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„Renormalons and R-evolution for Hadronic τ-Decays“

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Abstract

In this thesis we investigate a particular source of divergence of perturbative expansions in QCD which leads to divergent series that can at best be considered asymptotic. The asymptotic behaviour these divergences are causing is related to contributions from small and large loop momenta in perturbative calculations and manifests itself in poles of the corresponding Borel transform which are termed infrared (IR) and ultraviolet (UV) renormalons, depending on the momentum regions they are related to. Studies of renormalon divergences have become increasingly important in high-energy physics to achieve an ever higher precision in theoretical predictions. In this context we review the QCD description of hadronic τ decays and summarize what is known about ultraviolet and infrared renormalons. In particular we analyze the close connection between IR renormalons and non-perturbative power corrections in QCD and discuss how IR renormalons lead to ambiguities in the definition of the Borel integral. Furthermore we present another approach to deal with renormalon divergences and show how the introduction of an additional scale R can be used to improve the poor convergence behaviour of perturbative series suffering from renormalon ambiguities. From the solution of renormalization group equations with respect to the scale R, the so-called R-evolution equations, we deduce an analytic all-order expression for the Borel transform of perturbative series that can be used as a test for renormalon ambiguities. As a practical application we study the Borel transform of the Adler function in the context of the large-β_0 approximation.

Zusammenfassung

In dieser Arbeit wird eine bestimmte Art von Divergenzen betrachtet, welche in perturbativen Entwicklungen in QCD auftreten und zu divergenten Reihen führen, die bestenfalls asymptotisch sind. Das asymptotische Verhalten dieser Reihen wird sowohl von Bereichen mit niedrigen als auch von Bereichen mit hohen Impulsen in Schleifenintegralen verursacht und spiegelt sich in Form von Singularitäten in der entsprechenden Boreltransformierten wider, welche als Renormalons bezeichnet werden. In Abhängigkeit von ihrem Ursprung von Bereichen mit niedrigen oder hohen Impulsen, wird dabei zwischen infraroten (IR) bzw. ultravioletten (UV) Renormalons unterschieden. Genaue Untersuchungen dieser Renormalon Divergenzen werden zunehmend bedeutender, um theoretische Vorhersagen in der Hochenergie-Physik zu verbessern. In diesem Zusammenhang betrachten wir zunächst hadronische τ Zerfälle und geben einen allgemeinen Überblick über ultraviolette (UV) sowie infrarote (IR) Renormalons. Im Speziellen diskutieren wir die enge Verknüpfung zwischen IR Renormalons und nicht-perturbativen Powerkorrekturen in QCD und zeigen, dass IR Renormalons zu einer Ambiguität in der Definition des Borel-Integrals führen. Des Weiteren präsentieren wir einen alternativen Ansatz, um Renormalon Divergenzen zu behandeln und zeigen wie mit Hilfe einer zusätzlichen Skala R das schlechte Konvergenzverhalten von Störungsreihen verbessert werden kann. Die Lösung von Renormierungsgruppen-Gleichungen bezüglich dieser neuen Skala R, welche R-evolution Gleichungen genannt werden, liefert einen analytischen Ausdruck für die Boreltransformierte von Störungsreihen, die zum Testen von Renormalon Ambiguitäten verwendet werden kann. Als Anwendungsbeispiel betrachten wir die Boreltransformierte der Adlerfunktion in der Large-β_0 Näherung.

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Chapter 1

Introduction

It is a well-known fact that perturbative expansions in quantum field theory, and in particular in Quantum Chromodynamics (QCD), lead to divergent series for which one expects a factorial growth of the perturbative coefficients at high orders. An important source of divergence, which is directly related to small and large momentum behaviour in loop integrals, is referred to as *renormalons* and was investigated for the first time in the 1970s [1, 2, 3]. Ever since, studies of renormalons and large-order behaviour in perturbation series have become indispensable in high-energy physics, especially because theoretical predictions of fundamental parameters require an ever increasing precision in order to test the internal consistency of the Standard Model and find physics that goes beyond it.

In this context one of the central quantities of QCD is the strong coupling constant α_s and over the last decades major efforts, ranging from analyses of event shapes in e^+e^- annihilation (see e.g. [4, 5, 6, 7]) to lattice QCD (e.g. [8, 9, 10, 11]), have been made to improve its determination. The probably most important low-energy extraction of the strong coupling is provided by investigations of hadronic τ decays. Due to the mass $m_\tau = 1.78$ GeV of the τ lepton the strong coupling $\alpha_\mathrm{s}(M_\tau)$ at the τ mass scale is on the one side small enough such that perturbative expansions still converge, but on the other side large enough that the τ hadronic width R_τ is sensitive to it [12]. Besides, non-perturbative contributions to R_τ turn out to be rather small which makes an accurate theoretical prediction in the context of perturbation theory possible. At present the largest source of theoretical uncertainty in the determination of α_s from τ decays is related to the apparent discrepancy between different approaches used to improve the perturbative expansion of the τ decay rate by means of the renormalisation group. The two most commonly employed techniques, known as *fixed-order* (FOPT) and *contour-improved perturbation theory* (CIPT), lead to significant numerical differences and many studies have been motivated by the necessity to resolve this issue [13, 14, 15]. Detailed investigations [16] indicate that the preference for either FOPT or CIPT depends primarily on the assumptions made on the models used to predict the large-order behaviour of the perturbative series of R_τ.

Associated with these higher-order models are renormalons, which dominate the large-order behaviour and therefore play a crucial role. In order to deal with factorially divergent series related to renormalons it proves to be very useful to perform a *Borel transformation*. The Borel transform then encodes the information on the divergent behaviour in form of singularities that lie on the real axis in the complex Borel plane. It is actually these poles in the Borel transform along with the associated large order behaviour of the perturbative series which are called renormalons. Depending on their physical origin it is possible to separate renormalons into two different classes. Those related to the short-distance behaviour of QCD appear on the negative real axis and are termed ultraviolet (UV) renormalons, while the ones associated with long-distance physics are located on the positive real axis and are referred to as infrared (IR) renormalons.

IR renormalons represent a fundamental problem in the computation of the inverse Borel transform (see eq. (3.5)), which will be discussed in detail in chapter 3. This inverse transformation involves an integral over the positive real Borel axis (=Borel integral) and due to the IR renormalon poles, one is forced to regulate the integral in some way. The resulting ambiguities in the definition of the Borel integral are related to non-perturbative power corrections (cf. eq. (3.22)). To remove these ambiguities higher-dimensional operator corrections (QCD condensates) which arise in the framework of the operator product expansion (OPE) need to be considered.

The OPE is an important instrument for QCD predictions that separates short- and long distance physics into perturbatively computable Wilson coefficients and non-perturbative matrix elements of operators. As explained in section 3.3.1, the operator expansion is usually realised in the common $\overline{\text{MS}}$-scheme which is useful for multiloop calculations and preserves important properties, such as gauge and Lorentz invariance. In this scheme the Wilson coefficients do not suffer from explicit IR divergences but contain contributions from arbitrary small loop momenta and are therefore still IR sensitive. This sensitivity is reflected by IR renormalon divergences which lead to factorial growth of the perturbative Wilson coefficients (cf. (4.2)). These IR renormalons may cause poor convergence and are compensated by corresponding instabilities (UV renormalons) in the matrix elements of higher dimensional operators in the OPE. To avoid the problem of renormalon cancellations between different terms in the $\overline{\text{MS}}$-OPE, it is useful to switch to a scheme that involves explicit renormalon subtractions but retains the powerful computational properties of $\overline{\text{MS}}$ [17]. Associated with such a scheme is typically an additional cut-off scale R below which the renormalon contribution is subtracted. Treating the scale R as continuous variable leads to the definition of a new renormalisation group equation, the so-called *R-evolution* equation, that resums, at the same time, the asymptotic renormalon series and large logarithms in the difference of two subtractions at scales R_0 and R_1 in a renormalon free way. Moreover, we will see that the general solution of this equation yields an analytic expression for the Borel transform of the perturbative series that carries the information on a specific renormalon pole. As a byproduct we also obtain an analytic result for the normalization of the singular terms in the Borel transform. This expression for the normalization, called the *renormalon sum rule*, can in principle be applied to any perturbative series as a probe for renormalon ambiguities.

In order to demonstrate the usefulness of R-evolution and the renormalon sum rule we apply it to the Adler function which represents the central quantity in the perturbative description of hadronic τ decays. In addition, we prove that the solution of the R-evolution equation recovers an anlaytic expression for the Borel transform that matches exactly the ansatz for a physical model of the Adler function derived in [14]. The model for the Borel transform given there is based on common renormalisation group methods and allows one to predict the position and strength of the renormalon pole, but not its residue. As we will see, the renormalon sum rule resolves this issue and can, in principle, be used to gain additional information about the residues of renormalon poles.

This work is organized as follows: In chapter 2 we give an overview of hadronic τ decays. Chapter 3 provides an in-depth introduction to renormalons and Borel summation. In this section we discuss the simple example of the Adler function and investigate how renormalons emerge in loop calculations. Moreover we establish the connection between IR renormalons, ambiguities of the Borel integral and higher-dimensional operator corrections in the OPE. Chapter 4 is dedicated to R-evolution and the renormalon sum rule. We first discuss the implications of the R-evolution equation and then derive the analytic expressions for the Borel transform and the renormalon sum rule for a given perturbative series. The application of R-evolution to the Adler function in the context of the large-β_0 approximation is then presented in chapter 5 and, finally, chapter 6 summarises our findings.

Chapter 2

Hadronic τ Decays

Due to the fact that the τ is the only lepton heavy enough to also decay into hadrons, hadronic τ decays provide an ideal tool for precision tests of QCD and its fundamental quantities at low energies. One of the most important parameters in this respect is the strong coupling constant α_{s} which can be determined very accurately from investigations of the τ hadronic width

$$R_\tau = \frac{\Gamma\left(\tau^- \to \text{hadrons}\ \nu_\tau\,(\gamma)\right)}{\Gamma\left(\tau^- \to e^-\ \overline{\nu}_e\ \nu_\tau\,(\gamma)\right)}. \tag{2.1}$$

Theoretically, it is possible to divide R_τ into contributions from non-strange ($\overline{u}d$) and strange ($\overline{u}s$) quark currents, which can be further resolved into vector (V) and axial-vector (A) contributions. Since experimentally a separation of the strange decays into vector and axial-vector contributions is not feasible due to the lack of data for the Cabibbo-suppressed sector, it is convenient to decompose R_τ in the following form [12]

$$R_\tau = R_{\tau,V} + R_{\tau,A} + R_{\tau,S}, \tag{2.2}$$

where $R_{\tau,S}$ denotes the strange hadronic width. In general, the contribution from non-strange decays is given by [12, 14, 18]

$$R_{\tau,V/A} = \frac{N_C}{2} S_{\mathrm{EW}} |V_{ud}|^2 \left[1 + \delta_{\mathrm{EW}} + \delta^{(0)} + \sum_{D\geq 2} \delta^{(D)}_{ud,V/A}\right]. \tag{2.3}$$

In this decomposition S_{EW} and δ_{EW} comprise electroweak corrections, $\delta^{(0)}$ contains the perturbative QCD corrections to R_τ neglecting quark masses and $\delta^{(D)}_{ud,V/A}$ includes leading quark mass effects as well as corrections from higher dimensional terms in the operator product expansion (OPE) for current-current correlation functions. In this work we will mainly focus on the latter two corrections, $\delta^{(0)}$ and $\delta^{(D)}_{ud,V/A}$, electroweak corrections will not be covered. For a complete and detailed discussion of all contributions to the τ hadronic width, see [12].

2.1 Theoretical Framework for the Prediction of R_τ

Starting point for the analysis of the τ hadronic width is the two-point correlation function of vector and axial-vector currents, $J^{V/A}_{\mu,ij}(x) = [\overline{q}_j \gamma_\mu (\gamma_5) q_i](x)$, of massless quarks [14]:

$$\Pi^{V/A}_{\mu\nu,ij}(q) = i \int \mathrm{d}x\, \mathrm{e}^{iqx} \left\langle \Omega \right| T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^\dagger\} \left|\Omega\right\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\, \Pi^{V/A}_{ij}(q^2). \tag{2.4}$$

Here, $|\Omega\rangle$ denotes the full physical vacuum and the indices i, j characterize the relevant quark flavours up, down and strange. The hadronic width R_τ can be written as an integral of the spectral functions $\rho(s) \propto \operatorname{Im}\Pi(s)$ over the invariant mass-squared $s = q^2$ of the final state hadrons

[12, 13],

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)\mathrm{Im}\,\Pi_\tau(s), \tag{2.5}$$

where the correlator $\Pi_\tau(s)$ is given by

$$\Pi_\tau(s) = |V_{ud}|^2\left[\Pi_{ud}^V(s)+\Pi_{ud}^A(s)\right] + |V_{us}|^2\left[\Pi_{us}^V(s)+\Pi_{us}^A(s)\right]. \tag{2.6}$$

Exploiting the fact that $\Pi_\tau(s)$ is analytic in the entire complex s-plane except for the positive real axis, eq. (2.5) can be transformed by means of Cauchy's theorem into a contour integral running counter-clockwise around the circle $|s| = m_\tau^2$ [14]:

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{\mathrm{d}s}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)\Pi_\tau(s). \tag{2.7}$$

For a systematic calculation of the hadronic τ decay rate, it is furthermore convenient to introduce the so-called Adler function [19]

$$D(s) = 4\pi^2 s\,\frac{\mathrm{d}\Pi(s)}{\mathrm{d}s}, \tag{2.8}$$

which, in contrast to the correlators $\Pi(s)$, is a physical quantity as it is RG invariant [14]. In terms of the Adler function R_τ is given by [13]

$$R_\tau = -\frac{3\,i}{2\pi}\oint_{|x|=1}\frac{\mathrm{d}x}{x}\,(1-x)^3(1+x)\,D(m_\tau^2 x), \tag{2.9}$$

with the dimensionless integration variable $x = s/m_\tau^2$. Eq. (2.9) is especially useful for the evaluation of the τ hadronic width because at the energy scale $s = m_\tau^2$ non-perturbative effects in QCD are expected to be small (i.e. $\Lambda_{\mathrm{QCD}} \ll m_\tau$) and hence a series expansion in the framework of the operator product expansion can be applied. The OPE allows to perform a systematic separation of perturbative and non-perturbative contributions to $D(s)$ into a series in inverse powers of s. As will be explained in more detail later in this work, this concept is closely connected to renormalon divergences in perturbation series in QCD.

Due to the factor $(1-x)^3$ in the integrand of eq. (2.9), uncertainties in the OPE of the Adler function associated with contributions from the region near the positive real axis are strongly suppressed and the expansion can be regarded to be well-behaved along the complex contour.

The dominant contribution to the τ hadronic width is related to the purely perturbative QCD corrections $\delta^{(0)}$ in the chiral limit. In this limit vector and axial-vector correlation functions coincide and we therefore only need to consider the vector correlator $\Pi^V(s)$ in the massless case. Starting from the general structure [13],

$$\Pi^V(s) = \frac{N_c}{3}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=0}^{n+1} c_{n,k}\,\ln^{k-1}\left(\frac{-s}{\mu^2}\right), \tag{2.10}$$

the Adler function can be expressed in the form [13]

$$D(s) = \frac{N_c}{3}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\,c_{n,k}\,\ln^{k-1}\left(\frac{-s}{\mu^2}\right), \tag{2.11}$$

where we define $a_\mu \equiv \alpha_s(\mu)/\pi$ with μ being the renormalisation scale. The logarithms emerging in the above expression can be resummed with the choice $\mu^2 = -s = Q^2$, yielding[1] [13]:

$$D(Q^2) = \frac{N_c}{3} \sum_{n=0}^{\infty} c_{n,1}\, a_Q^n. \tag{2.12}$$

Thus, we see that only the coefficients $c_{n,1}$ are independent and all other coefficients $c_{n,k}$ with $k \geq 2$ can be expressed in terms of QCD β-function coefficients and the $c_{n,1}$ by means of renormalisation group arguments. (For further details see [14].)

Since 2008 analytic results for the coefficients of the Adler function up to $\mathcal{O}(\alpha_s^4)$ have been available [20]. In the $\overline{\text{MS}}$-scheme for $N_c = 3$ they read (see e.g. [20, 21, 22, 23]):

$$\begin{aligned}
c_{0,1} &= c_{1,1} = 1, \\
c_{2,1} &= \frac{365}{24} - 11\zeta_3 - \left(\frac{11}{12} - \frac{2}{3}\zeta_3\right) N_f = 1.64, \\
c_{3,1} &= \frac{87029}{288} - \frac{1103}{4}\zeta_3 + \frac{275}{6}\zeta_5 - \left(\frac{7847}{216} - \frac{262}{9}\zeta_3 + \frac{25}{9}\zeta_5\right) N_f + \left(\frac{151}{162} - \frac{19}{27}\zeta_3\right) N_f^2 = 6.37, \\
c_{4,1} &= \frac{144939499}{20736} - \frac{5693495}{864}\zeta_3 + \frac{5445}{8}\zeta_3^2 + \frac{65945}{288}\zeta_5 - \frac{7315}{48}\zeta_7 \\
&\quad + \left(-\frac{13044007}{10368} + \frac{12205}{12}\zeta_3 - 55\zeta_3^2 + \frac{29675}{432}\zeta_5 + \frac{665}{72}\zeta_7\right) N_f \\
&\quad + \left(\frac{1045381}{15552} - \frac{40655}{864}\zeta_3 + \frac{5}{6}\zeta_3^2 - \frac{260}{27}\zeta_5\right) N_f^2 \\
&\quad + \left(-\frac{6131}{5832} + \frac{203}{324}\zeta_3 + \frac{5}{18}\zeta_5\right) N_f^3 = 49.08.
\end{aligned} \tag{2.13}$$

The numerical results in (2.13) are given for $N_f = 3$. There also exist estimates for the next six-loop coefficient $c_{5,1}$ that employ methods such as *fastest apparent convergence* (FAC) or *principle of minimal sensitivity* (PMS) [20, 24]. For $N_f = 3$ the estimate used in [14] is given by:

$$c_{5,1} = 283 \pm 142. \tag{2.14}$$

2.1.1 Fixed-Order vs. Contour-Improved Perturbation Theory

One of the most important theoretical uncertainties in the prediction of the hadronic τ decay rate is related to different possibilities of performing renormalisaton group improvements of the perturbative correction $\delta^{(0)}$. In this section we compare the two most widely used techniques, namely fixed-order (FOPT) and contour-improved perturbation theory (CIPT), which in comparison to scale variations of the individual series seemingly lead to significantly different results.

Starting from eq. (2.9) and inserting the series expansion for $D(s)$ given in eq. (2.11), the perturbative QCD corrections can be expressed in the following form [14]

$$\delta^{(0)} = \sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k} \frac{1}{2\pi i} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (1-x)^3 (1+x) \ln^{k-1}\left(\frac{-M_\tau^2 x}{\mu^2}\right), \tag{2.15}$$

where the additional contribution coming from the axial-vector correlator has already been included and $N_c = 3$ was used. In fixed-order perturbation theory the perturbative series in eq. (2.15) is resummed with the choice $\mu^2 = m_\tau^2$, yielding [13]

$$\delta^{(0)}_{\text{FO}} = \sum_{n=0}^{\infty} a_{M_\tau}^n \sum_{k=1}^{n+1} k\, c_{n,k}\, J_{k-1}. \tag{2.16}$$

[1] Note that the series expansion in powers of α_s/π will only be used in this introductory chapter on hadronic τ decays in order to be consistent with the notation used in the literature. In chapter 5 the Adler function series will be expanded in powers of $\alpha_s/(4\pi)$. The corresponding coefficients a_n (see eq. (5.1)) are related to the coefficients $c_{n,1}$ in (2.12) via $c_{n,1} = a_n/4^n$.

Comparing this expansion with eq. (2.15) it is an easy task to derive the form of the contour integrals J_k:

$$J_k = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} \, (1-x)^3 \, (1+x) \ln^k(-x). \tag{2.17}$$

Details on how to solve these integrals analytically are given in [14].

In contrast to FOPT, contour-improved perturbation theory sums the logarithms appearing in eq. (2.15) by setting $\mu^2 = -M_\tau^2 x$, which leads to [14, 13]

$$\delta^{(0)}_{\mathrm{CI}} = \sum_{n=0}^{\infty} c_{n,1} \, J_n^a(M_\tau^2), \tag{2.18}$$

with $J_n^a(m_\tau^2)$ defined by

$$J_n^a(m_\tau^2) = \frac{1}{2\pi i} \oint\limits_{|x|=1} \frac{\mathrm{d}x}{x} \, (1-x)^3 \, (1+x) \left(\frac{\alpha_{\mathrm{s}}(-m_\tau^2 x)}{\pi} \right)^n. \tag{2.19}$$

Thus, we see that in CIPT the running coupling effects are resummed along the integration contour leading to an expansion for $\delta^{(0)}_{\mathrm{CI}}$ in which each perturbative order n only depends on a single coefficient $c_{n,1}$, while FOPT performs a systematic expansion in powers of the strong coupling at the τ mass scale.

With these characterizations in mind we can now compare the numerical results for $\delta^{(0)}$ obtained from employing the two approaches. Using the known corrections up to $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ of eq. (2.13) for $N_f = 3$ and $\alpha_{\mathrm{s}}(m_\tau) = 0.3186$ (corresponding to $\alpha_{\mathrm{s}}(M_Z) = 0.1184$), we find [15]

$$\begin{aligned} \delta^{(0)}_{\mathrm{FO}} &= 0.1959 \pm 0.0063, \\ \delta^{(0)}_{\mathrm{CI}} &= 0.1814 \pm 0.0033, \end{aligned} \tag{2.20}$$

where the given uncertainty is obtained from comparing the result for $\delta^{(0)}$ using the known coefficients with an approximation including the estimate $c_{5,1} = 283$ of the $\mathcal{O}(\alpha_{\mathrm{s}}^5)$ contribution. As can be seen, the CIPT result is considerably below the FOPT sum, which requires the strong coupling α_{s} to be larger in order to reproduce the experimental data for the τ hadronic width[2]. This behaviour is also reflected in Fig. 2.1, where the well-known PDG plot for the extraction of α_{s} from hadronic τ decays is shown. The thin dashed line and the yellow band represent the pre-average value of $\alpha_{\mathrm{s}}(M_Z^2)$ from studies which employ both, FOPT and CIPT, whereas the thick dashed line and grey-shaded band indicate the world-average value which includes determinations of the strong coupling from other sub-fields, such as lattice QCD and deep-inelastic lepton-nucleon scattering (DIS). Comparing the individual results of the different groups one can see a dominance of CIPT approaches because most values for α_{s} are on the high side of the world average.

Investigating the single contributions to $\delta^{(0)}$ at each perturbative order up to $\mathcal{O}(\alpha_{\mathrm{s}}^5)$ (see [14]), the CIPT series shows a faster convergence than FOPT and, therefore, CIPT is typically considered to be the method of choice. This argument, however, has to be taken with a grain of salt, since studies of the contour integrals (2.19) (see Fig. 1 in [14]) show that the numerical values for $J_n^a(m_\tau^2)$ are positive up to the sixth order and become progressively smaller before they turn negative around $n \sim 7$. Consequently, the CIPT series is always found to approach its minimal term around the seventh order, before the asymptotic behaviour[3] sets in (see Figs. 2.2 and 2.3 in the next section). Thus, the faster convergence to its minimal term does not necessarily indicate

[2]Note that this argument only holds, if the higher dimensional operator terms in (2.3) are the same for both approaches.

[3]The concept of asymptotic series will be explained in chapter 3.

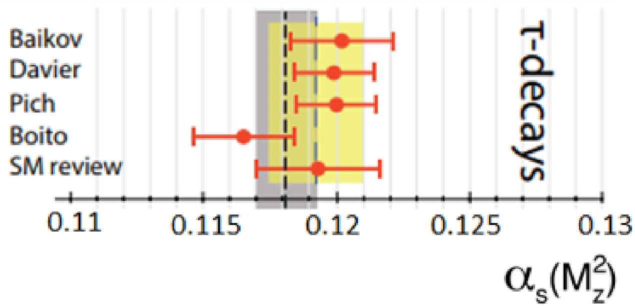

Fig. 2.1: Extraction of $\alpha_{\mathrm{s}}(M_Z^2)$ from hadronic τ-decays. The thin dashed line and the yellow band represent the pre-average value of $\alpha_{\mathrm{s}}(M_Z^2)$ from studies which employ both, FOPT and CIPT. The thick dashed line and grey-shaded band indicate the world-average value which includes determinations of the strong coupling from other sub-fields, such as lattice QCD. (Tanabashi et al. [26], p. 154)

that CIPT provides the better approximation to the 'true' result for the resummed perturbative series.

Up to now, the discrepancy between FOPT and CIPT (if treated as a theoretical uncertainty, see [25]) has developed into the largest numerical uncertainty of the α_{s} determination from hadronic τ decays and much effort has been expended to resolve this issue. An important question in this respect concerns the behaviour of the series $\delta^{(0)}_{\mathrm{FO}}$ and $\delta^{(0)}_{\mathrm{CI}}$ when higher-order perturbative coefficients are added. Intensive investigations [16] suggest that the preference for either FOPT or CIPT depends predominantly on the assumptions made for the higher-order contributions in the perturbative series.

2.2 Higher-Order Models

In order to gain some insight into the features favouring FOPT or CIPT we briefly present two different models for the higher-order terms in the perturbative expansion of $\delta^{(0)}$. The first one, the so-called large-β_0 approximation, can be considered as a toy model for the entire Adler function series and gives a first glance at the problems related to renormalons which dictate the large-order behaviour in perturbative series. This approximation will be analysed in much more detail later on.

The second model we discuss here is based on a physically motivated ansatz and represents the central model that will be important throughout this work. The study of R-evolution and the renormalon sum rule in the subsequent sections is to a great extent intended to gain more information on the higher-order contributions to the Adler function and, thus, to $\delta^{(0)}$ based on this physical model.

The following discussion of higher-order models (and especially of the physical model) is based on the work by Beneke and Jamin from 2008 [14].

2.2.1 The Large-β_0 Approximation

We first have a look at the implications of the large-β_0 approximation[4] on the perturbative expansion of $\delta^{(0)}$. Since the concepts introduced in the following will be explained at length in the introductory section on renormalons and Borel summation, we want to keep the discussion of this model brief and just outline the main features of the higher-order terms in this approximation.

The basic problem one has to face when dealing with higher-order models is related to the fact

[4]The origin of the term "large-β_0" approximation will be discussed in section 3.2.1. For now, it suffices to know that in the large-β_0 approximation only the contributions with the highest power of N_f in the coefficients $c_{n,1}$ are considered and N_f is replaced by $-3/2\,\beta_0$.

that perturbative series in QCD are typically divergent and can at best be considered asymptotic. In order to quantify the large-order behaviour of factorially divergent series, which we most often encounter, it is useful to perform a so-called Borel transformation. In contrast to other models, in the large-β_0 approximation a closed analytic result for the Borel transform of the Adler function and R_τ is available and the coefficients $c_{n,1}$ can in principle be deduced to all orders in perturbation theory (see e.g. [16, 27, 28, 29]).

In connection with Borel transformation in the large-β_0 approximation, one often defines the reduced Adler function $\widehat{D}$ by [14]

$$\frac{3}{N_c}D(s) = 1 + \widehat{D}(s) = 1 + \sum_{n=0}^{\infty} r_n\, \alpha_{\rm s}(\sqrt{s})^{n+1}, \tag{2.21}$$

where the relation between the coefficients of $D(s)$ and $\widehat{D}(s)$ is given by $c_{n,1} = \pi^n r_{n-1}$. The Borel transform of the reduced Adler function is defined as:

$$B[\widehat{D}](u) = \sum_{n=0}^{\infty} r_n \frac{u^n}{n!}. \tag{2.22}$$

To recover an expression for the perturbative series $\widehat{D}(s)$, we can perform the Borel integral (inverse Borel transformation)

$$\widehat{D}(\alpha_{\rm s}) = \frac{4\pi}{\beta_0}\int_0^\infty {\rm d}u\, {\rm e}^{\frac{-4\pi u}{\beta_0\alpha_{\rm s}}}\, B[\widehat{D}](u), \tag{2.23}$$

where β_0 denotes the first term in the QCD β-function (see appendix C.1). If this integral exists, i.e. if the Borel transform $B[\widehat{D}](u)$ has no singularities for $u > 0$, $\widehat{D}(s)$ is called Borel-summable and $\widehat{D}(\alpha_{\rm s})$ is considered to be the 'true' result for the perturbative series of the reduced Adler function $\widehat{D}(s)$ [27].

Computing a certain set of diagrams, the so-called *bubble-chain diagrams* (see Fig. 3.1), the Borel transform of the Adler function in the large-β_0 approximation takes the form [28]:

$$B[\widehat{D}](u) = \frac{32}{3\pi}\,\frac{{\rm e}^{-Cu}}{(2-u)}\sum_{k=2}^{\infty}\frac{(-1)^k\, k}{[k^2-(1-u)^2]^2}. \tag{2.24}$$

Here, C denotes a scheme-dependent constant which has the value $C = -5/3$ in the $\overline{\rm MS}$-scheme. Exploiting eq. (2.24), the perturbative coefficients r_n and $c_{n,1}$, respectively, can be recovered by simply Taylor expanding $B[\widehat{D}](u)$ in the variable u and afterwards evaluating the Borel integral (2.23) term by term.

When we look at the expression for the Borel transform given in eq. (2.24), we see that $B[\widehat{D}](u)$ has singularities at positive and negative integer values of u (except for $u = 0, 1$) that encode the information on the divergent behaviour of $\widehat{D}(s)$. These singularities can be viewed as manifestation of the asymptotic behaviour of the perturbative series and are referred to as infrared (IR) and ultraviolet (UV) renormalons, since they are related to small and large momentum regions in loop integrals. More specifically, IR renormalons are located at the positive real axis in the complex Borel plane and are closely connected to higher dimensional terms in the operator product expansion, whereas the UV renormalons appear on the negative real Borel axis with the leading UV renormalon at $u = -1$ dominating the characteristic sign-alternating large-order behaviour of the perturbative series.

We now recall the definition of the Borel integral given in eq. (2.23). Due to the IR renormalon poles at $u > 0$, the integral cannot be performed and we are forced to regulate the integral, e.g. by deforming the integration contour into the complex plane which introduces an arbitrary ambiguity in the definition of $\widehat{D}(\alpha)$. From the point of view that the regulation procedure of the Borel integral is not unique, the question concerning the different possibilities to quantify this ambiguity

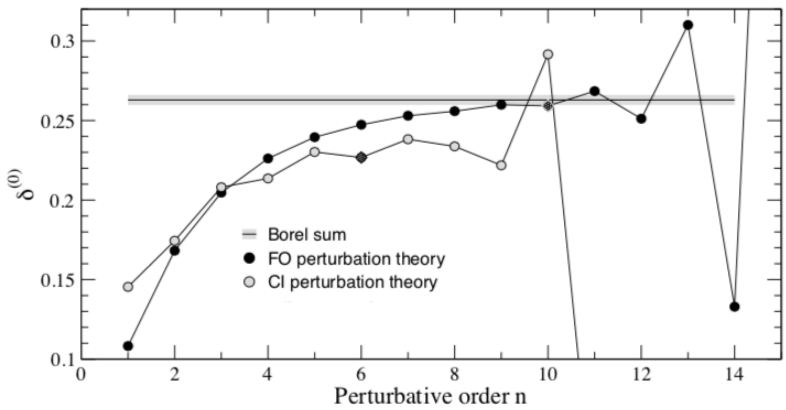

Fig. 2.2: Graphical representation of the results for $\delta^{(0)}$ at $\alpha_s(m_\tau) = 0.34$ in the large-β_0 approximation. The plot shows the partial sums for $\delta^{(0)}_{\rm FO}$ (black circles) and $\delta^{(0)}_{\rm FO}$ (grey circles) as a function of the order n. The horizontal line denotes the principal value of the Borel integral and the shaded region provides an estimate of the ambiguity. (M. Beneke and M. Jamin [14], p. 13)

arises. One frequently used method is the principle value prescription of the Borel integral and then simply takes the imaginary part of the result as an estimate for the ambiguity. In this sense the 'true' result for the perturbative series of the Adler function is expected to be consistent with the principal value of the Borel integral within an error band defined by the ambiguity of the integral. However, from the point of view of an arbitrary deformation of the integration contour, it is not so clear how well the Borel integral in connection with a principle value prescription actually quantifies the ambiguities related to IR renormalon singularities and the question remains whether there are alternative ways to approximate these ambiguities. This issue is one the questions that we want to address in this work and represents in part our motivation to apply the concept of R-evolution to hadronic τ-decays.

Returning to our discussion on the discrepancy between FOPT and CIPT, we can now calculate $\delta^{(0)}$ in the large-β_0 approximation employing the perturbative coefficients deduced from eq. (2.24). The results for $\delta^{(0)}_{\rm FO}$ and $\delta^{(0)}_{\rm CI}$ are depicted in Fig. 2.2 which shows the partial sums of the perturbative series[5] as a function of the order n. The FOPT sum $\delta^{(0)}_{\rm FO}$ is given by the full black circles, while the grey circles represent $\delta^{(0)}_{\rm CI}$. The solid straight line marks the principle value of the Borel integral obtained by using the well-known theorem (C.40),

$$\frac{1}{a-u\pm i\epsilon} = \mathrm{PV}\left(\frac{1}{a-u}\right) \mp i\pi\,\delta(a-u), \tag{2.25}$$

and the shaded band provides an estimate of the ambiguity given by the imaginary part of eq. (2.25)[6]. Analytic results for the calculation of the Borel integral can be found in appendix C.5.

If we start from the assumption that the 'true' result of the perturbative series is approximated reasonably well by the Borel sum, Fig. 2.2 shows that in the large-β_0 approximation the FOPT series smoothly approaches the value of the Borel integral before, eventually, the leading UV renormalon at $u = -1$ takes over and the sign-alternating divergent behaviour of the series sets in. CIPT, on the contrary, suffers from a much earlier onset of the divergent behaviour and therefore does not represent a good approximation. A more detailed investigation [14] reveals large cancellations between the Adler function series and additional contributions coming from the

[5] Note that for the calculation in the large-β_0 approximation the running of α_s at the one-loop order has been used in eqs. (2.16) and (2.18).

[6] The authors in [14] argue that the size of the ambiguity divided by π is of the order of the higher-dimensional corrections in the OPE. Thus, the shaded band in Fig. 2.2 is given by the imaginary part of eq. (2.25) divided by π.

integration along the contour in eq. (2.15). However, in [14] it is argued that these cancellations are missed by CIPT causing the poor convergence behaviour in the large-β_0 approximation.

2.2.2 Physical Model for the Adler Function

The second model we want to present is a physically motivated ansatz for the Borel transform of the Adler function series proposed by Beneke and Jamin in [14]. This model includes the exactly known coefficients up to $\mathcal{O}(\alpha_s^4)$ and comprises general features of renormalon singularities learned from the large-β_0 approximation and other toy models. The ansatz relies on the fact that the renormalon pole structure in the Borel transform needs to be modified when the running of α_s is implemented at higher-loop orders. In the case of the large-β_0 approximation, where a one-loop running coupling is used, the Borel transform (2.24) only consists of simple and double poles, but as soon as higher terms of the QCD β-function are incorporated in the running effects, the pole structure must be adapted in order to include the possibility of branch cuts.

To derive the appropriate structure for the IR renormalon singularities, one can exploit the close connection between IR renormalons and power-suppressed terms in the OPE which will be discussed extensively in the next sections. For now, the crucial point is that from the comparison of the scaling behaviour of a specific term in the OPE with the energy dependence of the ambiguity of the Borel integral, one can predict the structure of the renormalon singularity causing this ambiguity.

Starting point for the derivation of the IR renormalon pole structure, is a generic expression of a term in the OPE of the Adler function [14]

$$C_{O_d}(a_Q)\frac{\langle O_d\rangle}{Q^d} = [a_Q]^{\frac{2\gamma^{(1)}_{O_d}}{\beta_0}}\left[C^{(0)}_{O_d} + C^{(1)}_{O_d}a_Q + C^{(2)}_{O_d}a_Q^2 + \ldots\right]\frac{\langle O_d\rangle}{Q^d}, \tag{2.26}$$

where O_d denotes an operator of dimension d and $a_Q \equiv \alpha_s(Q)/\pi$. Skipping the details given in [14], the renormalon pole whose ambiguity matches the energy dependence of the term in eq. (2.26) is found to be[7] ($p > 0$):

$$B[D_p^{\rm IR}](u) = \frac{d_p^{\rm IR}}{(p-u)^{1+\tilde{\gamma}}}\left[1 + \tilde{b}_1(p-u) + \tilde{b}_2(p-u)^2 + \ldots\right]. \tag{2.27}$$

Here, $d_p^{\rm IR}$ is the residue of the pole and the following definitions have been used [14]

$$\begin{aligned} p &= \frac{d}{2}, & \tilde{\gamma} &= p\frac{\beta_1}{\beta_0^2} - \frac{2\gamma^{(1)}_{O_d}}{\beta_0}, \\ \tilde{b}_1 &= \frac{4(b_1+c_1)}{\beta_0\tilde{\gamma}}, & \tilde{b}_2 &= \frac{16(b_2 + b_1c_1 + c_2)}{\beta_0^2\,\tilde{\gamma}\,(\tilde{\gamma}-1)}, \end{aligned} \tag{2.28}$$

with:

$$\begin{aligned} b_1 &= \frac{d}{8\beta_0^3}(\beta_1^2 - \beta_0\,\beta_2), & b_2 &= \frac{b_1^2}{2} - \frac{d}{64\beta_0^4}(\beta_1^3 - 2\,\beta_0\,\beta_1\,\beta_2 + \beta_0^2\,\beta_3), \\ c_1 &= \frac{C^{(1)}_{O_d}}{C^{(0)}_{O_d}}, & c_2 &= \frac{C^{(2)}_{O_d}}{C^{(0)}_{O_d}}. \end{aligned} \tag{2.29}$$

$\gamma^{(1)}_{O_d}$ denotes the leading order anomalous dimension of the operator O_d defined by:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}O_d(\mu) = -\gamma_{O_d}(a_\mu)\,O_d(\mu) = -\left[\gamma^{(1)}_{O_d}\,a_\mu + \gamma^{(2)}_{O_d}\,a_\mu^2 + \gamma^{(3)}_{O_d}\,a_\mu^3\right]O_d(\mu). \tag{2.30}$$

[7] The structure of the renormalon pole given by eqs. (2.27) - (2.30) will be compared to our R-evolution approach in section 4.3. There we show that using R-evolution one can reproduce eq. (2.27) and furthermore determine the residue $d_p^{\rm IR}$ by means of the so-called renormalon sum rule (see eq. (4.47)).

From the point of view of eq. (2.23), UV renormalons are Borel summable, since they lie on the negative real axis in the complex plane and do not obstruct the naive calculation of the Borel integral. Nevertheless, this does not at all mean that the existence of UV renormalons is irrelevant on physical grounds. They indicate profound physics and are connected to the addition of higher dimensional operators in the context of the Standard Model Effective Field Theory (SMEFT). UV renormalons will be discussed in more detail in section 3.4.

In order to deduce an expression for UV renormalon singularities similar to the one in eq. (2.27), it is useful to change the sign of the coupling a_Q and consider a Borel-like integral ranging from zero to minus infinity [30]. In that way, the UV poles on the negative real axis lead to ambiguities which again can be linked to higher-dimensional operators. As a result, one obtains an expression for the Borel transform of UV renormalon singularities that resembles eq. (2.27)

$$B[D_{\tilde{p}}^{\mathrm{UV}}](u) = \frac{d_{\tilde{p}}^{\mathrm{UV}}}{(\tilde{p}+u)^{1+\bar{\gamma}}}\Big[1 + \bar{b}_1(\tilde{p}+u) + \bar{b}_2(\tilde{p}+u)^2 + \dots\Big], \tag{2.31}$$

where $\tilde{p} > 0$ and the appearing parameters are given by:

$$\bar{\gamma} = -\tilde{p}\frac{\beta_1}{\beta_0^2} + \frac{2\gamma_{\tilde{O}_d}^{(1)}}{\beta_0}, \qquad \bar{b}_1 = -\tilde{b}_1, \qquad \bar{b}_2 = \tilde{b}_2 \tag{2.32}$$

Here, $\gamma_{\tilde{O}_d}^{(1)}$ denotes the leading order anomalous dimension of the higher dimensional SMEFT operator $\tilde{O}_d$ corresponding to the UV renormalon at $u = -\tilde{p}$.

With these expressions for the renormalon singularities at hand, it is now possible to construct a physically motivated model for the Adler function that reproduces the analytically known coefficients and allows us to compare the higher-order contributions to the perturbative series in FOPT and CIPT based on the full QCD OPE.

From investigations of the large-β_0 approximation and the fact that in the $\overline{\mathrm{MS}}$-scheme the exact coefficients (2.13) of the Adler function display a fixed-sign behaviour we conclude that the low and intermediate orders of the perturbative series are governed by IR renormalons, while the UV renormalons dictate the sign-alternating behaviour at large orders. Thus, it was argued in [14] that it is reasonable to incorporate at least the two leading IR renormalons at $u = 2, 3$ as well as the leading UV singularity at $u = -1$ in the model to reproduce the exactly known low-order coefficients and to sufficiently characterise the large-order behaviour.

Taking these considerations into account, the authors in [14] proposed the following ansatz for a physical model of the Adler function:

$$B[\widehat{D}](u) = B[\widehat{D}_1^{\mathrm{UV}}](u) + B[\widehat{D}_2^{\mathrm{IR}}](u) + B[\widehat{D}_3^{\mathrm{IR}}](u) + d_0 + d_1\, u, \tag{2.33}$$

where the first three terms for the renormalon poles are obtained from eqs. (2.27) and (2.31). This ansatz depends on five independent parameters, the residues for the renormalons, d_1^{UV}, d_2^{IR} and d_3^{IR}, plus two additional polynomial coefficients, d_0 and d_1. The parameters are fixed in such a way that the perturbative expansion of the model matches the exactly known coefficients $c_{n,1}$, including the estimate for the six-loop coefficient $c_{5,1}$ in (2.14). The error of this estimate has been neglected in the matching procedure. The five parameters of the model (2.33) are found to be [14]:

- $d_1^{\mathrm{UV}} = -1.56 \cdot 10^{-2}$, $d_2^{\mathrm{IR}} = 3.16$, $d_3^{\mathrm{IR}} = -13.5$, (2.34)
- $d_0 = 0.781$, $d_1 = 7.66 \cdot 10^{-3}$.

In order to investigate the higher-order contributions to the perturbative series in FOPT and CIPT, we can now use the ansatz (2.33) to determine the coefficients $c_{n,1}$ needed for the computation of $\delta_{\mathrm{FO}}^{(0)}$ and $\delta_{\mathrm{CI}}^{(0)}$, respectively. Numerical values for the first few coefficients of the Adler function derived from the physical model can be found in [14].

Fig. 2.3 shows a graphical representation of the results for $\delta_{\mathrm{FO}}^{(0)}$ and $\delta_{\mathrm{CI}}^{(0)}$. Like in the plot for the large-β_0 approximation (see Fig. 2.2), the full black circles represent the partial sums of the

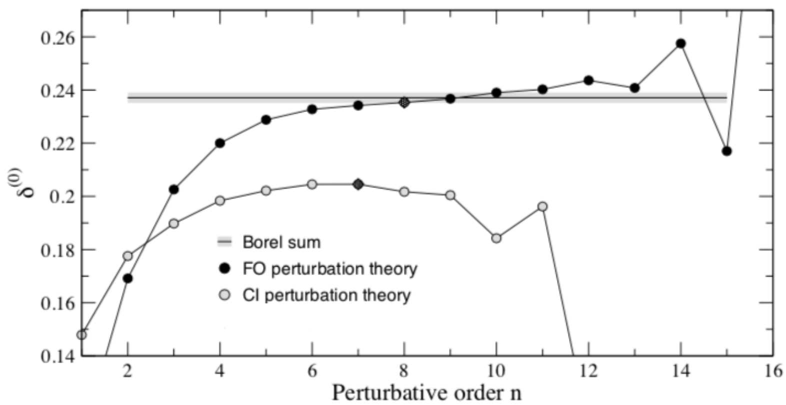

Fig. 2.3: Results for $\delta^{(0)}$ at $\alpha_s(m_\tau) = 0.34$ employing the physical model (2.33). The plot shows the partial sums for $\delta^{(0)}_{\text{FO}}$ (black circles) and $\delta^{(0)}_{\text{FO}}$ (grey circles) as a function of the order n. The horizontal line denotes the principal value of the Borel integral and the shaded region provides an estimate of the ambiguity. (M. Beneke and M. Jamin [14], p. 23)

FOPT series as a function of the perturbative order n and the grey circles give the corresponding result for the CIPT series. The horizontal line denotes again the principal value of the Borel integral and the shaded band provides an estimate for the complex ambiguity. As can be seen, the qualitative behaviour of the FOPT and CIPT series resembles basically the outcome of our analysis in the large-β_0 approximation. FOPT smoothly approaches the Borel sum with a later onset of the sign-alternating divergence as compared to the large-β_0 case, while the CIPT series always stays below the FOPT result. It was argued in [14] that using CIPT to extract the strong coupling from comparison to the experimental data for R_τ would result in a too large value for α_s and we thus conclude that FOPT prevails in the physical model.

The model described in (2.33) is based on the central idea that the structure of a given renormalon pole can be related to associated higher-dimensional terms in the OPE by means of the renormalisation group. In this way, the position and strength (i.e. the structure of the exponent) of renormalon poles can be determined, whereas it is not possible to predict the residues. Improving the form of the Borel transform used in physical models such as (2.33) is of utmost importance to resolve the discrepancy between FOPT and CIPT. In this context the open question is whether or not it is possible to gather more information on single renormalon poles. In particular, we want to know if one can gain knowledge about the residues of the singularities without having to rely on calculating bubble-chain diagrams in the large-β_0 approximation. Solving this issue represents the main motivation to deal with R-evolution and the renormalon sum rule in this work.

Before we start with a more detailed introduction to renormalons, let us briefly review the questions we want to address in the subsequent sections:

1. What can we tell about the ambiguities related to renormalon singularities?
   - How well does the Borel integral quantify these ambiguities?
   - Are there alternative ways to estimate the size of the ambiguities?
2. Is it possible to improve the form of the Borel transform used in physical models for the Adler function?
   - Can we gain more information on single renormalon poles?
   - Is it possible to predict the residues, if one does not rely on the large-β_0 approximation?

Chapter 3

Renormalons and Borel Summation

This chapter is intended to give a detailed introduction to the notion of renormalons which provide the basis for our discussion of R-evolution. Readers already familiar with the basic concepts related to renormalon divergences, especially with the connection between (IR) renormalons and non-perturbative power corrections in the OPE, may regard the following discussion as a brief review and skim through this section quickly.
The analysis given below draws primarily upon the comprehensive review in [27].

3.1 Basic Concepts and Terminology

As long as the fundamental interactions in quantum field theories (QFT) are weak, a generic observable F can usually be expressed in terms of a series expansion in the interaction coupling α:

$$F(\alpha) = \sum_{n=0}^{\infty} c_n \, \alpha^{n+1}. \tag{3.1}$$

However, it is well-known that such series are divergent and the coefficients c_n are typically expected to grow factorially [27]

$$c_n \overset{n\to\infty}{\sim} a^n n! \, n^b Z, \tag{3.2}$$

for some constants a, b, Z. A particular source causing this factorial growth in perturbative series is generally referred to as *renormalon* divergence [1, 2, 3].

Due to the divergent behaviour of the perturbative expansion it is not at all obvious how the series on the right-hand side of eq. (3.1) is related to the exact ('true') value of F on the left-hand side. From a mathematical point of view a divergent series can only be considered as a meaningful approximation to F, if it is asymptotic in a region $\mathcal{G}$ of the (complex) α-plane in the sense that [27]

$$\left| F(\alpha) - \sum_{n=0}^{N} c_n \, \alpha^{n+1} \right| < K_{N+1} \, \alpha^{N+2}, \tag{3.3}$$

for all $\alpha \in \mathcal{G}$ and some numbers K_{N+1}. For factorially divergent series these numbers most often also grow with the order N ($K_N \propto N!$) and the best approximation to the exact value of F is typically given when the perturbative series is truncated at its minimal term. Note, however, that perturbative expansions in QFT can only be assumed to be at best asymptotic. Rigorous proofs require a profound non-perturbative definition of observables which, in theories such as QCD, do generally not exist.

3.1.1 Borel Summation

In order to study the large-order behaviour of asymptotic series, we need to find a suitable way to deal with the divergences emerging in the perturbative expansion. As already mentioned in the discussion of the large-β_0 approximation in section 2.2.1, factorially divergent series are conveniently summed employing the Borel summation technique. Starting from a series expansion of the form of eq. (3.1), we first consider the *Borel transform* defined as

$$B[F](t) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\, t^n. \tag{3.4}$$

In this way, we introduce an equivalent series in the Borel variable t which is much better behaved than the original series due to the suppression factor $1/n!$. If $B[F](t)$ has no singularities for $t > 0$ and does not increase too rapidly for $t \to \infty$, we can recover a definition for the original divergent series $F(\alpha)$ by performing the *Borel integral* (inverse Borel transformation) [27]:

$$\tilde{F}(\alpha) = \int\limits_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-\frac{t}{\alpha}}\, B[F](t). \tag{3.5}$$

If this integral exists, $F(\alpha)$ is called Borel-summable and $\tilde{F}(\alpha)$ provides an expression for the sum of the perturbative series.

To investigate the concept of Borel summation in more detail, let us consider an asymptotic series similar to (3.2) with coefficients given by [27]

$$c_n = K\, a^n\, \Gamma(1 + n + b), \tag{3.6}$$

where K, a and b denote some arbitrary (real) constants. Assuming that $b > 0$, the Borel transform of this series takes the form [27]

$$B[F](t) = K\, \frac{\Gamma[1 + b]}{(1 - a\, t)^{1+b}}, \tag{3.7}$$

and one can observe that the information on the divergent behaviour originally contained in the Γ-function of eq. (3.6) is now encoded in the singularity at $t = 1/a$ of the Borel transform. As we are going to see in the next section, it is exactly these poles in the Borel transform of perturbative series that are referred to as renormalon divergences.

Besides this crucial aspect relating renormalons to singularities in the Borel plane, the illustrative example (3.6) also reveals some other general features of the Borel summation method. First of all, from the expression of the Borel transform in eq. (3.7) one notices that larger values of a, i.e. faster diverging series, lead to poles closer to the origin of the Borel plane. These singularities and the associated renormalons are thus regarded as the most severe ones[1].

Furthermore, the sign of a has a large impact on the definition of the Borel integral (3.5). While sign-alternating series ($a < 0$) yield singularities on the negative real axis in the Borel plane and are therefore Borel-summable, fixed-sign series ($a > 0$) produce poles at positive t and obstruct the naive calculation of the Borel integral. In this case the integral needs to be regulated and can for example still be performed by a deformation of the integration contour either above or below the pole. The regulation procedure introduces an ambiguity in the definition of the inverse Borel transform that is tightly connected to non-perturbative power corrections (cf. eq. (3.22)). The concept of the ambiguity of the Borel integral will be discussed in much more detail in connection with the operator product expansion (see section 3.3).

3.2 Bubble Chain Diagrams as a Probe for Renormalons

After the formal introduction to asymptotic series and Borel summation in the previous section, we now want to investigate how renormalon divergences emerge in loop calculations and how they

[1] At low orders the perturbative coefficients c_n in (3.6) also depend on the size of the constant K. Hence, the perturbative series can also be dominated at low orders by renormalon singularities farther away from the origin of the Borel plane provided that K is sufficiently large.

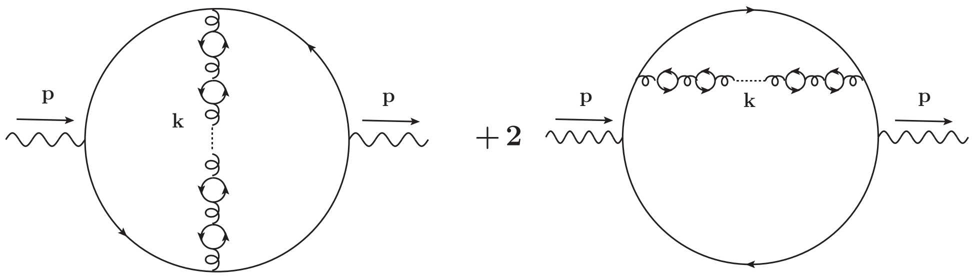

Fig. 3.1: Bubble chain diagrams for the calculation of the Adler function in the large-N_f limit. The factor 2 in front of the second diagram takes into account an identical contribution from the diagram in which the modified gluon propagator (= bubble chain) is attached to the lower antiquark line of the 'large' quark loop.

are related to poles in the Borel transform of perturbative series. For this purpose consider once again the Adler function $D(p^2)$ already discussed in chapter 2,

$$\begin{aligned} D(p^2) &= 4\pi^2 \frac{\mathrm{d}\Pi(p^2)}{\mathrm{d}p^2} = 1 + \widehat{D}(p^2) = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_\mathrm{s}}{\pi}\right)^n, \\ i\int \mathrm{d}x\, \mathrm{e}^{ipx} \langle\Omega|\, T\{J^{V/A}_{\mu,ij}(x) J^{V/A}_{\nu,ij}(0)^\dagger\}\, |\Omega\rangle &= (p_\mu p_\nu - p^2 g_{\mu\nu})\, \Pi^{V/A}_{ij}(p^2), \end{aligned} \tag{3.8}$$

where $J_\mu^{V/A}(x) = [\overline{q}\gamma_\mu(\gamma_5)q](x)$ denotes a vector/axialvector current of massless quarks. In order to study the large-order behaviour of the Adler function we cannot compute the entire perturbative series, but have to restrict ourselves to a certain subset of diagrams that can be calculated to all orders in perturbation theory. A very convenient choice in this respect is the set of so-called *bubble chain* diagrams of massless quarks depicted in Fig. 3.1. In the context of QCD these graphs represent the dominant contributions in the $1/N_f$-expansion in the limit $N_f \to \infty$, where N_f denotes the number of light quark flavours. To achieve an expansion in this limit, we define $a = \alpha_\mathrm{s} N_f/\pi$, and expand in the parameter $1/N_f$ with a held fixed. Even though this set of quark bubble graphs is not the only set of diagrams contributing to renormalon divergences in perturbative series, we still apply these diagrams as a reference probe for renormalons[2], since they are useful for explicit calculations.

When we look at the form of the bubble chain diagrams shown in Fig. 3.1, we notice that all information on the order in α_s is comprised in the number of quark bubble insertions into the gluon line. Thus, the summation of these diagrams reduces effectively to the evaluation of a simple two-loop diagram with a modified gluon propagator which contains the sum over any number of fermion bubble insertions. This modified gluon propagator is generally referred to as *bubble chain*. The only ingredient needed for the calculation of the bubble chain is the well-known result for the renormalized quark loop (see appendix A)

$$\hat{\pi}(p^2) = \frac{\alpha_\mathrm{s}\, N_f}{6\pi}\, \ln\left(\frac{\mu^2}{-p^2}\, \mathrm{e}^{-C}\right), \tag{3.9}$$

where C denotes a scheme-dependent constant. In $\overline{\mathrm{MS}}$, the renormalization scheme that will be used throughout this section, $C = -5/3$. Using this, the bubble chain in Landau gauge[3] is given in the handy form

$$\begin{aligned} D^{ab}_{\mu\nu}(k) &= \frac{-i}{k^2 + i0^+}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1+\hat{\pi}(k^2)}\,\delta^{ab} \\ &= \frac{-i}{k^2 + i0^+}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\sum_{n=0}^{\infty}\left[-\frac{\alpha_\mathrm{s}\, N_f}{6\pi}\,\ln\left(\frac{\mu^2}{-p^2}\,\mathrm{e}^{-C}\right)\right]^n, \end{aligned} \tag{3.10}$$

[2]Note that renormalons were originally discovered in bubble chain diagrams such as Fig. 3.1 [1, 2, 3].
[3]For the calculation of the diagrams in Fig. 3.1 Landau gauge proves to be very useful (see appendix A).

where the expression in the second line can be used to calculate the diagrams in Fig. 3.1. In the following, however, we mainly focus on the derivation of the leading renormalon divergences of the Adler function in the large-N_f approximation and therefore only sketch the bubble chain computation here. An in-depth calculation along with an derivation of the Borel transformed Adler function in this approximation can be found in Appendix A.

3.2.1 Calculating Bubble Chain Diagrams for the Adler Function

Starting from the diagrams in Fig. 3.1 and exploiting eq. (3.10) for the modified gluon propagator, we first integrate over the momentum of the 'large' quark loop that encloses the bubble chain. Introducing the dimensionless integration variable $\hat{k}^2 = -k^2/Q^2$ with $Q^2 = -p^2$, the Adler function (in the $\overline{\text{MS}}$-scheme) is given by [27]

$$D = \sum_{n=0}^{\infty} \alpha_{\rm s}(\mu) \int\limits_0^{\infty} \frac{\mathrm{d}\hat{k}^2}{\hat{k}^2}\, \omega(\hat{k}^2) \left[\frac{\alpha_{\rm s}(\mu)\, N_f}{6\pi} \, \ln\!\left(\hat{k}^2 \frac{Q^2}{\mu^2}\, \mathrm{e}^{-5/3} \right) \right]^n . \tag{3.11}$$

Here, the factor in the bracket arises from multiple insertions of renormalized fermion loops into the bubble chain and the additional factor $\alpha_{\rm s} = g^2/(4\pi)$ in front of the integral comes from attaching the end points of the gluon propagator to the quark lines of the large fermion loop. The weight function $\omega(\hat{k}^2)$ describes the gluon momentum distribution and is found to be [31]:

$$\omega(\hat{k}^2) = \frac{2\, C_F}{\pi^2} \sum_{s=2}^{\infty} (-1)^s \, \frac{\mathrm{d}}{\mathrm{d}s}\, \frac{1}{s^2-1} \int\limits_{-\infty}^{\infty} \mathrm{d}r\, \mathrm{e}^{ir \ln \hat{k}^2} \left(\frac{1+ir}{r^2+s^2} - \frac{1}{1-ir} \right). \tag{3.12}$$

Setting $\mu^2 = Q^2$, the integral receives large logarithmic enhancements from small ($\hat{k}^2 \ll 1$) and large ($\hat{k}^2 \gg 1$) momentum regions and we therefore only need to know the small and large momentum behaviour of $\omega(\hat{k}^2)$ [27]:

$$\begin{aligned} \omega(\hat{k}^2) &= \frac{3\, C_F}{2\pi}\, \hat{k}^4 + \mathcal{O}\big(\hat{k}^6 \ln \hat{k}^2\big) && (\hat{k}^2 \ll 1) \\ \omega(\hat{k}^2) &= \frac{C_F}{3\pi}\, \frac{1}{\hat{k}^2} \left(\ln \hat{k}^2 + \frac{5}{6} \right) + \mathcal{O}\!\left(\frac{\ln \hat{k}^2}{\hat{k}^4} \right) && (\hat{k}^2 \gg 1). \end{aligned} \tag{3.13}$$

Proceeding with the large-order evaluation of eq. (3.11), we now split the integral at $\hat{k}^2 = \mathrm{e}^{5/3}$ and perform the integration for small and large momenta employing the expressions in eq. (3.13). We finally obtain

$$D \sim \sum_{n=0}^{\infty} \alpha_{\rm s}^{n+1}(Q) \left[\mathrm{e}^{10/3} \left(-\frac{N_f}{12\pi} \right)^n n! + \mathrm{e}^{-5/3} \left(\frac{N_f}{6\pi} \right)^n n! \left(n + \frac{11}{6} \right) + \ldots \right], \tag{3.14}$$

where the first term comes from small momentum regions and the second one is due to the integration over large loop momenta. As one can see, the coefficients of the perturbative series exhibit the expected factorial growth and the corresponding divergences causing this behaviour are known as *infrared* (IR) and *ultraviolet* (UV) *renormalons* depending on the momentum region they are related to. To be more precise, the result in eq. (3.14) represents the leading IR and UV renormalons, while the sub-dominant divergent behaviour caused by other renormalons is not shown and only indicated by ellipses.

Despite its apparent usefulness for explicit calculations, the question remains whether the set of bubble graphs provides a suitable approximation to renormalon divergences encountered in perturbative expansions. In non-Abelian gauge theories, such as QCD, the N_f-terms coming from fermion bubble diagrams only give a small contribution to the complete perturbative coefficients and other contributions associated with non-Abelian diagrams, like gluon and ghost bubbles, are obviously missed. The absence of gluon self-couplings in the large N_f-limit has fundamental consequences, since one loses the QCD property of asymptotic freedom [32]. In order to restore this property at least approximately and estimate the non-Abelian contribution to renormalon divergences, it was suggested (see e.g. [16, 31]) to apply the replacement rule

$$N_f \to -\frac{3}{2}\left(\frac{11}{3}C_A - \frac{2}{3}N_f\right) = -\frac{3}{2}\beta_0 \tag{3.15}$$

to bubble chain calculations. This substitution, which is referred to as *Naive Non-Abelianization* (NNA) [33], turns out to give a major contribution to perturbative coefficients in almost all cases where a comparison with exact second order results is available and is thus expected to provide a reasonable large-order approximation for qualitative studies of perturbative series [16]. This observation along with the fact that fermion bubble graphs are much easier to evaluate than non-Abelian diagrams explains why there only exist very few analyses that go beyond the computation of bubble chains in this so-called *large-β_0 approximation*.

Replacing N_f with the full QCD β_0-function (see appendix C.1), the Adler function (3.14) in the large-β_0 approximation is given by:

$$D \sim \sum_{n=0}^{\infty}\left(\frac{\alpha_s(Q)\,\beta_0}{4\pi}\right)^{n+1}\left[\mathrm{e}^{10/3}\left(\frac{1}{2}\right)^n n! + \mathrm{e}^{-5/3}\,(-1)^n\, n!\left(n+\frac{11}{6}\right) + \ldots\right], \tag{3.16}$$

where the first contribution is again related to small loop momenta, whereas the second term arises from large momentum regions. From the form of the Adler function in the large-β_0 approximation we are now in the position to establish the connection between renormalon divergences and poles in the Borel transform. Using eq. (3.4) and the terms shown in (3.16) the Borel transform of the Adler function can be cast into the form,

$$B[D](u) \sim \mathrm{e}^{10/3}\,\frac{2}{2-u} + \mathrm{e}^{-5/3}\left(\frac{1}{(1+u)^2} + \frac{5}{6}\,\frac{1}{(1+u)}\right) + \ldots, \tag{3.17}$$

where the modified Borel variable $u \equiv t\beta_0/(4\pi)$ was used. The first term in the above expression exhibits a singularity at $u = 2$ and is associated with the leading IR renormalon divergence in (3.16). The singularity at $u = -1$ in the second term, on the other hand, is related to the leading UV renormalon divergence and consists of a simple as well as a double pole. More specifically, the simple pole is connected to the $n!$ contribution in the second term of eq. (3.16), while the double pole leads to the $(n+1)!$ growth of the coefficients and thus dominates. As can be seen, renormalon divergences in perturbative series related to small and large loop momentum behaviour are indeed connected to singularities in the corresponding Borel transform and in the following we will refer to renormalons as poles in the Borel plane.

The form of the Adler function in eq. (3.16) also corroborates our previous findings that poles closer to the origin of the Borel plane correspond to faster diverging series. The leading IR renormalon divergence is suppressed by a factor $(1/2)^n$ as compared to the leading UV renormalon at $u = -1$ which therefore dominates the large-order behaviour of the Adler function. Furthermore it now becomes obvious that UV renormalon poles located on the negative real Borel axis lead to sign-alternating factorial divergences and IR renormalons at $u > 0$ are related to series displaying a fixed-sign factorial growth.

Relying on the calculation of bubble chain diagrams, an analytic expression for the complete Borel transform of the reduced Adler function $\widehat{D}$ can be found [28] (see Appendix A for more details):

$$B[\widehat{D}] = \frac{32}{3\pi}\,\frac{\mathrm{e}^{-Cu}}{(2-u)}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{[k^2-(1-u)^2]^2}, \tag{3.18}$$

where C denotes the same scheme-dependent constant which also arises in eq. (3.9). Besides the leading UV and IR renormalons, the Borel transform also contains subleading (simple and double) renormalon poles at positive and negative integer values of u, except for $u = 0, 1$. The absence of the $u = 1$ renormalon, as we are going to see in the next section, is related to the fact that there is no dimension-2 operator in the OPE of the Adler function. The singularity at $u = 0$ is only present in the Borel transform of the correlation function $\Pi(Q^2)$ (see [34]) and vanishes in the case of the Adler function by taking the derivative with respect to Q^2 in (3.8).

The NNA prescription (3.15) reveals another important aspect of renormalon divergences in the context of the large-β_0 approximation. When we go back to the expression of the renormalized bubble chain given in eq. (3.10), the substitution of the Abelian N_f-terms with the full β_0-function yields:

$$\begin{aligned} D^{ab}_{\mu\nu}(k) &= \frac{-i}{k^2 + i0^+}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1+\hat{\pi}(k^2)}\,\delta^{ab} \\ &= \frac{-i}{k^2 + i0^+}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\sum_{n=0}^{\infty}\left(\frac{\alpha_s\,\beta_0}{4\pi}\right)^n\left[\ln\left(\frac{\mu^2}{-p^2}\,\mathrm{e}^{-C}\right)\right]^n, \end{aligned} \tag{3.19}$$

Using this expression in the calculation of the diagrams in Fig. 3.1, the Adler function in the $\overline{\mathrm{MS}}$-scheme is given by (compare with (3.11))

$$\begin{aligned} D &= \int_0^\infty \frac{\mathrm{d}\hat{k}^2}{\hat{k}^2}\,\omega(\hat{k}^2)\,\frac{\alpha_s(k)}{1+\hat{\pi}(\hat{k}^2)} = \int_0^\infty \frac{\mathrm{d}\hat{k}^2}{\hat{k}^2}\,\omega(\hat{k}^2)\;\alpha_s\big(k\,\mathrm{e}^{-5/6}\big) \\ &= \sum_{n=0}^{\infty}\alpha_s(\mu)\int_0^\infty \frac{\mathrm{d}\hat{k}^2}{\hat{k}^2}\,\omega(\hat{k}^2)\left[-\frac{\alpha_s(\mu)\,\beta_0}{4\pi}\;\ln\left(\hat{k}^2\frac{Q^2}{\mu^2}\,\mathrm{e}^{-5/3}\right)\right]^n, \end{aligned} \tag{3.20}$$

where the coupling $\alpha_s\big(k\,\mathrm{e}^{-5/6}\big)$ in the second term of the first line denotes the well-known one-loop QCD running coupling defined by:

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\frac{\alpha_s(\mu_0)}{4\pi}\,\beta_0\ln\left(\frac{\mu^2}{\mu_0^2}\right)}. \tag{3.21}$$

Hence, renormalons are closely connected to the concept of the running coupling and the set of bubble chain diagrams is actually equivalent to the evaluation of corresponding two-loop diagrams which use the one-loop running coupling at the vertices without any fermion bubble insertions into the gluon propagator. From a physical point of view it is exactly these running coupling effects in QCD which turn the small momentum behaviour in perturbative loop integrals into factorial divergences. Moreover it is now clearly evident that the large-β_0 limit only provides a useful approximation to the large-order behaviour of asymptotic series, if the contributions coming from the one-loop running dominate at high orders.

In this context the question arises how the structure of renormalon divergences and especially the positions of the poles on the real axis in the Borel plane change, when diagrams beyond the large-β_0 approximation are considered. The Borel transform (3.18) of the Adler function in the large-β_0 approximation consists of an infinite sequence of simple and double poles, but in the general case the renormalon pole structure will be much more complicated, involving cuts and higher terms of the QCD β-function (see e.g. the physical model for the Adler function presented in Chapter 2). Nevertheless, one can prove (see [27]) that the exact location of the renormalon poles is already given by the large-β_0 approximation.

Another crucial point we need to mention here is that even though the bubble chain diagrams in the large-β_0 approximation reproduce the correct positions for the renormalon poles, they do not yield all singularities that should be present in the Borel transform. An important example in this respect are instantons which are exponentially small effects in the $1/N_f$-expansion that lead to singularities far away from the origin in the Borel plane and thus can usually be neglected when it comes to the discussion of renormalons. A detailed review on instanton singularities can be found e.g. in [35].

3.3 Ambiguity of the Borel Integral

We have already alluded to the issue that the renormalon poles in the Borel transform have a large impact on the definition of the Borel integral (3.5). Since UV renormalons are located on the negative real axis in the Borel plane, they lie outside the integration range and the Borel integral is well-defined. Even though UV renormalons are Borel-summable in this sense, they are not at all irrelevant from a physical point of view. UV renormalons will be discussed in more detail in section 3.4.

IR renormalons, on the other hand, obstruct the integration and we are forced to regulate the integral in order to avoid the singularities. This introduces an ambiguity in the definition of the Borel integral which e.g. can be quantified by deforming the integration contour either above or below the pole. Thus, we can estimate the ambiguity caused by a generic IR renormalon pole at $u = p/2$ $(p > 0)$ in the Borel transform of the Adler function by a contour integration around the singularity:

$$\Delta D(Q^2) \sim \oint\limits_{u=p/2} \mathrm{d}u\, \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s(Q)}} \frac{1}{p/2 - u} \propto \left(\mathrm{e}^{-\frac{2\pi}{\beta_0 \alpha_s(Q)}} \right)^p \sim \left(\frac{\Lambda_{\mathrm{QCD}}}{Q} \right)^p. \tag{3.22}$$

Here, we used the definition of the intrinsic QCD scale, Λ_{QCD}, at the leading-logarithmic (LL) order in the last equality (see appendix C.1). Since physical observables, such as the Adler function, should not be affected by renormalon ambiguities, the result of eq. (3.22) indicates that additional non-perturbative power corrections[4] need to be considered in an adequate, i.e. unambiguous, definition of observables. This is a very crucial point we want to stress here: If QCD or any other theory is supposed to be of physical relevance, perturbation theory must be incomplete and the existence of IR (but also of UV) renormalons is just a reminder of this fact.

3.3.1 Infrared Renormalons and Operator Product Expansion

An important instrument to systematically incorporate non-perturbative effects in QCD calculations is the *operator product expansion* (OPE). The concept of the OPE was developed by Wilson [36] who proposed to replace a product of local operators evaluated at different space-time points that are close to each other by a linear combination of composite local operators:

$$\lim_{x \to y} A(x)\, B(y) = \sum_i C_i(x - y)\, O_i(x). \tag{3.23}$$

The sum on the right-hand side comprises all operators O_i which carry the same quantum numbers as the composite operator on the left-hand side and the complex-valued coefficient functions C_i are known as *Wilson coefficients*. Moreover one can associate mass dimensions d_i with the operators O_i and the sum can generally be organized in terms of a series of increasing dimensions.

Strict proofs of the OPE are only valid in perturbation theory [37, 38, 39] and there are many studies that investigate to which extent the operator expansion still holds when non-perturbative effects are included [40, 41, 42]. In particular, it is shown in [40] that there exists a critical dimension at which non-perturbative effects lead to a breakdown of the OPE. Nevertheless, the operator product expansion is an important tool for QCD predictions which allows us to separate short- and long-distance physics into perturbatively computable Wilson coefficients and matrix elements of operators that demand non-perturbative treatment.

Before we deal with the OPE of a generic observable, it is instructive to first consider the corresponding expansion for the Adler function in order to study the deep connection between IR renormalon ambiguities and non-perturbative effects. To this end recall that according to its definition

[4] From the result of (3.22) we conclude that the power corrections $(\Lambda_{\mathrm{QCD}}/Q)^p$ are given by exponential terms in the strong coupling α_s. In this sense they are also purely non-perturbative, since these terms vanish trivially in a perturbative expansion around $\alpha = 0$. Furthermore note that even though power corrections are sometimes referred to as *exponentially small*, they do not need to be small numerically.

in eq. (3.8) the Adler function is basically given by the time-ordered product $\langle 0| T\{J_\mu(x)J_\nu(0)^\dagger\}|0\rangle$ of two quark currents J_μ. In the limit in which the currents are evaluated at the same point x, we can apply the operator product expansion (3.23) and express the product of currents by means of a sum over local operators. Since we are interested in the vacuum expectation value of the Adler function, the possible set of operators appearing in the OPE can be restricted to the class of gauge-invariant Lorentz scalars. It turns out that it is not possible to construct any operators of dimension $d = 1, 2$ which meet these requirements and thus the first contributions to the OPE come from operators with dimension 3 and 4:

$$O_3 = \overline{q}q, \qquad O_4 = G^a_{\mu\nu}G^{a,\mu\nu}. \tag{3.24}$$

Here, $G^a_{\mu\nu}$ denotes the field strength tensor of the gluon field A^a_μ and the corresponding vacuum expectation values of these operators, $\langle\Omega|\,\overline{q}q\,|\Omega\rangle$ and $\langle\Omega|\,G^a_{\mu\nu}G^{a,\mu\nu}\,|\Omega\rangle$, are called *quark* and *gluon condensate*, respectively. Since the quark condensate violates the chiral symmetry, its associated Wilson coefficient comes with an additional factor of the quark mass m and the OPE of the Adler function receives non-perturbative contributions from operators starting effectively at dimension $d = 4$. Neglecting higher dimensional terms, the expansion of the Adler function reads

$$D(Q^2) = C_0(Q^2,\mu)\mathbb{1} + C_{GG}(Q^2,\mu)\frac{\langle (G^A_{\mu\nu})^2\rangle(\mu)}{Q^4} + C_{\overline{q}q}(Q^2,\mu)\, m\frac{\langle\overline{q}q\rangle(\mu)}{Q^4} + \mathcal{O}\left(\frac{\Lambda^6_{\mathrm{QCD}}}{Q^6}\right), \tag{3.25}$$

where we introduced the notation $\langle\Omega|\,O\,|\Omega\rangle \equiv \langle O\rangle$. Note that the first term in the expansion is related to the unit operator, whose corresponding Wilson coefficient C_0 comprises the purely perturbative contributions to the Adler function. In the large-β_0 approximation C_0 can be computed from the bubble chain diagrams in Fig. 3.1. In the massless case the quark condensate vanishes and we conclude that the dominant non-perturbative effects in the definition of the Adler function arise from the gluon condensate.

The contribution of the gluon condensate can be linked to the leading IR renormalon at $u = 2$ [27]. Starting from the observation that IR renormalons are related to small loop-momentum regions, the low-energy physics is described by a local operator, since the large external scale Q can be contracted to a single point with respect to the small momentum fluctuations. In addition, the Adler function is a Lorentz scalar and since no external hadrons exist, we need to compute vacuum expectation values. We also know that the dominant sequence of IR renormalons is due to a single soft gluon line in Fig. 3.1 which constrains the local operator to consist of low energy gauge fields A^a_μ attached to one local space-time point. Taking into account that gauge invariance rules out terms of the form $A^a_\mu A^{a,\mu}$, the lowest dimensional operator consistent with these constraints (in the limit of vanishing quark masses) is found to be the dimension-4 gluon condensate $\langle\Omega|\,G^a_{\mu\nu}G^{a,\mu\nu}\,|\Omega\rangle \sim \Lambda^4_{\mathrm{QCD}}$. This completes the connection between IR renormalons and higher dimensional terms in the operator product expansion we wished to establish. As is evident from eq. (3.22), the gluon condensate is precisely of the order of the ambiguity related to the leading $u = 2$ IR renormalon,

$$\Delta D(Q^2) \sim \left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)^4, \tag{3.26}$$

such that the Adler function is defined unambiguously in the sum of (3.25) up to power suppressed contributions of order $\mathcal{O}(\Lambda^6_{\mathrm{QCD}}/Q^6)$. In a similar fashion the dimension-6-operators in the OPE of the Adler function compensate the ambiguities caused by the IR renormalon at $u = 3$ and in the most general case an IR renormalon singularity located at $u = p/2 > 0$ in the Borel plane is related to non-perturbative matrix elements $\langle\Omega|\,O_p\,|\Omega\rangle \sim \Lambda^p_{\mathrm{QCD}}$ of p-dimensional operators[5]. Furthermore, it now becomes obvious how the missing of an $u = 1$ renormalon in the Borel transform of the Adler function is associated with the absence of a dimension-2-operator in the OPE, and later on we will also see that the simple pole structure of the $u = 2$ renormalon results from the

[5]To our knowledge the connection between IR renormalons and higher dimensional terms in the operator product expansion was noted for the first time by Parisi [43].

vanishing anomalous dimension of the gluon condensate in the large-β_0 approximation.

Before generalizing our previous findings to a generic observable a few comments need to be made. From the estimation in (3.22) we have seen that renormalon ambiguities are related to two different scales which typically involve a large, perturbative scale Q (e.g. a momentum transfer or mass) and $\Lambda_{\rm QCD}$, the scale at which perturbation theory breaks down. As indicated in eq. (3.25), the Wilsonian OPE introduces a factorization scale μ separating these two scales in such a way that $\Lambda_{\rm QCD} < \mu < Q$ and performs a consistent expansion in powers of $\Lambda_{\rm QCD}/Q$ for contributions from loop momenta $k < \mu$. The coefficient functions are perturbatively computable and only contain contributions $k \sim Q > \mu$, while the small momentum region $k \sim \Lambda_{\rm QCD} < \mu$ is inherent to the non-perturbative matrix elements. Therefore, the Wilson coefficient C_0, comprising the purely perturbative result for the Adler function, does not contain any IR renormalons in this picture and could in principle be defined unambiguously since UV renormalons are Borel summable anyway. However, since implementing a suitable real momentum cut-off μ and preserving important features, such as gauge symmetry and Lorentz invariance, at the same time is hard to achieve in calculations beyond the one-loop order, one usually uses dimensional regularization in connection with the $\overline{\rm MS}$-scheme in perturbative computations. In dimensional regularization the renormalization scale is introduced to realise the factorization in the OPE. In contrast to the Wilsonian picture, loop integrations are still performed over all momenta k, resulting in Wilson coefficients that are still IR sensitive and thus incorporate contributions from arbitrary small momenta. This in turn leads to IR renormalon divergences which manifest themselves in the factorial growth of the perturbative coefficients in C_0. In the $\overline{\rm MS}$-OPE the convergence behaviour of the Adler function gets improved by the addition of non-perturbative effects related to matrix elements of higher-dimensional operators. The cancellation of the renormalon ambiguities in C_0 with the corresponding instabilities in the matrix elements can be traced back to the fact that the $\overline{\rm MS}$-OPE does not provide a strict separation of momentum scales.

Note that the coefficient functions of higher dimensional terms in the OPE (e.g. C_{GG}, $C_{\overline{q}q}$ in (3.25)) in principle also contain IR renormalon series causing subleading divergent behaviour. Since there exists no consistent way of treating these renormalons (and since they are subleading compared to the most important ones in C_0), we do not consider any renormalon divergences in coefficient functions other than C_0 here.

The convergence of the Wilson coefficients in the Adler function and other physical observables obviously depends a lot on an appropriate scheme choice. As we are going to discuss in the next chapter, switching to a scheme that expresses observables in terms of perturbative series that are less IR sensitive will in general improve the convergence. The definition of such schemes typically requires an additional infrared scale R, below which the problematic fluctuations due to IR renormalons are subtracted. Similar to the case of the conventional renormalization scale μ, we can also study the renormalization group evolution with respect to the new scale R. This so-called *R-evolution* leads to corresponding RGEs which sum at the same time the asymptotic renormalon series and potential large logarithms in the difference of two subtractions defined at widely separated scales. Furthermore we will see that the solutions of these R-evolution equations yield expressions for a "*renormalon sum rule*" method that can be used as a probe for the existence of renormalons without relying on the calculation of bubble chain diagrams in the large-β_0 approximation. This renormalon sum rule provides the possibility to gain information on specific renormalons in perturbative series and allows us to address the second question raised at the end of chapter 2. Besides, the solution of the R-evolution equation is directly linked to Borel-type integrals and, hence, represents an alternative method to quantify the size and the structure of renormalon ambiguities.

Continuing with our discussion we now turn to the OPE of a generic observable σ. Assuming this observable to be dimensionless, its $\overline{\mathrm{MS}}$-*OPE* exhibits the general structure[6],

$$\begin{aligned}\sigma(Q) &= \overline{C}_0(Q) + \sum_i C_i(Q,\mu)\frac{\langle O_i(\mu)\rangle}{Q^p} + \ldots \\ &= \overline{C}_0(Q) + \vec{C}^{\mathsf{T}}(Q,\mu)\frac{\langle \vec{O}(\mu)\rangle}{Q^p} + \ldots,\end{aligned} \tag{3.27}$$

where $\overline{C}_0$ denotes the purely perturbative contributions to σ in the $\overline{\mathrm{MS}}$-scheme and the first non-vanishing non-perturbative matrix elements arise from operators with equal dimension p. According to our previous analysis of the Adler function, these matrix elements compensate the ambiguity caused by $u = p/2$ renormalons in $\overline{C}_0$. To facilitate the subsequent discussion, the set of dimension p operator matrix elements and their associated Wilson coefficients have been combined into the vectors $\langle \vec{O}(\mu)\rangle$ and $\vec{C}^{\mathsf{T}}$ in the second line.

In the following we are interested in investigating more closely the scale dependence of generic OPE correction terms due to operators of equal dimension p,

$$\begin{aligned}\sigma(Q) &= \overline{C}_0(Q) + \frac{F_O^p}{Q^p} + \ldots, \\ F_O^p &= \vec{C}^{\mathsf{T}}(\mu)\,\langle \vec{O}(\mu)\rangle.\end{aligned} \tag{3.28}$$

In the expression for the correction term F_O^p the renormalization scale μ is shown explicitly, while the dependence on other momentum scales is implicit. Since physical observables are independent of the renormalization scale, they satisfy a homogeneous RGE and we thus have,

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\Big(\vec{C}^{\mathsf{T}}(\mu)\,\langle \vec{O}(\mu)\rangle\Big) = 0, \tag{3.29}$$

which immediately leads to

$$\left[\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\vec{C}^{\mathsf{T}}(\mu)\right]\langle \vec{O}(\mu)\rangle = -\vec{C}^{\mathsf{T}}(\mu)\left[\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\langle \vec{O}(\mu)\rangle\right]. \tag{3.30}$$

Defining

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\vec{O}(\mu) = -\gamma_O[a_{\mathrm{s}}(\mu)]\,\vec{O}(\mu), \tag{3.31}$$

with the anomalous dimension matrix γ_O expanded in the form ($a_{\mathrm{s}} \equiv \alpha_{\mathrm{s}}/\pi$)

$$\gamma_O[a_{\mathrm{s}}(\mu)] = \gamma_O^{(1)} a_{\mathrm{s}} + \gamma_O^{(2)} a_{\mathrm{s}}^2 + \ldots, \tag{3.32}$$

the RGE for the Wilson coefficients is found to be:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\vec{C}(\mu) = \gamma_O^{\mathsf{T}}[a_{\mathrm{s}}(\mu)]\vec{C}(\mu). \tag{3.33}$$

In order to obtain a suitable expression for F_O^p, it is convenient to change to an operator basis in which the leading-order anomalous dimension matrix is diagonal. Upon diagonalizing $\gamma_O^{(1)}$ by a matrix V consisting of the eigenvectors of $\gamma_O^{(1)}$,

$$\gamma_D^{(1)} = V^{-1}\gamma_O^{(1)}V, \tag{3.34}$$

where the diagonal elements of $\gamma_D^{(1)}$ are now given by the eigenvalues of $\gamma_O^{(1)}$, the generic OPE term (3.28) can be rewritten in the following way,

$$F_O^p = \vec{C}^{\mathsf{T}}(\mu)\,\langle \vec{O}(\mu)\rangle = \vec{C}^{\mathsf{T}}(\mu)\,VV^{-1}\,\langle \vec{O}(\mu)\rangle = \vec{C}_D^{\mathsf{T}}(\mu)\,\langle \vec{O}_D(\mu)\rangle, \tag{3.35}$$

[6]To simplify the notation in the following discussion, only a generic set of correction terms associated with $u = p/2$ IR renormalons is displayed in the OPE (3.27). If $\overline{C}_0$ suffers from a series of IR renormalons for different p values, an appropriate sum over p is implied.

with

$$\langle \vec{O}_D(\mu)\rangle = V^{-1}\,\langle \vec{O}(\mu)\rangle,\qquad\qquad (3.36)$$
$$\vec{C}_D(\mu) = V^{\intercal}\,\vec{C}(\mu).$$

The corresponding leading order RGEs are given by:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\vec{O}_D(\mu) = -\gamma_D^{(1)}\,a_\mathrm{s}(\mu)\,\vec{O}_D(\mu),\qquad\qquad (3.37)$$
$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,\vec{C}_D(\mu) = \gamma_D^{(1),\intercal}\,a_\mathrm{s}(\mu)\,\vec{C}_D(\mu).$$

The leading logarithmic (LL) solution can be parameterized as

$$\vec{C}_D^{\intercal}(\mu)\,U_\mathrm{LL}(\mu,\mu')\,\langle \vec{O}_D(\mu')\rangle,\qquad\qquad (3.38)$$

which yields an RGE for the renormalization group evolution matrix $U_\mathrm{LL}(\mu,\mu')$ at the LL order ($\gamma_D^{(1)} \equiv \mathrm{diag}\big[\vec{\gamma}_D^{(1)}\big]$):

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,U_\mathrm{LL}(\mu,\mu') = -\mathrm{diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_\mathrm{s}(\mu)\,U_\mathrm{LL}(\mu,\mu').\qquad\qquad (3.39)$$

Exploiting the one-loop running of the strong coupling α_s (see eq. (C.1)), the LL solution eventually takes the form:

$$U_\mathrm{LL}(\mu,\mu') = \mathrm{diag}\left[\left(\frac{a_\mathrm{s}(\mu)}{a_\mathrm{s}(\mu')}\right)^{2\vec{\gamma}_D^{(1)}/\beta_0}\right],\qquad U_\mathrm{LL}(\mu,\mu) = \mathbb{1}.\qquad\qquad (3.40)$$

For the generalisation to the next-to-leading order, we proceed similar to the LL case and define $g := V^{-1}\gamma_O^{(2)}V$, which, in contrast to $\gamma_D^{(1)}$, is not diagonal in general. The RGE for the evolution matrix $U_\mathrm{NLL}(\mu,\mu')$ in the next-to-leading logarithmic approximation is given by:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,U_\mathrm{NLL}(\mu,\mu') = -\Big[\mathrm{diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_\mathrm{s}(\mu) + g\,a_\mathrm{s}^2(\mu)\Big]\,U_\mathrm{NLL}(\mu,\mu').\qquad\qquad (3.41)$$

This differential equation can be solved by using the ansatz,

$$U_\mathrm{NLL}(\mu,\mu') = \big[\mathbb{1} + a_\mathrm{s}(\mu)S\big]\,U_\mathrm{LL}(\mu,\mu')\,\big[\mathbb{1} - a_\mathrm{s}(\mu')S\big],\qquad\qquad (3.42)$$

which after a short calculation leads to the following expression for the elements of the matrix S:

$$\begin{aligned} S^{ij} &= \frac{\beta_1}{4\beta_0}\,\gamma_D^{(1),i}\,\delta^{ij}\,\frac{1}{(\gamma_D^{(1),i}-\gamma_D^{(1),j}-\beta_0/2)} - \frac{g^{ij}}{(\gamma_D^{(1),i}-\gamma_D^{(1),j}-\beta_0/2)} \\ &= -\frac{\beta_1}{2\beta_0^2}\,\gamma_D^{(1),i}\,\delta^{ij} + \frac{g^{ij}}{(\beta_0/2+\gamma_D^{(1),j}-\gamma_D^{(1),i})}. \end{aligned}\qquad\qquad (3.43)$$

Details on the calculation of S have been relegated to appendix C.2 and can also be found in [44, 45].

With these expressions for the evolution matrix at hand, we are now in the position to investigate generic OPE contributions at the NLL level. The subsequent derivation will be a bit technical including lots of redefinitions that may seem unnecessary at first sight. However, the notation introduced below greatly facilitates the discussion of the IR (and UV) renormalon structure, especially in view of the R-evolution formalism developed in the next chapter.

Using the NLL solution for the evolution matrix, the term F_O^p of (3.28) in the OPE can be rewritten in a form where all large logarithms involving the scales Q and Λ_IR are being resummed at NLL:

$$F_O^p = \vec{C}^{\intercal}(Q)\,U_\mathrm{NLL}(Q,\Lambda_\mathrm{IR})\,\langle \vec{O}(\Lambda_\mathrm{IR})\rangle,\qquad\qquad (3.44)$$

where we displayed explicitly the dependence on the large and low momentum scales with $\Lambda_{\rm IR} \ll Q$. Next, we exploit the ansatz (3.42), which yields:

$$
\begin{aligned}
F_O^p &= \underbrace{\vec{C}^\intercal(Q)\big[\mathbb{1} + a_{\rm s}(Q)S\big]\mathrm{diag}\Bigg[\left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{2\vec{\gamma}_D^{(1)}}{\beta_0}}\Bigg]}_{\equiv \hat{\vec{C}}_c^\intercal(Q)} \underbrace{\mathrm{diag}\Bigg[\left(\frac{a_{\rm s}(\Lambda_{\rm IR})}{c}\right)^{\frac{-2\vec{\gamma}_D^{(1)}}{\beta_0}}\Bigg]\big[\mathbb{1} - a_{\rm s}(\Lambda_{\rm IR})S\big]\langle\vec{O}(\Lambda_{\rm IR})\rangle}_{\equiv\langle\hat{\vec{O}}_c(\Lambda_{\rm IR})\rangle} \\
&= \sum_{i,j} C_i(Q)\big[\delta_{ij} + a_{\rm s}(Q)S_{ij}\big]\left(\frac{a_{\rm s}(Q)}{c}\right)^{2\gamma_{D,j}^{(1)}/\beta_0}\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle. \qquad (3.45)
\end{aligned}
$$

Here, $\hat{\vec{C}}_c^\intercal$ and $\langle\hat{\vec{O}}_c\rangle$ defined in the first line denote the Wilson coefficients and matrix elements, respectively, in the so-called renormalization group improved (RGI) scheme and the auxiliary constant c was introduced for later convenience. We emphasize that in the RGI-scheme the momentum scales Q and $\Lambda_{\rm IR}$ are strictly separated in the sense that the coefficient functions $\hat{C}_c(Q)$ only depend on the large scale Q, whereas $\langle\hat{O}_c(\Lambda_{\rm IR})\rangle$ does only depend on $\Lambda_{\rm IR}$. In the following we assume the coefficient functions in (3.45) to be renormalon free, that is we neglect all renormalons that may be contained in $C_i(Q)$, while the non-perturbative matrix elements carry the pure $\mathcal{O}(\Lambda_{\rm QCD}^p)$ renormalon. This will be important in the discussion of R-evolution in section 4.2.1.

Continuing with our analysis, we now decompose the Wilson coefficient C_i in the form

$$
C_i(Q) = a_{\rm s}^{\delta_i}(Q)\sum_{n=0}^{\infty} c_i^{(n)}\, a_{\rm s}^n(Q), \qquad (3.46)
$$

where δ_i represents the lowest power in the strong coupling of the coefficient function C_i. Thus, the OPE term F_O can be further rewritten as follows

$$
\begin{aligned}
F_O^p &= \sum_{i,j} a_{\rm s}^{\delta_i}(Q)\big[c_i^{(0)} + a_{\rm s}(Q)c_i^{(1)} + \ldots\big]\big[\delta_{ij} + a_{\rm s}(Q)S_{ij}\big]\left(\frac{a_{\rm s}(Q)}{c}\right)^{2\gamma_{D,j}^{(1)}/\beta_0}\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle \qquad (3.47)\\
&= \sum_j \left(\frac{a_{\rm s}(Q)}{c}\right)^{\delta_j}\big[c_j^{(0)} + a_{\rm s}(Q)\,c_j^{(1)}\big]\left(\frac{a_{\rm s}(Q)}{c}\right)^{2\gamma_{D,j}^{(1)}/\beta_0} c^{\delta_j}\,\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle \\
&\quad + \sum_{i,j} c_i^{(0)}\left(\frac{a_{\rm s}(Q)}{c}\right)^{\delta_i} a_{\rm s}(Q)S_{ij}\left(\frac{a_{\rm s}(Q)}{c}\right)^{2\gamma_{D,j}^{(1)}/\beta_0} c^{\delta_i-\delta_j}\, c^{\delta_j}\,\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle \\
&= \sum_j \left(\frac{a_{\rm s}(Q)}{c}\right)^{\delta_j+2\gamma_{D,j}^{(1)}/\beta_0}\left[c_j^{(0)} + a_{\rm s}(Q)\left(c_j^{(1)} + \sum_i \left(\frac{a_{\rm s}(Q)}{c}\right)^{\delta_i-\delta_j} c^{\delta_i-\delta_j}\, c_i^{(0)} S_{ij}\right)\right] c^{\delta_j}\,\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle \\
&= \sum_j c_j^{(0)}\left(\frac{a_{\rm s}(Q)}{c}\right)^{\delta_j+2\gamma_{D,j}^{(1)}/\beta_0}\left[1 + a_{\rm s}(Q)\big(c_j^{(0)}\big)^{-1}\left(c_j^{(1)} + \sum_i a_{\rm s}(Q)^{\delta_i-\delta_j}\, c_i^{(0)} S_{ij}\right)\right] c^{\delta_j}\,\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle,
\end{aligned}
$$

where we dropped terms of $\mathcal{O}(a_{\rm s}^2)$ in the expansion of the product of the square brackets in the first line. In addition, we stress that the exponent $\delta_i - \delta_j$ of $\alpha_{\rm s}$ in the second part of the last line can in principle take on any integer value greater than or equal to $-\delta_j$, since $\delta_i \geq 0$. In order to recover a suitable expression for F_O^p, we redefine the Wilson coefficients C_j and leading powers δ_j in the following way:

$$
F_O^p =: \sum_j \hat{c}_j^{(0)}\left(\frac{a_{\rm s}(Q)}{c}\right)^{\hat{\delta}_j+2\gamma_{D,j}^{(1)}/\beta_0}\left[1 + a_{\rm s}(Q)\,\hat{c}_j^{(1)} + \ldots\right] c^{\hat{\delta}_j}\,\langle\hat{O}_c^j(\Lambda_{\rm IR})\rangle. \qquad (3.48)
$$

The expression above is rearranged in such a way that the correction term $a_{\rm s}(Q)\,\hat{c}_j^{(1)}$ within the square brackets is truly $\mathcal{O}(\alpha_{\rm s})$-suppressed. In particular, note that depending on the values of δ_j and δ_i in (3.48), the newly defined leading powers $\hat{\delta}_j$ do not need to be identical with δ_j.

3.4 Ultraviolet Renormalons

In contrast to IR renormalons, UV renormalons in QCD are Borel summable and cannot be related to non-perturbative power corrections that arise in the framework of the operator product expansion. For this reason UV renormalons are sometimes regarded to be less problematic than their infrared counterparts and are often left aside in discussions. However, for some observables, like the Adler function, UV renormalons are closest to the origin in the Borel plane and therefore dominate the large-order behaviour, as can be seen from the sign-alternating factorial growth of the perturbative coefficients.

There is a striking similarity between UV renormalons and UV renormalization first noted by Parisi [46] who suggested that UV renormalons could be compensated by insertions of local higher-dimensional operators[7] into the Green functions from which the observable under consideration is derived. Parisi's conjecture is based on the observation that UV renormalons are connected to the large momentum expansion of the integrands in loop calculations. In the computation of bubble chain diagrams (see section 3.2) the Feynman integrand contains a large power of logarithms $\ln k^2/\mu^2$ coming from multiple fermion loop insertions into the gluon line. Potential logarithmic UV divergences in such computations are associated with integrations of terms like d^4k/k^4 over large loop momenta and are removed by counterterm insertions of operators of dimension four in the renormalization procedure. The UV renormalons then arise from integrating large powers of $\ln k^2/\mu^2$ in the remaining terms of the large momentum expansion of the Feynman integrand over the ultraviolet region. To be more specific, the leading UV renormalon $u = -1$ follows from the large-k behaviour of the d^4k/k^6-part and the subleading UV renormalons starting at $u = -2$ are due to the remaining terms such as d^4k/k^8. Similar to the removal of ordinary UV divergences by means of local counterterms, UV renormalons can be compensated by insertions of local higher-dimensional operators. In the particular case of the leading $u = -1$ renormalon we thus need to consider an additional term in the Lagrangian,

$$\Delta\mathcal{L}^{(6)} = \sum_i C_i^{(6)}(\mu)\,\frac{O_i^{(6)}(\mu)}{\Lambda_{\mathrm{UV}}^2}, \tag{3.49}$$

containing all local operators $O_i^{(6)}$ of dimension six. This additional term signifies the missing information from energy scales much larger than Q. In a similar fashion the subleading UV renormalon at $u = -2$ is related to local dimension-8 operators and, in general, a UV renormalon singularity located at $u = -\tilde{p}/2$ ($\tilde{p} > 0$) in the Borel plane can be remedied by adding further terms to the Lagrangian including local operators of dimension $k = \tilde{p} + 4$. Most works in the literature primarily deal with the leading UV renormalon and detailed discussions can be found e.g. in [30, 47].

Due to their analogy to UV renormalization, UV renormalons can in some sense be regarded universal, since once the coefficient functions C_i, which contain short distance information from scales $\Lambda_{\mathrm{UV}} \gg Q$, have been determined from Green functions for a specific process in a more UV complete theory, the UV renormalons for all other processes can be treated in the same way.

In the following we want to investigate in more detail the structure of additional terms in the Lagrangian needed to compensate the effects of UV renormalons located at $u = -\tilde{p}/2$ on the negative real Borel axis. Adopting the results from the discussion of IR renormalons in the previous section, a generic Lagrangian term for operators of dimension $k = \tilde{p} + 4$ is given by[8]

$$\Delta\mathcal{L} = \vec{C}^{\intercal}(\Lambda_{\mathrm{UV}})\,U_{\mathrm{NLL}}(\Lambda_{\mathrm{UV}}, Q)\,\frac{\vec{O}(Q)}{\Lambda_{\mathrm{UV}}^{\tilde{p}}} = \tag{3.50}$$

[7] Note that these higher-dimensional operators are considered local with respect to the scale Q.

[8] Analogous to (3.27), only a generic Lagrangian term for operators associated with $u = -\tilde{p}/2$ UV renormalons is displayed in (3.50). If $\overline{C}_0$ suffers form a series of UV renormalons for different $\tilde{p}$ values, an appropriate sum over $\tilde{p}$ is implied.

$$=\frac{1}{\Lambda_{\mathrm{UV}}^{\tilde{p}}}\underbrace{\vec{C}^{\intercal}(\Lambda_{\mathrm{UV}})\big[\mathbb{1}+a_{\mathrm{s}}(\Lambda_{\mathrm{UV}})S\big]\mathrm{diag}\Bigg[\left(\frac{a_{\mathrm{s}}(\Lambda_{\mathrm{UV}})}{c}\right)^{\frac{2\tilde{\gamma}_D^{(1)}}{\beta_0}}\Bigg]}_{\equiv\hat{\vec{C}}_c^{\intercal}(\Lambda_{\mathrm{UV}})}\underbrace{\mathrm{diag}\Bigg[\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{\frac{-2\tilde{\gamma}_D^{(1)}}{\beta_0}}\Bigg]\big[\mathbb{1}-a_{\mathrm{s}}(Q)S\big]\vec{O}(Q)}_{\equiv\hat{\vec{O}}_c(Q)},$$

where Λ_{UV} denotes e.g. a BSM scale with $\Lambda_{\mathrm{UV}}\gg Q$. In order to account for an $u=-\tilde{p}/2$ renormalon in the perturbative expansion of an observable $\sigma(Q)$ we have to consider Green functions for the SM/QCD process with a single insertion of the local operators O_i contributing to $\sigma(Q)$. Assuming that this calculation only involves a single scale Q, the contribution from each insertion of the operator O_i can be generically written as,

$$R_\sigma(O_i(Q))=Q^{\tilde{p}}\,a_{\mathrm{s}}^{\delta_i}(Q)\big[c_{\mathrm{UV},i}^{(0)}+a_{\mathrm{s}}(Q)\,c_{\mathrm{UV},i}^{(1)}+\ldots\big]. \tag{3.51}$$

Using this expression we can define (cf. eq. (3.28)),

$$\sigma(Q)=\overline{C}_0(Q)+\frac{F_O^{\tilde{p},\mathrm{UV}}}{\Lambda_{\mathrm{UV}}^{\tilde{p}}}+\ldots, \tag{3.52}$$

with,

$$\begin{aligned}
F_O^{\tilde{p},\mathrm{UV}}&=\sum_{i,j}\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0}\big[\delta_{ji}-a_{\mathrm{s}}(Q)\,S_{ji}\big]\,R_\sigma(O_i(Q)) \qquad (3.53)\\
&=\sum_{i,j}\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0}\big[\delta_{ji}-a_{\mathrm{s}}(Q)\,S_{ji}\big]\,Q^{\tilde{p}}\,a_{\mathrm{s}}^{\delta_i}(Q)\,\big[c_{\mathrm{UV},i}^{(0)}+a_{\mathrm{s}}(Q)\,c_{\mathrm{UV},i}^{(1)}+\ldots\big]\\
&=\sum_{i,j}c^{\delta_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0}\big[\delta_{ji}-a_{\mathrm{s}}(Q)\,S_{ji}\big]\\
&\qquad\times Q^{\tilde{p}}\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{\delta_i}c^{\delta_i-\delta_j}\big[c_{\mathrm{UV},i}^{(0)}+a_{\mathrm{s}}(Q)\,c_{\mathrm{UV},i}^{(1)}+\ldots\big]\\
&=\sum_{j}c^{\delta_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\,Q^{\tilde{p}}\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0+\delta_j}\big[c_{\mathrm{UV},j}^{(0)}+a_{\mathrm{s}}(Q)\,c_{\mathrm{UV},j}^{(1)}\big]\\
&\quad-\sum_{i,j}c^{\delta_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\,Q^{\tilde{p}}\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0+\delta_j}\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{(\delta_i-\delta_j)}c^{\delta_i-\delta_j}\,a_{\mathrm{s}}(Q)\,S_{ji}\,c_{\mathrm{UV},i}^{(0)}\\
&=Q^{\tilde{p}}\sum_{j}c^{\delta_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0+\delta_j}\left[c_{\mathrm{UV},j}^{(0)}+a_{\mathrm{s}}(Q)\left(c_{\mathrm{UV},j}^{(1)}-\sum_i a_{\mathrm{s}}(Q)^{\delta_i-\delta_j}\,c_{\mathrm{UV},i}^{(0)}\,S_{ji}\right)\right]\\
&=Q^{\tilde{p}}\sum_{j}c^{\delta_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{-2\gamma_{D,j}^{(1)}/\beta_0+\delta_j}c_{\mathrm{UV},j}^{(0)}\\
&\qquad\times\left[1+a_{\mathrm{s}}(Q)\,\big(c_{\mathrm{UV},j}^{(0)}\big)^{-1}\left(c_{\mathrm{UV},j}^{(1)}-\sum_i S_{ji}\,a_{\mathrm{s}}(Q)^{\delta_i-\delta_j}\,c_{\mathrm{UV},i}^{(0)}\right)\right].
\end{aligned}$$

Here, the process-independent and universal coefficients $\hat{C}_c(\Lambda_{\mathrm{UV}})$ compensate for $\mathcal{O}(\Lambda_{\mathrm{UV}}^{-\tilde{p}})$ UV renormalons. Similar to the discussion of the OPE corrections in the last section, it is possible to combine the terms in the last line such that one obtains an expression resembling the one in eq. (3.48):

$$F_O^{\tilde{p},\mathrm{UV}}=:\sum_j c^{\hat{\delta}_j}\,\hat{C}_{c,j}(\Lambda_{\mathrm{UV}})\,Q^{\tilde{p}}\left(\frac{a_{\mathrm{s}}(Q)}{c}\right)^{\frac{-2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j}\hat{c}_{\mathrm{UV},j}^{(0)}\big[1+a_{\mathrm{s}}(Q)\,\hat{c}_{\mathrm{UV},j}^{(1)}+\ldots\big]. \tag{3.54}$$

The expression above is rearranged in the same fashion as the generic IR-OPE correction in eq. (3.48) such that the first subleading term $a_\mathrm{s}(Q)\, c^{(\hat{1})}_{\mathrm{UV},j}$ in the square brackets is truly $\mathcal{O}(\alpha_\mathrm{s})$-suppressed. We stress again that $\hat{\delta}_j$ does not need to be equal to δ_j.

3.5 Renormalon Structure of a Generic Observable

After the rather elaborate notational setup in the previous two sections we briefly summarize and combine our knowledge about IR and UV renormalons. From the form of the Borel integral (3.5) we have seen that IR renormalons are associated with ambiguities in the definition of the perturbative series $\overline{C}_0$ of a physical observable $\sigma(Q)$, which indicate that low momentum non-perturbative corrections need to be considered[9]. These effects can be efficiently incorporated in the framework of the operator product expansion by adding higher-dimensional terms in the form of operators that are local with respect to momenta $\Lambda_\mathrm{IR} \ll Q$ (i.e. enconding interactions of Λ_IR-modes generated by the large scale Q). In this formalism IR renormalon ambiguities in the perturbative series of the coefficient $\overline{C}_0$ cancel with corresponding instabilities contained in vacuum matrix elements of these local operators which constitute non-perturbative condensate contributions.

UV renormalons, on the other hand, are related to missing high momentum information, but are Borel-summable and show remarkable similarities to ordinary renormalization of UV divergences. They are linked to terms in the large momentum expansion of integrands in loop calculations and can be compensated by insertions of higher-dimensional local operators into the Green function contributing to a specific observable. These operators are local with respect to scales $Q \ll \Lambda_\mathrm{UV}$, i.e. they encode interactions of Q-modes generated by the large scale Λ_UV.

Combining our previous findings, one can write down an OPE-like expression for a dimensionless observable $\sigma(Q)$ which compensates all IR and UV renormalon effects associated with power suppressed contributions of order $(\Lambda_\mathrm{IR}/Q)^p$ and $(Q/\Lambda_\mathrm{UV})^{\tilde{p}}$, respectively, that occur in the perturbative calculation of $\sigma(Q)$:

$$\sigma(Q) = \overline{C}_0(Q) + \vec{C}^\intercal(Q,\mu)\frac{\langle\vec{O}(\mu)\rangle}{Q^p} + \cdots + \vec{C}^\intercal_\mathrm{UV}(\mu)\,\frac{R_\sigma\big(\vec{O}_\mathrm{UV}(Q,\mu)\big)}{\Lambda^{\tilde{p}}_\mathrm{UV}} + \ldots \quad . \tag{3.55}$$

Here, the coefficient function $\overline{C}_0$ comprises the purely perturbative contributions to $\sigma(Q)$. In general, $\overline{C}_0$ suffers from an infinite series of IR and UV renormalons involving different p and $\tilde{p}$ values, so an appropriate sum over p and $\tilde{p}$ is implied[10].

At this point an important comment concerning the generic IR and UV correction terms in (3.55) is in order. In the IR case, as discussed in section 3.3.1, the Wilson coefficients can be computed perturbatively, while the matrix elements need non-perturbative treatment. For UV renormalon corrections, on the contrary, the role of Wilson coefficients and matrix elements is interchanged and only the matrix elements are perturbatively accessible. The UV Wilson coefficients contain information from energy scales much larger than Q and are computable in a more UV complete theory (BSM theories).

Exploiting the results of eqs. (3.48) and (3.54), the OPE for $\sigma(Q)$ is given by:

$$\sigma(Q) = \left[1+\sum_{n=1}^{\infty}\tilde{a}_n a^n_\mathrm{s}(Q)\right] + \frac{1}{Q^p}\sum_j \left(\frac{a_\mathrm{s}(Q)}{c}\right)^{\frac{2\gamma^{(1)}_{D,j}}{\beta_0}+\hat{\delta}_j} \underbrace{\hat{c}^{(0)}_j\left[1+a_\mathrm{s}(Q)\,\hat{c}^{(1)}_j\ldots\right]}_{=:\hat{C}_j(Q)}\langle c^{\hat{\delta}_j}\hat{O}^j_c\rangle + \ldots$$

[9] We assume that $\sigma(Q)$ involves only the single scale Q (i.e. Q is the scale of the measurement).

[10] Strictly speaking, the concept of the operator product expansion only applies to IR renormalons. However, in order to simplify the discussion in the following chapter, we will refer to the expression (3.55) including corrections for both, IR and UV renormalons, as the OPE for the observable $\sigma(Q)$.

$$+\frac{Q^{\tilde p}}{\Lambda_{\rm UV}^{\tilde p}}\sum_j\left[c^{\hat\delta_j}\hat C^{\rm UV}_{c,j}\right]\left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{-2\gamma^{(1)}_{D,j}}{\beta_0}+\hat\delta_j}\underbrace{\hat c^{(0)}_{{\rm UV},j}\left[1+a_{\rm s}(Q)\,\hat c^{(1)}_{{\rm UV},j}\ldots\right]}_{=:\hat C_{{\rm UV},j}(Q)}+\ldots$$

$$=\left[1+\sum_{n=1}^{\infty}\tilde a_n a_{\rm s}^n(Q)\right]\mathbb{1}+\frac{1}{Q^p}\sum_j\left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{2\gamma^{(1)}_{D,j}}{\beta_0}+\hat\delta_j}\hat C_j(Q)\,\langle c^{\hat{\hat\delta}_j}\hat O^j_c\rangle+\ldots$$

$$+\frac{Q^{\tilde p}}{\Lambda_{\rm UV}^{\tilde p}}\sum_j\left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{-2\gamma^{(1)}_{D,j}}{\beta_0}+\hat\delta_j}\hat C_{{\rm UV},j}(Q)\left[c^{\hat\delta_j}\hat C^{\rm UV}_{c,j}\right]+\ldots\quad. \tag{3.56}$$

It is essential to recall how the renormalon cancellation takes place in the $\overline{\rm MS}$-OPE. Since $\sigma(Q)$ is a physical observable, it must be renormalon free and hence certain cancellations happen between different contributions in the OPE. In particular, IR and UV renormalons causing factorial growth of the perturbative coefficients in $\overline{C}_0$ are compensated order-by-order in the $\alpha_{\rm s}$ expansion by corresponding instabilities in the process-independent and universal matrix elements $\langle c^{\hat{\hat\delta}_j}\hat O_{c,j}\rangle(\Lambda_{\rm IR})$ and coefficient functions $c^{\hat\delta_j}\hat C_{c,j}(\Lambda_{\rm UV})$, respectively. As already mentioned, these renormalon cancellations can be improved by switching to another scheme, called the MSR-scheme, that subtracts the renormalon contributions form $\overline{C}_0(Q)$ due to IR and UV renormalons at the new scales R and $R_{\rm UV}$, respectively. Similar to the renormalization scale μ in $\overline{\rm MS}$ one can also consider RG equations with respect to the new scales, R and $R_{\rm UV}$, which leads to the notion of R-evolution. This allows to choose R and $R_{\rm UV}$ of order Q for the subtracted versions of $\overline{C}_0$, called $C_0(Q,R,R_{\rm UV})$, which is free of large logarithms involving the scales Q,R and $R_{\rm UV}$. Furthermore, large logarithms involving the ratios R/Q and $Q/R_{\rm UV}$ can be summed systematically. The subtractions in the MSR-scheme and the concept of R-evolution will be formulated in the next chapter.

Note that it is in principle possible to decompose the perturbative coefficients $\tilde a_n$ into contributions from the various renormalons $\overline{C}_0$. Thus, we can write:

$$\tilde a_n=\sum_{p,j}\tilde a^p_{j,n}+\sum_{\tilde p,j}\tilde a^{-\tilde p}_{j,n}+\tilde a_{n,{\rm rem}}, \tag{3.57}$$

where the first part comprises the contributions due to IR renormalon poles at $u=p/2$ and the second sum takes into account all UV renormalon singularities at positions $u=-\tilde p/2$. The last term denotes a convergent remainder which includes all contributions to $\tilde a_n$ that do not arises from renormalon poles. In practice, however, it turns out that we cannot define this splitting uniquely, but only asymptotically.

Chapter 4

R-evolution and the Renormalon Sum Rule

In the previous section we discussed how large cancellations between the purely perturbative coefficient $\overline{C}_0$ and higher-dimensional IR and UV correction terms in the OPE-like expression (3.56) lead to a renormalon free description of a physical observable $\sigma(Q)$. However, the numerical cancellations at high orders in the perturbative expansion in α_s expressed in the conventional $\overline{\mathrm{MS}}$-scheme of the OPE is sometimes impractical for precise theoretical predictions of QCD observables and it can be useful to employ schemes for the IR matrix elements $\langle c^{\hat{\delta}_j}\hat{O}_{c,j}\rangle$ and UV Wilson coefficients $c^{\hat{\delta}_j}\hat{C}_{c,j}$ with explicit subtractions of low and high energy fluctuations involving new cut-off scales R and R_UV. In this chapter we lay out the basic formalism for these schemes and show in detail how the subtractions cancel the factorial growth due to IR and UV renormalon divergences in $\overline{C}_0$. Moreover, treating the parameters R and R_UV, respectively, as continuous variables yields a renormalization group equation [48], which connects subtractions at different scales without introducing renormalons. We study the implications of the renormalization group flow in R (and R_UV) and, in particular, present a detailed derivation of the so-called renormalon sum rule which allows us to probe renormalons in perturbative series.

At the beginning of the next section 4.1 we focus only on the discussion of IR renormalons to motivate the definition of the so-called MSR-scheme. UV renormalons will be included from section 4.1.1 onwards.

4.1 The MSR-Scheme

We have already argued that using a rigid momentum cut-off in the Wilsonian OPE provides a very intuitive physical picture for the factorization into coefficient functions and matrix elements, but is hard to implement and preserve gauge symmetry and Lorentz invariance in calculations beyond the one-loop order. Switching to the $\overline{\mathrm{MS}}$-scheme represents a convenient alternative which allows us to perform loop calculations in a gauge and Lorentz invariant way. However, as mentioned in section 3.3.1, $\overline{\mathrm{MS}}$ calculations involve integrations over all loop momenta which results in the known factorially divergent behaviour of the perturbative coefficients in $\overline{C}_0$ caused by renormalon poles. In the context of the $\overline{\mathrm{MS}}$-OPE these renormalon effects are compensated order-by-order in the α_s expansion by corresponding instabilities in the matrix elements.

In order to avoid the resulting large cancellations between $\overline{C}_0$ and higher-dimensional terms in the OPE, one can employ appropriate schemes for the Wilson coefficient $\overline{C}_0$ and the non-perturbative matrix elements that take explicit renormalon subtractions into account. Such schemes usually depend on an additional momentum scale, denoted as R, at which the renormalon contributions are removed. To illustrate how these subtractions can generally be defined, recall the expression for the Adler function in the $\overline{\mathrm{MS}}$-scheme quoted in section 3.2:

$$\overline{C}_0(Q,\mu) = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_\mathrm{s}(\mu)}{\pi}\right)^n , \tag{4.1}$$

$$c_n = \sum_{k=0}^{\infty} c_{n,k}\, \ln^k\left(\frac{\mu}{Q}\right).$$

From the computation of bubble chain diagrams (see eq. (3.16)) we see that renormalon singularities give rise to factorial growth of the perturbative coefficients c_n in (4.1). More precisely, the large-order behaviour of the coefficients due to a renormalon located at $u = p/2$ in the Borel plane is given by [17],[27]:

$$c_{n+1} \sim \left(\frac{\mu}{Q}\right)^p n! \left[\frac{2\beta_0}{p}\right]^n, \tag{4.2}$$

where the factor $(\mu/Q)^p$ arises at large n from the contribution of powers of $\ln(\mu/Q)$ in the coefficients c_n. A sensible definition of a scheme with explicit renormalon subtractions needs to account for cancellations of this factorial growth order by order. In general, $\overline{C}_0$ can be connected to the corresponding coefficient function $C_0(R)$ in a new R-scheme by a relation of the form [17]:

$$\begin{aligned}\overline{C}_0(Q) &= C_0(Q,R) + \delta C_0(Q,R,\mu) \\ &= C_0(Q,R) + \left(\frac{R}{Q}\right)^p \sum_{n=1}^{\infty} b_n\, a_s^n(\mu).\end{aligned} \tag{4.3}$$

Here, the perturbative series in δC_0 must have the same bad large-order behaviour as the series in $\overline{C}_0$, i.e. $b_{n+1} \sim (\mu/R)^p\, n!\, [2\beta_0/p]^n$, such that the coefficients in $C_0(Q,R)$ do not anymore exhibit factorial growth for large n [17]:

$$C_0(Q,R) \sim \left[\left(\frac{\mu}{Q}\right)^p - \left(\frac{R}{Q}\right)^p \frac{\mu^p}{R^p}\right] \sum_n n! \left[\frac{2\beta_0}{p}\right]. \tag{4.4}$$

Thus, we see that the factor $(R/Q)^p$ introduced in the definition (4.3) causes power law dependence of the series in δC_0 on the new cut-off scale which cancels exactly the contributions due to the $u = p/2$ renormalon in $\overline{C}_0$. The scheme change in $\overline{C}_0$ is associated with a corresponding scheme change in the vacuum matrix elements and one can construct subtractions similar to those in eq. (4.3) that compensate the instabilities in the definition of the matrix elements (see [17]).

So far the only constraint we put on the subtraction terms in the new R-scheme was related to the fact that the coefficients b_n need to have the same large-order behaviour as the coefficients in $\overline{C}_0$. This leaves substantial freedom for the definition of the subtractions. For our analysis in the following sections it is convenient to take the subtraction coefficients b_n to coincide with the contributions $\tilde{a}_n^p$ in (3.57) due to the $u = p/2$ renormalon in the $\overline{\text{MS}}$ coefficient function $\overline{C}_0$. The scheme thus defined preserves the computational features of $\overline{\text{MS}}$, like gauge and Lorentz invariance, and is called the *MSR-scheme.*

4.1.1 MSR-OPE for a Generic Observable

Returning to the discussion in section 3.5, we can now avoid the problem of large cancellations between $\overline{C}_0$ and higher-dimensional terms in the OPE for a dimensionless observable $\sigma(Q)$ by switching to the MSR-scheme. Similar to (4.3) one can define subtractions for both the universal low energy matrix elements $\langle \hat{O}_c^j \rangle$ and the high energy Wilson coefficients $\hat{C}_{c,j}^{\text{UV}}$:

- $$\langle c^{\hat{\delta}_j} \hat{O}_c^j(R)\rangle = \langle c^{\hat{\delta}_j} \hat{O}_c^j\rangle + R^p \left(\frac{a_s(R)}{c}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0} - \hat{\delta}_j} \left(\hat{C}_j(R)\right)^{-1} \sum_{n=1}^{\infty} \tilde{a}_{j,n}^p\, a_s^n(R) \tag{4.5}$$
- $$\frac{c^{\hat{\delta}_j}\, \hat{C}_{c,j}^{\text{UV}}(R_{\text{UV}})}{\Lambda_{\text{UV}}^{\tilde{p}}} = \frac{c^{\hat{\delta}_j}\, \hat{C}_{c,j}^{\text{UV}}}{\Lambda_{\text{UV}}^{\tilde{p}}} + \frac{1}{R_{\text{UV}}^{\tilde{p}}} \left(\frac{a_s(R_{\text{UV}})}{c}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0} - \hat{\delta}_j} \left(\hat{C}_{\text{UV},j}(R_{\text{UV}})\right)^{-1} \sum_{n=1}^{\infty} \tilde{a}_{j,n}^{-\tilde{p}}\, a_s^n(R_{\text{UV}}), \tag{4.6}$$

From the discussion at the end of the previous chapter (below eq. (3.55)) recall that in the IR OPE correction terms the Wilson coefficients are perturbatively computable, while in the UV case the matrix elements are perturbatively accessible. Therefore, in (4.5) the matrix element $\langle c^{\hat{\delta}_j}\hat{O}_c^j\rangle$ contains the pure $\mathcal{O}(\Lambda_{\rm QCD}^p)$ ambiguity which cancels against a corresponding ambiguity in the series of the second term on the right-hand side, such that the MSR matrix element $\langle c^{\hat{\delta}_j}\hat{O}_c^j(R)\rangle$ is ambiguity free.

In (4.6), on the other hand, the high energy Wilson coefficient $c^{\hat{\delta}_j}\,\hat{C}_{c,j}^{\rm UV}$ has a pure $\mathcal{O}(\Lambda_{\rm UV}^{-\tilde{p}})$ ambiguity[1] which is cancelled by an ambiguity of the same order in the second term on the right-hand side. Since both $\langle c^{\hat{\delta}_j}\hat{O}_c^j(R)\rangle$ and $c^{\hat{\delta}_j}\,\hat{C}_{c,j}^{\rm UV}(R_{\rm UV})$ do not contain an ambiguity, taking the derivative with respect to R and $R_{\rm UV}$ yields perturbative series on the RHS of eqs. (4.5) and (4.6) that are also ambiguity-free. This fact will be used in the derivation of the R-evolution equation in section 4.2.1.

Using the subtractions in (4.5) and (4.6) the OPE in the MSR-scheme takes the form:

$$\begin{aligned}\sigma(Q) =&\, C_0(Q,R,R_{\rm UV}) + \frac{1}{Q^p}\sum_j \left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j} \hat{C}_j(Q)\,\langle c^{\hat{\delta}_j}\hat{O}_c^j(R)\rangle + \ldots \\ &+ Q^{\tilde{p}}\sum_j \left(\frac{a_{\rm s}(Q)}{c}\right)^{\frac{-2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j} \hat{C}_{{\rm UV},j}(Q)\left[\frac{c^{\hat{\delta}_j}\,\hat{C}_{c,j}^{\rm UV}(R_{\rm UV})}{\Lambda_{\rm UV}^{\tilde{p}}}\right]\ldots,\end{aligned} \tag{4.7}$$

where the purely perturbative contributions in the Wilson coefficient $C_0(Q,R,R_{\rm UV})$ are now given by:

$$\begin{aligned}C_0(Q,R,R_{\rm UV}) =&\, \overline{C}_0(Q) - \sum_j \frac{R^p}{Q^p}\left(\frac{a_{\rm s}(Q)}{a_{\rm s}(R)}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j} \frac{\hat{C}_j(Q)}{\hat{C}_j(R)}\sum_{n=1}^{\infty}\tilde{a}_{j,n}^p\,a_{\rm s}^n(R) - \ldots \\ &- \sum_j \frac{Q^{\tilde{p}}}{R_{\rm UV}^{\tilde{p}}}\left(\frac{a_{\rm s}(Q)}{a_{\rm s}(R_{\rm UV})}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j} \frac{\hat{C}_{{\rm UV},j}(Q)}{\hat{C}_{{\rm UV},j}(R_{\rm UV})}\sum_{n=1}^{\infty}\tilde{a}_{j,n}^{-\tilde{p}}\,a_{\rm s}^n(R_{\rm UV}) - \ldots\quad .\end{aligned} \tag{4.8}$$

Note that we need to treat IR and UV renormalons separately and therefore have to introduce two additional momentum scales R and $R_{\rm UV}$. Setting $c = 2/\beta_0$ for later convenience and defining $\tilde{a}_n = a_n/4^n$ for the coefficients in (3.57) as well as $\overline{a}_{\rm s} = a_{\rm s}/4 = \alpha_{\rm s}/(4\pi)$, the MSR coefficient function $C_0(Q,R,R_{\rm UV})$ can be rewritten in the following form,

$$\begin{aligned}&C_0(Q,R,R_{\rm UV}) = \\ &= \overline{C}_0(Q) - \sum_j\left\{\frac{1}{Q^p}\left(\frac{\alpha_{\rm s}(Q)\,\beta_0}{2\pi}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j}\left[1+4\,\hat{c}_j^{(1)}\,\overline{a}_{\rm s}(Q)+\ldots\right]\right. \\ &\qquad\left.\times R^p\left(\frac{\alpha_{\rm s}(R)\,\beta_0}{2\pi}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j}\left[1+4\,\hat{c}_j^{(1)}\,\overline{a}_{\rm s}(R)+\ldots\right]^{-1}\sum_{n=1}^{\infty}a_{j,n}^p\,\overline{a}_{\rm s}^n(R)\right\} - \ldots \\ &\quad - \sum_j\left\{Q^{\tilde{p}}\left(\frac{\alpha_{\rm s}(Q)\,\beta_0}{2\pi}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j}\left[1+4\,\hat{c}_{{\rm UV},j}^{(1)}\,\overline{a}_{\rm s}(Q)+\ldots\right]\right. \\ &\qquad\left.\times\frac{1}{R_{\rm UV}^{\tilde{p}}}\left(\frac{\alpha_{\rm s}(R_{\rm UV})\,\beta_0}{2\pi}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j}\left[1+4\,\hat{c}_{{\rm UV},j}^{(1)}\,\overline{a}_{\rm s}(R_{\rm UV})+\ldots\right]^{-1}\sum_{n=1}^{\infty}a_{j,n}^{-\tilde{p}}\,\overline{a}_{\rm s}^n(R_{\rm UV})\right\} - \ldots\end{aligned} \tag{4.9}$$

[1]UV renormalons are Borel summable and therefore do not cause an ambiguity in the definition of the Borel integral (3.5) – at least formally. In practice, however, a UV renormalon may be as severe as an IR renormalon. So, in order to unify the discussion of IR and UV renormalons in this chapter, we will adopt the notion that UV renormalons at $u = \tilde{p}/2$ lead to ambiguities of $\mathcal{O}(\Lambda_{\rm UV}^{-\tilde{p}})$.

$$= \overline{C}_0(Q) + \sum_j \frac{1}{Q^p}\left(\frac{\alpha_s(Q)\,\beta_0}{2\pi}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j}\left[1+4\,\hat{c}_j^{(1)}\,\overline{a}_s(Q)+\ldots\right]\theta_{p,j}(R)+\ldots$$
$$+\sum_j Q^{\tilde{p}}\left(\frac{\alpha_s(Q)\,\beta_0}{2\pi}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0}+\hat{\delta}_j}\left[1+4\,\hat{c}_{\mathrm{UV},j}^{(1)}\,\overline{a}_s(Q)+\ldots\right]\theta^{\mathrm{UV}}_{-\tilde{p},j}(R_{\mathrm{UV}})+\ldots,$$

with the series $\theta_{p,j}(R)$ and $\theta_{-\tilde{p},j}(R_{\mathrm{UV}})$ defined as:

- $$\theta_{p,j}(R) = -R^p\left(\frac{\alpha_s(R)\,\beta_0}{2\pi}\right)^{-\frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j}\left[1+4\,\hat{c}_j^{(1)}\,\overline{a}_s(R)+\ldots\right]^{-1}\sum_{n=1}^{\infty}a^p_{j,n}\,\overline{a}_s^n(R) \tag{4.10}$$
- $$\theta^{\mathrm{UV}}_{-\tilde{p},j}(R_{\mathrm{UV}}) = -R_{\mathrm{UV}}^{-\tilde{p}}\left(\frac{\alpha_s(R_{\mathrm{UV}})\,\beta_0}{2\pi}\right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j}\left[1+4\,\hat{c}_j^{(1)}\,\overline{a}_s(R_{\mathrm{UV}})+\ldots\right]^{-1}\sum_{n=1}^{\infty}a^{-\tilde{p}}_{j,n}\,\overline{a}_s^n(R_{\mathrm{UV}}). \tag{4.11}$$

Remember that the perturbative series $\theta_{p,j}(R)$ and $\theta^{\mathrm{UV}}_{-\tilde{p},j}(R_{\mathrm{UV}})$ contain pure renormalon ambiguities of $\mathcal{O}(\Lambda^p_{\mathrm{QCD}})$ and $\mathcal{O}(\Lambda^{-\tilde{p}}_{\mathrm{UV}})$, respectively, which cancel the corresponding ambiguities in $\overline{C}_0(Q)$. Moreover, we emphasize that in order to consistently implement all information needed to remove the renormalon ambiguity at each order in the series expansion, we see from eq. (4.9) that the coefficients $\hat{c}_j^{(n)}$ have in principle to be available to one order less than the perturbative coefficients a_n of $\overline{C}_0(Q)$. The anomalous dimensions, on the other hand, need to be known to the same order as the coefficients a_n.

As one can see, the subtractions for both the IR and UV renormalons look very much alike and for the discussion in the subsequent sections it will be often convenient to consider a generic form for the subtraction terms,

$$\theta_{p,\alpha}(R) = -R^p\left(\frac{\alpha_s(R)\,\beta_0}{2\pi}\right)^{\alpha}\sum_{n=1}^{\infty}a^p_n\,\overline{a}_s^n(R), \tag{4.12}$$

where IR and UV renormalons are distinguished by means of the exponent α and the sign of p:

$$\alpha = \begin{cases} -\frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j, & \text{IR } (p>0) \\ \frac{2\gamma_{D,j}^{(1)}}{\beta_0}-\hat{\delta}_j, & \text{UV } (p=-\tilde{p}<0). \end{cases} \tag{4.13}$$

4.2 Renormalization Group Evolution in R – The R-Evolution

Remember that the scales R and R_{UV} in eq. (4.8) were introduced in order to remove the renormalons from the $\overline{\mathrm{MS}}$ perturbative series $\overline{C}_0(Q)$ and therefore give it a well-defined, i.e. unambiguous meaning. However, we have not yet talked about proper choices for the scales in the OPE (4.7) which are mainly constrained by the potential occurrence of large logarithms. First considering the IR-OPE terms, the low energy matrix elements $\langle\hat{O}_c^j(R)\rangle$ require $R \gtrsim \Lambda_{\mathrm{QCD}}$, that is a value close to the Landau pole but where perturbation theory is still valid. On the contrary, the perturbative calculation of the Wilson coefficient $C_0(Q,R)$ demands $R\sim Q$. Thus, we conclude that no choice for R avoids large logarithms in both, $C_0(Q,R)$ and the higher-dimensional matrix elements.

The very same is true for the UV renormalon correction terms in eq. (4.7). Here, the universal high-energy Wilson coefficients $\hat{C}^{\mathrm{UV}}_{c,j}(R_{\mathrm{UV}})$ require $R_{\mathrm{UV}}\sim\Lambda_{\mathrm{UV}}\gg Q$ and once again we cannot find appropriate values for R_{UV} that minimize the logarithms in $C_0(Q,R_{\mathrm{UV}})$ and $\hat{C}^{\mathrm{UV}}_{c,j}(R_{\mathrm{UV}})$ at the same time.

In the $\overline{\mathrm{MS}}$-OPE a similar problem arises already with respect to the renormalization scale μ and is avoided by performing an RGE in the scale μ which sums large logarithms between widely separated scales. In the case of the MSR-OPE we can use a similar approach and consider a renormalization group equation in the variable R. The corresponding RG evolution[2] connecting subtractions at different values for R is known as R-*evolution* [48].

4.2.1 R-Evolution Setup

In order to motivate the R-evolution equation for $C_0(Q, R, R_{\mathrm{UV}})$, we first consider the expression for the coefficient function at $R = R_{\mathrm{UV}} = Q$:

$$C_0(Q,Q,Q) = \overline{C}_0(Q) - \sum_j \sum_{n=1}^{\infty} a_{j,n}^{p}\, \overline{a}_{\mathrm{s}}^{n}(Q) - \cdots \sum_j \sum_{n=1}^{\infty} a_{j,n}^{-\tilde{p}}\, \overline{a}_{\mathrm{s}}^{n}(Q) - \ldots \quad . \tag{4.14}$$

Comparing this result to eq. (3.57), where the coefficients of $\overline{C}_0(Q)$ were split into contributions from the various renormalon poles, we see that $C_0(Q,Q,Q) = \sum_n a_{n,\mathrm{rem}}\, \overline{a}_{\mathrm{s}}^{n}(Q)$ describes a convergent series[3]. Thus, starting from this series we can now define an evolution in the scales R and R_{UV} in the following way,

$$\begin{aligned}
C_0(Q,\Lambda,\Lambda_{\mathrm{UV}}) &= C_0(Q,Q,Q) - \int_{\Lambda}^{Q} \mathrm{d}\ln R\, \frac{\mathrm{d}}{\mathrm{d}\ln R}\, C_0(Q,R,R_{\mathrm{UV}}) - \ldots \\
&\quad - \int_{\Lambda_{\mathrm{UV}}}^{Q} \mathrm{d}\ln R_{\mathrm{UV}}\, \frac{\mathrm{d}}{\mathrm{d}\ln R_{\mathrm{UV}}}\, C_0(Q,R,R_{\mathrm{UV}}) - \ldots, \\
&= C_0(Q,Q,Q) + \sum_j \left\{ \frac{1}{Q^p} \left(\frac{\alpha_{\mathrm{s}}(Q)\,\beta_0}{2\pi} \right)^{\frac{2\gamma_{D,j}^{(1)}}{\beta_0} + \hat{\delta}_j} \left[1 + 4\,\hat{c}_j^{(1)}\, \overline{a}_{\mathrm{s}}(Q) + \ldots\right] \int_{\Lambda}^{Q} \mathrm{d}\ln R\, \frac{\mathrm{d}}{\mathrm{d}\ln R}\, \theta_{p,j}(R) \right\} + \ldots \\
&\quad + \sum_j \left\{ Q^{\tilde{p}} \left(\frac{\alpha_{\mathrm{s}}(Q)\,\beta_0}{2\pi} \right)^{\frac{-2\gamma_{D,j}^{(1)}}{\beta_0} + \hat{\delta}_j} \left[1 + 4\,\hat{c}_{\mathrm{UV},j}^{(1)}\, \overline{a}_{\mathrm{s}}(Q) + \ldots\right] \int_{\Lambda_{\mathrm{UV}}}^{Q} \mathrm{d}\ln R_{\mathrm{UV}}\, \frac{\mathrm{d}}{\mathrm{d}\ln R_{\mathrm{UV}}}\, \theta_{-\tilde{p},j}^{\mathrm{UV}}(R_{\mathrm{UV}}) \right\} + \ldots,
\end{aligned} \tag{4.15}$$

where in the second equality we used the fact that $\overline{C}_0(Q)$ is independent of R. Note that the scale Λ must at least satisfy $\Lambda \gtrsim \Lambda_{\mathrm{QCD}}$ such that perturbation theory still converges and Λ_{UV} can be of order the scale where the more general underlying more UV-complete theory needs to be employed.

According to our previous definition of generic MSR subtraction terms in (4.12), the R-evolution equation for C_0 treating either a specific IR ($p > 0$) or UV ($p < 0$) renormalon then exhibits the general form:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\ln R}\, \theta_{p,\alpha}(R) &= -R^p \left(\frac{\alpha_{\mathrm{s}}(R)\,\beta_0}{2\pi} \right)^{\alpha} \gamma^R[\alpha_{\mathrm{s}}(R)] \\
&= -R^p \left(\frac{\alpha_{\mathrm{s}}(R)\,\beta_0}{2\pi} \right)^{\alpha} \sum_{n=0}^{\infty} \gamma_n^R\, \overline{a}_{\mathrm{s}}^{n+1}(R),
\end{aligned} \tag{4.16}$$

where the coefficients of the R-anomalous dimension $\gamma^R[\alpha_{\mathrm{s}}(R)]$ are given by (see Appendix C.3)

$$\gamma_n^R = p\, a_{n+1} - 2 \sum_{j=0}^{n-1} (\alpha + n - j)\, \beta_j\, a_{n-j}. \tag{4.17}$$

[2] An RGE for R was formulated in the context of heavy quark physics in [49, 50], its implications were discussed for the first time in [48].

[3] Strictly speaking, this statement is only true upon summing over all IR and UV renormalons on the RHS of eq. (4.14) (cf. (3.57)). Note that this sum is only implied in (4.14) and not written explicitly.

In the discussion of the subtraction terms (4.5) and (4.6) in section 4.1.1 we argued that the perturbative series $\theta_{p,\alpha}(R)$ contains a pure renormalon ambiguity of $\mathcal{O}(\Lambda_{\rm QCD}^p)$ or $\mathcal{O}(\Lambda_{\rm UV}^{-\tilde{p}})$, respectively, which drops out after taking the derivative in eq. (4.16). Thus, $\gamma^R[\alpha_{\rm s}(R)]$ is by construction free from the $u = p/2$ renormalon. This fact actually provides the most crucial feature of the R-evolution equation: Its solution sums at the same time the asymptotic renormalon series and large logarithms that arise in the difference of two subtractions at widely separated scales in a way avoiding the $\mathcal{O}(\Lambda_{\rm QCD/UV}^p)$ renormalon [51]. Due to its power law dependence on R, the solution of the R-evolution equation is fundamentally different from common RGEs which only have logarithmic scale dependence.

Up to now we have considered generic IR and UV subtraction terms in the definition of the MSR Wilson coefficient $C_0(Q, R, R_{\rm UV})$ in (4.9). In practice, usually only one or a few renormalons are relevant in explicit calculations. We therefore assume in the following, for simplicity, that the perturbative series in $\overline{C}_0(Q)$ is dominated by contributions of a single renormalon located at $u = p/2$ in the Borel plane. The generalization to several renormalons (either at the same location $u = p/2$ or at different locations) is straightforward.

4.2.2 Connection to the Borel Integral

There is a deep connection between the solution of the R-evolution equation and the Borel integral (3.5) which we want to briefly discuss here. To illustrate this relation it is instructive to consider the leading logarithmic (LL) solution of eq. (4.16) which is obtained from calculations in the large-β_0 approximation. In this approximation the perturbative series in the R-anomalous dimension reduces exactly to a single non-vanishing term $\gamma^R[\alpha_{\rm s}(R)] = \gamma_0^R\,\overline{a}_{\rm s}(R)$. Furthermore, assuming $\alpha = 0$ for simplicity[4] we obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}\ln R}\,\theta_{p,0}(R)\right]_{\rm LL} = -\gamma_0^R\,R^p\left(\frac{\alpha_{\rm s}(R)}{4\pi}\right), \tag{4.18}$$

and the solution of the R-evolution equation connecting subtractions at the two scales R_0 and R_1 is given by:

$$\big[\theta_{p,0}(R_1) - \theta_{p,0}(R_0)\big]_{\rm LL} = -\gamma_0^R \int\limits_{\ln R_0}^{\ln R_1} \mathrm{d}\ln R\; R^p\left(\frac{\alpha_{\rm s}(R)}{4\pi}\right). \tag{4.19}$$

Exploiting the QCD β-function at the LL level together with the LL relation for $\Lambda_{\rm QCD}$, that is $\Lambda_{\rm QCD}^{\rm LL} = R\exp[-2\pi/(\beta_0\alpha_{\rm s}(R))]$, the integration on the right-hand side can be expressed as an integral over $\alpha_{\rm s}(R)$:

$$\big[\theta_{p,0}(R_1) - \theta_{p,0}(R_0)\big]_{\rm LL} = \frac{\gamma_0^R}{2\,\beta_0} \int\limits_{\alpha_{\rm s}(R_0)}^{\alpha_{\rm s}(R_1)} \frac{\mathrm{d}\alpha_{\rm s}}{\alpha_{\rm s}}\,\left(\Lambda_{\rm QCD}^{\rm LL}\right)^p\,\mathrm{e}^{\frac{2\pi p}{\beta_0\alpha_{\rm s}}}. \tag{4.20}$$

Next, changing the integration variable to

$$t = -\frac{2\pi}{\alpha_{\rm s}\,\beta_0}, \tag{4.21}$$

we can further rewrite the integral,

$$\begin{aligned}\big[\theta_{p,0}(R_1) - \theta_{p,0}(R_0)\big]_{\rm LL} &= -\frac{\gamma_0^R}{2\,\beta_0}\left(\Lambda_{\rm QCD}^{\rm LL}\right)^p \int\limits_{t_0}^{t_1}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t\,p} \\ &= -\frac{\gamma_0^R}{2\,\beta_0}\left(\Lambda_{\rm QCD}^{\rm LL}\right)^p\left[\int_{t_0}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t\,p} - \int_{t_1}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t\,p}\right],\end{aligned} \tag{4.22}$$

[4] The value $\alpha = 0$ corresponds e.g. to an operator with vanishing leading order anomalous dimension, whose associated Wilson coefficient starts at order $\alpha_{\rm s}^0$ (cf. (4.13)).

where the integration limits are defined as $t_i = -2\pi/(\beta_0 \alpha_s(R_i))$. In the second line we split the integral into a difference of two terms for reasons that will become evident soon. Each of the two pieces is itself ill-defined, since for $R_{0,1} > \Lambda_{\rm QCD}$ we have $t_{0,1} < 0$ and one is forced to integrate over the Landau singularity at the point $t = 0$ (which encodes the information on the renormalon ambiguity) to the limit $t \to \infty$, which corresponds to $\alpha_s(\mu \to 0)$. However, these ambiguities cancel in the difference of the two integrals such that $\tilde{\theta}_{p,j}(R_1) - \tilde{\theta}_{p,j}(R_0)$ remains renormalon free.

In order to complete the connection to the Borel integral we finally switch to the Borel variable $u = -p(t/t_i - 1)/2$ and use the relation for $\Lambda_{\rm QCD}^{\rm LL}$ mentioned above in both integrals. A straightforward calculation yields:

$$\big[\theta_{p,0}(R_1) - \theta_{p,0}(R_0)\big]_{\rm LL} = \frac{\gamma_0^R}{2\,\beta_0} \int\limits_0^\infty {\rm d}u \left[R_1^p \left(\frac{\mu}{R_1}\right)^{2u} \frac{1}{u - \frac{p}{2}} - R_0^p \left(\frac{\mu}{R_0}\right)^{2u} \frac{1}{u - \frac{p}{2}} \right] {\rm e}^{-\frac{4\pi u}{\beta_0 \alpha_s(\mu)}}, \tag{4.23}$$

where the right-hand side represents the Borel integral over the difference of two Borel transforms of the $u = p/2$ renormalon contribution in the large-β_0 approximation.

Eq. (4.23) proves that R-evolution and the concept of Borel integration are closely related. Furthermore, R-evolution between two perturbative scales and Borel integration over the difference of Borel transforms sum the same logarithms. However, from a conceptual point of view one may argue ([51]) that R-evolution is more natural in the sense that it can be applied to any perturbative series without requiring additional approximations, while the Borel integral method relies on the knowledge of explicit expressions for the Borel transform which is often hard to achieve.

An important implication of the connection between R-evolution and the Borel integral sheds light on the first question raised at the end of chapter 2. We already know that IR renormalons in QCD inevitably lead to imaginary ambiguities in the definition of the Borel integral and the relation (4.23) suggests that these ambiguities may also be quantified by means of the R-evolution equation. Now consider one restores the ambiguity on the left-hand side of eq. (4.22) by taking the limit[5],

$$\lim_{R_0 \to 0} t_0 = \infty. \tag{4.24}$$

Upon changing to the Borel variable u, the second Borel transform on the right-hand side of eq. (4.23) vanishes in the limit[6] $R_0 \to 0$ and we are left with the usual Borel integral over a single Borel transform[7]. Using the relations (4.15) and (4.16) we thus obtain,

$$\overline{C}_0(Q) \simeq -\frac{1}{Q^p} \lim_{R_0 \to 0} \big[\theta_{p,0}(Q) - \theta_{p,0}(R_0)\big]_{\rm LL} = \int\limits_0^\infty {\rm d}u \, B[Q,\mu](u)\, {\rm e}^{-\frac{4\pi u}{\beta_0 \alpha_s(\mu)}}, \tag{4.25}$$

with

$$B[Q,\mu](u) = \frac{\gamma_0^R}{2\,\beta_0} \left(\frac{\mu}{Q}\right)^{2u} \frac{1}{u - \frac{p}{2}}. \tag{4.26}$$

The sign $\simeq$ in (4.25) implies that the right-hand side only contains the $u = p/2$ IR renormalon contribution of $\overline{C}_0$ as indicated in (4.14). An IR renormalon in the Borel transform obstructs the naive calculation of the integral and the resulting ambiguity can be related to the solution of the R-evolution equation in the limit $R_0 \to 0$. Note, however, that taking this limit is problematic since it pushes the perturbative running of the strong coupling $\alpha_s(R_0)$ into a momentum region

[5]Note that the limit in (4.24) is consistent with the one-loop running coupling $\alpha_s(R) = 2\pi/(\beta_0 \ln(R/\Lambda_{\rm QCD}))$. The limit is also valid when higher orders of the QCD β-function are included.

[6]The limit in R is related to a corresponding limit in u. Taking the limit $R_0 \to 0$ forces us to integrate over the Landau singularity at $t = 0$. Using the substitution $u = -p(t/t_i - 1)/2$ this translates to an integration over the renormalon pole at $u = p/2$ in the Borel integral.

[7]In this section we only discuss the connection between R-evolution and Borel integration in the large-β_0 approximation. In appendix C.6 we relate the general solution of the R-evolution equation (see section 4.3.1) to the Borel integral.

where it does not apply anymore. The ambiguity arises from integrating over the Landau pole at $R_0 = \Lambda_{\rm QCD}$. However, R-evolution requires $R_0 \gtrsim \Lambda_{\rm QCD}$ in order to stay within the perturbative regime. Thus, we can in principle use small variations in the lower limit R_0 close to, but larger than $\Lambda_{\rm QCD}$, to estimate the value of the Borel integral and quantify the size of its ambiguity. Typically one exploits the principal value prescription to evaluate the Borel integral and takes the imaginary part as an estimate for the ambiguity, but eq. (4.25) reveals that R-evolution provides an useful alternative to this procedure. We discuss this issue in much more detail in the next chapter where we investigate the implications of R-evolution on the Adler function in the large-β_0 approximation.

4.3 Renormalon Sum Rule

Another striking feature of the R-evolution equation is provided by the fact that its solution can be used to find an analytic expression for the Borel transform of any perturbative series which encodes exactly the information on a particular renormalon pole at $u = p/2$. Since this Borel transform only focuses on a single renormalon, the normalization of its singluar and non-analytic terms is characteristic for the "strength" of the $u = p/2$ renormalon. It turns out that the solution of the R-evolution equation provides an analytic expressions for this normalization in terms of the coefficients of the R-evolution equation (and thus of the original series). This analytic expression can be employed as a probe for the existence of renormalons in any perturbative series and is called the *renormalon sum rule*.

The sum rule was first studied for the $u = 1/2$ renormalon in the pole mass of heavy quarks in the letter [48] and details on the derivation can be found in [51]. A more general result for arbitrary IR poles at $u = p/2$ is given in [52], where the anomalous dimension of the operators in the OPE is not incorporated and the Wilson coefficients are assumed to start at tree-level. In this section we review the derivation of the renormalon sum rule for the IR case including the anomalous dimension and extend the discussion to also comprise UV renormalons.

4.3.1 General Solution of the R-Evolution Equation

Starting point for the derivation of an analytic expression for the Borel transform of the perturbative series $\theta_{p,\alpha}$ is the general solution of the R-evolution equation (4.16) which we want to discuss in the following. Integrating on both sides of the equation yields:

$$\begin{aligned}\theta_{p,\alpha}(R_1) - \theta_{p,\alpha}(R_0) &= -\int_{R_0}^{R_1} \mathrm{d}R\, R^{p-1}\, \gamma^R[\alpha_s(R)] \left(\frac{\alpha_{\rm s}(R)\,\beta_0}{2\pi}\right)^{\alpha} \\ &= \Lambda_{\rm QCD}^p \int_{t_0}^{t_1} \mathrm{d}t\, \gamma^R[t]\, \hat{b}(t)\, \mathrm{e}^{-G(t)\,p}\,(-t)^{-\alpha},\end{aligned} \tag{4.27}$$

where

$$\gamma^R[t] = \sum_{n=0}^{\infty} \gamma_n^R \left(-\frac{1}{2\beta_0\, t}\right)^{n+1}. \tag{4.28}$$

The function $G(t)$ is given in (C.5) in the appendix. In the second line of eq. (4.27) we switched to the integration variable $t = -2\pi/(\beta_0 \alpha_{\rm s})$ defined in (4.21) and used the result for $\Lambda_{\rm QCD}$ derived from the RGE for the strong coupling $\alpha_{\rm s}$ given in appendix C.1. In order to simplify the general solution we rewrite

$$\gamma^R[t]\,\hat{b}(t)\,\mathrm{e}^{-G(t)\,p} = \mathrm{e}^{-t\,p}\,(-t)^{-\hat{b}_1 p} \sum_{k=0}^{\infty} \frac{S_k^{(p)}}{(-t)^{k+1}}, \tag{4.29}$$

where explicit expressions for the first few coefficients $S_k^{(p)}$ can be found in (C.25) in appendix C.4. Next, inserting eq. (4.29) into eq. (4.27) we find,

$$\theta_{p,\alpha}(R_1) - \theta_{p,\alpha}(R_0) = \Lambda_{\rm QCD}^p \sum_{k=0}^{\infty} S_k^{(p)} \int_{t_0}^{t_1} \mathrm{d}t\,(-t)^{-\alpha-\hat{b}_1 p-k-1}\,\mathrm{e}^{-tp} \tag{4.30}$$

$$= \Lambda_{\rm QCD}^p \sum_{k=0}^{\infty} S_k^{(p)}\, p^{\alpha+\hat b_1 p+k} \int_{t_0\,p}^{t_1\,p} {\rm d}t\,(-t)^{-\alpha-\hat b_1 p-k-1}\,{\rm e}^{-t}.$$

Finally, splitting the integral above in the same fashion as we did in eq. (4.22), we can exploit the definition of the incomplete gamma function,

$$\Gamma(s,t) = \int_t^{\infty} {\rm d}x\, x^{s-1}\,{\rm e}^{-x}, \tag{4.31}$$

along with the standard phase convention $(-1) = {\rm e}^{-i\pi}$ to obtain the following all-order expression for the solution of the R-evolution equation:

$$\begin{aligned}&\theta_{p,\alpha}(R_1) - \theta_{p,\alpha}(R_0) = \\ &= \Lambda_{\rm QCD}^p \sum_{k=0}^{\infty} S_k^{(p)}\, \frac{{\rm e}^{i\pi(\hat b_1 p+k+\alpha)}}{p^{-k-\hat b_1 p-\alpha}} \Big\{\Gamma(-\hat b_1 p - k - \alpha,\, t_1\,p) - \Gamma(-\hat b_1 p - k - \alpha,\, t_0\,p)\Big\}.\end{aligned} \tag{4.32}$$

Before we proceed with the derivation of the renormalon sum rule, let us take a closer look at the analytic properties of the result (4.32). Returning to the integral in (4.30) and recalling the definition of the substitution variable, $t = -2\pi/(\beta_0\alpha_{\rm s}(R))$, we have that $t = 0$ for $R = \Lambda_{\rm QCD}$ and $t_{0,1} < 0$ for $R_{0,1} > \Lambda_{\rm QCD}$. Since the integrand has a singularity at the point $t = 0$, the integral is well-defined as long as the integration range lies outside the singular region and therefore the difference $\theta_{p,\alpha}(R_1)-\theta_{p,\alpha}(R_0)$ is renormalon free provided that $R_{0,1} > \Lambda_{\rm QCD}$. In eq. (4.32), however, we rewrote the integral in terms of the difference of two incomplete gamma functions $\Gamma(s,t)$, each having a cut for $t < 0$. The ambiguities these cuts[8] are causing cancel for each k in the sum and, thus, the difference $\theta_{p,\alpha}(R_1) - \theta_{p,\alpha}(R_0)$ remains free of the $u = p/2$ renormalon.

4.3.2 Analytic Borel Transform and Sum Rule for IR Renormalons

We first discuss the derivation of the sum rule for IR renormalons and deal with the sum rule for the UV case in the next section. In order to obtain an analytic expression for the Borel transform carrying the information on an $\mathcal{O}(\Lambda_{\rm QCD}^p)$ renormalon we need to restore the renormalon ambiguity in the solution of the R-evolution equation by again taking the limit,

$$\lim_{R_0\to 0} t_0 = \infty, \tag{4.33}$$

in (4.32). In this limiting procedure the second incomplete gamma function in eq. (4.32) vanishes. Using the relations (4.15) and (4.16) we thus obtain,

$$\overline{C}_0(Q) \simeq -\frac{1}{Q^p}\,(-t_{\rm Q})^{\alpha}\,\big[1 + 4\,\hat c^{(1)}\,\overline{a}_{\rm s}(Q) + 16\,\hat c^{(2)}\,\overline{a}_{\rm s}^2(Q) + \ldots\big]\,\big[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big], \tag{4.34}$$

with

$$\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0) = \Lambda_{\rm QCD}^p \sum_{k=0}^{\infty} S_k^{(p)}\, \frac{{\rm e}^{i\pi(\hat b_1 p+k+\alpha)}}{p^{-k-\hat b_1 p-\alpha}}\, \Gamma(-\hat b_1 p - k - \alpha,\, t_{\rm Q}\,p), \tag{4.35}$$

and

$$t_{\rm Q} = -\frac{2\pi}{\beta_0\,\alpha_{\rm s}(Q)}. \tag{4.36}$$

The sign $\simeq$ in (4.34) again implies that the right-hand side only contains the $u = p/2$ IR renormalon contribution of $\overline{C}_0$ as indicated in (4.14) with α given by (4.13). The ambiguity contained in the

[8]Note that the incomplete gamma functions in eq. (4.32) only have a cut when dealing with IR renormalons, since $p > 0$ and $t_{0,1}p < 0$ in this case. In the derivation of the general solution for UV renormalons ($p < 0$ and $t_{0,1}p > 0$), the integral in eq. (4.30) is split into two well-defined pieces such that the resulting gamma functions do not exhibit any cuts. This actually reflects the fact that UV renormalons are Borel summable and do not lead to ambiguities (compare eq. (4.25)).

perturbative series in $\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)$ is now visible through the cut in the remaining incomplete gamma functions for $t_{\rm Q} p < 0$. As one can see, the sum in k in the all-order result (4.35) provides systematically a reordering of the terms in the series $\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)$ related to the $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon ambiguity into leading and subleading contributions. This important feature represents the reason why the solution of the R-evolution equation allows one to find an analytic expression for the Borel transform focusing on a particular renormalon pole.

We proceed with an asymptotic expansion of the incomplete gamma function according to

$$\Gamma(s,t) \overset{t\to\infty}{=} \mathrm{e}^{-t}\, t^{s-1} \sum_{m=0}^{\infty} \frac{\Gamma(1-s+m)}{(-t)^m\, \Gamma(1-s)}, \tag{4.37}$$

which together with eq. (C.9) yields[9]:

$$\begin{aligned} &\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0) = \\ &\quad = -\, Q^p\, \mathrm{e}^{G(t_{\rm Q})p}\, \mathrm{e}^{-t_{\rm Q} p}\, (-t_{\rm Q})^{-\hat b_1 p} \sum_{k,m=0}^{\infty} S_k^{(p)} \frac{\Gamma(1+\hat b_1 p + k + \alpha + m)}{\Gamma(1+\hat b_1 p + k+\alpha)} (-t_{\rm Q})^{-1-m-k-\alpha}\, p^{-1-m}. \end{aligned} \tag{4.38}$$

Using the definition (see appendix C.4)

$$\mathrm{e}^{G(t)p}\, \mathrm{e}^{-tp}\, (-t)^{-\hat b_1 p} =: \sum_{l=0}^{\infty} g_l^{(p)}\, (-t)^{-l} \tag{4.39}$$

we can further rewrite this result in the following form:

$$\begin{aligned} &\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0) = \\ &\quad = -Q^p \sum_{l=0}^{\infty} g_l^{(p)} \sum_{k=0}^{\infty} S_k^{(p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+\hat b_1 p + k + \alpha + m)}{\Gamma(1+\hat b_1 p + k + \alpha)} (-t_{\rm Q})^{-1-m-k-\alpha-l}\, p^{-1-m}. \end{aligned} \tag{4.40}$$

To obtain the Borel transform related to the $u = p/2$ renormalon (and the corresponding α value) one must convert eq. (4.34) into the Borel space. The leading term (i.e. neglecting the $4\,\hat c^{(1)}\,\overline{a}_{\rm s}(Q)$ and $16\,\hat c^{(2)}\,\overline{a}_{\rm s}^2(Q)$ terms in (4.34)) has the form:

$$\begin{aligned} &-\frac{1}{Q^p}\,(-t_{\rm Q})^{\alpha}\big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big) = \\ &\quad = \sum_{l=0}^{\infty} g_l^{(p)} \sum_{k=0}^{\infty} S_k^{(p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+\hat b_1 p + k + \alpha + m)}{\Gamma(1+\hat b_1 p + k + \alpha)} (-t_{\rm Q})^{-1-m-k-l}\, p^{-1-m}. \end{aligned} \tag{4.41}$$

Performing the Borel transform of this series amounts to the substitution $(-t_{\rm Q})^{-n-1} \to 2\,(2u)^n/\Gamma(n+1)$. This replacement emerges from our definition of the (inverse) Borel transform which leads to the rule:

$$\left(\frac{\alpha_{\rm s}\beta_0}{4\pi}\right)^{n+1} \leftrightarrow \frac{u^n}{\Gamma(n+1)}. \tag{4.42}$$

The Borel transform of eq. (4.41) is therefore given by

$$\begin{aligned} B_0(u) &\equiv B\Big[-\frac{1}{Q^p}\,(-t_{\rm Q})^{\alpha}\big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big)\Big](u) = \\ &= 2 \sum_{k,m,l=0}^{\infty} S_k^{(p)}\, g_l^{(p)}\, \frac{\Gamma(1+\hat b_1 p + k + \alpha + m)}{\Gamma(1+\hat b_1 p + k + \alpha)\,\Gamma(1+m+k+l)}\, (2u)^{m+k+l}\, p^{-1-m} \\ &= 2 \sum_{k,l=0}^{\infty} S_k^{(p)}\, g_l^{(p)}\, \frac{{}_2F_1(1, k+\hat b_1 p + \alpha + 1; 1+k+l; \frac{2u}{p})}{p\,\Gamma(1+k+l)}\, (2u)^{k+l}, \end{aligned} \tag{4.43}$$

[9] Eq. (4.37) represents an asymptotic expansion and therefore the imaginary part originating from the cut in the incomplete gamma function must not occur on the right-hand side of eq. (4.38).

where in the last line we used

$$\sum_{m=0}^{\infty} p^{-1-m}\,\frac{\Gamma(1+\hat{b}_1 p+k+\alpha+m)}{\Gamma(1+\hat{b}_1 p+k+\alpha)\,\Gamma(1+m+k+l)}\,(2u)^{m+k+l} = \tag{4.44}$$
$$= \frac{{}_2F_1(1,k+\hat{b}_1 p+\alpha+1;1+k+l;\frac{2u}{p})}{p\,\Gamma(1+k+l)}\,(2u)^{k+l}.$$

Finally exploiting the following relation for hypergeometric functions

$${}_2F_1(1,a;b;z) = \frac{b-1}{b-a-1}\,{}_2F_1(1,a;2+a-b;1-z) + \frac{\Gamma(b)\,\Gamma(1+a-b)\,z^{1-b}}{\Gamma(a)\,(1-z)^{1+a-b}}, \tag{4.45}$$

the Borel transform can be written in the form:

$$B_0(u) = P^{\alpha}_{p/2}\sum_{l=0}^{\infty} g_l^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}}\,2^{-\hat{b}_1 p-\alpha+l} + \sum_{l=0}^{\infty} g_l^{(p)}\,Q_l^{(p)}(u), \tag{4.46}$$

with the normalization

$$P^{\alpha}_{p/2} = \sum_{k=0}^{\infty}\frac{S_k^{(p)}\,p^{k+\hat{b}_1 p+\alpha}}{\Gamma(1+k+\hat{b}_1 p+\alpha)}, \tag{4.47}$$

and

$$Q_l^{(p)}(u) = 2\sum_{k=0}^{\infty} S_k^{(p)}\,\frac{{}_2F_1(1,1+k+\hat{b}_1 p+\alpha;2+\hat{b}_1 p+\alpha-l;1-\frac{2u}{p})}{p\,(l-\hat{b}_1 p-\alpha-1)\,\Gamma(k+l)}\,(2u)^{k+l} \tag{4.48}$$
$$= -2\sum_{k=0}^{\infty} S_k^{(p)}\sum_{j=0}^{l+k-1}\frac{\Gamma(1+j+\alpha+\hat{b}_1 p-l)\,p^{l+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p+k)\,\Gamma(j+1)}\,(2u)^{j}.$$

The Borel transform (4.46) is divided into two different parts. The first one represents the singular terms which are related to the $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon ambiguity. The ambiguity caused by these non-analytic contributions can be easily checked by calculating the imaginary part of the corresponding Borel integral. Using

$$B_0^{\rm sing}(u) \equiv B_0(u)\big|_{\rm singular} = P^{\alpha,\rm IR}_{p/2}\sum_{l=0}^{\infty} g_l^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}}\,2^{-\hat{b}_1 p-\alpha+l}, \tag{4.49}$$

the ambiguity is given by:

$$\mathrm{Im}\left[\int_0^{\infty}\mathrm{d}u\,\mathrm{e}^{2t_{\rm Q}u}\,B_0^{\rm sing}(u)\right] = \tag{4.50}$$
$$= \mathrm{Im}\left[P^{\alpha}_{p/2}\sum_{l=0}^{\infty} g_l^{(p)}\,\Gamma(1+\hat{b}_1 p+\alpha-l)\,2^{-\hat{b}_1 p-\alpha+l}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2t_Q u}}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}}\right] =$$
$$= -\pi\,P^{\alpha}_{p/2}\,\mathrm{e}^{pt_Q}\,(-t_Q)^{\hat{b}_1 p+\alpha}\sum_{l=0}^{\infty} g_l^{(p)}\,(-t_Q)^{-l} = -\pi\,P^{\alpha}_{p/2}\,(-t_{\rm Q})^{\alpha}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)^{p}.$$

The details of the calculation have been relegated to appendix C.5. As one can see, the ambiguity is consistent with (4.34) and the starting equations (3.56) where $\langle (2/\beta_0)^{\hat{\delta}_j}\hat{O}_c^j\rangle$ contains a pure $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon ambiguity. Since the functions $S_k^{(p)}$ in the normalization $P^{\alpha}_{p/2}$ involve the coefficients γ_k^R of the R-anomalous dimension which in turn contain the perturbatively computable coefficients, it is possible to apply eq. (4.47) to any perturbative series as a probe for a $u=p/2$ renormalon and therefore the normalization $P^{\alpha}_{p/2}$ was called the *renormalon sum rule* by the authors in [48]. (In [48] only the case $p=1$, $\alpha=0$ was treated.)

An important issue in connection with the analytic Borel transform (4.46) concerns the convergence of the renormalon sum rule. As we are going to discuss in section 5.2, $P^{\alpha}_{p/2}$ only converges, if the variables p and α correspond to the strongest renormalon contained in the perturbative series. The dominant renormalon is given by smallest p (i.e. renormalon is located closest to the origin of the Borel plane) and largest α value.

The physical significance of the second term in the Borel transform (4.46) can be explained by looking at eqs. (4.43) and (4.44). In (4.44) we performed the sum over m and then used the relation (4.45) to split the Borel transform into a singular part $B_0^{\rm sing}(u)$ containing the renormalon sum rule and another term involving the function $Q_l^{(p)}$ in (4.48). In contrast to eq. (4.44), which represents a series in u starting at order u^{k+l}, $B_0^{\rm sing}(u)$ generates a series already starting at $\mathcal{O}(1)$. Thus, the singular part of the Borel transform encodes additional information that is not contained in the original perturbative series. It turns out that the second part of the Borel transform[10] (4.46) precisely removes these terms, i.e. it represents a series that terminates beyond order u^{k+l-1} (see also appendix C.6).

Eq. (4.46) is quite remarkable, since it demonstrates that starting from a perturbative series that is supposed to suffer from an $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon ambiguity, one can use the general solution of the R-evolution equation to derive an analytic expression for the Borel transform of this series that does not rely on any simplifying model, like e.g. the calculation of fermion bubble diagrams. For this reason the renormalon sum rule provides a very useful tool to gain information on specific renormalons and allows us to tackle the questions raised at the end of chapter 2. In order to illustrate its capability to probe (the dominant) renormalons in perturbative series, we apply the sum rule to the Adler function in the large-β_0 approximation in the next chapter.

Before we proceed, a few more comments are in order. First, we emphasize that eq. (4.46) does not represent the exact Borel transform, since it only encodes the information on a single renormalon pole located at $u = p/2$ in the complex Borel plane. In order to test for other poles we need to study the solutions of the R-evolution equation with different powers p of R and different parameters[11] α. Remember that the parameter α involves the leading order anomalous dimension of a specific operator in the OPE and the leading power in $\alpha_{\rm s}$ of the corresponding Wilson coefficient. Thus, changing its value corresponds to calculating the contribution associated with another operator of dimension $d = p$. Furthermore, we usually only know very few perturbative coefficients for most realistic cases, that is eq. (4.47) represents the all-order result and applying the truncated series,

$$P^{\alpha,(\mathrm{N}^n\mathrm{LL})}_{p/2} = \sum_{k=0}^{n} \frac{S_k^{(p)}\, p^{k+\hat{b}_1 p+\alpha}}{\Gamma(1+k+\hat{b}_1 p+\alpha)}, \tag{4.51}$$

provides an estimate with an uncertainty. From a mathematical point of view renormalons are connected to asymptotic high-order behaviour and strict proofs are very rare. However, the more coefficients of a perturbative series are known, the more precise becomes the renormalon sum rule.

To complete the discussion of the sum rule for IR renormalons, let us briefly check that the singular part of the analytic Borel transform (4.46) is consistent with the expression (2.27) in section 2.2.2 obtained previously by the authors in [14]. Our main concern is to improve the form of the Borel transform used in physical models for the Adler function and therefore it should be possible to reproduce the results given in section 2.2.2. For this purpose we rearrange the singular terms in our analytic Borel transform in the following way:

$$B_0^{\rm sing}(u) = P^{\alpha}_{p/2} \sum_{l=0}^{\infty} g_l^{(p)}\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}}\, 2^{-\hat{b}_1 p-\alpha+l} \tag{4.52}$$

[10]Note that the second line in (4.48) indeed represents the series expansion of $B_0^{\rm sing}(u)$ up to $\mathcal{O}(u^{k+l-1})$.

[11]We stress that it is possible to have various renormalon poles of different order located at the same position in the Borel plane. For example the Borel transform of the Adler function in the large-β_0 approximation contains an infinite series of simple and double IR renormalons singularities for positive integer values $u \geq 3$ (see section 3.2). We stress again that the renormalon sum rule $P^{\alpha}_{p/2}$ can only be used to probe a single renormalon whose pole structure is characterized by a specific parameter α.

$$= P^{\alpha}_{p/2}\, 2^{-\hat{b}_1 p-\alpha}\, \Gamma(1+\hat{b}_1 p+\alpha) \sum_{l=0}^{\infty} \frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\Gamma(1+\hat{b}_1 p+\alpha)} \, \frac{2^l\, g^{(p)}_l}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}}$$
$$= \frac{N^{\alpha}_{p/2}}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha}} \left[1 + \frac{2\, g^{(p)}_1}{(\hat{b}_1 p+\alpha)} \left(\frac{p}{2}-u\right) + \frac{4\, g^{(p)}_2}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)} \left(\frac{p}{2}-u\right)^2 + \ldots \right],$$

where $N^{\alpha}_{p/2}$ denotes an alternative normalization of the singular terms given by:

$$N^{\alpha}_{p/2} = P^{\alpha}_{p/2}\, 2^{-\hat{b}_1 p-\alpha}\, \Gamma(1+\hat{b}_1 p+\alpha). \tag{4.53}$$

Comparing eq. (4.52) with the corresponding expression (2.27), one immediately deduces[12],

$$\tilde{\gamma} = \hat{b}_1 p+\alpha = p\, \frac{\beta_1}{2\beta_0^2} - \frac{2\gamma^{(1)}_{D,j}}{\beta_0} - \hat{\delta}_j, \tag{4.54}$$

which is consistent with the findings in section 2.2.2 up to the additional factor $\hat{\delta}_j$ that describes the leading power in $\alpha_{\rm s}$ of the Wilson coefficient for the operator O_j. (In [14] it was assumed that $\hat{\delta}_j = 0$.) Exploiting the explicit expressions for the functions $\hat{b}_k$ and $g^{(p)}_l$ given in appendix C.1 and C.4, we obtain

- $$\frac{2\, g^p_1}{(\hat{b}_1 p+\alpha)} = \frac{p}{2\beta_0^4\, \tilde{\gamma}} \left(\beta_1^2 - \beta_0\, \beta_2\right) = \frac{4\, b_1}{\beta_0 \tilde{\gamma}}, \tag{4.55}$$
- $$\frac{4\, g^p_2}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)} = \frac{4}{\tilde{\gamma}\,(\tilde{\gamma}-1)}\, \frac{p}{2} \left(\hat{b}_2^2 p - \hat{b}_3\right)$$
$$= \frac{16}{\tilde{\gamma}\,(\tilde{\gamma}-1)\,\beta_0^2} \left[\frac{1}{2} \left(\frac{p}{8\beta_0^3}\left(\beta_1^2 - \beta_0\, \beta_2\right)\right)^2 - \frac{p}{64\beta_0^4}\left(\beta_1^3 - 2\,\beta_0\,\beta_1\,\beta_2 + \beta_0^2\,\beta_3\right)\right] = \frac{16\, b_2}{\tilde{\gamma}\,(\tilde{\gamma}-1)\,\beta_0^2},$$

which agrees with the first terms of the coefficients $\tilde{b}_1$ and $\tilde{b}_2$ in the Borel transform (2.27) (with the Wilson coefficient correction terms c_1 and c_2 set to zero). In order to reproduce the remaining parts of $\tilde{b}_1$ and $\tilde{b}_2$ we need to consider the subleading parts in (4.34) involving the term $4\,\hat{c}^{(1)}\,\overline{a}_{\rm s}(Q)$ and beyond. Using the solution of the R-evolution equation given in eq. (4.40), the series we have to deal with exhibit the generic form:

$$-\frac{1}{Q^p}(-t_{\rm Q})^{\alpha}\, \overline{a}^n_{\rm s}(Q)\, \big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big) = -\frac{(2\beta_0)^{-n}}{Q^p}\, (-t_{\rm Q})^{\alpha-n} \big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big) = \tag{4.56}$$
$$= (2\beta_0)^{-n} \sum_{l=0}^{\infty} g^{(p)}_l \sum_{k=0}^{\infty} S^{(p)}_k \sum_{m=0}^{\infty} \frac{\Gamma(1+\hat{b}_1 p+k+\alpha+m)}{\Gamma(1+\hat{b}_1 p+k+\alpha)}\, \frac{(-t_{\rm Q})^{-1-m-k-l-n}}{p^{-1-m}}.$$

Except for the prefactor $(2\beta_0)^{-n}$ the above expression is identical with eq. (4.52) if one substitutes $l+n \to l$. Thus, upon Borel transforming the series in (4.56) and repeating the manipulations we performed in the derivation of the analytic Borel transform (4.46), the Borel transform for the subleading terms can be written as:

$$B_n(u) \equiv B\Big[-\frac{1}{Q^p}\,(2\beta_0)^{-n}\,(-t_{\rm Q})^{\alpha-n}\,\big(\theta_{p,\alpha}(Q)-\theta_{p,\alpha}(0)\big)\Big](u) = \tag{4.57}$$
$$= (2\beta_0)^{-n}\, P^{\alpha}_{p/2} \sum_{l=0}^{\infty} g^{(p)}_l\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l-n}}\, 2^{-\hat{b}_1 p-\alpha+l+n} + (2\beta_0)^{-n} \sum_{l=0}^{\infty} g^{(p)}_l\, Q^{(p)}_{l,n},$$

[12] We emphasize that the Borel transform in section 2.2.2 has the form $B(u) \propto \frac{1}{(p-u)^{1+\tilde{\gamma}}}\left[1+\mathcal{O}(p-u)\right]$, while the analytic expression for the Borel transform derived in this chapter refers to a slightly different notation, $B^{\rm sing}(u) \propto \frac{1}{(p/2-u)^{1+\tilde{\gamma}}}\left[1+\mathcal{O}(p/2-u)\right]$. Hence, we need to substitute $p/2 \to p$ in order to relate the results of this section to those in 2.2.2.

with

$$Q_{l,n}^{(p)}(u) = -2\sum_{k=0}^{\infty} S_k^{(p)} \sum_{j=0}^{l+n+k-1} \frac{\Gamma(1+j+\alpha+\hat{b}_1 p - l - n)\, p^{l+n+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p + k)\,\Gamma(j+1)}\,(2u)^j. \tag{4.58}$$

We can once again check the ambiguity associated with the non-analytic terms contained in (4.57) by calculating the imaginary part of the inverse Borel transformation. Defining

$$B_n^{\rm sing}(u) \equiv B_n(u)\big|_{\rm singular} = (2\beta_0)^{-n}\, P_{p/2}^{\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\, 2^{-\hat{b}_1 p-\alpha+l+n}, \tag{4.59}$$

we find:

$$\begin{aligned}
&{\rm Im}\bigg[\int_0^\infty {\rm d}u\, {\rm e}^{2t_R u}\, B_n^{\rm sing}(u)\bigg] = \qquad\qquad (4.60)\\
&= {\rm Im}\bigg[(2\beta_0)^{-n}\, P_{p/2}^{\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{2^{-\hat{b}_1 p-\alpha+l+n}} \int_0^\infty {\rm d}u\, \frac{{\rm e}^{2t_R u}}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\bigg]\\
&= -\pi\, P_{p/2}^{\alpha}\, (2\beta_0)^{-n}\, {\rm e}^{p t_Q}\, (-t_Q)^{\hat{b}_1 p+\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\, (-t_Q)^{-l-n} = -\pi\, P_{p/2}^{\alpha}\, (-t_Q)^{\alpha}\, \overline{a}_s^n(Q) \left(\frac{\Lambda_{\rm QCD}}{Q}\right)^p,
\end{aligned}$$

which is again consistent with (3.56).

For the purpose of comparing eq. (4.57) with the corresponding expression given in (2.27), it is again helpful to rewrite the singular Borel transform using the alternative normalization $N_{p/2}^{\alpha}$ defined in (4.53):

$$\begin{aligned}
B_n^{\rm sing}(u) &= (2\beta_0)^{-n}\, P_{p/2}^{\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\, 2^{-\hat{b}_1 p-\alpha+l+n} \qquad (4.61)\\
&= P_{p/2}^{\alpha}\, 2^{-\hat{b}_1 p-\alpha}\, \Gamma(1+\hat{b}_1 p+\alpha) \sum_{l=0}^{\infty} \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\Gamma(1+\hat{b}_1 p+\alpha)}\, \frac{2^l\, \beta_0^{-n}\, g_l^{(p)}}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\\
&= N_{p/2}^{\alpha} \sum_{l=0}^{\infty} 2^l\, \beta_0^{-n}\, g_l^{(p)}\, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\Gamma(1+\hat{b}_1 p+\alpha)}\, \frac{1}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}.
\end{aligned}$$

For the first subleading contribution (including the coefficient $\hat{c}_j^{(1)}$) we therefore have,

$$\begin{aligned}
4\,\hat{c}_j^{(1)}\, B_1^{\rm sing}(u) &= 4\,\hat{c}_j^{(1)} N_{p/2}^{\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\, \frac{2^l}{\beta_0}\, \frac{\Gamma(\hat{b}_1 p+\alpha-l)}{\Gamma(1+\hat{b}_1 p+\alpha)}\, \frac{1}{\big(\frac{p}{2}-u\big)^{\hat{b}_1 p+\alpha-l}} \qquad (4.62)\\
&= 4\,\hat{c}_j^{(1)}\, \frac{N_{p/2}^{\alpha}}{\big(\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha}} \bigg[\frac{1}{\beta_0}\, \frac{1}{(\hat{b}_1 p+\alpha)} \left(\frac{p}{2}-u\right)\\
&\qquad + \frac{2}{\beta_0}\, \frac{g_1^{(p)}}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)} \left(\frac{p}{2}-u\right)^2 + \ldots\bigg],
\end{aligned}$$

where the terms in the square brackets are indeed found to agree with the corresponding terms in the coefficients $\tilde{b}_1$ and $\tilde{b}_2$ of (2.27):

- $\displaystyle \frac{1}{\beta_0}\, \frac{4\,\hat{c}_j^{(1)}}{(\hat{b}_1 p+\alpha)} = \frac{4\,\hat{c}_j^{(1)}}{\beta_0\,\tilde{\gamma}},$ (4.63)
- $\displaystyle \frac{2}{\beta_0}\, \frac{4\,\hat{c}_j^{(1)}\, g_1^{(p)}}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)} = \frac{2}{\beta_0}\, \frac{4\,\hat{c}_j^{(1)}\, \hat{b}_2\, p}{\tilde{\gamma}\,(\tilde{\gamma}-1)} = \frac{2\,\hat{c}_j^{(1)}\, p}{\beta_0^5\, \tilde{\gamma}\,(\tilde{\gamma}-1)}\, \big(\beta_1^2 - \beta_0\,\beta_2\big) = \frac{16\,\hat{c}_j^{(1)}\, b_1}{\beta_0^2\, \tilde{\gamma}\,(\tilde{\gamma}-1)}.$

The next subleading contribution is given by:

$$16\,\hat{c}_j^{(2)}\,B_2^{\rm sing}(u) = -R^p\,16\,\hat{c}_j^{(2)} N_{p/2}^{\alpha} \sum_{l=0}^{\infty} g_l^{(p)}\,\frac{2^l}{\beta_0^2}\,\frac{\Gamma(\hat{b}_1 p+\alpha-l-1)}{\Gamma(1+\hat{b}_1 p+\alpha)}\,\frac{1}{\left(\frac{p}{2}-u\right)^{\hat{b}_1 p+\alpha-l-1}}$$
$$= -R^p\,16\,\hat{c}_j^{(2)}\,\frac{N_{p/2}^{\alpha}}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha}}\left[\frac{1}{\beta_0^2}\,\frac{1}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)}\left(\frac{p}{2}-u\right)^2+\dots\right], \tag{4.64}$$

with,

$$\frac{1}{\beta_0^2}\,\frac{16\,\hat{c}_j^{(2)}}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)} = \frac{16\,\hat{c}_j^{(2)}}{\beta_0^2\,\tilde{\gamma}\,(\tilde{\gamma}-1)}. \tag{4.65}$$

As one can see, the expression for the analytic Borel transform derived from the general solution of the R-evolution equation is fully consistent with the one in section 2.2.2. However, the Borel transform (2.27) is based on the fact that the renormalisation group can be used to connect the pole structure of renormalon singularities to the leading order anomalous dimensions of the associated operators in the OPE. This allows one to determine the position and strength of renormalon poles, while the overall norm (i.e. the "weight" of this renormalon) cannot be predicted.

Our explicit derivation of the Borel transform from the solution of the R-evolution equation resolves this problem as it explicitly derives the norm as well. It is just the term $N_{p/2}^{\alpha}$ (or $P_{p/2}^{\alpha}$, respectively) and can be computed directly from the renormalon sum rule which, as shown above, is equivalent to the term $d_p^{\rm IR}$ in (2.27). Hence, R-evolution can, in principle, be applied in order to calculate the norm of renormalon singularities (and non-analytic terms) and allows us to gain knowledge about the Borel transform which is unaccessible using common RG methods.

4.3.3 Analytic Borel Transform and Sum Rule for UV Renormalons

We now turn to the discussion of the renormalon sum rule for the UV case. Even though UV renormalons are Borel summable and thus do not cause any ambiguities at the formal level, we can still use R-evolution to find an analytic sum rule which can be used as a probe for UV renormalons in perturbative series. Furthermore, taking the derivative with respect to $R_{\rm UV}$ in eq. (4.11) removes the sign-alternating divergent behaviour, leading to a renormalon-free R-evolution equation. The subsequent derivation is very similar to the one in the previous section for the IR case, except for the fact that we now have $p=-\tilde{p}<0$, since UV renormalons in QCD are located on the negative real Borel axis. In addition we also have to adjust the parameter α according to eq. (4.13).

To obtain an analytic Borel transform encoding the information on a specific UV renormalon pole we once again need to restore the renormalon in the general solution of the R-evolution equation. This time, however, we must consider the limit $R_0\to\infty$ which implies $t_0\to-\infty$. Using the relations (4.15) and (4.16) we thus obtain[13]

$$\overline{C}_0(Q) \simeq -Q^{\tilde{p}}\,(-t_Q)^{\alpha}\left[1+4\,\hat{c}_{\rm UV}^{(1)}\,\overline{a}_{\rm s}(Q)+16\,\hat{c}_{\rm UV}^{(2)}\,\overline{a}_{\rm s}^2(Q)+\dots\right]\left[\theta_{p,\alpha}(Q)-\theta_{p,\alpha}(\infty)\right], \tag{4.66}$$

where,

$$\theta_{p,\alpha}(Q)-\theta_{p,\alpha}(\infty) = \frac{1}{\Lambda_{\rm QCD}^{\tilde{p}}}\sum_{k=0}^{\infty} S_k^{(p)}\int_{-\infty}^{t_Q}{\rm d}t\,(-t)^{-\alpha-\hat{b}_1 p-k-1}\,{\rm e}^{-tp} \tag{4.67}$$
$$= \frac{1}{\Lambda_{\rm QCD}^{\tilde{p}}}\sum_{k=0}^{\infty} S_k^{(p)}\int_{t'_{\rm Q}}^{\infty}{\rm d}t'\,(t')^{-\alpha-\hat{b}_1 p-k-1}\,{\rm e}^{-(-p)t'}$$
$$= \frac{1}{\Lambda_{\rm QCD}^{\tilde{p}}}\sum_{k=0}^{\infty} S_k^{(p)}\,(-p)^{\hat{b}_1 p+k+\alpha}\int_{-pt'_{\rm Q}}^{\infty}{\rm d}t\,{\rm e}^{-t}\,t^{-\alpha-\hat{b}_1 p-k-1}.$$

[13]Since both variables p and $\tilde{p}$ will be used in the following derivation we stress again that $p=-\tilde{p}<0$ for UV renormalons.

The formula for the terms $S_k^{(p)}$ is the same as for IR renormalons and is given in eq. (C.25). In the second line we changed to the integration variable $t' = -t$. Rewriting the result above in terms of incomplete gamma functions, we obtain,

$$\begin{aligned}\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(\infty) &= \frac{1}{\Lambda_{\rm QCD}^{\tilde{p}}} \sum_{k=0}^{\infty} S_k^{(p)}\,(-p)^{\hat{b}_1 p+k+\alpha}\,\Gamma(-\hat{b}_1 p - \alpha - k, -p\,t'_{\rm Q}) \\ &= \frac{1}{\Lambda_{\rm QCD}^{\tilde{p}}} \sum_{k=0}^{\infty} S_k^{(p)}\,(-p)^{\hat{b}_1 p+k+\alpha}\,\Gamma(-\hat{b}_1 p - \alpha - k, p\,t_{\rm Q}),\end{aligned} \tag{4.68}$$

which is analogous to the expression in eq. (4.32). Note that the incomplete gamma functions in the sum over k do not exhibit any cuts, since $t_{\rm Q} p > 0$. This reflects the fact that UV renormalons do not cause ambiguities in the definition of the Borel integral. Next, we asymptotically expand the incomplete gamma function according to (4.37) and exploit eq. (C.9). This yields:

$$\begin{aligned}&\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(\infty) = \\ &= -\frac{1}{Q^{\tilde{p}}}\,{\rm e}^{G(t_Q)p}\,{\rm e}^{-t_Q p}\,(-t_Q)^{-\hat{b}_1 p} \sum_{k,m=0}^{\infty} S_k^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p + k + \alpha + m)}{\Gamma(1+\hat{b}_1 p + k + \alpha)}\,(-t_{\rm Q})^{-1-m-k-\alpha}\,p^{-1-m}.\end{aligned} \tag{4.69}$$

Finally using eq. (4.39) for the coefficients $g_l^{(p)}$ the solution of the R-evolution equation can be cast into the form:

$$\begin{aligned}&\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(\infty) = \\ &= -\frac{1}{Q^{\tilde{p}}} \sum_{l=0}^{\infty} g_l^{(p)} \sum_{k=0}^{\infty} S_k^{(p)} \sum_{m=0}^{\infty} \frac{\Gamma(1+\hat{b}_1 p + k + \alpha + m)}{\Gamma(1+\hat{b}_1 p + k + \alpha)}\,(-t_{\rm Q})^{-1-m-k-\alpha-l}\,p^{-1-m}.\end{aligned} \tag{4.70}$$

The further steps follow exactly the derivation of the IR renormalon sum rule in the previous section. First, taking into account the factor $(-t)^{-\alpha}$ for the leading term (4.12), we can apply the substitution rule $(-t_1)^{-n-1} \to 2(2u)^n/\Gamma(n+1)$ to perform the Borel transform of the perturbative series $-Q^{\tilde{p}}(-t_{\rm Q})^{\alpha}\big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(\infty)\big)$:

$$\begin{aligned}B_0^{\rm UV} &= B\big[-Q^{\tilde{p}}\,(-t_{\rm Q})^{\alpha}\big(\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(\infty)\big)\big](u) = \\ &= 2\sum_{k,m,l=0}^{\infty} S_k^{(p)}\,g_l^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p + k + \alpha + m)}{\Gamma(1+\hat{b}_1 p + k + \alpha)\,\Gamma(1+m+k+l)}\,(2u)^{m+k+l}\,p^{-1-m}.\end{aligned} \tag{4.71}$$

Exploiting the relations (4.44) and (4.45) the analytic Borel transform can be written as:

$$B_0^{\rm UV}(u) = P_{p/2}^{\alpha,{\rm UV}} \sum_{l=0}^{\infty} (-1)^l\,g_l^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p + \alpha - l)}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha-l}}\,2^{-\hat{b}_1 p - \alpha + l} + \sum_{l=0}^{\infty} g_l^{(p)}\,Q_l^{(p),{\rm UV}}(u), \tag{4.72}$$

where the renormalon sum rule for the UV renormalons is now given by:

$$P_{p/2}^{\alpha,{\rm UV}} = \sum_{k=0}^{\infty} \frac{S_k^{(p)}\,(-1)^{k+1}\,(-p)^{k+\hat{b}_1 p+\alpha}}{\Gamma(1+k+\hat{b}_1 p+\alpha)}. \tag{4.73}$$

Except for the sign of p and the additional sign-alternating factor $(-1)^{k+1}$ the UV renormalon sum rule coincides with the corresponding expression for IR renormalons given by eq. (4.47).

The functions $Q_l^{(p),{\rm UV}}$ in the second part of the Borel transform (4.72) are found to be:

$$Q_l^{(p),{\rm UV}}(u) = -2\sum_{k=0}^{\infty} S_k^{(p)} \sum_{j=0}^{l+k-1} (-1)^{l+k-j-1}\,\frac{\Gamma(1+j+\alpha+\hat{b}_1 p - l)\,(-p)^{l+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p + k)\,\Gamma(j+1)}\,(2u)^j. \tag{4.74}$$

Similar to the IR case these terms remove the additional information encoded in the singular part of the Borel transform (4.72) that is not contained in the original perturbative series.

Analogous to the consistency check we did in the previous section it is also important in the UV case to test whether the singular part of eq. (4.72) is in agreement with the Borel transform (2.31) for UV renormalons given in section 2.2.2. To that end it is again useful to rewrite the singular terms in the analytic Borel transform,

$$\begin{aligned} B_0^{\mathrm{UV,sing}}(u) &= P_{p/2}^{\alpha,\mathrm{UV}} \sum_{l=0}^{\infty} (-1)^l \, g_l^{(p)} \, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha-l}} \, 2^{-\hat{b}_1 p-\alpha+l} \\ &= N_{p/2}^{\alpha,\mathrm{UV}} \sum_{l=0}^{\infty} (-2)^l \, g_l^{(p)} \, \frac{\Gamma(1+\hat{b}_1 p+\alpha-l)}{\Gamma(1+\hat{b}_1 p+\alpha)} \, \frac{1}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha-l}} \\ &= \frac{N_{p/2,\,\mathrm{UV}}^{\alpha}}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha}} \left[1 - \frac{2\,g_1^{(p)}}{(\hat{b}_1 p+\alpha)}\Big(-\frac{p}{2}+u\Big) + \frac{4\,g_2^{(p)}}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)}\Big(-\frac{p}{2}+u\Big)^2 + \ldots\right] \\ &= \frac{N_{p/2,\,\mathrm{UV}}^{\alpha}}{\big(\frac{\tilde{p}}{2}+u\big)^{1-\hat{b}_1 \tilde{p}+\alpha}} \left[1 - \frac{2\,g_1^{(p)}}{(-\hat{b}_1 \tilde{p}+\alpha)}\Big(\frac{\tilde{p}}{2}+u\Big) + \frac{4\,g_2^{(p)}}{(-\hat{b}_1 \tilde{p}+\alpha)\,(-\hat{b}_1 \tilde{p}+\alpha-1)}\Big(\frac{\tilde{p}}{2}+u\Big)^2 + \ldots\right], \end{aligned} \tag{4.75}$$

where the alternative normalization $N_{p/2}^{\alpha,\mathrm{UV}}$ is given by:

$$N_{p/2}^{\alpha,\mathrm{UV}} = P_{p/2}^{\alpha,\mathrm{UV}} \, 2^{-\hat{b}_1 p-\alpha} \, \Gamma(1+\hat{b}_1 p+\alpha). \tag{4.76}$$

The last line in eq. (4.75) (where we used $\tilde{p} = -p$) matches exactly the alternative form of the Borel transform (4.52) for IR renormalons derived in the previous section except for the alternating sign behaviour of the terms in the square brackets due to the factor $(-1)^l$. The same distinction between IR and UV renormalons was found in section 2.2.2 which proves the consistency of eq. (4.75) and the results given there.

The subleading contributions containing information on the Wilson coefficients of the associated operators take the form,

$$\begin{aligned} B_n^{\mathrm{UV}}(u) &\equiv B\big[-Q^{\tilde{p}}\,(-t_{\mathrm{Q}})^{\alpha}\,\overline{a}_{\mathrm{s}}^{n}(Q)\,\big(\theta_{p,\alpha}(Q)-\theta_{p,\alpha}(\infty)\big)\big](u) = \\ &= (2\beta_0)^{-n}\,P_{p/2}^{\alpha}\sum_{l=0}^{\infty}(-1)^{l+n}\,g_l^{(p)}\,\frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\big(-\frac{p}{2}-u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\,2^{-\hat{b}_1 p-\alpha+l+n} + (2\beta_0)^{-n}\sum_{l=0}^{\infty} g_l^{(p)}\,Q_{l,n}^{(p),\mathrm{UV}}, \end{aligned} \tag{4.77}$$

where:

$$\begin{aligned} Q_{l,n}^{(p),\mathrm{UV}}(u) &= \\ &= -2\sum_{k=0}^{\infty} S_k^{(p)} \sum_{j=0}^{l+n+k-1} (-1)^{l+n+k-j-1}\,\frac{\Gamma(1+j+\alpha+\hat{b}_1 p-l-n)\,(-p)^{l+n+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p+k)\,\Gamma(j+1)}\,(2u)^j. \end{aligned} \tag{4.78}$$

The singular terms in (4.77) are given by,

$$\begin{aligned} B_n^{\mathrm{UV,sing}}(u) &\equiv B_n^{\mathrm{UV}}(u)\big|_{\mathrm{singular}} = \\ &= (2\beta_0)^{-n}\,P_{p/2}^{\alpha,\mathrm{UV}}\sum_{l=0}^{\infty} g_l^{(p)}\,(-1)^{l+n}\,\frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha-l-n}}\,2^{-\hat{b}_1 p-\alpha+l+n} \\ &= N_{p/2}^{\alpha,\mathrm{UV}}\sum_{l=0}^{\infty} 2^l\,\beta_0^{-n}\,g_l^{(p)}\,(-1)^{l+n}\,\frac{\Gamma(1+\hat{b}_1 p+\alpha-l-n)}{\Gamma(1+\hat{b}_1 p+\alpha)}\,\frac{1}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha-l-n}}. \end{aligned} \tag{4.79}$$

and also agree with the findings in section 2.2.2:

- $$4\,\hat{c}_j^{(1)}\,B_1^{\mathrm{UV,sing}}(u) = N_{p/2}^{\alpha,\mathrm{UV}}\sum_{l=0}^{\infty} g_l^{(p)}\,\frac{2^l}{\beta_0}\,\frac{\Gamma(\hat{b}_1 p+\alpha-l)}{\Gamma(1+\hat{b}_1 p+\alpha)}\,\frac{(-1)^{l+1}\,4\,\hat{c}_j^{(1)}}{\big(-\frac{p}{2}+u\big)^{\hat{b}_1 p+\alpha-l}} \tag{4.80}$$

$$= 4\,\hat{c}_j^{(1)}\,\frac{N_{p/2}^{\alpha,\mathrm{UV}}}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha}}\bigg[-\frac{1}{\beta_0}\,\frac{1}{(\hat{b}_1 p+\alpha)}\Big(-\frac{p}{2}+u\Big) + \frac{2}{\beta_0}\,\frac{g_1^{(p)}}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)}\Big(-\frac{p}{2}+u\Big)^2+\ldots\bigg],$$

- $$16\,\hat{c}_j^{(2)}\,B_2^{\mathrm{UV,sing}}(u) = N_{p/2}^{\alpha,\mathrm{UV}}\sum_{l=0}^{\infty} g_l^{(p)}\,\frac{(-2)^l}{\beta_0^2}\,\frac{\Gamma(\hat{b}_1 p+\alpha-l-1)}{\Gamma(1+\hat{b}_1 p+\alpha)}\,\frac{16\,\hat{c}_j^{(2)}}{\big(-\frac{p}{2}+u\big)^{\hat{b}_1 p+\alpha-l-1}}$$

$$= 16\,\hat{c}_j^{(2)}\,\frac{N_{p/2}^{\alpha,\mathrm{UV}}}{\big(-\frac{p}{2}+u\big)^{1+\hat{b}_1 p+\alpha}}\bigg[\frac{1}{\beta_0^2}\,\frac{1}{(\hat{b}_1 p+\alpha)\,(\hat{b}_1 p+\alpha-1)}\Big(-\frac{p}{2}+u\Big)^2+\ldots\bigg].$$

The consistency check completes our introduction to the concepts of R-evolution and the renormalon sum rule. As a practical application we use the methods developed here to analyze the Adler function in the large-β_0 approximation in the following chapter. In particular, we will show that the renormalon sum rule can indeed be employed as a powerful tool to probe renormalon poles. In appendix D, we additionally verify that the pole structure of our analytic expression for the Borel transform agrees with the structure of the dimension-4 gluon condensate and the dimension-6 four quark operator corrections found in the literature. Furthermore we also demonstrate how the leading order results for the coefficient functions of the gluon and four quark condensates can be obtained from asymptotic expansions of the diagrams in Fig. 3.1 in the context of the *expansion-by-region* method.

Chapter 5

Applications of R-Evolution in the Large-β_0 Approximation

So far we have seen how switching to appropriate schemes for IR matrix elements $\langle \hat{O}_c^j \rangle$ and UV Wilson coefficients $\hat{C}_{c,j}^{\mathrm{UV}}$ (see eqs. (4.5) and (4.6)) that account for explicit renormalon subtractions involving additional scales R and R_{UV}, respectively, can avoid large cancellations between $\overline{C}_0$ and higher-dimensional terms in the OPE for a physical observable $\sigma(Q)$. Furthermore we discussed the RG evolution for the scales R and R_{UV} and looked at important features of the R-evolution equation that allow us to address the questions raised in chapter 2. In this chapter we study the Adler function in the large-β_0 approximation in order to investigate the applications of R-evolution in more detail. More specifically, we illustrate that the relation (4.25) can be used as an alternative method to estimate IR renormalon ambiguities and demonstrate how the renormalon sum rule (see eq. (4.47)) can be employed to probe specific renormalons in the perturbative expansion of the Adler function. The reason why we choose to work in the large-β_0 approximation is simply given by the fact that in this approximation a closed analytic expression for the Borel transform of the Adler function is known which allows us to test very easily to which extent the methods based on R-evolution are consistent with the known exact results.

5.1 Adler Function in the Large-β_0 Approximation

Before we deal with the applications of R-evolution, let us briefly review the most important results for the Adler function in the large-β_0 approximation which we already discussed in section 3.2. In order to be consistent with the notation used in the previous chapter, we express the Adler function in the following way[1]

$$D(p^2) = 4\pi^2\, p^2\, \frac{\mathrm{d}\Pi(p^2)}{\mathrm{d}p^2} = 1 + \widehat{D}(p^2) = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{\alpha_{\mathrm{s}}}{4\pi}\right)^n. \tag{5.1}$$

Computing the set of bubble chain diagrams in Fig. 3.1 an analytic result for its Borel transform can be found [28]:

$$B[\widehat{D}](u)\Big|_{\mu^2=Q^2} = \frac{32}{3\pi}\,\frac{\mathrm{e}^{-Cu}}{(2-u)}\sum_{k=2}^{\infty}\frac{(-1)^k\, k}{[k^2-(1-u)^2]^2}, \tag{5.2}$$

where $Q^2 = -p^2$ and C denotes a scheme-dependent constant which takes the value $C = -5/3$ in the $\overline{\mathrm{MS}}$ scheme. Using this convention for the Borel transform the reduced Adler function $\widehat{D}$ is given by the inverse Borel transformation (3.5),

$$\widehat{D}(\alpha_{\mathrm{s}}) = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-\frac{t}{\alpha_{\mathrm{s}}^C}}\, B[\widehat{D}](t) = \frac{4\pi}{\beta_0}\int_0^{\infty}\mathrm{d}u\, \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_{\mathrm{s}}^C}}\, B[\widehat{D}](u), \tag{5.3}$$

[1] Note that we used a different notation for the Adler function series in chapter 2. The expansion coefficients $c_{n,1}$ (see eq. (2.12) in section 2.1) are related to the coefficients a_n in (5.1) via $c_{n,1} = a_n/4^n$.

n	a_n	$u=-1$	$u=2$	$u=-2$
1	4	1.1752	112.13	-0.0436
2	24.904	-13.599	504.57	0.2319
3	1005.5	299.18	4541.1	-2.408
4	6356.9	-9546.5	$6.1305 \cdot 10^4$	36.842
5	$8.0674 \cdot 10^5$	$3.9655 \cdot 10^5$	$1.1035 \cdot 10^6$	-741.18
6	$-8.1567 \cdot 10^6$	$-2.0224 \cdot 10^7$	$2.4829 \cdot 10^7$	18432
7	$1.6150 \cdot 10^9$	$1.2206 \cdot 10^9$	$6.7037 \cdot 10^8$	$-5.4506 \cdot 10^5$
8	$-7.0623 \cdot 10^{10}$	$-8.4997 \cdot 10^{10}$	$2.1117 \cdot 10^{10}$	$1.8662 \cdot 10^7$

Table 5.1: Perturbative coefficients of the Adler function in the large β_0 approximation. The results have been calculated in the $\overline{\text{MS}}$-scheme for $\mu^2 = -p^2$ and $N_f = 3$ light quark flavours. The last three columns show the breakdown of a_n into the contributions a_n^p from the leading renormalon poles (see eq. (5.6)). The results for the UV renormalons contain the combined contributions from simple and double poles.

where the modified Borel variable $u \equiv t\beta_0/(4\pi)$ is used in the second equality and α_s^C denotes the strong coupling in the scheme specified by the constant C. The relation between α_s^C for arbitrary C and the $\overline{\text{MS}}$ coupling $\overline{\alpha}_s \equiv \alpha_s^{C=-5/3}$ is given by:

$$\alpha_s^C = \frac{\overline{\alpha}_s}{1 - \frac{\overline{\alpha}_s \beta 0}{4\pi}\left(\frac{5}{3} + C\right)}. \tag{5.4}$$

According to eq. (5.2) the Borel transform of the Adler function consists of an infinite series of renormalon poles located at positive (IR) and negative (UV) integer values of u, except for $u = 0, 1$. We already know that the $u = 1$ renormalon is absent, since it would correspond to a dimension-2 operator in the OPE of the Adler function. The singularity at $u = 0$, on the other hand, is only present in the Borel transform of the RG-scale dependent correlation function $\Pi(Q^2)$ (see [34]) and vanishes in the case of the Adler function by taking the derivative with respect to Q^2 in (5.1). All UV and IR renormalons are double poles apart from the IR pole at $u = 2$ which is simple due to the vanishing anomalous dimension of the gluon condensate in the large-β_0 approximation.

The perturbative coefficients of the Adler function in the large-β_0 approximation can be obtained from Taylor expanding the Borel transform in the variable u and subsequently calculating the Borel integral (5.3) term by term. Numerical values for the first few coefficients in the $\overline{\text{MS}}$-scheme are shown in the first column of Table 5.1. One can observe that the sign-alternating asymptotic behaviour caused by the dominant $u = -1$ UV renormalon is delayed and sets in not until order $n \sim 6$. Low and intermediate orders are governed by contributions from IR renormalon poles, which are enhanced in the $\overline{\text{MS}}$-scheme due to the exponential factor $e^{5/3\,u}$. However, if we instead choose to work in other schemes (e.g. $C = 0$), we observe an earlier onset of the asymptotic behaviour.

In order to analyze in more detail how fast the asymptotic regime is reached it is sometimes useful to separate the Borel transform (5.2) into contributions associated with individual renormalon poles (see appendix C.7). Focusing on the leading renormalons the decomposition of the Borel transform in the $\overline{\text{MS}}$-scheme is given by:

$$\begin{aligned} B[\widehat{D}](u)\Big|_{\mu^2=Q^2} &= \frac{e^{-5/3}}{\pi}\left(\frac{4}{9}\frac{1}{(1+u)^2} + \frac{10}{9}\frac{1}{(1+u)}\right) + \frac{e^{10/3}}{\pi}\frac{2}{(2-u)} \\ &\quad - \frac{e^{-10/3}}{\pi}\left(\frac{2}{9}\frac{1}{(2+u)^2} + \frac{1}{2}\frac{1}{(2+u)}\right) + \ldots \quad . \end{aligned} \tag{5.5}$$

The terms shown in the decomposition above arise from an expansion of the Borel transform (5.2) with $C = -5/3$ about the locations of the individual renormalons at $u = p/2$. The first contribution, for instance, represents the two singular terms in the series expansion at $u = -1$. In the following we refer to these singular terms as the contributions of the double and simple pole, respectively, of the UV renormalon at $u = -1$. All other (non-singular) terms of the expansion at $u = -1$ are only indicated and not shown explicitly in (5.5). In a similar fashion, the second and the third term in (5.5) give the singular contributions of the renormalons at $u = 2$ and $u = -2$.

Taylor expanding each term in (5.5) and performing the Borel integral (5.3) we obtain the contributions of the individual renormalons to the series coefficients a_n. This way we can find a decomposition of the coefficients a_n similar to eq. (3.57):

$$a_n = \sum_{P_{\rm IR}} a_n^{P_{\rm IR}} + \sum_{P_{\rm UV}} a_n^{P_{\rm UV}} + a_{n,{\rm rem}}, \tag{5.6}$$

where the first sum contains all singular contributions due to IR renormalons, while the second sum accounts for UV poles. The last term represents a remainder which comprises all non-singular contributions in (5.5). Numerical values for the contributions a_n^p of the individual renormalons to the perturbative coefficients a_n are shown in Table 5.1 and explicit expressions for the decomposition in (5.6) (for the scheme $C = 0$) can be found in appendix C.7.

5.2 Renormalon Sum Rule as a Probe for Renormalon Ambiguities

The first feature of R-evolution we want to address here is the renormalon sum rule derived in section 4.3. As already explained, the renormalon sum rule denotes the normalization of the singular terms in the analytic Borel transform (4.46) that are characteristic for a particular renormalon and is hence suited as a probe for renormalon ambiguities in perturbative series.

In general one can distinguish three different cases when the sum rule is applied to test for the existence of a renormalon at $u = p/2$ with specific pole structure characterized by the parameter α (see eq. (4.13)):

1. $P^{\alpha}_{p/2}$ converges to zero as we go to higher orders in perturbation theory. In this case the corresponding $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon (characterized by the values of p and α) is absent but higher order renormalons further away from the origin in the Borel plane might still be present. Moreover it is also possible that another renormalon located at the same position $u = p/2$ exists which is given by a smaller variable $\alpha' < \alpha$.

2. $P^{\alpha}_{p/2}$ converges to a non-zero value. In this case we conclude that an $\mathcal{O}(\Lambda^p_{\rm QCD})$ renormalon ambiguity related to the value of α exists in the series.

3. $P^{\alpha}_{p/2}$ diverges which either suggests that an ambiguity of $\mathcal{O}(\Lambda^q_{\rm QCD})$ with $q < p$ is present (i.e. in the Borel transform there is a renormalon pole closer to the origin of the Borel plane) or indicates the existence of a second more dominant renormalon singularity located at the same position with another pole structure specified by a parameter $\alpha' > \alpha$ that leads to a faster factorial growth in the perturbative coefficients. (Compare e.g. the contributions of the double and the simple pole at $u = -1$ in eq. (3.16).)

In order to elaborate on the capability of the sum rule for probing renormalon ambiguities we apply it to the Adler function as an illustrative example in this section.

5.2.1 Analytic Borel Transform and Renormalon Sum Rule in the Large-β_0 Approximation

Before we study the renormalon structure of the Adler function with the methods of R-evolution, it is instructive to investigate the analytic Borel transform and the renormalon sum rule in the context of the large-β_0 approximation. Since this approximation results from taking the large-N_f limit and then replacing $N_f \to -3/2\,\beta_0$, the coefficients $\hat{b}_k$ (see eq. (C.8)) are suppressed by powers of $1/N_f$ and consequently vanish. In addition to the $\hat{b}_k$ also the functions $g_l^{(p)}$, with the exception of $g_0^{(p)} = 1$, drop out and the analytic expression of the Borel transform (4.46) reduces to,

$$\begin{aligned}
B_{\beta_0}^{\mathrm{sing}}(u) &= P^{\alpha}_{p/2,\beta_0}\,\frac{\Gamma(1+\alpha)\,2^{-\alpha}}{\left(\frac{p}{2}-u\right)^{1+\alpha}} = \frac{N^{\alpha}_{p/2,\beta_0}}{\left(\frac{p}{2}-u\right)^{1+\alpha}},\\
N^{\alpha}_{p/2,\beta_0} &= P^{\alpha}_{p/2,\beta_0}\,\Gamma(1+\alpha)\,2^{-\alpha},\\
P^{\alpha}_{p/2,\beta_0} &= \sum_{k=0}^{\infty}\frac{S_k^{(p)}\,p^{k+\alpha}}{\Gamma(1+k+\alpha)},
\end{aligned} \tag{5.7}$$

for IR renormalons ($p > 0$) and,

$$\begin{aligned}
B_{\beta_0}^{\mathrm{sing}}(u) &= P^{\alpha,\mathrm{UV}}_{p/2,\beta_0}\,\frac{\Gamma(1+\alpha)\,2^{-\alpha}}{\left(-\frac{p}{2}+u\right)^{1+\alpha}} = \frac{N^{\alpha,\mathrm{UV}}_{p/2,\beta_0}}{\left(-\frac{p}{2}+u\right)^{1+\alpha}},\\
N^{\alpha,\mathrm{UV}}_{p/2,\beta_0} &= P^{\alpha,uv}_{p/2,\beta_0}\,\Gamma(1+\alpha)\,2^{-\alpha},\\
P^{\alpha,\mathrm{UV}}_{p/2,\beta_0} &= \sum_{k=0}^{\infty}\frac{S_k^{(p)}\,(-1)^{k+1}\,(-p)^{k+\alpha}}{\Gamma(1+k+\alpha)},
\end{aligned} \tag{5.8}$$

for UV renormalons ($p < 0$). In both cases the functions $S_k^{(p)}$ simplify to:

$$\begin{aligned}
S_k^{(p)} &= \frac{\gamma_k^R}{(2\beta_0)^{k+1}},\\
\gamma_k^R &= p\,a_{k+1} - 2\,(\alpha+k)\,\beta_0\,a_k.
\end{aligned} \tag{5.9}$$

Note that if the renormalon sum rule in the large-β_0 approximation is applied to probe the series of an exact renormalon, all coefficients of the R-anomalous dimension, except for γ_0^R, vanish exactly. Consequently, $P^{\alpha}_{p/2,\beta_0}$ reduces to a single (non-zero) term which, according to point 2 in the discussion at the beginning of section 5.2, indicates that an $\mathcal{O}(\Lambda^p_{\mathrm{QCD}})$ renormalon characterized by the value of α exists in the series.

Eqs. (5.7) and (5.8) give the all-order result for the normalization $N^{\alpha}_{p/2,\beta_0}$ of the renormalon singularity at $u = p/2$. The corresponding result needed for the order-by-order study of the convergence of the sum rule in the next section is given by[2]:

$$\begin{aligned}
N^{\alpha,(n)}_{p/2} &= P^{\alpha,(n)}_{p/2}\,\Gamma(1+\alpha)\,2^{-\alpha} = \Gamma(1+\alpha)\,2^{-\alpha}\sum_{k=0}^{n}\frac{S_k^{(p)}\,p^{k+\alpha}}{\Gamma(1+k+\alpha)},\\
N^{\alpha,(n)}_{p/2,\mathrm{UV}} &= P^{\alpha,(n)}_{p/2,\mathrm{UV}}\,\Gamma(1+\alpha)\,2^{-\alpha} = \Gamma(1+\alpha)\,2^{-\alpha}\sum_{k=0}^{n}\frac{S_k^{(p)}\,(-1)^{k+1}\,(-p)^{k+\alpha}}{\Gamma(1+k+\alpha)}.
\end{aligned} \tag{5.10}$$

As we see, the analytic Borel transform carrying the information on a specific renormalon has a much simpler form in the large-β_0 approximation and reduces to a single term whose pole structure is determined by the parameter α defined in (4.13). In the case of the Adler function α can take on the values zero (simple pole) or one (double pole).

5.2.2 Probing the Leading Renormalon Divergence in the Adler Function

We already know that the Borel transform of the Adler function in the large-β_0 approximation consists of an infinite series of simple and double poles at integer values of u (expect for $u = 0, 1$) and from the extensive analysis in chapter 3 we deduced that the most severe renormalon is the one closest to the origin of the Borel plane. Moreover we have seen (cf. eq. (3.16)) that the factorial growth at high orders in the perturbative coefficients of a series $\sum_n a_n(\alpha_s/(4\pi))^{n+1}$ caused

[2] Since we only work in the large-β_0 approximation in this chapter, the subscript β_0 will be dropped.

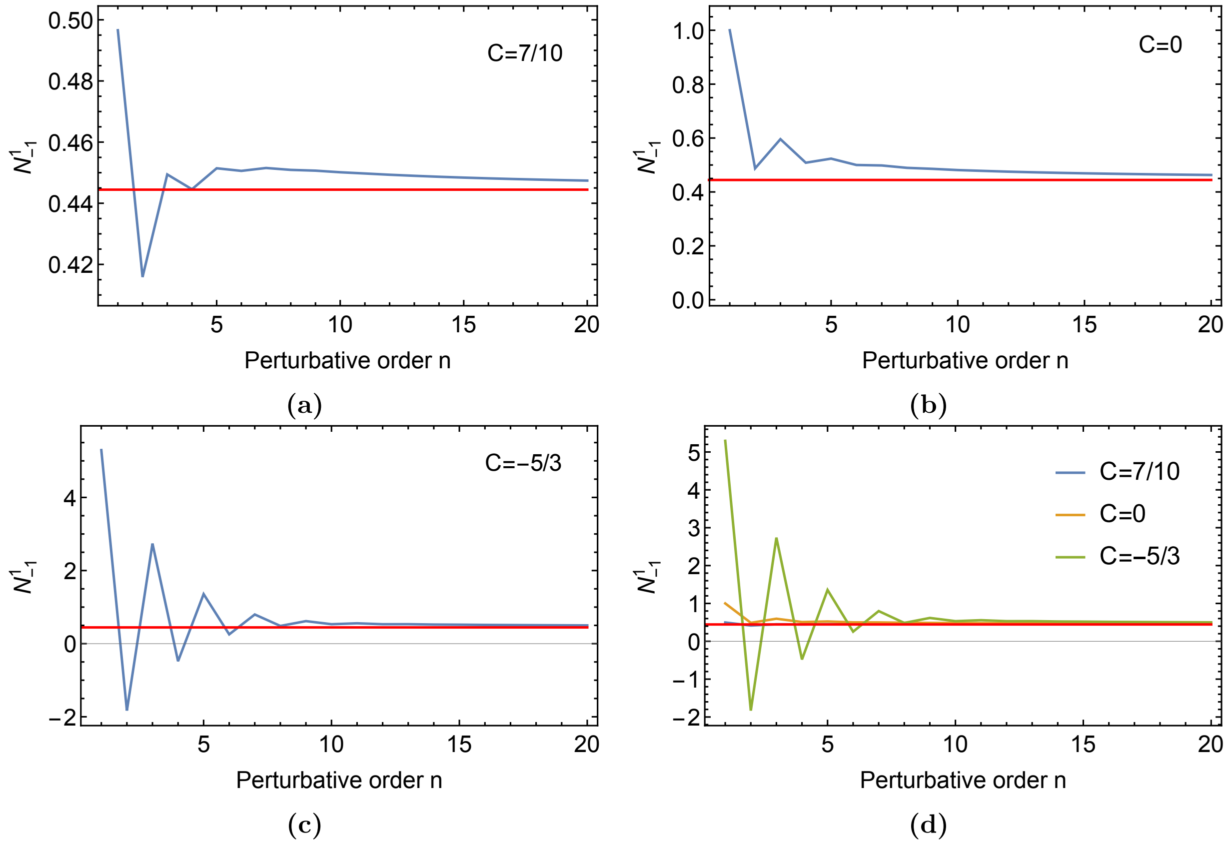

Fig. 5.1: Application of the renormalon sum rule to probe the double pole at $u = -1$ in the Adler function series. Graphs (a)-(c) show the result of eq. (5.10) (blue curves) for the values $p = -2$ and $\alpha = 1$ as a function of the perturbative order n for the three schemes $C = 7/10$, $C = 0$ and $C = -5/3$ ($\overline{\text{MS}}$-scheme). The red line denotes the exact value of the norm determined from the Borel transform (5.2). Graph (d) combines the plots (a)-(c) to illustrate the different convergence behaviour of the sum rule for the various schemes.

by a double pole at $u = p/2$ is of the type $a_n \sim (n+1)!\,(2\beta_0/p)^n$, whereas simple poles lead to $a_n \sim n!\,(2\beta_0/p)^n$. Thus the dominant renormalon divergence in the Adler function series is due to the double pole at $u = -1$ in the Borel transform.

For the purpose of probing the leading renormalon of the Adler function we simply need to generate its perturbative coefficients from the Borel transform (5.2) and insert them into the renormalon sum rule (5.8) with the choice $p = -2$ and $\alpha = 1$. A graphical representation of the results for the application of the renormalon sum rule is given in Fig. 5.1. The blue curves in the plots show the oder-by-order normalization $N^{1,(n)}_{-1,\text{UV}}$ for the $u = -1$ double pole obtained from eq. (5.10) as a function of the order n for three different schemes $C = 7/10$, $C = 0$ and $C = -5/3$ ($\overline{\text{MS}}$-scheme). The solid red line in each plot marks the exact value for the norm[3] of the renormalon pole determined from the known Borel transform expression given in eq. (5.2). In order to simplify the comparison of the outcomes for the different schemes the exponential normalization factor $\mathrm{e}^{-C\cdot(-1)}$ of the renormalon pole in the Borel transform (see e.g. (5.5)) has been removed in the final results. Thus, the norm of the $u = -1$ double pole takes on the same value in all plots.

The blue curves in Fig. 5.1 show that the renormalon sum rule indeed converges to the expected value of the norm as we go to higher orders in perturbation theory. This proves that the sum rule cannot only be employed to test whether a specific renormalon ambiguity is present, but

[3] We refer to norm of the $u = -1$ double pole as the over-all factor multiplying the singular structure $(1+u)^{-2}$ in the Borel transform. In this sense the norm of the double pole at $u = -1$ in the $\overline{\text{MS}}$ Borel transform (5.5) is e.g. given by $\mathrm{e}^{-5/3}\,4/(9\pi)$.

also allows one to determine the overall norm of the corresponding pole in the Borel transform at the same time. However, the convergence of the renormalon sum rule depends strongly on the scheme choice for the Adler function series. While the blue curve approaches the exact value rather fast within a small uncertainty band for schemes with a positive value for the constant C (Fig. 5.1a), we see that the convergence gets worse and sets in later as we go to schemes characterized by a negative value of C (Fig. 5.1b and Fig. 5.1c). The observed convergence behaviour of the renormalon sum rule is consistent with our findings in section 5.1. Since we are probing the leading UV renormalon at $u = -1$ the exponential factor $\mathrm{e}^{-C\cdot(-1)}$ multiplying the renormalon pole in the Borel transform (5.2) supports its dominance already at low orders for schemes with positive C, whereas schemes with negative C enhance IR renormalons such that the effects due to the leading UV renormalon are delayed. To illustrate the difference in the convergence for the different schemes, the plots of Fig. 5.1a - Fig. 5.1c have been combined to a single graph in Fig. 5.1d. In contrast to the $\overline{\mathrm{MS}}$-scheme ($C = -5/3$), where the exact value is reached reasonably well around order $n \sim 10$, the convergence in the scheme $C = 7/10$ sets in much earlier and the norm can already be predicted with the same accuracy around $n \sim 5$.

From our analysis one may suppose that it is possible to gradually improve the convergence of the sum rule by choosing schemes with ever larger positive values of C. This, however, turns out to be not the case, since increasing C also enhances all other UV renormalons at $u = p/2 < 0$ (and not just the leading one) due to the exponential factor e^{-Cu}. Therefore, in such schemes low and intermediate orders are governed by strongly enhanced UV renormalons far away from the origin of the Borel plane and consequently the dominant behaviour of the leading UV renormalon is only observed at higher orders. The renormalon sum rule for the $u = -1$ double pole shows the fastest convergence for the scheme with $C = 7/10$.

Once the norm of the double pole at $u = -1$ has been determined, the leading renormalon in the Adler function series can be removed from the Borel transform (5.2) and we can use the renormalon sum rule to test the next-to-leading renormalon, which corresponds to the simple pole at $u = -1$. For this purpose we first need to generate the perturbative coefficients associated with the $u = -1$ double pole from the analytic Borel transform expression,

$$B^{\mathrm{sing}}(u) = \frac{N_{-1}^{1,\mathrm{UV}}}{(1+u)^2}, \tag{5.11}$$

and subtract them from the coefficients a_n of the full Adler function series obtained from the Borel transform (5.2). The subtracted series can then be inserted into the renormalon sum rule (with the values $p = -2$ and $\alpha = 0$) to probe the simple pole at $u = -1$. The results are depicted in Fig. 5.2a and Fig. 5.2b. Similar to our previous analysis of the leading UV renormalon, we observe that the norm[4] of the simple pole (red line) is approached faster in schemes with positive C (Fig. 5.2a) as compared to schemes with negative C (Fig. 5.2b).

At this point a short comment is in order. Even though the exponential factor $\mathrm{e}^{-C\cdot(-1)}$ of the renormalon pole has once again been removed from the results of the sum rule calculation, the norm of the simple pole for different schemes does not coincide in the plots. The reason for this observation, which contradicts our findings in the investigation of the double pole, can be traced back to the decomposition of the Borel transform (5.2) into contributions from individual renormalon poles (see e.g. eq. (5.5)). In order to obtain the contribution due to the UV renormalon at $u = -1$, one needs to compute the corresponding Laurent series of the Borel transform about the point $u = -1$. Since the simple pole is the second term in the expansion, it does not only have an exponential factor $\mathrm{e}^{-C\cdot(-1)}$ like the double pole but also receives an additional contribution linear in C coming from the first derivative of the factor e^{-Cu} in (5.2). Hence, the norm of the simple pole in different schemes still does not coincide after the exponential factor has been accounted for.

We want to stress again that the renormalon sum rule for the simple pole at $u = -1$ does

[4]We stress that the norm of the simple pole at $u = -1$ is given by the over-all factor multiplying the singular term $(1+u)^{-1}$ in the Borel transform.

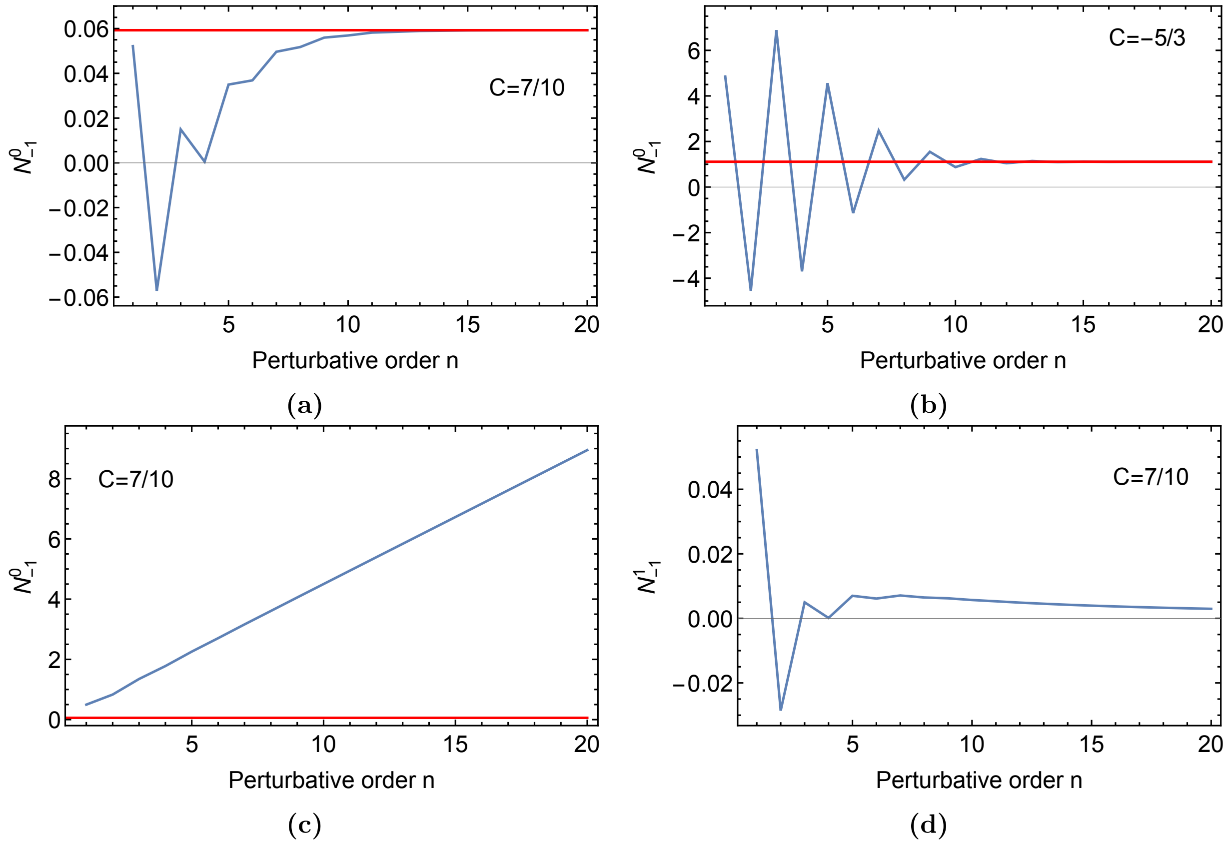

Fig. 5.2: Application of the renormalon sum rule $N^{0,(n)}_{-1,\mathrm{UV}}$ (blue curves) to probe the $u = -1$ simple pole in different schemes (graphs (a)-(c)). The red line denotes again the exact value of the norm given by the Borel transform (5.2). In (a) and (b) the contributions of the $u = -1$ double pole have been cancelled, whereas in (c) the sum rule is applied to the complete Adler function series. In (d) the double pole is probed after its contributions have been removed from the Adler function.

only converge to the exact value for the norm of the corresponding renormalon pole in the Borel transform (5.2), if the contributions due to the dominant double pole have been removed from the Adler function series. The situation, in which the sum rule for the simple pole is applied to the complete Adler function series (including the contributions from the leading renormalon pole), is depicted in Fig. 5.2c. In this case the renormalon sum rule diverges, as explained in point 3 of the discussion at the beginning of section 5.2, since a more severe renormalon is present. To complete the discussion of the $u = -1$ renormalon, let us also investigate what happens, if we eliminate the double pole but still try to probe this renormalon using the sum rule. This situation is shown in Fig. 5.2d, where the expected convergence to zero can be seen.

In order to probe the yet next-to leading renormalon ambiguity, which is the $u = -2$ double pole, we would have to remove both the contributions of the $u = -1$ double and simple pole from the perturbative series of the Adler function and repeat the entire procedure. This way it is in principle possible to find a decomposition of the Borel transform into contributions from various renormalon poles similar to the one given in eq. (5.5).

Finally, we briefly mention a subtlety of the renormalon sum rule that arises in situations where the large-order behaviour of a perturbative series is governed by two renormalon poles with the same pole structure (i.e. the same value of α) located equidistantly from the origin of the Borel plane on the positive and negative real axis. In such cases it is not possible to tell which one is the dominant renormalon. To illustrate this situation we consider a toy model (within the large-β_0 approximation) in which a series F is generated by a Borel transform that consists of only two

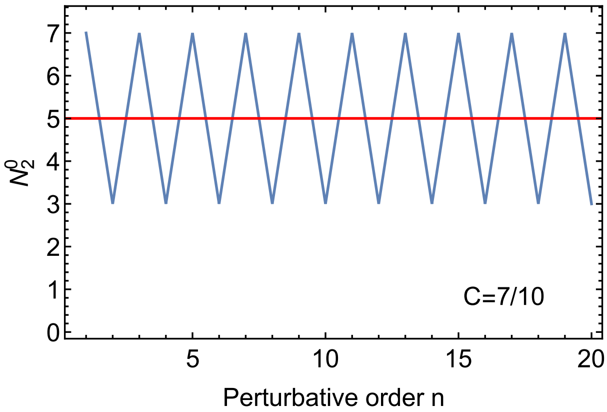

Fig. 5.3: Oscillation of the renormalon sum rule when applied to a perturbative series generated by the Borel transform (5.12). The graph shows the oscillation for the case $b_{\rm UV} = 2$ and $b_{\rm IR} = 5$. The exact value of the norm for the IR renormalon (red line) lies between the two accumulation points $b_{\rm IR} + b_{\rm UV} = 7$ and $b_{\rm IR} - b_{\rm UV} = 3$.

poles, a simple pole at $u = -2$ and another one at $u = 2$:

$$B[F](u) = \frac{b_{\rm UV}}{(2+u)} + \frac{b_{\rm IR}}{(2-u)}. \tag{5.12}$$

If we compute the perturbative coefficients of the series F and attempt to probe the ambiguity of the IR renormalon, we observe an oscillation of the sum rule around the norm $b_{\rm IR}$ of the renormalon pole. For the special case of $b_{\rm UV} = 2$ and $b_{\rm IR} = 5$ this oscillation behaviour is shown in Fig. 5.3, where the two accumulation points are given by $b_{\rm IR} + b_{\rm UV} = 7$ and $b_{\rm IR} - b_{\rm UV} = 3$. For a series F generated by the Borel transform (5.12) one can analytically calculate the renormalon sum rule (5.7) and prove that,

$$P^{0,\rm IR}_{2,\beta_0} = \sum_{k=0}^{\infty} \frac{S_k^{(4)}\, 4^k}{\Gamma(1+k)} = b_{\rm IR} + b_{\rm UV} + 2\, b_{\rm UV} \sum_{k=0}^{\infty} (-1)^k, \tag{5.13}$$

which is consistent with the accumulation points in Fig. 5.3.

Even though this feature of the renormalon sum rule seems to limit its usefulness, we want to emphasize that this situation is only encountered once the sub-leading renormalon ambiguities of the simple poles at $u = -2$ and $u = 2$ are probed in the Adler function series. The study of the leading $u = -1$ renormalon is not affected by this complication and the sum rule correctly predicts the norm in this case as shown in Fig. 5.1.

5.3 Estimation of IR Renormalon Ambiguities

The second application of R-evolution we want to address is related to the connection between the R-evolution equation and the Borel integral discussed in section 4.2.2. Assuming that the perturbative series in $\overline{C}_0(Q)$ is dominated by contributions of a single renormalon we showed that one can express the solution of the R-evolution equation in terms of a Borel integral over the difference of two Borel transforms defined at distinct scales R_1 and R_0 (see (4.23)). Moreover, we also addressed an important implication of this connection when the lower scale is set to zero. In this limit we found (see eq. (4.25)):

$$\overline{C}_0(Q) \simeq -\frac{1}{Q^p} \lim_{R_0\to 0} \big[\theta_{p,0}(Q) - \theta_{p,0}(R_0)\big]_{\rm LL} = \int\limits_0^\infty {\rm d}u\, B[Q,\mu](u)\, {\rm e}^{-\frac{4\pi u}{\beta_0 \alpha_{\rm s}(\mu)}}, \tag{5.14}$$

which proves that the information on the ambiguity caused by an (IR) renormalon in the Borel transform on the right-hand side can be related to the solution of the R-evolution equation for the corresponding perturbative series. Thus, R-evolution provides an alternative method to estimate the size of renormalon ambiguities which is not based on the commonly used principle value prescription of the Borel integral.

As eq. (5.14) indicates, the solution of the R-evolution equation only reduces exactly to the definition of the Borel integral in the limit $R_0 \to 0$. However, in order to stay within the perturbative regime we cannot integrate over the Landau pole $R_0 = \Lambda_{\rm QCD}$ in the R-evolution equation and hence need to choose values $R_0 \gtrsim \Lambda_{\rm QCD}$ such that perturbation theory is still valid (i.e. the perturbative prediction of R-evolution depends on the IR cutoff R_0). This suggests that using small variations in the lower scale R_0 close to $\Lambda_{\rm QCD}$ may allow one to find an estimate for the value of the Borel integral and quantify the size of its ambiguity[5].

5.3.1 Ambiguity of the $u = 2$ Renormalon in the Adler Function

In order to illustrate how R-evolution can be used to estimate renormalon ambiguities we want to apply eq. (5.14) to the Adler function in the large-β_0 approximation and compare the principle value prescription of the Borel integral to the prediction of R-evolution. For this purpose let us consider the contribution of the $u = 2$ IR renormalon in the decomposition of the $\overline{\rm MS}$ Borel transform (5.5):

$$B[D](u)\bigg|_{\mu^2=Q^2} \simeq \mathrm{e}^{10/3}\,\frac{2}{(2-u)}. \tag{5.15}$$

Using this expression we can easily generate the series $\sum_n a_n^{p=4}(\alpha_{\rm s}/(4\pi))^n$ containing only the contributions $a_n^{p=4}$ due to the $u = 2$ simple pole in Adler function series. The generated series then serves as a starting point for the R-dependent subtraction terms $\theta_{4,0}$ in our R-evolution ansatz[6] (cf. eq. (4.12)). Since we are working in the large-β_0 approximation this series reduces exactly to a single term when taking the derivative to obtain the R-evolution equation[7]. We thus obtain,

$$\left[\frac{\mathrm{d}}{\mathrm{d}\ln R}\,\theta_{4,0}(R)\right]_{\beta_0} = -R^4\,\gamma_0^R\left(\frac{\alpha_{\rm s}(R)}{4\pi}\right) \tag{5.16}$$

which is consistent with the expression in eq. (4.18). Following the derivation in section 4.2.2 the solution of the R-evolution equation is found to be,

$$\begin{aligned}\big[\theta_{4,0}(R_1)-\theta_{4,0}(R_0)\big]_{\beta_0} &= -\frac{\gamma_0^R}{2\,\beta_0}\left(\Lambda_{\rm QCD}^{\rm LL}\right)^4\left[\int_{t_0}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-4t}-\int_{t_1}^{\infty}\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-4t}\right]\\ &= -\frac{\gamma_0^R}{2\,\beta_0}\left(\Lambda_{\rm QCD}^{\rm LL}\right)^4\left[\Gamma\left(0,-\frac{2\pi\cdot 4}{\beta_0\,\alpha_{\rm s}(R_0)}\right)-\Gamma\left(0,-\frac{2\pi\cdot 4}{\beta_0\,\alpha_{\rm s}(R_1)}\right)\right],\end{aligned} \tag{5.17}$$

where the integration limits in the first line are defined as $t_i = -2\pi/(\beta_0\alpha_{\rm s}(R_i))$ and the LL relation for $\Lambda_{\rm QCD}$ is given by $\Lambda_{\rm QCD}^{\rm LL} = R\exp[-2\pi/(\beta_0\alpha_{\rm s}(R))]$. In the second line we adopted the definition (4.31) to rewrite the result in terms of incomplete gamma functions. The information on the renormalon ambiguity originally contained in the perturbative series $\theta_{4,0}$ is now encoded in the analytic structure of the appearing incomplete gamma functions and cancels in the difference of eq. (5.17) such that $\theta_{4,0}(R_1) - \theta_{4,0}(R_0)$ becomes ambiguity-free.

Eq. (5.17) represents the desired expression for the solution of the R-evolution equation which

[5]Note that the variations in the IR cutoff lead to changes in the perturbation theory part, which in principle need to be compensated by corresponding changes of the R-dependent condensates. The discussion of R-dependent condensates is beyond the scope of this work.

[6]Since the $u = 2$ renormalon corresponds to a simple pole in the Borel transform, we have $p = 4$ and $\alpha = 0$.

[7]Exploiting the analytic expressions (C.61) for the contributions of the various renormalons one can explicitly check that the R-anomalous dimension for a perturbative series in the large-β_0 approximation generated by a particular renormalon pole indeed collapses to a single non-vanishing term.

can now be used to compute the Borel integral according to eq. (5.14):

$$D(Q^2) \simeq -\frac{1}{Q^4} \lim_{R_0 \to 0} \left[\theta_{4,0}(Q) - \theta_{4,0}(R_0)\right]_{\mathrm{LL}} = \int_0^\infty \mathrm{d}u\, B[Q](u)\, \mathrm{e}^{-\frac{4\pi u}{\beta_0 \alpha_s(Q)}}, \tag{5.18}$$

where

$$B[Q](u) = \frac{\gamma_0^R}{2\,\beta_0} \frac{1}{u-2}. \tag{5.19}$$

The "$\simeq$" sign in eq. (5.18) indicates that we do only consider the contribution of the $u = 2$ renormalon of the Adler function. Note again that taking the limit $R_0 \to 0$ reintroduces the renormalon ambiguity once we integrate over the Landau pole at $R_0 = \Lambda_{\mathrm{QCD}}$. Therefore, varying the lower limit R_0 close to Λ_{QCD} in the perturbative regime provides an estimate for the Borel integral and the corresponding ambiguity which is independent of the common principal value prescription.

A graphical representation of this procedure is depicted in Fig. 5.4, which shows the solution of the R-evolution equation (orange curves) as a function of R_0 for the fixed scales $\mu = Q = 10\,\mathrm{GeV}$ (Fig. 5.4a) and $\mu = Q = 2\,\mathrm{GeV}$ (Fig. 5.4b). The solid blue line in both graphs represents the principle value result of the Borel integral and the blue-shaded bands provide an estimate of the renormalon ambiguity given by the imaginary part of the principle value prescription. The small peak in the orange curves at 0.5 GeV marks the position of the Landau pole and we see that the solution of the R-evolution equation agrees with the principal value result in an analytic continuation for scales $R_0 < \Lambda_{\mathrm{QCD}}$. In order to quantify the ambiguity of the Borel integral using R-evolution we choose to vary the scale R_0 in a narrow region between $0.8\,\mathrm{GeV} \leq R_0 \leq 1.2\,\mathrm{GeV}$ marked by the vertical red dashed lines. The resulting central value for the Borel integral is represented by the solid black line and the surrounding red-shaded error band gives the R-evolution estimate of the ambiguity.

Let us take a closer look at the graphs of Fig. 5.4 and compare the results of the principal value and the R-evolution method. In Fig. 5.4a, showing the situation for a hard scale $Q = 10$ GeV, the central values for both approaches are found to be,

$$\begin{aligned} \sigma_{\mathrm{PV}} &= 2.2939 \pm 0.0002 \\ \sigma_{\mathrm{R}} &= 2.2925 \pm 0.0007, \end{aligned} \tag{5.20}$$

and we conclude that both procedures yield quite compatible results for the Borel integral and its ambiguity. The relative deviation of the R-evolution central value σ_{R} compared to the principal value result amounts to $\Delta\sigma \approx 0.06\,\%$ in this case. Fig. 5.4b, on the other hand, depicts the situation for $Q = 2$ GeV and yields:

$$\begin{aligned} \sigma_{\mathrm{PV}} &= 5.89 \pm 0.15 \\ \sigma_{\mathrm{R}} &= 4.99 \pm 0.46, \end{aligned} \tag{5.21}$$

Here, we find a relative deviation $\Delta\sigma \approx 15\,\%$ which shows that the R-evolution prediction is now significantly below the principal value result. In view of the determination of α_s from hadronic τ decays the outcome of Fig. 5.4b is quite remarkable, since we see that for energy regions close to τ mass scale $M_\tau = 1.78\,\mathrm{GeV}$ R-evolution represents an alternative method to compute the Borel integral that differs considerably from the results of the common principal value prescription. Therefore, using R-evolution in investigations of hadronic τ decays could provide new insight into the long-standing discussion of the discrepancy between FOPT and CIPT and consequently improve the accuracy of the α_s extraction from R_τ.

5.4 Discussion

We conclude this chapter by summarizing the most important results. Our main goal was to investigate to what extend R-evolution can be used to address the questions raised in the discussion

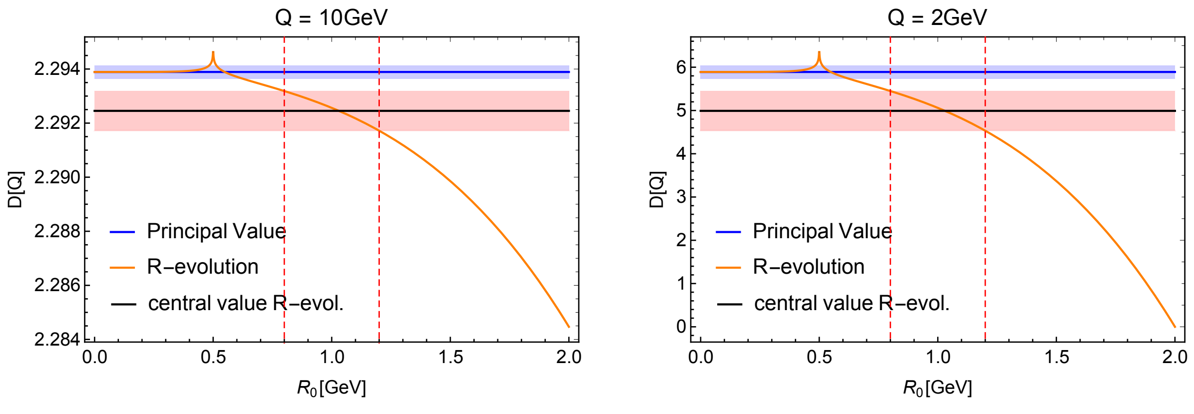

Fig. 5.4: Computation of the Borel integral (5.18) using R-evolution for fixed hard scales $Q = 10$ GeV (a) and $Q = 2$ GeV (b). The orange curves represent the solution of the R-evolution equation and the blue solid lines denote the principal value of the Borel integral with an ambiguity estimation given by the blue-shaded region. The red dashed lines mark the boundaries of the R_0 variation between $0.8\,\mathrm{GeV} \leq R_0 \leq 1.2\,\mathrm{GeV}$. The resulting R-evolution central value and the corresponding ambiguity are given by the black solid line and the red-shaded error band.

of higher order models for hadronic τ decays at the end of chapter 2. For this purpose we studied the Adler function in the large-β_0 approximation in which a simple analytic expression of its Borel transform is known.

The first purpose of this chapter was to illustrate how R-evolution can be used to gain more information on specific renormalons in perturbative series. To that end we applied the renormalon sum rule to the Adler function to demonstrate its capability to test renormalon ambiguities. In particular, we probed the leading $u = -1$ renormalon and showed that the sum rule converged to the expected value of the residue. In view of the renormalon models for the Adler function presented in section 2.2 this feature of R-evolution is very promising to lead to an improved ansatz for physical models used in the determination of the strong coupling constant α_{s}. The sum rule only relies on the knowledge of the QCD β-function and the coefficients of the perturbative series and, provided that the series coefficients are known to sufficiently high order, allows one to construct a Borel transform focusing on a single renormalon pole without making any approximations. For most realistic cases, e.g. the Adler function in full QCD, only a few perturbative coefficients are available which currently makes the application of the renormalon sum rule quite challenging.

Second, we compared the solution of the R-evolution equation with the principal value prediction for the Borel integral and in particular looked at the ambiguity caused by the leading $u = 2$ IR renormalon. The result of this comparison is shown in Fig. 5.4 and proves that R-evolution coincides with the principle value prescription in the limit $R_0 \to 0$. However, for scales $R_0 \gtrsim \Lambda_{\mathrm{QCD}}$ which are perturbatively accessible we see a considerable deviation between these two approaches and the variation of the R-evolution solution in the region $0.8\,\mathrm{GeV} \leq R_0 \leq 1.2\,\mathrm{GeV}$ leads to a new central value for the Borel integral that is significantly below the principle value result. We stress that for a full analysis one also needs to account for the R-dependent gluon condensate, but this is beyond the scope of this thesis. Besides, one may argue that taking these variation limits is a completely arbitrary choice and choosing values more closely to Λ_{QCD} would result in central values and corresponding ambiguities which are in better agreement with the principal value result. This is a totally valid objection, but the point we want to make here is the following. Computing the Borel integral by means of the principle value method and estimating the ambiguities by taking the imaginary part is also an arbitrary choice. We commonly expect the 'true' result of the Adler function series to be consistent with the principal value of the Borel integral within an error given by its ambiguity. However, since the ambiguities arise from deformations of the integration contour in the evaluation of the Borel integral, it is not at all obvious how well they are determined by the imaginary part of the principle value result. R-evolution provides an alternative possibility to quantify the ambiguity of the Borel integral and clearly leads to different results. This, in turn, sheds new light on the discussion of the mechanisms favouring FOPT or CIPT and also raises the question how much the extraction of the strong coupling from hadronic τ decays is affected, if one does not rely on the principle value computation of the Borel integral.

Chapter 6

Summary and Outlook

In this thesis we reviewed the basic problem of asymptotic series in QCD and studied a very promising approach, called R-evolution, to deal with the implications related to a particular source of divergence known as renormalons. Renormalon divergences have received increasing attention over the last few decades, especially because it has been realised that they often reveal far-reaching physics and indicate that non-perturbative effects need to be considered in order to define physical theories unambiguously.

Starting out from an exact, i.e. non-perturbative, definition of physical observables in the framework of the operator product expansion, we established the close connection between IR renormalons and higher-dimensional operator terms in the OPE in chapter 3 and discussed how the addition of these corrections account for renormalon ambiguities in the definition of the Borel integral (3.22). UV renormalons, on the other hand, are Borel summable and hence do not cause ambiguities – at least formally. Nevertheless, they also indicate the existence of profound UV physics and can be accounted for by insertions of local higher-dimensional operators in the context of the Standard Model Effective Field Theory (SMEFT).

In chapter 4 we derived the basic formalism for schemes with explicit renormalon subtractions involving additional scales R and R_{UV} and discussed in detail the implications of the renormalization group evolution when the parameter R is treated as a continuous variable. In particular, we showed that the solution of the R-evolution equation leads to an analytic expression for the Borel transform of a perturbative series that encodes the singular and non-analytic contributions which are characteristic for a particular renormalon pole. For the normalization of the singular renormalon terms we obtained an analytic sum rule which can be determined from the knowledge of the series coefficients and the QCD β-function and does not rely on simplifying approximations like the calculation of bubble chains. Therefore, the renormalon sum rule (4.47) can be applied to any perturbative series and serves as a probe for renormalon ambiguities. Early works on this topic [17, 48] focused primarily on the description of the leading $u = 1/2$ renormalon in the heavy quark pole mass and an extension for arbitrary IR poles at $u = p/2$ was derived in [52] where, however, the anomalous dimensions of the operators in the OPE are not considered and the Wilson coefficients are assumed to start at tree-level. A major purpose of this thesis was to review the derivation of the renormalon sum rule including operator anomalous dimensions and generalize it to also comprise UV renormalons.

Our main motivation for dealing with renormalons and R-evolution was provided by investigations of hadronic τ decays which enable the determination of the strong coupling constant α_{s} at low energies in a rather clean environment [12]. As discussed in chapter 2 the dominant source of uncertainty resides in the significant numerical discrepancy between different ways to improve the perturbative expansion through renormalization group methods, namely CIPT and FOPT. Since the preference for either CIPT or FOPT depends primarily on higher-order perturbative QCD corrections governed by renormalons, studies of adequate renormalon models for the Adler function series have become increasingly important. A particular useful model, in which the perturbative coefficients can be calculated to all orders in perturbation theory, is the large-β_0 approximation based on the computation of bubble chain diagrams. In this approximation a simple analytic expression for the Borel transform of the Adler function is known which allows us to test the predictions of R-evolution. As an application of the renormalon sum rule, we investigated the Adler function in chapter 5 and successfully quantified the residue of the leading $u = -1$ renormalon pole. Moreover we were also able to prove the compatibility of the analytic Borel transform expression derived from the R-evolution equation with the renormalon model for the Adler function given in [14] and showed that R-evolution provides an alternative method to estimate the size of renormalon ambiguities in the Borel integral.

The results of chapter 5 suggest that R-evolution is indeed a promising approach capable of improving the ansatz for physical models of the Adler function. However, an improved determination of α_{s} using R-evolution methods without any additional approximations is still hard to achieve, since the renormalon sum rule is restricted to the series coefficients of the Adler function which are currently known up to $\mathcal{O}\left(\alpha_{\mathrm{s}}^4\right)$. With only four coefficients available the sum rule displays a poor convergence behaviour and the prediction for the residues is unreliable.
Finally we want to stress again that the results for the analytic Borel transform and the renormalon sum rule derived in this thesis are not restricted to the study of the Adler function in hadronic τ decays but can also be applied to other fields to investigate renormalon divergences and are therefore extremely powerful.

Appendix A

Calculation of Bubble Chain Diagrams

We provide a detailed computation of the Adler function in the flavour expansion up to order $1/N_f$ in this part of the appendix. As already mentioned in section 3.2, the dominant contributions to the Adler function in this expansion arise from bubble chain diagrams with any number of massless quark bubble insertions into the internal gluon propagator (see Fig. A.1). All calculations in this section will be done using dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions.

Starting point for our analysis is the definition of the Adler function given in eq. (3.8). According to this, we write

$$\Pi_{\mu\nu}(p) = \left(p^\mu p^\nu - g_{\mu\nu} p^2\right) \Pi(p^2), \tag{A.1}$$

where we can decompose the correlation function $\Pi(p^2)$ at order $1/N_f$ in the form

$$\Pi(p^2) = \frac{1}{N_f} \sum_{n=0}^{\infty} \Pi^{(n)} a^{n+1}, \tag{A.2}$$

with $a = \alpha_\mathrm{s} N_f/\pi$. The superscript (n) denotes the number of fermion bubble insertions into the internal gluon line and the total number of loops in the corresponding diagrams is obviously given by $n+2$. We begin with the simple fermion loop diagram needed for the renormalization of the Adler function (see Fig. A.2). A straightforward calculation yields the well-known result (compare [53]):

$$\pi_{\mu\nu}(p) = \left(p^\mu p^\nu - g_{\mu\nu} p^2\right) a \left(\frac{-p^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{[\Gamma(2-\epsilon)]^2\, \Gamma(\epsilon)}{\Gamma(4-2\epsilon)}. \tag{A.3}$$

The renormalized quark loop is thus given by

$$\hat{\pi}(p^2) = \frac{\alpha_\mathrm{s}}{6\pi} N_f \ln\left(\frac{\mu^2}{-p^2} \mathrm{e}^{-C}\right), \tag{A.4}$$

where C denotes a scheme-dependent constant. In $\overline{\mathrm{MS}}$, the renormalization scheme that will be used throughout this section, $C = -5/3$. Note, that the factor $N_f/(6\pi)$ corresponds to the fermionic contribution of the one-loop QCD β-function (see Appendix C).

In the next step we are going to calculate the two-loop contribution $\Pi^{(0)}$, i.e. the diagrams depicted in Fig. A.1 without any insertion of fermion bubbles into the internal gluon line. A full derivation of the two-loop calculation is given in Appendix B. Here we are only interested in stating the final result:

$$\Pi^{(0)} = -\,C_F \frac{C_A}{16\pi^2} \cdot \frac{(1-\epsilon)}{\epsilon\,(3-2\epsilon)} \left(\frac{-p^2 - i0^+}{4\pi\mu^2}\right)^{-2\epsilon} [\Gamma(1-\epsilon)]^3 \cdot \left\{ \left[\frac{\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\right]^2 \Gamma(1-\epsilon) \cdot (2 - \epsilon + 2\epsilon^2) - \frac{\Gamma(2\epsilon)}{\Gamma(3-3\epsilon)} \frac{(2-2\epsilon)^2 + 4}{\epsilon} \right\}. \tag{A.5}$$

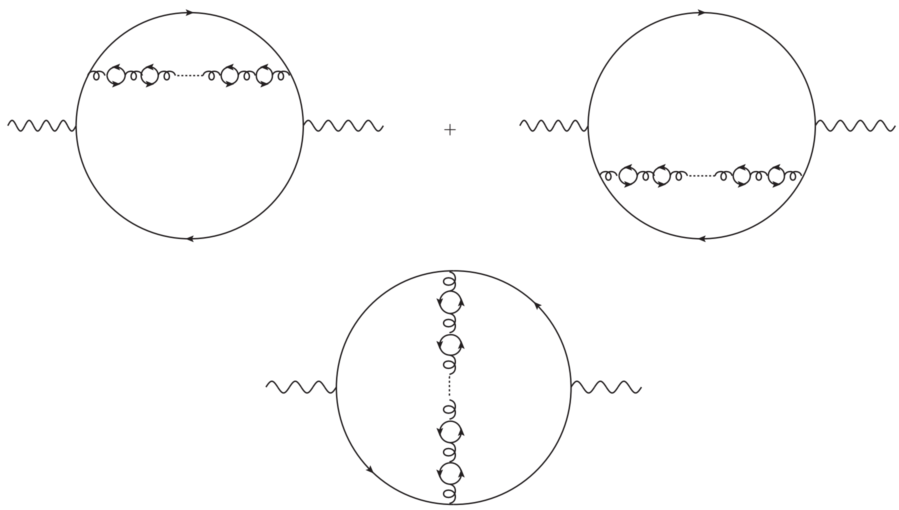

Fig. A.1: Fermion bubble diagrams that contribute to the leading term in the $1/N_f$-expansion of the Adler function.

Exploiting some well-known identities of the $\Gamma(z)$-function, one can easily show that eq. (A.5) can be rewritten in the form [53]

$$\begin{aligned}\Pi^{(0)} = &- C_F\,\frac{C_A}{4\pi^2}\left(\frac{-p^2}{4\pi\mu^2}\right)^{-2\epsilon}[\Gamma(2-\epsilon)]^2\left\{\frac{3+2\epsilon}{2}\,\frac{[\Gamma(1-\epsilon)\,\Gamma(\epsilon)]^2}{\Gamma(2-2\epsilon)\,\Gamma(4-2\epsilon)}\right. \\ &+\frac{\Gamma(\epsilon)\,\Gamma(-\epsilon)\,\Gamma(2\epsilon)\,\Gamma(-3+2\epsilon)}{\Gamma(1-3\epsilon)\,\Gamma(3\epsilon)}-\frac{\Gamma(2\epsilon)\,\Gamma(-3+2\epsilon)}{\epsilon} \\ &\left.-\,2\,(1-2\epsilon)\,(2-2\epsilon-\epsilon^2)\,\frac{\Gamma(-\epsilon)\,\Gamma(-3+2\epsilon)}{\epsilon\,\Gamma(3-3\epsilon)}\right\}.\end{aligned} \tag{A.6}$$

The calculation of the bubble diagrams with an arbitrary number of fermion bubbles in the gluon line is very tedious and, unfortunately, cannot be solved using 'standard' computational tricks such as *integration-by-parts techniques* (IBP). As pointed out in [54], one runs into difficulties when trying to calculate the third diagram in Fig. A.1, where the gluon is exchanged between the upper fermion and the lower antifermion line, if there are subdivergences nested within the internal gluon line. In such cases, the corresponding loop momentum integration involves two-loop integrals of the form (see Appendix B)

$$I(p^2, n\epsilon, d) = \frac{\left(\tilde{\mu}^2\right)^{(1+n)\epsilon}}{(2\pi)^{2d}}\iint\frac{\mathrm{d}^dk\,\mathrm{d}^dq}{[k^2]_+\,[q^2]_+\,[(k-p)^2]_+\,[(q-p)^2]_+\,[(k-q)^2]_+^{1+n\epsilon}}, \tag{A.7}$$

where we used the notation $[x]_+ \equiv [x + i0^+]$ and the non-integer exponent of the last propagator arises from the insertion of n fermion loops into the gluon line. In order to calculate this integral one can e.g. use the so-called *Gegenbauer polynomial x-space technique* [55] to reduce the occurring loop momentum integrals to a form that can be expanded in the infinitesimal regulator ϵ.

Applying Gegenbauer integration to our present problem of fermion bubble diagrams with at least one insertion of a fermion loop in the gluon line ($n \geq 1$), yields the result [53]

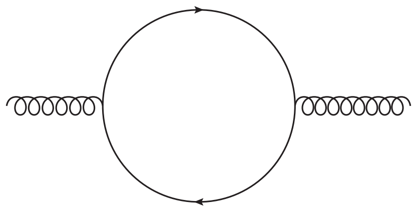

Fig. A.2: Fermion loop diagram needed for the renormalization of the Adler function in the $1/N_f$-expansion.

$$
\begin{aligned}
\Pi^{(n)} =& - C_F\,\frac{N_c}{4\pi^2}\,\frac{1}{2^n}\left(\frac{-p^2}{4\pi\mu^2}\right)^{-\epsilon(n+2)}\cdot\left\{-2\frac{\Gamma(1+\epsilon)\,[\Gamma(2-\epsilon)]^2}{\epsilon\,\Gamma(4-2\epsilon)}\right\}^n \qquad \text{(A.8)}\\
&\cdot\frac{1-\epsilon}{3-2\epsilon}\cdot\frac{[\Gamma(1-\epsilon)]^2}{\Gamma(1+n\epsilon)\Gamma(3-3\epsilon-n\epsilon)}\cdot\frac{\Gamma(1-\epsilon-n\epsilon)}{-\epsilon(1+n)}\\
&\cdot\Big\{\left(6-14\epsilon-8n\epsilon+\epsilon^2(4+10n+4n^2)+\epsilon^3(4+3n+n^2)\right)\Gamma(-2+n\epsilon+2\epsilon)\\
&+\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\cdot\Big[(2-n\epsilon-3\epsilon)\,(1-n\epsilon-3\epsilon)\,F(0)\\
&-(2+\epsilon)\,(n+1)\,n\,\epsilon^2\,(2-n\epsilon-3\epsilon)\,F(1)\\
&+(1-n\epsilon-\epsilon)\,(1-n\epsilon)\,(n+1)\,n\,\epsilon^2\,F(2)\Big]\Big\},
\end{aligned}
$$

where the occurring function $F(s)$ is given by [53]

$$
\begin{aligned}
F(s) =& \sum_{m,k=0}^{\infty}\frac{\Gamma(m-s+3\epsilon+n\epsilon)\,\Gamma(k+2-2\epsilon)\,\Gamma(m+k+1-s+n\epsilon+2\epsilon)}{m!\,k!\,\Gamma(m+k+2-\epsilon)} \qquad \text{(A.9)}\\
&\cdot\left\{\frac{1}{(m+k+1-s+\epsilon+n\epsilon)\,(k+1-s+n\epsilon)}\right.\\
&\left.-\frac{1}{(m+k+1)\,(k+1+s-n\epsilon-2\epsilon)}\right\}\cdot\frac{1}{\Gamma(3\epsilon+n\epsilon-s)}.
\end{aligned}
$$

For the purpose of analyzing the renormalization of the bubble diagrams, it is useful to study the structure of eq. (A.8) in more detail. If we look at the second curly bracket, which involves all terms containing the function $F(s)$ ($s = 0, 1, 2$), and perform an expansion in ϵ, we notice that this bracket contributes only to $\mathcal{O}(\epsilon^0)$. Thus, the divergent behaviour of the diagrams coming from the integration over the loop momenta is completely incorporated in the first part of eq. (A.8) and one can immediately see that the lowest order term in the Laurent series of $\Pi^{(n)}$ is proportional to $1/\epsilon^{n+1}$.[1] Allowing for a normalization factor we can therefore rewrite eq. (A.8) in the form [28, 53]

$$
\Pi^{(n)} = (-1)^n\,C_F\,\frac{N_c}{16\pi^2\cdot 6^n}\cdot\frac{1}{n+2}\cdot\frac{1}{\epsilon^{n+1}}\cdot G(p^2/\mu^2,\epsilon,n\epsilon), \qquad \text{(A.10)}
$$

with

$$
G\left(p^2/\mu^2,\epsilon,\delta=\epsilon\,(n+2)\right)=\sum_{j=0}^{\infty}G_j(\epsilon)\,(\epsilon\,(n+2))^j\,,\quad G_j(\epsilon)=\sum_{i=0}^{\infty}g_i^{(j)}\epsilon^i, \qquad \text{(A.11)}
$$

where $G(p^2/\mu^2,\epsilon,\delta)$ is analytic near $\epsilon=\delta=0$. According to eq. (A.3), the counterterms for the fermion loops in the bubble chain needed for the renormalization procedure are given by

[1] One would naively expect that the Laurent series of the regularized polarization function $\Pi(p^2)$ starts at order $1/\epsilon^{n+2}$. Each fermion loop in the bubble chain gives a factor $1/\epsilon$ and the remaining loop integrations are supposed to contribute $1/\epsilon^2$. But since the 2-loop vacuum polarization graphs in Landau gauge only have simple poles in the regularization parameter ϵ (see eq. (A.6) and Appendix B), the lowest order term in the Laurent series is proportional to $1/\epsilon^{n+1}$.

$-\alpha_s N_f/(6\pi\epsilon)$ in the $\overline{\text{MS}}$ scheme. Furthermore, in order to properly subtract all subdivergences we need to take a combinatorial factor into account which determines the number of ways, we can choose k renormalized loops from the n available loops in the bubble chain. Combining the previous results, we obtain the following expression [53, 34]

$$
\begin{aligned}
\overline{\Pi}^{(n)}_{\text{part.ren}} =& \sum_{k=0}^{n} \binom{n}{k} \Pi^{(n-k)} \left(-\frac{1}{6\epsilon}\right)^k \\
=& (-1)^n\, C_F\, \frac{N_c}{16\pi^2 \cdot 6^n} \sum_{j=0}^{\infty} \frac{G_j(\epsilon)}{\epsilon^{n+1-j}} \sum_{k=0}^{n} \binom{n}{k} (n+2-k)^{j-1} (-1)^k\,,
\end{aligned}
\tag{A.12}
$$

where $\overline{\Pi}^{(n)}_{\text{part.ren}}$ denotes the partially renormalized coefficient of the bubble diagrams at order a^{n+1} after subtraction of the subdivergences in the $\overline{\text{MS}}$ scheme. Since we are only interested in the finite terms that do not vanish in the limit $\epsilon \to 0$, we truncate the j-summation at $j = n+1$. Using the astonishing fact that [53]

$$
\sum_{k=0}^{n} \binom{n}{k} (n+2-k)^{j-1} (-1)^k = 0 \qquad \text{for } 1 \leq j \leq n, \tag{A.13}
$$

eq. (A.12) is non-zero only for $j = 0$ and $j = n+1$ and hence simplifies to

$$
\overline{\Pi}^{(n)}_{\text{part.ren}} = (-1)^n\, C_F\, \frac{N_c}{16\pi^2 \cdot 6^n} \left(\frac{(-1)^n}{(n+1)\cdot(n+2)\,\epsilon^{n+1}} G_0(\epsilon) + n! \cdot G_{n+1}(\epsilon) \right). \tag{A.14}
$$

Now we can add the counterterm allowing for the subtraction of the whole diagrams and finally take the limit $\epsilon \to 0$. Applying eq. (A.11), the $\overline{\text{MS}}$-renormalized coefficient is eventually given by [28]

$$
\overline{\Pi}^{(n)}_{\text{ren.}} = C_F\, \frac{N_c}{16\pi^2 \cdot 6^n} \left(\frac{1}{(n+1)\cdot(n+2)} g^{(0)}_{n+1} + (-1)^n\, n! \cdot G_{n+1}(0) \right). \tag{A.15}
$$

As one can see, in order to determine the fully renormalized coefficients of the bubble diagrams in Fig. A.1 we need to know only two terms of the n-loop generating function $G(p^2/\mu^2, \epsilon, n\epsilon)$ given in eq. (A.11). The relevant contributions come from $G_0(\epsilon)$, which corresponds to an analytic continuation to the zero-loops case of Fig. A.1, and $G(p^2/\mu^2, 0, \epsilon(n+2))$ which, as we are going to see, is associated with the Borel limit $\epsilon \to 0$ when $\delta = \epsilon\,(n+2)$ is held fixed [56].

To identify the term $G(p^2/\mu^2, \epsilon, 0) = G_0(\epsilon)$ it is once again useful to take a closer look at the result of eq. (A.8), but this time in the limit $n\epsilon \to 0$. Since we know from our analysis above that $G_0(\epsilon)$ contributes to order $1/\epsilon^{n+1}$, we need to extract the lowest order term of the Laurent series of $\Pi^{(n)}$. As it is evident from our observations concerning eq. (A.8), we thus have to investigate the ϵ-expansion of all terms involving the function $F(s)$ $(s = 0, 1, 2)$ defined in eq. (A.9). The crucial point at this stage is to realize that even though the function $F(s)$ is given by an infinite double series, just a very few terms of these series contribute to order $1/\epsilon^{n+1}$ in eq. (A.8)[2]. Combining the relevant terms, one obtains after a quite lengthy calculation [53, 28]

$$
G_0(\epsilon) = \frac{(1+\epsilon)(1-2\epsilon)(3-2\epsilon)\Gamma(4-2\epsilon)}{9\left[\Gamma(2-\epsilon)\right]^2 \Gamma(3-\epsilon)\Gamma(1+\epsilon)} \tag{A.16}
$$

In contrast to $G_0(\epsilon)$, however, it is not possible to express $G(p^2/\mu^2, 0, \epsilon(n+2))$ entirely in terms of simple Γ-functions. A full derivation of the analytic expression for $G(p^2/\mu^2, 0, \epsilon(n+2))$ that, to our knowledge, was first obtained by Broadhurst in 1993 [28] is far beyond the scope of this work and we will therefore just sketch the key aspects of the derivation here. First of all, we need to

[2] In fact there are only four significant terms. The first one comes from $F(1)$ when the running indices, k and m, take on the values $k = m = 0$ and the other three are associated with $F(2)$ for the values (k,m)=(0,0),(1,0),(0,1). The function $F(0)$ does not contribute to the lowest order $1/\epsilon^{n+1}$.

remind ourselves of the fact that the major problem in calculating the bubble diagrams arises from the graph, where the modified gluon line is exchanged between the fermion and the antifermion line. The corresponding two-loop integral (see eq. (A.7)) involves a propagator with a non-integer exponent which makes the calculation of the integral very challenging.

Over the last decades, many attempts have been made to find a suitable expansion for this and other related integrals. One of the first techniques, developed to resolve this issue, was the Gegenbauer integration technique [55] which we applied to derive the result given in eq. (A.8). Exploiting the symmetries of the integral, Broadhurst, Gracey and Kreimer were able to set up a recurrence relation and proved that the function $G(p^2/\mu^2, \epsilon, \delta)$ could be reduced to a Saalschützian ${}_3F_2$ hypergeometric series [57]. Applying systematically various identities for hypergeometric series one can show that it is possible to rewrite this Saalschützian ${}_3F_2$ series in the form [28, 27]

$$G(p^2/\mu^2, 0, \delta) = \frac{32}{3}\left(-\frac{p^2}{\mu^2}\mathrm{e}^C\right)^{-\delta}\frac{1}{1-(1-\delta)^2}\sum_{k=2}^{\infty}\frac{(-1)^k k}{(k^2-(1-\delta)^2)^2}, \tag{A.17}$$

where C is the same scheme-dependent constant that we already encountered in eq. (A.4). Remember, in the $\overline{\mathrm{MS}}$-scheme $C = -5/3$. The significance of eq. (A.17) for the remainder of this section is reflected by the fact that this expression for $G(p^2/\mu^2, 0, \delta)$ is consistent with the Borel transform of the correlation function $\Pi(p^2)$. To demonstrate this connection we need to consider that all information on the order in $a = \alpha_s N_f/\pi$ of the bubble diagrams is comprised in the number of fermion loop insertions into the gluon line. Thus, instead of summing all different graphs at order $1/N_f$, we can look at the gluon propagator and first calculate the sum of n fermion bubble insertions. The resulting effective propagator, which is usually called a *bubble chain*, has the form [27, 34]

$$\begin{aligned} D^{ab}_{\mu\nu}(k) &= \frac{-i}{k^2+i0^+}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1+\hat{\pi}(k^2)}\,\delta^{ab} \\ &= \frac{-i}{k^2+i0^+}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\sum_{n=0}^{\infty}\left[-\frac{\alpha_s\,N_f}{6\pi}\ln\left(\frac{\mu^2}{-k^2}\,\mathrm{e}^{-C}\right)\right]^n, \end{aligned} \tag{A.18}$$

where we have used Landau gauge and the renormalized fermion loop $\hat{\pi}(k^2)$ is given by eq. (A.4). Now we can easily perform the Borel transformation, which according to eq. (3.4) is given by [27, 34]

$$\begin{aligned} B[4\pi\,\alpha_s D^{ab}_{\mu\nu}](k,u) &= \frac{-i}{k^2+i0^+}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\left(-\frac{24\pi^2}{N_f}\right)\sum_{n=0}^{\infty}\frac{u^n}{n!}\left[\ln\left(\frac{\mu^2}{-k^2}\,\mathrm{e}^{-C}\right)\right]^n \\ &= \frac{-i}{k^2+i0^+}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\left(-\frac{24\pi^2}{N_f}\right)\exp\left[u\,\ln\left(\frac{\mu^2}{-k^2}\,\mathrm{e}^{-C}\right)\right] \\ &= \frac{-i}{k^2+i0^+}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\left(-\frac{24\pi^2}{N_f}\right)\left(\frac{\mu^2}{-k^2}\,\mathrm{e}^{-C}\right)^u \\ &= \frac{-i\,\delta^{ab}}{(-k^2-i0^+)^{2+u}}\left(g_{\mu\nu}\,k^2-k_\mu k_\nu\right)\left(-\frac{24\pi^2}{N_f}\right)\left(\mu^2\,\mathrm{e}^{-C}\right)^u. \end{aligned} \tag{A.19}$$

In the expression above, we added a factor $g^2 = 4\pi\alpha_s$ in the definition of our Borel transform, which takes into account the vertices at the end points of our bubble chain when the gluon propagator is attached to the large fermion loop of the diagrams in Fig. A.1. Eq. (A.19) describes the Borel transform of the fully renormalized bubble chain, which is essentially identical with the renormalization of the effective coupling in the exponent of the formula for the inverse Borel transformation (see eq. (3.5)).

Up to now, we have only taken into account the fermionic part of the one loop QCD β-function $\propto N_f/(6\pi)$, but did not consider the contribution coming from gluon and ghost bubbles. Thus, to order $1/N_f$ in the flavour expansion QCD is not asymptotically free and essentially identical to QED. Nevertheless, almost all cases in which exact higher order result are available suggest that we can recover the expected positions of UV and IR renormalons in QCD, once we substitute N_f with the full value of β_0 [33]:

$$N_f \to -\frac{3}{2}\left(\frac{11}{3}C_A - \frac{2}{3}N_f\right) = -\frac{3}{2}\beta_0. \tag{A.20}$$

Due to this substitution it is therefore possible to partly consider gluonic contributions attributed to diagrams that are gauge-dependent and much harder to calculate exactly. This simple procedure, which is usually referred to as *Naive Non-Abelianization* (NNA) or *large-β_0 approximation*, is the main reason why we consider the fermion bubble diagrams in the first place.

If we replace the gluon propagator with the modified propagator

$$B[4\pi\,\alpha_{\mathrm{s}} D^{ab}_{\mu\nu}](k,u) = \frac{-i\,\delta^{ab}}{(-k^2 - i0^+)^{2+u}}\Big(g_{\mu\nu}\,k^2 - k_\mu k_\nu\Big)\left(\frac{16\pi^2}{\beta_0}\right)\left(\mu^2\,\mathrm{e}^{-C}\right)^u. \tag{A.21}$$

and calculate the sum of the diagrams in Fig. A.1, the Borel transformed correlation function to order $1/N_f$ is found to be [34]

$$B[\Pi](p^2,u) = \frac{32}{3\pi}\left(-\frac{p^2}{\mu^2}\mathrm{e}^C\right)^{-u}\frac{1}{1-(1-u)^2}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{\left(k^2-(1-u)^2\right)^2}, \tag{A.22}$$

which confirms the expression given in eq. (A.17). As one can see, the Borel transform of the correlation function exhibits an undesirable pole at $u = 0$, which is due to the fact that in the derivation of eq. (A.22) we considered only the renormalization of the fermion loops in the bubble chain, but did not incorporate the counterterm for the subtraction of the whole diagrams. In the Appendix of Ref. [34] it is shown that a proper renormalization amounts to the cancellation of the singularity at $u = 0$.

Alternatively, we can also take one derivative with respect to the external momentum p^2 to remove this pole of the Borel transformed correlation function. Since the derivative of $\Pi(p^2)$ with respect to p^2 gives the Adler function (see eq. (3.8)), we immediately obtain an expression for the Borel tranform of the Adler function in the large-β_0 approximation. [27, 56]

$$\begin{aligned} B[D](p^2,u) &= \frac{32}{3\pi}\left(-\frac{p^2}{\mu^2}\mathrm{e}^C\right)^{-u}\frac{u}{1-(1-u)^2}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{\left(k^2-(1-u)^2\right)^2} \\ &= \frac{32}{3\pi}\left(-\frac{p^2}{\mu^2}\mathrm{e}^C\right)^{-u}\frac{1}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{\left(k^2-(1-u)^2\right)^2}. \end{aligned} \tag{A.23}$$

Appendix B

The Massless Two-Loop Two-Point Function

In this appendix we calculate the massless two-loop two-point function, i.e. the two-loop contribution $\Pi^{(0)}$ to the Adler function in the $1/N_f$-expansion (see Fig. A.1), and verify the result of eq. (A.5). In contrast to appendix A, however, we deal here with the similar QED calculation, that is the exchanged gluon is replaced by a corresponding photon line. To obtain the result for the QCD case one simply needs to account for the corresponding colour factor. All subsequent results in this section will be derived using dimensional regularization in $d = 4 - 2\epsilon$ space-time dimensions.

Diagrams Containing a Self-energy Subgraph

We begin our analysis with the computation of the two diagrams depicted in the first row of Fig. A.1. Since it makes no difference whether the photon (gluon) is attached to the fermion or antifermion line of the loop, both diagrams yield exactly the same result. According to the usual QED Feynman rules the amplitude $\mathcal{M}_1^{\mu\nu}$ of either graph is given by

$$i\mathcal{M}_1^{\mu\nu} = (ie)^2\,\tilde{\mu}^{2\epsilon}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{(-1)\,\mathrm{Tr}\left[\gamma^\mu\, i\not{k}\,(-i\Sigma(\not{k}))\, i\not{k}\,\gamma^\nu\, i(\not{k}+\not{p})\right]}{\left[k^2+i0^+\right]^2\,\left[(k+p)^2+i0^+\right]}, \tag{B.1}$$

where $\Sigma(\not{k})$ denotes the electron self-energy subgraph (see Fig. B.1b), which can be calculated in the following way:

$$\begin{aligned}
-i\Sigma(\not{k}) &= (ie)^2\,\tilde{\mu}^{2\epsilon}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{\gamma^\rho\, i(\not{k}-\not{q})\,\gamma^\sigma}{\left[(k-q)^2+i0^+\right]}\cdot\frac{-i\left(g_{\rho\sigma}q^2-(1-\xi)q_\rho q_\sigma\right)}{\left[q^2+i0^+\right]^2} \\
&= -e^2\,\tilde{\mu}^{2\epsilon}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\left\{\frac{(2-d)(\not{k}-\not{q})}{[(k-q)^2]_+\,[q^2]_+} - (1-\xi)\frac{2\not{q}(k\cdot q)-q^2(\not{q}+\not{k})}{[(k-q)^2]_+\,[q^2]_+^2}\right\} \\
&= -e^2\,\tilde{\mu}^{2\epsilon}\left[\Sigma_a(\not{k})-(1-\xi)\,\Sigma_b(\not{k})\right].
\end{aligned} \tag{B.2}$$

In eq. (B.2) above, we introduced the abbreviation $[p^2]_\pm \equiv [p^2 \pm i0^+]$, which will be used throughout this section. Before we proceed with the calculation of the electron self-energy, it is convenient to consider a specific class of loop integrals that will often appear during our analysis of the massless two-loop two-point function. The integrals of this class have the generic form

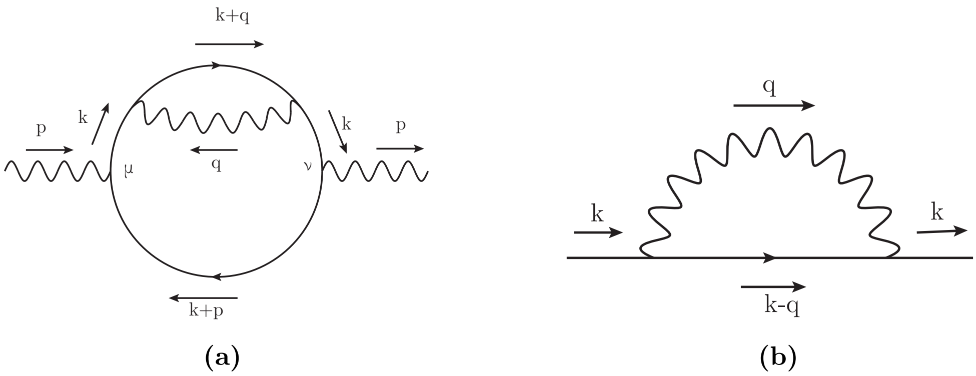

Fig. B.1: Calculation of vacuum polarization at two-loops. (a) Two-loop diagram containing a fermion self-energy subgraph. (b) Representation of the fermion self-energy subgraph nested within the diagram depicted in (a).

$$
\begin{aligned}
Z(p^2,\alpha,\beta,d) &\equiv \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{[k^2]_+^\alpha\;[(k-p)^2]_+^\beta} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{[k^2]_+^\alpha\;[(k+p)^2]_+^\beta} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{(1-x)^{\alpha-1}\,x^{\beta-1}}{[k^2(1-x)+(k^2-2k\cdot p+p^2)x+i0^+]^{\alpha+\beta}} \\
&= i\,\frac{\Gamma(\alpha+\beta-d/2)}{\Gamma(\alpha)\,\Gamma(\beta)}\cdot\frac{(-1)^{\alpha+\beta}}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\,\frac{(1-x)^{\alpha-1}\,x^{1-\beta}}{(-p^2x(1-x)-i0^+)^{\alpha+\beta-d/2}} \\
&= i\,\frac{(-1)^{\alpha+\beta}}{(4\pi)^{d/2}}\cdot\frac{\Gamma(\alpha+\beta-d/2)\,\Gamma(d/2-\alpha)\,\Gamma(d/2-\beta)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(d-\alpha-\beta)\,(-p^2-i0^+)^{\alpha+\beta-d/2}},
\end{aligned}
\tag{B.3}
$$

where α and β can take on arbitrary positive integer numbers. For later convenience it is furthermore helpful to define

$$
\begin{aligned}
N(\alpha,\beta,d) := &\frac{Z(p^2,\alpha,\beta,d)}{(-p^2-i0^+)^{d/2-\alpha-\beta}} \\
&= i\,\frac{(-1)^{\alpha+\beta}}{(4\pi)^{d/2}}\cdot\frac{\Gamma(\alpha+\beta-d/2)\,\Gamma(d/2-\alpha)\,\Gamma(d/2-\beta)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(d-\alpha-\beta)}.
\end{aligned}
\tag{B.4}
$$

Another handy expression that will be used a lot in the following relates integrals of different arguments with each other:

$$
\begin{aligned}
Z(p^2,\alpha+1,\beta,d) &= Z(p^2,\alpha,\beta+1,d) = \\
&= \frac{i\,(-1)^{d/2}}{(4\pi)^{d/2}}\,\frac{\Gamma(\alpha+\beta-d/2+1)\,\Gamma(d/2-\alpha-1)\,\Gamma(d/2-\beta)}{\Gamma(\alpha+1)\,\Gamma(\beta)\,\Gamma(d-\alpha-\beta-1)\,(p^2+i0^+)^{\alpha+\beta+1-d/2}} \\
&= Z(p^2,\alpha,\beta,d)\,\frac{(\alpha+\beta-d/2)\,(d-\alpha-\beta-1)}{(d/2-\alpha-1)\,\alpha\,(p^2+i0^+)}.
\end{aligned}
\tag{B.5}
$$

Since we are going to rewrite all subsequent results in terms of the integrals defined in eq. (B.3), we will refer to this class of loop integrals as *one-loop master integral*.

Applying these definitions to the calculation of the electron self-energy, the first integral in eq. (B.2) yields

$$\begin{aligned}
\Sigma_a(\not{k}) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(2-d)(\not{k}-\not{q})}{[(k-q)^2]_+ \, [q^2]_+} \\
&= (2-d)\left[\not{k}\, Z(k^2,1,1,d) - \frac{\not{k}}{2}\, \frac{Z(k^2,1,1,d)}{k^2}\right] \\
&= (2-d)\,\frac{\not{k}}{2} \cdot Z(k^2,1,1,d),
\end{aligned} \tag{B.6}$$

In order to sketch how this result can be derived, let us consider the following tensor integral which occurs in the calculation of Σ_a above:

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^\mu}{[(k-q)^2]_+ \, [q^2]_+}. \tag{B.7}$$

An easy way to solve this integral is to perform a *tensor reduction* and decompose the loop integral into scalar integrals. Since the result of the integral given in eq. (B.7) can only depend on the momentum k, it has the Lorentz decomposition

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^\mu}{[(k-q)^2]_+ \, [q^2]_+} = k^\mu B_1(q^2;0,0), \tag{B.8}$$

where B_1 can be expressed through scalar integrals only. Contracting both sides with k_μ gives

$$\begin{aligned}
k^2 B_1 &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q \cdot k}{[(k-q)^2]_+ \, [q^2]_+} \\
&= \frac{1}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\left[k^2 + q^2 - (k-q)^2\right]}{[(k-q)^2]_+ \, [q^2]_+}.
\end{aligned} \tag{B.9}$$

Neglecting all terms that lead to 'scaleless' integrals we get

$$k^2 B_1 = \frac{1}{2} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{k^2}{[(k-q)^2]_+ \, [q^2]_+} = \frac{1}{2}\, k^2\, Z(k^2,1,1,d) \tag{B.10}$$

and, thus, eq. (B.7) is given by

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^\mu}{[(k-q)^2]_+ \, [q^2]_+} = \frac{1}{2}\, k^\mu\, Z(k^2,1,1,d). \tag{B.11}$$

The second integral of eq. (B.2), $\Sigma_b(\not{k})$, can be solved in a similar fashion. A straightforward calculation yields

$$\begin{aligned}
\Sigma_b(\not{k}) &= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{2\not{q}(k\cdot q) - q^2(\not{q}+\not{k})}{[(k-q)^2]_+ \, [q^2]_+^2} \\
&= \frac{\not{k}}{2}\left[k^2\, Z(k^2,2,1,d) - Z(k^2,1,1,d)\right].
\end{aligned} \tag{B.12}$$

Combining Σ_a and Σ_b and exploiting the relation (B.5), we obtain

$$-i\Sigma(\not{k}) = e^2\, \tilde{\mu}^{2\epsilon}\, \frac{\not{k}}{2}\, \xi\,(d-2)\, Z(k^2,1,1,d) = \frac{\not{k}}{[-k^2 - i0^+]^{2-d/2}} \cdot \xi \cdot f(\epsilon), \tag{B.13}$$

with

$$f(\epsilon) = e^2\, \tilde{\mu}^{2\epsilon}\, \frac{(d-2)}{2} \cdot N(1,1,d) = e^2\, \tilde{\mu}^{2\epsilon}\, \frac{(d-2)}{2} \cdot \frac{i}{(4\pi)^{d/2}}\, \frac{\Gamma(2-d/2)\, \left[\Gamma(d/2-1)\right]^2}{\Gamma(d-2)}. \tag{B.14}$$

As one can see, the amplitude of the electron self-energy diagram depends a lot on the choice of the gauge parameter ξ and vanishes in Landau gauge ($\xi = 0$). Consequently, all diagrams containing $\Sigma(\not{k})$ as subgraph are also identical to zero and do not need to be considered. This fact provides the main reason for the usefulness of Landau gauge in the calculation of the massless two-loop two-point function.

Even though it is now apparent why Landau gauge represents the most suitable gauge choice, we will continue our analysis in general R_ξ-gauge and keep the gauge parameter arbitrary. As we are going to see, due to gauge invariance the ξ-dependent terms cancel anyway in the sum of all three diagrams that contribute to the massless two-loop two-point function.

Using the result of eq. (B.13), the amplitude $\mathcal{M}_1^{\mu\nu}$ of the diagram depicted in Fig. B.1 is given by

$$\begin{aligned} i\mathcal{M}_1^{\mu\nu} &= -\, i\, e^2\, \tilde{\mu}^{2\epsilon}\, f(\epsilon)\, \xi \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{(-1)\,\mathrm{Tr}\left[\gamma^\mu\, \not{k}\, \not{k}\, \not{k}\, \gamma^\nu\, (\not{k}+\not{p})\right]}{[k^2]_+^{4-d/2}\ [(k+p)^2]_+} \\ &= -\, i\, e^2\, \tilde{\mu}^{2\epsilon}\, f(\epsilon)\, \xi \cdot 4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-k^2 g^{\mu\nu} - g^{\mu\nu}(k\cdot p)}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} \\ &\quad -\, i\, e^2\, \tilde{\mu}^{2\epsilon}\, f(\epsilon)\, \xi \cdot 4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu p^\nu + p^\mu k^\nu + 2k^\mu k^\nu}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} \end{aligned} \tag{B.15}$$

where the simplifications of the Dirac structure have been carried out by means of the *FeynCalc* Mathematica package [58],[59]. The easiest way to proceed is to decompose the tensor integrals into scalar integrals as demonstrated in the calculation of the electron self-energy above. One finally obtains:

- $$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-k^2 g^{\mu\nu}}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} = -g^{\mu\nu}\, Z(k^2, 2-d/2, 1, d) \tag{B.16}$$
- $$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-(k\cdot p)\, g^{\mu\nu}}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} = \frac{g^{\mu\nu}}{2}\left[Z(k^2, 2-d/2, 1, d) - p^2 Z(k^2, 3-d/2, 1, d)\right]$$
- $$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu p^\nu}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} = -\frac{p^\mu p^\nu}{2}\left[Z(p^2, 3-d/2, 1, d) + \frac{Z(p^2, 2-d/2, 1, d)}{p^2}\right]$$
- $$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{p^\mu k^\nu}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} = -\frac{p^\mu p^\nu}{2}\left[Z(p^2, 3-d/2, 1, d) + \frac{Z(p^2, 2-d/2, 1, d)}{p^2}\right].$$

The last term in eq. (B.15) can be decomposed in the form

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} = g^{\mu\nu} p^2 B_{00} + p^\mu p^\nu B_{11}. \tag{B.17}$$

In order to determine the coefficients B_{00} and B_{11} we need to contract both sides with $g_{\mu\nu}$ and $p_\mu p_\nu$, respectively. After a short calculation, we get

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2]_+^{3-d/2}\ [(k+p)^2]_+} &= \left(g^{\mu\nu} - d\,\frac{p^\mu p^\nu}{p^2}\right) \frac{1}{2\,(1-d)}\left[\frac{Z(p^2, 1-d/2, 1, d)}{2p^2}\right. \\ &\quad + \frac{p^2\, Z(p^2, 3-d/2, 1, d)}{2} - Z(p^2, 2-d/2, 1, d) \\ &\quad \left. + \frac{p^\mu p^\nu}{p^2}\, Z(p^2, 2-d/2, 1, d)\right]. \end{aligned} \tag{B.18}$$

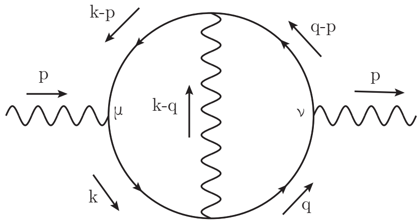

Fig. B.2: Second diagram needed for the calculation of the two-loop vacuum polarization function. This graph contains an overlapping divergence.

Putting all these results together, the amplitude $\mathcal{M}_1^{\mu\nu}$ is given by

$$\begin{aligned} i\mathcal{M}_1^{\mu\nu} =& -i\,e^2\,\tilde{\mu}^{2\epsilon}\,f(\epsilon)\,\xi\cdot 4\,\frac{Z(p^2,2-d/2,1,d)}{6\,(1-d)\,(4-d)} \\ &\cdot\left\{g^{\mu\nu}\left(6d^2-22d+16\right)-\frac{p^\mu p^\nu}{p^2}\left(8d^2-32d+24\right)\right\} \\ =& -i\,e^4\,\tilde{\mu}^{4\epsilon}\,\xi\cdot\frac{2}{3}\frac{(d-2)}{(d-4)}\,N(1,1,d)\cdot Z(p^2,2-d/2,1,d)\left\{(3d-8)\,g^{\mu\nu}-4\,(d-3)\,\frac{p^\mu p^\nu}{p^2}\right\}. \end{aligned} \tag{B.19}$$

Diagram Containing a Vertex Subgraph

The calculation of the second diagram needed for the computation of the two-loop vacuum polarization function (see Fig. B.2) requires some more effort due to the fact that it contains an overlapping divergence. In contrast to the previous section, where we were able to reduce a two-loop diagram with a nested subdivergence to one-loop computations only, we now have to face the problem that such a reduction is not always possible, since the internal photon line is common to both subdivergences. Nevertheless, we will see that integration-by-parts (IBP) techniques prove very helpful in this context to express the occurring loop momentum integrals in terms of the one-loop integral defined in eq. (B.3).

The amplitude $\mathcal{M}_2^{\mu\nu}$ of the diagram depicted in Fig. B.2 is given by

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu} = (ie)^4\,\tilde{\mu}^{4\epsilon}\iint\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{d}^dq}{(2\pi)^d}&\left\{\frac{(-1)\,\mathrm{Tr}\left[\gamma^\mu\,i(\not{k}-\not{p})\,\gamma^\rho\,i(\not{q}-\not{p})\,\gamma^\nu\,i\not{q}\,\gamma^\sigma\,i\not{k}\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+}\right. \\ &\left.\cdot\frac{-i\left[g_{\rho\sigma}(k-q)^2-(1-\xi)(k-q)_\rho(k-q)_\sigma\right]}{[(k-q)^2]_+^2}\right\}. \end{aligned} \tag{B.20}$$

Since the amplitude depends only on the external momentum p, we can make the following ansatz consistent with Lorentz covariance:

$$i\mathcal{M}_2^{\mu\nu} = g^{\mu\nu}p^2\,A + p^\mu p^\nu\,B, \tag{B.21}$$

where the coefficients A and B can once again be expressed through scalar integrals only. In order to determine these coefficients, it is appropriate to contract our ansatz with both $g_{\mu\nu}$ and $p_\mu p_\nu$:

- $i\mathcal{M}_2^{\mu\nu}\,g_{\mu\nu} = p^2\,(A\,d+B)$ (B.22)
- $i\mathcal{M}_2^{\mu\nu}\,p_\mu p_\nu = \left(p^2\right)^2(A+B).$ (B.23)

Contraction with $g_{\mu\nu}$

We start our analysis with the computation of the contraction of the amplitude $\mathcal{M}_2^{\mu\nu}$ with $g_{\mu\nu}$. The contracted amplitude is given by

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu}\, g_{\mu\nu} = i\, e^4\, \tilde{\mu}^{4\epsilon} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} &\Bigg\{ \frac{\mathrm{Tr}\left[\gamma^\mu\, (\not{k}-\not{p})\, \gamma^\rho\, (\not{q}-\not{p})\, \gamma_\mu\, \not{q}\, \gamma_\rho\, \not{k}\right]}{[(k-p)^2]_+\, [(q-p)^2]_+\, [q^2]_+\, [k^2]_+\, [(k-q)^2]_+} \\ &- (1-\xi)\, \frac{\mathrm{Tr}\left[\gamma^\mu\, (\not{k}-\not{p})\, (\not{k}-\not{q})\, (\not{q}-\not{p})\, \gamma_\mu\, \not{q}\, (\not{k}-\not{q})\, \not{k}\right]}{[(k-p)^2]_+\, [(q-p)^2]_+\, [q^2]_+\, [k^2]_+\, [(k-q)^2]_+^2} \Bigg\} \\ = i\, e^4\, \tilde{\mu}^{4\epsilon} \left[I_{g,1} - (1-\xi)\, I_{g,2}\right]. \end{aligned} \tag{B.24}$$

Calculation of $I_{g,1}$

We will first concentrate on the calculation of the scalar integral

$$I_{g,1} = \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu\, (\not{k}-\not{p})\, \gamma^\rho\, (\not{q}-\not{p})\, \gamma_\mu\, \not{q}\, \gamma_\rho\, \not{k}\right]}{[(k-p)^2]_+\, [(q-p)^2]_+\, [q^2]_+\, [k^2]_+\, [(k-q)^2]_+}. \tag{B.25}$$

The numerator of this integral involves a trace over a product of eight γ-matrices that can be simplified in the following way

$$\begin{aligned} \mathrm{num}(I_{g,1}) &= \mathrm{Tr}\left[\gamma^\mu\, (\not{k}-\not{p})\, \gamma^\rho\, (\not{q}-\not{p})\, \gamma_\mu\, \not{q}\, \gamma_\rho\, \not{k}\right] \\ &= 4\,(d-2) \Big\{ (d-4)\, k^2\, (p\cdot q - q^2) \\ &\quad + (k\cdot p)\left[(d-4)\, q^2 + 4\,(k\cdot q) - 4\,(p\cdot q)\right] \\ &\quad + (k\cdot q)\left[4(p\cdot q) - (d-4)p^2\right] - 4(k\cdot q)^2 \Big\}, \end{aligned} \tag{B.26}$$

where we again used FeynCalc to perform the simplifications [58],[59]. For the purpose of decomposing $I_{g,1}$ into simpler integrals, it is convenient to rewrite the occurring scalar products using the identities:

- $$k\cdot q = \frac{1}{2}\left[k^2 + q^2 - (k-q)^2\right] \tag{B.27}$$
- $$k\cdot p = \frac{1}{2}\left[k^2 + p^2 - (k-p)^2\right] \tag{B.28}$$
- $$p\cdot q = \frac{1}{2}\left[p^2 + q^2 - (q-p)^2\right]. \tag{B.29}$$

Replacing all scalar products in eq. (B.26) by the expressions given above yields

$$\begin{aligned} \mathrm{num}(I_{g,1}) = 4\,(d-2) \Big\{ &\frac{(4-d)}{2}\left[k^2\,(q-p)^2 + q^2\,(k-p)^2\right] \\ &+ \frac{(d-8)}{2}\, p^2\, (k-q)^2 + (k-q)^2\,(q-p)^2 \\ &+ k^2\,(k-q)^2 - \left[(k-q)^2\right]^2 \\ &+ q^2\,\left[p^2 + (k-q)^2 - k^2 - (q-p)^2\right] \\ &+ (k-p)^2\,\left[p^2 + (k-q)^2 - k^2 - (q-p)^2\right] \\ &+ p^2\,\left[(q-p)^2 - p^2 + k^2\right] \Big\} \end{aligned} \tag{B.30}$$

Since the diagram depicted in Fig. B.2 is symmetric in both loop momenta, we see that the structure of the numerator is invariant under the exchange of k and q, as well as under the simultaneous exchange of $k \leftrightarrow (k-p)$ and $q \leftrightarrow (q-p)$. Exploiting this symmetry, we obtain

$$\begin{aligned} \mathrm{num}(I_{g,1}) = 4\,(d-2) \Big\{ &(4-d)\, k^2\,(q-p)^2 - \left(p^2\right)^2 \\ &+ \frac{(d-8)}{2}\, p^2\,(k-q)^2 - \left[(k-q)^2\right]^2 \\ &+ 4\, p^2\,(q-p)^2 - 2\, k^2\, q^2 + 4\, k^2\,(k-q)^2 \Big\}, \end{aligned} \tag{B.31}$$

where the last two terms lead to vanishing integrals ('scaleless' integrals) and do not contribute to the final result. Considering the structure of the denominator, the integral $I_{g,1}$ is thus given by

$$\begin{aligned}
I_{g,1} =& 4\,(d-2) \Bigg\{ \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(4-d)}{[(k-p)^2]_+ \, [q^2]_+ \, [(k-q)^2]_+} \\
&+ \frac{(d-8)}{2} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{p^2}{[(k-p)^2]_+ \, [(q-p)^2]_+ \, [q^2]_+ \, [k^2]_+} \\
&- \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(k-q)^2}{[(k-p)^2]_+ \, [(q-p)^2]_+ \, [q^2]_+ \, [k^2]_+} \\
&+ 4\,p^2 \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[(k-p)^2]_+ \, [q^2]_+ \, [k^2]_+ \, [(k-q)^2]_+} \\
&- \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\left(p^2\right)^2}{[(k-p)^2]_+ \, [(q-p)^2]_+ \, [q^2]_+ \, [k^2]_+ \, [(k-q)^2]_+} \Bigg\} \\
=& 4\,(d-2) \left\{ (4-d)\, I^a_{g,1} + \frac{(d-8)\,p^2}{2}\, I^b_{g,1} - I^c_{g,1} + 4\,p^2\, I^d_{g,1} - \left(p^2\right)^2 I^e_{g,1} \right\}.
\end{aligned} \tag{B.32}$$

The first four two-loop integrals of the equation above can easily be reduce to one-loop integrals of the form given in eq. (B.3). To demonstrate this decomposition consider the first integral $I^a_{g,1}$:

$$I^a_{g,1} = \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[(k-p)^2]_+ \, [q^2]_+ \, [(k-q)^2]_+}. \tag{B.33}$$

First performing the integration over the loop momentum q gives

$$\begin{aligned}
I^a_{g,1} &= \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{1}{[(k-p)^2]_+ \, [q^2]_+ \, [(k-q)^2]_+} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[(k-p)^2]_+} \cdot Z(k^2, 1, 1, d) \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{[(k-p)^2]_+ \; [k^2]_+^{2-d/2}} \cdot N(1,1,d) \\
&= N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d),
\end{aligned} \tag{B.34}$$

where we made use of the definition introduced in eq. (B.4). The other integrals of eq. (B.32) except for $I^e_{g,1}$ can be solved in a similar fashion, yielding

- $I^b_{g,1} = \left[Z(p^2,1,1,d)\right]^2$ (B.35)
- $I^c_{g,1} = -\dfrac{1}{2}\,p^2\,\left[Z(p^2,1,1,d)\right]^2$ (B.36)
- $I^d_{g,1} = N(1,1,d) \cdot Z(p^2, 3-d/2, 1, d).$ (B.37)

In order to sketch how the result for $I^c_{g,1}$ was derived, note that in the numerator, $(k-q)^2 = k^2 + q^2 - 2\,k\cdot q$, only the last term leads to a non-vanishing integral:

$$\begin{aligned}
I^c_{g,1} =& \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(k-q)^2}{[(k-p)^2]_+ \, [(q-p)^2]_+ \, [q^2]_+ \, [k^2]_+} \\
=& \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{-2\,k\cdot q}{[(k-p)^2]_+ \, [(q-p)^2]_+ \, [q^2]_+ \, [k^2]_+} \\
=& -2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^\mu}{[(k-p)^2]_+ \, [k^2]_+} \cdot \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q_\mu}{[(q-p)^2]_+ \, [q^2]_+}.
\end{aligned} \tag{B.38}$$

Both integrals in the last line can then be decomposed according to eq. (B.8).

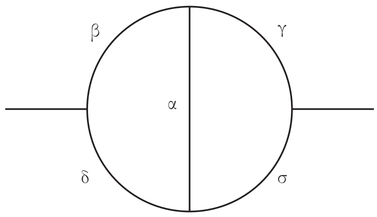

Fig. B.3: Topology of the two-loop two-point function diagram depicted in Fig. B.2.

Integration-by-Parts Technique

In contrast to the integrals $I^a_{g,1}$ to $I^d_{g,1}$, it is not possible to reduce the last integral $I^e_{g,1}$ to one-loop computations, since $I^e_{g,1}$ is associated with the case of an overlapping divergence.
Consider a generic two-loop integral of the form

$$I(\alpha, \beta, \gamma, \delta, \sigma, d) = \iint \frac{\mathrm{d}^d k\, \mathrm{d}^d q}{[(k-q)^2]^\alpha_+\ [(k-p)^2]^\beta_+\ [(q-p)^2]^\gamma_+\ [k^2]^\delta_+\ [q^2]^\sigma_+}, \tag{B.39}$$

that corresponds to the graph given in Fig. B.3. If one of the exponents vanished, one would be able to express eq. (B.39) in terms of nested one-loop integrals, which could then be solved by means of the one-loop master integrals given in eq. (B.3).

To achieve such a reduction it is convenient to use *integration-by-parts* (IBP), where one exploits the fact that in dimensional regularization the integral of a total derivative is zero, if the integrand vanishes at the boundaries:

$$\int \mathrm{d}^d k \, \frac{\partial}{\partial k^\mu} \, F(k) = 0. \tag{B.40}$$

In the equation above $F(k)$ denotes an arbitrary function that depends on the loop momentum k. Since in our case of the massless two-loop two-point function we are dealing with scalar integrals, we contract the derivative with the momentum $(k-q)^\mu$ of the internal photon line. Thus, we obtain

$$\iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \left\{ \frac{(k-q)^\mu}{[(k-q)^2]^\alpha_+\ [(k-p)^2]^\beta_+\ [(q-p)^2]^\gamma_+\ [k^2]^\delta_+\ [q^2]^\sigma_+} \right\} = 0. \tag{B.41}$$

Performing the derivatives

- $$\frac{\partial}{\partial k^\mu}(k-q)^\mu = \delta^\mu_\mu = d \tag{B.42}$$
- $$(k-q)^\mu \frac{\partial}{\partial k^\mu} \left[(k-q)^2\right]^{-\alpha} = \frac{-2\alpha}{[(k-q)^2]^\alpha} \tag{B.43}$$
- $$(k-q)^\mu \frac{\partial}{\partial k^\mu} \left[(k-p)^2\right]^{-\beta} = -\beta \left\{ \frac{1}{[(k-p)^2]^\beta} + \frac{(k-q)^2 - (q-p)^2}{[(k-p)^2]^{\beta+1}} \right\} \tag{B.44}$$
- $$(k-q)^\mu \frac{\partial}{\partial k^\mu} \left[k^2\right]^{-\delta} = -\delta \left\{ \frac{1}{[k^2]^\delta} - \frac{q^2 - (k-q)^2}{[k^2]^{\delta+1}} \right\} \tag{B.45}$$

gives

$$\begin{aligned} 0 &= \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\partial}{\partial k^\mu} \left\{ \frac{(k-q)^\mu}{[(k-q)^2]^\alpha_+\ [(k-p)^2]^\beta_+\ [(q-p)^2]^\gamma_+\ [k^2]^\delta_+\ [q^2]^\sigma_+} \right\} \\ &= \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{(d - 2\alpha - \beta - \delta)}{[(k-q)^2]^\alpha_+\ [(k-p)^2]^\beta_+\ [(q-p)^2]^\gamma_+\ [k^2]^\delta_+\ [q^2]^\sigma_+} \end{aligned} \tag{B.46}$$

$$
\begin{aligned}
&+\beta\bigg\{\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-q)^2]_+^{\alpha}\,[(k-p)^2]_+^{\beta+1}\,[(q-p)^2]_+^{\gamma-1}\,[k^2]_+^{\delta}\,[q^2]_+^{\sigma}}\\
&-\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-q)^2]_+^{\alpha-1}\,[(k-p)^2]_+^{\beta+1}\,[(q-p)^2]_+^{\gamma}\,[k^2]_+^{\delta}\,[q^2]_+^{\sigma}}\bigg\}\\
&+\delta\bigg\{\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-q)^2]_+^{\alpha}\,[(k-p)^2]_+^{\beta}\,[(q-p)^2]_+^{\gamma}\,[k^2]_+^{\delta+1}\,[q^2]_+^{\sigma-1}}\\
&-\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-q)^2]_+^{\alpha-1}\,[(k-p)^2]_+^{\beta}\,[(q-p)^2]_+^{\gamma}\,[k^2]_+^{\delta+1}\,[q^2]_+^{\sigma}}\bigg\}.
\end{aligned}
$$

The integral in eq. (B.39) is therefore given by

$$
\begin{aligned}
I(\alpha,\beta,\gamma,\delta,\sigma,d) =&\frac{\beta\,[I(\alpha-1,\beta+1,\gamma,\delta,\sigma,d)-I(\alpha,\beta+1,\gamma-1,\delta,\sigma,d)]}{(d-2\alpha-\beta-\delta)}\\
&+\frac{\delta\,[I(\alpha-1,\beta,\gamma,\delta+1,\sigma,d)-I(\alpha,\beta,\gamma,\delta+1,\sigma-1,d)]}{(d-2\alpha-\beta-\delta)}.
\end{aligned}
\tag{B.47}
$$

Using this result for the computation of the diagram depicted in Fig. B.2, the two-loop integral $I^e_{g,1}$ can be decomposed in the form

$$
\begin{aligned}
I^e_{g,1}\equiv I(1,1,1,1,1,d) =&\frac{I(0,2,1,1,1,d)-I(1,2,0,1,1,d)}{d-4}\\
&+\frac{I(0,1,1,2,1,d)-I(1,1,1,2,0,d)}{d-4}\\
=&\frac{2\;[I(0,2,1,1,1,d)-I(1,2,0,1,1,d)]}{d-4}.
\end{aligned}
\tag{B.48}
$$

In the last line we once again exploited the fact that the diagram is invariant under the exchange of k and q, as well as under the simultaneous exchange of $k\leftrightarrow(k-p)$ and $q\leftrightarrow(q-p)$. As indicated above, both integrals in eq. (B.48) can be solved in terms of the one-loop master integrals given in eq. (B.3):

$$
\begin{aligned}
I(0,2,1,1,1,d) &= \iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-p)^2]_+^2\,[(q-p)^2]_+\,[k^2]_+\,[q^2]_+}\\
&= Z(p^2,1,2,d)\cdot Z(p^2,1,1,d),\\
I(1,2,0,1,1,d) &= \iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-q)^2]_+\,[(k-p)^2]_+^2\,[k^2]_+\,[q^2]_+}\\
&= N(1,1,d)\cdot Z(p^2,3-d/2,2,d).
\end{aligned}
\tag{B.49}
$$

Thus, we finally get

$$
I^e_{g,1} = \frac{2}{d-4}\Big\{Z(p^2,1,2,d)\cdot Z(p^2,1,1,d) - N(1,1,d)\cdot Z(p^2,3-d/2,2,d)\Big\}. \tag{B.50}
$$

Combining all previous results, the integral $I_{g,1}$ is given by

$$
\begin{aligned}
I_{g,1} = 4\,(d-2)\Bigg\{ & (4-d)\,N(1,1,d)\cdot Z(p^2,2-d/2,1,d) \\
& + \frac{1}{2}\,(d-8)\,p^2\,\big[Z(p^2,2-d/2,1,d)\big]^2 \\
& + \frac{1}{2}\,p^2\,\big[Z(p^2,2-d/2,1,d)\big]^2 \\
& + 4\,p^2\,N(1,1,d)\cdot Z(p^2,3-d/2,1,d) \\
& - \left(p^2\right)^2\cdot\frac{2}{(d-4)}\,Z(p^2,1,2,d)\cdot Z(p^2,1,1,d) \\
& + \left(p^2\right)^2\cdot\frac{2}{(d-4)}\,N(1,1,d)\cdot Z(p^2,3-d/2,2,d)\Bigg\} \\
= 4\,(d-2)\Bigg\{ & (4-d)\,N(1,1,d)\cdot Z(p^2,2-d/2,1,d) \\
& + p^2\;\big[Z(p^2,2-d/2,1,d)\big]^2\;\frac{(d-4)^2+d}{2\,(d-4)} \\
& + p^2\,N(1,1,d)\cdot Z(p^2,3-d/2,1,d)\;\frac{2\,(2-d)}{d-4}\Bigg\}
\end{aligned}
\tag{B.51}
$$

Calculation of $I_{g,2}$

We now turn to the computation of the second scalar integral of the contracted amplitude:

$$
I_{g,2} = \iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\;\frac{\mathrm{Tr}\left[\gamma^\mu\,(\not{k}-\not{p})\,(\not{k}-\not{q})\,(\not{q}-\not{p})\,\gamma_\mu\,\not{q}\,(\not{k}-\not{q})\,\not{k}\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+\;[(k-q)^2]_+^2}. \tag{B.52}
$$

Using FeynCalc to simplify the Dirac structure, the numerator of the integral becomes [58],[59]

$$
\begin{aligned}
\mathrm{num}(I_{g,2}) &= \mathrm{Tr}\left[\gamma^\mu\,(\not{k}-\not{p})\,(\not{k}-\not{q})\,(\not{q}-\not{p})\,\gamma_\mu\,\not{q}\,(\not{k}-\not{q})\,\not{k}\right] \\
&= 4\,(d-2)\Big\{\left(k^2\right)^2\;(p\cdot q-q^2)+q^2\big[p^2(k\cdot q)+2\,(k\cdot p)^2 \\
&\quad+(k\cdot p)\left(q^2-2(k\cdot q+p\cdot q)\right)\big]+k^2\big[q^2(k\cdot p-2p^2+p\cdot q) \\
&\quad+(k\cdot q)(p^2-2\,(p\cdot q)+2q^2)+2\,(p\cdot q)(p\cdot q-k\cdot p)-\left(q^2\right)^2\big]\Big\}.
\end{aligned}
\tag{B.53}
$$

In order to further reduce $I_{g,2}$ into simpler integrals, we can rewrite the scalar products in the numerator by means of eqs. (B.27 - B.29):

$$
\begin{aligned}
\mathrm{num}(I_{g,2}) = 2\,(d-2)\Big\{ & k^2\left[(q-p)^2\right]+q^2\left[(k-p)^2\right]^2+\left(k^2\right)^2(q-p)^2 \\
& +\left(q^2\right)^2(k-p)^2-k^2\,p^2(q-p)^2-q^2\,p^2(k-p)^2 \\
& +k^2\,p^2(k-p)^2+q^2\,p^2(q-p)^2-k^2\,q^2(q-p)^2 \\
& -k^2\,q^2(k-p)^2-k^2(q-p)^2(k-p)^2-q^2(q-p)^2(k-p)^2 \\
& -k^2(k-q)^2(q-p)^2-q^2(k-q)^2-(k-p)^2\Big\}.
\end{aligned}
\tag{B.54}
$$

Exploiting the symmetry of the diagram, we finally get

$$
\begin{aligned}
\mathrm{num}(I_{g,2}) = 4\,(d-2)\Big\{ & 2\left(k^2\right)^2(q-p)^2-k^2\,p^2(q-p)^2-k^2(k-q)^2(q-p)^2 \\
& +k^2\,p^2(k-p)^2-k^2\,q^2(k-p)^2-k^2(q-p)^2(k-p)^2\Big\},
\end{aligned}
\tag{B.55}
$$

where the last three terms give vanishing contributions. Taking the structure of the denominator into account, the integral $I_{g,2}$ is given by

$$\begin{aligned} I_{g,2} =\ & 4\,(d-2)\Bigg\{ \iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{2\,k^2}{[(k-p)^2]_+\,[q^2]_+\,[(k-q)^2]_+^2} \\ & -\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{p^2}{[(k-p)^2]_+\,[q^2]_+\,[(k-q)^2]_+^2} \\ & -\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d q}{(2\pi)^d}\,\frac{1}{[(k-p)^2]_+\,[q^2]_+\,[(k-q)^2]_+}\Bigg\} \\ =\ & 4\,(d-2)\left\{2\,I^a_{g,2} - p^2\,I^b_{g,2} - I^c_{g,2}\right\}. \end{aligned} \tag{B.56}$$

In contrast to the calculation of $I_{g,1}$, where we had to apply integration-by-parts to solve the occurring integrals, $I_{g,2}$ can easily be reduced to the one-loop master integrals defined in eq. (B.3) without further knowledge. Following a similar approach as in the previous section, we obtain

- $I^a_{g,2} = N(1,2,d)\cdot Z(p^2, 2-d/2, 1, d)$ (B.57)
- $I^b_{g,2} = N(1,2,d)\cdot Z(p^2, 3-d/2, 1, d)$ (B.58)
- $I^c_{g,2} = N(1,1,d)\cdot Z(p^2, 2-d/2, 1, d).$ (B.59)

Inserting these results back into eq. (B.56), the integral $I_{g,2}$ takes the form

$$\begin{aligned} I_{g,2} =\ & 4\,(d-2)\Big\{2\,N(1,2,d)\cdot Z(p^2,2-d/2,1,d) \\ & - p^2\,N(1,2,d)\cdot Z(p^2,3-d/2,1,d) \\ & - N(1,1,d)\cdot Z(p^2,2-d/2,1,d)\Big\} \\ =\ & 4\,(d-2)\,Z(p^2,2-d/2,1,d)\Big\{N(1,2,d)\,\frac{d}{4-d} - N(1,1,d)\Big\} \\ =\ & 4\,(d-2)\,Z(p^2,2-d/2,1,d)\cdot N(1,1,d)\,\frac{(d-2)^2}{d-4}. \end{aligned} \tag{B.60}$$

Thus, we finally obtain the following expression for the contraction of the amplitude $\mathcal{M}_2^{\mu\nu}$ with $g_{\mu\nu}$:

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu} g_{\mu\nu} &= ie^4\,\tilde{\mu}^{4\epsilon}\,[I_{g,1} - (1-\xi)I_{g,2}] \\ &= ie^4\,\tilde{\mu}^{4\epsilon}\,4\,\frac{(d-2)}{(d-4)}\Bigg\{\left[Z(p^2,1,1,d)\right]^2\,p^2\,\frac{(d-4)^2+d}{2} \\ &\quad + N(1,1,d)\cdot Z(p^2,2-d/2,1,d)\,\frac{2\,(24-20d+7d^2-d^3)}{(d-4)} \\ &\quad + \xi\,(d-2)^2\,N(1,1,d)\cdot Z(p^2,2-d/2,1,d)\Bigg\} \end{aligned} \tag{B.61}$$

Contraction with $p_\mu p_\nu$

The computation of the second conditional equation (see eq. (B.23)) needed to determine the coefficients A and B of our ansatz given in eq. (B.21), follows basically the same procedure as the calculation we have done in the previous sections.

The contraction of $\mathcal{M}_2^{\mu\nu}$ with $p_\mu p_\nu$ yields

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu}\, p_\mu p_\nu = i\, e^4\, \tilde{\mu}^{4\epsilon} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \Bigg\{ & \frac{\mathrm{Tr}\left[\not p\,(\not k - \not p)\,\gamma^\rho\,(\not q - \not p)\,\not p\,\not q\,\gamma_\rho\,\not k\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+\,[(k-q)^2]_+} \\ & - (1-\xi)\, \frac{\mathrm{Tr}\left[\not p\,(\not k - \not p)\,(\not k - \not q)\,(\not q - \not p)\,\not p\,\not q\,(\not k - \not q)\,\not k\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+\,[(k-q)^2]_+^2} \Bigg\} \\ = i\, e^4\, \tilde{\mu}^{4\epsilon} \left[I_{p,1} - (1-\xi)\, I_{p,2}\right]. \end{aligned} \tag{B.62}$$

where the numerators of both integrals, $I_{p,1}$ and $I_{p,2}$, involve a trace over a product of eight γ-matrices.

Calculation of $I_{p,1}$

First, we deal with the calculation of the scalar integral

$$I_{p,1} = \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{Tr}\left[\not p\,(\not k - \not p)\,\gamma^\rho\,(\not q - \not p)\,\not p\,\not q\,\gamma_\rho\,\not k\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+\,[(k-q)^2]_+}. \tag{B.63}$$

After performing the Dirac algebra and substituting all occurring scalar products with the identities given in eq. (B.27 - B.29), the numerator takes the form

$$\begin{aligned} \mathrm{num}(I_{p,1}) =& \,\mathrm{Tr}\left[\not p\,(\not k - \not p)\,\gamma^\rho\,(\not q - \not p)\,\not p\,\not q\,\gamma_\rho\,\not k\right] \\ =& -2\,(d-2)\Big\{k^2\,p^2(q-p)^2 + q^2\,p^2(k-p)^2 - q^2\left[(k-p)^2\right]^2 \\ & - k^2\left[(q-p)^2\right]^2 + q^2\,k^2(k-p)^2 + k^2\,q^2(q-p)^2 \\ & - \left(k^2\right)^2(q-p)^2 - \left(q^2\right)^2(k-p)^2 - (q-p)^2(k-p)^2(k-q)^2 \\ & - q^2\,k^2(k-q)^2 + k^2(k-q)^2(q-p)^2 + q^2(k-q)^2(k-p)^2 \\ & + q^2(q-p)^2(k-p)^2 + k^2(k-p)^2(q-p)^2\Big\}. \end{aligned} \tag{B.64}$$

Exploiting the fact that the diagram depicted in Fig. B.2 is symmetric in both loop momenta, the numerator structure can be further simplified to

$$\begin{aligned} \mathrm{num}(I_{p,1}) = \; 4\,(d-2)\Big\{& 2\left(k^2\right)^2(q-p)^2 - k^2\,p^2(q-p)^2 - 2\,q^2\,k^2(k-p)^2 \\ & + q^2\,k^2(k-q)^2 - k^2(q-p)^2(k-q)^2\Big\}, \end{aligned} \tag{B.65}$$

where the last three terms can be omitted, since they lead to vanishing integrals. Combining this result for the numerator with the denominator of eq. (B.63), we obtain

$$\begin{aligned} I_{p,1} = \; 4\,(d-2)\Bigg\{& \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{2\,k^2}{[(k-p)^2]_+\,[(k-q)^2]_+\,[q^2]_+} \\ & - \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{p^2}{[(k-p)^2]_+\,[(k-q)^2]_+\,[q^2]_+} \Bigg\} \\ = \; 4\,(d-2)\,& \left[2\,I_{p,1}^a - p^2\,I_{p,1}^b\right], \end{aligned} \tag{B.66}$$

where both integrals, $I_{p,1}^a$ and $I_{p,1}^b$, can be expressed in terms of one-loop master integrals. A straightforward calculation yields

- $I_{p,1}^a = N(1,1,d)\cdot Z(p^2, 1-d/2, 1, d)$ (B.67)
- $I_{p,1}^b = N(1,1,d)\cdot Z(p^2, 2-d/2, 1, d).$ (B.68)

Hence, $I_{p,1}$ is given by

$$I_{p,1} = \; 4\,(d-2)\cdot N(1,1,d)\left\{2\cdot Z(p^2, 1-d/2, 1, d) - p^2\, Z(p^2, 2-d/2, 1, d)\right\}. \tag{B.69}$$

Calculation of $I_{p,2}$

The second integral of eq. (B.62),

$$I_{p,2} = \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{Tr}\left[\not{p}\,(\not{k}-\not{p})\,(\not{k}-\not{q})\,(\not{q}-\not{p})\,\not{p}\,\not{q}\,(\not{k}-\not{q})\,\not{k}\right]}{[(k-p)^2]_+\,[(q-p)^2]_+\,[q^2]_+\,[k^2]_+\,[(k-q)^2]_+^2}, \tag{B.70}$$

can be calculated in exactly the same way. First performing some Dirac algebra, then rewriting all scalar products and finally exploiting the symmetry of the diagram, the numerator of $I_{p,2}$ becomes:

$$\begin{aligned} \mathrm{num}(I_{p,2}) &= \mathrm{Tr}\left[\not{p}\,(\not{k}-\not{p})\,(\not{k}-\not{q})\,(\not{q}-\not{p})\,\not{p}\,\not{q}\,(\not{k}-\not{q})\,\not{k}\right] \\ &= 4\Big\{ \left(k^2\right)^2 \left[(q-p)^2\right]^2 - k^2\,p^2(q-p)^2(k-q)^2 - k^2\,q^2(k-p)^2(q-p)^2 \Big\}. \end{aligned} \tag{B.71}$$

Neglecting the last term, that gives a vanishing contribution and taking the denominator structure into consideration, we get

$$\begin{aligned} I_{p,2} = \;& 4\left\{ \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{k^2(q-p)^2}{[(k-p)^2]_+\,[q^2]_+\,[(k-q)^2]_+^2} \right. \\ & \left. - \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{p^2}{[(k-p)^2]_+\,[q^2]_+\,[(k-q)^2]_+} \right\} \\ = \;& 4\left[I^a_{p,2} - p^2\,I^b_{p,2}\right]. \end{aligned} \tag{B.72}$$

where the integrals, $I^a_{p,2}$ and $I^b_{p,2}$, are given by

- $$\begin{aligned} I^a_{p,2} =& \frac{p^2}{2}\,Z(p^2, 2-d/2, 1, d)\big[N(1,1,d) + N(1,2,d)\big] \\ & + \frac{1}{2}\,Z(p^2, 1-d/2, 1, d)\big[N(1,1,d) - N(1,2,d)\big] \end{aligned} \tag{B.73}$$
- $$I^b_{p,2} = N(1,1,d)\cdot Z(p^2, 2-d/2, 1, d). \tag{B.74}$$

Inserting these results back into eq. (B.72) yields

$$\begin{aligned} I_{p,2} = \;& 4\left\{ \frac{p^2}{2}\,Z(p^2, 2-d/2, 1, d)\big[N(1,2,d) - N(1,1,d)\big] \right. \\ & \left. + \frac{1}{2}\,Z(p^2, 1-d/2, 1, d)\big[N(1,1,d) - N(1,2,d)\big] \right\}. \end{aligned} \tag{B.75}$$

We hence obtain the following expression for the contraction of $\mathcal{M}_2^{\mu\nu}$ with $p_\mu p_\nu$:

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu}\,p_\mu p_\nu &= i\,e^4\,\tilde{\mu}^{4\epsilon}\left[I_{p,1} - (1-\xi)\,I_{p,2}\right] \\ &= i\,e^4\,\tilde{\mu}^{4\epsilon}\,4\,\xi\left\{ \frac{p^2}{2}\,Z(p^2, 2-d/2, 1, d)\big[N(1,2,d) - N(1,1,d)\big] \right. \\ &\qquad \left. + \frac{1}{2}\,Z(p^2, 1-d/2, 1, d)\big[N(1,1,d) - N(1,2,d)\big] \right\} \\ &= i\,e^4\,\tilde{\mu}^{4\epsilon}\,\frac{4}{3}\,\xi\cdot p^2\,Z(p^2, 2-d/2, 1, d)\big[N(1,2,d) - N(1,1,d)\big]. \end{aligned} \tag{B.76}$$

Since the contraction with $p_\mu p_\nu$ is explicitly ξ-dependent, eq. (B.76) has a major impact on the form of the amplitude $\mathcal{M}_2^{\mu\nu}$. If we choose to work in Landau gauge ($\xi = 0$), the contracted amplitude vanishes which, according to the conditional eq. (B.23), implies that $A = -B$. Therefore, the amplitude $\mathcal{M}_2^{\mu\nu}$ is transversal in Landau gauge:

$$i\mathcal{M}_2^{\mu\nu} = (g^{\mu\nu}p^2 - p^\mu p^\nu)\,A \qquad \text{(for } \xi = 0\text{)}. \tag{B.77}$$

In the following, however, we will work in R_ξ-gauge and will not make a specific choice for the value of the gauge parameter. Combining all previously obtained results for the contraction with both, $g_{\mu\nu}$ and $p_\mu p_\nu$, the conditional equations

$$1) \quad i\mathcal{M}_2^{\mu\nu}\, g_{\mu\nu} = p^2\,(A\,d + B) \tag{B.78}$$

$$2) \quad i\mathcal{M}_2^{\mu\nu}\, p_\mu p_\nu = \left(p^2\right)^2 (A + B) \tag{B.79}$$

provide a system of equations for the coefficients A and B. Solving this system, the coefficients are given by

$$\begin{aligned} A =& \frac{i\,e^4\,\tilde{\mu}^{4\epsilon}}{p^2\,(d-1)}\,\frac{(d-2)}{(d-4)} \cdot 4 \Bigg\{ \left[Z(p^2,1,1,d)\right]^2\, p^2\, \frac{(d-4)^2 + d}{2} \\ &+ N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d)\, \frac{2\,(24 - 20d + 7d^2 - d^3)}{(d-4)} \\ &+ \xi\, N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d) \left[(d-2)^2 + \frac{(d-4)}{3}\right] \Bigg\} \end{aligned} \tag{B.80}$$

$$B = i\,e^4\,\tilde{\mu}^{4\epsilon}\,\frac{4\,(2-d)}{3\,p^2} - A \tag{B.81}$$

and the amplitude $\mathcal{M}_2^{\mu\nu}$ finally takes the form

$$\begin{aligned} i\mathcal{M}_2^{\mu\nu} =& i\,e^4\,\tilde{\mu}^{4\epsilon} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \cdot 4\, \frac{(d-2)}{(d-4)\,(d-1)} \Bigg\{ \left[Z(p^2,1,1,d)\right]^2\, p^2\, \frac{(d-4)^2 + d}{2} \\ &+ N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d)\, \frac{2\,(24 - 20d + 7d^2 - d^3)}{(d-4)} \Bigg\} \\ &+ \xi\, i\, e^4\, \tilde{\mu}^{4\epsilon}\, \frac{4}{3}\, \frac{(d-2)}{(d-4)}\, N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d) \\ &\cdot \left\{ (3d-8)\, g^{\mu\nu} - 4\,(d-3)\, \frac{p^\mu p^\nu}{p^2} \right\}. \end{aligned} \tag{B.82}$$

Sum of all Diagrams

If one compares the results of eq. (B.19) and eq. (B.82), the expected cancellation of the ξ-dependent terms becomes obvious. Thus, in the sum of all three diagrams that contribute to the vacuum polarization function at the two-loop level, the amplitudes $\mathcal{M}_1^{\mu\nu}$ of the two diagrams involving a self-energy subgraph cancel exactly the ξ-dependent term of the amplitude $\mathcal{M}_2^{\mu\nu}$ and we obtain

$$\begin{aligned} i\mathcal{M}^{\mu\nu} =& 2\, i\mathcal{M}_1^{\mu\nu} + i\mathcal{M}_2^{\mu\nu} \\ =& i\,e^4\,\tilde{\mu}^{4\epsilon} \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \cdot 4\, \frac{(d-2)}{(d-4)\,(d-1)} \Bigg\{ \left[Z(p^2,1,1,d)\right]^2\, p^2\, \frac{(d-4)^2 + d}{2} \\ &+ N(1,1,d) \cdot Z(p^2, 2-d/2, 1, d)\, \frac{2\,(24 - 20d + 7d^2 - d^3)}{(d-4)} \Bigg\} \\ =& -i\,e^4\,\tilde{\mu}^{4\epsilon} \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \cdot \frac{4\,(d-2)}{(d-4)\,(d-1)}\, \frac{(-p^2 - i0^+)^{d-4}}{(4\pi)^d}\, \left[\Gamma(d/2-1)\right]^3 \\ &\cdot \left\{ \left[\frac{\Gamma(2-d/2)}{\Gamma(d-2)}\right]^2 \Gamma(d/2-1) \cdot \frac{(d-4)^2 + d}{2} + 2 \cdot \frac{\Gamma(4-d)}{\Gamma(3d/2-3)}\, \frac{(d-2)^2 + 4}{(d-4)} \right\} \\ =& \frac{\alpha^2}{(4\pi)^2} \left(g^{\mu\nu} p^2 - p^\mu p^\nu\right) \cdot \frac{4\,(1-\epsilon)}{\epsilon\,(3-2\epsilon)} \left(\frac{-p^2 - i0^+}{4\pi\,\tilde{\mu}^2}\right)^{-2\epsilon} \left[\Gamma(1-\epsilon)\right]^3 \\ &\cdot \left\{ \left[\frac{\Gamma(\epsilon)}{\Gamma(2-2\epsilon)}\right]^2 \Gamma(1-\epsilon) \cdot (2 - \epsilon + 2\epsilon^2) - \frac{\Gamma(2\epsilon)}{\Gamma(3-3\epsilon)}\, \frac{(2-2\epsilon)^2 + 4}{\epsilon} \right\}. \end{aligned} \tag{B.83}$$

In the last line we used $\alpha = e^2/(4\pi)$ and replaced the dimension d by $d = 4 - 2\epsilon$. Note again that all QED results we have derived in this appendix, are also valid in QCD (where the internal photon line gets substituted with a gluon), if we consider the QCD colour factors. Therefore, we see that eq. (B.83) is consistent with the expression for the two-loop contribution $\Pi^{(0)}$ of the Adler function given in eq. (A.5), if we take into account the notation introduced in appendix A.

Appendix C

QCD β-Function and Renormalon Sum Rule Coefficients

In this appendix we provide the details on important relations and coefficients needed for the derivation of the renormalon sum rule described in section 4.3.

C.1 QCD β-Function

First, let us consider the RG equation for the strong coupling α_s in the $\overline{\mathrm{MS}}$-scheme for which we use the convention:

$$\frac{\mathrm{d}\alpha_\mathrm{s}}{\mathrm{d}\ln R} = \beta[\alpha_\mathrm{s}(R)] = -2\alpha_\mathrm{s}(R)\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1}. \tag{C.1}$$

The coefficients of the QCD β-function are currently known up to the five-loop term β_4 [60] and, in the notation used here, they read:

$$\begin{aligned}
\beta_0 &= 11 - \frac{2}{3}N_f, \\
\beta_1 &= 102 - \frac{38}{3}N_f, \\
\beta_2 &= \frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2, \\
\beta_3 &= \frac{149753}{6} + 3564\,\zeta_3 - \left(\frac{1078361}{162} + \frac{6508}{27}\zeta_3\right)N_f + \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right)N_f^2 + \frac{1093}{729}N_f^3, \\
\beta_4 &= \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\,\zeta_5 \\
&\quad + \left(-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right)N_f \\
&\quad + \left(\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right)N_f^2 \\
&\quad + \left(-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right)N_f^3 \\
&\quad + \left(\frac{1205}{2916} - \frac{152}{81}\zeta_3\right)N_f^4.
\end{aligned} \tag{C.2}$$

In numerical form the coefficients of the β-function are given by,

$$\begin{aligned}
\beta_0 &\approx 11 - 0.6667\,N_f, \\
\beta_1 &\approx 102 - 12.6667\,N_f, \\
\beta_2 &\approx 1428.5 - 279.611\,N_f + 6.0185\,N_f^2,
\end{aligned} \tag{C.3}$$

$$\beta_3 \approx 29243 - 6946.29\,N_f + 405.089\,N_f^2 + 1.4993\,N_f^3,$$
$$\beta_4 \approx 537148 - 186162\,N_f + 17567.8\,N_f^2 - 231.278\,N_f^3 - 1.8425\,N_f^4.$$

The general solution of the RGE can be written in the form [48],[51]:

$$\ln\frac{R_1}{R_0} = \int_{\alpha_0}^{\alpha_1} \frac{\mathrm{d}\alpha_s}{\beta[\alpha_s]} = \int_{t_1}^{t_0} \mathrm{d}t\,\hat{b}(t) = G(t_0) - G(t_1), \tag{C.4}$$

where we used $\alpha_i = \alpha_s(R_i)$ and changed to the integration variable $t = -2\pi/(\beta_0\alpha_s)$ with $t_i = -2\pi/(\beta_0\alpha_i)$ in the second equality. We find

$$G(t) = t + \hat{b}_1\ln(-t) - \sum_{k=2}^{\infty}\frac{\hat{b}_k}{(k-1)\,t^{k-1}}, \tag{C.5}$$

$$\hat{b}(t) = G'(t) = 1 + \sum_{k=1}^{\infty}\frac{\hat{b}_k}{t^k}, \tag{C.6}$$

where the first few coefficients are given by:

$$\begin{aligned}
\hat{b}_1 &= \frac{\beta_1}{2\beta_0^2}, \\
\hat{b}_2 &= \frac{1}{4\beta_0^4}\,(\beta_1^2 - \beta_0\beta_2), \\
\hat{b}_3 &= \frac{1}{8\beta_0^6}\,(\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3), \\
\hat{b}_4 &= \frac{1}{16\beta_0^8}\,(\beta_1^4 - 3\beta_0\beta_1^2\beta_2 + \beta_0^2\beta_2^2 + 2\beta_0^2\beta_1\beta_3 - \beta_0^3\beta_4).
\end{aligned} \tag{C.7}$$

Note that there also exists a recursion relation for the coefficients $\hat{b}_k$ [51]:

$$\hat{b}_{n+1} = 2\sum_{i=0}^{n}\frac{\hat{b}_{n-i}\,\beta_{i+1}}{(-2\beta_0)^{i+2}}, \qquad \hat{b}_0 = 1. \tag{C.8}$$

From eq. (C.4) we finally obtain the well-known definition of the scale Λ_{QCD},

$$R_1 \mathrm{e}^{G(t_1)} = R_0 \mathrm{e}^{G(t_0)} = \Lambda_{\mathrm{QCD}}, \tag{C.9}$$

that allows us to determine an expression for $\Lambda_{\mathrm{QCD}}^{\mathrm{N}^k\mathrm{LL}}$ by truncating the series $G(t)$ after the k-term. In particular, the LL expression for Λ_{QCD} is found to be:

$$\Lambda_{\mathrm{QCD}}^{\mathrm{LL}} = R\,\mathrm{e}^{t_R} = R\,\mathrm{e}^{-\frac{2\pi}{\beta_0\alpha_s(R)}}. \tag{C.10}$$

C.2 OPE Corrections at NLO

In this part of the appendix we provide a detailed derivation of the NLL expression for the OPE corrections,

$$F_O = \vec{C}^{\intercal}(\mu)\,\langle\vec{O}(\mu)\rangle, \tag{C.11}$$

to a generic observable σ due to operators of equal dimensions (see section 3.3.1). Starting from the RGE for the evolution matrix $U_{\mathrm{NLL}}(\mu,\mu')$ in the next-to-leading logarithmic approximation,

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\,U_{\mathrm{NLL}}(\mu,\mu') = -\Big[\mathrm{diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_s(\mu) + g\,a_s^2(\mu)\Big]\,U_{\mathrm{NLL}}(\mu,\mu'), \tag{C.12}$$

where the matrix g is defined by

$$g := V^{-1}\gamma_O^{(2)}V, \tag{C.13}$$

it proves useful to make the following ansatz for the NLL solution:

$$\begin{aligned}U_{\rm NLL}(\mu,\mu') &= \big[\mathbb{1}+a_{\rm s}(\mu)S\big]\,U_{\rm LL}(\mu,\mu')\,\big[\mathbb{1}-a_{\rm s}(\mu')S\big] \\ &= \big[\mathbb{1}+a_{\rm s}(\mu)S\big]\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu').\end{aligned} \tag{C.14}$$

Inserting this ansatz into eq. (C.12), we find:

$$\begin{aligned}\frac{{\rm d}\,U_{\rm NLL}(\mu,\mu')}{{\rm d}\ln\mu} =& \Big[\mathbb{1}+a_{\rm s}(\mu)S\Big]{\rm diag}\Big[\frac{2\vec{\gamma}_D^{(1)}}{\beta_0}\Big(-\frac{\beta_0}{2}a_{\rm s}^2(\mu)-\frac{\beta_1}{8}a_{\rm s}^3(\mu)\Big)\big(a_{\rm s}(\mu)\big)^{2\vec{\gamma}_D^{(1)}/\beta_0-1}\Big]M(\mu') \\ &+S\Big(-\frac{\beta_0}{2}a_{\rm s}^2(\mu)-\frac{\beta_1}{8}a_{\rm s}^3(\mu)\Big){\rm daig}\Big[\big(a_{\rm s}(\mu)\big)^{2\vec{\gamma}_D^{(1)}/\beta_0}\Big]\,M(\mu')\end{aligned} \tag{C.15}$$

Neglecting all terms that go beyond the NLL level, we obtain:

$$\begin{aligned}\frac{{\rm d}\,U_{\rm NLL}(\mu,\mu')}{{\rm d}\ln\mu} =& -a_{\rm s}(\mu)\,\Big[\mathbb{1}+a_{\rm s}(\mu)S\Big]\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &-\frac{\beta_1}{4\beta_0}\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_{\rm s}^2(\mu)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &-\frac{\beta_0}{2}\,a_{\rm s}^2(\mu)\,S\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ =& -{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_{\rm s}(\mu)\,\big[\mathbb{1}+a_{\rm s}(\mu)S\big]\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &-g\,a_{\rm s}^2(\mu)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &+a_{\rm s}^2(\mu)\,\Big({\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,S-S\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\Big)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &-\frac{\beta_1}{4\beta_0}\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,a_{\rm s}^2(\mu)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &-\frac{\beta_0}{2}\,S\,a_{\rm s}^2(\mu)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu') \\ &+g\,a_{\rm s}^2(\mu)\,{\rm diag}\big[a_{\rm s}(\mu)^{2\vec{\gamma}_D^{(1)}/\beta_0}\big]\,M(\mu'),\end{aligned} \tag{C.16}$$

where the first two lines in the last equality reproduce the NLL RG equation given in eq. (C.12), while all other terms must vanish. This condition yields an defining equation for the matrix S,

$${\rm diag}\big[\vec{\gamma}_D^{(1)}\big]\,S-S\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]-\frac{\beta_1}{4\beta_0}\,{\rm diag}\big[\vec{\gamma}_D^{(1)}\big]-\frac{\beta_0}{2}\,S+g=0, \tag{C.17}$$

whose elements are therefore found to be:

$$\begin{aligned}S^{ij} &= \frac{\beta_1}{4\beta_0}\,\gamma_D^{(1),i}\,\delta^{ij}\,\frac{1}{(\gamma_D^{(1),i}-\gamma_D^{(1),j}-\beta_0/2)}-\frac{g^{ij}}{(\gamma_D^{(1),i}-\gamma_D^{(1),j}-\beta_0/2)} \\ &= -\frac{\beta_1}{2\beta_0^2}\,\gamma_D^{(1),i}\,\delta^{ij}+\frac{g^{ij}}{(\beta_0/2+\gamma_D^{(1),j}-\gamma_D^{(1),i})}.\end{aligned} \tag{C.18}$$

In the second line we exploited the fact that the first term in the definition of S^{ij} is symmetric under exchange of indices i and j. Furthermore, note that in contrast to $\gamma_D^{(1)}$ the matrix g^{ij} will not be diagonal in general. For more details on the derivation of the RG evolution matrix beyond the leading order approximation see [44, 45].

C.3 Derivation of the R-Anomalous Dimension

Exploiting the QCD β-function

$$\frac{\mathrm{d}}{\mathrm{d}\ln R}\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right) = -2\sum_{n=0}^{\infty}\beta_n\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+2}, \tag{C.19}$$

it is an easy task to derive the expression (4.17) for the coefficients of the R-anomalous dimension $\gamma^R[\alpha_\mathrm{s}]$ given in section 4.2.1. Starting from a perturbation series of the form

$$\theta_{p,\alpha}(R) = -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\sum_{n=1}^{\infty}a_n\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n}, \tag{C.20}$$

its R-evolution equation is then given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\ln R}\,\theta_{p,\alpha}(R) &= \frac{\mathrm{d}}{\mathrm{d}\ln R}\left[-R^p\,(2\beta_0)^{\alpha}\sum_{n=1}^{\infty}a_n\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+\alpha}\right] \\
&= -R^p\,(2\beta_0)^{\alpha}\left[\sum_{n=1}^{\infty}p\,a_n\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+\alpha} + \sum_{n=1}^{\infty}(-2)(\alpha+n)\,a_n\sum_{j=0}^{\infty}\beta_j\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{\alpha+n+j+1}\right] \\
&= -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\left[\sum_{n=0}^{\infty}p\,a_{n+1}\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1} + \sum_{k=0}^{\infty}(-2)(\alpha+k+1)\,a_{k+1}\sum_{j=0}^{\infty}\beta_j\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{k+j+2}\right] \\
&= -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\left[\sum_{n=0}^{\infty}p\,a_{n+1}\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1} + \sum_{n=0}^{\infty}(-2)\sum_{j=0}^{n-1}(\alpha+n-j)\,\beta_j\,a_{n-j}\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1}\right] \\
&= -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\sum_{n=0}^{\infty}\left[p\,a_{n+1} - 2\sum_{j=0}^{n-1}(\alpha+n-j)\,\beta_j\,a_{n-j}\right]\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1}.
\end{aligned} \tag{C.21}$$

If we now define the R-anomalous dimension as

$$\frac{\mathrm{d}}{\mathrm{d}\ln R}\,\theta_{p,\alpha}(R) = -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\gamma^R[\alpha_\mathrm{s}(R)] = -R^p\left(\frac{\alpha_\mathrm{s}(R)\,\beta_0}{2\pi}\right)^{\alpha}\sum_{n=0}^{\infty}\gamma_n^R\left(\frac{\alpha_\mathrm{s}(R)}{4\pi}\right)^{n+1}, \tag{C.22}$$

we see that the coefficients γ_n^R are indeed given by

$$\gamma_n^R = p\,a_{n+1} - 2\sum_{j=0}^{n-1}(\alpha+n-j)\,\beta_j\,a_{n-j}. \tag{C.23}$$

C.4 Coefficients $S_k^{(p)}$ and $g_l^{(p)}$

The coefficients $S_k^{(p)}$ arise in the general solution of the R-evolution equation (4.32) and are defined via the series:

$$\gamma^R[t]\,\hat{b}(t)\,\mathrm{e}^{-G(t)\,p}\,\mathrm{e}^{t\,p}\,(-t)^{\hat{b}_1 p} = \sum_{k=0}^{\infty}\frac{S_k^{(p)}}{(-t)^{k+1}}. \tag{C.24}$$

The exact form of the $S_k^{(p)}$ can be determined by expanding the left-hand side for large t and subsequently comparing the coefficients for the various powers of t on both sides of the equation. The first few coefficients read

$$\begin{aligned}
S_0^{(p)} &= \tilde{\gamma}_0^R, \\
S_1^{(p)} &= \tilde{\gamma}_1^R - \left(\hat{b}_1 + p\,\hat{b}_2\right)\tilde{\gamma}_0^R,
\end{aligned} \tag{C.25}$$

$$
\begin{aligned}
S_2^{(p)} &= \tilde{\gamma}_2^R - \big(\hat{b}_1 + p\,\hat{b}_2\big)\,\tilde{\gamma}_1^R + \left[\big(1 + p\,\hat{b}_1\big)\hat{b}_2 + \frac{p}{2}\big(p\,\hat{b}_2 + \hat{b}_3\big)\right]\tilde{\gamma}_0^R, \\
S_3^{(p)} &= \tilde{\gamma}_3^R - \big(\hat{b}_1 + p\,\hat{b}_2\big)\,\tilde{\gamma}_2^R + \left[\big(1 + p\,\hat{b}_1\big)\hat{b}_2 + \frac{p}{2}\big(p\,\hat{b}_2 + \hat{b}_3\big)\right]\tilde{\gamma}_1^R \\
&\quad - \left[p\left(1 + \frac{p}{2}\,\hat{b}_1 + \frac{p^2}{6}\,\hat{b}_2\right)\hat{b}_2^2 + \left(1 + \frac{p}{2}\,\hat{b}_1 + \frac{p^2}{2}\,\hat{b}_2\right)\hat{b}_3 + \frac{p}{3}\,\hat{b}_4\right]\tilde{\gamma}_0^R,
\end{aligned}
$$

where

$$
\tilde{\gamma}_n^R = \frac{\gamma_n^R}{(2\beta_0)^{n+1}}. \tag{C.26}
$$

The coefficients $g_l^{(p)}$ are defined by the series:

$$
\mathrm{e}^{G(t)p}\,\mathrm{e}^{-tp}\,(-t)^{-\hat{b}_1 p} = \sum_{l=0}^{\infty} g_l^{(p)}\,(-t)^{-l}, \tag{C.27}
$$

and can be determined in a similar fashion as the functions $S_k^{(p)}$. Up to $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ the coefficients $g_l^{(p)}$ are given by

$$
\begin{aligned}
g_0^{(p)} &= 1, \qquad\qquad (C.28)\\
g_1^{(p)} &= p\,\hat{b}_2, \\
g_2^{(p)} &= \frac{p}{2}\left(p\,\hat{b}_2^2 - \hat{b}_3\right), \\
g_3^{(p)} &= \frac{p}{6}\left(p^2\,\hat{b}_2^3 - 3p\,\hat{b}_2\,\hat{b}_3 + 2\hat{b}_4\right),
\end{aligned}
$$

which is in agreement with the recursion relation [51]:

$$
g_{n+1}^{(p)} = \frac{1}{n+1}\,p\sum_{i=0}^{n}(-1)^i\,\hat{b}_{i+2}\,g_{n-i}^{(p)}, \qquad g_0^{(p)} = 1. \tag{C.29}
$$

C.5 Borel Integral

In this section we provide the details on the calculation of the Borel integral (= inverse Borel transformation) of generic renormalon poles and also discuss the complex ambiguities associated with IR renormalons.

We first consider the Borel transform of a generic UV-renormalon-type contribution of the form:

$$
B[D_{\mathrm{UV}}](u) = \frac{d_{\tilde{p}}^{\mathrm{UV}}}{\left(\frac{\tilde{p}}{2} + u\right)^{1+\gamma}}, \tag{C.30}
$$

with $\tilde{p} > 0$ and $\gamma \in \mathbb{R}$. The corresponding inverse transformation is given by the Borel integral,

$$
D_{\mathrm{UV}}(\alpha_{\mathrm{s}}) = d_{\tilde{p}}^{\mathrm{UV}} \int_0^{\infty} \mathrm{d}u\,\frac{\mathrm{e}^{2tu}}{\left(\frac{\tilde{p}}{2} + u\right)^{1+\gamma}}, \tag{C.31}
$$

where we introduced the variable $t = -2\pi/(\beta_0\alpha_{\mathrm{s}}) < 0$ for convenience. Since the renormalon pole lies outside the integration range, we can easily calculate the Borel integral using the substitution $x = -2t\,(\tilde{p}/2 + u)$:

$$
D_{\mathrm{UV}}(\alpha_{\mathrm{s}}) = d_{\tilde{p}}^{\mathrm{UV}} \int_0^{\infty} \mathrm{d}u\,\frac{\mathrm{e}^{2tu}}{\left(\frac{\tilde{p}}{2} + u\right)^{1+\gamma}} \tag{C.32}
$$

$$
\begin{aligned}
&= d_{\tilde{p}}^{\mathrm{UV}}\,\mathrm{e}^{-t\,\tilde{p}}\,(-2t)^{\gamma}\int\limits_{-t\,\tilde{p}}^{\infty}\mathrm{d}x\,\mathrm{e}^{-x}\,x^{-1-\gamma}\\
&= d_{\tilde{p}}^{\mathrm{UV}}\,\mathrm{e}^{-t\,\tilde{p}}\,(-2t)^{\gamma}\,\Gamma(-\gamma,-t\,\tilde{p})\\
&= d_{\tilde{p}}^{\mathrm{UV}}\,\mathrm{e}^{\frac{2\pi\,\tilde{p}}{\beta_0\alpha_{\mathrm{s}}}}\left(\frac{4\pi}{\beta_0\alpha_{\mathrm{s}}}\right)^{\gamma}\Gamma\Big(-\gamma,\frac{2\pi\,\tilde{p}}{\beta_0\alpha_{\mathrm{s}}}\Big).
\end{aligned}
$$

We stress that $-t\,\tilde{p} = 2\pi\tilde{p}/(\beta_0\alpha_{\mathrm{s}}) > 0$ such that the Borel integral for UV renormalons is defined unambiguously and does not lead to any ambiguities.

For the discussion of IR-renormalon-type contributions, we consider Borel integrals of the generic form:

$$
B[D_{\mathrm{IR}}](u) = \frac{d_p^{\mathrm{IR}}}{\left(\frac{p}{2}-u\right)^{1+\gamma}}, \tag{C.33}
$$

where we now have $p > 0$ and $\gamma \in \mathbb{R}$. In contrast to UV case, the integrand has a singularity at $u = p/2$ as well as a corresponding branch cut for $u > p/2$ and we therefore need to move the integration contour either above or below the cut. Equivalently, we can also shift the singularity into the complex Borel plane by means of the '$i\epsilon$' prescription, i.e. $p \to p \pm i\epsilon$. Forcing the singularity into the lower half-plane ($p = p - i\epsilon$), which corresponds to taking the integration contour above the cut, yields:

$$
\begin{aligned}
D_{\mathrm{IR}}(\alpha_{\mathrm{s}}) &= d_p^{\mathrm{IR}}\int\limits_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2tu}}{\left(\frac{p}{2}-u-i\epsilon\right)^{1+\gamma}} = d_p^{\mathrm{IR}}\int\limits_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2tu}}{\mathrm{e}^{(1+\gamma)\,\ln\left(\frac{p}{2}-u-i\epsilon\right)}}\\
&= d_p^{\mathrm{IR}}\int\limits_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2tu}\,\theta(p/2-u)}{\left(\frac{p}{2}-u\right)^{1+\gamma}} + (-d_p^{\mathrm{IR}})\int\limits_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2tu}\,\theta(u-p/2)}{\mathrm{e}^{-i\pi\gamma}\left(u-\frac{p}{2}\right)^{1+\gamma}},
\end{aligned} \tag{C.34}
$$

where in the second integral of the last line we used:

$$
\ln(x \pm i\epsilon) = \ln|x| \pm i\pi \quad \text{for} \;\; x < 0. \tag{C.35}
$$

The integrals in the second line of eq. (C.34) can be calculated similarly to eq. (C.32). We finally obtain:

$$
\begin{aligned}
D_{\mathrm{IR}}(\alpha_{\mathrm{s}}) &= -d_p^{\mathrm{IR}}\,\mathrm{e}^{i\pi\gamma}\,\mathrm{e}^{t\,p}\,(-2t)^{\gamma}\,\Gamma(\gamma,t\,p)\\
&= -d_p^{\mathrm{IR}}\,\mathrm{e}^{i\pi\gamma}\,\mathrm{e}^{-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}}\left(\frac{4\pi}{\beta_0\alpha_{\mathrm{s}}}\right)^{\gamma}\Gamma\Big(-\gamma,-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}\Big).
\end{aligned} \tag{C.36}
$$

Exploiting the identity,

$$
E_n(x) = x^{n-1}\,\Gamma(1-n,x), \tag{C.37}
$$

one can also express this result in terms of the exponential integral $E_n(x)$:

$$
D_{\mathrm{IR}}(\alpha_{\mathrm{s}}) = -d_p^{\mathrm{IR}}\left(\frac{2}{p}\right)^{\gamma}\mathrm{e}^{-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}}\,E_{1+\gamma}\Big(-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}\Big). \tag{C.38}
$$

Instead of pushing the singularity into the lower half-plane, we can also choose $p = p + i\epsilon$ which is equivalent to moving the contour below the cut. In this case the Borel integral is given by,

$$
D_{\mathrm{IR}}(\alpha_{\mathrm{s}}) = -d_p^{\mathrm{IR}}\left(\frac{2}{p}\right)^{\gamma}\mathrm{e}^{-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}}\,E_{1+\gamma}\Big(-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}\Big) - 2\pi i\,\frac{\mathrm{e}^{-\frac{2\pi\,p}{\beta_0\alpha_{\mathrm{s}}}}}{\Gamma(1+\gamma)}\left(\frac{4\pi}{\beta_0\alpha_{\mathrm{s}}}\right)^{\gamma}, \tag{C.39}
$$

which differs from (C.38) only by the discontinuity of the branch cut.

In order to find an expression for the ambiguity caused by IR renormalons, one can easily compute the imaginary part of eq. (C.38) or (C.39) and take the outcome as an estimate for the corresponding ambiguity. For $\gamma = 1$ we can also directly calculate the imaginary part of the Borel integral using the well-known *Sokhotski-Plemelj theorem*:

$$\lim_{\epsilon\to 0}\frac{1}{x\pm i\epsilon} = \mathrm{PV}\left(\frac{1}{x}\right) \mp i\pi\,\delta(x). \tag{C.40}$$

The ambiguity for arbitrary $\gamma \in \mathbb{R}$ is found to be:

$$\begin{aligned}\mathrm{Amb}(D_{\mathrm{IR}}) &= \mathrm{Im}\left[\int_0^\infty \mathrm{d}u\,\frac{\mathrm{e}^{2tu}}{\left(\frac{p}{2}-u\right)^{1+\gamma}}\right] \\ &= \pm\pi\,\frac{(-2t)^\gamma}{\Gamma(1+\gamma)}\,\mathrm{e}^{t\,p} = \pm\pi\,\frac{\mathrm{e}^{-\frac{2\pi\,p}{\beta_0\alpha_s}}}{\Gamma(1+\gamma)}\left(\frac{4\pi}{\beta_0\alpha_s}\right)^\gamma,\end{aligned} \tag{C.41}$$

where the positive sign refers to the case of moving the integration contour above the cut, while the minus sign denotes the solution for moving the contour below the cut.

C.6 Relation between R-Evolution and Borel Integration

In this part of the appendix we establish the connection between R-evolution and the (inverse) Borel integration by means of a variable transformation from the R-evolution variable $t = -2\pi/(\alpha_s\beta_0)$ to the Borel parameter u. In the following we will only focus on IR renormalons, but similar relations for UV renormalons can also be found.

Starting point for our derivation is the solution of the R-evolution equation given by (4.30) in section 4.3.1:

$$\theta_{p,\alpha}(R_1) - \theta_{p,\alpha}(R_0) = \Lambda_{\mathrm{QCD}}^p\sum_{k=0}^\infty S_k^{(p)}\int_{t_0}^{t_1}\mathrm{d}t\,(-t)^{-\alpha-\hat{b}_1p-k-1}\,\mathrm{e}^{-tp}. \tag{C.42}$$

Recall that in order to obtain an expression for the Borel transform carrying the information of an $\mathcal{O}(\Lambda_{\mathrm{QCD}}^p)$ renormalon we need to restore the IR renormalon ambiguity in the solution of the R-evolution equation by taking the limit,

$$\lim_{R_0\to 0} t_0 = \infty. \tag{C.43}$$

Using the relations (4.15) and (4.16) we then obtained ($t_{\mathrm{Q}} = -2\pi/(\beta_0\alpha_s(Q))$),

$$\overline{C}_0(Q) \simeq -\frac{1}{Q^p}\,(-t_{\mathrm{Q}})^\alpha\left[1 + 4\,\hat{c}^{(1)}\,\overline{a}_s(Q) + 16\,\hat{c}^{(2)}\,\overline{a}_s^2(Q) + \ldots\right]\left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right], \tag{C.44}$$

where the right-hand side only contains the $u = p/2$ renormalon contribution of $\overline{C}_0$. According to (C.42) the perturbative series of the leading term (i.e. neglecting the corrections $\hat{c}^{(1)}$ and $\hat{c}^{(2)}$ in (C.44)) is given by,

$$\begin{aligned}-\frac{1}{Q^p}\,(-t_{\mathrm{Q}})^\alpha\left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right] &= \\ &= \frac{\Lambda_{\mathrm{QCD}}^p}{Q^p}\,(-t_{\mathrm{Q}})^\alpha\sum_{k=0}^\infty S_k^{(p)}\int_{t_{\mathrm{Q}}}^\infty \mathrm{d}t\,(-t)^{-\alpha-\hat{b}_1p-k-1}\,\mathrm{e}^{-tp} \\ &= \mathrm{e}^{t_{\mathrm{Q}}p}\sum_{l=0}^\infty\sum_{k=0}^\infty g_l^{(p))}\,S_k^{(p)}\,(-t_{\mathrm{Q}})^{\alpha+\hat{b}_1p-l}\int_{t_{\mathrm{Q}}}^\infty \mathrm{d}t\,(-t)^{-\alpha-\hat{b}_1p-k-1}\,\mathrm{e}^{-tp},\end{aligned} \tag{C.45}$$

where in the second equality (C.9) and (C.27) were used. The integral on the right-hand side can now be written in the form of a Borel integral by changing the integration variable to the $u = -p\,(t/t_{\mathrm{Q}} - 1)/2$:

$$
\begin{aligned}
-\frac{1}{Q^p}\,(-t_{\mathrm{Q}})^{\alpha}\,\big[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\big] &= \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} g_l^{(p))}\,S_k^{(p)}\left(\frac{2}{p}\right)^{-\alpha-\hat{b}_1 p - k}(-t_{\mathrm{Q}})^{-l-k}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p + k}}.
\end{aligned}
\tag{C.46}
$$

Note that the upper limit of the integral is $u = +\infty$, since $t_{\mathrm{Q}} < 0$ for $Q > \Lambda_{\mathrm{QCD}}$. The remaining task is to prove that the Borel transform on the right-hand side of (C.46) can be rewritten in terms of the analytic Borel transform $B_0(u)$ in eq. (4.46) of section 4.3.2. To that end we rearrange the terms in the Borel integral using integration by parts,

$$
\begin{aligned}
(-t_{\mathrm{Q}})^{-l-k}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma+k}} &= \\
&= -\left(\frac{2}{p}\right)^{\gamma+k}\frac{(-t_{\mathrm{Q}})^{-l-k}}{(\gamma+k)} + \frac{2(-t_{\mathrm{Q}})^{-l-k+1}}{(\gamma+k)}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{\gamma+k}},
\end{aligned}
\tag{C.47}
$$

where we introduced $\gamma = \alpha + \hat{b}_1 p$ to simplify the notation in the following. Repeating this procedure for the Borel integral $(l+k)$-times yields:

$$
\begin{aligned}
(-t_{\mathrm{Q}})^{-l-k}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma+k}} &= \\
&= -\left(\frac{2}{p}\right)^{\gamma+k}\frac{(-t_{\mathrm{Q}})^{-l-k}}{(\gamma+k)} - \cdots - 2^{l+k-1}\left(\frac{2}{p}\right)^{\gamma-l+1}\frac{(-t_{\mathrm{Q}})^{-1}}{\prod_{i=0}^{l+k-1}(\gamma+k-i)} \\
&\quad + \frac{2^{l+k}}{\prod_{i=0}^{l+k-1}(\gamma+k-i)}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma-l}}.
\end{aligned}
\tag{C.48}
$$

The products in the denominators on the right-hand side can be conveniently organized in terms of Γ-functions by multiplying the equation above with $1 = \Gamma(\gamma+k)/\Gamma(\gamma+k)$:

$$
\begin{aligned}
&(-\,t_{\mathrm{Q}})^{-l-k}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma+k}}\,\frac{\Gamma(\gamma+k)}{\Gamma(\gamma+k)} = \\
&= -\left(\frac{2}{p}\right)^{\gamma+k}\frac{\Gamma(\gamma+k)}{\Gamma(1+\gamma+k)}\,(-t_{\mathrm{Q}})^{-l-k} - \cdots - 2^{l+k-1}\left(\frac{2}{p}\right)^{\gamma-l+1}\frac{\Gamma(1+\gamma-l)}{\Gamma(1+\gamma+k)}\,(-t_{\mathrm{Q}})^{-1} \\
&\quad + 2^{l+k}\,\frac{\Gamma(1+\gamma-l)}{\Gamma(1+\gamma+k)}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma-l}} \\
&= -\sum_{j=1}^{l+k} 2^{l+k-j}\left(\frac{2}{p}\right)^{\gamma-l+j}\frac{\Gamma(j+\gamma-l)}{\Gamma(1+\gamma+k)}\,(-t_{\mathrm{Q}})^{-j} + 2^{l+k}\,\frac{\Gamma(1+\gamma-l)}{\Gamma(1+\gamma+k)}\int_0^{\infty}\mathrm{d}u\,\frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\,u}}{\left(\frac{p}{2}-u\right)^{1+\gamma-l}}
\end{aligned}
\tag{C.49}
$$

Next, we rewrite the polynomial terms in $(-t_{\mathrm{Q}})^{-n} \sim \alpha_s^n(Q)$ as Borel integrals of polynomials in u which amounts to the replacements $(-t_{\mathrm{Q}})^{-n-1} \to 2\,(2u)^n/\Gamma(n+1)$ (see eq. (4.42) in section 4.3.2):

$$
\begin{aligned}
&(-t_{\mathrm{Q}})^{-l-k} \int\limits_0^\infty \mathrm{d}u\, \frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\, u}}{\left(\frac{p}{2}-u\right)^{1+\gamma+k}} \frac{\Gamma(\gamma+k)}{\Gamma(\gamma+k)} = \qquad \text{(C.50)}\\
&= \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{2\,t_{\mathrm{Q}} p\, u} \left[\frac{\Gamma(1+\gamma-l)}{\Gamma(1+\gamma+k)} \frac{2^{l+k}}{\left(\frac{p}{2}-u\right)^{1+\gamma-l}} - 2 \sum_{j=1}^{l+k} 2^{l+k-j} \left(\frac{2}{p}\right)^{\gamma-l+j} \frac{\Gamma(j+\gamma-l)\,(2u)^{j-1}}{\Gamma(1+\gamma+k)\,\Gamma(j)} \right]
\end{aligned}
$$

Inserting this expression into eq. (C.46) finally yields,

$$
\begin{aligned}
&-\frac{1}{Q^p}\,(-t_{\mathrm{Q}})^\alpha \left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right] = \qquad \text{(C.51)}\\
&= \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{2\,t_{\mathrm{Q}} p\, u} \left[\sum_{k=0}^\infty S_k^{(p)} \frac{p^{k+\alpha+\hat{b}_1 p}}{\Gamma(1+\alpha+\hat{b}_1 p)} \sum_{l=0}^\infty g_l^{(p)} \frac{\Gamma(1+\alpha+\hat{b}_1 p - l)}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p - l}}\, 2^{-\alpha-\hat{b}_1 p + l} \right.\\
&\qquad \left. -2 \sum_{l=0}^\infty g_l^{(p)} \sum_{k=0}^\infty S_k^{(p)} \sum_{j=0}^{l+k-1} \frac{\Gamma(1+j+\alpha+\hat{b}_1 p - l)\, p^{l+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p + k)\,\Gamma(j+1)}\,(2u)^j \right]\\
&= \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{2\,t_{\mathrm{Q}} p\, u} \left[P_{p/2}^{\alpha,\mathrm{IR}} \sum_{l=0}^\infty g_l^{(p)} \frac{\Gamma(1+\alpha+\hat{b}_1 p - l)}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p - l}}\, 2^{-\alpha-\hat{b}_1 p + l} + \sum_{l=0}^\infty g_l^{(p)}\, Q_l^{(p)} \right],
\end{aligned}
$$

which agrees precisely with the expression of the analytic Borel transform $B_0(u)$ given in eq. (4.46) in section 4.3.2. We stress that the second term of the Borel transform including the functions $Q_l^{(p)}$ represents the the series expansion of the singular terms in the Borel transform up to order u^{k+l-1} (see discussion in section 4.3.2 below eq. (4.50)).

For the subleading terms in (C.44) we have to consider series of the generic form,

$$
\begin{aligned}
&-\frac{1}{Q^p}\,(-t_{\mathrm{Q}})^\alpha\, \overline{a}_s^n(Q) \left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right] = -\frac{1}{Q^p}\,(2\beta_0)^{-n}\,(-t_{\mathrm{Q}})^{\alpha-n} \left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right] = \qquad \text{(C.52)}\\
&= (2\beta_0)^{-n} \sum_{l=0}^\infty \sum_{k=0}^\infty g_l^{(p))}\, S_k^{(p)} \left(\frac{2}{p}\right)^{-\alpha-\hat{b}_1 p - k} (-t_{\mathrm{Q}})^{-l-n-k} \int\limits_0^\infty \mathrm{d}u\, \frac{\mathrm{e}^{2\,t_{\mathrm{Q}} p\, u}}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p + k}}.
\end{aligned}
$$

Repeating the manipulations above with the substitution $l \to l+n$ one obtains,

$$
\begin{aligned}
&-\frac{1}{Q^p}\,(2\beta_0)^{-n}\,(-t_{\mathrm{Q}})^{\alpha-n} \left[\theta_{p,\alpha}(Q) - \theta_{p,\alpha}(0)\right] = \qquad \text{(C.53)}\\
&= \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{2\,t_{\mathrm{Q}} p\, u}\,(2\beta_0)^{-n} \left[\sum_{k=0}^\infty S_k^{(p)} \frac{p^{k+\alpha+\hat{b}_1 p}}{\Gamma(1+\alpha+\hat{b}_1 p)} \sum_{l=0}^\infty g_l^{(p)} \frac{\Gamma(1+\alpha+\hat{b}_1 p - l - n)}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p - l - n}}\, 2^{-\alpha-\hat{b}_1 p + l + n} \right.\\
&\qquad \left. -2 \sum_{l=0}^\infty g_l^{(p)} \sum_{k=0}^\infty S_k^{(p)} \sum_{j=0}^{l+n+k-1} \frac{\Gamma(1+j+\alpha+\hat{b}_1 p - l - n)\, p^{l+n+k-j-1}}{\Gamma(1+\alpha+\hat{b}_1 p + k)\,\Gamma(j+1)}\,(2u)^j \right]\\
&= \int\limits_0^\infty \mathrm{d}u\, \mathrm{e}^{2\,t_{\mathrm{Q}} p\, u}\,(2\beta_0)^{-n} \left[P_{p/2}^{\alpha,\mathrm{IR}} \sum_{l=0}^\infty g_l^{(p)} \frac{\Gamma(1+\alpha+\hat{b}_1 p - l - n)}{\left(\frac{p}{2}-u\right)^{1+\alpha+\hat{b}_1 p - l - n}}\, 2^{-\alpha-\hat{b}_1 p + l + n} + \sum_{l=0}^\infty g_l^{(p)}\, Q_{l,n}^{(p)} \right],
\end{aligned}
$$

which is also in line with the formluae for the analytic Borel transforms $B_n(u)$ in eq. (4.57).

C.7 Perturbative Coefficients of the Adler Function in the Large-β_0 Approximation

In this part of the appendix we derive analytic expressions for the series coefficients of the Adler function discussed in section 5.1. For the subsequent analysis the following notation will be used:

$$\bullet\; D(p^2) = 4\pi^2\,\frac{\mathrm{d}\Pi(p^2)}{\mathrm{d}\ln p^2} = 1 + \widehat{D}(p^2),$$
$$\bullet\; \widehat{D}(p^2) = \sum_{n=1}^{\infty} d_n \left(\frac{\alpha_s\,\beta_0}{4\pi}\right)^n = \sum_{n=1}^{\infty} a_n \left(\frac{\alpha_s}{4\pi}\right)^n \quad\Rightarrow\quad a_n = \beta_0^n\, d_n, \tag{C.54}$$
$$\bullet\; \widehat{D}(p^2) = \frac{4\pi}{\beta_0}\int_0^\infty \mathrm{d}u\; \mathrm{e}^{-\frac{4\pi u}{\alpha_s \beta_0}}\, B[\widehat{D}](u).$$

In the large-β_0 approximation the Borel transform of the Adler function can be written in the closed form [28]

$$B[\widehat{D}](u) = \frac{32}{3\pi}\left(\frac{-p^2}{\mu^2}\,\mathrm{e}^C\right)^{-u} \frac{1}{2-u}\sum_{k=2}^{\infty}\frac{(-1)^k\,k}{(k^2-(1-u)^2)^2}, \tag{C.55}$$

where C denotes a scheme-dependent constant. ($C = -5/3$ in the $\overline{MS}$-scheme.) In the following we will set $\mu^2 = -p^2$ and work in the renormalization scheme in which the constant C vanishes. Using partial fraction decomposition one can separate the individual contributions of the various renormalon poles and rewrite eq.(C.55) in the form:

$$\begin{aligned} B[\widehat{D}](u) = \frac{1}{\pi}\Bigg\{ & \frac{2}{2-u} \\ & + \frac{32}{3}\sum_{k=2}^{\infty}\frac{(-1)^k}{4\,k^2}\Bigg[\frac{2k+1}{(k+1)^2}\,\frac{1}{(u+k-1)} + \frac{k}{(k+1)\,(u+k-1)^2} \\ & + \frac{2k-1}{(k-1)^2}\,\frac{1}{(u-k-1)} - \frac{k}{(k-1)\,(u-k-1)^2}\Bigg]\Bigg\}. \end{aligned} \tag{C.56}$$

The sum over k includes all IR and UV renormalons except for the simple pole at u=2. For later convenience it will be useful to further decompose this sum into two different parts, one which contains the information on the IR renormalons and another one which comprises all UV renormalons:

$$\begin{aligned} B[\widehat{D}](u) = \frac{1}{\pi}\Bigg\{ & \frac{2}{2-u} \\ & + \frac{32}{3}\sum_{k=-\infty}^{-1}\frac{(-1)^{-k+1}}{4\,(-k+1)^2}\Bigg[\frac{-2k+3}{(-k+2)^2}\,\frac{1}{(u-k)} + \frac{(-k+1)}{(-k+2)\,(u-k)^2}\Bigg] \\ & + \frac{32}{3}\sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{4\,(k-1)^2}\Bigg[\frac{2k-3}{(k-2)^2}\,\frac{1}{(u-k)} - \frac{k-1}{(k-2)\,(u-k)^2}\Bigg]\Bigg\}. \end{aligned} \tag{C.57}$$

From the two sums over k we can easily read off the coefficients of all simple and double poles contained in the Borel transform. To obtain an analytic expression for the coefficients a_n of the Adler function given in (C.54), we use the generalized binomial theorem to express the contributions of the renormalon poles in terms of infinite series:

$$B[\widehat{D}](u) = \frac{1}{\pi}\sum_{n=0}^{\infty} u^n \Bigg\{ \frac{1}{2^n} \tag{C.58}$$
$$+\frac{32}{3}\sum_{k=-\infty}^{-1} \frac{(-1)^{-k+1}}{4\,(-k+1)^2}\left[-\frac{-2k+3}{(-k+2)^2}\,\frac{1}{k^{n+1}} + \frac{(-k+1)}{(-k+2)}\,\frac{(n+1)}{k^{n+2}}\right]$$
$$-\frac{32}{3}\sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{4\,(k-1)^2}\left[\frac{2k-3}{(k-2)^2}\,\frac{1}{k^{n+1}} + \frac{k-1}{(k-2)}\,\frac{(n+1)}{k^{n+2}}\right]\Bigg\}.$$

Subsequently performing the inverse Borel transformation according to (C.54) yields,

$$\widehat{D}(Q) = \frac{4}{\beta_0}\sum_{n=0}^{\infty} n!\left(\frac{\alpha_s(Q)\,\beta_0}{4\pi}\right)^{n+1} \Bigg\{ \frac{1}{2^n} \tag{C.59}$$
$$+\frac{32}{3}\sum_{k=-\infty}^{-1} \frac{(-1)^{-k+1}}{4\,(-k+1)^2}\left[-\frac{-2k+3}{(-k+2)^2}\,\frac{1}{k^{n+1}} + \frac{(-k+1)}{(-k+2)}\,\frac{(n+1)}{k^{n+2}}\right]$$
$$-\frac{32}{3}\sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{4\,(k-1)^2}\left[\frac{2k-3}{(k-2)^2}\,\frac{1}{k^{n+1}} + \frac{k-1}{(k-2)}\,\frac{(n+1)}{k^{n+2}}\right]\Bigg\}$$
$$= \frac{4}{\beta_0}\sum_{n=1}^{\infty} (n-1)!\left(\frac{\alpha_s(Q)\,\beta_0}{4\pi}\right)^{n} \Bigg\{ \frac{1}{2^{n-1}}$$
$$+\frac{32}{3}\sum_{k=-\infty}^{-1} \frac{(-1)^{-k+1}}{4\,(-k+1)^2}\left[-\frac{-2k+3}{(-k+2)^2}\,\frac{1}{k^{n}} + \frac{(-k+1)}{(-k+2)}\,\frac{n}{k^{n+1}}\right]$$
$$-\frac{32}{3}\sum_{k=3}^{\infty} \frac{(-1)^{k-1}}{4\,(k-1)^2}\left[\frac{2k-3}{(k-2)^2}\,\frac{1}{k^{n}} + \frac{k-1}{(k-2)}\,\frac{n}{k^{n+1}}\right]\Bigg\}.$$

As one can see, it is possible to express the coefficients d_n of the Adler function in the following form;

$$a_n = a_n^{2,(1)} + \sum_{k_{\rm IR}=3}^{\infty}\left[a_n^{k_{\rm IR},(1)} + a_n^{k_{\rm IR},(2)}\right] + \sum_{k_{\rm UV}=-\infty}^{-1}\left[a_n^{k_{\rm UV},(1)} + a_n^{k_{\rm UV},(2)}\right], \tag{C.60}$$

where the superscript in parentheses denotes simple (1) or double (2) poles. Recalling that $a_n = \beta_0^n\, d_n$, the individual coefficients are found to be:

$$\begin{aligned}
&\bullet\ a_n^{2,(1)} = \frac{4}{\beta_0}\,\beta_0^n\,\frac{(n-1)!}{2^{n-1}},\\
&\bullet\ a_n^{k_{\rm IR},(1)} = \frac{4}{\beta_0}\,\frac{32}{3}\,\beta_0^n\,\frac{(-1)^{k_{\rm IR}}}{4(k_{\rm IR}-1)^2}\,\frac{2k_{\rm IR}-3}{(k_{\rm IR}-2)^2}\,\frac{(n-1)!}{k_{\rm IR}^n},\\
&\bullet\ a_n^{k_{\rm IR},(2)} = \frac{4}{\beta_0}\,\frac{32}{3}\,\beta_0^n\,\frac{(-1)^{k_{\rm IR}}}{4\,(k_{\rm IR}-1)^2}\,\frac{(k_{\rm IR}-1)}{(k_{\rm IR}-2)}\,\frac{n!}{k_{\rm IR}^{n+1}},\\
&\bullet\ a_n^{k_{\rm UV},(1)} = \frac{4}{\beta_0}\,\frac{32}{3}\,\beta_0^n\,\frac{(-1)^{-k_{\rm UV}}}{4\,(-k_{\rm UV}+1)^2}\,\frac{-2k_{\rm UV}+3}{(-k_{\rm UV}+2)^2}\,\frac{(n-1)!}{k_{\rm UV}^n},\\
&\bullet\ a_n^{k_{\rm UV},(2)} = \frac{4}{\beta_0}\,\frac{32}{3}\,\beta_0\,\frac{(-1)^{-k_{\rm UV}+1}}{4\,(-k_{\rm UV}+1)^2}\,\frac{(-k_{\rm UV}+1)}{(-k_{\rm UV}+2)}\,\frac{n!}{k_{\rm UV}^{n+1}}.
\end{aligned} \tag{C.61}$$

Appendix D

OPE of the Adler Function

This last appendix is dedicated to a thorough investigation of the dimension-4 and dimension-6 operator corrections in the OPE of the Adler function in the large-β_0 approximation. The subsequent analysis is divided into two different sections. In the first one we collect the results for the coefficient functions of the gluon and four-quark condensates distributed throughout the literature and check whether the structure of these contributions in the OPE can be reproduced by the analytic expression of the Borel transform derived in section 4.3. In particular, we want to study which four-quark operators are connected to the simple and double pole structure of the $u = 3$ renormalon in the large-β_0 Borel transform (3.18).

In the second part we present a detailed leading order computation of the coefficient functions using the expansion-by-regions method.

D.1 Adler Function Revisited

The OPE of the Adler function and its connection with IR renormalons has been discussed extensively in section 3.3.1 and will not be reviewed in the following. Here, the important point to remember is that the relevant quantity in the QCD analysis of hadronic τ decays is given by the two-point function of flavour non-diagonal vector and axial-vector currents, $J_\mu^{V/A}(x) = [\overline{u}\gamma_\mu(\gamma_5)d](x)$, of massless quarks [61],

$$\Pi_{\mu\nu}^{V/A}(q) = i\int \mathrm{d}x\, \mathrm{e}^{iqx}\, \langle\Omega|\, T\{J_\mu^{V/A}(x) J_\nu^{V/A}(0)^\dagger\}\, |\Omega\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu})\, \Pi^{V/A}(q^2). \tag{D.1}$$

where $|\Omega\rangle$ denotes the full physical vacuum. In the Euclidean region ($Q^2 = -q^2 > 0$), the concept of the OPE then allows one to organize the scalar correlator $\Pi^{V/A}$ in a series expansion in inverse powers of Q^2 [61]:

$$\Pi^{V/A}(Q^2) = C_0(\mu, Q^2) + C_4^i(\mu, Q^2)\, \frac{\langle O_4^i\rangle(\mu)}{Q^4} + C_6^{V/A,i}(\mu, Q^2)\, \frac{\langle O_6^i\rangle(\mu)}{Q^6} + \dots \tag{D.2}$$

where $\langle O\rangle$ represents the vacuum matrix elements of the operators arising in the OPE. Both, the coefficient functions and the local matrix elements, depend on the renormalization scale μ, while only the coefficient functions additionally depend on the momentum Q. In the chiral limit, the contributions from vector and axial-vector currents coincide for the purely perturbative QCD correction C_0 as well as for the dimension-4 term, which in this case reduces to a single operator given by the gluon condensate $\langle G_{\mu\nu}^a G^{a,\mu\nu}\rangle$. The coefficient functions of the gluon condensate and other operators in the OPE of current-current correlators have been calculated at leading order for the first time in [40] and computations beyond leading order can be found e.g. in [62, 63]:

$$C_4\, \langle O_4\rangle = \frac{1}{12}\left[1 - \frac{11}{18}\, \frac{\alpha_s(Q)}{\pi}\right] \left\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a,\mu\nu} \right\rangle. \tag{D.3}$$

Besides the gluon condensate, we are mainly interested in investigating the dimension-6 corrections to (D.2), which consist of the three-gluon condensate $\langle f_{abc} G_{\mu\nu}^a G^{b,\nu}{}_{\rho}\, G^{c,\rho\mu}\rangle$ and a set of four-quark operators. However, since the three-gluon condensate does not contribute at leading order, we will focus our study on the four-quark condensates only. The next-to-leading order results for the coefficient functions of the four-quark operators were calculated in [64] and for $N_f = 3$ light flavours and $N_c = 3$ colour degrees of freedom one obtains (in the $\overline{\mathrm{MS}}$-scheme) [65],

$$\begin{aligned} C_6^{V,i}\,\langle O_6^i\rangle = -4\pi^2 a_\mathrm{s} \Bigg\{ & \left(\frac{55}{48}+\frac{5}{4}\,L\right) a_\mathrm{s}\,\langle O_V^o\rangle + \left[2+\left(\frac{85}{16}+\frac{9}{4}\,L\right)a_\mathrm{s}\right]\langle O_A^o\rangle + \left(\frac{11}{18}+\frac{2}{3}\,L\right)a_\mathrm{s}\,\langle O_V^s\rangle \\ & + \left[\frac{2}{9}+\left(\frac{37}{72}+\frac{95}{324}\,L\right)a_\mathrm{s}\right]\langle O_3\rangle + \left(\frac{35}{219}+\frac{5}{36}\,L\right)a_\mathrm{s}\,\langle O_4\rangle \\ & + \left(\frac{7}{81}+\frac{2}{27}\,L\right)a_\mathrm{s}\,\langle O_6\rangle - \left(\frac{1}{81}-\frac{2}{27}\,L\right)a_\mathrm{s}\,\langle O_7\rangle \Bigg\}, \end{aligned} \tag{D.4}$$

for the vector correlation function and,

$$\begin{aligned} C_6^{A,i}\,\langle O_6^i\rangle = -4\pi^2 a_\mathrm{s} \Bigg\{ & \left[2+\left(\frac{85}{16}+\frac{9}{4}\,L\right)a_\mathrm{s}\right]\langle O_V^o\rangle + \left(\frac{55}{48}+\frac{5}{4}\,L\right) a_\mathrm{s}\,\langle O_A^o\rangle + \left(\frac{11}{18}+\frac{2}{3}\,L\right)a_\mathrm{s}\,\langle O_A^s\rangle \\ & + \left[\frac{2}{9}+\left(\frac{37}{72}+\frac{95}{324}\,L\right)a_\mathrm{s}\right]\langle O_3\rangle + \left(\frac{35}{219}+\frac{5}{36}\,L\right)a_\mathrm{s}\,\langle O_4\rangle \\ & + \left(\frac{7}{81}+\frac{2}{27}\,L\right)a_\mathrm{s}\,\langle O_6\rangle - \left(\frac{1}{81}-\frac{2}{27}\,L\right)a_\mathrm{s}\,\langle O_7\rangle \Bigg\}, \end{aligned} \tag{D.5}$$

for the axial-vector correlator. In eqs. (D.4) and (D.5) we used $a_\mathrm{s} = \alpha_\mathrm{s}(\mu)/\pi$, $L = \ln \mu^2/Q^2$ and the occurring operators define a subset of a complete basis needed for their one-loop renormalization[1] [61]:

$$\begin{aligned} O_V^o &= \left(\bar{u}\gamma_\mu T^a d\bar{d}\gamma^\mu T^a u\right), \quad O_V^s = \left(\bar{u}\gamma_\mu d\bar{d}\gamma^\mu u\right), \\ O_A^o &= \left(\bar{u}\gamma_\mu\gamma_5 T^a d\bar{d}\gamma^\mu\gamma_5 T^a u\right), \quad O_A^s = \left(\bar{u}\gamma_\mu\gamma_5 d\bar{d}\gamma^\mu\gamma_5 u\right), \\ O_3 &= \left(\bar{u}\gamma_\mu T^a u + \bar{d}\gamma_\mu T^a d\right) \sum_{q=u,d,s} \left(\bar{q}\gamma^\mu T^a q\right), \\ O_4 &= \left(\bar{u}\gamma_\mu\gamma_5 T^a u + \bar{d}\gamma_\mu\gamma_5 T^a d\right) \sum_{q=u,d,s} \left(\bar{q}\gamma^\mu\gamma_5 T^a q\right), \\ O_5 &= \left(\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d\right) \sum_{q=u,d,s} \left(\bar{q}\gamma^\mu q\right), \\ O_6 &= \left(\bar{u}\gamma_\mu\gamma_5 u + \bar{d}\gamma_\mu\gamma_5 d\right) \sum_{q=u,d,s} \left(\bar{q}\gamma^\mu\gamma_5 q\right), \\ O_7 &= \sum_{q=u,d,s} \left(\bar{q}\gamma_\mu T^a q\right) \sum_{q'=u,d,s} \left(\bar{q}'\gamma^\mu T^a q'\right), \\ O_8 &= \sum_{q=u,d,s} \left(\bar{q}\gamma_\mu\gamma_5 T^a q\right) \sum_{q'=u,d,s} \left(\bar{q}'\gamma^\mu\gamma_5 T^a q'\right), \\ O_9 &= \sum_{q=u,d,s} \left(\bar{q}\gamma_\mu q\right) \sum_{q'=u,d,s} \left(\bar{q}'\gamma^\mu q'\right), \\ O_{10} &= \sum_{q=u,d,s} \left(\bar{q}\gamma_\mu\gamma_5 q\right) \sum_{q'=u,d,s} \left(\bar{q}'\gamma^\mu\gamma_5 q'\right). \end{aligned} \tag{D.6}$$

D.1.1 Analytic Borel Transform and Structure of the Coefficient Functions

Our main concern in this section will be to establish the connection between the higher-dimensional operator contributions to the OPE of the Adler function and its Borel transform given by the analytic R-evolution expression (4.46). More specifically, we want to study whether the general

[1]The discussion of the dimension-6 four quark operators in this section is based on [61], in which the leading order anomalous dimension matrix is calculated and its influence on the pole structure of the corresponding $u = 3$ IR renormalon is studied.

formula,

$$B_0^{\rm sing}(u) = -R_1^p \left[P_{p/2}^{\alpha,{\rm IR}} \sum_{l=0}^{\infty} g_l^{(p)} \, \frac{\Gamma(1+\hat{b}_1 p + \alpha - l)}{\left(\frac{p}{2}-u\right)^{1+\hat{b}_1 p+\alpha-l}} \, 2^{-\hat{b}_1 p - \alpha + l} \right], \tag{D.7}$$

allows one to reproduce the correct structure of the $u = 2$ and $u = 3$ renormalons provided that the operator corrections (D.3) and (D.4) as well as the corresponding anomalous dimensions are given. To that end, it is once again useful to consider the large-β_0 approximation of the Adler function, where the pole structure of its Borel transform is completely known (see eq. (3.18)). In particular, the Borel transform consists of a linear pole at $u = 2$ in this limit, whereas the $u = 3$ renormalon has both, a linear and a quadratic pole. The $u = 2$ singularity obviously corresponds to the dimension-4 gluon condensate and the question arises which four-quark operators are connected to pole structure of the $u = 3$ renormalon.

Before we proceed, let us reflect about the form of eq. (D.7) in the large-β_0 approximation (see (5.7)),

$$B_{\beta_0}^{\rm sing}(u) = -R^p \, P_{p/2,\beta_0}^{\alpha,{\rm IR}} \, \frac{\Gamma(1+\alpha)\, 2^{-\alpha}}{\left(\frac{p}{2}-u\right)^{1+\alpha}}, \tag{D.8}$$

$$\alpha = -\frac{2\gamma_{D,j}^{(1)}}{\beta_0} - \hat{\delta}_j .$$

From the simple and double pole structure of the Adler function Borel transform we can infer that the parameter α is either zero or one. Therefore, since $\hat{\delta}_j \geq 0$, $2\gamma_{D,j}^{(1)}/\beta_0{}^2$ needs to be either zero or takes on negative integer values:

$$\frac{2\gamma_{D,j}^{(1)}}{\beta_0} = 0, -1, -2, -3 \ldots \quad . \tag{D.9}$$

It is not at all obvious why we observe this behaviour in the large-β_0 limit, but the crucial point is that we can predict these values because we know the pole structure of the renormalons in the Borel transform. According to eq. (D.8) the pole structure of the singular terms in the Borel transform is completely determined by the leading order anomalous dimensions of the operators in the OPE and the leading powers in $\alpha_{\rm s}$ of the associated Wilson coefficients, but does not depend on the purely perturbative coefficient C_0. This is a remarkable fact: only the normalization of the singular terms, i.e. the renormalon sum rule $P_{p/2}^{\alpha_{\rm s}}$, relies on the knowledge of the purely perturbative QCD corrections, but as long as we are only interested in the pole structure of the Borel transform we do not need to have any information on C_0. This is also in line with the assumptions on the physical Borel model made in [14]. From the generic form of a term in the OPE and renormalisation group arguments one can deduce the position and structure of an infrared renormalon pole corresponding to an operator O_j, but it is not possible to predict the normalization (i.e. the residue) in this way. R-evolution and the renormalon sum rule provide us a handy tool to determine these residues.

Given the possible values of $\hat{\delta}_j$ and $\gamma_{D,j}^{(1)}$ in the large-β_0 approximation, we are now in the position to investigate the generic form of the corresponding terms in the OPE of the Adler function. For this purpose consider the general expression (3.48) for an OPE contribution due to a set of operators of equal dimension ($a_{\rm s} \equiv \alpha_{\rm s}/\pi$),

$$F_O = \sum_j \hat{C}_j^{(0)} \left(\frac{a_{\rm s}(Q)}{c}\right)^{\hat{\delta}_j + 2\gamma_{D,j}^{(1)}/\beta_0} \left[1 + a_{\rm s}(Q)\, \hat{C}_j^{(1)} + \ldots \right] c^{\hat{\delta}_j} \, \langle \hat{O}_c^j(\Lambda_{\rm IR}) \rangle, \tag{D.10}$$

[2] Recall that $\hat{\delta}_j$ gives the leading power in $\alpha_{\rm s}$ of the coefficient function corresponding to the operator O_j and $\gamma_{D,j}^{(1)}$ denotes the leading order anomalous dimension in the diagonal basis (see section 3.3.1 for more details).

where c denotes an arbitrary constant that will be irrelevant in the following[3]. The large-β_0 approximation of the Adler function is associated with the set of bubble chain diagrams which contain any number of fermion bubble insertions into the internal gluon line (see Fig. 3.1). Thus, assuming that the coefficient functions are given in the form (cf. eq. (3.46)),

$$\hat{C}_j(Q) = a_s^{\hat{\delta}_j}(Q) \sum_{n=0}^{\infty} \hat{c}_j^{(n)}\, a_s^n(Q), \tag{D.11}$$

we can decompose the perturbative coefficients $\hat{c}_j^{(n)}$ according to,

$$\hat{c}_j^{(n)} = r_{n0}^j + r_{n1}^j N_f + \cdots + r_{nn}^j N_f^n, \tag{D.12}$$

where the contributions r_{nn}^j with the largest power of N_f can be obtained from the calculation of bubble chain diagrams. Inserting this decomposition into eq. (D.10) yields,

$$F_O = \sum_j r_{00}^j \left(\frac{a_s(Q)}{c}\right)^{\hat{\delta}_j + 2\gamma_{D,j}^{(1)}/\beta_0} \left[1 + a_s(Q)\,\left(r_{10}^j + r_{11}^j N_f\right) + \ldots\right] c^{\hat{\delta}_j}\, \langle \hat{O}_c^j(\Lambda_{\mathrm{IR}})\rangle, \tag{D.13}$$

where, depending on the values of $\hat{\delta}_j$ and $\gamma_{D,j}^{(1)}$, each term in the expansion will contribute to a specific IR pole in the Borel transform of the Adler function. In particular, we find that terms $\sim (\alpha_s N_f)^n$ contribute to simple poles, while terms of the form $\sim (\alpha_s N_f)^n N_f$ give rise to a double pole structure. Other contributions, such as $\alpha_s(\alpha_s N_f)^n$, will not lead to any singularities in the Borel transform.

In order to make our analysis more concrete, let us eventually consider the coefficient functions of the gluon condensate and the four-quark operators given in the previous section and check whether our findings are consistent with these results.

Starting with the gluon condensate, we first take a closer look at its coefficient function given in eq. (D.3). Since one power of α_s is part of the matrix element, we have that $\hat{\delta}_{\mathrm{GG}} = 0$ in this case. Moreover, we already know that the leading order anomalous dimension of the gluon condensate vanishes in the large-β_0 approximation (see [27]). Therefore, according to our expansion (D.13), the coefficient function can be cast into the generic form,

$$C_4 \sim r_{00}\, a_s^0(Q) \left[1 + a_s(Q)\,(r_{10} + r_{11} N_f) + \ldots\right], \tag{D.14}$$

where the term $\sim r_{10}\, a_s(Q)$ does not yield any singularities in the Borel transform, whereas the second term $\sim r_{11}\, a_s(Q) N_f$ contributes to a simple pole. This pole structure is precisely in line with the known structure of the $u = 2$ renormalon in the Borel transform of the Adler function given by eq. (3.18).

For the investigation of the pole structure of the $u = 3$ renormalon, we first need to deal with the anomalous dimension matrix of the corresponding four-quark operators in (D.6). The calculation of the anomalous dimension is quite standard and results can be found e.g. in [66, 67]. Here, we follow primarily the discussion in [61] where the leading order anomalous dimension matrices for the $V + A$ and $V - A$ cases have been studied. Since the vector and axial-vector current contributions to C_0 coincide, the purely perturbative QCD corrections in the $V - A$ case cancel and consequently no renormalon ambiguity emerges. Hence, we are only interested in $V + A$ and for the closed set of operators $(O_+^o, O_+^s, O_3, O_4, O_6, O_7, O_8, O_9, O_{10})$ the leading order anomalous

[3]In the derivation of the R-evolution equation in chapter 4 the constant c was set to $c = 2/\beta_0$.

dimension matrix for $N_c = 3$ colours takes the form [61],

$$\gamma_+^{(1)} = \begin{pmatrix} -1 & \frac{2}{3} & -\frac{1}{9} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{N_f}{3} - \frac{85}{36} & \frac{5}{4} & \frac{2}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 2 & -\frac{2}{3} & \frac{113}{36} & -\frac{1}{4} & -\frac{2}{3} & -1 & -1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{11}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2N_f}{3} - \frac{85}{36} & \frac{5}{4} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{41}{36} & -\frac{9}{4} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{11}{3} & 0 & 0 & 0 \end{pmatrix}, \tag{D.15}$$

where the operators O_+^o and O_+^s are defined as [61]:

$$O_+^o = O_V^o + O_A^o, \qquad O_+^s = O_V^s + O_A^s. \tag{D.16}$$

The anomalous dimension in the form (D.15) is especially convenient for our purposes, since the explicit dependence on N_f allows us to easily connect to the large-β_0 approximation and address the question which four-quark operators are connected to the simple and double pole structure of the $u = 3$ renormalon.

If we take the large-N_f limit of (D.15) and use the replacement rule (3.15) to obtain the large-β_0 result, only two entries of the anomalous dimension matrix remain on the diagonal corresponding to the operators O_3 and O_7, respectively:

$$\gamma_{O_3}^{(1)} = -\frac{\beta_0}{2}, \qquad \gamma_{O_7}^{(1)} = -\beta_0. \tag{D.17}$$

These two operators will be responsible for the pole structure in the Borel transform of the Adler function. For the operator O_3 we find $2\gamma_{O_3}^{(1)}/\beta_0 = -1$ and since its coefficient function starts at order α_s (i.e. $\hat{\delta}_{O_3} = 1$), we can exploit the findings in (D.14) to argue that this operator is connected to the simple pole at $u = 3$.
For O_7, on the other hand, we have $2\gamma_{O_7}^{(1)}/\beta_0 = -2$ and looking at the generic form of its coefficient function,

$$C_{O_7} \sim r_{00}^7 \, a_s^{\hat{\delta}_{O_7} - 2} \left[1 + a_s \left(r_{10}^7 + r_{11}^7 N_f \right) + \ldots \right], \tag{D.18}$$

we expect $\hat{\delta}_{O_7} = 1$ in order that the operator contributes to the double pole in the Borel transform. However, from (D.4) we see that the coefficient function C_{O_7} starts at $\mathcal{O}(\alpha_s^2)$ which only gives rise to a simple pole structure.

At first sight the analytic Borel transform (D.8) seems to contradict the form of the four-quark condensate corrections in the OPE of the Adler function, but going back to the very beginning of the derivation of (D.8) the inconsistencies can easily be resolved. At the end of section 3.3.1 we started out from a generic OPE term of the form,

$$F_O = \vec{C}_6^{\intercal} \, \langle \vec{O}_6 \rangle, \tag{D.19}$$

and argued to work in an operator basis in which the leading order anomalous dimension matrix is diagonal,

$$\gamma_D^{(1)} = V^{-1} \gamma_+^{(1)} V. \tag{D.20}$$

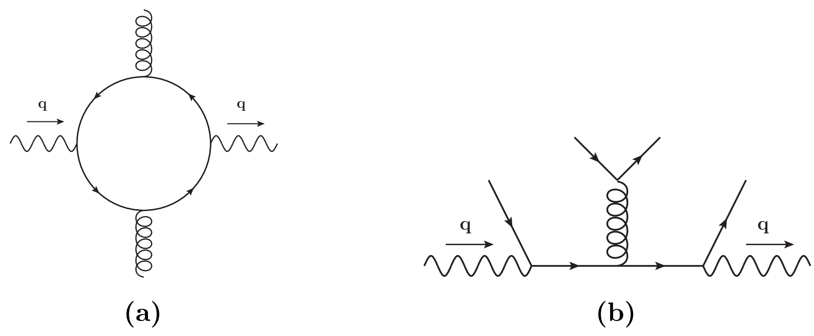

Fig. D.1: Exemplary leading order diagrams relevant for the computation of the gluon condensate (a) and the four-quark operator O_3 (b).

Here, $\gamma_D^{(1)}$ contains the eigenvalues of $\gamma_+^{(1)}$ on its diagonal and the transformation matrix V is used to construct the new operator basis:

$$\begin{aligned} F_O = \vec{C}_6^\intercal \, \langle \vec{O}_6 \rangle = \vec{C}_6^\intercal \, V V^{-1} \, \langle \vec{O}_6 \rangle = \vec{C}_{6,D}^\intercal \, \langle \vec{O}_{6,D} \rangle, \\ \langle \vec{O}_{6,D} \rangle = V^{-1} \, \langle \vec{O}_6 \rangle, \\ \vec{C}_{6,D}^\intercal = \vec{C}_6^\intercal \, V. \end{aligned} \tag{D.21}$$

Solving the RGE (3.41) to resum the logarithms in $\vec{C}_6$, the OPE term can then be cast into the form of eq. (D.10). Thus, we see that the inconsistencies in our investigation of the four-quark operators originated from the fact that we naively took the large-N_f limit of $\gamma_+^{(1)}$ without first changing to the diagonal basis. This is the crucial point: only the eigenvalues of $\gamma_+^{(1)}$, i.e. the entries of the diagonal matrix $\gamma_D^{(1)}$, enter in the analytic Borel transform (D.8), but we cannot simply consider the large-N_f limit to achieve a diagonal form of the anomalous dimension matrix. We need to do the diagonalization properly such that the structure of the coefficient functions in the new basis is adjusted appropriately and only then we can take the large-N_f limit.

If we consider once again the anomalous dimension matrix in (D.15) and perform the diagonalization, we will find that the coefficient function of O_7 receives additional contributions from other operators, including O_3 and O_+^o, with the result that the operator combination in the diagonal basis corresponding to the eigenvalue $-\beta_0$ has a coefficient function starting at order[4] α_s. This then leads to the expected double pole structure of the $u = 3$ renormalon in the Borel transform and, furthermore, shows that the double pole contribution is connected to the off-diagonal entries of the anomalous dimension matrix $\gamma_+^{(1)}$.

At the end of this section, we want to discuss a subtle issue that arises in the investigation of the dimension-4 and dimension-6 operator condensates and their coefficient functions. If we compare the matrix elements we notice that in the case of the gluon condensate one power of α_s is part of the matrix element, whereas for the four-quark operators the information on the strong coupling is completely contained in the coefficient functions. It seems there is some freedom in the definition of the matrix elements and the question arises if it is possible to move powers of α_s between the condensates and their coefficient functions.

To resolve this issue, let us have a look at Fig. D.1 which shows the leading order diagrams for the coefficient functions of the gluon condensate and the four-quark operator O_3. In both graphs the hard momentum q enters through the left current and exits on the right-hand side without passing through the gluon lines. Thus, the gluon lines are soft and it is reasonable to include the corresponding factor of α_s in the definition of the matrix element, as is done for the gluon condensate. For the operator O_3, on the other hand, the factor of α_s is part of the coefficient function

[4]The $\mathcal{O}(\alpha_s)$ contribution to this coefficient function will be actually proportional to $1/N_f$ and would again vanish in the limit $N_f \to \infty$. However, we can factor out this $1/N_f$-dependence into the coefficient $\hat{c}^{(0)}$ of (D.10) and in the derivation of our analytic Borel transform expression this coefficient cancels (cf. eq. (4.9)).

which seems to be inconsistent. In this case, however, the anomalous dimension matrix given by (D.15) has exactly the appropriate form needed such that the O_3-condensate correction in the OPE is still in line with the $u = 3$ renormalon structure in the Borel transform of the Adler function. Hence, depending on the definition of the matrix elements, the anomalous dimension matrix will always be adjusted properly. So, if for example one power of α_s was moved from the coefficient functions of the four-quark operators into the matrix elements, the anomalous dimension matrix would change accordingly in order to be still consistent with the renormalon structure of the Borel transform.

D.2 Leading Order Computation of the Coefficient Functions

The second part of the appendix provides a detailed leading order computation of the coefficient functions of the gluon condensate and some four-quark operators that belong to the complete basis given in (D.6). As already mentioned, the leading order results for the operator expansion of various currents can be found in [40] and [42], where diagrams like the ones in Fig. D.1 are calculated. In this section we reflect upon these computations and show in detail how the coefficient functions of the condensates can be obtained from an asymptotic expansion of the Feynman integrals related to the leading order diagrams depicted in Fig. 3.1 (i.e. the diagrams with an exchange of a simple gluon line).

To begin with, let us briefly review the operator product expansion of the Adler function and particularly discuss how the non-perturbative condensate corrections arise in this framework. Starting point for the investigation of the OPE in section 3.3.1 is the vacuum polarization induced by T-ordered products of vector and axial-vector currents. At large Q^2, in the asymptotically free regime, $\alpha_s(Q^2)$ is numerically small and standard perturbation theory can be applied. As we move to larger distances confinement effects become sizable and the asymptotic freedom starts to break down. This breakdown is typically reflected by the appearance of non-vanishing vacuum expectation values (condensates) of higher dimensional operators due to non-perturbative effects that change the nature of the QCD vacuum[5] [42]. The emergence of these matrix elements signals that the free particle propagators are modified strongly at large distances. Therefore, the propagators cannot be reliably described within perturbation theory and it is assumed that they are related to soft non-perturbative fields in the vacuum. This observation is key to the understanding of the computation in this section.

To become more concrete, let us consider the diagrams in Fig. D.2 and investigate how the emergence of the gluon condensate is connected to soft gluon lines. First, assuming that all quark and gluon lines carry a momentum of order q, the purely perturbative contributions to the vacuum polarization function can be calculated by applying the usual Feynman rules. However, it is also possible that the large momentum is carried only by the quark lines, while the gluon is soft. In this case we run into problems, since the soft gluon line cannot be described any longer by the perturbative form for its propagator. In order to account for the non-perturbative gluon fields we need to introduce a new unknown parameter given by the vacuum matrix element of the gluon condensate[6]. Diagramatically, this situation is shown on the right-hand side of Fig. D.2. The crossed blob contains the hard quark lines and currents at short distances where everything is computable, while the gluon line is soft and constrained to start and end at the same point. This soft gluon line is then substituted by the gluon condensate. At this point the natural question arises how we can make sure that only long-distance physics is contained in the definition of the matrix element. To that end, we have to evaluate the gluon condensate in perturbation theory and subtract this contribution from the perturbative calculation of the diagrams on the left-hand side in Fig. D.2. How this is done in practice, will be shown in the computation below.

Returning to the problem of soft non-perturbative fields, we can also study how other oper-

[5] Note that these vacuum expectation values vanish per definition in standard perturbation theory.

[6] In connection with soft gluon lines, the gluon condensate represents the simplest expression compatible with Lorentz and gauge invariance (see discussion in section 3.3.1).

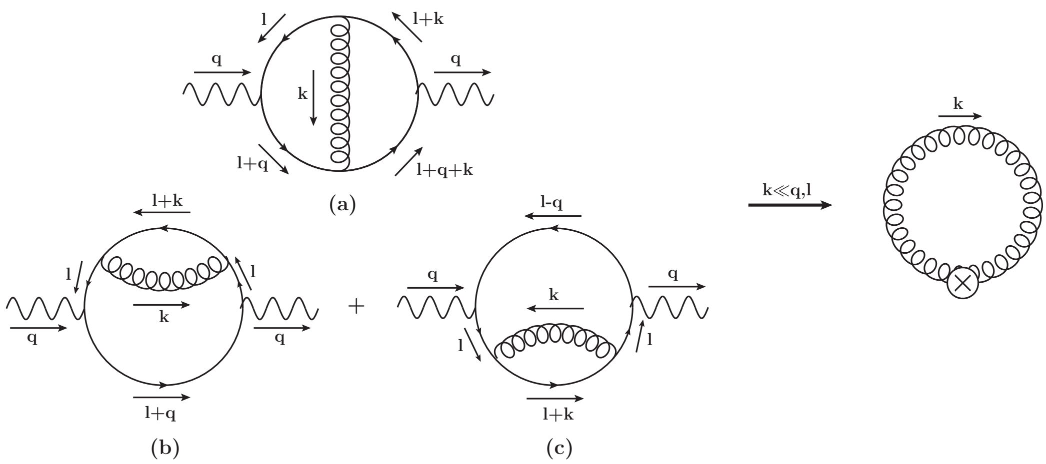

Fig. D.2: Momentum routing for the one-gluon exchange diagrams that gives rise to the gluon condensate in the OPE of the Adler function. The crossed vertex in the diagram on the right-hand side contains all hard quark lines and the currents and only the gluon line with momentum k is soft. This soft gluon line is substituted by the vacuum matrix element of the gluon condensate.

ator condensates in the OPE of the Adler function are characterized by different routings of the large momentum q. In Fig. D.3, for instance, we consider the same diagrams with a single gluon exchange, but this time we have two soft quark lines that will be related to the matrix element of a four quark-operator.

In the following we are going to compute the diagrams for various momentum routings and demonstrate how the coefficient functions for the resulting operator condensates can be obtained.

D.2.1 Gluon Condensate $\left\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a,\mu\nu} \right\rangle$

Starting with the gluon condensate, we consider once again the diagrams in Fig. D.2. Since we already know that the gluon condensate is connected to soft gluon fields, we choose the momentum routing in such a way that only the momentum k of the gluon line is soft, while the other loop momentum l of the quark bubble is of the order of the large momentum q. Before we deal with the calculation of these diagrams, let us first take a closer look at the perturbative expression for the gluon condensate represented by the one-loop diagram on the right-hand side of Fig. D.2. In $d = 4 - 2\epsilon$ dimensions the T-ordered product corresponding to this graph is given by,

$$
\begin{aligned}
\left\langle G^a_{\mu\nu} G^{a,\mu\nu} \right\rangle &\equiv \langle 0|\, T\{G^a_{\mu\nu}(x)\, G^{a,\mu\nu}(x)\}\, |0\rangle \\
&= 2\left[\, \langle 0|\, T\{(\partial_\mu A^a_\nu)\, (\partial^\mu A^{a,\nu})\}\, |0\rangle - \langle 0|\, T\{(\partial_\mu A^a_\nu)\, (\partial^\nu A^{a,\mu})\}\, |0\rangle \,\right] \\
&= -2i \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left\{ \frac{d\, k^2\, \delta^{aa}}{[k^2 + i0^+]} - \frac{\delta_\nu{}^\mu\, k_\mu\, k^\nu\, \delta^{aa}}{[k^2 + i0^+]} \right\} \\
&= -2i\, (d-1) \left(N_c^2 - 1\right) \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, \frac{k^2}{[k^2 + i0^+]},
\end{aligned}
\tag{D.22}
$$

where we did not cancel the k^2 in the numerator to emphasize that this factor is due to the derivatives in the definition of the gluon field strength $G^a_{\mu\nu}$.

Using this expression for the gluon condensate we can now compute the diagrams on the left-hand side. For the amplitude of the first graph (a) we obtain,

$$
\begin{aligned}
i\mathcal{M}^{\mu\nu}_a &= i\, g_s^2\, C_F\, C_A\, \tilde{\mu}^{4-d} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu\, \not{l}\, \gamma^\rho\, (\not{l} + \not{k})\, \gamma^\nu\, (\not{l} + \not{k} + \not{q})\, \gamma_\rho\, (\not{l} + \not{k})\right]}{[l^2]_+\, [k^2]_+ [(l+k)^2]_+\, [(l+q)^2]_+\, [(l+k+q)^2]_+} \\
&= 2\pi\, i\, \alpha_s \left(N_c^2 - 1\right)\, \tilde{\mu}^{4-d} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\mathrm{Tr}\left[\gamma^\mu\, \not{l}\, \gamma^\rho\, (\not{l} + \not{k})\, \gamma^\nu\, (\not{l} + \not{k} + \not{q})\, \gamma_\rho\, (\not{l} + \not{k})\right]}{[l^2]_+\, [k^2]_+ [(l+k)^2]_+\, [(l+q)^2]_+\, [(l+k+q)^2]_+},
\end{aligned}
\tag{D.23}
$$

where we introduced the notation $[p^2]_\pm \equiv [p^2 \pm i0^+]$ and used $C_F = (N_c^2-1)/(2N_c)$, $C_A = N_c$ in the second line. Comparing the dimensions of the loop integrals in (D.22) and (D.23) in $d=4$, we find that the two expressions are not compatible with each other. While the gluon condensate displays the expected four dimensional form, the dimensional analysis of (D.23) gives rise to a dimension-2 operator condensate. Thus, in order to obtain the contribution corresponding to the gluon condensate we need to expand the integrand in (D.23) in the soft gluon momentum and keep only those terms which are bilinear in k. Performing this expansion generates all possible terms containing two powers of k in the numerator and subsequently applying the substitution,

$$k^\alpha\, k^\beta \to \frac{1}{d}\, g^{\alpha\beta} k^2, \tag{D.24}$$

allows us to factor out the perturbative expression for the gluon condensate in (D.22). In particular, this reveals that the colour factor of the amplitude in (D.23) is not part of the coefficient function, but is rather contained in the definition of the matrix element. Finally, we can perform the integration over the loop momentum l and in the limit $d \to 4$ we obtain,

$$i\mathcal{M}_a^{\mu\nu} = \frac{i}{(q^2)^2}\left[\frac{q_\mu\, q_\nu}{18}\left(\frac{3}{\epsilon}+5+3\,\ln\!\left(-\frac{\mu^2}{q^2}\right)\right)-\frac{1}{4}\,q^2\, g_{\mu\nu}\right]\left\langle \frac{\alpha_\mathrm{s}}{\pi} G^a_{\mu\nu} G^{a,\mu\nu}\right\rangle, \tag{D.25}$$

where we already included the factor α_s/π in the definition of the matrix element. The very same procedure can now be applied to the other two diagrams in Fig. D.2, leading to:

$$\begin{aligned}
&\bullet\ i\mathcal{M}_b^{\mu\nu} = -\frac{i}{(q^2)^2}\left[\frac{q_\mu\, q_\nu}{72}\left(\frac{6}{\epsilon}+7+6\,\ln\!\left(-\frac{\mu^2}{q^2}\right)\right)-\frac{1}{12}\,q^2\, g_{\mu\nu}\right]\left\langle \frac{\alpha_\mathrm{s}}{\pi} G^a_{\mu\nu} G^{a,\mu\nu}\right\rangle,\\
&\bullet\ i\mathcal{M}_c^{\mu\nu} = -\frac{i}{(q^2)^2}\left[\frac{q_\mu\, q_\nu}{72}\left(\frac{6}{\epsilon}+7+6\,\ln\!\left(-\frac{\mu^2}{q^2}\right)\right)-\frac{1}{12}\,q^2\, g_{\mu\nu}\right]\left\langle \frac{\alpha_\mathrm{s}}{\pi} G^a_{\mu\nu} G^{a,\mu\nu}\right\rangle.
\end{aligned} \tag{D.26}$$

Adding the contributions of all three diagrams we eventually obtain,

$$\Pi^{\mu\nu} = \mathcal{M}_a^{\mu\nu} + \mathcal{M}_b^{\mu\nu} + \mathcal{M}_c^{\mu\nu} = \frac{(q_\mu\, q_\nu - q^2 g_{\mu\nu})}{12\,(q^2)^2}\left\langle \frac{\alpha_\mathrm{s}}{\pi} G^a_{\mu\nu} G^{a,\mu\nu}\right\rangle, \tag{D.27}$$

which is consistent with the transverse structure of the vacuum polarization tensor in (D.1) and gives the correct leading order result for the coefficient function of the gluon condensate (see eq. (D.3)).

D.2.2 Vector Octet Operator $\langle O_V^o \rangle$

Next, we want to study the coefficient function of the vector octet operator $O_V^o = \left(\overline{u}\gamma_\mu T^a d\, \overline{d}\gamma^\mu T^a u\right)$ given in (D.6). For this purpose, we consider the diagrams depicted in Fig. D.3 where we now have two soft quark lines that give rise to a four-quark condensate. Note that for the calculation of the coefficient function we need to take into account all possible routes of the large momentum q which lead to the same form of the matrix element represented by the diagram on the right-hand side. In particular, this means that diagrams (a) and (b) are not equivalent topologically and both need to be accounted for.

Following the logic of the investigation in the previous section, we first start with the perturbative computation of the matrix element for O_V^o which, to lowest order, is given by:

$$\begin{aligned}
\left\langle \overline{u}\gamma_\mu T^a d\, \overline{d}\gamma^\mu T^a u\right\rangle &\equiv \langle 0|\, T\{(\overline{u}\gamma_\mu T^a d)(x)\, (\overline{d}\gamma^\mu T^a u)(x)\}\, |0\rangle \\
&= C_F\, C_A \iint \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\mathrm{d}^d l}{(2\pi)^d}\, \frac{(-1)\,\mathrm{Tr}\left[i\not{k}\,\gamma_\mu\, i\not{l}\,\gamma^\mu\right]}{[k^2]_+\,[l^2]_+} \\
&= 2\,(2-d)\left(N_c^2-1\right)\iint \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{k\cdot l}{[k^2]_+\,[l^2]_+}.
\end{aligned} \tag{D.28}$$

In $d=4$ this matrix element leads to a dimension-6 expression for the four-quark operator O_V^o and in order to be compatible with this result on dimensional grounds, we again have to expand the

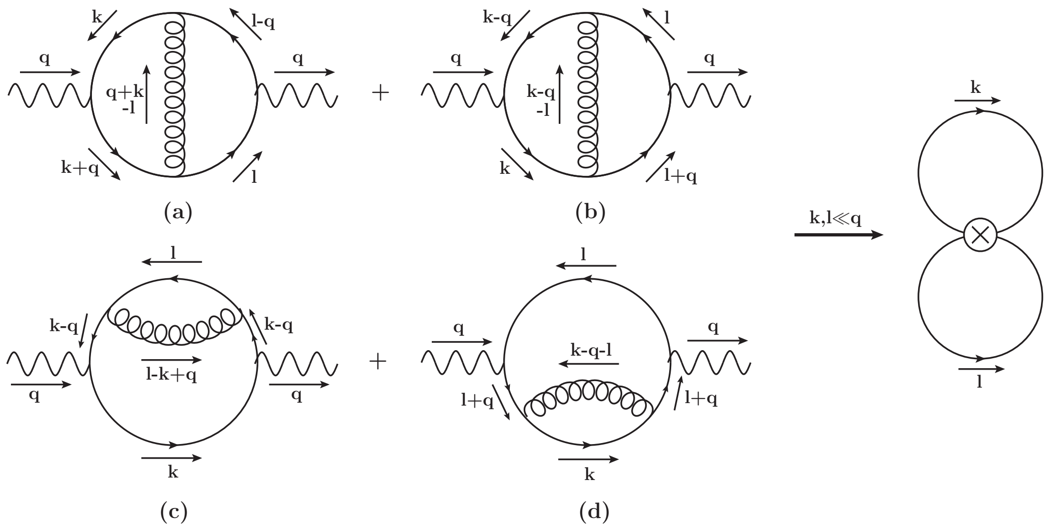

Fig. D.3: Diagrams relevant for the leading order computation of the operator O_V^o. One needs to take into account all momentum routes with two soft quark lines that lead to the same form of the matrix element on the right-hand side.

propagators in the diagrams on the left-hand side in the soft quark momenta. This time, however, we need to do the expansion consistently in both soft momenta, k and l, keeping only those terms that are either linear in k and l or bilinear in k or bilinear in l. Of course, this step again generates a lot of terms, but luckily, all contributions bilinear in k or l cancel leaving only the terms linear in both momenta[7]. Finally, we can perform the substitution,

$$k^\alpha\, l^\beta \to \frac{1}{d}\, g^{\alpha\beta}\, (k\cdot l), \tag{D.29}$$

and factor out the definition of the matrix element given by (D.28) to obtain the contribution to the coefficient function. In contrast to the computation of the gluon condensate we do not need to perform any loop integration here, since both integrals are part of the matrix element. In the limit $d \to 4$ the results for the four diagrams are then found to be:

$$\begin{aligned}
&\bullet\ i\mathcal{M}_a^{\mu\nu} = 4\pi i\, \alpha_\mathrm{s}\, \frac{g_{\mu\nu}}{(q^2)^2}\, \langle O_V^o\rangle\,,\\
&\bullet\ i\mathcal{M}_b^{\mu\nu} = 4\pi i\, \alpha_\mathrm{s}\, \frac{g_{\mu\nu}}{(q^2)^2}\, \langle O_V^o\rangle\,,\\
&\bullet\ i\mathcal{M}_c^{\mu\nu} = -4\pi i\, \alpha_\mathrm{s}\, \frac{q_\mu\, q_\nu}{(q^2)^3}\, \langle O_V^o\rangle\,,\\
&\bullet\ i\mathcal{M}_d^{\mu\nu} = -4\pi i\, \alpha_\mathrm{s}\, \frac{q_\mu\, q_\nu}{(q^2)^3}\, \langle O_V^o\rangle\,.
\end{aligned} \tag{D.30}$$

Adding all contributions finally yields,

$$\Pi^{\mu\nu} = \mathcal{M}_a^{\mu\nu} + \mathcal{M}_b^{\mu\nu} + \mathcal{M}_c^{\mu\nu} + \mathcal{M}_d^{\mu\nu} = \frac{\left(q_\mu\, q_\nu - q^2 g_{\mu\nu}\right)}{(q^2)^3}\,(-8\pi\,\alpha_\mathrm{s})\,\langle O_V^o\rangle\,, \tag{D.31}$$

which is consistent with the leading order contribution of the coefficient function given in (D.5).

D.2.3 Operator $\langle O_3\rangle$

The study of the operator O_3 is a bit more involved than the computations in the previous two sections. First, note that the flavour sum of this four-quark operator is usually rewritten by means

[7]Note that these cancellations take place for all four diagrams separately.

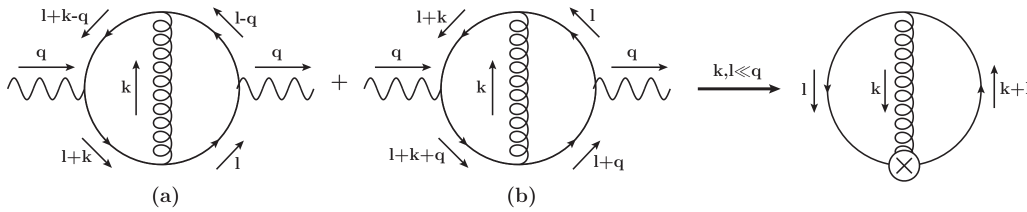

Fig. D.4: Possible routes of the large momentum q that lead to the operator O_3. Only the diagram in which the gluon is exchanged between the upper antiquark and lower quark line contributes in this case to the matrix element on the right-hand side.

of the equation of motion,

$$\begin{aligned} D_\mu \, G^{a,\mu\nu} + g_{\mathrm{s}} \sum_q \left(\overline{q}\,\gamma^\nu T^a q\right) = 0 \qquad \text{(D.32)} \\ \Rightarrow O_3 = -\frac{1}{g_{\mathrm{s}}} \left(\overline{u}\gamma_\mu T^a u + \overline{d}\gamma_\mu T^a d\right) D_\nu \, G^{a,\nu\mu}. \end{aligned}$$

The resulting form of the operator is more suitable for the perturbative evaluation of its matrix element. To lowest order, this matrix element is given by the two-loop graph on the right-hand side of Fig. D.4 and yields:

$$\begin{aligned} \langle O_3 \rangle &\equiv -\frac{1}{g_{\mathrm{s}}} \, \langle\Omega|\, T\{(\overline{u}\gamma_\mu T^a u + \overline{d}\gamma_\mu T^a d)(x)\,(D_\nu \, G^{a,\nu\mu})(x)\}\,|\Omega\rangle \qquad \text{(D.33)} \\ &= 2\, C_F \, C_A \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \, \frac{(-1)\,\mathrm{Tr}\left[i\not{l}\,\gamma_\mu\, i(\not{l}+\not{k})\,\gamma^\mu\right] k^2}{[l^2]_+\,[(l+k)^2]_+\,[k^2]_+} \\ &= 4\,(2-d)\left(N_c^2-1\right) \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \, \frac{k\cdot l}{[k^2]_+\,[l^2]_+}. \end{aligned}$$

It is not a coincidence that the perturbative expression for the matrix element of O_3 agrees with the one for O_V^o in (D.28)[8]. If we do not apply the equation of motion (D.32) and directly compute the matrix element for O_3 given in its four-quark form, we obtain the two-loop graph depicted in Fig. D.3. However, it is more convenient to use (D.33), since the form of the corresponding diagram in Fig. D.4 allows us to specify the momentum routings of the two-loop vacuum polarization diagrams that lead to the condensate corrections for O_3. We find that only the diagram in which the gluon is exchanged between the quark and the antiquark lines contributes in this case. More specifically, we can distinguish two different routes of the large momentum q, shown on the left-hand side of Fig. D.4, that give rise to the matrix element of O_3.

In the following we will concentrate on the computation of diagram (a) to demonstrate how the Wilson coefficient can be extracted,

$$i\mathcal{M}_a^{\mu\nu} = -i\,g_{\mathrm{s}}^2\,C_F\,C_A\,\tilde{\mu}^{4-d} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \, \frac{\mathrm{Tr}\left[\gamma^\mu\,(\not{l}+\not{k}-\not{q})\,\gamma^\rho\,(\not{l}-\not{q})\,\gamma^\nu\,\not{l}\,\gamma_\rho\,(\not{l}+\not{k})\right]}{[l^2]_+\,[k^2]_+[(l+k)^2]_+\,[(l-q)^2]_+\,[(l+k-q)^2]_+}. \qquad \text{(D.34)}$$

From the dimensional analysis of the matrix element (see second line in (D.33)) we can infer that we need to keep all terms in the asymptotic expansion that give four additional powers of the soft momenta[9] k and l. The enormous amount of terms this expansion is generating makes the computation very cumbersome.

Next, we can start to rewrite the terms of order $k\,l^3$, $k^2\,l^2$ and $k^3\,l$ by reducing the tensor integrals over one of the loop momenta, e.g. k, to scalar ones. Since the (perturbative) matrix

[8] The additional factor 2 in the definition of (D.33) compared with (D.28) is due to the fact that the operator O_3 contains the sum of the currents $(\overline{u}\gamma_\mu T^a u)$ and $(\overline{d}\gamma_\mu T^a d)$.

[9] We stress that one has to take into account all possible contributions in this expansion, i.e. all terms of order k^4, $k^3\,l$, $k^2\,l^2$, $k\,l^3$ and l^4.

element of O_3 is scaleless, we need to be careful about the decomposition and, in particular, are not allowed to drop any scaleless terms that usually vanish in dimensional regularization. A long but straightforward calculation, including also the terms of order k^4 and l^4, then leads to,

$$
\begin{aligned}
i\mathcal{M}_a^{\mu\nu} =& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(D.35)}\\
=&\frac{-i\,g_s^2\,C_F\,C_A\,\tilde{\mu}^{4-d}}{(d-1)}\Bigg\{-\frac{16\,(d^2-6d-4)}{d\,(q^2)^3}\iint\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{(k\cdot l)\,(q_\mu\,q_\nu-q^2\,g_{\mu\nu})}{[k^2]_+\,[l^2]_+}\\
&+\frac{32\,(d^3-7d^2+6d-16)}{d\,(d+2)\,(q^2)^3}\iint\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{l^2\,q_\mu\,q_\nu}{[k^2]_+\,[l^2]_+}\\
&+\frac{4\,(d^4-13d^3-56d^2-28d+112)}{d\,(d+2)\,(q^2)^3}\iint\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{l^2\,q^2\,g_{\mu\nu}}{[k^2]_+\,[l^2]_+}\\
&-\frac{32\,(d+1)}{(q^2)^4}\iint\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{(l\cdot q)^2\,\left[l_\mu\,l_\nu\,q^2-(l\cdot q)\,\left(l_\mu\,q_\nu+l_\nu q_\mu-(l\cdot q)\,g_{\mu\nu}\right)\right]}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+}\Bigg\},
\end{aligned}
$$

where we already applied the substitutions (D.24) and (D.29) wherever possible. This result is not yet in a suitable form and still contains undesired terms that we need to get rid of. For this purpose we use a "dirty" trick. Consider for example a two-loop integral of the following form:

$$
\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(l\cdot q)\,(k\cdot q)\,k_\mu\,k_\nu}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+} = \iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{l_\alpha\,k_\beta\,k_\mu\,k_\nu\,q^\alpha\,q^\beta}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+}. \tag{D.36}
$$

To decompose this integral we have two possibilities. On the one hand we can start with the integral over l and use,

$$
\int \mathrm{d}^d l\,\frac{l_\alpha}{[l^2]_+\,[(k+l)^2]_+} = -\frac{1}{2}\,k_\alpha\int \mathrm{d}^d l\,\frac{1}{[l^2]_+\,[(k+l)^2]_+}, \tag{D.37}
$$

to rewrite (D.36) in the form,

$$
\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(l\cdot q)\,(k\cdot q)\,k_\mu\,k_\nu}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+} = -\frac{1}{2}\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(k\cdot q)^2\,k_\mu\,k_\nu}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+}. \tag{D.38}
$$

On the other hand, we can first consider the k-integral and decompose the tensor structure $k_\beta k_\mu k_\nu$ according to the general ansatz:

$$
\int \mathrm{d}^d k\,\frac{k_\beta\,k_\mu\,k_\nu}{[k^2]_+\,[(k+l)^2]_+} = l_\beta\,l_\mu\,l_\nu\,B_{111} + (l_\beta\,g_{\mu\nu}+l_\mu\,g_{\beta\nu}+l_\nu\,g_{\mu\beta})\,B_{001}. \tag{D.39}
$$

Keeping all scaleless integrals in the evaluation of the coefficients B_{111} and B_{001} one then finds,

$$
\begin{aligned}
\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(l\cdot q)\,(k\cdot q)\,k_\mu\,k_\nu}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+} = \iint \mathrm{d}^d k\,\mathrm{d}^d l\,\Bigg[&\frac{(d+2)\,(l\cdot q)^2\,l_\mu\,l_\nu}{8\,(1-d)\,[k^2]_+\,[l^2]_+\,[(k+l)^2]_+}\\
&+\frac{3\,(d-2)\,\left(k\cdot l-l^2\right)\left(2q_\mu\,q_\nu+q^2\,g_{\mu\nu}\right)}{4\,d\,(d-1)\,(d+2)\,[k^2]_+\,[l^2]_+}\Bigg],
\end{aligned} \tag{D.40}
$$

where we again applied (D.24) and (D.29) wherever possible. If the decomposition is done properly, the right-hand sides of eqs. (D.38) and (D.40) need to coincide which allows us to derive a handy relation:

$$
\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(l\cdot q)^2\,l_\mu\,l_\nu}{[k^2]_+\,[l^2]_+\,[(k+l)^2]_+} = \frac{2\,\left(2q_\mu\,q_\nu+q^2\,g_{\mu\nu}\right)}{d\,(d+2)}\iint \mathrm{d}^d k\,\mathrm{d}^d l\,\frac{(l^2-k\cdot l)}{[k^2]_+\,[l^2]_+}. \tag{D.41}
$$

In a similar fashion one can also find other relations that can be used to rewrite the result in (D.35),

$$\begin{aligned}
&\bullet \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{(l\cdot q)^3 \, l_\mu \, q_\nu}{[k^2]_+ \, [l^2]_+ \, [(k+l)^2]_+} = \frac{6\,(d-2)\, q_\mu \, q_\nu \, q^2}{d\,(d+2)} \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{\left(l^2 - k\cdot l\right)}{[k^2]_+ \, [l^2]_+}, \\
&\bullet \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{(l\cdot q)^3 \, l_\nu \, q_\mu}{[k^2]_+ \, [l^2]_+ \, [(k+l)^2]_+} = \frac{6\,(d-2)\, q_\mu \, q_\nu \, q^2}{d\,(d+2)} \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{\left(l^2 - k\cdot l\right)}{[k^2]_+ \, [l^2]_+}, \\
&\bullet \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{(l\cdot q)^4 \, g_{\mu\nu}}{[k^2]_+ \, [l^2]_+ \, [(k+l)^2]_+} = \frac{6\,(d-2)\, g_{\mu\nu} \left(q^2\right)^2}{d\,(d+2)} \iint \mathrm{d}^d k \, \mathrm{d}^d l \, \frac{\left(l^2 - k\cdot l\right)}{[k^2]_+ \, [l^2]_+}.
\end{aligned} \tag{D.42}$$

Applying these identities to (D.35) then gives,

$$\begin{aligned}
i\mathcal{M}_a{}^{\mu\nu} =& \\
= -& \frac{4\,(d-2)\, i\, g_\mathrm{s}^2 \, C_F \, C_A \, \tilde{\mu}^{4-d}}{d\,(d+2)\, \left(q^2\right)^3} \left\{ \frac{4\,(4 + d\,(2-d))}{(d-1)} \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(k\cdot l)\, \left(q_\mu \, q_\nu - q^2 \, g_{\mu\nu}\right)}{[k^2]_+ \, [l^2]_+} \right. \\
&\left. + (d-4) \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{l^2 \left(8\, q_\mu \, q_\nu + (d-6)\, q^2 \, g_{\mu\nu}\right)}{[k^2]_+ \, [l^2]_+} \right\},
\end{aligned} \tag{D.43}$$

and in the limit $d \to 4$ we obtain:

$$i\mathcal{M}_a{}^{\mu\nu} = \frac{16\, i\, g_\mathrm{s} \, C_F \, C_A}{9\, \left(q^2\right)^3} \left(q_\mu \, q_\nu - q^2 \, g_{\mu\nu}\right) \iint \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{(k\cdot l)}{[k^2]_+ \, [l^2]_+}. \tag{D.44}$$

In the last step, we can finally account for the matrix element of O_3 given in (D.33) which leads to,

$$i\mathcal{M}_a{}^{\mu\nu} = -i\, 4\pi \, \alpha_\mathrm{s} \, \frac{\left(q_\mu \, q_\nu - q^2 \, g_{\mu\nu}\right)}{9\, \left(q^2\right)^3} \langle O_3 \rangle. \tag{D.45}$$

Diagram (b) can be calculated in the very same way and is found to yield an identical contribution. Finally adding the results for both diagrams gives the correct leading order contribution for the coefficient function of O_3 (see (D.4)):

$$\Pi^{\mu\nu} = \mathcal{M}_a^{\mu\nu} + \mathcal{M}_b^{\mu\nu} = \frac{\left(q_\mu \, q_\nu - q^2 g_{\mu\nu}\right)}{\left(q^2\right)^3} \, 4\pi \, \alpha_\mathrm{s} \left(-\frac{2}{9}\right) \langle O_3 \rangle. \tag{D.46}$$

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