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Abstract

Based on Archimedes' principle, we model and investigate the behaviour of immersed hyperelastic bodies. We derive an energy functional, and use the Direct Method of the calculus of variations to prove existence of minimizers of this energy functional. In the first part of the thesis, we will study the theoretical foundations, namely introduce the Direct Method, polyconvex, and quasiconvex materials and summarize the most important and well-known results. In the second part, we prove existence of minimizers in the case of Dirichlet boundary conditions and in the case of the specimen being tied to a fixed anchor by an elastic rope. Moreover, we examine the case, where the specimen can move freely, and give an existence result for slightly compressible materials. Lastly, we prove existence of local minima, regardless of the choice of density parameters.

Zusammenfassung

In Anlehnung an das Prinzip des Archimedes untersuchen wir das Verhalten von schwimmenden, hyperelastischen Körpern. Dazu leiten wir ein Energie-Funktional her, welches anschließend mit Hilfe der direkten Methode der Variationsrechnung minimiert wird. Im ersten Teil dieser Masterarbeit legen wir die mathematischen Grundlagen und studieren die direkte Methode, sowie polykonvexe und quasikonvexe Materialien. Im zweiten, angewandten, Teil beweisen wir die Existenz von Minima in diversen Gegebenheiten. Wir betrachten den Fall von Dirichlet Randbedingnungen, sowie die Situation, in der das Objekt mit einem elastischen Seil an einem fixen Punkt gebunden ist. Weiters untersuchen wir das Verhalten des Objekts, wenn sich dieses frei bewegen kann und geben ein Existenzresultat im Falle, dass das Objekt nur geringfügig kompressibel ist. Des Weiteren zeigen wir die Existenz eines lokalen Minimums, unabhängig von der Wahl der Dichte des Fluids und des Körpers.

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Contents

Introduction

Consider an object submerged in some fluid, e.g., think of a submarine under water, which, therefore, is subjected to two forces, gravity and buoyancy. We know by Archimedes' principle that the buoyancy is the weight of the displaced fluid. Thus, if the object is incompressible, we can simply compare the density of the fluid to the density of the gravity to determine, whether the object sinks, stays or floats.

If, however, the object is compressible, the situation changes entirely. Now, the volume of the displaced fluid depends on the current deformation. Although the gravitational force always stays the same, the buoyancy will change with the deformation. For example, if the object sinks, the water pressure rises, the volume of the body may decrease, which then leads to a decrease of buoyancy.

To examine the behaviour of the body, we derive an energy functional from physical considerations. We then minimize this energy functional by the Direct Method of the calculus of variations. We consider hyperelastic, and polyconvex materials. For these we try to give a comprehensive theoretical background, which we will rely on, when giving the existence results.

We will prove existence of minimizers in the case of Dirichlet boundary conditions and give a new existence result in the case, where the body is tied to a fixed anchor by an elastic rope. Furthermore, we consider the problem of a freely moving body, and introduce a material locking condition yielding an existence result for bodies rising to the surface. This material locking condition is already well-known in the literature (e.g. refer to [3]), but has been considered as additional side constraint so far, whereas here, it is crucially used to prove the result. At last, we will check the existence of local minimizers in the case of compressible bodies.

In Chapter 1, we introduce our main mathematical tool, the Direct Method, guaranteeing the existence of a minimizer of a coercive and lower semicontinuous functional. A prototypical existence result will be given. After having introduced the mathematical tools, in Chapter 2 we look at deformations, and how the specimen reacts to applied forces. This will also lead to the introduction of the Cauchy stress tensor and hyperelastic materials, where the Cauchy stress tensor can be obtained from a stored energy function. To ensure that the deformation energy is weakly lower semicontinuous, we define in Chapter 3 the notion of polyconvexity, a weaker form of convexity, which incorporates the minors of the deformation gradient. This chapter is dedicated to prove that polyconvexity is sufficient for weak lower semicontinuity, relying crucially on weak convergence of minors. A first general existence result is given here. In Chapter 4, we introduce an even weaker form of convexity, the so-called, quasiconvexity, which turns out to be also necessary for weak lower semicontinuity under polynomial growth. As the main tool, we work with Young measures, hint at open problems and current research. The last Chapter 5 is of theoretical nature and dedicated to examine the invertibility of deformations, which is ensured by the so-called Ciarlet-Nečas condition. Chapters 6 to 9 are of more applied nature and deal with specific problems. In Chapter 6, we derive the energy functional from physical considerations and prove that the terms, which model the applied forces, are weak lower semicontinuous. Then, in Chapter 7, we give an existence result for a classical Dirichlet problem and the case, where the specimen is fixed along an inner beam. In Chapter 8, we will consider the situation, where the specimen is tied to a fixed anchor with an elastic rope. This calls for checking a Poincaré-type inequality and eventually proving the existence of minimizers. The final Chapter 9 is dedicated to the case of a freely moving object, where the energy is not bounded from below. We illustrate the difficulties and give and existence result in the case of a slightly compressible specimen rising to the surface. In the general case, we prove the existence of local minima.

1. The Direct Method

In this section, we introduce the main tool to prove existence and uniqueness of minimizers: the Direct Method of the calculus of variations. After defining the main notions, namely coercivity and lower semicontinuity, and working out the central idea of the Direct Method, we will present a prototypical existence result. Moreover, this exemplary result will point out certain difficulties, which will also arise during the later sections of this thesis. The main sources of this chapter are [8], and [27], although we sometimes refer to [9] as well.

1.1. The Direct Method of the calculus of variations

In many physical problems, as including elasticity, one seeks to minimize a functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$, mapping from a complete metric space X into the extended real numbers, under given boundary values or other constraints (for a comprehensive overview of applications, see [27], Chap. I). The Direct Method guarantees the existence of minimizers, if $\mathcal{F}$ satisfies the following conditions:

- Coercivity: $\mathcal{F}$ is *coercive*, i.e., there exists $\Lambda \in \mathbb{R}$ such that

  $$\{y \in X \,:\, \mathcal{F}(y) \leq \Lambda\} \qquad \text{is nonempty and sequentially precompact,}$$

  i.e., the sequence $(y_j) \subset X$ with $\mathcal{F}(y_j) \leq \Lambda$ has a converging subsequence.

- Lower semicontinuity (l.c.s.): $\mathcal{F}$ is *lower semicontinuous*, i.e., for all sequences $(y_j) \subset X$ with $y_j \to y$ we have that

  $$\mathcal{F}(y) \leq \liminf \mathcal{F}(y_j).$$

Theorem 1.1.1 (The Direct Method)**.** *If $\mathcal{F}$ is coercive and lower semicontinuous, then the minimization problem*

$$\textit{Minimize } \mathcal{F} \textit{ over } X$$

has at least one solution, i.e., there is a $y^* \in X$ *with* $\mathcal{F}(y^*) = \min_{y\in X} \mathcal{F}(y)$*.*

Proof. Assume that there is a y such that $\mathcal{F}(y) < \infty$, as otherwise all $y \in X$ would be a solution. By the definition of the infimum there is a minimizing sequence $(y_j) \subset X$, i.e., $\lim_{j\to\infty} \mathcal{F}(y_j) = \alpha := \inf_{y\in X} \mathcal{F}(y) < \infty$. Therefore, $\alpha \leq \Lambda \in \mathbb{R}$ and $\mathcal{F}(y_j) \leq \Lambda$ for all j, which in turn implies, by coercivity, the existence of a subsequence (not extra relabelled) and a $y^* \in X$, with

$$y_j \to y^*.$$

Thus, by the lower semicontinuity, we conclude

$$\alpha \leq \mathcal{F}(y^*) \leq \liminf_{j\to\infty} \mathcal{F}(y_j) = \alpha.$$

Therefore, $\mathcal{F}(y^*) = \alpha$ and y^* is a minimizer. □

Remark 1.1.1. We defined the notions above and proved the Direct Method in a sequential form, which is not necessary, but will be in accordance to the rest of the thesis. One could, however, do this in arbitrary topological spaces, with only slight modifications, as long as it is ensured, that the coercivity and the lower semicontinuity are with respect to the same topology. To find a suitable topology is here the main issue, as these two notions oppose each other (to establish coercivity one would prefer a coarse topology, whereas verifying lower semicontinuity is easier in a fine topology). As a rule of thumb, the weak topology is a good candidate, if X is a Banach space. For more on this topic, see [27], Sect. 2.1.

The gist of the Direct Method is that one has to prove coercivity and lower semicontinuity, which can be done separately. We will see that coercivity for functionals of the form $\mathcal{F}(y) = \int f(y(x))dx$ is closely related to certain growth-conditions on f, whereas lower semicontinuity is connected with some convexity of the integrand f, which are entirely different assumptions. Thus, one can develop different tools for each of the problems, independent of the fact, that one actually wants to minimizes a functional. Note, that by the Direct Method, coercivity and lower semicontinuity are sufficient, but not necessary for the existence of minimizers. Still non of these conditions can be dispensed of.

Example 1.1.1 (Not coercive). Consider the exponential function $exp : \mathbb{R} \to \mathbb{R}$, having no minimizer. It is bounded from below by the x-axis, but it is clearly not coercive, as for all $\Lambda > 0$, there is a sequence x_n with $\exp(x_n) \leq \Lambda$ with no converging subsequence (e.g. take $x_n := -n + \ln \Lambda$, $n \in \mathbb{N}$). △

Intuitively speaking, coercivity means that the function-values are "large", if the argument is "far" outside. This is obviously violated by the example above.

Example 1.1.2 (Not lower semicontinuous)*.* Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ x+1 & \text{if} \quad x \geq 0 \end{cases}$$

This function is not lower semicontinuous in $x_0 = 0$, since for an infimizing sequence $x_n = -\frac{1}{n} \to 0$, we have

$$0 = \liminf_{n\to\infty} f(x_n) \not\geq f(0) = 1.$$

Hence, although we have an infimizing sequence, the function value at point, where the minimum "should" be, is not minimal. △

We give two examples of existence of minimizers where the Direct Method can not be applied.

Example 1.1.3*.* Consider the function $f(x) = x^2 \exp(-x^2)$. Clearly, f is not coercive, but still has a minimum at the origin. △

Example 1.1.4*.* The function

$$f(x) = \mathbb{1}_{\mathbb{R}\backslash\mathbb{Q}}(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{Q} \\ 1 & \text{if} \quad x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

has a minimum at each $x \in \mathbb{Q}$, but is not lower semicontinuous. △

In Banach spaces, which are dual to Banach spaces, we know that bounded sets are sequentially pre-compact in the *weak topology* by the Banach-Alaoglu theorem (for more details, see Appendix, Rem A.4.1). For coercivity, we demand that the set of points in the space with bounded energy, is sequentially precompact. Therefore, if the space X is the dual of a Banach space, we coercivity follows by verifying that this set is bounded in norm. This is in general easier. In this thesis, we always will verify that the set $\{y : \mathcal{F}(y) \leq \Lambda\}$ is bounded in norm. Obviously, this implies, that we also need a weak form of sequential lower semicontinuity to apply the Direct Method.

We mainly work with the Sobolev space $W^{1,p}$, which is the dual of a Banach space for $1 < p < \infty$ (for the definition of Sobolev spaces and the most important results, see Appendix, Sec. A.2). Thus, to establish coercivity, one seeks to bound the minimizing sequence in the $W^{1,p}$-norm. In our applications this can usually be done, having the physical background of the problem in mind.

We will now state the Direct Method, for the weak form of coercivity and lower semicontinuity, which will used for the rest of the thesis.

Theorem 1.1.2 (Direct Method)**.** *Let X be a Banach space or a closed affine subset of a Banach space and let $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$, satisfying*

(i) Weak coercivity of $\mathcal{F}$: for some $\Lambda \in \mathbb{R}$ the sublevel set

$$\{y \in X \,:\, \mathcal{F}(y) \leq \Lambda\} \text{ is nonempty sequentially weakly precompact,}$$

i.e., the sequence $(y_j) \subset X$ with $\mathcal{F}(y_j) \leq \Lambda$ has a weakly convergent subsequence, and

(ii) Weak lower semicontinuity of $\mathcal{F}$: If a sequence $(y_j) \subset X$ is weakly convergent to $y \in X$, $y_j \rightharpoonup y$, then

$$\mathcal{F}(y) \leq \liminf_{j \to \infty} \mathcal{F}(y_j).$$

Then, there is a minimizer y^* of $\mathcal{F}$.

The proof is analogous to the one of Theorem 1.1.1. Often, we minimize over a set of functions, which also should satisfy some additional side constraints. Therefore, we also have to ensure that the minimizer fulfils these side constraints, which requires some kind of closure of the side constraints and has to be proven extra.

1.2. The Direct Method in the scalar case

In this thesis we only consider functionals of the form

$$\mathcal{F}(y) = \int_\Omega f(x, y(x), \nabla y(x))dx,$$

where $y \in W^{1,p}(\Omega)$ and $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary.

In order to highlight the most important ideas and difficulties arising when applying the Direct Method, we present at general (but clearly not the most general) version of an existence theorem (cf. [9], Thm. 3.3), and prove it by the Direct Method. Before formulating the theorem, we need to introduce the some important notions and results.

As we already mentioned, we bound the minimizing sequence in terms of the $W^{1,p}$-norm. The main tool to do so is the Poincaré inequality (cf. [9], Thm. 1.47).

Theorem 1.2.1 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, Lipschitz set and $1 \leq p \leq \infty$. Then, there is a $\gamma = \gamma(\Omega, p) > 0$ such that*

$$\|y\|_{L^p} \leq \gamma \|\nabla y\|_{L^p}, \qquad \forall y \in W_0^{1,p}(\Omega),$$

or, equivalently, a $\tilde{\gamma}$ such that

$$\|y\|_{W^{1,p}} \leq \tilde{\gamma} \|\nabla y\|_{L^p}, \qquad \forall y \in W_0^{1,p}(\Omega).$$

Moreover, we need weak the lower semicontinuity of $\mathcal{F}$. The main ingredient of establishing this is convexity of the integrand f.

Definition 1.2.1. (i) A set $\Omega \subset \mathbb{R}^n$ is called *convex*, if for every $x, y \in \Omega$ and every $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda) y \in \Omega$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be convex. The function $f : \Omega \to \mathbb{R}$ is called *convex*, if for every $x, y \in \Omega$ and every $\lambda \in [0, 1]$, the following inequality holds

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

Theorem 1.2.2. *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathscr{C}^1(\Omega)$. Then the function f is convex, if and only if*

$$f(y) \geq f(x) + \langle \nabla f(y), y - x \rangle \quad \forall x, y \in \mathbb{R}^n. \tag{1.2.1}$$

Proof. If f is convex, we have after rewriting the definition

$$f(y + \lambda(x - y)) \leq f(y) + \lambda \left(f(x) - f(y) \right) \quad \forall x, y \in \Omega \, \forall \lambda \in [0, 1].$$

Thus, we get

$$\frac{f\left(y + \lambda(x - y)\right) - f(y)}{\lambda} \leq f(x) - f(y),$$

which yields

$$\nabla f^T(y)(x - y) \leq f(x) - f(y)$$

after letting $\lambda \to 0$.

Assume that (1.2.1) holds for all $x, y \in \Omega$ and take $\lambda \in [0, 1]$ arbitrary. Since Ω is convex $z = \lambda x + (1 - \lambda) y$ belongs to Ω. We have

$$f(x) \geq f(z) + \nabla f^T(z)(x - z) \tag{1.2.2}$$

$$f(y) \geq f(z) + \nabla f^T(z)(y - z). \tag{1.2.3}$$

Multiplying (1.2.2) by λ, (1.2.3) by $(1-\lambda)$, and adding, we obtain

$$\begin{aligned}\lambda f(x)+(1-\lambda)f(y) &\geq f(z)+\nabla f^T(z)\underbrace{(\lambda x+(1-\lambda)y-z)}_{=0}\\ &= f(z)=f(\lambda x+(1-\lambda)y).\end{aligned}$$

□

Now, we state and prove the prototypical result providing the existence of minimizers in the scalar case (cf. [9], Sec. 3.3). We highlight all important notions by underlining them.

Theorem 1.2.3. *Let $\Omega\subset\mathbb{R}^n$ be bounded, and open with a Lipschitz boundary and let $f\in\mathscr{C}^1(\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n)$ such that*

H1+ Convexity: $(y,F)\mapsto f(x,y,F)$ is convex for all x

H2+ Coercivity: $\exists p>1, a_1>0, a_3\in\mathbb{R}:\quad f(x,y,F)\geq a_1|F|^p+a_3$

H3+ $\exists b\geq 0$ such that for all (x,y,F)

$$|f_y(x,y,F)|,|f_F(x,y,F)|\leq b(1+|y|^{p-1}+|F|^{p-1}).$$

Let

$$m:=\inf\left\{\mathcal{F}(y)=\int_\Omega f(x,y(x),\nabla y(x))dx\;:\;y\in y_0+W_0^{1,p}(\Omega)\right\}$$

for a given $y_0\in W^{1,p}(\Omega)$ with $\mathcal{F}(y_0)<\infty$. Then, there is a minimizer y^* of $\mathcal{F}$.

Proof. Step 1: Coercivity

By assumption, we have that $-\infty\overset{H2+}{<}m\leq I(u_0)\overset{assump.}{<}\infty$. Thus, there exists a minimizing sequence $(y_n)\subset y_0+W_0^{1,p}$, i.e. $\mathcal{F}(y_n)\to\inf\mathcal{F}(y)$. By the coercivity condition [H2+] one can find n large enough such that

$$m+1\geq\mathcal{F}(y_n)\overset{H2+}{\geq}a_1\|\nabla y_n\|_{L^p}^p-|a_3|\text{meas}(\Omega).$$

Therefore, there exists $a_4>0$ such that $\|\nabla y_n\|_{L^p}\leq a_4$. By employing the Poincaré inequality, we infer that there exist $a_5,a_6>0$ such that

$$a_5\|y_n\|_{W^{1,p}}-a_6\|y_0\|_{W^{1,p}}\leq\|\nabla y_n\|_{L^p}\leq a_4,$$

so that

$$\|y_n\|_{W^{1,p}}\leq a_7,$$

for some $a_7 > 0$. Hence, we have shown that (y_n) is bounded in the separable and reflexive Banach space $W^{1,p}$. Therefore, there exists a $y^* \in y_0 + W_0^{1,p}$ and a (not-relabelled) subsequence (y_n) such that

$$y_n \rightharpoonup y^* \text{ in } W^{1,p},$$

which verifies that $\mathcal{F}$ indeed is coercive in the sense of Definition 1.1.

Step 2: Lower semicontinuity

We will show that $y_n \rightharpoonup y^*$ in $W^{1,p}$ implies $\liminf \mathcal{F}(y_n) \geq \mathcal{F}(y^*)$.

Since f is convex and $f \in \mathscr{C}^1$, Theorem 1.2.2 is applicable, and thus,

$$f(x, y_n, \nabla y_n) \geq f(x, y^*, \nabla y^*) + f_u(x, y^*, \nabla y^*) \underbrace{(y_n - y^*)}_{\in L^p} + \langle f_F(x, y^*, \nabla y^*), \underbrace{\nabla y_n - \nabla y^*}_{\in L^p} \rangle.$$

Integrating the above relation yields

$$\mathcal{F}(y_n) \geq \mathcal{F}(y^*) + \int_\Omega f_y(x, y, \nabla y^*)(y_n - y^*) dx + \int_\Omega \langle f_F(x, y^*, \nabla y^*), \nabla y_n - \nabla y^* \rangle dx.$$

We now would like to invoke the weak convergence of $y_n \rightharpoonup y^*$, to deduce $\liminf \mathcal{F}(y_n) \geq \mathcal{F}(y^*)$, but we need to check that the integrals above are defined. Therefore, we need to verify

1. $f_y, f_F \in L^{p'}$, where p' is the Hölder conjugate of p, and use this to prove
2. $f_y \cdot (y_n - y^*), \langle f_F, \nabla y_n - \nabla y^* \rangle \in L^1$.

Let us prove $f_y(x, y^*, \nabla y^*) \in L^{p'}$, as the other statement analogously follows. To do so, we use the growth estimate [H3+] and that $y^* \in W^{1,p}$ to obtain

$$\int_\Omega |f_y(x, y^*, \nabla y^*)|^{p'} dx \leq b \int \left(1 + |y^*|^{p-1} + |\nabla y^*|^{p-1}\right)^{\frac{p}{p-1}} dx \leq b_1(1 + \|y^*\|^p_{W^{1,p}}) < \infty.$$

The second assertion follows directly from Hölder's inequality (see A.1.4).

Step 3: Combining steps 1 and 2

Now are we in the position to apply the Direct Method as in Thm. 1.1.1. Since (y_n) is a minimizing sequence, i.e. $\mathcal{F}(y_n) \to m = \inf \mathcal{F}(y)$ and we have established lower semicontinuity, we deduce $\liminf \mathcal{F}(y_n) \geq \mathcal{F}(y^*)$ and thus $\mathcal{F}(y^*) = m$. □

Remark 1.2.1. We know want to summarize the proof above and recapitulate.

- To prove coercivity we use growth conditions and the Poincaré inequality.

- To prove the l.s.c. we use convexity, weak convergence of the minimizing sequence, and a bound on f.

In the following, all of these notions will reappear in a more general setting. We need to find appropriate generalizations of convexity, weak enough to be applicable to a large class of functions, but still strong enough to allow us to infer weak lower semicontinuity. Also, the classical Poincaré inequality, as used in this proof, requires given boundary values. We will need to generalize this later on for such a boundary condition will not be available in some of the problems attacked in later sections.

1.3. Weak lower semi-continuity - A first result

As mentioned, one problem in the vector-valued case is to establish a form of lower semicontinuity using a proper notion of convexity. We conclude this section by presenting a result for vector valued functions $y : \Omega \to \mathbb{R}^m$ which indicates the difficulties occurring in the vector-valued case. To make our life easier at first, we assume that our integrand only depends on the Jacobian of y, i.e. is of the form

$$\mathcal{F}(y) = \int_\Omega f(x, \nabla y(x))dx.$$

Eventually, it is our goal to minimize a functional $\mathcal{F}(y) := \int_\Omega f(x, y(x), \nabla y(x))dx$ with certain additional conditions and $y \in W^{1,p}(\Omega, \mathbb{R}^d)$. To be able to treat this mathematically we will introduce so called Carathéodory functions (cf. [8], Definition 3.5).

Definition 1.3.1. The function

$$f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$$

is called a *Carathéodory function*, if

(i) $x \mapsto f(x, y, z)$ is measurable for all (y, z) and
(ii) $(y, z) \mapsto f(x, y, z)$ is continuous $\forall_{aa} x \in \Omega$.

Here we abbreviate "for almost all x" with the symbol $\forall_{aa}$.

The statement is due to Tonelli and Serrin and can be found in [27], Theorem 2.6.

Theorem 1.3.1. *Let $f : \Omega \times \mathbb{R}^{m\times d} \to [0, \infty)$ be a Carathéodory integrand such that $f(x, \cdot)$ is convex for almost all $x \in \Omega$. Then $\mathcal{F}(y) = \int_\Omega f(x, \nabla y(x))dx$ is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$ for any $p \in (1, \infty)$.*

Proof. The proof consist of two steps, in the first step we establish strong l.s.c., which we will use in the second step to conclude weak l.s.c.

Step 1: Claim: $\mathcal{F}$ is (strongly) lower semicontinuous.

Let $y_k \to y$ in $W^{1,p}$. By the Rellich-Kondrachov theorem, see Thm. A.2.3, we can select a subsequence (nonrelabelled) such that $\nabla y_k \to \nabla y$ almost everywhere. By assumption $M \mapsto f(x, M)$ is continuous, and thus, $f(x, \nabla y_k) \to f(x, \nabla y)$ a.e. Furthermore, we assumed f to be non-negative, and therefore, Fatou's lemma, Thm. A.1.1, is applicable, and we get

$$\mathcal{F}(y) = \int f(x, \nabla y) = \int \liminf f(x, \nabla y_k) \leq \liminf \int f(x, \nabla y_k).$$

Note that we passed to a subsequence, and hence, have proved the inequality above only for this subsequence. Fortunately, this does not spoil the argument, as is shown in proposition A.6.1.

Step 2: Claim: $\mathcal{F}$ is weakly l.s.c.

Let $y_n \rightharpoonup y$ in $W^{1,p}$. We need to show $\mathcal{F}(y) \leq \liminf \mathcal{F}(y_n) =: \alpha$. Take a subsequence (nonrelabelled) realizing the lim inf, i.e. $\mathcal{F}(y_n) \to \alpha$. By Mazur's lemma A.4.11 there are convex combinations

$$v_n = \sum_{k=n}^{N_n} \lambda_{k,n} y_k, \qquad \sum_{k=n}^{N_n} \lambda_{k,n} = 1,$$

such that $v_n \to u$ strongly in $W^{1,p}$. By assumption we know that $f(x, \cdot)$ is convex. Therefore, we can apply Jensen's inequality 8.1.2 to obtain

$$\mathcal{F}(v_n) = \int f\left(x, \sum_{k=n}^{N_n} \lambda_{k,n} \nabla y_k\right) dx \leq \int \sum_{k=n}^{N_n} \lambda_{k,n} f(x, \nabla y_k) dx = \sum_{k=n}^{N_n} \lambda_{k,n} \mathcal{F}(y_k).$$

Since $\mathcal{F}(y_k) \to \alpha$ as $k \to \infty$ and $\sum_{k=n}^{N_n} \lambda_{k,n} = 1$ for all n, passing to the limit yields

$$\liminf_n \mathcal{F}(v_n) \leq \alpha.$$

Because we have that $v_n \to u$ strongly we can use the Step 1 to obtain $\mathcal{F}(y) \leq \liminf_n \mathcal{F}(v_n)$ and finally

$$\mathcal{F}(y) \leq \liminf_n \mathcal{F}(v_n) \leq \alpha = \liminf_n \mathcal{F}(u_n),$$

which finishes the proof. $\square$

Remark 1.3.1. For integrands also depending on y the proof is more involved. The problem is that we still want to assume convexity in the last argument only. But then, we cannot “pull out” the sum coming from Mazur’s lemma

$$\int f\left(x, \sum \lambda_{k,n} y_k, \sum \lambda_{k,n} \nabla y_k\right).$$

Quite a bit of measure theory is needed in order to do an elementary proof (cf. [8], Chap. 3.2.6). A way to circumvent this is to introduce Young measures, which we will do in chapter 4.

In the scalar case, one can show that convexity is also a necessary condition for weak lower semicontinuity.

Theorem 1.3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded set and $\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$, $p \in [1, \infty)$ be an integral functional with continuous integrand $f : \mathbb{R}^{m \times d} \to \mathbb{R}$ (not x-dependent). If $\mathcal{F}$ is weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$, and if either $d = 1$ or $m = 1$ (i.e. the one-dimensional case, and the scalar case), then f is convex.*

For the proof see [27], Prop. 2.9. At this point, we only want to mention, that if $d, m \neq 1$, convexity is not necessary for weak lower semicontinuity. The naturally arising question is, what is the right weaker notion of convexity, which preserve this result in higher dimensions. In fact, it turns out that the appropriate condition is quasiconvexity, a weaker form of convexity, which we will examine in chapter 4.

2. Elements of continuum mechanics of solids

In the previous section, we discussed the minimization of an energy functional. The energy functional, which is considered later on in this thesis, depends on the actual deformation of a solid. Therefore, we will denote this chapter to introduce the most important notions to describe deformations and examine how the material react to applied forces. The central definition is that of the Cauchy stress tensor, which is used to describe the internal stress of a specimen in relation to the external applied forces. Furthermore, elastic materials will be introduced, with the focus on hyperelasticity, where the internal stresses can be described via a energy function. We follow [7] and [17].

2.1. Deformations

Let us consider a bounded domain (:= open and connected) $\Omega \subset \mathbb{R}^d$, where we assume that either $d = 2$ or $d = 3$, and that $\mathbb{R}^d$ is equipped with a right-handed orthonormal bases $e_1, \dots, e_d$. Furthermore, we assume the boundary $\Gamma := \partial\Omega$ of Ω to be smooth enough, and specify this wherever needed.

The closure $\bar{\Omega}$ describes the body before it is deformed, and is therefore called *reference configuration*. A point $x \in \bar{\Omega}$ in the reference configuration is called *material point*.

A *deformation* of $\bar{\Omega}$ is a mapping $y : \bar{\Omega} \to \mathbb{R}^d$ that is smooth enough, injective except possibly on the boundary of Ω, and orientation-preserving, see Fig. 2.1. The reason why we exclude injectivity at the boundary is that we allow self-contact at the boundary. We denote the *deformed configuration* by $y(\bar{\Omega})$ and introduce the notation

$$x^y := y(x), \qquad \bar{\Omega}^y := y(\bar{\Omega})$$

to distinguish between material points and *spatial points* $x^y \in \bar{\Omega}^y$. Throughout the whole thesis we will stick to this convention and mark quantities defined in the deformed configuration with a superscript y. The description in terms of material coordinates

$x \in \bar{\Omega}$ is called *Lagrangian description*, the description in terms of spatial coordinates is called *Eulerian description*.

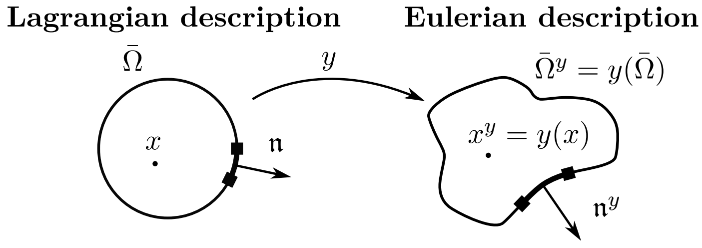


Figure 2.1.: Deformation

A central notion is the *deformation gradient*, defined as the Jacobian of y,

$$\nabla y(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} y_1 & \cdots & \frac{\partial}{\partial x_d} y_1 \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} y_d & \cdots & \frac{\partial}{\partial x_d} y_d \end{pmatrix}.$$

Orientation preservation is the condition

$$\det \nabla y(x) > 0 \quad \text{for all } x \in \bar{\Omega},$$

if y is smooth enough. Under this condition ∇y is invertible. It's sometimes convenient to introduce the *displacement* as $u : \bar{\Omega} \to \mathbb{R}^d$

$$u(x) := y(x) - x,$$

with *displacement gradient*

$$\nabla u(x) = \nabla y(x) - \mathbb{I},$$

where $\mathbb{I}$ denotes the identity matrix.

2.2. The Piola transform

We have now seen two configurations of the specimen: the deformed configuration and the reference configuration. Later on, when dealing with applied forces we will work in the deformed configuration, whereas working in the reference configuration is often more convenient, as it is a fixed domain. Therefore, we will need a tool to transform quantities form one configuration into the other. This is the Piola transform.

Definition 2.2.1. If $T^y(x^y)$ denotes a tensor field over $y(\bar{\Omega})$, then its *Piola transform* is the matrix-valued map $T : \bar{\Omega} \to \mathbb{R}^{d\times d}$ defined by

$$T(x) := (\det \nabla y(x)) T^y(x^y) \, (\nabla y(x))^{-T} = T^y(x^y) \operatorname{Cof}(\nabla y(x)).$$

Here, the cofactor matrix and the determinant are defined as usual. They are particular examples of minors, whose general definition can be found e.g. in [27], Sec. 5.2, or [8], Sec. 5.4.

We need the divergence of a matrix-valued map is used, which is defined as follows.

Definition 2.2.2. If $M : \Omega \to \mathbb{R}^{d\times d}$ is a smooth matrix-valued map, then the *divergence of M*, denoted by $\operatorname{div}(M)$, is a vector, defined by

$$(\operatorname{div}(M))_i := \sum_{j=1}^{d} \partial_{x_j} M_{ij}. \tag{2.2.1}$$

Let's have a look at properties of the Piola transform which will be used later on.

Lemma 2.2.1 (Piola's identity)**.** *If $y \in \mathscr{C}^2(\bar{\Omega};\mathbb{R}^d)$, then for all $x \in \bar{\Omega}$ we have*

$$\operatorname{div}\left(\operatorname{Cof}(\nabla y(x))\right) = 0.$$

Proof. We only prove this in the three-dimensional case, as for $d = 2$ its even simpler. Note that we can write

$$\operatorname{Cof}(\nabla y)_{ij} = \partial_{i+1} y_{j+1} \partial_{i+2} y_{j+2} - \partial_{i+1} y_{j+2} \partial_{i+2} y_{j+1},$$

if we count the indices modulo 3, i.e. $4 \mapsto 1, 5 \mapsto 2$. We abbreviated the partial derivative w.r.t. the j-th coordinate by writing $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$.

Then, formula (2.2.1) yields the claim. □

Theorem 2.2.2 (Properties of the Piola transform)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $y \in \mathscr{C}^2(\bar{\Omega};\mathbb{R}^d)$ be injective, and let $T : \bar{\Omega} \to \mathbb{R}^{d\times d}$ be the Piola transform of a tensor field $T^y \in \mathscr{C}(y(\bar{\Omega});\mathbb{R}^{d\times d})$. Then*

(i) $\operatorname{div} T(x) = (\det \nabla y(x)) \operatorname{div}^y T^y(x^y)$ *for all $x^y = y(x)$, $x \in \bar{\Omega}$, and*

(ii) For all subsets $\omega \subset \bar{\Omega}$ with smooth boundary we have

$$\int_{\partial\omega} T(x)\mathfrak{n} dS = \int_{\partial\omega^y} T^y(x^y)\mathfrak{n}^y dS^y, \tag{2.2.2}$$

where $\mathfrak{n}$ and $\mathfrak{n}^y$ are outer unit normals to $\partial\omega$ and $\partial\omega^y$, respectively. In particular, the area elements dS and dS^y at the points $x \in \partial\Omega$ and $x^y \in \partial\Omega^y$ are related by setting $T^y = \mathrm{id}$ in equation (2.2.2)*, i.e.*

$$\det \nabla y(x) |\nabla y(x)^{-T}\mathfrak{n}| dS = |\operatorname{Cof}(\nabla y(x))\mathfrak{n}| dS = |\mathfrak{n}^y| dS^y = dS^y. \tag{2.2.3}$$

The proof of the first claim can be obtained by elementary yet trivial calculations, the one for the second claim is based on a change of variables and Gauß' divergence theorem. It can be found in [17], Thm. 1.1.9.

Remark 2.2.1 (Relating normal vectors)*.* The calculations above imply the following formula relating the normal vectors of the deformed configuration to the normal vectors of the reference configuration. If y is a deformation and $x \in \partial\Omega$, $x^y = y(x) \in \partial y(\Omega)$, then the following equation holds

$$\mathfrak{n}^y(x^y) = \frac{\operatorname{Cof} \nabla y(x)\mathfrak{n}(x)}{|\operatorname{Cof} \nabla y(x)\mathfrak{n}(x)|} = \frac{(\nabla y(x))^{-T})\mathfrak{n}(x)}{|(\nabla y(x))^{-T})\mathfrak{n}(x)|}. \tag{2.2.4}$$

2.3. Volume, area, and length elements in the deformed configuration

The goal of this section is to give a correspondence between quantities defined in the deformed configuration and quantities defined in the reference configuration by employing the Piola transform. In particular, this will lead to the introduction of the strain.

If $y : \bar{\Omega} \to \mathbb{R}^d$ is a deformation and dx is volume element around x in the reference configuration, then the volume element dx^y in the deformed configuration is formally given by

$$dx^y = \det \nabla y(x) dx. \tag{2.3.1}$$

We can use this to calculate volumes of deformed regions. So, if $\omega \subset \bar{\Omega}$ is measurable, then the volumina of ω and of its deformation $\omega^y := y(\omega)$ are given by

$$\begin{aligned}
\operatorname{vol}(\omega) &:= \operatorname{meas}_d(\omega) = \int_\omega dx, \\
\operatorname{vol}(\omega^y) &:= \operatorname{meas}_d(\omega^y) = \int_{\omega^y} dx^y = \int_\omega \det \nabla y(x) dx.
\end{aligned}$$

The last identity is due to the change of variables formula A.3.5, and thus, only holds under certain assumptions. This is the reason why the relation (2.3.1) is only formal. For brevity we use the notation $|\cdot|$ to denote the d-dimensional Lebesgue-measure of a set, i.e. $|\omega| = \mathrm{meas}_d(\omega)$.

As we have seen in Thm. 2.2.2, the following correspondence holds for area elements (see also Fig. 2.1)

$$\det \nabla y(x) |\nabla y(x)^{-T} \mathfrak{n}| dS = |\operatorname{Cof} \nabla y(x) \mathfrak{n}| dS = |\mathfrak{n}^y| dS^y = dS^y.$$

Therefore, a measurable subset A of the boundary $\partial\Omega$ and its deformation $A^y := y(A)$ fulfil

$$\begin{aligned} \mathrm{area}(A) &:= \mathrm{meas}_{d-1}(A) = \int_A dS \\ \mathrm{area}(A^y) &:= \mathrm{meas}_{d-1}(A^y) = \int_{A^y} dS^y = \int_A (\det \nabla y) |\nabla y^{-T} \mathfrak{n}| dS = \int_A |\operatorname{Cof} \nabla y \mathfrak{n}| dS. \end{aligned}$$

We write dS to indicate that the integrals above are surface integrals.

Finally, considerations of how a length element is deformed under a sufficiently smooth deformation will lead to the introduction of strain tensors. If y is differentiable at a point $x \in \bar{\Omega}$, we can write for all points $x + x' \in \bar{\Omega}$ for a suitable $x' \in \Omega$

$$y(x') - y(x) = \nabla y(x)(x' - x) + o(|x' - x|),$$

and therefore,

$$|y(x') - y(x)|^2 = (x' - x)^T \nabla y(x)^T \nabla y(x)(x' - x) + o(|x' - x|).$$

Definition 2.3.1. The symmetric tensor

$$C := \nabla y^T \nabla y$$

is called the *right Cauchy-Green strain tensor*.

Since by assumption $\nabla y(x)$ is invertible, the quadratic form associated to the Cauchy-Green strain tensor is positive definite:

$$(\xi, \xi) \mapsto \xi^T C(x) \xi = |\nabla y(x) \xi^2| \geq 0, \qquad \forall \xi \neq 0.$$

This quadratic form appears when calculating the length of deformed curves. Let

$\gamma : I \to \bar{\Omega}$ be a curve in the reference configuration. The length of γ is given by

$$|\gamma| := \int_I |\gamma'(t)| dt = \int_I \left(\sum \gamma_i' \gamma_i'\right)^{1/2} dt.$$

Hence the length of the deformed curve $\gamma^y = y(\gamma)$ is

$$|\gamma^y| = \int_I |(y \circ \gamma)'(t)| dt = \int_I \left((\gamma')^T C(\gamma)\gamma'\right)^{1/2} dt.$$

The tensor C is indeed a measure of strain and can be used to measure how much the underlying deformation y differs from being only a rotation and translation.

Definition 2.3.2. A deformation $y : \bar{\Omega} \to \mathbb{R}^d$ is called *rigid*, if there is a $c \in \mathbb{R}^d$ and a $R \in SO(d)$ such that

$$y(x) = c + Rx,$$

i.e. it is only a rotation followed by a translation. We denote by $\mathrm{SO}(d) := \{A \in \mathrm{GL}(d) : A^T A = AA^T = \mathbb{I}, \det(A) = 1\}$ the special orthogonal group.

Obviously, if y is rigid, then $\nabla y = R$ and hence $C = \nabla y^T \nabla y = \mathrm{id}$. The converse statement is also true: If $C = \mathrm{id}$ on $\bar{\Omega}$ and $\det \nabla y > 0$, then y is necessarily rigid (cf. [7], Thm. 1.8.-1). This tells us that y is rigid, if and only if $C = \mathrm{id}$, and therefore the tensor

$$E := \frac{1}{2}(C - \mathrm{id})$$

measures the "deviation" of y from being a rigid deformation. The tensor E is called *Green-Lagrange* or *Green-St. Vernant strain tensor*.

There is another remarkable property of the Cauchy-Green strain tensor: It completely determines the deformation up to composition with rigid motions.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and connected and assume that the two mappings $y, \tilde{y} \in \mathscr{C}^1(\Omega; \mathbb{R}^d)$ fulfil*

$$\nabla y(x)^T \nabla y(x) = \nabla \tilde{y}(x)^T \nabla \tilde{y}(x) \quad \forall x \in \Omega,$$

$\tilde{y}$ is injective, and $\det y(x) \neq 0$ for all $x \in \Omega$. Then, there is a vector $c \in \mathbb{R}^d$ and an orthogonal matrix $R \in O(d)$ such that

$$y(x) = c + R\tilde{y}(x) \quad \forall x \in \Omega.$$

For the proof refer to [7], Thm. 1.8-2.

Remark 2.3.1 (Summary). As we have seen, volume elements in the deformed and the reference configuration are related by the *determinant* of the deformation gradient, and surface elements are related by the *cofactor matrix*, both minors of ∇y. This indicates the importance of the behaviour of minors, which will reappear in the Chapter 3 in relation with polyconvexity.

2.4. Applied forces and the Cauchy stress tensor

Forces acting on the body cause stresses and deformations. In this section, we will elaborate on the applied forces and how the specimen will react on them.

We will consider two kind of applied forces

1. *applied body forces* defined through a force density $f^y : \bar{\Omega}^y \to \mathbb{R}^d$ per unit volume in the deformed configuration in the physical unit $\mathrm{N\,m^{-3}}$;

2. *applied surface forces* defined by $g^y : \Gamma_N^y \to \mathbb{R}^d$ on a measurable subset (w.r.t. the surface measure) $\Gamma_N^y \subset \Gamma^y$ as density per unit area in the physical unit $\mathrm{Pa} = \mathrm{N\,m^{-2}}$.

Under the following axioms due to Euler and Cauchy on can deduce the existence of the Cauchy stress tensor.

Axiom (Stress principle of Euler and Cauchy). *Let $\bar{\Omega}^y$ be the deformed configuration of a body subjected to applied forces represented by $f^y : \bar{\Omega}^y \to \mathbb{R}^d$ and $g^y : \Gamma_N^y \to \mathbb{R}^d$. Let $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ be the unit sphere. We assume the existence of a vector field*

$$t^y : \bar{\Omega}^y \times \mathbb{S}^{d-1} \to \mathbb{R}^d,$$

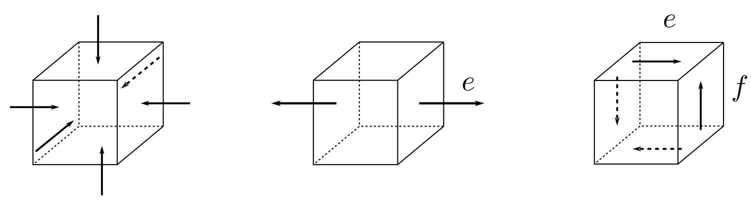
called Cauchy's stress vector*, such that*

1. *For any subdomain $\omega^y \subset \bar{\Omega}^y$ and any point $x^y \in \Gamma_N^y \cap \partial\omega^y$ where the joint outer unit vector $\mathfrak{n}^y$ exists, it holds*

$$t^y(x^y, \mathfrak{n}^y) = g^y(x^y).$$

2. *Axiom of balance of forces: For any subdomain $\omega^y \subset \bar{\Omega}^y$, it holds*

$$\int_{\omega^y} f^y(x^y)dx^y = \int_{\partial\omega^y} t^y(x^y, \mathfrak{n}^y)dS^y = 0.$$



(a) Isotropic pressure (b) Pure tension (c) Pure shear

Figure 2.2.: Examples of loads

3. *Axiom of balance of momenta: For any subdomain $\omega^y \subset \bar{\Omega}^y$ with the outer unit normal $\mathfrak{n}^y$, it holds*

$$\int_{\omega^y} x^y \times f^y(x^y)dx^y + \int_{\partial\omega^y} x^y \times t^y(x^y, \mathfrak{n}^y)dS^y = 0.$$

The axioms of balance of forces and momenta express that the deformed configuration is in static equilibrium.

Theorem 2.4.1 (Cauchy's theorem)**.** *Let $\Omega^y \subset \mathbb{R}^d$ be open and let the applied force density $f^y : \bar{\Omega}^y \to \mathbb{R}^d$ be continuous. Furthermore, let $t^y(\cdot, \mathfrak{n}) \in \mathscr{C}^1(\bar{\Omega}^y; \mathbb{R}^d)$ for every $\mathfrak{n} \in \mathbb{S}^{d-1}$ and $t^y(x^y, \cdot) \in \mathscr{C}(\mathbb{S}^{d-1}; \mathbb{R}^d)$ for any $x^y \in \bar{\Omega}^y$. Then, there is a symmetric tensor $T^y : \bar{\Omega}^y \to \mathbb{R}^{d\times d}$ belonging to $\mathscr{C}^1(\bar{\Omega}^y; \mathbb{R}^{d\times d})$ with*

$$\begin{aligned} t^y(x^y, \mathfrak{n}) &= T^y(x^y)\mathfrak{n} && \forall x^y \in \bar{\Omega}^y,\ \forall \mathfrak{n} \in \mathbb{S}^{d-1}, \\ -\operatorname{div} T^y(x^y) &= f^y(x^y) && \forall x^y \in \bar{\Omega}^y, \\ T^y(x^y)\mathfrak{n}^y &= g^y(x^y) && \forall x^y \in \Gamma_N^y, \end{aligned}$$

where $\mathfrak{n}^y$ is the outer unit normal to Γ_N^y. The tensor T^y is called Cauchy's stress tensor*.*

The proof relies on the axioms of Euler and Cauchy and can be found in [17], Thm. 1.2.2.

Example 2.4.1. Let us consider three basic examples, illustrated in Fig. 2.2, to understand the Cauchy stress tensor (cf. [7], Sect. 2.3).

First, for a $p \in \mathbb{R}$ we set

$$T^y(x^y) = -p\mathbb{I} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix},$$

where p is a *pressure*. Then, also $t^y(x^y, \mathfrak{n}^y) = -p\mathfrak{n}^y$, and thus, the Cauchy stress vector is always normal to the elementary surface element. This defines a *pressure load* on Ω^y.

Second, let $e \in \mathbb{R}^3$, with $|e| = 1$ be a unit vector, $\tau \in \mathbb{R}$, and recall the following notation $(a \otimes b)_{ij} = a_i b_j$. If we set

$$T^y(x^y) = \tau e \otimes e,$$

then the Cauchy stress tensor is called a *pure tension* if $\tau > 0$ or a *pure compression* if $\tau < 0$ in the direction e. For the stress vector we get $t^y(x^y, \mathfrak{n}^y) = T^y(x^y)\mathfrak{n}^y = \tau(e \cdot \mathfrak{n}^y)e$, which is always parallel to e and is directed outward (resp. inward) for $\tau > 0$ (resp. $\tau < 0$) on the faces with normals $\mathfrak{n}^y = \pm e$ and vanishes on the faces orthogonal to e. Furthermore, if we assume that $e = e_1$ equals the first basis vector, then T^y takes the form

$$T^y = \begin{pmatrix} \tau & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As third example, let $e, f \in \mathbb{R}^3$, both unit vectors, orthogonal to each other, $e \cdot f = 0$, and $\sigma \in \mathbb{R}$. Then

$$T^y(x^y) = \sigma(e \otimes f + f \otimes e)$$

is called *pure shear*, with *shear stress* σ relative to directions e and f. The Cauchy stress vector takes the form $t^y(x^y, \mathfrak{n}^y) = \sigma\left((e \cdot \mathfrak{n}^y)f + (f \cdot \mathfrak{n}^y)e\right)$. If we now assume that $e = e_1$ and $f = e_2$, then T^y reads as follows

$$T^y = \begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \triangle$$

Since the choice of the reference configuration is arbitrary, the description of the equilibrium should be independent of the chosen reference. Formally, this is described in the *axiom of frame-indifference* (or *frame-invariance*), which states that, if a deformation z of $\bar{\Omega}^y$ is $z(x) := Ry(x)$ for all $x \in \bar{\Omega}$ and some rotation $R \in SO(d)$, then for all $x \in \bar{\Omega}$ and any $\mathfrak{n} \in \mathbb{S}^{d-1}$, it holds that

$$t^z(x^z, R\mathfrak{n}) = Rt^y(x^y, \mathfrak{n}).$$

Since

$$t^z(x^z, R\mathfrak{n}) = T^z(x^z)R\mathfrak{n} = Rt^y(x^y, \mathfrak{n}) = RT^y(x^y)\mathfrak{n}$$

one immediatelly gets by setting $R\mathfrak{n} = \tilde{\mathfrak{n}} \in \mathbb{S}^{d-1}$ the identity

$$T^z(x^z) = RT^y(x^y)R^T.$$

2.5. Equilibrium equations and the principle of virtual work

Before we start, we clarify some notation. The simple dot denotes the Euclidean product $u \cdot v = \sum u_i v_i$ and the colon symbolizes the matrix inner product $A : B = \sum_{ij} A_{ij}B_{ij} = \operatorname{tr}\left(A^T B\right)$.

We need the following lemma, cf. [24] Sec. 3.1.2.

Lemma 2.5.1 (Green's formula)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded, measurable, and with Lipschitz boundary $\partial\Omega$. Furthermore, let $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ where $1/p + 1/q \leq (d+1)/d$, if $d > p \geq 1$, $d > q \geq 1$ with $q > 1$ if $p \geq d$, and with $p > 1$ if $q \geq d$. Then,*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = \int_{\partial\Omega} uv\mathfrak{n}_i dS - \int_\Omega u \frac{\partial v}{\partial x_i} dx, \tag{2.5.1}$$

where $\mathfrak{n} = (\mathfrak{n}_1, \ldots, \mathfrak{n}_d)$ is the exterior normal.

Using the formula above we can prove an analogous result for tensor fields.

Lemma 2.5.2 (Green's formula for tensor fields)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded, measurable, and with Lipschitz boundary $\partial\Omega$. Let $T \in W^{1,p}(\Omega; \mathbb{R}^{d\times d})$ be a tensor field and $v \in W^{1,q}(\Omega; \mathbb{R}^d)$ be a vector field where $1/p + 1/q \leq (d+1)/d$, if $d > p \geq 1$, $d > q \geq 1$ with $q > 1$ if $p \geq d$, and with $p > 1$ if $q \geq d$. Then the following relation holds*

$$\int_\Omega \operatorname{div} T \cdot v dx = -\int_\Omega T : \nabla v dx + \int_{\partial\Omega} (T\mathfrak{n}) \cdot v dx, \tag{2.5.2}$$

where $\mathfrak{n}$ is the exterior normal. Note that $\operatorname{div} T$ is a vector defined via

$$(\operatorname{div} T)_i := \sum_{j=1}^{d} \partial_j T_{ij}.$$

Proof. By the definition of vector-valued Sobolev functions we can apply (2.5.2) componentwise and obtain

$$\begin{aligned}
\int_\Omega \operatorname{div} T \cdot v dx &= \int \sum_i \left(\sum_j \partial_j T_{ij} v_i \right) dx = \sum_{ij} \int_\Omega \partial_j T_{ij} v_i \\
&\overset{(2.5.2)}{=} \sum_{ij} \int_{\partial\Omega} T_{ij} \mathfrak{n}_j v_i dS - \sum_{ij} \int_\Omega T_{ij} \partial_j v_i dx \\
&= \sum_i \int_{\partial\Omega} (T\mathfrak{n})_i v_i dS - \int_\Omega T : \nabla v dx.
\end{aligned}$$
□

Theorem 2.5.3 (Principle of virtual work in the deformed configuration)**.** *Let $\Omega^y \subset \mathbb{R}^d$ be bounded, measurable, and with Lipschitz boundary $\partial\Omega^y$. Let $T^y \in W^{1,p}(\Omega^y; \mathbb{R}^{d\times d})$ be a tensor field. Furthermore, let $f^y \in L^p(\Omega^y; \mathbb{R}^d)$ and $g^y \in L^p(\partial\Omega; \mathbb{R}^d)$. If T^y is a weak solution of the following equations*

$$\begin{aligned}
-\operatorname{div}^y T^y &= f^y \qquad \textit{in } \Omega^y, \\
T^y \mathfrak{n}^y &= g^y \qquad \textit{on } \Gamma_N^y,
\end{aligned} \tag{2.5.3}$$

i.e., satisfies the identity

$$\int_{\Omega^y} T^y : \nabla^y v dx^y = \int_{\Omega^y} f^y \cdot v dx^y + \int_{\Gamma_N^y} g^y \cdot v dS^y \tag{2.5.4}$$

for all vector fields $v \in W^{1,q}(\Omega; \mathbb{R}^d)$ where $1/p + 1/q \leq (d+1)/d$, if $d > p \geq 1$, $d > q \geq 1$ with $q > 1$ if $p \geq d$, and with $p > 1$ if $q \geq d$ with $v = 0$ on $\Gamma \backslash \Gamma_N^y$, then it also satisfies (2.5.3) *in a weak sense.*

Proof. Let v be as in the theorem. Then integrating the product of $\operatorname{div}^y T^y + f^y = 0$ with v over Ω^y yields

$$0 = \int_{\Omega^y} (\operatorname{div}^y T^y + f^y) \cdot v dx^y \overset{(2.5.2)}{=} \int_{\Omega^y} -T^y : \nabla^y v + f^y \cdot v dx^y + \int_{\Gamma_N^y} T^y \mathfrak{n}^y \cdot v dS^y,$$

which yields the integral identity after recalling that $T^y \mathfrak{n}^y = g^y$ on Γ_N^y. Conversely, (2.5.4) reduces to

$$\int_{\Omega^y} T^y : \nabla^y v dx^y = \int_{\Omega^y} f^y \cdot v dx^y,$$

if $v = 0$ on Γ^y, and hence by (2.5.2) we obtain $\operatorname{div}^y T^y + f^y = 0$. Using this equation and the Green's formula once again, the integral identity reduces to

$$\int_{\Gamma_N^y} T^y \mathfrak{n}^y \cdot v dS^y = \int_{\Gamma_N^y} g^y \cdot v dS^y,$$

which implies that the boundary condition $T^y \mathfrak{n}^y = g^y$ holds on Γ_N^y. □

Remark 2.5.1. To prove Thm. 2.5.3 we relied on Lemma 2.5.2. Its assumptions, however, are not realistic even for very regular bodies under regular loadings. Therefore, the principle of virtual work is sometimes stated in a formal way, see for instance [17], Thm. 1.2.4, [7], Thm. 2.4-1.

Definition 2.5.1. The equations

$$\begin{aligned} -\operatorname{div}^y T^y &= f^y && \text{in } \Omega^y, \\ T^y \mathfrak{n}^y &= g^y && \text{on } \Gamma_N^y, \\ T^y &= (T^y)^T && \text{in } \Omega^y \end{aligned}$$

are called the *equations of equilibrium in the deformed configuration.* The variational

We now have formulated equilibrium equations in the deformed configuration, which is not known a priori, but part of the sought solution. To resolve this issue, we will rewrite these equations in Lagrangian variables. This can be done by mapping the Cauchy stress tensor to the reference configuration with the aid of the Piola transform. We will see that the boundary value in the reference configuration has the same form as in the deformed configuration.

Definition 2.5.2. We shall define the *1st Piola-Kirchhoff stress tensor* $S : \bar{\Omega} \to \mathbb{R}^d$ as the Piola transform of the Cauchy stress tensor T^y, i.e.

$$S(x) := T^y(x^y) \operatorname{Cof} \nabla y(x) = (\det \nabla y(x)) T^y(x^y) \nabla y(x)^{-T},$$

where $x^y = y(x)$, $x \in \bar{\Omega}$.

The properties of the Piola transform, Thm. 2.2.2, imply

$$\operatorname{div} S(x) = (\det \nabla y(x)) \operatorname{div}^y T^y(x^y).$$

This means that the equilibrium equations of the deformed configuration are still of "divergence structure" when being transformed into the equations over the reference configuration. Since this was the crucial ingredient to apply Green's formula and establish the equivalence between the variational formulation and the formulation as boundary value problem, we will have a similar result for the principle of virtual work in the reference configuration.

Note that, while the Cauchy stress tensor is symmetric, the 1^{st} Piola-Kirchhoff tensor is not symmetric in general. Therefore, one defines the *2^{nd} Piola-Kirchhoff stress tensor* Σ by

$$\Sigma(x) = \nabla y(x)^{-1} S(x) = (\nabla y(x))^{-1} T^y(x^y) \operatorname{Cof} \nabla y(x),$$

which is symmetric.

As we have already transformed the Cauchy stress tensor, we only need to transform the applied force densities to formulate the principle of virtual work in the reference configuration. Firstly, if we are given a body force density $f^y : \bar{\Omega}^y \to \mathbb{R}^d$ we are looking for a force density $f : \Omega \to \mathbb{R}^d$ such that for every subdomain $\omega \subset \Omega$ it holds that

$$\int_\omega f(x)dx = \int_{\omega^y} f^y(x^y)dx^y,$$

i.e. the total force acting on subsets of the specimen must be the same. Since we already have derived $dx^y = \det \nabla y(x)dx$, we obtain

$$f(x) = f^y(x^y) \det \nabla y(x). \tag{2.5.5}$$

Similarly, we have for mass densities $\rho : \Omega \to \mathbb{R}$ and $\rho^y : \Omega^y \to \mathbb{R}$ the following correspondence

$$\rho(x) = \rho^y(x^y) \det \nabla y(x). \tag{2.5.6}$$

Note that this implies

$$\int_\Omega \rho(x)dx = \int_{\Omega^y} \rho^y(x^y)dx^y,$$

and thus, that the total mass of the body is conserved.

Secondly, for a given surface force density $g^y : \Gamma_N^y \to \mathbb{R}^d$, we look for $g : \Gamma_N \to \mathbb{R}^d$, $y(\Gamma_N) =: \Gamma_N^y$, such that for all $\gamma \subset \Gamma_N$ we have that

$$\int_\gamma g(x)dS = \int_{\gamma^y} g^y(x^y)dS^y.$$

By the properties of the Piola transform, Thm. 2.2.2, and in particular the correspondence between area elements, we arrive at

$$g(x) = g^y(x^y)|\operatorname{Cof}\nabla y(x)\mathfrak{n}(x)|, \qquad x \in \Gamma_N. \tag{2.5.7}$$

The description of the forces in the reference configuration enables us to introduce a new notion.

Definition 2.5.3. An applied body force f^y is a *dead load*, if its associated density in the in reference configuration f is independent of the deformation y.

This is, for instance, the case of gravity field, for which the body force in the reference configuration is given by

$$f(x) = -g\rho(x)e_3 = (0, 0, -g\rho(x)).$$

Then, for the body force in the deformed configuration we have $f^y(x^y) = (0, 0, -g\rho^y(x^y))$. Analogously, an applied surface force is a dead load, if its associated density in the reference configuration is independent of the deformation y. Note that, applied forces are very rarely dead loads in reality, but instead the force densities f, g usually appear not only as functions of $x \in \Omega$, but also of the deformation itself. As an example consider the *pressure load*, where the surface force in the deformed configuration is given by

$$g^y(x^y) = -p\mathfrak{n}^y(x^y),$$

for a $x^y \in \Gamma_N^y$, and $p \in \mathbb{R}$, called *pressure*. The minus sign indicates that the vector g^y points inwards for $p > 0$. To show that the pressure load cannot be a dead load, recall the correspondence between g and g^y, equation (2.5.7), and the respective outer unit normals, equation (2.2.4), to derive

$$\begin{aligned} g(x) &= |\operatorname{Cof}\nabla y(x)\mathfrak{n}(x)|g^y(x^y) \\ &= -p|\operatorname{Cof}\nabla y(x)\mathfrak{n}(x)|\mathfrak{n}^y(x^y) = -p\operatorname{Cof}\nabla y(x)\mathfrak{n}(x) \\ &= -p(\det\nabla y(x))\nabla y(x)^{-T}\mathfrak{n}(x), \end{aligned} \tag{2.5.8}$$

for a $x \in \Gamma_N$. Thus, g takes the form

$$g(x) := \hat{g}(x, \nabla y(x)),$$

where the mapping $\hat{g} : \Gamma_N \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ is given by $\hat{g}(x, F) = -p\operatorname{Cof}(F)\mathfrak{n}(x)$.

Having all transformations at hand, we can formulate the principle of virtual work in the reference configuration.

Theorem 2.5.4 (Principle of virtual work in the reference configuration)**.** *Let* $\Omega \subset \mathbb{R}^d$ *be bounded, measurable, and with Lipschitz boundary* $\partial\Omega$*. If 1*st *Piola-Kirchhoff tensor* $S(x) = (\det \nabla y(x)) T^y(x^y) \nabla y(x)^{-T}$ *belong to* $W^{1,p}(\Omega; \mathbb{R}^{d\times d})$*, then it satisfies the following equations in the reference configuration:*

$$\begin{aligned} -\operatorname{div} S &= f \qquad \text{in } \Omega, \\ S\mathfrak{n} &= g \qquad \text{on } \Gamma_N. \end{aligned} \tag{2.5.9}$$

Moreover, if $v \in W^{1,q}(\Omega; \mathbb{R}^d)$ *is a vector field with* $v = 0$ *on* $\Gamma \backslash \Gamma_N$ *where* $1/p + 1/q \leq (d+1)/d$*, if* $d > p \geq 1$*,* $d > q \geq 1$ *with* $q > 1$ *if* $p \geq d$*, and with* $p > 1$ *if* $q \geq d$*, then the equations* (2.5.9) *are equivalent to the variational formulation*

$$\int_\Omega S : \nabla v dx = \int_\Omega f \cdot v dx + \int_{\Gamma_N} g \cdot v dS.$$

Proof. This follows from the equations (2.5.3) and the definitions of f, g and S. The assertion on the equivalence is then established as in the proof of Thm. 2.5.3. ☐

Remark 2.5.2 (on the terminology)*.* The equation on Γ_N is called a *boundary condition of traction.* Later *boundary condition of place* of the form

$$y = y_D \quad \text{on } \Gamma_D,$$

where y_D is a given mapping, will appear. If this is the case, one can define the set of admissible configurations as $\mathcal{A} := \{y : \bar{\Omega} \to \mathbb{R}^3 \,:\, \det \nabla y > 0;\, y = y_D \text{ on } \Gamma_0\}$ of which the tangent space at y_0 is given by $T_{y_0}\mathcal{A} := \{v : \bar{\Omega} \to \mathbb{R}^3 \,:\, v = 0 \text{ on } \Gamma_D\}$. This means the vector fields occurring in the principle of virtual work are to be understood as variations, and are essentially mathematical, “virtual” quantities, which explains the name of the principle. For additional considerations, see [7], Sec. 2.6. and further sources mentioned there.

Remark 2.5.3 (Summary)*.* The axioms of Euler and Cauchy imply the existence of the Cauchy stress tensor T^y, which unifies all applied forces into one tensor. The divergence structure of the Cauchy Stress tensor $\operatorname{div} T = f$ allows us to give a variational formulation, called the principle of virtual work. Unfortunately, the forces (and thus, the Cauchy stress tensor) are defined in the deformed configuration in terms of the unknown deformation y.

To resolve this issue, one needs the Piola transform to express T^y in terms of reference configuration. Here, the nice properties of the Piola transform come into play, which tell us that the also T^y maintains its divergence structure. This allows us to obtain the analogous variational formulas over the reference configuration as well as over the deformed configuration.

2.6. Conservative forces

Recall from the example of the pressure load, that the density in the reference configuration was of the form $g(x) = \hat{g}(x, \nabla y(x))$. This serves as motivation to only consider applied forces, which are either dead loads, or are of the form

$$\begin{aligned} f(x) &= \tilde{f}(x, y(x), \nabla y(x)), \quad x \in \Omega, \\ g(x) &= \tilde{g}(x, y(x), \nabla y(x)), \quad x \in \Gamma_N, \end{aligned}$$

where $\tilde{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ and $\tilde{g} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}^d$ are given.

Definition 2.6.1. An applied body force with density $f : \Omega \to \mathbb{R}^d$ in the reference configuration is called *conservative*, if for all smooth $v : \bar{\Omega} \to \mathbb{R}^d$ vanishing on $\Gamma_D = \Gamma \backslash \Gamma_N$ the integral

$$\int_\Omega f(x) \cdot v(x) dx = \int_\Omega \tilde{f}(x, y(x), \nabla y(x)) \cdot v(x) dx$$

can be written as Gâteaux derivative

$$F'(y)v = \int_\Omega \tilde{f}(x, y(x), \nabla y(x)) \cdot v(x) dx$$

of a functional of the form

$$\begin{aligned} &F : \{y : \bar{\Omega} \to \mathbb{R}^d\} \to \mathbb{R} \\ &F(y) = \int_\Omega \hat{F}(x, y(x), \nabla y(x)) dx. \end{aligned}$$

Then, the function $\hat{F} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to$ is called *potential of the applied body force.*

Example 2.6.1. A dead load is conservative, with

$$\hat{F}(x, \eta, \xi) = f(x) \cdot \eta.$$

Thus, the gravitational force density $f(x) = -g\rho_S(x)e_3$ is a conservative force. More generally, a density $f(x) = \tilde{f}(x, y(x))$ is conservative, if

$$\tilde{f}(x,\eta) = \operatorname{grad}_\eta \hat{F}(x,\eta) \quad \forall x \in \Omega, \eta \in \mathbb{R}^d. \qquad \triangle$$

Analogously, an applied surface force with density $g : \Gamma_N \to \mathbb{R}^d$ in the reference configuration is *conservative*, if the integral

$$\int_{\Gamma_N} g(x) \cdot v(x) dS = \int_{\Gamma_N} \tilde{g}(x, y(x), \nabla y(x)) \cdot v(x) dS$$

can be written as the Gâteaux derivative

$$G'(y)v = \int_{\Gamma_N} \tilde{g}(x, y(x), \nabla y(x)) \cdot v(x) dS$$

of a functional

$$G : \{y : \Gamma_N \to \mathbb{R}^d\} \to \mathbb{R}$$
$$G(y) = \int_{\Gamma_N} \hat{G}(x, y(x), \nabla y(x)) dS,$$

where $\hat{G} : \Gamma_N \times \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R}$ is called *potential* of g.

Next, we will show that the pressure load is conservative. Thinking of a submerged object, which experiences an increasing pressure as it sinks further, the assumption that p is constant is not very realistic. Instead, we will assume that p depends on the position in the deformed configuration.

Theorem 2.6.1. *Let $p : \mathbb{R}^d \to [0;\infty)$ be smooth and let $y : \bar{\Omega} \to \bar{\Omega}^y$ be a given deformation. Then, then pressure force $g^y(x^y) := -p(x^y)\mathfrak{n}^y(x^y)$, $x^y \in \Gamma^y$ is conservative (after being transformed to the reference configuration).*

Proof. As already derived in equation (2.5.8), we have

$$g(x) = \tilde{g}(x, y(x), \nabla y(x)) = -p(y(x)) \operatorname{Cof} \nabla y(x) \mathfrak{n}(x).$$

Consider $\Gamma_N := \Gamma$ and the functional

$$F(y) = -\int_\Omega p(y(x)) \det \nabla y(x) dx. \tag{2.6.1}$$

Then, via tedious calculations, the use of Green's formula, and Piola identity, one can show that

$$F'(y)v = -\int_\Omega \int_\Gamma p(y(x)) \operatorname{Cof} \nabla y(x)\mathfrak{n}(x) \cdot v(x) dS$$

and thus that the Gâteaux derivative can indeed be written as integral over $\tilde{g}(x, y, \nabla y) \cdot v(x)$. The details can be found in A.6 in full rigour. $\square$

2.7. Physical properties and elastic materials

Elastic materials

The balance of forces in the deformed configuration (2.5.3) consists of d equations, whereas we have to find $d(d+3)/2$ unknowns in total, namely d components of y and, exploiting the symmetry of T^y, $d(d+1)/2$ components of T^y. In order the problem to be solvable, we complete the system with a constitutive model for the material response. In particular, we will consider elastic materials.

Definition 2.7.1. A material is called *elastic* (or sometimes *Cauchy-elastic*), if the Cauchy stress tensor at any point $x^y = y(x) \in \bar{\Omega}^y$ is completely determined by the deformation gradient $\nabla y(x)$ at the corresponding point $x \in \bar{\Omega}$. Formally, the material is called elastic, if there is a mapping

$$\tilde{T}^D : \bar{\Omega} \times \mathrm{GL}^+(d) \to \mathbb{R}^{d \times d}_{\mathrm{sym}},$$

such that

$$T^y(x^y) = \tilde{T}^D(x, \nabla y(x)) \tag{2.7.1}$$

for all $x \in \bar{\Omega}$. In this definition the mapping $\tilde{T}^D$ is called *response function* to the stress tensor, and the relation given by (2.7.1) is the so-called *constitutive equation* of the material. The superscript D indicates that we consider the Cauchy stress tensor in the deformed configuration. A material is called *homogeneous* if its response function does not depend on x. Otherwise it is called *inhomogeneous*.

Remark 2.7.1. By definition, the response function at any point must be defined for *all* matrices in $\mathrm{GL}^+(d)$. This implicitly means that the definitions only holds for materials, with the property that for a given $x \in \bar{\Omega}$ and a $F \in \mathrm{GL}^+(d)$ there is a deformation y such that $F = \nabla y(x)$. Therefore, this definition rules out materials subjected to internal constraints such as incompressible materials (see [7], Sec. 5.7).

By virtue of the Piola-transform one can also define response function $\tilde{S}$ for the 1st Piola-Kirchhoff stress tensor in terms of the response function of T^y

$$\tilde{S}(x, F) = (\det F)\tilde{T}^D(x, F)F^{-T}, \tag{2.7.2}$$

with $x \in \bar{\Omega}$ and $F \in \mathrm{GL}^+(d)$. The 1st Piola-Kirchhoff stress tensor can be written as

$$S(x) = \tilde{S}(x, \nabla y(x)). \tag{2.7.3}$$

Remark 2.7.2. For some materials one needs more refined models. For instance, sometimes its necessary to relate the Cauchy stress tensor not only to ∇y, but to higher order gradients. Such materials are called *nonsimple materials*. A different approach is to relate the Cauchy stress tensor to the gradient ∇y in the whole deformed configuration. This is called *nonlocal elasticity*. A brief introduction and various examples can be found in [17], Sec. 2.5.

The response function of an elastic material does not depend on the choice of a particular reference configuration. Therefore, as in the case of the Cauchy stress tensor, we want the response function to be invariant under rotations, i.e., to be *frame-indifferent*. Recall that for the Cauchy stress tensor the axiom of frame-indifference means that for a deformation y and a rotation $R \in \mathrm{SO}(d)$ the Cauchy stress tensor of the deformation z given by $z(x) := Ry(x)$ satisfies $T^z(x^z) = RT^y(x^y)R^T$. Therefore, we have for all $x \in \bar{\Omega}$, and for all $R \in \mathrm{SO}(d)$ and any deformation y that

$$RT^y(x^y)R^T = T^z(x^z) = \tilde{T}^D(x, \nabla Ry(x)) = \tilde{T}^D(x, R\nabla y(x)),$$

and thus the response function satisfies for any $F \in \mathrm{GL}^+(d)$ the relation

$$\tilde{T}^D(x, RF) = R\tilde{T}^D(x, F)R^T.$$

Consequently, the axiom of frame-indifference implies the identity

$$R^T\tilde{S}(x, RF) = \tilde{S}(x, F).$$

Hyperelastic materials

Now we introduce the notion of hyperelasticity. This captures the idea that deformations are reversible, in the sense that the deformation energy can be stored in the material and used later to do work without any loss. One could imagine dropping a rubber ball.

When the ball hits the ground, it gets deformed, and deformation energy will be stored inside the ball, which then causes the ball to regain its original form and bounce back into the air.

Definition 2.7.2 (Hyperelasticity, stored energy)**.** An elastic material is called *hyperelastic*, if there exits a *stored energy function* $\varphi : \bar{\Omega} \times GL^+(d) \to [0, \infty)$ such that

$$\tilde{S}(x, F) = \partial_F \varphi(x, F).$$

Proposition 2.7.1 (Frame-indifference for hyperelastic materials)**.** *The axiom of frame-indifference is equivalent the following identity*

$$\varphi(x, RF) = \varphi(x, F), \tag{2.7.4}$$

for all rotations $R \in \mathrm{SO}(d)$ *and all* $F \in \mathrm{GL}^+(d)$.

Proof. We prove the assertion in two steps. First, we show that frame-indifference is equivalent to $\partial_F \varphi(x, RF) = \partial_F \varphi(x, F)$. In the second step we check that $\partial_F \varphi(x.RF) = \partial_F \varphi(x, F)$ is equivalent to $\varphi(x, RF) = \varphi(x, F)$.

Step 1. By the definition of hyperelasticity, $\tilde{S}$ is frame-indifferent, namely $R^T \tilde{S}(x, RF) = \tilde{S}(x, F)$, if and only if $R^T \partial_F \varphi(x, RF) = \partial_F \varphi(x, F)$ holds for any $R \in \mathrm{SO}(d)$. To compute the partial derivative of the mapping $\varphi_R : F \mapsto \varphi_R(x, F) := \varphi(x, RF)$ we fix $R \in \mathrm{SO}(d)$ and use Taylor's theorem to obtain

$$\begin{aligned} \varphi_R(x, F + G) &= \varphi(x, RF + RG) \\ &= \varphi(x, RF) + \partial_F \varphi(x, RF) : RG + o(|RG|) \\ &= \varphi_R(x, F) + R^T \partial_F \varphi(RF) : G + o(|G|). \end{aligned}$$

Therefore, we get $\partial_F \varphi_R(x, F) = R^T \partial_F \varphi(x, RF)$ for all $F \in \mathrm{GL}^+(d)$. Thus, we have

$$R^T \partial_F \varphi(x, RF) = \partial_F \varphi(x, F) \iff \partial_F \left(\varphi_R(x, F) - \varphi(x, F) \right) = 0. \tag{2.7.5}$$

This proves the first claim.

Step 2. Clearly, if (2.7.4) is satisfied, then also $\partial_F \left(\varphi_R(x, F) - \varphi(x, F) \right) = 0$ holds for all $F \in \mathrm{GL}^+(d)$. For the converse statement, note that the set $\mathrm{GL}^+(d)$ is connected (cf. [17], p. 29). Therefore, the relation $\partial_F \left(\varphi_R(x, F) - \varphi(x, F) \right) = 0$ implies that $\varphi_R(x, F) - \varphi(x, F)$ is a constant $K = K(R)$ depending on R. Hence, there exists a mapping $K : \mathrm{SO}(d) \to \mathbb{R}$

such that we have

$$\varphi(x, RF) - \varphi(x, F) = K(R) \qquad \forall F \in \mathrm{GL}^+(d). \tag{2.7.6}$$

Testing (2.7.6) successively for $F = \mathbb{I}$, $F = R$, $F = R^2$, etc. we find that

$$\varphi(x, R^n) = \varphi(x, \mathbb{I}) + nK(R)$$

and thus, $|\varphi(x, R^n)| \geq n|K(R)| - |\varphi(x, \mathbb{I})|$. If $K(R) \neq 0$, we would get that $\lim_{n\to\infty} |\varphi(x, R^n)| = \infty$. Since $\{R^n\}$ is compact and φ is continuous for almost all x (since $\varphi(x, \cdot)$ is differentiable), it takes a maximum on $\{R^n\}$, which yields a contradiction. Thus, $K(R) = 0$. $\square$

Since we do not allow the material to deform to a point, or a plane, or even intersect itself, we set up the natural condition on φ

$$\varphi(x, F) \begin{cases} \to +\infty & \text{if} \quad \det F \to 0_+ \\ = +\infty & \text{if} \quad \det F \leq 0. \end{cases} \tag{2.7.7}$$

One can extend $\varphi(x, \cdot)$ by $+\infty$ to the set of matrices with nonpositive determinants. This extension makes φ continuous as a map $\mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$.

For the rest of the thesis we will assume a special form of φ, which is in accordance with (2.7.7). Let us suppose there are constants $\varepsilon, p, q, r > 0$, such that, for all $x \in \bar{\Omega}$ and all $F \in \mathbb{R}^{d\times d}$, φ satisfies the following inequality

$$\varphi(x, F) \geq \begin{cases} \varepsilon\left(|F|^p + |\operatorname{Cof} F|^q + (\det F)^r\right) & \text{if} \quad \det F > 0 \\ +\infty & \text{if} \quad \text{otherwise.} \end{cases} \tag{2.7.8}$$

Again, this means that large deformation gradients and changes of volume and surface contribute to the energy stored in the material.

This assumption on φ covers examples such as neo-Hookean, or Ogden materials, which are used to model materials like rubber, polymers, and similar biological tissue (for a definition refer to [17], Sec. 2.4).

Furthermore, the assumption (2.7.8) allows us to conclude coercivity of the functional $\mathcal{E}(y) = \int_\Omega \varphi(x, \nabla y)dx$, which is necessary to apply the direct method (the precise argument will be given in Sec. 3.2).

An issue here, however, is that such a natural physical property clashes with convexity of the stored energy function, which is desirable property as it implies lower semicontinuity. Therefore, we cannot apply the direct method naively and have to circumvent this nonconvexity of φ by other, more suitable, notions of convexity. Let us make this point precise (cf. [7], Thm. 4.8-1) via the following

Theorem 2.7.2. *Let $d \geq 2$.*

(i) There is no function $\varphi : \bar{\Omega} \times \mathbb{R}^{d \times d} \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(x, \cdot)$ is convex and finite on $\mathrm{GL}^+(d)$ for $x \in \bar{\Omega}$ and satisfies (2.7.7).

(ii) Let $x \in \bar{\Omega}$ such that the function

$$\begin{aligned} \varphi(x, \cdot) : GL^+(3) &\to \mathbb{R} \\ F &\mapsto \varphi(x, F) \end{aligned}$$

is convex. The axiom of frame-indifference implies that for any deformation y of the reference configuration $\bar{\Omega}$, the eigenvalues τ_i of the Cauchy stress tensor $T^y(x^y)$ at a point $x^y = y(x)$ of the deformed configuration satisfy the inequalities

$$\begin{aligned} \tau_1 + \tau_2 &\geq 0, \\ \tau_2 + \tau_3 &\geq 0, \\ \tau_3 + \tau_1 &\geq 0. \end{aligned} \tag{2.7.9}$$

The proof can be found in [7], Thm. 4.8-1. For the first assertion only, see [17], Prop. 2.3.4.

The first statement rules out the convexity of stored energy function with explicit dependence on $\det F$. The second assertion says that it cannot be expected that the eigenvalues of the Cauchy stress tensor satisfy the inequalities (2.7.9) at all points in all deformed configurations. This fails even in very simple examples: for instance in the case of an object subjected to uniform pressure (cf. Ex. 2.4.1), in which case the Cauchy stress tensor is of the form $T^y(x^y) = -p\mathbb{I}$.

In order to stress that the stored energy function cannot be convex, in the mathematical elasticity literature (e.g. [7]) the symbol W is used instead of φ, as it resembles the graph of a non-convex function. We will adopt this notion.

The goal of the next section is to find a suitable, weaker notion of convexity respecting the coercivity assumption (2.7.8) but still allowing us to conclude (weak) lower semicontinuity.

3. Existence of minimizers for polyconvex materials

As discussed in chapter 2, our goal is to minimize a functional of the form

$$\mathcal{E}(y) = \int_\Omega \varphi(x, \nabla y(x))dx - \mathcal{F}.$$

To do so, we are aiming at employing the direct method. However, we have that the stored energy φ is in general not convex, so we have to use another notion of convexity ensuring weak lower semicontinuity. The right concept is polyconvexity, which we introduce in Section 3.1. We examine the advantages of polyconvexity and prove that polyconvexity implies weak lower semicontinuity. These results will be used in section 3.2, where we eventually prove the existence of minimizers of the energy functional for polyconvex, hyperelastic materials.

3.1. Polyconvexity

The definition of polyconvexity was originally introduced by Morrey in [21] but then used by Ball [2] in the frame of elasticity. For defining polyconvex functions we need the minors of a matrix. As we mainly work in dimension $d = 3$, and thus, only have to deal with 3×3-matrices, we will give the definition of polyconvexity in this particular case.

Definition 3.1.1. Let $F \in \mathbb{R}^{d\times d}$ be a square matrix and denote the vector of all minors of F with $M(F)$. In particular, we have $M(F) = (F, \operatorname{Cof} F, \det F)$ for $d = 3$.

A function $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$ is said to be *polyconvex*, if there exists a convex and continuous function $\tilde{\varphi} : \mathbb{R}^{2^{d^2}+1} \to \mathbb{R} \cup \{+\infty\}$, such that

$$\varphi(F) = \tilde{\varphi}(M(F)). \tag{3.1.1}$$

An elastic material, whose stored energy functional is given by a polyconvex function, is also called *polyconvex*.

Remark 3.1.1. One can generalize the definition of polyconvexity to higher dimensions. To do so, one keeps the defining property (3.1.1), but considers minors of higher order, check [8], Def. 1.5.

We want to remind the reader of the following fact: The cofactor matrix can be used to calculate the inverse of an invertible matrix

$$M^{-1} = \frac{1}{\det M} \operatorname{Cof} M^T.$$

In other words: The product of a matrix with the transposed cofactor matrix yields a diagonal matrix with the determinant as entries

$$M \operatorname{Cof}(M) = \det(M) I.$$

We saw in Section 2.3 that given a deformation y, the cofactor matrix and the determinant of the Jacobian ∇y of this deformation describe how the area and volume change under the deformation.

Example 3.1.1. Convex functions are clearly polyconvex. The map $F \mapsto \det F$ is polyconvex, but not convex. Thus, polyconvexity is indeed a weaker notion of convexity. $\triangle$

Similar to the scalar case of the direct method, we are given a weakly converging minimizing sequence (y_n) and need to conclude (weak) lower semicontinuity of the functional. For a sufficiently nice function $\varphi(x, z, v)$ one can show that, if $z_k \to z$ almost everywhere and $v_k \rightharpoonup v$ weakly in L^1, then the functional $(z, v) \mapsto \int \varphi(x, z, v)$ is lower semicontinuous (using Mazur's Lemma). Although we only know that $y_n \rightharpoonup y$, we can show that also $\operatorname{Cof}(\nabla y_n)$ and $\det \nabla y_n$ converge in a weak sense. This is due to the fact that we can rewrite the minors of ∇y_n in divergence form and then apply the Gauß divergence theorem. Setting $z_k := y_k$, $v_k := M(\nabla y_k) = (\nabla y_k, \operatorname{Cof}(\nabla y_k), \det(\nabla y_k))$ and $\varphi(x, z, y)$ as in the definition of polyconvexity yields the fact that the energy functional is weak lower semicontinuous. Let's make the above statement precise (see also [17], Thm. 3.3.1).

Theorem 3.1.1 (Weak lower semicontinuity)**.** *Let $\xi : \Omega \times \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:*

(i) $\xi(\cdot, z, v) : \Omega \to \mathbb{R} \cup \{+\infty\}$ *is measurable for all* $(z, v) \in \mathbb{R}^s \times \mathbb{R}^\sigma$*,*

(ii) $\xi(x, \cdot, \cdot) : \mathbb{R}^s \times \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ *is continuous for almost every* $x \in \Omega$*,*

(iii) $\xi(x, z, \cdot) : \mathbb{R}^\sigma \to \mathbb{R} \cup \{+\infty\}$ *is convex. Assume further that for all* $(z, v) \in \mathbb{R}^s \times \mathbb{R}^\sigma$ $\xi(\cdot, z, v) \geq \psi$ *for some* $\psi \in L^1(\Omega)$*.*

Furthermore, let $z_k \to z$ almost everywhere in Ω and $v_k \rightharpoonup v$ weakly in $L^1(\Omega;\mathbb{R}^\sigma)$. Then

$$\int_\Omega \xi(x, z(x), v(x))dx \leq \liminf_{k\to\infty} \int_\Omega \xi(x, z_k(x), v_k(x))dx.$$

The proof can be found in [17], Thm. 3.3.1. We will only sketch it here.

Proof-Sketch. W.l.o.g. one can assume $\xi \geq 0$, otherwise just use $\xi - \psi$. Let (z_k, v_k) be a (nonrelabelled) subsequence realizing the $\liminf$, i.e.,

$$\lim_{k\to\infty} \int_\Omega \xi(x, z_k(x), v_k(x))dx =: \alpha.$$

Set $g_k(x) := \xi(x, z_k(x), v_k(x)) - \xi(x, z(x), v_k(x))$. Then g_k converges to zero in measure. We will not proof this claim here, but refer to [17]. By the convergence in measure, there is a subsequence (g_k) (again not relabelled) with $g_k \to 0$ almost everywhere.

Apply Mazur's lemma A.4.11 to $v_k \rightharpoonup v$ in L^1 to obtain a sequence of convex combinations

$$w_k = \sum_{j=k}^{N_k} \lambda_{k,j} v_j, \qquad \sum_{j=k}^{N_k} \lambda_{k,j} = 1,$$

such that $w_k \to w$ strongly in L^1. By applying Jensen's inequality to the concave function $-\xi(x, z, \cdot)$ one obtains

$$\xi(x, z(x), w_k(x)) + \sum_j \lambda_{k,j} g_j(x) \leq \sum_j \lambda_{k,j} \xi(x, z_j(x), v_j(x)).$$

Passing to the limit for $k \to \infty$, integrating over Ω and applying Fatou's lemma A.1.1 yields the result. □

Remark 3.1.2. We only sketched the proof for two reasons. First, we want to point out the similarity to the proof of Theorem 1.3.1. However, ξ depending on z makes everything much more problematic. We simplified the proof by claiming that g_k converges to 0 in measure. This construction takes care of, or cancels out respectively, the z-dependence and allows us to proceed in the already familiar manner. Secondly, we will give a rigorous proof later on, where we will show that polyconvexity implies quasiconvexity and establish that the functional is weak lower semicontinuous if the integrand is quasiconvex. To do so, we will need new tools, in particular Young measures, and therefore we postpone this to chapter 4.

Eventually, we want to apply this theorem in the case of a polyconvex integrand. In particular, we set $z_k = y_k$ and $v_k = M(\nabla y_k) = (\nabla y_k, \mathrm{Cof}\,\nabla y_k, \det \nabla y_k)$. Hence, we need to make sure that we have weak convergence of the minors if $y_k \rightharpoonup y$ in $W^{1,p}$. This is the goal of the next theorems, i.e. we now are about to prove that the cofactor matrix as well as the determinant of ∇y_n converge nicely, if we only know that $y_n \rightharpoonup y$ weakly in $W^{1,p}$. To illustrate the idea of the proof, we sketch it for the case $y : \mathbb{R}^2 \to \mathbb{R}^2$ before proving everything in detail. The proof is based on the fact, that minors can be written in a divergence form, which allows us to use the Gauß divergence theorem. We refer to [27], Chap. 5.2, for a thorough general treatment.

Suppose $y : \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$\nabla y = \begin{pmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{pmatrix},$$

where $y_{i,j} = \frac{\partial y_i}{\partial x_j}$. Then

$$\det \nabla y = y_{1,1}y_{2,2} - y_{1,2}y_{2,2} = \mathrm{div}(y_1 y_{2,2}; -y_1 y_{2,1}) = \nabla \cdot (y_1 y_{2,2}; -y_1 y_{2,1}),$$

where $(\,\cdot\,;\,\cdot\,)$ denotes the components of a vectorfield. If $y^n \rightharpoonup y$ weakly in $W^{1,p}$, then multiplying with a test function ϕ and employing the Gauß divergence theorem yields

$$\begin{aligned} &\int \det \nabla y^n \phi = \int \nabla \cdot (y_1^n y_{2,2}^n; -y_1^n y_{2,1}^n)\phi = -\int (y_1^n y_{2,2}^n; -y_1^n y_{2,1}^n)\nabla\phi \\ &\to -\int (y_1 y_{2,2}; -y_1 y_{2,1})\nabla\phi = \int \det \nabla y \phi. \end{aligned}$$

The general theorem (and the respective proof) for arbitrary minors can be found in [27], Lemma 5.10 (or [8], Theorem 8.20) and reads as:

Theorem 3.1.2 (General weak convergence of minors)**.** *Let $\Omega \subset \mathbb{R}^d$ be a domain and $M : \mathbb{R}^{m\times d} \to \mathbb{R}$ be an $(r \times r)$-minor, $r \in \{1, \dots, \min\{d, m\}\}$, and let $(y_k) \subset W^{1,p}(\Omega, \mathbb{R}^m)$, where $p \in (r, \infty)$. If*

$$y_k \rightharpoonup y \text{ in } W^{1,p},$$

then

$$M(\nabla y_k) \rightharpoonup M(\nabla y) \text{ in } L^{p/r}.$$

Remark 3.1.3. In the proof we are going to use the Rellich-Kondrachov theorem, Thm. A.2.3, to conclude (strong) L^p-convergence from the given weak $W^{1,p}$-convergence. To do so, one has to select a subsequence (because of the compact embedding). Therefore, the conclusion of the theorem has to be stated as follows: there is a subsequence y_{k_n}

such that

$$M(\nabla y_{k_n}) \rightharpoonup M(\nabla y).$$

Since we eventually apply this theorem in the context of the direct method, where we nevertheless pass to subsequences, it does not matter and we use the same formulation as in [27].

Let us state and prove the result rigorously in the case $\Omega \subset \mathbb{R}^3$ for the cofactor matrix ([7], Thm. 7.5-1. and [17], Thm. 3.2.1) and the determinant ([7], Thm. 7.6-1 and [17], Thm. 3.2.2).

Theorem 3.1.3 (Weak convergence of Cof)**.** *Let $\Omega \subset \mathbb{R}^3$ be a domain and let $p \geq 2$. Then the mapping $y \in W^{1,p} \mapsto \operatorname{Cof} \nabla y$ is well-defined and continuous. Furthermore, if $y_k \rightharpoonup y$ weakly in $W^{1,p}(\Omega; \mathbb{R}^3)$ and $\operatorname{Cof} \nabla y_k \rightharpoonup H$ in $L^q(\Omega)$, for some $q \geq 1$, then*

$$\operatorname{Cof} \nabla y = H.$$

Proof. For the proof a number technical lemmata will be needed. These are either stated and proved either directly after the proof or in the appendix.

Step 1: Claim: the bilinear mapping

$$\begin{aligned} A : (L^p(\Omega))^2 &\to L^{p/2}(\Omega) \\ (\xi, \eta) &\mapsto \xi\eta \end{aligned}$$

is well-defined and continuous.

To prove that A is well-defined one uses Hölder's inequality

$$\int |\xi\eta|^{p/2} = \int |\xi|^{p/2}|\eta|^{p/2} \leq \left(\int \left(|\xi|^{p/2}\right)^2\right)^{1/2} \left(\int \left(|\eta|^{p/2}\right)^2\right)^{1/2} = \|\xi\|_p^{p/2}\|\eta\|_p^{p/2} < \infty.$$

With the step above, we also have proved that the bilinear map A is bounded, which implies by Lemma 3.1.4 the continuity of A.

Step 2: Claim: the map

$$\begin{aligned} W^{1,p} &\to L^{p/2}(\Omega) \\ y &\mapsto \operatorname{Cof} \nabla y \end{aligned}$$

is well defined and continuous. The Cofactor matrix in $3d$ has the form (do not sum over repeated indices and count them modulo three, [17], Thm. 3.2.1)

$$(\operatorname{Cof}\nabla y)_{i,j} = \frac{\partial y_{j+2}}{\partial x_{i+2}}\frac{\partial y_{j+1}}{\partial x_{i+1}} - \frac{\partial y_{j+2}}{\partial x_{i+1}}\frac{\partial y_{j+1}}{\partial x_{i+2}}.$$

Since $y \in W^{1,p}$ we know that $\frac{\partial y_k}{\partial x_l} \in L^p$ for all $1 \leq k,l \leq 3$. Therefore, the Cofactor matrix is a linear combination of products of L^p-functions, and thus, the first step implies the claim.

Step 3: Now we use the representation of Cof as divergence. Let y be smooth enough, e.g. $y \in \mathscr{C}^2(\Omega)$. By Schwarz Theorem we can write

$$(\operatorname{Cof}\nabla y)_{ij} = \partial_{x_{i+2}}\left(y_{j+2}\partial_{x_{i+1}}y_{j+1}\right) - \partial_{x_{i+1}}\left(y_{j+2}\partial_{x_{i+2}}y_{j+1}\right),$$

and consequently, by Gauß-Green Theorem (integration by parts) we have for all $y \in \mathscr{C}^2$ and all test functions $\phi \in \mathscr{D}$ that

$$\begin{aligned}\int_\Omega (\operatorname{Cof}\nabla y)_{ij}\phi dx &= \int_\Omega \left(\partial_{x_{i+2}}\left(y_{j+2}\partial_{x_{i+1}}y_{j+1}\right) - \partial_{x_{i+1}}\left(y_{j+2}\partial_{x_{i+2}}y_{j+1}\right)\right)\phi dx \\ = -\int_\Omega \left(y_{j+2}\partial_{x_{i+1}}y_{j+1}\right)\partial_{x_{i+2}}\phi &+ \int_\Omega \left(y_{j+2}\partial_{x_{i+2}}y_{j+1}\right)\partial_{x_{i+1}}\phi dx. \end{aligned} \tag{3.1.2}$$

Moreover, we have that

$$\begin{aligned}\left|\int_\Omega (\operatorname{Cof}\nabla y)_{i,j}\phi dx\right| &\leq \|(\operatorname{Cof}\nabla y)_{i,j}\|_{L^1}\|\phi\|_{L^\infty} \overset{(*)}{\leq} c_1(\phi)\|y\|^2_{W^{1,2}} \\ \left|\int_\Omega y_i\partial_j y_k\partial_l\phi dx\right| &\leq \|\partial_l\phi\|_{L^\infty}\int_\Omega |y_j\partial_j y_k|dx \leq c_2(\phi)\|y\|^2_{W^{1,2}}.\end{aligned}$$

The estimate $(*)$ is not a problem, because one could argue with (using Cauchy's estimate and $L^2 \subset L^1$ since Ω bounded)

$$\int |\partial_i y_j \partial_k y_k| \leq c\left(\int |\partial_i y_j|^2\right)\left(\int |\partial_l y_k|^2\right) \leq c'\|\partial_i y_j\|^2_{L^2} \leq c''\|\nabla y\|^2_{W^{1,2}}.$$

Since $\mathscr{C}^2$ is dense in $W^{1,2}$ and we have continuity, the equation (3.1.2) is valid in $W^{1,2}$ as well, and by Sobolev embeddings, even in $W^{1,p}$, for $p \geq 2$.

Step 4: Claim: for $p \geq 2$ and $\phi \in \mathscr{D}$ fixed, we have

$$y^n \rightharpoonup y \text{ in } W^{1,p} \implies \int_\Omega y_i^n \partial_j y_k^n \partial_m\phi dx \to \int y_i\partial_j y_k\partial_m\phi.$$

This would then imply by step 3

$$y^n \rightharpoonup y \text{ in } W^{1,p} \implies \int_\Omega (\operatorname{Cof} \nabla y_n)_{ij} \phi dx \to \int (\operatorname{Cof} \nabla y)_{ij} \phi dx.$$

To prove the claim consider the bilinear mapping

$$(\xi, \chi) \in L^r(\Omega) \times W^{1,p}(\Omega) \to \int \xi \partial_j \chi \partial_m \phi dx.$$

Now, if $\frac{1}{p} + \frac{1}{r} = 1$, we can apply Hölder's inequality to obtain

$$\left| \int \xi \partial_j \chi \partial_m \phi dx \right| \leq \|\partial_m \phi\|_{L^\infty} \int |\xi \partial_j \chi| \leq c \left(\int |\xi|^r \right)^{1/r} \left(\int |\partial_j \chi|^p \right)^{1/p}$$

Since, if $\tilde{r} < r$ and $L^r \subset L^{\tilde{r}}$, the calculation above also holds if $\frac{1}{p} + \frac{1}{r} \leq 1$, and thus, the bilinear mapping is continuous. Therefore, Prop. 3.1.5 is applicable, and we have

$$\left.\begin{array}{lll} \xi^n \to \xi & \text{in} & L^r(\Omega) \\ \chi^n \rightharpoonup \xi & \text{in} & W^{1,p}(\Omega) \end{array}\right\} \implies \int \xi^n \partial_j \chi^n \partial_m \phi dx \to \int \xi \partial_j \chi \partial_m \phi dx.$$

From the Rellich-Kondrachov compact embedding Theorem A.2.3 we obtain

$$W^{1,p} \Subset L^r \text{ for all } 1 \leq r < p^* = \begin{cases} \frac{3p}{3-p} & \text{if} \quad p < 3 \\ +\infty & \text{if} \quad p \geq 3. \end{cases}$$

Thus, we get that, if $y_n \rightharpoonup y$ in $W^{1,p}$, then $y_n \to y$ in L^r for all $1 \leq r < p^*$ (possibly passing to a subsequence).

Since it is possible to find a number r that simultaneously satisfies

$$\frac{1}{p} + \frac{1}{r} \leq 1 \quad \text{and} \quad r < p^*$$

for $p \leq 2$, our claim is proved.

Step 5: We will combine the steps above to prove the assertion of the theorem. Let $(y_n) \subset W^{1,p}$, $p \geq 2$ such that $\operatorname{Cof} \nabla y_n \in L^q$, $q \geq 1$ and such that $y_n \rightharpoonup y$ in $W^{1,p}$ and $\operatorname{Cof} \nabla y_n \rightharpoonup H$ in L^q.

By Step 4, we know

$$\int (\operatorname{Cof} \nabla y_n)_{ij} \phi dx \to \int (\operatorname{Cof} \nabla y)_{ij} \phi \quad \forall \phi \in \mathscr{D},$$

and by assumption

$$\int (\operatorname{Cof} \nabla y_n)_{ij} \phi dx \to \int H_{ij} \phi dx.$$

Therefore $(\operatorname{Cof} \nabla y - H) \in L^1(\Omega)$ and

$$\int (\operatorname{Cof} \nabla y - H) \phi dx = 0 \quad \phi \in \mathscr{D}.$$

Hence, by the fundamental theorem of calculus of variations (cf. [28], Lemma 10.21), we get $\operatorname{Cof} \nabla y = H$ almost everywhere in Ω and the proof is complete. □

We state and prove the necessary lemmata now.

Lemma 3.1.4. *Let X, Y, Z be normed vector spaces and $B : X \times Y \to Z$ be a bilinear mapping. Then the following are equivalent:*

(i) B is continuous
(ii) B is continuous at $(0,0)$
(iii) B is bounded, i.e. there is a $c > 0$ such that

$$\|B(x,y)\|_Z \leq c\|x\|_X \|y\|_Y \quad \forall (x,y) \in X \times Y$$

Moreover, if at least one of the spaces X, Y is a Banach Space, then the above properties are equivalent to:

(i) B is separately continuous, i.e. continuous in each coordinate.

Proof. $(i) \implies (ii)$ is trivial.

$(ii) \implies (iii)$ by contradiction. Suppose (iii) is false. Then, for each $n \in \mathbb{N}$ there is $(0,0) \neq (x_n, y_n) \in X \times Y$ such that $\|B(x_n, y_n)\|_Z > n^2 \|x_n\|_X \|y_n\|_Y$. Set

$$\tilde{x}_n := \frac{x_n}{n\|x_n\|} \to 0, \qquad \tilde{y}_n := \frac{y_n}{n\|y_n\|} \to 0.$$

By the bilinearity of B we have

$$\|B(\tilde{x}_n, \tilde{y}_n)\|_Z = \frac{1}{n^2\|x_n\|_X\|y_n\|_Y}\|B(x_n, y_n)\|_Z > \frac{1}{n^2\|x_n\|_X\|y_n\|_Y} n^2\|x_n\|_X\|y_n\|_Y = 1 \nrightarrow 0,$$

and thus, a contradiction.

$(iii) \implies (i)$: Let (iii) hold and assume $x_n \to x$, $y_n \to y$. Then, there is a $M > 0$

such that $\|x_n\|_X \leq M$ and $\|y_n\|_Y \leq M$. Therefore, we have

$$\begin{aligned}
\|B(x_n, y_n) - B(x, y)\|_Z &\leq \|B(x_n, y_n) - B(x_n, y)\|_Z + \|B(x_n, y) - B(x, y)\|_Z \\
&= \|B(x_n, y_n - y)\|_Z + \|B(x_n - x, y)\|_Z \\
&\leq c\|x_n\|_X \|y_n - y\|_Y + c\|x_n - x\|_X \|y\|_Y \\
&< cM\left(\|x_n - x\|_X + \|y_n - y\|_Y\right) \to 0,
\end{aligned}$$

which proves the claim.

The last statement can be proved by using the Banach-Steinhaus Theorem. □

We also will need the following fact about continuous bilinear forms, cf. [7], Thm. 7.1-5, for the proof.

Proposition 3.1.5. *Let V be a normed space and W be a Banach space. Furthermore, let $B : V \times W \to \mathbb{R}$ be a continuous bilinear mapping. Then*

$$v_k \to v \text{ in } V \text{ and } w_k \rightharpoonup w \text{ in } W \implies B(v_k, w_k) \to B(v, k).$$

Next, we are going to establish a result similar to Thm. 3.1.3 for the determinant of ∇y. Notice that

$$\det \nabla y = \frac{1}{6} \sum_{i,j,k,l,m,n=1}^{3} \varepsilon_{ijk}\varepsilon_{lmn}\partial_l y_i \partial_m y_j \partial_n y_k$$

and Hölder's inequality implies that the trilinear mapping

$$(\xi, \eta, \zeta) \in (L^p(\Omega))^3 \mapsto \xi\eta\zeta \in L^{p/3}(\Omega)$$

is continuous. This suggests that we need at least $p \geq 3$ for the mapping

$$\begin{aligned}
W^{1,p} &\to L^1(\Omega) \\
y &\mapsto \det \nabla y
\end{aligned}$$

to be well-defined and continuous. However, we can even do better! If we have additional information on the Cof, we can weaken the requirement on p, by realizing that

$$\det \nabla y = \sum_j \partial_j y_1 \left(\operatorname{Cof} \nabla y\right)_{1j}.$$

Then Hölder's inequality shows that $\det \nabla y$ is well defined in L^s, if $y \in W^{1,p}$ for $p \geq 2$

and $\operatorname{Cof}\nabla y \in L^q$ with

$$\frac{1}{s} := \frac{1}{p} + \frac{1}{q} \leq 1.$$

Under these assumptions we can get the desired convergence properties for the determinant as well, as stated in the following theorem.

Theorem 3.1.6 (Weak convergence of determinant). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. For each $p \geq 2$ and each q such that $1/s = 1/p + 1/q \leq 1$ is satisfied, the mapping*

$$(y, \operatorname{Cof}\nabla y) \in W^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^{3\times 3}) \mapsto \det\nabla y := \sum_j \partial_j y_1 \left(\operatorname{Cof}\nabla y\right)_{1j} \in L^s(\Omega)$$

is well-defined and continuous. Moreover, if $y_k \rightharpoonup y$ in $W^{1,p}$, $\operatorname{Cof}\nabla y_k \rightharpoonup H$ in L^q and $\det\nabla y_k \rightharpoonup \delta$ in L^r for $r \geq 1$, then $\operatorname{Cof}\nabla y = H$ and $\det\nabla y = \delta$.

We only present the proof ideas for $p \geq 3$, the whole prove can be found in [17], Thm. 3.2.2 in full rigour.

Proof-sketch. By Thm. 3.1.3 we have the convergence result for the Cof. The continuity of the mapping follows by Hölder's inequality similarly to the proof of Thm. 3.1.3. The identity $\det\nabla y := \sum_j \partial_j y_1 \left(\operatorname{Cof}\nabla y\right)_{1j}$ can be proved for $y \in \mathscr{C}^2$ using the identity $(\det A)\mathbb{I} = A(\operatorname{Cof} A)^T$ and Piola's identity, and thus, we get for every test function $\phi \in \mathscr{D}$

$$\int_\Omega \partial_j y_1 \left(\operatorname{Cof}\nabla y\right)_{1j} \phi dx = -\int_\Omega y_1 \left(\operatorname{Cof}\nabla y\right)_{1j} \partial_j \phi dx.$$

Now, if $p \geq 3$ the mapping $y \mapsto \int_\Omega \partial_j y_1 \left(\operatorname{Cof}\nabla y\right)_{1j} \phi dx$ is continuous w.r.t the $W^{1,p}$-norm. Therefore, one can proceed as in the proof of Thm. 3.1.3. □

Corollary 3.1.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $p > 3$. If $y_k \rightharpoonup y$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, then $\det\nabla y_k \rightharpoonup \det\nabla y$ weakly in $L^{p/3}(\Omega)$.*

Remark 3.1.4 (On the choice of p). As opposed to the general case of Thm. 3.1.2, where we need $p > r$ strictly to conclude weak continuity of a $r \times r$-minor, we could choose $p \geq 2$, with equality allowed, in the case of the Cof in Thm. 3.1.3. However, this does not work for the determinant, where in general, we need $p > 3$. A counterexample for $p = 3$ can be found in [17], directly after Cor. 3.2.3., showing that the sequential continuity breaks down if $\{|\nabla y_k|^d\} \subset L^1$ concentrates at the boundary of the half plane. If one assumes, that $\det\nabla y_k > 0$ almost everywhere, for all k, then one *can* choose $p = 3$. In fact, one could prove an even better result. These remarks show, that the optimal choice of p is a very subtle issue. A rigorous treatment is not in the scope of this thesis, but instead we refer to the papers of Müller, [22], [23], and further resources mentioned there.

Summarizing the above results, we have that polyconvexity implies weak lower semicontinuity of the functional.

Corollary 3.1.8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and let $\phi : \bar{\Omega} \times \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ be a polyconvex stored energy function, i.e. assume there is a Carathéodory integrand $\tilde{\varphi} : \bar{\Omega} \times \mathbb{R}^{9\times 9\times 1} \to \mathbb{R} \cup \{\infty\}$ such that $\tilde{\varphi}(x, \cdot)$ is convex $\forall_{aa} x \in \Omega$ and such that*

$$\forall_{aa} x \in \Omega \forall F \in \mathbb{R}^{3\times 3} : \quad \varphi(x, F) = \tilde{\varphi}(M(F)).$$

Furthermore, assume $p > 3$. Then, the functional given by $\mathcal{E}(y) := \int_\Omega \varphi(x, \nabla y(x)) dx$ is sequentially lower semicontinuous with respect to the weak $W^{1,p}(\Omega; \mathbb{R}^3)$-topology.

Proof. Let $(y_k) \subset W^{1,p}(\Omega; \mathbb{R}^3)$ be a weakly convergent subsequence, i.e., let $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $y_k \rightharpoonup y$ in $W^{1,p}$. By Thm. 3.1.2 we know that also $\mathrm{Cof}(\nabla y_k) \rightharpoonup \mathrm{Cof}(\nabla y)$ in $L^{p/2}$ and $\det \nabla y_k \rightharpoonup \det \nabla y$ in $L^{p/3}$. Now, we can apply Thm. 3.1.1, setting $v_k := M(\nabla y_k)$, $z_k := y_k$ and $\xi(x, z, v) := \tilde{\varphi}(x, M(\nabla y)) = \varphi(x, \nabla y)$. $\square$

Remark 3.1.5. Noting that polyconvexity is a sufficient condition for weak lower semicontinuity, we can ask ourself the question: is it also a necessary condition? The answer is no and the search for a necessary condition will lead to the notion of quasiconvexity as we will see in Chapter 4.

3.2. Existence result

We have all the ingredients now to state our first existence result for hyperelastic, polyconvex materials.

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, and $\varphi : \bar{\Omega} \times \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{+\infty\}$ a stored energy function satisfying*

(i) Polyconvexity: let there be a Carathéodory integrand $\tilde{\varphi} : \bar{\Omega} \times \mathbb{R}^{9\times 9\times 1} \to \mathbb{R} \cup \{+\infty\}$ such that

$$\tilde{\varphi}(x, \cdot) \text{ is convex for almost all } x \in \Omega \text{ and}$$
$$\varphi(x, F) = \tilde{\varphi}(x, M(F)) \quad \forall_{aa} x \in \Omega \, \forall F \in \mathbb{R}^{3\times 3}.$$

(ii) Coercivity and frame-indifference:

$$\varphi(x,F)=\begin{cases}\to\infty & \textit{if} \quad \det F\to 0\\ =\infty & \textit{if} \quad \det F\le 0\end{cases} \quad \forall_{aa}x\in\Omega, \textit{ and} \tag{3.2.1}$$
$$\varphi(x,RF)=\varphi(x,F) \quad \forall R\in SO(d).$$

Furthermore assume that $\exists c>0$, $p\ge 2$, $q\ge p/(p-1)$, $r>1$ *such that*

$$\varphi(x,F)\ge\begin{cases} c\left(|F|^p+|\operatorname{Cof}F|^q+(\det F)^r\right) & \textit{if} \quad \det F>0\\ +\infty & \textit{if} \quad \textit{otherwise.}\end{cases}$$

(iii) Admissibility: Let $\partial\Omega=\Gamma=\Gamma_D\cup\Gamma_N$ *a measurable partition of the boundary with* $meas_2(\Gamma_D)>0$, *let* $y_D\in W^{1,p}(\Omega,\mathbb{R}^d)$ *be given and let*

$$\mathcal{A}=\{y\in W^{1,p}(\Omega,\mathbb{R}^d):\operatorname{Cof}\nabla y\in L^q(\Omega,\mathbb{R}^{d\times d}),\, \det\nabla y\in L^r(\Omega),$$
$$y=y_D \textit{ on } \Gamma_D, \det\nabla y>0 \textit{ a.e. }\}\neq\emptyset.$$

(iv) $y\mapsto\mathcal{F}(y)$ *be such that* $-\mathcal{F}$ *is weak lower semicontinuous, and* $\mathcal{F}(y)\le\tilde{K}\left(\|y\|^s_{W^{1,p}}+1\right)$ *for* $\le s<p$.

(v) Let there be a $y\in\mathcal{A}$ *such that* $\mathcal{E}(y)<\infty$.

Then the minimum of

$$\mathcal{E}(y)=\int_\Omega\varphi(x,\nabla y(x))dx-\mathcal{F}(y)$$

over $\mathcal{A}$ *exists.*

Proof. Since $\tilde{\varphi}$ is a Carathéodory integrand, the mapping $x\mapsto\tilde{\varphi}(x,M(\nabla y))$ is measurable. By the coercivity assumption and the growth condition on $\mathcal{F}$, we get

$$\mathcal{E}(y)\ge c\int_\Omega|\nabla y|^p+|\operatorname{Cof}\nabla y|^q+(\det\nabla y)^r-\tilde{K}\|y\|^s_{W^{1,p}}-\tilde{K}.$$

Applying Poincaré's inequality A.2.7 yields

$$\mathcal{E}(y)\ge c'\left(\|y\|^p_{W^{1,p}}+\|\operatorname{Cof}\nabla y\|^q_{L^q}+\|\det\nabla y\|^r_{L^r}\right)-\tilde{K},$$

for a constant $c'>0$ and all $y\in\mathcal{A}$. Let $(y_k)\subset\mathcal{A}$ be a minimizing sequence, i.e., satisfying $\lim\mathcal{E}(y_k)=\inf_{\mathcal{A}}\mathcal{E}<\infty$. The previous coercivity estimates yield that the sequence $(\nabla y_k,\operatorname{Cof}\nabla y_k,\det\nabla y_k)$ is bounded in the reflexive Banach space $W^{1,p}\times L^q\times L^r$, and thus, has subsequence weakly converging to some element $(y,A,\delta)\in W^{1,p}\times L^q\times L^r$.

By the Thms. 3.1.3 and 3.1.6, we can conclude $\operatorname{Cof}\nabla y_k \rightharpoonup A = \operatorname{Cof}\nabla y$ in L^q and $\det\nabla y_k \rightharpoonup \delta = \det\nabla y$ in L^r. Theorem 3.1.1 implies that $\mathcal{E}$ is sequentially weakly lower semicontinuous. We are left with verifying that y is admissible. The boundary conditions are satisfied by the continuity of the trace, Thm. A.2.1. Moreover, $\det\nabla y > 0$ is satisfied, for if not, then $\mathcal{E}(y) = \infty$, because the stored energy would be ∞, by (3.2.1). But this contradicts $\mathcal{E}(y) = \liminf \mathcal{E}(y_k) < \infty$. $\square$

4. Quasiconvexity and existence results for quasiconvex materials

As we already mentioned, polyconvexity of the integrand is sufficient, but not necessary for weak lower semicontinuity of an integral functional. In this section, we will treat the notion of quasiconvexity, first introduced by Morrey in [21]. As it will turn out, quasiconvexity is necessary for weak lower semicontinuity. To prove this we will introduce a general tool, so called Young measures (or parametrized measures). Furthermore, the relation between polyconvexity and quasiconvexity is examined, which makes, as a by-product, the assertion about polyconvexity and weak lower semicontinuity, i.e. Thm. 3.1.1, rigorous.

4.1. Quasiconvexity

We follow [17], cf. Definition 4.1.1. (as opposed to the definition provided in [27], Sec. 5.1).

Definition 4.1.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We say that $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ is *quasiconvex*, if for any $A \in \mathbb{R}^{d\times d}$ and every $y \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ it holds that

$$\varphi(A) \leq \frac{1}{\text{meas}_d(\Omega)} \int_\Omega \varphi(A + \nabla y(x))dx,$$

whenever the integral on the right hand side exists.

Remark 4.1.1. The definition of quasiconvexity is independent of the choice of a particular Lipschitz domain, i.e. in the above definition Ω could be replaced by an arbitrary bounded Lipschitz domain and the set of quasiconvex functions would still be the same. For the proof of this claim see [17], Section 4.1 or [27], Lemma 5.2.

Moreover, in the definition of quasiconvexity, if φ satisfies a p-growth condition, one could use $W_0^{1,p}$ test functions instead of $W_0^{1,\infty}$, cf. [27], Lemma 5.2.

Remark 4.1.2. An issue in the definition of quasiconvexity is whether to allow φ to take the value ∞ or not. If we allow φ to take ∞, we could run into problems. In fact, it cannot be shown that for such functions quasiconvexity is equivalent to weak lower semicontinuity. It has not yet been proven, that quasiconvexity for such functions is a sufficient condition, cf. [8], Rem. 5.2. Furthermore, quasiconvex functions taking the value $+\infty$ are not necessarily rank-one convex as this would be in the case for quasiconvex functions into $\mathbb{R}$, see [17], Prop. 4.1.6. However, defining quasiconvexity only for $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R}$ conflicts with the choice we made to extend the stored energy function to $\{\infty\}$ to make it continuous (cf. formula (2.7.7) and the following paragraph). To solve this issue, and in accordance with the focus on mathematical elasticity, we follow [17] and define quasiconvexity for functions into the extended real numbers.

Before we start discussing the properties of quasiconvex functions, we want to give a intuition of why quasiconvexity is somehow a natural notion. Quasiconvexity of an energy functional means that affine deformation amount to less energy than internally distorted deformations. Let

$$\mathcal{F}(y) = \int_\Omega \varphi(\nabla y(x))dx$$

be given with quasiconvex integrand φ. Furthermore, let y be an affine deformation, i.e. y_a is of the form $y_a(x) = y_0 + Ax$, for $y_0 \in \mathbb{R}^3$ and $A \in \mathbb{R}^{3\times 3}$, and thus, $\nabla y_a = A$. Then, quasiconvexity of φ implies that

$$\mathcal{F}(y_a) = \int_\Omega \varphi(A) \leq \int_\Omega \varphi(A + \nabla\psi(x))dx = \mathcal{F}(y_a + \psi).$$

As quasiconvexity is not a pointwise notion, it can be difficult to verify if a given function is quasiconvex or not. To circumvent this problem we will introduce another, weaker concept of convexity, which can be easily checked.

Definition 4.1.2. A function $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$ is called *rank-one convex*, if for all $A, B \in \mathbb{R}^{d\times d}$ with $\operatorname{rank}(A - B) \leq 1$ and all $\lambda \in [0, 1]$ the following inequality is satisfied

$$\varphi(\lambda A + (1 - \lambda)B) \leq \lambda\varphi(A) + (1 - \lambda)\varphi(B).$$

Note, that $\operatorname{rank}(A - B) \leq 1$ if and only if there are $a, b \in \mathbb{R}^d$ with $A - B = a \otimes b$.

Rank-one convexity is related to ellipticity of partial differential equations. In fact, if $\varphi \in \mathscr{C}^2(\mathbb{R}^{d\times d})$, then rank-one convexity is equivalent to the so-called *Legendre-Hadamard condition*

$$\sum_{i,j=1}^{d} \sum_{\alpha,\beta=1}^{d} \frac{\partial^2\varphi(\xi)}{\partial_{i,\alpha}\partial\xi_{j,\beta}} \lambda_i\lambda_j\mu_\alpha\mu_\beta \geq 0,$$

for every $\lambda, \mu \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{d\times d}$. We will not go into detail but instead refer to [8], Chapter 5.

Whereas in the one-dimensional case all notions of convexity introduced so far are equivalent, this is not the case for $d \leq 2$. For higher dimensions only the following implications hold.

Theorem 4.1.1. *Let $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R}$. Then the following chain of implications holds true*

$$\varphi \text{ convex} \implies \varphi \text{ polyconvex} \implies \varphi \text{ quasiconvex} \implies \varphi \text{ rank-one convex.}$$

For the proof, see [27], Prop. 6.1, or [8], Theorem 5.3. The converse implications do not hold in general.

As it may cause trouble considering functions which can have the value ∞ (see Rem. 4.1.2), we stated the previous result for functions into $\mathbb{R}$. For functions $\varphi : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$, the implications

$$\varphi \text{ convex} \implies \varphi \text{ polyconvex} \implies \varphi \text{ rank-one convex,}$$

still hold true (refer to [8], Thm. 5.3).

Theorem 4.1.1 provides a necessary and sufficient condition for quasiconvexity, which are in general easier to verify.

4.2. Young measures

Young measures, named after its inventor L.C. Young [30], are a major tool in nowadays theory of calculus of variations. We are going to use them to prove that quasiconvexity of the integrand is a necessary condition for weak lower semicontinuity. In the first section, we want to lay out the main ideas of the proof and the importance of Young measures in a very informal fashion, which we will make more precise in the following parts.

Introduction

Suppose we are given a functional $\mathcal{F}(v) := \int_\Omega f(x, v(x))dx$ where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Consider a sequence $v_j \rightharpoonup v$ in $L^2(\Omega)$. Then, $(\mathcal{F}(v_j))$ is bounded and contains a convergent subsequence. But how to compute this limit for an arbitrary integrand f? Or, equivalently, what is the weak-* limit in L^∞ of the sequence $(f(x, v_j(x)))_j$? The problem is that, although $v_j \rightharpoonup v$, in general $f(x, v_j) \not\rightharpoonup^* f(x, v)$. Formulated differently, if we have $v_j \rightharpoonup v$ and $f(v_j) \rightharpoonup^* g$, then in general $g \neq f(v)$. Thus, we have no chance to pass to the limit for nonlinear f! As an example consider

$$v_j(x) := \begin{cases} -1 & \text{if} \quad jx - \lfloor jx \rfloor \in (0, 1/2] \\ 1 & \text{if} \quad jx - \lfloor jx \rfloor \in (1/2, 1) \end{cases}$$

on $\Omega = (0,1)$. Then $v_j \rightharpoonup v \equiv 0$, but for $f(x) = x^2$, we get that $f(v_j) \equiv 1 \neq 0 = f(v)$.

Young measures contribute in clarifying this, for they provide a finer description of the limit of the sequence $(\mathcal{F}(v_j))_j$. Informally, this due to the fact that Young measures encode the oscillations, so that the information about the oscillations does not get lost.

We give a precise definition below. For the moment let a Young measure be a family of probability measures $\nu = (\nu_n)_{x\in\Omega}$ associated to the sequence v_j such that for any continuous function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ we have

$$\bar{f}(x) = \int f(x, A) d\nu_x(A)$$

is measurable. The crucial observation is that the weak-* limit of f_j is exactly $\bar{f}$, i.e.

$$\lim_{j\to\infty} \int f(x, v_j)\psi(x)dx = \int \psi(x) \int f(x, A)d\nu_x(A)dx = \int \psi(x)\bar{f}(x). \tag{4.2.1}$$

If we absorb the test function into f and set $f(x, A) = A$, then (4.2.1) implies that $v_j \rightharpoonup \int A d\nu_x(A)$.

The idea is the following: Consider a highly oscillating sequence (e.g., take a sequence jumping between -1 and 1 with increasing frequency as above) and a fixed point $x_0 \in \Omega$. Then, the value of the function at x_0 is 1 with probability 1/2 or -1 with probability 1/2. This concept of introducing a measure at a point, telling us the probability of a function value to be attained, yields the Young measures.

Using Young measures we show that quasiconvexity is sufficient for weak lower semicontinuity. The proof relies on two properties of Young measures, namely a lower semicontinuity result resembling Fatou's lemma, presented in Prop. 4.3.1, and a Jensen-type inequality for quasiconvex functions, given in Prop. 4.3.2. We give an idea how to apply these results to prove that quasiconvexity implies weak lower semicontinuity, for the rigorous result see Thm. 4.3.4. If we assume that the following lower semicontinuity result for Young measures hold

$$\liminf \int_\Omega f(V_n(x))dx \geq \int_\Omega \left(\int_{\mathbb{R}^d} f(\xi) d\nu_x(\xi) \right) dx, \tag{4.2.2}$$

we get weak lower semicontinuity of $v \mapsto \int_\Omega f(v(x))dx$ by

$$\begin{aligned}\liminf \int_\Omega f(v_n)dx &\overset{(4.2.2)}{\geq} \int_\Omega \left(\int_{\mathbb{R}^d} f(\xi)d\nu_x(\xi) \right) dx \\ &\overset{Jensen}{\geq} \int_\Omega f\left(\int \xi d\nu_x(\xi) \right) dx \overset{(4.2.1)}{=} \int_\Omega f(v(x))dx.\end{aligned}$$

In the next sections we make these arguments precise.

Fundamental properties of Young measures

Before we start with the treatment of Young measures we need to introduce the notion of equiintegrability (from [27], A.3 Measure Theory).

Definition 4.2.1. A family $(f_j) \subset L^p(\Omega; \mathbb{R}^m)$ is called *L^p-equiintegrable* (sometimes also called *uniformly integrable*), if one of the following equivalent conditions hold

(i) $\lim_{R \nearrow \infty} \sup_{j \in \mathbb{N}} \int_{\{|f_j| > R\}} |f_j|^p dx = 0$,
(ii) $\lim_{R \nearrow \infty} \limsup_{j \to \infty} \int_{\{|f_j| > R\}} |f_j|^p dx = 0$,
(iii) for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all Borel sets $B \subset \Omega$ with $\mathrm{meas}(B) < \delta$ we have
$$\sup_{j \in \mathbb{N}} \int_B |f_j|^p dx < \varepsilon.$$

The following theorem gives us an equivalent description of equiintegrability for the particular case of $p = 1$.

Theorem 4.2.1 (Dunford-Pettis)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and open. A norm-bounded family $(f_j) \subset L^1(\Omega)$ is equiintegrable if and only if it is weakly sequentially precompact in $L^1(\Omega)$.*

For the proof refer to [4], Thm. 4.7.18 and note that we additionally employed the Eberlein-Šmulian Theorem A.4.4 here.

The following theorem states the existence of a Young measure, as well as the convergence result, we claimed in the introduction.

Theorem 4.2.2 (Fundamental theorem of Young measures)**.** *Let $(v_j) \subset L^p(\Omega, \mathbb{R}^N)$ be a norm-bounded sequence, where $p \in [1, \infty]$. Then, there exists a subsequence (non-relabelled) of (v_j) and a family of probability measures*

$$(\nu_x)_{x \in \Omega} \subset \mathcal{M}^1(\mathbb{R}^N),$$

called the (L^p-)Young measure generated by the (sub)sequence (v_j), such that the following assertions are true:

(i) The family $(\nu_x)_{x\in\Omega}$ is weakly- measurable, that is, for all Carathéodory integrands $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$, the compound function*

$$x \mapsto \langle f(x,\cdot), \nu_x\rangle := \int f(x,A)d\nu_x(A)$$

is Lebesgue-measurable.

(ii) If $p \in [1,\infty)$, then

$$\int_\Omega \int |A|^p d\nu_x(A)dx < \infty,$$

or, in the case of $p=\infty$, there is a compact set $K \subset \mathbb{R}^N$ such that

$$\operatorname{supp}\nu_x \subset K \qquad \forall_{aa} x \in \Omega.$$

(iii) For all Carathéodory integrands $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ with the property that the family $\big(f(x,v_j)\big)_j$ is uniformly L^1-bounded and equiintegrable, it holds that

$$f(x,v_j) \rightharpoonup \left(x \mapsto \int f(x,A)d\nu_x(A)\right) \quad \text{in } L^1. \tag{4.2.3}$$

Parametrized measures $\nu = (\nu_x)_x$ satisfying items (i) and (ii) above are called *Young measures* and we write $\nu = (\nu_x)_x \in \mathbb{Y}^p(\Omega,\mathbb{R}^N)$.

Writing out (iii) yields

$$f(x,v_j) \rightharpoonup \left(x \mapsto \int f(x,A)d\nu_x(A)\right) \quad \text{in } L^1$$

$$\Longleftrightarrow$$

$$\int_\Omega f(x,v_j(x))\psi(x)dx \to \int_\Omega \int \psi(x)f(x,A)d\nu_x(A)dx \quad \forall \psi \in L^\infty.$$

Since $(f(x,v_j))_j$ is uniformly L^1-bounded and equiintegrable if and only if $(\psi(x)f(x,v_j))_j$ is L^1-bounded and equiintegrable, we can absorb the test function ψ into f. Then, we can express (4.2.3) equivalently as

$$\int_\Omega f(x,v_j(x))dx \to \int_\Omega \int f(x,A)d\nu_x(A)dx = \int_\Omega \langle f(x,\cdot);\nu_x\rangle =: \langle\!\langle f,\nu\rangle\!\rangle.$$

We do not present the proof of the Fundamental Theorem here, as it would go beyond the scope of this thesis. Instead, the reader may refer to [27], Chapter 4.1.

Definition 4.2.2. Young measures generated by gradients of L^p functions are called *gradient Young measures*. More precisely, let $\nu \in \mathbb{Y}^p(\Omega;\mathbb{R}^{m\times d})$, $p \in [1,\infty]$. We say that ν is a $W^{1,p}$*-gradient Young measure*, and write $\nu \in \mathbb{GY}^p(\Omega,\mathbb{R}^{m\times d})$, if there is a norm-bounded sequence $(u_j)_j \subset W^{1,p}(\Omega,\mathbb{R}^m)$ such that (∇u_j) generates ν.

Note, that not every Young measure is a gradient Young measure. This leads to the question whether we can characterize all gradient Young measures. This question is positively answered by the theorem of Kinderlehrer and Pedregal, which will be stated later in this section.

Now we summarize some properties of (gradient) Young measures, which will be needed to prove that quasiconvexity implies lower semicontinuity.

Lemma 4.2.3. *Let $(v_j) \subset L^p(\Omega;\mathbb{R}^N)$, $p \in (1,\infty)$, be a sequence generating the Young measure $\nu \in \mathbb{Y}^p(\Omega;\mathbb{R}^N)$. Then,*

$$v_j \rightharpoonup v \quad \text{ in } L^p,$$

where $v(x) := [\nu](x) = [\nu_x] = \langle \mathrm{id}, \nu_x \rangle = \int A d\nu_x(A)$.

Proof. Since L^p is reflexive ($p \in (1,\infty)$ by assumption), bounded sequences are weakly compact, and thus, by the Dunford-Pettis theorem 4.2.1 also L^1-equiintegrable. Therefore, we can apply the assertion (iii) in the Fundamental Theorem of Young measures 4.2.2 for the integrand $f(x,A) = \mathrm{id}(A) = A$. $\square$

4.3. Lower semicontinuity

Now, we are about to prove the equivalence of weak lower semicontinuity and quasiconvexity of the integrand. Recall that in the proof of Thm. 1.3.1, where we established weak lower semicontinuity for a convex integrand, we used two main ingredients: Fatou's Lemma and Jensen's inequality. This will also be the goal in the case of quasiconvex integrands, as we already indicated in the introduction. Therefore, we want to state and prove similar results for Young measures.

We start with a lower semicontinuity result for the duality pairing, which in some sense will replace Fatou's Lemma.

Proposition 4.3.1 (Lower semicontinuity result for Young measures)**.** *Let $(v_j) \subset L^p(\Omega;\mathbb{R}^N)$, $p \in [1,\infty)$, be a norm-bounded sequence, generating the Young measure*

$\nu \in \mathbb{Y}^p(\Omega;\mathbb{R}^N)$ and let $f:\Omega\times\mathbb{R}^N \to [0,\infty)$ be a Carathéodory integrand (not necessarily equiintegrable). Then,

$$\liminf_{j\to\infty}\int_\Omega f(x,v_j(x))dx = \liminf_{j\to\infty}\langle\!\langle f,\delta(v_j)\rangle\!\rangle \geq \langle\!\langle f,\nu\rangle\!\rangle.$$

Proof. For $k\in\mathbb{N}$ set $f_k(x,A) := \min\{f(x,A),k\}$. This ensures equiintegrability of (f_k) and thus (iii) of the Fundamental Theorem 4.2.2 is applicable, which yields

$$\int_\Omega f_k(x,v_j(x))dx \to \langle\!\langle f_k,\nu\rangle\!\rangle = \int_\Omega\int f_k(x,A)d\nu_x(A)dx.$$

Since $f\geq f_k$, we have

$$\liminf_{j\to\infty}\int_\Omega f(x,v_j(x))dx \geq \langle\!\langle f_k,\nu\rangle\!\rangle.$$

By letting $k\to\infty$ and using the monotone convergence theorem A.1.2, we obtain the assertion. □

Now, we state a Jensen-type inequality. Whereas the classical Jensen inequality holds for convex integrands only (and therefore, also for quasiconvex integrands), this result will work for quasiconvex, but not necessarily convex, functions as well. In this sense, it extends the classical Jensen inequality. Still, one needs a certain growth condition on the integrand.

Definition 4.3.1. We say a Carathéodory function $f:\Omega\times\mathbb{R}^{m\times d}\to\mathbb{R}$ has *p-growth*, if there is an $M>0$ such that

$$|f(x,A)| \leq M\left(1+|A|^p\right).$$

Additionally, one needs to assume the following property on the Young measure.

Definition 4.3.2. A Young measure $(\nu_x)_x = \nu \in \mathbb{Y}^p(\Omega;\mathbb{R}^N)$ is called *homogeneous*, if ν_x is constant almost everywhere in $x\in\Omega$.

Proposition 4.3.2 (Jensen-type inequality)**.** *Let $\nu\in\mathbb{GY}^p(B(0,1),\mathbb{R}^{m\times d})$ with $p\in(1,\infty)$ be a homogeneous gradient Young measure. Then, for all quasiconvex functions $f:\mathbb{R}^{m\times d}\to\mathbb{R}$ with p-growth it holds that*

$$f([\nu]) \leq \int f d\nu.$$

We only want to sketch the proof here. For all details, refer to [27], Lemma 5.11.

Proof. For convex integrands the conclusion holds by the classical Jensen inequality. Set $F := [\nu]$.

We claim, that there is a $(u_j) \subset W^{1,p}_{Fx}(B(0,1);\mathbb{R}^m)$ (meaning $u|_{\partial B(0,1)} = Fx$ in a trace sense), such that (∇u_j) generates ν, and that this sequence (∇u_j) is L^p-equiintegrable (cf. [27], Lemma 4.13).

Note: $u_j \in W^{1,p}_{Fx} \iff u_j - Fx \in W^{1,p}_0$ and $\nabla F = F$. By the definition of quasiconvexity (with $y = u_j - Fx$) we have

$$h(F) \leq \frac{1}{\text{meas}(\Omega)} \int_\Omega h(\nabla u_j(x))dx$$

for all $j \in \mathbb{N}$.

The growth assumption on h yields that $(h(\nabla u_j))$ is equiintegrable, and thus, by passing to the Young measure limit as $j \to \infty$, we obtain for the right-hand side

$$h(F) \leq \frac{1}{\text{meas}(\Omega)} \int_\Omega \int h d\nu dx = \int h d\nu.$$

(Here, the homogeneity of ν plays an important role.) $\square$

To apply the Jensen-type inequality above, one needs a homogeneous Young measure. Therefore, the last ingredient for proving lower semicontinuity is a localization (also called blow-up) technique, which allows us to work with homogeneous measures.

Proposition 4.3.3 (Blow-up technique)**.** *Let $\nu = (\nu_x)_x \in \mathbb{GY}^p(\Omega;\mathbb{R}^{m\times d})$, $p \in [1,\infty)$, be a gradient Young measure. Then, for almost all $x_0 \in \Omega$ the probability measure ν_{x_0} is a homogeneous gradient Young measure, $\nu_{x_0} \in \mathbb{GY}^p(B(1,0);\mathbb{R}^{m\times d})$.*

We will not repost the proof here, but instead refer to [27], Prop. 5.14.

After we collected the necessary tools, we are now able to verify that quasiconvexity implies weak lower semi continuity, a result which was proved by Morrey in 1952 in [21] under stronger assumptions, and later by Acerbi and Fusco in 1984 in [1] using different methods. For simplicity, we will only consider functionals <u>not</u> depending on u.

Theorem 4.3.4 (Quasiconvexity $\implies$ w.l.s.c)**.** *Let $p \in (1,\infty)$ and let $f : \Omega \times \mathbb{R}^{m\times d} \to [0,\infty)$ be a Carathéodory integrand with p-growth and such that $f(x,\cdot)$ is quasiconvex for almost every $x \in \Omega$. Then, the functional $\mathcal{F}(u) := \int_\Omega f(x,\nabla u(x))dx$ is weakly lower semicontinuous on $W^{1,p}$.*

For u-dependent functionals we will need an additional lemma, stated in [27], Lemma 5.19.

Proof. Let $(u_j) \subset W^{1,p}(\Omega, \mathbb{R}^m)$ such that $u_j \rightharpoonup u$ in $W^{1,p}$. This means $\nabla u_j \rightharpoonup \nabla u$ in L^p. Therefore, (∇u_j) is bounded (cf. Thm. A.4.5). Thus, it generates a gradient Young measure $\nu \in \mathbb{GY}^p(\Omega; \mathbb{R}^{m\times d})$. By Lemma 4.2.3 we have that $\nabla u_j \rightharpoonup [\nu]$ and by the uniqueness of the weak limit $\nabla u = [\nu]$. From the result on lower semicontinuity for Young measures, Prop. 4.3.1, we get

$$\liminf_{j\to\infty} \int_\Omega f(x, \nabla u_j(x))dx \geq \langle\langle f, \nu \rangle\rangle = \int_\Omega \int f(x, A) d\nu_x(A) dx. \tag{4.3.1}$$

Now, by the blow-up technique from Prop. 4.3.3, we can consider $\nu_x \in \mathbb{GY}^p(B(0;1); \mathbb{R}^{m\times d})$ as homogeneous Young measure for almost all $x \in \Omega$. Thus, the Jensen-type inequality Prop. 4.3.2 applies and yields

$$\int f(x, A) d\nu_x(A) \geq f(x, \nabla u(x)) \quad \forall_{aa} x \in \Omega. \tag{4.3.2}$$

Combining (4.3.1) and (4.3.2), we obtain

$$\liminf_{j\to\infty} \mathcal{F}(u_j) \geq \mathcal{F}(u),$$

and thus, have proved the assertion. □

As already mentioned, quasiconvexity is actually equivalent to weak lower semicontinuity.

Theorem 4.3.5 (w.l.s.c $\implies$ quasiconvexity)**.** *Let $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ be continuous with $p-$growth. If the functional*

$$\mathcal{F}(y) = \int_\Omega f(\nabla y(x)) dx,$$

where $y \in W^{1,p}(\Omega, \mathbb{R}^m)$, is weakly lower semicontinuous, then f is quasiconvex.

Proof. By Remark 4.1.1 we have to verify that

$$f(A) \leq \frac{1}{|B(0,1)|} \int_{B(0,1)} f(A + \nabla \psi(z)) dz,$$

where $A \in \mathbb{R}^{m\times d}$, $\psi \in W_0^{1,\infty}(B(0,1), \mathbb{R}^m)$, and $B(0,1)$ denotes the ball with radius 1 centred at 0. One can assume (after a possible translation and scaling of the domain) that $B(0,1) \Subset \Omega$ holds.

By Thm. A.5.5, there exists for all $j \in \mathbb{N}$ a Vitali covering of $B(0,1)$ in disjoint balls, i.e. $a_k^{(j)} \in B(0,1)$ and $r_k^{(j)} > 0$, with the additional property that $r_k^{(j)} \leq 1/j$, such that

$$B(0,1) = Z^{(j)} \cup \bigcup_{k=1}^{\infty} B(a_k^{(j)}, r_k^{(j)}),$$

where $|Z^{(j)}| = 0$. Since $B(0,1) \Subset \Omega$ we can find a smooth function $h : \Omega \backslash B(0,1) \to \mathbb{R}^m$ with $h(x) = Ax$ for $x \in \partial B(0,1)$. Define

$$y_j(x) := \begin{cases} Ax + a_k^{(j)} \psi\left(\frac{x - a_k^{(j)}}{r_k^{(j)}}\right) & \text{if} \quad x \in B(a_k^{(j)}, r_k^{(j)}) \\ h(x) & \text{if} \quad x \in \Omega \backslash B(0,1). \end{cases}$$

Since ψ is bounded, we can conclude that $y_j \rightharpoonup y$ in $W^{1,p}$, where

$$y(x) := \begin{cases} Ax & \text{if} \quad x \in B(a_k^{(j)}, r_k(j)) \\ h(x) & \text{if} \quad x \in \Omega \backslash B(0,1). \end{cases}$$

Therefore, by the weak lower semicontinuity, we get

$$\begin{aligned} \int_{B(0,1)} f(A)dx &\leq \liminf_{j\to\infty} \int_{B(0,1)} f(\nabla y_j(x))dx \\ &= \liminf_{j\to\infty} \sum_{k=1}^{\infty} \int_{B(a_k^{(j)}, r_k^{(j)})} f\left(A + \nabla\psi\left(\frac{x - a_k^{(j)}}{r_k^{(j)}}\right)\right) dx \\ &= \liminf_{j\to\infty} \sum_{k=1}^{\infty} (r_k^{(j)})^d \int_{B(0,1)} f(A + \nabla\psi(x'))dx' \\ &= \int_{B(0,1)} f(A + \nabla\psi(x'))x', \end{aligned} \tag{4.3.3}$$

where we used a change of variables, indicated by a change in notation, and the fact that $\sum_{k=1}^{\infty} (r_k^{(j)})^d = 1$. To see this, we write out the equality of the volume of the ball and the volume of the Vitali covering. Denoting with ω_d the volume of the unitary d-sphere, we get

$$\omega_d = |B(0,1)| = \bigcup_{k=1}^{\infty} |B(a_k^{(j)}, r_k^{(j)})| = \sum_{k=1}^{\infty} \omega_d (r_k^{(j)})^d.$$

Looking at the result of the calculation in (4.3.3), we see that this is exactly the definition of quasiconvexity. □

At this point, we want to give an example of a Young measure, which is not a gradient

Young measure. As the barycenter of a gradient Young measure is again a gradient, and thus, curl-free, it is enough to consider a the Young measure $\delta(v)$ for a v with $\operatorname{curl} v \not\equiv 0$, to get a counterexample. Although it is outside the scope of this thesis, we want to emphasize the importance of the Jensen-type inequality, since it characterizes all gradient Young measures in the class of Young measures.

Theorem 4.3.6 (Kinderlehrer-Pedregal)**.** *Assume that $\nu \in \mathbb{Y}^p(\Omega;\mathbb{R}^{m\times d})$, for $p \in (1,\infty]$, is a Young measure with $[\nu] = \nabla y$ for some $y \in W^{1,p}(\Omega;\mathbb{R}^m)$. Then, ν is a gradient Young measure, i.e. $\nu \in \mathbb{GY}^p(\Omega;\mathbb{R}^{m\times d})$, if and only if for almost all $x \in \Omega$ and all quasiconvex functions $h : \mathbb{R}^{m\times d} \to \mathbb{R}$ with p-growth (no growth condition for $p = \infty$), the Jensen-type inequality*

$$h(\nabla y(x)) \leq \int h d\nu_x$$

holds.

The proof can be found in [27], Thm. 7.15.

Eventually, we summarize the results of this chapter by giving the following existence result for quasiconvex integrands.

Theorem 4.3.7. *Let $f : \Omega \times \mathbb{R}^{m\times d} \to [0,\infty)$ be a Carathéodory integrand satisfying*

(i) f has p-growth, for $p \in (1,\infty)$,

(ii) there exists $c > 0$ such that the following p-coercivity estimate holds

$$c|A|^p \leq f(x,A),$$

(iii) f is quasiconvex in the second argument.

Then, the functional

$$\mathcal{F}(y) = \int_\Omega f(x,\nabla y(x))dx$$

has a minimizer in

$$W^{1,p}_g(\Omega,\mathbb{R}^m) = \left\{ y \in W^{1,p}(\Omega;\mathbb{R}^m) \,:\, u|_{\partial\Omega} = g \right\},$$

with $g \in W^{1-1/p,p}(\partial\Omega;\mathbb{R}^m)$ and the equality at the boundary in the trace sense.

Proof. By the work done above, leading ultimately to Thm. 4.3.4, the claim follows from the direct method, as soon as we have established coercivity. Therefore, we have to show that any sequence $(y_j) \subset W^{1,p}_g(\Omega;\mathbb{R}^m)$ such that $\sup_j \mathcal{F}(y_j) < \infty$ is weakly precompact, i.e. admits a weak $W^{1,p}$-convergent subsequence. By the p-growth condition, we get

$$\infty > \sup_j \mathcal{F}(y_j) \geq c \sup_j \int_\Omega |\nabla y_j|^p dx.$$

If we fix $y_0 \in W_g^{1,p}(\Omega; \mathbb{R}^m)$, then $y_j - y_0 \in W_0^{1,p}$ and thus, by the Poincaré inequality A.2.5, we get

$$\sup_j \|y_j\|_{W^{1,p}} \leq \sup_j \|y_j - y_0\|_{W^{1,p}} + \|y_0\|_{W^{1,p}} < \infty.$$

Since $W^{1,p}(\Omega; \mathbb{R}^m)$ is a separable and reflexive Banach space for $p \in (1, \infty)$, this uniform bound on $\|y_j\|_{W^{1,p}}$ implies the existence of a weakly convergent subsequence, and we have verified coercivity. The direct method implies the existence of a minimizer and the continuity of the trace operator ensures that the minimizer satisfies the boundary condition. $\square$

5. Invertibility of deformations

By the Inverse Function Theorem, the condition of orientation-preservation $\det \nabla y > 0$ implies that y is at least locally invertible. This, however, does not ensure global invertibility as the following example shows.

Example 5.0.1. Consider the set $\Omega := (1,2)\times(0,4\pi) \subset \mathbb{R}^2$ and the deformation $y : \Omega \to \mathbb{R}^2$ given by $y(x_1, x_2) := (x_1 \cos(x_2), x_1 \sin(x_2))$. Then $(-x_1, 0) = y(x_1, \pi) = y(x_1, 3\pi)$ and therefore y is not injective and cannot be invertible everywhere. Still the orientation is preserved

$$\det \nabla y = \det \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2 \end{pmatrix} = x_1 \cos^2 x_2 + x_1 \sin^2 x_2 = x_1 > 0. \qquad \triangle$$

The so called Ciarlet-Nečas condition entail no self-penetration instead.

Definition 5.0.1. We say a deformation y satisfies the *Ciarlet-Nečas*-condition if

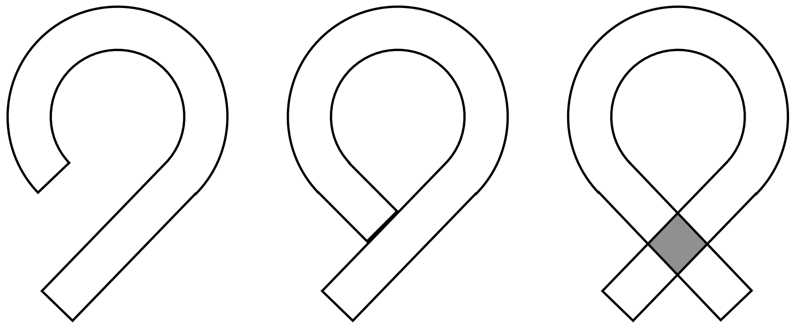
$$\int_\Omega \det \nabla y(x) dx \leq \mathrm{meas}_d(y(\Omega)) \tag{5.0.1}$$

holds.

We sometimes abbreviate “Ciarlet-Nečas-condition” by CN.

Remark 5.0.1. To see that this condition indeed prevents self-penetration, consider Fig. 5.1, where the Ciarlet-Nečas condition is violated. The right-hand side of (5.0.1) gives us the area of $y(\Omega)$, where the grey part is counted once. But since the determinant of the Jacobian describes the local stretching and rotation caused by the deformation y, the left hand side of the Ciarlet-Nečas condition (considering the integral as “infinitesimal sum”) adds up all of these local changes and thus, gives us the area of the deformed configuration if we would “unbend” $y(\Omega)$. This means, the grey part is counted twice, which makes the left-hand side larger and violates (5.0.1).

The Ciarlet-Nečas condition, combined with orientation-preservation, ensures global injectivity almost everywhere.



(a) No self-penetration CN holds

(b) Self-contact, but no self-penetration CN holds

(c) Self-penetration CN violated

Figure 5.1.: Ciarlet-Nečas condition.

The proof relies on the change of variables formula, which uses the so-called Banach indicatrix.

Definition 5.0.2. For any $z \in \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ the *Banach indicatrix* $N(z, y, \Omega)$ is the number of elements in Ω, which are mapped to z by y, formally

$$N(z, y, \Omega) := \#\{x \in \Omega \,:\, y(x) = z\},$$

where the right-hand side is the counting measure.

Theorem 5.0.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, let $p > d$ and $y \in W^{1,p}(\Omega, \mathbb{R}^d)$ satisfying*

(i) Orientation-preservation: $\det \nabla y > 0$ *almost everywhere in* Ω*, and*

(ii) Ciarlet-Nečas:
$$\int_\Omega \det \nabla y(x) dx \leq meas_d(y(\Omega)).$$

Then, for almost every $x^y \in \Omega^y$ there is only one $x \in \Omega$ satisfying $y(x) = x^y$. Using the Banach indicatrix (defined in 5.0.2), this means $N(x^y, y, \Omega) = 1$ for almost all $x^y \in \Omega^y$.

Proof. By the change of variables formula from Thm. A.3.4 and the Ciarlet-Nečas condition, we have

$$\int_{y(\Omega)} N(x^y, y, \Omega) dx^y = \int_\Omega \det \nabla y(x) dx \leq \mathrm{meas}_d(y(\Omega)) = \int_{y(\Omega)} 1 dx^y.$$

Since $N(x^y, y, \Omega) \leq 1$ must hold for all $x^y \in \Omega^y$, we get $N(x^y, y, \Omega) = 1$ for almost all $x^y \in \Omega^y$. $\square$

Fortunately, the CN condition is compatible with the existence result established in the previous sections. This means, when imposing the CN condition, in addition to the assumptions of the existence result Thm. 3.2.1, yields almost everywhere invertible minimizers of the energy functional. Before we prove this claim, we will make precise the notion of almost everywhere invertibility.

Definition 5.0.3. A deformation $y : \Omega \to \mathbb{R}^d$ is *injective almost everywhere* in a bounded domain $\Omega \subset \mathbb{R}^d$, if there is $\omega \subset \Omega$ such that $\text{meas}_d(\omega) = 0$ and $y(x_1) \neq y(x_2)$ for every $x_1, x_2 \in \Omega \backslash \omega$, with $x_1 \neq x_2$.

Another important notion in this context is Lusin's N-condition.

Definition 5.0.4 (Lusin's conditions)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $y : \Omega \to \mathbb{R}^d$ is said to satisfy *Lusin's condition N*, if for every $\omega \subset \Omega$ with $\text{meas}_d(\omega) = 0$ it holds that $\text{meas}_d(y(\omega)) = 0$.

Of course, one can consider functions whose pre-image of null sets as again a null set. The function $y : \Omega \to \mathbb{R}^d$ is said to satisfy *Lusin's condition* N^{-1}, if for every $\tilde{\omega} \subset y(\Omega)$ with $\text{meas}(\tilde{\omega}) = 0$ it holds that $\text{meas}_d(y^{-1}(\tilde{\omega})) = 0$.

Regular Sobolev functions on bounded sets automatically satisfy Lusin's condition N.

Lemma 5.0.2. *Let $\Omega \subset \mathbb{R}^3$ be bounded and $y : \Omega \to \mathbb{R}^3$ such that $y \in W^{1,p}(\Omega; \mathbb{R}^3)$, with $p > 3$. Then y satisfies Lusin's condition N.*

The proof can be found in [20], Cor. 1.

If an almost everywhere injective deformation y satisfies Lusin's condition N, then $\text{meas}_d(y(\Omega)) = \text{meas}_d(y(\Omega\backslash\omega))$, and y as a map $y : \Omega\backslash\omega \to y(\Omega\backslash\omega)$ is injective.

Now we have all tools to prove the main theorem of this section.

Theorem 5.0.3 (Injectivity almost everywhere)**.** *Let all assumptions of Theorem 3.2.1 hold and $p > d$. Furthermore, let*

$$\mathcal{A}_{inj} := \mathcal{A} \cap \{y \in W^{1,p}(\Omega; \mathbb{R}^d) \,:\, y \textit{ satisfies the Ciarlet-Nečas condition } (5.0.1)\}.$$

Then, there is a minimizer of $\mathcal{E}$ on $\mathcal{A}_{inj}$, which is injective almost everywhere in Ω. In particular, this minimizer also satisfies the Ciarlet-Nečas condition.

The core of the proof of this statement is to verify weak continuity of the Ciarlet-Nečas condition, i.e. if $(y_n) \subset W^{1,p}$ satisfies CN and weakly converges to some y, $y_n \rightharpoonup y$, then y satisfies CN. One way to prove this, is to employ the regularity of the Lebesgue measure and compactness of the set $y(\bar{\Omega})$ as presented here (cf. [17], Thm. 3.4.6).

Proof. We will show that a minimizer of $\mathcal{E}$ is also injective. Since $p > d$, by Morrey's inequality in Thm. A.2.2, we can embed $W^{1,p} \subset \mathcal{C}^{0,\gamma}$, with $\gamma = 1 - \frac{d}{p}$. By Arzelà-Ascoli's Theorem (Thm. A.2.4), we have the compact embedding $\mathscr{C}^{0,\gamma} \Subset \mathscr{C}$, and thus, $W^{1,p} \Subset \mathscr{C}$. This means that a minimizing sequence $y_k \rightharpoonup y$, which is of course bounded, contains a subsequence (not relabelled) converging uniformly, i.e. $\|y_k - y\|_{L^\infty} \to 0$. By Lemma 5.0.2, y satisfies Lusin's condition N, and thus, $\text{meas}_d(y(\partial\Omega)) = 0$ as $\text{meas}_d(\partial\Omega) = 0$, since Ω is a Lipschitz domain. Therefore, one gets $|y(\Omega)| = |y(\bar{\Omega})|$. Moreover, the set $y(\bar{\Omega})$ is compact, and hence, by the regularity of the Lebesgue measure, for any $\varepsilon > 0$ there exists an open set O_ε with $y(\bar{\Omega}) \subset O_\varepsilon$ and

$$|O_\varepsilon \backslash y(\bar{\Omega})| < \varepsilon. \tag{5.0.2}$$

Next we want to show that there is a $N \in \mathbb{N}$, such that $y_k(\bar{\Omega}) \subset O_\varepsilon$ for all $k \geq N$. To do so, we claim that there is a $\delta = \delta(\varepsilon)$ such that

$$\bigcup_{x \in y(\bar{\Omega})} B(x, \delta) \subset O_\varepsilon. \tag{5.0.3}$$

If this was not the case, we could find some $\varepsilon > 0$ and sequences δ_k with $\delta_k \to 0$ as $k \to \infty$, and $(x_k) \subset y(\bar{\Omega})$, such that there is a $z_k \in B(x_k, \delta_k)$, but $z_k \notin O_\varepsilon$. Since $y(\bar{\Omega})$ is compact, there is some $x \in y(\bar{\Omega})$ and a (nonrelabelled) subsequence, such that $x_k \to x$ and thus, also $z_k \to x$. But $\mathbb{R}^d \backslash O_\varepsilon$ is closed, and therefore, $x \in \mathbb{R}^d \backslash O_\varepsilon$, which contradicts the fact that $y(\bar{\Omega}) \subset O_\varepsilon$. Hence, inclusion (5.0.3) holds. We infer, by the definition of uniform continuity, the existence of a N such that for all $|y_k(x) - y(x)| \leq \delta$ for all $x \in \bar{\Omega}$ and all $k \geq N$. Thus, by (5.0.3), also $y_k(\bar{\Omega}) \subset O_\varepsilon$ for all $k \geq N$. Since y_k satisfies the CN-condition, we can conclude

$$\int_\Omega \det \nabla y_k dx \leq |y_k(\bar{\Omega})| \leq |O_\varepsilon|,$$

for all $k \geq N$. By the weak convergence of the determinant Thm. 3.1.6, we get

$$\int_\Omega \det \nabla y = \lim \int_\Omega \det \nabla y_k \leq O_\varepsilon. \tag{5.0.4}$$

Since $|O_\varepsilon| = |y(\bar{\Omega})| + |O_\varepsilon \backslash y(\bar{\Omega})|$ (Lemma A.5.6), we obtain by (5.0.2) and (5.0.4) and the fact that ε was arbitrary

$$\int_\Omega \det \nabla y \leq |y(\bar{\Omega})| = |y(\Omega)|.$$

Therefore, we have proved that also the weak limit satisfies the Ciarlet-Nečas condition.

The last step is to use this to show that the limit is also injective almost everywhere. This follows from the change of variable formula Thm. A.3.4

$$|y(\Omega)| = \int_{y(\Omega)} dx^y \leq \int_{y(\Omega)} N(x^y, y, y(\Omega))dx^y = \int_{\Omega} \det \nabla y(x)dx \leq |y(\Omega)|,$$

which implies that $N(x^y, y, y(\Omega)) = 1$ for almost all $x^y \in y(\Omega)$. Now, if $\omega \subset y(\Omega)$ is a null-set on which $N(x^y, y, \omega) > 1$, then also $\{x \in \Omega : y(x) \in \omega\}$ is a null-set, since y satisfies Lus □

The idea of the proof above can be used to prove another useful result.

Lemma 5.0.4. *Let $y_n, y \in W^{1,p}(\Omega, \mathbb{R}^3)$, with $p > 3$, such that $y_n \rightharpoonup y$. Then, there is a (non-relabelled) subsequence y_n such that*

$$|\Omega^{y_n} \Delta \Omega^y| \to 0 \quad \textit{as } n \to \infty.$$

The idea of the proof is as follows: since $p > 3$ we can assume uniform convergence and thus can bound $\Omega^{y_n} \Delta \Omega^y$ between two sets whose measure are arbitrarily close.

Proof. As in the proof of Thm. 5.0.3, we can assume that $y_n \to y$ uniformly (passing to a subsequence), and for each $\varepsilon > 0$ we find an open set O_ε such that $|O_\varepsilon \backslash y(\bar{\Omega})| < \varepsilon$. Furthermore, there is a $N \in \mathbb{N}$ such that $y_k(\Omega) \subset O_\varepsilon$ for all $k \geq N$.

By a similar argument one can also find a closed set A_ε such that $|y(\bar{\Omega}) \backslash A_\varepsilon| < \varepsilon$ and such that $A_\varepsilon \subset y_k(\Omega)$ for k large enough.

Thus, one concludes (using Lemma A.5.6 and the monotonicity of the measure) with

$$\begin{aligned} |\Omega^{y_n} \Delta \Omega^y| &= |(\Omega^{y_n} \cup \Omega^y) \setminus (\Omega^{y_n} \cap \Omega^y)| \\ &= |\Omega^{y_n} \cup \Omega^y| - |\Omega^{y_n} \cap \Omega^y| \leq |O_\varepsilon| - |A_\varepsilon| = |O_\varepsilon \backslash A_\varepsilon| < 2\varepsilon. \end{aligned}$$

Note, that we need to assume $|y(\Omega)| < \infty$ to apply Lemma A.5.6. However, this is satisfied, because $p > 3$, by the lemma below. □

Lemma 5.0.5. *Let $\Omega \subset \mathbb{R}^d$ and assume that $y \in W^{1,p}(\Omega)$, with $p > 3$. Then, we can bound the measure of $y(\Omega)$ in terms of the $W^{1,p}$-norm of y, i.e. there is a $c > 0$, not depending on y, such that*

$$|y(\Omega)| \leq c\|y\|^3_{W^{1,p}}.$$

Proof. Recall the identity

$$\det \nabla y = \sum \varepsilon_{jkn} \partial_j y_i \partial_k y_l \partial_n y_m,$$

where ε_{jkn} denotes the Levi-Civita symbol. Note that, because $y \in W^{1,p}(\Omega; \mathbb{R}^3)$ and $p \geq 3$, we automatically get $\partial_j y_k \in L^3(\Omega)$, for all $1 \leq j, k \leq 3$. Thus, generalized Hölder's inequality, Thm. A.1.5, yields

$$|y(\Omega)| = \int_{y(\Omega)} dx^y = \int_\Omega |\det \nabla y| dx \leq c \int |\partial_j y_i||\partial_k y_l||\partial_n y_m| \leq c \|\nabla y\|_{L^3}^3 = c \int |\nabla y|^3.$$

By Hölder's inequality, we get $\|\nabla y\|_{L^3}^3 \leq c \|\nabla y\|_{L^p}^3$ for any $p > 3$ and thus by the Sobolev Embedding Theorem A.2.2, we get

$$|y(\Omega)| \leq c \|\nabla y\|_{W^{1,p}}^3. \qquad \square$$

Remark 5.0.2. One could proceed differently to prove the weak continuity of the Ciarlet-Nečas condition, after obtaining the result above.

First, note the trivial statements

$$\begin{aligned} A \cap B \subset B &\implies |A \cap B| \leq |B|, \\ (A \cap B^c) \subset (A \cap B^c) \cup (B \cap A^c) &\implies |A \cap B^c| \leq |A \Delta B|, \end{aligned}$$

which leads us to

$$|A| = |(A \cap B) \cup (A \cap B^c)| \leq |A \cap B| + |A \cap B^c| \leq |B| + |A \Delta B|,$$

and thus,

$$|A| - |B| \leq |A \Delta B|.$$

Because of symmetry, one can interchange A and B in the above calculations.

This preliminary considerations allow us to conclude

$$\begin{aligned} |y_n(\Omega)| &\leq |y(\Omega)| + |y_n(\Omega) \Delta y(\Omega)| \text{ and} \\ |y(\Omega)| &\leq |y_n(\Omega)| + |y_n(\Omega) \Delta y(\Omega)|. \end{aligned}$$

Lemma 5.0.4 implies that $|y_n(\Omega) \Delta y(\Omega)| \to 0$ as $n \to \infty$, which implies $\liminf |y_n(\Omega)| \geq$

$|y(\Omega)|$ and hence, by symmetry,

$$\liminf_{n\to\infty} |y_n(\Omega)| = |y(\Omega)|.$$

The theorem on convergence of minors, Thm. 3.1.2, implies that $\det \nabla y_n \rightharpoonup \det \nabla y$ in $L^{p/d}$, i.e. $\int_\Omega \det \nabla y_n g \to \int_\Omega \det \nabla y g$ for all $g \in (L^{p/d})^*$. In particular this must hold for $g = \mathrm{id}$ (assuming $p \neq \infty$, to be sure that g is an L^q-function).

Therefore, we have

$$\int \det \nabla y_n \to \int \det \nabla y \leq \liminf |y_n(\Omega)| = |y(\Omega)|$$

and have proved that also y satisfies the Ciarlet-Nečas condition.

6. The model

In this section, we will consider the two applied forces, which appear in our problem: gravity and buoyancy. In particular, we will model the buoyancy and derive an integral formulation of the work done by the applied forces, using our calculations from Section 2.4. This will eventually lead to a functional $\mathcal{E}$ incorporating all necessary information.

6.1. Starting Point: Archimedes' principle

Consider a compressible object submerged under water (or a different medium). We consider two forces acting on this object: gravity and buoyancy. Gravity can be modelled by the gravitational force density in the deformed configuration $f^y : \Omega^y \to \mathbb{R}^3$ given by

$$f^y(x^y) = -g\rho_S^y(x^y)e_3,$$

where g is the gravitational acceleration, $\rho_S^y(x^y)$ the density of the specimen at a point $x^y \in \Omega^y$ and e_3 denotes the vertical unit vector of the standard basis in $\mathbb{R}^3$, which means that e_3 points "upwards", and thus, f^y points "downwards". To calculate the corresponding force in the reference configuration, we recall the formulas (2.5.5) and (2.5.6), which were derived in Section 2.5:

$$\begin{aligned} f(x) &= f^y(x^y)\det\nabla y(x), \\ \rho(x) &= \rho^y(x^y)\det\nabla y(x). \end{aligned}$$

For simplicity, we choose the density in the reference configuration $\rho_S : \Omega \to \mathbb{R}$ to be constant, i.e, $\rho_S(x) \equiv \rho_S$. Thus, we get $\rho_S^y(y(x))\det\nabla y(x) = \rho_S$, which implies

$$f(x) = f^y(x^y)\det\nabla y(x) = -g\rho_S^y(y(x))\det\nabla y(x)e_3 = -g\rho_S e_3.$$

Recall from Section 2.6 that f is a dead load, and thus a conservative body force with potential $\hat{F}(x,y) = f(x)\cdot y(x)$. For dead loads the corresponding functional describing the work is given by

$$\mathcal{F}_g(y) = \int_\Omega \hat{F}(x, y(x))dx = \int_\Omega f(x) \cdot y(x)dx = -\int_\Omega g\rho_S y_3(x)dx.$$

To model the buoyancy (cf. [29], Sec. 2.6), we note that the pressure conditions in the fluid (due to gravity) cause a force. To determine this force consider the fluid in static equilibrium. In this case, each portion of the fluid is in equilibrium, i.e. the buoyancy exactly compensates the gravitational force. If we replace the portion of the fluid by some object, then the buoyancy is unchanged and only the gravitational force changes. Therefore, we arrive at the following

Axiom (Archimedes' Principle)**.** *The buoyancy force that is exerted on an immersed body is in absolute value equal to the weight of the displaced fluid.*

It has been suggested by the famous antique mathematician, physicist, and inventor Archimedes of Syracuse in his treatise "On Floating Bodies", Book I, cf. [15]. By the Archimedes' principle, the buoyancy would then be given by $\mathcal{F}_b = \rho_W V_{displ} g e_3$, where V_{displ} is the volume of the displaced fluid and $\rho_W > 0$ is the density of the fluid. The volume of the displaced fluid is the volume of the deformed configuration, i.e. $V_{displ} = |y(\Omega)| = \int_{y(\Omega)} dx^y$. Consequently, one could guess that the work could be given by

$$\mathcal{F}_b(y) = \int_{y(\Omega)} \rho_W g x_3^y dx^y.$$

To make this precise, we have to take a different approach (following [19]), starting from the *hydrostatic equation* (cf. [26], eq. (3.3))

$$\nabla p(x^y) = -\rho_W g e_3. \tag{6.1.1}$$

Here, p denotes the pressure and we additionally assume that the gravitational field is uniform and vertically pointing downwards. Note that we have to work in Eulerian (spatial) coordinates, as we want to determine the pressure experienced by the actual deformed specimen. Equation (6.1.1) can be deduced from physical considerations and Newton's second law (see [26], Chap. 3.1). If we write (6.1.1) component wise we obtain

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho_W g.$$

Let us now integrate over Ω^y, to get

$$p(x^y) = -\rho_W g x_3^y.$$

Since the pressure load is conservative, with work functional given by formula (2.6.1), we obtain

$$\mathcal{F}_b(y) = \int_\Omega \rho_W g y_3(x) \det \nabla y(x) dx = \int_{\Omega^y} \rho_W g x_3^y dx^y.$$

Thus, all forces experienced by the submerged object can be modelled by the following functional

$$\mathcal{F} = \mathcal{F}_g + \mathcal{F}_b = -\int_\Omega g \rho_S y_3(x) dx + \int_{\Omega^y} \rho_W g x_3^y dx^y.$$

Remark 6.1.1 (on ρ_S, ρ_W). In this thesis we assume ρ_S and ρ_W to be constant, as it simplifies the calculations in next chapters. For more evolved models, however, this may be too restricting. In particular, for inhomogeneous materials one needs to look at ρ_S depending on $x \in \Omega$. Moreover, for a gaseous medium a density increasing in direction $-e_3$ may be considered. However, these interesting cases are not in the scope of the thesis.

6.2. Well-definedness

We now check that the functional $\mathcal{F}$ is well-defined and make sure that $\mathcal{F}$ is indeed weak lower semicontinuous, which is crucial to apply the direct method.

We prove that both integrals are well-defined separately. Whereas in the buoyancy integral the integrand f (omitting the constants) is of the form $f_b(y) = \mathbb{1}_{y(\Omega)} y_3 \det \nabla y$, the integrand in the gravitational integral looks like $f_g(y) = y_3(x)$.

Well-definedness for $\mathcal{F}_g$ follows from a measure-theoretic result, which is here just sketched. For the detailed discussion, we refer to [13], Sec. 5.1, where the proof can be found (check Thm. 5.1).

Theorem 6.2.1 (Well-definedness). *Let $\Omega \subset \mathbb{R}^d$ be a measurable set with finite measure and let $1 \leq p < \infty$. Furthermore, assume $f : \mathbb{R}^d \to [-\infty, \infty]$ be a measurable function. Then,*

$$\int_\Omega (f(x))^- dx < \infty$$

for every $y \in L^p(\Omega; \mathbb{R}^d)$ if and only if there is a constant $c > 0$ such that

$$f(z) \geq -c \left(|z|^p + 1\right) \quad \forall z \in \mathbb{R}^d. \tag{6.2.1}$$

The superscript minus sign indicates the negative part of the function, defined by $f^+ := \max\{+f, 0\}$ and $f^{+-} := \max\{-f, 0\}$, respectively. This convention implies that $f^-, f^+ \geq 0$, and $f = f^+ - f^-$.

Setting $f(z) := z_3$, and omitting the constants, we get

$$f(x, z) = z_3 \geq -|z_3| \geq -\frac{1}{p}|z|^p - \frac{1}{q}, \tag{6.2.2}$$

after applying Young's inequality with $1/p + 1/q = 1$. Thus, we get (6.2.1).

To check well-definedness of $\mathcal{F}_b$, we will verify that $\mathbb{1}_{y(\Omega)} y_3 \det \nabla y \in L^1(\Omega)$ directly. By assumption $y \in W^{1,p}(\Omega; \mathbb{R}^3)$, and thus $\det \nabla y \in L^{p/3}(\Omega)$. If $p > d$ we can use the Sobolev Embedding Theorem, Thm. A.2.2 and obtain

$$\begin{aligned}\int_\Omega |\mathbb{1}_{y(\Omega)}||y_3(x)|| \det \nabla y| &\leq \|y\|_{L^\infty} \|\det \nabla y\|_{L^1} \\ &\leq c\|y\|_{W^{1,p}} \|\det \nabla y\|_{L^{p/3}(\Omega)} \leq c\|y\|^2_{W^{1,p}} \infty.\end{aligned}$$

6.3. Weak lower semicontinuity

In the following chapters, we will try to find a minimizer to the functional $\mathcal{E}(y) = \int_\Omega W(\nabla y) - \mathcal{F}(y)$ by employing the direct method. Therefore, we need to check, that $-\mathcal{F}$ is indeed weakly lower semicontinuous. It it will prove useful to be able to treat both integrals separately, which is admissible by the superadditivity of the $\liminf$.

Lemma 6.3.1 (Superadditivity of $\liminf$). *For any two sequences a_n, b_n, the following inequality holds*

$$\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n),$$

whenever the left-hand side is well-defined.

For a proof check [11], Thm. 3.127.

Thus, we will show that $-\mathcal{F}_i(y) \leq \liminf_n -\mathcal{F}_i(y_n)$, for $i \in \{b, g\}$ and a sequence $y_n \rightharpoonup y$ in $W^{1,p}$ and conclude

$$\begin{aligned}-\mathcal{F}(y) &= -\mathcal{F}_g(y) - \mathcal{F}_b(y) \\ &\leq \liminf_n -\mathcal{F}_g(y_n) - \liminf_n \mathcal{F}_b(y_n) \\ &\leq \liminf_n \left(-\mathcal{F}_g(y_n) - \mathcal{F}_b(y_n)\right) \leq \liminf_n -\mathcal{F}(y_n).\end{aligned}$$

To prove weak lower semicontinuity of $-\mathcal{F}_g(y) = \int_\Omega g\rho_S y_3(x)dx$, we use the following theorem.

Theorem 6.3.2 (Weak lower semicontinuity)**.** *Let $\Omega \subset \mathbb{R}^3$ be a measurable set, $1 \leq p < \infty$, and $f : \mathbb{R}^3 \to (-\infty, \infty]$ be a measurable and lower semicontinuous function. Furthermore, assume that there is a constant $c > 0$ such that*

$$f(z) \geq -c\,(1 + |z|^p) \quad \forall z \in \mathbb{R}^3. \tag{6.3.1}$$

Then, the functional

$$y \in L^p(\Omega;\mathbb{R}^3) \mapsto \int_\Omega f(y(x))dx$$

is L^p-weakly sequentially lower semicontinuous if and only if f is convex.

The proof can be found in [13], Sec. 5.2.2.

The theorem provides us with weak lower semicontinuity with respect to the weak L^p-topology, whereas we actually want to conclude w.l.s.c. for the weak-$W^{1,p}$-topology. This, however, is not a problem, since by Thm. A.4.8, $y_n \rightharpoonup y$ in $W^{1,p}$ implies that $y_n \rightharpoonup y$ in L^p. Therefore, we only have to check whether the assumptions of the previous theorem are satisfied for $f(y) := y_3(x)$, again omitting the constants. First, notice that the calculation in (6.2.2) implies (6.3.1). Moreover, the mapping $y \mapsto f(y)$ is linear, and thus, convex and continuous. Hence, we can apply Thm. 6.3.2 and get the weak lower semicontinuity of $-\mathcal{F}_g$.

For $-\mathcal{F}_b$ we use that $\mathbb{1}_{y_n(\Omega)} \to \mathbb{1}_{y(\Omega)}$ in $L^1(\Omega)$, provided $y_n \rightharpoonup y$, which can be proved using the Lemma 5.0.4, which states that $|y_n(\Omega)\Delta y(\Omega)| \to 0$. Thus, we have

$$\begin{aligned}\|\mathbb{1}_{y_n(\Omega)} - \mathbb{1}_{y(\Omega)}\|_{L^1} &= \int |\mathbb{1}_{y_n(\Omega)} - \mathbb{1}_{y(\Omega)}| \\ &= \int_{\mathbb{R}^3} \mathbb{1}_{y_n(\Omega)\Delta y(\Omega)} = |y_n(\Omega)\Delta y(\Omega)| \to 0.\end{aligned}$$

Therefore, we obtain for the $\liminf$ (which is even a lim in this case)

$$\begin{aligned}\liminf \left(- \int_{y_n(\Omega)} \rho_W g x_3 dx \right) &= \liminf \left(- \int_{\mathbb{R}^3} \rho_w g x_3 \mathbb{1}_{y_n(\Omega)}(x) dx \right) \\ &= - \int_{\mathbb{R}^3} \rho_W g x_3 \mathbb{1}_{y(\Omega)}(x) dx = - \int_{y(\Omega)} \rho_w g x_3 dx.\end{aligned}$$

We summarize the results of this section in the following

Corollary 6.3.3. *The functional*

$$-\mathcal{F} = -\mathcal{F}_g - \mathcal{F}_b = \int_\Omega g\rho_S(x)y_3(x)dx - \int_{y(\Omega)} \rho_M g x_3^y dx^y$$

is lower semicontinuous with respect to the weak $W^{1,p}$-topology.

6.4. The barycentre

We introduce the barycentre of Ω^y, which will turn up in different occasions during this thesis.

Definition 6.4.1. The *barycentre* $S = (s_1, s_2, s_3)$ of a body $B \subset \mathbb{R}^3$ is defined via

$$s_i = \frac{1}{\mathrm{vol}(B)} \int_B x_i dx,$$

where $i = 1, 2, 3$ and $\mathrm{vol}(B) = \int_B dx = |B|$ is the volume and dx indicates that the integrals have to be understood as volume integrals.

Notation: We will often denote the barycentre of y_i with the bar, i.e. $\bar{y}_i = |\Omega|^{-1} \int_\Omega y_i(x)dx$.

The barycentre is often called centroid, or centre of mass, because it is the particle equivalent of the object for the application of Newton's laws. Formulated differently, at the barycentre we would apply the gravitational force to get a linear acceleration with no angular acceleration. Note, that this definition assumes that B has uniform density.

Now we want to calculate the barycentre of a deformed body. Let $y : \bar{\Omega} \to \mathbb{R}^3$ be a deformation and $y(\bar{\Omega})$ be the deformed body. If we set $x^y := y(x)$, then the coordinates s_i^y of the barycentre of $y(\bar{\Omega})$ are given by

$$s_i^y = \frac{1}{\mathrm{vol}(y(\bar{\Omega}))} \int_{y(\Omega)} x_i^y dx^y.$$

By the change of variables formula, we obtain

$$s_i^y = \frac{1}{\mathrm{vol}(y(\bar{\Omega}))} \int_{y(\Omega)} x_i^y dx^y = \frac{1}{|\bar{\Omega}^y|} \int_\Omega y_i(x) \det \nabla y(x) dx.$$

Remark 6.4.1. In the formula above $\det \nabla y$ appears. This is to be expected, as also the density of Ω^y is not uniform anymore (as it was in the reference configuration). The density of the medium, however, is assumed to be constant, which means that the centre of mass of the deformed configuration, i.e. the point where the gravitational force applies and the centre of mass of the displaced fluid, i.e. the point where the buoyancy would apply, are *not* the same. If they are not vertically aligned, one would have an angular acceleration, which would rotate the object and drive the object towards a position, where the application point of the forces are vertically aligned. We therefore assume for the rest of the thesis that no such rotation occurs, which is part of the Archimedes' principle.

Moreover, the energy functional given by

$$\mathcal{E}(y) = \int_{\Omega} W(\nabla y(x))dx + \int_{\Omega} \rho_S g e_3 \cdot y(x)dx - \int_{y(\Omega)} \rho_W g x_3^y dx^y,$$

is invariant under e_1 and e_2 translations, i.e., for a deformation with additional translation in directions e_1, e_2, expressed as $\tilde{y}(x) = y(x) + (t_1, t_2, 0)$, we obtain

$$\mathcal{E}(\tilde{y}) = \mathcal{E}(y). \tag{6.4.1}$$

To see this, note that the second and third term in $\mathcal{E}$, describing the potential energy coming from gravity and buoyancy, do not depend on y_1 and y_2. Thus, we only have to check the claim for the first term. Since $\nabla\tilde{y}(x) = \nabla(y(x) + (t_1, t_2, 0) = \nabla y(x) + 0$, also $\int \nabla\tilde{y} = \int \nabla y$ and therefore (6.4.1) holds.

This means, that we can w.l.o.g. choose t_1, t_2 in such a way that

$$s_1^y = s_2^y = 0,$$

i.e. the barycentre lies on the z-axis.

Thus, by the result above and Remark 6.4.1 we are left with the question of how s_3^y behaves. We impose additional conditions, guaranteeing that $\bar{y}_3$ stays bounded.

7. Existence results for fixed conditions

In the previous sections, we have collected a variety of mathematical tools, we want to bring to application in this section. The problem is as follows: Consider a hyperelastic specimen fully submerged into water, but fixed at the boundary. How will this object behave? To answer this question, we will minimize the energy functional describing the deformation energy and the potential energy coming from gravity and buoyancy. The existence of minimizers will be guaranteed by the direct method as soon as we have verified coercivity and lower semicontinuity of the corresponding energy functional.

7.1. Dirichlet boundary conditions

At first, we consider an object with prescribed boundary values. In particular, we fix a part of the boundary. Problems with prescribed Dirichlet boundary values, as this one, are called *problems of place* (cf. [7], Sec. 2.6.) or *pure displacement problems* (cf. [17], Chap. 3), whereas for Neumann boundary conditions, on speaks of a *problems of traction.* Mixed boundary conditions are called *displacement-traction problems.*

We start with defining the energy functional we want to minimize. We assume $\Omega \subset \mathbb{R}^3$ with $|\Omega| < \infty$, where $|\cdot|$ denotes the d-dimensional Lebesgue measure. Furthermore, we assume Ω to have a Lipschitz boundary $\Gamma = \partial\Omega$, (cf. [18], Def. 9.57 for the definition), with subset $\Gamma_D \subset \Gamma$ such that Γ_D has positive surface measure as in Fig. 7.1. For $y \in W^{1,p}(\Omega;\mathbb{R}^3)$ define

$$\mathcal{E}(y) := \int_\Omega W(\nabla y(x))dx + \int_\Omega \rho_S g\vec{e}_3 \cdot y(x)dx - \int_{y(\Omega)} \rho_W g x_3^y dx^y,$$

where $\rho_S > 0$ denotes the density of the solid and $\rho_W > 0$ the density of the surrounding medium (e.g. water), respectively. The first term of $\mathcal{E}$ describes the deformation energy, the second term models the energy coming from the gravitational force acting on the material and the third term corresponds to buoyancy.

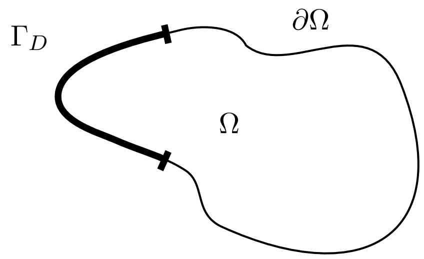


Figure 7.1.: Boundary Value Problem

Let $p > 4$ and set

$$\mathcal{A} := \left\{ y \in W^{1,p}(\Omega) \, : \, y = \mathrm{id} \text{ on } \Gamma_D, \, CN \right\},$$

to be the set of admissible functions. In the above definition we abbreviated CN to denote that y satisfies the Ciarlet-Nečas condition. Note that $\mathcal{A}$ is not empty since the identity satisfies all given conditions. The choice $p > 4$ may seem to be arbitrary at the first glance, but it will be apparent why we need to choose $p > 4$ in the proof.

Moreover, we assume that the material is hyperelastic and polyconvex, i.e., there is a stored energy function $W : GL^+(d) \to [0, \infty)$ satisfying

(i) W is polyconvex, (7.1.1)

(ii) $W(F) \to +\infty$ as $\det F \searrow 0+$, and (7.1.2)

(iii) $W(F) \geq c_1 |F|^p - c_2$, (7.1.3)

for some constants $c_1, c_2 > 0$.

Remark 7.1.1 (on dealing with constants). Over the course of some lengthy calculations many different constants will appear, for instance from Sobolev embeddings, Hölder's inequality, etc. We will not keep track of all these constants and we will denote different constants with the symbol c, possibly changing from line to line. However, when ever necessary, we will add subscripts. For example, we would write for the coercivity condition above (7.1.3)

$$W(F) \geq c|F|^p - c,$$

although the two appearing constants are not necessarily the same.

Theorem 7.1.1. *Under the assumptions above the minimization problem*

$$\textit{Minimize } \mathcal{E}(y) \textit{ for } y \in \mathcal{A}$$

has a solution.

Before we start with the proof, we want to point out a major tool of establishing coercivity, namely the Poincaré inequality.

Lemma 7.1.2 (Generalized Poincaré inequality)**.** *Let* $\Omega \subset \mathbb{R}^3$ *as above and* $p \geq 1$*. Then, there is a constant* $c > 0$ *such that*

$$\|y\|_{W^{1,p}} \leq c\left(\|\nabla y\|_{L^p(\Omega)} + \|y - \mathrm{id}\,\|_{L^p(\Gamma_D)}\right). \tag{7.1.4}$$

The proof can be found in [16], Lemma 3.3. Since we assume $y = \mathrm{id}$ on Γ_D, in our case relation (7.1.4) reduces to

$$\|y\|_{W^{1,p}} \leq c\|\nabla y\|_{L^p(\Omega)}. \tag{7.1.5}$$

Proof of Thm. 7.1.1. We have that $\mathrm{id} \in \mathcal{A}$ and $\mathcal{E}(\mathrm{id}) < \infty$. Notice that $\nabla\,\mathrm{id} = \mathbb{I}$ and hence, $\det \nabla\,\mathrm{id} = 1$, which yields

$$\mathcal{E}(\mathrm{id}) = \int_\Omega W(\mathbb{I}) + \int_\Omega \rho_S g x_3 dx - \int_\Omega \rho_W g x_3 dx \leq \int_\Omega W(\mathbb{I}) + |\Omega| g(\rho_S - \rho_W).$$

By assumption 7.1.2, we know that $W(\mathbb{I}) < \infty$, which implies that also $\int_\Omega W(\mathbb{I}) dx < \infty$.

We aim to apply the direct method, so let $y_n \in \mathcal{A}$ realizing the inf, i.e. $\mathcal{E}(y_n) \to \inf_{\mathcal{A}} \mathcal{E}$. We now are going to prove coercivity and weak lower semicontinuity.

Step 1: Coercivity: We need to show that the set $\{\mathcal{E} \leq \Lambda\} \Subset \mathcal{A}$ w.r.t. to the weak $W^{1,p}$ topology. Since $W^{1,p}$ is reflexive, it is sufficient to show that the set $\{\mathcal{E} \leq \Lambda\}$ is bounded in the $W^{1,p}$-norm. As we will do throughout the thesis, we will consider each term of the energy functional separately, i.e. set

$$\mathcal{E}(y) = \underbrace{\int_\Omega W(\nabla y(x)) dx}_{=:I_1} + \underbrace{\int_\Omega \rho_S g e_3 \cdot y(x) dx}_{=:I_2} - \underbrace{\int_{y(\Omega)} \rho_W g x_3^y dx^y}_{=:I_3}.$$

By the coercivity condition (7.1.3) and the reduced Poincaré inequality (7.1.5) we get that

$$I_1 = \int_\Omega W(\nabla y(x)) dx \geq \int_\Omega c|\nabla y(x)|^p dx - c|\Omega| \geq c_P \|y\|^p_{W^{1,p}(\Omega)} - c.$$

For the second term, we use the Cauchy-Schwarz inequality and Young's inequality (with $1/p + 1/q = 1$) to conclude

$$\begin{aligned} I_2 = \int_\Omega \rho_S g e_3 \cdot y(x) dx &\geq -\int_\Omega c|e_3 \cdot y(x)| dx \\ &\geq -c \int_\Omega |y(x)| dx \\ &\geq -c \int_\Omega \left(|y(x)|^p \frac{\delta^p}{p} + \frac{1}{q\delta^q} \right) \\ &\geq -c\delta^p \|y\|_{L^p}^p - c. \end{aligned}$$

We will need the notion of the *diameter of a nonempty set* $M \subset \mathbb{R}^d$, given by

$$\varnothing(M) := \sup_{x,y \in M} |x - y|,$$

with $|x| := \left(\sum_{i=1}^d x_i^2\right)^{1/2}$ denoting the Euclidean distance in $\mathbb{R}^d$.

For the third term I_3, we use a geometrical argument. As the points on Γ_D are fixed by the boundary condition, a point in the deformed configuration cannot get mapped further afar than the diameter of $y(\Omega)$. Thus, for a fixed point $x_0 \in \Gamma_D$, and *any* point $x^y \in y(\Omega)$ we have

$$|x^y - x_0| \leq \varnothing(y(\Omega)).$$

Since for any two points $x, z \in \Omega$, we have $|y(x) - y(z)| \leq |y(x) - x| + |x - z| + |z - y(z)| \leq 2\|y - \mathrm{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega)$, we can take the supremum and obtain

$$\varnothing(y(\Omega)) \leq 2\|y - \mathrm{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega).$$

In particular, for $x^y \in y(\Omega)$ one arrives at

$$|x_3^y| \leq |x_0| + 2\|y - \mathrm{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega).$$

Recall that $x_0 \in \Gamma_D$ is fixed. It hence only depends on the reference configuration, as well as $\varnothing(\Omega)$. Therefore, we can further simplify the equation above to

$$|x_3^y| \leq c + 2\|y - \mathrm{id}\|_{L^\infty(\Omega)},$$

which holds true for all $x^y \in y(\Omega)$.

Using this geometrical observation, we get

$$I_3 \geq -\int_{y(\Omega)} \rho_w g |x_3^y| dx^y \geq -c|y(\Omega)| - c|y(\Omega)| \|y - \mathrm{id}\,\|_{L^\infty}.$$

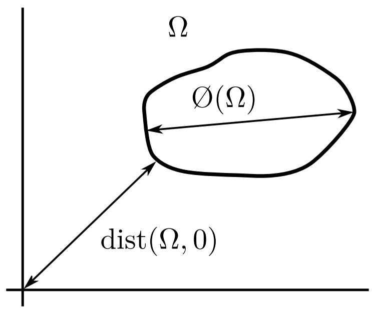


Figure 7.2.: Estimate $\|\,\mathrm{id}\,\|_{L^\infty}$

By Lemma 5.0.5, we know that $|y(\Omega)| \leq c\|y\|^3_{W^{1,p}}$. Moreover, we can estimate $\|\,\mathrm{id}\,\|_{L^\infty(\Omega)} \leq \text{Ø}(\Omega) + \mathrm{dist}(\Omega, 0)$ (see Fig. 7.2), which is a constant only depending on the reference configuration. W.l.o.g we can assume that $\mathrm{dist}(\Omega, 0) = 0$, and thus $\|\,\mathrm{id}\,\|_{L^\infty(\Omega)} \leq \text{Ø}(\Omega)$. We use the Sobolev Embedding Theorem A.2.2 and obtain

$$\begin{aligned} \|y - \mathrm{id}\,\|_{L^\infty(\Omega)} &\leq \|y\|_{L^\infty(\Omega)} + \|\,\mathrm{id}\,\|_{L^\infty(\Omega)} \\ &\leq \|y\|_{L^\infty(\Omega)} + \text{Ø}(\Omega) \\ &\leq c + c\|y\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Therefore, we bound I_3 as follows

$$I_3 \geq -c|y(\Omega)| - c|y(\Omega)| \left(c + \|y\|_{W^{1,p}}\right) \geq -c\|y\|^3_{W^{1,p}} - c\|y\|^4_{W^{1,p}}.$$

Applying Young's inequality, Lemma A.5.3, to each of the summands, would eventually yield the desired result. This is the point, where $p > 4$ comes into play, and so we are going to present the precise argument for the second summand to emphasize our assertion. Set $\tilde{p} := \frac{p}{4}$. Then $\tilde{p} > 1$, since $p > 4$ and so Young's inequality is applicable with $1/\tilde{p} + 1/q = 1$. Thus, we get

$$\|y\|^4_{W^{1,p}} \leq \frac{1}{\tilde{p}} \delta^{\tilde{p}} \left(\|y\|^4_{W^{1,p}}\right)^{p/4} + \frac{1}{\delta^q q} = c\delta^{p/4} \|y\|^p_{W^{1,p}} + c.$$

The analogous result holds for the first summand.

Putting everything together, we arrive at

$$\begin{aligned}\Lambda > \mathcal{E}(y) &= I_1 + I_2 + I_3\\ &\geq c_P\|y\|^p_{W^{1,p}(\Omega)} - c - c\delta^p\|y\|^p_{L^p} - c - c\delta^{p/3}\|y\|^p_{W^{1,p}(\Omega)} - c\delta^{p/4}\|y\|^p_{W^{1,p}(\Omega)}\\ &\geq (c_P + \delta^p c)\|y\|^p_{W^{1,p}(\Omega)} - c.\end{aligned}$$

Since $\delta > 0$ is arbitrary, we can make it sufficiently small, so that $(c_P - \delta^p c)$ is positive, and thus, we have verified that $\{\mathcal{E} < \Lambda\}$ is bounded in the Sobolev-norm, and coercivity follows.

Step 2: Now we are going to prove weak lower semicontinuity, i.e., we verify that for a weakly convergent sequence $y_n \subset W^{1,p}(\Omega)$, $y_n \rightharpoonup y$, the following inequality

$$\mathcal{E}(y) \leq \liminf \mathcal{E}(y_n)$$

is satisfied. We will prove weak lower semicontinuity for the different integral terms separately, and then add them up again. This procedure is legitimate, by the super-additivity of the $\liminf$, Lemma 6.3.1, i.e., for two sequences a_n, b_n one has that $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n)$, whenever this expression is defined.

For the first term, we make use of the results from the theory Section 3, and get

$$\int_\Omega W(\nabla y) \leq \liminf \int_\Omega W(\nabla y_n),$$

because of polyconvexity of W, which allows us to use Cor. 3.1.8.

The integral term related to the forces were treated in Sec. 6, where we proved weak lower semicontinuity for $I_2 + I_3$ in Cor. 6.3.3.

Step 3: By coercivity and weak lower semicontinuity, we can apply the Direct Method of Thm. 1.1.1 to prove the existence of a minimizer.

This minimizer also satisfies the Ciarlet-Nečas condition by Thm. 5.0.3.

Moreover, the minimizer fulfils the boundary condition by the continuity of the trace operator A.2.1. □

7.2. Existence result for internally fixed bodies

In the previous section we considered an object fixed at a part of its boundary. Now we look at objects, which are fixed internally.

Again, we prove the existence of minimizers for the following functional

$$\mathcal{E}(y) = \int_{\Omega} W(\nabla y(x))dx + \int_{\Omega} \rho_S g e_3 \cdot y(x)dx - \int_{y(\Omega)} \rho_W g x_3^y dx^y,$$

but now, the set Ω is assumed to have a certain structure. Let $\omega \subset \mathbb{R}^2$ be a bounded, open domain and $\omega_D \subset \omega$, with $\text{meas}_2(\omega_D) > 0$. We consider the 3-dimensional set $\Omega := \omega \times [a,b]$, where $a, b \in \mathbb{R}, 0 < |b-a| < \infty$ and assume that $y = \text{id}$ on the set $\Omega_D = \omega_D \times [a,b] \subset \Omega$, see Fig. 7.3. By this definition, the set Ω is bounded (and therefore, has finite measure), and has a Lipschitz boundary.

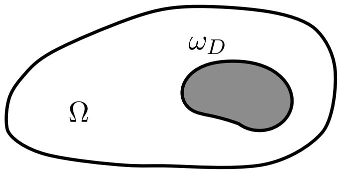


Figure 7.3.: Structure of Ω

Furthermore, we again assume that the stored energy $W : GL^+(d) \to [0,\infty)$ satisfies

(i) W is polyconvex,
(ii) $W(F) \to +\infty$ as $\det F \searrow 0+$ and,
(iii) $W(F) \geq c|F|^p - c$.

Theorem 7.2.1. *Under the assumptions above, the functional $\mathcal{E}$ has a minimum in the set of admissible functions $\mathcal{A} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) \,:\, y = \text{id} \text{ on } \Omega_D, CN\}$, where CN denotes that, $y \in \mathcal{A}$ fulfils the Ciarlet-Nečas condition.*

Proof. Note that $\text{id} \in \mathcal{A}$ and thus $\mathcal{A} \neq \emptyset$. Moreover, $\mathcal{E}(\text{id}) < \infty$ by the same argumentation as before.

As in the previous section, we will split the proof in three parts: Establishing coercivity, proving weak lower semicontinuity, and applying the direct method. Moreover, we have to verify that the minimizer is an admissible function. For simplification, we will split the energy functional in three separate terms, which we can treat on their own, whenever it is convenient to do so.

$$\mathcal{E}(y) = \underbrace{\int_{\Omega} W(\nabla y(x))dx}_{=:I_1} + \underbrace{\int_{\Omega} \rho_S g e_3 \cdot y(x)dx}_{=:I_2} - \underbrace{\int_{y(\Omega)} \rho_W g x_3^y dx^y}_{=:I_3}.$$

<u>Step 1: Coercivity.</u>

Again, we have to show that the set $\{\mathcal{E} < \Lambda\}$ is sequentially precompact in the weak-$W^{1,p}$-topology. As in the previous section, we will treat each term separately, starting with the second one. The reason is, that for this term everything is exactly the same as before, and we can write without more ado

$$I_2 = \int_\Omega \rho_S g e_3 \cdot y(x) dx \geq -c\delta^p \|y\|_{L^p}^p - c.$$

By definition of Ω we have $\omega_D \subset \partial\Omega$ and $\text{meas}_2(\omega_D) > 0$. This puts us in the same position as in Section 7, allowing us to apply Poincaré's inequality, Lemma 7.1.2, and the coercivity condition to obtain for the term I_1

$$I_1 = \int_\Omega W(\nabla y(x)) dx \geq \int_\Omega c|\nabla y|^p dx - c|\Omega| \geq c_P \|y\|_{W^{1,p}(\Omega)}^p - c.$$

The third term is more difficult, although the geometrical idea is very simple: Since the $y = \text{id}$ on Ω_D, each point $x^y = y(x) \in \Omega^y$ is at most $\varnothing(y(\Omega))$-far away from Ω_D. Thus, we can say

$$|x^y| \leq \text{dist}(0, \Omega_D) + \varnothing(\Omega_D) + \varnothing(\Omega^y),$$

where $\text{dist}(0, \Omega_D)$ and $\varnothing(\Omega_D)$ only depend on the reference configuration, and therefore, can be considered as constant. The hardest part here, is to estimate $\varnothing(\Omega^y)$ in terms of the norm of y. To see this, note that by Morrey's inequality and the Poincaré inequality (see Section A.2, in the appendix), we have

$$\|y - \bar{y}\|_{L^\infty} \leq c\|y - \bar{y}\|_{W^{1,p}} \leq c\|y - \bar{y}\|_{L^p} + \|\nabla y\|_{L^p} \leq c\|\nabla y\|_{L^p},$$

and thus,

$$\varnothing(y(\Omega)) = \sup_{x,z \in \Omega} |y(x) - y(z)| \leq 2 \sup_{x \in \Omega} |y(x) - \bar{y}| \leq 4c\|\nabla y\|_{L^p} \leq c\|y\|_{W^{1,p}}.$$

By Lemma 5.0.5, we have $|y(\Omega)| \leq c\|y\|_{W^{1,p}}^3$, which allows us to conclude

$$\begin{aligned} I_3 &\geq -\int_{\Omega^y} g\rho_W |x_3^y| dx^y \geq -\rho_W g |x_3^y| \, |y(\Omega)| \\ &\geq -c \left(\varnothing(y(\Omega)) + c\right) \|y\|_{W^{1,p}}^3 \geq -c\|y\|_{W^{1,p}}^3 - c\|y\|_{W^{1,p}}^4. \end{aligned}$$

Applying Young's inequality to each term separately as in Section 7.1, we finally get

$$\begin{aligned}\Lambda > \mathcal{E}(y) &= I_1 + I_2 + I_3 \\ &\geq c_P \|y\|^p_{W^{1,p}(\Omega)} - c - c\delta^p \|y\|^p_{L^p} - c - c\delta^{p/3} \|y\|^p_{W^{1,p}(\Omega)} - c\delta^{p/4} \|y\|^p_{W^{1,p}(\Omega)} \\ &\geq (c_P + \delta^p c) \|y\|^p_{W^{1,p}(\Omega)} - c.\end{aligned}$$

The coefficient $(c_P + \delta^p c)$ can be made positive, by choosing δ arbitrarily small, and thus, we have shown that $\{E < \Lambda\}$ is sequentially precompact. Therefore, every sequence $(y_n) \subset \mathcal{A}$ realizing the $\inf \mathcal{E}$ has a subsequence weakly converging in $W^{1,p}$.

Step 2: Weak lower semicontinuity. By assumption, W is polyconvex, and thus, I_1 is weakly lower semicontinuous, by Cor. 3.1.8. Moreover, $I_2 + I_3$ is also weakly lower semicontinuous, as has been shown in Sec. 6.3. The superadditivity of the $\liminf$, Lemma 6.3.1, yields the claim.

Step 3: The Direct Method. After establishing coercivity and weak lower semicontinuity, we can apply the direct method and obtain the existence of a minimizer y_{min}, where we still have to make sure, that it belongs to the set of admissible functions.

By Thm. 5.0.3, y_{min} satisfies the Ciarlet-Nečas condition, so we are left with proving that $y_{min} = \mathrm{id}$ on Ω_D. However, this is not difficult. By Morrey's inequality, Thm. A.2.2, and the Arzelà-Ascoli theorem, Thm. A.2.4, we know that there is a subsequence y_{n_k} converging uniformly to y_{min}, in Ω. Thus, also $y_{min} = \mathrm{id}$ on $\omega_D \times (a, b)$. For the parts on the boundary, we conclude with the continuity of the trace operator, see A.2.1. $\square$

8. Existence results with elastic conditions

As we saw in the previous sections, we need some additional information of the problem to be able to apply Poincaré's inequality and conclude coercivity. In this section, we provide this information by considering bodies tied to a fixed point or some kind of an anchor by an elastic rubber band. Thus, the specimen gets pulled towards the fixed anchor. For example, one could imagine a helium balloon held by a children. Therefore, we will add a new term to the energy functional modelling the spring energy of the rubber band. This term, coming from physical considerations, will allow us to prove a Poincaré-type inequality from which we eventually can conclude coercivity.

8.1. Poincaré inequality

As we have already seen in the previous chapter, we need a Poincaré-type inequality to be able to prove coercivity. The following theorem provides such inequality, incorporating the term $\|y - \mathrm{id}\,\|_{L^2(\omega)}$ which is relevant as elastic energy of the rubber band. This means, the Poincaré inequality is already tailored in such a way, that we can easily apply it in the coming sections.

Theorem 8.1.1 (Poincaré-type inequality)**.** *Assume that $\Omega \subset \mathbb{R}^d$ is non-empty, bounded, open, and connected. Moreover, let $\omega \subset \Omega$ with meas$(\omega) > 0$. If $p > 2$, then for all $y \in W^{1,p}(\Omega)$ the following inequality holds*

$$\|y\|_{L^p(\Omega)} \leq c\left(\|\nabla y\|_{L^p(\Omega)} + \|y - \mathrm{id}\,\|_{L^2(\omega)}\right). \tag{8.1.1}$$

Proof. Aiming at a contradiction, let us assume the opposite, i.e., that there exists a sequence $(y_k)_k \subset W^{1,p}(\Omega;\mathbb{R}^d)$, such that

$$\|\nabla y_k\|_{L^p(\Omega)} + \|y_k - \mathrm{id}\,\|_{L^2(\omega)} < \frac{1}{k}\|y_k\|_{L^p(\Omega)}. \tag{8.1.2}$$

Claim 1: $\|y_k\|_{L^p(\Omega)} \to \infty$.

Assume otherwise, i.e., $\|y_k\|_{L^p(\Omega)} \leq M$ for all $k \in \mathbb{N}$. This implies by (8.1.2) that

$$\|y_k\|_{W^{1,p}(\Omega)} = \|y_k\|_{L^p(\Omega)} + \|\nabla y_k\|_{L^p(\Omega)} \leq M + \frac{1}{k}\|y_k\|_{L^p(\Omega)} \leq M\left(1 + \frac{1}{k}\right).$$

But since $1 + \frac{1}{k} \leq 2$ for all $k \in \mathbb{N}$, we conclude

$$\|y_k\|_{W^{1,p}(\Omega)} \leq 2M.$$

Therefore, there is a (not relabelled) subsequence such that $y_k \rightharpoonup y$ in $W^{1,p}$. In particular, by Thm. A.4.8, this implies that $\nabla y_k \rightharpoonup \nabla y$ in $L^p(\Omega)$ and by the weak lower semicontinuity of the norm, Thm. A.4.7, we obtain

$$\|\nabla y\|_{L^p(\Omega)} \leq \liminf_k \|\nabla y_k\|_{L^p(\Omega)}.$$

We know that $\|y_k\|_{L^p(\omega)} \leq \|y_k\|_{L^p(\Omega)} < M$. By Hölder's inequality, the boundedness of Ω, and the assumption $p > 2$, we can conclude

$$\|y_k\|_{L^2(\omega)} \leq c\|y_k\|_{L^p(\omega)} \leq c\|y_k\|_{L^p(\Omega)} < M.$$

Thus, we are able to select a subsequence (again not relabelled) such that $y_k \rightharpoonup y$ in $L^2(\omega)$, and hence $(y_k - \mathrm{id}) \rightharpoonup (y - \mathrm{id})$ in $L^2(\omega)$. Again, by the weak lower semicontinuity of norms in Banach spaces, we eventually arrive at

$$\|y - \mathrm{id}\,\|_{L^2(\omega)} \leq \liminf \|y_k - \mathrm{id}\,\|_{L^2(\omega)}.$$

Recall the superadditivity of the lim inf: $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n)$. Putting everything together and using the boundedness of y_k yields

$$\begin{aligned}
\|\nabla y\|_{L^p(\Omega)} + \|y - \mathrm{id}\,\|_{L^2(\omega)} &\leq \liminf \|\nabla y_k\|_{L^p(\Omega)} + \liminf \|y_k - \mathrm{id}\,\|_{L^2(\omega)} \\
&\leq \liminf \left(\|\nabla y_k\|_{L^p(\Omega)} + \|y_k - \mathrm{id}\,\|_{L^2(\omega)}\right) \\
&\leq \liminf \left(\frac{1}{k}\|y_k\|_{L^p(\Omega)}\right) \leq 0.
\end{aligned}$$

This means that $\nabla y = 0$ almost everywhere, and by the connectedness of Ω, we get that y is constant. On the other hand, we also have $\|y - \mathrm{id}\,\|_{L^2(\omega)} = 0$, and hence, $y = \mathrm{id}$ almost everywhere on ω, a contradiction.

Since Claim 1 holds, we can divide y_k by $\|y_k\|_{L^p(\Omega)}$ for k large enough. Thus,

$$w_k := \frac{y_k}{\|y_k\|_{L^p(\Omega)}}$$

is well-defined and has norm $\|w_k\|_{L^p(\Omega)} = 1$. By dividing into (8.1.2) by $\|y_k\|_{L^p(\Omega)}$ we obtain

$$\|\nabla w_k\|_{L^p(\Omega)} + \left\| w_k - \frac{1}{\|y_k\|_{L^p(\Omega)}} \operatorname{id} \right\|_{L^2(\omega)} < \frac{1}{k}. \tag{8.1.3}$$

Since $(w_k)_k$ is bounded in $W^{1,p}(\Omega)$, there is a (not relabelled) subsequence (w_k) with $w_k \rightharpoonup w$ in $W^{1,p}$. In particular, we have $\nabla w_k \rightharpoonup \nabla w$ in L^p (by Thm. A.4.8), and thus, by the weak lower semicontinuity of the norm

$$\|\nabla w\| \leq \liminf \|\nabla w_k\|.$$

Moreover, $w_k \rightharpoonup w$ in $W^{1,p}$ implies the existence of a strongly converging subsequence in $L^p(\Omega)$ (by the compact embedding theorem). Therefore, this subsequence also satisfies

$$\|w_k - w\|_{L^2(\omega)} \leq \|w_k - w\|_{L^2(\Omega)} \leq c\|w_k - w\|_{L^p(\Omega)} \to 0.$$

Since $\frac{1}{\|y_k\|_{L^p(\Omega)}} \to 0$, we have

$$\left\| \left(w_k - \frac{1}{\|y_k\|_{L^p(\Omega)}} \operatorname{id} \right) - w \right\|_{L^2(\omega)} \leq \|w_k - w\|_{L^2(\omega)} + \left\| \frac{1}{\|y_k\|_{L^p(\Omega)}} \operatorname{id} \right\|_{L^2(\omega)} \to 0.$$

Altogether, this subsequence (w_k) fulfils

$$\begin{aligned} \|\nabla w\|_{L^p(\Omega)} + \|w\|_{L^2(\omega)} &\leq \liminf \left(\|\nabla w_k\|_{L^p(\Omega)} + \left\| w_k - \frac{1}{\|y_k\|_{L^p(\Omega)}} \operatorname{id} \right\|_{L^2(\omega)} \right) \\ &\overset{(8.1.3)}{\leq} \liminf \frac{1}{k} = 0. \end{aligned}$$

Thus, we arrive at

$$\|\nabla w\|_{L^p(\Omega)} + \|w\|_{L^2(\omega)} \leq 0,$$

and therefore, $\nabla w = 0$ almost everywhere on Ω, and $w = 0$ almost everywhere on ω. Since Ω is connected, $\nabla w = 0$ implies that w is constant and we can conclude that $w = 0$ almost everywhere on Ω, which is in contradiction to $\|w_k\|_{L^p(\Omega)} = 1$. This proves the assertion. $\square$

Remark 8.1.1. To be completely rigorous one also has to verify the following details:

- Is $\nabla w_k = \frac{1}{\|y_k\|_{L^p}} \nabla y_k$? Yes, because

$$\int \partial_j w_k \phi = -\int w_k \partial_j \phi = -\int \frac{y_k}{\|y_k\|_{L^p}} \partial_j \phi = -\frac{1}{\|y_k\|_{L^p}} \int y_k \partial_j \phi = \frac{1}{\|y_k\|_{L^p}} \nabla y_k.$$

- Why can we conclude $w_k \to w$ in L^p and $\|w_k\|_{L^p} = 1$ is a contradiction to $\|w\| = 0$? We have the following trivial conclusion: Convergence *in* norm implies convergence *of* the norms. The proof is the reverse triangle inequality:

$$|\|w_k\| - \|w\|| \leq \|w_k - w\| \to 0.$$

Before we start using the inequality above to proof coercivity, we want to recall Jensen's inequality and one particular implication.

Theorem 8.1.2 (Jensen)**.** *For a convex function $f : \mathbb{R}^d \to \mathbb{R}$, $x_i \in \mathbb{R}^d$, and nonnegative λ_i with $\sum_{i=1}^n \lambda_i = 1$ we have*

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i).$$

The proof for a general version of this inequality can be found in [4], Thm. 2.12.19.

Recall Definition 1.2.1: A function $f : C \to \mathbb{R}$ is convex, if C is convex and, if for all $x, y \in C$, $t \in [0,1]$, the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

As a particularly useful example, we want to mention the power function. The map $x \mapsto x^p$ is convex for $x \in \mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ and all $p \geq 1$.

Hence, we can conclude using Jensen's inequality

$$(|a| + |b|)^p = 2^{1/p} \left(\frac{1}{2}|a| + \frac{1}{2}|b|\right)^p \leq 2^{1/p} \left(\frac{1}{2}|a|^p + \frac{1}{2}|b|^p\right) = c\left(|a|^p + |b|^p\right), \qquad (8.1.4)$$

where $c = 2^{1/p-1} > 0$.

To be precise, (8.1.4) follows directly from the definition of convexity, because it only includes two summands. One can get the analogous result by Jensen's inequality for an arbitrary number of summands.

8.2. First existence result

As a first result, we consider the case where a part of the material $\omega \subset \Omega$ is tied to a fixed anchor by a rubber band, i.e., only this part will be pulled back and contributes to the energy. This example is illustrated in Fig. 8.1 where ω is the anchor and the connection via the rubber band is indicated by the dashed line.

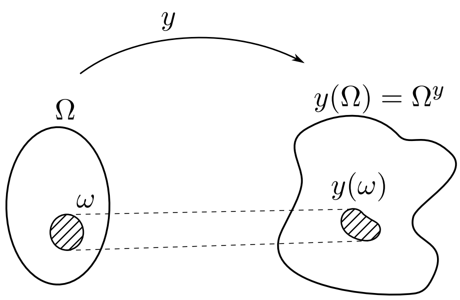


Figure 8.1.: Elastic Conditions

Formally, let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with Lipschitz boundary and $\omega \subset \Omega$ nonempty, open, and with positive measure, as in Fig. 8.1. The goal is to prove the existence of minimizers for the following functional

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))dx + \int_\Omega \rho_S g e_3 \cdot y(x)dx - \int_{y(\Omega)} \rho_W g x_3^y dx^y + \frac{k}{2}\int_\omega |y(x) - x|^2 dx,$$

where g is the constant of gravity, ρ_S, ρ_W are the densities of the solid and the water (or a different medium), respectively, and k denotes the elastic modulus of the rubber band.

As usually, we additionally assume $W : GL^+(d) \to [0, \infty)$ satisfies

(i) W is polyconvex,
(ii) $W(F) \to +\infty$ as $\det F \searrow 0+$, and
(iii) $W(F) \geq c|F|^p - \frac{1}{c}$.

Theorem 8.2.1. *Let $p > 4$. Under the assumptions above, the functional $\mathcal{E}$ takes a minimum in the set of admissible functions $\mathcal{A} := \{y \in W^{1,p}(\Omega; \mathbb{R}^3) : CN\}$.*

Note that $\mathrm{id} \in \mathcal{A}$ and thus $\mathcal{A} \neq \emptyset$ and $\mathcal{E}(\mathrm{id}) < \infty$. As in the previous sections we will establish coercivity and verify weak lower semicontinuity of each term in $\mathcal{E}$. Moreover, we consider each integral term of the energy separately

$$E(y) = \underbrace{\int_\Omega W(\nabla y(x))dx}_{=:I_1} + \underbrace{\int_\Omega \rho_S g e_3 \cdot y(x)dx}_{=:I_2} - \underbrace{\int_{y(\Omega)} \rho_W g x_3 dx}_{=:I_3} + \underbrace{\frac{k}{2}\int_\omega |y(x) - x|^2 dx}_{=:I_4}.$$

Step 1: Coercivity. Note that by Poincaré's inequality (8.1.1) we get

$$\|y\|_{W^{1,p}(\Omega)} \leq c\left(\|\nabla y\|_{L^p(\Omega)} + \|y - \operatorname{id}\|_{L^2(\omega)}\right),$$

and thus, by Jensen's inequality (8.1.4),

$$\|y\|^p_{W^{1,p}(\Omega)} \leq c\left(\|\nabla y\|_{L^p(\Omega)} + \|y - \operatorname{id}\|_{L^2(\omega)}\right)^p \leq c'\|\nabla y\|^p_{L^p\Omega} + c'\|y - \operatorname{id}\|^p_{L^2(\omega)}.$$

Using this, we can infer

$$\begin{aligned} I_1 &= \int_\Omega W(\nabla y)dx \geq c_1 \int_\Omega |\nabla y|^p - c_2 \\ &= c_1\|\nabla y\|^p_{L^p(\Omega)} - c_2 \geq c_3\|y\|^p_{W^{1,p}(\Omega)} - c_4\|y - \operatorname{id}\|^p_{L^2(\omega)} - c_2. \end{aligned} \tag{8.2.1}$$

The term I_2 can be estimated as

$$\begin{aligned} I_2 &= \int_\Omega \rho_S g e_3 \cdot y(x)dx \geq -\int c|e_3 \cdot y(x)|dx \geq -c\int |y(x)|dx \\ &\overset{Young}{\geq} -c\int\left(|y(x)|^p\frac{\delta^p}{p} + \frac{1}{q\delta^q}\right) \geq -c|\Omega| - c\delta^p\|y\|^p_{L^p} = -c\delta^p\|y\|^p_{L^p(\Omega)} - c' \end{aligned}$$

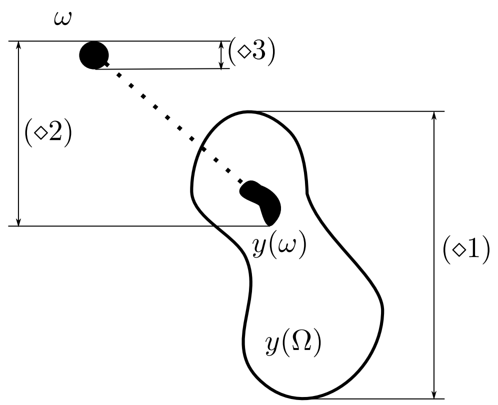


Figure 8.2.: Measuring the maximal displacement

For the term I_3, we use a geometrical argument, see Fig. 8.2.

Basically, x_3^y expresses how "deep" a point $x^y \in y(\Omega)$ is, regarding the fixed anchor point ω (this is where the virtual spring is attached). We are going to estimate the depth of $y(\Omega)$ in terms of y and ω based on a purely geometrical argument: A point $x \in \Omega$ can, at most, be as deep as the "depth", i.e. the vertical expanse, of the deformed object ($\diamond$1), plus the length of the string ($\diamond$2), plus possibly the length of the anchor ($\diamond$3). This yields the mathematical expression

$$|x_3^y| \leq \underbrace{\varnothing(y(\Omega))}_{(\diamond 1)} + \underbrace{\|y - \operatorname{id}\|_{L^\infty(\Omega)}}_{(\diamond 2)} + \underbrace{\varnothing(\omega)}_{(\diamond 3)}.$$

Therefore, for $x, z \in \Omega$ we have

$$|y(x) - y(z)| \leq |y(x) - x| + |x - z| + |z - y(z)| \leq 2\|y - \operatorname{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega),$$

and thus $\varnothing(y(\Omega)) = \sup|y(x) - y(z)| \leq 2\|y - \operatorname{id}\|_{L^\infty} + \varnothing(\Omega)$. Without more ado, we can assume $\varnothing(\omega) \leq \varnothing(\Omega)$, which gives us

$$|x_3^y| \leq 3\|y - \operatorname{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega),$$

and hence,

$$\int_{y(\Omega)} x_3^y dx \leq \int_{y(\Omega)} 3\|y - \operatorname{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega) \leq \left(3\|y - \operatorname{id}\|_{L^\infty(\Omega)} + \varnothing(\Omega)\right) |y(\Omega)|.$$

By Lemma 5.0.5, we have $|y(\Omega)| \leq c\|y\|_{W^{1,p}}^3$. Therefore, we can proceed as in Chap. 7 and eventually conclude

$$I_3 \geq -c|y(\Omega)| - c|y(\Omega)|\,(c + \|y\|_{W^{1,p}}) \geq -c\|y\|_{W^{1,p}}^3 - c\|y\|_{W^{1,p}}^4,$$

which yields after applying Young's inequality

$$I_3 \geq -c\left(\delta^{p/3} + \delta^{p/4}\right)\|y\|_{W^{1,p}(\Omega)}^p - c.$$

For the last term we use the triangle inequality, Jensen's inequality for the particular case of the power function, Young' inequality, and the fact, that

$$\int_\omega |x|^2 dx = \|\operatorname{id}\|_{L^2(\omega)}^2 \leq c(\omega).$$

To see this, note that for all $x \in \Omega$ we have that $|x| \leq \varnothing(\Omega) + \operatorname{dist}(\Omega, 0)$ and, thus, $\|\operatorname{id}\|_{L^2(\omega)}^2 = \int_\omega |x|^2 dx \leq \sup_{x\in\omega}|x|^2\,|\omega| \leq |\omega|\,(\varnothing(\omega) + \operatorname{dist}(\omega, 0))^2 =: c(\omega)$. The term I_4

can be now handled as follows.

$$
\begin{aligned}
I_4 = \frac{k}{2}\int_\omega |y(x)-x|^2 dx = c\|y-\mathrm{id}\,\|^2_{L^2(\omega)} & & \text{triangle ineq.}\\
\geq -c\int_\omega \left(|y(x)|+|x|\right)^2 dx & & \text{Jensen}\\
\geq -c\int_\omega \left(|y(x)|^2+|x|^2\right) dx & & \text{Young}\\
\geq -c\delta^{p/2}\int |y(x)|^p - c(\omega) \geq -c\delta^{p/2}\|y\|^p_{L^p(\Omega)} - c. & & \qquad (8.2.2)
\end{aligned}
$$

We will estimate the term $\|y-\mathrm{id}\,\|^p_{L^2(\omega)}$, also appearing in (8.2.1), using Young's inequality and bring it to the same form as in (8.2.2).

$$
\begin{aligned}
& -\|y-\mathrm{id}\,\|^p_{L^2(\omega)} & & \text{Young}\\
\geq & -\frac{p}{2}\|y-\mathrm{id}\,\|^2_{L^2(\omega)} - \frac{p}{p-2} & & (8.2.2)\\
\geq & -c\delta^{p/2}\|y\|^p_{L^p(\Omega)} - c.
\end{aligned}
$$

Recall, that we always can estimate $\|y\|^p_{L^p(\Omega)} \leq \|y\|^p_{W^{1,p}(\Omega)}$.

Now we are able to combine the estimates on I_i, $i = 1,2,3,4$, to obtain

$$
\begin{aligned}
\mathcal{E}(y) &= I_1 + I_2 + I_3 + I_4\\
&\geq c_P\|y\|^p_{W^{1,p}(\Omega)} - c\|y-\mathrm{id}\,\|^p_{L^2(\omega)} - c\delta^p\|y\|^p_{L^p(\Omega)} - c\left(\delta^{p/3}+\delta^{p/4}\right)\|y\|^p_{W^{1,p}(\Omega)} - c\delta^{p/2}\|y\|^p_{L^p(\Omega)} - C\\
&\geq \left(c_P - c\delta^p\right)\|y\|^p_{W^{1,p}} - C.
\end{aligned}
$$

By choosing δ sufficiently small, we obtain a positive coefficient of $\|y\|_{W^{1,p}}$ and, thus, coercivity.

Step 2: Weak lower semicontinuity. Again, by the superadditivity of the lim inf, one can verify weak lower semicontinuity for each integral separately and then combine these results. Weak lower semicontinuity for $I_1 + I_2 + I_3$ is already proved, cf. Chap. 7.

A proof for weak lower semicontinuity of I_4 can be given, using Fatou's Lemma A.1.1. Let $y_n \rightharpoonup y$ in $W^{1,p}(\Omega)$. Then, by the compact embedding, there is a subsequence (not relabelled) $y_n \to y$ pointwise almost everywhere. Set $f_n(x) := |y_n(x)-x|^2$ as a function from Ω into the real numbers. Then f_n is nonnegative and measurable. Furthermore,

$f_n(x) \to f(x) := |y(x) - x|^2$ almost everywhere. Thus, by Fatou's Lemma

$$\begin{aligned} I_4(y) &= c \int_\Omega |y(x) - x_0|^2 dx = c \int_\Omega \liminf f_n(x) dx \\ &\leq c \liminf \int_\Omega f_n(x) dx = \liminf I_4(y). \end{aligned}$$

By the argument in Section A.6, where we have verified that passing to a subsequence does not destroy weak lower semicontinuity, we have proved the assertion.

Step 3:

The direct method implies the existence of a minimizer, and Thm. 5.0.3 confirms that this minimizer also satisfies the Ciarlet-Nečas condition.

8.3. Second existence result - rope with clearance

Now we are going to consider a slight variation of the previous result. Instead of requiring that a certain portion of the reference configuration $\omega \subset \Omega$ serves as anchor, we only take one point $x_0 \in \Omega$ to fix the imaginary rubber band. Moreover, we consider the case where this rubber band has a certain clearance, i.e., it develops a retraction force only if it gets extended further than a certain length l, see Fig. 8.3. Since this problem is a variation of the one above, we also need a slightly different Poincaré inequality.

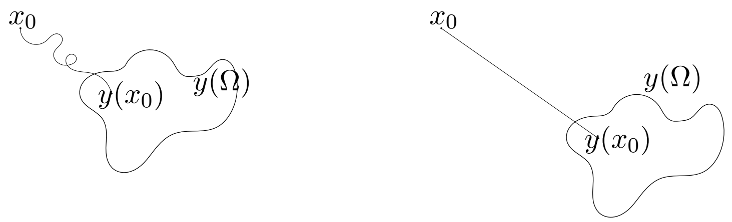
Theorem 8.3.1 (Poincaré-type inequality)**.** *Assume that $\Omega \subset \mathbb{R}^d$ is non-empty, bounded, open, and connected and let $\omega \subset \Omega$ be a nonempty, measurable set. Moreover, let $x_0 \in \Omega$. If $p > 2$, then for all $y \in W^{1,p}(\Omega)$ the following inequality holds*

$$\|y\|_{L^p(\Omega)} \leq c \left(\|\nabla y\|_{L^p(\Omega)} + \|y - x_0\|_{L^2(\omega)} \right).$$

The proof is analogous to the one of Thm. 8.1.1.

Remark 8.3.1. In the theorem above one identifies x_0 with the class of functions being equal to x_0 almost everywhere. Since Ω has finite measure, $x_0 \in L^2(\Omega)$. In principle, one could generalize this result to arbitrary function in L^2 in the same manner.

Theorem 8.3.2. *Under the same assumptions as in Thm. 8.2.1 the following energy*



(a) Rope not outstreched

(b) Outstrechted rope, retraction force applies

Figure 8.3.: Elastic condition

functional has a minimizer

$$\mathcal{E}(y) = \underbrace{\int_\Omega W(\nabla y(x))dx}_{=:I_1} + \underbrace{\int_\Omega \rho_S g e_3 \cdot y(x)dx}_{=:I_2}$$
$$- \underbrace{\int_{y(\Omega)} \rho_W g x_3^y dx^y}_{=:I_3} + \underbrace{\frac{k}{2}\int_\omega \Big((|y(x)-x_0|-l)^+\Big)^2 dx}_{=:\tilde{I}_4},$$

where $l \geq 0$ is some constant, $f^+(x) := \max\{f(x), 0\}$ and $x_0 \in \Omega$.

This functional models the potential energy of a balloon, filled with a gas with density ρ_S (e.g., Helium), surrounded by some medium with density ρ_W (e.g., air), which is attached to a rubber band fixed a x_0, which pulls back only if the band is stretched out more than l, see Fig. 8.3.

Proof. For the proof, we proceed as for Theorem 8.2.1. We will only mention the major changes.

Step 1: Coercivity

The terms I_1 and I_2 can be handled as above. The term I_3 is also not a problem, since we can simply add the length l in our geometric considerations, which only adds an additional constant to the estimate. For the term I_4 we consider

$$(|y(x)-x_0|-l)^+ = \max\{|y(x)-x_0|-l, 0\} \leq ||y(x)-x_0|-l| \leq |y(x)-x_0|+l.$$

By the convexity of the mapping $x \mapsto x^2$ we get

$$\Big((|y(x)-x_0|-l)^+\Big)^2 \leq (|y(x)-x_0|+l)^2 \leq c|y(x)-x_0|^2 + c'l^2.$$

Therefore, one gets

$$\tilde{I}_4 = c\int \left((|y(x) - x_0| - l)^+\right)^2 dx \geq -c\int |y(x) - x_0|^2 dx - c' \underbrace{\int l^2 dx}_{const.}.$$

Thus, we are able to estimate the both "elasticity"-terms, appearing from Poincaré and the last integral, in a similar fashion as we did in the example above.

Step 2: Weak lower semicontinuity.

Weak lower semicontinuity for the integrals I_1, I_2, I_3 follows from the same calculations as in the proof of Thm. 8.2.1. The only thing left to check is that the different choice of the integrand in I_4 does not spoil lower semicontinuity.

Fortunately this is not the case, as we will show by employing the Fatou's Lemma, Thm. A.1.1. Let $y_n \rightharpoonup y$ in $W^{1,p}(\Omega)$. Then, by the compact embedding, there is a subsequence (not relabelled) $y_n \to y$ pointwise almost everywhere. Set $f_n(x) := (|y_n(x) - x_0| - l)^2$ as a function from Ω into the real numbers. Then f_n is non-negative and measurable. Furthermore, $f_n(x) \to f(x) = (|y(x) - x_0| - l)^2$ almost everywhere, because everything is continuous here. Thus, by Fatou's Lemma

$$\begin{aligned} I_4(y) = c\int_\Omega (|y(x) - x_0| - l)^2 \, dx &= c\int_\Omega \liminf f_n(x) dx \\ &\leq c \liminf \int_\Omega f_n(x) dx = \liminf I_4(y_n). \end{aligned}$$

Therefore, the direct method is applicable, and we conclude the existence of an admissible minimizer. □

9. Existence results without boundary conditions

So far we have controlled the position of the body by extended conditions, as boundary conditions or the elastic condition. In this section, we will treat the case, where we do not have such external conditions, i.e., the specimen is able to move freely in the medium. As it turns out, this will cause trouble, letting $\inf \mathcal{E} = -\infty$ and, thus, preventing us from proving existence of minimizers. The first goal is to make this statement precise.

Then, we investigate the easy case of incompressible objects and study under which assumptions we still can give an existence result. Secondly, we will turn to the case of compressible objects, where we introduce the concept of slightly compressible objects, which then allows us to prove existence of minimizers. The condition of slight compressibility is a particular example of material locking, which is also studied in [3] or [14]. There, the authors introduce a way of variationally characterising this condition. However, this approach cannot be applied in our case, as we will see. While $\inf \mathcal{E} = -\infty$, it is still possible that there are local minima of $\mathcal{E}$. This is the content of Sec. 9.3, where we eventually prove the existence of such a local minimum.

9.1. Incompressible bodies

Definition 9.1.1. A deformation y is called *incompressible* if, for all $x \in \Omega$ the deformation gradient equals to 1, i.e., $\det \nabla y(x) = 1$.

The volume stays constant under incompressible deformations: $|\Omega| = |y(\Omega)|$, following from the change of variables formula. Naturally, also the density stays unchanged under incompressible deformations, $\rho^y(y(x)) = \rho(x)$.

If the body is incompressible, Archimedes' law is applicable, which tells us that the buoyancy equals to the gravitational force of the displaced medium.

For now, we also assume that the whole space is filled by the fluid, which simplifies the argument. Then, by Archimedes' law, the forces acting on the body are $g\rho_W|\Omega|e_3 - g\rho_S|\Omega|e_3$. This term entirely describes the behaviour of the body:

1. If $g\rho_W|\Omega| - g\rho_S|\Omega| < 0$, or equivalently $\rho_W < \rho_S$, then the body sinks (i.e., the energy decreases for transitions in the $-e_3$ direction).

2. If $g\rho_W|\Omega| - g\rho_S|\Omega| = 0$, or equivalently $\rho_W = \rho_S$, then the body stays at its position (i.e., the energy is invariant by transitions in the e_3 direction).

3. If $g\rho_W|\Omega| - g\rho_S|\Omega| > 0$, or equivalently $\rho_W > \rho_S$, then the body rises (i.e., the energy decreases by translating the body in the e_3 direction).

Note, that the behaviour of the body is completely determined by the relation of ρ_W and ρ_S.

In the first case, namely for $\rho_W < \rho_S$, the energy does not have a minimum. More rigorously, suppose $\hat{y}$ is a minimum of $\mathcal{E}$, then we can show that $\mathcal{E}(\hat{y} - ce_3) < \mathcal{E}(\hat{y})$, a contradiction. For proving this claim in the case, we use the change of variables formula, and obtain

$$\begin{aligned}
\mathcal{E}(\hat{y} - ce_3) &= \int_\Omega W(\nabla(\hat{y} - ce_3)) + \int_\Omega \rho_S g(\hat{y}_3 - c) - \int_{y(\Omega)} \rho_W g(x_3^y - c) dx^y \\
&= \mathcal{E}(\hat{y}) - \int_\Omega \rho_S gc + \int_{y(\Omega)} \rho_W gc dx^y \\
&= \mathcal{E}(\hat{y}) - \int_\Omega \rho_S gc + \int_\Omega \rho_W gc \\
&= \mathcal{E}(\hat{y}) + \int_\Omega \underbrace{\rho_W gc - \rho_S gc}_{<0} < \mathcal{E}(\hat{y}).
\end{aligned}$$

Note that the energy can be lowered arbitrarily, by choosing c sufficiently large. For any $M \in \mathbb{R}$, we can get $\mathcal{E}(\hat{y} - ce_3) < \mathcal{E}(\hat{y}) - M$, by letting $c > \frac{M}{g|\Omega|(\rho_W - \rho_S)}$. Thus, we have that $\inf_{y \in \mathcal{A}} \mathcal{E}(y) = -\infty$, with $\mathcal{A} := \{y \in W^{1,p}(\Omega; \mathbb{R}^3) \ : \ \det \nabla y = 1\}$. The analogous result holds in the third case, namely for $\rho_W > \rho_S$, but taking $\hat{y} + ce_3$ instead. In the equilibrium case, $\rho_W = \rho_S$, the energy no longer depends on y_3, i.e., we can choose the barycentre arbitrarily, w.l.o.g. we set $\bar{y}_3 = 0$. Therefore, one has the additional condition of $\bar{y}_i = 0$, for $i = 1, 2, 3$, and thus can solve minimization problem.

Proposition 9.1.1. *Under the usual assumptions on the stored energy function W, as stated in Sec. 7, and the in the case of $\rho_W = \rho_S$ and $\det \nabla y = 1$, there exists a minimizer of the energy functional*

$$\mathcal{E}(y) = \int_\Omega W(\nabla y) - \int_{\Omega^y} g x_3^y \rho_W dx^y + \int_\Omega y_3(x) g \rho_S dx.$$

Proof. By the assumption $\rho_W = \rho_S$ and $\det \nabla y = 1$, we see that $\mathcal{E}$ reduces to $\mathcal{E}(y) = \int W(\nabla y)$. Furthermore, we can assume that $(\bar{y}_1, \bar{y}_2, \bar{y}_3) = (0,0,0)$, since the energy is invariant under translations of y_3. By the coercivity assumptions, we get $\mathcal{E} \geq c\|\nabla y\|^p_{L^p} - c$ and, due to Poincaré's inequality A.2.6, we can conclude $\mathcal{E}(y) \geq c\|y\|^p_{W^{1,p}} - c$, which yields coercivity of $\mathcal{E}$. Since W is assumed to be polyconvex, $\mathcal{E}$ is also $W^{1,p}$-weakly lower semicontinuous by the results from Chapter 3. Hence, the direct method applies. $\square$

We now change our assumption and introduce a transition between the medium and air. Recall that the buoyancy is the weight of the displaced fluid, and therefore, if the body is only partially submerged, only the submerged part will be taken into account. We assume the fluid to cover the whole lower half space $H^- := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$. It is easy to calculate the buoyancy, as we only have to consider those spacial points, which also lie in H^-, i.e., $x^y \in \Omega^y \cap H^-$. Hence, after a change of variable, and using $\det \nabla y = 1$, the energy functional will take the form

$$\mathcal{E}(y) = \int_\Omega W(\nabla y) + \int_\Omega y_3(x) g\rho_S dx + \int_\Omega g y_3^-(x) \rho_W dx,$$

where $f^\pm = \max(0, \pm f)$, and hence, $f = f^+ - f^-$. If the fluid does not fill the whole space, it is not enough to compare the densities only, but we also have to take into account the volume of the deformed configuration, which does depend on the deformation. We therefore have the following behaviour:

1. If $g\rho_W|\Omega^y \cap H^-| - g\rho_S|\Omega| < 0$, then the body sinks.

2. If $g\rho_W|\Omega^y \cap H^-| - g\rho_S|\Omega| = 0$, then the body stays at its position.

3. If $g\rho_W|\Omega^y \cap H^-| - g\rho_S|\Omega| > 0$, then the body rises.

Again, in the case of the sinking body we have $\inf \mathcal{E}(y) = -\infty$. The proof is analogous to the one of Prop. 9.2.1. However, due to the fluid-air-transition, we are able to show existence of minimizers for the third case, where $\rho_W > \rho_S$, i.e., the object floats. Intuitively, this is clear, because if $\min y_3 \geq 0$ (namely, the body is fully above water), the energy would decrease by partially submerging the solid. If the object was submerged deeply, it would rise to the surface. We can also see this, when looking at the energy $\mathcal{E}$, neglecting deformation energy. Then, if $\inf y_3 > 0$ (the specimen flies), we can assume $y_3^- = 0$, and hence, $\mathcal{F}(y) = \int_\Omega g\rho_S y_3(x)dx + \int_\Omega g\rho_W y_3^-(x)dx = g\rho_S|\Omega|\bar{y}_3 > 0$. On the other hand, if $\sup y_3 < 0$ (the specimen is deep under water), we can assume $(y_3)^- = -y_3$, which yields $\mathcal{F}(y) = \int_\Omega g\rho_S y_3(x)dx + \int_\Omega g\rho_W y_3^-(x)dx = g|\Omega|\underbrace{(\rho_S - \rho_W)}_{<0}\underbrace{\bar{y}_3}_{<0} > 0$. We see that the energy $\mathcal{F}$ is proportional to $\bar{y}_3$, or $-\bar{y}_3$, respectively. To make this argument precise, we will bound $|\bar{y}_3|$ in terms of the energy, which means that $\bar{y}_3$ can only be large, if also $\mathcal{E}$ is large. This enables us to prove coercivity and eventually the existence of minimizers. We claim there is a $c > 0$ such that for all deformations with $\det \nabla y = 1$ the following inequality holds

$$|\bar{y}_3| \leq c\mathcal{E}(y). \tag{9.1.1}$$

Clearly, $\rho_W |\Omega^y \cap H^-| > \rho_S |\Omega|$ implies $\rho_W > \rho_S$. Just divide by $|\Omega^y \cap H^-|$ and use $|\Omega^y \cap H^-| \leq |\Omega^y| = |\Omega|$. Therefore, we get

$$\begin{aligned}
\mathcal{E}(y) &= \underbrace{\int_\Omega W(\nabla y)}_{\geq 0} + \int_\Omega y_3(x) g\rho_S dx + \int_\Omega g y_3^-(x)\rho_W dx \\
&\geq \int_\Omega y_3^+(x) g\rho_S dx - \int_\Omega y_3^-(x) g\rho_S dx + \int_\Omega g\rho_S y_3^-(x) dx \\
&\geq \int_\Omega g\rho_S y_3^+(x) + \int_\Omega g \underbrace{(\rho_W - \rho_S)}_{>0} y_3^-(x) dx. \qquad (9.1.2)
\end{aligned}$$

Moreover, we have $|y_3| = y_3^+ + y_3^-$, and thus

$$\begin{aligned}
|\bar{y}_3| &\leq \frac{1}{|\Omega|}\int_\Omega |y_3| = \frac{1}{|\Omega|}\int y_3^+ dx + \frac{1}{|\Omega|}\int_\Omega y_3^- dx \\
&\leq \frac{1}{g\rho_S|\Omega|}\int_\Omega g\rho_S y_3^+(x) dx + \frac{1}{|\Omega| g(\rho_W - \rho_S)}\int_\Omega g(\rho_W - \rho_S) y_3^-(x) dx = (*)
\end{aligned}$$

Setting $K := \max\{\frac{1}{g\rho_S|\Omega|}, \frac{1}{|\Omega| g(\rho_W - \rho_S)}\} > 0$ we get

$$(*) \leq K \int_\Omega g\rho_S y_3^+(x) dx + K\int_\Omega g(\rho_W - \rho_S) y_3^-(x) dx \overset{(9.1.2)}{\leq} K\mathcal{E}(y).$$

Proposition 9.1.2. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with Lipschitz boundary and assume that the material is hyperelastic and polyconvex, i.e., there exists a stored energy function W satisfying* (7.1.1) - (7.1.3). *Furthermore, let $\rho_W > \rho_S$ and set $\mathcal{A} := \{y \in W^{1,p}(\Omega;\mathbb{R}^3) \,:\, \det \nabla y = 1\}$. Then, the minimization problem*

$$\textit{Minimize } \mathcal{E}(y) \textit{ for } y \in \mathcal{A}$$

has a solution.

Proof. We aim to apply the direct method. By the analysis of Chapter 3 and 6, we know that $\mathcal{E}$ is weakly lower semicontinuous. Since we assume $\rho_W > \rho_S$, we have

$$\begin{aligned} &\int_\Omega y_3(x) g\rho_S dx + \int_\Omega g y_3^-(x)\rho_W dx \\ \overset{\rho_W>\rho_S}{\geq} &\int_\Omega y_3(x) g\rho_S dx + \int_\Omega g\rho_S y_3^-(x) dx = g\rho_S \int_\Omega \underbrace{(y_3 + y_3^-)}_{=y_3^+\geq 0} dx \geq 0. \end{aligned} \tag{9.1.3}$$

To show coercivity, we apply Poincaré's inequality, Thm. A.2.6, and get

$$\begin{aligned} \Lambda \geq \mathcal{E}(y) &\overset{(9.1.3)}{\geq} c\|\nabla y\|_{L^p}^p - c \\ &\overset{A.2.6}{\geq} c\|y\|_{W^{1,p}}^p - c|\bar{y}_3|^p - c. \end{aligned}$$

By (9.1.1), we obtain $-c|\bar{y}_3|^p \geq -c\mathcal{E}(y)^p \geq -c\Lambda^p$, and hence, all deformations in the set $\{\mathcal{E} \leq \Lambda\}$ satisfy

$$\|y\|_{W^{1,p}}^p \leq c\,(\Lambda + \Lambda^p)\,,$$

i.e., are bounded. Therefore, we have shown coercivity and the direct method is applicable. □

9.2. The compressible case

Consider now the compressible case, i.e., $0 < \det \nabla y$, not necessarily $\det \nabla y = 1$. In this case, the buoyancy will also depend on the deformation, because the volume of the deformed body changes with y.

In particular, if $g\rho_W|\Omega^y| - g\rho_S|\Omega| < 0$ or equivalently, $|\Omega^y|\rho_W < |\Omega|\rho_S$ for a certain deformation y, then the body will sink. Therefore, we will expect $\inf \mathcal{E} = -\infty$ in this case. The next proposition will describe this phenomenon rigorously.

Proposition 9.2.1. *If the body is compressible and $|\Omega^{y^1}|\rho_W < |\Omega|\rho_S$ for a deformation y^1 with $\max y_3^1(x) \leq 0$, then there is another deformation y^2, such that $\mathcal{E}(y^2) < \mathcal{E}(y^1)$ and $|\Omega^{y^2}|\rho_W < |\Omega|\rho_S$. In particular, we have*

$$\inf \mathcal{E}(y) = -\infty.$$

Proof. Set $y^2 := y^1 - ce_3$, with $c > 0$ constant. Note that $\det \nabla y^1 = \det \nabla y^2$. Then, using the change of variables formula twice, we obtain

$$\begin{aligned}\rho_W|\Omega^{y^2}| = \rho_W \int_{y^2(\Omega)} dx^y = \rho_W \int_\Omega \det \nabla(y^1 - ce_3)dx \\ = \rho_W \int_\Omega \det \nabla y^1(x)dx = \rho_W|\Omega^{y^1}| < \rho_S|\Omega|.\end{aligned}$$

We can use the calculation above in the following computation and get

$$\begin{aligned}\mathcal{E}(y^2) &= \int_\Omega W(\nabla(y^1 - ce_3)) + \int_\Omega \rho_S g(y_3^1 - c) - \int_{y^2(\Omega)} \rho_W g(x_3^y - c)dx^y \\ &= \int_\Omega W(\nabla y^1) + \int_\Omega \rho_S g y_3^1 - \int_\Omega \rho_S g c - \int_\Omega \rho_W g(y_3^1(x) - c)(\det \nabla y^2)dx \\ &= \int_\Omega W(\nabla y^1) + \int_\Omega \rho_S g y_3^1 - \int_\Omega \rho_S g c - \int_{y^1(\Omega)} g\rho_W x_3^y dx^y + \int_{y^1(\Omega)} g\rho_W c dx^y \\ &= \mathcal{E}(y^1) - \underbrace{\Big(|\Omega|\rho_S g c - |\Omega^{y^1}|\rho_W c g\Big)}_{>0} < \mathcal{E}(y^1).\end{aligned}$$

As before, the difference between the energies of y^1 and y^2 can be made arbitrarily large by choosing c sufficiently large. □

Remark 9.2.1. The analogous result holds true, if $|\Omega^{y^1}|\rho_W > |\Omega|\rho_S$ and the whole space is filled with the fluid, i.e., the energy is of the form

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))dx + \int_\Omega y_3(x) g\rho_S dx - \int_\Omega g y_3(x)\rho_W dx.$$

To see this, just define $y^2 := y^1 + ce_3$. Invoking the physical picture, this amounts to the body floating, driven by the buoyancy. Therefore, the only case, where the energy is bounded, is if $|\Omega^y|\rho_W = |\Omega|\rho_S$ holds true for all deformations. Then, we are again in the case of incompressible materials and the equilibrium of forces.

If we suppose, that not the whole space is filled by the fluid, but there is a transition between the fluid and air, then we can show that the energy is bounded from below, provided the material is only slightly compressible. In this case, we can prove the existence of minimizers.

Slightly compressible materials

We introduce the notion of slightly compressible materials and give an existence result, when $|\Omega^y|\rho_W > |\Omega|\rho_S$. From now on, we will write $J := \det \nabla y$ for brevity.

Definition 9.2.1. We say a deformation is a *slight compression*, if there is a constant $c > 0$, such that for all deformations satisfy

$$J = \det \nabla y \geq c > 0. \tag{9.2.1}$$

This condition means that the deformations cannot compress arbitrarily. Using this, we can show that the energy is bounded from below, if the specimen satisfies $|\Omega^y|\rho_W > |\Omega|\rho_S$.

Proposition 9.2.2. *Let $c > 0$ and define $\mathcal{A} := \{y \in W^{1,p}(\Omega, \mathbb{R}^3) : |\Omega^y|\rho_W > |\Omega|\rho_S, \text{ and } \det \nabla y \geq c > 0\}$ the set of admissible functions. Then $\inf_{y\in\mathcal{A}} \mathcal{E}(y) > -\infty$.*

Proof. By the change of variables A.3.4, we get

$$|\Omega^y| = \int_{\Omega^y} dx^y = \int_\Omega \det \nabla y(x) dx > c|\Omega|.$$

Moreover, we have that $\rho_S \leq \rho_W J$ holds, for if $\rho_S > \rho_W J$, then $|\Omega|\rho_S = \int_\Omega \rho_S dx > \int_\Omega \rho_W J(x) dx = |\Omega^y|\rho_W$, which is a contradiction. Thus,

$$\rho_S \leq \rho_W J \leq \rho_W c, \tag{9.2.2}$$

and hence,

$$\begin{aligned}
\mathcal{E}(y) &= \underbrace{\int_\Omega W(\nabla y)}_{\geq 0} + \int_\Omega y_3(x) g\rho_S dx + \int_\Omega g y_3(x)^- \rho_W J dx \\
&\overset{(9.2.2)}{\geq} \int_\Omega y_3(x) g\rho_S dx + \int_\Omega g\rho_S y_3(x)^- dx = g\rho_S \int_\Omega \underbrace{(y_3 + y_3^-)}_{=y_3^+ \geq 0} dx \geq 0.
\end{aligned}$$

□

Remark 9.2.2 (on condition (9.2.1)). The condition $\det \nabla y \geq c > 0$ is a very specific case of *material locking*, which originally was introduced by Prager in [25]. The specific condition (9.2.1) was studied by Benešová, Kručík, and Schlömerkemper in [3], and ultimately leads to the introduction of *gradient polyconvexity.* There, the authors showed that Hölder continuity of the $\det \nabla y$ implies the existence of such a $c = c(y) > 0$ with $\det \nabla y > c > 0$, which will yield a uniform bound on c, and thus, the condition (9.2.1), if a energy functional of the form $I(y) := \int_\Omega W(\nabla y(x))dx + \int_\Omega (\det \nabla y(x))^{-s} dx$ is bounded.

However, this approach cannot be used in our case, because, for the given energy

$$\begin{aligned}
&\mathcal{E}(y) + \int_\Omega (\det \nabla y(x))^{-s} dx \\
=&\underbrace{\int_\Omega W(\nabla y)}_{\geq 0} + \int_\Omega (\det \nabla y(x))^{-s} dx + \int_\Omega y_3(x) g \rho_S dx + \int_\Omega g y_3^-(x) \rho_W J dx,
\end{aligned}$$

we cannot conclude that, if $\mathcal{E}(y) \leq \Lambda$, then $\int_\Omega (\det \nabla y(x))^{-s} dx < C$, because the term modelling the forces may may be unbounded from below.

Another way to obtain (9.2.1) by bounding a certain energy functional is provided by Healey and Krömer in [14]. They, however, assume second gradients and Hölder continuous boundary conditions, which is not compatible with the setting of a freely moving specimen.

Proposition 9.2.3. *Let $\Omega \subset \mathbb{R}^3$ be an open, bounded set with Lipschitz boundary and assume that the material is hyperelastic and polyconvex, i.e., there exists a stored energy function W satisfying* (7.1.1) - (7.1.3)*. Furthermore, assume $|\Omega^y| \rho_W > |\Omega| \rho_S$ and set $\mathcal{A}_c := \{y \in W^{1,p}(\Omega; \mathbb{R}^3) \, : \, \det \nabla y \geq c > 0\}$. Then the minimization problem*

$$\textit{Minimize } \mathcal{E}(y) \textit{ for } y \in \mathcal{A}$$

has a solution.

Proof. The proof is analogous to the one of Prop. 9.1.2. First, we claim that $|\bar{y}_3| \leq k\mathcal{E}(y)$. We have that

$$\begin{aligned}
|\bar{y}_3| &\leq \frac{1}{|\Omega|} \int_\Omega |y_3| = \frac{1}{|\Omega|} \int y_3^+ dx + \frac{1}{|\Omega|} \int_\Omega y_3^- dx \\
&\leq \frac{1}{g\rho_S |\Omega|} \int_\Omega g \rho_S y_3^+(x) dx + \frac{1}{|\Omega| g (c\rho_W - \rho_S)} \int_\Omega g(c\rho_W - \rho_S) y_3^-(x) dx = (*).
\end{aligned}$$

By (9.2.2) we have $K := \max\{\frac{1}{g\rho_S|\Omega|}, \frac{1}{|\Omega| g(c\rho_W - \rho_S)}\} > 0$ and, hence, get

$$\begin{aligned}
(*) &\leq K \int_\Omega g \rho_S y_3^+(x) dx + K \int_\Omega g(c\rho_W - \rho_S) y_3^-(x) dx \\
&\overset{(9.2.1)}{\leq} K \int_\Omega g\rho_S y_3^+(x) dx - K \int_\Omega g \rho_S y_3^-(x) dx + K \int_\Omega g \rho_W y_3^-(x) \det \nabla y(x) dx \\
&\leq K \mathcal{E}(y).
\end{aligned}$$

Now we can apply the Poincaré inequality and conclude coercivity as in Prop. 9.1.2.

The weak lower semicontinuity follows by the analysis of Chapters 3 and 6. □

9.3. Local Minimum

So far, the results we obtained are of global nature, where we considered minima over *all* admissible deformations. Moreover, we saw, that if $|\Omega^y|\rho_W < |\Omega|\rho_S$, i.e., the specimen sinks, the infimum of the energy will be $-\infty$. It is anyhow considerable, that local minima may exist. Consider for example the case, where the specimen is located barely under the water surface. In this case compressing the specimen so that it sinks, costs a lot of energy, and thus, it may be more favourable for the specimen to float. In this section, we make these considerations above precise and prove the existence of such local minimizers. To be able to do so, we specify further the structure of the energy.

still not optimal –> just reformulate

The energy functional considered now, will be of the form

$$\mathcal{E}(y) = \int_\Omega W(\nabla y(x))dx + \int_\Omega \frac{k}{J^s} + g\int_\Omega \left(\rho_S y_3 + \rho_W J {y_3}^-\right),$$

for a constant $k > 0$ and $s > 0$.

Let $y_0 : \Omega \to \mathbb{R}^3$ be the deformation chosen in such a way, that it only consists of a rotation and a translation, i.e. $y_0(x) = Rx + t$, for $R \in \mathrm{SO}(d)$, $t \in \mathbb{R}$, and such that $\max y_3 = 0$, as depicted in Fig. 9.1. By definition of y_0 we have $\det \nabla y_0 = 1$ and, therefore, $|y_0(\Omega)| = |\Omega|$.

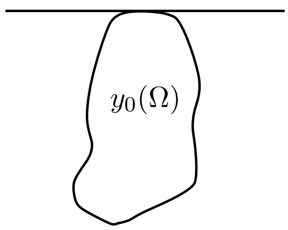


Figure 9.1.: Position of y_0

If we furthermore assume that $|\Omega^{y_0}|\rho_W > |\Omega|\rho_S$, or equivalently $\rho_W > \rho_S$ (note $\nabla y = R$, $\det R = 1$), i.e. the body rises, it is clear that the energy of y_0 is not optimal, and there are more favourable states when the object rises above the surface. Introducing a new notation

$$⨍_\Omega f(x)dx := \frac{1}{|\Omega|}\int_\Omega f(x)dx,$$

we can rewrite the condition $|\Omega^y|\rho_W > |\Omega|\rho_S$ as

$$\fint J > \frac{\rho_S}{\rho_W}. \tag{9.3.1}$$

In this case, we already have shown existence of minimizers. However, this condition is not closed, and therefore, we use a slight modification of it:

$$\fint J \geq \tau > \frac{\rho_S}{\rho_W}, \tag{9.3.2}$$

for $0 < \tau < 1$. This choice is in accordance with our assumptions, since we are assuming that $\rho_W > \rho_S$, and therefore, $\rho_S/\rho_W < 1$. This new assumption implies the old one (9.3.1), under which we already proved the existence of a minimum in Section 9.2. Let us denote this minimum by y^*.

We will prove that all deformations close to this minimum, will admit the condition (9.3.2), even if do not assume that $|\Omega^y|\rho_W > |\Omega|\rho_S$. Hence, this minimum is a local one, considered over admissible deformations. More precisely, we state the following lemma.

Lemma 9.3.1. *There exists a $\varepsilon > 0$ such that all deformations $\hat{y}$ with $\mathcal{E}(\hat{y}) \leq \mathcal{E}(y_0)$, and $\|\hat{y}_3 - y_3^*\|_{L^\infty} < \varepsilon$, automatically satisfy* (9.3.2), *i.e., setting $\hat{J} := \det \nabla \hat{J}$, we have*

$$\fint_\Omega \hat{J} \geq \tau > \frac{\rho_S}{\rho_W}.$$

Proof. Aiming for a contradiction, we assume $\fint \hat{J} \leq \tau < 1$. Then,

$$\mathcal{E}(y_0) \geq \mathcal{E}(\hat{y}) = \underbrace{\int W(\nabla \hat{y})}_{\geq 0} + \int_\Omega \frac{k}{\hat{J}^s} + g\int_\Omega (\rho_S \hat{y}_3 + \rho_W \hat{J}\hat{y}_3^-)dx = (*)$$

Since, $\|\hat{y} - y^*\|_{W^{1,p}} < \varepsilon$, we can estimate the term above using y^* with an error of order ε. The application of Jensen's inequality and the assumption on τ yields

$$\begin{aligned}(*) &\geq \int_\Omega \frac{k}{\hat{J}^s} + \underbrace{g\int_\Omega \left(\rho_S y_3^* + \rho_W J^*(y_3^*)^-\right) dx}_{=:G} + O(\varepsilon)\\
&\overset{Jensen}{\geq} |\Omega| \frac{k}{\left(\fint \hat{J}\right)^s} + G + O(\varepsilon)\\
&\overset{assumpt.}{\geq} k|\Omega| \underbrace{\frac{1}{\tau^s}}_{>1} + G + O(\varepsilon).\end{aligned} \tag{9.3.3}$$

Moreover $\mathcal{E}(y_0) = k|\Omega| + g \int_\Omega (\rho_S - \rho_W)(y_0)_3 dx$, as $\nabla y_0 = R \in \mathrm{SO}(d)$, with $\det \nabla y_0 = 1$. This implies, that $g \int (\rho_S - \rho_W) y_{03} dx = c = c(g, \rho_S, \rho_W, \Omega, y_0)$ can be considered to be a constant, which does *not* depend on k. Hence, we can write $\mathcal{E}(y_0) = k|\Omega| + c$.

If $G = 0$, we would have found a contradiction, since

$$\mathcal{E}(y_0) \geq k|\Omega| \underbrace{\frac{1}{\tau^s}}_{>1} + O(\varepsilon) > k|\Omega| + c = \mathcal{E}(y_0),$$

choosing k large enough.

However, if $G \neq 0$, we have to bound G in terms of the energy to get to a contradiction. First, note that by the coercivity assumption on W and Poincaré's inequality, we get $\mathcal{E}(y^*) \geq c\|\nabla y^*\|_p^p - c \geq c\|y^*\|_\infty^p$. Trivially, we can bound $\varnothing(\Omega^y) \leq 2\|y\|_{L^\infty}$. Therefore, we have

$$G \leq |\Omega| g \rho_S \|y^*\|_\infty + \varnothing(\Omega^y) g \rho_W \leq c\|y^*\|_\infty \leq c\mathcal{E}(y^*)^{1/p}.$$

Since $\mathcal{E}(y^*) \leq \mathcal{E}(y_0)$ we can infer

$$G \geq -c\mathcal{E}(y^*)^{1/p} \geq -c\mathcal{E}(y_0)^{1/p}.$$

Relation (9.3.3) entails that

$$\begin{aligned}
\mathcal{E}(y_0) &> k|\Omega| \frac{1}{\tau^s} - c\mathcal{E}(y^*)^{1/p} + O(\varepsilon) \\
&\geq k|\Omega| \frac{1}{\tau^s} - c\mathcal{E}(y_0)^{1/p} + O(\varepsilon) \\
&= k|\Omega| \frac{1}{\tau^s} - c\,(k|\Omega| + c)^{1/p} + O(\varepsilon).
\end{aligned}$$

This is a contradiction: Since $\mathcal{E}(y_0)$ is linear in k, and $k^{\frac{1}{p}}$ tends to infinity much slower than k, as $k \to \infty$, we can make the right hands side arbitrarily large, choosing k large enough. □

Proposition 9.3.2. *Under the usual assumptions, there exists a local minimizer of $\mathcal{E}$ in*

$$\left\{ \fint J \geq \tau > \frac{\rho_S}{\rho_W} \right\}.$$

Proof. Step 1: Energy is bounded from below.

Note that $\fint J \geq \tau > \rho_S/\rho_W$ implies that $\rho_W \fint J - \rho_S > 0$. Since we can write $\rho_S = \fint \rho_S$, we obtain

$$0 < \rho_W \fint J - \rho_S = \frac{1}{|\Omega|} \int_\Omega \rho_W J - \rho_S.$$

Thus, we have that $\rho_S \leq \rho_W J$ holds, for if $\rho_S > \rho_W J$, then $|\Omega|\rho_S = \int_\Omega \rho_S dx > \int_\Omega \rho_W J(x) dx = |\Omega^y|\rho_W$, which is a contradiction. We make use of the identity

$$y_3 = y_3^+ - y_3^-,$$

and get

$$\begin{aligned}\mathcal{E}(y) &= \underbrace{\int_\Omega W(\nabla y)}_{\geq 0} + \int_\Omega y_3(x) g \rho_S dx + \int_\Omega g y_3(x)^- \rho_W J dx \\ &\geq \int_\Omega y_3(x) g \rho_S dx + \int_\Omega g \rho_S y_3(x)^- dx = g\rho_S \int_\Omega \underbrace{(y_3 + y_3^-)}_{= y_3^+ \geq 0} dx \geq 0.\end{aligned}$$

Step 2: Local Minimum.

Next, we want to put ourselves in the setting of Lemma 9.3.1. Any deformation y with $\mathcal{E}(y) > \mathcal{E}(y_0) \geq \mathcal{E}(y^*)$ cannot be a minimizer. Thus, we can w.l.o.g assume that $\mathcal{E}(y) \leq \mathcal{E}(y_0)$, and therefore, by Lemma 9.3.1, we know that there exists $\varepsilon > 0$ such that for such deformations with additionally $\|y - y^*\|_{L^\infty} < \varepsilon$, the condition $J \geq \tau > {}^{\rho_S}/_{\rho_W}$ holds. Therefore, y^* is indeed a minimizer over all deformations y with $\|y - y^*\|_{L^\infty} < \varepsilon$. $\square$

A. Appendix

We provide an overview of notions and results used over the course of the thesis.

Let $\Omega \subset \mathbb{R}^d$ be open, connected, and with Lipschitz-boundary (for a definition refer to [18], Def. 9.57). Furthermore, we assume Ω to be of finite d-dimensional Lebesgue measure, denoted as $|\Omega| < \infty$. From now on we will use these assumptions on Ω, although further generalizations would be possible.

A.1. L^p-spaces

We define L^p-functions in the case of $p \in [1, \infty]$, and recall the most important results used in the thesis. Let $f : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}$, be a Lebesgue-measurable function. We set

$$\|f\|_{L^p(\Omega;\mathbb{R}^m)} := \begin{cases} \sqrt[p]{\int_\Omega |f(x)|^p dx} & \text{for} \quad 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & \text{for} \quad\quad p = \infty, \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$ and the integral is to be understood as Lebesgue integral. The class of all measurable functions with $\|f\|_{L^p(\Omega;\mathbb{R}^m)} < \infty$, up to a.e. identification, is denoted with $L^p(\Omega; \mathbb{R}^m)$. The essential supremum is defined as

$$\operatorname*{ess\,sup}_{x\in\Omega} f := \inf_{|N|=0} \sup_{x\in\Omega\setminus N} f(x).$$

Here, the $|\cdot|$ denote the d-dimensional Lebesgue measure. It should not lead to confusion with the Euclidean norm, as in this case, we are considering sets.

Remark A.1.1 (on the notation). During the thesis, for brevity we often omit to specify the domain or the target space, when subscripting the norm, whenever it is clear what is meant. Moreover, when needed to distinguish the d and $(d-1)$-dimensional Lebesgue measure (e.g. when considering the boundary of Ω), we will denote the measure of the set with $\text{meas}_d(\Omega)$, or $\text{meas}_{d-1}(\partial\Omega)$ respectively.

It will often be useful to interchange the limit and integral, which is allowed under the conditions of the following important theorems.

Theorem A.1.1 (Fatou)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. For each sequence (f_n) of non-negative, measurable functions $f_n : \Omega \to \mathbb{R} \cup \{\infty\}$ it holds that*

$$\int_\Omega \liminf_n f_n d\mu \leq \liminf_n \int_\Omega f_n d\mu.$$

Theorem A.1.2 (Monotone convergence)**.** *Let (f_n) be a sequence of Lebesgue-integrable functions with $f_n \leq f_{n+1}$ almost everywhere, for all $n \in \mathbb{N}$. Moreover, suppose that $\sup_n \int_\Omega f_n(x)dx < \infty$. Then,*

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

Theorem A.1.3 (Dominated convergence)**.** *Let (f_n) be a sequence of Lebesgue-integrable functions, converging almost everywhere to f. If there exists a $g \in L^1(\Omega)$, with $|f_n| \leq g$ almost everywhere, then f is integrable and*

$$\lim_{n\to\infty} \int_\Omega f_n(x)dx = \int_\Omega f(x)dx.$$

The proofs can be found in [4], Sec. 2.8.

Another important tool is Hölder's inequality.

Theorem A.1.4 (Hölder's inequality)**.** *Let $1 \leq p \leq \infty$ and let $f \in L^p(\Omega, \mathbb{R}^m)$ and $g \in L^q(\Omega, \mathbb{R}^m)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the function $f \cdot g$ is in $L^1(\Omega, \mathbb{R})$ and*

$$\|f \cdot g\|_{L^1(\Omega;\mathbb{R})} \leq \|f\|_{L^p(\Omega;\mathbb{R}^m)} \|g\|_{L^q(\Omega,\mathbb{R}^m)}.$$

Theorem A.1.5 (Generalized Hölder inequality)**.** *Let $1 \leq p_1, \cdots, p_N, p \leq \infty$ such that $\frac{1}{p_1} + \cdots \frac{1}{p_N} = \frac{1}{p}$ and let $f_i \in L^{p_i}(\Omega)$ for $i = 1, \ldots, N$. Then, the following inequality holds*

$$\left\| \prod_{i=1}^N f_i \right\|_{L^p} \leq \prod_{i=1}^N \|f_i\|_{L^{p_i}}.$$

The proof of the Generalized Hölder inequality follows by induction.

A.2. Sobolev spaces

We recall the basic definitions and most important results regarding Sobolev spaces, such as the embedding theorems, the existence of the trace, and compact embeddings. See also [17], B.3, and [18], [12].

Basic Definitions

Infinitely differentiable functions with compact support are called *test functions*, denoted by $\mathscr{D} = \mathscr{C}_c^\infty$. For a multi-index $\alpha \in \mathbb{N}_0^d$, with order $|\alpha| := \alpha_1 + \cdots + \alpha_d$, and a function $y \in L^p(\Omega)$, we define the α-*distributional derivative* of y as the distribution satisfying

$$\left\langle \frac{\partial^\alpha y}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}, \phi \right\rangle = (-1)^{|\alpha|} \int_\Omega y \frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} dx,$$

for all test functions $\phi \in \mathscr{D}$. We call d-tuple of distributional derivatives of order 1 the *gradient* and write $\nabla y = \left(\frac{\partial}{\partial x_1} y, \dots, \frac{\partial}{\partial x_d} y \right)$. For $p < \infty$ we define the *Sobolev space*

$$W^{1,p}(\Omega) := \{ y \in L^p(\Omega) \, : \, \nabla y \in L^p(\Omega; \mathbb{R}^d) \},$$

which is a Banach space, when equipped with the norm

$$\|y\|_{W^{1,p}(\Omega)} := \left(\|y\|_{L^p(\Omega)}^p + \|\nabla y\|_{L^p(\Omega;\mathbb{R}^d)}^p \right)^{1/p}.$$

The spaces of $\mathbb{R}^m$-valued Sobelev functions are defined as $W^{1,p}(\Omega; \mathbb{R}^m) := \{ y = (y_1, \dots, y_m) \, : \, y_i \in W^{1,p}(\Omega), \, i = 1, \dots, m \}$.

Lebesgue spaces L^p and Sobolev spaces $W^{1,p}$ are separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$.

Since Sobolev functions are L^p functions, and thus, insensitive to changes of sets of measure zero, we cannot evaluate these functions at the boundary. This, however, prevents us from considering problems with prescribed boundary values. A way out is given by the following trace theorem.

Theorem A.2.1 (Trace theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, connected, of finite measure, and with Lipschitz boundary $\Gamma := \partial\Omega$. Then, there exists a unique linear continuous operator, called trace operator,*

$$T : W^{1,p}(\Omega) \to L^1(\Gamma)$$

such that, for any $y \in \mathscr{C}(\bar{\Omega})$, it holds that

$$Ty = y|_\Gamma.$$

Moreover, the operator T

$$T : W^{1,p}(\Omega) \rightarrow L^q(\Gamma) \quad \begin{array}{c} \\ y \mapsto u|_\Gamma \end{array} \quad \text{is} \begin{cases} \text{continuous} & \text{if} \quad 1 \leq q \leq p^\sharp, \\ \text{compact} & \text{if} \quad 1 \leq q < p^\sharp, \end{cases}$$

where $p^\sharp$ is the so-called Sobolev trace exponent with values

$$p^\sharp := \begin{cases} \frac{dp-p}{d-p} & \text{for} \quad p < d, \\ \text{an arbitrary } r \in \mathbb{R} & \text{for} \quad p = d, \\ \infty & \text{for} \quad p > d. \end{cases}$$

See [17], Theorem B.3.6, [12], Chap. 5.5. Theorem 1, or [18], Chap. 18.1. We consistently consider boundary values of Sobolev functions in the trace sense, given by the trace operator above. Moreover, we define $W_0^{1,p}(\Omega)$ to be the set of Sobolev functions with zero trace (cf. [12], Sec. 5.5, Thm. 2).

Embedding theorems

Definition A.2.1. For $1 \leq p < n$ we define the *Sobolev conjugate $p*$* as

$$p* := \frac{pn}{n-p}.$$

We summarize the embedding theorem of Gagliardo-Nierenberg-Sobolev ([12], Sec. 5.6, Thm. 1) and the embedding theorem of Morrey ([12], Sec. 5.6, Thm. 4) into the following Theorem ([12], Sec. 5.6, Thm. 6)

Theorem A.2.2 (General Sobolev inequalities)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a $\mathscr{C}^1$ boundary. Assume $u \in W^{k,p}(\Omega)$.*

(i) If

$$k < \frac{n}{p},$$

then $u \in L^q(\Omega)$, where

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

and we have the estimate

$$\|u\|_{L^q(\Omega)} \leq C(k,p,n,\Omega)\|u\|_{W^{k,p}}.$$

(ii) *If*

$$k > \frac{n}{p},$$

then $u \in \mathscr{C}^{k - \lfloor \frac{n}{p} \rfloor - 1, \gamma}(\bar{\Omega})$, *where*

$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number } < 1 & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

Additionally, we have the estimate

$$\|u\|_{\mathscr{C}^{k - \lfloor \frac{n}{p} \rfloor - 1, \gamma}(\bar{\Omega})} \leq C(k, p, n, \gamma, \Omega) \|u\|_{W^{k,p}}.$$

Theorem A.2.3 (Rellich-Kondrachov, [12], Sec. 5.7, Thm. 1)**.** *Assume* Ω *to be a bounded open subset of* $\mathbb{R}^n$ *with* $\mathscr{C}^1$ *boundary* $\partial\Omega$. *Suppose* $1 \leq p < n$. *Then we have the compact embedding*

$$W^{1,p}(\Omega) \Subset L^q(\Omega)$$

for each $1 \leq q < p*$.

A family of continuous functions $F \subset \mathscr{C}(X, Y)$ between metric spaces is called *equicontinuous*, if for every $\varepsilon > 0$ and every $x \in X$, there is a neighbourhood $U(x)$ of x such that

$$d_Y(f(x), f(y)) < \varepsilon \quad \forall y \in U(x), \forall f \in F.$$

Theorem A.2.4 (Arzelà-Ascoli)**.** *Let* X *be a compact metric space and* Y *be a metric space, satisfying the Heine-Borel property. Let* $F \subset \mathscr{C}(X, Y)$ *be a family of continuous functions. Then every sequence from* F *has a uniformly convergent subsequence, if and only if* F *is equicontinuous and the set* $\{f(x) : f \in F\}$ *is bounded for every* $x \in X$.

For the proof refer to [28], Thm. B.39.

Poincaré inequalities

Theorem A.2.5 ([12], Sec. 5.6, Thm. 3)**.** *Assume* Ω *to be a bounded, open subset of* $\mathbb{R}^n$ *and* $1 \leq p < n$. *Then, there exists* $c > 0$ *such that*

$$\|u\|_{L^q} \leq c \|\nabla u\|_{L^p} \quad \forall u \in W_0^{1,p}(\Omega)$$

for each $q \in [1, p^*]$, *the constant* c *depending on* $p, q, n,$ *and* Ω.

Notation: Introducing the barycentre of a function u over its domain Ω

$$\bar{u} := \frac{1}{|\Omega|} \int_\Omega u(x)dx,$$

we can formulate the following Poincaré inequality.

Theorem A.2.6 ([12], Sec. 5.8, Thm. 1)**.** *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$ with a $\mathscr{C}^1$ boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then, there exists a constant c, depending only on n, p, and Ω such that*

$$\|u - \bar{u}\|_{L^p} \leq c\|\nabla u\|_{L^p} \quad \forall u \in W^{1,p}.$$

A powerful generalization is the following (cf. Theorem B.3.15, [17], p. 519).

Theorem A.2.7 (Generalized Poincaré)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded Lipschitz domain and let $1 \leq p < \infty$. Let further $\Gamma_D \subset \partial\Omega$ be such that $meas_{d-1}(\Gamma_D) > 0$. Then, there exists a constant $k > 0$ such that for all $v \in W^{1,p}(\Omega, \mathbb{R}^n)$ the following inequalities hold*

$$\int_\Omega |v(x)|^p dx \leq k \left(\int_\Omega |\nabla v(x)|^p dx + \left| \int_\Omega v(x) dx \right| \right).$$

And if $\Gamma \subset \partial\Omega$ is measurable and such that $meas_{d-1}(\Gamma) > 0$, then

$$\int_\Omega |v(x)|^p dx \leq k \left(\int_\Omega |\nabla v(x)|^p dx + \left| \int_{\Gamma_D} v(x) dS \right| \right).$$

A.3. Integral identities

We will collect a few integral formulas and introduce further necessary notions, like Lusin's condition. Aside the commonly known Green formula, we will mention two change of variable formulas and preliminary notions to formulate them.

Theorem A.3.1 (Green's formula)**.** *Let Ω be a Lipschitz domain and $\mathfrak{n} = \mathfrak{n}(x) \in \mathbb{R}^d$ the outward unit normal to the boundary $\Gamma = \partial\Omega$ at the point $x \in \Gamma$. Then, for all $v \in W^{1,p}(\Omega)$ and $z \in W^{1,q}(\Omega; \mathbb{R}^d)$, the following formula holds*

$$\int_\Omega (v(\operatorname{div} z) + z \cdot \nabla v)\, dx = \int_\Gamma v(z \cdot \mathfrak{n}) dS.$$

Before we get to the change of variables, we introduce the Lusin's conditions and the Banach indicatrix.

Definition A.3.1 (Lusin's conditions)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded measurable domain. Then, $y : \Omega \to \mathbb{R}^d$ is said to satisfy *Lusin's condition* N if for every $\omega \subset \Omega$ with $\text{meas}_d(\omega) = 0$ it holds that $\text{meas}_d(y(\omega)) = 0$.

Of course, one can consider functions whose pre-image of null sets as again a null set. The function $y : \Omega \to \mathbb{R}^d$ is said to satisfy *Lusin's condition* N^{-1} if for every $\tilde{\omega} \subset y(\Omega)$ with $\text{meas}(\tilde{\omega}) = 0$ it holds that $\text{meas}_d(y^{-1}(\tilde{\omega})) = 0$.

We recall that smooth Sobolev functions on bounded sets automatically satisfy Lusin's condition N.

Lemma A.3.2. *Let $\Omega \subset \mathbb{R}^3$ be bounded and $y \in W^{1,p}(\Omega;\mathbb{R}^3)$, with $p > 3$. Then y satisfies Lusin's condition N.*

The proof can be found in [20], Cor. 1. A result on Lusin's condition N^{-1} is the following (cf. [5], Lemma 8.3).

Theorem A.3.3. *Let Ω be a bounded domain and $y : \Omega \to \mathbb{R}^d$ be a continuous mapping satisfying Lusin's condition N. Assume that y is differentiable almost everywhere in Ω and that $\det \nabla y$ is integrable in $\mathbb{R}^d$ and positive a.e. in Ω. Then y satisfies Lusin's condition N^{-1}. In particular, if $y \in W^{1,p}(\Omega;\mathbb{R}^d)$ for some $p > d$ is a continuous representative of the equivalence class and $\det \nabla y > 0$ a.e. in Ω then y satisfies Lusin's condition N^{-1}.*

To formulate the change of variables formula, we need one additional ingredient, the Banach indicatrix.

Definition A.3.2. For any $z \in \mathbb{R}^d$ and $\Omega \subset \mathbb{R}^d$ the *Banach indicatrix* $N(z, y, \Omega)$ is the number of elements in Ω, which are mapped to z by y, formally

$$N(z, y, \Omega) := \#\{x \in \Omega \,:\, y(x) = z\},$$

where the right hand side is the counting measure.

Theorem A.3.4 (Change of Variables)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $y : \Omega \to \mathbb{R}^d$ be continuous, satisfying Lusin's condition N. Assume that y is differentiable a.e. in Ω and that $\det \nabla y$ is integrable in $\mathbb{R}^d$. Then, the Banach indicatrix $N(\cdot, y, \Omega)$ is integrable and we have*

$$\int_\Omega |\det \nabla y(x)| dx = \int_{\mathbb{R}^d} N(z, y, \Omega) dz = \int_{y(\Omega)} N(z, y, \Omega) dz.$$

Note that any $y \in W^{1,p}(\Omega;\mathbb{R}^d)$ satisfies the hypothesis of the above theorem. For a proof of this assertion and Thm. A.3.4 refer to [5].

Theorem A.3.5 (Change of Variables 2)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $y \in W^{1,d}(\Omega;\mathbb{R}^d)$ be continuous, satisfying Lusin's condition N, and such that* $\det \nabla y > 0$ $\forall_{aa} x \in \Omega$*. Then for every $f \in L^\infty(y(\Omega))$ it holds that*

$$\int_\Omega f(y(x)) \det \nabla y(x) dx = \int_{y(\Omega)} f(z) N(z, y, \Omega) dz.$$

The proof can be found in [5], Thm. 8.4.

A.4. Weak Topology

Given a real Banach Space X we define its dual space X' as the set of all bounded linear functional into $\mathbb{R}$. The weak topology is the initial topology on X with respect to X', i.e., it is the coarsest topology on X such that all functionals $f \in X'$ are continuous.

Definition A.4.1. A sequence $(x_n) \subset X$ converges weakly to $x \in X$, in symbols: $x_n \rightharpoonup x$, if

$$\lim f(x_n) = f(x) \quad \forall f \in X'.$$

A sequence $(f_n) \subset X'$ converges weak-* to $f \in X'$, if

$$f_n(x) \to f(x) \qquad \forall x \in X.$$

Proposition A.4.1. *If $(f_n) \subset X'$ converges weakly, then it also converges weak-*, i.e. weak convergence implies weak-* convergence. So, weak-* convergence is indeed weaker than weak convergence.*

If, however, the Banach Space X is reflexive these to notions of convergence are equivalent. We will make this precise later on.

Theorem A.4.2. *Let X be a Banach space. Then X is reflexive, if and only if each bounded sequence has a weakly convergent subsequence.*

This theorem is a special case of the

Theorem A.4.3 (Banach-Alaoglu)**.** *Let E be a normed space and E' its topological dual space. Then the unit sphere $D := \{f \in E' : \|f\|_{E'} \leq 1\}$ is compact with respect to the weak-* topology.*

This means that every bounded set $S \subset E'$ contains a bounded sub*net* $(f_\iota)_{\iota \in I}$ such that $f_\iota(x) \to f(x)$ for all $x \in E$. Important: For general metric spaces this does not imply sequential compactness!

Example A.4.1. The unit ball in the dual of l^∞ is weak-* compact by the Banach-Alaoglu theorem. It is, however, not sequentially compact. To see this consider

$$\begin{aligned} p_n : l^\infty(\mathbb{N}) &\to \mathbb{R} \\ (x_k)_k &\mapsto x_n \end{aligned}$$

Then $(p_k)_k$ does not have a convergent subsequence. △

On the contrary, for a Banach space weak-compactness and weak-sequential compactness are the same. This is not obvious, as the weak topology is not metrizable.

Theorem A.4.4 (Eberlein-Šmulian)**.** *Let X be a Banach space. Then the following are equivalent:*

(i) Weak sequential compactness: Every sequence has a weakly convergent subsequence.
(ii) Weak compactness: Every weakly open (open sets in the weak topology) cover contains a finite subcover.

For a proof, see [10], Chap. V.6. This result is incredibly useful, as it guarantees that we can work with sequences in weak compact sets.

Remark A.4.1 (Putting all together). Recall: A Banach space is reflexive if $X = (X')' = X''$, i.e. when the Banach space and its bidual are isometrically isomorphic via the Riesz isomorphism. In general, the weak-* topology on X' is strictly weaker than the weak topology on X' (cf. Prop. A.4.1). If, however, the Banach space is reflexive, then the weak topology on X is identical to the weak-* topology on $(X')' = X''$. By the Banach-Alaoglu theorem, we know that the unit sphere is weak-* compact in the dual of a normed space, so applying this to the dual of X', we have that the unit sphere in $(X')'$ is weak-* compact, and thus, it is weak compact in X! This means that every bounded subset of a reflexive Banach space is weak compact! Since, by Eberlein-Smulian, this is equivalent to being weak sequential compact, we have: In a reflexive Banach space every bounded sequence has a convergent subsequence! In fact, this is even an “if and only if”: If the every bounded sequence has a convergent subsequence the Banach space is reflexive.

Boundedness of weak converging sequences and lower semicontinuity

Similar to the simple case of converging sequences of real numbers, weak converging sequences in Banach spaces are norm-bounded.

Theorem A.4.5 (Boundedness of weak converging sequences)**.** *Let X be a Banach space and $(x_n) \subset X$ converging weakly to $x \in X$, i.e. $x_n \rightharpoonup x$. Then (x_n) is bounded.*

Proof. Let x_n be weakly convergent: $x_n \rightharpoonup x \Leftrightarrow f(x_n) \to f(x)$ for all $f \in X'$. Define $T_n \in X''$ by $T_n(f) = f(x_n)$ for all $f \in X'$, and a fixed $f \in X'$. Since x_n is weakly convergent $f(x_n)$ is bounded, and thus, $(T_n(f))_n$ is in a bounded set. By the Uniform Boundedness principle we obtain

$$\sup \|x_n\| = \sup \|T_n\| < \infty.$$

Hence, (x_n) is bounded. □

Before we state another important property of the norm, we recall the following consequence of the Hahn-Banach Theorem.

Lemma A.4.6. *Let X be Banach space. For every $x \in X$, there is a $f \in X'$ such that $\|f\|_{X'} = \|x\|_X$ and $f(x) = \|x\|^2$.*

The proof can be found in [6], Corollary 1.3.

Theorem A.4.7 (Norm is w.l.s.c.)**.** *Let X ba a Banach space and $(x_n) \subset X$ an weakly converging sequence $x_n \rightharpoonup x$. Then the following holds*

$$\|x\| \leq \liminf \|x_n\|.$$

Proof. By Lemma A.4.6, there is a functional $f_0 \in X'$ such that $\|f_0\|_{X'} = 1$ and $|f_0(x)| = \|x\|$. Since $x_n \rightharpoonup x$, we have $f(x_n) \to f(x)$ for all $f \in X'$. In particular, $f_0(x_n) \to f_0(x)$ and thus $|f_0(x_n)| \to |f_0(x)| = \|x\|$. Putting all this together we obtain

$$\|x\| = |f_0(x)| = \lim |f_0(x_n)| \leq \liminf |f_0(x_n)| \leq \liminf \|f_0\|_{X'} \|x_n\| = \liminf \|x_n\|. \quad \square$$

Results on weak and strong convergence

We now recall different notions of convergence and collect a few statements on the connection of weak and strong convergence in Sobolev and L^p spaces.

Weak convergence of a sequence (u_j) means that for all elements f in the dual space the sequence $f(u_j)$ converges. For the space $W^{1,p}$ it is not very handy to work with this definition, but we would like to have a characterization of weak convergence in terms of u_j and ∇u_j.

Theorem A.4.8. *In the space $W^{1,p}$, $1 \leq p < \infty$ the following are equivalent:*

(i) $u_j \rightharpoonup u$ *in* $W^{1,p}$*, i.e.,* $f(u_j) \to f(u)$ *for all* $f \in (W^{1,p})^\star$

(ii) $u_j \rightharpoonup u$ *in* L^p *and* $\nabla u_j \rightharpoonup \nabla u$ *in* L^p*.*

Theorem A.4.9 (Weak Sobolev convergence implies strong L^q convergence)**.** *Suppose* $y_k \rightharpoonup y$ *weakly in* $W^{1,p}(\Omega)$*,* $\Omega \subset \mathbb{R}^n$*. Then the sequence converges strongly in* L^p*, that is,* $y_{k_n} \to y$ *as* $n \to \infty$*.*

Proof. $y_k \rightharpoonup y$ in $W^{1,p}$ implies, that (y_k) is bounded in $W^{1,p}$. By the compact embedding (Rellich-Kondrachov) $W^{1,p} \subset\subset L^q$ for $1 \leq q \leq p^* = \frac{np}{n-p}$ there is a subsequence y_{k_n} converging strongly in L^q. By Thm. A.4.8 we also have $y_k \rightharpoonup y$ in L^p. Therefore, the limit of y_{k_n} is y, as strong convergence implies weak convergence and weak limit are unique. The following fact on Hausdorff spaces concludes the proof: If y, y_k are such that every subsequence of y_k has a subsubsequence converging to y, then also y_k converges to y. □

Corollary A.4.10. *If* $y_n \rightharpoonup y$ *weakly in* $W^{1,p}$*, then it converges almost everywhere to* y*.*

Theorem A.4.11 (Mazur)**.** *Let* X *be a normed vector space and* (x_n) *a sequence weakly converging to* x*. Then, there is a sequence* (y_n) *of convex combinations of* x_n*, i.e.,* $y_n = \sum_{i=1}^{N_n} \lambda_{i,n} x_i$ *with* $\sum_{i=1}^{N_n} \lambda_{i,n} = 1$*, such that* (y_n) *convergences strongly to* x *(namely,* $\|y_n - x\| \to 0$*).*

A proof can be found in [28], Cor. 5.12.

A.5. A Mathematician's toolbox

We collect some important tools, including some useful inequalities.

Inequalities

Lemma A.5.1 (Cauchy's inequality)**.** *We have*

$$2ab \leq a^2 + b^2 \quad \forall a, b \in \mathbb{R}.$$

Using this one can now prove:

Proposition A.5.2. *If* $f, g \in L^p(\Omega)$*, then their product* fg *belongs to* $L^{p/2}(\Omega)$*.*

In fact, we have that

$$\int |fg|^{p/2} \leq \frac{1}{2} \int \left(|f|^{p/2}\right)^2 + \frac{1}{2} \int \left(|g|^{p/2}\right)^2 \leq c\|f\|_p + c\|g\|_p < \infty$$

Another way to prove this result is using Hölder's inequality:

$$\int |fg|^{p/2} = \int |f|^{p/2}|g|^{p/2} \leq \left(\int \left(|f|^{p/2}\right)^2\right)^{1/2} \left(\int \left(|g|^{p/2}\right)^2\right)^{1/2} = \|f\|_p^{p/2}\|g\|_p^{p/2} < \infty$$

Lemma A.5.3 (Young's inequality)**.** *Let $a, b, \delta > 0$ and $p, q \geq 1$ such that $1/p + 1/q = 1$. Then, the following inequality holds*

$$ab \leq \frac{\delta^p a^p}{p} + \frac{b^q}{\delta^q q}.$$

Proof. By the identities $\exp(\ln(x)) = x$ for $x > 0$ and $\ln(ab) = \ln(a) + \ln(b)$, we get

$$\begin{aligned} ab &= \exp(\ln(\delta a\, 1/\delta b)) = \exp\left(\frac{1}{p} p \ln(\delta a) + \frac{1}{q} q \ln(b/\delta)\right) \\ &= \exp\left(\frac{1}{p}\ln(\delta^p a^p) + \frac{1}{q}\ln(b^q/\delta^q)\right) \leq \frac{1}{p}\delta^p a^p + \frac{1}{\delta^q q} b^q, \end{aligned}$$

where we used in the last step the definition of convexity applied to the strictly convex function exp. □

For two matrices $A, B \in \mathbb{R}^{d\times d}$ we define the matrix dot-product by $A : B := \sum_{i,j=1}^d A_{ij}B_{ij}$. The Frobenius norm of a matrix $A \in \mathbb{R}^{d\times d}$ is the norm induced by this product, i.e., $|A|_F^2 := A : A$.

Lemma A.5.4 (Hadamard's inequality)**.** *Let $|\cdot|_F$ denote the Frobenius norm and let $A \in \mathbb{R}^{d\times d}$. Then,*

$$|\det A| \leq d^{d/2} {|A|_F}^d.$$

Measure theory

Theorem A.5.5 (Vitali convering theorem)**.** *Let $\Omega, D \subset \mathbb{R}^d$ be open and bounded. Then, there exist $a_k \in \Omega, r_k > 0$, for $k \in \mathbb{N}$, such that Ω is the disjoint union*

$$\Omega = Z \cup \bigcup_{k=1}^{\infty} \overline{D(a_k, r_k)},$$

with $D(a_k, r_k) := a_k + r_k D$, and $Z \subset \Omega$ is a Lebesgue-null set, i.e. $|Z| = 0$. Moreover, is for almost all $x \in \Omega$, we are give an real number $r(x) > 0$, then one can require that $r_k < r(a_k)$ for all $k \in \mathbb{N}$.

Lemma A.5.6. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $A, B \in \mathcal{A}$ such that $A \subset B$. Then,*

$$\mu(B \backslash A) + \mu(A) = \mu(B).$$

Moreover, if $\mu(A) < \infty$, we have $\mu(B \backslash A) = \mu(B) - \mu(A)$.

Proof. Clearly, by the inclusion,

$$B = (B \backslash A) \cup (B \cap A) = (B \backslash A) \cap A.$$

Since $B \backslash A$ and A are disjoint, we get

$$\mu(B) = \mu(B \backslash A) + \mu(A). \qquad \square$$

A.6. Auxiliary Calculations

Passing to subsequences when proving weak lower semicontinuity

We often prove lower semicontinuity for some subsequence. Fortunately, this also imply the statement for the original sequence, as we will show now.

Assume we have already proved

$$u_n \to u \implies \exists \text{ subsequence of } (u_n) : f(u) \leq \liminf_{j \to \infty} f(u_{n_j}), \tag{A.6.1}$$

then also

$$f(u) \leq \liminf_{n} f(u_n).$$

In other words: If we know that f is lower semicontinuous for *a particular* subsequence of *an arbitrary* converging sequence, we know that f is lower semicontinuous. This result is very important, as the described procedure is exactly the one, we carry out for proving lower semicontinuity in the case of the existence results. There, we often consider an arbitrary sequence $y_n \rightharpoonup y$ in $W^{1,p}$, and then apply embedding theorems which give us subsequences of y_n for which we prove lower semicontinuity. If this would not hold, proving lower semicontinuity would become extremely cumbersome.

We state this again with all needed assumptions (cf. [27], Problem 2.1).

Proposition A.6.1. *Let X be a complete metric space and $f : X \to \mathbb{R}$. If for every* *sequence* $u_n \to u$ *there is a subsequence* (u_{n_j}) *such that*

$$f(u) \leq \liminf_j f(u_{n_j}),$$

then also

$$f(u) \leq \liminf_n f(u_n).$$

Proof. Recall the definition of the limes inferior: $\liminf f(x_n)$ is the smallest accumulation point of the sequence $f(x_n)$. Therefore, there is subsequence x_{n_k} such that

$$\liminf_n f(x_n) = \lim_k f(x_{n_k}).$$

Step 1: We know by the definition of the $\liminf$, that there is a subsequence of $(f(u_n))_n$ realizing this $\liminf$, i.e.

$$\alpha = \liminf_n f(u_n) = \lim_j f(u_{n_j}). \tag{A.6.2}$$

So, although we do not know that $f(u_n) \to f(u)$, we do know that $f(u_{n_j}) \to \alpha$ and $u_{n_j} \to u$ (since it is a subsequence).

Step 2: By applying A.6.1 to the sequence (u_{n_j}), we also know that there is a subsequence $(u_{n_{j_k}})_k$ of (u_{n_j}) such that

$$f(u) \leq \liminf_k f(u_{n_{j_k}}). \tag{A.6.3}$$

Step 3: Since $(u_{n_{j_k}})$ is a subsequence of (u_{n_j}), we have that $f(u_{n_{j_k}})$ is a subsequence of $f(u_{n_j})$, and therefore, also converges to α, i.e. $f(u_{n_{j_k}}) \to \alpha = \lim_j f(u_{n_j})$.

Step 4: Combining these steps:

$$f(u) \overset{A.6.3}{\leq} \liminf_k f(u_{n_{j_k}}) = \lim_j f(u_{n_j}) \overset{A.6.2}{=} \liminf_n f(u_n). \qquad \square$$

Pressure Load is conservative

We start by defining the Gâteaux differential (cf. [13], Definition 4.60). Let V be a Banach space. A function $F : V \to (-\infty, +\infty]$ is *Gâteaux differentiable* at $v_0 \in V$ if $F(v_0) \in \mathbb{R}$ and there exists a $v' \in V'$ such that for every $v \in V$,

$$\lim_{\varepsilon \to 0+} \frac{F(v_0 + \varepsilon v) - F(v_0)}{\varepsilon} = v'(v) =: F'(v_0)v.$$

The element $v' =: F'(v_0)$ is called *Gâteaux differential* of F at v_0.

Some authors prefer to define the Gâteaux differential as $F'(v_0)v := \lim_{\varepsilon \to 0+} \frac{f(v_0+\varepsilon v)-f(v)}{\varepsilon}$, given the limit exists. In this case, however, the map $F'(v_0)$ is not necessarily linear. For a counterexample check [28], Example 16.3.

Now, we prove that the pressure load is conservative. We are given the functional

$$F(y) = -\int_\Omega p(y(x)) \det \nabla y(x) dx.$$

Thus, the Gâteaux derivative of F at y in direction v is given by

$$F'(y)v = \left(\frac{d}{dt}F(y+tv)\right)_{t=0} = \left(-\frac{d}{dt}\int_\Omega p(y(x)+tv(x)) \det \nabla(y(x)+tv(x))dx\right)_{t=0} = (*).$$

Differentiating under the integral, we get

$$\begin{aligned}
(*) &= \left(-\int_\Omega \left(\frac{d}{dt}p(y+tv)\right) \det \nabla(y+tv) + p(y+tv)\frac{d}{dt}\det(\nabla y + t\nabla v)dx\right)_{t=0} \\
&= \left(-\int_\Omega (\det \nabla(y+tv))\nabla^y p(y+tv)\cdot v + p(y+tv)\operatorname{Cof}\nabla(y+tv) : \nabla v dx\right)_{t=0} \\
&= -\int (\det \nabla y)\nabla^y p(y)\cdot v + P(y)\operatorname{Cof}\nabla y : \nabla v dx = (**).
\end{aligned}$$

The first identity follows from the formula for the Gateaux derivative of the determinant $\det'(A)H = \operatorname{Cof} A : H$, which can be found in [7], Sec. 1.2. This enables us to use the Gauß-Green Theorem, which already appeared in the principle of virtual work:

$$\int_\Omega H : \nabla v dx = -\int_\Omega (\operatorname{div} H)\cdot v dx + \int_\Gamma H\mathfrak{n}\cdot v dS,$$

where the divergence of a tensor field is a vector field given by $(\operatorname{div} H)_i = \sum_{j=1}^d \partial_{x_j} H_{ij}$. Therefore, setting $H := p(y)\operatorname{Cof}\nabla y$ in the formula, we get

$$(**) = -\int (\det \nabla y)\nabla^y p(y)\cdot v + \int_\Omega \operatorname{div}(p(y)\operatorname{Cof}\nabla y)\cdot v + \int_\Gamma p(y)\operatorname{Cof}\nabla y\mathfrak{n}\cdot v dS = (***).$$

Now, we can use the formula for the divergence of a tensor field and Piola identity

$\operatorname{div}(\operatorname{Cof} \nabla y) = 0$ to calculate

$$
\begin{aligned}
(\operatorname{div}(p(y(x)) \operatorname{Cof} \nabla y(x)))_i &= \sum_j \partial_{x_j} (p(y(x))(\operatorname{Cof} \nabla y(x))_{ij}) \\
&= \sum_j \Big(\partial_{x_j} p(y(x))\Big) (\operatorname{Cof} \nabla y(x))_{ij} + \sum_j p(y(x)) \partial_{x_j} (\operatorname{Cof} \nabla y(x))_{ij} \\
&= ((\operatorname{Cof} \nabla y) \nabla p(y))_i + p(y(x)) \underbrace{\operatorname{div}(\operatorname{Cof} \nabla y)}_{=0},
\end{aligned}
$$

and thus, finally get

$$
(***) = -\int (\det \nabla y) \nabla^y p(y) \cdot v + \int_\Omega (\operatorname{Cof} \nabla y) \nabla p(y) \cdot v + \int_\Gamma p(y) \operatorname{Cof} \nabla y \mathfrak{n} \cdot v dS. \quad (\bigstar)
$$

We are left with proving that the right-hand side of $(\bigstar)$ equals $-\int_\Gamma p(y) \operatorname{Cof} \nabla y \mathfrak{n} \cdot v dS$.

$$
\begin{aligned}
&\int_\Omega ((\operatorname{Cof} \nabla y) \nabla p(y)) \cdot v dx = \int_\Omega \sum_i ((\operatorname{Cof} \nabla y) \nabla p(y))_i v_i \\
&= \int_\Omega \sum_i \sum_j (\operatorname{Cof} \nabla y)_{ij} \partial_j p(y) v_i = \int_\Omega \sum_{i,j} (\operatorname{Cof} \nabla y)_{ij} \sum_k (\nabla^y p(y))_k \partial_j y_k v_i \\
&= \int_\Omega \sum_{i,j,k} (\operatorname{Cof} \nabla y)_{ij} (\nabla^y p(y))_k \partial_j y_k v_i = \int_\Omega \sum_{i,k} (\nabla^y p(y))_k v_i \sum_j (\operatorname{Cof} \nabla y)_{ij} \partial_j y_k \\
&= \int_\Omega \sum_{i,k} (\nabla^y p(y))_k v_i \Big((\operatorname{Cof} \nabla y)(\nabla y)^T\Big)_{ik} \overset{(2)}{=} \int_\Omega (\nabla^y p(y))_k v_i (\det \nabla y) \delta_{ik} \\
&= \int_\Omega \sum_i (\det \nabla y)(\nabla^y p(y))_i v_i = \int_\Omega (\det \nabla y) \nabla^y p(y) \cdot v dx.
\end{aligned}
$$

In the step (2), we use the identity $\operatorname{Cof} F F^T = F^T \operatorname{Cof} F = (\det F)\mathbb{I}$.

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