\Psi(W \cdot f) &= V(\mu) \circ \psi^{(1)} \circ U\left(\frac{W(x)f(x)-W(\tilde{x})f(\tilde{x})}{x-\tilde{x}}\right) \\
&= V(\mu) \circ \psi^{(1)} \circ U\left(\frac{W(x)-W(\tilde{x})}{x-\tilde{x}} f(x) + W(\tilde{x}) \frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right) \\
&= V(W) \circ V(\mu) \circ \psi^{(0)} \circ U\left(\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right) \\
&= W' \cdot V(\mu) \circ \psi^{(0)} \circ U\left(\frac{f(x)-f(\tilde{x})}{x-\tilde{x}}\right),
\end{aligned} \tag{4.49}$$

where we used in the last step that $V(W) = W' \cdot \mathbf{1}$. This shows that the surjectivity holds in the fermionic case, too.

In addition we have seen that it is enough to consider only a morphism of degree 0 (for the bosonic morphisms) and of degree 1 (for the fermionic morphisms). This matches with Theorem 3.4.1 that all cohomologies of degree greater than one are trivial. In total we have shown that Ξ^{I_W} is a surjective map on the morphism space of matrix factorisations, i.e. we can obtain any morphism between matrix factorisation from a suitable morphism between fusion functors, which motivates the physical relevance of the thesis.

4.4.3 Injectivity in the one variable case

Again set $R = \mathbb{C}[x]$. In the following we analyse the injectivity of the map $\Xi^{I_W}_{U,V} : U(M_{I_W}) \to V(M_{I_W})$. Let $\phi^{(n)}_{UV}$ be a morphism of degree n between the fusion functors U and V that is mapped to an exact morphism $\delta\psi^{(i)}$ where $\psi^{(i)}$ is bosonic for $i = 0$ or fermionic for $i = 1$.

The map $\Xi^{I_W}_{U,V}$ is injective if we can show that the morphism $\phi^{(n)}_{U,V}$ mapped to an exact morphism $\delta\psi^{(i)}$ is exact itself. This means there is a morphism $\chi^{(n-1)}_{U,V}$ of degree $n-1$ such that $d\chi^{(n-1)}_{U,V} = \phi^{(n)}_{U,V}$.
For $n = 0$ the condition of injectivity implies that $\phi^{(0)}_{UV} = 0$ is the null-morphism.

In the following we construct a counter example from which we see that the map $\Xi_{UV}^{I_W}$ fails to be injective.

Set $U = V = id$ the identity functors and $i = 1$ such that $\psi^{(1)}$ is a fermionic morphism. Now consider the following equation:

$$\Xi_{U,V}^{I_W}\left(\phi_{UV}^{(0)}\right) = \phi_{UV}^{(0)}(M_{I_W}). \tag{4.50}$$

What we have to show now is that we can represent this morphism by an exact bosonic morphism $\delta\psi_{I_W,I_W}^{(1)} : M_{I_W} \to M_{I_W}$. Since $\phi^{(0)}$ is an element of the zeroth cohomology group it is closed, and as shown in A.2.5 we obtain the following commutative diagram for arbitrary R-modules N and M:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi^{(0)}(M)"'] & N \arrow[d, "\phi^{(0)}(N)"] \\ M \arrow[r, "f"'] & N, \end{tikzcd}$$

where f is a arbitrary module homomorphism. When we choose $M = N = R$ we obtain the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "r"] \arrow[d, "\phi^{(0)}(R)"'] & R \arrow[d, "\phi^{(0)}(R)"] \\ R \arrow[r, "r"'] & R \end{tikzcd}$$

where we can represent the module homomorphism f by a polynomial r. This together with the fact that $\psi^{(1)}$ is a fermionic morphism and therefore can be represented as

$$\psi^{(1)} = \begin{pmatrix} 0 & \tilde{\psi}_1 \\ \tilde{\psi}_2 & 0 \end{pmatrix}, \tag{4.51}$$

implies that

$$\begin{aligned} &\phi^{(0)}(M_{I_W}) = \delta\psi^{(1)} = I_W \circ \psi^{(1)} + \psi^{(1)} \circ I_W \\ &= \begin{pmatrix} 0 & \tilde{\psi}_1 \\ \tilde{\psi}_2 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & x - \tilde{x} \\ \frac{W(x)-W(\tilde{x})}{x-\tilde{x}} & 0 \end{pmatrix} + \begin{pmatrix} 0 & x - \tilde{x} \\ \frac{W(x)-W(\tilde{x})}{x-\tilde{x}} & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & \tilde{\psi}_1 \\ \tilde{\psi}_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (x-\tilde{x})\tilde{\psi}_2 + \frac{W(x)-W(\tilde{x})}{x-\tilde{x}}\tilde{\psi}_1 & 0 \\ 0 & (x-\tilde{x})\tilde{\psi}_2 + \frac{W(x)-W(\tilde{x})}{x-\tilde{x}}\tilde{\psi}_1 \end{pmatrix}, \end{aligned} \tag{4.52}$$

is represented by an expression of the form

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \tag{4.53}$$

whit $r = \phi^{(0)}(R)$ i.e. a polynomial. The polynomial r depends only on the variable x of the first factor of M_{I_W} due to the action of functors on bimodules

$$\begin{aligned} \phi^{(0)}(M_{I_W}) : (R \otimes R) \oplus (R \otimes R) &\to (R \otimes R) \oplus (R \otimes R) \\ s_1 \otimes s_2 + t_1 \otimes t_2 &\mapsto r s_1 \otimes s_2 + r t_1 \otimes t_2. \end{aligned} \tag{4.54}$$

In order to find a suitable $\psi^{(1)}$ we demand that

$$\tilde{\psi}_1 = \tilde{\psi}_2. \tag{4.55}$$

Choosing $W(x) = x^2$ and plug this into (**??**) finally leads to

$$r(x) = ((x - \tilde{x}) + (x + \tilde{x}))\, \tilde{\psi}_1(x, \tilde{x}) = 2x\tilde{\psi}_1(x, \tilde{x}), \tag{4.56}$$

where we choose $\tilde{\psi}_1(x, \tilde{x}) = \tilde{\psi}_1(x)$. The choice

$$\psi^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.57}$$

then implies

$$\delta\psi^{(1)} = \begin{pmatrix} 2x & 0 \\ 0 & 2x \end{pmatrix}. \tag{4.58}$$

This means that we constructed a nontrivial bosonic morphism $\delta\psi^{(1)}$ which lies in the image of $\phi^{(0)}$ under the map Ξ^{I_W}. In other words we have shown that Ξ^{I_W} fails to be injective.

4.5 The case of two variables

This chapter originates from handwritten notes of Prof. Stefan Fredenhagen, my thesis supervisor, where he did several calculations and discussed the bosonic case for two variables.

When we consider the case of two variable modules we have to modify our identity defect in a non trivial way. To derive it formally would lie outside the scope of this work. So we follow [9] and define the identity defect as

$$I_W = \begin{pmatrix} 0 & 0 & (x_1 - \tilde{x_1}) & (x_2 - \tilde{x_2}) \\ 0 & 0 & -\frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}} & \frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}} \\ \frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}} & -(x_2 - \tilde{x_2}) & 0 & 0 \\ \frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}} & (x_1 - \tilde{x_1}) & 0 & 0 \end{pmatrix}, \tag{4.59}$$

In the following we replace $\tilde{x}$ by x' for readability reasons and introduce the shorthand notation $I_W^{(0)}$ for the upper right block of I_W and $I_W^{(1)}$ for the lower left block. Because the identity defect has been changed, the maps λ_Q and $\lambda_Q^{(-1)}$ change too. Following again [9] the map $\lambda_Q : M_{I_W} \otimes M_Q \to M_Q$ is given by

$$\lambda_Q = \begin{pmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \end{pmatrix}. \tag{4.60}$$

According to the work above the inverse map $\lambda_Q^{(-1)} : M_Q \to M_{I_W} \otimes M_Q$ has the following form

$$\lambda_Q^{(-1)} = \begin{pmatrix} \mathbb{1} & 0 \\ * & 0 \\ * & 0 \\ * & 0 \\ 0 & * \\ 0 & * \\ 0 & \mathbb{1} \\ 0 & * \end{pmatrix}. \tag{4.61}$$

The entries $*$ can be determined by the constraints (4.15) and (4.16). One ends up with the following

$$\lambda_Q^{(-1)} = \begin{pmatrix} \left(\lambda_Q^{(-1)}\right)^{(0)} & 0 \\ 0 & \left(\lambda_Q^{(-1)}\right)^{(1)} \end{pmatrix}. \tag{4.62}$$

Where

$$\left(\lambda_Q^{(-1)}\right)^{(0)} = \begin{pmatrix} \mathbb{1} \\ \frac{Q^{(1)}(x_1',x_2)-Q^{(1)}(x_1',x_2')}{x_2-x_2'} \; \frac{Q^{(0)}(x_1,x_2)-Q^{(0)}(x_1',x_2)}{x_1-x_1'} \\ \frac{Q^{(0)}(x_1,x_2)-Q^{(0)}(x_1',x_2)}{x_1-x_1'} \\ \frac{Q^{(0)}(x_1',x_2)-Q^{(0)}(x_1',x_2')}{x_2-x_2'} \end{pmatrix}, \tag{4.63}$$

and

$$\left(\lambda_Q^{(-1)}\right)^{(1)} = \begin{pmatrix} \frac{Q^{(1)}(x_1,x_2)-Q^{(1)}(x_1',x_2)}{x_1-x_1'} \\ \frac{Q^{(1)}(x_1',x_2)-Q^{(1)}(x_1',x_2')}{x_2-x_2'} \\ \mathbb{1} \\ \frac{Q^{(0)}(x_1',x_2)-Q^{(0)}(x_1',x_2')}{x_2-x_2'} \; \frac{Q^{(1)}(x_1,x_2)-Q^{(1)}(x_1',x_2)}{x_1-x_1'} \end{pmatrix}. \tag{4.64}$$

4.5.1 Surjectivity in two variables in the fermionic case

In the fermionic case we consider an odd morphism

$$\phi = \begin{pmatrix} 0 & \psi^{(1)} \\ \psi^{(0)} & 0 \end{pmatrix}, \tag{4.65}$$

By analogous treatment as in the one variable case we can construct an induced morphism $\phi_Q : U(Q) \to V(Q)$ by

$$\phi_Q = U(\lambda_Q) \circ (\phi \otimes \mathbb{1}_{M_Q}) \circ V(\lambda_Q^{(-1)}) \tag{4.66}$$

which translates to

$$
\begin{aligned}
\psi_Q =& V \begin{pmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \end{pmatrix} \\
& \circ \begin{pmatrix} 0 & 0 & \psi^{(1)} \otimes \mathbf{1}_{M_{Q,0}} & 0 \\ 0 & 0 & 0 & \psi^{(0)} \otimes \mathbf{1}_{M_{Q,1}} \\ \psi^{(0)} \otimes \mathbf{1}_{M_{Q,0}} & 0 & 0 & 0 \\ 0 & \psi^{(1)} \otimes \mathbf{1}_{M_{Q,1}} & 0 & 0 \end{pmatrix} \\
& \circ U \lambda_Q^{(-1)} \begin{pmatrix} \left(\lambda_Q^{(-1)}\right)^{(0)} & 0 \\ 0 & \left(\lambda_Q^{(-1)}\right)^{(1)} \end{pmatrix} \\
=& \begin{pmatrix} 0 \\ (V(\mu) \;\; 0) \circ \psi^{(1)} \circ \begin{pmatrix} U\left(\frac{Q^{(0)}(x_1,x_2) - Q^{(0)}(x_1',x_2)}{x_1 - x_1'}\right) \\ U\left(\frac{Q^{(0)}(x_1',x_2) - Q^{(0)}(x_1',x_2')}{x_2 - x_2'}\right) \end{pmatrix} \\ \\ (V(\mu) \;\; 0) \circ \psi^{(1)} \circ \begin{pmatrix} U\left(\frac{Q^{(1)}(x_1,x_2) - Q^{(1)}(x_1',x_2)}{x_1 - x_1'}\right) \\ U\left(\frac{Q^{(1)}(x_1',x_2) - Q^{(1)}(x_1',x_2')}{x_2 - x_2'}\right) \end{pmatrix} \\ 0 \end{pmatrix}
\end{aligned}
$$

This has an equivalent form as in the one variable case. Following the same arguments as in the one variable case this induced morphism defines a morphism between matrix factorisations.

4.5.2 Surjectivity in two variables in the bosonic case

The situation is completely different in the bosonic case. This is due to the quadratic expressions in λ_Q and $\lambda_Q^{(-1)}$ which come into effect since we now consider an even morphism

$$\phi = \begin{pmatrix} \phi^{(0)} & 0 \\ 0 & \phi^{(1)} \end{pmatrix}, \tag{4.67}$$

satisfying

$$V(I_W) \circ \phi + \phi \circ U(I_W) = 0. \tag{4.68}$$

Again we can determine the induced morphism between $U(Q)$ and $V(Q)$ as

$$\phi_Q = V(\lambda_Q) \circ (\phi \otimes \mathbf{1}) \circ U\left(\lambda_Q^{-1}\right). \tag{4.69}$$

One obtains

$$
\begin{aligned}
\psi_Q =& V \begin{pmatrix} \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 \end{pmatrix} \\
& \circ \begin{pmatrix} \phi^{(0)} \otimes \mathbf{1}_{M_{Q,0}} & 0 & 0 & 0 \\ 0 & \phi^{(1)} \otimes \mathbf{1}_{M_{Q,1}} & 0 & 0 \\ 0 & 0 & \phi^{(1)} \otimes \mathbf{1}_{M_{Q,0}} & 0 \\ 0 & 0 & 0 & \phi^{(0)} \otimes \mathbf{1}_{M_{Q,1}} \end{pmatrix} \\
& \circ U \begin{pmatrix} \left(\lambda_Q^{(-1)}\right)^{(0)} & 0 \\ 0 & \left(\lambda_Q^{(-1)}\right)^{(1)} \end{pmatrix} \\
=& \begin{pmatrix} *_0 & 0 \\ 0 & *_1 \end{pmatrix},
\end{aligned}
\tag{4.70}
$$

where

$$
*_0 = (V(\mu) \ \ 0) \circ \phi^{(0)} \circ \begin{pmatrix} U(\mathbb{1}) \\ U\left(\frac{Q^{(1)}(x_1',x_2)-Q^{(1)}(x_1',x_2')}{x_2-x_2'} \ \frac{Q^{(0)}(x_1,x_2)-Q^{(0)}(x_1',x_2)}{x_1-x_1'} \right) \end{pmatrix},
\tag{4.71}
$$

and

$$
*_1 = (V(\mu) \ \ 0) \circ \phi^{(0)} \circ \begin{pmatrix} U(\mathbb{1}) \\ U\left(\frac{Q^{(0)}(x_1',x_2)-Q^{(0)}(x_1',x_2')}{x_2-x_2'} \ \frac{Q^{(1)}(x_1,x_2)-Q^{(1)}(x_1',x_2)}{x_1-x_1'} \right) \end{pmatrix}.
\tag{4.72}
$$

Now we have a closer look on the upper left entry $*_0$. We write the components of $\phi^{(0)}$ as

$$
\phi^{(0)} = \begin{pmatrix} \phi_a^{(0)} & \phi_b^{(0)} \\ \phi_c^{(0)} & \phi_d^{(0)} \end{pmatrix}.
\tag{4.73}
$$

The upper left entry becomes

$$
\begin{aligned}
*_0 =& V(\mu) \circ \phi^{(0)} \circ \begin{pmatrix} U(\mathbb{1}) \\ U\left(\frac{Q^{(1)}(x_1',x_2)-Q^{(1)}(x_1',x_2')}{x_2-x_2'} \ \frac{Q^{(0)}(x_1,x_2)-Q^{(0)}(x_1',x_2)}{x_1-x_1'} \right) \end{pmatrix} \\
=& \left(V(\mu)\phi_a^{(0)} \quad V(\mu)\phi_b^{(0)} \right) U \begin{pmatrix} \mathbb{1} \\ \frac{Q^{(1)}(x_1',x_2)-Q^{(1)}(x_1',x_2')}{x_2-x_2'} \ \frac{Q^{(0)}(x_1,x_2)-Q^{(0)}(x_1',x_2)}{x_1-x_1'} \end{pmatrix} \\
=& V(\mu) \circ \phi_a^{(0)} \circ \mathbb{1} + V(\mu) \circ \phi_b^{(0)} \\
& \circ U \left(\frac{Q^{(1)}(x_1',x_2) - Q^{(1)}(x_1',x_2')}{x_2 - x_2'} \, \frac{Q^{(0)}(x_1,x_2) - Q^{(0)}(x_1',x_2)}{x_1 - x_1'} \right)
\end{aligned}
\tag{4.74}
$$

Due to the defining property of matrix factorisations Q, we see that in the above expression the terms of order two and the terms of order zero in Q start to "communicate". Since we demand that the the morphisms are closed, we obtain a constraint on the induced morphism. The equation

$$\phi Q_1 - Q_2 \phi = 0, \tag{4.75}$$

with $\phi : Q_1 \to Q_2$, reads

$$\phi^{(0)} Q_1^{(1)} - Q_2^{(1)} \phi^{(1)} = 0. \tag{4.76}$$

Thus we calculate

$$\begin{aligned}
&\begin{pmatrix} \phi_a^{(0)} & \phi_b^{(0)} \\ \phi_c^{(0)} & \phi_d^{(0)} \end{pmatrix} U \begin{pmatrix} x_1 - x_1' & x_2 - x_2' \\ -\frac{W(x_1',x_2)-W(x_1',x_2')}{x_2-x_2'} & \frac{W(x_1,x_2)-W(x_1',x_2)}{x_1-x_1'} \end{pmatrix} - \\
&V \begin{pmatrix} x_1 - x_1' & x_2 - x_2' \\ -\frac{W(x_1',x_2)-W(x_1',x_2')}{x_2-x_2'} & \frac{W(x_1,x_2)-W(x_1',x_2)}{x_1-x_1'} \end{pmatrix} \begin{pmatrix} \phi_a^{(1)} & \phi_b^{(1)} \\ \phi_c^{(1)} & \phi_d^{(1)} \end{pmatrix} = 0
\end{aligned} \tag{4.77}$$

When we consider again the upper left entry we obtain the following relation

$$\begin{aligned}
\phi_a^{(0)} \left(U(x_1) - x_1' \right) + \phi_b^{(0)} U \left(-\frac{W(x_1', x_2) - W(x_1', x_2')}{x_2 - x_2'} \right) \\
= \left(V(x_1) - x_1' \right) \phi_a^{(1)} + \left(V(x_2) - x_2' \right) \phi_c^{(1)}.
\end{aligned} \tag{4.78}$$

When we apply $V(\mu)$ to the equation above, we obtain the following expression

$$V(\mu) \circ \left(\phi_a^{(0)} (U(x_1) - x_1') + \phi_b^{(0)} U \left(\frac{W(x_1', x_2) - W(x_1', x_2')}{x_1 - x_1'} \right) \right) = 0. \tag{4.79}$$

Analogously we find

$$V(\mu) \circ \left(\phi_a^{(0)} (U(x_2) - x_2') + \phi_b^{(0)} U \left(\frac{W(x_1, x_2) - W(x_1', x_2)}{x_1 - x_1'} \right) \right) = 0. \tag{4.80}$$

From this point it is not clear if we can separate this into a $\phi_a^{(0)}$ and a $\phi_b^{(0)}$ part. One thing that has to be satisfied is that the induced morphism is closed with respect to $\delta_{U_1(Q),U_2(Q)}$, but in general the separation of the induced morphism into a degree zero and a degree two morphism is not possible so that there are two remaining options. First there are morphisms which do not lead to functorial morphism on the matrix factorisations, i.e. Ξ^{I_W} fails to be surjective. If this is not true one could ingestive if every morphism with $\phi_b^{(0)} = 0$ induces a degree zero morphism and a morphism with $\phi_a^{(0)} = 0$ induces a morphism of degree two. For now let us assume this and perform a plausibility check. The hope is to gain some more constraints for the morphism ϕ.

It we know that the induced morphism has to be $\delta_{U_1(Q),U_2(Q)}$-closed. One can calculate this explicitly but we will omit to write this fact down here in detail since the calculation would take several pages.

Now we "guess" such a separated morphism and check if it is closed under the differential d. Note that the main ingredient to show that the guessed morphism in the one variable case is closed was the relation (4.39). Without doing the calculation here one can show a similar equation for the two variable case

$$\begin{aligned}
V(\mu)\circ\phi_a^{(0)}\circ p(x_1',x_2') &= V(\mu)\circ\phi_a^{(0)}\circ U(p(x_1,x_2)) \\
&+ V(\mu)\circ\phi_b^{(0)}\circ U\left(-\frac{W(x_1',x_2)-W(x_1',x_2')}{x_2-x_2'}\frac{p(x_1,x_2)-p(x_1',x_2)}{x_1-x_1'}\right) \\
&+ V(\mu)\circ\phi_b^{(0)}\circ U\left(\frac{W(x_1,x_2)-W(x_1',x_2)}{x_2-x_2'}\frac{p(x_1',x_2)-p(x_1',x_2')}{x_2-x_2'}\right).
\end{aligned} \tag{4.81}$$

This could help us to construct a suitable morphism ϕ and just to check whether it is closed. For a degree zero morphism we make the following choice

$$\phi^{(0)} = V(\mu)\circ\phi_a^{(0)} + V(\mu)\circ\phi_b^{(0)}\circ U(g(x_1,x_2,x_1',x_2')). \tag{4.82}$$

Here g is a suitable R-module homomorphism. Now let us check if this morphism is closed, by acting on an arbitrary module homomorphism f:

$$\begin{aligned}
V(f)\phi^{(0)}-\phi^{(0)}U(f) &= V(\mu)\circ\phi_a^{(0)}\circ f(x_1',x_2') + V(\mu)\circ\phi_b^{(0)}\circ f(x_1',x_2')\circ U(g) \\
&- V(\mu)\circ\phi_a^{(0)}\circ U(f(x_1,x_2)) - V(\mu)\circ\phi_b^{(0)}\circ U(g)U(f(x_1,x_2)) \\
&= V(\mu)\circ\phi_b^{(0)}\circ U\Bigg(f(x_1',x_2')g - gf(x_1,x_2) \\
&+ \frac{W(x_1,x_2)-W(x_1',x_2)}{x_1-x_1'}\frac{f(x_1',x_2)-f(x_1',x_2')}{x_2-x_2'} \\
&- \frac{W(x_1',x_2)-W(x_1',x_2')}{x_2-x_2'}\frac{f(x_1,x_2)-f(x_1',x_2)}{x_1-x_1'}\Bigg) \\
&= V(\mu)\circ\phi_b^{(0)}\circ U\Bigg(f(x_1',x_2')g - gf(x_1,x_2) \\
&+ \frac{1}{(x_1-x_1')(x_2-x_2')}\big(W(x_1,x_2)f(x_1',x_2) - W(x_1',x_2)f(x_1',x_2) \\
&- W(x_1,x_2)f(x_1',x_2') + W(x_1',x_2)f(x_1',x_2') - W(x_1',x_2)f(x_1,x_2) \\
&+ W(x_1',x_2')f(x_1,x_2) + W(x_1',x_2)f(x_1',x_2) - W(x_1',x_2')f(x_1',x_2)\big)\Bigg)
\end{aligned} \tag{4.83}$$

Which simplifies to

$$
\begin{aligned}
V(f)\phi^{(0)} - \phi^{(0)}U(f) = V(\mu) \circ \phi_b^{(0)} \circ U \Big\{ & f(x_1', x_2')g - gf(x_1, x_2) \\
& + \frac{1}{(x_1 - x_1')(x_2 - x_2')} \Big(f(x_1', x_2')\big(- W(x_1, x_2) + W(x_1', x_2)\big) \\
& - f(x_1, x_2)\big(W(x_1', x_2) - W(x_1', x_2')\big) \\
& + f(x_1', x_2)\big(W(x_1, x_2) - W(x_1', x_2')\big)\Big)\Big\}.
\end{aligned}
\tag{4.84}
$$

but here we are again not able to formulate suitable conditions for $\phi_b^{(0)}$ that ϕ become closed.

In this section we tried to analyse the surjectivity of Ξ^{I_W} in the two variable case which turned to be extremely difficult because of the mixture of terms of even order in Q and the terms of zeroth order in Q. It is not obvious that there are suitable morphisms satisfying our derived constraints. In the next chapter we will see that a consequence by computing the kernel of Ξ^{I_W} is that this already shows that all morphisms between the corresponding matrix factorisations are induced by a degree-0 morphism in the case $U = V = id$. Maybe one can use this result for further considerations.

4.6 The kernel of Ξ^{I_W} and the Jacobi ring

As announced in the earlier chapters our aim is to to describe the kernel of Ξ^{I_W}, unfortunately this cannot be done in general in a trivial way. But when we restrict ourselves to bosonic morphisms one knows from [9] that the following holds for the n-variable case

$$
H^0_\delta(I_W, I_W) \cong \frac{\mathbb{C}[x_1, ..., x_n]}{\langle \partial_1 W, ..., \partial_n W \rangle}. \tag{4.85}
$$

As mentioned in section 3.4 we can identify $H^0(x, M)$ with R when $U = V = id$ and $M \cong R$. We will use this restriction in the following section to give a compact expression of the kernel of Ξ^{I_W} in this special case.

The one variable case

Now set $R = \mathbb{C}[x]$, then we have $Hom(R, R) \cong R$, and since we know that Ξ^{I_W} is surjective in the one variable case, this give us a strong hint that the kernel is spanned by the ideal $\langle \partial_x W \rangle$.

To perform a plausibility check of this relation set $W = x^2$. Let $\phi^{(0)} \in Hom(R,R) \cong R$, i.e. we can identify the closed degree zero morphisms with polynomials in R. When we let act this morphism to the module M_{I_W} we obtain

$$\phi^{(0)}(M_{I_W}) = \mathbb{1}_{M_{I_W}} \cdot \phi^{(0)}(R). \tag{4.86}$$

It remains to check which degree zero morphisms are mapped to an exact morphism between matrix factorisations. Therefore we consider fermionic morphisms which satisfy

$$\delta_{I_W,I_W}\psi^{(1)}_{I_W} = I_W \circ \psi^{(1)}_{I_W} + \psi^{(1)}_{I_W} \circ I_W = \phi^{(0)}_{I_W}. \tag{4.87}$$

The fermionic morphism $\psi^{(1)}_{I_W}$ is of the form

$$\psi^{(1)}_{I_W} = \begin{pmatrix} 0 & \psi^{(1)} \\ \psi^{(0)} & 0 \end{pmatrix} \tag{4.88}$$

and the identity defect is given by

$$I_W = \begin{pmatrix} 0 & x - \tilde{x} \\ \frac{W(x)-W(\tilde{x}}{x-\tilde{x}} & 0 \end{pmatrix}. \tag{4.89}$$

Considering the upper left entry (the lower right entry yields a similar constraint) we obtain the following constraint

$$\phi^{(0)} = (x-\tilde{x})\psi^{(0)} + \psi^{(1)}\frac{W(x)-W(\tilde{x})}{x-\tilde{x}} \in R, \tag{4.90}$$

which simplifies to

$$\phi^{(0)} = (x-\tilde{x})\psi^{(0)} + \psi^{(1)}(x+\tilde{x}) \in R, \tag{4.91}$$

since we set $W = x^2$. The condition that the above expression lies in R s that

$$\psi^{(0)} = \psi^{(1)} + \tilde{\psi}. \tag{4.92}$$

To proceed note that we can write every polynomial $\tilde{\psi}$ in two variables in the following form

$$\tilde{\psi}(x,\tilde{x}) = \tilde{\psi}(\tilde{x})_0 + (x+\tilde{x})\tilde{\psi}_1(x,\tilde{x}), \tag{4.93}$$

for suitable $\tilde{\psi}_0$ and $\tilde{\psi}_1$. We can absorb the second term of $\tilde{\psi}$ into the term multiplied $(x-\tilde{x})$ which means that we finally end up with a constraint of the form

$$\phi^{(0)} = (x-\tilde{x})\psi^{(0)} + \psi^{(0)}(x+\tilde{x}) \in R. \tag{4.94}$$

What one can obverse from the relation above is that the entries of $\phi^{(0)}_R$ has to be at least polynomials of degree one in x. In particular this means that the relation above spans an ideal in $\mathbb{C}[x,\tilde{x}]$ whose elements lie in the kernel, we have

$$I = \langle (x-\tilde{x}), (x+\tilde{x}) \rangle = \langle (x+\tilde{x}) \rangle \subset \mathbb{C}[x,\tilde{x}]. \tag{4.95}$$

Intersecting the ideal I with $\mathbb{C}[x]$ gives

$$I \cap \mathbb{C}[x, \tilde{x}] = \langle x \rangle. \tag{4.96}$$

Therefore the kernel of Ξ^{I_W} is given by $\langle x \rangle$ which is equal $\langle 2x \rangle = \langle \partial_x W \rangle$, the Jacobi ideal.

General case: Now our aim is to state this result for arbitrary W. Our first observation is that we can assume that $deg(W) \geq 1$ without loss of generality since constant factors cancel out in (4.90). The next step is to calculate $\frac{W(x)-W(\tilde{x})}{x-\tilde{x}}$, therefore let $W = a_n x^n + ... + a_1 x$. We state the obtained expression in the following lemma.

Lemma 4.6.1. *The expression*

$$\frac{W(x) - W(\tilde{x})}{x - \tilde{x}} \tag{4.97}$$

with $W = a_n x^n + ... + a_1 x + a_0$ *is given by*

$$a_n(x^{n-1} + \tilde{x}x^{n-2} + ... + \tilde{x}^{n-1}) + ... + a_2(x + \tilde{x}) + a_1 \tag{4.98}$$

Proof. The proof follows from the polynomial division algorithm. Let us look at the first step

$$\begin{array}{rr|l}
- & a_n(x^n - \tilde{x}^n) + ... + a_1(x - \tilde{x}) & a_n x^{n-1} \\
 & a_n x^n - a_n x^{n-2}\tilde{x} & \overline{a_n(x^{n-1} + x^{n-2}\tilde{x})} \\
\hline
- & a_n(x^n - x^{n-2}\tilde{x}) + ... + a_1(x - \tilde{x}) & \\
 & a_n x^{n-1}\tilde{x} - a_n x^{n-2}\tilde{x} & \\
\hline
 & a_n(x^{n-2}\tilde{x} - x^{\widetilde{n-1}}) + ... + a_1(x - \tilde{x}) &
\end{array}$$

We see that at each step of the division algorithm a term of the form $x^{n-i}.\tilde{x}^{i-1}$ for $i \in \{1, ..., n-k\}$ occur, where k is the power of the variable x, which can be compensated by a factor of the form $x^{n-i-1}\tilde{x}^i$. It easy to see that this process terminates when $k = 1$, since the remaining factor is $a_1(x - y)$. $\square$

The next key ingredient is to note that the constraint (4.90) can be equally written as the following intersection

$$\{\phi^{(0)} \in H^0_d(id, id), \Xi^{I_W}(\phi^{(0)}) = 0\} = \underbrace{\langle (x - \tilde{x}), \frac{W(x) - W(\tilde{x})}{x - \tilde{x}} \rangle}_{=:I} \cap \mathbb{C}[x]. \tag{4.99}$$

Here we view the ideal I as a subset of $\mathbb{C}[x, \tilde{x}]$. To show that $I \cap \mathbb{C}[x] = \langle \partial_x W \rangle \subset \mathbb{C}[x]$ we need to write the ideal I in another form. First observe that

$$I = \{q(x - \tilde{x}) + p(a_n(x^{n-1} + \tilde{x}x^{n-2} + ... + \tilde{x}^{n-1}) + ... + a_2(x + \tilde{x}) + a_1) \,|\, p, q \in \mathbb{C}[x, \tilde{x}]\}. \tag{4.100}$$

By writing y as $x-(x-y)$ we see that we can split the second polynomial which is multiplied with p as

$$\underbrace{na_nx^{n-1}+...+2a_2x+a_1}_{=\partial_x W}+(x-\tilde{x})(-a_n(x^{n-2}+(x-\tilde{x})x^{n-3}+...)-...-a_2). \tag{4.101}$$

We can neglect the terms with a $(x-\tilde{x})$ in front by redefining q with $q \to \tilde{q}+p(-a_n(x^{n-2}+(x-\tilde{x})x^{n-3}+...)-...-a_2)$ which leads to

$$I=\langle \tilde{q}(x-\tilde{x}), p(\partial_x W)\rangle. \tag{4.102}$$

Without loss of generality we can write an arbitray polynomial in two variables $p(x,\tilde{x})$ as

$$p(x,\tilde{x})=p_1(x)+(x-\tilde{x})p_2(x,\tilde{x}). \tag{4.103}$$

This implies

$$I=\langle q(x-\tilde{x}), p_1(x)(\partial_x W)\rangle \tag{4.104}$$

Intersecting this ideal with $\mathbb{C}[x]$ we see that only the term which only contain x remain and therefore we finally obtain

$$I\cap\mathbb{C}[x]=\langle\partial_x W\rangle. \tag{4.105}$$

This result then immediately implies that the kernel of Ξ^{I_W} is given by $\langle\partial_x W\rangle$ which is exactly the Jacobi ideal.

The two variable case

Let us switch to the two variable case $R=\mathbb{C}[x_1,x_2]$:

$$I_W=\begin{pmatrix} 0 & 0 & (x_1-\tilde{x_1}) & (x_2-\tilde{x_2}) \\ 0 & 0 & -\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}} & \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}} \\ \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}} & -(x_2-\tilde{x_2}) & 0 & 0 \\ \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}} & (x_1-\tilde{x_1}) & 0 & 0 \end{pmatrix}, \tag{4.106}$$

and the fermionic morphism is given by

$$\psi^{(1)}_{I_W}=\begin{pmatrix} 0 & 0 & \psi^{(1)}_a & \psi^{(1)}_b \\ 0 & 0 & \psi^{(1)}_c & \psi^{(1)}_d \\ \psi^{(0)}_a & \psi^{(0)}_b & 0 & 0 \\ \psi^{(0)}_c & \psi^{(0)}_d & 0 & 0 \end{pmatrix}. \tag{4.107}$$

Calculating the differential

$$\delta_{I_W,I_W}\psi^{(1)}_{I_W}=I_W\circ\psi^{(1)}_{I_W}+\psi^{(1)}_{I_W}\circ I_W, \tag{4.108}$$

leads to a block diagonal matrix of the form

$$\delta_{I_W,I_W}\psi^{(1)}_{I_W} = \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}. \tag{4.109}$$

When we take into account (4.86) that the induced morphism is represented by a diagonal matrix. We find that four constraints has to be equal zero and the other constraints force that the expression has to lie R. Note that the expressions on the diagonal has to be equal in R. Straightforward calculation yields the following constraints

1.
$$(x_1-\tilde{x_1})\psi_a^{(0)} + (x_2-\tilde{x_2})\psi_c^{(0)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_a^{(1)} + \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\psi_b^{(1)} = f \in R,$$

2.
$$(x_1-\tilde{x_1})\psi_b^{(0)} + (x_2-\tilde{x_2})\psi_d^{(0)} + (x_1-\tilde{x_1})\psi_b^{(1)} - (x_2-\tilde{x_2})\psi_a^{(1)} = 0,$$

3.
$$\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_c^{(0)} - \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\psi_a^{(0)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_c^{(1)} + \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\psi_d^{(1)} = 0,$$

4.
$$(x_1-\tilde{x_1})\psi_d^{(1)} - (x_2-\tilde{x_2})\psi_c^{(1)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_d^{(0)} - \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\psi_b^{(0)} = f \in R,$$

5.
$$(x_1-\tilde{x_1})\psi_a^{(0)} - (x_2-\tilde{x_2})\psi_c^{(1)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_a^{(1)} - \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\psi_b^{(0)} = f \in R,$$

6.
$$-(x_2-\tilde{x_2})\psi_d^{(1)} + (x_2-\tilde{x_2})\psi_a^{(0)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_b^{(1)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\psi_b^{(0)} = 0,$$

7.

$$(x_1 - \tilde{x_1})\psi_c^{(0)} + (x_1 - \tilde{x_1})\psi_c^{(1)} + \frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}}\psi_a^{(1)} - \frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}}\psi_d^{(0)} = 0,$$

8.

$$(x_1 - \tilde{x_1})\psi_d^{(1)} + (x_2 - \tilde{x_2})\psi_c^{(0)} + \frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}\psi_d^{(0)} + \frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}}\psi_b^{(1)} = f \in R.$$

First we analyse the equality constraints 2,6 and 7. From them we can read of the general from of $\psi_a^{(0)}$, $\psi_a^{(0)}$, $\psi_a^{(0)}$ and $\psi_a^{(0)}$. For each equation we obtain two relations, for 2 we have

$$\psi_b^{(0)} = -\psi_b^{(1)} + (x_2 - \tilde{x_2})q_2, \qquad \psi_d^{0)} = \psi_a^{(1)} + (x_1 - \tilde{x_1})p_2, \tag{4.110}$$

with q_2 and p_2 are polynomials in $x_1, \tilde{x_1}, x_2$ and $\tilde{x_2}$. From 6 ant 7 we obtain in a analogous way

$$\psi_a^{(0)} = \psi_d^{(1)} + \frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}q_6, \qquad \psi_b^{(0)} = -\psi_b^{(1)} + (x_2 - \tilde{x_2})p_6, \tag{4.111}$$

$$\psi_c^{(0)} = -\psi_c^{(1)} + \frac{W(\tilde{x_1}, x_2) - W(\tilde{x_1}, \tilde{x_2})}{x_2 - \tilde{x_2}}q_7, \qquad \psi_d^{(0)} = \psi_a^{(1)} + (x_1 - \tilde{x_1})p_7, \tag{4.112}$$

By directly comparing the relations containing the same constellations of terms we immediately see that several relations are equivalent. We have that $q_2 = p_6$ and $p_2 = p_7$. When we plug in the relation for $\psi_b^{(0)}$ (4.110) and the relation for $\psi_d^{(0)}$ in (4.110) into 6., we obtain

$$\frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}(x_2 - \tilde{x_2})q_2 + (x_2 - \tilde{x_2})\frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}q_3 = 0, \tag{4.113}$$

which implies $q_2 = -q_3$. The same procedure, plugging in the relation for $\psi_c^{(0)}$ and the relation for $\psi_d^{(0)}$ into 7 leads to

$$-\frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}(x_1 - \tilde{x_1})p_7 + (x_1 - \tilde{x_1})\frac{W(x_1, x_2) - W(\tilde{x_1}, x_2)}{x_1 - \tilde{x_1}}q_6 = 0, \tag{4.114}$$

which is equivalent to $p_7 = q_6$. Finally when we plug in the relation for $\phi_c^{(0)}$ and $\phi_a^{(0)}$ into 3. we obtain

$$\begin{aligned} &\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}q_7 \\ -&\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}q_6 = 0, \end{aligned} \tag{4.115}$$

which leads to $q_6 = q_7$. In total we end up with the following relations

$$p_6 = q_2 = -q_6 = -p_7 = -p_2 = -q_7. \tag{4.116}$$

This reduces the relations to

$$\begin{aligned} \psi_a^{(0)} &= \psi_d^{(1)} + \frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}(-q_2), \\ \psi_b^{(0)} &= -\psi_b^{(1)} + (x_2-\tilde{x_2})q_2, \\ \psi_c^{(0)} &= -\psi_c^{(1)} + \frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}(-q_2), \\ \psi_d^{(0)} &= \psi_a^{(1)} + (x_1-\tilde{x_1})(-q_2). \end{aligned} \tag{4.117}$$

Our aim is it to use these relations to reduce 1.,4.,5. and 8. to a single constraint, i.e. we show they are equivalent. To do so we consider the differences of the constraints. Begin with 4 and 5. When we use the relations (4.117) one can easily compute that the difference is identical zero which means that 4 and 5 are equal constraints. We can do the same for the constraints 1 and 8 and finally we obtain that 4 and 8 are equal. So indeed the number of non equal constraints reduces to one. In the following we consider the constraint 1., following the reasoning outlined in the one variable case we have that the ideal I is given by

$$\begin{aligned} I = \Bigg\{&\psi_a^{(0)}(x_1-\tilde{x_1}) + \psi_c^{(0)}(x_2-\tilde{x_2}) + \psi_a^{(1)}\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}} \\ +&\psi_b^{(1)}\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}} | \psi_a^{(0)},\psi_c^{(0)},\psi_a^{(1)},\psi_b^{(1)} \in \mathbb{C}[x_1,\tilde{x_1},x_2,\tilde{x_2}]\Bigg\}. \end{aligned} \tag{4.118}$$

Let us write the superpotential $W(x_1,x_2)$ as

$$W(x_1,x_2) = a_0 + b_1x_1 + c_1x_2 + d_{11}x_1x_2 + b_2x_1^2 + c_2x_2^2 + d_{12}x_1x_2^2 + d_{21}x_1^2x_2 + ..., \tag{4.119}$$

where the coefficients b are in front of the pure terms in x_1, the coefficients c are in front of the pure terms in x_2 and the coefficients d indicate the mixed terms. Without loss of generality we can assume that $a_0 = 0$ since it would cancel out anyway. When we consider the term $\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}}$, we can apply our lemma from above to the polynomial with factors b and factors d separately (note that the c terms cancel out) and obtain

$$\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}} = b_n(x_1^{n-1}+x_1^{n-2}\tilde{x_1}+...+\tilde{x_1}^{n-1})+...+b_2(x_1-\tilde{x_1})+b_1$$
$$+d_{nn}x_2^n(x_1^{n-1}+x_1^{n-2}\tilde{x_1}+...+\tilde{x_1}^{n-1})$$
$$+d_{nn-1}x_2^{n-2}(x_1^{n-1}+x_1^{n-2}\tilde{x_1}+...+\tilde{x_1}^{n-1})+....+d_{11}. \tag{4.120}$$

Again we can rewrite this expression in the way we did in the one variable case

$$\frac{W(x_1,x_2)-W(\tilde{x_1},x_2)}{x_1-\tilde{x_1}} = \partial_{x_1}W(x_1,x_2)+(x_1-\tilde{x_1})(...). \tag{4.121}$$

By the same argument we find a similar expression for $\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}}$ which is given by

$$\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}} = \partial_{x_2}W(\tilde{x_1},x_2)+(x_2-\tilde{x_2})(...). \tag{4.122}$$

As we did it before we can express this as

$$\frac{W(\tilde{x_1},x_2)-W(\tilde{x_1},\tilde{x_2})}{x_2-\tilde{x_2}} = \partial_{x_2}W(x_1,x_2)+(x_2-\tilde{x_2})(...)+(x_1-\tilde{x_1})(...). \tag{4.123}$$

Thus our Ideal I now has the form

$$I=\Big\{\psi_a^{(0)}(x_1-\tilde{x_1})+\psi_c^{(0)}(x_2-\tilde{x_2})+\tilde{\psi}_a^{(1)}(x_1,x_2)\partial_{x_1}W(x_1,x_2)$$
$$+\tilde{\psi}_b^{(1)}(x_1,x_2)\partial_{x_2}W(x_1,x_2)|\psi_a^{(0)},\psi_c^{(0)},\psi_a^{(1)},\psi_b^{(1)}\in\mathbb{C}[x_1,\tilde{x_1},x_2,\tilde{x_2}]\Big\}, \tag{4.124}$$

where we absorbed the terms which are proportional to $(x_1-\tilde{x_1})$ and $(x_2-\tilde{x_2})$ in the first two terms respectively. To avoid cancellations in the mixed terms $\psi_a^{(0)}(x_1-\tilde{x_1})+\psi_c^{(0)}(x_2-\tilde{x_2})$ to pure terms in x_1 and x_2 we can use the same argument again and write $\phi_c^{(0)}$ as

$$\phi_c^{(0)}=\phi_{c_0}^{(0)}(x_1,x_2)+(x_1-\tilde{x_1})\phi_{c_1}^{(0)}(x_1,\tilde{x_1},x_2,\tilde{x_2})+(x_2-\tilde{x_2})\phi_{c_2}^{(0)}(x_1,\tilde{x_1},x_2,\tilde{x_2}), \tag{4.125}$$

and absorb without loss of generality the component proportional to $(x_1-\tilde{x_1})$ of $\psi_c^{(0)}$ in the first term $\psi_a^{(0)}(x_1-\tilde{x_1})$. Hence we find that the ideal I is given by

$$I=\Big\{\psi_a^{(0)}(x_1-\tilde{x_1})+\tilde{\psi}_c^{(0)}(x_1,x_2,\tilde{x_2})(x_2-\tilde{x_2})+\tilde{\psi}_a^{(1)}(x_1,x_2)\partial_{x_1}W(x_1,x_2)$$
$$+\tilde{\psi}_b^{(1)}(x_1,x_2)\partial_{x_2}W(x_1,x_2)|\psi_a^{(0)},\psi_c^{(0)},\psi_a^{(1)},\psi_b^{(1)}\in\mathbb{C}[x_1,\tilde{x_1},x_2,\tilde{x_2}]\Big\}, \tag{4.126}$$

and therefore the intersection of I with $\mathbb{C}[x_1, x_2]$ is indeed the Jacobi ideal

$$I \cap \mathbb{C}[x_1, x_2] = \langle \partial_{x_1} W(x_1, x_2), \partial_{x_2} W(x_1, x_2) \rangle. \tag{4.127}$$

Note that this result also shows that we achieve surjectivity when we only consider a morphism of degree zero. In fact any morphism between matrix factorisations is induced by a bosonic degree zero morphism between U and V, where $U = V = id$. This show that there is at least one suitable choice of $\phi_b^{(0)}$ in (4.84), maybe this gives us some hinst how we can argue in the case of general U and V.

4.7 Outlook

We have shown that our functor Ξ^{I_W} is indeed surjective on the morphism spaces in the one variable case and also for the two variable case where we set $U = V = id$. By knowing the kernel one can introduce an equivalence relation on the morphism space of functors such that Ξ^{I_W} becomes an isomorphism, which would mean that one has a one-to-one correspondence between morphisms of fusion functors and morphisms between matrix factorisations. Maybe one can calculate the kernel of Ξ^{I_W} in other situations too, to obtain hints how to prove or disprove the surjectivity of Ξ^{I_W} in the general two variable case.

The content of the thesis can help us understanding the process of fusing defects. Since finding factorisations of defects obtained by fusion is equivalent finding morphisms between the corresponding fusion functors. The first question one has to investigate is if every defect can be obtained from fusion functors. In further consequence this can maybe used to formulate a correspondence between two-dimensional conformal field theories with defects in it and our two-dimensional Landau-Ginzburg models.

A surprising fact is that the the cohomologies of morphisms between matrix factorisations also appear in the context of invariants of knots. So maybe this is a hint that there is a connection between defects in two-dimensional Landau Ginzburg theories and knot theory. Finding such connections often helps to generalize things in physics as we have more room to interpret our observations.

Appendices

Appendix A

Formal definitions

A.1 Modules

Definition A.1.1 (Module). *Let R be a unital ring and 1_R its multiplicative identity. A left R-module M consists of an abelian group $(M, +)$ and a map $\cdot : R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$, we have:*

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1_R \cdot x = x$.

The map is called the scalar multiplication and we denote it by rx instead of $r \cdot x$ when the context is clear.
A right R-module is defined in a similar way, except that the ring acts on the right i.e. the scalar multiplication takes the form $\cdot : M \times R \to M$ with the above axioms are written with scalars $r, s \in R$ on the right side of $x, y \in M$.

Definition A.1.2 (Bimodule). *Let R and S be two rings, then an ${}_SM_R$ bimodule is an abelian group such that:*

1. *${}_SM_R$ is a left-S-module and a right R-module.*
2. *For all $r \in R$, $s \in S$ and $m \in_S M_R$ we have*

$$(rm) \cdot s = r \cdot (ms).$$

A.2 Category Theory

Definition A.2.1 (Category). *A category $\mathscr{C}$ consists of the following data:*

1. *A class $Ob(\mathscr{C})$, which elements we call objects.*

2. *For all $A, B \in Ob(\mathscr{C})$ there is a set $Hom_{\mathscr{C}}(A, B)$ with elements $f : A \to B$. We call them morphisms from A to B.*

3. *For all $A, B, C \in Ob(\mathscr{C})$ there is a map*

$$Hom_{\mathscr{C}}(A, B) \times Hom_{\mathscr{C}}(B, C) \to Hom_{\mathscr{C}}(A, C),$$

which we denote with $(f, g) \to g \circ f$ and call this map composition of morphisms. This composition of morphisms is associative i.e. for all $f : A \to B$, $g : B \to C$ and $h : C \to D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$.

4. *For every $A \in Ob(\mathscr{C})$ there is a unique morphism $id_A \in Hom_{\mathscr{C}}(A, A)$ which we call identity. Such that we have for all $f \in Hom_{\mathscr{C}}(A, B)$, $1_B \circ f = f = f \circ 1_A$.*

In this thesis we work with so called differential graded categories. They are defined in the following way:

Definition A.2.2 (Differential graded category)**.** *A category $\mathscr{C}$ is called differential graded when for any two objects $A, B \in \mathscr{C}$ we have $Hom(A, B) = \bigoplus_{n=0}^{\infty} Hom_n(A, B)$ and there exists a map $d : Hom_n(A, B) \to Hom_{n+1}(A, B)$, called differential, which satisfies $d \circ d = 0$.*

When we work with categories we also need to introduce functors between two categories. This leads us to the next definition.

Definition A.2.3 (Functor)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories. A functor*

$$F : \mathscr{C} \to \mathscr{D}$$

from $\mathscr{C}$ to $\mathscr{D}$ consists of the following data:

1. *For all objects $A \in \mathscr{C}$ there is an object $F(A) \in \mathscr{D}$.*

2. *For all morphisms $f : A \to B$ in $\mathscr{C}$ there is a morphism*

$$F(f) : F(A) \to F(B)$$

in $\mathscr{D}$.

Such that

1. *For all objects $A \in \mathscr{C}$ is $F(id_A) = id_{F(A)}$.*

2. *For each pair of morphisms $f : A \to B$, $g : B \to C$ in $\mathscr{C}$ holds that*

$$F(g \circ f) = F(g) \circ F(f)$$

in $\mathscr{D}$.

We also need an exact definition of composing functors.

Definition A.2.4 (Composition of functors). *Let be $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ two functors. The composition $G \circ F : \mathscr{C} \to \mathscr{E}$ is given by $(G \circ F)(A) = G(F(A))$ for all objects $A \in \mathscr{C}$ and $(G \circ F)(f) = G(F(f))$ for all morphisms $f : A \to B$.*

Next we want to define what a natural transformation between two categories means. This is one of the most important concepts in category theory.

Definition A.2.5 (Natural Transformation). *Let be $F, G : \mathscr{C} \to \mathscr{D}$ two functors bewteen the categories $\mathscr{C}$ and $\mathscr{D}$. A natural transformation t from $F$to G associates to every object $X \in \mathscr{C}$ a morphism $t_X : F(X) \to G(X)$ in $\mathscr{D}$ such that for every morphism $f : X \to Y$ in $\mathscr{C}$, we have $t_Y \circ F(f) = G(f) \circ t_X$; this means that the following diagram is commutative:*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "t_x"'] & F(Y) \arrow[d, "t_y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y). \end{tikzcd}$$

A.3 Homological Algebra

Let R be a unital ring and A_i, with $i \in \mathbb{Z}$ modules over this ring.

Definition A.3.1 (Chain complexes). *A chain complex (A_i, d) is a sequence of modules A_i connected by homomorphisms, called boundary operators or differentials, $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is the zero map, i.e. $d_n \circ d_{n-1}$ for all $n \in \mathbb{Z}$, they chain complex is written as*

$$\cdots \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} A_{n-2} \to \cdots \tag{A.1}$$

In an analogous way we define cochain complexes:

Definition A.3.2 (Cochain complexes). *A cochain complex (A_i, d) is a sequence of modules A_i connected by homomorphisms, called boundary operators or differentials, $d_n : A_n \to A_{n-1}$, such that the composition of any two consecutive maps is the zero map, i.e. $d_n \circ d_{n+1}$ for all $n \in \mathbb{Z}$, the cochain complex is written as*

$$\cdots \to A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \xrightarrow{d_{n+1}} A_{n+2} \to \cdots \tag{A.2}$$

Definition A.3.3 (Homology). *The kernel $Z_n = ker(d_n) \subset A_{n-1}$ is a submodule of A_{n-1} and $B_n = im(d_{n+1}) \subset A_n$ is a submodule of A_n. Since $d_n \circ d_{n+1}$=0 we have $B_n \subset Z_n$. The n-th Homology group of A is then defined as*

$$H_n = Z_n / B_n. \tag{A.3}$$

Again in the same way we can define cohomologies

Definition A.3.4 (Cohomology)**.** *The kernel* $Z_n = ker(d_n) \subset A_{n+1}$ *is a submodule of* A_{n+1} *and* $B_n = im(d_{n-1}) \subset A_n$ *is a submodule of* A_n*. Since* $d_n \circ d_{n-1}$*=0 we have* $B_n \subset Z_n$*. The n-th Cohomology group of A is then defined as*

$$H_n = Z_n/B_n. \tag{A.4}$$

In the thesis we recognize that the structure of the morphism spaces are isomorphic to the Hochschild cohomology which is defined in the following way.

Definition A.3.5 (Hochschild)**.** *Let R be a commutative ring, A be an associative R-algebra and M a R-bimodule. The chain complex of the Hochschild homology is given by*

$$C_n(A, M) := M \otimes A^{\otimes n}. \tag{A.5}$$

With the differential defined by

$$\begin{aligned} d_0(m \otimes a_1 \otimes ... \otimes a_n) &= ma_1 \otimes a_2 \otimes ... \otimes a_n, \\ d_i(m \otimes a_1 \otimes ... \otimes a_n) &= m \otimes a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n, \\ d_n(m \otimes a_1 \otimes ... \otimes a_n) &= a_n m \otimes a_1 \otimes ... \otimes a_{n-1}. \end{aligned} \tag{A.6}$$

The Hochschild cohomology is then defined in an analogous way where we replace $C_n(A, M) = Hom(A^{\otimes n}, M)$.

For our considerations we need a few result from involved homological algebra which we want to state here without the proofs for the sake of completeness.

Lemma A.3.1. *The Hochschild homology and cohomology are isomorphic to relative Tor and Ext for the ring map* $k \to R^e = R \otimes_{\mathbb{C}} R^{op}$*:*

$$H_*(R, M) \cong Tor_*^{R^e/k}(M, R); \text{ and } H^*(R, M) \cong Ext^*_{R^e/k}(R, M), \tag{A.7}$$

where k is any field.

The scond theorem is the so called Koszul resolution which is very helpful for investigating the structure of the cohomology.

Theorem A.3.2 (Koszul resolution)**.** *If x is a regular sequence in a ring R, then K(x) is a free solution of* R/I *with* $I = (x_1, ..., x_n)R$*. Then the following sequence is exact*

$$0 \to \Lambda^n((R)^n) \to ... \to \Lambda^2((R)^n) \to (R)^n \to R \to R/I \to 0. \tag{A.8}$$

Appendix B

Proofs

B.1 Proof of lemma 3.2.1

Proof. First note that we can rewrite the action of the differential in a more compact form. We have

$$\begin{aligned}
&\left(d\phi_{UV}^{(n)}\right) = V(f_{n+1}) \circ \phi_{UV}^{(n)} \left(M_{n+1} \xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1} ... \xleftarrow{f_1} M_1\right) \\
&+ \sum_{i=1}^{n} (-1)^i \phi_{UV}^{(n)} \left(M_{n+2} \xleftarrow{f_n+1} M_{n+1} ... M_{i+2} \xleftarrow{f_{i+1} \circ f_i} M_i ... M_2 \xleftarrow{f_1} M_1\right) \\
&+ (-1)^{n+1} \phi_{UV}^{(n)} \left(M_{n+2} \xleftarrow{f_{n+1}} M_{n+1} \xleftarrow{f_n} M_n ... \xleftarrow{f_2} M_2\right) \circ U(f_1).
\end{aligned} \tag{B.1}$$

When we apply d twice on $\phi^{(n)}_{UV}$ we obtain

$$
\begin{aligned}
&(d\circ d)\phi^{(n)}_{UV} = \\
&\quad = V(f_{n+2})\circ V(f_{n+1})\circ \phi^{(n)}_{UV}\phi^{(n)}_{UV}\left(M_{n+1}\xleftarrow{f_n} M_n \xleftarrow{f_{n-1}} M_{n-1}... \xleftarrow{f_1} M_1\right)\\
&\quad + \sum_{i=1}^{n+1} V(f_{n+2})\circ \phi^{(n)}_{UV}\left(M_{n+2}\xleftarrow{f_n+1} M_{n+1}...M_{i+2}\xleftarrow{f_{i+1}\circ f_i} M_i...M_2\xleftarrow{f_1} M_1\right)\\
&\quad + (-1)^{n+2}V(f_{n+2})\circ \phi^{(n)}_{UV}\left(M_{n+2}\xleftarrow{f_{n+1}} M_{n+1}\xleftarrow{f_n} M_n...\xleftarrow{f_2} M_2\right)\circ U(f_1)\\
&\quad + \sum_{j=1}^{n+1}(-1)^j\sum_{i=1}^{n}(-1)^i\phi^{(n)}_{UV}\left(...M_{j+2}\xleftarrow{f_{j+1}\circ f_j} M_j...M_{i+2}\xleftarrow{f_{i+1}\circ f_i} M_i...\right)\\
&\quad + (-1)^{n+1}V(f_{n+2})\circ \phi^{(n)}_{UV}\left(M_{n+2}\xleftarrow{f_{n+1}} M_{n+1}\xleftarrow{f_n} M_n...\xleftarrow{f_2} M_2\right)\circ U(f_1)\\
&\quad + \sum_{i=1}^{n+1}\phi^{(n)}_{UV}\left(M_{n+2}\xleftarrow{f_n+1} M_{n+1}...M_{i+2}\xleftarrow{f_{i+1}\circ f_i} M_i...M_3\xleftarrow{f_2} M_2\right)\circ U(f_1)\\
&\quad + (-1)^{n+1+n+2}\phi^{(n)}_{UV}\left(M_{n+2}\xleftarrow{f_{n+1}} M_{n+1}\xleftarrow{f_n} M_n...\xleftarrow{f_3} M_3\right)\circ U(f_2)\circ U(f_1)\\
&\quad = 0
\end{aligned}
\tag{B.2}
$$

where the term with summation over i and j vanishes. To see this take the pair (j, i) with $1 \le i \le n$ and $1 \le j \le n+1$. If we first compose the maps "around" M_i and then "around" M_j will produce an opposite sign as to when we first compose the maps "around" M_j and then "around" M_i. Note that the sum decays into an even number of terms since the product of n with $n+1$ has to be even.

The Leibniz-rule identity follows easily by direct calculation. □

B.2 Proof of lemma 3.3.1

Proof. Let $\phi^{(n)} \in Hom_n(U,V)$ be a closed morphism of degree n between functors U and V. Define a morphism $\phi^{(n-1)} \in Hom_{n-1}(U,V)$ on a sequence of R-module homomorphisms $M_n \xleftarrow{f_{n-1}} M_{n-1} \xleftarrow{f_{n-2}} M_{n-2}... \xleftarrow{f_1} M_1$, as argued above it is equivalent to consider homomorphisms between R-modules R^{m_i} instead of the modules M_i. Let the morphism $\phi^{(n-1)} \in Hom_{n-1}(U,V)$ given by

$$\phi^{(n-1)} = \sum_{j=1}^{n}(-1)^{n-j+1}\phi^{(n-1)}_{[j]}, \tag{B.3}$$

with

$$\phi^{(n-1)}_{[1]}(f_{n-1},...,f_1) = V(\iota_{k_n})\circ\phi^{(n)}(p_{n-1},...,p_2,\pi_{k_2}\circ f_1,\iota_{k_1})\circ U(\pi_{k_1})$$
$$\phi^{(n-1)}_{[2]}(f_{n-1},...,f_1) = V(\iota_{k_n})\circ\phi^{(n)}(p_{n-1},...,p_3,\pi_{k_3}\circ f_2,\iota_{k_2},\pi_{k_2}\circ f_1)$$
$$\vdots$$
$$\phi^{(n-1)}_{[n-2]}(f_{n-1},...,f_1) = V(\iota_{k_n})\circ\phi^{(n)}(p_{n-1},\pi_{k_{n-1}}\circ f_{n-2},\iota_{k_{n-2}},\pi_{k_{n-2}}$$
$$\circ f_{n-3},f_{n-4},...,f_1)$$
$$\phi^{(n-1)}_{[n-1]}(f_{n-1},...,f_1) = V(\iota_{k_n})\circ\phi^{(n)}(\pi_{k_n}\circ f_{n-1},\iota_{k_{n-1}},\pi_{k_{n-1}}\circ f_{n-2},f_{n-3},...,f_1)$$
$$\phi^{(n-1)}_{[n]}(f_{n-1},...,f_1) = \phi^{(n)}(\iota_{k_n},\pi_{k_n}\circ f_{n-1},f_{n-2},...,f_1), \tag{B.4}$$

where $p_j := \pi_{k_{j+1}}\circ f_j\circ\iota_{k_j} : R\to R$, and it is understood that on the right hand side a summation over all appearing labels $k_j = 1,...,m_j$ is performed. We want to show that

$$\phi^{(n)}(f_n,...,f_1) = V(\iota_{k_{n+1}}\circ\phi^{(n)}(p_n,...,p_1)\circ U(\pi_{k_1}) + d\phi^{(n-1)}(f_{n-1},...,f_1). \tag{B.5}$$

For the readability reasons replace ι_{k_j} by g_j^{-1} and π_j by g_j and the summation is meant over the k_j.

Where $g_m := g_{M_m}$ and $p_m := g_{m+1}\circ f_m\circ g_m^{-1}$, using the notation of section 3.3. Our aim is to show that

$$\phi^{(n)}(f_n,...,f_1) = V(g_n^{-1})\circ\phi^{(n)}(p_n,...,p_1)\circ U(g_1) + d\phi^{(n-1)}(f_n,...,f_1). \tag{B.6}$$

Therefore we first calculate $d\phi^{(n-1)}_{[n]}$.

$$\begin{aligned}
d\phi^{(n-1)}_{[n]}(f_n, ..., f_1) =& V(f_n) \circ \phi^{(n-1)}_{[n]}(f_{n-1}, ..., f_1) \\
& - \phi^{(n-1)}_{[n]}(f_n \circ f_{n-1}, f_{n-2}, ..., f_1) + ... \\
& ... + (-1)^{n-1}\phi^{(n-1)}_{[n]}(f_n, ..., f_3, f_2 \circ f_1) \\
& + (-1)^n \phi^{(n-1)}_{[n]}(f_n, ..., f_2) \circ U(f_1) \\
& = V(f_n) \circ \phi^{(n)}(g_n^{-1}, g_n \circ f_{n-1}, f_{n-2}, ..., f_1) \\
& - \phi^{(n)}(g_{n+1}^{-1}, g_{n+1} \circ f_n \circ f_{n-1}, f_{n-2}, ..., f_1) \\
& + \phi^{(n)}(g_{n+1}^{-1}, g_{n+1} \circ f_n, f_{n-1} \circ f_{n-2}, f_{n-3}, ..., f_1) + ... \\
& ... + (-1)^{n-1}\phi^{(n)}(g_{n+1}^{-1}, g_{n+1} \circ f_n, f_{n-1}, ..., , f_3, f_2 \circ f_1) \\
& + (-1)^n \phi^{(n)}(g_{n+1}^{-1}, g_{n+1} \circ f_n, f_{n-1}, ..., f_3, f_2) \circ U(f_1) \\
& = V(f_n) \circ \phi^{(n)}(g_n^{-1}, g_n \circ f_{n-1}, f_{n-2}, ..., f_1) \\
& + \Big[V(g_{n+1}^{-1}) \circ \phi^{(n)}(g_{n+1} \circ f_n, f_{n-1}, ..., f_1) \\
& - \phi^{(n)}(f_n, ..., f_1) \Big].
\end{aligned} \tag{B.7}$$

In the last step we used that

$$d\phi^{(n)}(g_{n+1}^{-1}, g_{n+1} \circ f_n, f_{n-1}, ..., f_1) = 0, \tag{B.8}$$

since $\phi^{(n)}$ is closed. Now we consider $\phi^{(n-1)}_{[j]}$ for $1 < j < n$. We obtain

$$
\begin{aligned}
&d\phi^{(n-1)}_{[j]}(f_n, ..., f_1) = V(g_{n+1}^{-1})\circ \\
&\Big(\big[\, V(p_n) \circ \phi^{(n)}(p_{n-1}, ..., p_{j+1}, g_{j+1} \circ f_j, g_j^{-1}, g_j \circ f_{j-1}, f_{j-2}, ..., f_1) \\
&- \phi^{(n)}(p_n \circ p_{n-1}, p_{n-2}, ..., p_{j+1}, g_{j+1} \circ f_j, g_j^{-1}, g_j \circ f_{j-1}, f_{j-2}, ..., f_1) \\
&+ ... \\
&+ (-1)^{n-j-1}\phi^{(n)}(p_n, ..., p_{j+3}, p_{j+2} \circ p_{j+1}, g_{j+1} \circ f_j, g_j^{-1}, g_j \circ f_{j-1}, f_{j-2}, ..., f_1) \\
&+ (-1)^{n-j}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1} \circ f_j, g_j^{-1}, g_j \circ f_{j-1}, f_{j-2}, ..., f_1)\,\big] \\
&+ \big[\, (-1)^{n-j+1}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j \circ f_{j-1}, f_{j-2}, ..., f_1) \\
&+ (-1)^{n-j+2}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1} \circ f_{j-2}, f_{j-3}, ..., f_1) \\
&+ ... \\
&+ (-1)^{n-1}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_3, f_2 \circ f_1) \\
&+ (-1)^{n}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_3, f_2) \circ U(f_1)\,\big]\Big) \\
&= V(g_{n+1}^{-1}) \circ \Big[A(n,j) + B(n,j)\Big],
\end{aligned}
\tag{B.9}
$$

where $A(n,j)$ combines the first $n-j+1$ summands and $B(n,j)$ the remaining

j summands. Since $\phi^{(n)}$ is closed we know that for $1 < j < n-1$ we obtain

$$
\begin{aligned}
0 =& d\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
=& V(p_n) \circ \phi^{(n)}(p_{n-1}, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
& - \phi^{(n)}(p_n \circ p_{n-1}, p_{n-2}, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
& + ... \\
& + (-1)^{n-j-2}\phi^{(n)}(p_n, ..., p_{j+4}, p_{j+3} \circ p_{j+2}, g_{j+2} \\
& \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
& + (-1)^{n-j-1}\phi^{(n)}(p_n, ..., p_{j+3}, g_{j+3} \circ f_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
& + (-1)^{n-j}\phi^{(n)}(p_n, ..., p_{j+1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
& + (-1)^{n-j+1}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, f_j, ..., f_1) \\
& + (-1)^{n-j+2}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j \circ f_{j-1}, f_{j-2}, ..., f_1) \\
& + (-1)^{n-j+3}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \\
& \circ f_j, f_{j-1} \circ f_{j-2}, f_{j-3}, ..., f_1) \\
& + ... \\
& (-1)^n\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_3, f_2 \circ f_1) \\
& (-1)^{n+1}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_2) \circ U(f_1).
\end{aligned}
\tag{B.10}
$$

With the definition from above we have

$$
\begin{aligned}
0 =& d\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, g_{j+1}^{-1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
&= A(n, j+1) - B(n, j) \\
&+ (-1)^{n-j}\phi^{(n)}(p_n, ..., p_{j+1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1) \\
&+ (-1)^{n-j+1}\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, f_j, ..., f_1).
\end{aligned}
\tag{B.11}
$$

This holds also for $j = 1$ and $j = n-1$ if we set

$$
\begin{aligned}
B(n, 1) &= (-1)^n\phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2, g_2^{-1}) \circ U(g_2 \circ f_1) \\
A(n, n) &= V(g_{n+1} \circ f_n) \circ \phi^{(n)}(g_n^{-1}, g_n \circ f_{n-1}, f_{n-2}, ..., f_1).
\end{aligned}
\tag{B.12}
$$

Therefore the contribution of the $\phi^{(n-1)}_{[j]}$ for $1 < j < n$ to $d\phi^{(n-1)}$ is

$$
\begin{aligned}
&\sum_{j=2}^{n-1}(-1)^{n-j+1} d\phi^{(n-1)}_{[j]}(f_n, ..., f_1) = \\
&\qquad = \sum_{j=2}^{n-1}(-1)^{n-j+1} V(g_{n+1}^{-1}) \circ \big[A(n,j) + B(n,j)\big] \\
&\qquad = V(g_{n+1}^{-1}) \circ \Bigg[A(n,n) - (-1)^n B(n,1) \\
&\qquad + \sum_{j=1}^{n-1}\big[\phi^{(n)}(p_n, ..., p_{j+2}, g_{j+2} \circ f_{j+1}, f_j, ..., f_1) \\
&\qquad - \phi^{(n)}(p_n, ..., p_{j+1}, g_{j+1} \circ f_j, f_{j-1}, ..., f_1)\big]\Bigg] \\
&\quad = V(f_n) \circ \phi^{(n)}(g_n^{-1}, g_n \circ f_{n-1}, f_{n-2}, ..., f_1) \\
&\qquad - V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2, g_2^{-1}) \circ U(g_2 \circ f_1) \\
&\qquad + V(g_{n+1}^{-1}) \circ \phi^{(n)}(g_{n+1} \circ f_n, f_{n-1}, ..., f_1) \\
&\qquad - V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_2, g_2 \circ f_1).
\end{aligned}
\tag{B.13}
$$

Combing this result with our result of (B.7) we obtain

$$
\begin{aligned}
&\sum_{j=2}^{n-1}(-1)^{n-j+1} d\phi^{(n-1)}_{[j]}(f_n, ..., f_1) = \\
&\quad = \phi^{(n)}(f_n, ..., f_1) \\
&\qquad - V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2, g_2^{-1}) \circ U(g_2 \circ f_1) \\
&\qquad - V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_2, g_2 \circ f_1).
\end{aligned}
\tag{B.14}
$$

Now we calaculate the contribution of $\phi^{(n)}_{[1]}$.

$$
\begin{aligned}
&d\phi^{(n)}_{[1]}(f_n, ..., f_1) \\
&\quad = V(f_n \circ g_{n+1}^{-1}) \circ \phi^{(n)}(p_{n-1}, ..., p_2, g_2 \circ f_1, g_1^{-1}) \circ U(g_1) \\
&\qquad - V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n \circ p_{n-1}, p_{n-2}, ..., p_2, g_2 \circ f_1, g_1^{-1}) \circ U(g_1) \\
&\qquad + ... \\
&\qquad + (-1)^{n-2} V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_4, p_3 \circ p_2, g_2 \circ f_1, g_1^{-1}) \circ U(g_1) \\
&\qquad + (-1)^{n-1} V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2 \circ f_1, g_1^{-1}) \circ U(g_1) \\
&\qquad + (-1)^{n} V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2, g_2^{-1}) \circ U(g_2 \circ f_1)
\end{aligned}
\tag{B.15}
$$

Since $\phi^{(n)}$ is closed we conclude that

$$\begin{aligned} 0 &= d\phi^{(n)}(p_n, ..., p_2, g_2 \circ f_1, g_2^{-1}) \\ &= V(p_n) \circ \phi^{(n)}(p_{n-1}, ..., p_2, g_2 \circ f_1, g_2^{-1}) \\ &- \phi^{(n)}(p_n \circ p_{n-1}, p_{n-2}, ..., p_2, g_2 \circ f_1, g_2^{-1}) \\ &+ ... \\ &+ (-1)^{n-2}\phi^{(n)}(p_n, ..., p_4, p_3 \circ p_2, g_2 \circ f_1, g_2^{-1}) \\ &+ (-1)^{n-1}\phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2 \circ f_1, g_2^{-1}) \\ &+ (-1)^{n}\phi^{(n)}(p_n, ..., p_1) \\ &+ (-1)^{n+1}\phi^{(n)}(p_n, ..., p_2, g_2 \circ f_1) \circ U(g_2^{-1}). \end{aligned} \tag{B.16}$$

Hence that

$$\begin{aligned} 0 &= d\phi^{(n)}(f_n, ..., f_1) \\ &= -(-1)^n V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_1) \circ U(g_1) \\ &- (-1)^{n+1} V(g_{n+1}^{-1}) \circ \phi^{(n)}(p_n, ..., p_2, g_2 \circ f_1) \\ &+ (-1)^n V(g_{n+1}^{-1})^{-1} \circ \phi^{(n)}(p_n, ..., p_3, g_3 \circ f_2, g_2^{-1}) \circ U(g_2 \circ f_1). \end{aligned} \tag{B.17}$$

Thus we finally obtain the desired result

$$\begin{aligned} 0 = d\phi^{(n)}(f_n, ..., f_1) = \sum_{j=1}^{n} (-1)^{n-j+1} d\phi_{[j]}^{(n-1)}(f_n, ..., f_1) \\ = \phi^{(n)}(f_n, ..., f_1) \\ - V(g_{n+1}^{-1}) \circ \phi(n)(p_n, ..., p_1) \circ U(g_1). \end{aligned} \tag{B.18}$$

□

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