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Maximilian Löschner, MSc BSc

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Contents

Preface

The structure of this thesis is as follows: in Part I, we lay down the main theoretical concepts of importance to this work and provide an overview of the relevant experimental observations. The focus lies on the field of neutrino physics which motivated an abundance of theoretical investigations into the generation of lepton masses and mixings.

Part II is the culmination of publications produced in the past three years. Each chapter shows the publications' status or journal of publication, followed by a table of contents[1] and the original preprint versions as available online (www.arXiv.org). Firstly, in Chapter 3, we provide an inspection of well-known on-shell renormalization conditions in theories with inter-family mixing wherein we developed a simple and non-ad hoc way to derive them. This served as an introductory study on the interplay of one-loop corrections and mixing. Chapter 4 is devoted to radiative corrections in a toy model with an arbitrary number of Majorana or Dirac fermions and real scalars. This model was a simple example used for the development of a renormalization scheme to be finalized in more realistic models. In Chapter 5, we apply our findings to the multi-Higgs Standard Model (mHDSM), in order to carry out a renormalization of the scalar and leptonic sector, eventually arriving at analytic expressions for one-loop lepton masses. The focus in this chapter lies on the discussion of gauge-parameter dependencies, tadpole contributions and the renormalization of vacuum expectation values.

Lastly, in Part III, we summarize the main findings of this thesis and discuss some early results from an attempt at the numerical evaluation of the general one-loop results of Chapter 5.

[1] The page references in the tables of content in Part II refer to the page numbers in the respective papers, shown at the bottom of the pages.

Part I

Introduction

CHAPTER 1

Preliminaries

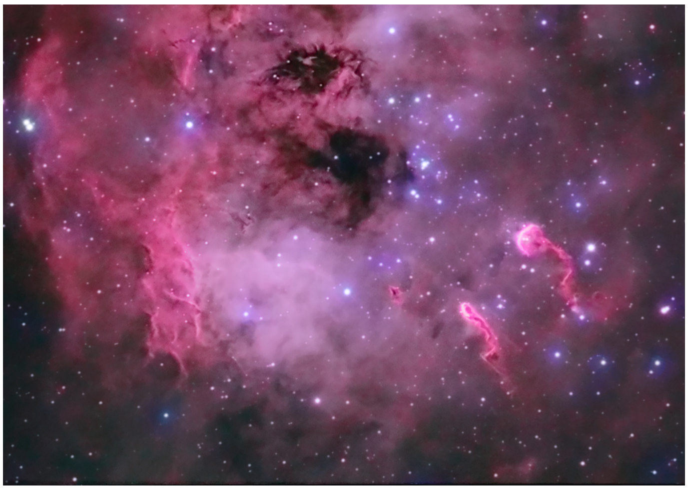


Figure 1.1: Astrophysical tadpoles of the emission nebula IC 410. Source: [1].

Our current understanding of the fundamental interactions and building blocks of nature at the smallest scales is described by the *Standard Model of particle physics* (SM). The SM is a theory based on local gauge-invariance under transformations belonging to the group

$$\mathcal{G}_{\text{SM}} = SU(3)_C \times SU(2)_L \times U(1)_Y, \tag{1.1}$$

where $(S)U(N)$ stand for (special) unitarity groups of dimension N. The particle content as well as the interactions between particles are described in terms of a Lagrangian density, $\mathcal{L}_{\text{SM}}$, consisting of all quantum-field theoretic hermitian operators of energy-dimension ≤ 4 that are invariant under said transformations and moreover Lorentz-invariant.

In Eq. (1.1), the subscript C stands for *color* and describes the force between strongly interacting fermions, the color-charged quarks. This force is mediated by eight massless color-charged gluons, corresponding to the $3^2 - 1 = 8$ generators of the Lie-group $SU(3)$. In nature, quarks appear as bound states called hadrons with the most prominent examples being the nuclei of atoms in the matter surrounding us.

The $SU(2)_L$- and $U(1)_Y$-components of Eq. (1.1) describe the electroweak part of the SM, representing one of the major successes of the SM: the unification of the electromagnetic and the weak force [2, 3, 4]. Interestingly enough, we find that only left-handed fermions and right-handed anti-fermions take part in the weak interactions, mediated by the massive W- and Z-gauge bosons. In the SM, this is described by left- and right-chiral fields transforming differently under $SU(2)$ which is the origin of the subscript L.[1] This was

[1] Note that the *chirality of a field* is determined by the action of the chiral projectors $\gamma_L = (\mathbb{1} - \gamma^5)/2$ and $\gamma_R = (\mathbb{1} + \gamma^5)/2$ on it while the *handedness* or *helicity* is given by the projection of the spin onto the momentum of a particle.

also the cause for a complication in the development of the SM: one can not generate the mass of a fermion Ψ via the naive inclusion of the Lorentz-invariant term $-m\overline{\Psi}_L\Psi_R + \text{H.c.}$ in $\mathcal{L}_{\text{SM}}$ because Ψ_L and Ψ_R transform differently under $SU(2)_L$. Therefore, mass terms of this kind spoil gauge-invariance and must not be included in the SM Lagrangian. A similar argument holds for the mass terms of the gauge bosons. Instead, the generation of mass is achieved via couplings to a scalar field and the mechanism of spontaneous symmetry breaking (SSB), to be further explained in Section 1.1.

Finally, the subscript Y in Eq. (1.1) stands for the electroweak hypercharge defined as

$$Y \equiv 2\,(Q - T_3)\,, \tag{1.2}$$

where Q stands for the electric charge and T_3 for the third component of the weak isospin, which is simply $+1/2$ for the upper component of a $SU(2)$-doublet and $-1/2$ for the lower one. After SSB, the $SU(3)_C$ symmetry remains intact while $SU(2)_L \times U(1)_Y$ is broken down to a residual $U(1)_Q$ invariance, describing the electromagnetic force with the massless photon as mediator.

The predictive power of the SM in describing or explaining experimental data is unparalleled. A well-known example is the anomalous magnetic moment of the electron $a_e = (g_e - 2)/2$. It describes the deviation of the electrons actual magnetic moment g_e from the value 2, as expected from the Dirac equation. In the SM, corrections to the Dirac-value can nowadays be calculated up to order ten in perturbation theory, resulting in an astonishing agreement between the experimentally measured and theoretically expected values of [5]

$$a_{e,\text{exp}} = 1\,159\,652\,180.73(28) \times 10^{-12}, \tag{1.3a}$$

$$a_{e,\text{theory}} = 1\,159\,652\,182.032(720) \times 10^{-12}. \tag{1.3b}$$

Despite its accomplishments, there are several reasons to believe in physics *Beyond the SM* (BSM). The most relevant for this thesis is the large number of freely adjustable values of theory defining parameters. The SM with three massive neutrinos[2] contains 26 of these if neutrinos are solely of Dirac nature: six quark and six lepton masses, three quark mixing and three lepton mixing angles, two phases parameterizing CP-violation, two gauge boson masses, the Higgs mass, one gauge coupling for the strong and one for the electroweak sector. Moreover, one should include a CP-violating operator built from the $SU(3)$-gauge boson field strength tensor in $\mathcal{L}_{\text{SM}}$, parameterized by θ_{QCD}. This operator is allowed on gauge symmetry considerations, yet θ_{QCD} is experimentally known to be minuscule if non-zero at all. Therefore, BSM physics is possibly needed to explain the absence of this term [6]. Lastly, two additional Majorana phase parameters need to be included if neutrinos are of Majorana nature. Note that the fermion masses and mixings represent the largest proportion of the set of free parameters.

Another intriguing fact is that the SM contains three families of leptons and quarks. Concerning the quantum numbers of the corresponding fields, each family represents an identical copy of the others, the only fundamental difference lying in the values of the respective masses. Up to this point, neither conclusive experimental evidence for additional families, nor definite theoretical reasoning for the non-existence of such has been found.

The main motivation for the work carried out in this thesis is the simple question whether there exist yet to be discovered principles behind the values of these parameters and the

[2]Note that in the literature, the term SM oftentimes refers to the model only containing massless left-chiral neutrinos. In our terminology, we refer to the SM as containing three massive neutrinos, which is due to the convincing experimental evidence of their non-vanishing masses and the straightforward way of their inclusion in the SM via the addition of right-chiral Dirac neutrinos.

organization into three families, specifically concerning the lepton masses and mixings. The discovery of neutrinos carrying non-zero but light masses and the measurement of the lepton mixing parameters have inspired a variety of mass and mixing models trying to explain the patterns found. An introduction to this subject in terms of minimal additions to the SM is given in Section 1.1 and further approaches are shown in Chapter 2. More specifically, the intent of our work is not to propose yet another model, but rather to provide tools for studying a larger portion of already proposed ones in terms of perturbative stability, possibly falsifying some current models or resuscitating models which have otherwise already been discarded.

Throughout this thesis we will work in *natural units* as is typically done in particle physics. Effectively this means setting

$$\hbar = c = \epsilon_0 = 1. \tag{1.4}$$

The result is that lengths, times and masses all carry the unit of energy to some power, *i.e.*

$$[\text{length}] = [\text{time}] = [\text{mass}]^{-1} = [\text{energy}]^{-1}. \tag{1.5}$$

For later reference, we derive the energy dimensions of vector boson, fermion and scalar fields from the action functional S which is dimensionless in natural units. Thus, for the energy dimension of a Lagrangian $\mathcal{L}$, *i.e.* all terms therein, it follows for d dimensions

$$\begin{aligned} 0 = [S] &= \left[\int \frac{\mathrm{d}^d x}{(2\pi)^d}\right] + [\mathcal{L}] \\ \Rightarrow [\mathcal{L}] &= d. \end{aligned} \tag{1.6}$$

The dimension of vector boson fields A^μ can be derived via their kinetic terms, *viz.*

$$\begin{aligned} 0 &= [(\partial_\mu A^\mu)\, \partial_\nu A^\nu] = 2 + 2\,[A^\mu] \\ \Rightarrow [A^\mu] &= \frac{d-2}{2}. \end{aligned} \tag{1.7}$$

In the same way, the dimension of scalar fields φ and fermions ψ are found to be

$$[\varphi] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}. \tag{1.8}$$

Subsequently, the energy dimension of an operator in $\mathcal{L}$ is defined as the sum over the dimensions of the fields which the operator consists of.

Lastly, we mention that we will make use of the *Feynman slash notation* as in

$$\not{A} \equiv \gamma^\mu A_\mu, \tag{1.9}$$

and moreover, emphasize that summation over repeated indices is implied, if not mentioned otherwise.

1.1 Introduction to masses and mixings

In this section we want to discuss the emergence of lepton masses and mixings in terms of the simplest possible generic SM extensions. For didactic reasons, the mass generation of the vector bosons is also included. We will use the same notation as in Chapter 5 and apply it to the case of having a single Higgs doublet, as in the SM.

It is now a firmly established fact that there are at least two non-zero neutrino masses and a strong inter-family mixing in the lepton sector, but we still lack the knowledge about the possible Majorana nature of neutrinos. Therefore, Dirac- and Majorana-type mass generation will be discussed. In the literature, similar versions are sometimes dubbed as νMSM, as it was introduced in [7, 8].

1.1.1 Spontaneous symmetry breaking

The masses of the SM fermions and vector bosons can be introduced via the mechanism of *spontaneous symmetry breaking* (SSB) or the so called *Brout-Englert-Higgs mechanism* [9, 10]. This mechanism states that masses of the SM fields can be generated via couplings of these to a scalar $SU(2)$-doublet field

$$\phi(x) = \begin{pmatrix} \varphi^+(x) \\ \varphi^0(x) \end{pmatrix}, \tag{1.10}$$

whose neutral component[3] acquires a non-vanishing vacuum expectation value (VEV)

$$\langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{1.11}$$

The VEV represents the global minimum of the scalar potential

$$V(\phi) = \mu^2 \phi^\dagger \phi + \lambda \left(\phi^\dagger \phi\right)^2, \tag{1.12}$$

which is included in $\mathcal{L}_{\text{SM}}$. Note that the existence of a non-vanishing VEV depends on the values of μ^2 and λ. The usual choice is $\mu^2 < 0$ and $\lambda > 0$ such that the potential is bounded from below and the global minimum lies at $v^2 = -\mu^2/\lambda$ and $v \in \mathbb{R}$. The vector bosons acquire their mass via the scalar kinetic term

$$\mathcal{L}_{\phi,\text{kin}} = (D_\mu \phi)^\dagger D^\mu \phi, \quad \text{with } D_\mu = \partial_\mu + i\frac{g}{2}\tau^a W^a_\mu + i\frac{g'}{2} B_\mu, \tag{1.13}$$

where τ^a are the Pauli matrices. Here, we shift the neutral component

$$\phi^0(x) \to (v + h(x))/\sqrt{2}, \tag{1.14}$$

which can be viewed as an expansion in terms of small field excitations $h(x) \in \mathbb{R}$ around the minimum[4] and collect the terms quadratic in the vector boson fields in Eq. (1.13), *i.e.*

$$\mathcal{L}_{V,\text{mass}} = -\frac{1}{2}\frac{v^2}{4} \begin{pmatrix} W^1_\mu & W^2_\mu & W^3_\mu & B_\mu \end{pmatrix} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{1,\mu} \\ W^{2,\mu} \\ W^{3,\mu} \\ B^\mu \end{pmatrix}, \tag{1.15}$$

giving three distinct mass eigenvalues

$$m_W = \frac{g}{2}v, \quad m_Z = \frac{\sqrt{g^2 + g'^2}}{2}v, \quad m_\gamma = 0. \tag{1.16}$$

These are the masses of the physical $W^\pm$-bosons, the Z-boson and the photon, which we can define as linear combinations of the fields in Eq. (1.13), *viz.*

$$W^\pm_\mu \equiv \frac{1}{\sqrt{2}} \left(W^1_\mu \mp i W^2_\mu\right), \quad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \equiv \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \quad \text{with } \tan\theta_w = \frac{g'}{g}. \tag{1.17}$$

The angle θ_w is the so called Weinberg angle. In the following, we will use the abbreviations

$$s_w \equiv \sin\theta_w \quad \text{and} \quad c_w \equiv \cos\theta_w. \tag{1.18}$$

[3]This choice guarantees that the residual $U(1)_Q$-symmetry remains unbroken, meaning that electric charge is a conserved quantity.

[4]Note that implicitly, here we have chosen the unitary gauge.

Note that the vector boson masses only depend on the modulus of v and therefore do not change in the multi-Higgs Standard Model (mHDSM) of Chapter 5. Comparing the coupling constants in the weak charged current interactions[5] with Fermis effective four-fermion theory and using Eq. (1.16), we can infer the relation [11]

$$\frac{4G_F}{\sqrt{2}} = \frac{g^2}{2m_W^2} = \frac{2}{v^2}. \tag{1.19}$$

The Fermi constant G_F can be measured rather precisely from the muon lifetime and has a value of [12]

$$G_F = 1.166\,378\,7(6) \times 10^{-5}\,\mathrm{GeV}^{-2} \simeq (292.8\,\mathrm{GeV})^{-2}\,, \tag{1.20}$$

which, using Eq. (1.19), gives the well known SM vacuum expectation value

$$v = \left(\sqrt{2}G_F\right)^{-1/2} \simeq 246.2\,\mathrm{GeV}. \tag{1.21}$$

We now turn to the masses of the leptons. Contrary to the vector boson, these arise from so called *Yukawa*-couplings to ϕ, which are constructed as follows. One can build two distinct gauge- and Lorentz-invariant terms of couplings of ϕ to leptons from the left-chiral $SU(2)$-doublets

$$D_{L,\alpha} = \begin{pmatrix} \nu_{L,\alpha} \\ e_{L,\alpha} \end{pmatrix}, \quad \alpha = e, \mu, \tau, \tag{1.22}$$

and the right-chiral $SU(2)$-singlets $e_{R,\alpha}$ and $\nu_{R,\alpha}$, which read

$$\mathcal{L}_{Y,\mathrm{lep}} = -\overline{e}_{R,\alpha}\,(\Gamma)_{\alpha\beta}\,\phi^\dagger D_{L,\beta} - \overline{\nu}_{R,\alpha}\,(\Delta)_{\alpha\beta}\,\tilde{\phi}^\dagger D_{L,\beta} + \mathrm{H.c.}, \tag{1.23}$$

Here we have used the definition

$$\tilde{\phi} = \varepsilon\phi^* = \begin{pmatrix} (\varphi^0)^* \\ -\varphi^- \end{pmatrix}, \quad \text{with } \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.24}$$

The gauge-invariance of the second term in Eq. (1.23) is realized by the fact that

$$U^T \varepsilon\, U = \det(U)\varepsilon \quad \text{for } U \in \mathbb{C}^{2\times 2}, \tag{1.25}$$

and the knowledge that in our case $U \in SU(2)$ with $\det(U) = 1$. In Eq. (1.23), we have defined the Yukawa coupling matrices for charged leptons, $\Gamma \in \mathbb{C}^{3\times 3}$, and neutral leptons, $\Delta \in \mathbb{C}^{n_R\times 3}$, where n_R is the number of right-chiral neutrinos. This number can differ from three, depending on the possible Majorana nature of neutrinos and the number of non-zero neutrino masses. If neutrinos turn out to be of only Dirac type and the lightest one is massless, which is still allowed by current experimental data (see Section 1.2), then $n_R = 2$ would suffice.

By inserting Eq. (1.11) in Eq. (1.23), we find the mass matrices of the charged leptons and neutrinos to be[6]

$$M_l \equiv \frac{v}{\sqrt{2}}\,\Gamma, \quad M_D \equiv \frac{v}{\sqrt{2}}\,\Delta, \tag{1.26}$$

which are of the same structure as the Yukawa coupling matrices. The subscript D in Eq. (1.26) stands for *Dirac-type* masses. The type of mass generation in Eq. (1.23) could

[5] See Eq. (1.35).

[6] In a general multi-Higgs doublet model, the VEVs can be complex and M_l is usually defined in terms of the complex conjugate of the VEVs.

be considered as the minimal extension of the SM allowing for an accommodation of massive neutrinos. This is to be understood in the sense that the masses for neutral and charged leptons would be generated in complete analogy to the ones of the up- and down-type quarks, respectively.

Next, we will diagonalize the mass matrices via the introduction of the unitary 3×3 matrices W_L and W_R, as well as $\tilde{U}_L$ and $\tilde{U}_R$ which are 3×3 and $n_R \times n_R$ unitary matrices, respectively.[7] The transformations to the mass eigenfields $l_{L/R}$ and $\tilde{\chi}_{L/R}$ read

$$e_L = W_L l_L, \quad e_R = W_R l_R, \quad \nu_L = \tilde{U}_L \tilde{\chi}_L, \quad \nu_R = \tilde{U}_R \tilde{\chi}_R. \tag{1.27}$$

The physical masses subsequently result from

$$W_R^\dagger M_l W_L \equiv \hat{m}_l = \mathrm{diag}(m_e, m_\mu, m_\tau), \tag{1.28a}$$

$$\tilde{U}_R^\dagger M_D \tilde{U}_L \equiv \hat{m}_\nu = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \\ 0 & 0 & 0 \\ & \vdots & \\ 0 & 0 & 0 \end{pmatrix} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} n_R - 3 \quad . \tag{1.28b}$$

The potential Majorana nature of neutrinos allows for two additional types of mass terms, *i.e.*

$$\mathcal{L}_{\mathrm{Maj},\nu_R} = -\frac{1}{2}\overline{(\nu_R)^c} M_R^* \nu_R + \mathrm{H.c.} = \frac{1}{2}\nu_R^T C^{-1} M_R^* \nu_R + \mathrm{H.c.} \tag{1.29a}$$

$$\mathcal{L}_{\mathrm{Maj},\nu_L} = -\frac{1}{2}\overline{(\nu_L)^c} M_L \nu_L + \mathrm{H.c.} = \frac{1}{2}\nu_L^T C^{-1} M_L \nu_L + \mathrm{H.c.} \tag{1.29b}$$

Note that the appearance of the complex conjugate of M_R in Eq. (1.29a) is purely conventional. The superscript c represents the operation of charge conjugation, *i.e.*

$$\psi^c = C(\gamma^0)^T \psi^*. \tag{1.30}$$

Here, C is the charge conjugation matrix which acts on the Dirac matrices as

$$C^{-1}\gamma^\mu C = -(\gamma^\mu)^T. \tag{1.31}$$

Note that $C^{-1}\gamma^5 C = (\gamma^5)^T$. Moreover, it fulfills the relation

$$C^T = -C. \tag{1.32}$$

Additionally, in a basis where γ^0 is hermitian and $\gamma^{i=1,2,3}$ are anti-hermitian, we have

$$C^\dagger = C^{-1}. \tag{1.33}$$

M_R and M_L are complex symmetric $n_R \times n_R$ and 3×3 matrices, respectively. The symmetry results from the anti-symmetry of C and the anti-commutativity of the fermion fields.[8]

[7] Note that these definitions slightly differ from the ones in Chapter 5 which we signal by the tilde notation. The difference is that in Chapter 5, Dirac and Majorana degrees of freedom are combined into the $(n_L + n_R)$-dimensional vector of mass eigenfields χ. Here, we keep left- and right-chiral degrees of freedom separate for a more direct translation from $\tilde{U}_L$ to the full lepton mixing matrix.

[8] The positive sign of the r.h.s. of Eq. (1.29a) appears because $\overline{(\nu_R)^c} = (C(\gamma^0)^T \nu_R^*)^\dagger \gamma^0 = -\nu_R^T C^{-1}$.

With ν_R being a singlet under $\mathcal{G}_{\text{SM}}$, Eq. (1.29a) does not break local gauge-invariance. Therefore, the standard mechanism of SSB is not necessary in order to generate these masses. This is in contrast to the term in Eq. (1.29b). The fields ν_L form part of the $SU(2)_L$-doublets, and therefore mass terms of this kind will violate gauge-invariance. Similar to the generation of the SM fermion masses, this can be cured by using the mechanism of SSB, but only by adding either a scalar gauge singlet or triplet field to the SM particle content with appropriate Yukawa interactions between these scalars and the $SU(2)_L$-doublets, as *e.g.* in [13, 14].[9] Both kinds of Majorana mass terms share the appealing feature that they are decoupled from the Higgs-VEV v, and are therefore not necessarily expected to have their origin at this energy scale. The consequences thereof are further explored in Section 2.1.

Lastly, it is worth mentioning that both terms in Eq. (1.29) violate lepton number conservation, because they break an *accidental* global $U(1)$-symmetry with the total lepton number L as conserved charge, which would be present without them. As originally pointed out in [16], this fact could be of relevance in explaining the matter-antimatter asymmetry of our universe via the mechanism of *leptogenesis*.

1.1.2 Weak interactions and lepton mixing

The mixing matrices we have introduced in Section 1.1 do not represent physical observables. Instead, we will see in this section that only a specific combination of $\tilde{U}_L$ and W_L is experimentally accessible. We will again only discuss the lepton sector, but the same arguments hold for the quark sector.

The relevant part of $\mathcal{L}_{\text{SM}}$ for this discussion is the kinetic term of the lepton doublets D_L.[10] By inserting the Pauli-matrices τ^a as well as using Eq. (1.17), this term reads

$$\mathcal{L}_{D,\text{kin}} = \overline{D_L}\, i\left(\partial\!\!\!/ + i\frac{g}{2}\tau^a W\!\!\!\!/^a + i\frac{g'}{2}B\!\!\!\!/\right) D_L \tag{1.34a}$$

$$= \begin{pmatrix}\overline{\nu_L} & \overline{e_L}\end{pmatrix}\left[i\partial\!\!\!/\mathbb{1} - \frac{g}{\sqrt{2}}\begin{pmatrix} 0 & W\!\!\!\!/^+ \\ W\!\!\!\!/^- & 0\end{pmatrix} + \frac{g}{2c_w}Z\!\!\!/\begin{pmatrix} s_w^2 - c_w^2 & 0 \\ 0 & 1\end{pmatrix} + eA\!\!\!/\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}. \tag{1.34b}$$

We put the focus on the charged current interaction represented by the second term of Eq. (1.34b)—the only instance where upper and lower components of the lepton doublets are joined. By inserting the mass-basis transformation of Eq. (1.27), we arrive at the *charged current Lagrangian* in terms of mass eigenfields

$$\mathcal{L}_{cc} = -\frac{g}{\sqrt{2}}\left[W_\mu^+ \overline{\tilde{\chi}_L}\left(\tilde{U}_L^\dagger W_L\right)\gamma^\mu l_L + W_\mu^- \overline{l_L}\gamma^\mu\left(W_L^\dagger \tilde{U}_L\right)\tilde{\chi}_L\right]. \tag{1.35}$$

The charged current interactions represent the only instances where mixing matrices can neither be absorbed by virtue of a redefinition of other SM parameters nor do they cancel, because W_L and $\tilde{U}_L$ are in general not related. Therefore, the mixing generated by the combination

$$U_{\text{PMNS}} \equiv W_L^\dagger \tilde{U}_L \tag{1.36}$$

is physically observable in processes involving charged current interactions. The matrix in Eq. (1.36) is usually called *Pontecorvo-Maki-Nakagawa-Sakata mixing matrix*. A popular

[9] Note that neutrino masses can also be induced via the effective dimension five *Weinberg operator* [15]. However, we chose to put the focus of this thesis on *renormalizable* theories of neutrino masses and mixings.

[10] Note that D_L as well as the other lepton fields are vectors in flavor space. We suppress the flavor indices for readability at this point and choose the kinetic terms flavor-diagonal.

parametrization is [12]

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\,\mathrm{e}^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$\times\,\mathrm{diag}(1, \mathrm{e}^{-i\alpha_1}, \mathrm{e}^{-i\alpha_2}), \tag{1.37}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. This parametrization is the combination of three rotations about the Euler angles θ_{12}, θ_{13} and θ_{23}. Moreover, the CP-violating phases δ and the Majorana phases α_1 and α_2 are included. The latter two phases can not be absorbed by redefinitions of the fields should a Majorana mass term of the kind Eq. (1.29b) exist, because it would not be invariant under transformations of the kind

$$\nu_L \to \mathrm{diag}\left(1, \mathrm{e}^{-i\alpha_1}, \mathrm{e}^{-i\alpha_2}\right)\nu_L \tag{1.38}$$

Therefore, we see that the number of relevant parameters in lepton mixing is four if neutrinos are of Dirac-only type and six if they are of Majorana nature. The quark mixing matrix V_{CKM}, called *Cabibbo-Kobayashi-Maskawa* matrix, is fundamentally of the same nature as U_{PMNS} with the only differences lying in the pure Dirac nature of quarks and therefore non-existent Majorana phases. Other than that, both only differ by the fact that they turn out to show rather dissimilar structures which is shown and further discussed in Section 1.2.

Two equivalently valid viewpoints of the nature of mixing in charged-current interactions can be defined. One can either think of absorbing the mixing in the coupling of the interaction, consequently generating transitions between mass-eigenfields of different families. The other option is to redefine the neutrino fields as

$$\tilde{\nu}_L = U_{\text{PMNS}}\,\tilde{\chi}_L, \tag{1.39}$$

and view the interaction as if it was taking place between the mass eigenfields of the charged leptons and the interaction eigenfields $\tilde{\nu}_L$, a linear combination of the neutrino mass eigenfields. Note that these are not to be confused with the states ν_L, as originally introduced in Eq. (1.23). The first viewpoint represents an equal treatment of upper and lower components of an $SU(2)_L$-doublet and is commonly used in the quark sector. The latter one has evolved as the more common point of view for the leptonic sector. This is owing to the fact that in the lepton sector, flavor-changing interactions are typically viewed as a result of *neutrino oscillations* (see Section 1.1.4). Due to these different treatments, we would like to stress the point that both, lepton and quark mixing, follow the same fundamental principles.

The interpretation of flavor-changing charged interactions of the quarks as *quark oscillations* is not the most helpful, for which at least two reasons can be named. The first is that quarks do not propagate freely over long distances because they take part in strong interactions and are therefore subject to *confinement*. The second is that the quark mass spectrum shows a much stronger hierarchy, meaning larger mass squared differences between the quarks of different flavors than in the lepton sector. This leads to very short oscillation lengths which will hardly be accessible in experiments. Nevertheless, the phenomenon of *meson oscillations*, *e.g.* loop-induced oscillations between K^0 and $\overline{K^0}$, play an important role in the determination some of the V_{CKM}-elements [12].

1.1.3 Soft symmetry breaking

For later reference, we will give a concise introduction of the notion *soft symmetry breaking*. It is a well-known method widely used *e.g.* in the Minimal Supersymmetric Model or the Two-Higgs-Doublet Model and moreover in some of the SM extensions that we will describe in Chapter 2. Heuristically speaking, this term reflects the idea that symmetries of a Lagrangian can be broken explicitly but only show the breaking at comparably low energies. In other words, a softly broken symmetry might be restored in processes at very high energies. This is achieved by introducing symmetry breaking operators solely of energy dimension $d_{\text{soft}} < 4$.

A major benefit of the soft symmetry breaking approach is that a theory remains renormalizable *without* the inclusion of symmetry breaking operators with energy dimension $d > d_{\text{soft}}$. If, for instance, an explicitly symmetry breaking operator of dimension $d = 2$ is introduced, it suffices for the renormalizability of the theory to include solely all other symmetry breaking operators of dimension $d \leq 2$. A heuristic argument for this fact can be given along the lines of Ref. [17]. For a coupling with positive mass dimension, as introduced via soft symmetry breaking, only a finite number of superficially divergent Feynman diagrams is introduced. All these can be made finite by solely introducing a counterterm to the coupling responsible for said divergencies. In a theory containing only couplings of positive mass dimensions, this behavior is usually described in terms of super-renormalizability [18].

In fact, we will see this very behavior explicitly in Chapter 4 and 5. The counterterms of $d = 0$, the scalar quartic and the Yukawa couplings, do not contain any divergencies proportional to the couplings of $d > 0$ in both models. Therefore, they need not be altered when introducing a symmetry breaking via the couplings of dimension $d > 0$. In other words, via the soft breaking no divergencies come about that need to be absorbed in the counterterms of the dimensionless couplings. The opposite is true if a symmetry is broken via the dimensionless couplings, *e.g.* the scalar quartic couplings. These enter the result for the counterterms of the scalar bilinear couplings (see *e.g.* Eq. (77) of Chapter 5) which are therefore also altered via the symmetry breaking.

An instructive example is given by the general Two-Higgs-Doublet Model discussion of Ref. [19]. Here, the general version of the scalar potential contains the terms

$$- m_{12}^2 \, \phi_1^\dagger \phi_2 + \lambda_6 \left(\phi_1^\dagger \phi_1\right) \left(\phi_1^\dagger \phi_2\right) + \text{H.c.}, \tag{1.40}$$

In order to avoid flavor-changing neutral interactions at tree-level, the discrete symmetry $\phi_1 \to -\phi_1$ can be introduced which would provoke $m_{12}^2 = 0$ as well as $\lambda_6 = 0$. If we now choose $m_{12}^2 \neq 0$, the mechanism of soft symmetry breaking tells us that despite the explicit breaking, neither λ_6 nor the corresponding counterterm need to be reintroduced for a consistent and renormalizable theory. The reason for introducing the breaking in this case is the wish to generate *finite* flavor-changing neutral interactions [19].

1.1.4 Neutrino oscillations

In this section, we want to briefly introduce the standard theoretical approach to the phenomenon of neutrino oscillations along the lines of Ref. [20]. For our purposes, it will be sufficient to consider a simple plane wave approach for freely propagating neutrinos in vacuum. In other treatments, more realistic wave-packet descriptions for the propagation are used as in Ref. [21] and references therein. In principle, even a full quantum field theoretical description where neutrino production, propagation and detection are discussed in detail in terms of charged-current interactions of Section 1.1.2 can be carried out, but it eventually leads to very similar conclusions as the simple quantum mechanical approach [22]. Lastly, matter effects play an important role, but go beyond the scope of this thesis. A well-known example is the neutrino conversion in the sun, called *Mikheev-Smirnov-Wolfenstein-* or *MSW-effect* [23, 24].

For brevity, we drop the subscript of U_{PMNS} in this section and denote it simply by U.

In the simplest case of a 3×3 unitary mixing matrix U, the neutrino interaction eigenstates are given by

$$|\tilde{\nu}_\alpha\rangle = \sum_{j=1}^{3} U^*_{\alpha j}|\tilde{\chi}_j\rangle, \tag{1.41}$$

where $|\tilde{\chi}_j\rangle$ are the mass eigenstates fulfilling the orthogonality relation

$$\langle\tilde{\chi}_j|\tilde{\chi}_k\rangle = \delta_{jk}. \tag{1.42}$$

Note that the complex conjugate of U appears in Eq. (1.41). This is because a Dirac field $\psi(x)$ has the Fourier decomposition

$$\psi(x) = \sum_s \int \frac{\mathrm{d}^3p}{\sqrt{2E_{\boldsymbol{p}}(2\pi)^3}} \left[u(\boldsymbol{p},s)\,\mathrm{e}^{-ix\cdot p}\,b(\boldsymbol{p},s) + v(\boldsymbol{p},s)\,\mathrm{e}^{ix\cdot p}d^\dagger(\boldsymbol{p},s)\right], \tag{1.43}$$

where $b(\boldsymbol{p},s)$ is the annihilation operator for a particle state of momentum $\boldsymbol{p}$ and spin s and $d^\dagger(\boldsymbol{p},s)$ the creation operator for the corresponding anti-particle state. Therefore, in order to create a neutrino state $|\tilde{\chi}\rangle$, we let $\tilde{\chi}^\dagger(x)$ act on the vacuum $|0\rangle$ and use Eq. (1.39) to get Eq. (1.41).

We define $|\tilde{\nu}_\alpha, x=0\rangle = |\tilde{\nu}_\alpha\rangle$ and set $\boldsymbol{p}_j = p_j\hat{\boldsymbol{e}}_x$. Letting time evolution act on the mass eigenstates, we get

$$|\tilde{\nu}_\alpha, x, t\rangle = \sum_{j=1}^{3} U^*_{\alpha j}\mathrm{e}^{-i(Et-p_jx)}|\tilde{\chi}_j\rangle. \tag{1.44}$$

Experimentally, we know that detectable neutrinos are relativistic, therefore $E \gg m_j$ holds and we can expand the neutrino momentum as

$$p_j = \sqrt{E^2 - m_j^2} \simeq E - \frac{m_j^2}{2E}. \tag{1.45}$$

Eventually, using Eq. (1.44) and Eq. (1.45), we arrive at the standard formula for the probability of a conversion from $|\tilde{\nu}_\alpha, x=0\rangle$ to a state $\tilde{\nu}_\beta$ after the travel distance $x = L$, *i.e.*

$$P_{\tilde{\nu}_\alpha\to\tilde{\nu}_\beta}(L/E) = \left|\sum_{j=1}^{3} U_{\beta j}U^*_{\alpha j}\mathrm{e}^{-im_j^2\frac{L}{2E}}\right|^2. \tag{1.46}$$

The following properties hold for Eq. (1.46):

1. From Eq. (1.41) it is apparent that oscillation probabilities for anti-neutrinos are achieved by the replacement $U^* \to U$.

2. $P_{\tilde{\nu}_\alpha\to\tilde{\nu}_\beta}$ is invariant under $U_{\alpha j} \to \mathrm{e}^{i\phi_\alpha}U_{\alpha j}\mathrm{e}^{i\phi_j}$ for arbitrary phases ϕ_α. Therefore, Majorana phases are not observable in oscillation experiments.

3. Oscillation probabilities for neutrinos and anti-neutrinos coincide if the CP-violating phase $\delta = 0, \pi$.

4. $P_{\tilde{\nu}_\alpha\to\tilde{\nu}_\beta}$ does not depend on the absolute mass values, but only the mass-squared differences.

The last item of this list becomes more apparent by another way of writing Eq. (1.46), *i.e.* [12]

$$P_{\tilde{\nu}_\alpha\to\tilde{\nu}_\beta} = \sum_{j=1}^{3} |U_{\alpha j}|^2\,|U_{\beta j}|^2 + 2\sum_{j>k=1}^{3} \left|U_{\alpha j}U^*_{\beta j}U_{\beta k}U^*_{\alpha k}\right|\cos\left(\Delta m^2_{jk}\frac{L}{2E} - \phi_{\alpha\beta;jk}\right), \tag{1.47}$$

where $\phi_{\alpha\beta;jk} \equiv \arg U_{\alpha j} U^*_{\beta j} U_{\beta k} U^*_{\alpha k}$.[11] Looking at the argument of the cosine in Eq. (1.47), it provides further insight to define oscillation lengths $L^{\rm osc}_{jk}$ via

$$2\pi \frac{L}{L^{\rm osc}_{jk}} \equiv \Delta m^2_{jk} \frac{L}{2E}. \tag{1.48}$$

Typical length scales for neutrino oscillations can subsequently be read off from

$$L^{\rm osc}_{jk} = 2.48\,{\rm m} \times \left(\frac{E}{1\,{\rm MeV}}\right) \times \left(\frac{1\,{\rm eV}^2}{\Delta m^2_{jk}}\right). \tag{1.49}$$

From Eq. (1.49), we can understand part of the neutrino terminology: the oscillation length for neutrinos of $E \simeq 13\,{\rm GeV}$ and a mass-squared difference of $\Delta m^2_{3l} = 2.524 \times 10^{-3}\,{\rm eV}^2$ (see Section 1.2) approximately coincides with the diameter of the earth. Therefore, muon-neutrinos produced in atmospheric showers in the GeV-range show a strong oscillatory behavior when traversing the earth with the amplitude of the oscillation being determined by the entries of $U_{\rm PMNS}$ according to Eq. (1.47). This so called *ν_μ-disappearance* has been demonstrated in impressive experimental efforts by the Super-Kamiokande experiment [25] and Δm^2_{3l} and θ_{23} are consequently called *atmospheric* mass-squared difference and mixing angle, respectively. The phenomenon of neutrino oscillations in general was eventually confirmed a few years later by the Sudbury Neutrino Observatory (SNO) [26] via the measurement of the total neutrino flux from ^8B decays in the sun. By using two different detection channels, one only sensitive to electron-neutrinos, the other to all neutrino flavors, a significant deficit in the arrival of electron-neutrinos was detected and could be explained by a non-electron flavor active neutrino component in the solar flux [26]. For these findings, the two experiments were awarded the Nobel prize in physics of 2015. As a final remark, we note that the experimental evidence for neutrino oscillations also implies the non-conservation of lepton family numbers.

1.2 Experimental status of neutrino physics

The field of experimental neutrino physics has been a thriving one throughout the past decades and is growing at an even faster pace since the detection of neutrino oscillations and the awarding of the Nobel prize in physics of 2015. There are too many experiments to name and do justice in the briefness of this thesis. Therefore, we focus on a few instructive and well-known examples in the following and guide the interested reader towards a constantly updated web resource (www.nu.to.infn.it/exp/), an exhaustive list of neutrino-related experiments currently curated by Stefano Gariazzo, Carlo Giunti and Marco Laveder.

1.2.1 Neutrino oscillation parameters

Here we list some of the main experimental sources for the determination of the neutrino oscillation parameters:

- As already mentioned in Section 1.1.4, the *atmospheric* mass-squared difference and mixing angle Δm^2_{3l} and θ_{23} have originally been determined in the Super-Kamiokande experiment [25] from the disappearance of atmospheric muon-neutrinos and via the measurement of electron- and non-electron flavor components of the solar neutrino flux in the Sudbury Neutrino Observatory (SNO) [26].

[11] In [12], the neutrino momentum p is used instead of the energy E in Eq.(1.47). Nevertheless, we have seen that $p \simeq E$ up to $\mathcal{O}(m_j^2/2E)$.

- The most precisely determined mixing angle, the *reactor mixing angle* θ_{13}, is derived from experiments like Double Chooz [27], Daya Bay [28] or RENO [29] which study the disappearance of electron anti-neutrinos from nuclear reactors on length-scales of a few kilometers. The precision is achieved by using two detectors ("near" and "far") to eliminate uncertainties in the total neutrino flux. The measurement of a non-zero θ_{13} came as a surprise to the neutrino theory community, challenging a variety of mixing models "predicting" $\theta_{13} = 0$.

- Δm_{21}^2 and θ_{12}, the *solar* mass-squared difference and mixing angle derive their name from their original determination via the disappearance of solar electron neutrinos in the Homestake [30], GALLEX [31] and SAGE [32] experiments. Note that the cause of the disappearance is not an oscillation over the sun-earth distance, but instead the matter-related effects briefly mentioned in the beginning of Section 1.1.4.

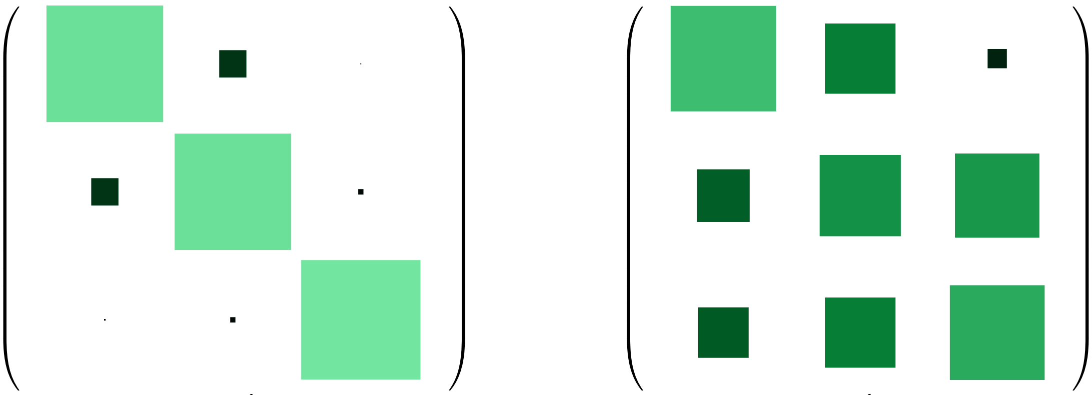
- The CP-violating phase δ is the least well determined oscillation parameter. At the 1σ-level, values around $\delta \simeq 1.45\pi \stackrel{\wedge}{=} 261°$ are favored, while the range around $\pi/2$ or $90°$ is disfavored. Nevertheless, at the 3σ-level, δ can still take any value in $[0, 2\pi)$. The main data source for the determination of δ currently is the T2K-experiment, which investigates neutrino beams sent from the J-PARC accelerator in Tokai, Japan to the Super-Kamiokande detector around 295 km away.

Generally speaking, the most precise determinations of neutrino oscillation parameters are achieved by global fits to a combination of measurements from a wide variety of experiments, *e.g.* as carried out by the NuFIT group in [33]. In table 1.1, we present a list of the current best fit values for all neutrino oscillation parameters.

parameter	**best fit value**	**3σ-range**
$\sin^2\theta_{12}$	$0.306^{+0.012}_{-0.012}$	$[0.271, 0.345]$
$\sin^2\theta_{23}$	$0.441^{+0.027}_{-0.021}$	$[0.385, 0.635]$
	$0.587^{+0.020}_{-0.024}$	$[0.393, 0.640]$
$\sin^2\theta_{13}$	$0.02166^{+0.00075}_{-0.00075}$	$[0.01934, 0.02392]$
	$0.02179^{+0.00076}_{-0.00076}$	$[0.01934, 0.02397]$
δ/π	$1.45^{+0.28}_{-0.33}$	$[0, 2]$
	$1.54^{+0.22}_{-0.26}$	$[0, 2]$
$\Delta m_{21}^2/10^{-5}\,\mathrm{eV}^2$	$7.5^{+0.19}_{-0.17}$	$[7.03, 8.09]$
$\Delta m_{3l}^2/10^{-3}\,\mathrm{eV}^2$	$+2.524^{+0.039}_{-0.040}$	$[2.407, 2.643]$
	$-2.514^{+0.038}_{-0.041}$	$[-2.635, -2.399]$

Table 1.1: Best fit values and 3σ-ranges for the lepton mixing parameters from a global fit to neutrino oscillation data available as of fall 2016 [33]. Two different values are shown if the parameter depends on the ordering in the neutrino mass spectrum (upper line: normal ordering, lower line: inverted ordering; see Sec. 1.2.3 for clarification of these terms).

1.2.2 Lepton mixing vs. quark mixing



(a) $V_{\mathrm{CKM}}^{\mathrm{abs}}$ (b) $U_{\mathrm{PMNS}}^{\mathrm{abs}}$

Figure 1.2: Graphical representation of quark and lepton mixing. The area of each square is equal to the absolute values squared of the corresponding mixing matrix entries (best fit values of table 1.1 for normal ordering). Moreover, the lightness-levels are determined by the size of the absolute values squared. The arrangement of the squares in this figure is in direct correspondence to the labeling in Eq. (1.50). Inspired by the representation in Ref. [34].

In this section we present a comparison of experimental data for quark and lepton mixing. Using the absolute values for the CKM-matrix entries of [12] and the best fit values for the neutrino oscillations parameters of Ref. [33] for the calculation of the absolute values of the PMNS-matrix entries, we find

$$V_{\mathrm{CKM}}^{\mathrm{abs}} = \begin{array}{c} \\ u \\ c \\ s \end{array} \!\!\begin{array}{c} \begin{array}{ccc} d & s & b \end{array} \\ \begin{pmatrix} 0.9743 & 0.2254 & 0.0036 \\ 0.2252 & 0.9734 & 0.0414 \\ 0.0089 & 0.0405 & 0.9991 \end{pmatrix} \end{array}, \quad U_{\mathrm{PMNS}}^{\mathrm{abs}} = \begin{array}{c} \\ \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \tilde{\chi}_1 & \tilde{\chi}_2 & \tilde{\chi}_3 \end{array} \\ \begin{pmatrix} 0.8240 & 0.5471 & 0.1472 \\ 0.4088 & 0.6336 & 0.6568 \\ 0.3923 & 0.5470 & 0.7395 \end{pmatrix} \end{array}. \tag{1.50}$$

The labeling is to be understood as follows: as explained in Section 1.1.2, the mixing is usually absorbed in a redefinition of the fields in the case of the leptons. Therefore, U_{PMNS} is represented as the transformation from mass eigenfields $\tilde{\chi}_i$ to interaction eigenfields $\tilde{\nu}_\alpha$. The V_{CKM}-labels hint at the interpretation of weak interactions inducing transitions between different families of up- and down-type quark mass eigenfields. Note that the roles of up- and down-type entries of the respective $SU(2)_L$-doublets is transposed in this way of denoting the quark and lepton mixing matrices. A graphical representation of the two matrices is shown in Fig. 1.2.

Obviously, the structures of quark and lepton mixing matrices differ quite strongly. The first is oftentimes considered as *almost diagonal* while the latter is rather *close to maximal* mixing. These very different mixing structures formed the motivation for a variety of approaches in model building, especially in the leptonic sector.

1.2.3 Neutrino masses

The absolute scale of the neutrino masses is yet to be determined experimentally. There is a variety of direct and indirect methods for its measurement, but so far only upper bounds can be given. Still, we are rather certain that it lies at least six orders of magnitude below the mass of the next lightest fermion, the electron. Neutrino masses below the eV-scale correspond to Yukawa-couplings of the order $\sim 10^{-12}$ if the Standard Model mechanism of SSB with the sole addition of right-chiral Dirac neutrinos is responsible for their emergence. In the end, it is a matter of taste whether this is to be considered an unnaturally small value for couplings, yet it serves as a strong new physics hint for many.

As a side remark, we note that argumentation based on *naturalness* has had both, positive and negative examples in the scientific history [35]. Still, considering the lack of other guiding principles in terms of model building, we consider it as a good form of motivation for following one pursuit rather than another of the many possible choices. Moreover, we emphasize the role of *technical naturalness* in model building which is not to be confused with the former. Technical naturalness, as originally defined by 't Hooft in Ref. [36], states that small numbers, such as small couplings in a field theory, are not to be considered unnatural if the symmetry of the theory is enhanced by letting these numbers go to zero. An example is letting the electron mass vanish, resulting in the restoration of a chiral symmetry, *i.e.* the separate conservation of left- and right-chiral electron-like currents. This symmetry in turn provokes that radiative corrections to the electron mass are always proportional to the mass itself [36]. In fact, this is true for all SM particles except the Higgs. Therefore, small values for the Yukawa-couplings of the neutral leptons are not necessarily problematic, yet this is neither a definite argument against potential interesting dynamics behind these numbers.

Independent of the discussions around naturalness and the like, the light neutrino masses together with other surprises in neutrino phenomenology have spawned an abundance of neutrino mass models and ideas to explain the smallness of their masses, among them the well-known *seesaw mechanism*. The theory-landscape relevant to this thesis is shown and further explored in Chapter 2.

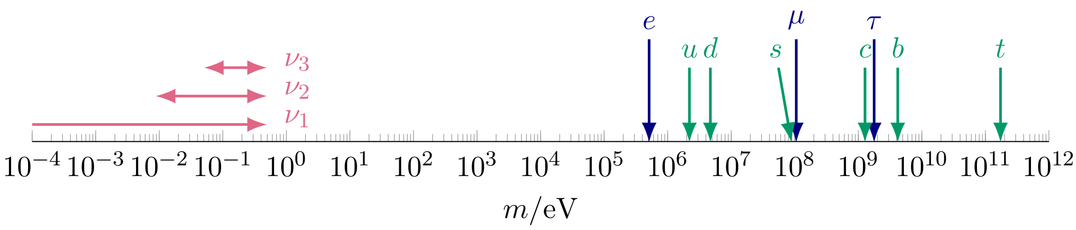


Figure 1.3: The masses of the SM fermions for the case of normal neutrino mass ordering. Adapted from a figure in [37]. We show a fairly conservative upper bound for the neutrino masses of 0.5 eV. Moreover, the one-sided arrow for ν_1 indicates the still valid possibility of the lightest neutrino being massless.

Absolute mass scale

The only feasible way of a direct neutrino mass determination up to now is the investigation of Tritium beta decays—via measuring the endpoint of the electron-energy spectrum. Tritium has a relatively long half-life of ~ 12.3 yr, guaranteeing a stable measuring environment, and low energy release of $Q_\beta \simeq 18.6$ keV, making possible the use of an electrostatic spectrometer [38]. The energy released in a Tritium decay is split between the emitted electron and anti-electron neutrino. If the anti-neutrino were massless, the maximal energy E^e_{max} of the electron would be equal to Q_β. Though, knowing that neutrinos are massive, the spectrums endpoint E^e_{max} must be shifted to a lower value. The difference $Q_\beta - E^e_{\text{max}}$

then gives an upper bound for the neutrino mass scale [12]

$$m_{\nu_e} \equiv \left(\sum_{i=1}^{3} |U_{ei}|^2 \, m_i^2 \right)^{\frac{1}{2}} . \tag{1.51}$$

The current strongest bound of this determination type was achieved by the Troitsk experiment [38], *viz.*

$$m_{\nu_e}^{(\mathrm{Tri})} < 2.05\,\mathrm{eV}, \tag{1.52}$$

at the 95% confidence level. Currently, KATRIN [39] is the only experiment continuing the direct measurement efforts. It is of similar type as the Troitsk experiment, but trying to achieve around an order of magnitude improvement compared to their results. After an extensive construction phase, it started measurements in June 2018 and is hoping to reach design-sensitivity after three years of data taking. With KATRIN being the only *direct* neutrino mass measurement in the foreseeable future, their efforts are of high value even though indirect bounds on m_{ν_e} can reach beyond their projected sensitivity.

Coming to the indirect determinations of the neutrino mass scale, the most stringent bounds arise from studies of cosmological probes, most prominently from the *cosmic microwave background* (CMB) radiation. The strongest current bound on the sum of neutrino masses of Dirac nature (which for our purposes is to be considered as an equivalent general measure as m_{ν_e}) is reported in Ref. [40] with

$$\sum_i m_i < 0.118\,\mathrm{eV}, \tag{1.53}$$

at 95% confidence level. The effects investigated are the background evolution and spectra of matter perturbations in the CMB. These are altered by the presence of neutrinos, firstly because in the very early universe, neutrinos behaved relativisticly due to their light mass and therefore contributed to the radiation energy density. Secondly, due to the expansion of the universe, neutrinos turned non-relativistic at later cosmological times and consequently contributed to the total matter density. Both densities have an imprint on measures such as the time of matter-radiation equality or the CMB lensing. The specific effects of massive neutrinos on the CMB are discussed in great detail in [40].

Note that values as in Eq. (1.53) are to be taken with a grain of salt though, because the systematical errors in indirect determinations are still a matter of discussion [41]. The values for the sum of neutrino masses differ rather strongly between different collaborations, due to their systematically different approaches. This might also be the reason why the Particle Data Group is not presenting a world average for this sum, in contrast to the direct bounds on m_{ν_e}.

Mass ordering

Next to the ignorance about m_{ν_e}, current neutrino experiments have not been able to determine the sign of the atmospheric mass-squared difference Δm_{3l}^2. Therefore, two different *orderings* of the neutrino mass spectrum are currently allowed by experiment:[12]

1. *Normal ordering* (NO) signifies the case $m_1 < m_2 < m_3$, $\Delta m_{3l}^2 \equiv \Delta m_{31}^2 > 0$.
2. *Inverted ordering* (IO) signifies the case $m_3 < m_1 < m_2$, $\Delta m_{3l}^2 \equiv \Delta m_{32}^2 < 0$.

[12] Note that a negative sign of the smaller or solar mass squared difference Δm_{21}^2 can be absorbed via relabeling the fields and would not change the mass ordering in an observable way. Hence, it is set to a positive value.

Note that Δm^2_{3l} simply stands for the largest possible mass splitting and therefore carries different indices depending on the ordering (see Fig. 1.4 for clarification). In the literature, the term *hierarchy* is sometimes used synonymously to ordering, but we reserve the former to describe only the relative sizes of the differences between masses or masses-squared. In Section 1.2.4, we will see how the determination of the possible Majorana nature of neutrinos can also help determining the ordering of the masses.

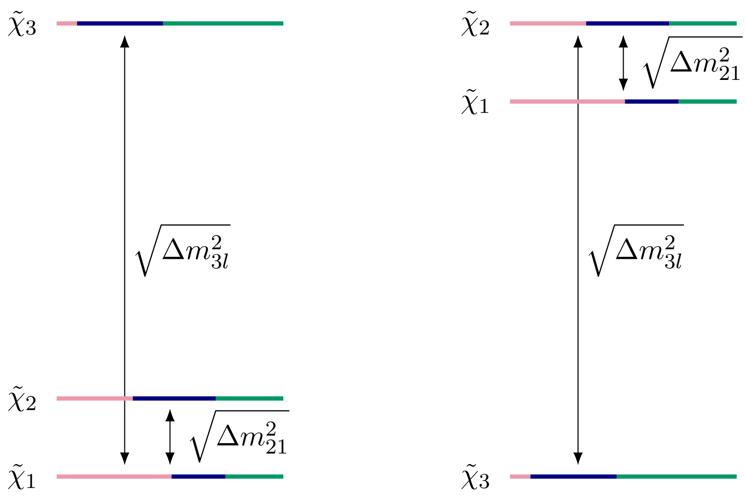


Figure 1.4: Possible neutrino mass orderings. Left: normal ordering, right: inverted ordering. The colors represent the admixture of interaction eigenstates contributing to the corresponding mass eigenstate (light red: $\tilde{\nu}_e$, dark blue: $\tilde{\nu}_\mu$, green: $\tilde{\nu}_\tau$).

1.2.4 Possible Majorana nature of neutrinos

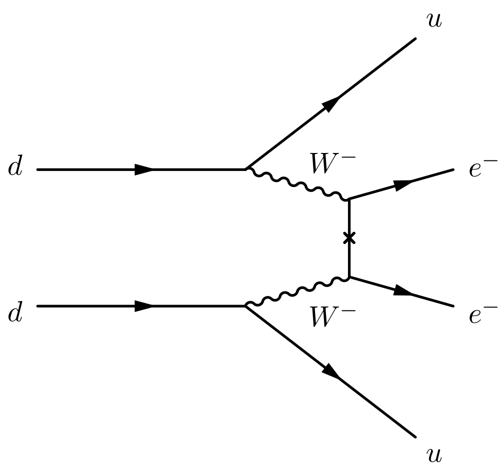


Figure 1.5: Feynman diagram for a $(\beta\beta)_{0\nu}$-decay. The crossed line represents a contraction of Majorana neutrino fields of the kind $\overbrace{\chi_i(x_1)\,\chi_i^T}(x_2) = -S_i(x_1 - x_2)C$, where $S_i(x_1 - x_2)$ is the usual fermionic Feynman propagator [42].

As of writing this thesis, it remains an open question whether neutrinos are of Dirac or Majorana nature. Although the theoretical treatment of both types in terms of mass-generation and the implications thereof are fairly different, it is difficult to determine experimental signatures for their distinction. Exemplarily, one can mention the independence of neutrino oscillations of the Majorana phases α_1 and α_2, as discussed in Section 1.1.2. Currently, the only types of experiments in reach of a possible discrimination are the ones searching for *neutrinoless double beta* or $(\beta\beta)_{0\nu}$-decays. The Feynman diagram for processes of this kind is shown in Fig. 1.5. Two examples for such experiments are GERDA [43] at the Gran Sasso underground laboratory in Italy, a detector containing a large mass of ^{76}Ge and the KamLAND-Zen detector [44] in Kamioka, Japan, holding several tons of ^{136}Xe-enriched liquid Xenon.

Double beta decay *with* the emission of two neutrinos is a well studied phenomenon which can be explained via the empirical Bethe-Weizsäcker formula [45]. It describes the binding energy of a nucleus in terms of its total number of protons Z and number of neutrons N. The formula tells us that nuclei with even Z and N are usually more stable than the ones with Z and N uneven. Typically, the single beta decay of an even-even nucleus $A(Z,N)$ to an uneven-uneven $B(Z',N')$, *i.e.*

$$A(2n,2n) \to B(2n+1,2n-1) + e^- + \overline{\nu}_e, \quad n \in \mathbb{N}, \tag{1.54}$$

is energetically forbidden. Still, for some isotopes like the aforementioned ^{76}Ge and ^{136}Xe, transitions to isobar states $C(2n+2,2n-2)$ via a double beta decay gain binding energy $Q_{\beta\beta}$ and are therefore allowed. Such decays have been observed in various experiments. The associated half-lifes lie in the range between $\sim 10^{18}$ yr and $\sim 10^{24}$ yr [46]. In the usual double beta decay, the released energy is shared between the two electrons and neutrinos. Therefore, the sum of the electron energies shows a continuous spectrum with $Q_{\beta\beta}$ as the endpoint. The signature of a $(\beta\beta)_{0\nu}$-decay would be visible as a sharp peak around $Q_{\beta\beta}$ in the electron energy spectrum.

Up to this point, no such peak has been observed in the experiments. The GERDA collaboration reports a limit of 8.0×10^{25} yr for the half-life of the $(\beta\beta)_{0\nu}$-decay of ^{76}Ge at the 90% confidence level [43]. The lower bound for ^{136}Xe lies at 1.07×10^{26} yr [44].

The non-observation of $(\beta\beta)_{0\nu}$-decays in these experiments can give information on the neutrino mass spectrum. This is due to the fact, that the amplitude corresponding to the Feynman diagram in Fig. 1.5 is proportional to the effective mass [12]

$$m_{\beta\beta} \equiv \left| \sum_{i=1}^{3} m_i U_{ei}^2 \right| \tag{1.55a}$$

$$= \left| \left(m_1 c_{12}^2 + m_2 s_{12}^2 e^{i\alpha_1} \right) c_{13}^2 + m_3 s_{13}^2 e^{i(\alpha_2 - 2\delta)} \right|. \tag{1.55b}$$

A detailed derivation of the appearance of this factor is given in [42]. The masses m_i can be expressed in terms of the mass-squared differences and the lightest neutrino mass m_{lightest}. Figure 1.6 shows the regions excluded by experiment. It nicely illustrates that a further investigation of $(\beta\beta)_{0\nu}$-decays can potentially lead to an exclusion of the inverted neutrino mass ordering in addition to delivering competitive bounds on the absolute neutrino mass scale. At 90% confidence level, the strongest upper bound achieved by the KamLAND-Zen collaboration is [44]

$$m_{\text{lightest}}^{(\text{KamLAND})} < (0.18 - 0.48)\,\text{eV}. \tag{1.56}$$

It is amusing to note that in principle, similar signatures as in Fig. 1.5 could be visible in vector boson processes at hadron colliders involving leptonic final states (see Fig. 1.7). An example is the same sign W-scattering as observed in the ATLAS and CMS experiments at CERN [47, 48]. Unfortunately though, reaching the sensitivity needed to measure Majorana-neutrino induced sub-contributions to this processes is most probably out of question in the foreseeable future. An alternative could be given by investigating Drell-Yan processes at hadron colliders, where the resonant production of heavy Majorana neutrinos is claimed to be in reach of the LHC [49]. Nevertheless, up to this point no discovery was made and only upper bounds on the potential heavy Majorana mass and mixing with lepton flavor eigenstates have been reported.

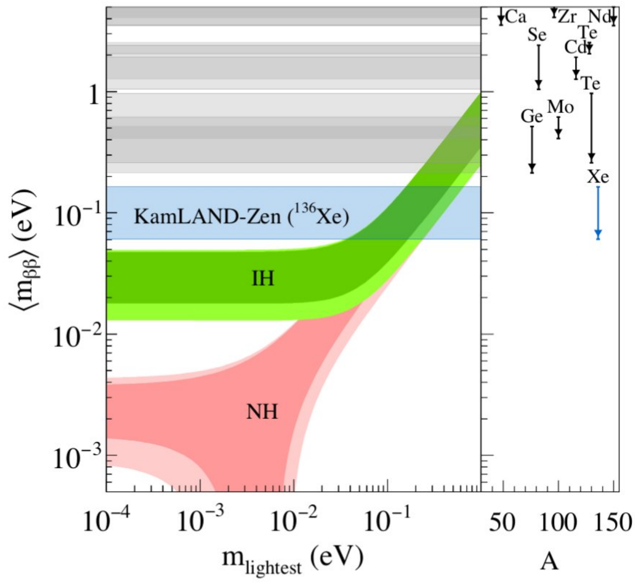


Figure 1.6: Plot of the effective Majorana mass $m_{\beta\beta}(m_{\text{lightest}})$ of Ref. [44]. The darker green (inverted mass ordering) and red (normal mass ordering) areas correspond to the allowed regions of $m_{\beta\beta}$ under a variation of the phase parameters in Eq. (1.55b) and using the best-fit values for the rest of the parameter set. The lighter regions correspond to the 3σ-ranges of the oscillation parameters. Experimentally excluded regions are shown in grey and light blue for the KamLAND-Zen experiment. The right panel shows exclusion limits in terms of the mass number A of the material investigated.

1.2.5 Flavor changing neutral interactions

As a final remark on experimental data relevant for the neutrino landscape, we want to shortly discuss the subject of *flavor-changing neutral interactions* (FCNIs). Most beyond the SM theories of the lepton sector contain additional scalar fields. An important example is the inclusion of multiple Higgs doublets, as we will discuss in Chapter 5, which go hand in hand with additional Yukawa couplings to the SM fermions. In general, these Yukawa couplings are non-diagonal in flavor space, therefore allowing transitions of leptons between different families, *i.e.* FCNIs via loop-induced flavor transitions in processes involving the neutral gauge bosons or decays via intermediate neutral scalars as shown in Fig. 1.8. This is in contrast to the SM, where the diagonalization of the mass matrices procures diagonal Yukawa coupling matrices and therefore no FCNIs as in Fig. 1.8 are expected. Moreover, other loop-induced FCNIs possible in the SM are suppressed via the GIM-mechanism [50].

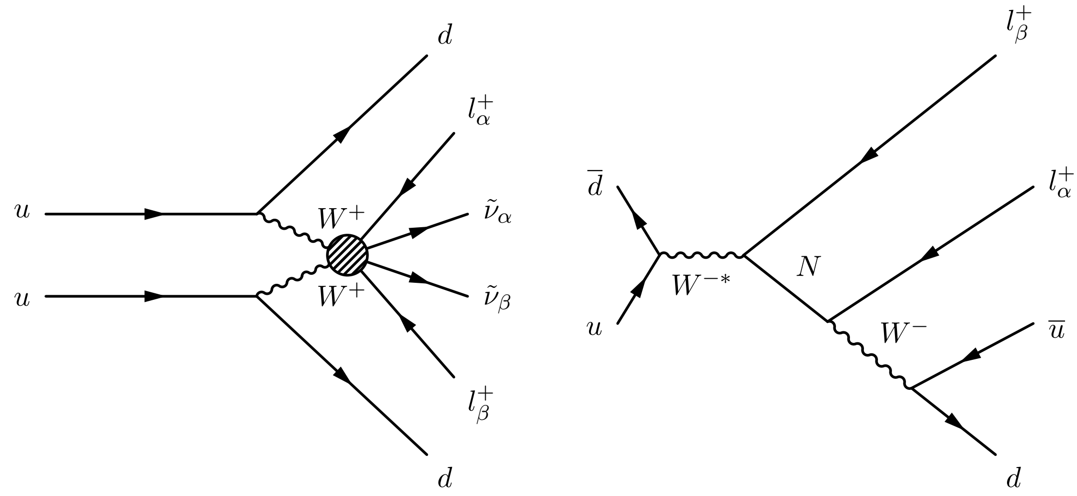
There are strong experimental constraints on processes involving FCNIs. An important example is the potential decay of muons into electrons and positrons. Two contributions to processes of this kind are shown in Fig. 1.8. The strongest current bounds at the 90%-confidence level are [12]

$$\text{BR}\left(\mu^- \to e^-\gamma\right) \quad < \quad 4.2\times 10^{-13}, \tag{1.57a}$$
$$\text{BR}\left(\mu^- \to e^-e^+e^-\right) \quad < \quad 1.0\times 10^{-12}, \tag{1.57b}$$

where BR is the branching ratio, *i.e.* , in terms of decay widths Γ,

$$\text{BR}(X \to Y) \equiv \frac{\Gamma(X \to Y)}{\Gamma(X \to \text{anything})}. \tag{1.58}$$

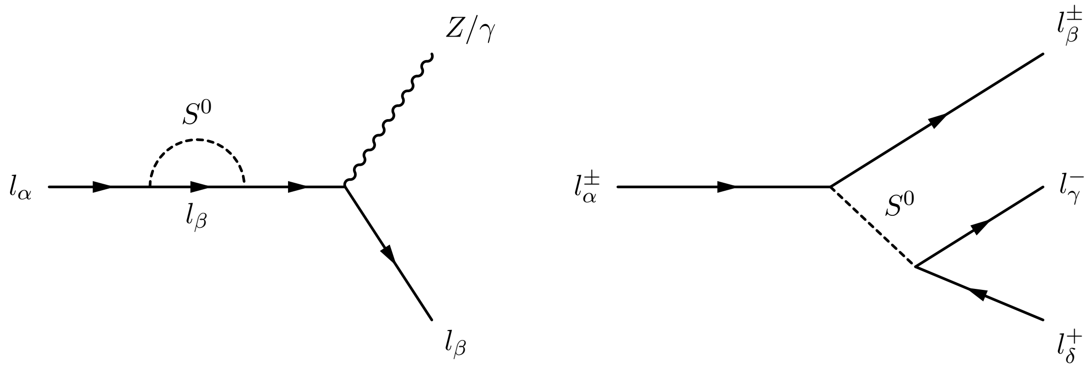
Bounds of this kind can present a major hurdle for model building while at the same time they provide the potential for discoveries of new physics, should the measurements



(a) Feynman diagram for same-sign W scattering with leptonic final state as observed by ATLAS and CMS [47, 48]. Connecting the neutrino lines would give a similar final state as in Fig. 1.5.

(b) Production of intermediate heavy Majorana neutrinos N in a Drell-Yan process.

Figure 1.7: Hadron collider processes as rather unlikely, though conceivable alternatives to the $(\beta\beta)_{0\nu}$-decays for a Majorana neutrino discovery.



(a) Loop-induced contribution.

(b) Tree-level contribution.

Figure 1.8: Exemplary contributions to flavor-changing neutral interactions as generally expected in models with a modified scalar sector, *e.g.* the one of Chapter 5.

violate SM expectations. It is a highly non-trivial task to create a SM extension of predictive power respecting all experimental constraints such as the impressively strong bounds in Eq. (1.57). This is even more so the case if one wants to avoid manually adjusted cancellations of potential FCNIs via the *fine-tuning* of parameters.

An example of a predictive model with an emphasis on respecting experimental bounds is given in the two-Higgs doublet model with heavy right-handed Majorana neutrinos of Ref. [51], originally investigated in Ref. [52]. The model makes use of soft symmetry breaking as explained in Section 1.1.3. All Yukawa couplings are flavor-diagonal, implying conservation of lepton family numbers, yet broken *softly* by Majorana mass terms. This means that operators with lepton family number violating Yukawa couplings (which are of dimension 4) need not be included for acquiring a renormalizable theory, even though the symmetry is explicitly broken by the Majorana mass term. For acquiring a renormalizable theory, it suffices to take into account solely the symmetry breaking operators of dimension ≤ 3 of which only the Majorana mass term can be constructed. Effectively, a strong suppression of FCNIs can be achieved for large Majorana masses, even in the presence of multiple Higgs doublets, while keeping the benefits of the seesaw mechanism and, for instance, means to explain discrepancies for the anomalous magnetic moment of the muon between theory and experiment [51].

CHAPTER 2

Further beyond the Standard Model

In order to discuss the various approaches in beyond the SM model building concerning the lepton mixings and masses, let us recapitulate the main questions to be dealt with, along the lines of Ref. [20].

1. *Can one build a model for accommodating neutrino masses and lepton mixings?*
   The answer is of course yes. We have already discussed this question and seen three different possibilities of neutrino mass generating operators in Section 1.1. The simplest way, in the sense that it is in complete analogy to the quark sector, is the addition of three right-chiral gauge-singlet Dirac neutrinos to the SM particle content. The other two possibilities arise if neutrinos are of Majorana nature and masses are generated either via Eq. (1.29b) or Eq. (1.29a).
2. *Why are neutrinos so much lighter than the other SM fermions?*
   Possible approaches to answering this question are discussed in Section 2.1.
3. *Can one find explanations for the peculiar features of neutrino mixing?*
   Ideas for answers are shown in Section 2.3.
4. *What are the implications of the relatively weak mass hierarchy in the neutrino mass spectrum, if any?*
   The ratio of the neutrino mass-squared differences $\Delta m^2_{21}/|\Delta m^2_{3l}| \sim 3 \times 10^{-2}$ is relatively large compared to the other SM fermions: for the charged leptons we have $\Delta m^2_{\mu e}/\Delta m^2_{\tau e} \sim 3 \times 10^{-3}$, for the down-type quarks $\Delta m^2_{sd}/\Delta m^2_{bd} \sim 5 \times 10^{-4}$ and for the up-type quarks $\Delta m^2_{cu}/\Delta m^2_{tu} \sim 5 \times 10^{-5}$. It is unclear, whether one can find an explanation for this fact and moreover, if there might be any connection to the differences between quark and lepton mixing.

2.1 Lightness of neutrino masses

The relative lightness of neutrinos compared to the other SM fermions might simply be an amusing choice of nature. Nevertheless, the difference of at least six orders of magnitude between the neutrino mass scale and the mass of the electron (see Fig. 1.3) served as a strong motivation to the theory community for trying to explain this fact dynamically.

For understanding the various approaches to neutrino mass generation, it is worthwhile to study which leptonic bilinears can be obtained in the standard model. The SM lepton doublets D_L living in the $SU(2)_L$ isospin-1/2 representation and SM lepton $SU(2)_L$-singlets l_R of the isospin-0 representation can be combined as

$$\overline{D}_L \otimes l_R : \qquad \underline{\tfrac{1}{2}} \otimes \underline{0} \cong \underline{\tfrac{1}{2}} \qquad \text{with } Y = -1, \tag{2.1a}$$

$$D_L \otimes D_L : \qquad \underline{\tfrac{1}{2}} \otimes \underline{\tfrac{1}{2}} \cong \underline{0} \oplus \underline{1}, \quad \text{with } Y = -2, \tag{2.1b}$$

$$l_R \otimes l_R : \qquad \underline{0} \otimes \underline{0} \cong \underline{0}, \qquad \text{with } Y = -4. \tag{2.1c}$$

Therefore, we find the following:

1. Gauge-singlet operators can be built via the SM Higgs-doublet ϕ with $Y = 1$ in Eq. (2.1a), leading to the well know SM Yukawa couplings. The introduction of right-chiral Dirac neutrino fields leads to a mass generation similar to the one of the charged leptons.
2. In Eq. (2.1b), either a charged gauge-singlet or a gauge-triplet with $Y = 2$ can be introduced for mass generation.
3. Finally, one can add a doubly charged scalar singlet with $Y = 4$ to the SM field content for generating a gauge-singlet in Eq. (2.1c).

This leads to various approaches for extensions of the SM, three well-known ones being:

Type I seesaw: Heavy Majorana neutrinos, *i.e.* right-chiral singlet fields ν_R as seen in Eq. (1.29a) with a large mass scale m_R are introduced in addition to couplings of the type Eq. (2.1a). They are responsible for light physical neutrino masses solely upon the diagonalization of the joint neutrino mass matrix $\mathcal{M}_\nu$. This is the relevant type of seesaw mechanism for the thesis at hand and therefore explained in detail in Section 2.2. As mentioned before, it is interesting to note that Eq. (1.29a) is a gauge-invariant term and in principle, no introduction of additional scalars is needed to explain its presence.

Type II seesaw: Light physical neutrino masses are achieved by the addition of a scalar triplet with $Y = 2$ and an independent VEV $v_T \ll v$ of its neutral component, determining the neutrino masses. The term seesaw comes about because $v_T \simeq -\mu_T^* v^2/M_T^{-2}$, where M_T is the scalar triplets mass scale and μ_T is the coupling between SM-Higgs and the scalar triplet. Therefore, large values of M_T provoke diminishing v_T and neutrino masses [53].

Radiative masses: One can use the aforementioned charged scalar singlet with $Y = 2$ to generate neutrino masses only at the one-loop level in the so called *Zee-model* [54], or even at the two loop-level level by the further addition of a charged scalar singlet with $Y = 4$ in the so called *Zee-Babu model* [55]. One obvious source of the neutrinos lightness in such models is given simply by the proportionality of the masses to the loop factors $1/(16\pi^2)^l$, where l is the loop-level at which neutrinos gain their masses.

2.2 Seesaw mechanism

The *type I seesaw mechanism* [56, 57, 58] is a widely used approach for the dynamic generation of light neutrino masses. It can be implemented in a variety of extension of the SM, among them the ones treated in Chapter 5. We therefore want to illustrate the workings of this mechanism in the following few paragraphs, exploiting the notation used in Chapter 5. Note that the Majorana fields ν_R represent the only addition to the SM particle content needed in type I seesaw models.

Firstly, we introduce the joint neutrino mass matrix of Dirac- and and Majorana-type masses by combining left- and right-chiral degrees of freedom, *viz.*

$$\mathcal{L}_{\nu,\text{mass}} = \frac{1}{2} \begin{pmatrix} \nu_L^T & (\nu_R^c)^T \end{pmatrix} C^{-1} \begin{pmatrix} 0_{n_L \times n_L} & (M_D^T)_{n_L \times n_R} \\ (M_D)_{n_R \times n_L} & (M_R)_{n_R \times n_R} \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{H.c.} \tag{2.2}$$

Note that we leave the number of left-chiral fields n_L open for this general discussion. Lorentz invariance in Eq. (2.2) is realized by the fact that charge conjugation flips the chirality. Using the chiral projectors $\gamma_L = (\mathbb{1} - \gamma^5)/2$ and $\gamma_R = (\mathbb{1} + \gamma^5)/2$, we have

$$\gamma_L \left(\nu_R\right)^c = \left(\nu_R\right)^c , \quad \gamma_R \left(\nu_R\right)^c = 0, \tag{2.3}$$

and by a way of rewriting

$$\overline{\nu_R} = - \left[(\nu_R)^c\right]^T C^{-1}. \tag{2.4}$$

The combined symmetric mass matrix is consequently defined as

$$\mathcal{M}_\nu \equiv \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{2.5}$$

It can be diagonalized yielding positive eigenvalues by virtue of a unitary $(n_L + n_R) \times (n_L + n_R)$ matrix $\mathcal{U}$, as in [59]

$$\mathcal{U}^T \mathcal{M}_\nu \mathcal{U} = \mathrm{diag}(m_1, m_2, \ldots, m_{n_L+n_R}). \tag{2.6}$$

We denote the typical scale of mass eigenvalues of M_R by m_R and the scale of eigenvalues of the Dirac mass matrix M_D by m_D. Then we infer $m_D \ll m_R$. Now, the diagonalization Eq. (2.6) can be carried out in terms of an expansion in m_D/m_R of a two-step process, *cf.* [60]. We decompose the mixing in

$$\mathcal{U} = \mathcal{W}\, \mathrm{diag}(\tilde{V}_L, \tilde{V}_R), \tag{2.7}$$

with the unitary matrices $\mathcal{W}$, $\tilde{V}_L$ and $\tilde{V}_R$ of dimensions

$$\begin{aligned} \mathcal{W}: &\quad (n_L + n_R) \times (n_L + n_R), \\ \tilde{V}_L: &\quad n_L \times n_L, \\ \tilde{V}_R: &\quad n_R \times n_R. \end{aligned}$$

The matrix $\mathcal{W}$ is supposed to disentangle light from heavy scales, *i.e.*

$$\mathcal{W}^T \mathcal{M}_\nu \mathcal{W} = \begin{pmatrix} \mathcal{M}_{\text{light}} & 0 \\ 0 & \mathcal{M}_{\text{heavy}} \end{pmatrix}. \tag{2.8}$$

This can be achieved by making the ansatz

$$\mathcal{W} = \begin{pmatrix} \sqrt{1 - BB^\dagger} & B \\ -B^\dagger & \sqrt{1 - B^\dagger B} \end{pmatrix}, \tag{2.9}$$

where B can be expanded in orders of m_D/m_R. To first order it reads

$$B = M_D^\dagger \left(M_R^{-1}\right)^* + \mathcal{O}\left(\frac{m_D^2}{m_R^2}\right). \tag{2.10}$$

The benefit of this ansatz is that no specific form of M_D or M_R has to be known. However, we see that M_R must be non-singular in this approach. The result is

$$\mathcal{M}_{\text{light}} = -M_D^T M_R^{-1} M_D + \mathcal{O}\left(\frac{m_D^2}{m_R^2}\right), \tag{2.11a}$$

$$\mathcal{M}_{\text{heavy}} = M_R + \mathcal{O}\left(\frac{m_D^2}{m_R^2}\right). \tag{2.11b}$$

Eventually, we notice that the heavy masses remain at the scale m_R while the light masses are of the order m_D^2/m_R. With the heavy Majorana neutrinos pushing down the masses of the light ones, the term *seesaw* presents itself as a fitting description.

An appealing, yet possibly coincidental fact is that if the Dirac mass scale m_D is of the order of the only relevant energy scale of the electroweak SM, the Higgs-VEV v, and the Yukawa couplings are of order one, then the heavy Majorana mass scale m_R must lie almost in the energy regime of Grand Unified Theories (GUTs) of $M_{\text{GUT}} \sim 10^{14} - 10^{15}\,\text{GeV}$ in order to explain the small values of the measured physical neutrino masses. This fact implies that neutrino masses fit well into the GUT-picture [61] and partly explains the strong interest of the theory community.

Another appealing property of the seesaw mechanism, or rather the presence of a Majorana mass term, is that such a term violates lepton number conservation by two units. As mentioned before, this might give means to explain the matter-antimatter asymmetry of our universe [16].

Without major changes to the approach explained so far, one can also include contributions of a mass term of the kind Eq. (1.29b). Then the $n_L \times n_L$ mass matrix M_L of the scale m_L replaces the upper left zero in Eq. (2.5). Inferring $m_L \ll m_R$, we simply find [62]

$$\mathcal{M}_{\text{light}} = M_L - M_D^T M_R^{-1} M_D + \mathcal{O}\left(\frac{m_D^2}{m_R^2}\right), \tag{2.12}$$

as opposed to Eq. (2.11a).

2.3 Models of lepton mixing

In addition to neutrinos exhibiting non-zero masses, three main features of lepton mixing captured the model builders attention: a large, but non-maximal $\theta_{12} \sim 34°$, a consistent with being maximal $\theta_{23} \simeq 45°$ and a small, but non-zero $\theta_{13} \sim 8.5°$. These values came as a surprise when compared to the almost diagonal quark mixing (see Section 1.2.2). An abundance of models trying to describe the observations—mainly of the lepton sector—has been created over the past decades. Many of these rely on the use of *non-Abelian discrete groups* acting on the three different families or flavors of leptons as well as quarks, hence the widely used term *flavor symmetries*. Continuous symmetry groups are not as widely used, because if these are broken spontaneously together with the gauge symmetries, additional Goldstone bosons would appear [63]. Similarly, Abelian discrete symmetry groups are more rarely discussed, because they have only a limited capacity in enforcing specific mixing patterns. They can, for instance, not be used to enforce the so called tri-bimaximal mixing pattern [64], which was the most prominent candidate for lepton mixing before the measurement of a non-zero reactor mixing angle. For comprehensive reviews on the use of discrete symmetries in model building, see *e.g.* [63, 65, 66, 67].

Usually, the aforementioned models contain an extended scalar sector. Heuristically speaking, this is due to the fact that a lot of freedom is gained model building wise, *e.g.* in order to generate the experimentally known mass hierarchies of the leptons [65]. Moreover, it can be shown that in presence of only a single Higgs doublet which transforms as a singlet under the respective flavor symmetry, no discrete unbroken symmetry can produce the observed lepton mixing pattern [68]. These facts lead us to the study of the mHDSM of Chapter 5.

It is worth mentioning that other than the mixing models discussed here, the theory community also accommodates diametral approaches labeled as *neutrino anarchy*. Originally in Ref. [69] and more recently *e.g.* in Ref. [70], the generation of the observed fermion mixing and mass hierarchies from random Yukawa couplings is investigated. Although it is perfectly viable that nature might have simply gambled to give us the known values for masses and mixing angles, it is difficult to judge the explanatory power of anarchy models. A major drawback of such models is the heavy dependence on the choice of probability distributions for the generation of random input values for the models. The choices for these distributions are carried out in very reasonable ways. Still, experimental access to these is foreseeably out of the question and therefore it remains disputable how satisfactory such explanations can be. Moreover, while most anarchy approaches succeed in generating mixing patterns with comparably large mixing angles, they usually fail to give an explanation for the almost diagonal structure of the quark mixing matrix or vice versa [67].

2.4 A model for maximal atmospheric mixing

An illustrative example of a discrete flavor symmetry[1] in a predictive and renormalizable mixing model is the *generalized CP-model* of Ref. [71, 72], making use of the so called μ-τ-symmetry introduced in Ref. [73]. Models of this kind formed a major motivation for the main part of this thesis and we therefore want to showcase this example to some detail.

The goal is to explain some of the peculiar features of the lepton mixing matrix, in this case the potentially *maximal atmospheric mixing* angle θ_{23}. It has been realized in Refs. [74, 75] that if the mass matrix of the experimentally observed light neutrinos $\mathcal{M}_{\text{light}}$ exhibits the structure

$$\mathcal{M}_{\text{light}} = \begin{pmatrix} a & r & r^* \\ r & s & b \\ r^* & b & s^* \end{pmatrix}, \quad \text{with } r, s \in \mathbb{C}, \quad a, b \in \mathbb{R}, \tag{2.13}$$

then it can be shown that the diagonalization of Eq. (2.13) with a unitary matrix U always yields

$$|U_{\mu j}| = |U_{\tau j}|, \quad j = 1,2,3 \quad \text{and} \quad \delta = \pm\frac{\pi}{2}, \tag{2.14}$$

for $\theta_{13} \neq 0$—hence the name *μ-τ-symmetry* [72]. Another way of prescribing the peculiar structure of Eq. (2.13) is to demand the transformation behavior

$$\mathcal{M}^*_{\text{light}} = S\mathcal{M}_{\text{light}}S, \quad \text{with } S = \begin{matrix} \\ \tilde{\nu}_e \\ \tilde{\nu}_\mu \\ \tilde{\nu}_\tau \end{matrix} \begin{matrix} \begin{matrix} \tilde{\chi}_1 & \tilde{\chi}_2 & \tilde{\chi}_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{matrix}. \tag{2.15}$$

It is now the question how to formulate this behavior in terms of a renormalizable theory.

Lepton sector of the generalized CP-model

An example for a model achieving the former was given in [72]. The setup is as follows:[2] the field content of the electroweak SM is supplemented by three right-chiral Majorana

[1] An overview of finite flavor groups for fermions is given in Ref. [63].

[2] Note that in [72], the terminology *a model* is used. This is owing to the fact that the presented model is not necessarily expected to be the simplest or sole possibility for describing lepton mixing, but rather acts as an illustrative, yet potentially physical example.

neutrinos ν_R and two additional Higgs-doublets ϕ_2 and ϕ_3. The neutral components of the three Higgs doublets acquire the VEVs $\langle 0|\varphi_j^0|0\rangle = v_j/\sqrt{2}$ where $v = \sqrt{\sum_j v_j v_j^*} \simeq 246\,\text{GeV}$. The theory is built according to the following symmetry considerations

1. The global $U(1)$ lepton-number symmetries L_α are introduced, resulting in flavor-diagonal Yukawa couplings.
2. The $U(1)$ symmetries are broken softly via a Majorana-mass term of the right-chiral neutrinos, providing enough freedom for generating lepton mixing.
3. A $\mathbb{Z}_2$-symmetry is introduced, under which μ_R, τ_R, ϕ_2 and ϕ_3 change sign. This decouples the mass generation of the electron and the neutrinos from the muon and tau.

Consequently, the Yukawa-Lagrangian reads

$$\begin{aligned}\mathcal{L}_\text{Y} = &-\frac{\sqrt{2}}{v_1}\sum_{\alpha=e,\mu,\tau} f_\alpha \overline{\nu_{R,\alpha}}\tilde{\phi}_1^\dagger D_{L,\alpha} - \frac{\sqrt{2}m_e}{v_1^*}\,\overline{e_R}\phi_1^\dagger D_{L,e} \\ &- g_{2\mu}\,\overline{\mu_R}\phi_2^\dagger D_{L,\mu} - g_{2\tau}\,\overline{\tau_R}\phi_2^\dagger D_{L,\tau} - g_{3\mu}\,\overline{\mu_R}\phi_3^\dagger D_{L,\mu} - g_{3\tau}\,\overline{\tau_R}\phi_3^\dagger D_{L,\tau} + \text{H.c.}, \end{aligned} \tag{2.16}$$

where f_α and $g_{j\alpha}$ are complex numbers and m_e is, without loss of generality, the electron mass. We see that the symmetries provoke a flavor-diagonal Yukawa Lagrangian. After inserting the VEVs of the scalar doublets, *i.e.*

$$\phi_i \to \frac{v_i}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.17}$$

the neutrino Dirac-mass matrix consequently reads

$$M_D = \text{diag}(f_e, f_\mu, f_\tau), \tag{2.18}$$

and the masses of the muon and tau are given by

$$m_\mu = \frac{1}{\sqrt{2}}\left|g_{2\mu}v_2^* + g_{3\mu}v_3^*\right|, \quad m_\tau = \frac{1}{\sqrt{2}}\left|g_{2\tau}v_2^* + g_{3\tau}v_3^*\right|. \tag{2.19}$$

Additionally, the Majorana mass term

$$\mathcal{L}_\text{Maj} = \frac{1}{2}\nu_R^T C^{-1} M_R^* \nu_R + \text{H.c.}, \tag{2.20}$$

is introduced, which represents the soft symmetry breaking as described in Section 1.2.5 and will be the sole source of lepton mixing.

We now invoke the seesaw mechanism, which in this case means assuming a Majorana mass scale $m_R \gg v$.[3] As explained in Section 2.2, this generates the light neutrino mass matrix

$$\mathcal{M}_\text{light} = -M_D^T M_R^{-1} M_D, \tag{2.21}$$

of Eq. (2.8). In order to achieve the sought after structure of $\mathcal{M}_\text{light}$, we demand

$$S M_R S = M_R^*, \quad \text{and} \quad S M_D S = M_D^*, \tag{2.22}$$

from which Eq. (2.15) follows by virtue of Eq. (2.21), leading to maximal atmospheric mixing $\theta_{23} = 45°$ and a maximally CP-violating phase $\delta = \pm\pi/2$ in the lepton mixing matrix.[4]

[3] As described in Section 2.2, in the context of neutrino mass models, it is common practice to choose the SM-VEV v as the relevant scale for the Dirac-masses, hence this choice for the scale m_R.

[4] Due to the flavor-diagonal Yukawa Lagrangian, we have $W_{L,R} = \mathbb{1}_{3\times 3}$ for the charged leptons and therefore the mixing of the neutrino sector directly translates to the full lepton mixing matrix U_PMNS.

The transformation behavior of Eq. (2.22) can be generated in terms of a symmetry via demanding invariance of the model under the *generalized CP-transformations*

$$\begin{aligned} \nu_{L,\alpha} &\to iS_{\alpha\beta}\gamma^0 C\,\overline{\nu}_{L,\beta}^T, \\ \nu_{R,\alpha} &\to iS_{\alpha\beta}\gamma^0 C\,\overline{\nu}_{R,\beta}^T, \\ \alpha_L &\to iS_{\alpha\beta}\gamma^0 C\,\overline{\beta}_L^T, \\ \alpha_R &\to iS_{\alpha\beta}\gamma^0 C\,\overline{\beta}_R^T, \quad \text{for } \alpha = e,\mu,\tau, \end{aligned} \tag{2.23a}$$

$$\begin{aligned} \phi_{1,2} &\to \phi_{1,2}^*, \\ \phi_3 &\to -\phi_3^*, \end{aligned} \tag{2.23b}$$

with S of Eq. (2.15).[5] It follows that f_e must be real, $f_\mu = f_\tau^*$, $g_{2\mu} = g_{2\tau}^*$ and $g_{3\mu} = -g_{3\tau}^*$. Without loss of generality, it is assumed that v_1 is real and positive, eventually yielding Eq. (2.22). Another consequence is that the masses of the muon and tau are now given by

$$m_\mu = \frac{1}{\sqrt{2}}\left|g_{2\mu}v_2^* + g_{3\mu}v_3^*\right|, \quad m_\tau = \frac{1}{\sqrt{2}}\left|g_{2\mu}^*v_2^* - g_{3\mu}^*v_3^*\right|. \tag{2.24}$$

In order to avoid fine-tuning[6] in achieving the relation $m_\mu \ll m_\tau$, one final symmetry is introduced, *viz.*

$$K: \quad \mu_R \to -\mu_R,\ \phi_2 \leftrightarrow \phi_3, \tag{2.25}$$

provoking the relation $g_{2\mu} = -g_{3\mu}$. Then, the μ- and τ-masses are given by

$$m_\mu = \frac{|g_{2\mu}|}{\sqrt{2}}\left|v_2 - v_3\right|, \quad m_\tau = \frac{|g_{2\mu}|}{\sqrt{2}}\left|v_2 + v_3\right|. \tag{2.26}$$

Under exact K-symmetry, one has $v_2 = v_3$ and therefore a vanishing mass of the muon. A more realistic setting is achieved by introducing terms in the scalar potential which break K *softly*, giving an explanation for the relation $m_\mu \ll m_\tau$ in a technically natural way.

Scalar sector of the generalized CP-model

The scalar sector of the generalized CP-model of Ref. [72] is investigated to greater detail in Ref. [71]. Here, we present a culmination of the results therein which are necessary for a numerical evaluation of the general one-loop formulae of Chapter 5.

The complete scalar potential of the 3HDM in question reads

$$\begin{aligned} V_\phi = &- \mu_1\phi_1^\dagger\phi_1 - \mu_2\left(\phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\right) \\ &+ \mu_{\text{soft}}\left(\phi_2^\dagger\phi_2 - \phi_3^\dagger\phi_3\right) + i\mu'_{\text{soft}}\left(\phi_2^\dagger\phi_3 - \phi_3^\dagger\phi_2\right) \\ &+ \lambda_1\left(\phi_1^\dagger\phi_1\right)^2 + \lambda_2\left[\left(\phi_2^\dagger\phi_2\right)^2 + \left(\phi_3^\dagger\phi_3\right)^2\right] \\ &+ \lambda_3\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\right) + \lambda_4\left(\phi_2^\dagger\phi_2\right)\left(\phi_3^\dagger\phi_3\right) \\ &+ \lambda_5\left[\left(\phi_1^\dagger\phi_2\right)\left(\phi_2^\dagger\phi_1\right) + \left(\phi_1^\dagger\phi_3\right)\left(\phi_3^\dagger\phi_1\right)\right] + \lambda_6\left(\phi_2^\dagger\phi_3\right)\left(\phi_3^\dagger\phi_2\right) \end{aligned}$$

[5] A thorough discussion of standard- and non-standard CP-transformations is given in Ref. [76].

[6] One would have to choose two products of unrelated complex quantities such that they nearly cancel which is considered as a more severe fine-tuning than *e.g.* in the SM case, where one simply chooses small Yukawa couplings [71].

$$
\begin{aligned}
&+ \lambda_7 \left[\left(\phi_2^\dagger \phi_3 \right)^2 + \left(\phi_3^\dagger \phi_2 \right)^2 \right] \\
&+ \lambda_8 \left[\left(\phi_1^\dagger \phi_2 \right)^2 + \left(\phi_2^\dagger \phi_1 \right)^2 + \left(\phi_1^\dagger \phi_3 \right)^2 + \left(\phi_3^\dagger \phi_1 \right)^2 \right] \\
&+ i\lambda_9 \left[\left(\phi_1^\dagger \phi_2 \right) \left(\phi_1^\dagger \phi_3 \right) - \left(\phi_2^\dagger \phi_1 \right) \left(\phi_3^\dagger \phi_1 \right) \right] \\
&+ i\lambda_{10} \left(\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_2 \right) \left(\phi_2^\dagger \phi_2 - \phi_3^\dagger \phi_3 \right) . \qquad (2.27)
\end{aligned}
$$

All couplings therein are real. The quantities $\mu_{\text{soft}}^{(\prime)}$ represent the soft breaking of the K-symmetry of Eq. (2.25). The determination of the minimum is carried out in terms of

$$
\frac{v_1}{\sqrt{2}} = u_1, \quad \frac{v_2}{\sqrt{2}} = u\, \mathrm{e}^{i\alpha} \cos\sigma, \quad \frac{v_3}{\sqrt{2}} = u\, \mathrm{e}^{i\beta} \sin\sigma, \qquad (2.28)
$$

where, without loss of generality, u_1 and u are chosen positive and σ lies in the first quadrant. Inserting the replacement of Eq. (2.17) into Eq. (2.27), the VEV of the scalar potential can be determined from

$$
\begin{aligned}
\langle 0|V_\phi|0\rangle = &- \mu_1 u_1^2 - \mu_2 u^2 + \mu_{\text{soft}} u^2 \cos 2\sigma + \mu'_{\text{soft}} u^2 \sin 2\sigma \sin(\alpha - \beta) \\
&+ \lambda_1 u_1^4 + \lambda_2 u^4 + (\lambda_3 + \lambda_5)\, u_1^2 u^2 \\
&+ \left[\tilde{\lambda} - 4\lambda_7 \sin^2(\alpha - \beta) \right] u^4 \cos^2\sigma \sin^2\sigma \\
&+ 2\lambda_8 u_1^2 u^2 \left[\cos^2\sigma \cos(2\alpha) + \sin^2\sigma \cos(2\beta) \right] \\
&- \lambda_9 u_1^2 u^2 \sin 2\sigma \sin(\alpha + \beta) + \lambda_{10} u^4 \cos 2\sigma \sin 2\sigma \sin(\alpha - \beta) , \qquad (2.29)
\end{aligned}
$$

where $\tilde{\lambda} \equiv -2\lambda_2 + \lambda_4 + \lambda_6 + 2\lambda_7$. An exact solution to the minimization of this function is not available. Nevertheless, for the case $\tilde{\lambda} < 0$ and $\lambda_7 < 0$, the minimum can be determined in terms of an expansion in $(\mu_{\text{soft}}/u_1^2)$ and $(\mu'_{\text{soft}}/u_1^2)$ in the following way. To zeroth order in these quantities, the minimum lies at

$$
\sigma = \frac{\pi}{4}, \quad \alpha = \beta = \omega, \qquad (2.30)
$$

where ω is the solution to the equation

$$
2\lambda_8 \sin 2\omega + \lambda_9 \cos 2\omega = 0. \qquad (2.31)
$$

Switching on the soft symmetry breaking terms, the minimum gets shifted. Introducing the small quantities δ_0, δ_+ and δ_-, this shift can be parameterized as

$$
\sigma = \frac{\pi}{4} - \frac{\delta_0}{2}, \quad \alpha = \omega + \delta_+ + \frac{\delta_-}{2}, \quad \beta = \omega + \delta_+ - \frac{\delta_-}{2}. \qquad (2.32)
$$

The relevant quantity for the determination of the minimum shift is achieved by expanding the quartic coupling terms of Eq. (2.29) to second order and the soft breaking terms to first order in the δ_a and reads

$$
\frac{1}{2} \sum_{a,b} \mathcal{F}_{ab} \delta_a \delta_b + \sum_a f_a \delta_a, \qquad (2.33)
$$

where the components of $\mathcal{F}$ are given by

$$
\mathcal{F}_{00} = -\frac{1}{2} \tilde{\lambda} u^4 + \lambda_9 u_1^2 u^2 \sin 2\omega, \qquad (2.34a)
$$

$$
\mathcal{F}_{++} = (-8\lambda_8 \cos 2\omega + 4\lambda_9 \sin 2\omega)\, u_1^2 u^2, \qquad (2.34b)
$$

$$\mathcal{F}_{--} = -2\lambda_7 u^4 - 2\lambda_8 u_1^2 u^2 \cos 2\omega, \tag{2.34c}$$

$$\mathcal{F}_{0-} = \mathcal{F}_{-0} = -2\lambda_8 u_1^2 u^2 \sin 2\omega + \lambda_{10} u^4, \tag{2.34d}$$

$$\mathcal{F}_{0+} = \mathcal{F}_{0+} = 0, \tag{2.34e}$$

$$\mathcal{F}_{+-} = \mathcal{F}_{-+} = 0. \tag{2.34f}$$

and the contributions of the soft breaking terms read

$$f_0 = \mu_{\text{soft}} u^2, \quad f_+ = 0, \quad f_- = \mu'_{\text{soft}} u^2. \tag{2.35}$$

By defining the vectors $\boldsymbol{\delta} \equiv (\delta_0, \delta_+, \delta_-)^T$ and $\boldsymbol{f} \equiv (f_0, f_+, f_-)^T$, the minimum of V_ϕ to first order in the δ_a can eventually be written as

$$\boldsymbol{\delta} = -\mathcal{F}^{-1} \boldsymbol{f}. \tag{2.36}$$

At this order, the ratio of muon to tau mass can now be written as

$$\frac{m_\mu}{m_\tau} = \left| \frac{v_2 - v_3}{v_2 + v_3} \right| \simeq \frac{1}{2} \left| \delta_0 + i\delta_- \right|. \tag{2.37}$$

Therefore, we find that different phases in v_2 and v_3 are sufficient for generating a non-zero muon mass in this setup.

This concludes the technical side of the discussion of the generalized CP-model. We have presented all necessary ingredients for a potential numerical evaluation of this exemplary model in terms of the general results to be presented in Chapter 5. In App. C, a conversion list from the quantities of this section to the general expressions of Chapter 5 is given. Moreover, we present a numerical example for this model in App. D and discuss some conclusions we draw therefrom in Chapter 6.

Domain walls

As an aside, it should be mentioned that in theories with spontaneously broken discrete symmetries, the semi-classical problem of domain walls can arise [77]. This effect appears whenever several disconnected vacuum states are present in a theory. More specifically, if vacua cannot be continuously transformed into one another under some symmetry transformation that determines the structure of the scalar potential of a theory, then *domain walls* carrying a potentially high energy density can arise between these separate states. This is obviously the case in the presence of discrete symmetry groups. Domain walls are to be expected between causally disjoint regions of the early universe, where each vacuum state would be populated randomly. The energy density arising with the domain walls would have major consequences for the evolution of the universe, meaning there are strong experimental bounds on their potential existence [78]. As noted in [79] and references therein, there are at least two ways to reconcile the problems posed by the potential existence of domain walls. If the scale of symmetry breaking is high enough, *i.e.* around the scale of grand unification $\sim \mathcal{O}(10^{16})$ GeV, then domain walls are formed before the end of inflation and are therefore "inflated away". If however the breaking scale is lower, their effects should persist on cosmologically relevant time scales, but might be avoided by a small explicit breaking of the discrete symmetry responsible for their existence.

Part II

Publications

CHAPTER 3

On-shell renormalization conditions

Walter Grimus and Maximilian Löschner

Revisiting on-shell renormalization conditions in theories with flavour mixing

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Revisiting on-shell renormalization conditions in theories with flavour mixing

W. Grimus* and M. Löschner‡

University of Vienna, Faculty of Physics
Boltzmanngasse 5, A–1090 Vienna, Austria

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Abstract

In this review, we present a derivation of the on-shell renormalization conditions for scalar and fermionic fields in theories with and without parity conservation. We also discuss the specifics of Majorana fermions. Our approach only assumes a canonical form for the renormalized propagators and exploits the fact that the inverse propagators are non-singular in $\varepsilon = p^2 - m_n^2$, where p is the external four-momentum and m_n is a pole mass. In this way, we obtain full agreement with commonly used on-shell conditions. We also discuss how they are implemented in renormalization.

*E-mail: walter.grimus@univie.ac.at
‡E-mail: maximilian.loeschner@univie.ac.at

1 Introduction

On-shell renormalization conditions in theories with inter-family or flavour mixing [1, 2, 3, 4, 5, 6] are quite important in view of the experimentally established quark and lepton mixing matrices [7] and the mixing between the photon and the Z boson in the Standard Model—see for instance [8, 9]. In theories beyond the Standard Model, mixing of new fermions and scalar mixing might occur as well. However, we feel that several aspects of the derivation of the on-shell renormalization conditions remain a bit vague for the general reader of the relevant literature and should be discussed in more detail. In this review, we present a consistent way of deriving the on-shell conditions, first for scalar, then for fermionic fields in theories with and without parity conservation and finally for Majorana fields, for the mixing of N fields. Our approach is solely based on the pole structure of the $N \times N$ propagator matrix and relies on the fact that the inverse propagator has no singularity in p^2, where p is the external momentum. We also review the counting of the number of renormalization conditions and demonstrate that this number coincides with the number of degrees of freedom in the counterterms, except for overall phase factors that remain free for the fermionic fields in the case of Dirac fermions. Our results agree, of course, with the ones derived in [2]. Explicit formulas for the field strength renormalization constants and mass counterterms in the above-mentioned theories are given for the lowest non-trivial order.

Our discussion is based on two assumptions:

1. All N physical masses m_i are different.
2. The N poles in the propagator matrix are located in a region where absorptive parts are absent or can be neglected.

Some remarks relating to these assumptions are in order. Equality of two or more masses would require a "flavour" symmetry, but dealing with on-shell renormalization in the presence of such a symmetry is beyond the scope of this review.[1] The on-shell renormalization in this review deals with the dispersive parts of the scalar and fermionic propagators. If large imaginary parts appear in the higher order corrections to the propagators, a treatment of their absorptive parts becomes relevant, but lies beyond the scope of this review as well. Still, the renormalization conditions derived here are fully applicable in the regions of p^2 where the absorptive parts vanish. Elsewhere, they can be used by simply inserting only the dispersive parts of the self-energy functions into the conditions. Another possible approach is to use complex masses as well as complex counterterms in the so-called complex-mass scheme. A treatment of this approach can *e.g.* be found in [10, 11]. Since our paper is intended as a pedagogical review and the distinction between dispersive an absorptive parts in the propagator matrix plays an important role in our presentation, we have included, for the sake of completeness, a discussion of this issue as an appendix.

The treatment of on-shell conditions in our paper is based on an expansion in $p^2 - m_n^2$ around each pole mass m_n, for both propagator and inverse propagator, whereas the

[1] This would, for instance, change the aforementioned counting of number of renormalization conditions and degrees of freedom in the counterterms.

authors of [6] make use of exact matrix relations between the propagator matrix and its inverse. In this sense, our treatment of fermions is complementary to that of [6].

The plan of the paper is as follows. In section 2 we discuss mixing of real scalar fields, whereas fermions are treated in section 3 in the case of parity conservation. The complications which arise when parity is violated are elaborated in section 4. This section contains also a subsection on the on-shell renormalization of Majorana fermions. After a summary in section 5, the emergence of dispersive and absorptive contributions to the propagator is covered in appendix A in the framework of the Källén–Lehmann representation. Some computational details in the treatment of fermions are deferred to appendix B. The condition on the propagator matrix which arises in the case of Majorana nature of the fermions is derived in appendix C.

In the following, we will use k, l as summation indices whereas i, j, n do not imply summation.

2 Scalar propagator

2.1 On-shell conditions

We first study the scalar propagator for N real scalar fields, which is a simple and instructive case to begin with. Here the propagator is an $N \times N$ matrix

$$\Delta(p^2) = \left(\Delta_{ij}(p^2)\right), \tag{1}$$

where p^2 is the Minkowski square of the four-momentum p. We assume that all masses m_n of the scalars are different. We stress that $\Delta(p^2)$ is the *renormalized* propagator. Defining

$$\varepsilon \equiv p^2 - m_n^2, \tag{2}$$

the on-shell renormalization conditions consist of the requirement [2]

$$\Delta_{ij}(p^2) \overset{\varepsilon \to 0}{\longrightarrow} \frac{\delta_{in}\delta_{nj}}{\varepsilon} + \Delta_{ij}^{(0)} + \mathcal{O}(\varepsilon) \tag{3}$$

for all $n = 1, \ldots, N$. In this formula and in the following, the symbol δ_{rs} always signifies the Kronecker symbol. The coefficients $\Delta_{ij}^{(0)}$ are of order one *viz.* ε^0. In the following, the superscripts (0) and (1) will always indicate order ε^0 and ε^1, respectively.

The complication comes from the fact that we actually want to impose on-shell renormalization conditions on the *inverse* propagator, which we denote by $A = (A_{ij})$.[2] Accordingly, we have to translate equation (3) into conditions on A. The inverse propagator fulfills

$$\Delta_{ik}A_{kj} = A_{ik}\Delta_{kj} = \delta_{ij} \quad \text{with} \quad A_{ij} = A_{ij}^{(0)} + \varepsilon A_{ij}^{(1)} + \mathcal{O}(\varepsilon^2), \tag{4}$$

where the latter relation states that A has no singularity in ε. Equation (4) is reformulated as

$$\Delta_{ik}A_{kj} = \frac{1}{\varepsilon}\delta_{in}A_{nj}^{(0)} + \delta_{in}A_{nj}^{(1)} + \Delta_{ik}^{(0)}A_{kj}^{(0)} + \mathcal{O}(\varepsilon) = \delta_{ij}, \tag{5a}$$

[2]From now on, for the sake of simplicity of notation, we skip the dependence on p^2 in all quantities, whenever this dependence is obvious.

$$A_{ik}\Delta_{kj} = \frac{1}{\varepsilon}A_{in}^{(0)}\delta_{nj} + A_{in}^{(1)}\delta_{nj} + A_{ik}^{(0)}\Delta_{kj}^{(0)} + \mathcal{O}(\varepsilon) = \delta_{ij}. \tag{5b}$$

Avoiding the singularity in $1/\varepsilon$ requires

$$A_{in}^{(0)} = 0 \;\; \forall\, i = 1, \ldots, N \quad \text{and} \quad A_{nj}^{(0)} = 0 \;\; \forall\, j = 1, \ldots, N. \tag{6}$$

For $i = j = n$ we obtain the further condition

$$A_{nn}^{(1)} = 1. \tag{7}$$

Note that, because of equation (6), $\Delta_{nk}^{(0)}$ and $\Delta_{kn}^{(0)}$ do not occur in the second condition.

The remaining coefficients in equation (5), which have not yet been fixed by equations (6) and (7), are determined by the orthogonality conditions at order ε^0:

$$\begin{aligned} i \neq n,\ j \neq n :&\quad \Delta_{ik}^{(0)} A_{kj}^{(0)} = A_{ik}^{(0)} \Delta_{kj}^{(0)} = \delta_{ij}, \\ i = n,\ j \neq n :&\quad A_{nj}^{(1)} + \Delta_{nk}^{(0)} A_{kj}^{(0)} = 0, \\ i \neq n,\ j = n :&\quad A_{in}^{(1)} + A_{ik}^{(0)} \Delta_{kn}^{(0)} = 0. \end{aligned} \tag{8}$$

However, these conditions have nothing to do with on-shell renormalization.

In summary, for the inverse propagator we have derived the on-shell conditions

$$A_{in}(m_n^2) = A_{nj}(m_n^2) = 0 \;\; \forall\, i,j = 1, \ldots, N \quad \text{and} \quad \left.\frac{\mathrm{d}A_{nn}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} = 1, \tag{9}$$

in agreement with the result of [2]. Of course, for every n there is such a set of conditions.

2.2 Renormalization and parameter counting

The dispersive part of the inverse propagator is a real symmetric matrix, *i.e.*

$$\left(\Delta^{-1}(p^2)\right)^T = \Delta^{-1}(p^2). \tag{10}$$

A derivation of this symmetry relation in terms of the Källén–Lehmann representation of the scalar propagator is given in appendix A.1. Using the fact that on-shell renormalization conditions are imposed on the dispersive part, we can reformulate them as

$$i \neq j:\ A_{ij}(m_j^2) = 0, \quad i = j:\ A_{ii}(m_i^2) = 0,\ \left.\frac{\mathrm{d}A_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_i^2} = 1. \tag{11}$$

The number of on-shell conditions is thus

$$2\binom{N}{2} + 2N = N^2 + N. \tag{12}$$

The factor 2 in front of the binomial coefficient originates from the two pairs (i,j) and (j,i) for $i \neq j$.

In terms of the self-energy, the renormalized inverse propagator has the form

$$A(p^2) = p^2 - \hat{m}^2 - \Sigma^{(r)}(p^2). \tag{13}$$

The bare fields $\varphi_i^{(b)}$ are related to the renormalized ones via the field strength renormalization matrix:

$$\varphi_i^{(b)} = \left(Z^{(1/2)}\right)_{ik} \varphi_k. \tag{14}$$

In this way, the renormalized self-energy $\Sigma^{(r)}(p^2)$ is related to the unrenormalized one by

$$\Sigma^{(r)}(p^2) = \Sigma(p^2) + \left(\mathbb{1} - \left(Z^{(1/2)}\right)^T Z^{(1/2)}\right) p^2 + \left(Z^{(1/2)}\right)^T \left(\hat{m}^2 + \delta\hat{m}^2\right) Z^{(1/2)} - \hat{m}^2. \tag{15}$$

For real scalar fields, $Z^{(1/2)}$ is a general real $N \times N$ matrix. Therefore, it contains N^2 real parameters. Assuming to be in a basis where

$$\hat{m}^2 = \mathrm{diag}\left(m_1^2, \ldots, m_N^2\right), \tag{16}$$

then $\delta\hat{m}^2$ is diagonal too. Thus, there are N real parameters in $\delta\hat{m}^2$. In summary, we have $N^2 + N$ free parameters at disposal for implementing $N^2 + N$ on-shell renormalization conditions. The condition $A_{ii}(m_i^2) = 0$ is imposed with the help of $\delta\hat{m}_i^2$, while for the rest the field strength renormalization matrix $Z^{(1/2)}$ is responsible.

It is instructive to perform the renormalization of the inverse propagator at the lowest non-trivial order in $Z^{(1/2)}$. In this case we write

$$Z^{(1/2)} = 1 + \frac{1}{2} z \tag{17}$$

and

$$\Sigma^{(r)}(p^2) = \Sigma(p^2) - \frac{1}{2}\left(z + z^T\right) p^2 + \frac{1}{2}\left(\hat{m}^2 z + z^T \hat{m}^2\right) + \delta\hat{m}^2. \tag{18}$$

Then it is straightforward to derive

$$i \neq j\colon\ \frac{1}{2} z_{ij} = -\frac{\Sigma_{ij}(m_j^2)}{m_i^2 - m_j^2}, \quad i = j\colon\ \delta\hat{m}_i^2 = -\Sigma_{ii}(m_i^2),\ z_{ii} = \left.\frac{\mathrm{d}\Sigma_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2 = m_i^2} \tag{19}$$

from equation (11).

3 Fermion propagator with parity conservation

3.1 On-shell conditions

For fermions the situation is more involved, but we can nevertheless proceed in analogy to the scalar case. We write the propagator as

$$S = C\not{p} - D \quad \text{with} \quad C = \left(C_{ij}(p^2)\right), \quad D = \left(D_{ij}(p^2)\right) \tag{20}$$

being $N \times N$ matrices. It is now expedient to formulate for fermions the condition analogous to equation (3). Using again ε of equation (2), we have now

$$S_{ij} \xrightarrow{\varepsilon \to 0} \frac{\delta_{in}\delta_{nj}}{\not{p} - m_n} + \tilde{S}_{ij}, \tag{21}$$

where $\tilde{S}_{ij}$ is non-singular in ε. First of all, we have to work out what equation (21) means for C and D. Multiplying S by $\not{p} - m_n$ and exploiting equation (21), we find

$$\left(\not{p} - m_n\right) S_{ij} = \delta_{in}\delta_{nj} + \left(\not{p} - m_n\right) \tilde{S}_{ij} = \varepsilon C_{ij} - \left(\not{p} - m_n\right) \left(D_{ij} + m_n C_{ij}\right), \tag{22}$$

whence we conclude

$$C_{ij} = \frac{\delta_{in}\delta_{nj}}{\varepsilon} + C_{ij}^{(0)} + \mathcal{O}(\varepsilon), \quad D_{ij} = -\frac{m_n \delta_{in}\delta_{nj}}{\varepsilon} + D_{ij}^{(0)} + \mathcal{O}(\varepsilon). \tag{23}$$

The second relation follows from the non-singularity of $\tilde{S}_{ij}$. We see that in the fermion case we have two relations instead of one, equation (3), in the scalar case.

Now we have to formulate the conditions of equation (23) for the inverse propagator

$$S^{-1} = A\not{p} - B. \tag{24}$$

The relation between A, B and C, D is given by

$$\left(SS^{-1}\right)_{ij} = \delta_{ij} \quad \Rightarrow \quad C_{ik}A_{kj}\,p^2 + D_{ik}B_{kj} = \delta_{ij}, \quad C_{ik}B_{kj} + D_{ik}A_{kj} = 0, \tag{25a}$$

$$\left(S^{-1}S\right)_{ij} = \delta_{ij} \quad \Rightarrow \quad A_{ik}C_{kj}\,p^2 + B_{ik}D_{kj} = \delta_{ij}, \quad B_{ik}C_{kj} + A_{ik}D_{kj} = 0. \tag{25b}$$

Since the inverse propagator is non-singular for $\varepsilon \to 0$, we have the expansion

$$A_{ij} = A_{ij}^{(0)} + \varepsilon A_{ij}^{(1)} + \mathcal{O}(\varepsilon^2), \quad B_{ij} = B_{ij}^{(0)} + \varepsilon B_{ij}^{(1)} + \mathcal{O}(\varepsilon^2) \tag{26}$$

with

$$A_{ij}^{(0)} = A_{ij}(m_n^2), \quad A_{ij}^{(1)} = \left.\frac{\mathrm{d}A_{ij}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} \tag{27}$$

and the analogous relations for B. We have to plug the relations of equation (26) into equation (25) and invoke the expansion for the propagator presented in equation (23). For the details, we refer the reader to appendix B.1.

Summarizing the computation in appendix B.1, the on-shell renormalization conditions on the renormalized inverse propagator are given by

$$B_{in}(m_n^2) = m_n A_{in}(m_n^2) \;\; \forall\, i = 1, \ldots, N; \tag{28a}$$

$$B_{nj}(m_n^2) = m_n A_{nj}(m_n^2) \;\; \forall\, j = 1, \ldots, N; \tag{28b}$$

$$A_{nn}(m_n^2) + 2m_n^2 \left.\frac{\mathrm{d}A_{nn}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} - 2m_n \left.\frac{\mathrm{d}B_{nn}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} = 1. \tag{28c}$$

They are in agreement with the conditions derived in [2]. Relations (28a) and (28b) follow from the cancellation of the terms with $1/\varepsilon$, whereas relation (28c) stems from the terms with zeroth power in ε.

3.2 Renormalization and parameter counting

As in the scalar case, we can make use of a symmetry of the dispersive part of the propagator, namely

$$\gamma_0 \left(S^{-1}(p)\right)^\dagger_{\text{disp}} \gamma_0 = S^{-1}(p)_{\text{disp}}, \tag{29}$$

which shows us that

$$A^\dagger = A, \quad B^\dagger = B \tag{30}$$

is valid for the dispersive parts. A derivation of equation (29) is given in appendix A.2. Therefore, the on-shell conditions can be rewritten as

$$i \neq j: \qquad B_{ij}(m_j^2) = m_j A_{ij}(m_j^2), \tag{31a}$$

$$i = j: \qquad B_{ii}(m_i^2) = m_i A_{ii}(m_i^2), \tag{31b}$$

$$A_{ii}(m_i^2) + 2m_i^2 \left.\frac{\mathrm{d}A_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_i^2} - 2m_i \left.\frac{\mathrm{d}B_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_i^2} = 1. \tag{31c}$$

For each pair $i \neq j$ we have four conditions, because relation (31a) is complex and contains $i < j$ and $i > j$. For every $i = j$ there are two real conditions, one from equation (31b) and one from equation (31c). Thus, the number of on-shell conditions amounts to

$$4 \binom{N}{2} + 2N = 2N^2. \tag{32}$$

In analogy to the scalar case, we write

$$S^{-1}(p) = \not{p} - \hat{m} - \Sigma^{(r)}(p). \tag{33}$$

We introduce the field strength renormalization via

$$\psi_i^{(b)} = \left(Z^{(1/2)}\right)_{ik} \psi_k \tag{34}$$

with bare and renormalized fields $\psi_i^{(b)}$ and ψ_k, respectively. Then, the renormalized self-energy $\Sigma^{(r)}(p)$ is written as

$$\Sigma^{(r)}(p) = \Sigma(p) + \left(\mathbb{1} - \left(Z^{(1/2)}\right)^\dagger Z^{(1/2)}\right)\not{p} + \left(Z^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z^{(1/2)} - \hat{m} \tag{35}$$

with the unrenormalized self-energy $\Sigma(p)$. From this equation it is obvious that the transformation

$$Z^{(1/2)} \to e^{i\hat{\alpha}} Z^{(1/2)} \tag{36}$$

by a diagonal matrix $e^{i\hat{\alpha}}$ of phase factors leaves $\Sigma^{(r)}(p)$ invariant. Though $Z^{(1/2)}$ is a general complex $N \times N$ matrix, we only have $2N^2 - N$ parameters in this matrix at our disposal. Adding to this number the N parameters in $\delta\hat{m}$, we arrive at $2N^2$ renormalization parameters, which equals the number of renormalization conditions in equation (32).

To perform the renormalization, we decompose the unrenormalized self-energy into

$$\Sigma(p) = \Sigma^{(A)}(p^2)\not{p} + \Sigma^{(B)}(p^2). \tag{37}$$

This leads to

$$A = \left(Z^{(1/2)}\right)^\dagger Z^{(1/2)} - \Sigma^{(A)} \quad \text{and} \quad B = \Sigma^{(B)} + \left(Z^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z^{(1/2)}. \tag{38}$$

Let us—just as in the scalar case—do the one-loop renormalization to illustrate the above discussion. Again we set $Z^{(1/2)} = \mathbb{1} + \frac{1}{2}\, z$ and obtain

$$A = \mathbb{1} - \Sigma^{(A)} + \frac{1}{2}\left(z + z^\dagger\right), \tag{39a}$$

$$B = \hat{m} + \Sigma^{(B)} + \frac{1}{2}\left(\hat{m} z + z^\dagger \hat{m}\right) + \delta\hat{m}. \tag{39b}$$

We assume a diagonal mass matrix

$$\hat{m} = \operatorname{diag}(m_1, \ldots, m_N). \tag{40}$$

Inserting these quantities A, B and $\delta\hat{m}$ into equation (31), it is straightforward to find the solution

$$\frac{1}{2} z_{ij} = -\frac{1}{m_i - m_j}\left(m_j \Sigma^{(A)}_{ij}(m_j^2) + \Sigma^{(B)}_{ij}(m_j^2)\right) \tag{41}$$

for $i \neq j$ and

$$\delta m_i = -m_i \Sigma^{(A)}_{ii}(m_i^2) - \Sigma^{(B)}_{ii}(m_i^2), \tag{42a}$$

$$\operatorname{Re} z_{ii} = \Sigma^{(A)}_{ii}(m_i^2) + 2m_i^2 \left.\frac{\mathrm{d}\Sigma^{(A)}_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_i^2} + 2m_i \left.\frac{\mathrm{d}\Sigma^{(B)}_{ii}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_i^2} \tag{42b}$$

for $i = j$. We notice that $\operatorname{Im} z_{ii}$ is not determined as a consequence of the phase freedom expressed in equation (36).

3.3 Formal derivation of the fermionic on-shell conditions

An interesting aspect of the on-shell conditions for fermions is that they can be derived by formally considering $\not{p}$ as a variable [2] and expanding in

$$\eta \equiv \not{p} - m_n. \tag{43}$$

Then one can exactly imitate the computation for scalars. We begin with the condition

$$S_{ij} \xrightarrow{\eta \to 0} \frac{\delta_{in}\delta_{nj}}{\eta} + S^{(0)}_{ij} \tag{44}$$

for the propagator. Then with

$$T = S^{-1} \quad \text{and} \quad T_{ij} = T^{(0)}_{ij} + \eta T^{(1)}_{ij} + \mathcal{O}(\eta^2) \tag{45}$$

we obtain

$$S_{ik} T_{kj} = \frac{1}{\eta}\delta_{in} T^{(0)}_{nj} + \delta_{in} T^{(1)}_{nj} + S^{(0)}_{ik} T^{(0)}_{kj} + \mathcal{O}(\eta) = \delta_{ij}, \tag{46a}$$

$$T_{ik}S_{kj} = \frac{1}{\eta}T^{(0)}_{in}\delta_{nj} + T^{(1)}_{in}\delta_{nj} + T^{(0)}_{ik}S^{(0)}_{kj} + \mathcal{O}(\eta) = \delta_{ij}. \tag{46b}$$

In this way we arrive at conditions completely analogous to the scalar conditions:

$$T^{(0)}_{in} = 0 \;\; \forall i = 1,\ldots,N, \quad T^{(0)}_{nj} = 0 \;\; \forall j = 1,\ldots,N, \quad T^{(1)}_{nn} = 1. \tag{47}$$

But we know that the inverse propagator has the decomposition

$$T_{ij}(\not{p}) = A_{ij}(p^2)\not{p} - B_{ij}(p^2), \tag{48}$$

where $p^2 = \not{p}^2$. Therefore, equation (47) translates into

$$T_{in}(\not{p} = m_n) = A_{in}(m_n^2)m_n - B_{in}(m_n^2) = 0, \tag{49a}$$
$$T_{nj}(\not{p} = m_n) = A_{nj}(m_n^2)m_n - B_{nj}(m_n^2) = 0. \tag{49b}$$

In order to tackle $T^{(1)}_{nn} = 1$, we note that

$$\frac{\mathrm{d}p^2}{\mathrm{d}\not{p}} = 2\not{p}. \tag{50}$$

Therefore, we end up with

$$\left.\frac{\mathrm{d}T_{nn}(\not{p})}{\mathrm{d}\not{p}}\right|_{\not{p}=m_n} = A_{nn}(m_n^2) + 2m_n^2\left.\frac{\mathrm{d}A_{nn}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} - 2m_n\left.\frac{\mathrm{d}B_{nn}(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} = 1. \tag{51}$$

Thus we recover relation (28c).

4 Fermion propagator without parity conservation

4.1 On-shell conditions

With the chiral projectors

$$\gamma_L = \frac{\mathbb{1} - \gamma_5}{2}, \quad \gamma_R = \frac{\mathbb{1} + \gamma_5}{2}, \tag{52}$$

the propagator is now given by

$$S = \not{p}\left(C_L\gamma_L + C_R\gamma_R\right) - \left(D_L\gamma_L + D_R\gamma_R\right). \tag{53}$$

The on-shell condition (21) is again valid. Since

$$\left(\not{p} - m_n\right)S_{ij} = \varepsilon\left(C_L\gamma_L + C_R\gamma_R\right)_{ij} - \left(\not{p} - m_n\right)\left[\left(D_L + m_nC_L\right)\gamma_L + \left(D_R + m_nC_R\right)\gamma_R\right]_{ij}, \tag{54}$$

we have the following behaviour for $\varepsilon \to 0$:

$$(C_L)_{ij} = \frac{\delta_{in}\delta_{nj}}{\varepsilon} + (C^{(0)}_L)_{ij} + \mathcal{O}(\varepsilon), \qquad (D_L)_{ij} = -\frac{m_n\delta_{in}\delta_{nj}}{\varepsilon} + (D^{(0)}_L)_{ij} + \mathcal{O}(\varepsilon), \tag{55a}$$

$$(C_R)_{ij} = \frac{\delta_{in}\delta_{nj}}{\varepsilon} + (C_R^{(0)})_{ij} + \mathcal{O}(\varepsilon), \qquad (D_R)_{ij} = -\frac{m_n\delta_{in}\delta_{nj}}{\varepsilon} + (D_R^{(0)})_{ij} + \mathcal{O}(\varepsilon). \tag{55b}$$

We define the inverse propagator as

$$S^{-1} = \not{p}\,(A_L\gamma_L + A_R\gamma_R) - (B_L\gamma_L + B_R\gamma_R)\,. \tag{56}$$

Since the set of matrices

$$\{\gamma_L,\, \gamma_R,\, \not{p}\gamma_L,\, \not{p}\gamma_R\} \tag{57}$$

is linearly independent for $p \neq 0$, we obtain the relations

$$\left(SS^{-1}\right)_{ij} = \delta_{ij} \quad\Rightarrow\quad \left(C_RA_L\,p^2 + D_LB_L\right)_{ij} = \delta_{ij}, \quad (C_LB_L + D_RA_L)_{ij} = 0, \tag{58a}$$

$$\left(C_LA_R\,p^2 + D_RB_R\right)_{ij} = \delta_{ij}, \quad (C_RB_R + D_LA_R)_{ij} = 0, \tag{58b}$$

$$\left(S^{-1}S\right)_{ij} = \delta_{ij} \quad\Rightarrow\quad \left(A_RC_L\,p^2 + B_LD_L\right)_{ij} = \delta_{ij}, \quad (B_RC_L + A_LD_L)_{ij} = 0, \tag{58c}$$

$$\left(A_LC_R\,p^2 + B_RD_R\right)_{ij} = \delta_{ij}, \quad (B_LC_R + A_RD_R)_{ij} = 0. \tag{58d}$$

Because the inverse propagator is non-singular in ε, we are allowed to assume the expansion

$$(A_L)_{ij} = (A_L^{(0)})_{ij} + \varepsilon(A_L^{(1)})_{ij} + \mathcal{O}(\varepsilon^2), \qquad (B_L)_{ij} = (B_L^{(0)})_{ij} + \varepsilon(B_L^{(1)})_{ij} + \mathcal{O}(\varepsilon^2), \tag{59a}$$

$$(A_R)_{ij} = (A_R^{(0)})_{ij} + \varepsilon(A_R^{(1)})_{ij} + \mathcal{O}(\varepsilon^2), \qquad (B_R)_{ij} = (B_R^{(0)})_{ij} + \varepsilon(B_R^{(1)})_{ij} + \mathcal{O}(\varepsilon^2). \tag{59b}$$

For the details of evaluating equations (55), (58) and (59), we refer the reader to appendix B.2.

In summary, the on-shell conditions on the inverse propagator are given by equations (B9) and (B13). The first equation reads

$$(B_L)_{in}(m_n^2) = m_n(A_R)_{in}(m_n^2), \quad (B_R)_{in}(m_n^2) = m_n(A_L)_{in}(m_n^2) \qquad \forall\, i, \tag{60a}$$

$$(B_L)_{nj}(m_n^2) = m_n(A_L)_{nj}(m_n^2), \quad (B_R)_{nj}(m_n^2) = m_n(A_R)_{nj}(m_n^2) \qquad \forall\, j. \tag{60b}$$

Equation (B13) can be brought into the form

$$\begin{aligned}(A_L)_{nn}(m_n^2) + m_n^2\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((A_L)_{nn}(p^2) + (A_R)_{nn}(p^2)\right)\Big|_{p^2=m_n^2}& \\ -m_n\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((B_L)_{nn}(p^2) + (B_R)_{nn}(p^2)\right)\Big|_{p^2=m_n^2} &= 1,\end{aligned} \tag{61a}$$

$$\begin{aligned}(A_R)_{nn}(m_n^2) + m_n^2\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((A_L)_{nn}(p^2) + (A_R)_{nn}(p^2)\right)\Big|_{p^2=m_n^2}& \\ -m_n\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((B_L)_{nn}(p^2) + (B_R)_{nn}(p^2)\right)\Big|_{p^2=m_n^2} &= 1.\end{aligned} \tag{61b}$$

Note that in these two equations the parts with the derivatives are identical, therefore, we are lead to the consistency condition

$$(A_L)_{nn}(m_n^2) = (A_R)_{nn}(m_n^2). \tag{62}$$

Plugging this into equation (60), the further consistency condition

$$(B_L)_{nn}(m_n^2) = (B_R)_{nn}(m_n^2) \tag{63}$$

follows.

4.2 Renormalization and parameter counting

Turning to the counting of the number of renormalization conditions, we note that the relations analogous to the ones of equation (30) are now

$$A_L^\dagger = A_L, \quad A_R^\dagger = A_R, \quad B_L^\dagger = B_R. \tag{64}$$

These relations cut the number of independent relations in equation (60) in half and it suffices to consider only equation (60a). Written in indices i, j, this equation reads

$$(B_L)_{ij}(m_j^2) = m_j (A_R)_{ij}(m_j^2), \quad (B_R)_{ij}(m_j^2) = m_j (A_L)_{ij}(m_j^2). \tag{65}$$

In counting the number of independent on-shell conditions we proceed as in section 3.2. We first consider $i \neq j$. The relations in equation (65) are not symmetric under the exchange $i \leftrightarrow j$. Furthermore, each of the two independent relations in equation (65) delivers one complex condition. Therefore, each pair of indices gives eight real conditions. For $i = j$, the problem is more subtle. We know from equation (64) that $\mathrm{Im}\,(A_L)_{ii} = \mathrm{Im}\,(A_R)_{ii} = 0$ and $(B_R)_{ii}(m_i^2) = ((B_L)_{ii}(m_i^2))^*$. Clearly, these are no renormalization conditions. Discounting them, there remain the three real renormalization conditions

$$(A_L)_{ii}(m_i^2) = (A_R)_{ii}(m_i^2), \quad \mathrm{Im}\,(B_L)_{ii}(m_i^2) = 0, \quad \mathrm{Re}\,(B_L)_{ii}(m_i^2) = m_i (A_R)_{ii}(m_i^2) \tag{66}$$

plus one condition containing the derivatives. Actually, only the second and third relation in equation (66) follow from equation (65) with $i = j$, whereas the first one ensues from that equation only for $m_i \neq 0$. However, the first one also derives from equation (61) without the necessity of assuming $m_i \neq 0$ and, because of this, the same equation gives only one real condition containing the derivatives. Thus we end up with a total of

$$8 \binom{N}{2} + 4N = 4N^2 \tag{67}$$

real on-shell conditions.

Next we switch to renormalization. Equation (33) holds again, but now we have to introduce a field strength renormalization matrix for each set of chiral fields, *i.e.*

$$\psi_{iL}^{(b)} = \left(Z_L^{(1/2)}\right)_{ik} \psi_{Lk}, \quad \psi_{iR}^{(b)} = \left(Z_R^{(1/2)}\right)_{ik} \psi_{Rk}. \tag{68}$$

Then, the renormalized self-energy $\Sigma^{(r)}(p)$ is written as

$$\begin{aligned} \Sigma^{(r)}(p) &= \Sigma(p) + \left(\mathbb{1} - \left(Z_L^{(1/2)}\right)^\dagger Z_L^{(1/2)}\right) \not{p}\gamma_L + \left(\mathbb{1} - \left(Z_R^{(1/2)}\right)^\dagger Z_R^{(1/2)}\right) \not{p}\gamma_R \\ &\quad + \left(Z_R^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z_L^{(1/2)} \gamma_L + \left(Z_L^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z_R^{(1/2)} \gamma_R - \hat{m} \end{aligned} \tag{69}$$

with the unrenormalized self-energy $\Sigma(p)$. In order to satisfy the $4N^2$ renormalization conditions, we have $Z_L^{(1/2)}$, $Z_R^{(1/2)}$ and $\delta\hat{m}$ at our disposal. Just as in the case of parity conservation, we have to take the phase freedom

$$Z_L^{(1/2)} \to e^{i\hat{\alpha}} Z_L^{(1/2)}, \quad Z_R^{(1/2)} \to e^{i\hat{\alpha}} Z_R^{(1/2)} \tag{70}$$

into account, because $\Sigma^{(r)}(p)$ is invariant under this rephasing. Note that both field strength renormalization matrices are multiplied with the *same* diagonal matrix of phase factors. Each field strength renormalization matrix is a general complex matrix and thus has $2N^2$ real parameters. Taking into account the N parameters in $\delta\hat{m}$, we end up with total number of $(4N^2 - N) + N = 4N^2$ real parameters, which we have at our disposal for renormalization, which exactly matches the number of independent real renormalization conditions in equation (67).

For renormalization, we decompose the bare self-energy into

$$\Sigma(p) = \not{p}\left(\Sigma_L^{(A)}(p^2)\gamma_L + \Sigma_R^{(A)}(p^2)\gamma_R\right) + \Sigma^{(B)}(p^2)\gamma_L + \Sigma_R^{(B)}(p^2)\gamma_R, \tag{71}$$

which in turn gives

$$A_L = \left(Z_L^{(1/2)}\right)^\dagger Z_L^{(1/2)} - \Sigma_L^{(A)}, \quad B_L = \Sigma_L^{(B)} + \left(Z_R^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z_L^{(1/2)}, \tag{72a}$$

$$A_R = \left(Z_R^{(1/2)}\right)^\dagger Z_R^{(1/2)} - \Sigma_R^{(A)}, \quad B_R = \Sigma_R^{(B)} + \left(Z_L^{(1/2)}\right)^\dagger (\hat{m} + \delta\hat{m})\, Z_R^{(1/2)}. \tag{72b}$$

Note that

$$\left(\Sigma_L^{(A)}\right)^\dagger = \Sigma_L^{(A)}, \quad \left(\Sigma_R^{(A)}\right)^\dagger = \Sigma_R^{(A)}, \quad \left(\Sigma_L^{(B)}\right)^\dagger = \Sigma_R^{(B)} \tag{73}$$

due to the discussion in appendix A.

Let us now perform explicit renormalization at the lowest non-trivial order. To this end, we assume that $\hat{m}$ is diagonal, with the pole masses on its diagonal. Then $\delta\hat{m}$ is diagonal too. We write $Z_L^{(1/2)} = \mathbb{1} + \frac{1}{2}z_L$ and $Z_R^{(1/2)} = \mathbb{1} + \frac{1}{2}z_R$ and obtain

$$A_L = \mathbb{1} - \Sigma_L^{(A)} + \frac{1}{2}z_L + \frac{1}{2}z_L^\dagger, \tag{74a}$$

$$A_R = \mathbb{1} - \Sigma_R^{(A)} + \frac{1}{2}z_R + \frac{1}{2}z_R^\dagger, \tag{74b}$$

and

$$B_L = \hat{m} + \delta\hat{m} + \Sigma_L^{(B)} + \frac{1}{2}\hat{m}z_L + \frac{1}{2}z_R^\dagger\hat{m}, \tag{75a}$$

$$B_R = \hat{m} + \delta\hat{m} + \Sigma_R^{(B)} + \frac{1}{2}\hat{m}z_R + \frac{1}{2}z_L^\dagger\hat{m}. \tag{75b}$$

With this identification, it is straightforward to compute $(z_L)_{ij}$ and $(z_R)_{ij}$ for $i \neq j$ from equation (60a). The result is (see, for instance, [4, 12])

$$\frac{1}{2}(z_L)_{ij} = \tag{76a}$$
$$-\frac{1}{m_i^2 - m_j^2}\left[m_j^2\left(\Sigma_L^{(A)}\right)_{ij} + m_i m_j\left(\Sigma_R^{(A)}\right)_{ij} + m_j\left(\Sigma_R^{(B)}\right)_{ij} + m_i\left(\Sigma_L^{(B)}\right)_{ij}\right]_{p^2=m_j^2},$$

$$\frac{1}{2}(z_R)_{ij} = \tag{76b}$$
$$-\frac{1}{m_i^2 - m_j^2}\left[m_i m_j\left(\Sigma_L^{(A)}\right)_{ij} + m_j^2\left(\Sigma_R^{(A)}\right)_{ij} + m_i\left(\Sigma_R^{(B)}\right)_{ij} + m_j\left(\Sigma_L^{(B)}\right)_{ij}\right]_{p^2=m_j^2}.$$

For $i = j$, we have the three conditions in equation (66) plus one of the conditions of equation (61), *i.e.* there are four equations for five unknowns, which are $\mathrm{Re}\,(z_L)_{ii}$, $\mathrm{Re}\,(z_R)_{ii}$, $\mathrm{Im}\,(z_L)_{ii}$, $\mathrm{Im}\,(z_R)_{ii}$ and δm_i. This reflects the rephasing invariance of equation (70). To solve for δm_i, the conditions in equation (66) are sufficient, leading to the result

$$2\,\delta\hat{m}_{ii} = -m_i\left((\Sigma_L^{(A)})_{ii}(m_i^2) + (\Sigma_R^{(A)})_{ii}(m_i^2)\right) - \left((\Sigma_L^{(B)})_{ii}(m_i^2) + (\Sigma_R^{(B)})_{ii}(m_i^2)\right). \tag{77}$$

Note that, due to equation (73), the second expression on the right-hand side is real. The remaining task is the determination of the diagonal entries of the field strength renormalization constants. The real parts can be completely fixed:

$$\begin{aligned}\mathrm{Re}(z_L)_{ii} = (\Sigma_L^{(A)})_{ii}(m_i^2) &+ m_i^2\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((\Sigma_L^{(A)})_{ii}(p^2) + (\Sigma_R^{(A)})_{ii}(p^2)\right)\Bigg|_{p^2=m_i^2}\\ &+ m_i\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((\Sigma_L^{(B)})_{ii}(p^2) + (\Sigma_R^{(B)})_{ii}(p^2)\right)\Bigg|_{p^2=m_i^2},\end{aligned} \tag{78a}$$

$$\begin{aligned}\mathrm{Re}(z_R)_{ii} = (\Sigma_R^{(A)})_{ii}(m_i^2) &+ m_i^2\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((\Sigma_L^{(A)})_{ii}(p^2) + (\Sigma_R^{(A)})_{ii}(p^2)\right)\Bigg|_{p^2=m_n^2}\\ &+ m_i\,\frac{\mathrm{d}}{\mathrm{d}p^2}\left((\Sigma_L^{(B)})_{ii}(p^2) + (\Sigma_R^{(B)})_{ii}(p^2)\right)\Bigg|_{p^2=m_i^2}.\end{aligned} \tag{78b}$$

Only $\mathrm{Im}\,(B_L)_{ii}(m_i^2) = 0$ in equation (66) involves a non-trivial imaginary part, from which we deduce [12]

$$m_i\,\mathrm{Im}\,(z_L - z_R)_{ii} = -2\,\mathrm{Im}\,(\Sigma_L^{(B)})_{ii}(m_i^2) = 2\,\mathrm{Im}\,(\Sigma_R^{(B)})_{ii}(m_i^2). \tag{79}$$

The latter equality results from equation (73).

4.3 Comparison with Aoki *et al.*

In [2], the inverse propagator is parameterized as

$$S^{-1} = K_1\mathbb{1} + K_5\gamma_5 + K_\gamma\not{p} + K_{5\gamma}\not{p}\gamma_5. \tag{80}$$

Comparison with equation (56) leads to the translation table

$$\begin{aligned} K_1 &= -\tfrac{1}{2}\,(B_L + B_R)\,,\\ K_5 &= \tfrac{1}{2}\,(B_L - B_R)\,,\\ K_\gamma &= \tfrac{1}{2}\,(A_L + A_R)\,,\\ K_{5\gamma} &= -\tfrac{1}{2}\,(A_L - A_R)\,. \end{aligned} \tag{81}$$

Inverting it, we have

$$\begin{aligned} A_L &= K_\gamma - K_{5\gamma},\\ A_R &= K_\gamma + K_{5\gamma},\\ B_L &= -K_1 + K_5,\\ B_R &= -K_1 - K_5. \end{aligned} \tag{82}$$

With this information, equation (60) is translated into

$$\begin{array}{ll}(K_1)_{in}(m_n^2) + m_n(K_\gamma)_{in}(m_n^2) = 0, & (K_5)_{in}(m_n^2) - m_n(K_{5\gamma})_{in}(m_n^2) = 0 \quad \forall i, \\ (K_1)_{nj}(m_n^2) + m_n(K_\gamma)_{nj}(m_n^2) = 0, & (K_5)_{nj}(m_n^2) + m_n(K_{5\gamma})_{nj}(m_n^2) = 0 \quad \forall j.\end{array} \tag{83}$$

For $i = j = n$, we find

$$(K_1)_{nn}(m_n^2) + m_n(K_\gamma)_{nn}(m_n^2) = 0, \quad (K_5)_{nn}(m_n^2) = 0, \quad (K_{5\gamma})_{nn}(m_n^2) = 0, \tag{84a}$$

$$(K_\gamma)_{nn}(m_n^2) + 2m_n^2 \left.\frac{\mathrm{d}K_\gamma(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} + 2m_n \left.\frac{\mathrm{d}K_1(p^2)}{\mathrm{d}p^2}\right|_{p^2=m_n^2} = 1. \tag{84b}$$

Therefore, we have full agreement with the on-shell conditions in [2].

4.4 On-shell renormalization of Majorana fermions

In the case of Majorana fermions, the propagator matrix is severely restricted because each fermion field is its own charge-conjugate field—see appendix C. For instance, the bare fields can represented as

$$\psi_i^{(b)} = \psi_{Li}^{(b)} + \left(\psi_{Li}^{(b)}\right)^c, \tag{85}$$

where the superscript c denotes charge conjugation. This formula expresses the fact that for each field only one chiral component is independent. Therefore,

$$Z_R^{(1/2)} = \left(Z_L^{(1/2)}\right)^* \tag{86}$$

holds [5]. The freedom of rephasing expressed by equation (70) is lost, except for those indices i for which $m_i + \delta m_i = 0$. However, in the following discussion, we will exclude such cases. Therefore, real parameters we have at disposal for renormalization are those in $Z_L^{(1/2)}$ and $\delta\hat{m}$. This amounts to $2N^2 + N$ parameters.

The Majorana condition

$$S^{-1}(p) = C\left(S^{-1}(-p)\right)^T C^{-1} \tag{87}$$

on the propagator matrix is derived in appendix C. Applying this condition to the inverse propagator as parameterized in equation (56), one readily finds

$$A_L^T = A_R, \quad B_L^T = B_L, \quad B_R^T = B_R. \tag{88}$$

These conditions hold, in the Majorana case, in addition to those of equation (64). Therefore, in summary we have here

$$A_R = A_L^* \text{ with } A_L^\dagger = A_L \quad \text{and} \quad B_R = B_L^* \text{ with } B_L^T = B_L. \tag{89}$$

It is important not to confuse these relations with renormalization conditions.

In order to count the number of the latter ones in the case of Majorana fermions, we first consider $i \neq j$. As discussed in section 4.2, it suffices to consider equation (65).

However, using equation (89) to eliminate B_R and A_L in the second relation of this equation, we find that in the Majorana case the two relations are equivalent. Thus, for $i \neq j$ there are $4\binom{N}{2}$ independent real renormalization conditions. For $i = j$, we reconsider equation (66). The first relation in this equation now follows from equation (89) and must not be considered as a renormalization condition. Therefore, in summary, together with the residue condition of equation (61), we have

$$4\binom{N}{2} + 3N = 2N^2 + N \tag{90}$$

renormalization conditions, which matches the number of independent renormalization constants determined above.

Equation (89) holds for the unrenormalized self-energies as well, in particular, the relations

$$\Sigma_R^{(A)} = \left(\Sigma_L^{(A)}\right)^* \quad \text{and} \quad \Sigma_R^{(B)} = \left(\Sigma_L^{(B)}\right)^* \tag{91}$$

are valid. Using these to eliminate the self-energy parts $\Sigma_R^{(A)}$ and $\Sigma_R^{(B)}$ in the one-loop renormalization conditions of section 4.2, it is easy to see their compatibility with the Majorana condition. Indeed, one finds that equation (76b) is the complex conjugate of equation (76a) and the same is true for equations (78a) and (78b). Equation (79) now reads

$$m_i \,\mathrm{Im}\,(z_L)_{ii} = -\mathrm{Im}\,(\Sigma_L^{(B)})_{ii}(m_i^2) \tag{92}$$

because of $\mathrm{Im}\,(z_R)_{ii} = -\mathrm{Im}\,(z_L)_{ii}$. As discussed above, there is no phase freedom in the Majorana case.

5 Summary

In this review we have given a very explicit presentation of on-shell renormalization. In particular, we have taken pains to dispel any unclear point in the derivation of the on-shell renormalization conditions imposed on the inverse propagator matrix. We have extensively discussed mixing of N fields in the case of real scalar fields and fermion fields in parity-conserving and parity-violating theories. We have also distinguished between Dirac and Majorana fermions and described how the renormalization scheme gets modified in the latter case as compared to the more familiar Dirac case. We have not treated here mixing of complex scalar fields, but with the methods explained in this review this should pose no problem. Moreover, we have omitted mixing of vector fields, in order to avoid the complications of gauge theories—for photon–Z boson mixing we refer the reader to [8, 9], for instance. The main motivation for this review originates from extensions of the Standard Model in the fermion and scalar sector, with special focus on Majorana neutrinos. For self-consistency and because of the important role in our discussion, we have also supplied a derivation of the emergence of dispersive and absorptive parts in the propagator matrix and a derivation of the restrictions on the propagator matrix in the case of Majorana fermions.

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A Dispersive and absorptive parts in the propagator

In this review we stick to hermitian counterterms in the Lagrangian. In the present appendix, for the sake of completeness, we want to demonstrate that the dispersive parts of the propagator matrix fulfill hermiticity conditions matching those of the counterterms. This is a necessary prerequisite for imposing on-shell conditions. Dispersive and absorptive parts arise through the well-known relation in distribution theory

$$\frac{1}{p^2-\mu^2+i\epsilon} = \mathrm{P}\frac{1}{p^2-\mu^2} - i\pi\delta(p^2-\mu^2), \tag{A1}$$

from the principle value and the delta function, respectively. This can be verified quite easily in the Källén–Lehmann representation of the propagator matrix. By and large we follow the derivation in [13].

A.1 Real scalar fields

The total propagator matrix is defined by

$$i\left(\Delta'(x-y)\right)_{ij} = \langle 0|\mathrm{T}\varphi_i(x)\varphi_j(y)|0\rangle, \tag{A2}$$

where T denotes time ordering. The propagator Δ' could be renormalized or unrenormalized. Inserting a complete system of four-momentum eigenstates and exploiting energy momentum conservation leads to

$$\begin{aligned} i\left(\Delta'(x-y)\right)_{ij} \;=\; & \sum_n \big\{\Theta(x^0-y^0)\,\langle 0|\varphi_i(0)|n\rangle\langle n|\varphi_j(0)|0\rangle e^{-ip_n\cdot(x-y)} \\ & +\Theta(y^0-x^0)\,\langle 0|\varphi_j(0)|n\rangle\langle n|\varphi_i(0)|0\rangle e^{ip_n\cdot(x-y)}\big\}\,. \end{aligned} \tag{A3}$$

It is then useful to define the density

$$(2\pi)^3\sum_n \delta^{(4)}(q-p_n)\langle 0|\varphi_i(0)|n\rangle\langle n|\varphi_j(0)|0\rangle \equiv \rho_{ij}(q^2)\Theta(q^0), \tag{A4}$$

where the Heaviside function Θ indicates that only $q^0 \geq 0$ gives a contribution.

Next we invoke CPT invariance, which holds in any local, Lorentz-invariant field theory. The transformation of the real scalar fields is given by

$$(\mathcal{CPT})\varphi_i(x)(\mathcal{CPT})^{-1} = \varphi_i(-x). \tag{A5}$$

Since $(\mathcal{CPT})$ is an antiunitary operator and $(\mathcal{CPT})|0\rangle = |0\rangle$, we have the relation

$$\langle 0|\varphi_i(0)|n\rangle = \big(\langle 0|\varphi_i(0)(\mathcal{CPT})|n\rangle\big)^*. \tag{A6}$$

Clearly, if $|n\rangle$ is a complete system, then $(\mathcal{CPT})|n\rangle$ is a complete system as well. Application of equation (A6) to $\rho_{ij}(q^2)$ gives the two relations

$$\rho_{ij}(q^2) = \left(\rho_{ij}(q^2)\right)^* = \rho_{ji}(q^2). \tag{A7}$$

The second one allows us to write the propagator matrix as

$$i\Delta'(x-y) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\,\Theta(q^0)\rho(q^2)\left(\Theta(x^0-y^0)e^{-iq\cdot(x-y)} + \Theta(y^0-x^0)e^{iq\cdot(x-y)}\right). \tag{A8}$$

By insertion of $1 = \int_0^\infty \mathrm{d}\mu^2\,\delta(q^2-\mu^2)$ one ends up with the Källén–Lehmann representation

$$i\Delta'(x-y) = i\int_0^\infty \mathrm{d}\mu^2\rho(\mu^2)\Delta_F(x-y;\mu), \tag{A9}$$

where

$$\Delta_F(x-y;\mu) = -i\int \frac{\mathrm{d}^3q}{(2\pi)^3 2\sqrt{\mu^2+\vec{q}^2}}\left(\Theta(x^0-y^0)e^{-iq\cdot(x-y)} + \Theta(y^0-x^0)e^{iq\cdot(x-y)}\right) \tag{A10}$$

is the free Feynman propagator of a single real scalar field with mass μ and $q^2 = \mu^2$. In momentum space, this formula reads

$$\Delta'(p^2) = \int_0^\infty \mathrm{d}\mu^2\rho(\mu^2)\frac{1}{p^2-\mu^2+i\epsilon}. \tag{A11}$$

If we identify $\Delta'(p^2)$ with the renormalized Feynman propagator, we see that $\Delta_{ij}(p^2)$ is real and symmetric in the region of p^2 where the absorptive part vanishes. This justifies the assumption made in section 2.2.

A.2 Fermion fields

In the fermionic case we have

$$\begin{aligned} i\left(S'(x-y)\right)_{ia,jb} &= \langle 0|\mathrm{T}\psi_{ia}(x)\bar{\psi}_{jb}(y)|0\rangle \qquad\qquad (A12)\\ &\equiv \Theta(x^0-y^0)\langle 0|\psi_{ia}(x)\bar{\psi}_{jb}(y)|0\rangle - \Theta(y^0-x^0)\langle 0|\bar{\psi}_{jb}(y)\psi_{ia}(x)|0\rangle, \end{aligned}$$

where for the sake of clarity we have also introduced Dirac indices a, b for each field. With the first term on the right-hand side of equation (A12) one can, just as in the scalar case, define a density:

$$(2\pi)^3\sum_n \delta^{(4)}(q-p_n)\langle 0|\psi_{ia}(0)|n\rangle\langle n|\bar{\psi}_{jb}(0)|0\rangle \equiv \rho_{ia,jb}(q)\Theta(q^0). \tag{A13}$$

Lorentz-invariance permits the decomposition

$$\rho(q) = \not{q}\left(c_L(q^2)\gamma_L + c_R(q^2)\gamma_R\right) + d_L(q^2)\gamma_L + d_R(q^2)\gamma_R. \tag{A14}$$

In this formula, $c_{L,R}$, $d_{L,R}$ are $N\times N$ matrices in family space, while $\not{q}\gamma_{L,R}$, $\gamma_{L,R}$ carry the Dirac indices.

In order to relate the second term on the right-hand side of equation (A12) to the density ρ, CPT-invariance has to be invoked [13]. To be as explicit as possible, we present all our conventions. We use the defining relation

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{A15}$$

for the charge conjugation matrix, whence $C^T = -C$ ensues. As for the Dirac gamma matrices, we assume the hermiticity conditions $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$ for $i = 1, 2, 3$. In this basis, $C^\dagger = C^{-1}$ without loss of generality. Charge conjugation, parity and time reversal transformation for fermion fields are formulated as

$$\mathcal{C}\psi_i(x)\mathcal{C}^{-1} = C\gamma_0^T\psi_i^*(x), \;\; \mathcal{P}\psi_i(x)\mathcal{P}^{-1} = \gamma_0\psi_i(x^0, -\vec{x}\,), \;\; \mathcal{T}\psi_i(x)\mathcal{T}^{-1} = iC^{-1}\gamma_5\psi_i(-x^0, \vec{x}\,), \tag{A16}$$

respectively. Combination of the three discrete transformations gives the CPT transformation property [13, 14]

$$(\mathcal{CPT})\psi_i(x)(\mathcal{CPT})^{-1} = -i\gamma_5^T\psi_i^*(-x) \quad \text{and} \quad (\mathcal{CPT})\bar{\psi}_i(x)(\mathcal{CPT})^{-1} = i\psi_i^T(-x)(\gamma_5\gamma_0)^*, \tag{A17}$$

where the overall sign is convention, but the i is necessary in Yukawa couplings.[3]

Now we apply equation (A17) to the second term on the right-hand side of the propagator (A12). A straightforward computation, taking into account antiunitary of $(\mathcal{CPT})$ and the above conventions for C and the gamma matrices, leads to the simple result [13]

$$\langle 0|\bar{\psi}_{jb}(y)\psi_{ia}(x)|0\rangle = -\left(\gamma_5\right)_{ad}\langle 0|\psi_{id}(-x)\bar{\psi}_{jc}(-y)|0\rangle\left(\gamma_5\right)_{cb}. \tag{A18}$$

The minus sign is used to cancel the minus from the time ordering when $y^0 > x^0$. Thus, for the contribution to the propagator of equation (A18), we obtain the density

$$\gamma_5\rho(q)\gamma_5 = -\not{q}\left(c_L(q^2)\gamma_L + c_R(q^2)\gamma_R\right) + d_L(q^2)\gamma_L + d_R(q^2)\gamma_R, \tag{A19}$$

with $\rho(q)$ being identical to that of equation (A14).

The remaining steps are the same as in the scalar case. We finally arrive at

$$iS'(x-y) = i\int_0^\infty \mathrm{d}\mu^2\left[i\not{\partial}_x\left(c_L(\mu^2)\gamma_L + c_R(\mu^2)\gamma_R\right) + d_L(\mu^2)\gamma_L + d_R(\mu^2)\gamma_R\right]\Delta_F(x-y;\mu), \tag{A20}$$

where the subscript x indicates derivative with respect to x. In momentum space the result is

$$S'(p) = \int_0^\infty \mathrm{d}\mu^2\left[\not{p}\left(c_L(\mu^2)\gamma_L + c_R(\mu^2)\gamma_R\right) + d_L(\mu^2) + d_R(\mu^2)\right]\frac{1}{p^2-\mu^2+i\epsilon}. \tag{A21}$$

[3] The reason is that the transformation (A5) together with (A17) leaves the most general Yukawa coupling

$$\sum_{r,s,k}\Gamma_{rs}^k\psi_r^T C^{-1}\psi_s\,\varphi_k + \text{H.c.} \quad \text{with} \quad \Gamma_{rs}^k = \Gamma_{sr}^k\;\forall\, r,s$$

invariant; in this Yukawa coupling, all fermion fields are, for instance, left-chiral. Therefore, equations (A5) and (A17) together form a true CPT transformation.

From the definition of the density $\rho(q)$, equation (A14), the following property is easy to prove:

$$\gamma_0\rho^\dagger(q)\gamma_0 = \rho(q) \quad \Rightarrow \quad c_L^\dagger(q^2) = c_L(q^2),\ c_R^\dagger(q^2) = c_R(q^2),\ d_L^\dagger(q^2) = d_R(q^2). \tag{A22}$$

Transferring the latter relations to the dispersive part of S', we obtain

$$\gamma_0(S'(p))^\dagger_{\text{disp}}\gamma_0 = (S'(p))_{\text{disp}}. \tag{A23}$$

This is the justification of equation (29).

B Computational details

B.1 Theories with parity conservation

We first consider the second relation in each line of equation (25). The expansion in ε furnishes

$$\frac{\delta_{in}}{\varepsilon}\left(B^{(0)}_{nj} - m_n A^{(0)}_{nj}\right) + \delta_{in}\left(B^{(1)}_{nj} - m_n A^{(1)}_{nj}\right) + C^{(0)}_{ik}B^{(0)}_{kj} + D^{(0)}_{ik}A^{(0)}_{kj} + \mathcal{O}(\varepsilon) = 0, \tag{B1a}$$

$$\left(B^{(0)}_{in} - m_n A^{(0)}_{in}\right)\frac{\delta_{nj}}{\varepsilon} + \left(B^{(1)}_{in} - m_n A^{(1)}_{in}\right)\delta_{nj} + B^{(0)}_{ik}C^{(0)}_{kj} + A^{(0)}_{ik}D^{(0)}_{kj} + \mathcal{O}(\varepsilon) = 0. \tag{B1b}$$

Removing the singularity, we obtain the conditions

$$B^{(0)}_{in} = m_n A^{(0)}_{in} \ \ \forall\, i = 1,\dots,N \quad \text{and} \quad B^{(0)}_{nj} = m_n A^{(0)}_{nj} \ \ \forall\, j = 1,\dots,N. \tag{B2}$$

Next we consider the first relation in each line of equation (25). Taking into account $p^2 = \varepsilon + m_n^2$, we find

$$\begin{aligned} &-\frac{\delta_{in}m_n}{\varepsilon}\left(B^{(0)}_{nj} - m_n A^{(0)}_{nj}\right) \\ &+ \delta_{in}\left(A^{(0)}_{nj} + m_n^2 A^{(1)}_{nj} - m_n B^{(1)}_{nj}\right) + m_n^2 C^{(0)}_{ik}A^{(0)}_{kj} + D^{(0)}_{ik}B^{(0)}_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}, \end{aligned} \tag{B3a}$$

$$\begin{aligned} &-\left(B^{(0)}_{in} - m_n A^{(0)}_{in}\right)\frac{m_n\delta_{nj}}{\varepsilon} \\ &+ \left(A^{(0)}_{in} + m_n^2 A^{(1)}_{in} - m_n B^{(1)}_{in}\right)\delta_{nj} + m_n^2 A^{(0)}_{ik}C^{(0)}_{kj} + B^{(0)}_{ik}D^{(0)}_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}. \end{aligned} \tag{B3b}$$

The singularity in these relations leads again to equation (B2).

Up to now we have only considered the singular terms. In order to obtain a renormalization condition from the terms of order ε^0, we choose the indices $i = j = n$. In this case, equation (B3a) leads to

$$A^{(0)}_{nn} + m_n^2 A^{(1)}_{nn} - m_n B^{(1)}_{nn} + m_n^2 C^{(0)}_{nk}A^{(0)}_{kn} + D^{(0)}_{nk}B^{(0)}_{kn} = 1, \tag{B4a}$$

$$A^{(0)}_{nn} + m_n^2 A^{(1)}_{nn} - m_n B^{(1)}_{nn} + m_n^2 A^{(0)}_{nk}C^{(0)}_{kn} + B^{(0)}_{nk}D^{(0)}_{kn} = 1. \tag{B4b}$$

Invoking equation (B2), we rewrite this equation as

$$A^{(0)}_{nn} + m_n^2 A^{(1)}_{nn} - m_n B^{(1)}_{nn} + \left(m_n C^{(0)}_{nk} + D^{(0)}_{nk}\right)A^{(0)}_{kn}m_n = 1, \tag{B5a}$$

$$A_{nn}^{(0)} + m_n^2 A_{nn}^{(1)} - m_n B_{nn}^{(1)} + m_n A_{nk}^{(0)} \left(m_n C_{kn}^{(0)} + D_{kn}^{(0)} \right) \;=\; 1. \tag{B5b}$$

In the same way equation (B1) gives

$$B_{nn}^{(1)} - m_n A_{nn}^{(1)} + \left(m_n C_{nk}^{(0)} + D_{nk}^{(0)} \right) A_{kn}^{(0)} \;=\; 0, \tag{B6a}$$

$$B_{nn}^{(1)} - m_n A_{nn}^{(1)} + A_{nk}^{(0)} \left(m_n C_{kn}^{(0)} + D_{kn}^{(0)} \right) \;=\; 0. \tag{B6b}$$

These relations allow us to eliminate in equation (B5) the terms in parentheses. We obtain one further renormalization condition

$$A_{nn}^{(0)} + 2m_n^2 A_{nn}^{(1)} - 2m_n B_{nn}^{(1)} = 1. \tag{B7}$$

B.2 Theories without parity conservation

Inserting the expansions (55) and (59) of S and S^{-1}, respectively, into equation (58), right column, leads to

$$\begin{aligned} &\frac{\delta_{in}}{\varepsilon} \left((B_L^{(0)})_{nj} - m_n (A_L^{(0)})_{nj} \right) + \delta_{in} \left((B_L^{(1)})_{nj} - m_n (A_L^{(1)})_{nj} \right) \\ &\qquad + (C_L^{(0)})_{ik} (B_L^{(0)})_{kj} + (D_R^{(0)})_{ik} (A_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) \;=\; 0, \end{aligned} \tag{B8a}$$

$$\begin{aligned} &\frac{\delta_{in}}{\varepsilon} \left((B_R^{(0)})_{nj} - m_n (A_R^{(0)})_{nj} \right) + \delta_{in} \left((B_R^{(1)})_{nj} - m_n (A_R^{(1)})_{nj} \right) \\ &\qquad + (C_R^{(0)})_{ik} (B_R^{(0)})_{kj} + (D_L^{(0)})_{ik} (A_R^{(0)})_{kj} + \mathcal{O}(\varepsilon) \;=\; 0, \end{aligned} \tag{B8b}$$

$$\begin{aligned} &\left((B_R^{(0)})_{in} - m_n (A_L^{(0)})_{in} \right) \frac{\delta_{nj}}{\varepsilon} + \left((B_R^{(1)})_{in} - m_n (A_L^{(1)})_{in} \right) \delta_{nj} \\ &\qquad + (B_R^{(0)})_{ik} (C_L^{(0)})_{kj} + (A_L^{(0)})_{ik} (D_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) \;=\; 0, \end{aligned} \tag{B8c}$$

$$\begin{aligned} &\left((B_L^{(0)})_{in} - m_n (A_R^{(0)})_{in} \right) \frac{\delta_{nj}}{\varepsilon} + \left((B_L^{(1)})_{in} - m_n (A_R^{(1)})_{in} \right) \delta_{nj} \\ &\qquad + (B_L^{(0)})_{ik} (C_R^{(0)})_{kj} + (A_R^{(0)})_{ik} (D_R^{(0)})_{kj} + \mathcal{O}(\varepsilon) \;=\; 0. \end{aligned} \tag{B8d}$$

The poles in ε must vanish, which gives the conditions

$$(B_L^{(0)})_{in} = m_n (A_R^{(0)})_{in}, \quad (B_R^{(0)})_{in} = m_n (A_L^{(0)})_{in} \qquad \forall\, i = 1, \ldots, N, \tag{B9a}$$

$$(B_L^{(0)})_{nj} = m_n (A_L^{(0)})_{nj}, \quad (B_R^{(0)})_{nj} = m_n (A_R^{(0)})_{nj} \qquad \forall\, j = 1, \ldots, N. \tag{B9b}$$

Now we can insert the same expansions into equation (58), left column, set $p^2 = \varepsilon + m_n^2$ and arrive at

$$\begin{aligned} &-\frac{\delta_{in} m_n}{\varepsilon} \left((B_L^{(0)})_{nj} - m_n (A_L^{(0)})_{nj} \right) + \delta_{in} \left((A_L^{(0)})_{nj} + m_n^2 (A_L^{(1)})_{nj} - m_n (B_L^{(1)})_{nj} \right) \\ &+ m_n^2 (C_R^{(0)})_{ik} (A_L^{(0)})_{kj} + (D_L^{(0)})_{ik} (B_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}, \end{aligned} \tag{B10a}$$

$$\begin{aligned} &-\frac{\delta_{in} m_n}{\varepsilon} \left((B_R^{(0)})_{nj} - m_n (A_R^{(0)})_{nj} \right) + \delta_{in} \left((A_R^{(0)})_{nj} + m_n^2 (A_R^{(1)})_{nj} - m_n (B_R^{(1)})_{nj} \right) \\ &+ m_n^2 (C_L^{(0)})_{ik} (A_R^{(0)})_{kj} + (D_R^{(0)})_{ik} (B_R^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}, \end{aligned} \tag{B10b}$$

$$- \left((B_L^{(0)})_{in} - m_n (A_R^{(0)})_{in} \right) \frac{m_n \delta_{nj}}{\varepsilon} + \left((A_R^{(0)})_{in} + m_n^2 (A_R^{(1)})_{in} - m_n (B_L^{(1)})_{in} \right) \delta_{nj}$$
$$+ m_n^2 (A_R^{(0)})_{ik} (C_L^{(0)})_{kj} + (B_L^{(0)})_{ik} (D_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}, \tag{B10c}$$
$$- \left((B_R^{(0)})_{in} - m_n (A_L^{(0)})_{in} \right) \frac{m_n \delta_{nj}}{\varepsilon} + \left((A_L^{(0)})_{in} + m_n^2 (A_L^{(1)})_{in} - m_n (B_R^{(1)})_{in} \right) \delta_{nj}$$
$$+ m_n^2 (A_L^{(0)})_{ik} (C_R^{(0)})_{kj} + (B_R^{(0)})_{ik} (D_R^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}. \tag{B10d}$$

The terms with $1/\varepsilon$ lead again to equation (B9). Next, we investigate equation (B10) for $i = j = n$ at order ε^0 and make use of equation (B9). In this way we obtain

$$(A_L^{(0)})_{nn} + m_n^2 (A_L^{(1)})_{nn} - m_n (B_L^{(1)})_{nn} + m_n^2 (C_R^{(0)})_{nk} (A_L^{(0)})_{kn} + m_n (D_L^{(0)})_{nk} (A_R^{(0)})_{kn} = 1, \tag{B11a}$$
$$(A_R^{(0)})_{nn} + m_n^2 (A_R^{(1)})_{nn} - m_n (B_R^{(1)})_{nn} + m_n^2 (C_L^{(0)})_{ik} (A_R^{(0)})_{kj} + m_n (D_R^{(0)})_{nk} (A_L^{(0)})_{kn} = 1, \tag{B11b}$$
$$(A_R^{(0)})_{nn} + m_n^2 (A_R^{(1)})_{nn} - m_n (B_L^{(1)})_{nn} + m_n^2 (A_R^{(0)})_{ik} (C_L^{(0)})_{kj} + m_n (A_L^{(0)})_{nk} (D_L^{(0)})_{kn} = 1, \tag{B11c}$$
$$(A_L^{(0)})_{nn} + m_n^2 (A_L^{(1)})_{nn} - m_n (B_R^{(1)})_{nn} + m_n^2 (A_L^{(0)})_{nk} (C_R^{(0)})_{kn} + m_n (A_R^{(0)})_{nk} (D_R^{(0)})_{kn} = 1. \tag{B11d}$$

Via equation (B8) for $i = j = n$ at order ε^0, *i.e.*

$$(B_L^{(1)})_{nn} - m_n (A_L^{(1)})_{nn} + m_n (C_L^{(0)})_{nk} (A_R^{(0)})_{kn} + (D_R^{(0)})_{nk} (A_L^{(0)})_{kn} = 0, \tag{B12a}$$
$$(B_R^{(1)})_{nn} - m_n (A_R^{(1)})_{nn} + m_n (C_R^{(0)})_{nk} (A_L^{(0)})_{kn} + (D_L^{(0)})_{nk} (A_R^{(0)})_{kn} = 0, \tag{B12b}$$
$$(B_R^{(1)})_{nn} - m_n (A_L^{(1)})_{nn} + m_n (A_R^{(0)})_{nk} (C_L^{(0)})_{kn} + (A_L^{(0)})_{nk} (D_L^{(0)})_{kn} = 0, \tag{B12c}$$
$$(B_L^{(1)})_{nn} - m_n (A_R^{(1)})_{nn} + m_n (A_L^{(0)})_{nk} (C_R^{(0)})_{kn} + (A_R^{(0)})_{nk} (D_R^{(0)})_{kn} = 0, \tag{B12d}$$

we can eliminate the terms with $C_{L,R}$ and $D_{L,R}$ in equation (B11). Eventually, this leads to the final two on-shell conditions

$$(A_L^{(0)})_{nn} + m_n^2 \left((A_L^{(1)})_{nn} + (A_R^{(1)})_{nn} \right) - m_n \left((B_L^{(1)})_{nn} + (B_R^{(1)})_{nn} \right) = 1, \tag{B13a}$$
$$(A_R^{(0)})_{nn} + m_n^2 \left((A_L^{(1)})_{nn} + (A_R^{(1)})_{nn} \right) - m_n \left((B_L^{(1)})_{nn} + (B_R^{(1)})_{nn} \right) = 1. \tag{B13b}$$

C Majorana condition for the propagator matrix

In the case of Majorana fields, a condition on the propagator matrix arises from the fact that each fermion field ψ_n is identical to its charge-conjugate field $(\psi_n)^c$. Dealing with four-component spinors, this reads

$$C \gamma_0^T \psi_n^*(x) = \psi_n(x), \tag{C1}$$

where C is the charge-conjugation matrix. The star means that one has to take the hermitian conjugate of each component ψ_{na} $(a = 1, \ldots, 4)$ of ψ_n such that ψ_n^* is a column vector.

The starting point for the derivation of a propagator condition in the case of Majorana fermions is the identity

$$\langle 0|\mathrm{T}\psi_{ia}(x)\psi_{jb}(y)|0\rangle = -\langle 0|\mathrm{T}\psi_{jb}(y)\psi_{ia}(x)|0\rangle. \tag{C2}$$

With $\psi_j^T = -\bar{\psi}_j C$ and $\psi_i = C\bar{\psi}_i^T$, which follow from equation (C1), this identity is rewritten as

$$\sum_{c=1}^{4} \langle 0|\mathrm{T}\psi_{ia}(x)\bar{\psi}_{jc}(y)|0\rangle(-C_{cb}) = -\sum_{d=1}^{4} C_{ad}\, \langle 0|\mathrm{T}\psi_{jb}(y)\bar{\psi}_{id}(x)|0\rangle. \tag{C3}$$

Therefore, the Majorana condition on the propagator matrix is

$$S'(x-y) = CS'^T(y-x)C^{-1}. \tag{C4}$$

Note that the transposition refers to both Dirac and family indices. In momentum space, equation (C4) reads [2, 15]

$$S'(p) = CS'^T(-p)C^{-1}. \tag{C5}$$

There is the analogous condition for the inverse propagator.

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CHAPTER 4

Toy model renormalization

Maarten Fox, Walter Grimus and Maximilian Löschner

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Contents

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Renormalization and radiative corrections to masses in a general Yukawa model

M. Fox*, W. Grimus‡ and M. Löschner§

University of Vienna, Faculty of Physics
Boltzmanngasse 5, A–1090 Vienna, Austria

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Abstract

We consider a model with arbitrary numbers of Majorana fermion fields and real scalar fields φ_a, general Yukawa couplings and a $\mathbb{Z}_4$ symmetry that forbids linear and trilinear terms in the scalar potential. Moreover, fermions become massive only after spontaneous symmetry breaking of the $\mathbb{Z}_4$ symmetry by vacuum expectation values (VEVs) of the φ_a. Introducing the shifted fields h_a whose VEVs vanish, $\overline{\text{MS}}$ renormalization of the parameters of the unbroken theory suffices to make the theory finite. However, in this way, beyond tree level it is necessary to perform finite shifts of the tree-level VEVs, induced by the finite parts of the tadpole diagrams, in order to ensure vanishing one-point functions of the h_a. Moreover, adapting the renormalization scheme to a situation with many scalars and VEVs, we consider the *physical* fermion and scalar masses as derived quantities, *i.e.* as functions of the coupling constants and VEVs. Consequently, the masses have to be computed order by order in a perturbative expansion. In this scheme we compute the selfenergies of fermions and bosons and show how to obtain the respective one-loop contributions to the tree-level masses. Furthermore, we discuss the modification of our results in the case of Dirac fermions and investigate, by way of an example, the effects of a flavour symmetry group.

*E-mail: a0905865@unet.univie.ac.at
‡E-mail: walter.grimus@univie.ac.at
§E-mail: maximilian.loeschner@univie.ac.at

1 Introduction

Thanks to the results of the neutrino oscillation experiments—see for instance [1, 2]—it is now firmly established that at least two light neutrinos have a nonzero mass and that there is a non-trivial lepton mixing matrix or PMNS matrix in analogy to the quark mixing matrix or CKM matrix. The surprisingly large mixing angles in the PMNS matrix have given a boost to model building with spontaneously broken flavour symmetries—for recent reviews see [3]. Many interesting results have been discovered, however, no favoured scenario has emerged yet. Moreover, predictions of neutrino mass and mixing models refer frequently to tree-level computations. It would thus be desirable to check the stability of such predictions under radiative corrections. In the case of renormalizable models one has a clear-cut and consistent method to remove ultraviolet (UV) divergences and to compute such corrections.

However, there is the complication that the envisaged models always have a host of scalars and often complicated spontaneous symmetry breaking (SSB) of the flavour group. This makes it impossible to replace all Yukawa couplings by ratios of masses over vacuum expectation values (VEVs), as done for instance in the renormalization of the Standard Model. Of course, one could replace part of the Yukawa coupling constants by masses, but this would make the renormalization procedure highly asymmetric. In this paper we suggest to make such models finite by $\overline{\text{MS}}$ renormalization of the parameters of the unbroken model and to perform finite VEV shifts at the loop level in order to guarantee vanishing scalar one-point functions of the shifted scalar fields [4]. Additionally, we introduce finite field strength renormalization for obtaining on-shell selfenergies. In this way, all fermion and scalar masses are derived quantities and functions of the parameters of the model.

In the usual approach to renormalization of theories with SSB and mixing [5] one has counterterms for masses, quark and lepton mixing matrices—see for instance [6, 7]—and tadpoles—see for instance [8, 9].[1] We stress that in our approach there are no such counterterms because we use an alternative approach tailored to the situation with a proliferation of scalars and VEVs.

In order to present the renormalization scheme in a clear and compact way, we consider a toy model which has

- an arbitrary number of Majorana or Dirac fermions,
- an arbitrary number of neutral scalars,
- a $\mathbb{Z}_4$ ($\mathbb{Z}_2$) symmetry which forbids Majorana (Dirac) fermion masses before SSB[2] and
- general Yukawa interactions.

[1]There are other treatments of tadpoles adapted to the theory where they occur, see for instance reference [10] for the MSSM and [11, 12] where the issue of gauge invariance is discussed.

[2]This is motivated by the Standard Model where—before SSB—fermion masses as well as linear and trilinear terms in the scalar potential are absent due to the gauge symmetry.

We put particular emphasis on the treatment of tadpoles. Since radiative corrections in this model are already finite due to $\overline{\text{MS}}$ renormalization with the counterterms of the unbroken theory, also the sum of all tadpole contributions, *i.e.* the loop contributions and those induced by the counterterms of the unbroken theory, is finite. However, tadpoles introduce finite VEV shifts which have to be taken into account for instance in the selfenergies. Eventually, the finite VEV shifts also contribute to the radiative corrections of the tree-level masses.[3] We also focus on Majorana fermions, having in mind that neutrinos automatically obtain Majorana nature through the seesaw mechanism [13].

An attempt at a renormalization scheme—with one fermion and one scalar field—along the lines discussed here has already been made in [14]; however, the treatment of the VEV in this paper cannot be generalized to the case of more than one scalar field.

The paper is organized as follows. In section 2 we introduce the Lagrangian, define the counterterms and discuss SSB. Section 3 is devoted to the explanation of our renormalization scheme, while in section 4 we explicitly compute the selfenergies of fermions and scalars at one-loop order. We present an example of a flavour symmetry in section 5 and study how the symmetry teams up with the general renormalization scheme. In section 6 we describe the changes when one has Dirac fermions instead of Majorana fermions. Finally, in section 7 we present the conclusions. Some details which are helpful for reading the paper can be found in the three appendices.

2 Toy model setup

In this section, we give the specifics of the investigated model and discuss the generation of masses via SSB. We focus on Majorana fermions. Throughout this paper we always use the sum convention, if not otherwise stated.

2.1 Bare and renormalized Lagrangian

The bare Lagrangian is given by

$$\begin{aligned} \mathcal{L}_B \;=\;& i\bar{\chi}_{iL}^{(B)}\gamma^\mu\partial_\mu\chi_{iL}^{(B)} + \frac{1}{2}\left(\partial_\mu\varphi_a^{(B)}\right)\left(\partial^\mu\varphi_a^{(B)}\right) && \text{(1a)}\\ &+\left(\frac{1}{2}\left(Y_a^{(B)}\right)_{ij}\,{\chi_{iL}^{(B)}}^T C^{-1}\chi_{jL}^{(B)}\varphi_a^{(B)} + \text{H.c.}\right) && \text{(1b)}\\ &-\frac{1}{2}(\mu_B^2)_{ab}\varphi_a^{(B)}\varphi_b^{(B)} - \frac{1}{4}\lambda_{abcd}^{(B)}\varphi_a^{(B)}\varphi_b^{(B)}\varphi_c^{(B)}\varphi_d^{(B)}. && \text{(1c)} \end{aligned}$$

The charge-conjugation matrix C acts only on the Dirac indices. We assume n_χ chiral Majorana fermion fields $\chi_{iL}^{(B)}$ and n_φ real scalar fields $\varphi_a^{(B)}$. This Lagrangian exhibits the $\mathbb{Z}_4$ symmetry

$$\mathcal{S}:\quad \chi_L^{(B)} \to i\chi_L^{(B)}, \quad \varphi^{(B)} \to -\varphi^{(B)}, \tag{2}$$

[3] After SSB, these shifts have to be taken into account everywhere in the Lagrangian where VEVs appear in order to obtain a consistent set of counterterms.

with

$$\chi_L^{(B)} = \begin{pmatrix} \chi_{1L}^{(B)} \\ \vdots \\ \chi_{n_\chi L}^{(B)} \end{pmatrix}, \quad \varphi^{(B)} = \begin{pmatrix} \varphi_1^{(B)} \\ \vdots \\ \varphi_{n_\varphi}^{(B)} \end{pmatrix}. \tag{3}$$

Note that

$$(Y_a^{(B)})^T = Y_a^{(B)} \;\; \forall a = 1, \ldots, n_\chi, \quad (\mu_B^2)_{ab} = (\mu_B^2)_{ba} \tag{4}$$

and $\lambda_{abcd}^{(B)}$ is symmetric in all indices.[4]

We define the renormalized fields by

$$\chi_L^{(B)} = Z_\chi^{(1/2)}\, \chi_L, \quad \varphi^{(B)} = Z_\varphi^{(1/2)}\, \varphi, \tag{5}$$

where χ_L and φ are the vectors of the renormalized fermion and scalar fields, respectively. The quantity $Z_\chi^{(1/2)}$ is a general complex $n_\chi \times n_\chi$ matrix, while $Z_\varphi^{(1/2)}$ is a real but otherwise general $n_\varphi \times n_\varphi$ matrix. Since we use dimensional regularization with dimension

$$d = 4 - \varepsilon, \tag{6}$$

we also introduce an arbitrary mass parameter $\mathcal{M}$ which renders the renormalized Yukawa and quartic coupling constants dimensionless. We split the bare Lagrangian into

$$\mathcal{L}_B = \mathcal{L} + \delta\mathcal{L}, \tag{7}$$

where the renormalized Lagrangian is given by

$$\begin{aligned} \mathcal{L} \;=\; & i\bar{\chi}_{iL}\gamma^\mu \partial_\mu \chi_{iL} + \frac{1}{2}\left(\partial_\mu \varphi_a\right)\left(\partial^\mu \varphi_a\right) & \text{(8a)} \\ & + \left(\frac{1}{2}\,\mathcal{M}^{\varepsilon/2}\left(Y_a\right)_{ij}\, \chi_{iL}^T C^{-1} \chi_{jL} \varphi_a + \text{H.c.}\right) & \text{(8b)} \\ & -\frac{1}{2}\mu_{ab}^2 \varphi_a \varphi_b - \frac{1}{4}\mathcal{M}^\varepsilon \lambda_{abcd}\, \varphi_a \varphi_b \varphi_c \varphi_d & \text{(8c)} \end{aligned}$$

and

$$\begin{aligned} \delta\mathcal{L} \;=\; & i\delta_{ij}^{(\chi)}\, \bar{\chi}_{iL}\gamma^\mu \partial_\mu \chi_{jL} + \frac{1}{2}\delta_{ab}^{(\varphi)}\left(\partial_\mu \varphi_a\right)\left(\partial^\mu \varphi_b\right) & \text{(9a)} \\ & + \left(\frac{1}{2}\,\mathcal{M}^{\varepsilon/2}\left(\delta Y_a\right)_{ij}\, \chi_{iL}^T C^{-1} \chi_{jL} \varphi_a + \text{H.c.}\right) & \text{(9b)} \\ & -\frac{1}{2}\delta\mu_{ab}^2 \varphi_a \varphi_b - \frac{1}{4}\mathcal{M}^\varepsilon \delta\lambda_{abcd}\, \varphi_a \varphi_b \varphi_c \varphi_d & \text{(9c)} \end{aligned}$$

contains the counterterms. In $\delta\mathcal{L}$, the counterterms corresponding to the parameters in $\mathcal{L}$ are given by

$$\begin{aligned} \mathcal{M}^{\varepsilon/2}\delta Y_a \;&=\; \left(Z_\chi^{(1/2)}\right)^T Y_b^{(B)}\, Z_\chi^{(1/2)} \left(Z_\varphi^{(1/2)}\right)_{ba} - \mathcal{M}^{\varepsilon/2} Y_a, & \text{(10a)} \\ \mathcal{M}^\varepsilon \delta\lambda_{abcd} \;&=\; \lambda_{a'b'c'd'}^{(B)} \left(Z_\varphi^{(1/2)}\right)_{a'a} \left(Z_\varphi^{(1/2)}\right)_{b'b} \left(Z_\varphi^{(1/2)}\right)_{c'c} \left(Z_\varphi^{(1/2)}\right)_{d'd} - \mathcal{M}^\varepsilon \lambda_{abcd}, & \text{(10b)} \end{aligned}$$

[4]One can show that the number of independent elements of $\lambda_{abcd}^{(B)}$ is $\binom{n_\varphi+3}{4}$.

$$\delta\mu^2 = \left(Z_\varphi^{(1/2)}\right)^T \mu_B^2 \, Z_\varphi^{(1/2)} - \mu^2. \tag{10c}$$

Note that, whenever possible, we use matrix notation, as done in equations (10a) and (10c). Moreover we have defined

$$\delta^{(\chi)} = \left(Z_\chi^{(1/2)}\right)^\dagger Z_\chi^{(1/2)} - \mathbb{1}, \quad \delta^{(\varphi)} = \left(Z_\varphi^{(1/2)}\right)^T Z_\varphi^{(1/2)} - \mathbb{1}. \tag{11}$$

The renormalized parameters have the same symmetry properties as the unrenormalized ones, *i.e.*

$$Y_a^T = Y_a \quad \forall a = 1, \ldots, n_\chi, \quad \mu_{ab}^2 = \mu_{ba}^2 \tag{12}$$

and λ_{abcd} is symmetric in all indices. The same applies to the corresponding counterterms.

2.2 Spontaneous symmetry breaking

We introduce the shift

$$\varphi_a = \mathcal{M}^{-\varepsilon/2} \bar{v}_a + h_a \quad \text{with} \quad \bar{v}_a = v_a + \Delta v_a. \tag{13}$$

For convenience we have split the shift into v_a and Δv_a; below we will identify the v_a with the tree-level VEVs of the scalar fields φ_a, while the Δv_a indicate further finite shifts effected by loop corrections. Throughout our calculations, the symbol δ signifies UV divergent counterterms, while with the symbol Δ we denote finite shifts. A one-loop discussion of Δv_a will be presented in section 3. The shift leads to the scalar potential, including counterterms,

$$V + \delta V - V_0 = \mathcal{M}^{-\varepsilon/2} \left(t_a + \Delta t_a + \delta\mu_{ab}^2 \bar{v}_b + \delta\lambda_{abcd} \, \bar{v}_b \bar{v}_c \bar{v}_d\right) h_a \tag{14a}$$

$$+\frac{1}{2} \left(\left(M_0^2\right)_{ab} + \left(\Delta M_0^2\right)_{ab} + \delta\mu_{ab}^2 + 3\delta\lambda_{abcd} \, \bar{v}_c \bar{v}_d\right) h_a h_b \tag{14b}$$

$$+\mathcal{M}^{\varepsilon/2} \left(\lambda_{abcd} + \delta\lambda_{abcd}\right) \bar{v}_d h_a h_b h_c \tag{14c}$$

$$+\frac{1}{4} \mathcal{M}^{\varepsilon} \left(\lambda_{abcd} + \delta\lambda_{abcd}\right) h_a h_b h_c h_d, \tag{14d}$$

with V as in equation (9c),

$$t_a = \mu_{ab}^2 v_b + \lambda_{abcd} v_b v_c v_d, \quad \Delta t_a = \mu_{ab}^2 \bar{v}_b + \lambda_{abcd} \bar{v}_b \bar{v}_c \bar{v}_d - t_a, \tag{15}$$

V_0 being the constant term,

$$\left(M_0^2\right)_{ab} \equiv \mu_{ab}^2 + 3\lambda_{abcd} v_c v_d \quad \text{and} \quad \left(\Delta M_0^2\right)_{ab} \equiv \mu_{ab}^2 + 3\lambda_{abcd} \bar{v}_c \bar{v}_d - \left(M_0^2\right)_{ab}. \tag{16}$$

The quantities Δt_a and $(\Delta M_0^2)_{ab}$ will become useful when we go beyond the tree level because they will be induced by the shifts Δv_a. We will drop V_0 in the rest of the paper since it does not alter the dynamics of the theory.

From now on we choose the v_a as the tree-level vacuum expectation values (VEVs) of the scalars, *i.e.* as the values of the φ_a at the minimum of $V(\varphi)$. Taking the derivative of the scalar potential V, we obtain

$$\frac{\partial V}{\partial \varphi_a} = \mu_{ab}^2 \varphi_b + \mathcal{M}^{\varepsilon} \lambda_{abcd} \varphi_b \varphi_c \varphi_d. \tag{17}$$

Therefore, the conditions that the v_a ($a = 1, \dots, n_\varphi$) correspond to a stationary point of V are given by

$$t_a = 0 \quad \text{for} \quad a = 1, \dots, n_\varphi. \tag{18}$$

SSB occurs if the minimum $\varphi_1 = v_1, \dots, \varphi_{n_\varphi} = v_{n_\varphi}$ of V is non-trivial, *i.e.* different from $v_1 = \cdots = v_{n_\varphi} = 0$. In any case, whether there is SSB or not, M_0^2 of equation (16) is the tree level mass matrix of the scalars.

The mass matrix of the fermions is given by

$$m_0 = \sum_{a=1}^{n_\varphi} v_a Y_a. \tag{19}$$

The subscript 0 in m_0 and M_0^2 indicates tree level mass matrices. The tree-level mass matrices and fermions and scalars are diagonalized by

$$U_0^T m_0 U_0 = \hat{m}_0 \equiv \operatorname{diag}\left(m_{01}, \dots, m_{0n_\chi}\right), \tag{20a}$$

$$W_0^T M_0^2 W_0 = \hat{M}_0^2 \equiv \operatorname{diag}\left(M_{01}^2, \dots, M_{0n_\varphi}^2\right), \tag{20b}$$

where U_0 is unitary [15] and W_0 is orthogonal.

The diagonalization matrices U_0 and W_0 allow us to introduce mass eigenfields $\hat{\chi}_{jL}$ and $\hat{h}_a$ via

$$\chi_{iL} = (U_0)_{ij}\, \hat{\chi}_{jL} \quad \text{and} \quad h_a = (W_0)_{ab} \hat{h}_b, \tag{21}$$

respectively. Rewriting the Lagrangian in terms of the mass eigenfields amounts to the replacements

$$\delta^{(\chi)} \to \hat{\delta}^{(\chi)} = U_0^\dagger \delta^{(\chi)} U_0, \tag{22a}$$

$$Y_a \to \hat{Y}_a = \left(U_0^T Y_b U_0\right)(W_0)_{ba}, \tag{22b}$$

$$\delta^{(\varphi)} \to \hat{\delta}^{(\varphi)} = W_0^T \delta^{(\varphi)} W_0, \tag{22c}$$

$$v_a \to \hat{v}_a = (W_0)_{ba} v_b, \tag{22d}$$

$$t_a \to \hat{t}_a = (W_0)_{ba} t_b, \tag{22e}$$

$$\mu^2 \to \hat{\mu}^2 = W_0^T \mu^2 W_0, \tag{22f}$$

$$\lambda_{abcd} \to \hat{\lambda}_{abcd} = \lambda_{a'b'c'd'} (W_0)_{a'a} (W_0)_{b'b} (W_0)_{c'c} (W_0)_{d'd}, \tag{22g}$$

such that the form of the Lagrangian is preserved. Therefore, without loss of generality we assume that we are in the mass bases of fermions and scalars, when we perform the one-loop computation of the selfenergies. Note that $\hat{\bar{v}}_a$ and $\Delta\hat{v}_a$ are defined analogously to $\hat{v}_a$.

In the mass basis it is useful to rewrite the Yukawa interaction as

$$\mathcal{L}_Y = -\frac{1}{2} \bar{\hat{\chi}} \left(\hat{Y}_a \gamma_L + \hat{Y}_a^* \gamma_R\right) \hat{\chi} \left(\mathcal{M}^{\varepsilon/2} \hat{h}_a + \hat{\bar{v}}_a\right) \tag{23}$$

with

$$\gamma_L = \frac{\mathbb{1} - \gamma_5}{2}, \quad \gamma_R = \frac{\mathbb{1} + \gamma_5}{2}, \quad \hat{\chi} = \begin{pmatrix} \hat{\chi}_1 \\ \vdots \\ \hat{\chi}_{n_\chi} \end{pmatrix} \quad \text{and} \quad \hat{\chi}_i = \hat{\chi}_{iL} + \left(\hat{\chi}_{iL}\right)^c, \tag{24}$$

where the superscript c indicates charge conjugation.

3 Renormalization

General outline: Our objective is to describe the general renormalization procedure and to work out a prescription for the computation of the one-loop contribution to the *physical* fermion and scalar masses. For this purpose we have to compute the selfenergies. Clearly, the manner in which the selfenergies—and thus the quantities we aim at—depend on the parameters of our toy model is renormalization-scheme-dependent. It is, therefore, expedient to clearly expound the scheme we want to use and how we plan to reach our goal.

We proceed in three steps:

i. $\overline{\mathrm{MS}}$ renormalization for the determination of $\delta\hat{Y}_a$, $\delta\hat{\lambda}_{abcd}$, $\delta\hat{\mu}^2_{ab}$, $\hat{\delta}^{(\chi)}$ and $\hat{\delta}^{(\varphi)}$.

ii. Finite shifts $\Delta\hat{v}_a$ such that the scalar one-point functions of the $\hat{h}_a$ are zero. These two steps allow us to compute *renormalized* one-loop selfenergies $\Sigma(p)$ and $\Pi(p^2)$ for fermions and scalars, respectively.

iii. Finite field strength renormalization in order to switch from the $\overline{\mathrm{MS}}$ selfenergies $\Sigma(p)$ and $\Pi(p^2)$ to on-shell selfenergies[5] $\widetilde{\Sigma}(p)$ and $\widetilde{\Pi}(p^2)$.

Several remarks are in order to concretize this outline. $\overline{\mathrm{MS}}$ renormalization, *i.e.* subtraction of terms proportional to the constant

$$c_\infty = \frac{2}{\varepsilon} - \gamma + \ln(4\pi), \tag{25}$$

where γ is the Euler–Mascheroni constant, is realized in the following way:

(a) $\delta\hat{\lambda}_{abcd}$ is determined from the quartic scalar coupling,

(b) $\delta\hat{Y}_a$ is obtained from the Yukawa vertex,

(c) $\delta\hat{\mu}^2_{ab}$ removes c_∞ from the p^2-independent part of the scalar selfenergy,

(d) $\hat{\delta}^{(\chi)}$ and $\hat{\delta}^{(\varphi)}$ are determined from the momentum-dependent parts of the respective selfenergies.

With the prescriptions (a)–(d) above, all correlation functions and all physical quantities computed in our toy model must be finite. This applies in particular to the selfenergies.

Fermion selfenergy: Let us first consider the renormalized fermion selfenergy $\Sigma(p)$, defined via the inverse propagator matrix

$$S^{-1}(p) = \not{p} - \hat{m}_0 - \Sigma(p), \tag{26}$$

[5]Note that here the term *on-shell* refers to field strength renormalization only. We have no mass counterterms, because in our approach masses are derived quantities and, therefore, functions of the parameters of the model—see the discussion at the end of this section.

where $\Sigma(p)$ has the chiral structure

$$\Sigma(p) = \not{p}\left(\Sigma_L^{(A)}(p^2)\gamma_L + \Sigma_R^{(A)}(p^2)\gamma_R\right) + \Sigma_L^{(B)}(p^2)\gamma_L + \Sigma_R^{(B)}(p^2)\gamma_R. \tag{27}$$

For the relationships between $\Sigma_L^{(A)}$ and $\Sigma_R^{(A)}$ and between $\Sigma_L^{(B)}$ and $\Sigma_R^{(B)}$ in the case of Dirac and Majorana fermions we refer the reader to appendix A. At one-loop order, $\Sigma(p)$ has the terms

$$\begin{aligned}\Sigma(p) &= \Sigma^{\text{1-loop}}(p) - \not{p}\left[\hat{\delta}^{(\chi)}\gamma_L + \left(\hat{\delta}^{(\chi)}\right)^*\gamma_R\right] \\ &\quad +\hat{v}_a\left[\delta\hat{Y}_a\gamma_L + (\delta\hat{Y}_a)^*\gamma_R\right] + \Delta\hat{v}_a\left[\hat{Y}_a\gamma_L + \hat{Y}_a^*\gamma_R\right],\end{aligned} \tag{28}$$

where $\Sigma^{\text{1-loop}}$ corresponds to the diagram of figure 1. Since $\delta\hat{Y}_a$ is already determined by the Yukawa vertex, the corresponding term in $\Sigma(p)$ must automatically make $\Sigma_{L,R}^{(B)}$ in equation (28) finite. As for $\Sigma_{L,R}^{(A)}$ in $\Sigma(p)$, we note that these matrices are hermitian—see also appendix A, therefore, the counterterms with the hermitian matrix $\hat{\delta}^{(\chi)}$ suffice for finiteness. The last term in equation (28) is induced by the finite VEV shifts.

Scalar selfenergy: Now we address the inverse scalar propagator matrix

$$\Delta^{-1}(p^2) = p^2 - \hat{M}_0^2 - \Pi(p^2). \tag{29}$$

The scalar selfenergy $\Pi(p^2)$ has the structure

$$\Pi_{ab}(p^2) = \Pi_{ab}^{\text{1-loop}}(p^2) - \hat{\delta}_{ab}^{(\varphi)}p^2 + \delta\hat{\mu}_{ab}^2 + 3\delta\hat{\lambda}_{abcd}\hat{v}_c\hat{v}_d + 6\hat{\lambda}_{abcd}\hat{v}_c\Delta\hat{v}_d \tag{30}$$

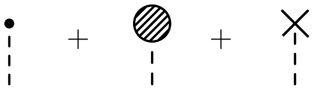
at one-loop order. With an argument analogous to the fermionic case we find that the symmetric matrix $\hat{\delta}^{(\varphi)}$ suffices for making the derivative of $\Pi(p^2)$ finite. According to our renormalization prescription, $\delta\hat{\lambda}_{abcd}\hat{v}_c\hat{v}_d$ is already fixed, but we have $\delta\hat{\mu}_{ab}^2$ at our disposal to cancel the infinity in the p^2-independent term in $\Pi(p^2)$. The last term in the scalar selfenergy, equation (30), stems from the finite mass corrections ΔM_0^2—see equation (14b)—expressed in terms of the finite VEV shifts induced by tadpole contributions.

Another commonly used approach for the renormalization of $\hat{\mu}^2$, *e.g.* in [12], is to express its diagonal entries via the tadpole parameters $\hat{t}_a$ as of equation (15), resulting in renormalization conditions more closely related to physical observables. However, there are simply not enough tadpole parameters available to replace *all* parameters in the $n_\varphi \times n_\varphi$ symmetric matrix $\hat{\mu}^2$ and we have two main reasons for dismissing this choice in our case. One is that expressing $\hat{\mu}_{aa}^2$ in terms of the tadpole parameters involves the inverses of the VEVs $\hat{v}_a$. In the general case, some of these can be zero, leading to ill-defined expressions for $\delta\hat{\mu}_{aa}^2$. The other one is that the diagonal and off-diagonal entries of $\hat{\mu}^2$ can be treated on an equal footing in our approach, leading to a more compact description.

One-point function: These shifts derive from the linear term in the scalar potential. For simplicity we stick to the lowest non-trivial order, where it is given by

$$\mathcal{M}^{-\varepsilon/2}\left(\hat{t}_a + \Delta\hat{t}_a + \delta\hat{\mu}_{ab}^2\hat{v}_b + \delta\hat{\lambda}_{abcd}\,\hat{v}_b\hat{v}_c\hat{v}_d\right)\hat{h}_a. \tag{31}$$

Diagrammatically, the one-point function pertaining to $\hat{h}_a$ has the contributions[6]



$$= \mathcal{M}^{-\varepsilon/2} \frac{i}{-M_{0a}^2} \times (-i) \left(\hat{t}_a + T_a + \Delta \hat{t}_a + \delta \hat{\mu}_{ab}^2 \hat{v}_b + \delta \hat{\lambda}_{abcd}\, \hat{v}_b \hat{v}_c \hat{v}_d \right) = 0, \tag{32}$$

where $i/(-M_{oa}^2)$ is the external scalar propagator at zero momentum. The requirement that the one-point function is zero is identical with the requirement that the VEV of $\hat{h}_a$ is zero. The first diagram in equation (32) represents the scalar tree-level one-point function corresponding to $\hat{t}_a$, which vanishes identically due to equation (18); we have included it only for illustrative purposes. The second diagram, which represents the one-loop tadpole contributions, corresponds to T_a. The third diagram represents the sum of $\Delta \hat{t}_a$ and the two counterterm contributions. We can decompose T_a into an infinite and a finite part, *i.e.*

$$T_a = (T_\infty)_a + (T_{\text{fin}})_a \,. \tag{33}$$

Since with the imposition of conditions (a)–(d) the theory becomes finite, in equation (31) we necessarily have

$$\delta \hat{\mu}_{ab}^2 \hat{v}_b + \delta \hat{\lambda}_{abcd}\, \hat{v}_b \hat{v}_c \hat{v}_d + (T_\infty)_a = 0. \tag{34}$$

An explicit check of this relation is presented in section 4.3. Moreover, we translate the finite tadpole contributions $(T_{\text{fin}})_a$ to shifts of the VEVs $\Delta \hat{v}_b$, similar to the approach of [9]. At one-loop order this is effected by

$$\Delta \hat{t}_a = \hat{\mu}_{ab}^2 \Delta \hat{v}_b + 3 \hat{\lambda}_{abcd} \hat{v}_c \hat{v}_d \Delta \hat{v}_b = \left(\hat{M}_0^2 \right)_{ab} \Delta \hat{v}_b, \tag{35}$$

where we have used equation (15). Therefore, equation (32) leads to the finite shift

$$\Delta \hat{v}_a = - \left(\hat{M}_0^2 \right)_{ab}^{-1} (T_{\text{fin}})_b \,. \tag{36}$$

Note that these finite shifts eventually contribute to the finite mass corrections because they contribute to the two-point functions of the fermions and scalars—see equations (28) and (30), respectively. Further clarifications concerning the VEV shifts Δv_a are found in appendix B.

Pole masses and finite field strength renormalization: It remains to perform a finite field strength renormalization in order to transform the one-loop selfenergies $\Sigma(p)$ and $\Pi(p^2)$ to on-shell selfenergies $\widetilde{\Sigma}(p)$ and $\widetilde{\Pi}(p^2)$, respectively. Immediately the question arises why we cannot use the $Z_\chi^{(1/2)}$ and $Z_\varphi^{(1/2)}$ defined in section 2.1 for this purpose. Note that we have incorporated these matrices into δY_a and $\delta \lambda_{abcd}$ at the respective interaction vertices. Therefore, in $\delta \mathcal{L}$ the field strength renormalization matrices $Z_\chi^{(1/2)}$ and $Z_\varphi^{(1/2)}$ occur solely in the hermitian matrix $\delta^{(\chi)}$ and the symmetric matrix $\delta^{(\varphi)}$, respectively.

[6] We stress again that we do not introduce tadpole counterterms.

Obviously, the latter matrices have fewer parameters than the original ones and it is impossible to perform on-shell renormalization with $\delta^{(\chi)}$ for more than one fermion field and with $\delta^{(\varphi)}$ for more than one scalar field. What happens if we do not incorporate $Z_\chi^{(1/2)}$ and $Z_\varphi^{(1/2)}$ into the Yukawa and quartic couplings, respectively? Let us consider the Yukawa interaction for definiteness and denote by $\check{\delta} Y_a$ the Yukawa counterterm where $Z_\chi^{(1/2)}$ is not incorporated. Obviously, the relation between δY_a and $\check{\delta} Y_a$ is given by

$$\delta Y_a = \left(Z_\chi^{(1/2)}\right)^T \left(Y_b + \check{\delta} Y_b\right) Z_\chi^{(1/2)} \left(Z_\varphi^{(1/2)}\right)_{ba} - Y_a. \tag{37}$$

Actually, the quantity that is determined by the $\overline{\mathrm{MS}}$ Yukawa vertex renormalization is δY_a and not $\check{\delta} Y_a$. Moreover, since we generate mass terms by SSB, the fermion mass term is induced by the shift of equation (13) and has the form

$$\frac{1}{2}\chi_L^T C^{-1} \delta Y_a \bar{v}_a \chi_L + \frac{1}{2}\chi_L^T C^{-1} Y_a \bar{v}_a \chi_L + \text{H.c.} \tag{38}$$

Thus it is clearly the same δY_a that occurs in both the mass term and the vertex renormalization. Consequently, with the counterterms of the unbroken theory we always end up with δY_a and $\delta^{(\chi)}$ as independent quantities and we can in general not perform on-shell renormalization. Therefore, we need, in addition to $Z_\chi^{(1/2)}$ and $Z_\varphi^{(1/2)}$, finite field strength renormalization matrices $\mathring{Z}_\chi^{(1/2)}$ and $\mathring{Z}_h^{(1/2)}$ for fermions and bosons, respectively, inserted into the *broken* Lagrangian, in order to perform on-shell renormalization. In this way, the $\sqrt{Z}$-factors of the external lines in the LSZ formalism are exactly one [5].

We denote the one-loop contributions to $\mathring{Z}_\chi^{(1/2)}$ and $\mathring{Z}_h^{(1/2)}$ by $\frac{1}{2}\mathring{z}_\chi$ and $\frac{1}{2}\mathring{z}_h$, respectively. For the details of the computation and the results for these quantities we refer the reader to appendix A. Here we only state the masses [16, 17]

$$m_i = m_{0i} + m_{0i}\left(\Sigma_L^{(A)}\right)_{ii}(m_{0i}^2) + \mathrm{Re}\left(\Sigma_L^{(B)}\right)_{ii}(m_{0i}^2), \tag{39}$$

$$M_a^2 = M_{0a}^2 + \Pi_{aa}(M_{0a}^2) \tag{40}$$

at one-loop order. There is no summation in these two formulas over equal indices.

Eventually we remark that one could decompose $\mathring{z}_\chi$ into a hermitian and an antihermitian matrix. One could be tempted to conceive the antihermitian part as a correction to the tree-level diagonalization matrix U_0. However, we think that in our simple model such a decomposition has no physical meaning; in essence, we have no PMNS mixing matrix at disposal where it could become physical. Of course, a similar remark applies to $\mathring{z}_h$—see also [18] for a recent discussion in the context of the two-Higgs-doublet model.

4 Renormalization at the one-loop level

In this section we concretize, at the one-loop level, the renormalization procedure introduced in the previous section. For the relevant integrals needed for these computations see appendix C.

4.1 One-loop results for selfenergies and tadpoles

Here we display the results for the one-loop contributions $\Sigma^{\text{1-loop}}(p)$ and $\Pi^{\text{1-loop}}_{ab}(p^2)$ to the fermion and scalar selfenergies, respectively, and also for the one-loop tadpole expression T_a.

Fermion selfenergy: The only direct one-loop contribution to the fermionic self-energy is given by the diagram of figure 1. Then, the definitions

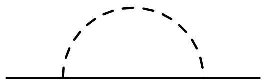


Figure 1: One-loop fermion selfenergy diagram.

$$\Delta_{a,k} = xM_{0a}^2 + (1-x)m_{0k}^2 - x(1-x)p^2, \tag{41}$$

$$D_{a,k} = \int_0^1 \mathrm{d}x\, x \ln\frac{\Delta_{a,k}}{\mathcal{M}^2}, \quad E_{a,k} = \int_0^1 \mathrm{d}x \ln\frac{\Delta_{a,k}}{\mathcal{M}^2} \tag{42}$$

and

$$\hat{D}_a = \operatorname{diag}\left(D_{a,1},\dots,D_{a,n_\chi}\right), \quad \hat{E}_a = \operatorname{diag}\left(E_{a,1},\dots,E_{a,n_\chi}\right) \tag{43}$$

allow us to write the one-loop contribution to the fermionic selfenergy as

$$\begin{aligned}
\Sigma^{\text{1-loop}} = \frac{1}{16\pi^2}\Big\{ & \not{p}\gamma_L \left[-\frac{1}{2}c_\infty \hat{Y}_a^* \hat{Y}_a + \hat{Y}_a^* \hat{D}_a \hat{Y}_a \right] & \text{(44a)}\\
& + \not{p}\gamma_R \left[-\frac{1}{2}c_\infty \hat{Y}_a \hat{Y}_a^* + \hat{Y}_a \hat{D}_a \hat{Y}_a^* \right] & \text{(44b)}\\
& + \gamma_L \left[-c_\infty \hat{Y}_a \hat{m}_0 \hat{Y}_a + \hat{Y}_a \hat{m}_0 \hat{E}_a \hat{Y}_a \right] & \text{(44c)}\\
& + \gamma_R \left[-c_\infty \hat{Y}_a^* \hat{m}_0 \hat{Y}_a^* + \hat{Y}_a^* \hat{m}_0 \hat{E}_a \hat{Y}_a^* \right] \Big\}. & \text{(44d)}
\end{aligned}$$

Scalar selfenergy: In the following, the superscripts (a), (b), (c) refer to the Feynman diagrams of figure 2. Thus the selfenergy has the contributions

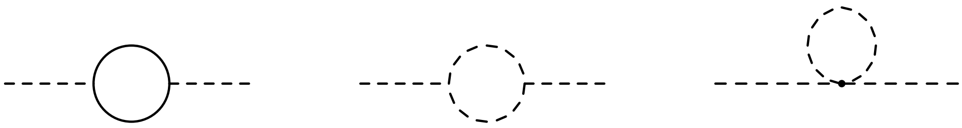
$$\Pi^{\text{1-loop}}_{ab}(p^2) = \Pi^{(a)}_{ab}(p^2) + \Pi^{(b)}_{ab}(p^2) + \Pi^{(c)}_{ab}(p^2). \tag{45}$$

We define

$$\Delta_{ij} = xm_{0i}^2 + (1-x)m_{0j}^2 - x(1-x)p^2 \quad \text{and} \quad \tilde{\Delta}_{rs} = xM_{0r}^2 + (1-x)M_{0s}^2 - x(1-x)p^2. \tag{46}$$

With these definitions we obtain

$$\Pi^{(a)}_{ab}(p^2) = \frac{1}{16\pi^2}\Big\{ c_\infty \operatorname{Tr}\left[\hat{Y}_a \hat{m}_0 \hat{Y}_b \hat{m}_0 + \hat{Y}_a^* \hat{m}_0 \hat{Y}_b^* \hat{m}_0 + 2\hat{Y}_a \hat{Y}_b^* \hat{m}_0^2 + 2\hat{Y}_a^* \hat{Y}_b \hat{m}_0^2 \right]$$



(a) (b) (c)

Figure 2: The Feynman diagrams of the one-loop contributions to the scalar selfenergy.

$$
\begin{aligned}
&-\frac{1}{2}c_\infty \operatorname{Tr}\left[\hat{Y}_a\hat{Y}_b^* + \hat{Y}_a^*\hat{Y}_b\right] p^2 \\
&+\operatorname{Tr}\left[\left(\hat{Y}_a\hat{Y}_b^* + \hat{Y}_a^*\hat{Y}_b\right)\left(\hat{m}_0^2 - \frac{1}{6}p^2\right)\right] \\
&-\int_0^1 \mathrm{d}x \left[\left((\hat{Y}_a)_{ij}(\hat{Y}_b)^*_{ji} + (\hat{Y}_a)^*_{ij}(\hat{Y}_b)_{ji}\right)\left(2\Delta_{ij} - x(1-x)p^2\right)\right. \\
&\left.+(\hat{Y}_a)_{ij}m_{0j}(\hat{Y}_b)_{ji}m_{0i} + (\hat{Y}_a)^*_{ij}m_{0j}(\hat{Y}_b)^*_{ji}m_{0i}\right] \ln\frac{\Delta_{ij}}{\mathcal{M}^2}\Bigg\}, \qquad (47a)
\end{aligned}
$$

$$\Pi^{(b)}_{ab}(p^2) = -\frac{18}{16\pi^2}\,\hat{\lambda}_{acrs}\hat{v}_c\hat{\lambda}_{bdrs}\hat{v}_d\left(c_\infty - \int_0^1 \mathrm{d}x\,\ln\frac{\tilde{\Delta}_{rs}}{\mathcal{M}^2}\right), \tag{47b}$$

$$\Pi^{(c)}_{ab}(p^2) = -\frac{3}{16\pi^2}\,\hat{\lambda}_{abrr}M^2_{0r}\left(c_\infty + 1 - \ln\frac{M^2_{0r}}{\mathcal{M}^2}\right). \tag{47c}$$

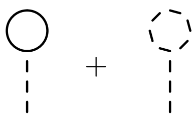
For the following discussion, it is useful to introduce a separate notation for the divergent p^2-independent parts of $\Pi^{\text{1-loop}}_{ab}$:

$$\left(\Pi^{(a)}_\infty\right)_{ab} = \frac{1}{16\pi^2}c_\infty \operatorname{Tr}\left[\hat{Y}_a\hat{m}_0\hat{Y}_b\hat{m}_0 + \hat{Y}_a^*\hat{m}_0\hat{Y}_b^*\hat{m}_0 + 2\hat{Y}_a\hat{Y}_b^*\hat{m}_0^2 + 2\hat{Y}_a^*\hat{Y}_b\hat{m}_0^2\right], \tag{48a}$$

$$\left(\Pi^{(b)}_\infty\right)_{ab} = -\frac{18}{16\pi^2}c_\infty\,\hat{\lambda}_{acrs}\hat{v}_c\hat{\lambda}_{bdrs}\hat{v}_d, \tag{48b}$$

$$\left(\Pi^{(c)}_\infty\right)_{ab} = -\frac{3}{16\pi^2}c_\infty\,\hat{\lambda}_{abrr}M^2_{0r}. \tag{48c}$$

Tadpoles: There are two one-loop tadpole contributions to the scalar one-point function, namely



$$= \mathcal{M}^{-\varepsilon/2}\frac{i}{-M^2_{0a}} \times (-i)\left(T^{(\chi)}_a + T^{(h)}_a\right). \tag{49}$$

We find the following result for tadpole terms:

$$T^{(\chi)}_a = \frac{1}{16\pi^2}\operatorname{Tr}\left[\left(\hat{Y}_a\hat{m}_0^3 + \hat{Y}_a^*\hat{m}_0^3\right)\left(c_\infty + 1 - \ln\frac{\hat{m}_0^2}{\mathcal{M}^2}\right)\right], \tag{50}$$

$$T^{(h)}_a = -\frac{3}{16\pi^2}\hat{\lambda}_{abrr}\hat{v}_b M^2_{0r}\left(c_\infty + 1 - \ln\frac{M^2_{0r}}{\mathcal{M}^2}\right). \tag{51}$$

We denote the divergences in the tadpole expressions by $\left(T^{(\chi)}_\infty\right)_a$ and $\left(T^{(h)}_\infty\right)_a$.

4.2 Determination of the counterterms

Counterterms of Yukawa and quartic scalar couplings: Using $\overline{\text{MS}}$ renormalization, it is straightforward to compute these counterterms. For the Yukawa couplings we obtain

$$\delta\hat{Y}_a = \frac{1}{16\pi^2} c_\infty \hat{Y}_b \hat{Y}_a^* \hat{Y}_b. \tag{52}$$

The $\delta\hat{\lambda}_{abcd}$ can be split into

$$\delta\hat{\lambda}_{abcd} = \delta\hat{\lambda}^{(\chi)}_{abcd} + \delta\hat{\lambda}^{(\varphi)}_{abcd}, \tag{53}$$

generated by fermions and scalars, respectively, in the loop. The first case yields

$$\begin{aligned}\delta\hat{\lambda}^{(\chi)}_{abcd} \;=\; & -\frac{1}{3} \times \frac{1}{16\pi^2} c_\infty \mathrm{Tr}\left[\hat{Y}_a \hat{Y}_b^* \hat{Y}_c \hat{Y}_d^* + \hat{Y}_a \hat{Y}_c^* \hat{Y}_d \hat{Y}_b^* + \hat{Y}_a \hat{Y}_d^* \hat{Y}_b \hat{Y}_c^* \right. \\ & \left. + \hat{Y}_a^* \hat{Y}_b \hat{Y}_c^* \hat{Y}_d + \hat{Y}_a^* \hat{Y}_c \hat{Y}_d^* \hat{Y}_b + \hat{Y}_a^* \hat{Y}_d \hat{Y}_b^* \hat{Y}_c \right].\end{aligned} \tag{54}$$

In this formula we have taken into account that the Yukawa coupling matrices are symmetric. The scalar contribution is

$$\delta\hat{\lambda}^{(\varphi)}_{abcd} = \frac{3}{16\pi^2} c_\infty \left(\hat{\lambda}_{abrs}\hat{\lambda}_{rscd} + \hat{\lambda}_{adrs}\hat{\lambda}_{rsbc} + \hat{\lambda}_{acrs}\hat{\lambda}_{rsbd}\right). \tag{55}$$

Counterterms pertaining to field strength renormalization: Cancellation of the divergence in equation (44b) determines $\hat{\delta}^{(\chi)}$ as

$$\hat{\delta}^{(\chi)} = -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty \hat{Y}_a^* \hat{Y}_a. \tag{56}$$

Considering the scalar selfenergy, we find that only diagram (a) of figure 2 has a divergence proportional to p^2. Therefore, we obtain from equation (47a)

$$\hat{\delta}^{(\varphi)}_{ab} = -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty \mathrm{Tr}\left[\hat{Y}_a \hat{Y}_b^* + \hat{Y}_a^* \hat{Y}_b\right]. \tag{57}$$

Counterterm pertaining to $\hat{\mu}^2_{ab}$: The counterterm $\delta\hat{\mu}^2_{ab}$ has to be determined by the cancellations of the divergences of equations (48b) and (48c). Thus we demand

$$\begin{aligned} 0 &= \delta\hat{\mu}^2_{ab} + 3\delta\hat{\lambda}^{(\varphi)}_{abcd}\hat{v}_c\hat{v}_d + \left(\Pi^{(b)}_\infty\right)_{ab} + \left(\Pi^{(c)}_\infty\right)_{ab} \\ &= \delta\hat{\mu}^2_{ab} + \frac{3}{16\pi^2} c_\infty \left[3\hat{\lambda}_{abrs}\hat{\lambda}_{rscd}\hat{v}_c\hat{v}_d - \hat{\lambda}_{abrr} M^2_{0r}\right] \\ &= \delta\hat{\mu}^2_{ab} + \frac{3}{16\pi^2} c_\infty \left[\hat{\lambda}_{abrs}\left(\hat{\mu}^2_{rs} + 3\hat{\lambda}_{rscd}\hat{v}_c\hat{v}_d - \hat{\mu}^2_{rs}\right) - \hat{\lambda}_{abrr} M^2_{0r}\right] \\ &= \delta\hat{\mu}^2_{ab} - \frac{3}{16\pi^2} c_\infty \, \hat{\lambda}_{abrs}\hat{\mu}^2_{rs}. \end{aligned} \tag{58}$$

Therefore, $\delta\hat{\mu}^2_{ab}$ is fixed as

$$\delta\hat{\mu}^2_{ab} = \frac{3}{16\pi^2} c_\infty \, \hat{\lambda}_{abrs}\hat{\mu}^2_{rs}. \tag{59}$$

4.3 Cancellation of divergences

Having fixed all available counterterms, the remaining UV divergences in the selfenergies and tadpoles have to drop out. This is what we want to show in this subsection.

Fermion selfenergy: With $\delta\hat{Y}_a$ of equation (52) and

$$\hat{v}_a \delta\hat{Y}_a = \frac{1}{16\pi^2} c_\infty \hat{Y}_b \hat{m}_0 \hat{Y}_b, \tag{60}$$

we find that Σ is finite without any mass renormalization, as it has to be.

Scalar selfenergy: We have already treated the divergences (48b) and (48c), but there is still the divergence of equation (48a). However, it is easy to see that its cancellation in the selfenergy (30) is simply effected by

$$\left(\Pi_\infty^{(a)}\right)_{ab} + 3\delta\hat{\lambda}^{(\chi)}_{abcd}\hat{v}_c\hat{v}_d = 0. \tag{61}$$

Tadpoles: It remains to verify equation (34). First we consider the result of the fermionic tadpole in equation (50). Contracting the counterterm $\delta\hat{\lambda}^{(\chi)}_{abcd}$ of equation (54) with the VEVs and adding to it $\left(T_\infty^{(\chi)}\right)_a$ yields

$$\left(T_\infty^{(\chi)}\right)_a + \delta\hat{\lambda}^{(\chi)}_{abcd}\hat{v}_b\hat{v}_c\hat{v}_d = \left(T_\infty^{(\chi)}\right)_a - \frac{1}{16\pi^2} c_\infty \operatorname{Tr}\left[\left(\hat{Y}_a \hat{m}_0^3 + \hat{Y}_a^* \hat{m}_0^3\right)\right] = 0. \tag{62}$$

Similarly, using equations (55) and (59), the scalar tadpole contribution of equation (51) is found to be finite via

$$\begin{aligned}
&\left(T_\infty^{(h)}\right)_a + \delta\hat{\mu}^2_{ab}\hat{v}_b + \delta\hat{\lambda}^{(\varphi)}_{abcd}\hat{v}_b\hat{v}_c\hat{v}_d \\
&= \left(T_\infty^{(h)}\right)_a + \frac{3}{16\pi^2} c_\infty \left(\hat{\lambda}_{abrs}\hat{\mu}^2_{rs}\hat{v}_b + 3\hat{\lambda}_{abrs}\hat{\lambda}_{rscd}\hat{v}_b\hat{v}_c\hat{v}_d\right) \\
&= \left(T_\infty^{(h)}\right)_a + \frac{3}{16\pi^2} c_\infty \hat{\lambda}_{abrr}\hat{v}_b M_{0r}^2 = 0.
\end{aligned} \tag{63}$$

4.4 Counterterms and UV divergences in a general basis

The results for the selfenergies and counterterms shown in the previous sections are given in the mass bases. However, for a check of the cancellation of divergences it might be advantageous to have the divergences in a general basis. Such expressions can be obtained by using the parameter transformations (22).

As an example, let us do this transformation in the case of $\hat{\delta}^{(\chi)}$ of equation (56), where one has to apply

$$\begin{aligned}
\delta^{(\chi)} &= U_0 \hat{\delta}^{(\chi)} U_0^\dagger \\
&= -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty U_0 \hat{Y}_a^* \hat{Y}_a U_0^\dagger \\
&= -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty U_0 \left(U_0^\dagger Y_b^* U_0^* \left(W_0\right)_{ba}\right)\left(U_0^T Y_c U_0 \left(W_0\right)_{ca}\right) U_0^\dagger
\end{aligned}$$

$$= -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty Y_a^* Y_a. \tag{64}$$

In the case of the divergence in $\Sigma_L^{(B)}$—see equation (44c), we have to use the slightly different transformation

$$U_0^* \hat{Y}_a \hat{m}_0 \hat{Y}_a U_0^\dagger = Y_a m_0^* Y_a \tag{65}$$

This explains that we have to be careful when a fermion mass term occurs because in general

$$v_a Y_a^* = m_0^* \neq v_a Y_a = m_0. \tag{66}$$

This complication only arises in

$$\left(T_\infty^{(\chi)}\right)_a = \frac{1}{16\pi^2} c_\infty \operatorname{Tr}\left[Y_a {m_0}^* m_0 {m_0}^* + Y_a^* m_0 {m_0}^* m_0\right] \tag{67}$$

and

$$\left(\Pi_\infty^{(a)}\right)_{ab} = \frac{1}{16\pi^2} c_\infty \operatorname{Tr}\left[Y_a {m_0}^* Y_b m_0^* + Y_a^* m_0 Y_b^* m_0 + 2 Y_a Y_b^* m_0 {m_0}^* + 2 Y_a^* Y_b {m_0}^* m_0\right]. \tag{68}$$

The divergences $\left(T_\infty^{(h)}\right)_a$, $\left(\Pi_\infty^{(b)}\right)_{ab}$ and $\left(\Pi_\infty^{(c)}\right)_{ab}$ are obtained in a general basis by simply removing the hats from all quantities and the same is true for all counterterms.

5 An example of a flavour symmetry

Motivated by flavour models of the lepton sector [3], we will now consider a Lagrangian with a simple flavour symmetry and study how renormalization is affected in this case.

5.1 Symmetry group and Lagrangian

We assume the same number of Majorana and scalar fields, *i.e.* $n_\chi = n_\varphi \equiv n$. In addition, we require $n \geq 2$. Instead of the $\mathbb{Z}_4$ symmetry of equation (2), which acts at the same time on all fields, we will now postulate a $\mathbb{Z}_4$ symmetry for every index $a = 1, \ldots, n$:

$$(\mathbb{Z}_4)_a: \quad \chi_{aL}^{(B)} \to i\chi_{aL}^{(B)}, \quad \varphi_a^{(B)} \to -\varphi_a^{(B)}, \quad \chi_{bL}^{(B)} \to \chi_{bL}^{(B)}, \quad \varphi_b^{(B)} \to \varphi_b^{(B)} \; \forall\, b \neq a. \tag{69}$$

This has the consequence that scalar fields with the same index occur in pairs in the scalar potential. Note that now it is reasonable to use the same indices for both fermions and scalars. In addition, we assume that the Lagrangian is invariant under simultaneous permutations of fermion and scalar fields. Therefore, group-theoretically the symmetry group of the Lagrangian can be conceived as

$$G_n = (\mathbb{Z}_4)^n \rtimes S_n. \tag{70}$$

With this flavour group, the bare Lagrangian has the form

$$\mathcal{L}_B = \sum_{a=1}^{n} \left[i\overline{\chi}_{aL}^{(B)} \partial\!\!\!/ \chi_{aL}^{(B)} + \frac{1}{2}\partial_\mu \varphi_a^{(B)} \partial^\mu \varphi_a^{(B)} + \frac{1}{2} y^{(B)} \left(\chi_{aL}^{(B)} C^{-1} \chi_{aL}^{(B)} \varphi_a^{(B)} + \text{H.c.} \right) \right] - V_B, \tag{71}$$

where the bare scalar potential can be written as

$$V_B = \frac{1}{2}\mu^2 \sum_{a=1}^{n} \left(\varphi_a^{(B)}\right)^2 + \frac{1}{4}\lambda \left(\sum_{a=1}^{n} \left(\varphi_a^{(B)}\right)^2\right)^2 + \frac{1}{4}\lambda' \sum_{a,b=1}^{n} \left(\varphi_a^{(B)}\right)^2 \left(\varphi_b^{(B)}\right)^2 (1-\delta_{ab}), \tag{72}$$

where δ_{ab} is the Kronecker delta.

5.2 Relation to the general model

Due to the symmetry group G_n, we only have one Yukawa coupling constant and two quartic couplings. In order to use the general one-loop results, we have to establish the relation between the general model of section (2.1) and the present example. For simplicity we now drop the superscript (B) and keep in mind that the following list applies not only to the renormalized coupling constants but also to the counterterms and the bare coupling constants:

$$(Y_a)_{bc} = y\,\delta_{ab}\delta_{ac} \quad \forall a, \tag{73a}$$

$$\left(\mu^2\right)_{ab} = \mu^2\delta_{ab}, \tag{73b}$$

$$\lambda_{aaaa} = \lambda \quad \forall a \quad \text{and} \quad \lambda_{aabb} = \frac{1}{3}(\lambda+\lambda') \quad \forall a \neq b. \tag{73c}$$

Note that now we just have one mass parameter μ^2. Moreover, quartic couplings λ_{abbb} with $a \neq b$ and those with three or four different indices are zero. Without loss of generality we assume $y > 0$. In addition, we have to consider equation (11), which now reads

$$\delta_{ab}^{(\chi)} = \delta^{(\chi)}\delta_{ab}, \quad \delta_{ab}^{(\varphi)} = \delta^{(\varphi)}\delta_{ab}, \tag{74}$$

because due to the symmetry group G_n only one field strength renormalization constant is allowed for each type of fields.

The results of section 4, found for the general Yukawa model, can directly be used for the present case by applying equation (73). In this way we obtain the counterterms

$$\delta y = \frac{1}{16\pi^2} c_\infty y^3, \tag{75a}$$

$$\delta\lambda^{(\chi)} = -\frac{2}{16\pi^2} c_\infty y^4, \tag{75b}$$

$$\delta\lambda^{(\varphi)} = \frac{1}{16\pi^2} c_\infty \left[9\lambda^2 + (n-1)(\lambda+\lambda')^2\right], \tag{75c}$$

$$(\delta\lambda + \delta\lambda')^{(\chi)} = 0, \tag{75d}$$

$$(\delta\lambda + \delta\lambda')^{(\varphi)} = \frac{1}{16\pi^2} c_\infty \left[6\lambda(\lambda+\lambda') + (n+2)(\lambda+\lambda')^2\right], \tag{75e}$$

$$\delta\mu^2 = \frac{\mu^2}{16\pi^2} c_\infty \left[3\lambda + (n-1)(\lambda+\lambda')\right], \tag{75f}$$

where the superscripts (χ) and (φ) indicate fermions and scalars in the loop, respectively, in analogy to the notation in section 4.2. Field strength renormalization yields

$$\delta^{(\chi)} = -\frac{1}{2} \times \frac{1}{16\pi^2} c_\infty y^2 \quad \text{and} \quad \delta^{(\varphi)} = -\frac{1}{16\pi^2} c_\infty y^2. \tag{76}$$

5.3 Spontaneous symmetry breaking

In order to have SSB we assume $\mu^2 < 0$. For the vacuum expectation values we introduce the notation

$$v^2 = \sum_{a=1}^{n} v_a^2. \tag{77}$$

Obviously, for the scalar potential to be bounded from below we must have $\lambda > 0$, but λ' can be positive or negative.

Case $\lambda' > 0$: Here, the minimum of the scalar potential is achieved when only one VEV is nonzero. Without loss of generality we assume

$$v_1 = v, \quad v_2 = \cdots = v_n = 0 \quad \Rightarrow \quad v^2 = -\frac{\mu^2}{\lambda}. \tag{78}$$

The symmetry breaking can be formulated as

$$G_n \stackrel{\text{SSB}}{\longrightarrow} G_{n-1}, \tag{79}$$

where G_{n-1} is the residual symmetry group. This residual symmetry is reflected in the mass spectrum

$$M_{01}^2 = 2\lambda v^2,\ M_{02}^2 = \cdots M_{0n}^2 = \lambda' v^2, \quad m_{01} = yv,\ m_{02} = \cdots = m_{0n} = 0. \tag{80}$$

Since the mass matrices of both fermions and scalars are diagonal at tree level, it is straightforward to compute the one-loop corrections to equation (80). It easy to see that at one-loop order the VEV shifts fulfill $\Delta v_2 = \cdots = \Delta v_n = 0$, only Δv_1 will in general be nonzero. It is also obvious that the nonzero masses in equation (80) receive one-loop corrections. However, $m_2 = \cdots = m_n$ is still valid because the unbroken symmetry group G_{n-1} forbids such masses.

Case $\lambda' < 0$: For negative λ', the condition

$$|\lambda'| < \frac{n}{n-1}\lambda \tag{81}$$

is necessary for the scalar potential to be bounded from below. In this case the minimum is given by

$$v_1^2 = \cdots = v_n^2 = \frac{v^2}{n} \quad \Rightarrow \quad v^2 = \frac{-\mu^2}{\lambda + \frac{n-1}{n}\lambda'}. \tag{82}$$

In principle, the VEVs v_a could have different signs. However, since arbitrary sign changes of the scalar fields are part of G_n, we can assume $v_a > 0\ \forall a$ without loss of generality. Therefore, we have the symmetry breaking

$$G_n \stackrel{\text{SSB}}{\longrightarrow} S_n, \tag{83}$$

where the permutation group is given by its "natural permutation representation" corresponding to $n \times n$ permutation matrices. This representation decays into the trivial one-dimensional and a $(n-1)$-dimensional irreducible representation. Defining n vectors w_a $(a = 1, \ldots, n)$ such that

$$w_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad w_a \cdot w_b = \delta_{ab}\ \forall a, b, \tag{84}$$

then w_1 is invariant under all permutation matrices and belongs, therefore, to the trivial irreducible representation, while the vectors $w_2, \ldots w_n$ span the space pertaining to the $(n-1)$-dimensional one. This is borne out by the tree-level masses. The scalars have the mass matrix

$$M_0^2 = A\mathbb{1} + B w_1 w_1^T \quad \text{with} \quad A = -\frac{2\lambda' v^2}{n}, \quad B = 2(\lambda + \lambda')v^2. \tag{85}$$

Hence, the diagonalization matrix is given by

$$W_0 = (w_1, \ldots, w_n) \tag{86}$$

and we find

$$M_{01}^2 = A + B, \quad M_{02}^2 = \cdots = M_{0n}^2 = A. \tag{87}$$

However, the fermion masses are all equal at tree level:

$$m_{01} = \cdots = m_{0n} = \frac{yv}{\sqrt{n}}. \tag{88}$$

At one-loop order, the scalar masses of equation (87) will receive radiative corrections, but—due to the unbroken symmetry group S_n—the relation $M_2^2 = \cdots = M_n^2$ will still hold.

One might expect that the total degeneracy of the fermion masses, as expressed in equation (88), will be lifted because of radiative corrections such that m_1 is different from the rest. However, as we will demonstrate now, this is not the case.

First we discuss the contribution from the finite one-loop VEVs shifts to the fermion masses. Since the fermion mass matrix is diagonal, we have

$$\hat{Y}_a = Y_b \left(W_0\right)_{ba} = y \operatorname{diag}\left(\left(W_0\right)_{1a}, \ldots, \left(W_0\right)_{na}\right). \tag{89}$$

In particular,

$$\hat{Y}_1 = \frac{y}{\sqrt{n}} \mathbb{1} \quad \text{and} \quad \operatorname{Tr} \hat{Y}_a = 0 \quad \text{for } a = 2, \ldots, n \tag{90}$$

due to w_1 of equation (84). Therefore, it follows from equation (50) that

$$T_a^{(\chi)} = 0 \quad \text{for } a = 2, \ldots, n. \tag{91}$$

Moreover, from equations (22d) and (86) we find

$$\hat{v}_1 = v, \quad \hat{v}_2 = \cdots = \hat{v}_n = 0. \tag{92}$$

With this the tadpole expression $T_a^{(h)}$ of equation (51) has the structure

$$T_a^{(h)} = \hat{\lambda}_{abrr} \hat{v}_b X_r = \hat{\lambda}_{a1rr} v X_r. \tag{93}$$

According to equation (73c), this expression can only be nonzero for $a = 1$. Therefore,

$$T_a^{(h)} = 0 \quad \text{for } a = 2, \ldots, n \tag{94}$$

as well and $\Delta \hat{v}_a \hat{Y}_a = \Delta \hat{v}_1 \hat{Y}_1 \propto \mathbb{1}$. This proves that the finite VEV shifts cannot remove the total fermion mass degeneracy.

Next we consider $\Sigma^{\text{1-loop}}$ of equation (44). We note that both $\hat{D}_a$ and $\hat{E}_a$ are proportional to the unit matrix because of equation (88). In addition, because of equation (87),

$$\hat{D}_2 = \ldots = \hat{D}_n \quad \text{and} \quad \hat{E}_2 = \ldots = \hat{E}_n. \tag{95}$$

Thus we can write $\hat{D}_a = f_a \mathbb{1}$ with $f_2 = \cdots = f_n$, but $f_1 \neq f_2$ in general. There are the analogous relations for the $\hat{E}_a$. Considering now the b-th entry of the (diagonal) finite parts of $\Sigma^{\text{1-loop}}$ and taking into account that the Yukawa coupling matrices are given by equation (89), we have the generic sum

$$\sum_{a=1}^{n} (W_0)_{ba} \, f_a \, (W_0)_{ba} = (W_0)_{b1} \, (f_1 - f_2) \, (W_0)_{b1} + \sum_{a=1}^{n} (W_0)_{ba} \, f_2 \, (W_0)_{ba} = \frac{1}{n} (f_1 - f_2) + f_2. \tag{96}$$

(Note that there is no summation over the index b in this equation.) This result does not depend on b and, therefore, $\Sigma^{\text{1-loop}}$ is proportional to the unit matrix. Consequently, the fermion mass degeneracy cannot be lifted by one-loop contributions, as stated above.

5.4 Soft symmetry breaking

It is possible to lift any mass degeneracies by explicit breaking of G_n. The model remains renormalizable, if we have soft breaking, for instance, by terms of dimension two. This is done by admitting in equation (73) a general mass matrix μ^2_{ab}, whereas the Yukawa and quartic couplings are still restricted by G_n. This breaks the symmetry group G_n down to

$$G \xrightarrow{\dim 2} (\mathbb{Z}_4)_{\text{diag}} \tag{97}$$

with

$$(\mathbb{Z}_4)_{\text{diag}}: \quad \chi^{(B)}_{aL} \to i\chi^{(B)}_{aL}, \quad \varphi^{(B)}_a \to -\varphi^{(B)}_a \;\forall\, a, \tag{98}$$

i.e. this $\mathbb{Z}_4$ acts simultaneously on all fields and agrees with equation (2). In this way, the scalar mass spectrum will be completely non-degenerate already at tree level, but also the fermion mass spectrum because a general matrix μ^2_{ab} will induce general VEVs v_a. It is easy to understand why this modified model remains renormalizable; allowing for a general matrix μ^2_{ab}, we also allow for a general counterterm matrix $\delta\mu^2_{ab}$ and we can cancel the divergences related to the scalar mass terms as handled by equation (59).

It is natural that soft symmetry breaking is small. We can easily incorporate this by taking one large mass parameter μ^2 and setting

$$\mu^2_{ab} = \mu^2 \delta_{ab} + \sigma_{ab} \tag{99}$$

such that $\sum_{a=1}^{n} \sigma_{aa} = 0$ and $|\sigma_{ab}| \ll \mu^2 \;\forall\, a, b$. In this case the previously degenerate masses will now become slightly different and we can produce quasi-degenerate mass spectra.

6 Dirac fermions

So far, we have put the focus on Majorana fermions. We have done so because in the long run we are interested in studying radiative corrections in neutrino mass models which typically feature the seesaw mechanism and, therefore, neutrinos of Majorana nature. However, it is straightforward to switch from Majorana to Dirac fermions. How this is done will be explained in this section—see also [14, 19].

Lagrangian, diagonalization of Dirac mass matrices, and renormalization: In the Dirac setup, we can in general have n_{χ_L} chiral fields $\chi_{iL}^{(B)}$ and n_{χ_R} independent chiral fields $\chi_{iR}^{(B)}$, while the scalar sector remains the same as in the Majorana case. Then, the bare Lagrangian is given by

$$\begin{aligned} \mathcal{L}_B \;=\; & i\bar{\chi}_{iL}^{(B)}\gamma^\mu\partial_\mu\chi_{iL}^{(B)} + i\bar{\chi}_{iR}^{(B)}\gamma^\mu\partial_\mu\chi_{iR}^{(B)} + \frac{1}{2}\left(\partial_\mu\varphi_a^{(B)}\right)\left(\partial^\mu\varphi_a^{(B)}\right) && \text{(100a)} \\ & - \left(\left(Y_a^{(B)}\right)_{ij}\,\bar{\chi}_{iR}^{(B)}\chi_{jL}^{(B)}\varphi_a^{(B)} + \text{H.c.}\right) && \text{(100b)} \\ & -\frac{1}{2}(\mu_B^2)_{ab}\varphi_a^{(B)}\varphi_b^{(B)} - \frac{1}{4}\lambda_{abcd}^{(B)}\,\varphi_a^{(B)}\varphi_b^{(B)}\varphi_c^{(B)}\varphi_d^{(B)}, && \text{(100c)} \end{aligned}$$

where the $Y_a^{(B)}$ now are n_φ general complex $n_{\chi_R} \times n_{\chi_L}$ matrices. In principle, n_{χ_L} could be different from n_{χ_R}, in which case one has $|n_{\chi_L} - n_{\chi_R}|$ massless Weyl fermions. However, for simplicity we assume $n_{\chi_L} = n_{\chi_R} \equiv n_\chi$ in the following. A possible modification of the transformation of the fermions in equation (2) is the $\mathbb{Z}_2$ symmetry

$$\mathcal{S}': \quad \chi_L^{(B)} \to -\chi_L^{(B)}, \quad \chi_R^{(B)} \to \chi_R^{(B)}, \quad \varphi^{(B)} \to -\varphi^{(B)}, \tag{101}$$

in order to forbid fermion tree-level mass terms and linear and trilinear terms in the scalar potential.

The renormalization of the fermionic fields now becomes

$$\chi_L^{(B)} = Z_{\chi_L}^{(1/2)}\,\chi_L, \quad \chi_R^{(B)} = Z_{\chi_R}^{(1/2)}\,\chi_R, \tag{102}$$

involving two independent general complex matrices $Z_{\chi_L}^{(1/2)}$ and $Z_{\chi_R}^{(1/2)}$. Inserting this into equation (100) yields a renormalized Lagrangian with Yukawa coupling matrices Y_a and counterterms similar to the Majorana case. The main changes lie in the definition of the Yukawa counterterm

$$\mathcal{M}^{\varepsilon/2}\delta Y_a = \left(Z_{\chi_R}^{(1/2)}\right)^\dagger Y_b^{(B)}\, Z_{\chi_L}^{(1/2)}\left(Z_\varphi^{(1/2)}\right)_{ba} - \mathcal{M}^{\varepsilon/2}Y_a, \tag{103}$$

and the need for the definition of two independent hermitian matrices

$$\delta^{(\chi_L)} = \left(Z_{\chi_L}^{(1/2)}\right)^\dagger Z_{\chi_L}^{(1/2)} - \mathbb{1}, \quad \delta^{(\chi_R)} = \left(Z_{\chi_R}^{(1/2)}\right)^\dagger Z_{\chi_R}^{(1/2)} - \mathbb{1}. \tag{104}$$

Via SSB we obtain the tree-level Dirac mass matrix

$$m_0 = \sum_{a=1}^{n_\varphi} v_a Y_a. \tag{105}$$

This mass matrix is bi-diagonalized with two unitary matrices U_{L0} and U_{R0}:

$$U_{R0}^\dagger m_0 U_{L0} = \hat{m}_0 \equiv \mathrm{diag}\left(m_{01}, \ldots, m_{0n_\chi}\right). \tag{106}$$

Due to the left and right diagonalization matrices, there are now left and right chiral mass eigenfields

$$\hat{\chi}_L = U_{L0}^\dagger \chi_L, \quad \hat{\chi}_R = U_{R0}^\dagger \chi_R. \tag{107}$$

Moreover, equations (22a) and (22b) are modified to

$$\delta^{(\chi_L)} \to \hat{\delta}^{(\chi_L)} = U_{L0}^\dagger \delta^{(\chi_L)} U_{L0}, \qquad \delta^{(\chi_R)} \to \hat{\delta}^{(\chi_R)} = U_{R0}^\dagger \delta^{(\chi_R)} U_{R0}, \tag{108a}$$

$$Y_a \to \hat{Y}_a = \left(U_{R0}^\dagger Y_b U_{L0}\right)(W_0)_{ba}, \tag{108b}$$

respectively.

In analogy to equation (24), we define Dirac mass eigenfields

$$\hat{\chi}_i = \hat{\chi}_{iL} + \hat{\chi}_{iR} \tag{109}$$

and the corresponding vector of eigenfields $\hat{\chi}$. In terms of mass eigenfields, the Yukawa interaction reads

$$\mathcal{L}_Y = -\bar{\hat{\chi}}\left(\hat{Y}_a \gamma_L + \hat{Y}_a^\dagger \gamma_R\right)\hat{\chi}\left(\mathcal{M}^{\varepsilon/2}\hat{h}_a + \hat{\bar{v}}_a\right). \tag{110}$$

Formally, Dirac and Majorana Yukawa terms look the same [19]. Note that the only difference of this $\mathcal{L}_Y$ to that of equation (23) is the factor 1/2 which we do not introduce in the Dirac case. It will become clear in the last paragraph of this section why we prefer this definition.

Fermion selfenergy: With the above definitions, the renormalization programme of section 3 goes through with only minor modifications. The renormalized fermion selfenergy for Dirac fermions is given by

$$\begin{aligned}\Sigma(p) \;=\;& \Sigma^{\text{1-loop}}(p) - \not{p}\left[\hat{\delta}^{(\chi_L)}\gamma_L + \hat{\delta}^{(\chi_R)}\gamma_R\right] \\ &+ \hat{v}_a\left[\delta\hat{Y}_a\gamma_L + \left(\delta\hat{Y}_a\right)^\dagger\gamma_R\right] + \Delta\hat{v}_a\left[\hat{Y}_a\gamma_L + \hat{Y}_a^\dagger\gamma_R\right].\end{aligned} \tag{111}$$

Eventually, the one-loop Dirac masses read

$$m_i = m_{0i} + \frac{1}{2}m_{0i}\left[\left(\Sigma_L^{(A)}\right)_{ii}(m_{0i}^2) + \left(\Sigma_R^{(A)}\right)_{ii}(m_{0i}^2)\right] + \mathrm{Re}\left(\Sigma_L^{(B)}\right)_{ii}(m_{0i}^2). \tag{112}$$

Note that $\mathrm{Re}\left(\Sigma_L^{(B)}\right)_{ii} = \mathrm{Re}\left(\Sigma_R^{(B)}\right)_{ii}$ because of the symmetry relation (A1).

Computation of amplitudes: There are two changes when we switch from Majorana to Dirac fermions [14]:

i. $\hat{Y}_a^* \to \hat{Y}_a^\dagger$ and

ii. a factor of two for every closed Dirac fermion loop compared to the corresponding Majorana fermion loop.

As discussed above, the first change simply comes from the fact that for Dirac neutrinos the Yukawa coupling matrices are not symmetric. The reason for the factor of two is the following. In the Majorana case we have defined the Yukawa Lagrangian with a factor 1/2—see equation (23). If a Majorana fermion line in a Feynman diagram is not closed, then all factors of 1/2 are cancelled because, whenever a fermion line is connected to a vertex, there are two possible Wick contractions; however, in a closed loop one factor 1/2 is left over because, when closing the loop, there is only one contraction. In the Dirac case, we have omitted the factor 1/2 in the Yukawa Lagrangian (110) because, when we connect a Dirac fermion line to a vertex, there is exactly one Wick contraction. Therefore, when a closed fermion loop occurs, there is a factor of two for Dirac fermions relative to Majorana fermions. Finally, whenever we have made a simplification in a trace by exploiting $Y_a^T = Y_a$ in the Majorana case, as done in equations (47a) and (54), we have to revoke it in the Dirac case.

Consequently, in the Dirac case, $\Pi^{(a)}(p^2)$ is given by

$$\begin{aligned}\Pi^{(a)}_{ab}(p^2) \;=\; & \frac{2}{16\pi^2}\Big\{c_\infty\,\mathrm{Tr}\Big[\hat{Y}_a\hat{m}_0\hat{Y}_b\hat{m}_0 + \hat{Y}_a^\dagger\hat{m}_0\hat{Y}_b^\dagger\hat{m}_0 + \hat{Y}_a\hat{Y}_b^\dagger\hat{m}_0^2 + \hat{Y}_a^\dagger\hat{Y}_b\hat{m}_0^2 \\ & +\hat{Y}_a\hat{m}_0^2\hat{Y}_b^\dagger + \hat{Y}_a^\dagger\hat{m}_0^2\hat{Y}_b\Big] - \frac{1}{2}c_\infty\,\mathrm{Tr}\Big[\hat{Y}_a\hat{Y}_b^\dagger + \hat{Y}_a^\dagger\hat{Y}_b\Big]p^2 \\ & +\frac{1}{2}\,\mathrm{Tr}\Big[\Big(\hat{Y}_a\hat{Y}_b^\dagger + \hat{Y}_b^\dagger\hat{Y}_a + \hat{Y}_a^\dagger\hat{Y}_b + \hat{Y}_b\hat{Y}_a^\dagger\Big)\Big(\hat{m}_0^2 - \frac{1}{6}p^2\Big)\Big] - \cdots\Big\}. \quad (113)\end{aligned}$$

The dots refer to the integral in equation (47a) where merely Y_a^* has to be substituted by $Y_a^\dagger$. From equation (113), $\big(\Pi^{(a)}_\infty\big)_{ab}$ can be read off. Equation (54) is modified to

$$\delta\hat{\lambda}^{(\chi)}_{abcd} = -\frac{1}{3}\times\frac{1}{16\pi^2}c_\infty\mathrm{Tr}\left[\hat{Y}_a\hat{Y}_b^\dagger\hat{Y}_c\hat{Y}_d^\dagger + \cdots + \hat{Y}_a^\dagger\hat{Y}_b\hat{Y}_c^\dagger\hat{Y}_d + \cdots\right], \quad (114)$$

where the dots indicate the five non-trivial permutations of the indices b, c, d. No complications arise in equations (50), (57), (67) and (68); for Dirac fermions one simply has to multiply the right-hand side by a factor of two and replace complex conjugation by hermitian conjugation.

7 Conclusions

In this paper we have presented a versatile and simple renormalization procedure which is adapted to models which have SSB and a multitude of scalars. This renormalization programme takes seriously the nature of masses as functions of the parameters of the underlying model; therefore, physical masses have an expansion in perturbation theory just like any other observable. We have exemplified our renormalization procedure by discussing a general Yukawa model with an arbitrary number of fermion fields of Majorana or Dirac nature and an arbitrary number of real scalar fields; moreover, this toy model has the feature that tree-level fermion masses are generated by SSB of a cyclic group. In particular, we have explicitly computed the fermionic and scalar selfenergies and studied radiative corrections at the one-loop level to tree-level masses.

The main idea discussed in this paper is to split renormalization into a step in which UV divergent parts are cancelled by $\overline{\mathrm{MS}}$ renormalization of the parameters of the unbroken theory and a subsequent step in which finite corrections are performed to make the scalar one-point functions vanish and to obtain one-loop pole masses. We have presented the details of the cancellation of UV divergences and elucidated the role of tadpole diagrams in our renormalization procedure and their contributions to the masses. We have also applied our findings to a showcase model furnished with a non-Abelian flavour symmetry group.

A typical example where the renormalization procedure put forward in this paper can be applied is the lepton sector of the multi-Higgs-doublet Standard Model with an arbitrary number of right-handed neutrino singlets and flavour symmetries; this comprises the seesaw mechanism as well as light sterile neutrinos. A derivation of general formulae which permit to compute radiative corrections to tree-level predictions of masses and mixing angles in this rather general class of flavour models is in preparation.

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A Selfenergies and on-shell renormalization

Since for fermions the general relations

$$\left(\Sigma_L^{(A)}\right)^\dagger = \Sigma_L^{(A)}, \quad \left(\Sigma_R^{(A)}\right)^\dagger = \Sigma_R^{(A)}, \quad \left(\Sigma_L^{(B)}\right)^\dagger = \Sigma_R^{(B)} \tag{A1}$$

are valid,[7] we see that for the finiteness of $\Sigma_L^{(A)}$ and $\Sigma_R^{(A)}$ the counterterm with the hermitian $\delta^{(\chi)}$ suffices. In addition, we remark that in the case of Majorana fermions the further conditions [19, 20]

$$\left(\Sigma_L^{(A)}\right)^T = \Sigma_R^{(A)}, \quad \left(\Sigma_L^{(B)}\right)^T = \Sigma_L^{(B)}, \quad \left(\Sigma_R^{(B)}\right)^T = \Sigma_R^{(B)} \tag{A2}$$

hold. This is a general condition, but can also be seen explicitly in our one-loop result.

In order to switch from the renormalized Majorana selfenergy $\Sigma(p)$ and the bosonic selfenergy $\Pi(p^2)$ to the on-shell selfenergies $\widetilde{\Sigma}(p)$ and $\widetilde{\Pi}(p^2)$, respectively, we must allow for finite field strength renormalization matrices. Denoting these by

$$\overset{\circ}{Z}{}_\chi^{(1/2)} = \mathbb{1} + \frac{1}{2}\overset{\circ}{z}_\chi \quad \text{and} \quad \overset{\circ}{Z}{}_h^{(1/2)} = \mathbb{1} + \frac{1}{2}\overset{\circ}{z}_h, \tag{A3}$$

we have at one-loop order

$$\begin{aligned} \widetilde{\Sigma}(p) &= \Sigma(p) - \frac{1}{2}\not{p}\left[\left(\left(\overset{\circ}{z}_\chi\right)^\dagger + \overset{\circ}{z}_\chi\right)\gamma_L + \left(\left(\overset{\circ}{z}_\chi\right)^\dagger + \overset{\circ}{z}_\chi\right)^*\gamma_R\right] \\ &\quad + \frac{1}{2}\left[\left(\left(\overset{\circ}{z}_\chi\right)^T \hat{m}_0 + \hat{m}_0\,\overset{\circ}{z}_\chi\right)\gamma_L + \left(\left(\overset{\circ}{z}_\chi\right)^T \hat{m}_0 + \hat{m}_0\,\overset{\circ}{z}_\chi\right)^*\gamma_R\right], \end{aligned} \tag{A4}$$

$$\widetilde{\Pi}_{ab}(p^2) = \Pi_{ab}(p^2) - \frac{1}{2}\left[\left(\overset{\circ}{z}_h\right)^T + \overset{\circ}{z}_h\right]_{ab} p^2 + \frac{1}{2}\left[\left(\overset{\circ}{z}_h\right)^T \hat{M}_0^2 + \hat{M}_0^2\,\overset{\circ}{z}_h\right]_{ab}. \tag{A5}$$

It is important to note that we have no freedom for mass renormalization because in our scheme the masses are computed in terms of the renormalized parameters of the model. Due to the Majorana nature of the fermions under consideration, the relation

$$\overset{\circ}{z}_\chi \equiv \left(\overset{\circ}{z}_L\right)_{ij} = \left(\overset{\circ}{z}_R\right)^*_{ij} \tag{A6}$$

holds for left and right-chiral fields. In $\widetilde{\Sigma}(p)$ this fact has been taken into account. Using the second relation in equation (A1) and the first relation in equation (A2), the on-shell conditions lead for $i \neq j$ to [6, 16, 17, 21]

$$\begin{aligned} &\frac{1}{2}(\overset{\circ}{z}_\chi)_{ij} = \\ &-\frac{1}{m_{0i}^2 - m_{0j}^2}\left[m_{0j}^2\left(\Sigma_L^{(A)}\right)_{ij} + m_{0i}m_{0j}\left(\Sigma_L^{(A)}\right)_{ji} + m_{0j}\left(\Sigma_L^{(B)}\right)^*_{ji} + m_{0i}\left(\Sigma_L^{(B)}\right)_{ij}\right]_{p^2=m_{0j}^2}. \end{aligned} \tag{A7}$$

[7]Strictly speaking these relations hold only for the dispersive part of the selfenergy.

Furthermore, for $i = j$ we obtain

$$\mathrm{Re}(\overset{\circ}{z}_\chi)_{ii} = \tag{A8}$$
$$(\Sigma_L^{(A)})_{ii}(m_{0i}^2) + 2m_{0i}^2 \frac{\mathrm{d}}{\mathrm{d}p^2}\left(\Sigma_L^{(A)})_{ii}(p^2)\right)\Big|_{p^2=m_{0i}^2} + 2m_{0i}\frac{\mathrm{d}}{\mathrm{d}p^2}\,\mathrm{Re}\left(\Sigma_L^{(B)})_{ii}(p^2)\right)\Big|_{p^2=m_{0i}^2}$$

and

$$m_{0i}\,\mathrm{Im}\,(\overset{\circ}{z}_\chi)_{ii} = -\mathrm{Im}\,(\Sigma_L^{(B)})_{ii}(m_{0i}^2). \tag{A9}$$

It is characteristic of Majorana fermions that there is no phase freedom in the determination of the field strength renormalization matrix, *i.e.* not only the real part but also the imaginary part of $(z_\chi)_{ii}$ is fixed.

Finally, in the scalar scalar case we are lead to

$$a \neq b\colon\ \frac{1}{2}\left(\overset{\circ}{z}_h\right)_{ab} = -\frac{\Pi_{ab}(M_{0b}^2)}{M_{0a}^2 - M_{0b}^2}, \quad a = b\colon\ \left(\overset{\circ}{z}_h\right)_{aa} = \frac{\mathrm{d}\Pi_{aa}(p^2)}{\mathrm{d}p^2}\Big|_{p^2=M_{0a}^2} \tag{A10}$$

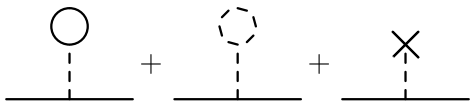
for on-shell renormalization.

B Finite tadpole contributions

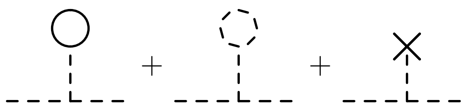
Throughout this appendix the discussion refers to the one-loop order. In the fermionic as well as the scalar selfenergy, tadpole diagrams contribute indirectly via the finite shift (36), even though in both cases the condition $t_a = 0$ of equation (18) and the requirement that the scalar one-point function is zero—see equation (32)—procure the vanishing of the sum of tadpole diagrams and the term

$$\Delta\hat{t}_a + \delta\hat{\mu}_{ab}^2\hat{v}_b + \delta\hat{\lambda}_{abcd}\,\hat{v}_b\hat{v}_c\hat{v}_d. \tag{B1}$$

Diagrammatically, this can be written as


$= 0$ (B2)

and


$= 0,$ (B3)

where the cross symbolizes the contribution of equation (B1). Still, the finite parts of the tadpole diagrams generate, via the finite VEV shifts Δv_a, the mass shifts

$$\Delta\hat{m}_0 = \hat{Y}_a\Delta\hat{v}_a \tag{B4}$$

for the fermions—see equation (28)—and

$$(\Delta\hat{M}_0^2)_{ab} = 6\hat{\lambda}_{abcd}\hat{v}_c\Delta\hat{v}_d \tag{B5}$$

for the real scalars—see equation (30). These add to the counterterms of the fermionic and scalar two-point functions. In terms of diagrams, this can be symbolized as

$$= -i\left(\delta\hat{Y}_a\hat{v}_a + \Delta\hat{m}_0\right) \tag{B6}$$

for the fermions and

$$= -i\left(\delta\hat{\mu}^2_{ab} + 3\delta\hat{\lambda}_{abcd}\hat{v}_c\hat{v}_d + (\Delta\hat{M}^2_0)_{ab}\right) \tag{B7}$$

for the scalars.

C Integrals

$$\mathcal{M}^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\Delta+i\epsilon} = \frac{i}{16\pi^2}\Delta\left(c_\infty + 1 - \ln\frac{\Delta}{\mathcal{M}^2}\right), \tag{C1}$$

$$\mathcal{M}^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{16\pi^2}\left(c_\infty - \ln\frac{\Delta}{\mathcal{M}^2}\right), \tag{C2}$$

$$\mathcal{M}^\varepsilon \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^2}{(k^2-\Delta+i\epsilon)^2} = \frac{i}{16\pi^2}\Delta\left(2c_\infty + 1 - 2\ln\frac{\Delta}{\mathcal{M}^2}\right). \tag{C3}$$

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CHAPTER 5

One-loop lepton masses in the mHDSM

Walter Grimus and Maximilian Löschner

Renormalization of the multi-Higgs-doublet Standard Model and one-loop lepton mass corrections

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Renormalization of the multi-Higgs-doublet Standard Model and one-loop lepton mass corrections

W. Grimus* and M. Löschner‡

University of Vienna, Faculty of Physics
Boltzmanngasse 5, A–1090 Vienna, Austria

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Abstract

Motivated by models for neutrino masses and lepton mixing, we consider the renormalization of the lepton sector of a general multi-Higgs-doublet Standard Model with an arbitrary number of right-handed neutrino singlets. We propose to make the theory finite by $\overline{\text{MS}}$ renormalization of the parameters of the unbroken theory. However, using a general R_ξ gauge, in the explicit one-loop computations of one-point and two-point functions it becomes clear that—in addition—a renormalization of the vacuum expectation values (VEVs) is necessary. Moreover, in order to ensure vanishing one-point functions of the physical scalar mass eigenfields, finite shifts of the tree-level VEVs, induced by the finite parts of the tadpole diagrams, are required. As a consequence of our renormalization scheme, physical masses are functions of the renormalized parameters and VEVs and thus derived quantities. Applying our scheme to one-loop corrections of lepton masses, we perform a thorough discussion of finiteness and ξ-independence. In the latter context, the tadpole contributions figure prominently.

*E-mail: walter.grimus@univie.ac.at
‡E-mail: maximilian.loeschner@univie.ac.at

1 Introduction

In this paper we propose a renormalization scheme for the multi-Higgs-doublet Standard Model (mHDSM). We are motivated by models for neutrino masses and lepton mixing, which all have an extended scalar sector. In the simplest cases, they have several Higgs doublets and a number of right-handed neutrino gauge singlet fields and permit in this way to incorporate the seesaw mechanism [1, 2, 3, 4, 5].

Our framework is the following. We consider the lepton sector[1] of an extended Standard Model (SM), comprising $n_L = 3$ left-handed lepton gauge doublets,[2] n_R right-handed neutrino gauge singlet fields and n_H Higgs doublets. We assume

$$n_R \geq 1 \quad \text{and} \quad n_H \geq 1, \tag{1}$$

but otherwise these numbers are arbitrary. We furthermore postulate that spontaneous gauge-symmetry breaking in the mHDSM happens in the same way as in the SM, *i.e.* the SM gauge group is broken down to $U(1)_{\text{em}}$. We do not discuss conditions on the scalar potential $V(\phi)$ which make this symmetry breaking possible. Though above we have mentioned the seesaw mechanism, in the renormalization of the mHDSM we do not assume anything about the scale of the right-handed neutrino masses; our setting is completely general with respect to fermion mass scales, but the seesaw mechanism is included. One-loop corrections to the seesaw mechanism have been computed earlier in [6, 7, 8] (see also [9, 10]) and at the end of the present paper we will comment on the relationship between our renormalization scheme here and the radiative corrections of ref. [8].

Before we lay out the renormalization scheme, we have to discuss some of our notation. For the precise definitions of the parameters of the Lagrangian we refer the reader to section 2. We choose positive gauge coupling constants g and g' of $SU(2)_L$ and $U(1)_Y$, respectively, and thus the sine and cosine of the Weinberg angle, s_w and c_w, respectively, are positive as well.[3] In the discussion in the present paper we do not need to renormalize g and g'. We will always use renormalized parameters of the Lagrangian; these include the Yukawa coupling matrices Δ_k and Γ_k ($k = 1, \ldots, n_H$), the parameters of the scalar potential $V(\phi)$, μ^2_{ij} and λ_{ijkl}, and the Majorana mass matrix M_R of the right-handed neutrino singlets. The corresponding counterterm parameters are denoted by $\delta\Delta_k$, $\delta\Gamma_k$, $\delta\mu^2_{ij}$, $\delta\lambda_{ijkl}$ and δM_R. The vacuum expectation values v_k (VEVs) are in the notation of our paper pure tree-level quantities, in principle expressible in terms of μ^2_{ij} and λ_{ijkl} by finding the minimum of the (tree level) scalar potential $V(\phi)$, written in terms of the renormalized parameters.

We will work in the R_ξ or 't Hooft gauge [14, 15, 16] with general parameters ξ in almost all our computations[4] and use dimensional regularization in the one-loop computations. We will not go beyond the one-loop level.

[1]We do not consider quarks in our discussion though they could be included in a straightforward way.

[2]Actually, the value $n_L = 3$ comes solely from the known three families of fermions, but has otherwise no bearing on our discussion.

[3]Note the sign difference to [11, 12, 13] where $-g$ occurs in the covariant derivative.

[4]When we write ξ we mean ξ_W, ξ_Z and ξ_A, which occur in the propagators of the $W^\pm$, Z and photon, respectively.

Physically, only the spontaneously broken mHDSM makes sense, because otherwise all fermions and vector bosons would be massless. The renormalization scheme for the *broken* theory we propose consists of $\overline{\text{MS}}$ renormalization of the parameters of the *unbroken* theory plus a VEV renormalization with renormalization parameters δv_k. That the latter is necessary in a gauge theory quantized in an R_ξ gauge with $\xi \neq 0$ has been proven in [17]. Note that this δv_k is a renormalization in addition to the scalar wave-function renormalization already included in the VEVs v_k. Complying with our proposed renormalization scheme, we use dimensional regularization in $d = 4 - \varepsilon$ dimensions. Therefore, at one-loop order the renormalization parameters $\delta\Delta_k$, $\delta\Gamma_k$, $\delta\mu^2_{ij}$, $\delta\lambda_{ijkl}$, δM_R and δv_k are all proportional to

$$c_\infty = \frac{2}{\varepsilon} + \gamma_E + \ln(4\pi). \tag{2}$$

(Note that in the present paper we use the symbol "δ" solely for the purpose of indicating quantities proportional to c_∞.) We will show that the proposed scheme, with counterterms induced by renormalization parameters listed here, allows to remove all divergences at the one-loop level and that the divergences uniquely determine the counterterm parameters.

Previously, avoiding the intricacies of gauge theories, this very fact has been demonstrated in [18] for a general Yukawa model with an arbitrary number of real scalars. (For an early attempt with only one scalar see [19].)

In detail, we proceed as follows:

1. We determine $\delta\lambda_{ijkl} + \delta\lambda_{ilkj}$ from the divergence of the neutral-scalar four-point function of the unbroken theory.[5]

2. Plugging $\delta\lambda_{ijkl} + \delta\lambda_{ilkj}$ into the counterterm of the scalar two-point function of the broken phase, the remaining divergencies uniquely fix $\delta\mu^2_{ij}$ and δv_k.

3. With the so far obtained renormalization parameters we compute the counterterm for the scalar one-point function and, as a check, we prove its finiteness.

4. We determine $\delta\Delta_k$ and $\delta\Gamma_k$ from the divergencies of vertex corrections of the neutral-scalar couplings to neutrinos and charged leptons, respectively. For simplicity, this is also done in the unbroken theory.

5. Having obtained $\delta\Delta_k$, $\delta\Gamma_k$, δv_k and the counterterm of the scalar one-point function, all ingredients required for the counterterms of the fermion self-energies are at hand and can be determined. We demonstrate that these make indeed the neutrino self-energy Σ_ν and the charged-lepton self-energy Σ_ℓ finite.[6]

6. Finally, having in mind formulas for the extraction of corrections to the tree-level pole masses from Σ_ν and Σ_ℓ—see for instance [20, 21], we discuss radiative corrections to the tree-level physical neutrino and charged-lepton masses. In particular, we carefully examine the ξ-independence of these physical quantities.

[5] This combination is sufficient for our purposes—see section 3.1.

[6] This constitutes another independent cross check of our renormalization scheme.

We emphasize that in our renormalization scheme there is no mass renormalization because both scalar and fermion masses are derived quantities and the mass counterterms are, therefore, derived quantities as well. This is a consequence of renormalizing the parameters of the *unbroken* theory, which is owing to the fact that, for an arbitrary number of Higgs doublets, the number of Yukawa coupling constants is in general much larger than the number of fermion masses.

Since we are discussing the lepton sector of the mHDSM, we have both Dirac and Majorana fermions in the theory. When we deem it helpful for the reader, we stress the differences in the treatment of both types of fermions and dwell on the field-theoretical specifics for Majorana neutrinos.

The $2n_H$ neutral scalar mass eigenfields have by definition vanishing VEVs. These have to be re-adjusted, order by order, by finite VEV shifts Δv_k such that the scalar one-point functions vanish [22, 23].[7] In an n-point function with $n \geq 2$ one can either take into account these VEV shifts or, equivalently, include all tadpole diagrams instead, as shown for the SM in [22, 23]. We show this explicitly in the mHDSM at the one-loop level for the neutrino and charged-lepton self-energies. Moreover, tadpole diagrams play an important role with respect to ξ-independence of physical observables [24]. We present a thorough discussion of this role in the context of radiative fermion mass corrections. Other methods for the treatment of tadpole contributions are carried out in [25, 26, 27] for variants of the two-Higgs doublet model.

Concerning the notation, we have already explained that the parameters of the Lagrangian are always considered as being renormalized quantities. Furthermore, $-i\Sigma_\nu(p)$, $-i\Sigma_\ell(p)$ and $-i\Pi(p^2)$ denote the two-point functions of neutrinos, charged fermions and neutral scalars, respectively, as obtained in perturbation theory, including all counterterms. Thus the corresponding quantities with $(-i)$ removed denote the *renormalized* self-energies.

In order to enhance legibility of the paper we list here the definition and notation of all masses which occur in the paper:

- The tree-level neutrino masses are m_i $(i = 1, 2, \ldots, n_L + n_R)$.
- The tree-level charged-lepton masses are m_α $(\alpha = e, \mu, \tau)$.
- The *finite* radiative corrections to tree-level fermion masses are denoted by Δm_i and Δm_α.
- Then the total fermion masses are given by $m_{\text{tot},i} = m_i + \Delta m_i$ for neutrinos and $m_{\text{tot},\alpha} = m_\alpha + \Delta m_\alpha$ for charged leptons.
- The scalar masses are denoted by M_{+a} $(a = 1, \ldots, n_H)$ and M_b $(b = 1, \ldots, 2n_H)$ for charged and neutral scalars, respectively.
- The vector boson masses are denoted by m_W and m_Z for $W^\pm$ and Z boson, respectively.

[7] We emphasize once more that, in our notation, δv_k is infinite while Δv_k is finite. In the rest of the paper we always reserve the symbol "Δ" for finite quantities.

- Charged and neutral Goldstone bosons correspond to the indices $a = 1$ and b =1, respectively, with eigenvalues $M^2_{+1} = M^2_1 = 0$ of the respective mass matrices.
- However, due to the R_ξ gauge the Goldstone bosons have squared masses $\xi_W m^2_W$ and $\xi_Z m^2_Z$ in the respective propagators.

All boson masses are tree-level masses. If in an expression several summations occur referring to charged-scalar mass eigenfields or masses, then the indices a, a' or $a_1, a_2, \ldots$ are used. In the case of neutral scalars, b, b' or $b_1, b_2, \ldots$ is utilized.

The paper is organized as follows. In section 2 we write down all interaction Lagrangians of the mHDSM needed for the computation of one- and two-point scalar functions and the self-energies of the charged leptons and Majorana neutrinos. This section also includes important relations concerning the diagonalization of the $(n_L+n_R)\times(n_L+n_R)$ neutrino mass matrix. In section 3 we discuss the counterterms of the scalar one- and two-point functions and determine all counterterm parameters, including δv_k of the scalar sector. Section 4 is devoted to a thorough examination of the ξ-independence of the one-loop fermion masses. In section 5 we prove the finiteness of the fermion self-energies in our renormalization scheme. In section 6 we present formulas for these self-energies in Feynman gauge, by listing the individual contributions originating from charged-scalar, neutral-scalar, W and Z exchange, and and discuss the special case of the seesaw mechanism. Finally, our conclusions are found in section 7. In appendix A we show how the charged and neutral-scalar mass matrices are obtained from the scalar potential, while in appendix B we discuss properties of the diagonalization matrices of the scalar mass terms. A short consideration of on-shell contributions to fermion self-energies is found in appendix C. Lastly, in appendix D we convert the loop functions that we use in section 6 to other functions commonly used in the literature.

2 Lagrangians

The formalism for the mHDSM has been developed in [11, 12, 28] (see also [8]).

2.1 Yukawa Lagrangian and lepton mass matrices

In this subsection we follow the notation of [11] and repeat some material from this paper. As mentioned in the introduction, we assume that the electric charge remains conserved after spontaneous symmetry breaking. Therefore, we can parameterize the Higgs doublets and their VEVs as

$$\phi_k = \begin{pmatrix} \varphi_k^+ \\ \varphi_k^0 \end{pmatrix}, \quad \tilde{\phi}_k = \begin{pmatrix} \varphi_k^{0*} \\ -\varphi_k^- \end{pmatrix}, \quad \langle \phi_k \rangle_0 = \frac{v_k}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3}$$

with

$$v = \sqrt{\sum_k |v_k|^2} \simeq 246\,\text{GeV}. \tag{4}$$

The Yukawa Lagrangian may be written as

$$\mathcal{L}_\mathrm{Y} = -\sum_{k=1}^{n_H} \left[\left(\begin{array}{cc} \varphi_k^-, & \varphi_k^{0*} \end{array} \right) \bar{e}_R \, \Gamma_k + \left(\begin{array}{cc} \varphi_k^0, & -\varphi_k^+ \end{array} \right) \bar{\nu}_R \, \Delta_k \right] \left(\begin{array}{c} \nu_L \\ e_L \end{array} \right) + \mathrm{H.c.}, \tag{5}$$

where the Yukawa coupling matrices Γ_k and Δ_k are $n_L \times n_L$ and $n_R \times n_L$, respectively. The lepton mass terms are given by

$$\mathcal{L}_\mathrm{mass} = -\bar{e}_R M_\ell e_L - \bar{\nu}_R M_D \nu_L + \frac{1}{2} \nu_R^T C^{-1} M_R^* \nu_R + \mathrm{H.c.} \tag{6}$$

with[8]

$$M_\ell = \frac{1}{\sqrt{2}}\, v_k^* \Gamma_k \quad \text{and} \quad M_D = \frac{1}{\sqrt{2}}\, v_k \Delta_k. \tag{7}$$

The $n_R \times n_R$ matrix M_R is in general complex and symmetric. With the chiral projectors

$$\gamma_L = \frac{1}{2}\left(\mathbb{1} - \gamma_5\right) \quad \text{and} \quad \gamma_R = \frac{1}{2}\left(\mathbb{1} + \gamma_5\right), \tag{8}$$

the fermion mass eigenfields ℓ_α ($\alpha = e, \mu, \tau$) and χ_i ($i = 1, \ldots, n_L + n_R$) are obtained from the weak chiral eigenfields $e_{L,R}$ and $\nu_{L,R}$ by the transformations

$$e_L = W_L \gamma_L \ell, \quad e_R = W_R \gamma_R \ell, \quad \nu_L = U_L \gamma_L \chi, \quad \nu_R = U_R \gamma_R \chi. \tag{9}$$

The matrices W_L and W_R are unitary $n_L \times n_L$ matrices such that[9]

$$W_R^\dagger M_\ell W_L \equiv \hat{m}_\ell = \mathrm{diag}\left(m_e, m_\mu, m_\tau\right). \tag{10}$$

The matrices U_L and U_R are $n_L \times (n_L + n_R)$ and $n_R \times (n_L + n_R)$, respectively, such that the matrix

$$\mathcal{U} \equiv \left(\begin{array}{c} U_L \\ U_R^* \end{array} \right) \tag{11}$$

is $(n_L + n_R) \times (n_L + n_R)$ unitary. The unitarity of $\mathcal{U}$ is expressed as

$$U_L U_L^\dagger = \mathbb{1}_{n_L}, \tag{12a}$$
$$U_R U_R^\dagger = \mathbb{1}_{n_R}, \tag{12b}$$
$$U_L U_R^T = 0_{n_L \times n_R}, \tag{12c}$$

and

$$U_L^\dagger U_L + U_R^T U_R^* = \mathbb{1}_{n_L + n_R}. \tag{13}$$

$\mathcal{U}$ diagonalizes the $(n_L + n_R) \times (n_L + n_R)$ Majorana neutrino mass matrix, *i.e.*

$$\mathcal{U}^T \left(\begin{array}{cc} 0 & M_D^T \\ M_D & M_R \end{array} \right) \mathcal{U} \equiv \hat{m}_\nu = \mathrm{diag}\left(m_1, m_2, \ldots, m_{n_L + n_R}\right), \tag{14}$$

[8]Here and in the following we use the summation convention.

[9]We deviate slightly in notation from that of [11] where a basis has been assumed with $W_L = W_R = \mathbb{1}$. In the present paper, for the sake of clarity, we stick to general unitary matrices W_L and W_R.

with real and non-negative m_i [29]. Therefore,

$$\begin{aligned} U_L^* \hat{m}_\nu U_L^\dagger &= 0_{n_L \times n_L} \,, && (15a)\\ U_R \hat{m}_\nu U_L^\dagger &= M_D \,, && (15b)\\ U_R \hat{m}_\nu U_R^T &= M_R \,. && (15c) \end{aligned}$$

A further relation is given by [8]

$$U_R^\dagger M_D = \hat{m}_\nu U_L^\dagger. \tag{16}$$

Now we turn to the scalar mass eigenfields S_a^+ ($a = 1, \ldots, n_H$) and S_b^0 ($b = 1, \ldots, 2n_H$), related to φ_k^+ and φ_k^0 by

$$\varphi_k^+ = U_{ka} S_a^+ \quad \text{and} \quad \varphi_k^0 = \frac{1}{\sqrt{2}} \left(v_k + V_{kb} S_b^0 \right), \tag{17}$$

respectively. For the definition and properties of the matrices U and V we refer the reader to appendix B.

Now we are in a position to formulate the Yukawa interactions in terms of mass eigenfields. Since we perform computations with Majorana neutrinos, *i.e.* $\chi^c = \chi$, it is useful to have at hand both the interaction Lagrangians of the charged-lepton fields ℓ and of the charge-conjugated fields ℓ^c [30]. The neutral-scalar Yukawa interaction Lagrangian may be written as

$$\mathcal{L}_\mathrm{Y}(S^0) = -\frac{1}{\sqrt{2}} S_b^0 \left\{ \bar{\chi} \left[F_b \gamma_L + F_b^\dagger \gamma_R \right] \chi + \bar{\ell} \left[G_b \gamma_L + G_b^\dagger \gamma_R \right] \ell \right\}. \tag{18}$$

Note that

$$\bar{\ell} \left[G_b \gamma_L + G_b^\dagger \gamma_R \right] \ell = \overline{\ell^c} \left[G_b^T \gamma_L + G_b^* \gamma_R \right] \ell^c. \tag{19}$$

The charged-scalar Yukawa interaction Lagrangian can be formulated as

$$\begin{aligned} \mathcal{L}_\mathrm{Y}(S^\pm) &= S_a^- \bar{\ell} \left[R_a \gamma_R - L_a \gamma_L \right] \chi + S_a^+ \bar{\chi} \left[R_a^\dagger \gamma_L - L_a^\dagger \gamma_R \right] \ell && (20a)\\ &= S_a^- \bar{\chi} \left[R_a^T \gamma_R - L_a^T \gamma_L \right] \ell^c + S_a^+ \overline{\ell^c} \left[R_a^* \gamma_L - L_a^* \gamma_R \right] \chi. && (20b) \end{aligned}$$

Then for these Lagrangians the coupling matrices are given by [11]

$$\begin{aligned} F_b &= \frac{1}{2} \left(U_R^\dagger \Delta_k U_L + U_L^T \Delta_k^T U_R^* \right) V_{kb}, && (21a)\\ G_b &= \left(W_R^\dagger \Gamma_k W_L \right) V_{kb}^*, && (21b)\\ R_a &= \left(W_L^\dagger \Delta_k^\dagger U_R \right) U_{ka}^*, && (21c)\\ L_a &= \left(W_R^\dagger \Gamma_k U_L \right) U_{ka}^*. && (21d) \end{aligned}$$

Since we identify the scalars carrying index 1 with the Goldstone bosons, we have $S_1^0 \equiv G^0$ and $S_1^+ \equiv G^+$. Using the matrix elements V_{k1} and U_{k1}, required for the Goldstone boson couplings, of equation (B.14) in appendix B, we obtain

$$\Delta_k V_{k1} = i \Delta_k U_{k1} = i \frac{\sqrt{2}}{v} M_D, \quad \Gamma_k V_{k1}^* = -i \Gamma_k U_{k1}^* = -i \frac{\sqrt{2}}{v} M_\ell, \tag{22}$$

with v being defined in equation (4). Then, exploiting the formulas for the diagonalization of the fermion mass matrices, the coupling matrices of G^0 and $G^\pm$ can be converted into

$$F_1 = \frac{i}{\sqrt{2}v}\left(\hat{m}_\nu U_L^\dagger U_L + U_L^T U_L^* \hat{m}_\nu\right), \quad G_1 = -i\frac{\sqrt{2}}{v}\hat{m}_\ell \tag{23}$$

and [11]

$$R_1 = \frac{\sqrt{2}}{v} W_L^\dagger U_L \hat{m}_\nu, \quad L_1 = \frac{\sqrt{2}}{v}\hat{m}_\ell W_L^\dagger U_L, \tag{24}$$

respectively.

2.2 Charged and neutral current interactions

The squares of vector boson masses are

$$m_W^2 = \frac{g^2 v^2}{4}, \quad m_Z^2 = \frac{g^2 v^2}{4c_w^2}, \tag{25}$$

with $c_w = m_W/m_Z$ being the cosine of the weak mixing or Weinberg angle.

In terms of the lepton mass eigenfields, we obtain the charged-current Lagrangian

$$\mathcal{L}_{\rm cc} = -\frac{g}{\sqrt{2}}\left[W_\mu^- \bar{\ell}\left(W_L^\dagger U_L\right)\gamma^\mu\gamma_L\chi + W_\mu^+\bar{\chi}\left(U_L^\dagger W_L\right)\gamma^\mu\gamma_L\ell\right] \tag{26a}$$

$$= +\frac{g}{\sqrt{2}}\left[W_\mu^- \bar{\chi}\left(W_L^\dagger U_L\right)^T\gamma^\mu\gamma_R\ell^c + W_\mu^+\overline{\ell^c}\left(U_L^\dagger W_L\right)^T\gamma^\mu\gamma_R\chi\right], \tag{26b}$$

and the neutral-current Lagrangians [8, 11]

$$\mathcal{L}_{\rm nc} = -\frac{g}{4c_w} Z_\mu\bar{\chi}\gamma^\mu F_{LR}\chi - \frac{g}{c_w} Z_\mu\bar{\ell}\gamma^\mu\left[\left(s_w^2 - \frac{1}{2}\right)\gamma_L + s_w^2\gamma_R\right]\ell \tag{27}$$

with

$$F_{LR} = \left(U_L^\dagger U_L\right)\gamma_L - \left(U_L^T U_L^*\right)\gamma_R. \tag{28}$$

Finally, the electromagnetic interaction Lagrangian of the charged leptons with charge $-e$ is

$$\mathcal{L}_{\rm em} = e\, A_\mu\bar{\ell}\gamma^\mu\ell. \tag{29}$$

Concerning the vector boson propagators in the R_ξ gauge, they have the form

$$\Delta_V^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 - m_V^2 + i\epsilon} + \frac{k^\mu k^\nu}{m_V^2}\left(\frac{1}{k^2 - m_V^2 + i\epsilon} - \frac{1}{k^2 - \xi_V m_V^2 + i\epsilon}\right) \tag{30a}$$

$$= -\frac{g^{\mu\nu}}{k^2 - m_V^2 + i\epsilon} + (1-\xi_V)\frac{k^\mu k^\nu}{(k^2 - m_V^2 + i\epsilon)(k^2 - \xi_V m_V^2 + i\epsilon)} \tag{30b}$$

with $V = Z, W, A$. For photons only the second form of the propagator is meaningful.

2.3 Vector boson–scalar interactions

Here we only display those interaction Lagrangians which we need in the present paper. For the complete set of vector boson–scalar interaction Lagrangians see [12, 28]. Derivative couplings of the vector bosons to scalars are given by

$$\begin{aligned} \mathcal{L}_\partial &= \frac{ig}{2} V_{kb} U_{ka}^* W_\mu^+ \left(S_b^0 \partial^\mu S_a^- - S_a^- \partial^\mu S_b^0 \right) && \text{(31a)} \\ &+ \frac{ig}{2} V_{kb}^* U_{ka} W_\mu^- \left(S_a^+ \partial^\mu S_b^0 - S_b^0 \partial^\mu S_a^+ \right) && \text{(31b)} \\ &- \frac{g}{4c_w} \operatorname{Im} \left(V_{kb}^* V_{kb'} \right) Z_\mu \left(S_b^0 \partial^\mu S_{b'}^0 - S_{b'}^0 \partial^\mu S_b^0 \right). && \text{(31c)} \end{aligned}$$

In the case of non-derivative couplings, we will only need those to the neutral scalars:

$$\mathcal{L}_{\text{non}-\partial} = \left(\frac{g^2}{4} W_\mu^+ W^{-\,\mu} + \frac{g^2}{8c_w^2} Z_\mu Z^\mu \right) \left[\left(v_k^* V_{kb} + V_{kb}^* v_k \right) S_b^0 + V_{kb}^* V_{kb'} S_b^0 S_{b'}^0 \right]. \tag{32}$$

2.4 Scalar–ghost interactions

Defining

$$\omega_k = \frac{v_k}{v}, \tag{33}$$

we can write the interaction of the scalar mass eigenfields S_b^0 with the ghost fields c^+, c^- and c_Z as

$$\begin{aligned} \mathcal{L}(S^0 \bar{c} c) &= -\frac{g m_W \xi_W}{2} \sum_{b=2}^{\infty} S_b^0 \left(\omega_k^* V_{kb}\, \overline{c^+} c^+ + \omega_k V_{kb}^*\, \overline{c^-} c^- \right) && \text{(34a)} \\ &- \frac{g m_Z \xi_Z}{2c_w} \sum_{b=2}^{\infty} S_b^0 \operatorname{Re} \left(\omega_k^* V_{kb} \right) \bar{c}_Z c_Z. && \text{(34b)} \end{aligned}$$

2.5 Triple scalar interactions

The triple-scalar interactions [28] follow straightforwardly from the scalar potential, equation (A.1):

$$\mathcal{L}(S^0 S^0 S^0) = -\frac{1}{2} \lambda_{ijkl} \left(v_i^* V_{jb_1} + V_{ib_1}^* v_j \right) V_{kb_2}^* V_{lb_3} S_{b_1}^0 S_{b_2}^0 S_{b_3}^0, \tag{35}$$

$$\mathcal{L}(S^0 S^- S^+) = -\lambda_{ijkl} \left(v_i^* V_{jb} + V_{ib}^* v_j \right) U_{ka_1}^* U_{la_2} S_b^0 S_{a_1}^- S_{a_2}^+. \tag{36}$$

For the properties of λ_{ijkl} see equation (A.2). If one of the scalars is a Goldstone boson, then the difference of the squares of the masses of the other two scalars is involved in the triple scalar coupling [11]. Specializing to the coupling of S^0 to the Goldstone bosons leads to [28]

$$\mathcal{L}(S^0 G G) = \frac{1}{v} \sum_{b=2}^{2n_H} M_b^2 \operatorname{Im} \left(V^\dagger V \right)_{1b} S_b^0 \left(G^+ G^- + \frac{1}{2} G^0 G^0 \right). \tag{37}$$

2.6 Quartic scalar interactions

The quartic scalar couplings which we need in the following are given by [28]

$$\begin{aligned} \mathcal{L}(S^0S^0S^0S^0) &= -\frac{1}{4}\lambda_{ijkl}V^*_{ib_1}V_{jb_2}V^*_{kb_3}V_{lb_4}S^0_{b_1}S^0_{b_2}S^0_{b_3}S^0_{b_4}, & (38)\\ \mathcal{L}(S^0S^0S^-S^+) &= -\lambda_{ijkl}V^*_{ib_1}V_{jb_2}U^*_{ka_1}U_{la_2}S^0_{b_1}S^0_{b_2}S^-_{a_1}S^+_{a_2}. & (39)\end{aligned}$$

2.7 Scale factors in dimensional regularization

As mentioned in the introduction, we will be using dimensional regularization for the one-loop integrals in $d = 4 - \varepsilon$ dimensions. Introducing the mass scale $\mathcal{M}$, in order to keep the coupling constants dimensionless in d dimensions, we have to make the replacements

$$g \to \mathcal{M}^{\varepsilon/2}g, \quad \Delta_k \to \mathcal{M}^{\varepsilon/2}\Delta_k, \quad \Gamma_k \to \mathcal{M}^{\varepsilon/2}\Gamma_k, \quad \lambda_{ijkl} \to \mathcal{M}^{\varepsilon}\lambda_{ijkl}. \tag{40}$$

Similarly, the VEVs have to be scaled by

$$v_i \to \mathcal{M}^{-\varepsilon/2}v_i, \tag{41}$$

so that they have the dimension of a mass.

3 The scalar sector

In this section we discuss the one- and two-point functions of the *neutral* scalars.

3.1 The counterterms for the one- and two-point scalar functions

The scalar potential is defined in equation (A.1). The Lagrangian of the scalar potential plus its counterterm parameters is given by

$$-V(\phi) - \delta V(\phi) = -\left(\mu^2_{ij} + \delta\hat{\mu}^2_{ij}\right)\phi^\dagger_i\phi_j - \mathcal{M}^\varepsilon\left(\lambda_{ijkl} + \delta\hat{\lambda}_{ijkl}\right)\phi^\dagger_i\phi_j\phi^\dagger_k\phi_l. \tag{42}$$

Here the components of the Higgs doublets ϕ_i are meant to be bare fields. For the neutral components of the Higgs doublets we make the ansatz[10]

$$\varphi^0_i = \frac{1}{\sqrt{2}}\left(Z^{(1/2)}_\varphi\right)_{ij}\left[\mathcal{M}^{-\varepsilon/2}\left(v_j + \Delta v_j + \delta v_j\right) + V_{jb}S^0_b\right]. \tag{43}$$

The VEVs v_j are in our notation pure tree-level quantities defined as the solution of the set of n_H equations

$$\left(\mu^2_{ij} + \lambda_{ijkl}v^*_kv_l\right)v_j = 0. \tag{44}$$

By definition, the fields S^0_b have vanishing VEVs, which is guaranteed beyond tree level by the finite VEV shifts Δv_j. In addition, the VEV renormalization δv_j is needed in the

[10]Note that the symbol "Δv_j" used in [25] refers to the total VEV shifts of the bare scalar fields and has, therefore, a meaning different from our Δv_j.

R_ξ gauge in the case of $\xi \neq 0$ [17]. The complex $n_H \times 2n_H$ matrix (V_{jb}) is connected to the orthogonal $2n_H \times 2n_H$ diagonalization matrix of the mass matrix of the neutral scalars. For its definition and properties we refer the reader to appendix B.

Since we only perform one-loop computations, we write

$$\left(Z_\varphi^{(1/2)}\right)_{ij} = \delta_{ij} + \frac{1}{2}\, z_{ij}^{(\varphi)} \tag{45}$$

for the wave-function renormalization of the neutral scalars. It is convenient to absorb the wave-function renormalization into the counterterm parameters of equation (42). Thus we define

$$\delta\mu_{ij}^2 = \delta\hat{\mu}_{ij}^2 + \frac{1}{2}\left(\mu_{i'j}^2 \left(z_{i'i}^{(\varphi)}\right)^* + \mu_{ij'}^2\, z_{j'j}^{(\varphi)}\right) \tag{46}$$

and

$$\delta\lambda_{ijkl} = \delta\hat{\lambda}_{ijkl} + \frac{1}{2}\left(\lambda_{i'jkl}\left(z_{i'i}^{(\varphi)}\right)^* + \lambda_{ij'kl} z_{j'j}^{(\varphi)} + \cdots\right). \tag{47}$$

With these definitions the counterterms for the one- and two-point functions are induced by $\delta\mu_{ij}^2$, $\delta\lambda_{ijkl}$ and δv_j.

In writing down the counterterm for the scalar one-point function of S_b^0, we "truncate" it by removing $\mathcal{M}^{-\varepsilon/2} i/(-M_b^2)$. Then the counterterm reads

$$\begin{aligned} \underset{S_b^0}{\otimes} \quad = \quad & -\frac{i}{2}\left[\delta\mu_{ij}^2 + \frac{1}{2}\,\delta\tilde{\lambda}_{ijkl} v_k^* v_l\right]\left(v_i^* V_{jb} + V_{ib}^* v_j\right) && \text{(48a)} \\ & -\frac{i}{2}\left(\delta v_i^* V_{ib} + V_{ib}^* \delta v_i\right) M_b^2. && \text{(48b)} \end{aligned}$$

Here we have introduced the definition

$$\tilde{\lambda}_{ijkl} \equiv \lambda_{ijkl} + \lambda_{ilkj}. \tag{49}$$

and, consequently,

$$\delta\tilde{\lambda}_{ijkl} \equiv \delta\lambda_{ijkl} + \delta\lambda_{ilkj}. \tag{50}$$

In order to achieve the form of equation (48b) with the neutral scalar masses M_b, we have taken into account equation (B.12). Note that, for later reference, we have split the counterterm of equation (48) into a part induced by $\delta\mu_{ij}^2$ and $\delta\tilde{\lambda}_{ijkl}$ and a part induced by δv_i.

The counterterm pertaining to the scalar self-energy $-i\Pi_{bb'}(p^2)$ is given by

$$\begin{aligned} S_b^0 \text{ - - - -} \otimes \text{- - - - } S_{b'}^0 \;=\; & -\frac{i}{2}\left[\delta\mu_{ij}^2 + \delta\tilde{\lambda}_{ijkl} v_k^* v_l\right]\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right) \\ & -\frac{i}{4}\delta\tilde{\lambda}_{ijkl}\left[v_i^* v_k^* V_{jb} V_{lb'} + v_j v_l V_{ib}^* V_{kb'}^*\right] && \text{(51a)} \\ & -\frac{i}{2}\tilde{\lambda}_{ijkl}\left(\delta v_k^* v_l + v_k^* \delta v_l\right)\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right) \\ & -\frac{i}{2}\lambda_{ijkl}\left[\left(\delta v_i^* v_k^* + v_i^* \delta v_k^*\right) V_{jb} V_{lb'} + \left(\delta v_j v_l + v_j \delta v_l\right) V_{ib}^* V_{kb'}^*\right]. && \text{(51b)} \end{aligned}$$

In this counterterm we have done a splitting analogous to the case of the one-point function. Note that the second line in equation (51a) comes about because

$$\delta\lambda_{ijkl} v_i^* v_k^* V_{jb} V_{lb'} = \frac{1}{2}\, \delta\tilde{\lambda}_{ijkl} v_i^* v_k^* V_{jb} V_{lb'}, \tag{52}$$

cf. equation (A.2). A similar argument applies to the second line of equation (51b).

We anticipate here that the counterterm of equation (48) connects via the scalar propagator to a neutral or charged fermion line and contributes thus to the counterterms of the fermion self-energy, and the VEV renormalization δv_j contributes directly via $v_j \to v_j + \delta v_j$ to the fermion mass counterterms—see figure 6 for a graphical rendering. These counterterms will play an important role in sections 4.3 and 5.

Now we proceed as announced in the introduction in items 1-3.

3.2 Renormalization of the quartic scalar couplings

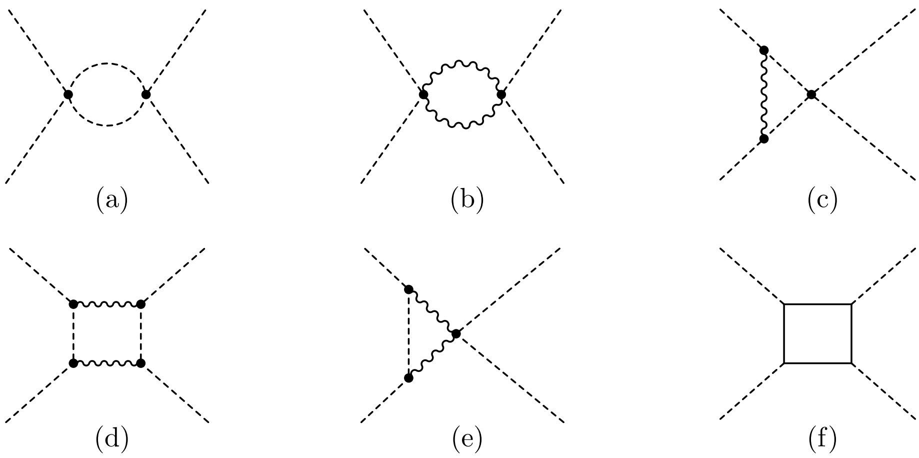


Figure 1: The Feynman diagrams that determine $\delta\tilde{\lambda}_{ijkl}$ defined in equation (49). The lines have the usual meaning: full, wiggly and dashed lines indicate fermions, vector bosons and scalars, respectively.

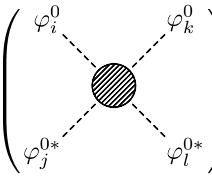
In the $\overline{\text{MS}}$ renormalization scheme at one-loop order the terms proportional to c_∞ of equation (2) have to be cancelled by the respective counterterms.

Since we are only interested in the divergencies of the scalar four-point function, we can stick to the unbroken theory for the computation of $\delta\lambda_{ijkl}$. Moreover, as discussed in the previous subsection, it suffices to compute $\delta\tilde{\lambda}_{ijkl}$ instead of $\delta\lambda_{ijkl}$. This leads us to consider the four-point function

$$\langle 0|T\varphi_i^0\, \varphi_j^{0*} \varphi_k^0\, \varphi_l^{0*}|0\rangle. \tag{53}$$

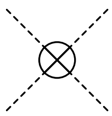
The Feynman diagrams from which we compute the divergencies are displayed in figure 1. There is a one-to-one correspondence between the labels of the subfigures and those of the subequations of equation (54). Moreover, every subequation contains the contributions of both charged and neutral inner lines of the diagrams. We obtain the following result for the divergencies:[11]

$$iL_{ijkl} \equiv \Bigg(\quad \text{(diagram with external lines } \varphi_i^0, \varphi_k^0, \varphi_j^{0*}, \varphi_l^{0*}) \quad \Bigg)_{c_\infty}$$



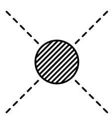
$$= \frac{i}{16\pi^2}\, c_\infty \Big\{ 4 \Big[\tilde{\lambda}_{ijmn}\tilde{\lambda}_{klnm} + \tilde{\lambda}_{ilmn}\tilde{\lambda}_{kjnm}$$
$$+ \lambda_{ijmn}\lambda_{klnm} + \lambda_{ilmn}\lambda_{kjnm} + \lambda_{imkn}\lambda_{mjnl} + \lambda_{imkn}\lambda_{mlnj} \Big] \tag{54a}$$
$$+ \left[\frac{g^4}{4}\left(3+\xi_W^2\right) + \frac{g^4}{8c_w^4}\left(3+\xi_Z^2\right) \right] \left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right) \tag{54b}$$
$$- \left(2g^2\xi_W + \frac{g^2}{c_w^2}\xi_Z \right) \tilde{\lambda}_{ijkl} \tag{54c}$$
$$+ \left(\frac{g^4}{4}\xi_W^2 + \frac{g^4}{8c_w^4}\xi_Z^2 \right) \left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right) \tag{54d}$$
$$- \left(\frac{g^4}{2}\xi_W^2 + \frac{g^4}{4c_w^4}\xi_Z^2 \right) \left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}\right) \tag{54e}$$
$$- 2\,\mathrm{Tr}\left(\Gamma_i\Gamma_j^\dagger\Gamma_k\Gamma_l^\dagger + \Gamma_i\Gamma_l^\dagger\Gamma_k\Gamma_j^\dagger \right) - 2\,\mathrm{Tr}\left(\Delta_i^\dagger\Delta_j\Delta_k^\dagger\Delta_l + \Delta_i^\dagger\Delta_l\Delta_k^\dagger\Delta_j \right) \Big\}. \tag{54f}$$

The counterterm pertaining to the four-point function of equation (53) is given by

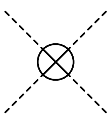


$$= -2i\left(\delta\lambda_{ijkl} + \delta\lambda_{ilkj}\right) = -2i\,\delta\tilde{\lambda}_{ijkl}. \tag{55}$$

Then the $\overline{\mathrm{MS}}$ condition is



$$\Big(\quad \Big)_{c_\infty} + $$



$$= iL_{ijkl} - 2i\delta\tilde{\lambda}_{ijkl} = 0 \quad \text{or} \quad \delta\tilde{\lambda}_{ijkl} = \frac{1}{2}\,L_{ijkl}. \tag{56}$$

Note that $L_{ijkl} = L_{ilkj}$, as it has to be for consistency.

For the further discussion it is convenient to decompose $\delta\tilde{\lambda}_{ijkl}$ as

$$\delta\tilde{\lambda}_{ijkl} = \delta\tilde{\lambda}_{ijkl}(S) + \delta\tilde{\lambda}_{ijkl}(\ell^\pm) + \delta\tilde{\lambda}_{ijkl}(\chi) + \delta\tilde{\lambda}_{ijkl}(\xi^0) + \delta\tilde{\lambda}_{ijkl}(\xi^1) + \delta\tilde{\lambda}_{ijkl}(\xi^2). \tag{57}$$

The first three terms correspond to the contributions of the scalars, the charged leptons and the neutrinos, respectively, *i.e.* to those diagrams which do not have a vector boson line. Vector boson contributions can be characterized by powers in the gauge parameters

[11] Here and in the following, a full blob means a sum over one-loop diagrams, whereas a circle with a cross refers to a counterterm. When specific parts of these entities are addressed, they are put within parentheses with subscripts indicating the specifics.

(ξ^ν with $\nu = 0, 1, 2$)—see equation (54). Actually, diagrams with two vector boson lines are proportional to g^4 and have parts with ξ^0 and ξ^2, whereas diagrams with one vector boson line are proportional to $g^2\xi^1$ and a quartic scalar coupling.

However, inspection of equation (54) reveals that the ξ^2-terms cancel each other, *i.e.*

$$\delta\tilde{\lambda}_{ijkl}(\xi^2) = 0. \tag{58}$$

3.3 Divergencies of the neutral-scalar self-energy

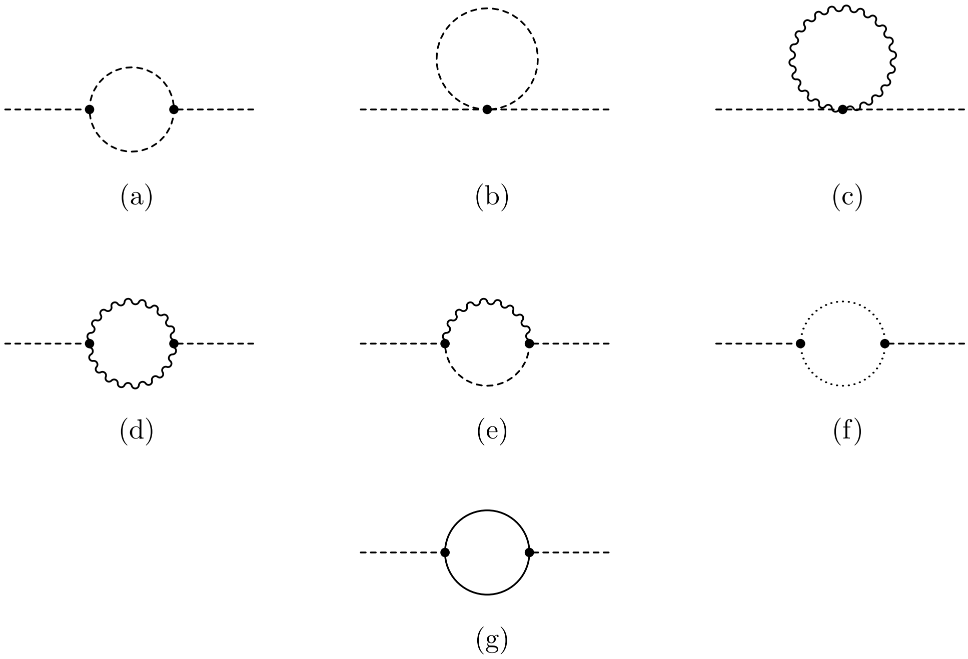


Figure 2: The Feynman diagrams contributing the scalar self-energy in the broken phase. In addition to the lines explained in figure 1, in diagram (f) we have dotted lines indicating ghost propagators.

Now we turn to the divergencies of the scalar two-point function. Having obtained the result for $\delta\tilde{\lambda}_{ijkl}$, we need $\delta\mu^2_{ij}$ and δv_i for the full counterterm of the two-point function, equation (51). The aim of the present section is not only to compute $\delta\mu^2_{ij}$ and δv_i but our computations also serve as a consistency check that the scalar self-energy can indeed be made finite by a suitable choice of these parameters.

In the presentation of the one-loop results for the divergencies we use the vector boson masses of equation (25), the definitions of the matrices Λ, K and K' given in appendix A, and equations (B.2) and (B.11). The divergencies refer to those of $-i\Pi_{bb'}(p^2)$, where $\Pi_{bb'}(p^2)$ is the self-energy matrix of the neutral scalar mass eigenfields S^0_b.

Now we list the momentum-independent divergencies belonging to the diagrams of figure 2. In the individual results we indicate the nature of the particle (or particles) in the loop.

Diagram (a), charged scalars:

$$\frac{i}{16\pi^2}\, c_\infty \lambda_{ijmn}\lambda_{nmkl}\left(v_i^* V_{jb} + V_{ib}^* v_j\right)\left(v_k^* V_{lb'} + V_{kb'}^* v_l\right), \tag{59}$$

diagram (a), neutral scalars:

$$\frac{i}{16\pi^2}\, c_\infty \Big\{ \left(V_{kb}^* V_{lb'} + V_{kb'}^* V_{lb}\right)\left[\tilde\lambda_{ilmn}\tilde\lambda_{kjnm} + \lambda_{imkn}\left(\lambda_{mjnl} + \lambda_{njml}\right)\right] v_i^* v_j \\ + \left(V_{jb} V_{lb'} v_i^* v_k^* + V_{ib}^* V_{kb'}^* v_j v_l\right)\tilde\lambda_{ijmn}\tilde\lambda_{klnm}\Big\}, \tag{60}$$

diagram (b), charged scalars:

$$\frac{i}{16\pi^2}\, c_\infty \left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\left[\frac{1}{4}\, g^2 \xi_W\, \Lambda_{ij} + \lambda_{ijkl}\left(\mu^2 + \Lambda\right)_{lk}\right], \tag{61}$$

diagram (b), neutral scalars:

$$\frac{i}{16\pi^2}\, c_\infty \left\{ \frac{g^2\xi_Z}{8c_w^2}\left[\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\left(\Lambda + K'\right)_{ij} - \lambda_{ijkl}\left(V_{jb}V_{lb'}v_i^* v_k^* + V_{ib}^* V_{kb'}^* v_j v_l\right)\right] \right. \\ \left. + \tilde\lambda_{ijkl}\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\left(\mu^2 + \Lambda + K'\right)_{lk} + \lambda_{ijkl} V_{jb} V_{lb'} K_{ik}^* + \lambda_{ijkl} V_{ib}^* V_{kb'}^* K_{jl} \right\}, \tag{62}$$

diagram (c), $W^\pm$ and Z bosons:

$$\frac{i}{16\pi^2}\, c_\infty \delta_{bb'}\left[\frac{g^4 v^2}{8}\left(3 + \xi_W^2\right) + \frac{g^4 v^2}{16 c_w^4}\left(3 + \xi_Z^2\right)\right], \tag{63}$$

diagram (d), $W^\pm$ and Z bosons:

$$\frac{i}{16\pi^2}\, c_\infty \left(v_k^* V_{kb} + V_{kb}^* v_k\right)\left(v_l^* V_{lb'} + V_{lb'}^* v_l\right)\left[\frac{g^4}{16}\left(3 + \xi_W^2\right) + \frac{g^4}{32 c_w^4}\left(3 + \xi_Z^2\right)\right], \tag{64}$$

diagram (e), $W^\pm$ boson and charged scalars:

$$-\frac{i}{16\pi^2}\, c_\infty \left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\left[\frac{1}{16}\, g^4 \xi_W^2\left(\delta_{ij} v^2 + v_i v_j^*\right) + \frac{1}{4}\, g^2 \xi_W\left(\mu^2 + \Lambda\right)_{ij}\right], \tag{65}$$

diagram (e), Z boson and neutral scalars:

$$\frac{i}{16\pi^2}\, c_\infty \left\{ -\frac{g^4\xi_Z^2}{64 c_w^4}\left[2\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\delta_{ij} v^2 + \left(v_k^* V_{kb} + V_{kb}^* v_k\right)\left(v_l^* V_{lb'} + V_{lb'}^* v_l\right)\right] \right. \\ \left. + \frac{g^2\xi_Z}{8c_w^2}\left[V_{ib} V_{jb'} K_{ij}^* + V_{ib}^* V_{jb'}^* K_{ij} - \left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right)\left(\mu^2 + \Lambda + K'\right)_{ij}\right] \right\}, \tag{66}$$

diagram (f), charged ghosts:

$$-\frac{i}{16\pi^2}\, c_\infty \frac{g^4 v^2}{16}\, \xi_W^2\left(\omega_k^* V_{kb}\, \omega_l^* V_{lb'} + \omega_k V_{kb}^*\, \omega_l V_{lb'}^*\right), \tag{67}$$

diagram (f), neutral ghost:

$$-\frac{i}{16\pi^2}\, c_\infty \frac{g^4 v^2}{16 c_w^4}\, \xi_Z^2 \,\mathrm{Re}\,(\omega_k^* V_{kb})\,\mathrm{Re}\,(\omega_l^* V_{lb'}) \tag{68}$$

diagram (g), charged leptons:

$$\begin{aligned} -\frac{i}{16\pi^2}\, c_\infty \;\times\; \mathrm{Tr}\Big\{ &\Big(\Gamma_k^\dagger \Gamma_j M_\ell^\dagger M_\ell + \Gamma_j \Gamma_k^\dagger M_\ell M_\ell^\dagger\Big)\left(V_{jb}^* V_{kb'} + V_{jb'}^* V_{kb}\right) \\ &+ \Gamma_k^\dagger M_\ell \Gamma_j^\dagger M_\ell\, V_{jb'} V_{kb} + \Gamma_k M_\ell^\dagger \Gamma_j M_\ell^\dagger\, V_{jb'}^* V_{kb}^* \Big\}, \end{aligned} \tag{69}$$

diagram (g), neutrinos:

$$\begin{aligned} -\frac{i}{16\pi^2}\, c_\infty \;\times\; \mathrm{Tr}\Big\{ &\Big[M_D^\dagger M_D \Delta_j^\dagger \Delta_k + \Big(M_D M_D^\dagger + M_R M_R^\dagger\Big)\Delta_k \Delta_j^\dagger\Big]\left(V_{jb}^* V_{kb'} + V_{jb'}^* V_{kb}\right) \\ &+ \Delta_k M_D^\dagger \Delta_j M_D^\dagger\, V_{jb'} V_{kb} + \Delta_k^\dagger M_D \Delta_j^\dagger M_D\, V_{jb'}^* V_{kb}^* \Big\}. \end{aligned} \tag{70}$$

In the last two equations we have exploited the mass relations for the leptons, as presented in section 2.1.

Some remarks concerning the ξ-dependence of the divergencies are in order. In equations (61) and (62) the linear ξ-dependence comes from the Goldstone bosons in the loop because, due to the R_ξ gauge, the Goldstone boson masses $M_{+1}^2 = M_1^2 = 0$ are replaced by $\xi_W m_W^2$ and $\xi_Z m_Z^2$, respectively. The vector boson loops of equation (63) and (64) lead to a quadratic ξ-dependence, stemming from the ξ-dependence of the vector boson propagators. Finally, the mixed vector boson–scalar loops have a linear ξ-dependence originating in the vector boson propagator, however, an additional factor ξ comes into play in the case of Goldstone bosons in the loop.

3.4 Determination of $\delta\mu_{ij}^2$ and δv_k

Having computed $\delta\tilde{\lambda}_{ijkl}$ in section 3.2, we are now in a position to determine $\delta\mu_{ij}^2$ and δv_k from the divergencies of the scalar self-energy as presented in the previous subsection. In a graphical presentation, $\delta\mu_{ij}^2$ and δv_k are to be computed from

$$\left(S_b^0 \text{----}\bullet\text{----} S_{b'}^0\right)_{c_\infty} + S_b^0 \text{----}\otimes\text{----} S_{b'}^0 = 0. \tag{71}$$

It can be checked straightforwardly that the divergencies of the scalar self-energy given by equations (59)–(70) have the same types of terms as those in the decomposition of $\delta\tilde{\lambda}_{ijkl}$ in equation (57). Therefore, $\delta\mu_{ij}^2$ can be decomposed in the same way:

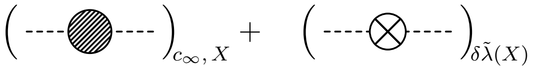
$$\delta\mu_{ij}^2 = \delta\mu_{ij}^2(S) + \delta\mu_{ij}^2(\ell^\pm) + \delta\mu_{ij}^2(\chi) + \delta\mu_{ij}^2(\xi^0) + \delta\mu_{ij}^2(\xi^1) + \delta\mu_{ij}^2(\xi^2). \tag{72}$$

It will turn out that, after insertion of $\delta\tilde{\lambda}_{ijkl}$ of equation (56) into the counterterm of equation (51a) and adding it to the divergencies of the scalar self-energy, the determination of $\delta\mu_{ij}^2$ and δv_k is unique for the following reasons:

1. As proven in [17], δv_k is a linear function in ξ.
2. Therefore, with the exception of the terms proportional to ξ^1, for the cancellation of the divergencies we only have the counterterm containing $\delta\mu^2_{ij}$ at our disposal.
3. As we will see, both the divergencies proportional to ξ^1 and the counterterm induced by δv_k are linear combinations of the two linearly independent matrices $\delta_{bb'} M^2_{b'}$ and $(V^*_{ib} V_{jb'} + V^*_{ib'} V_{jb})\, \mu^2_{ij}$, while the counterterm induced by $\delta\mu^2_{ij}(\xi^1)$ is proportional to the second matrix. Therefore these two counterterms are linearly independent and a unique combination of the two cancels the divergencies.

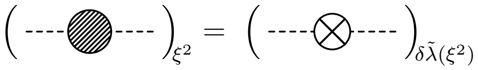
One might think that the usage of the scalar one-point function is appropriate to fix δv_k, but this does not offer any advantage because one would need $\delta\mu^2_{ij}$ anyway since it occurs in the counterterm—see equation (48).

Inserting $\delta\tilde{\lambda}_{ijkl}$ of equation (56) into the counterterm of equation (51a), we find



$$= 0 \quad \text{for} \quad X = l^\pm, \xi^0. \tag{73}$$

Moreover,

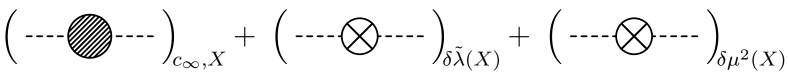


$$= 0. \tag{74}$$

Therefore, in these cases we obtain

$$\delta\mu^2_{ij}(\ell^\pm) = \delta\mu^2_{ij}(\xi^0) = \delta\mu^2_{ij}(\xi^2) = 0. \tag{75}$$

However, in the cases of $X = S, \chi, \xi^1$ the counterterm parameter $\delta\mu^2_{ij}(X)$ is non-trivial. From



$$= 0 \quad \text{for} \quad X = S, \chi, \tag{76}$$

we compute

$$\delta\mu^2_{ij}(S) = \frac{2}{16\pi^2}\, c_\infty \left(2\lambda_{ijkl} + \lambda_{ilkj}\right) \mu^2_{lk} \tag{77}$$

and

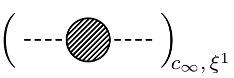
$$\delta\mu^2_{ij}(\chi) = -\frac{2}{16\pi^2}\, c_\infty \mathrm{Tr}\left(M_R M_R^\dagger \Delta_j \Delta_i^\dagger\right). \tag{78}$$

It is amusing to notice that the latter equation is the only instance where M_R, the mass matrix of the right-handed neutrino singlets, appears in a counterterm.

The linear ξ-terms need a special treatment and we will be very detailed in their discussion. Our aim is to determine the remaining counterterm parameters $\delta\mu^2_{ij}(\xi^1)$ and δv_k from the divergencies linear in ξ of the scalar self-energy. In order to streamline the notation, we define

$$A_1 = \frac{c_\infty}{16\pi^2}\left(\frac{g^2\xi_W}{4} + \frac{g^2\xi_Z}{8c_w^2}\right), \tag{79}$$

where the index 1 indicates linearity in ξ. In terms of this quantity, the sum over all divergencies linear in ξ of the scalar self-energy—see section 3.3—can be written as



$$= -i\left(V^*_{ib} V_{jb'} + V^*_{ib'} V_{jb}\right) \mu^2_{ij} A_1. \tag{80}$$

Now we turn to the counterterm of equation (51) and discuss the various contributions linear in ξ. It is easy to see from equations (54) and (56) that $\delta\tilde{\lambda}(\xi^1)_{ijkl}$, the part proportional to ξ^1 of $\delta\tilde{\lambda}_{ijkl}$, can be written in terms of A_1 as well:

$$\delta\tilde{\lambda}(\xi)_{ijkl} = -4A_1\tilde{\lambda}_{ijkl}. \tag{81}$$

Plugging this expression into the counterterm formula of equation (51a) and using equation (B.13), we obtain

$$\left(\text{----}\otimes\text{----}\right)_{\delta\tilde{\lambda}(\xi^1)} = 4iA_1\delta_{bb'}M_{b'}^2 - 2i\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\mu_{ij}^2 A_1. \tag{82}$$

The remaining terms linear in ξ in equation (51) consist of the $\delta\mu_{ij}^2(\xi^1)$-part of equation (51a) and the counterterm induced by δv_i, equation (51b), and contain thus the parameters we want to determine.

Adding up all terms linear in ξ, divergence and the three counterterm contributions, we have

$$\begin{aligned}
&\left(\text{----}\bullet\text{----}\right)_{c_\infty,\,\xi^1} + \left(\text{----}\otimes\text{----}\right)_{\xi^1}\\
&= -i\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\mu_{ij}^2 A_1 && \text{(83a)}\\
&\quad + 4iA_1\delta_{bb'}M_{b'}^2 - 2i\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\mu_{ij}^2 A_1 && \text{(83b)}\\
&\quad - \frac{i}{2}\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\delta\mu_{ij}^2(\xi^1) && \text{(83c)}\\
&\quad - \frac{i}{2}\tilde{\lambda}_{ijkl}\left(\delta v_k^* v_l + v_k^*\delta v_l\right)\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\\
&\quad - \frac{i}{4}\tilde{\lambda}_{ijkl}\left[\left(\delta v_j v_l + v_j\delta v_l\right)V_{ib}^*V_{kb'}^* + \left(\delta v_i^* v_k^* + v_i^*\delta v_k^*\right)V_{jb}V_{lb'}\right]. && \text{(83d)}
\end{aligned}$$

We know that the sum of these terms must be zero. Since in equations (83a) and (83b) these gauge parameters only occur in A_1 and taking into account that δv_k is linear in ξ_W and ξ_Z, we are lead to the ansatz $\delta v_k = cA_1 v_k$, where c is a constant to be determined by the cancellation of the divergencies. Plugging this ansatz into equation (83d) (last two lines of equation (83)) and using equation (B.13), after some computation these two lines are rewritten as

$$-icA_1\left[2\delta_{bb'}M_{b'}^2 - \left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\mu_{ij}^2\right]. \tag{84}$$

Obviously, with $c = 2$ or

$$\delta v_k = 2A_1 v_k \tag{85}$$

the A_1-part of the counterterm induced by δv_k, equation (84), cancels the term $4iA_1\delta_{bb'}M_{b'}^2$ of equation (83b).

Having thus determined δv_k, we consider the sum of the remaining terms in equation (83) which amounts to

$$-i\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\mu_{ij}^2 A_1 - \frac{i}{2}\left(V_{ib}^*V_{jb'} + V_{ib'}^*V_{jb}\right)\delta\mu_{ij}^2(\xi^1). \tag{86}$$

Thus we find that

$$\delta\mu_{ij}^2(\xi^1) = -2A_1\mu_{ij}^2 \tag{87}$$

together with δv_k of equation (85) induce counterterms which cancel the terms linear in ξ in the scalar self-energy. With this, we have finally determined the complete set of parameters, $\delta\mu_{ij}^2$, $\delta\tilde{\lambda}_{ijkl}$ and δv_k that make the scalar self-energy finite.

3.5 Finiteness of the scalar one-point function

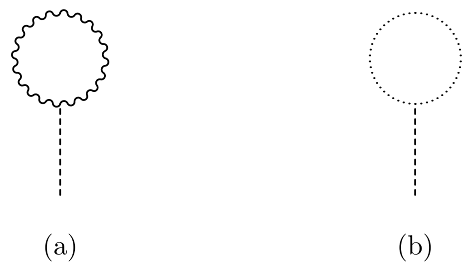


Figure 3: The tadpole diagrams involving the vector bosons and the ghost fields.

As in the case of the counterterm of equation (48), we consider the "truncated" one-point function of S_b^0, where the external propagator $i/(-M_b^2)$ and the factor $\mathcal{M}^{-\varepsilon/2}$ are removed. Also we emphasize that the scalar one-point function referring to $S_1^0 \equiv G^0$ is zero—see discussion after equation (97) in section 3.6. Thus we consider the truncated one-point functions of S_b^0 with $b = 2, \ldots, 2n_H$. The counterterms for the one-point function induced by $\delta\mu_{ij}^2$, $\delta\tilde{\lambda}_{ijkl}$ and δv_i are obtained by application of equation (48).

Vector boson and ghost loops: The tadpole diagrams involving the vector boson and ghost loops—see figure 3—deserve a special treatment because the ghost loops cancel exactly that part of the vector boson loops deriving from the gauge-dependent part of the propagator in equation (30a) [24]. Concretely, we are going to demonstrate that the c_Z loop cancels the ξ_Z-dependent contribution of the Z propagator and the c^+ and c^- loops cancel the ξ_W-dependent contribution of the W propagator.

First we consider the Z and c_Z loops in diagrams (a) and (b) of figure 3, respectively. According to the Lagrangians (32) and (34b), we obtain the loop integrals

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\{ \frac{ig^2}{8c_w^2} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) \frac{k^2}{m_Z^2} \frac{-i}{k^2 - \xi_Z m_Z^2 + i\epsilon} + (-i) \frac{g\xi_Z m_Z}{2c_w} \operatorname{Re}\left(\omega_i V_{ib}\right) \frac{-i}{k^2 - \xi_Z m_Z^2 + i\epsilon} \right\}. \tag{88}$$

Note that the minus sign in the numerator of the ghost propagator takes into account the anticommuting nature of the ghost fields. Now we use

$$\frac{k^2}{k^2 - \xi_Z m_Z^2} = 1 + \frac{\xi_Z m_Z^2}{k^2 - \xi_Z m_Z^2} \quad \text{and} \quad \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, 1 = 0, \tag{89}$$

the formula for the Z mass of equation (25), the definition of ω_k in equation (33), and the relation

$$v_i^* V_{ib} + V_{ib}^* v_i = 2v \, \mathrm{Re} \left(\omega_i^* V_{ib} \right). \tag{90}$$

It is then easy to see that the integral (88) is zero.

In the case of the W boson in diagram (a) and the ghost fields c^+ and c^- in diagram (b) of figure 3, Lagrangians (32) and (34a) lead to the loop integral

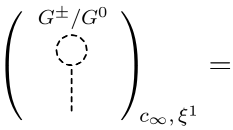
$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\{ \frac{ig^2}{4} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) \frac{k^2}{m_W^2} \frac{-i}{k^2 - \xi_W m_W^2 + i\epsilon} \right. \\ \left. +(-i) \frac{g \xi_W m_W}{2} \left(\omega_i^* V_{ib} + \omega_i V_{ib}^* \right) \frac{-i}{k^2 - \xi_Z m_W^2 + i\epsilon} \right\} \tag{91}$$

With the same arguments as before we conclude that it is zero.

Divergencies of the Goldstone boson loops and the ξ-dependent terms: The ξ-dependent divergencies of the tadpoles come from the Goldstone-boson loops. The relevant Lagrangian is displayed in equation (37). Using the identity

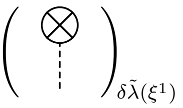
$$\mathrm{Im} \left(V^\dagger V \right)_{1b} = -\frac{1}{2v} \left(v_i^* V_{ib} + V_{ib}^* v_i \right), \tag{92}$$

these are given by



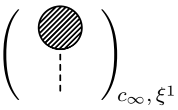
$$= \frac{i}{2} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) M_b^2 A_1. \tag{93}$$

The counterterm of equation (48a) associated with $\delta\tilde{\lambda}_{ijkl}(\xi^1)$ of equation (54c) can be put into the form

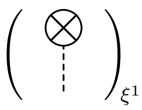


$$= \frac{i}{2} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) M_b^2 A_1 - i \left(v_i^* V_{jb} + V_{ib}^* v_j \right) \mu_{ij}^2 A_1. \tag{94}$$

Now we add up all terms of the scalar one-point function proportional to ξ^1, *i.e.* equations (93) and (94) and the remaining counterterms of equation (48), namely those with $\delta\mu_{ij}^2(\xi^1)$ and δv_i:



$+$



$$= \frac{i}{2} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) M_b^2 A_1 \tag{95a}$$

$$+ \frac{i}{2} \left(v_i^* V_{ib} + V_{ib}^* v_i \right) M_b^2 A_1 - i \left(v_i^* V_{jb} + V_{ib}^* v_j \right) \mu_{ij}^2 A_1 \tag{95b}$$

$$- \frac{i}{2} \left(v_i^* V_{jb} + V_{ib}^* v_j \right) \delta\mu_{ij}^2(\xi^1) \tag{95c}$$

$$- \frac{i}{2} \left(\delta v_i^* V_{ib} + V_{ib}^* \delta v_i \right) M_b^2. \tag{95d}$$

Taking δv_i from equation (85) and $\delta\mu^2_{ij}(\xi^1)$ from equation (87), the terms in equation (95) add up to zero. Note that the blob in equation (95) is identical with the Goldstone loops in equation (93) because, as explained in the beginning of this subsection, the ξ-dependence of vector boson propagators is cancelled by the ghost loops.

The remaining tadpole diagrams: A tedious but straightforward computation demonstrates that the ξ-independent divergencies of the tadpole diagrams are cancelled by the counterterm (48a) by plugging in the expressions for $\delta\mu^2_{ij}(X)$ and $\delta\tilde{\lambda}_{ijkl}(X)$ with $X = S, \ell^\pm, \chi, \xi^0$.

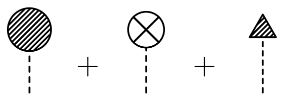
Summarizing, we have found that the counterterms determined by $\overline{\text{MS}}$ renormalization of the scalar four-point function and the scalar self-energy make the scalar one-point function finite.

3.6 The VEV shift Δv_i and the tadpoles

Using equation (B.12), we find that a finite VEV shift Δv_i induces the term

$$-\mathcal{M}^{-\varepsilon/2}\frac{1}{2}M_b^2\left(\Delta v_i^* V_{ib} + V_{ib}^*\Delta v_i\right)S_b^0 \equiv -\mathcal{M}^{-\varepsilon/2}\Delta t_b S_b^0 \tag{96}$$

in the scalar potential. As announced in section 1, we will now show that it is possible to choose Δv_i such that the scalar one-point function is zero at the one-loop level. There are three contributions to the truncated one-point function: the loop integrals $-iT_b$, the sum of the counterterms (95b)–(95d) denoted by $-i\mathcal{C}_b$, and the contribution of equation (96). Thus we require, in order to achieve a vanishing one-point function,



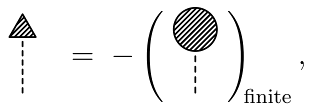
$$= -i\left(T_b + \mathcal{C}_b + \Delta t_b\right) \equiv 0 \quad \text{or} \quad \Delta t_b = -T_b - \mathcal{C}_b. \tag{97}$$

The triangle represents the term $-i\Delta t_b$ induced by equation (96).

Before we derive a formula for VEV shift Δv_k, let us dwell a little bit on the one-point function of G^0. That G^0 is an unphysical field is suggestive of its vanishing. This is indeed borne out by explicit one-loop considerations: all couplings of G^0 to bosons, including the ghost field c_Z, vanish, the c^+ and c^- loops cancel each other exactly, and the fermion loops give zero when the trace is taken in flavour and Dirac space. Concerning the counterterms (48) and taking into account equation (85), we see that all counterterms of the one-point function of G^0 contain the factor $v_i^* V_{i1} + V_{i1}^* v_i$. Since $V_{i1} = iv_i/v$—see equation (B.14), this factor obviously vanishes.

To proceed further, some remarks are in order:

i. From the considerations in section 3.5 we know that the scalar one-point functions, *i.e.* the quantities $T_b + \mathcal{C}_b = -\Delta t_b$ are finite, thus the Δv_i are finite as well.[12] Diagrammatically, this can be expressed as



[12] We remind the reader that we use the symbol "Δ," occurring in Δv_k and Δt_b, for finite quantities.

where the subscript "finite" indicates that all terms proportional to c_∞ have been subtracted.

ii. Since Δt_b belongs to the real scalar field S_b^0, this quantity must be real.

iii. In the quantity Δt_b, the masses M_b derive from the mass matrix of the neutral scalars where the Goldstone boson has zero mass, while the Goldstone mass-squared $\xi_Z m_Z^2$ derives from the R_ξ-gauge condition and occurs only in the propagator.

We can summarize this discussion in the following way:

$$\Delta t_1 = 0, \quad (\Delta t_b)^* = \Delta t_b, \quad \text{and} \quad \frac{1}{2}\left(\Delta v_i^* V_{ib} + V_{ib}^* \Delta v_i\right) = \frac{\Delta t_b}{M_b^2} \quad \text{for} \quad b = 2, \ldots, 2n_H. \tag{98}$$

Eventually, our aim is to obtain Δv_k from the Δt_b, but there is the obstacle that $\Delta v_i^* V_{i1} + V_{i1}^* \Delta v_i$ is not determined, because it is multiplied by $M_1^2 = 0$ in Δt_1. However, as we will shortly see, the only consistent value of this quantity is

$$\Delta v_i^* V_{i1} + V_{i1}^* \Delta v_i = 0. \tag{99}$$

Taking this relation into account, the first two relations of equation (B.5) allow to derive the VEV shifts

$$\Delta v_k = \sum_{b=2}^{2n_H} \frac{\Delta t_b}{M_b^2} V_{kb}. \tag{100}$$

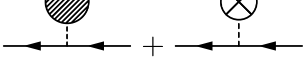
This means that it is indeed possible to make the scalar one-point function vanish by a finite VEV shift Δv_k.

Let us check now that equation (100) is indeed consistent with equation (99). We plug the result for Δv_k into equation (99) and utilize the third relation in equation (B.5). In this way we obtain

$$\Delta v_i^* V_{i1} + V_{i1}^* \Delta v_i = \sum_{b=2}^{2n_H} \frac{\Delta t_b}{M_b^2} \left(V_{ib}^* V_{i1} + V_{i1}^* V_{ib}\right) = 2 \sum_{b=2}^{2n_H} \frac{\Delta t_b}{M_b^2} \delta_{b1} = 0,$$

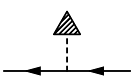
which is the desired result.

Now we want to demonstrate that the insertion of all tadpole contributions, including the tadpole counterterm, on a fermion line is equivalent to make the shift $v_k \to v_k + \Delta v_k$ in the mass term of the respective fermion in the Lagrangian. We begin with the charged-lepton lines. The corresponding expression for these diagrams is



$$= -\frac{i}{\sqrt{2}} \mathcal{M}^{\varepsilon/2} \sum_{b=2}^{2n_H} \left[G_b \gamma_L + G_b^\dagger \gamma_R\right] \times \frac{i}{-M_b^2} \times (-i) \mathcal{M}^{-\varepsilon/2} \left(T_b + \mathcal{C}_b\right). \tag{101}$$

Replacing $T_b + \mathcal{C}_b$ by $-\Delta t_b$, *c.f.* equation (97), and using the expression for Δt_b given in equation (96), we obtain



$$= -\frac{i}{\sqrt{2}} \sum_{b=2}^{2n_H} \left[G_b \gamma_L + G_b^\dagger \gamma_R\right] \times \frac{i}{-M_b^2} \times \frac{i}{2} M_b^2 \left(\Delta v_i^* V_{ib} + V_{ib}^* \Delta v_i\right). \tag{102}$$

Here the scalar squared masses M_b^2 cancel. Taking into account equation (99), we can—after this cancellation—take the sum from $b = 1$ to $2n_H$.[13] Since G_b contains the factor V_{kb}^*—see equation (21b), we apply the first two relations in equation (B.5). This gives the final form of equation (101), namely

$$= -\frac{i}{\sqrt{2}} \left(W_R^\dagger \Gamma_k \Delta v_k^* W_L \gamma_L + W_L^\dagger \Gamma_k^\dagger \Delta v_k W_R \gamma_R \right). \tag{103}$$

This expression has exactly the form of the contribution of a mass term to the fermion self-energy, however, with Δv_k instead of v_k. We have thus proven the above statement.

Now we turn to neutrino lines. We proceed as before and obtain the intermediate expression

$$\begin{aligned} & 2 \times \left(-\frac{i}{\sqrt{2}} \right) \mathcal{M}^{\varepsilon/2} \sum_{b=2}^{2n_H} \left[F_b \gamma_L + F_b^\dagger \gamma_R \right] \times \frac{i}{-M_b^2} \times (-i) \mathcal{M}^{-\varepsilon/2} \left(T_b + \mathcal{C}_b \right) \\ &= -i\sqrt{2} \sum_{b=2}^{2n_H} \left[F_b \gamma_L + F_b^\dagger \gamma_R \right] \times \frac{i}{-M_b^2} \times \frac{i}{2} M_b^2 \left(\Delta v_i^* V_{ib} + V_{ib}^* \Delta v_i \right). \end{aligned} \tag{104}$$

Note that the factor 2 comes about due to the Majorana nature of the neutrinos. Then we plug in F_b of equation (21a) and employ again equation (B.5). We finally arrive at the expression

$$-\frac{i}{\sqrt{2}} \left\{ \left(U_R^\dagger \Delta_k U_L + U_L^T \Delta_k^T U_R^* \right) \Delta v_k \gamma_L + \left(U_L^\dagger \Delta_k^\dagger U_R + U_R^T \Delta_k^* U_L^* \right) \Delta v_k^* \gamma_R \right\} \tag{105}$$

This contribution to $-i\Sigma$ corresponds precisely to a term

$$-\frac{1}{\sqrt{2}} \bar{\chi} \left(U_R^\dagger \Delta_k U_L \, \Delta v_k \gamma_L + U_L^\dagger \Delta_k^\dagger U_R \, \Delta v_k^* \gamma_R \right) \chi \tag{106}$$

in the Lagrangian—compare with the M_D term in equation (6) after utilizing equation (9). This proves that taking into account the tadpole diagrams on the neutrino lines corresponds to making a finite VEV shift in the Dirac-type neutrino mass term in the Lagrangian.

4 Gauge-parameter independence of the one-loop fermion masses

4.1 Two decompositions of the fermion self-energy

We denote by Σ the *renormalized* fermion self-energy. It can be decomposed as

$$\Sigma(p) = \not{p} \left(\Sigma_L^{(A)}(p^2) \gamma_L + \Sigma_R^{(A)}(p^2) \gamma_R \right) + \Sigma_L^{(B)}(p^2) \gamma_L + \Sigma_R^{(B)}(p^2) \gamma_R. \tag{107}$$

[13] In this context, equation (99) means that the Goldstone boson, *i.e.* $b = 1$ does not contribute, which we know already from an earlier discussion in the present subsection. This is further evidence that equation (99) is correct.

For definiteness we use the index i for the fermion masses in this subsection though, in the light of our notation convention laid out in section 1, for charged fermions we should be using α instead. For n fermions the quantities $\Sigma_L^{(A)}$, $\Sigma_R^{(A)}$, $\Sigma_L^{(B)}$, $\Sigma_R^{(B)}$ are $n \times n$ matrices that fulfill the matrix relations

$$\left(\Sigma_L^{(A)}\right)^\dagger = \Sigma_L^{(A)}, \quad \left(\Sigma_R^{(A)}\right)^\dagger = \Sigma_R^{(A)}, \quad \left(\Sigma_L^{(B)}\right)^\dagger = \Sigma_R^{(B)}. \tag{108}$$

Strictly speaking these relations hold only for the dispersive part of the self-energy. In the case of Majorana fermions, one has in addition

$$\left(\Sigma_L^{(A)}\right)^T = \Sigma_R^{(A)}, \quad \left(\Sigma_L^{(B)}\right)^T = \Sigma_L^{(B)}, \quad \left(\Sigma_R^{(B)}\right)^T = \Sigma_R^{(B)}. \tag{109}$$

If there are no degeneracies in the tree-level masses m_i, the diagonal elements of the coefficient matrices in equation (107) allow to express—at lowest non-trivial order—the mass shifts as

$$\Delta m_i = \frac{1}{2}\left\{m_i\left[\left(\Sigma_L^{(A)}\right)_{ii}(m_i^2) + \left(\Sigma_R^{(A)}\right)_{ii}(m_i^2)\right] + \left(\Sigma_L^{(B)}\right)_{ii}(m_i^2) + \left(\Sigma_R^{(B)}\right)_{ii}(m_i^2)\right\}. \tag{110}$$

Therefore, the pole masses, comprising tree-level plus radiative corrections, are given by

$$m_{\text{tot},i} = m_i + \Delta m_i. \tag{111}$$

As we will see, another useful decomposition of Σ is given by

$$\begin{aligned} \Sigma \;=\; & A_L\gamma_L + A_R\gamma_R + \left(\not{p} - \hat{m}\right)\left(B_L^{(r)}\gamma_L + B_R^{(r)}\gamma_R\right) + \left(B_L^{(l)}\gamma_L + B_R^{(l)}\gamma_R\right)\left(\not{p} - \hat{m}\right) \\ & + \left(\not{p} - \hat{m}\right)\left(C_L\gamma_L + C_R\gamma_R\right)\left(\not{p} - \hat{m}\right), \end{aligned} \tag{112}$$

where, for our purposes, $\hat{m}$ is either $\hat{m}_\nu$ or $\hat{m}_\ell$. For simplicity of notation we have dropped the p^2-dependence in the coefficient matrices $A_{L,B}$, $B_{L,R}^{(r)}$, $B_{L,R}^{(l)}$, $C_{L,R}$. Of course, one can convert equation (112) into the form of equation (107), in which case one obtains the identifications

$$\Sigma_L^{(A)} = B_L^{(r)} + B_R^{(l)} - C_L\hat{m} - \hat{m}C_R, \tag{113a}$$
$$\Sigma_R^{(A)} = B_R^{(r)} + B_L^{(l)} - C_R\hat{m} - \hat{m}C_L, \tag{113b}$$
$$\Sigma_L^{(B)} = A_L - \hat{m}B_L^{(r)} - B_L^{(l)}\hat{m} + p^2 C_R + \hat{m}C_L\hat{m}, \tag{113c}$$
$$\Sigma_R^{(B)} = A_R - \hat{m}B_R^{(r)} - B_R^{(l)}\hat{m} + p^2 C_L + \hat{m}C_R\hat{m}. \tag{113d}$$

The nice feature of the form of equation (112) is that the radiative mass shifts to the tree-level masses are simply given by

$$\Delta m_i = \frac{1}{2}\left\{\left(A_L(m_i^2)\right)_{ii} + \left(A_R(m_i^2)\right)_{ii}\right\}. \tag{114}$$

Of course, this is to be expected but can also be checked explicitly by plugging the expressions of equation (113) into equation (110).

At this point we want to stress that the discussion in the last paragraph holds also for any part of the fermion self-energy. If such a part is decomposed as in equation (112), then the coefficient matrices $B^{(r)}_{L,R}$, $B^{(l)}_{L,R}$, $C_{L,R}$ of this part will not contribute to the physical mass shifts Δm_i. Therefore, any gauge dependence in $B^{(r)}_{L,R}$, $B^{(l)}_{L,R}$, $C_{L,R}$ is irrelevant for the masses. This will be utilized in the following. The ξ-independence of the one-loop neutrino masses for the model put forward in [6] has recently been shown in [31].

4.2 Gauge-parameter cancellation in fermion self-energy loops

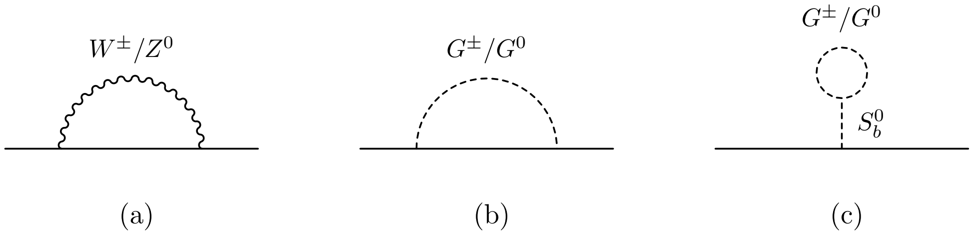


Figure 4: Loop contributions to the fermion self-energies that depend on ξ. In diagram (c) only physical neutral scalars contribute, *i.e.* $b = 2, \ldots, 2n_H$.

The diagrams in figure 4 are those loop diagrams which have ξ-dependent boson propagators. Using equation (30a) for the vector boson propagator, the gauge dependence resides in

$$-\frac{k_\mu k_\nu}{m_V^2}\frac{1}{k^2 - \xi_V m_V^2 + i\epsilon} \quad \text{with} \quad V = Z, W. \tag{115}$$

In this subsection we will demonstrate that, when we include only this part of the vector boson propagator in diagram (a), then the sum of the diagrams in figure 4 has the form of equation (112) with vanishing A_L and A_R. In other words, of the diagrams in figure 4 only diagram (a) with the vector boson propagator

$$\frac{1}{k^2 - m_V^2 + i\epsilon}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2}\right) \tag{116}$$

contributes to Δm_i and the sum over the one-loop diagrams gives a ξ-independent contribution to Δm_i, as it must be on physical grounds. We will prove this separately for charged leptons and neutrinos and for charged and neutral boson exchange. The discussion presented here does not apply to photon exchange. This case will be treated separately at the end of this subsection.

We first consider the contribution of diagram (a) of figure 4, with the vector boson propagator of equation (115), to the fermion self-energies $-i\Sigma_f(p)$ ($f = \chi, \ell$). In order to streamline the discussion, we have to introduce some notation. We define

$$\mathcal{A} \equiv \mathcal{A}_L\gamma_L + \mathcal{A}_R\gamma_R \quad \text{and} \quad \tilde{\mathcal{A}} \equiv \mathcal{A}_L\gamma_R + \mathcal{A}_R\gamma_L \tag{117}$$

such that the coupling matrices of the vector bosons V to the fermions f have the structure $\gamma^\mu\mathcal{A}$, where $\mathcal{A}$ also includes the flavour part. The matrices $\mathcal{A}$ can be read off from the respective Lagrangians. Here is a list of all matrices $\mathcal{A}$ under discussion, with f being the incoming fermion:

$$\mathcal{A} = -\frac{g}{4c_w} F_{LR} \qquad \text{for} \quad V = Z,\ f = \chi, \tag{118}$$

$$\mathcal{A} = -\frac{g}{\sqrt{2}} W_L^\dagger U_L \gamma_L \qquad \text{for} \quad V = W,\ f = \chi, \tag{119}$$

$$\mathcal{A}_c = +\frac{g}{\sqrt{2}} \left(W_L^\dagger U_L \right)^* \gamma_R \qquad \text{for} \quad V = W,\ f = \chi, \tag{120}$$

$$\mathcal{A} = -\frac{g}{c_w} \left[\left(s_w^2 - \frac{1}{2} \right) \gamma_L + s_w^2 \gamma_R \right] \qquad \text{for} \quad V = Z,\ f = \ell, \tag{121}$$

$$\mathcal{A} = -\frac{g}{\sqrt{2}} U_L^\dagger W_L \gamma_L \qquad \text{for} \quad V = W,\ f = \ell. \tag{122}$$

The matrix F_{LR} is defined in equation (28). The expression $\mathcal{A}_c$ occurs in the Lagrangian of equation (26b), its usage will be explained below. Moreover, we stipulate that $\hat{m}$ is the diagonal mass matrix of the fermions with momentum p on the external line, while $\underline{\hat{m}}$ denotes the diagonal mass matrix of the fermions with momentum $p - k$ on the internal line of diagram (a) in figure 4. Eventually, with the abbreviation

$$\mathcal{P} \equiv \frac{1}{\not{p} - \not{k} - \underline{\hat{m}} + i\epsilon}, \tag{123}$$

we find for the contribution to $-i\Sigma_f(p)$, pertaining to the propagator of equation (115), the expression

$$-\frac{1}{m_V^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \xi_V m_V^2 + i\epsilon} \not{k} \mathcal{A}^\dagger \mathcal{P} \not{k} \mathcal{A}. \tag{124}$$

Since we are dealing with Majorana neutrinos, it has to be taken into account that χ cannot only be Wick-contracted with $\bar{\chi}$ but also with χ, and the analogue applies to $\bar{\chi}$. Dealing with one-loop computations, this complication arises only in the case of the self-energy $\Sigma_\chi(p)$. In this context a very convenient identity is given by [30]

$$\bar{f}_1 \mathcal{O} f_2 = \overline{(f_2)^c}\, C \mathcal{O}^T C^{-1} (f_1)^c, \tag{125}$$

where f_1 and f_2 are any vectors of fermion fields and $\mathcal{O}$ is product of an arbitrary matrix in Dirac space times a matrix in flavour space or a sum over matrices of this type; the superscript c indicates the charge-conjugated field and C is the charge-conjugation matrix, which acts only in Dirac space. Thus if one contracts χ with an interaction term $\bar{f}_1 \mathcal{O} f_2$ where $f_2 = \chi$,[14] because of $\chi^c = \chi$ one can simply take advantage of this identity and the aforementioned contraction becomes the ordinary contraction of χ with $\bar{\chi}$, however, at the expense of transforming $\mathcal{O}$ into $C\mathcal{O}^T C^{-1}$. Actually, in our one-loop computations two cases[15] cover all possible situations:

[14] For simplicity of notation we assume that $\mathcal{O}$ also contains the boson fields.

[15] As a side remark, for neutral-scalar vertex corrections a third case occurs which requires the Lagrangian of equation (19).

i. In the couplings of S^0_b and Z—see equations (18) and (27), respectively—we have $f_1 = f_2 = \chi$ and our respective coupling matrices are defined in such a way that they are invariant under the transformation of equation (125):

$$C\left(\gamma^\mu F_{LR}\right)^T C^{-1} = \gamma^\mu F_{LR}, \quad C\left(F_b\gamma_L + F_b^\dagger\gamma_R\right)^T C^{-1} = F_b\gamma_L + F_b^\dagger\gamma_R. \tag{126}$$

Consequently, in our formalism, a contraction of χ with an interaction term of the type $\bar{\chi}\mathcal{O}\chi$ simply gives a factor of 2.

ii. In the case of charged-boson interactions, we display both versions of the Lagrangian interaction density, the common one with the charged-lepton fields ℓ and, using equation (125), the one with the fields ℓ^c—see equations (20) and (26), in order to clearly spell out both contractions of the external χ or $\bar{\chi}$.

For $W^\pm$ exchange, *i.e.* diagram (a) of figure 4, the second case applies. In the light of the discussion presented here there are two contributions to $\Sigma_\chi(p)$ to be taken into account, stemming from $\mathcal{A}$ of equation (119) and $\mathcal{A}_c$ of equation (120).

Now we return to a discussion of equation (124). We follow [24] and make the decomposition

$$\not{k}\mathcal{A}^\dagger\mathcal{P}\not{k}\mathcal{A} = -\tilde{\mathcal{A}}^\dagger\not{k}\mathcal{A} + A' + B', \tag{127}$$

where $\tilde{\mathcal{A}}$ is defined in equation (117), and

$$\begin{aligned} A' &= -\frac{1}{2}\left(\not{p}-\hat{m}\right)\mathcal{A}^\dagger\mathcal{A} - \frac{1}{2}\tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}}\left(\not{p}-\hat{m}\right) \\ &\quad -\frac{1}{2}\hat{m}\mathcal{A}^\dagger\mathcal{A} - \frac{1}{2}\tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}}\hat{m} + \tilde{\mathcal{A}}^\dagger\underline{\hat{m}}\mathcal{A}, \end{aligned} \tag{128}$$

$$\begin{aligned} B' &= \left(\not{p}-\hat{m}\right)\mathcal{A}^\dagger\mathcal{P}\tilde{\mathcal{A}}\left(\not{p}-\hat{m}\right) \\ &\quad + \left(\not{p}-\hat{m}\right)\mathcal{A}^\dagger\mathcal{P}\left(\tilde{\mathcal{A}}\hat{m} - \underline{\hat{m}}\mathcal{A}\right) + \left(\hat{m}\mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger\underline{\hat{m}}\right)\mathcal{P}\tilde{\mathcal{A}}\left(\not{p}-\hat{m}\right) \\ &\quad + \left(\hat{m}\mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger\underline{\hat{m}}\right)\mathcal{P}\left(\tilde{\mathcal{A}}\hat{m} - \underline{\hat{m}}\mathcal{A}\right). \end{aligned} \tag{129}$$

Obviously, the first term on the right-hand side of equation (127) drops out when performing the integration in equation (124) and all terms in equations (128) and (129) that have $\not{p}-\hat{m}$ do not contribute to Δm_i or Δm_α. Therefore, it useful to introduce the definitions

$$A \equiv -\frac{1}{2}\hat{m}\mathcal{A}^\dagger\mathcal{A} - \frac{1}{2}\tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}}\hat{m} + \tilde{\mathcal{A}}^\dagger\underline{\hat{m}}\mathcal{A}, \quad B \equiv \left(\hat{m}\mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger\underline{\hat{m}}\right)\mathcal{P}\left(\tilde{\mathcal{A}}\hat{m} - \underline{\hat{m}}\mathcal{A}\right), \tag{130}$$

which refer to the last line in equation (128) and in equation (129), respectively.

When we use in the following the notions A-term and B-term, we mean the contribution of A and B, respectively, to the loop integral of equation (124). We will now prove the following [24]:

1. The A-term contribution of equation (130) to diagram (a) of figure 4 is exactly cancelled by the sum over all physical neutral scalars S^0_b in diagram (c).

2. The B-term contribution of equation (130) to diagram (a) of figure 4 is exactly cancelled by diagram (c).

These cancellations occur separately for both neutrino and charged-leptons on the external lines and for both neutrino and charged-leptons on the inner lines. Therefore, in total there are eight cancellations.

Neutrinos and the cancellation of the A-term: Firstly we consider Z exchange and neutrinos on the internal line of diagram (a) in figure 4. With equation (118) and using some formulas of section 2.1, we find in this case

$$A = -\frac{g^2}{32c_w^2}\left[\hat{m}_\nu\left(U_L^\dagger U_L\gamma_L + U_L^T U_L^*\gamma_R\right) + \left(U_L^\dagger U_L\gamma_R + U_L^T U_L^*\gamma_L\right)\hat{m}_\nu\right]. \tag{131}$$

Taking into account a factor 4 from the Majorana contractions, we obtain for the loop integral of equation (124) the A-term

$$4\times\frac{1}{m_Z^2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon} \times\frac{g^2}{32c_w^2}\left[\hat{m}_\nu\left(U_L^\dagger U_L\gamma_L + U_L^T U_L^*\gamma_R\right) + \left(U_L^\dagger U_L\gamma_R + U_L^T U_L^*\gamma_L\right)\hat{m}_\nu\right]. \tag{132}$$

The expression for diagram (c) of figure 4 is given by

$$\sum_{b=2}^{2n_H}(-i\sqrt{2})\left(F_b\gamma_L + F_b^\dagger\gamma_R\right)\times\frac{i}{-M_b^2}\times\frac{i}{2v}M_b^2\times\mathrm{Im}\left(V^\dagger V\right)_{1b}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{i}{k^2-\xi_Z m_Z^2+i\epsilon}. \tag{133}$$

Note the factor 1/2 on the vertex of S_b^0 coupling to G^0—see equation (37). Since here M_b^2 cancels, we can use equation (B.7) to perform the summation

$$\sum_{b=2}^{2n_H}F_b\,\mathrm{Im}\left(V^\dagger V\right)_{1b} = -\frac{1}{\sqrt{2}v}\left(\hat{m}_\nu U_L^\dagger U_L + U_L^T U_L^*\hat{m}_\nu\right). \tag{134}$$

To obtain this expression, we have also utilized equation (16). Note that in this sum we can include $b=1$ because $\mathrm{Im}\left(V^\dagger V\right)_{1b}=0$. Finally, we end up with the expression

$$-\frac{1}{2v^2}\left[\left(\hat{m}_\nu U_L^\dagger U_L + U_L^T U_L^*\hat{m}_\nu\right)\gamma_L + \left(U_L^\dagger U_L\hat{m}_\nu + \hat{m}_\nu U_L^T U_L^*\right)\gamma_R\right]\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}. \tag{135}$$

for diagram (c). Because of

$$\frac{g^2}{m_Z^2 c_w^2} = \frac{4}{v^2} \tag{136}$$

it exactly cancels the A-term.

Secondly we consider W exchange and charged leptons on the internal line of diagram (a) in figure 4. Here we obtain

$$A = -\frac{g^2}{4}\left(\hat{m}_\nu U_L^\dagger U_L\gamma_L + U_L^\dagger U_L\hat{m}_\nu\gamma_R\right). \tag{137}$$

However, due to the Majorana nature, we also have the contribution from $\mathcal{A}_c$ of equation (120), leading to

$$A_c = -\frac{g^2}{4}\left(\hat{m}_\nu U_L^T U_L^* \gamma_R + U_L^T U_L^* \hat{m}_\nu \gamma_L\right). \tag{138}$$

The full A-term is, therefore,

$$\frac{1}{m_W^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_W m_W^2+i\epsilon} \times\frac{g^2}{4}\left[\left(\hat{m}_\nu U_L^\dagger U_L\gamma_L + U_L^\dagger U_L \hat{m}_\nu \gamma_R\right)+\left(\hat{m}_\nu U_L^T U_L^*\gamma_R + U_L^T U_L^*\hat{m}_\nu\gamma_L\right)\right]. \tag{139}$$

As for diagram (c) we can take over the previous result, equation (135), with minor modifications:

$$-\frac{1}{v^2}\left[\left(\hat{m}_\nu U_L^\dagger U_L + U_L^T U_L^* \hat{m}_\nu\right)\gamma_L + \left(U_L^\dagger U_L \hat{m}_\nu + \hat{m}_\nu U_L^T U_L^*\right)\gamma_R\right]\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_W m_W^2+i\epsilon}. \tag{140}$$

Note there is no factor 1/2 on the vertex of S_b^0 coupling to $G^\pm$—see equation (37). With

$$\frac{g^2}{m_W^2}=\frac{4}{v^2} \tag{141}$$

we see that also here the tadpoles cancel the A-term.

Neutrinos and the cancellation of the B-term: Firstly we discuss Z and χ exchange in the loop diagram (a) of figure 4. For the the computation of B we need

$$\tilde{\mathcal{A}}\hat{m}_\nu - \hat{m}_\nu\mathcal{A} = -\frac{g}{4c_w}\left(T\gamma_R - T^\dagger\gamma_L\right) \quad \text{with} \quad T = U_L^\dagger U_L \hat{m}_\nu + \hat{m}_\nu U_L^T U_L^*. \tag{142}$$

Since here $\mathcal{A}$ is hermitian, we have

$$\hat{m}_\nu\mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger\hat{m}_\nu = -\left(\tilde{\mathcal{A}}\hat{m}_\nu - \hat{m}_\nu\mathcal{A}\right). \tag{143}$$

Thus the B-term is given by

$$-\frac{4}{m_Z^2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}\times(-1)\times\frac{g^2}{16c_w^2}\left(T\gamma_R - T^\dagger\gamma_L\right)\mathcal{P}\left(T\gamma_R - T^\dagger\gamma_L\right). \tag{144}$$

Turning to diagram (b) of figure 4, the G^0-coupling matrix F_1 of equation (23) can be written with the help of T as

$$F_1 = \frac{i}{\sqrt{2}v}T^\dagger. \tag{145}$$

This leads to the following expression for diagram (b):

$$-\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}\times\frac{1}{v^2}\left(T^\dagger\gamma_L - T\gamma_R\right)\mathcal{P}\left(T^\dagger\gamma_L - T\gamma_R\right). \tag{146}$$

Using again equation (136) we see that the expression for diagram (b) exactly cancels the B-term.

Secondly we discuss $W^\pm$ and charged-fermion exchange in the loop diagram (a) of figure 4. In the light of the discussion concerning Majorana fermions, we have to take into account both W^+ and ℓ^- exchange and W^- and ℓ^+ exchange. Defining for simplicity of notation the $n_L \times (n_L + n_R)$ matrix

$$V_L = W_L^\dagger U_L, \tag{147}$$

for the quantity B of equation (130) we require the expressions

$$\begin{aligned}
\tilde{\mathcal{A}}\hat{m}_\nu - \hat{m}_\ell \mathcal{A} &= -\frac{g}{\sqrt{2}} \left(V_L \hat{m}_\nu \gamma_R - \hat{m}_\ell V_L \gamma_L \right), & \text{(148a)}\\
\hat{m}_\nu \mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger \hat{m}_\ell &= -\frac{g}{\sqrt{2}} \left(\hat{m}_\nu V_L^\dagger \gamma_L - V_L^\dagger \hat{m}_\ell \gamma_R \right), & \text{(148b)}\\
\tilde{\mathcal{A}}_c \hat{m}_\nu - \hat{m}_\ell \mathcal{A}_c &= +\frac{g}{\sqrt{2}} \left(V_L^* \hat{m}_\nu \gamma_L - \hat{m}_\ell V_L^* \gamma_R \right), & \text{(148c)}\\
\hat{m}_\nu \mathcal{A}_c^\dagger - \tilde{\mathcal{A}}_c^\dagger \hat{m}_\ell &= +\frac{g}{\sqrt{2}} \left(\hat{m}_\nu V_L^T \gamma_R - V_L^T \hat{m}_\ell \gamma_L \right). & \text{(148d)}
\end{aligned}$$

Then, the corresponding B-term is given by

$$\begin{aligned}
& -\frac{1}{m_W^2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \xi_W m_W^2 + i\epsilon} \times \frac{g^2}{2} \\
& \times \left\{ \left(\hat{m}_\nu V_L^\dagger \gamma_L - V_L^\dagger \hat{m}_\ell \gamma_R \right) \mathcal{P} \left(V_L \hat{m}_\nu \gamma_R - \hat{m}_\ell V_L \gamma_L \right) \right. \\
& \left. + \left(\hat{m}_\nu V_L^T \gamma_R - V_L^T \hat{m}_\ell \gamma_L \right) \mathcal{P} \left(V_L^* \hat{m}_\nu \gamma_L - \hat{m}_\ell V_L^* \gamma_R \right) \right\}.
\end{aligned} \tag{149}$$

In equation (20) we have formulated the couplings of $G^\pm$ to the fermions with the help of the matrices of equation (24). With the matrix V_L they are simply

$$R_1 = \frac{\sqrt{2}}{v} V_L \hat{m}_\nu, \quad L_1 = \frac{\sqrt{2}}{v} \hat{m}_\ell V_L. \tag{150}$$

Just as diagram (a) of figure 4, diagram (b) has two contributions as well—see the Lagrangians of equations (20a) and (20b), given by

$$\begin{aligned}
& \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - \xi_W m_W^2 + i\epsilon} \times \frac{2}{v^2} \\
& \times \left\{ \left(\hat{m}_\nu V_L^\dagger \gamma_L - V_L^\dagger \hat{m}_\ell \gamma_R \right) \mathcal{P} \left(V_L \hat{m}_\nu \gamma_R - \hat{m}_\ell V_L \gamma_L \right) \right. \\
& \left. + \left(\hat{m}_\nu V_L^T \gamma_R - V_L^T \hat{m}_\ell \gamma_L \right) \mathcal{P} \left(V_L^* \hat{m}_\nu \gamma_L - \hat{m}_\ell V_L^* \gamma_R \right) \right\}.
\end{aligned} \tag{151}$$

Obviously, using equation (141), diagram (c) cancels the B-term.

Charged leptons and the cancellation of the A-term: For Z and neutrino exchange A is simply given by

$$A = -\frac{g^2}{8c_w^2} \hat{m}_\ell \tag{152}$$

and the A-term is thus

$$\frac{1}{m_Z^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}\times\frac{g^2}{8c_w^2}\,\hat{m}_\ell. \tag{153}$$

The sum over the tadpoles in diagram (c) of figure 4 now reads

$$\sum_{b=2}^{2n_H}\frac{-i}{\sqrt{2}}\left(G_b\gamma_L+G_b^\dagger\gamma_R\right)\times\frac{i}{-M_b^2}\times\frac{i}{2v}M_b^2\times\mathrm{Im}\left(V^\dagger V\right)_{1b}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{i}{k^2-\xi_Z m_Z^2+i\epsilon}. \tag{154}$$

With

$$\sum_{b=2}^{2n_H} G_b\,\mathrm{Im}\left(V^\dagger V\right)_{1b}=-\frac{\sqrt{2}}{v}\,\hat{m}_\ell \tag{155}$$

the tadpole contributions are thus

$$-\frac{1}{2v^2}\,\hat{m}_\ell\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{i}{k^2-\xi_Z m_Z^2+i\epsilon}. \tag{156}$$

Applying equation (136), we see that they exactly cancel the A-term.

Though for charged $W^\pm$ and charged-lepton exchange $\mathcal{A}$ of equation (122) looks very different from that of equation (121), the result for A is quite similar:

$$A=-\frac{g^2}{4}\,\hat{m}_\ell. \tag{157}$$

It gives the A-term

$$\frac{1}{m_W^2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2-\xi_W m_W^2+i\epsilon}\times\frac{g^2}{4}\,\hat{m}_\ell, \tag{158}$$

while the tadpole contributions are

$$-\frac{1}{v^2}\,\hat{m}_\ell\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{i}{k^2-\xi_W m_W^2+i\epsilon}. \tag{159}$$

Both contributions exactly cancel each other, when taking into account equation (141).

Charged leptons and the cancellation of the B-term: Firstly we consider Z and ℓ exchange in diagram (a) of figure 4. According to equation (130), the expressions

$$\tilde{\mathcal{A}}\hat{m}_\ell-\hat{m}_\ell\mathcal{A}=\frac{g}{2c_w}\,\hat{m}_\ell\left(\gamma_R-\gamma_L\right)\quad\text{and}\quad\hat{m}_\ell\mathcal{A}^\dagger-\tilde{\mathcal{A}}^\dagger\hat{m}_\ell=-\frac{g}{2c_w}\,\hat{m}_\ell\left(\gamma_R-\gamma_L\right) \tag{160}$$

leads to B and thus to the B-term

$$\frac{1}{m_Z^2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}\times\frac{g^2}{4c_w^2}\,\hat{m}_\ell\left(\gamma_L-\gamma_R\right)\mathcal{P}m_\ell\left(\gamma_L-\gamma_R\right). \tag{161}$$

With G_1 of equation (23) and the Lagrangian of equation (18), diagram (b) of figure 4 gives

$$-\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2-\xi_Z m_Z^2+i\epsilon}\times\frac{1}{v^2}\,\hat{m}_\ell\left(\gamma_L-\gamma_R\right)\mathcal{P}m_\ell\left(\gamma_L-\gamma_R\right), \tag{162}$$

which exactly cancels the B-term.

Secondly we consider W and χ exchange in diagram (a) of figure 4. Here we have

$$\tilde{\mathcal{A}}\hat{m}_\ell - \hat{m}_\nu \mathcal{A} = -\frac{g}{\sqrt{2}}\left(V_L^\dagger \hat{m}_\ell \gamma_R - \hat{m}_\nu V_L^\dagger \gamma_L\right), \tag{163a}$$

$$\hat{m}_\ell \mathcal{A}^\dagger - \tilde{\mathcal{A}}^\dagger \hat{m}_\nu = -\frac{g}{\sqrt{2}}\left(\hat{m}_\ell V_L \gamma_L - V_L \hat{m}_\nu \gamma_R\right) \tag{163b}$$

and the B-term

$$-\frac{1}{m_W^2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2 - \xi_W m_W^2 + i\epsilon} \times \frac{g^2}{2}\left(\hat{m}_\ell V_L \gamma_L - V_L \hat{m}_\nu \gamma_R\right)\mathcal{P}\left(V_L^\dagger \hat{m}_\ell \gamma_R - \hat{m}_\nu V_L^\dagger \gamma_L\right). \tag{164}$$

Considering diagram (b) of figure 4, we need the Lagrangian of equation (20a) and the matrices R_1 and L_1 of equation (24). The expression for this diagram is then

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{k^2 - \xi_W m_W^2 + i\epsilon} \times \frac{2}{v^2}\left(V_L \hat{m}_\nu \gamma_R - \hat{m}_\ell V_L \gamma_L\right)\mathcal{P}\left(\hat{m}_\nu V_L^\dagger \gamma_L - V_L^\dagger \hat{m}_\ell \gamma_R\right), \tag{165}$$

which precisely cancels the B-term.

Photon exchange: This is only possible for charged leptons. Moreover, diagrams (b) and (c) in figure 4 do not exist in this case. However, here $\mathcal{A} = e\mathbb{1}$ and the decomposition of equation (127) is simply given by

$$\not{k}\mathcal{P}\not{k} = -\not{k} - \left(\not{p} - \hat{m}_\ell\right) + \left(\not{p} - \hat{m}_\ell\right)\mathcal{P}\left(\not{p} - \hat{m}_\ell\right). \tag{166}$$

This demonstrates that the part of the photon propagator proportional to ξ_A—*c.f.* equation (30b)—does not contribute to Δm_α.

4.3 Yukawa coupling renormalization and mass counterterms

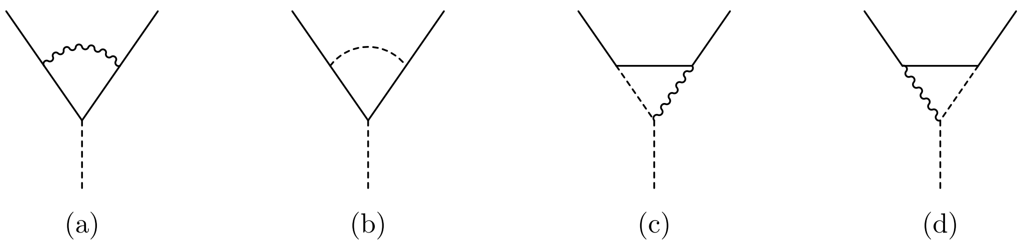


Figure 5: Vertex corrections to the couplings of neutral scalars to fermions.

Yukawa coupling renormalization: Vertex corrections to the S_b^0 coupling to neutrinos can effectively be written as counterterms to the Yukawa coupling matrices Δ_k or can be computed in the unbroken theory as a correction to the φ_k^0 vertex. The result is

$$\delta\Delta_k = -2A_1\Delta_k - \frac{1}{16\pi^2}\, c_\infty \Delta_j \Gamma_k^\dagger \Gamma_j, \tag{167}$$

where the first term stems from diagrams (c) and (d) in figure 5 and the second one from diagram (b) with charged-scalar exchange. Note that the contributions of diagram (a) with neutral and charged vector boson exchange and of diagram (b) with neutral scalar exchange are zero separately.

Now we discuss vertex corrections of the S_b^0 coupling to leptons. Those can be subsumed as $\delta\Gamma_k$. The result is

$$\begin{aligned}\delta\Gamma_k &= -2A_1\Gamma_k - \frac{1}{16\pi^2}\, c_\infty \Gamma_j \Delta_k^\dagger \Delta_j \\ &\quad - \frac{g^2}{16\pi^2 c_w^2}\, c_\infty \left(3+\xi_Z\right) s_w^2 \left(s_w^2 - \frac{1}{2}\right)\Gamma_k - \frac{e^2}{16\pi^2}\, c_\infty \left(3+\xi_A\right)\Gamma_k,\end{aligned} \tag{168}$$

where the two terms in the first line originate, just as before, from diagrams (c) and (d) in figure 5 and from diagram (b) with charged-scalar exchange. As before, diagram (a) with $W^\pm$ exchange and diagram (b) with $S_{b'}^0$ exchange give vanishing contributions. However, Z and photon exchange in diagram (a) are non-zero, leading to the two terms in the second line.

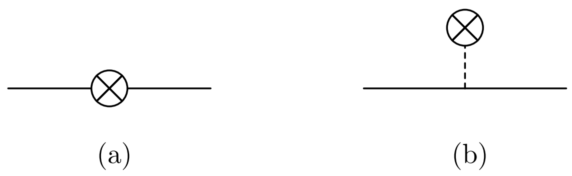


Figure 6: Counterterms on fermion lines.

Mass counterterms: There are two types of counterterms for the fermion self-energies. The first type, depicted in diagram (a) of figure 6, originates in δv_k, $\delta\Delta_k$ and δM_R for neutrinos, while for charged leptons it stems from δv_k and $\delta\Gamma_k$. Both have also wave-function renormalization counterterms. Defining

$$\delta M_\ell \equiv \frac{1}{\sqrt{2}} \sum_k \left(\delta v_k^*\, \Gamma_k + v_k^*\, \delta\Gamma_k\right) \quad \text{and} \quad \delta M_D \equiv \frac{1}{\sqrt{2}} \sum_k \left(\delta v_k\, \Delta_k + v_k\, \delta\Delta_k\right), \tag{169}$$

diagram (a) thus refers to the terms

$$i\not{p}\,\delta^{(\ell)} - i\left[W_R^\dagger\, \delta M_\ell\, W_L \gamma_L + W_L^\dagger\, \delta M_\ell^\dagger\, W_R \gamma_R\right] \quad \text{with} \quad \delta^{(\ell)} = \delta_L^{(\ell)}\gamma_L + \delta_R^{(\ell)}\gamma_R \tag{170}$$

in the case of charged leptons and to

$$
\begin{aligned}
& i\not{p}\left(\delta^{(x)}\gamma_L + \left(\delta^{(x)}\right)^* \gamma_R\right) && \text{(171a)}\\
& -i\left[\left(U_R^\dagger\,\delta M_D\,U_L + U_L^T\,\delta M_D^T\,U_R^*\right)\gamma_L + \left(U_L^\dagger \delta M_D^\dagger U_R + U_R^T\,\delta M_D^*\,U_L^*\right)\gamma_R\right] && \text{(171b)}\\
& -i\left[U_R^\dagger\,\delta M_R\,U_R^*\,\gamma_L + U_R^T\,\delta M_R^*\,U_R\,\gamma_R\right] && \text{(171c)}
\end{aligned}
$$

for neutrinos. The second type of counterterm is symbolized by diagram (b) of figure 6 and connects the tadpole counterterm to the fermion line.

In order to discuss the ξ-dependence of the counterterms it is expedient to have the explicit ξ-dependence of the wave-function renormalization matrices as well. For neutrinos the result is

$$\delta^{(x)}(\xi) = -2A_1 U_L^\dagger U_L, \tag{172}$$

whereas for charged lepton we find the more involved result

$$\delta^{(\ell)}(\xi) = -\frac{1}{16\pi^2}\,c_\infty\left\{\frac{g^2}{2}\xi_W\gamma_L + \frac{g^2}{c_w^2}\xi_Z\left[\left(s_w^2 - \frac{1}{2}\right)^2\gamma_L + s_w^4\gamma_R\right] + e^2\xi_A\right\}. \tag{173}$$

Using the definition of A_1, equation (79), we can recast this equation into

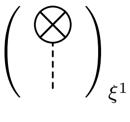
$$\delta^{(\ell)}(\xi) = -2A_1\gamma_L - \frac{1}{16\pi^2}\,c_\infty\left\{\frac{g^2}{c_w^2}\xi_Z\left[\left(s_w^4 - s_w^2\right)\gamma_L + s_w^4\gamma_R\right] + e^2\xi_A\right\}. \tag{174}$$

Gauge-parameter independence of mass counterterms: Let us take stock of the ξ-dependence in the counterterms.

(a) In the counterterms pertaining to diagram (a) of figure 6, ξ-dependence occurs in

- (a-i) δv_k—see equation (85)—of δM_ℓ and δM_D,
- (a-ii) the A_1-term of $\delta\Delta_k$ and $\delta\Gamma_k$,
- (a-iii) the residual ξ-dependence of $\delta\Gamma_k$ not contained in A_1,
- (a-iv) the A_1-term of $\delta^{(x)}(\xi)$ and of $\delta^{(\ell)}(\xi)$ and
- (a-v) the residual ξ-dependence of $\delta^{(\ell)}(\xi)$ not contained in A_1.

(b) Finally, the ξ-dependence that resides in the tadpole counterterms symbolized by diagram (b) of figure 6 has to be accounted for in the present discussion.[16] The relevant expression can be read off from equation (95), where the first line is the divergence of the tadpole loops. Thus the sum of the other three lines, given by



$$\left(\;\cdot\;\right)_{\xi^1} = -\frac{i}{2}\left(v_i^* V_{ib} + V_{ib}^* v_i\right) M_b^2 A_1, \tag{175}$$

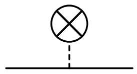
are the ξ-dependent tadpole counterterms to be taken into account here.

[16] Note that in section 4.2 we have already treated the tadpole loops and their ξ-dependence–see figure 4, but that discussion did not include the ξ-dependence of the tadpole counterterms.

In the following we will show that the ξ-dependent counterterms listed here do not contribute to Δm_i and Δm_α.

With equations (85), (167) and (168) it is obvious that the terms stemming from (a-i) and (a-ii) cancel each other in both δM_ℓ and δM_D.

Next we consider the terms originating in (a-iv) and diagram (b). In the case of neutrinos the contribution of diagram (b) of figure 6 to $-i\Sigma_\chi(p)$ gives

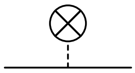


$$= \sum_{b=2}^{2n_H}(-i\sqrt{2})\left(F_b\gamma_L + F_b^\dagger\gamma_R\right)\times\frac{i}{-M_b^2}\times\left(-\frac{i}{2}\right)\left(v_i^* V_{ib} + V_{ib}^* v_i\right) M_b^2 A_1. \quad (176)$$

Since the expression in equation (175) is zero for $b = 1$—*c.f.* equation (B.14)—and the scalar masses cancel, we can include $b = 1$ in the sum. Moreover, taking advantage of the first two relations of equation (B.5) and using

$$\left(U_R^\dagger \Delta_k U_L + U_L^T \Delta_k^T U_R^*\right) v_k = -2viF_1, \quad (177)$$

which follows from equations (21a) and (23), the contribution of diagram (b) of figure 6 finally has the form



$$= iA_1\left[\left(\hat{m}_\nu U_L^\dagger U_L + U_L^T U_L^* \hat{m}_\nu\right)\gamma_L + \left(U_L^\dagger U_L \hat{m}_\nu + \hat{m}_\nu U_L^T U_L^*\right)\gamma_R\right]. \quad (178)$$

Adding this to the A_1-term of $\delta^{(\chi)}(\xi)$ and introducing the abbreviation $\mathcal{Z} \equiv U_L^\dagger U_L$, we arrive at

$$\begin{aligned} & -iA_1\left[2\not{p}\left(\mathcal{Z}\gamma_L + \mathcal{Z}^T\gamma_R\right) - \left(\hat{m}_\nu\mathcal{Z} + \mathcal{Z}^T\hat{m}_\nu\right)\gamma_L + \left(\mathcal{Z}\hat{m}_\nu + \hat{m}_\nu\mathcal{Z}^T\right)\gamma_R\right] \\ = & -iA_1\left[\left(\not{p} - \hat{m}_\nu\right)\mathcal{Z}\gamma_L + \mathcal{Z}^T\gamma_L\left(\not{p} - \hat{m}_\nu\right) + \left(\not{p} - \hat{m}_\nu\right)\mathcal{Z}^T\gamma_R + \mathcal{Z}\gamma_R\left(\not{p} - \hat{m}_\nu\right)\right]. \quad (179) \end{aligned}$$

According to section (4.1), we can read off from this expression that it does not contribute to Δm_i. With this we have concluded the discussion of counterterms to the neutrino self-energy.

Now we proceed in an analogous fashion in the case of charged leptons. Skipping here all details, the A_1-term of $\delta^{(\ell)}(\xi)$ together with the contribution of diagram (b) of figure 6 leads to

$$-iA_1\left(2\not{p}\gamma_L - \hat{m}_\ell\right) = -iA_1\left[\left(\not{p} - \hat{m}_\ell\right)\gamma_L + \gamma_R\left(\not{p} - \hat{m}_\ell\right)\right] \quad (180)$$

in $-i\Sigma_\ell(p)$. Therefore, this does not contribute to Δm_α.

It remains to consider the terms (a-iii) and (a-v), which refer solely to charged leptons. These are

$$\begin{aligned} & -\frac{i}{16\pi^2}\, c_\infty \left\{\frac{g^2}{c_w^2}\xi_Z\left[\left(s_w^4 - s_w^2\right)\gamma_L + s_w^4\gamma_R\right] + e^2\xi_A\right\}\not{p} \\ & +\frac{i}{16\pi^2}\, c_\infty \left[\frac{g^2}{c_w^2}\xi_Z s_w^2\left(s_w^2 - \frac{1}{2}\right) + e^2\xi_A\right)\hat{m}_\ell. \quad (181) \end{aligned}$$

Obviously, the ξ_A-terms combine to give $\not{p}-\hat{m}_\ell$. That the ξ_Z-terms can also be decomposed into expressions having external factors $\not{p}-\hat{m}_\ell$, can be concluded from the discussion in appendix C. Explicitly, one such decomposition is given by

$$\begin{aligned} & \left[\left(s_w^4 - s_w^2\right)\gamma_L + s_w^4 \gamma_R\right]\not{p} - s_w^2\left(s_w^2 - \frac{1}{2}\right)\hat{m}_\ell \\ = \; & \left(s_w^4 - \frac{1}{2}s_w^2\right)\left(\not{p}-\hat{m}_\ell\right)\gamma_L - \frac{1}{2}s_w^2\gamma_R\left(\not{p}-\hat{m}_\ell\right) + s_w^4\left(\not{p}-\hat{m}_\ell\right)\gamma_R. \end{aligned} \tag{182}$$

Summarizing, we have found that all ξ-dependent counterterms to the neutrino or charged-lepton self-energies have external factors $\not{p}-\hat{m}_\nu$ or $\not{p}-\hat{m}_\ell$, respectively. Therefore, in our renormalization scheme there are no ξ-dependent counterterms for the one-loop radiative masses Δm_i ($i = 1, \ldots, n_L + n_R$) and Δm_α ($\alpha = e, \mu, \tau$).

5 Finiteness of the fermion self-energies

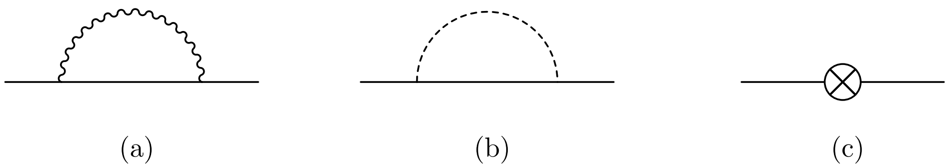


Figure 7: Fermion self-energy diagrams.

Here we want to demonstrate that in our renormalization scheme the fermion self-energies are finite, while the explicit formulas are given later in section 6.1. The corresponding diagrams are displayed in figure 7. Since the wave-function renormalization matrices can always be chosen such that the divergent terms proportional to $\not{p}$ cancel, it is sufficient to consider $\Sigma^{(B)}_{L,R}$, defined in equation (107), of the fermion self-energies. Moreover, due to equation (108) we only need to examine $\Sigma^{(B)}_L$. According to our renormalization scheme, diagram (c) of figure 7 is induced by the renormalization of the Yukawa coupling matrices and the VEVs. The only genuine mass renormalization we have refers to the renormalization of the mass matrix M_R. However, as we will see shortly, at one-loop level it is not needed.

5.1 Neutrinos

With δv_k of equation (85) and $\delta\Delta_k$ of equation (167), the part of the counterterm of equation (171) that is supposed to make $-i\left(\Sigma^{(B)}_{\nu L}\gamma_L + \Sigma^{(B)}_{\nu R}\gamma_R\right)$ finite reads

$$\left(\;—\!\otimes\!—\;\right)_{\not{p}=0} = \frac{ic_\infty}{16\pi^2}\left\{\left(U_R^\dagger \Delta_j M_\ell^\dagger \Gamma_j U_L + U_L^T \Gamma_j^T M_\ell^* \Delta_j^T U_R^*\right)\gamma_L + \right.$$

$$+ \left(U_L^\dagger \Gamma_j^\dagger M_\ell \Delta_j^\dagger U_R + U_R^T \Delta_j^* M_\ell^T \Gamma_j^* U_L^* \right) \gamma_R \Big\}$$
$$- i \left[U_R^\dagger \, \delta M_R \, U_R^* \, \gamma_L + U_R^T \, \delta M_R^* \, U_R \, \gamma_R \right]. \tag{183}$$

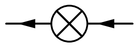
Beginning with the Z contribution to the neutrino self-energy, it is easy to see that equation (15a) makes it finite. Concerning neutral scalar exchange, it also does not give a divergence because of $V_{kb} V_{lb} = 0$—see equation (B.5). As for $W^\pm$ exchange, here the chiral projector is the reason that both divergent and finite contributions to $\Sigma_{\nu L}^{(B)}$ vanish. However, the charged-scalar exchange has a non-vanishing divergence, which agrees, apart from the sign, with the counterterms in the first and second line of equation (183). Therefore, the ξ-independent Yukawa counterterm in equation (167) indeed cancels the divergence of $\Sigma_{\nu L}^{(B)}$. Consequently, the δM_R counterterm of equation (171c) must be zero because it is not needed. In other words, at the one-loop level we find

$$\delta M_R = 0. \tag{184}$$

For explicit formulas for the Z and scalar contributions to $\Sigma_{\nu L}^{(B)}$ see equation (193) and equations thereafter.

5.2 Charged leptons

We have to plug δv_k of equation (85) and $\delta \Gamma_k$ of equation (168) into the $\not{p}$-independent counterterm of equation (170). In this way, we obtain the complete counterterm pertaining to $-i \left(\Sigma_{\ell L}^{(B)} \gamma_L + \Sigma_{\ell R}^{(B)} \gamma_R \right)$ as

$\Big($  $\Big)_{\not{p}=0}$

$$= \frac{i c_\infty}{16\pi^2} \left\{ \left[\frac{g^2}{c_w^2} \left(3 + \xi_Z\right) s_w^2 \left(s_w^2 - \frac{1}{2} \right) + e^2 \left(3 + \xi_A\right) \right] \hat{m}_\ell \right.$$
$$\left. + W_R^\dagger \Gamma_j M_D^\dagger \Delta_j W_L \gamma_L + W_L^\dagger \Delta_j^\dagger M_D \Gamma_j^\dagger W_R \gamma_R \right\}. \tag{185}$$

It is straightforward to check that the ξ_Z, ξ_A and M_D terms in this formula cancel the divergences of the Z, photon and charged-lepton loops in $\Sigma_{\ell L}^{(B)}$. As in the case of neutrinos, the $W^\pm$ loop is zero and the sum over the neutral-lepton loops is finite because of the second relation in equation (B.5). Therefore, we find that δv_k and $\delta \Gamma_k$ indeed make the self-energy of the charged leptons finite. For explicit formulas for the gauge boson and scalar contributions to $\Sigma_{\ell L}^{(B)}$ see equation (193) and equations thereafter.

6 One-loop fermion self-energy formulas in Feynman gauge

In this section, our goal is to present formulas for the fermion self-energies that allow to compute the one-loop radiative corrections to the tree-level masses. Since we have proven that these corrections are ξ-independent, we can resort to a specific gauge. In order to have the most simple form of the vector boson propagators—see equation (30b), we choose the Feynman gauge with

$$\xi_W = \xi_Z = \xi_A = 1 \quad \text{and} \quad M_{+1}^2 = m_W^2, \quad M_1^2 = m_Z^2. \tag{186}$$

Consequently, in our formulas for the lepton self-energies, summations over scalar contributions always include the Goldstone bosons. At the end we use the self-energies to investigate radiative corrections to the seesaw mechanism.

6.1 Self-energies

The mass formula, equation (110), can be adapted to the type of fermion by taking into account equations (108) and (109). In this way we find the radiative fermion mass corrections

$$\Delta m_i = m_i \left(\Sigma_{\nu L}^{(A)}\right)_{ii}(m_i^2) + \mathrm{Re}\left(\Sigma_{\nu L}^{(B)}\right)_{ii}(m_i^2) \quad (i = 1, \ldots, n_L + n_R) \tag{187}$$

and

$$\Delta m_\alpha = \frac{m_\alpha}{2}\left[\left(\Sigma_{\ell L}^{(A)}\right)_{\alpha\alpha}(m_\alpha^2) + \left(\Sigma_{\ell R}^{(A)}\right)_{\alpha\alpha}(m_\alpha^2)\right] + \mathrm{Re}\left(\Sigma_{\ell L}^{(B)}\right)_{\alpha\alpha}(m_\alpha^2) \quad (\alpha = e, \mu, \tau) \tag{188}$$

for neutrinos and charged leptons, respectively.

Because we are dealing with Majorana neutrinos, according to equation (109) we only need $\Sigma_{\nu L}^{(A)}$ and $\Sigma_{\nu L}^{(B)}$ for Σ_ν, while for Σ_ℓ, the charged-lepton self-energy, $\Sigma_{\ell L}^{(A)}$, $\Sigma_{\ell R}^{(A)}$ and $\Sigma_{\ell L}^{(B)}$ are required for the determination of the total self-energy. The self-energies can be formulated with the two functions[17]

$$\mathcal{F}_0(r,s,t) = \int_0^1 \mathrm{d}x \ln \frac{\Delta(r,s,t)}{\mathcal{M}^2} \quad \text{and} \quad \mathcal{F}_1(r,s,t) = \int_0^1 \mathrm{d}x\, x \ln \frac{\Delta(r,s,t)}{\mathcal{M}^2}, \tag{189}$$

where

$$\Delta(r,s,t) = xr + (1-x)s - x(1-x)t. \tag{190}$$

In order to write the $(n_L + n_R) \times (n_L + n_R)$ matrices $\Sigma_{\nu L}^{(A)}$ and $\Sigma_{\nu L}^{(B)}$ in a compact way, we define the following diagonal matrices:

$$\begin{aligned}
\widehat{\mathcal{F}}_{0,a,\nu} &= \mathrm{diag}\left(\mathcal{F}_0(M_{+a}^2, m_1^2, p^2), \ldots, \mathcal{F}_0(M_{+a}^2, m_{n_L+n_R}^2, p^2)\right), && (191a)\\
\widehat{\mathcal{F}}_{0,b,\nu} &= \mathrm{diag}\left(\mathcal{F}_0(M_b^2, m_1^2, p^2), \ldots, \mathcal{F}_0(M_b^2, m_{n_L+n_R}^2, p^2)\right), && (191b)\\
\widehat{\mathcal{F}}_{0,W,\nu} &= \mathrm{diag}\left(\mathcal{F}_0(m_W^2, m_1^2, p^2), \ldots, \mathcal{F}_0(m_W^2, m_{n_L+n_R}^2, p^2)\right), && (191c)\\
\widehat{\mathcal{F}}_{0,Z,\nu} &= \mathrm{diag}\left(\mathcal{F}_0(m_Z^2, m_1^2, p^2), \ldots, \mathcal{F}_0(m_Z^2, m_{n_L+n_R}^2, p^2)\right). && (191d)
\end{aligned}$$

In addition, we need the analogous matrices with $\mathcal{F}_1$. For the charged leptons, we have $n_L \times n_L = 3 \times 3$ matrices

$$\widehat{\mathcal{F}}_{0,a,\ell} = \mathrm{diag}\left(\mathcal{F}_0(M_{+a}^2, m_e^2, p^2), \mathcal{F}_0(M_{+a}^2, m_\mu^2, p^2), \mathcal{F}_0(M_{+a}^2, m_\tau^2, p^2)\right), \tag{192}$$

[17] A conversion to other widely used loop functions is given in appendix D.

etc. In the case of charged fermions, photon exchange occurs as well, therefore, we also introduce the matrices $\widehat{\mathcal{F}}_{0,A,\ell}$ and $\widehat{\mathcal{F}}_{1,A,\ell}$, which have no analogue in the neutrino sector.

In the case of $\Sigma^{(B)}_{\nu L}$ and $\Sigma^{(B)}_{\ell L}$ we distinguish between that part stemming from the diagrams of figure 7 and those from the VEV shift—see section 3.6. The first one we indicate by the superscript "proper" and the second one by "shift":

$$\Sigma^{(B)}_{\nu L} = \Sigma^{(B,\,\text{proper})}_{\nu L} + \Sigma^{(B,\,\text{shift})}_{\nu L} \quad \text{and} \quad \Sigma^{(B)}_{\ell L} = \Sigma^{(B,\,\text{proper})}_{\ell L} + \Sigma^{(B,\,\text{shift})}_{\ell L}. \tag{193}$$

For the neutrinos, the result for the self-energy is

$$\begin{aligned} 16\pi^2\,\Sigma^{(A)}_{\nu L} &= L_a^\dagger \widehat{\mathcal{F}}_{1,a,\ell} L_a + R_a^T \widehat{\mathcal{F}}_{1,a,\ell} R_a^* + 2F_b^\dagger \widehat{\mathcal{F}}_{1,b,\nu} F_b \\ &\quad + \frac{g^2}{2} V_L^\dagger \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{1,W,\ell} \right) V_L + \frac{g^2}{4c_w^2} \left(U_L^\dagger U_L + 2U_L^\dagger U_L \widehat{\mathcal{F}}_{1,Z,\nu} U_L^\dagger U_L \right), \quad (194) \\ 16\pi^2\,\Sigma^{(B,\,\text{proper})}_{\nu L} &= -R_a^\dagger \hat{m}_\ell \widehat{\mathcal{F}}_{0,a,\ell} L_a - L_a^T \hat{m}_\ell \widehat{\mathcal{F}}_{0,a,\ell} R_a^* + 2F_b \hat{m}_\nu \widehat{\mathcal{F}}_{0,b,\nu} F_b \\ &\quad + \frac{g^2}{c_w^2} U_L^T U_L^* \hat{m}_\nu \widehat{\mathcal{F}}_{0,Z,\nu} U_L^\dagger U_L. \quad (195) \end{aligned}$$

The corresponding coupling matrices are found in sections 2.1 and 2.2. In the $W^\pm$ term, the matrix V_L of equation (147) occurs, which will also be relevant for the charged-lepton self-energy. In the computation of the Z contribution, we have employed

$$F_{LR}^2 = U_L^\dagger U_L \gamma_L + U_L^T U_L^* \gamma_R \tag{196}$$

and

$$F_{RL} \hat{m}_\nu F_{LR} = 0 \quad \text{with} \quad F_{RL} = U_L^\dagger U_L \gamma_R - U_L^T U_L^* \gamma_L. \tag{197}$$

These relations follow from equation (15a). The matrix F_{RL} occurs in the Z contribution because of changing the position of the Dirac matrices: $\gamma^\mu F_{LR} = F_{RL}\gamma^\mu$.

Turning to charged leptons, the self-energy is given by

$$\begin{aligned} 16\pi^2\,\Sigma^{(A)}_{\ell L} &= R_a \widehat{\mathcal{F}}_{1,a,\nu} R_a^\dagger + \frac{1}{2}\, G_b^\dagger \widehat{\mathcal{F}}_{1,b,\ell} G_b + e^2 \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{1,A,\ell} \right) \\ &\quad + \frac{g^2}{2} \left(\mathbb{1} + 2V_L \widehat{\mathcal{F}}_{1,W,\nu} V_L^\dagger \right) + \frac{g^2}{c_w^2} \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{1,Z,\ell} \right) \left(s_w^2 - \frac{1}{2} \right)^2, \quad (198) \\ 16\pi^2\,\Sigma^{(A)}_{\ell R} &= L_a \widehat{\mathcal{F}}_{1,a,\nu} L_a^\dagger + \frac{1}{2}\, G_b \widehat{\mathcal{F}}_{1,b,\ell} G_b^\dagger + e^2 \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{1,A,\ell} \right) \\ &\quad + \frac{g^2}{c_w^2} \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{1,Z,\ell} \right) s_w^4, \quad (199) \\ 16\pi^2\,\Sigma^{(B,\,\text{proper})}_{\ell L} &= -L_a \hat{m}_\nu \widehat{\mathcal{F}}_{0,a,\nu} R_a^\dagger + \frac{1}{2}\, G_b \hat{m}_\ell \widehat{\mathcal{F}}_{0,b,\ell} G_b - 2e^2 \hat{m}_\ell \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{0,A,\ell} \right) \\ &\quad - \frac{2g^2}{c_w^2} \hat{m}_\ell \left(\mathbb{1} + 2\widehat{\mathcal{F}}_{0,Z,\ell} \right) s_w^2 \left(s_w^2 - \frac{1}{2} \right). \quad (200) \end{aligned}$$

The parts of the fermion self-energies induced by the VEV shift are read off from equations (105) and (103), respectively:

$$\Sigma^{(B,\,\text{shift})}_{\nu L} = \frac{1}{\sqrt{2}} \left(U_R^\dagger \Delta_k U_L + U_L^T \Delta_k^T U_R^* \right) \Delta v_k \quad \text{and} \quad \Sigma^{(B,\,\text{shift})}_{\ell L} = \frac{1}{\sqrt{2}} W_R^\dagger \Gamma_k W_L \Delta v_k^*. \tag{201}$$

The shifts Δv_k are related to the Δt_b of equation (96) via equation (100), while the Δt_b are given by the sum over the finite parts of the tadpole diagrams—*c.f.* equation (97).

The individual results obtained from the tadpole diagrams are

$$\Delta t_b^{(W)} = \frac{1}{16\pi^2}\left(v_j^* V_{jb} + V_{jb}^* v_j\right)\frac{g^2 m_W^2}{4}\left(1 - 3\ln\frac{m_W^2}{\mathcal{M}^2}\right), \tag{202a}$$

$$\Delta t_b^{(Z)} = \frac{1}{16\pi^2}\left(v_j^* V_{jb} + V_{jb}^* v_j\right)\frac{g^2 m_Z^2}{8c_w^2}\left(1 - 3\ln\frac{m_Z^2}{\mathcal{M}^2}\right), \tag{202b}$$

$$\Delta t_b^{(S^\pm)} = \frac{1}{16\pi^2}\lambda_{ijkl}\left(v_i^* V_{jb} + V_{ib}^* v_j\right)U_{ka}^* U_{la} M_{+a}^2\left(1 - \ln\frac{M_{+a}^2}{\mathcal{M}^2}\right), \tag{202c}$$

$$\Delta t_b^{(S^0)} = \frac{1}{16\pi^2}\left[\tilde{\lambda}_{ijkl}\left(v_i^* V_{jb} + V_{ib}^* v_j\right)V_{kb'}^* V_{lb'} + \lambda_{ijkl}\left(v_i^* V_{jb'} V_{kb}^* V_{lb'} + V_{ib'}^* v_j V_{kb'}^* V_{lb}\right)\right] \times \frac{M_{b'}^2}{2}\left(1 - \ln\frac{M_{b'}^2}{\mathcal{M}^2}\right), \tag{202d}$$

$$\Delta t_b^{(\ell)} = -\frac{\sqrt{2}}{16\pi^2}\,\mathrm{Tr}\left[\hat{m}_\ell^3\left(\mathbb{1} - \ln\frac{\hat{m}_\ell^2}{\mathcal{M}^2}\right)\left(G_b + G_b^\dagger\right)\right], \tag{202e}$$

$$\Delta t_b^{(\chi)} = -\frac{\sqrt{2}}{16\pi^2}\,\mathrm{Tr}\left[\hat{m}_\nu^3\left(\mathbb{1} - \ln\frac{\hat{m}_\nu^2}{\mathcal{M}^2}\right)\left(F_b + F_b^\dagger\right)\right], \tag{202f}$$

where the superscripts indicate the particles in the loop. In the $W^\pm$ and Z contributions the respective ghost loops are contained.

6.2 Seesaw mechanism

Our computation of the fermion self-energies did not assume anything about the scales of the neutrino masses. However, in the seesaw mechanism [1, 2, 3, 4, 5] one stipulates that the M_R is non-singular and the eigenvalues of $M_R^\dagger M_R$ are of a scale m_R^2 such that m_R is much larger than all entries in M_D. With this assumption, there are n_L light neutrinos and n_R heavy neutrinos with approximate mass matrices

$$M_{\text{light}} = -M_D^T M_R^{-1} M_D \quad \text{and} \quad M_{\text{heavy}} = M_R. \tag{203}$$

The matrix $\mathcal{U}$, as occurring in equation (14), is approximated by

$$\mathcal{U} \simeq \begin{pmatrix} \mathbb{1}_{n_L} & \left(M_R^{-1} M_D\right)^\dagger \\ -M_R^{-1} M_D & \mathbb{1}_{n_R} \end{pmatrix} \begin{pmatrix} S_{\text{light}} & 0_{n_L \times n_R} \\ 0_{n_R \times n_L} & S_{\text{heavy}} \end{pmatrix} \tag{204}$$

with

$$S_{\text{light}}^T M_{\text{light}} S_{\text{light}} \simeq \mathrm{diag}\left(m_1, \ldots, m_{n_L}\right), \tag{205a}$$

$$S_{\text{heavy}}^T M_{\text{heavy}} S_{\text{heavy}} \simeq \mathrm{diag}\left(m_{n_L+1}, \ldots, m_{n_L+n_R}\right). \tag{205b}$$

As demonstrated in [8], the dominant radiative corrections to the seesaw mechanism reside in the left upper corner of the $(n_L + n_R) \times (n_L + n_R)$ Majorana neutrino mass matrix of equation (14):

$$\begin{pmatrix} (M_L)_{\text{1-loop}} & M_D^T \\ M_D & M_R \end{pmatrix}. \tag{206}$$

Since in the mHDSM gauge symmetry forbids Majorana mass terms of the ν_L, there is a zero at tree level in the left upper corner of this mass matrix and no counterterms are allowed for $(M_L)_{\text{1-loop}}$. Therefore, this radiative $n_L \times n_L$ mass matrix must be finite. Moreover, the light neutrino mass matrix is modified to

$$M_{\text{light}} = (M_L)_{\text{1-loop}} - M_D^T M_R^{-1} M_D. \tag{207}$$

Examination of the neutrino self-energy and using the seesaw approximation of $\mathcal{U}$ of equation (204) leads to the conclusion that $(M_L)_{\text{1-loop}}$ is determined by the contributions of the neutral scalars and the Z boson to $\Sigma_{\nu L}^{(B,\,\text{proper})}$ of equation (195) [8]. As discussed in section 5.1, these are indeed finite without any renormalization. Moreover, the dominant corrections are induced by heavy neutrino exchange and, therefore, we can neglect light neutrino masses in $\mathcal{F}_0$. In this way, we can define an effective neutrino mass matrix given by

$$\begin{aligned}
(M_L)_{\text{1-loop}} &= \frac{1}{16\pi^2} U_L^* \left[2F_b\, \hat{m}_\nu \mathcal{F}(M_b^2, \hat{m}_\nu^2) F_b + \frac{g^2}{c_w^2} U_L^T U_L^*\, \hat{m}_\nu \mathcal{F}(m_Z^2, \hat{m}_\nu^2) U_L^\dagger U_L \right] U_L^\dagger \\
&= \frac{1}{16\pi^2} \left[\frac{1}{2}\, \Delta_k^T V_{kb} U_R^* \hat{m}_\nu \mathcal{F}(M_b^2, \hat{m}_\nu^2) U_R^\dagger \Delta_l V_{lb} + \frac{g^2}{c_w^2} U_L^* \hat{m}_\nu \mathcal{F}(m_Z^2, \hat{m}_\nu^2) U_L^\dagger \right]
\end{aligned} \tag{208}$$

with

$$\mathcal{F}(a,b) \equiv \mathcal{F}_0(a,b,0) = \int_0^1 \mathrm{d}x \ln \frac{[xa + (1-x)b]}{\mathcal{M}^2} = -\ln \mathcal{M}^2 - 1 + \frac{a \ln a - b \ln b}{a - b}. \tag{209}$$

Obviously, the function has the symmetry $\mathcal{F}(a,b) = \mathcal{F}(b,a)$. Since $(M_L)_{\text{1-loop}}$ must be given in the same basis as the mass matrix of equation (14), we have performed the corresponding basis transformation, given by U_L^* to the left and $U_L^\dagger$ to the right in the first line of equation (208).

To proceed further, we convert $\mathcal{F}$ in the two forms

$$\mathcal{F}(a,b) = -\ln \mathcal{M}^2 - 1 + \ln a + \frac{\ln \frac{a}{b}}{\frac{a}{b} - 1} = -\ln \mathcal{M}^2 - 1 + \ln a + \frac{\frac{b}{a} \ln \frac{b}{a}}{\frac{b}{a} - 1}. \tag{210}$$

Next we define the diagonal matrices

$$\hat{r}_b \equiv \frac{\hat{m}_\nu^2}{M_b^2} \quad \text{and} \quad \hat{r}_Z \equiv \frac{\hat{m}_\nu^2}{m_Z^2}. \tag{211}$$

Then we can write the neutral-scalar contribution to equation (208) as

$$\frac{1}{2}\, \Delta_k^T V_{kb} U_R^* \hat{m}_\nu \left[-\left(\ln \mathcal{M}^2 + 1 \right) \mathbb{1} + \ln \hat{m}_\nu^2 + \frac{\ln \hat{r}_b}{\hat{r}_b - \mathbb{1}} \right] U_R^\dagger \Delta_l V_{lb}. \tag{212}$$

Summing in this equation from $b = 1$ to $b = 2n_H$, the second relation of equation (B.5) tells us that only the last term in the square brackets contributes. Similarly, the Z contribution can be formulated as

$$\frac{g^2}{c_w^2} U_L^* \hat{m}_\nu \left[-\left(\ln \mathcal{M}^2 + 1 - \ln m_Z^2 \right) \mathbb{1} + \frac{\hat{r}_Z \ln \hat{r}_Z}{\hat{r}_Z - \mathbb{1}} \right] U_L^\dagger. \tag{213}$$

In this case, because of equation (15a), again only the last term in the square brackets contributes. Now we consider the Goldstone boson contribution to equation (212) separately. Since $\hat{r}_1 = \hat{r}_Z$, it is suggestive to add it to the Z contribution. Indeed, using equations (B.14) and (16), we obtain

$$U_R^\dagger \Delta_k V_{k1} = \frac{i}{v}\, U_R^\dagger \Delta_k v_k = \frac{i\sqrt{2}}{v}\, U_R^\dagger M_D = \frac{i\sqrt{2}}{v}\, \hat{m}_\nu U_L^\dagger. \tag{214}$$

Plugging this into the part with $b = 1$ of equation (212), we find that the G^0 contribution differs from the Z contribution solely by the numerical factor $-1/4$. Eventually, we arrive at the result [8]

$$(M_L)_{\text{1-loop}} = \frac{1}{32\pi^2} \sum_{b=2}^{2n_H} \Delta_k^T V_{kb}\, U_R^* \,\hat{m}_\nu \frac{\ln \hat{r}_b}{\hat{r}_b - \mathbb{1}} U_R^\dagger \,\Delta_l V_{lb} + \frac{3g^2}{64\pi^2 m_W^2}\, U_L^* \,\hat{m}_\nu^3 \frac{\ln \hat{r}_Z}{\hat{r}_Z - \mathbb{1}} U_L^\dagger. \tag{215}$$

For $n_H = 1$ it agrees with the result in [7]. We observe that the Goldstone plus Z contribution to $(M_L)_{\text{1-loop}}$ is universal, *i.e.* independent of n_H.[18] After employing once more equation (16), we find that it is of the same order of magnitude as that of the physical neutral scalars. Moreover, the light neutrino masses are completely negligible in these one-loop corrections, which amounts to setting

$$U_R^* = (0, S_{\text{heavy}})\,. \tag{216}$$

It has been pointed out in [6, 7, 8, 32] that numerically these corrections can be sizeable.

7 Conclusions

Extensions of the scalar sector play an important role in lepton mass and mixing models. However, predictions of such models have mostly been computed at tree level and their stability under radiative corrections has not been tested.

In this paper we have considered an important class of such models, the mHDSM, which has an arbitrary number n_H of Higgs doublets and an arbitrary number n_R of right-handed neutrino singlets with Majorana mass terms. Using the R_ξ gauge for the quantization of the mHDSM, we have proposed a simple renormalization scheme which gives a straightforward recipe for the computation of radiative corrections. The idea is that all counterterms are induced by the parameters of the unbroken theory, with the exception of the VEV renormalization δv_k ($k = 1, \ldots, n_H$). Since all masses in the mHDSM are obtained by spontaneous gauge-symmetry breaking, masses are derived quantities and there is no mass renormalization in our scheme. The removal of the corresponding divergencies is procured by the renormalization of VEVs and Yukawa couplings.

In addition to the infinite counterterm parameters δv_k, there are the well-known finite VEV shifts Δv_k induced by the finite parts of tadpole diagrams, which guarantee that beyond tree level the VEVs of the physical neutral scalars are still vanishing.

[18] We thank A. Pilaftsis for drawing our attention to this fact.

We have demonstrated, by determination of all counterterm parameters, that at the one-loop level our renormalization scheme is capable of removing all divergences and we have elucidated the prescription for computing the VEV shifts Δv_k. Moreover, we have shown analytically for the one-loop fermion self-energies that including the VEV shifts is equivalent to the insertion of all tadpole diagrams on the fermion line.

As an application of the renormalization scheme we have presented the full fermion mass corrections at the one-loop level. In this context, we have identified all the mechanisms and performed analytically the necessary computations to show that these corrections are ξ-independent, in both the loop diagrams and the counterterms. In the case of loop diagrams we have closely followed ref. [24]. We have also demonstrated that in the seesaw limit the radiative corrections to the seesaw mechanism computed in ref. [8] derive from our much more general framework.

We conclude with a speculation about the importance of tadpole contributions to one-loop fermion masses. Tadpoles induce VEV shifts Δv_k via equation (100). The potentially largest contribution comes from heavy neutrinos in the tadpole loop, *i.e.* from $\Delta t_b^{(\chi)}$, equation (202f), because this quantity roughly scales with the third power in the seesaw scale m_R. A very crude estimate suggests that a scale $m_R \gg 10\,\mathrm{TeV}$ induces huge VEV shifts, much larger than the electroweak scale, while for $m_R \lesssim 10\,\mathrm{TeV}$—a scale at which heavy neutrinos can possibly be seen at large colliders—the induced VEV shifts are below $10\,\mathrm{GeV}$. Whether this apparent incompatibility of a generic mHDSM (or the SM), *i.e.* without any suppression mechanism for $\Delta t_b^{(\chi)}$, with large seesaw scales is physical or an artefact of our renormalization scheme remains to be investigated.

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A The scalar mass matrices

In this section we discuss the tree-level scalar masses. The scalar potential is given by

$$V(\phi) = \mu_{ij}^2 \phi_i^\dagger \phi_j + \lambda_{ijkl} \phi_i^\dagger \phi_j \phi_k^\dagger \phi_l \tag{A.1}$$

with

$$\mu_{ij}^2 = \left(\mu_{ji}^2\right)^*, \quad \lambda_{ijkl} = \lambda_{klij} = \lambda_{jilk}^*. \tag{A.2}$$

We allow for an arbitrary number n_H of scalar doublets ϕ_k ($k = 1, 2, \ldots, n_H$) and use the notation

$$\phi_k = \begin{pmatrix} \varphi_k^+ \\ \varphi_k^0 \end{pmatrix}, \quad \text{with} \quad \left\langle 0 \left| \varphi_k^0 \right| 0 \right\rangle = \frac{v_k}{\sqrt{2}}. \tag{A.3}$$

We then write

$$\varphi_k^0 = \frac{1}{\sqrt{2}} \left(v_k + \rho_k + i\sigma_k\right), \quad \text{hence} \quad \langle 0 \left|\rho_k\right| 0\rangle = \langle 0 \left|\sigma_k\right| 0\rangle = 0, \quad \rho_k^\dagger = \rho_k, \quad \sigma_k^\dagger = \sigma_k. \tag{A.4}$$

The quadratic terms in the scalar potential are written as

$$V_{\text{mass}} = \sum_{i,j} \varphi_i^- \left(\mathcal{M}_+^2\right)_{ij} \varphi_j^+ + \frac{1}{2} \left[A_{ij}\, \rho_i \rho_j + B_{ij}\, \sigma_i \sigma_j + 2C_{ij}\, \rho_i \sigma_j\right]. \tag{A.5}$$

The mass matrix of the charged scalars [11],

$$\mathcal{M}_+^2 = \mu^2 + \Lambda, \quad \text{where} \quad \Lambda_{ij} = \sum_{k,l} \lambda_{ijkl}\, v_k^* v_l, \tag{A.6}$$

is complex and hermitian, with Λ being hermitian as well. The matrices A and B are real and symmetric; C is real but otherwise arbitrary. All matrices defined so far are $n_H \times n_H$ matrices.

The mass matrix of the neutral real scalar fields ρ_i and σ_j may then be formulated as the real $2n_H \times 2n_H$ matrix

$$\mathcal{M}_0^2 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{A.7}$$

Defining complex $n_H \times n_H$ matrices

$$K_{ik} = \sum_{j,l} \lambda_{ijkl}\, v_j v_l \quad \text{and} \quad K'_{il} = \sum_{j,k} \lambda_{ijkl}\, v_j v_k^*, \tag{A.8}$$

where K is symmetric and K' is hermitian. The matrices A, B, C are obtained as [11]

$$A = \operatorname{Re}\left(\mu^2 + \Lambda + K'\right) + \operatorname{Re} K, \tag{A.9a}$$
$$B = \operatorname{Re}\left(\mu^2 + \Lambda + K'\right) - \operatorname{Re} K, \tag{A.9b}$$
$$C = -\operatorname{Im}\left(\mu^2 + \Lambda + K'\right) + \operatorname{Im} K, \tag{A.9c}$$

respectively.

B The diagonalization matrices of the charged and neutral scalar mass terms

Let $\hat{M}_+^2$ be the diagonal matrix of the squares of charged scalar scalar masses. Furthermore, we denote the unitary $n_H \times n_H$ matrix which diagonalizes $\mathcal{M}_+^2$ by U. Thus we have

$$U^\dagger \mathcal{M}_+^2 U = \hat{M}_+^2. \tag{B.1}$$

Inverting this relation, we obtain

$$U \hat{M}_+^2 U^\dagger = \mu^2 + \Lambda. \tag{B.2}$$

Let $\hat{M}_0^2$ be the diagonal matrix of the squares of the neutral scalar masses and $\tilde{V}$ the orthogonal $2n_H \times 2n_H$ matrix $\tilde{V}$ that diagonalizes the mass matrix of the neutral scalars, *i.e.*

$$\tilde{V}^T \mathcal{M}_0^2 \tilde{V} = \hat{M}_0^2. \tag{B.3}$$

Without loss of generality we can write

$$\tilde{V} = \begin{pmatrix} \operatorname{Re} V \\ \operatorname{Im} V \end{pmatrix}, \tag{B.4}$$

where V is a complex $n_H \times 2n_H$ matrix. In terms of V, orthogonality of $\tilde{V}$ reads [8, 12, 28]

$$V_{jb}^* V_{kb} = 2\delta_{jk}, \quad V_{jb} V_{kb} = 0, \quad V_{jb}^* V_{jb'} + V_{jb'}^* V_{jb} = 2\delta_{bb'}. \tag{B.5}$$

Notice that

$$V_{jb}^* V_{jb'} = \delta_{bb'} + i \operatorname{Im} \left(V^\dagger V \right)_{bb'} \tag{B.6}$$

is in general not diagonal, but it is easy to see that $\operatorname{Im} \left(V^\dagger V \right)_{bb'}$ is antisymmetric in the indices b, b' [28]. A further relation, useful in one-loop computations, is

$$V_{kb'} \operatorname{Im} \left(V^\dagger V \right)_{b'b} = -i V_{kb}. \tag{B.7}$$

The matrices U and V allow to write the Higgs doublets ϕ_k in terms of the mass eigenfields S_a^+ and S_b^0:

$$\phi_k = \begin{pmatrix} U_{ka} S_a^+ \\ \frac{1}{\sqrt{2}} \left(v_k + V_{kb} S_b^0 \right) \end{pmatrix}. \tag{B.8}$$

There is no straightforward analogue to equation (B.2). But rewriting equation (B.3) as

$$\mathcal{M}_0^2 = \tilde{V} \hat{M}_0^2 \tilde{V}^T = \begin{pmatrix} \operatorname{Re} V \hat{M}_0^2 \operatorname{Re} V^T & \operatorname{Re} V \hat{M}_0^2 \operatorname{Im} V^T \\ \operatorname{Im} V \hat{M}_0^2 \operatorname{Re} V^T & \operatorname{Im} V \hat{M}_0^2 \operatorname{Im} V^T \end{pmatrix} \tag{B.9}$$

and using subsequently equation (A.7) leads to

$$V \hat{M}_0^2 V^T = A - B + iC + iC^T, \tag{B.10a}$$
$$V \hat{M}_0^2 V^\dagger = A + B - iC + iC^T. \tag{B.10b}$$

Eventually, application of equation (A.9) gives the useful results [28]

$$V\hat{M}_0^2 V^T = 2K, \tag{B.11a}$$

$$V\hat{M}_0^2 V^\dagger = 2\left(\mu^2 + \Lambda + K'\right). \tag{B.11b}$$

We denote the masses of the charged and neutral scalars by M_{+a} ($a = 1, \ldots, n_H$) and M_b ($b = 1, \ldots, 2n_H$), respectively. Obviously, the a-th column of U is an eigenvector of $\mathcal{M}_+^2$ with eigenvalue M_{+a}^2. By definition, the b-th column of $\tilde{V}$ is an eigenvector of $\mathcal{M}_0^2$ with eigenvalue M_b^2, which in terms of the columns of V reads [11]

$$\left(\mu^2 + \Lambda + K'\right)_{ij} V_{jb} + K_{ij} V_{jb}^* = M_b^2 V_{ib}. \tag{B.12}$$

Taking into account equation (B.5), this equation can be cast into the very useful form

$$\frac{1}{2}\left(\mu_{ij}^2 + \tilde{\lambda}_{ijkl} v_k^* v_l\right)\left(V_{ib}^* V_{jb'} + V_{ib'}^* V_{jb}\right) + \frac{1}{2}\lambda_{ijkl}\left(v_i^* v_k^* V_{jb} V_{lb'} + v_j v_l V_{ib}^* V_{kb'}^*\right) = \delta_{bb'} M_b^2. \tag{B.13}$$

The mass matrices $\mathcal{M}_+^2$ and $\mathcal{M}_0^2$ also contain one eigenvalue zero each, referring to the Goldstone bosons. Allocating the indices $a = 1$ and $b = 1$ to the charged and the neutral Goldstone boson, respectively, the corresponding columns in U and V are given by [11]

$$U_{k1} = \frac{v_k}{v} \quad \text{and} \quad V_{k1} = i\frac{v_k}{v}, \tag{B.14}$$

respectively.

C On-shell contributions to the fermion self-energies

Here we confine ourselves to one fermion field. Suppose a contribution $\sigma(p)$ to the total (renormalized) self-energy $\Sigma(p)$ is given by

$$\sigma(p) = \not{p}\left(a_L \gamma_L + a_R \gamma_R\right) - b_L \gamma_L - b_R \gamma_R. \tag{C.1}$$

What is the condition that $\sigma(p)$ vanishes on-shell? With this we mean that it has the form

$$\sigma(p) = \left(c_L \gamma_L + c_R \gamma_R\right)\left(\not{p} - m\right) + \left(\not{p} - m\right)\left(d_L \gamma_L + d_R \gamma_R\right). \tag{C.2}$$

It is easy to prove that for $m \neq 0$ this is the case if and only if

$$\frac{1}{m}\left(b_L + b_R\right) = a_L + a_R. \tag{C.3}$$

Notice, however, that the coefficients $c_{L,R}$ and $d_{L,R}$ are not uniquely determined by $a_{L,R}$ and $b_{L,R}$, because the shift $c'_{L,R} = c_{L,R} + s$ and $d'_{L,R} = d_{L,R} - s$ with an arbitrary (real) s does not change $\sigma(p)$ of equation (C.2).

D Conversion to scalar Feynman integrals

The loop functions $\mathcal{F}_0$ and $\mathcal{F}_1$ of section 6 can easily be converted into the well-known Feynman integral representations B_0 and B_1 as defined in [33, 34] and numerically evaluated using freely available computer algebra software such as *LoopTools* [35]. The conversion reads

$$\mathcal{F}_0(m_1^2, m_2^2, p^2) = c_\infty - B_0(p^2; m_1, m_2), \tag{C.4a}$$

$$\mathcal{F}_1(m_1^2, m_2^2, p^2) = \frac{c_\infty}{2} - \left[B_0(p^2; m_1, m_2) + B_1(p^2; m_1, m_2)\right], \tag{C.4b}$$

where $B_1(p^2; m_1, m_2)$ is related to the vector two-point integral $B^\mu(p; m_1, m_2)$ via

$$B_1(p^2; m_1, m_2) = \frac{1}{p^2}\, p_\mu B^\mu(p; m_1, m_2), \quad p^2 \neq 0. \tag{C.5}$$

The loop-integrals referred to are

$$B_0(p^2; m_1, m_2) = \frac{16\pi^2}{i}\, \mathcal{M}^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[k^2 - m_1^2]\left[(p+k)^2 - m_2^2\right]}, \tag{C.6a}$$

$$B^\mu(p; m_1, m_2) = \frac{16\pi^2}{i}\, \mathcal{M}^{4-d} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu}{[k^2 - m_1^2]\left[(p+k)^2 - m_2^2\right]}. \tag{C.6b}$$

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Part III

Evaluation

CHAPTER 6

Conclusions

The goal of this thesis was to create a tool for investigating radiative corrections in mass and mixing models of the leptonic sector. We have presented the three main publications created in the course of the dissertation with which we approached this goal.

In Chapter 3, we presented a detailed study of on-shell renormalization conditions in theories with flavor-mixing. The purpose of this study was to create a non-ad-hoc way of deriving the results which have been known for a few decades [80], in order to get a deeper understanding of the contribution of these conditions to loop-corrected flavor mixing. We were able to derive the conditions in a new way solely by carrying out an expansion of the respective (inverse) propagators in the small quantities $p^2 - m_n^2$ when going on-shell for $n = 1 \dots N$, where N is the number of flavors. In this way, the known conditions could be derived *e.g.* without the use of on-shell spinor relations in the fermionic sector, as it is usually done in the literature. One of the main insights gained via this work was the fact that off-diagonal components of renormalized propagators in flavor space can contain non-singular parts in $p^2 - m_n^2$. This was in contrast to our prior general perception that all off-diagonal elements of the renormalized propagator need to vanish when going on-shell. An obvious, yet unexpected benefit of our investigations was that via the derivation of on-shell mass counterterms, we implicitly derived the one-loop expressions for the finite mass corrections needed in our later work.

Our second publication, as presented in Chapter 4, gave the means to develop and study a renormalization procedure for the scalar and fermionic sector in terms of a toy model, which we eventually elevated to a realistic case in our third publication. In this toy model with an arbitrary number of real scalars and either Majorana or Dirac fermions, masses are generated via spontaneous symmetry breaking similar to the SM case. Yet, without the introduction of gauge-symmetries, we avoided dealing with intricacies such as the discussion of gauge-dependencies of counterterms and renormalized quantities. We showed in a very explicit manner that the renormalization of the parameters of the *unbroken* theory suffices for obtaining finite scalar and fermionic self-energies in the *broken* phase. The finiteness of the renormalized scalar one-point function served as a cross check for the correctness of our renormalization procedure and, more importantly, showed that no independent renormalization of the VEVs was needed. Moreover, we made use of the idea of absorbing the finite parts of the scalar one-point function into finite shifts of the VEVs, anticipating the need for such a practice in a realistic gauge theory. In hindsight, this study turned out to be very helpful in leading the way to a complete renormalization program for the scalar and fermionic sector of the mHDSM. The drawback in studying this simplistic toy model was the fact that flavor-mixing would not appear in physical observables whatsoever, because a change of basis of the fields could always be absorbed via a redefinition of the parameters of the theory. Therefore, studying perturbative corrections to flavor-mixing was also not possible.

The potential to overcome the aforementioned drawback was given in the study of the mHDSM, as presented in Chapter 5. Here, we applied and extended the renormalization procedure of the toy model to the case of a gauge theory with an arbitrary number of

Higgs doublets and right-chiral neutrinos, the SM being a special case thereof. We took pains[1] in creating a detailed study of the convoluted gauge-parameter dependencies in the counterterms of the model and in one-loop corrections to lepton masses using a general R_ξ gauge. In contrast to the toy model, we found that a renormalization of the parameters of the unbroken theory is not sufficient to make the broken theory finite. In addition to the counterterms of the scalar potential, it became evident that additional VEV-counterterms are necessary. Using an ansatz for the VEV-counterterms inspired by Ref. [82], we were able to choose these and the counterterms of the scalar potential such that the renormalized scalar one- and two-point functions are simultaneously rendered finite. Without the use of VEV-counterterms, this was unachievable. A remarkable cross-check of our choice of the latter was presented by the finiteness of the renormalized leptonic self-energies. The respective mass counterterms were derived via the renormalization of the Yukawa-couplings in the unbroken theory, *i.e.* via demanding finiteness of the three-point correlation function of two leptons and a scalar, and via the already fixed VEV-counterterms of the scalar sector.[2] Therefore, no freedom was left in the choice of the infinite parts of the charged lepton mass counterterms, yet they were sufficient for providing the finiteness of the respective self-energies. Moreover, it is amusing to note that the counterterms of the neutral lepton Yukawa-couplings together with the VEV-counterterms suffice to make the neutral lepton self-energies finite, *i.e.* the additional freedom in the choice of the Majorana mass counterterms is not needed for finite self-energies at the one-loop level.

The next part of the discussion focused on proving the gauge-parameter independence of the one-loop lepton masses via an appropriate treatment of tadpole contributions. Inspired by the discussion in [83], we presented the proof that all gauge-parameter dependent terms contributing to the one-loop masses cancel. Of major importance for this proof are the contributions of the Goldstone boson tadpole diagrams, showing the importance of properly taking these into account in the definition of the one-loop lepton masses. We continued in explaining how all finite tadpole contributions can be absorbed into finite VEV-shifts, guaranteeing gauge-independence of the one-loop lepton masses, while having vanishing renormalized scalar one-point functions. We suggest that tadpole contributions to n-point one-loop correlation functions can subsequently be taken into account not by an explicit insertion of the respective diagrams, but simply by inserting the shifted VEVs at every instance of a VEV appearing in the calculation. Having shown the gauge-parameter independence, the discussion is concluded by presenting the complete analytic one-loop lepton mass corrections of the mHDSM in Feynman gauge.

As a final exercise, a first attempt at the numerical evaluation of our general formulae for the leptonic one-loop mass corrections was carried out for the generalized CP-model presented in Section 2.4. An independent check of the results is still left open which is why we opt out of presenting explicit numerical results for the one-loop lepton masses. Nevertheless, we are able to draw some important general conclusions from the findings. It became evident that the VEV-shifts induced by the neutral lepton tadpole diagrams, *cf.* Eq. (202f) of Chapter 5 become problematic if heavy Majorana neutrinos are present, while the rest of the contributions seem well-behaved. We want to discuss this behavior in terms of the finite one-loop corrections to charged lepton masses induced via said shifts. Common examples of SM extensions with heavy Majorana neutrinos feature masses ranging from the TeV-range almost up to the scale of grand unification. For our following argumentation, we set $m_R \simeq 10^{14}\,\mathrm{GeV}$. An exemplary set of numerical input values from which we draw

[1] It might be worth noting that due to the general nature of the investigated model and the non-existence of ready-to-use computer algebra systems for such a scenario, all calculations were carried out by hand. Generalizing already available software packages, *e.g.* FeynRules [81], for applications in such general settings could present a worthwhile endeavor for the theory community.

[2] The momentum-dependent infinities are absorbed via an appropriate choice of the wave function renormalization, but are of no further importance to this discussion.

some of our conclusions is given in App. D. Moreover, we repeat the definition of the problematic contributions for clarity, *i.e.*

$$\Delta t_b^{(\chi)} = -\frac{\sqrt{2}}{16\pi^2}\,\mathrm{Tr}\left[\hat{m}_\nu^3\left(\mathbb{1} - \ln\frac{\hat{m}_\nu^2}{\mathcal{M}^2}\right)\left(F_b + F_b^\dagger\right)\right], \tag{6.1}$$

and also the definition of F_b, *cf.* Eq. (21a) of Chapter 5, *i.e.*

$$F_b = \frac{1}{2}\left(U_R^\dagger \Delta_k U_L + U_L^T \Delta_k^T U_R^*\right) V_{kb}. \tag{6.2}$$

The tadpole contributions of Eq. (6.1) induce the VEV-shifts

$$\Delta v_k^{(\chi)} = \sum_{b=2}^{2n_H} \frac{\Delta t_b^{(\chi)}}{M_b^2}\, V_{kb}. \tag{6.3}$$

We note that in Eq. (6.1), the full neutrino mass matrix to the third power enters, containing the potentially heavy Majorana masses. The reason for the problematic behavior is that in general, there is no significant suppression of these large factors, neither in the VEV- nor the subsequent mass-shifts for the following reasons:

(i) A general feature of the seesaw mechanism is that no small Yukawa couplings are needed in order to explain the lightness of the neutrinos. Therefore, the Δ_k entering Eq. (6.2) can not be considered as small.

(ii) The mixing matrices U_L and U_R in Eq. (6.2) appear in such a way that their potentially small entries (as mentioned at the end of App. D) do not compensate for all large factors induced by m_R.

(iii) Neither the scalar masses appearing in Eq. (6.3), nor the mixing matrix V of the scalar sector are in general related to the leptonic sector and the seesaw scale.

(iv) The contribution of Eq. (6.3) to the self-energies of the charged leptons is given by Eq. (103) of Chapter 5. Here, only the mixing matrices of the charged leptons, W_L and W_R, appear which are not related to the parameters of the neutral lepton sector, *i.e.* the parameters relevant for the respective VEV-shifts.

(v) The investigation of the scale dependence of the neutral lepton tadpole contributions suggested that one might be able to find a renormalization scale at which they vanish. However, this turned out to be a numerical fluke. We suspect that a severe fine-tuning would be needed in setting the numerical value of the renormalization scale such that the tadpole contributions truly vanish. Moreover, Eq. (6.1) shows that one can not expect to have all neutral lepton tadpole contributions vanish at just one scale for each $b = 2, \ldots, 2n_H$. Additionally, we note that using the input values of App. D, these scales lie at around $\sim 10^{13}$ GeV. Choosing such a high renormalization scale for determining radiative corrections to masses around the MeV- or GeV-scale in the case of the charged leptons would be, at the very least, a highly unconventional approach. Similarly high values for $\mathcal{M}$ have been found for all cases of randomly generated input values of the leptonic sector (for heavy Majorana masses). It is therefore suspected not to be coincidental.

Therefore, we conclude that the presence of very heavy Majorana neutrinos in the seesaw setup can spoil the stability of the perturbative expansion in the $\overline{\mathrm{MS}}$-scheme.

In addition to these findings, we note that only the tadpole induced self-energy contributions to the masses seem problematic. All other contributions showed a numerically stable behavior independent of the Majorana mass scale. A reasoning for this is given by the fact that—except in the tadpole induced mass shifts—the presence of heavy Majorana neutrinos in loops is usually accompanied by a strong suppression via the mixing matrix of the neutral leptons.

Moreover, no problematic behavior was found for $m_R \lesssim 10^3\,\text{TeV}$, in the sense that the contributions of Eq. (6.1) were in the same order of magnitude of the remaining tadpole contributions or smaller. It is also worth emphasizing that for these lower scales m_R, the use of our proclaimed VEV-shifts seemed generally well-behaved, meaning that the so induced mass shifts lied around the order of magnitude of the remaining self-energy contributions to the masses.

In our view, the problematic behavior is independent of the specific tadpole treatment, which is merely to be seen as different ways of bookkeeping. The tadpole contributions will either enter indirectly via an absorption in the definition of a renormalized quantity, in our case the VEVs, or directly via additional Feynman diagrams. The question remains whether the problematic behavior can be cured completely via choosing different renormalization conditions such as in the on-shell renormalization scheme. Still, we would like to emphasize that in the general case, there are fewer process-independent physical observables than parameters of the theory that need to be renormalized. Therefore, one would have to resort to process-dependent renormalization conditions in order to avoid $\overline{\text{MS}}$-prescriptions completely. The problem of numerical instabilities induced by finite tadpole contributions using $\overline{\text{MS}}$-renormalization has been discussed in Ref. [84], yet it is unclear whether the problematic heavy Majorana contributions fall into the same category or present an even more severe obstacle.

Some upper bounds of Majorana neutrino mass scales have been mentioned *e.g.* in Ref. [61], but to our knowledge, the inference of such a bound from one-loop mass corrections or shifts of the vacuum expectation values has not been investigated up to this point. Further studies of the physical relevance of our findings, *e.g.* via a comparison of different renormalization methods could therefore present a worthwhile endeavor.

As a final remark, we would like to mention again the subject of mixing angle renormalization. Unfortunately, the study thereof in the mHDSM went beyond the scope of this thesis. Still, we give a concise overview of some widely used approaches in App. B. In the view of this thesis' author, what remains to be studied concerning the renormalization of mixing angles is whether one can find a scheme fulfilling the requirements of

1. gauge-independence,
2. general theoretical applicability in the sense that a simple definition in terms of a self-energy decomposition can be found
3. and general practical applicability, *e.g.* in terms of numerical stability.

We hope to be able to carry out further investigations of this subject at a later point in time.

Appendix

A Gauge-fixing conditions and Faddeev-Popov ghosts in the mHDSM

In this section we present some additional prerequisites to the studies carried out in Chapter 5.

Gauge-fixing

The gauge-fixing Lagrangian in terms of the fixing conditions $\mathcal{G}_X$ reads

$$\mathcal{L}_{\rm GF} = -\frac{1}{2}\left(2\mathcal{G}_{W^+}\mathcal{G}_{W^-} + \mathcal{G}_Z^2 + \mathcal{G}_A^2\right). \tag{A.1}$$

Throughout our calculations, we work in a general R_ξ-gauge, sometimes also called 't Hooft gauge [85]. Here, the choice of gauge-fixing conditions provoke the cancellation of all bilinear mixing terms of vector bosons and scalars which result from the kinetic term of the scalar sector. Therefore, the gauge fixing terms are chosen as

$$\mathcal{G}_{W^\pm} = \frac{1}{\sqrt{\xi_W}}\left(\partial^\mu W_\mu^\pm \pm i\xi_W m_W \sum_{a=1}^{n_H}(\omega^\dagger U)_a S_a^\pm\right), \tag{A.2a}$$

$$\mathcal{G}_Z = \frac{1}{\sqrt{\xi_Z}}\left(\partial^\mu Z_\mu + \xi_Z m_Z \sum_{b=1}^{2n_H}\mathrm{Im}(\omega^\dagger V)_b S_b^0\right), \tag{A.2b}$$

$$\mathcal{G}_A = \frac{1}{\sqrt{\xi_A}}\partial^\mu A_\mu \quad \text{with} \quad \omega_k = \frac{v_k}{v}. \tag{A.2c}$$

Identifying the scalars S_1^0 and $S_1^\pm$ with the neutral and charged Goldstone bosons G^0 and $G^\pm$, respectively, and using the relations

$$\mathrm{Im}\left(\omega^\dagger V\right)_b = \mathrm{Im}\left(iV_{k1}^* V_{kb}\right) = \delta_{1b}, \tag{A.3a}$$

$$(\omega^\dagger U)_b = U_{k1}^* U_{kb} = \delta_{1b}, \tag{A.3b}$$

the gauge-fixing conditions of the mHDSM are found to be equivalent to the ones of the SM (see *e.g.* [76]), *viz.*

$$\mathcal{G}_{W^\pm} = \frac{1}{\sqrt{\xi_W}}\left(\partial^\mu W_\mu^\pm \pm i\xi_W m_W G^\pm\right), \tag{A.4a}$$

$$\mathcal{G}_Z = \frac{1}{\sqrt{\xi_Z}}\left(\partial^\mu Z_\mu + \xi_Z m_Z G^0\right), \tag{A.4b}$$

$$\mathcal{G}_A = \frac{1}{\sqrt{\xi_A}}\partial^\mu A_\mu. \tag{A.4c}$$

Gauge variation

In order to derive the Faddeev-Popov ghost Lagrangian, we need to know the transformation behavior of the gauge-fields under infinitesimal gauge-variations $\alpha^a(x)$ for the $SU(2)$-

and $\beta(x)$ for the $U(1)$-part. We have seen in Eq. (A.4) that only the Goldstone boson and vector boson fields appear in the gauge fixing conditions. The conditions are unchanged in the mHDSM compared to the SM, because the gauge-structure remains the same. Therefore we need

$$\delta W_\mu^\pm = \partial_\mu \alpha^\pm \mp igW_\mu^\pm \left(s_w\alpha^A + c_w\alpha^Z\right) \pm ig\left(s_w A_\mu + c_w Z_\mu\right)\alpha^\pm, \tag{A.5a}$$

$$\delta Z_\mu = \partial_\mu \alpha^Z + igc_w\left(\alpha^- W_\mu^+ - \alpha^+ W_\mu^-\right), \tag{A.5b}$$

$$\delta A_\mu = \partial_\mu \alpha^A + ie\left(\alpha^- W_\mu^+ - \alpha^+ W_\mu^-\right). \tag{A.5c}$$

These field variations are the same as the ones presented in Ref. [11] up to sign differences which appear due to a relative minus sign between the $SU(2)$- and the $U(1)$-part of the covariant derivative of Ref. [11] (the same convention is used in Ref. [76]). This relative minus sign also leads to a different basis transformation between the gauge bosons fields B_μ, W_μ^3 and the physical mass eigenfields A_μ, Z_μ. In Chapter 1 and Chapter 5, we use a covariant derivative with only positive signs, leading to the transformation of Eq. (1.17) of Chapter 1.

The variation of the Goldstone boson fields can be acquired from the behavior of the scalar doublets under infinitesimal gauge-variations, *viz.*

$$\delta\phi_k = \left(i\frac{g}{2}\alpha^a\tau^a + i\frac{g'}{2}\beta\right)\phi_k. \tag{A.6}$$

Just as the fields themselves, we identify the variation of the first components of the scalar mass eigenfields with the one of the Goldstone bosons, *i.e.* $\delta S_1^\pm \equiv \delta G^\pm$ and $\delta S_1^0 \equiv \delta G^0$. The relevant variations are therefore

$$\delta G^+ = im_W\alpha^+ + i\frac{g}{2}\alpha^+\left(\omega^\dagger V\right)_b S_b^0 + i\left(e\alpha^A + g\frac{c_w^2 - s_w^2}{2c_w}\alpha^Z\right)G^+, \tag{A.7a}$$

$$\delta G^- = -im_W\alpha^- - i\frac{g}{2}\alpha^-\left(\omega^T V^*\right)_b S_b^0 - i\left(e\alpha^A + g\frac{c_w^2 - s_w^2}{2c_w}\alpha^Z\right)G^-, \tag{A.7b}$$

$$\delta G^0 = -m_Z\alpha^Z - \frac{g}{2c_w}\alpha^Z \mathrm{Re}\left(\omega^\dagger V\right)_b S_b^0 + \frac{g}{2}\left[\alpha^-(\omega^\dagger U)_a S_a^+ + \alpha^+(\omega^T U^*)_a S_a^-\right]. \tag{A.7c}$$

Note that

$$\mathrm{Re}(\omega^\dagger V)_1 = \mathrm{Re}(-i\omega_b^*\omega_b) = 0. \tag{A.8a}$$

Therefore the neutral Goldstone boson G^0 does not appear in the variation δG^0 and sums over the indices of the neutral scalars can be taken from $b = 2, \ldots, 2n_H$. The SM case in the notation of Ref. [76] can be reproduced by choosing

$$U = 1, \quad V = \begin{pmatrix} i & 1 \end{pmatrix}, \tag{A.9a}$$

$$G^\pm \to \varphi^\pm, \quad G^0 \to \chi, \quad S_2^0 \to H. \tag{A.9b}$$

Faddeev-Popov ghosts

The Faddeev-Popov ghost Lagrangian can be derived via [11, 76, 86]

$$\mathcal{L}_{\mathrm{FP}} = -\bar{c}_a(x)\frac{\delta\mathcal{G}_a}{\delta\alpha_b(x)}c_b(x). \tag{A.10}$$

Note that we inserted a global minus sign such that the mass terms of the ghost fields carry the correct sign in $\mathcal{L}_{\mathrm{FP}}$. Moreover, factors of $\sqrt{\xi_a}$ which would appear by inserting the respective gauge-variations are implicitly absorbed in the ghost fields c_a via a rescaling. Using the results of the previous section, the complete Faddeev-Popov ghost Lagrangian $\mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{FP}}^W + \mathcal{L}_{\text{FP}}^Z + \mathcal{L}_{\text{FP}}^A$ for the mHDSM is derived as

$$\begin{aligned}\mathcal{L}_{\text{FP}}^W = &- \overline{c}^+ \left(\partial^2 + \xi_W m_W^2\right) c^+ \\ &+ ig \left(\partial^\mu \overline{c}^+\right) \left[\left(c_w Z^\mu + s_w A^\mu\right) c^+ - W_\mu^+ \left(c_w c^Z + s_w c^A\right)\right] \\ &- i\frac{g}{2}\xi_W m_W \, \overline{c}^+ G^0 c^+ - \xi_W m_W \, \overline{c}^+ G^+ \left(e c^A + g\frac{c_w^2 - s_w^2}{2c_w} c^Z\right) \\ &- \frac{g}{2}\xi_W m_W \, \overline{c}^+ \sum_{b=2}^{2n_H} \left(\omega^\dagger V\right)_b S_b^0 c^+ \\ &+ \left(c^+ \to c^-,\ W^+ \to W^-,\ G^+ \to G^-,\ \omega^\dagger V \to \omega^T V^*,\ i \to -i\right). \end{aligned} \tag{A.11a}$$

$$\begin{aligned}\mathcal{L}_{\text{FP}}^Z = &- \overline{c}^Z \left(\partial^2 + \xi_Z m_Z^2\right) c^Z \\ &+ igc_w \left(\partial^\mu \overline{c}^Z\right) \left(W_\mu^+ c^- - W_\mu^- c^+\right) \\ &+ \frac{g}{2}\xi_z m_z \, \overline{c}^Z \left(G^+ c^- + G^- c^+\right) \\ &+ \frac{g}{2}\xi_Z m_Z \, \overline{c}^Z \sum_{a=2}^{n_H} \left[\left(\omega^\dagger U\right)_a S_a^+ c^- + \left(\omega^T U^*\right)_a S_a^- c^+\right] \\ &- g\xi_Z \frac{m_Z}{2c_w} \, \overline{c}^Z \sum_{b=2}^{2n_H} \text{Re}\left(\omega^\dagger V\right)_b S_b^0 c^Z, \end{aligned} \tag{A.11b}$$

$$\mathcal{L}_{\text{FP}}^A = - \overline{c}^A \partial^2 c^A + ie \, \partial^\mu \overline{c}^A \left(W_\mu^+ c^- - W_\mu^- c^+\right). \tag{A.11c}$$

B Mixing angle renormalization

The original intent of this thesis was to carry out investigations of radiative corrections to mixing angles in addition to the masses, though it turned out to go beyond our scope for the time being. Nevertheless, we think it is worth presenting a quick overview of various methods known from the literature for a potential application in future research endeavors.

Usually, mixing angles are treated as free parameters receiving independent renormalization for rendering a theory finite. It was shown in [87] that this is indeed needed for non-diagonal mixing matrices, by investigating one-loop corrections to weak decays governed by the charged current interaction. Herein, after renormalizing all fields and free parameters other than the mixing, divergent combinations of the anti-hermitian parts of FSRCs are left over. These are absorbed while preserving unitarity of the mixing matrix via

$$\delta V_{ud} = -\frac{1}{4}\left[\left(\delta Z_{u'u}^{L*} - \delta Z_{uu'}^L\right) V_{u'd} + V_{ud'}\left(\delta Z_{d'd}^L - \delta Z_{dd'}^{L*}\right)\right], \tag{B.12}$$

where we denote by V_{ud} the generic mixing between up- and down- components of a fermionic weak doublet with counterterms δV_{ud} and by δZ^L the left-handed component of the FSRC of these fields. Since mixing angles are physical observables in the weak interaction, they ought to be gauge-independent.[1] Oftentimes, on-shell renormalization conditions are used to define δZ^L, but it can be shown that this leads to gauge-dependent

[1] This is discussed more formally in terms of the cohomology classes of Slavnonv-Taylor operators in [88].

contributions in the finite terms of (B.12) [88]. Heuristically speaking, this is because in on-shell renormalization, the parts of the self-energies absorbed in $\delta Z^L_{uu'}$ are evaluated at $p^2 = m^2_{u'}$. Then the hermitian conjugate $\delta Z^{L*}_{u'u}$ carries the argument m^2_u. This can be read off of the on-shell definition of δZ^L for $u \neq u'$, *i.e.* [80, 89]

$$\frac{1}{2}(\delta Z^L)_{uu'} = \frac{-1}{m_u^2 - m_{u'}^2} \left[m_{u'}^2 \Sigma^L_{uu'}(p^2) + m_u m_{u'} \Sigma^R_{uu'}(p^2) + m_u^2 \Sigma^S_{uu'}(p^2) + m_{u'}^2 \Sigma^S_{uu'}(p^2) \right]_{p^2 = m^2_{u'}}, \tag{B.13}$$

where we made use of the fermionic self-energy decomposition

$$\Sigma_{uu'}(p) = \Sigma^L_{uu'}(p^2) \not{p} \gamma_L + \Sigma^R_{uu'}(p^2) \not{p} \gamma_R + \Sigma^S_{uu'}(p^2) \left(m_u \gamma_L + m_{u'} \gamma_R \right), \tag{B.14}$$

The finite gauge-dependent terms do effectively not cancel out in $\delta Z^L_{uu'} - \delta Z^{L*}_{u'u}$ due to the different arguments appearing therein. This can be avoided by instead defining δZ^L at a *symmetric point* [11]. The choice made in [88] is setting $p^2 = 0$, yielding a gauge-independent definition of δV_{ud}. More specifically, the anti-hermitian part of the left-handed FSRC $\delta \mathcal{Z}^{L,A}_{uu'}$ for $u \neq u'$ is chosen as

$$\delta \mathcal{Z}^{L,A}_{uu'} = \frac{m_u^2 + m_{u'}^2}{m_u^2 - m_{u'}^2} \left[\Sigma^L_{uu'}(0) + 2\Sigma^S_{uu'}(0) \right], \tag{B.15}$$

which is then used to replace δZ^L in (B.12).

A technically rather intricate alternative is the so-called *pinch-technique* as used in [90, 91]. Here, one uses explicitly gauge-independent self energy decompositions in the R_ξ-gauge for the definition of renormalized mixing angles, *viz.* for scalar self-energies

$$\overline{\Sigma}(p^2) = \Sigma^{\text{tad}}\Big|_{\xi_V = 1}(p^2) + \Sigma^{\text{add}}(p^2), \tag{B.16}$$

where Σ^{tad} is the full self-energy containing tadpole contributions and Σ^{add} represents an additional explicitly gauge-independent part of the full self-energy. The latter is calculated by investigating a toy two-to-two scattering process (*e.g.* $\mu^+\mu^+ \to b\bar{b}$) involving intermediate scalars H_i for which one derives a decomposition of the form

$$\Gamma^{H_i XX} \frac{i}{p^2 - m^2_{H_i}} i\Sigma^{PT}_{H_i H_j}(p^2) \frac{i}{p^2 - m^2_{H_j}} \Gamma^{H_j YY}. \tag{B.17}$$

Herein, Γ represent vertex functions and Σ^{PT} is a self-energy-like function. The major insight is that

$$\overline{\Sigma}_{H_i H_j}(p^2) = \Sigma^{\text{tad}}_{H_i H_j}(p^2) + \Sigma^{\text{PT}}_{H_i H_j}(p^2) = \Sigma^{\text{tad}}_{H_i H_j}\Big|_{\xi_V = 1}(p^2) + \Sigma^{\text{add}}_{H_i H_j}(p^2), \tag{B.18}$$

yielding a gauge-independent function. The downside to this technique is that one merely trades gauge dependencies on a dependence on the prescription in the definition of $\overline{\Sigma}$ [92].

Oftentimes, the use of the pinch-technique is supplemented by the so called p^*-scheme, which merely states the use of another *symmetric point* at which the external momentum in the self-energies is evaluated, namely $(p^*)^2 = (M^2_{H_i} + M^2_{H_j})/2$. This is not restricted to the pinch-technique, but often used here because the final expressions for $\overline{\Sigma}$ then take an especially simple form.

Another possible choice for renormalizing mixing angles is the $\overline{\text{MS}}$-scheme. This typically is computationally much easier, also ensures gauge-independence and yields renormalization scale dependent results [84]. The latter is considered advantageous by some, because testing one-loop results under a variation of the scale can give a handle on the theoretical error of the calculation, *i.e.* the expected size of higher order corrections. The disadvantage of this method are possible numerical instabilities [84].

Lastly, one has the option of using physical decay processes for the imposition of renormalization conditions for the mixing. A typical choice is to demand equality between leading order and radiatively corrected decay widths in such processes. This technique yields manifestly gauge-independent and numerically stable results, but introduces a non-universality and flavor-dependence in the renormalized mixing angles and is computationally demanding when going beyond one-loop corrections [84].

Only quite recently, two independent comparison of the various mixing renormalization techniques have been carried out. In Ref. [93], the on-shell-, p^*- and $\overline{\text{MS}}$-mixing angle renormalization schemes are compared in the context of the Two-Higgs-Doublet Model. In the second recent e-print on this subject of Ref. [94], the same techniques and additional new approaches are discussed, also in the Two-Higgs-Doublet Model and moreover in the Higgs-Singlet Extension of the SM. Both groups find that the $\overline{\text{MS}}$-approach is technically the easiest, yet care needs to be taken in the inclusion of tadpole contributions concerning the treatment of gauge dependencies and the introduction of a strong dependence on the choice of the renormalization scale.

As a final remark we want to emphasize that to the knowledge of the author of this thesis, no exhaustive study of the renormalization of leptonic mixing has been carried out so far. An early study of mixing renormalization in the fermionic sector, namely the CKM matrix, has been carried out in Ref. [87] which found the numerical contribution to observables to be minuscule. This situation might change for the leptons due to the much stronger mixing in this sector.

C Conversion from the CP-model to the mHDSM

In this section, we list a conversion from the parameters of the generalized CP-model of [71, 72] to the ones of the mHDSM of Chapter 5. The bilinear terms of the scalar potentials are related by

$$\left(\mu_{ij}^2\right) = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & -\mu_2 + \mu_{\text{soft}} & i\mu'_{\text{soft}} \\ 0 & -i\left(\mu'_{\text{soft}}\right)^* & -\mu_2 - \mu_{\text{soft}} \end{pmatrix}. \tag{C.19}$$

The conversion for the quartic scalar couplings reads

$$\begin{aligned}
&\lambda_{1111} = \lambda_1, \quad \lambda_{2222} = \lambda_2, \quad \lambda_{3333} = \lambda_2, \quad \lambda_{1122} = \frac{\lambda_3}{2}, \quad \lambda_{1133} = \frac{\lambda_3}{2}, \quad \lambda_{2233} = \frac{\lambda_4}{2}, \\
&\lambda_{1221} = \frac{\lambda_5}{2}, \quad \lambda_{1331} = \frac{\lambda_5}{2}, \quad \lambda_{2332} = \frac{\lambda_6}{2}, \quad \lambda_{2323} = \lambda_7, \quad \lambda_{3232} = \lambda_7, \quad \lambda_{1212} = \lambda_8, \\
&\lambda_{1313} = \lambda_8, \quad \lambda_{2121} = \lambda_8, \quad \lambda_{3131} = \lambda_8, \quad \lambda_{1213} = i\frac{\lambda_9}{2}, \quad \lambda_{2131} = -i\frac{\lambda_9}{2}, \quad \lambda_{2322} = i\frac{\lambda_{10}}{2}, \\
&\lambda_{3233} = i\frac{\lambda_{10}}{2}.
\end{aligned} \tag{C.20}$$

All other entries of λ_{ijkl} are either given its symmetry relations (see Eq. (A.2) of Chapter 5) or they are zero. The Yukawa-couplings of the charged leptons are given by

$$\Gamma_1 = \frac{\sqrt{2}m_e}{v_1^*}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix}0 & 0 & 0\\ 0 & g_{2\mu} & 0\\ 0 & 0 & g_{2\mu}^*\end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix}0 & 0 & 0\\ 0 & -g_{2\mu} & 0\\ 0 & 0 & g_{2\mu}^*\end{pmatrix}, \tag{C.21}$$

and the ones of the neutral leptons by

$$\Delta_1 = \frac{\sqrt{2}}{v_1}\begin{pmatrix}f_e & 0 & 0\\ 0 & f_\mu & 0\\ 0 & 0 & f_\mu^*\end{pmatrix}, \quad \Delta_2 = \Delta_3 = 0_{3\times 3}. \tag{C.22}$$

The Majorana mass matrix of the generalized CP-model has the form

$$M_R = \begin{pmatrix}a & r & r^*\\ r & s & b\\ r^* & b & s^*\end{pmatrix}, \quad \text{with } r, s \in \mathbb{C}, \quad a, b \in \mathbb{R}. \tag{C.23}$$

Due to the flavor-diagonal structure of the generalized CP-model, the charged lepton mixing matrices are simply

$$W_L = W_R = \mathbb{1}_{3\times 3}. \tag{C.24}$$

Additionally, we list the further necessary ingredients for an evaluation of our general one-loop results and their respective determination with the help of computer algebra systems, namely

- v_k: via the method described in Section 2.4. The consistency of the VEV-determination can be checked via the vanishing of the linear terms of the scalar potential, *cf.* Chapter 5, Eq. (44).
- U, V: via the diagonalization of the scalar mass matrices.
- $\mathcal{U}$: from the singular value decomposition of $\mathcal{M}_\nu$. For widely separated scales of the Dirac and Majorana masses, it is numerically more stable to determine $\mathcal{U}$ in a two step process as explained in Sec. 2.2. The same holds for $\hat{m}_\nu$.
- U_L, U_R: these are given by the first n_L rows of $\mathcal{U}$ and the conjugate of the remaining rows, respectively.

D Input values for a numerical evaluation of one-loop mass corrections

In this section, we list the input values used in a first attempt at a numerical evaluation of the one-loop mass correction results of Chapter 5. In fairness, a somewhat complete numerical investigation and accompanying discussion goes beyond the scope of this thesis. Nevertheless, as a final exercise, we wanted to put the results to the test in an exemplary study of the generalized CP-model of Section 2.4. We refrain from presenting explicit numerical results for the one-loop lepton mass corrections because an independent cross-check of the results obtained could not be carried out as of writing this thesis. Still, we were able to draw some general conclusions going beyond what was anticipated from the work carried out in Chapter 5. We ask the reader to refer to Chapter 6 for the discussion of the results.

For the derivation of all necessary input values needed in the general one-loop expressions, we follow the methods described in Section 2.4 and App. C. The numbers of the scalar, left-chiral neutrino and right-chiral neutrino families are

$$n_H = 3, \quad n_L = 3, \quad n_R = 3. \tag{D.25}$$

As it turned out, finding sensible values for the couplings of the scalar sector in a random number scan is a non-trivial task due to the large number of free parameters. For the generation of a realistic setting, a conversion of the scalar couplings to physical observables such as the scalar masses would be necessary, but this approach had to be dropped simply due to the scarcity of time in finishing this thesis. Therefore, the parameters of the scalar sector have been chosen in a pseudo-random manner, meaning that several constraints went into their respective choices, *viz.*

1. tree-level unitarity, *i.e.* simply $|\lambda_i| < 2\pi$.
2. negative λ_7 and $\tilde{\lambda}$ in order to satisfy the constraints for the VEV-determination of Section 2.4.
3. choose $u = 172\,\text{GeV}$, as suggested in Ref. [71].
4. positive μ_1 and μ_2.
5. all mass eigenvalues are positive and the spectrum contains the massless Goldstone bosons.
6. the vanishing of the linear terms of the scalar potential, *cf.* Chapter 5, Eq. (44) is given to a reasonable degree, *i.e.* the result should be of the order $\lesssim 10^{-6}$.

Using these constraints, we fixed the values of the scalar bilinear couplings as

$$\begin{aligned} \mu_1 &= 41804.5\,\text{GeV}^2, & \mu_2 &= 91131.5\,\text{GeV}^2, \\ \mu_{\text{soft}} &= -0.122091\,\text{GeV}^2, & \mu'_{\text{soft}} &= -0.00914418\,\text{GeV}^2. \end{aligned} \tag{D.26}$$

The reason for choosing such small values for the soft-breaking terms is that otherwise the determination of the VEVs has been found to be numerically inconsistent regarding item 4 of the list above. This choice leads to an unphysically small ratio m_μ/m_τ which is one of the reasons why the numerical setting of this section is to be viewed solely as a numerical example. For a more realistic setting, one would need to go to higher orders in the determination of the VEVs as explained in Section 2.4.

Our choice for the scalar quartic couplings reads[2]

$$\begin{aligned} \lambda_1 &= 5.84763, & \lambda_2 &= 1.89129, & \lambda_3 &= 3.14676, & \lambda_4 &= 3.20913, \\ \lambda_5 &= -0.0138371, & \lambda_6 &= -0.634739, & \lambda_7 &= -0.124175, & \lambda_8 &= 0.992954, \\ \lambda_9 &= -0.0389144, & \lambda_{10} &= -0.0371923, & \tilde{\lambda} &= -1.45654. \end{aligned} \tag{D.27}$$

This gives the VEVs

$$v_1 = 36.7151\,\text{GeV}, \quad v_2 = (1.68511 - 171.992i)\,\text{GeV}, \quad v_3 = (1.68479 - 171.991i)\,\text{GeV}, \tag{D.28}$$

and related to these, the VEV shifts induced by the soft symmetry breaking, *i.e.*

$$\delta_0 = 5.75457 \times 10^{-6}, \quad \delta_+ = 0, \quad \delta_- = 1.76445 \times 10^{-6}. \tag{D.29}$$

[2] We remind the reader that this choice merely represents a numerical example. Choosing smaller values is unproblematic, but in our study we followed the common practice of the multi-Higgs community (see *e.g.* [95]).

The scalar mass matrices are derived as shown in Chapter 5, App. A and App. B. For the values of the couplings given above, the resulting masses of the charged scalars are

$$M_{+1}^2 \equiv M_{G^+}^2 < 10^{-11}\,\mathrm{GeV}^2, \quad M_{+2}^2 = (165.75\,\mathrm{GeV})^2\,, \quad M_{+3}^2 = (246.01\,\mathrm{GeV})^2\,, \tag{D.30}$$

and the ones of the neutral scalars read

$$\begin{aligned} &M_1^2 \equiv M_{G^0}^2 < 10^{-11}\,\mathrm{GeV}^2, && M_2^2 = (123.01\,\mathrm{GeV})^2\,, && M_3^2 = (131.12\,\mathrm{GeV})^2\,, \\ &M_4^2 = (208.01\,\mathrm{GeV})^2\,, && M_5^2 = (346.70\,\mathrm{GeV})^2\,, && M_6^2 = (425.85\,\mathrm{GeV})^2\,. \end{aligned} \tag{D.31}$$

The consistency of the scalar sectors mixing matrices is checked via the unitarity relations of Chapter 5, App. B.

Luckily, the determination of reasonable values for the leptonic sector turned out to be much less problematic. The generation of the Majorana mass matrix M_R can be carried out with a random number generation for the four input values in Eq. (C.23), or alternatively, for the scaled version

$$M_R = m_R \begin{pmatrix} \hat{a} & \hat{r} & \hat{r}^* \\ \hat{r} & \hat{s} & \hat{b} \\ \hat{r}^* & \hat{b} & \hat{s}^* \end{pmatrix}, \quad \text{with } \hat{r}, \hat{s} \in \mathbb{C}, \quad \hat{a}, \hat{b} \in \mathbb{R}, \tag{D.32}$$

where m_R is the characteristic scale of the Majorana neutrino masses. Effectively, we use the latter approach and generate the Majorana mass matrix by a random choice of $\hat{a}, \hat{b} \in (-1, 1)$ and $\hat{r}, \hat{s}$ in a square area of the complex plane with its corners at $-1 - i$ and $1 + i$. A numerical example is given by

$$\begin{aligned} &m_R = 10^{14}\,\mathrm{GeV}, && \\ &\quad \hat{a} = 0.502699, && \hat{b} = -0.425434, \\ &\quad \hat{r} = 0.0846253 + 0.384058\,i, && \hat{s} = -0.283465 + 0.538736\,i. \end{aligned} \tag{D.33}$$

In a similar manner, we choose the Dirac neutrino masses f_e and f_μ by setting the characteristic scale $m_D = v = 246\,\mathrm{GeV}$ and drawing randomly $\hat{f}_e \in (0, 1)$ and $\hat{f}_\mu$ in the same way as $\hat{r}$ above, *e.g.*

$$\hat{f}_e = \frac{f_e}{m_D} = 0.37019, \quad \hat{f}_\mu = \frac{f_\mu}{m_D} = -0.07186 + 0.79989\,i \tag{D.34}$$

with these choices, the resulting neutrino masses, *i.e.* the diagonal entries of $\hat{m}_\nu$ read

$$\begin{aligned} &m_{\nu,1} = 0.1352\,\mathrm{eV}, && m_{\nu,2} = 0.3563\,\mathrm{eV}, && m_{\nu,3} = 1.411\,\mathrm{eV}, \\ &m_{\nu,4} = 1.943 \times 10^{13}\,\mathrm{GeV}, && m_{\nu,5} = 9.041 \times 10^{13}\,\mathrm{GeV}, && m_{\nu,6} = 1.058 \times 10^{14}\,\mathrm{GeV}. \end{aligned} \tag{D.35}$$

In the charged lepton sector, the mass of the electron is simply given by its experimentally known value and we also choose to fix the tau mass in this way. The modulus of the muon mass is subsequently given by

$$m_\mu = \frac{m_\tau}{2}\,|\delta_0 + i\delta_-|\,, \tag{D.36}$$

cf. Eq. (2.37). The last free parameter is the Yukawa coupling $g_{2\mu}$, *i.e.* its complex phase which is drawn randomly in the interval $(0, 2\pi)$ while its modulus is given by

$$|g_{2\mu}| = \frac{\sqrt{2}m_\tau}{|v_2 + v_3|}. \tag{D.37}$$

The numerical expression we use in this example reads

$$g_{2\mu} = 0.00720707 + 0.00119053i. \tag{D.38}$$

Eventually, the resulting masses of the charged leptons are

$$m_e = 0.511\,\mathrm{MeV}, \quad m_\mu = 5.347 \times 10^{-6}\,\mathrm{GeV}, \quad m_\tau = 1.777\,\mathrm{GeV}. \tag{D.39}$$

As pointed put at the beginning of this section, due to the difficulties in determining the true VEVs of the scalar potential for large values of the soft symmetry breaking parameters $\mu^{(\prime)}_{\mathrm{soft}}$, only an unphysically small value for the muon mass could be achieved.

In order to conclude this section, we show the result for the absolute values of mixing parameters of the neutral leptons

$$\left(\, |\mathcal{U}_{ij}| \, \right) = \begin{pmatrix} 0.917 & 0.155 & 0.367 & 2.71\times10^{-12} & 8.11\times10^{-13} & 1.14\times10^{-13} \\ 0.282 & 0.699 & 0.658 & 5.87\times10^{-12} & 9.16\times10^{-13} & 1.31\times10^{-12} \\ 0.282 & 0.699 & 0.658 & 5.87\times10^{-12} & 9.16\times10^{-13} & 1.31\times10^{-12} \\ 1.36\times10^{-12} & 6.07\times10^{-13} & 5.68\times10^{-12} & 0.578 & 0.805 & 0.132 \\ 1.93\times10^{-13} & 1.26\times10^{-12} & 4.70\times10^{-12} & 0.577 & 0.419 & 0.701 \\ 1.93\times10^{-13} & 1.26\times10^{-12} & 4.70\times10^{-12} & 0.577 & 0.419 & 0.701 \end{pmatrix}. \tag{D.40}$$

It explicitly shows two important features:

1. The entries for the mixing of light to heavy degrees of freedom are small. This can be understood by inspection of Eq. (2.9). Here, we see that the entries of the upper left and lower right block of the mixing matrix $\mathcal{W}$ are of

$$\mathcal{O}\left(\sqrt{1 - m_D^2/m_R^2}\right) \simeq \mathcal{O}\left(1\right), \tag{D.41}$$

while the rest of the entries is of

$$\mathcal{O}\left(m_D/m_R\right) \simeq \mathcal{O}\left(10^{-12}\right). \tag{D.42}$$

The same orders of magnitude are carried over to the full mixing matrix $\mathcal{U}$.

2. The mixing matrix $\mathcal{U}$ clearly exhibits the desired μ-τ-symmetric structure which also translates to the mixing matrix of the charged current interaction due to Eq. (C.24) and Eq. (1.36).

E Exercises on symmetric matrices and tensors

In Chapter 4, we determined the counterterms to the quartic scalar couplings via the absorption of the $\overline{\mathrm{MS}}$-divergencies in the scalar four-point function. One type of contribution is represented by fermionic box diagrams of which one finds three topologically different variants. The overall result is given in terms of the sum of three traces over different products of four symmetric Yukawa matrices. Eventually we checked the symmetry of the result under all permutations of the four free indices accompanying the Yukawa matrices, corresponding to a permutation of the outer scalar fields. This check is equivalent to asking how many different traces one can build from the product of four symmetric matrices.

As a quick exercise, let us generalize this problem to N symmetric matrices A_i and study how many independent variants of traces we can build from their products. The trace we start from is

$$X \equiv \mathrm{Tr}\left[A_1 A_2 \dots A_N\right]. \tag{E.43}$$

Independent variants of X are such that we can not transform one into one another by transformations that leave the trace invariant. These transformations are

1. $N-1$ cyclic permutations $X \to \mathrm{Tr}\,[A_N A_1 \ldots A_{N-1}]$,
2. the inversion $X \to \mathrm{Tr}\,[A_N A_{N-1} \ldots A_1]$, which leaves the trace invariant if all A_i are symmetric,
3. $N-1$ cyclic permutations after the inversion $X \to \mathrm{Tr}\,[A_1 A_N A_{N-1} \ldots A_2]$.

Counting the original X and all variants, we can build

$$1 + (N-1) + 1 + (N-1) = 2N \tag{E.44}$$

equivalent versions of the same trace. The total number of possible permutations of N matrices is $N!$. Therefore, the number of independent variants of X for $N > 2$ is

$$\frac{N!}{2N} = \frac{(N-1)!}{2}. \tag{E.45}$$

The result for $N = 4$ is three as it should be, coinciding with the three topologically distinct contributions of Feynman diagrams we started with.

Another amusing combinatorial exercise was presented by counting the number of independent quartic couplings of the toy model scalars of Chapter 4. The model features a totally symmetric tensor λ_{abcd}. In the course of the renormalization program, we wanted to clarify how many independent elements this tensor has. As it turned out, this problem is equivalent to the question of how many possibilities exist for choosing four elements from a set of cardinality n (corresponding to the number of families of scalars), without taking into account the order and without removing entries from the latter set. In other words: how many different multisets with four elements exist for a basis set with n elements? This problem is easily generalizable to arbitrary symmetric tensors $\lambda_{i_1 \ldots i_p}$ of rank p, therefore we treat the general case first.

We start by writing every multiset in terms of placeholders for indices $\bullet$ and placeholders for separators $|$. After every separator, the following indices are raised by one. For $p = 4$ and $n = 3$ we have, *e.g.* $\{1,1,2,3\} \simeq \bullet\bullet|\bullet|\bullet$ or $\{1,2,2,2\} \simeq \bullet|\bullet\bullet\bullet|$. In the general case, we have p spots for placing indices, $n-1$ spots for separators and $p+n-1$ symbols in total, *i.e.*

$$\underbrace{\underbrace{\bullet\bullet\cdots\bullet}_{p}\underbrace{||\ldots|}_{n-1}}_{p+n-1}. \tag{E.46}$$

The number of all permutations of these symbols is then $(p+n-1)!$, but we need to divide out $p!$ indistinguishable permutations for the dots among themselves and $(n-1)!$ for the permutations of the separators among themselves. Therefore, the number of different multisets of cardinality p drawn from a basis set of cardinality n is

$$\frac{(p+n-1)!}{p!(n-1)!} = \binom{p+n-1}{p}. \tag{E.47}$$

This corresponds to the number of independent elements of a totally symmetric tensor of rank p, where each index can take a value $i_j = 1 \ldots n$ for $j = 1 \ldots p$. Therefore, in the toy model of Chapter 4, the number of independent quartic couplings is given by

$$\binom{3+n_\varphi}{4}, \tag{E.48}$$

where n_φ is the number of families in the scalar sector.

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Abstract

An abundance of models trying to explain the patterns in lepton masses and mixings has been created over the past few decades—many of them inspired by the rich phenomenology of the neutrino sector. In many cases, radiative corrections to the tree-level predictions for observables in such models are unknown or have not been studied systematically so far. Therefore, we have developed a generally applicable setup for studying radiative corrections in a variety of models in order to check the perturbative stability of their predictions. All of the models referred to inherit an extended scalar sector and oftentimes Majorana neutrinos. Therefore, we performed our studies in the Multi-Higgs Standard Model with an arbitrary number of Higgs doublets and right-handed Majorana neutrinos. Our work consists of exhaustive discussions on the $\overline{\text{MS}}$-renormalization of this general model, the treatment of tadpole contributions in terms of gauge-parameter dependencies, the renormalization of vacuum expectation values and a presentation of the complete analytic one-loop results for lepton mass corrections. Additionally, we present our earlier works in which we developed the methods for acquiring the mentioned results. These works contain a novel way of deriving on-shell renormalization conditions in a general setup and the discussion of a toy model which already exhibits some of the important features of our main study.

Zusammenfassung

Eine Vielzahl physikalischer Modelle, die versuchen, die experimentell bekannten Muster in Leptonmassen und Mischungswinkeln zu erklären, wurde im Laufe der letzten zwei Jahrzehnte erschaffen—viele davon inspiriert durch die reiche Phänomenologie des Neutrino-Sektors. In vielen Fällen sind radiative Korrekturen zu den Vorhersagen auf Baumgraphen-Niveau für Observablen nicht bekannt oder systematisch diskutiert. Aus diesem Grund haben wir ein vielseitig einsetzbares Werkzeug entwickelt um die störungstheoretische Stabilität solcher Vorhersagen in einer Vielzahl von Modellen untersuchen zu können. Alle für uns relevanten Modelle enthalten einen erweiterten skalaren Sektor und in vielen Fällen zusätzlich Majorana-Neutrinos. Deshalb haben wir unsere Untersuchungen im Rahmen des Multi-Higgs Doublet Standard Modells mit beliebiger Anzahl an Higgs-Doublets und rechts-händigen Majorana-Neutrinos durchgeführt. Unsere Arbeit besteht aus eingehenden Diskussionen der $\overline{\text{MS}}$-Renormierung jenes allgemeinen Modells, der Behandlung von sog. Tadpole-Beiträgen im Zusammenhang mit Eichparameterabhängigkeiten, der Renormierung von Vakuumerwartungswerten und einer Präsentation der vollständigen analytischen Ein-Schleifen-Ergebnisse der Korrekturen zu den Leptonmassen. Zusätzlich sind unsere früheren Arbeiten gezeigt, in denen die Methoden entwickelt wurden, um zu besagten Resultaten zu gelangen. Diese Arbeiten enthalten eine neuartige Art sog. on-shell-Renormierungsbedingungen in einer allgemein gefassten Theorie herzuleiten, sowie die Diskussion eines Spielzeugmodells, das bereits viele der auszeichnenden Merkmale des zuletzt studierten realistischen Modells aufzeigt.

Statement of the thesis advisor

I hereby confirm that Maximilian Löschner has provided the main contribution to the following three papers, which are part of his thesis:

- W. Grimus and M. Löschner, *Revisiting on-shell renormalization conditions in theories with flavor mixing*, Int. J. Mod. Phys. A **31** (2017) no.24, 1630038 [Erratum-*ibid.* **32** (2017) no.13, 1792001] [arXiv:1606.06191 [hep-ph]].
- M. Fox, W. Grimus and M. Löschner, *Renormalization and radiative corrections to masses in a general Yukawa model*, Int. J. Mod. Phys. A **33** (2018) no.03, 1850019 [arXiv:1705.09589 [hep-ph]].
- W. Grimus and M. Löschner, *Renormalization of the multi-Higgs-doublet Standard Model and one-loop lepton mass corrections*, arXiv:1807.00725 [hep-ph].

According to the common practice in particle physics, the authors are listed in alphabetical order. Nevertheless Maximilian Löschner is the main author of the above publications. In particular he has written the major part of each paper himself.

Ao. Univ.-Prof. Dr. Walter Grimus

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