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„Segregation as Network Formation in a Public Good Game"

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1 Introduction

In many occasions in our everyday life, we face situations in which individuals organize as groups. Those groups seemingly coordinate the actions of group members. Imagine political movements, any interest group, or even groups of friends within an extended set of people. Of course closeness of the coordinated group action to the individual preference will be attractive for potential members. Additionally, the coordination in greater, more relevant gatherings of individuals even makes the group relatively more appealing to individuals who support slightly different actions than those agreed upon in the group. What may be the reasons that agents find themselves supporting a group-action which differs considerably from their preferred choice? For example, consider that the positive outcome of an action can be amplified by being executed by more people or similarly that useful information is exchanged in the group. Information exchange and the underlying conversation with like-minded members of society can thus be seen as a property which provides value for those engaging in it.

Further motivation to organize in groups comes from the proliferation and use of information. There is a better chance of receiving a crucial bit of information if one individual can interact with more people. In modern societies defined by more flat hierarchies, groups which are size-wise more dominant also appear stronger within the entire society and may have a more considerable influence on the formulation of policies that affect the entire society around the group or also provide members with a feeling of security and protection against opportunism of other groups.

The motivation behind this work is to fit the propensity of individuals to assign themselves to groups into a model. Further, the model shall be used to give an intuition and base for the analysis of recent observations or strategies that have been applied in the context of manipulating group formation.

To account for a benefit that is created and proliferated within a segregated group, an approach of modeling voluntary segregation as a contribution to local public goods is used. Segregation in the context of this work describes the self-chosen separation of members of society into different, mutually exclusive groups. Further, I will investigate how specific parameters of the pure base model can affect later outcomes of the segregation profile. Also, under different parameters, members of society will optimally choose to modify the group-action to maximize outcome. In the context of efficiency, also a planning authority is considered.

The start of the paper is a model introducing local public goods. Joining or leaving a group represented by one public good is subject to individual outcome optimization. The public good derives benefits for members from interaction within while the cost imposed to members arises from the difference between individually optimal positions to the agreed group-action. The parameter defining attractivity of bigger groups and the distribution of preferences have implications on the state of segregation that prevails in society as a stable state. Stability is applied in the sense of a Nash-equilibrium in which no individual involved can improve by choosing another strategy as the current one. So in stability, no individual would want to switch to a different public good.

2 The model

2.1 Environment

In the modelled society, a population of agents is assumed. These Agents act as individual nodes in a network and decide to contribute to a local public good. Each agent has an individual preference value realization of x. The preferences are normalized to be between 0 and 1. If agents decide to contribute to a public good, they establish a connection to a joint hub representing the public good. This hub has a position in the same preference space as the agents which describes the "agreed" action situated on the preference value $\tilde{x}$. $\tilde{x}$ will be referred to as the core in this paper. This core position, which then all contributor agents share is, in general, different to the individual preferences and therefore imposes a perceived cost on the contributing agents. The value for $\tilde{x}$ may be imposed exogenously through a planning authority or as a focal point of the preferences, or alternatively, it may be decided endogenously by the group members.

The payoff is generated in a procedure after individuals have chosen the public good. During this step, agents interact with each other pairwise. Imagine that relevant information for the individuals can be shared, or any undefined, beneficial interaction takes place in this step. Only if there is a joint public good hub, interaction can take place, and θ units of individual payoff are created for each of the two connected players. If there is no connection, then nothing is created, independent of the relative preferences. This property is to capture the aspect of mutual exclusivity of the groups as the critical element in segregation.

The positive contribution in the public good is the same for all n members of a public good core since all agents get to interact with the same number of partners. The payoff is defined through the sum over all other members $n-1$ reflecting that payoff is only created through interaction with $n-1$ agents in the same cluster. The intuition for this kind of payoff setup is to explicitly model the perceived benefit of bigger groups in a linear manner.

Individual cost is imposed through the enforced, joint property of the group, which is the agreed action at the core-preference position. For the model, it is a function of the distance of the own preference value to the core. This function c is positive, increasing in its argument $x_i - \tilde{x}$ and has a positive first and second derivative, which adds non-linearity to the model. For the analysis in later sections, I let the cost be a quadratic function of the distance argument which fulfills the conditions stated above.

I further impose that there are no fringe agents between contributors to a public good, which means that if a person on a more disadvantageous preference value contributes with positive utility, then any person closer, on a more advantageous relative position will also contribute for the same positive contribution to payoff but a lower individual cost. In general, I assume two boundaries attributed to any public good, a, and b. For each core, those values define the most distant contributor in any direction on the axis of preference values. If multiple public goods provide potential payoff, then the agent will choose the option with the highest payoff, and if multiple public goods provide the same outcome, she may randomly choose one to become a member.

First, consider a model with N agents. The payoff when all n agents between a and b contribute with individual preference realization x_j is:

$$u_i = \theta * \sum_{x_j \in [a,b]} \frac{1}{n-1} - c(x_i - \tilde{x})$$

Without loss of generality, $b \geq a$ is assumed.

I also develop a continuous approach which facilitates static analysis and models large populations. The contribution accumulating sum then translates to an integral, accumulating the density of agents over the covered preference space. The density in the sense of a probability distribution f already expresses population weight as a proportion of the total population.

θ is now normalized to measure the potential utility if the entire society is hypothetically captured around one core, which allows for an analysis of populations of an undefined size.

$$u_i = \theta * \int_a^b f(x)dx - c(x_i - \tilde{x})dx$$

$$u_i = \theta * (F(b) - F(a)) - c(x_i - \tilde{x})$$

In the CDF formulation, one can see that the utility contribution is equal to the share of players contributing to the same public good core multiplied with θ as explained above.

2.2 Stability

Stability is reached if two conditions are met; no agent has an incentive to join a public good core, or in a continuous environment that no preference value with a positive density of agents can be assigned to a core with a positive utility in those preference values. Also, at the same time, no agent should have an incentive to leave a public good, or equivalently no preference value inside the reach of a public good may produce negative payoff or less payoff than switching to any other core.

Definition 1 *A single public good is stable in its expansion, denoted through a and b as positions in the preference space with* $0 \leq a \leq b \leq 1$ *if utility at the position of these boundaries is exactly 0 or at least 0 if the expansion is bound from that direction.*

To illustrate, I let a single, given public good space extend into one direction and include at least one additional member. Then I can account for the sign of the utility for the new member to predict whether she joins or not.

In the continuous framework, I can similarly calculate the boundaries as the spots that provide 0 utility but have to include another condition to allow for an initial growth when the utility provided is currently 0 from a lack of contributors. Therefore I let one of the bounds a or b extend outward for an infinitesimally small amount and compare whether the utility at the bound itself decreases or increases. If moving the barrier into one direction produces additional utility, which then is positive for those who joined, then it makes sense for the adjacent agents to contribute. So initially at any outer bound the marginal cost of increasing

a or b has to be less than the additional benefit to let the public good expand.

The a bound will therefore incrementally extend downward if:

$$-\frac{\partial(\theta * F(b) - \theta * F(a))}{\partial a} \geq -\frac{\partial c(a - \tilde{x})}{\partial a}$$

$$\theta * f(a) \geq -\frac{\partial c(a - \tilde{x})}{\partial a} \tag{1}$$

Similarly, the upper bound b will extend upward:

$$\theta * f(b) \geq \frac{\partial c(b - \tilde{x})}{\partial b} \tag{2}$$

One has to keep in mind that the bounds are not interchangeable. In the current configuration, a is of a lower value than $\tilde{x}$ and b and extends downward as the outward direction. This ordering produces the negative sign when comparing marginal cost and benefit in the above equation.

Assuming a positive first derivative and a non-zero second derivative of the cost with respect to distance, then for a core to gain initial support, the contribution side only needs to be bigger than 0. Thus the density on the preference space needs to be positive at any point to allow for public goods with positive extension at any point, which is a restriction I impose on the densities for the model in general to avoid cores failing at a level of zero contributions.

To check the stability of a public good expansion beyond the starting point, I derive the position of a (and b). For this, I set the position of an arbitrary x_i as a and define utility on that position as 0.

$$u_i = \theta * F(b) - \theta * F(a) - c(a - \tilde{x}) = 0$$

Depending on the formulation of the cumulative density of F and the cost function c, this might not be solvable or have a unique solution. To still present results and intuitions, I choose simple distributions and quadratic costs. In this situation, for $c(a - \tilde{x}) = (a - \tilde{x})^2$ and a uniform distribution over $[0; 1]$ with $F(x) = x$, I get a unique solution for the bounds. The positions of the boundaries are symmetric around the core.

$$\tilde{x} = \frac{a+b}{2}$$

$$a = \tilde{x} - 2\theta$$

$$b = \tilde{x} + 2\theta$$

The result indicates a total expansion of 2θ around the mean $\tilde{x}$ if the bounds can move freely. If the bounds cannot adapt because they meet the outer end of the density or border another public good space, the result for stability becomes the following which is derived in detail in the appendix:

$$a = \tilde{x} - \frac{\theta}{2} - \sqrt{\frac{\theta^2}{4} + \theta b - \theta \tilde{x}} \tag{3}$$

$$b = \tilde{x} + \frac{\theta}{2} + \sqrt{\frac{\theta^2}{4} - \theta a + \theta \tilde{x}} \tag{4}$$

In this case, one of the values for a or b has to be exogenously fixed outside this calculation for the public good expansion to be able to calculate the other bound. Consider as an example that the public good core is situated at the preference-position on the value 1 then b also has to take that position since it cannot extend beyond the extreme value 1. a then takes the value $\tilde{x} - \theta$ as calculated from equation 3.

2.3 Endogenous Movement

Imagine a segregated group is left without any persistent agreed action or disputed leadership which can be due to the group forming in an essentially organic way or existing for a long time such that any initial common value or common action may be adapted by the group members. The question is, how is it possible to model this behavior based on utility optimization. My approach is to place the core representing the agreed action always in the welfare-efficient spot within a group to reflect that within a group the individuals are inclined to determine a group action to be the most efficient, thus finding the spot for the agreed action that maximizes accumulated welfare within the group.

In reality, this can be commonly observed in older or grown political parties and movements in democratic countries. Over time the positions adapt to accommodate the preferences of the targeted part of the electorate represented by that party. Otherwise, in an evolutionary inspired argument, the faction will have little success in surviving against the new, position-wise more flexible competition in democratic environments.

For the model, the first step then is to define the optimal placement of a core within one public good so that it maximizes the cumulative payoff of all agents currently contributing. This total welfare of the contributing agents can be described and is then optimized over the positioning of $\tilde{x}$.

First, let the joint payoff of all members of a group m be the measure for efficiency in the discrete case:

$$U_m = n * \theta * \sum_1^j \frac{1}{N-1} - \sum_1^j c(x_j - \tilde{x})$$

With $\tilde{x}$ as the decision variable, the sum of costs needs to be minimized:

$$min \sum_1^j c(x_j - \tilde{x})$$

$$0 = \sum_1^j \frac{\partial c}{\partial \tilde{x}}(x_j - \tilde{x})$$

Using $c(x_j - \tilde{x}) = (x_j - \tilde{x})^2$ then $\tilde{x}$ becomes the arithmetic mean of the member preferences.

$$\tilde{x} = \frac{1}{n} \sum_1^n x_j$$

Then in the continuous frame, the welfare is given as:

$$\tilde{x} = \frac{1}{F(b) - F(a)} \int_a^b x * f(x)dx$$

This $\tilde{x}$ definition then can be inserted into calculations of the border positions to represent that with extension in one side also the core position gets updated accordingly.

3 Efficiency

Intuitively a variety of different configurations of public good cores can exist and may be stable in a given environment. Consider sets of friends within a group who want to gather as separate groups for activities. Depending on what each potential group plans, there is a different number of groups present and groups may also vary in sizes. The question that arises from here is whether one configuration can be seen as better as another. While in general, it is difficult up to impossible for individuals to influence the segregation pattern of the groups it is worth considering whether under Pareto efficiency or total welfare efficiency it makes sense to consolidate or split groups. If gains can be made, then there is value in both having a planning authority and in extension, being that very planning authority.

While it is certainly easy to argue for Pareto efficient measures, it may be difficult to find broad support for a total welfare optimization. Imagine consolidating two groups into a bigger one that is just as attractive to those at the outer boundaries after consolidating. Given that no group member expects to be worse off, this will receive full support. In contrast, imagine a big group is split in 2 because the members within the group are concentrated far from the agreed action in their preferred value. This split improves the overall situation, as measured by total welfare. It allows the members concentrated around the outer perimeter of the initial single core to freely join any of the successor groups, focusing on actions greatly different from the one before. On the downside of splitting, for positive densities there are always several individuals close to the former core position losing individual well-being in the process.

3.1 Core Interaction

To proceed in the analysis, definitions for states of the model as well as for welfare are needed.

Definition 2 *A state in this model is described through information on the position of all cores and their respective contributors through the boundaries of the spheres of influence, accumulating the members.*

For stability I refer to section 2.2. A state then is welfare efficient in utilitarian understanding if there is no other state of the cores that is better in cumulative utility terms. So one needs to compare the accumulated utility of all players to compare states for efficiency in this approach.

As an alternative, I also address efficiency under a Pareto optimality condition, so if no player can be made better off without making another worse off, a state is Pareto efficient. I will start by discussing the Pareto case.

3.2 Pareto Efficiency

If setups are considered in which the cores are placed exogenously, then a stable state occurs naturally through expanding the reach of the core over individual agents. However, this state is not necessarily efficient in neither the Pareto nor the utilitarian sense. For this consider the case of agents, with the majority existing only to a certain distance from the outer bounds of the game and a single core existing in the center. If the distance to the outer bounds is high enough then most agents would not join neither in the continuous case nor in the discrete case. However, if θ is great enough, then the joint contribution of all agents is enough to maintain the entire population inside the public good. A specific example for this is a distribution of agents consisting of uniform distributions in the intervals $[0; \frac{1}{3}]$ and $[\frac{2}{3}; 1]$ with high density and minimal density β in $[\frac{1}{3}; \frac{2}{3}]$. Further, $\theta = \frac{1}{4}$ with $\tilde{x} = \frac{1}{2}$ but no initial contributors. Initially, a certain range will join, but it will only extend to $2 * \frac{1}{4} * \beta$ from the middle. The $\theta = \frac{1}{4}$ would, however, support stable extension over the entire space, providing positive utility for everyone and higher utility for those close to the middle.

To check for Pareto optimality in a multi-core environment, I investigate dropping 1 of 2 neighboring stable public goods to incorporate all members of the dropped center into the remaining which must provide the same or better utility to everyone involved after adapting to the efficient position. If that is impossible, we have reached Pareto optimality.

Two possibilities arise when comparing the previous two cores with the single new one. We either have a Pareto improvement or not. To examine, I compare the old state to the new, altered state after the supposed improvement and explore the conditions with calculations in the appendix.

1. An alteration which covers at least the same space in its stable, maximal extension to one side is always beneficial for members on that specific side between the core and the boundary.

2. If considering smooth cost with always positive second derivative then to show potential

Pareto improvement, the furthest agent at the boundary which the core moved away from, has to receive at least the same utility. This guarantees at least the same payoff for all agents on this far side of the core.

3. If the alteration is a Pareto improvement, it has to cover at least the same fraction of the agents than before in a stable state.

4. Lastly, at least one preference position within the sphere of influence of the public good needs to experience an increased utility through the alteration of the state.

To construct a potentially Pareto improving alteration for a single core setup, I consider a preference space with a population density f, a public good with a core $\tilde{x}$ and maximal boundaries a and b. If that is stable according to the definition in the stability section, then the only way to potentially Pareto-improve the situation is to alter the position of the core toward a more efficient one. I utilize that if a benevolent planner was to install a public good, then the core should be placed at the population weighed preference average in the targeted sphere of influence to minimize the cost burden which uses the idea of ideal core placements with a quadratic cost function. A core shift may as a result in recruiting new contributors on one side but needs to not lose on the other, representing that at least the same utility has to persist for the original boundary agent which is now further away from the core. A core re-positioning therefore does not generally imply a Pareto improvement since the then covered space might not overlap with the original state.
In a multi-core situation with already optimized core positions, the only viable alterations are those in which 2 or more neighboring cores get unified and feature a re-positioning of the joint core. Consider two neighboring public goods in a stable state. If the situation when unifying these cores cannot extend its boundaries further in at least one direction, then the original state is Pareto optimal. Proposition 1 captures a way to check for optimality.

Lemma 1 *Imposing that one boundary position stays the same, using efficient endogenous core placement provides a potential extension increase for the second boundary. If the boundary value is further away than the original outer boundary on that side, then the new core provides higher utility for all original contributors.*

For lemma 1, consider that if one boundary stays the same then only through capturing more contributors, one can ensure utility improvement on the side of this bound.

3.3 Welfare Efficiency

The case of utilitarian efficiency, I need to compare the current state with an optimal state in which a planner has the ability to create and drop public good cores at any position with any range of contributors. Again, to ensure efficiency within the covered space of any core, it is necessary to utilize the result on optimal placement within the public good-sphere of influence derived in the endogenous movement section.

Welfare in the discrete agent case and continuous case for M cores with n_m contributors are:

$$W_d = \sum_{m=1}^{M} \sum_{n=1}^{n_m} (\theta n_m - c(x_{nm} - \tilde{x}_m))$$

$$W_c = \sum_{m=1}^{M} \left(\int_{a_m}^{b_m} (\theta - c(x - \tilde{x}_m)) * f(x) dx \right) \tag{5}$$

Depending on the starting state, a planner has a set of viable options to increase welfare in a system, including the addition of new cores or omitting existing ones and reassigning contributors. In any efficient state, all utility maximizing agents contribute to a public good with positive payoff since it is always at least as good to instantiate a public good at their position under positive densities. Otherwise one can allow the endogenous formation of a small public good which has a positive payoff for a defined set of contributors. Further, as soon as there is a positive number of expected contributors or a positive density around oneself, then it is strictly dominant to create the local core with expected positive utility gain.

From the equation 5 it is possible to discuss potential steps towards optimization. There is a trade-off between more cores with lower cumulative cost realization due to less density weighed cost and fewer cores with a higher positive contribution. Intuitively it makes sense to split into more cores if the distribution is higher on the more remote sections in the sphere of influence; that state being represented by the weighed payoff in the equation being smaller relative to a uniform distribution. However, any planner has to check total welfare in both, the original configuration and the targeted configuration to ensure an improvement.

3.4 Efficiency in Equilibria

An equilibrium formed is not necessarily efficient the sense of either, Pareto efficiency and welfare efficiency. Consider the following example with a uniform distribution, two cores placed at $\frac{1}{4}$ and $\frac{3}{4}$ respectively, and $\theta = \frac{1}{2}$. This state is stable with the value $\frac{1}{2}$ marking the barrier between the two cores with each core covering its side of the preference line. The welfare generated for each of the two groups totals to the social welfare of $\frac{41}{96}$ for this stable state. As an alternative, consider the same setup with only one core placed in the center at $\frac{1}{2}$. Under $\theta = \frac{1}{2}$ this core extends over the entire line in equilibrium. According to the restrictions for a Pareto improvement outlined in section 3.2 this is a Pareto improvement as compared to the situation with two cores. Social welfare now comes from only a single core and is $\frac{53}{48}$ which is greater than before, so the new state defines an improvement in Pareto efficiency and welfare efficiency terms, even though both described states are in equilibrium. Equilibria mark a super-set of Pareto efficient states since all Pareto efficient states are stable equilibria and welfare efficient states are a subset of Pareto efficient states Therefore efficient states are found by testing the states of the super-set if they satisfy the conditions for Pareto and welfare efficiency respectively.

4 Static Analysis

A sizeable proportion of observations about segregation dynamics is related to politics and elections; therefore the model fits best to those situations, in which the choice of being part of one group is mutually exclusive to being part of another as with interest groups and the choice to support one candidate in an election.
For analyses of observed patterns and how well the model can replicate the intuition behind observations, I investigate how model parameters produce different results as stable outcomes.

4.1 Exogenous Core

Consider the situation in which there is an exogenous planner behind the public good positioning. Without any direct influence on the planner, the core position is perceived as fixed from the individuals. This arises not only from a planner being agnostic on which individuals set a certain action in any planning-target in this work, but can also be due to restrictions on the valid and eligible actions as a subset of the full preference space in the first place. For instance, the possible actions may be limited to adoption or dismissal of a certain policy, and thus only the values of 0 and 1 are valid actions for the cores. Individual preferences, however, can still be in-between representing varying degrees of approval for the agents.

In the following, I construct examples with exogenously imposed symmetric positions of public good cores and a quadratic cost function which lets me present some results under immutable cores.

First consider a model world consisting of a uniform distribution of the population over the preference space and a single core placed in the middle at the position $\frac{1}{2}$. The utility of a player with preference x_i is then given as:

$$u_i = \theta(b-a) - \left(x_i - \frac{1}{2}\right)^2$$

The initial point of interest is to narrow down under which conditions the public good will have the entire space in its sphere of influence. I form an inequation on the utility of the agents placed exactly on the extreme boundaries of the preference space with having the positions $a = 0$ and $b = 1$. The distance to the core then is $\frac{1}{2}$ to both extremes, and since the entire space is supposed to be covered, the benefit is $1 * \theta$.

$$\theta - \left(\frac{1}{2}\right)^2 \geq 0$$
$$\theta \geq \frac{1}{4}$$

So θ has to have a specific size at minimum to increase the reach of the public good up to the point that allows the core to cover the entire preference space.

Even though this is only to start the analysis, it can already reflect that there are many agreed actions in societies that appear appealing enough, so all or almost all members of society decide to join. Consider culture-specific actions like dressing in a certain way or participating in cultural activities to some extent because the networking-benefit these actions provide lets people productively interact while greatly outweighing the perceived cost.

4.2 Interaction with Exogenous Cores

Exogenous cores serve to represent actions that cannot be altered by members of a group. Consider factions under strong leadership that, to some extent, stick to an ideological framework and cannot deviate freely. Alternatively one can also assume that an exogenous core may be planned to achieve another goal, exogenous to the game, which may be to acquire some control over the stable state outcome or obtain a sizeable but closely interlinked group of individuals for marketing purposes. It is important for such a purpose-driven group to center around preferences such that the potential product is relevant, but also for the group to be of sufficient size to be relevant as consumers.

If more than one core exists in the setup, then cores may interact which means that the cores extend to the point that allows for a number of agents to switch contribution to either good with positive payoff for either side. Now, for this set of agents, more than one group resulting from polarized preferences can be a realistic group to join in equilibrium.

The model setup for illustration of this property is symmetric and extended by a second core. Placing the cores at the arbitrary positions $\frac{3}{8}$ and $\frac{5}{8}$ gives enough space to let them interact before extending to the outer bounds. Through the cores extending freely initially, we can use the result for the extension under a uniform distribution from the section 2.2. The spheres of influence will then initially extend for $2 * \theta$.

To let the spheres around the cores meet at the center, θ has to be at least $\frac{1}{16}$. All beyond that will not alter the border preference-position, which produces indifference between the two cores. The formula below captures utility of the agent at the interaction point as the utility for joining either of the two cores:

$$u_i = \theta * \frac{1}{2} - \theta * a - c(x_i - \frac{1}{2})$$

Applying the formulas 3 and 4 for one-sided bounded expansion, I can also define the position of the non-interaction border of the cores that logically extends outward with higher θ which then gives the expression for each side with a of the core at $\frac{3}{8}$ as an example:

$$a = \frac{3}{8} - \frac{\theta}{2} - \sqrt{\frac{\theta^2}{4} + \theta\frac{1}{8}}$$

4.3 Exogenous Cores with Asymmetry

Asymmetry can be introduced into the model through non-symmetric distributions of the agent preferences or in the case of exogenous cores, through asymmetric placement. The former can be investigated through alternative or modified distributions. Any symmetric environment ultimately suggests public goods that all extend similarly, which however is not what is commonly observed. Uneven distributions of preferences therefore can help explain different realizations of group sizes relative to each other.

Let the underlying distribution density be $f(x) = 2 - 2x$ to represent higher densities toward the lower preference realizations, and assume one core at the left end with $\tilde{x} = 0$. The maximum extension can be calculated as usual from the utility of the border agent set to 0 and a CDF of $F(x) = 2x - x^2$ coming from the density.

The resulting position of the upper border then is:

$$b = \frac{2 * \theta}{1 + \theta}$$

The derived position is different in value and reaction to changes in θ as compared to the result of the same setup with a uniform distribution and $b = \theta$. Formulating the first derivatives with respect to θ lets me show the differing force of extension depending on underlying densities. Speed or force to increase in the context of my model setup is to be understood as the calculated benefit of a group for a potential new member at a given position. The underlying assumption is that under a constant cost structure cores with a higher payoff promise always extend at least as forceful than those with lower accumulated benefits. For the uniform case, the derivative is 1 representing a constant relationship on extension if the benefit of group size increases. The falling density produces a derivative of:

$$\frac{\partial b}{\partial \theta} = \frac{2}{(1 + \theta)^2}$$

This expression still is positive in general but decreasing in θ. So the expansion will extend while slowing down with a θ increase.

The interpretation is that with asymmetric densities, a core is extending further in sub-spaces of the preferences in which the covered density is comparably higher, which is further backed by another example of asymmetry through introducing a fixed fraction of the population being bound indefinitely to one core. While the first case describes an asymmetric distribution of agents depending on preference value which is something natural to observe, the latter case can model a group being bound to one core by external factors to the model such as laws, regulations or contracts. For the setup, I ignore utilities of the bound fraction of the population since the decisions are not inside the model, but I will still account for their contribution to the benefits for other group members.

Also in the context of an external force, it can also be that a planner lets one core appear bigger in the number of contributors to achieve certain goals. This planner-strategy can be useful to help start a new core that would otherwise not be very attractive and thus extend in smaller steps.

Assume a setup with one core at $\tilde{x} = \frac{1}{2}$ and initially an unmodified uniform distribution. Now I compare the situation in which a fraction p is now bound to be contributors to the core. The extension of the sphere of influence d in either direction can then be calculated as the distance to the core which allows for zero payoff:

$$\begin{aligned} a &= \tilde{x} - d \\ b &= \tilde{x} - d \\ 0 &= 2d\theta(1-p) + \theta p - d^2 \\ d &= \theta(1-p) + \sqrt{(\theta(1-p))^2 + \theta p} \end{aligned}$$

The part outside the square root and the first term within represent the extension due to the independent agent coverage of the public good which can be noticed by the factor $(1-p)$ and the last term in the square root is the contribution of the bound agents. If p is set to 0 one receives the known result for the extension of 2θ and if $p = 1$ the extension is considerably different, taking the value $\sqrt{\theta}$. It is noteworthy that, comparing these extreme cases, for a reasonable parameter as $\theta \in [0, \frac{1}{4}]$ the formula containing the square root produces a greater extension, taking the same value at $\theta = \frac{1}{4}$. This is due to the additional contribution when extending further which plays a bigger role if the bound share p is low. For low θ it is then beneficial to already have a fraction bound to the core even if that means any further extension will be less valuable as a contribution to the utilities of the members.

Thus there is a benefit of being able to let a core appear abundant in contributors. For this, one can consider corporate presence in social media. Obtaining a member base linked to the social media outlet can help initially share product information and experiences and will increase membership growth in situations where group membership is not especially attractive in general represented by a low θ.

It is however known practice from some social media presences to buy followers or let employees populate their social media outlets which lets the group appear bigger. Of course, there may be a backlash if the practice is found out, but it appears attractive to start the development of a presence until the desired amount of actual contributors is achieved.

4.4 Takeover

If two neighboring cores are interacting by having touching spheres of influence in one point and one has an advantage through maintaining a higher number of contributors, it is possible for that more powerful core to cannibalize the other. Cannibalizing means that area on the density space which was covered by the other core is now taken over through individual agents at the barrier between the two cores switching contribution. This process may start at any point between two cores and will end at a point which needs to satisfy the equilibrium-condition of providing the same utility as the best alternative choice in this point. This point is further referred to as the conflict point z.

First, I introduce a definition for the endpoint such that a stable state is reached as the outcome of a cannibalization process. It is also noteworthy that a cannibalization process does not necessarily need to end in one public good being entirely taking over by the other but can end at any point which was formerly within the sphere of influence of the less attractive core.

Definition 3 *The endpoint of the process of taking over space from a neighboring core is defined through an agent having equal utility prospects at either of the two cores.*

It is not possible to start cannibalizing if the conflict point is derived as a stable outcome. Thus I need to set a starting condition for the entire process. That also means that if the initial state was stable, then there need to be a disruption resulting in different relative attractiveness of the two neighboring cores to start the takeover process.

Lemma 2 *To start cannibalization of a neighboring core, the agent at the conflict point z has to have strictly higher utility prospects joining one of the cores.*

Ultimately To derive how far cannibalization takes place, it is sufficient to check for the starting condition and find the endpoint.

Proposition 1 *Only the beginning condition and the endpoint conditions are relevant for taking over space. If for a more distant agent, it is beneficial to switch sides from a less distant core then for all agents relatively more distant to the initial core, it is also beneficial to switch.*

Lemma 2 is straightforward. To extend into one direction, it has to be beneficial for the agent at the border to join.

Proposition 1 comes from comparing utilities of switching agents with increasing distance to the powerful core. The cost increases faster the higher the distance d to the referring core. I consult the derivatives to illustrate:

$$c = d^2$$
$$\frac{\partial c}{\partial d} = 2 * d$$

The additional contribution being 0 for agents in the continuous case and the cost for further agents being higher, it has to hold that if one agent has the incentive to switch to one core all closer agent have an even higher individual incentive to do so. This can be extended to the situation when the underlying density is falling with distance since then all closer preference values also jointly have a higher contribution aside from the lower cost.

I construct the following setup to determine the roles of distribution and group benefit θ for the aspect of taking over space from neighboring cores. Therefore, let there be two cores $\tilde{x}_1 = 0$ and $\tilde{x}_2 = 1$ with an underlying uniform distribution modified by a fraction $p > 0$ of the population being assigned to the first core $\tilde{x}_1$. If the borders of the cores meet then at the equidistant point at $\frac{1}{2}$ the core $\tilde{x}_1$ will be more attractive because of the additional fraction assigned to that core. If θ then is sufficiently large it is possible for core $\tilde{x}_1$ to extend to the endpoint $\tilde{x}_2$. The conditions for the core at 0 extending to the other side are therefore $\theta \geq 1$ for the endpoint condition and $p > 0$ for the starting condition. At any point between start and endpoint, this core will, under these conditions, be relatively more attractive.

5 Endogenous Core Static Analysis

In this section, I will analyze behavior in the model under endogenous core placement which implies that in all calculations $\tilde{x}$ is replaced by the formula resulting from an optimization of within-group welfare under quadratic cost.

In a real society, this would allow new members of the contributing group to equally affect the positioning of the agreed action which is a reasonable assumption if a group is not directed exogenously in a dictatorial manner. Imagine a setup in which the position of the core is updated regularly. So if the starting position is towards one extreme and there are more and more individuals with more moderate preferences joining, this will ultimately lead to the core becoming more moderate. For instance, consider an interest group which initially is catering to a rather extreme audience in terms of preferences and is currently neighboring a public good with a moderate agreed action. Now due to some reason like fashion or a scandal, the moderate public good exogenously loses attractivity and bordering individuals now switch to the more extreme public good. The updating process would then draw the extreme core more and more away from the outer edge of the preferences.

Due to fashion, certain positions tend to be relatively less attractive at any given period in time. If this development of preferences is to the disadvantage of moderate forces, the process of individuals switching attention to extreme opinions modifies the observed positions of extreme groups themselves. The result is that in many discussions featuring more extreme groups those extreme groups may attempt to compare their positions to the former more extreme positions and attempt to appear comparably moderate and therefore more attractive to individuals with less extreme views.

For the model, consider a simple setup with uniform distribution and a single core placed at the low boundary $\tilde{x} = 0$. Now let the update of the core position happen step-wise to emulate dynamics. For a given θ, the sphere of influence will naturally extend up to the position $b = \theta$. Note that any extension will then also alter the core position, which in turn modifies the reach of the public good. In the presented case, the extension and updating will continue until the utility of a border agent is 0.

$$
\begin{aligned}
u(x = b) &= 0 \\
0 &= \theta(b - a) - (b - \tilde{x})^2 \\
0 &= \theta(b - a) - (b - \frac{a + b}{2})^2 \\
0 &= \theta(b - a) - (\frac{-a - b}{2})^2
\end{aligned}
$$

The indefinitely repeated updating process leads to a position of $b = 4 * \theta$ with a core position $\tilde{x} = 2 * \theta$, letting a remain at 0. This result can be produced by calculating the value for the final b as the convergence value for infinitely many steps of updating. Alternatively, for a non-increasing distribution, there can not be overshooting when re-positioning the core which would result in a loss of covered space towards the initial position. Therefore I will calculate the border position of b as the maximum extension for the core while holding a constant at 0. The restriction for the difference is $b - a = 4 * \theta$. With a higher θ, both the reach will extend and the core position will move more towards the center of the preference space.

Any group, even if it is initially installed at an extreme endpoint-position, attracting agents close to the original position and slides toward a more moderate position on the preference space.

5.1 Interaction of Endogenous Cores

If there is no interference from the outside, the case with endogenous core placement is the more reasonable option, to which interaction of cores is now introduced. Consider groups of members of a society that come together regularly and discuss at a table. A certain popular opinion or agreed action evolves which is shared with differing degrees of approval. However, to be considered part of the group, this basic and therefore identifying opinion has to be supported. In this context, as a member one cannot act or express oneself entirely free. At this point, let another agent with slightly different preferences join the group. Even if the agent initially supports the agreed value, the slight differences in preference value will shine through in repeated interactions, and thus alter the agreed action slightly. Assume now that there is a second table with people discussing and agents having the freedom to switch between those tables at will. Intuitively, every time an individual switches table, the agreed action in both is updated. In a state of equilibrium we will likely observe two greatly different agreed actions due to the repeated updating of the core values and agent choices.

To model an interaction of endogenous cores, I consider a setup with uniform distribution and two cores $\tilde{x}_1$ and $\tilde{x}_2$ situated on top of the center with a marginally small offset to each other, so one core initially favors smaller preference values and the other bigger ones. Moreover, a low θ is imposed, which enforces that the cores do not cover the entire space jointly or alone in any relative positioning. The offset is imposed to avoid an equilibrium with overlapping, equally beneficial cores.

Each core will develop into one direction from the center following the offset, thus altering its position after accounting for the preference values of new members. The extension into the direction of either a or b will take place if the above results are fulfilled as the following inequations:

$$a > \tilde{x}_i - 2\theta$$

$$b < \tilde{x}_i + 2\theta$$

Note, that this is with the same argument as in section 5. Under a different underlying distribution one has to calculate the distance to a core as the value that enforces zero payoff for the respective preference value.

Since one of the directions is blocked for an extension for each of the cores, they will only develop one-sidedly. Assuming a low θ, there will be no other barriers than the existing other public good; therefore the cores will take positions $\tilde{x}_1 = \frac{1}{2} - 2\theta$ and $\tilde{x}_1 = \frac{1}{2} + 2\theta$ and extend their reach for full 2θ in either direction.

If θ is sufficiently large, the outer bound of the core extensions will touch the outer extremes of the preference space rendering further expansion impossible. These situations for which $\theta \geq \frac{1}{4}$ will be without a change in agent choice since any expansion is halted. For the uniform distribution case, the cores will always sit at the center of their respective sphere of influence; therefore, I will use d_i to measure the distance to any border of an extension from core $\tilde{x}_i$. This is especially useful when introducing more cores to the system.

Let me now increase the number of cores to n. From before I use the information that between any two cores the agent at the border is indifferent with d_i as the distance to the agent and at the same time $2d_i$ measuring the extension of good i under uniform distribution:

$$\theta * 2 * d_i - d_i^2 = \theta * 2 * d_j - d_j^2$$

This gives two possible results.

$$d_i = d_j$$
$$d_i + d_j = 2\theta$$

The first result highlights the intuitive result that in a possible equilibrium in a uniform case, the cores are equally extended. The second result produces an interesting configuration. To calculate it will be combined with the restriction that exactly the entire preference space will be covered:

$$1 = 2 * \sum_i d_i$$

Combining shows that for a specific θ for a given number n of cores, it is possible to observe stable, but unevenly extended public goods even under uniform distribution. For this, consider an example with two cores. The resulting θ with potentially special outcome is $\theta = \frac{1}{4}$. Now any combination of extensions that satisfy $d_1 + d_2 = 2\theta$ and $2(d_1 + d_2) = 1$ is an equilibrium. The solutions of this system of equations are $d_1 = \frac{1}{3}$ and thus $d_2 = \frac{1}{6}$ which let me calculate the payoff at a border position for both public goods. Payoff is as expected the same value $\frac{1}{18}$.

Only greater values for group size benefit θ will stabilize unequal share between the various cores active on a uniform distribution with the required size depending on the number of cores available. Regularity in public good expansions seems intuitively unnatural, therefore it makes sense to generally allow for less evenly distributed preferences.

If the underlying preferences are drawn from a general distribution f while the setup otherwise stays the same with two cores, one can derive the distance to the borders more generally. When calculating for the conflict points between 2 cores, let there be 2 different distances $d_{1,1}$, $d_{1,2}$

for one core into both directions and $d_{2,1}$, $d_{2,2}$ for the other with $d_{i,1}$ measured downward and $d_{i,2}$ measured upward. Without loss in generality, I assume for the underlying cores that if $i \leq j$ then $\tilde{x}_i \leq \tilde{x}_j$.

Note, that the contribution collected in each core may be different. Setting the utility equal at the conflict point allows for setting the distances into relation:

$$\begin{aligned}
\theta * (F(b_1) - F(a_1)) - d_{1,2}^2 &= \theta * (F(b_2) - F(a_2)) - d_{2,1}^2 \\
d_{1,2}^2 - d_{2,1}^2 &= \theta(F(b_1) - F(a_1) - F(b_2) + F(a_2)) \\
d_{1,2} &= \sqrt{\theta(F(b_1) - F(a_1) - F(b_2) + F(a_2)) + d_{2,1}^2}
\end{aligned}$$

Assigning a new variable for the contributor-mass difference;

$$D = F(b_1) - F(a_1) - F(b_2) + F(a_2)$$

$$d_{1,2} = \sqrt{d_{2,1}^2 + \theta D} \tag{6}$$

Respectively:

$$d_{2,1} = \sqrt{d_{1,2}^2 - \theta D}$$

Equation 6 expresses the relationship between two neighboring cores. Under the square root, the distance of the other core is then offset by the difference in contribution with the sign depending on which side relative to the selected core is calculated. The formula in 6 therefore expresses the distance to the right using the distance of the neighboring core to the left and the contribution difference.

With equation 6 it is possible to evaluate strategies to intentionally distort such an equilibrium. The possible ways to interfere are to make benefits appear different or manipulate the perception on the positions relative to a neighboring core.

Given the aim is to increase the reach for one core as measured by any d value, letting benefits appear more in favor of the preferred core has a non-linear translation into an effect as expressed by the relationship under the square root, similarly for efforts to let the distance to the neighboring core appear greater.

5.2 Endogenous Core Creation

In a considerable number of occasions, one can observe agents who are not part of a public good sphere of influence. In this case, it is likely that new groups may emerge if there are sufficiently many potential supporters without membership to an existing public good. Those new groups may be tiny but derive their value in catering very specifically to those part of the group. The actual agreed action therefore does not have to be of extreme nature but just very specific, such that a newly formed group would be interested in this action even if contribution size is minimal. Imagine any group of friends with very unevenly distributed tastes. For preference with a high density of prospective agents, there might already be a group forming, but as soon as there is a potentially stable group capturing unsatisfied members, then even the small group will receive members.

After discussing an endogenous placement of the cores, I now allow for the creation of entire public goods themselves by the agents within the model. Therefore, I set up conditions when it makes sense to create a new core as I will refer to as initiative.

Ultimately an equilibrium is supposed to be reached in which no agent has an incentive to change behavior which includes to not wanting to instantiate a new initiative. When allowing endogenous core creation, all agents will be contributing to a public good in equilibrium. A newly created initiative can develop to be of any size including minuscule extensions; therefore it is always dominating for an agent currently without public good to create an initiative that may provide a marginally greater payoff than 0 from not contributing to any core. The consequence is also that any core will have borders touching either the bounds of the preference space or the borders of other cores.

To construct a stable state, I need to state the position and extension of any core in the setup. The core position is taken from the endogenous core positioning in section 2.3. Equation 6 provides a necessary condition for stability of a state. To then construct a stable state I set up an initial public good adjacent to one outer barrier at 0 or 1 by defining both boundaries so that utility at the boundaries is positive but not necessarily equal 0. The endogenous core placement then provides the core position and in extension the values for d_{i_1} and d_{i_2}. From these values, the distance for the neighboring core to the conflict point can be calculated using both conditions as equations. Note that at this point the distance results for the neighboring core do not necessarily have to be unique. Assuming that one can determine a unique result to use further, this process of calculating distances can be repeated until the entire preference space is covered. Then it only needs to satisfy $\sum_i d_{i,1} + d_{i,2} = 1$ to have stability in the last public good.

As an example consider the case of a uniform distribution and $\theta = \frac{1}{4}$ as in section 5.1. Starting from the value 0, I set up the first public good with $a_1 = 0$ and $b_1 = \frac{2}{3}$ with $d_{1,1} = d_{1,2} = \frac{1}{3}$ and a resulting core position $\tilde{x}_1 = \frac{1}{3}$. The distance and core placement conditions then allow for the distance of the next public good to either be $\frac{1}{3}$ or $\frac{1}{6}$ symmetrically around the core. To satisfy the final condition $\sum_i d_{i,1} + d_{i,2} = 1$ only $\frac{1}{6}$ can be kept. Note that $\theta = \frac{1}{4}$ provides a special case for the uniform distribution, as mentioned in section 5.1 with the possibility of an asymmetric equilibrium.

5.3 Welfare

When discussing welfare, it is interesting how far segregation itself is a positive or negative effect. I assume that polarisation of preferences is something that occurs naturally which can then be translated into several stable segregation outcomes in my model. The question is now whether one segregation pattern can be seen as more beneficial to society in terms of welfare as compared to another pattern. Of course, this question is very much focused on potential improvements through a social planner, but it also can indicate when there is a benefit for a society in questioning the current segregation pattern and may encourage consolidating or splitting of existing groups.

Considering equation 5, I set up a model to illustrate a way to find a welfare efficient configuration. A uniform distribution is used with undefined θ. I compare how a single core fares against two optimally positioned cores and ultimately define the threshold θ to define when one setup emerges to be more welfare efficient than the other.

First, the core position is considered. I assume a single core environment, which has that core placed at the center. The reasoning is that ultimately, when the core captures the entire space and under efficient endogenous placement, this position will be assumed automatically. Similarly for 2 cores the positions $\frac{1}{4}$ and $\frac{3}{4}$ are the core locations with the same argument; Endogenous core placement and boundary updates will enforce those positions.

Note that initially for low values in θ, the solution with more cores will always be more efficient. Low θ allows for free extension without barriers in both setups, so two cores will grow to exactly the same size each as the single core and thus create twice the welfare together. Increasing θ so $\theta \geq \frac{1}{8}$ creates a situation in which the two core setup will have the cores at the positions $\frac{1}{4}$ and $\frac{3}{4}$ extending over the entire space together with the barrier between at $\frac{1}{2}$. This uses the result that in a uniform distribution, the barriers will generally extend for 2θ from the core center in each direction. The consequence is that the two core setup will, with higher θ, only increase the provided utility by that θ as a factor while the single core setup increases by the factor and the additional contribution from newly covered contributors. So the welfare of the setups is used to form an inequation. With this inequation it is possible to derive the threshold θ from which the single core provides higher total welfare as compared to a two core setup.

Note, that when $\theta \geq \frac{1}{4}$ then the single core environment also covers the entire preference space, thus negating the growth advantage to utility production. Setting the welfare equal in this situation results in a more simple inequation providing another threshold θ to consider simultaneously.

The results for the uniform distribution are: For $0 \leq \theta \leq 0.2075...$ a dual-core setup is strictly more welfare efficient. For $0.2075... \leq \theta \leq 1$ single core provides higher welfare. The last step, checking the case when both configurations cover the entire preference space, shows that for all $\theta \geq \frac{1}{4}$ a single core is beneficial. All necessary steps can be found in the appendix.

To summarize, a situation with more cores is always more efficient if fewer cores fail to cover the entire space on the preference spectrum. It is a reasonable outcome to observe many cores in situations with low attractivity of groups, as captured in a low value for θ. There is an intermediate case to consider when multiple cores lose efficiency through interacting at the boundaries. If both configurations fully capture the entire space the calculations become more simple since the provided utility in each core does not depend on further expansion anymore and give a clear value for a threshold θ indicating an advantage for fewer cores if the θ surpasses that value.

6 Application

In this section, I will investigate explanatory potential and applications of my model in the context of observations and publications.

6.1 Organ Donors in Austria

A 2003 paper, *Do Defaults Save Lives?* [2] compares the effective organ donor rate in various European countries and draws a connection to how consent for post-mortal organ donation is regulated in those countries. The key difference is between assumed consent or assumed non-consent, resulting in vastly differing effective donation rates. For instance, Austria pursues an assumed-consent approach with opt-out, obtaining 99.98% effective rate compared to Germany with an opt-in system and 12% effective rate as published in the paper.

The decision on whether a person is an organ donor cannot be due to rationally optimizing own payoff since as a decision it promises no difference in payoff for the individual. If fact agents would care for others to become organ donors in case of an accident but exercise no noteworthy effort to do so themselves since the expected material gain is 0.

There exists a plethora of other situation where the relative payoff of one to any other action in a decision has is minimal. Consider a runoff election with a clear expected outcome; then due to the low probability of having a pivotal vote, voting would make almost no difference in matters of own expected payoff from the election outcome.

I assume that a factor in the decision process of individuals in the current context is group identification asides from personal preference on the outcome. More clearly, an agent may consider setting the action close to their own ideal but also reacts to whether this decision is different from what the bigger group around the agent does. The outcome then can be modelled with the approach of this paper.

To explain the behavior of the individuals in the context of organ donations, I allow for a single core to exist at the specific action, which is not the default. This is to capture how groups need to be formed actively around one action while the default does not require any active contribution to sustain itself. Also, this is an example of a situation in which actions aside from 0 and 1 make no sense. The agreed action of a group is either being an organ donor or not. It is also possible to assume a second core in support of the default setup but measuring the actual number of members of such a group is not feasible since a default allows for tacit acceptance of the default action as well as active support.

Consider an opt-in system, then the conscious decision would be to opt in through joining a core at the extreme position representing consent. The observable result is that around 12% decide to opt-in in Germany. In Austria, under the opt-out system, 0.02% actively chose to opt out. Assuming comparability between Austrian and German societies it can be seen that there is a higher density of agents with preferences approving of organ donation as compared to actively opposing it. The group forming around opposing organ donation does not extend to a relevant size while the approving group extends and manages to capture comparably less enthusiastic supporters in terms of individual preferences of organ donation as well.

6.2 Election Meddling

Short after the surprising victory of Donald Trump in the presidential elections of the United States in 2016, a debate on foreign influences on the election process started. Also, the practices of Cambridge Analytica, a company engaging in data analysis and micro-targeting became public. Trying to manipulate public opinion is not a new phenomenon, especially with the presence and important role of mass media in modern societies. However, the intensity of suggestive reports on especially Hillary Clinton echoing in media reports throughout the majority of the election period is noteworthy. The question is, whether this can be explained as a prudent tactical move in a model. Ultimately this hopefully does not serve as a manual to replicate the observed strategies but helps mitigate the effect through manipulating perceived group size by understanding how a planner can provoke a certain reaction or abstention from voting of parts of the electorate.

While the discussion on Cambridge Analytica was mostly about how data was acquired from Facebook, raising massive privacy concerns and resulting in a unprecedented fine, in the context of this work the aspect on how micro-targeting might have influenced the election is of interest. Similarly to Cambridge Analytica, also Russian agencies have been accused of having interfered with the election. The Guardian accumulated information in the article *Russian propagandists targeted African Americans to influence 2016 US election* [11] about activities of the Internet Research Agency (IRA) which is known to be a Russian organization attempting to influence relevant decisions through social media. The practices of the IRA can be very well explained through my model, especially section 5.1 and equation 6 to quantify the expected success, and it also lets me make some assumptions about the distribution of political preferences in the electorate of the United States.

According to the article from The Guardian, the target was mostly to discourage potential electors of Hillary Clinton in the election as she was the expected winner. Hillary won the popular vote but through the peculiarities of the presidential election in the United Stated Trump assumed victory through winning by a lower margin but in more states. In this context, discouraging potentially pivotal voters makes much sense. In a greater environment, such as the entirety of the electorate, the effort would be enormous to manipulate the outcome. However, a focus on expected close ballot outcomes may be economically feasible.

Consider a model in which voting for one or the other candidate is represented as contributing to one core or the other core with not voting as the status-quo option. There are now multiple options to increase the relative chances of one candidate. One is to mobilize potential voters directly, another is to discourage voters of the other candidate. Discouraging can be accomplished by letting the group, defined through the agreed action of voting for the other candidate, appear weak in numbers. If the expected gains from joining that group diminish as a result, the sphere of influence will decline, causing amplified damage through a self fulfilling prophecy. From the information in the article, I assume discouraging was the preferred strategy. The voter density of Donald Trump, therefore, seems to have been more concentrated around the core position relative to Hillary Clinton which allowed for more efficient manipulation on voters close to the boundaries of not voting for Clinton. The impact from a decline or increase of the sphere of influence is highly dependent on how much is lost through a specific change in this sphere of influence. If there is a greater number towards the boundaries, then the effect is greater. If the members are already highly concentrated around the center, the change in the extension will not alter the group size as much.

The practices employed and outlined in the article was creating social media groups which aimed to let voting for Hillary Clinton appear less beneficial or encourage voting for irrelevant candidates by letting a bigger group show intention to do so and thus letting the support of Clinton appear smaller.

The *Voting and Registration Supplement* [13] from the Census Bureau in the United States supports this interpretation. When comparing the development of reported voting rates of different ethnicities, the election of 2016 shows an unexpected decrease, which is especially striking with voters categorized as black.

The *Report On The Investigation Into Russian Interference In The 2016 Presidential Election* [10], dubbed as the Mueller Report, has been submitted and published in a cut version. Aside from investigating the allegations against actors in the presidential election, the report finds channels used to influence the election outcome. Similar to earlier newspaper articles as in The Guardian [11], the IRA is identified as a source of active interference.

Further, the Mueller report outlines specific practices of IRA agents to exercise influence in the form of political rallies while avoiding direct interactions. Steps to organize those include forming groups in social media that appear as large as possible using IRA controlled accounts and acquiring interested U.S. voters. Then, the IRA accounts would retreat from organizing such rallies leaving only the accumulated group of local citizens, which then proceeds to organize the rally. Considering these tactics in the context of the model in this paper, the IRA agents managed to organize segregation around a predefined core position through initially letting the number of supporters appear larger than it is. The result of this interference proved to be stable after the IRA agents retreated and in extension, the number of followers for the supported candidate Donald Trump appeared relatively larger. Depending on the topic, the opposition to positions of Hillary Clinton for her targeted voters would appear stronger thus discouraging supporters, through spawning cores that express dissatisfaction with the policies of Clinton.

In the onset of the election for the members of the European Parliament in 2019 similar interference was expected as expressed in *Online disinformation and the EU's response* [8] in 2019 and just a few months before the election. This document outlines threats, greatly focusing on the spread of disinformation.

In the context of my model, the found tactics correspond to letting parameters and preference distributions on key topics appear different than the true distribution to agents. Such

interference then would benefit an outcome which is in the interest of an external force rather than the interest of individuals in the form of an at least Pareto Efficient outcome.
Online disinformation and the EU's response also outlines measures, which can be undertaken to mitigate the effect and efficiency of interference strategies. Those include measures improving detection of misinformation, coordinating the response, and a 2018 Code of Practice on Disinformation, which is co-signed by social media corporations. The code of conduct urges the signatories to identify automated social media bots, deleting fake accounts, and cooperate with fact checkers. All of the suggested measures of the code target fighting any induced bias on how irregularly attractive or popular one political group may appear to voters.
Facebook has, following the code, installed a so-called "War Room" to counteract interference in the European elections of 2019 as presented in the official release *Fighting Election Interference in Real Time* [4] from 2018. This body consists of a team from various fields within Facebook and aims to mitigate efforts of interference. As the release states, the targeted scenarios are harassment and voter suppression, both of which are potential tactics to bias the perceived parameters in the introduced model.

7 Literature

7.1 Stability in Competition

Harold Hotelling employs a model in his 1929 work, *Stability in Competition* [12], featuring potential consumers distributed along the Hotelling-line which measures their individually assigned position similar to the preference space used in this paper. The agents of Hotelling choose one of two sellers who can choose their location on the same line. The buyers will always choose the closest one and if the distance is equal, they will randomize. The result in the form of a Nash equilibrium is that the two sellers will place themselves in the center on equal positions. Otherwise one could outperform any current outcome by taking a position marginally close to the other seller but facing towards the side of the center of the line.
Interpreting this as a segregation pattern in my model, it would best compare to a setup with two cores and endogenous core placement. then it would make sense for a core to move towards the side where more members can be gained as compared to the number lost on the

other side. So as long as one core is not entirely enveloped by the sphere of influence of the other, then the cores would move apart which is against the intuition of the Hotelling paper. The only circumstances in which my model can replicate the outcome of Hotelling is when θ is such that both cores attract members from the entire Hotelling line. Both cores need to have the same member count as to avoid that one core is more attractive to the entire population. Also, the cores need to be placed in the population mean so by theoretically altering core position, one core cannot acquire more members. For a position planner of one of the cores, the placement in the center is therefore ideal. Any movement would mean to secure the fraction of the population on one side of the line at the higher cost of losing the entire other side and sharing the population density in-between the new position and the center-position still held by the other core.

In the sense of the model in this work, it only makes sense for producers to close distance to each other if they have a similarly big consumer base with the same reach over the entire population of potential buyers.

7.2 Echo Chambers

A 2018 paper, *Why Echo Chambers are Useful* [14] takes another approach to model behavior of agents in the context of segregation. For this, the authors design a cheap-talk game with groups, referred to as echo chambers. In the setup, the world has a certain state θ of which the players posses information with limited accuracy. Individuals set actions as a response to the state that affect themselves and others and have a bias b_i on what potential response they prefer. Within introduced echo chambers, players can give information as a signal to all members of the same chamber. The strategy on whether to be honest or lie about the information is influenced by individual bias, and the joint belief on the state of the world in the current chamber.

The individual payoff function used is:

$$u_i = -(a_i - b_i - \theta)^2 - \alpha \sum_{j \neq i} (a_j - b_i - \theta)^2$$

The formulation appears similar to the cost in my work; however, here payoff is accumulated over all members of the chamber with a discount factor for other players than the selected. In contrast to my work, the author focuses more on information rather than the segregation patterns.

For the information approach, every agent is initially endowed with a unit of information on the actual state of the world with certain accuracy. More information is accumulated within one echo chamber lets the agents respond more accurately and is therefore beneficial.

7.3 Residential Segregation

In his paper from 1971, *Dynamic models of segregation* [3], Thomas Schelling introduces a model of residential segregation based on neighborhood preferences. In his model, agents exist in two types and live in neighborhoods with a certain proportion of individuals of a different type. If this proportion passes a threshold, then the individual becomes discontented and might move to another neighborhood, which ultimately results in segregation into neighborhoods that are dominated by individuals of similar traits.

This concept is very close to selecting a public good with a core that is adapting to the preferences of its members, as in section 2.3. To illustrate I let two public good cores start close but not identical to each other around the position $\frac{1}{2}$ reflecting mixed neighborhoods and then let the core be moved endogenously adapting to the weighted average of member preferences. Again I consider a uniform distribution over the entire space and $\theta = \frac{1}{8}$. Each public good will extend outward from $\frac{1}{2}$ and adapt core positions since no members can be collected in the direction of the other core. As long as $\theta \geq \frac{1}{8}$ the core will ultimately extend to a reach of $\frac{1}{4}$ from its position in both directions. Since the other good blocks all density beyond $\frac{1}{2}$, the cores will take positions $\frac{1}{4}$ and $\frac{3}{4}$ representing segregation into two neighborhoods. The work of Schelling also stresses that even more moderate preferences for some agents will not prevent residential segregation. My model supports the result similarly. Agents or residents decide to contribute to one core as a representation of a neighborhood or move their contribution to another core which provides relatively bigger utility by being closer in its agreed action. One significant difference can arise if I allow for non-uniformly distributed preferences and generally more weight in the space around the center. This way if one core acquires a greater number of members close to the center, then the core will move closer to the bulk of agents through the endogenous placement at the density weighed preferences. Ultimately in my setup, it is perfectly possible for a θ high enough to observe unification in one single neighborhood with the core close to the center, given enough members of society with moderate preferences.

So in my model agents with smaller biases and roughly centered preferences about their neighbors can under assumed center-heavy distribution of preferences allow for one neighborhood to become less homogeneous with a wider reach. This, however, happens while forcing the other potential cores towards the extreme if not the entire space is covered.

7.4 Homophily

Marketing via Friends: Strategic Diffusion of Information in Social Networks with Homophily [5] investigates strategies for decision-makers interacting with social networks with certain properties. This work focuses on homophily, which is a measure of the degree of connections between similar agents. Depending on that degree, different strategies in marketing are more efficient than others in proliferating information about products from the point of view of a producer.

While this paper focuses on strategies depending on Network structure, I focus on setting up the network, resulting in segregation. In my setup, I introduced public goods as clusters that provide payoff by letting players interact if there is a connection. If we then see all clusters as a connection between similar players, I can derive a definition for the degree of homophily in my model. Under my specification only links between similar agents are present, but one agent is connected to all similar agents following the nature of a cluster.

My model setup allows for another layer to the strategies of the producers other than interacting with a network as a platform to proliferate information. A producer might plan to install a core which then collects members and can be used later to convey product information. An example of this is corporate Twitter activities in which corporations install clusters around a certain preference value. A network, of course, has a certain benefit depending on what proportion can be mobilized to ultimately purchase a product.

Modeling this in my model context requires the producer to place a core at the desired location $\tilde{x}$ and either naturally accumulate members or try to let the core appear more attractive in terms of θ. A commonly observed behaviour is posting humorous content or give away benefits against memberships which then has new members interact. Alternatively a producer may employ deceptive strategies, such as using bots to appear more popular which bares the risk of a potential scandal if found out.

8 Conclusion

When endogenizing network formation in the context of graph theory, a common approach is to define cost and benefit and let agents create a bond if the benefit outweighs the evaluated effort of creating and sustaining this bond. A prime example of this approach are network generating processes in computer science as in the paper *On a Network Creation Game* [7] from 2003.

Another approach is to not focus on the graph theoretic setup, but on the individual decision through emulating the mechanics from Hotelling's 1929 paper [12] and assume agents from a population will choose the option which is closest in a distance metric. The model in this paper relies greatly on the latter approach, while seeking a foundation for the payoff creating process in network theory.

A shortcoming of my model is predicting welfare-efficient states from only information about the distribution of agent preferences. One has to compare the social welfare of all possible Pareto efficient states which are sourced from the set of equilibria in a Nash equilibrium sense. The restrictions for states to be in equilibrium or to be Pareto efficient allows for potentially many possible outcomes to be compared, so analysis on efficiency is best conducted as a comparison between a number of specified cases of interest.

Generally my model can assess whether segregation in more or less and respectively bigger or smaller cores is beneficial in terms of welfare through comparison. Also, as in section 5.3, the intuition is supported that with greater attractivity of group size, fewer but bigger cores become more welfare efficient.

The model can be extended towards a foundation on how payoff and cost of a membership to one core are created in specific environments. For example, a model of how likely one core as a political faction is to be a relevant member of the government after an election can make sense. Ultimately only a faction that is part of the government may produce payoff for its members. Linear benefit from group size may then be changed against a measure to evaluate success probabilities based on group size and properties of neighbor cores. Also, the mode of the election and the way the government is then formed needs to be accounted in the new benefit creating function.

Another use of the model can be to predict the migration of voters or customers between political parties and producers due to product property decisions. For this, one can set up a discrete choice model with the relative size of the groups as one explanatory factor aside from a metric to account for distance of the offered core preference value. The result may be relevant to decide over parameters which can then increase the number of newly acquired costumers.

9 Appendix

2.2

Solving for boundaries:

$$\begin{aligned} u_i = \theta * F(b) - \theta * F(a) - c(a - \tilde{x}) = 0 \\ \theta b - \theta a - (a - \tilde{x})^2 = 0 \\ a^2 - 2\tilde{x}a + \theta a + \tilde{x}^2 - \theta b = 0 \\ a = \tilde{x} - \frac{\theta}{2} - \sqrt{\frac{\theta^2}{4} + \theta b - \theta \tilde{x}} \end{aligned}$$

I omitted the positive root since the more interesting value for a is the one far below $\tilde{x}$ which provides 0 utility. Similarly I only kept the positive root when formulating b

2.3

Optimal placement in continuous framework:

$$\begin{aligned}
min \int_a^b c(x-\tilde{x})f(x)dx \\
min \int_a^b (x-\tilde{x})^2 f(x)dx \\
min \int_a^b x^2 f(x)dx - \int_a^b 2x\tilde{x}f(x)dx + \int_a^b \tilde{x}^2 f(x)dx \\
-2\int_a^b xf(x)dx + 2\tilde{x}\int_a^b f(x)dx = 0 \\
\tilde{x} = \frac{1}{F(b)-F(a)}\int_a^b x * f(x)dx
\end{aligned}$$

3

In a potential Pareto improvement, the decreased distance implies less cost with at least the same benefit (x_i is now closer to the core):

$$c(x_i - \tilde{x}') \leq c(x_i - \tilde{x})$$

the positive second derivative of cost enforces for a greater distance (x_j is now farther to the core):

$$\frac{\partial c}{\partial \tilde{x}}(x_j - \tilde{x}') \geq \frac{\partial c}{\partial \tilde{x}}(x_j - \tilde{x})$$

Considering:

$$u_b \geq 0$$

then the cost derivative presents the increase in utility if one moves closer to the core and is always higher for further preference spots. Therefore if b has at least the same utility and the core moves further away then all spots in between will exhibit higher utility than before

5.3

Welfare for a single central core under uniform distribution while not touching the outer boundaries:

$$W = 4\theta^2 - \int_{\tilde{x}-d}^{\tilde{x}+d} (x-\tilde{x})^2 dx$$

Due to symmetry:

$$W = 4\theta^2 - 2\int_{\tilde{x}}^{\tilde{x}+d} (x-\tilde{x})^2 dx$$

with $\tilde{x} = \frac{1}{2}$ and $d = 2\theta$:

$$W = 4\theta^2 - 2(\frac{1}{3}(\frac{1}{2}+2\theta)^3 - \frac{1}{2}(\frac{1}{2}+2\theta)^2 + \frac{1}{4}(\frac{1}{2}+2\theta)) + \frac{1}{24}$$

The Welfare for a 2 core configuration with the cores placed efficiently at $\frac{1}{4}$ and $\frac{3}{4}$ while expanding over the entire space:

$$W = \frac{1}{2}\theta - \int_{0}^{\frac{1}{2}} (x-\frac{1}{4})^2 dx - \int_{\frac{1}{2}}^{1} (x-\frac{3}{4})^2 dx$$

For symmetry:

$$W = \frac{1}{2}\theta - 2\int_{0}^{\frac{1}{2}} (x-\frac{1}{4})^2 dx$$
$$W = \frac{1}{2}\theta - \frac{1}{48}$$

Setting the 2 welfare equations equal lets me derive corresponding values for θ when one configuration starts outperforming the other.

$$\frac{1}{2}\theta - \frac{1}{48} = 4\theta^2 - 2(\frac{1}{3}(\frac{1}{2}+2\theta)^3 - \frac{1}{2}(\frac{1}{2}+2\theta)^2 + \frac{1}{4}(\frac{1}{2}+2\theta)) + \frac{1}{24}$$
$$32\theta^3 - 24\theta^2 + 3\theta = -\frac{1}{8}$$

The real solutions are $\theta_{1,2,3} = \{-0.0327..., 0.2075..., 0.5752...\}$

Calculating for $\theta \geq \frac{1}{4}$ changes the welfare equation for the single core setup:

$$W = \theta - 2(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}) + \frac{1}{24}$$

The resulting θ from setting the welfare equations equal is $\theta = \frac{5}{24}$ implying that for all $\theta \geq \frac{1}{4}$ the single core configuration provides higher social welfare.

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English Abstract

Segregation can be observed in both consumer decisions and political affiliation through individuals in a group choosing similarly. When Individuals make decisions, an option appears more attractive depending on how many others have already chosen this very option even if the specific choice is noticeably different from the individually most intuitive. Assuming polarisation represented by an underlying preference distribution for individuals, I model the selection of actions as segregation into groups using a local public good to include positive effects from group size. In both, the U.S. presidential elections in 2016 and the election of the members of the European Parliament in 2019 concerns arose that foreign forces manipulated voter decisions. The model in this work provides intuitions on how the suspected tactics may tip an equilibrium as well as an assessment of different equilibria under segregation mechanics.

German Abstract

Segregation kann sowohl bei Verbraucherentscheidungen als auch bei Wahlentscheidungen beobachtet werden, wenn Individuen in einer Gruppe die gleiche Wahl treffen. Wenn Personen Entscheidungen treffen, erscheinen einige Optionen attraktiver, je nachdem, wie viele andere diese Option bereits gewählt haben, auch wenn sich die jeweilige Wahl teils deutlich von einer rein intuitiven unterscheidet. Diese Präferenzverteilung wird als Polarisierung angenommen und die Auswahl von Aktionen als Gruppen wird davon ausgehend als Segregation in Gruppen mit öffentlichen Gütern modelliert, um positive Effekte aus der Gruppengröße einzubeziehen. Sowohl bei den US-Präsidentschaftswahlen im Jahr 2016 als auch bei der Wahl der Mitglieder des Europäischen Parlaments im Jahr 2019 kam es zu Bedenken, dass ausländische Kräfte die Wahlentscheidungen manipulierten. Das Modell in dieser Arbeit liefert eine Intuition dazu, wie eine Taktik ein Gleichgewicht beeinflussen kann, sowie eine Analyse verschiedener Gleichgewichte unter Segregation.