MASTERARBEIT/ MASTER'S THESIS

Titel der Masterarbeit / Title of the Master‘s Thesis

Smooth regularity of CR maps into boundaries of classical symmetric domains

verfasst von / submitted by

Josef Eberhard Greilhuber, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020/ Vienna, 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Bernhard Lamel, PhD

Abstract [EN]

In this thesis, we study the smooth regularity of CR maps between CR submanifolds with targets that are foliated by complex manifolds. As an application, we consider CR-transversal CR maps from strongly pseudoconvex real hypersurfaces into the smooth part of the boundary of classical symmetric domains. For the first, second and third series of classical symmetric domains, we find sharp conditions on the CR-dimension of the source manifold which force any such CR map to be C^∞-smooth on a dense open subset of the source, as long as it satisfies an initial regularity condition. For classical symmetric domains of the fourth kind, we prove that any CR map of sufficient initial regularity is either smooth on a dense open subset or locally takes its values in a single complex line.

We begin by briefly introducing the necessary concepts of CR geometry before turning to a theorem of Lamel & Mir [8] which serves as the main instrument to establish regularity of CR maps. Subsequently, a setup for the analysis of CR maps into pseudoconvex hypersurfaces foliated by complex manifolds is developed and applied to CR maps into boundaries of classical symmetric domains.

Abstract

In dieser Arbeit untersuchen wir die C^∞-Regularität von CR-Abbildungen zwischen CR-Teilmannigfaltigkeiten, deren Zielmannigfaltigkeit mit komplexen Mannigfaltigkeiten foliert ist. Als Anwendungsbeispiel konzentrieren wir uns auf CR-transversale CR-Abbildungen, die eine stark pseudokonvexe reelle Hyperfläche in den glatten Teil des Randes eines klassischen symmetrischen Gebiets abbilden. Für die erste, zweite und dritte Reihe klassischer symmetrischer Gebiete finden wir scharfe Bedingungen an die CR-Dimension der Ursprungsmannigfaltigkeit, unter welchen eine solche CR-Abbildung bereits glatt ist, solange sie genügend viele Ableitungen besitzt. Für die klassischen symmetrischen Gebiete vom Typ Vier zeigen wir, dass jede genügend oft differenzierbare CR-Abbildung entweder glatt auf einer dichten offenen Teilmenge ist, oder lokal ihre Werte in einer einzigen komplexen Gerade annimmt.

Wir beginnen mit einer kurzen Einführung in die notwendigen Grundbegriffe der CR-Geometrie, und wenden uns dann einem Satz von Lamel & Mir [8] zu, der das Hauptwerkzeug für die Regularitätsbeweise bereitstellt. Danach untersuchen wir Abbildungen in pseudokonvexe, mit komplexen Mannigfaltigkeiten folierte Hyperflächen. Zuletzt wenden wir die entwickelten Methoden auf Abbildungen in den Rand klassischer symmetrischer Gebiete an.

Contents

1 A primer on CR geometry

When studying holomorphic functions of several complex variables, one quickly comes across real hypersurfaces in $\mathbb{C}^N$ (here considered as $\mathbb{R}^{2N}$). These arise as smooth parts of the boundary of domains of definition of holomorphic functions. Real submanifolds of higher codimension arise as skeletons of boundaries, e.g. as the edges and hyperedges of hypercubes or polydiscs.

Given this context, we are interested in the behavior of boundary limits to $\partial\Omega$ of holomorphic functions defined inside Ω. A natural question is whether there is an intrinsic way of deciding if a function $f \in C^1(\partial\Omega)$ can be extended to a holomorphic function on Ω. The answer, as we shall see, is yes, at least locally.

The key phenomenon appearing in higher complex dimensions is that there always exist complex curves tangential to $\partial\Omega$, contrary to the one-dimensional case. Let us consider a general real submanifold $M \subseteq \mathbb{C}^N$ of codimension d. At each point $p \in M$, the tangent space T_pM is a $(2N-d)$-dimensional real subspace of $T_p\mathbb{C}^N$. The map $z \to p + i(z-p)$ induces a linear isomorphism J on $T_p\mathbb{C}^N$ by its pushforward at p. Writing $z_j = x_j + iy_j$, J is determined by $J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$ and $J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}$. Evidently, $J \circ J = -\operatorname{id}$, and $T_p\mathbb{C}^N$ may be endowed with a complex vector space structure by setting $(a+ib)\cdot V = a\cdot V + b\cdot JV$.

The *complex tangent space* T_p^cM at p is given by $T_pM \cap J(T_pM)$. Since $J \circ J = -\operatorname{id}$, T_p^cM is invariant under action by J and therefore a complex subspace of $T_p\mathbb{C}^N$. From the definition of J one can also see that T_p^cM is the tangent space of the largest affine complex plane in $\mathbb{C}^N$ which is tangential to M at p. By elementary linear algebra, $\dim_{\mathbb{R}} T_p^cM \geq 2N-2d$, hence $\dim_{\mathbb{C}} T_p^cM \geq N-d$. If $\dim_{\mathbb{C}} T_p^cM$ is constant on M, M is called a *CR manifold*. In this case, $T^cM := \bigcup_{p\in M} T_p^cM$ is a distribution on M. If T_pM and $J(T_pM)$ lie in general position, i.e. if $\dim_{\mathbb{C}} T_p^cM = N-d$, M is called *generic*.

Example 1. *The real submanifold $M \in \mathbb{C}^2$ given by $M = \{(z,w) \in \mathbb{C}^2 : w = \bar{z}^2\}$ is not a CR manifold.*

Proof. The defining equations in real coordinates $x = \Re(z)$, $y = \Im(z)$, $u = \Re(w)$ and

$v = \Im(w)$ read $\rho(x, y, u, v) = (u - x^2 + y^2, v + 2xy) = (0, 0)$. We compute

$$\begin{aligned} T_pM &= \ker \begin{pmatrix} -2x & 2y & 1 & 0 \\ 2y & 2x & 0 & 1 \end{pmatrix} = \left\langle (1, 0, 2x, -2y)^T, (0, 1, -2y, 2x)^T \right\rangle_{\mathbb{R}} \\ &= \left\langle (1, 2\bar{z})^T, i(1, 2z)^T \right\rangle_{\mathbb{R}} \end{aligned}$$

The space $T_pM + J(T_pM) = \left\langle (1, 2\bar{z})^T, i(1, 2z)^T \right\rangle_{\mathbb{C}}$ has one complex dimension where $z = \bar{z}$, and two complex dimensions otherwise. Thus $T_p^cM = T_pM \cap J(T_pM)$ is one-dimensional where $z \in \mathbb{R}$, and trivial elsewhere, and M is not a CR manifold. □

Example 2. *Every real hypersurface $M \subset \mathbb{C}^n$ is a generic CR manifold.*

Proof. At any point $p \in M$, $\dim_{\mathbb{R}} T_pM = 2n - 1$. From the dimension formula,

$$\begin{aligned} 2n - 1 \geq \dim_{\mathbb{R}} (T_pM \cap J(T_pM)) &\geq \dim_{\mathbb{R}} T_pM + \dim_{\mathbb{R}} J(T_pM) - \dim_{\mathbb{R}} T_p\mathbb{C}^n \\ &= (2n - 1) + (2n - 1) - 2n = 2(n - 1), \end{aligned}$$

we infer that $2n - 1 \geq \dim_{\mathbb{R}} T_p^cM \geq 2(n - 1)$, implying $\dim_{\mathbb{C}} T_p^cM = n - 1$. □

1.1 The tangential Cauchy-Riemann equations

Consider now a holomorphic function f defined on an open neighborhood of $p \in \mathbb{C}^N$, and a CR manifold M containing p. For a vector $V \in T_p^cM$, the complex line given by $\gamma(t) = p + \Re(t)V + \Im(t)JV$ is tangential to M. The function $f \circ \gamma : \mathbb{C} \to \mathbb{C}$ is holomorphic by the chain rule, and thus $\Re(Vf) = \Im((JV)f)$ and $\Im(Vf) = -\Re((JV)f)$ by the one-dimensional Cauchy-Riemann equations. Equivalently, $(V + iJV)f = 0$, if we define the derivation iX for a tangent vector X simply by applying X and subsequently multiplying the resulting number with i.

This phenomenon leads us to consider the space of all operators $X + iY$, $X, Y \in T_p\mathbb{C}^N$, i.e. the complexification $\mathbb{C}T_p\mathbb{C}^N$. The real linear map J is simply extended to $\mathbb{C}T_p\mathbb{C}^N$ as a complex linear map. Note that since $J^2 = -\mathbb{I}$, the eigenvalues of J are i and $-i$, each occuring with multiplicity n. The space of all vectors $Z \in \mathbb{C}T_p\mathbb{C}^N$ with $JZ = iZ$ is denoted by $T_p^{1,0}\mathbb{C}^N$, and $T_p^{0,1}\mathbb{C}^N$ denotes the space of all Z with $JZ = -iZ$. The space $T_p^{1,0}M := \mathbb{C}T_pM \cap T_p^{1,0}\mathbb{C}^N$ consists then of all vectors $Z \in T_p^{1,0}\mathbb{C}^N$ such that both $\Re(Z)$ and $\Im(Z)$ are tangential to M. Analogously, we define $T_p^{0,1}M$. One readily checks that

$T_p^{1,0}M$ and $T_p^{0,1}M$ consist of all vectors $V-iJV$ and $V+iJV$, respectively, for $V \in T_p^cM$. As with the complex tangent bundle, we obtain the vector bundles $T^{1,0}M \subseteq \mathbb{C}TM$ and $T^{0,1}M \subseteq \mathbb{C}TM$ if M is a CR manifold. The bundle $T^{0,1}M$ is called the *CR bundle* of M, and $\dim_{\mathbb{C}} T^{0,1}M =: \dim_{CR} M$ the *CR-dimension* of M. A section $\bar{L}$ of the CR bundle is then called a *CR vector field*. The space of such vector fields is denoted by $\mathcal{V}(M)$.

A key advantage of working with $T^{0,1}M$ instead of T^cM is that $T_p^{0,1}M$ may be calculated directly from defining equations $\rho(z_1,\dots,z_n)=0$. We start with the exterior derivatives $dz_1,\dots,dz_n,d\bar{z}_1,\dots,d\bar{z}_n$, which form a basis of $\mathbb{C}T_p^*\mathbb{C}^n$. It is easy to check that the dual basis is given by $\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n},\frac{\partial}{\partial \bar{z}_1},\dots,\frac{\partial}{\partial \bar{z}_n}$, where $\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$. In coordinates, $Z \in T_p\mathbb{C}^n$ reads $\sum_{j=1}^n \left(X_j\frac{\partial}{\partial x_j} + Y_j\frac{\partial}{\partial y_j}\right)$, and $JZ = \sum_{j=1}^n \left(-Y_j\frac{\partial}{\partial x_j} + X_j\frac{\partial}{\partial y_j}\right)$. Writing $Z_j := X_j + iY_j$, we obtain

$$\begin{aligned}\frac{1}{2}(Z+iJZ) &= \frac{1}{2}\sum_{j=1}^n \left(X_j\frac{\partial}{\partial x_j} + Y_j\frac{\partial}{\partial y_j}\right) + \frac{i}{2}\sum_{j=1}^n \left(-Y_j\frac{\partial}{\partial x_j} + X_j\frac{\partial}{\partial y_j}\right)\\ &= \frac{1}{2}\sum_{j=1}^n (X_j - iY_j)\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) = \sum_{j=1}^n \bar{Z}_j\frac{\partial}{\partial \bar{z}_j}.\end{aligned}$$

A vector $\frac{1}{2}(Z+iJZ) \in T_p^{0,1}\mathbb{C}^n$ is tangential to M if and only if $Z\rho = JZ\rho = 0$. Because ρ is real valued, this is equivalent to $\frac{1}{2}(Z+iJZ)\rho = 0$, by taking real and imaginary parts. We obtain an alternative characterization of $T_p^{0,1}M$ as the set of all vectors $\sum_{j=1}^n \bar{Z}_j\frac{\partial}{\partial \bar{z}_j}$ satisfying $\sum_{j=1}^n \bar{Z}_j\frac{\partial}{\partial \bar{z}_j}\rho(z) = 0$.

Example 3. *A submanifold $M \in \mathbb{C}^n$ is a complex manifold if and only if $T^cM = TM$.*

Proof. If M is a complex manifold, then we can take a holomorphic parametrization ϕ of M, and observe that the range of $D\phi$ is a complex vector space at each point, essentially by definition, hence $TM = T^cM$.

For the other direction, let $2m = \dim TM = 2\dim_{CR} M$, and take $p \in M$. We choose suitable complex linear coordinates such that $p=0$ and $T_pM = \{(z_1,\dots,z_n)^T \in \mathbb{C}^n : z_{m+1} = \dots = z_n = 0\}$. Then we may locally write M as the graph of a smooth function $\phi : (z_1,\dots,z_m) \mapsto (\phi_1,\dots,\phi_{n-m})$. The projection of $T_q^{0,1}M$ onto $\left\langle \frac{\partial}{\partial \bar{z}_1},\dots,\frac{\partial}{\partial \bar{z}_m}\right\rangle_{\mathbb{C}}$ is the identity at p, hence it is invertible on a neighborhood of p, yielding a basis of CR vector fields $\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} + \bar{L}_j^{m+1}\frac{\partial}{\partial \bar{z}_{m+1}} + \dots + \bar{L}_j^n\frac{\partial}{\partial \bar{z}_n}$ for $j=1,\dots,m$. Applying $\bar{L}_j$ to the defining equations $\phi_k - z_k = 0$ for $m+1 \le k \le n$, we obtain $\frac{\partial \phi_k}{\partial \bar{z}_j} = 0$, and thus a holomorphic parametrization $(z_1,\dots,z_m) \mapsto (z_1,\dots,z_m,\phi_1,\dots,\phi_{n-m})$ of M around p. $\square$

The annihilator bundle to T^cM in T^*M is called the *characteristic bundle* and denoted T_p^oM. A functional $\theta_p \in T_p^*M$ lies in T_p^oM if and only if $\theta_p(V) = 0$ for any $V \in T_p^cM$, equivalently if $\theta_p(V+iJV) = \theta_p(V-iJV) = 0$ after extending θ_p as a linear functional to $\mathbb{C}T_pM$. This bundle is especially interesting on hypersurfaces, since it is one-dimensional there. Let the hypersurface M be defined by $\rho(z) = 0$. It is easily checked that

$$d\rho = \sum_{j=1}^{n} \left(\frac{\partial \rho}{\partial x_j} dx_j + \frac{\partial \rho}{\partial y_j} dy_j \right) = \sum_{j=1}^{n} \left(\frac{\partial \rho}{\partial z_j} dz_j + \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j \right).$$

Consider $\Theta = i \sum_{j=1}^{n} \left(\frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j - \frac{\partial \rho}{\partial z_j} dz_j \right)$. That ρ is real valued implies $\rho = \bar{\rho}$, and together with $\overline{dz_j} = d\bar{z}_j$ that $\bar{\Theta} = \Theta$, showing that Θ has real coefficients when expressed in terms of dx and dy. Since $dz_j(\frac{\partial}{\partial \bar{z}_j}) = 0$, we have that $\Theta(V+iJV) = id\rho(V+iJV) = 0$ for $V + iJV \in T^{0,1}M$, and analogously, $\Theta(V - iJV) = -id\rho(V - iJV) = 0$, implying that Θ is a characteristic form. Because $\frac{1}{2}(d\rho - i\Theta) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} dz_j$ and $\frac{1}{2}(d\rho + i\Theta) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{z}_j} d\bar{z}_j$ are clearly $\mathbb{C}$-linearly independent everywhere, $\Theta|_p$ and $d\rho|_p$ cannot have the same kernel, thus Θ never vanishes on TM. Since T^oM is just one-dimensional, this means that Θ spans T^oM.

1.1.1 CR functions

A function $f \in C^1(M, \mathbb{C})$ is called a *CR function* if $\bar{L}f = 0$ for all CR vector fields $\bar{L} \in \mathcal{V}(M)$. The prototypical CR function is given by the restriction of a holomorphic function to M. Such restrictions do not provide all CR functions, but locally they lie dense in the space of CR functions on M, by the approximation theorem of Baouendi-Treves [1, Chapter 2].

Theorem 1. *Let $M \subseteq \mathbb{C}^N$ be a CR submanifold, $p \in M$. There exists a compact neighborhood K of p in M such that any continuously differentiable CR function f on K may be approximated uniformly in K by restrictions of polynomials to K.*

With this theorem at hand, we now are interested in holomorphic maps $\varphi : \mathbb{D}^1 \to \mathbb{C}^N$ mapping the unit disc into $\mathbb{C}^N$, which extend at least continuously to $\overline{\mathbb{D}^1}$ and map $\partial\mathbb{D}^1$ into K. Consider a sequence of polynomials $(P_j)_{j=1}^{\infty}$ converging uniformly to a CR function f on K. The functions $P_j \circ \varphi : \mathbb{D}^1 \to \mathbb{C}$ are holomorphic on the unit disc and extend continuously to its boundary, thus by the one-dimensional maximum principle, $P_j(z)$ also converges for each $z \in \varphi(\mathbb{D}^1)$, with the same rate of convergence as we have on K. If we can fill an open set of points with such *analytic discs*, the sequence of polynomials thus converges uniformly to a uniquely defined holomorphic function depending only on f. A very careful study of the set of analytic discs attached to M, which can be equipped with a Banach space manifold structure, yields the following theorem of Tumanov (see [1, Chapter 8]).

Theorem 2. *Let $M \subseteq \mathbb{C}^N$ be a generic submanifold of codimension d. If M is* minimal *at a point $p \in M$, then for every neighborhood U of p there exists a wedge $\mathcal{W}$ with edge M centered at p such that every continuously differentiable CR function in U extends holomorphically to the wedge $\mathcal{W}$.*

A *wedge* with edge M centered at p is a set $\mathcal{W} \subseteq \mathbb{C}^N$ which is defined by a neighborhood $O \subseteq \mathbb{C}^N$ of p, an open convex cone $\Gamma \subseteq \mathbb{C}^d$ and a defining function $\rho : O \to \mathbb{C}^d$ of M as follows: $\mathcal{W} = \{z \in O : \rho(z) \in \Gamma\}$. Notably, wedges with hypersurface edges centered at p are just one-sided neighborhoods of p. A CR submanifold M is called *minimal* at $p \in M$ if there does not exist a CR submanifold $S \subset M$ of positive codimension in M satisfying $\dim_{CR} S = \dim_{CR} M$. The converse to Tumanov's theorem was proven by Baouendi & Rothschild: If M is not minimal at $p \in M$, then there exists a CR function defined on a neighborhood of p which does not extend to any wedge with edge M centered at p (cf. [1, Chapter 8]).

1.1.2 CR maps

A continuously differentiable map $h : M \to M'$ between CR submanifolds $M \subseteq \mathbb{C}^N$ and $M' \subseteq \mathbb{C}^{N'}$ is called a *CR map* if $h_* T_p^{0,1} M \subseteq T_{h(p)} T^{0,1} M'$ at each point $p \in M$, where h_* denotes the $\mathbb{C}$-linear extension to $\mathbb{C}TM$ of the standard pushforward map $h_* : TM \to TM'$. Let us calculate this pushforward of a CR vector $\bar{L}|_p = \sum_{j=1}^N \bar{L}_j \frac{\partial}{\partial \bar{z}_j}|_p$ along a CR map $h : M \to M'$ in coordinates $z_1, \ldots, z_N$ of $\mathbb{C}^N$ and $w_1, \ldots, w_{N'}$ of $\mathbb{C}^{N'}$. To simplify calculations, extend h to a C^1 function $\tilde{h} : \mathbb{C}^N \to \mathbb{C}^{N'}$. Writing $z_j = x_j + iy_j$ and $w_k = u_k + iv_k$, we extend the real pushforward map $h_* : T\mathbb{C}^N \to T\mathbb{C}^{N'}$ as a $\mathbb{C}$-linear map to the standard basis of $\mathbb{C}T\mathbb{C}^N$ to obtain

$$\begin{aligned}
\tilde{h}_* \frac{\partial}{\partial \bar{z}_j}\big|_p &= \frac{1}{2}\tilde{h}_* \frac{\partial}{\partial x_j}\big|_p + \frac{i}{2}\tilde{h}_* \frac{\partial}{\partial y_j}\big|_p \\
&= \frac{1}{2}\sum_{k=1}^{N'} \left(\frac{\partial (u_k \circ \tilde{h})}{\partial x_j}\big|_p \frac{\partial}{\partial u_k}\big|_{h(p)} + \frac{\partial (v_k \circ \tilde{h})}{\partial x_j}\big|_p \frac{\partial}{\partial v_k}\big|_{h(p)} \right) \\
&+ \frac{i}{2}\sum_{k=1}^{N'} \left(\frac{\partial (u_k \circ \tilde{h})}{\partial y_j}\big|_p \frac{\partial}{\partial u_k}\big|_{h(p)} + \frac{\partial (v_k \circ \tilde{h})}{\partial y_j}\big|_p \frac{\partial}{\partial v_k}\big|_{h(p)} \right) \\
&= \frac{1}{4}\sum_{k=1}^{N'} \left(\frac{\partial (u_k \circ \tilde{h})}{\partial x_j} + i\frac{\partial (v_k \circ \tilde{h})}{\partial x_j} + i\frac{\partial (u_k \circ \tilde{h})}{\partial y_j} - \frac{\partial (v_k \circ \tilde{h})}{\partial y_j} \right)\Big|_p \left(\frac{\partial}{\partial u_k}\big|_{h(p)} - i\frac{\partial}{\partial v_k}\big|_{h(p)} \right) \\
&+ \frac{1}{4}\sum_{k=1}^{N'} \left(\frac{\partial (u_k \circ \tilde{h})}{\partial x_j} - i\frac{\partial (v_k \circ \tilde{h})}{\partial x_j} + i\frac{\partial (u_k \circ \tilde{h})}{\partial y_j} + \frac{\partial (v_k \circ \tilde{h})}{\partial y_j} \right)\Big|_p \left(\frac{\partial}{\partial u_k}\big|_{h(p)} + i\frac{\partial}{\partial v_k}\big|_{h(p)} \right) \\
&= \sum_{k=1}^{N'} \left(\frac{\partial (w_k \circ \tilde{h})}{\partial \bar{z}_j}\big|_p \frac{\partial}{\partial w_k}\big|_{h(p)} + \frac{\partial (\bar{w}_k \circ \tilde{h})}{\partial \bar{z}_j}\big|_p \frac{\partial}{\partial \bar{w}_k}\big|_{h(p)} \right).
\end{aligned}$$

An analogous computation yields $\tilde{h}_* \frac{\partial}{\partial z_j}\big|_p = \sum_{k=1}^{N'} \left(\frac{\partial (w_k \circ \tilde{h})}{\partial z_j}\big|_p \frac{\partial}{\partial w_k}\big|_{h(p)} + \frac{\partial (\bar{w}_k \circ \tilde{h})}{\partial z_j}\big|_p \frac{\partial}{\partial \bar{w}_k}\big|_{h(p)} \right)$. By linearity, the formula $\tilde{h}_* X_p = \sum_{j=1}^{N'} \left(X_p(w_j \circ \tilde{h}) \frac{\partial}{\partial w_j}|_{h(p)} + X_p(\bar{w}_j \circ \tilde{h}) \frac{\partial}{\partial \bar{w}_j}|_{h(p)} \right)$ extends to general vectors $X_p \in \mathbb{C}T_p\mathbb{C}^N$. Since h and $\tilde{h}$ agree on M, $h_*\bar{L}|_p = \tilde{h}_*\bar{L}|_p = \sum_{j=1}^{N} \bar{L}_j \tilde{h}_* \frac{\partial}{\partial \bar{z}_j}|_p$, which yields

$$\begin{aligned}
h_*\bar{L}|_p &= \sum_{j=1}^{N} \bar{L}_j \sum_{k=1}^{N'} \left(\frac{\partial (w_k \circ \tilde{h})}{\partial \bar{z}_j}\big|_p \frac{\partial}{\partial w_k}\big|_{h(p)} + \frac{\partial (\bar{w}_k \circ \tilde{h})}{\partial \bar{z}_j}\big|_p \frac{\partial}{\partial \bar{w}_k}\big|_{h(p)} \right) \\
&= \sum_{k=1}^{N'} \left(\bar{L}|_p(w_k \circ \tilde{h}) \frac{\partial}{\partial w_k}\big|_{h(p)} + \bar{L}|_p(\bar{w}_k \circ \tilde{h}) \frac{\partial}{\partial \bar{w}_k}\big|_{h(p)} \right).
\end{aligned}$$

Therefore, $h_*\bar{L}|_p$ is a CR vector if and only if for each component w_j the derivative $\bar{L}|_p(w_j \circ \tilde{h}) = \bar{L}|_p(w_j \circ h)$ vanishes. A continuously differentiable map h between CR submanifolds is thus a CR map if and only if each of its components is a CR function. In particular, a CR function f on M is also a CR map $f : M \to \mathbb{C}$.

The composition of two CR maps $h : M \to M'$ and $g : M' \to M''$ is again a CR map, since $h_* T_p^{0,1}M \subseteq T_{h(p)}^{0,1}M'$ and $g_* T_{p'}^{0,1}M' \subseteq T_{g(p')}^{0,1}M''$ together imply $(g \circ h)_* T_p^{0,1}M = g_* h_* T_p^{0,1}M \subseteq T_{g(h(p))}^{0,1}M''$.

1.2 The Levi form

Heuristically, the curvature of a hypersurface $M \in \mathbb{R}^n$ is described by the way a normal vector n_p at p to the hypersurface has to "twist" around as we move along M. In Riemannian geometry, this thought leads to the shape tensor of a submanifold. In the case of a real hypersurface M in $\mathbb{C}^n$, the complex orthogonal complement to n_p is exactly the complex tangent space T^c_pM, suggesting that the curvature of M relates to the way the distribution T^c_pM "twists" around in TM. This way we will obtain an intrinsic invariant of M strongly influencing how M may be embedded into $\mathbb{C}^n$.

The extent of the non-integrability of a given distribution is encoded in the Lie brackets of vector fields taking values in the distribution. Indeed, if $E \subseteq TM$ is a distribution on M, the *Frobenius Theorem* tells us that there exists a foliation η of M such that $E_p = T_p\eta_p$ at each point p if and only if for any two vector fields $X, Y \in \Gamma(E)$, we have $[X, Y] \in \Gamma(E)$ (cf. [9]). Note that although the Lie bracket's value at a point p depends on the behavior of X and Y on a neighborhood of p, the relevant components only depend on the values of X and Y at p. To also exploit the complex structure on T^cM, we will work with the following object:

Definition 1. *The* Levi-Form $\mathcal{L} : \Gamma(T^{0,1}M) \times \Gamma(T^{0,1}M) \to \Gamma(\mathbb{C}TM/\mathbb{C}T^cM)$ *defined by* $\mathcal{L}(\bar{L}_1, \bar{L}_2) = \frac{1}{2i}[L_1, \bar{L}_2] + \mathbb{C}T^cM$ *is a Hermitian form on* $T^{0,1}M$.

By $\mathbb{C}T^cM$ we denote the subbundle of vectors in $\mathbb{C}TM$ with both real and imaginary part in T^cM. Note that $\mathbb{C}T^cM$ is endowed with two different complex structures induced by i and J, respectively, and that $T^{1,0}M \oplus T^{0,1}M = \mathbb{C}T^cM$. A Hermitian form is a tensor that is C^1-*antilinear* in the first slot and C^1-linear in the second slot, and exhibits the Hermitian symmetry $\mathcal{L}(\bar{L}_1, \bar{L}_2) = \mathcal{L}(\bar{L}_2, \bar{L}_1)$. Both are easy to check for the Levi form:

$$\begin{aligned} 2i\mathcal{L}(f\bar{L}_1, g\bar{L}_2) &= [\bar{f}L_1, g\bar{L}_2] + \mathbb{C}T^cM = g[\bar{f}L_1, \bar{L}_2] + \bar{f}(L_1 g)\bar{L}_2 + \mathbb{C}T^cM = \\ &= g\bar{f}[L_1, \bar{L}_2] - g(\bar{L}_2\bar{f})L_1 + \mathbb{C}T^cM = 2i\bar{f}g\mathcal{L}(f\bar{L}_1, g\bar{L}_2), \\ \mathcal{L}(\bar{L}_2, \bar{L}_1) &= \tfrac{1}{2i}[L_2, \bar{L}_1] + \mathbb{C}T^cM = -\tfrac{1}{2i}[\bar{L}_1, L_2] + \mathbb{C}T^cM \\ &= \overline{\tfrac{1}{2i}[L_1, \bar{L}_2] + \mathbb{C}T^cM} = \overline{\mathcal{L}(\bar{L}_1, \bar{L}_2)}. \end{aligned}$$

Given a form $\theta \in \Gamma(T^0M)$, which by definition annihilates $\mathbb{C}T^cM$ and thus descends onto $\mathbb{C}TM/\mathbb{C}T^cM$, we may define the respective *scalar Levi form* $\mathcal{L}_\theta : \Gamma(T^{0,1}M) \times$

$\Gamma(T^{0,1}M) \to C^\infty(M)$ by $\mathcal{L}_\theta(\bar{L}_1, \bar{L}_2) := \frac{1}{2i}\theta([L_1, \bar{L}_2])$. The advantage here is that a scalar Levi form is a proper Hermitian form in the usual sense. By the spectral theorem for Hermitian forms, $\mathcal{L}_\theta|_p$ has $\dim_{CR} M$ real eigenvalues and an orthonormal basis of corresponding eigenvectors at each point p.

In the case of a hypersurface M, all scalar Levi forms differ only by multiplication with real-valued scalar functions, since all characteristic forms do. Thus the triple (n_+, n_0, n_-) given by the numbers n_+, n_0 and n_- of positive, zero and negative eigenvalues of the scalar Levi forms at p is independent of the choice of characteristic form, as long as we choose the sign of the characteristic form such that $n_+ \geq n_-$. This triple is called the *signature* of the Levi form at p. If $n_- = 0$, M is called *pseudoconvex* at p, and if $n_- = n_0 = 0$, M is called *strongly pseudoconvex*.

Since $\theta(L_1) = \theta(\bar{L}_2) = 0$, and $d\theta(U,V) = V\theta(U) - U\theta(V) + \theta([U,V])$, we can rewrite $\mathcal{L}_\theta(\bar{L}_1, \bar{L}_2) = \frac{1}{2i}d\theta(L_1, \bar{L}_2)$. Let us now calculate the scalar Levi form of a hypersurface corresponding to the characteristic form constructed in section 1.1.

Lemma 1. *The scalar Levi form* $\mathcal{L}_\Theta$ *corresponding to* $\Theta = i\sum_{j=1}^n \left(\frac{\partial\rho}{\partial\bar{z}_j}d\bar{z}_j - \frac{\partial\rho}{\partial z_j}dz_j\right)$ *is given by*

$$\mathcal{L}_\Theta(\bar{L}_1, \bar{L}_2) = \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_j \partial\bar{z}_k} L_1^j \bar{L}_2^k,$$

where $\bar{L}_1 = \sum_{j=1}^n \bar{L}_1^j \frac{\partial}{\partial\bar{z}_j}$ *and* $\bar{L}_2 = \sum_{k=1}^n \bar{L}_2^k \frac{\partial}{\partial\bar{z}_j}$ *are CR vectors tangential to* M.

Proof. We compute the exterior derivative of $\Theta = i\sum_{j=1}^n \left(\frac{\partial\rho}{\partial\bar{z}_j}d\bar{z}_j - \frac{\partial\rho}{\partial z_j}dz_j\right)$.

$$d\Theta = i\sum_{j,k=1}^n \left(\frac{\partial^2\rho}{\partial z_k\partial\bar{z}_j}dz_k \wedge d\bar{z}_j - \frac{\partial^2\rho}{\partial\bar{z}_k\partial z_j}d\bar{z}_k \wedge dz_j\right) = 2i\sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_k\partial\bar{z}_j}dz_k \wedge d\bar{z}_j.$$

Since $(dz_k \wedge d\bar{z}_j)(\frac{\partial}{\partial z_\kappa}, \frac{\partial}{\partial\bar{z}_\iota}) = \delta_{k\kappa}\delta_{i\iota}$, we obtain $\frac{1}{2i}d\Theta(L_1, \bar{L}_2) = \sum_{j,k=1}^n \frac{\partial^2\rho}{\partial z_k\partial\bar{z}_j} L_1^j \bar{L}_2^k$. □

Although the Levi form's coordinate expression contains second derivatives of ρ, we obtain the following useful fact on complex curves merely tangential to M.

Lemma 2. *Let* $\gamma(t) = p + tA + t^2B + \mathcal{O}(|t|^3)$ *be a complex curve tangential to* M *at* p*. Then* $\frac{d^2}{dtd\bar{t}}|_{t=0}(\rho \circ \gamma) = \mathcal{L}_\Theta(\frac{1}{2}(A + iJA), \frac{1}{2}(A + iJA))$ *for a defining function* ρ *and its corresponding characteristic form* Θ.

Proof. A straightforward chain rule calculation reveals that

$$\begin{aligned}\frac{d^2}{dtd\bar{t}}\big|_{t=0}(\rho\circ\gamma) &= \frac{d}{dt}\big|_{t=0}\sum_{j=1}^{n}\left(\left(\frac{\partial\rho}{\partial z_j}\circ\gamma\right)\frac{d\gamma_j}{d\bar{t}}+\left(\frac{\partial\rho}{\partial\bar{z}_j}\circ\gamma\right)\frac{d\bar{\gamma}_j}{d\bar{t}}\right)\\ &= \frac{d}{dt}\big|_{t=0}\sum_{j=1}^{n}\left(\left(\frac{\partial\rho}{\partial\bar{z}_j}\circ\gamma\right)\left(\bar{A}_j+2\bar{t}\bar{B}_j+\mathcal{O}(|t|^2)\right)\right)=\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_k\partial\bar{z}_j}\big|_p A_k\bar{A}_j,\end{aligned}$$

proving the claim, since $\frac{1}{2}(A+iJA)=\sum_{j=1}^{n}\bar{A}_j\frac{\partial}{\partial\bar{z}_j}|_p$. □

The notion of pseudoconvexity is strongly tied to the usual notion of convexity, but due to its invariance under biholomorphic maps, this connection is somewhat intricate. Let us illustrate this with a few examples.

Example 4. *Let Ω be a (strongly) convex open subset of $\mathbb{C}^N$ such that $M := \partial\Omega$ is a smooth real hypersurface. Then M is a (strongly) pseudoconvex hypersurface. Furthermore, consider a holomorphic immersion $F:\mathbb{C}^K\to\mathbb{C}^N$ mapping $q\in\mathbb{C}^K$ to $p\in M$ such that $F_*T_q\mathbb{C}^K+T_pM=T_p\mathbb{C}^M$. Then $F^{-1}(M)\subset\mathbb{C}^K$ is also a (strongly) pseudoconvex hypersurface near q.*

Proof. Convexity of Ω means that near $p\in M$, we may choose complex coordinates $(z_1,\ldots,z_{n-1},z_n)=(x_1+iy_1,\ldots,x_n+iy_n)$ such that M is the graph of a (strongly) convex real function $f:\mathbb{C}^{n-1}\times\mathbb{R}\to\mathbb{R}$ with vanishing gradient at $(p_1,\ldots,p_{n-1},\Re(p_n))$. The function $\rho(z_1,\ldots,z_n)=f(z_1,\ldots,z_{n-1},x_n)-y_n$ then provides a defining equation for M. For any nonzero CR vector $\frac{1}{2}(V+iJV)\in T_p^{0,1}M$, the complex line $\gamma(t)=p+tV$ is tangential to M at p, and $V_n=0$. Since γ is a straight line, $f\circ(\gamma_1,\ldots,\gamma_{n-1},\Re(p_n))$ is still (strongly) convex. By Lemma 2, $\mathcal{L}_\Theta(\frac{1}{2}(V+iJV),\frac{1}{2}(V+iJV))=\frac{d^2}{dtd\bar{t}}|_{t=0}\rho\circ\gamma$, and because $\rho\circ\gamma=f(p_1+tV_1,\ldots,p_{n-1}+tV_{n-1},\Re(p_n))$ is (strongly) convex, we obtain $\mathcal{L}_\Theta(\frac{1}{2}(V+iJV),\frac{1}{2}(V+iJV))\geq 0$, holding strictly in the strongly convex case.

For the second claim, note that $\rho\circ F$ provides a defining equation for $S:=F^{-1}(M)$ by the transversality condition $F_*T_q\mathbb{C}^K+T_pM=T_p\mathbb{C}^M$. Choosing a nonzero CR vector $\bar{L}|_q=\frac{1}{2}(V+iJV)\in T_q^{0,1}S$, we again consider $\gamma(t)=q+tV$. By immersivity of F, $F\circ\gamma$ is a complex curve in $\mathbb{C}^N$ with nonzero CR tangent vector $h_*\bar{L}|_p$, hence $\frac{d^2}{dtd\bar{t}}|_{t=0}(\rho\circ F)\circ\gamma=\frac{d^2}{dtd\bar{t}}|_{t=0}\rho\circ(F\circ\gamma)=\mathcal{L}_\Theta(h_*\bar{L}|_p,h_*\bar{L}|_p)\geq 0$, which is again strict in the strongly convex case. It is worth noting that immersivity of F was only needed for strong pseudoconvexity. □

Strong pseudoconvexity is a condition on tangential complex curves, and thus only sensitive to structure that harmonizes with the complex structure on $\mathbb{C}^N$. As examples, consider the two hypersurfaces in $\mathbb{C}^2$ given by $|z_1|^2 = 0$ and $\Re(z_1)^2 + \Re(z_2)^2 = 0$, respectively. Even though they are isometric as real hypersurfaces, the former is not strongly pseudoconvex, but the latter is by the following fact.

Example 5. *Let D be a strongly convex open subset of $\mathbb{R}^N$ such that ∂D is a smooth real hypersurface. Then $M = \{z \in \mathbb{C}^N : \Re(z) \in \partial D\}$, called the* tube over *∂D, is a strongly pseudoconvex hypersurface.*

Proof. Near $p \in \mathbb{C}^N$, we may choose coordinates $z_j = x_j + iy_j$, $j = 1, \ldots, N$ such that $M = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : f(x_1, \ldots, x_{N-1}) - x_N = 0\}$ for a strongly convex function $f : \mathbb{R}^{N-1} \to \mathbb{R}^N$ with vanishing gradient at $(\Re(p_1), \ldots, \Re(p_{N-1}))$. Then V lies in $T_p^c M$ if and only if $V_N = 0$. Along the *real* curve $r \mapsto p + rV$, ρ is strongly convex unless $\Re(V_1) = \cdots = \Re(V_{N-1}) = 0$. Similarly, along the curve $s \mapsto p + sJV$, ρ is strongly convex unless $\Im(V_1) = \cdots = \Im(V_{N-1}) = 0$. If we consider the complex curve $\gamma(r + is) = p + rV + sJV$, this implies that

$$\frac{\partial^2}{\partial t \partial \bar{t}}\Big|_{t=0} \rho \circ \gamma = \frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}\right)\Big|_{t=0} \rho \circ \gamma > 0,$$

unless $V = 0$, proving that M is strongly pseudoconvex. □

2 Irregular CR maps and formal holomorphic foliations

Beginning with the classical Schwarz reflection principle, a central question in complex analysis is whether a holomorphic map between two given domains Ω and Ω', which extends to some degree of regularity to $\partial\Omega$ and maps it into $\partial\Omega'$, in fact extends analytically beyond $\partial\Omega$. That extensions of holomorphic maps to smooth domain boundaries give rise to CR maps suggests a natural boundary version of this question: Under which conditions is a finitely differentiable CR map $h : M \to M'$ between given C^∞-smooth, real analytic or algebraic CR submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ already C^∞-smooth, real analytic or algebraic, respectively?

If the source manifold M is a real hypersurface with two nonzero Levi eigenvalues of different sign, any continuous CR function extends analytically to either side of M

by Lewy's extension theorem (see [12, Chapter 7]), automatically forcing it to be C^∞-smooth on M. Since the components of a CR map are CR functions, the same holds for CR maps. On the other hand, strong pseudoconvexity of the *target* is good for regularity as well. For example, any $C^{N'-N+1}$-smooth CR map from $\mathbb{S}^{2N-1}$ to $\mathbb{S}^{2N'-1}$ extends to a rational map due to a result of Forstnerič [5]. In a stark contrast, in the case of a source that is strongly pseudoconvex and a target that is not there can be arbitrarily irregular CR maps, as has been shown by Berhanu & Xiao in [2]. We begin by considering CR functions, following [2].

Example 6. *Let $M \subset \mathbb{C}^N$ be a strongly pseudoconvex CR hypersurface and $p \in M$. Then there exists a neighborhood $O \subseteq \mathbb{C}^N$ of p such that for each $k \in \mathbb{N}_{\geq 1}$ there is a C^k-smooth CR function $\phi : O \cap M \to \mathbb{C}$ which is nowhere smooth on $O \cap M$.*

Proof. If O is chosen small enough, there exist holomorphic coordinates $z_1, \ldots, z_N$ defined on O such that M is a strongly convex real hypersurface (see [11, Page 61]). Let $U := O \cap M$ and fix a point $q \in U$. After a linear coordinate change, we may assume $q = (0, \ldots, 0) \in \mathbb{C}^N$ and $T_qM = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : \Im(z_N) = 0\}$. By strong convexity of M, $\Im(z_N) > 0$ on $U \setminus \{q\}$. For any $m \in \mathbb{N}$, we can thus define the function $\phi_q^m(z) = z_N^{m+\frac{1}{2}}$ on U by taking the branch cut where $\Re(z_N) = 0$ and $\Im(z_N) < 0$, since this set is disjoint of U. Outside of this set, $z_N^{m+\frac{1}{2}}$ is m times continuously differentiable, thus in particular $\phi_q^m \in C^m(U)$. Since $z_N^{m+\frac{1}{2}}$ is holomorphic on an open neighborhood of $U \setminus \{q\}$, ϕ_q^m is a CR function on U, and C^∞-smooth on $U \setminus \{q\}$. Along a smooth real curve $\gamma : (-\varepsilon, \varepsilon) \to M$ tangential to $(0, \ldots, 0, 1) \in T_qM$, the function ϕ_q^m behaves like $t \mapsto t^{m+\frac{1}{2}}$, showing that it is not $m+1$ times differentiable.

To construct a nowhere smooth function, take a sequence of different points $(q_j)_{j=0}^\infty$ which lie dense in U and consider the functions $\phi_j := \phi_{q_j}^{k+j}$ defined as before. Then take a sufficiently rapidly decreasing sequence $(\lambda_j)_{j=0}^\infty$ of positive reals such the sum $\sum_{j=m+1}^\infty \lambda_j \phi_j$ converges in the Banach space $C^{k+m+1}(U)$ for every $m \in \mathbb{N}$. Consequently, $\sum_{j=m+1}^\infty \lambda_j \phi_j$ is differentiable $k+m+1$ times at q_m, the finite sum $\sum_{j=0}^{m-1} \lambda_j \phi_j$ is as well, but ϕ_m is not. Therefore, $\phi := \sum_{j=0}^\infty \lambda_j \phi_j$ is C^k-smooth, but not $k+m+1$ times differentiable at any of the points q_m, hence ϕ is nowhere smooth. □

As an immediate conseqence, there exist nowhere smooth CR maps from M into M' if the target manifold M' contains a complex curve Γ. Indeed, any parametrization

$t \mapsto \gamma(t)$ of Γ is a smooth CR immersion of $\mathbb{C}$ into M', hence $\gamma \circ \phi : M \to M'$ provides a nowhere smooth CR function of regularity C^k. We obtain another, more general set of examples from targets of the form $M' = \hat{M} \times \mathbb{C} \subset \mathbb{C}^{N+1}$ and CR functions $\hat{h} : M \to \hat{M}$. Here, the map $(\hat{h}, \phi) : M \to \hat{M} \times \mathbb{C}$ is a CR map, since each of its components is a CR map, and it is nowhere smooth because ϕ is. In [8], Lamel and Mir prove a result in the other direction, essentially stating that near a generic point, any nowhere smooth CR map formally exhibits the structure of these latter examples.

2.1 The formal foliation theorem

Before stating the theorem, some concepts need to be introduced. A *formal holomorphic submanifold* Γ of dimension r at a point $p \in \mathbb{C}^{N'}$ is simply a formal power series $\Gamma \in \mathbb{C}[\![t_1, \ldots, t_r]\!]^{N'}$, $\Gamma = \sum_{\alpha \in \mathbb{N}^r} \gamma_\alpha t^\alpha$ satisfying $\gamma_0 = p$ and $\operatorname{rk}(\gamma_{t_1}, \ldots, \gamma_{t_r}) = r$. It is *tangential to infinite order* to a set $S \subseteq \mathbb{C}^{N'}$ if for any germ of a C^∞-smooth function ρ vanishing on S, the composition of Γ with the Taylor series of ρ at p vanishes to infinite order. If M is a CR manifold and $(\Gamma_q)_{q \in M}$ is a family of such formal holomorphic submanifolds, we call this family a *CR family* if each of its coefficients is a CR map $M \to \mathbb{C}^{N'}$.

In a similar vein, we might consider the maximal order of tangency of holomorphic curves to a given set $S \subseteq \mathbb{C}^{N'}$ at a point $p \in S$, since this is an obvious biholomorphic invariant of the pair (S, p). This leads to the *D'Angelo type* of S at p, given by

$$\Delta(S,p) = \sup_{\substack{\gamma : \mathbb{D}^1 \to \mathbb{C}^{N'}, \\ \gamma(0) = p, \gamma \not\equiv p}} \left(\inf_{\rho \in \mathscr{I}_S(p)} \frac{v_0(\rho \circ \gamma)}{v_0(\gamma)} \right).$$

Here, as in the rest of this section, we denote the ideal of germs of smooth functions at p which vanish on a given set S by $\mathscr{I}_S(p)$, and by $v_0(\rho \circ \gamma)$ the vanishing order of the composition of γ with the Taylor series of ρ at p. The set of points $p \in S$ such that $\Delta(S,p) = \infty$ is said to be of *D'Angelo infinite type*, and is denoted by $\mathcal{E}_S$.

It turns out that the structural property of the target which forces smoothness of CR maps is the number of different directions into which successive CR derivatives of gradients of defining functions can point. This motivates the introduction of the following numerical invariants. For a CR map $h : M \to \mathbb{C}^{N'}$, let

$$r_0(p) := \dim_{\mathbb{C}} \left\langle \left\{ \rho_w \circ h(p) : \rho \in \mathscr{I}_{h(M)}(h(p)) \right\} \right\rangle,$$
$$r_k(p) := \dim_{\mathbb{C}} \left\langle \left\{ \bar{L}_1 \dots \bar{L}_j (\rho_w \circ h)(p) : \rho \in \mathscr{I}_{h(M)}(h(p)), \bar{L}_1, \dots, \bar{L}_j \in \mathcal{V}_p(M), 0 \le j \le k \right\} \right\rangle,$$

where $\mathcal{V}_p(M)$ denotes the set of germs of CR vector fields at p. The *complex gradient* $\rho_w = \left(\frac{\partial \rho}{\partial w_1}, \dots, \frac{\partial \rho}{\partial w_{N'}} \right)$ is considered here as a vector in $\mathbb{C}^{N'}$. The function $q \mapsto r_k(q)$ is integer valued and lower semicontinuous as it is given by the rank of a collection of continuously varying vectors. Of course, $r_k(p)$ is only defined if $h \in C^k$, since $\rho_w \circ h$ is only as regular as h is. To extract a global invariant of h, let r_k be the maximum value such that $r_k(p) \ge r_k$ on a dense open subset of M. We are now in a position to state the formal foliation theorem of Lamel and Mir (Theorem 2.2 in [8]).

Theorem 3. *Let $M \subset \mathbb{C}^N$ be a C^∞-smooth minimal CR submanifold, $k, l \in \mathbb{N}$ with $0 \le k \le l \le N'$ and $N' - l + k \ge 1$ be given integers and $h : M \to \mathbb{C}^{N'}$ be a CR map of class $C^{N'-l+k}$. Assume that $r_k \ge l$ and that there exists a non-empty open subset M_1 of M where h is nowhere C^∞. Then there exists a dense open subset $M_2 \subseteq M_1$ such that for every $p \in M_2$, there exists a neighborhood $V \subseteq M_2$ of p, an integer $r \ge 1$ and a C^1-smooth CR family of formal complex submanifolds $(\Gamma_\xi)_{\xi \in V}$ of dimension r through $h(V)$ for which Γ_ξ is tangential to infinite order to $h(M)$ at $h(\xi)$, for every $\xi \in V$. In particular, there exists a dense open subset M_2 of M_1 with $h(M_2) \subseteq \mathcal{E}_{h(M)}$.*

Evidently, the rank r of the family of holomorphic manifolds in the statement of this theorem merely serves as a reminder that in concrete cases, one can hope for a rank of more than one. Since there is no condition given when this might occur, for black-box applications of this theorem we will have to be satisfied with CR families of holomorphic curves with nonvanishing derivative, which can always be obtained by simply restricting $\Gamma_q = \sum_{\alpha \in \mathbb{N}^r} \gamma_\alpha(q) t^\alpha$ to $t = (t_1, 0, \dots, 0)$.

Let us remark that if h is not C^∞-smooth on a dense open subset of M, there exists an open subset $O \subseteq M$ such that h is nowhere C^∞-smooth on O. The reason is simply that the set of all points $p \in M$ such that h is C^∞-smooth on a neighborhood of p is open. If this set is not dense, then the complement of its closure is a non-empty open subset of M, where, by definition, h is nowhere C^∞-smooth.

Another interesting point to note is that while the formal complex manifolds obtained from Theorem 3 are tangential to infinite order to the image $h(M)$, infinite tangency to a non-smooth set is not nearly as strong as one might think at first sight. As a toy example, take a nowhere smooth, but C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ and consider its graph $S := \{(x, \phi(x)), x \in \mathbb{R}\} \subset \mathbb{R}^2$. Then any function $\rho \in C^\infty(\mathbb{R}^2)$ must vanish to infinite order at S by the following argument: If either ρ_x or ρ_y did not vanish at a point $(x, \phi(x))$, the implicit function theorem would yield a smooth parametrization of S near that point, which does not exist. Thus both ρ_x and ρ_y vanish on S, and the argument may proceed at infinitum. The y-Axis is therefore tangential to infinite order to S in the sense of Theorem 3, while not even being tangential to first order in the usual sense. However, if $h(M) \subseteq M'$ for some smooth manifold M', then tangency to infinite order to $h(M)$ clearly implies tangency to infinite order to M'.

To apply Theorem 3, we need $0 \leq k \leq l \leq N'$ such that $r_k \geq l$. It is always possible to choose $k = l = 0$, but if h maps M into a CR submanifold $M' \subseteq \mathbb{C}^{N'}$, a slight improvement holds (Lemma 6.1 in [8]).

Lemma 3. *Let $M \subset \mathbb{C}^N$ be a C^∞-smooth CR submanifold and $h : M \to \mathbb{C}^{N'}$ be a continuous CR map. If there exists a C^∞-smooth CR submanifold $M' \subset \mathbb{C}^{N'}$ such that $h(M) \subseteq M'$, then $r_0 \geq N' - n'$, where $n' = \dim_{CR} M'$. In particular, if M' is maximally real, then $r_0 = N'$.*

If it is guaranteed that enough CR directions tangential to $h(M)$ exist along which M' behaves like a Levi nondegenerate manifold, we can say more about the first derivatives of gradients, yielding a bound on r_1. The resulting lemma is a slight adaptation of Lemma 6.2. in [8].

Lemma 4. *Consider a C^∞-smooth CR submanifold $M \subset \mathbb{C}^N$, a C^∞-smooth real hypersurface $M' \subset \mathbb{C}^{N'}$ and a continuously differentiable CR map $h : M \to M'$ mapping $p \in M$ to $p' \in M'$. If h is immersive at p and a scalar Levi form $\mathcal{L}_\Theta$ of M' restricts to a nondegenerate Hermitian form on $h_* T_p^{0,1} M$, then $r_1 \geq \dim_{CR} M + 1$ on a neighborhood of p.*

Proof. Since we are in a purely local setting, we may assume that $\mathcal{L}_\Theta$ arises from a defining function ρ of M' in the way described in Lemma 1, such that for any two CR vectors $\bar\Gamma = \sum_{j=1}^{N'} \bar\Gamma_j \frac{\partial}{\partial \bar w_j}|_{p'}$ and $\bar L = \sum_{k=1}^{N'} \bar L_k \frac{\partial}{\partial \bar w_k}|_{p'}$ we have

$$\mathcal{L}_\Theta(\bar\Gamma, \bar L) = \sum_{j,k=1}^{N'} \frac{\partial^2 \rho}{\partial w_j \partial \bar w_k}(p')\Gamma_j \bar L_k.$$

By definition $\bar L \rho_w = \sum_{j=1}^{N'} \bar L_k \frac{\partial^2\rho}{\partial w_j \partial \bar w_k}(p')$, so using the standard scalar product on $\mathbb{C}^{N'}$ we can express $\mathcal{L}_\Theta(\bar\Gamma, \bar L) = \big((\bar\Gamma_1, \ldots, \bar\Gamma_{N'})|\bar L\rho_w\big)_{\mathbb{C}^{N'}}$. Nondegeneracy of the restricted Levi form on $h_*T_p^{0,1}M$ precisely means that the map $h_*\bar L \mapsto \mathcal{L}_\Theta(\cdot, h_*\bar L)$ is an isomorphism of $h_*T_p^{0,1}M$ and the space of antilinear functionals on $h_*T_p^{0,1}M$. Since h is immersive, h_* is an isomorphism between $T_p^{0,1}M$ and $h_*T_p^{0,1}M$. The map associating to each $\bar L \in T_p^{0,1}M$ the antilinear functional $\mathcal{L}_\Theta(\cdot, h_*\bar L) = \big(\cdot\,|\bar L(\rho_w \circ h)\big)_{\mathbb{C}^{N'}}$ is thus an isomorphism, in particular implying that $\dim_{\mathbb{C}}\{\bar L(\rho_w \circ h) : \bar L \in T_p^{0,1}M\} = \dim_{CR} M$. Furthermore, the complex gradient $\rho_w(p')$ itself is linearly independent of $\bar L(\rho_w \circ h)$ for any nonzero $\bar L \in T_{p'}^{0,1}M'$ by the following argument. For any $\bar\Gamma = \sum_{j=1}^{N'} \bar\Gamma_j \frac{\partial}{\partial \bar w_j}|_{p'} \in T_{p'}^{0,1}M'$, tangency implies that

$$\Gamma\rho = \sum_{j=1}^{N'} \Gamma_j \frac{\partial \rho}{\partial w_j}(p') = \big((\bar\Gamma_1, \ldots, \bar\Gamma_{N'})|\rho_w(p')\big)_{\mathbb{C}^{N'}} = 0.$$

Thus $\rho_w(p')$ lies in the orthogonal complement of $\big\{(\bar\Gamma_1, \ldots, \bar\Gamma_{N'}) : \bar\Gamma \in T_{p'}^{0,1}M'\big\}$ while $\bar L(\rho_w \circ h)$ does not, showing linear independence. This implies $r_1(p) \geq \dim_{CR} M + 1$ and since r_1 is lower semicontinuous and integer valued, $r_1 \geq \dim_{CR} M + 1$ holds on a neighborhood of p as claimed. $\square$

3 Regularity of maps into pseudoconvex, Levi-degenerate hypersurfaces

As an example of a hypersurface foliated by complex manifolds, where an unconditional regularity result must necessarily fail, Lamel and Mir consider the *tube over the light cone* $M' := \{(z_1, \ldots, z_{N'-1}, z_{N'}) : \Re(z_1)^2 + \cdots + \Re(z_{N'-1})^2 = \Re(z_{N'})^2,\ z_{N'} \neq 0\}$. They obtain the following result (Corollary 2.6 in [8]).

Theorem 4. *Let $M \subset \mathbb{C}^N$ be a C^∞-smooth minimal CR submanifold and $M' \subseteq \mathbb{C}^{N'}$ be the tube over the light cone. Then every CR map $h : M \to M'$, of class $C^{N'-1}$ and of rank ≥ 3, is C^∞-smooth on a dense open subset of M.*

The proof given in [8] and [7] makes quite ingenious use of the simple structure of M'. However, this type of theorem extends readily to the more general situation of pseudoconvex hypersurfaces which are foliated by complex manifolds of lower dimension. This class of examples covers not only the tube over the light cone, but also the smooth part of the boundary of all classical irreducible symmetric domains. Mappings into such targets will be discussed in section 3.2.

3.1 Maps into pseudoconvex, holomorphically foliated hypersurfaces

Let us introduce the setting more precisely. Let $M' \subseteq \mathbb{C}^{N'}$ be a real hypersurface which is foliated by K-dimensional complex manifolds. For $q \in M'$, let η_q be the leaf of the foliation containing q. We denote by $T\eta := \bigcup_{q\in M'} T_q\eta_q$ the bundle of tangent spaces to the leaves of the foliation, and by $T^{0,1}\eta := \bigcup_{q\in M'} T_q^{0,1}\eta_q$ the bundle of CR tangent spaces of leaves.

If M' is pseudoconvex, the existence of the foliation implies that the Levi form has at least K zero eigenvalues at any point. As usual, the set of points where this minimum number of zero eigenvalues is attained is open.

Lemma 5. *Let $M' \subset \mathbb{C}^{N'}$ be a* pseudoconvex *hypersurface foliated by K-dimensional complex manifolds. Then the Levi form of M' has at least K zero eigenvalues, and if it has exactly K zero eigenvalues at a point $p' \in M'$, then this holds also on an open neighborhood of p'. In the latter case, the null space of the Levi form is given by $T^{0,1}\eta$.*

Proof. Since M' is pseudoconvex, there exists a characteristic form $\Theta \in \Gamma(T^oM')$ such that the respective scalar Levi form $\mathcal{L}(\bar L_1, \bar L_2) = \frac{1}{2i}\Theta([L_1, \bar L_2])$ is positive semidefinite. For a vector $\bar L_{p'} \in T^{0,1}_{p'}\eta_{p'}$, choose a CR vector field $\bar L$ with $\bar L|_{p'} = \bar L_{p'}$ that is tangential to $\eta_{p'}$. Then, $[L, \bar L]|_{p'} \in \mathbb{C}T_{p'}\eta_{p'}$, hence $\mathcal{L}(\bar L_{p'}, \bar L_{p'}) = 0$. Positive semidefiniteness implies that $\bar L_{p'}$ lies in the null space of $\mathcal{L}$. Therefore, the K-dimensional space $T^{0,1}_{p'}\eta_{p'}$ is contained in the null space of $\mathcal{L}$, and $\mathcal{L}$ has at least K zero eigenvalues. Writing out the Levi form with respect to a basis of CR vector fields, the assumption that it has only K zero eigenvalues at p' is equivalent to the existence of an $(N'-K-1) \times (N'-K-1)$-minor of $\mathcal{L}|_{p'}$ with nonzero determinant. Since the determinant of this minor is a smooth function on M', it then does not vanish on an open neighbourhood of p', implying that $\mathcal{L}$ has exactly K zero eigenvalues around p'. $\square$

Our main technical tool will be a tensorial quantity measuring obstructions to the existence of CR sections of $T\eta$. We denote by $T^{\perp}\eta := \bigcup_{p\in M'}(T_p\eta_p)^{\perp}$ the bundle of orthogonal complements in $T\mathbb{C}^{N'}$ of tangent spaces to leaves.

Lemma 6. *There exists a tensor field $R \in \mathcal{V}(M')^{*}\otimes\Gamma(T\eta)^{*}\otimes\Gamma(T^{\perp}\eta)$ such that for every $\bar{L} \in \mathcal{V}(M')$ and $\psi \in \Gamma(T\eta)$, we have $\mathbb{P}_{T_p^{\perp}\eta_p}(\bar{L}|_p\psi) = R_p(\bar{L}|_p, \psi|_p)$. For any $V_p \in T_p\eta_p$, the kernel of $R_p(\cdot, V_p)$ contains $T_p^{0,1}\eta_p$.*

Proof. Define $R(\bar{L}, \psi) = \mathbb{P}_{T^{\perp}\eta}(\bar{L}\psi)$. Evidently, R is C^1-linear in the first slot, since directional derivatives always are. For two sections V and W of $T\eta$ and $f \in C^1(M)$, we have that $\bar{L}|_p(V + fW) = \bar{L}|_pV + f\bar{L}|_pW + (\bar{L}|_pf)W$. The last term is canceled by the projection onto $T_p^{\perp}\eta_p$, thus R is also C^1-linear in the second slot, implying that R is a tensor.

Consider now $V_p \in T_p\eta_p$. We may construct a section $V \in \Gamma(T\eta)$ satisfying $V|_p = V_p$, which is holomorphic on η_p and smooth on M'. First, we choose a holomorphic parametrization ϕ for η_p, extend $\phi_*^{-1}V_p$ to a constant vector field $\tilde{V}$ and note that $\phi_*\tilde{V}$ is holomorphic, since $D\phi$ has holomorphic components and $\tilde{V}$ is constant. To obtain a vector field, we then simply extend the result smoothly to M'. But now, $\bar{L}|_pV = 0$ if $\bar{L} \in \Gamma(T^{0,1}\eta)$, since V is holomorphic on η_p and $\bar{L}|_p$ only takes derivatives along η. Therefore, $T_p^{0,1}\eta \subseteq \ker R_p(\cdot, V_p)$. □

Evidently, a CR section ψ of $T\eta$ has to satisfy $R(\cdot, \psi) \equiv 0$. Of course, Theorem 3 only yields a CR section of T^cM' along $h(M)$. But, if M' is pseudoconvex and its Levi form has at most K zero eigenvalues, the problem can be reduced to a study of sections of $T\eta$ along $h(M)$ by the following observation.

Lemma 7. *Let M' be a pseudoconvex hypersurface foliated by K-dimensional complex manifolds. Suppose that its Levi form has exactly K zero eigenvalues at a point p'. Suppose there exists a formal holomorphic curve $\gamma(t) = p' + t\gamma_t + t^2\gamma_{tt} + \dots$ tangential to second order to M' at p'. Then $\gamma_t \in T_{p'}\eta_{p'}$.*

Proof. A formal holomorphic curve $\gamma(t) = p' + t\gamma_t + t^2\gamma_{tt} + \dots$ is tangential to second order to M' if and only if the curve $\tilde{\gamma}(t) = p' + t\gamma_t + t^2\gamma_{tt}$ arising from the truncated power series is. Choosing a positive semidefinite scalar Levi form $\mathcal{L}_\Theta$ arising from a defining function ρ, we obtain by Lemma 2 that $\mathcal{L}_\Theta(\frac{1}{2}(\gamma_t + iJ\gamma_t), \frac{1}{2}(\gamma_t + iJ\gamma_t)) = \frac{d^2}{dtd\bar{t}}|_{t=0}\rho\circ\tilde{\gamma} = 0$, since $\rho \circ \tilde{\gamma}$ vanishes to second order. But by Lemma 5, the null space of $\mathcal{L}_\Theta|_{p'}$ is given by $T^{0,1}_{p'}\eta_{p'}$, implying that $\gamma_t \in T_{p'}\eta_{p'}$. $\square$

It turns out that the tensor R introduced in Lemma 6 also measures obstructions to the existence of CR sections of $T\eta$ along h, although h need not be an immersion.

Proposition 1. *Consider a pseudoconvex hypersurface M' as in Lemma 7, a CR manifold M and a continuously differentiable CR map $h : M \to M'$ mapping a point $p \in M$ to $p' \in M'$, and suppose that the Levi form of M' has exactly K zero eigenvalues at p'. Suppose there exists a C^1-smooth CR family of formal holomorphic curves $(\Gamma_q)_{q\in O}$ defined on a neighborhood $O \subseteq M$ of p such that Γ_q is tangential to second order to M' at $h(q)$ for each $q \in O$. Then $\gamma_t(p) \in T_{p'}\eta_{p'}$, and $h_* T^{0,1}_p M \subseteq \ker R_{p'}(\cdot, \gamma_t(p))$.*

Proof. By Lemma 7, we know that at each point $q \in O$, $\gamma_t(q) \in T_{h(q)}\eta_{h(q)}$, since Γ_q is a formal holomorphic curve tangential to second order to M' at $h(q)$.

Consider now $\bar{L} \in \mathcal{V}(M)$ such that $h_*\bar{L}|_p \neq 0$. Choosing a two-dimensional real submanifold $S \subseteq O$ such that $\bar{L}|_p$ is tangential to S, the derivative of $h|_S$ has full rank at p, and hence $h|_S$ is a local embedding around p. We may thus extend $\gamma_t \circ h^{-1}|_{h(S)}$, defined on $h(S)$, to a section $\tilde{\gamma}_t \in \Gamma(T\eta)$ defined on an open neighbourhood of p'. Since γ_t and $\tilde{\gamma}_t \circ h$ agree on S and $\bar{L}_p$ only takes derivatives along S, it follows that

$$R_{p'}(h_*\bar{L}|_p, \gamma_t(p)) = \mathbb{P}_{T^\perp_{p'}\eta_{p'}}\left(h_*\bar{L}|_p\tilde{\gamma}_t\right) = \mathbb{P}_{T^\perp_{p'}\eta_{p'}}\left(\bar{L}|_p(\tilde{\gamma}_t \circ h)\right) = \mathbb{P}_{T^\perp_{p'}\eta_{p'}}\left(\bar{L}|_p\gamma_t\right) = 0,$$

implying that $h_*\bar{L}|_p \in \ker R_{p'}(\cdot, \gamma_t(p))$. $\square$

Before we finally apply Theorem 3 to our situation, let us introduce a numerical quantity measuring the size of $\ker R(\cdot, V)$ as well as a method of computing it. For $p' \in M'$, let $\nu_{p'} := \max_{V \in T_{p'}\eta_{p'}\setminus\{0\}} \dim_{\mathbb{C}} \ker R_{p'}(\cdot, V) - K$.

It should be noted that $\nu_{p'}$ is an upper semicontinuous, integer valued function on M'. Indeed, if $\nu_{p'} \leq k$, then $\nu_q \leq k$ for all q in an open neighbourhood of p' by the following observation: If we express $R_q(\cdot, V)$ in smooth coordinates adapted to the foliation, the condition $\max_{V \in T_q\eta_q\setminus\{0\}} \dim_{\mathbb{C}} \ker R_q(\cdot, V) \leq K + k$ simply means that a certain $(N'-1-K-k) \times (N'-1-K-k)$-minor of the matrix representation of $R_q(\cdot, V)$ does not vanish for any $V \in \mathbb{S}^{2K-1}$. This minor is a smooth function of q and V, and because $\mathbb{S}^{2K-1}$ is compact, there exists a neighbourhood O of p' such that that the minor also does not vanish on $O \times \mathbb{S}^{2K-1}$.

To calculate $\nu_{p'}$, the following setup will be helpful.

Lemma 8. *Let M' be a pseudoconvex hypersurface foliated by complex manifolds of dimension K. Consider a point $p' \in M'$ and an $(N'-K)$-dimensional complex manifold Σ through p' such that $T_{p'}\eta_{p'} \oplus T_{p'}\Sigma_{p'} = T_{p'}\mathbb{C}^N$. If $S := M' \cap \Sigma$ is strongly pseudoconvex, then the Levi form of M' has exactly K zero eigenvalues at p'. Furthermore, exactly as in Lemma 6, the map $R^S \in \mathcal{V}(S)^* \otimes \Gamma(T\eta|_S)^* \otimes \Gamma(T^\perp\eta|_S)$ given by $R^S(\bar{L}, V) = \mathbb{P}_{T^\perp\eta}(\bar{L}V)$ is a tensor, and $\max_{V \in T_{p'}\eta_{p'}\setminus\{0\}} \dim_{\mathbb{C}} \ker R^S_{p'}(\cdot, V) = \nu_{p'}$.*

Proof. Let Θ be a characteristic form on M' such that the respective scalar Levi form $\mathcal{L}_\Theta(\bar{L}_1, \bar{L}_2) = \frac{1}{2i}\Theta([L_1, \bar{L}_2])$ is positive semidefinite. If S is strongly pseudoconvex at p', then $\mathcal{L}_\Theta|_{T^{0,1}_{p'}S}$ is strictly positive definite, hence, by elementary linear algebra, $\mathcal{L}_\Theta$ has at least $N' - 1 - K$ positive eigenvalues and as calculated in Lemma 5, the other K eigenvalues have to be zero. Consider now $R_{p'}(\bar{L}, V)$ for $\bar{L} \in \mathcal{V}(M)$ and $V \in T_{p'}\eta_{p'}$. Decompose $\bar{L}|_{p'} = U + W$ for $U \in T^{0,1}_{p'}S_{p'}$ and $W \in T^{0,1}_{p'}\eta_{p'}$. As proven in Lemma 6, $R_{p'}(W, V) = 0$, hence $R_{p'}(\bar{L}, V) = 0$ iff $R_{p'}(U, V) = R^S_{p'}(U, V) = 0$. This implies that $\ker R_{p'}(\cdot, V) = \ker R^S_{p'}(\cdot, V) \oplus T^{0,1}_{p'}\eta_{p'}$, proving the second claim. □

If the kernel of R is of minimal dimension even at a single point, Proposition 1 implies a very strong corollary, fully generalizing the result on the tube over the light cone.

Corollary 1. *Consider a pseudoconvex hypersurface $M' \subset \mathbb{C}^{N'}$ foliated by complex manifolds of dimension K, a* minimal *CR manifold M and a $C^{N'-1}$-regular CR map $h : M \to M'$ mapping a point $p \in M$ to $p' \in M'$. If the Levi form of M' has exactly K eigenvalues at p', and additionally, $\nu_{p'} = 0$, then there exists an open neighborhood O of p where h behaves as follows: Each connected open set $\tilde{O} \subseteq O$ where h is nowhere C^∞-smooth is mapped into a single leaf $\eta_{h(q)}$, $q \in \tilde{O}$. In particular, if h is of (real) rank $\geq 2K + 1$, h is C^∞-smooth on a dense open subset of O.*

Proof. By Lemma 5 and because ν is upper semicontinuous and integer-valued, there is an open neighborhood O' of p' where the Levi form of M' has exactly K zero eigenvalues, and where $\nu = 0$. Denote $h^{-1}(O')$ by O.

Consider now a connected open set $\tilde{O} \subseteq O$ where h is nowhere C^∞-smooth. Lemma 3 implies $r_0 \geq 1$, thus we may apply Theorem 3 to obtain a dense open subset $\tilde{O}_1$ of $\tilde{O}$ where for every point $q \in \tilde{O}_1$, there exists a neighborhood O_q and a C^1-smooth CR family of formal complex curves $(\Gamma_\xi)_{\xi\in O_q}$ such that Γ_ξ is tangential to M' to infinite order at $h(\xi)$. By Proposition 1, the fact that $\nu_q = 0$ implies that $h_* T^{0,1}_q M \subseteq T^{0,1}_{h(q)}\eta_{h(q)}$. In total, this implies $h_* T^{0,1}\tilde{O}_1 \subseteq T^{0,1}\eta$, and since this is a closed property and $\tilde{O}_1 \subseteq \tilde{O}$ lies dense, also $h_* T^{0,1}\tilde{O} \subseteq T^{0,1}\eta$.

On points $q \in M$ where h is regular enough, this means that $h^{-1}(\eta_{h(q)})$ integrates the complex tangent bundle and thus, by minimality of M, has to contain an open neighborhood of q. Indeed, as is worked out in detail in Corollary 3 in section 5, there is a family $(\tilde{O}_k)_{k=1}^\infty$ of open sets whose union is dense in $\tilde{O}$ such that the preimages $h^{-1}(\eta_{h(q)}) \cap \tilde{O}_k$, $q \in \tilde{O}_k$ are $C^{N'-1}$-smooth submanifolds of constant dimension which foliate $\tilde{O}_k$. Furthermore, $(h_*)^{-1} T_{h(q)}\eta_{h(q)} = T_q(h^{-1}(\eta_{h(q)}))$, which in our case implies $T^{0,1}M \subseteq \mathbb{C}T(h^{-1}\eta)$ on $\tilde{O}_k$, i.e. that the foliation $h^{-1}\eta$ with its bundle of tangent spaces $T(h^{-1}\eta)$ integrates the complex tangent bundle on $\tilde{O}_k$. But because M is minimal, any such foliation has to be trivial, hence $h(\tilde{O}_k) \subseteq \eta_{h(q_k)}$ for a $q_k \in \tilde{O}_k$. Thus, $h_* T\tilde{O}_k \subseteq T\eta_{h(q_k)}$, and by density of $\bigcup_{k=1}^\infty \tilde{O}_k$ and since h is continuously differentiable, $h_* T\tilde{O} \subseteq T\eta$. Because $\tilde{O}$ is connected, this finally implies $h(\tilde{O}) \subseteq \eta_{h(q)}$, $q \in \tilde{O}$.

In this case h is at most of rank $2K$ on $\tilde{O}$, proving the last claim. □

To convince ourselves that these conditions occur naturally and are verified by straight calculation, let us return to the tube over the light cone.

Example 7. *Let $M' \subseteq \mathbb{C}^{N'}$ be the tube over the light cone. It is foliated by complex lines, at any point $p' \in M'$ with $\Re(p') \neq 0$ the Levi form of M' has exactly one zero eigenvalue, and $\nu_{p'} = 0$.*

Proof. Recall that the tube over the light cone is defined as the set of points $z \in \mathbb{C}^{N'}$ such that $\Re(z_1)^2 + \dots \Re(z_{N'-1})^2 = \Re(z_{N'})^2$. It is a smooth real hypersurface where $\Re(z_{N'}) \neq 0$, and foliated by complex lines $q + t\,(\Re(q_1), \dots, \Re(q_{N'-1}), \Re(q_{N'}))$, $q \in M'$. Indeed, let us check that

$$\begin{aligned}
&\Re\left(q_1 + t\Re(q_1)\right)^2 + \dots + \Re\left(q_{N'-1} + t\Re(q_{N'-1})\right)^2 \\
&= (1+\Re(t))^2 \left(\Re(q_1)^2 + \dots + \Re(q_{N'-1})^2\right) \\
&= (1+\Re(t))^2 \Re(q_{N'})^2 = \Re\left(q_{N'} + t\Re(q_{N'})\right)^2.
\end{aligned}$$

The hypersurface M' is pseudoconvex, since the tube over the interior of the light cone is convex. The hypersurface $\Sigma = \{z \in \mathbb{C}^{N'} : z_{N'} = p'_{N'}\}$ through $p' \in M'$ is transversal to

$\eta_{p'}$ and intersects M' in $S = \{z \in \mathbb{C}^{N'} : z_{N'} = p'_{N'}, \Re(z_1)^2 + \cdots + \Re(z_{N'-1})^2 = \Re(p'_{N'})^2\}$, which is a strongly pseudoconvex CR submanifold of Σ because it is a tube over a strongly convex real manifold. To obtain the setup of Lemma 8, it now suffices to calculate $R^S_{p'}(\bar{L}|_{p'}, V|_{p'})$ for a single section V of $T\eta$ (since $T\eta$ is one-dimensional). Take $V(q) = (\Re(q_1), \ldots, \Re(q_{N'-1}), \Re(q_{N'}))$ for $q \in S$, and consider a CR vector $\bar{L}|_{p'} \in T^{0,1}_{p'}S$. Since $\bar{L}|_{p'}\Re(q_{N'}) = 0$, $L|_{p'}V \in T_{p'}\Sigma$, and because $T_{p'}\Sigma$ and $T_{p'}\eta$ lie in general position, $R^S_{p'}(\bar{L}|_{p'}, V|_{p'}) = 0$ if and only if $\bar{L}|_{p'}V = 0$, i.e. $\bar{L}|_{p'}(\Re(q_j)) = 0$ for $j = 1, \ldots, N'$. But since $\bar{L}_{p'}q_j = 0$, this is the case if and only if $\bar{L}_{p'}\bar{q}_j = 0$ as well, hence $\bar{L}|_{p'} \in T^{0,1}_{p'}S \cap T^{1,0}_{p'}S = \{0\}$, which proves that $\nu_{p'} = 0$. $\square$

If one knows that $h_*T^{0,1}_pM$ and $T_{p'}\eta_{p'}$ intersect trivially, and that $h_*T^{0,1}_pM$ has enough dimensions, a similar result holds for positive $\nu_{p'}$. This occurs if M is a strongly pseudoconvex hypersurface of sufficient dimensions, and h satisfies a commonly considered nondegeneracy condition, that of *CR-transversality*.

Definition 2 (CR-transversality)**.** *A CR map $h : M \to M'$ between hypersurfaces M and M' is called CR-transversal at $p \in M$ if $T^{0,1}_{h(p)}M' + T^{1,0}_{h(p)}M' + h_*\mathbb{C}T_pM = \mathbb{C}T_{h(p)}M'$.*

The point is that if M is strongly pseudoconvex and $h : M \to M'$ is CR-transversal, then $h_*T^{0,1}_pM$ has maximal dimensions and intersects $T^{0,1}_{p'}\eta_{p'}$ trivially. This is very well known and a key component of regularity proofs e.g. in [15].

Lemma 9. *Consider a pseudoconvex hypersurface M', a* strongly pseudoconvex *hypersurface M and a C^2-smooth* CR-transversal *CR map $h : M \to M'$ mapping $p \in M$ to $p' \in M'$. Then h is an immersion, $\dim h_*T^{0,1}_pM = \dim_{CR} M$, and $h_*T^{0,1}_pM \cap \mathcal{N}_{p'} = \{0\}$, where $\mathcal{N}_{p'} \subseteq T^{0,1}_{p'}M'$ denotes the null space of the Levi form of M' at p'.*

Proof. Because h is CR-transversal, the pull-back of a non-zero characteristic form $\Theta \in \Gamma(T^0M')$ is itself a non-zero characteristic form $\theta := h^*\Theta \in \Gamma(T^0M)$. Since M is strongly pseudoconvex, the respective scalar Levi form $\mathcal{L}_\theta(\bar{L}_1, \bar{L}_2) = \frac{1}{2i}\theta\left([L_1, \bar{L}_2]\right)$ is strictly positive definite (otherwise take $-\Theta$). Since L_1 and $\bar{L}_2$ are both contained in the kernel of θ, by definition, we have

$$2i\mathcal{L}_\theta(\bar{L}_1, \bar{L}_2) = \theta\left([L_1, \bar{L}_2]\right) = \bar{L}_2\theta(L_1) - L_1\theta(\bar{L}_2) + \theta\left([L_1, \bar{L}_2]\right) = d\theta(L_1, \bar{L}_2),$$

and similarly, $\mathcal{L}_\Theta(\bar{L}'_1, \bar{L}'_2) = \frac{1}{2i} d\Theta(L'_1, \bar{L}'_2)$ for CR vector fields $\bar{L}'_1, \bar{L}'_2$ on M'. Since pull-back along C^2 maps and exterior derivative commute, we obtain

$$\begin{aligned}\mathcal{L}_\theta(\bar{L}_1, \bar{L}_2)|_p &= \frac{1}{2i} d(h^*\Theta)|_p(L_1|_p, \bar{L}_2|_p) = \frac{1}{2i}(h^* d\Theta)|_p(L_1|_p, \bar{L}_2|_p) \\ &= \frac{1}{2i} d\Theta|_{p'}(h_* L_1|_p, h_* \bar{L}_2|_p) = \mathcal{L}_\Theta|_{p'}(h_*\bar{L}_1|_p, h_*\bar{L}_2|_p)\end{aligned}$$

If $L_1 \neq 0$, strict pseudoconvexity implies that $\mathcal{L}_\theta(\bar{L}_1, \bar{L}_1)|_p = \mathcal{L}_\Theta|_{p'}(h_*\bar{L}_1|_p, h_*\bar{L}_1|_p) > 0$. Thus $h_*\bar{L}_1|_p \neq 0$, which shows that the tangent map Dh has full rank on $T^{0,1}M$. By complex conjugation, Dh also has full rank on $T^{1,0}M$, and finally, because $h_*\mathbb{C}TM \not\subseteq T^{0,1}M' + T^{0,1}M'$, Dh has full rank on all of $\mathbb{C}TM$, i.e. h is an immersion. To prove the last claim, recall that if $h_*\bar{L}_1|_p \in \mathcal{N}_{p'}$, then $0 = \mathcal{L}_\Theta|_{p'}(h_*\bar{L}_1|_p, h_*\bar{L}_1|_p) = \mathcal{L}_\theta(\bar{L}_1, \bar{L}_1)|_p$ and thus $\bar{L}_1|_p = 0$. □

With this fact in mind, it is clear that strict pseudoconvexity of M and CR-transversality of h come together to imply that there are many CR directions available along $h(M)$ where some obstructions to the existence of CR families of infinitely tangential formal complex curves, encoded in R, might exist.

Corollary 2. *Consider a pseudoconvex hypersurface $M' \subset \mathbb{C}^{N'}$ foliated by complex manifolds of dimension K, a* strongly pseudoconvex *hypersurface $M \subset \mathbb{C}^N$ and a $C^{N'-N+1}$-regular* CR-transversal *CR map $h : M \to M'$ mapping a point $p \in M$ to $p' \in M'$. If the Levi form of M' has exactly K zero eigenvalues at p', and additionally, $\nu_{p'} < \dim_{CR} M$, then there exists an open neighborhood O of p such that h is C^∞-smooth on a dense open subset of O.*

Proof. Since ν is upper semicontinuous and integer-valued and secondly, by Lemma 5, there exists an open neighborhood O' of p' where $\nu < \dim_{CR} M$ and the Levi form of M' has exactly K zero eigenvalues. Let $O = h^{-1}(O')$ and suppose that h is nowhere C^∞-smooth on an open set $O_1 \subseteq O$.

Strong pseudoconvexity of M and CR-transversality of h imply by Lemma 9 that $h_*T^{0,1}M \cap T^{0,1}\eta = \{0\}$, and that $\dim_{\mathbb{C}} h_*T^{0,1}M = \dim_{CR} M$. Since the null space of the Levi form of M' is given by $T^{0,1}\eta$, any scalar Levi form restricts to a strictly positive or negative definite Hermitian form on $h_*T^{0,1}_pM$. By Lemma 4, $r_1 \geq N$, so we may apply Theorem 3 with $k = 1$, $l = N$ to obtain a point $q \in O_1$ (mapped to $q' \in O'$) and a neigborhood $O_2 \subseteq O_1$ of q such that there exists a C^1-smooth CR family of formal complex curves $(\Gamma_\xi)_{\xi \in O_2}$ through $h(O_2)$ for which Γ_ξ is tangent to infinite order to M' at $h(\xi)$.

By Proposition 1, this implies that $h_* T_q^{0,1} M \subseteq \ker R_{q'}(\cdot, \gamma_t(q))$. However, since we always have that $T_{q'}^{0,1}\eta_{q'} \subseteq \ker R_{q'}(\cdot, \gamma_t(q))$, and because $h_* T_q^{0,1} M \cap T_{q'}^{0,1}\eta_{q'} = \{0\}$ and $\dim_{\mathbb{C}} h_* T_q^{0,1} M = \dim_{CR} M$, we have $\dim_{\mathbb{C}} \ker R_{q'}(\cdot, \gamma_t(q)) \geq K + \dim_{CR} M$, contradicting $\nu_{q'} < \dim_{CR} M$. Hence, h has to be C^∞-smooth on a dense open subset of O. $\square$

3.2 Maps into boundaries of classical symmetric domains

We call a bounded domain $\Omega \subset \mathbb{C}^N$ a *bounded symmetric domain* if it exhibits a biholomorphic involution $h_p : \Omega \to \Omega$ for every point $p \in \Omega$ which has p as an *isolated* fixed point and which satisfies $Dh(p) = -\mathbb{I}_N$ (cf. [13]).

A bounded domain Ω may be equipped with the *Bergman metric*, a Hermitian metric with the property that each biholomorphism on Ω is an isometry. Considered together with this metric, a bounded symmetric domain Ω becomes a special case of a *Hermitian symmetric space*, i.e. a manifold equipped with a Hermitian metric such that each point is an isolated fixed point of some involutive isometry. It can be shown that the group of isometries of such manifolds acts transitively, therefore they can be expressed as the coset space of the the stabilizer group of Ω, defined as the group of isometries leaving a chosen point fixed, in the full isometry group of Ω (cf. [4]). This allows the classification of bounded symmetric domains by Lie group techniques.

According to [13], any bounded symmetric domain is biholomorphic to a direct product of *irreducible* bounded symmetric domains. Irreducible bounded symmetric domains fall into four series of classical symmetric domains as well as two exceptional cases (as classified by Cartan, cf. [3]). The study of proper holomorphic maps into classical symmetric domains, and consequently of CR maps into their boundaries, has been taken up by Xiao in [15]. We will adopt Xiao's naming convention, which differs from Cartan's original numbering only in swapping domains of the third and fourth kind.

Before delving into our study of the boundaries of Cartan's classical symmetric domains, let us briefly recall the singular value decomposition from linear algebra. A matrix $A \in \mathbb{C}^{m \times n}$, $m \leq n$ may always be decomposed as $A = U\Sigma V^*$, where

1. $U \in \mathbb{C}^{m\times m}$ is a unitary matrix, forming a basis of eigenvectors for AA^*,
2. $\Sigma \in \mathbb{C}^{m\times n}$ is a diagonal matrix with nonnegative entries, and
3. $V \in \mathbb{C}^{n\times n}$ is another unitary matrix, forming a basis of eigenvectors for A^*A.

The diagonal entries of Σ, $0 \leq \sigma_1 \leq \cdots \leq \sigma_m$ are called the *singular values* of A. They are given by the square roots of the eigenvalues of the (Hermitian, positive semidefinite) matrix AA^*, equivalently by the square roots of the m largest eigenvalues of A^*A. The largest singular value of A yields the operator norm of A with respect to the standard scalar product on $\mathbb{C}^m$ and $\mathbb{C}^n$. The matrix V of *right singular vectors* may be freely chosen among the orthonormal eigenvector bases of A^*A, which then fixes $\Sigma U = AV$, and therefore those columns of U corresponding to nonzero singular values, the *left singular vectors.*

3.2.1 Classical domains of the first kind

We will denote the examples in the first series by $D_I^{m,n}$ for $1 \leq m \leq n$. According to Cartan [3], they may be realized as

$$D_I^{m,n} = \{Z \in \mathbb{C}^{m\times n} : \mathbb{I}_m - ZZ^* \text{ is strictly positive definite.}\}$$

The condition $\mathbb{I}_m - ZZ^* > 0$ is equivalent to the largest singular value of Z being strictly bounded by one, i.e. $\|Z\|_{2\to 2} < 1$. The boundary of $D_I^{m,n}$ is thus given by the set of matrices of norm 1, equivalently, by those matrices which have 1 as their largest singular value. This set is a smooth manifold where only one singular value is 1. To see this, consider the characteristic polynomial $P(\lambda) = \det(\lambda\mathbb{I}_m - ZZ^*)$ of ZZ^*, which has a simple zero at 1 by assumption. Now $\rho(Z) := \det(\mathbb{I}_m - ZZ^*)$ has nonvanishing gradient, since $\rho(Z + \mu Z) = \det(\mathbb{I}_m - |1+\mu|^2 ZZ^*) = |1+\mu|^{2m} P(|1+\mu|^{-2})$ has nonvanishing derivative, providing us with a defining equation.

Let us denote this smooth piece of the boundary by $M_I^{m,n}$. Because $M_I^{m,n}$ bounds the convex region $D_I^{m,n}$, it is a pseudoconvex real hypersurface. The singular value decomposition will translate to a foliation of $M_I^{m,n}$ by complex (in fact, complex linear) manifolds, setting $M_I^{m,n}$ up as an interesting example case for applying Corollary 2. The following result should be compared to Proposition 1.2 in [15], where only hypersurfaces in $\mathbb{C}^{m+n-1}$ are considered.

Proposition 2. *Let $m \geq n \geq 2$ and M be a strongly pseudoconvex smooth hypersurface in $\mathbb{C}^N$ for $N \in \{m+n-2, m+n-1\}$. Then every CR-transversal CR map h of regularity C^{mn-N+1} from M into $M_I^{m,n}$ is C^∞-smooth on a dense open subset of M.*

This will be a consequence of the boundary orbit theorem, which states that the Lie group of biholomorphic automorphisms of $D_I^{m,n}$ also acts transitively on $M_I^{m,n}$ by ambient biholomorphisms (as is stated in paper [15], which refers to [14] and [10]). An elementary proof for this statement is outlined in section 4. This allows us to analyze $M_I^{m,n}$ around points which are particularly easy to understand from the matrix model alone, namely those matrices of rank one in $M_I^{m,n}$.

Indeed, suppose $h : M \to M_I^{m,n}$ is nowhere smooth on a neighborhood $O \subset M$ of a point $p \in M$. Any matrix ab^* for vectors $a \in \mathbb{C}^m$, $b \in \mathbb{C}^n$ of unit norm is contained in $M_I^{m,n}$, since it has a lone singular value 1. By the boundary orbit theorem, there exists a biholomorphic map $F_{h(p)}$ defined on a neighborhood of $h(p)$ mapping $h(p)$ to $ab^* \in M_I^{m,n}$ and $M_I^{m,n}$ into itself. Then $\tilde{h} := F_{h(p)} \circ h$ is a CR-transversal CR map taking p to ab^*, which is nowhere smooth on O as well. At ab^*, we check directly that the prerequisites to apply Corollary 2 are fulfilled.

Lemma 10. *Let $a \in \mathbb{C}^n, b \in \mathbb{C}^m$ be unit vectors. Around ab^*, the pseudoconvex hypersurface $M_I^{m,n}$ is foliated by $(n-1)\times(m-1)$-dimensional complex (linear) manifolds. Its Levi form has exactly $m + n - 2$ positive eigenvalues, and $\nu_{ab^*} = m + n - 4$.*

If $\tilde{h}$ was nowhere smooth around p, this would contradict Corollary 2 if $\nu_{ab^*} = m+n-4 < \dim_{CR} M$, which is indeed the case if $M \subset \mathbb{C}^{m+n-2}$ or $M \subset \mathbb{C}^{m+n-1}$. This proves Proposition 2.

Proof of Lemma 10. Let $\Sigma \subset \mathbb{C}^{m\times n}$ be the set of $m \times n$ matrices of rank (exactly) 1, which is an $(m+n-1)$-dimensional holomorphic manifold containing ab^*. In linear coordinates such that $a = (1, 0, \ldots, 0)^T \in \mathbb{C}^n$ and $b = (1, 0, \ldots, 0)^T \in \mathbb{C}^m$, Σ is parametrized holomorphically by $(z_1, \ldots, z_m, w_2, \ldots, w_n) \mapsto (z_1, \ldots, z_m)^T(1, w_2 \ldots, w_n)$ around $ab^* = (1, 0, \ldots, 0)^T(1, 0, \ldots, 0)$. To see explicitly that this map is one-on-one near ab^*, for a matrix $Z \in \Sigma$, let w be the (unique) intersection of $(\ker Z)^\perp$ and $b + \langle b\rangle^\perp$. Then $w^* = (1, w_2, \ldots, w_n)$ and $Z(w)/\|w\|^2 = (z_1, \ldots, z_m)^T$.

The hypersurface $S := \Sigma \cap M_I^{m,n}$ of rank one matrices with norm 1 is strongly pseudoconvex. Indeed, because $\|uv^*\|_{2\to 2} = \|u\|\|v\|$, a defining equation for S is given by

$$\rho(z_1, \ldots, z_m, w_2, \ldots, w_n) = (|z_1|^2 + \cdots + |z_m|^2)(1 + |w_2|^2 + \cdots + |w_n|^2) - 1 = 0,$$

with (real) Hessian $2\mathbb{I}_{2(m+n-1)}$ at ab^*, implying that S is actually strongly convex.

The singular value decomposition expresses any matrix $A \in M_I^{m,n}$ as $uv^* + B$, where u and v are unit singular vectors (unique up to simultaneous multiplication by $\lambda \in \mathbb{S}^1$) corresponding to the lone singular value 1, and the uniquely determined matrix $B \in \mathbb{C}^{m\times n}$ satisfies $Bv = 0$, $u^*B = 0$ and $\|B\| < 1$. Conversely, every matrix $uv^* + B$ of this type lies in $M_I^{m,n}$. The set of all $B \in \mathbb{C}^{m\times n}$ with $u^*B = 0$ and $Bv = 0$ is an $(m-1) \times (n-1)$-dimensional vector space, and thus the affine planes

$$\eta_{uv^*} := \{uv^* + B : B \in \mathbb{C}^{m\times n}, u^*B = 0, Bv = 0\}$$

for $uv^* \in S$ provide the desired foliation η of $M_I^{m,n}$ near ab^*. The tangent bundle $T\eta$ at S is just given by $T_{uv^*}\eta = \{B \in \mathbb{C}^{m\times n} : Bv = 0, u^*B = 0\}$.

Having established the setup from Lemma 8, all that remains is to compute the tensor R^S at ab^*. Take $B_0 \in T_{ab^*}\eta$, $B_0 \neq 0$. If we define $B(Z)$ for $Z \in \mathbb{C}^{m\times n}$ by

$$B(Z) = (\mathbb{I}_m - ZZ^*)B_0(\mathbb{I}_n - Z^*Z),$$

then $B(uv^*)$ provides a section of $T\eta$ along S satisfying $B(ab^*) = B_0$, since

$$\begin{aligned} u^*B(uv^*) &= u^*(\mathbb{I}_m - uu^*)B_0(\mathbb{I}_n - vv^*) = 0, \\ B(uv^*)v &= (\mathbb{I}_m - uu^*)B_0(\mathbb{I}_n - vv^*)v = 0 \text{ and} \\ B(ab^*) &= (\mathbb{I}_m - aa^*)B_0(\mathbb{I}_n - bb^*) = \mathbb{I}_m B_0 \mathbb{I}_n = B_0. \end{aligned}$$

Returning to S, we work out that $\mathfrak{V} := \{a\beta^* + \alpha b^* : \alpha \in \langle a\rangle^\perp \subset \mathbb{C}^m, \beta \in \langle b\rangle^\perp \subset \mathbb{C}^n\}$ is the complex tangent space of S at ab^*. To show that $\mathfrak{V}$ is tangential, we take two curves $\gamma_1 : (-\varepsilon, \varepsilon) \to \mathbb{S}^{2m-1}$ and $\gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{S}^{2n-1}$ through a and b, respectively, satisfying $\dot\gamma_1(0) = \alpha$ and $\dot\gamma_2(0) = \beta$. Now $\gamma_1\gamma_2^*$ defines a curve in S, and $\frac{d}{dt}|_{t=0}(\gamma_1(t)\gamma_2(t)^*) = a\beta^* + \alpha b^*$. The space $\mathfrak{V}$ is parametrized in a complex linear way by $(\alpha, \bar\beta) \mapsto a\bar\beta^T + \alpha b^*$, where $(\alpha, \bar\beta)$ lies in the $(m+n-2)$-dimensional complex subspace of $\mathbb{C}^{m+n}$ defined by $a^*\alpha = 0$, $\bar\beta^T b = \beta^* b = 0$. To check that this map is indeed injective, test $\alpha b^* + a\beta^*$ from right and left with b and a^*, respectively, to obtain α and $\bar{\beta}$ again. Since the complex tangent space of S has only $\dim_{\mathbb{C}} \Sigma - 1 = m + n - 2$ dimensions, $T^c_{ab^*} S = \mathfrak{V}$ follows.

Consider a CR vector $\bar{L}|_{ab^*} = \frac{1}{2}(X + iJX)$ for $X \in T^c_{ab^*} S$ and write $X = a\beta^* + \alpha b^*$. Then the holomorphic curve $\gamma(t) = (a + t\alpha)(b + \bar{t}\beta)^*$ is tangential to $\bar{L}|_{ab^*}$ at $t = 0$. Observing that both $\|a + t\alpha\|^2 = 1 + |t|^2\|\alpha\|^2$ and $\|b + \bar{t}\beta\|^2$ are constant to first order, we obtain

$$\begin{aligned}
\bar{L}|_{ab^*} B &= \frac{d}{d\bar{t}}\Big|_{t=0} B \circ \gamma(t) = \frac{d}{d\bar{t}}\Big|_{t=0} (\mathbb{I}_m - \gamma(t)\gamma(t)^*)\, B_0\, (\mathbb{I}_n - \gamma(t)^*\gamma(t)) \\
&= \frac{d}{d\bar{t}}\Big|_{t=0} \left(\mathbb{I}_m - \|b + \bar{t}\beta\|^2 (a + t\alpha)(a + t\alpha)^*\right) B_0 \left(\mathbb{I}_n - \|a + t\alpha\|^2 (b + \bar{t}\beta)(b + \bar{t}\beta)^*\right) \\
&= -a\alpha^* B_0 - B_0 \beta b^*.
\end{aligned}$$

Recall that the scalar product in $\mathbb{C}^{m\times n}$ may be written as $(A|B) = \operatorname{tr}(A^*B)$. By commuting matrices inside the trace we see that for any Z in $T_{ab^*}\eta$,

$$\operatorname{tr}\left((\bar{L}|_{ab^*} B)^* Z\right) = -\operatorname{tr}(B_0^* \alpha a^* Z + b\beta^* B_0^* Z) = -\operatorname{tr}(B_0^* \alpha a^* Z) - \operatorname{tr}(B_0^* Z b \beta^*) = 0,$$

since $a^* Z = Zb = 0$. This means that $R^S_{ab^*}(\bar{L}|_{ab^*}, B_0) = -a\alpha^* B_0 - B_0\beta b^*$, because the projection onto $T^\perp \eta$ is already taken care of, and $\bar{L}|_{ab^*} \in \ker R^S_{ab^*}(\cdot, B_0)$ if and only if $a\alpha^* B_0 + B_0 \beta b^* = 0$. Testing this with a^* and b from left and right, respectively, we obtain $\alpha \in \ker B_0^*$ and $\beta \in \ker B_0$. Since $B_0^* a = 0$ and $B_0 b = 0$ already, both kernels have codimension at least one in $\langle a \rangle^\perp$ and $\langle b \rangle^\perp$, respectively, thus $\dim_{\mathbb{C}} \ker R^S_{ab^*}(\cdot, B_0) \le m + n - 4$, implying $\nu_{ab^*} = m + n - 4$. $\square$

Proposition 2 gives all dimensions where a statement this simple is meaningful and possible. If M has more than $m + n - 2$ positive Levi eigenvalues, there is no CR-transversal map from M to $M_I^{m,n}$, since, by Lemma 9, the target manifold would need to have at least as many positive Levi eigenvalues as the source. If M has less than $m + n - 3$ positive Levi eigenvalues, there are nowhere smooth CR-transversal CR maps into $M_I^{m,n}$ of arbitrarily high regularity.

Example 8. *Let $\hat{S}$ be the strongly pseudoconvex hypersurface given by the $(m-1)\times(n-1)$ matrices of rank one and norm 1. Then $\hat{S}$ has $m+n-4$ positive Levi eigenvalues. Take a C^k, but nowhere C^∞-smooth CR function ϕ on $\hat{S}$ with $|\phi| < 1$. Then $h(Z) = \begin{pmatrix} Z & 0 \\ 0 & \phi \end{pmatrix}$ gives a nowhere smooth CR-transversal CR map $h : \hat{S} \to M_I^{m,n}$ of regularity C^k.*

Proof. Regularity is obvious from the component-wise definition. CR-transversality always holds for the graph map of a CR function, i.e. $h : M \to M \times \mathbb{C}$, $h(p) = (p, \phi(p))$, since $T^c(M \times \mathbb{C}) \cong T^cM \times \mathbb{C}$, $T(M \times \mathbb{C}) \cong TM \times \mathbb{C}$ and $\mathbb{P}_{TM} \circ h_* \cong \mathrm{id}$ together imply that any transversal vector $v \in TM \setminus T^cM$ maps into a transversal vector again. That $h(Z) \in M_I^{m,n}$ follows from the singular value computations in the proof of Lemma 10. $\square$

3.2.2 Classical domains of the second kind

These classical symmetric domains, denoted by D_{II}^m, $m \geq 2$, are given as the sets of *skew symmetric* complex $m \times m$ matrices with norm less than 1. Equivalently,

$$D_{II}^m = \left\{ Z \in \mathbb{C}^{m \times m} : Z^T = -Z, \mathbb{I}_m - Z^* Z > 0 \right\}.$$

Every nonzero singular value of a skew symmetric matrix Z occurs with even multiplicity. Suppose u is a right singular vector corresponding to a singular value σ, which is equivalent to $Z^*Zu = \sigma^2 u$. Then $v := \sigma^{-1}\overline{Zu}$ is another right singular vector corresponding to σ, since it follows from $Z^* = \bar{Z}^T = -\bar{Z}$ that $Z^*Zv = -\sigma^{-1}\bar{Z}Z\bar{Z}\bar{u} = -\sigma^{-1}\bar{Z}\overline{\bar{Z}Zu} = \sigma^{-1}\bar{Z}\overline{Z^*Zu} = \sigma\bar{Z}\bar{u} = \sigma^2 v$, and $v^*v = \sigma^{-2}u^TZ^T\bar{Z}\bar{u} = \sigma^{-2}\overline{u^*Z^*Zu} = 1$. Furthermore, v and u are orthogonal, and $u = -\sigma^{-1}\overline{Zv}$:

$$\begin{aligned} \sigma u^* v = u^* \bar{Z} \bar{u} = (u^* \bar{Z} \bar{u})^T = u^* \bar{Z}^T \bar{u} = -u^* \bar{Z} \bar{u} \Rightarrow u^* v = 0, \\ -\sigma^{-1}\overline{Zv} = -\sigma^{-2}\overline{Z\bar{Z}\bar{u}} = -\sigma^{-2}\bar{Z}Zu = \sigma^{-2}Z^*Zu = u. \end{aligned}$$

The boundary of D_{II}^m is given by those skew symmetric matrices with norm 1. It is a smooth manifold where exactly the largest two singular values are 1. We will denote this smooth piece of the boundary by M_{II}^m. Let us postpone checking that M_{II}^m is a manifold to the proof of Lemma 11.

Proposition 3. *Let $m \geq 4$ and M be a strongly pseudoconvex smooth hypersurface in $\mathbb{C}^N$ for $2m - 6 \leq N \leq 2m - 3$. Then every CR-transversal CR map h of regularity $C^{\frac{m(m-1)}{2} - N + 1}$ from M into M_{II}^m is C^∞-smooth on a dense open subset of M.*

Completely analogously to the situation of Proposition 2, this follows from the boundary orbit theorem for D_{II}^m, which allows us to map each point in M_{II}^m to $p' := ab^T - ba^T$ for orthonormal $a, b \in \mathbb{C}^m$ by an automorphism of M_{II}^m, and from the following structural properties.

Lemma 11. *Let $a, b \in \mathbb{C}^m$ be orthonormal vectors. Around $p' := ab^T - ba^T \in M_{II}^m$, the pseudoconvex hypersurface M_{II}^m is foliated by $\frac{(m-2)(m-3)}{2}$-dimensional complex (linear) manifolds. Its Levi form has exactly $2m-4$ positive eigenvalues, and $\nu_{p'} = 2m-8$.*

Proof. As the intersection of the linear subspace of skew symmetric matrices with the convex matrix norm unit ball, D_{II}^m is convex and M_{II}^m is a pseudoconvex hypersurface.

The set Σ of skew symmetric matrices of rank two is a $(2m-3)$-dimensional complex manifold around p'. In coordinates where $a = (1, 0, \dots, 0)^T$ and $b = (0, 1, 0, \dots, 0)^T$, it is parametrized around p' by

$$(z_3, ..., z_m, w_2, ..., w_m) \mapsto (1, 0, z_3, ..., z_m)^T(0, w_2, ..., w_m) - (0, w_2, ..., w_m)^T(1, 0, z_3, ..., z_m).$$

To check surjectivity, let $\bar{u}$ and $\bar{v}$ be two right singular vectors corresponding to the only nonzero singular value σ, chosen such that $Z\bar{u} = -\sigma v$ and $Z\bar{v} = \sigma u$. Then $Z = u(\sigma v)^T - (\sigma v)u^T$. Since $a^*(ab^T - ba^T)\bar{b} = 1$, $a^*Z\bar{b} \neq 0$ near p', implying that at least one of a^*u or a^*v is nonzero. By substituting $(-v, u)$ for (u, v) if necessary, we can arrange $a^*u \neq 0$. Let $\tilde{u} = u$, $\tilde{v} = \sigma(v - \frac{a^*v}{a^*u}u)$, then $a^*\tilde{v} = 0$ and $Z = \tilde{u}\tilde{v}^T - \tilde{v}\tilde{u}^T$. Note that $a^*Z\bar{b} \neq 0$ now implies $b^*\tilde{v} \neq 0$. Let $z = \frac{1}{a^*\tilde{u}}\tilde{u} - \frac{b^*\tilde{u}}{(a^*\tilde{u})(b^*\tilde{v})}\tilde{v}$ and $w = (a^*\tilde{u})\tilde{v}$. Then we have $a^*z = 1$, $b^*z = 0$, $a^*w = 0$ and $Z = zw^T - wz^T$, proving that Z is in the range of our parametrization. To check that it is an immersion, it suffices to calculate $\frac{\partial}{\partial z_j}(zw^T - wz^T) = e_j e_2^T - e_2 e_j^T$, $3 \leq j \leq m$ and $\frac{\partial}{\partial w_k}(zw^T - wz^T) = e_1 e_k^T - e_k e_1^T$, $2 \leq k \leq m$, since these are evidently $\mathbb{C}$-linearly independent matrices.

The set $S = \Sigma \cap M_{II}^m$ of skew symmetric rank two matrices with norm 1 is a strictly pseudoconvex hypersurface in Σ. To show this, first note that for orthogonal vectors $\alpha, \beta \in \mathbb{C}^m$, we have

$$\begin{aligned}\|\alpha\beta^T - \beta\alpha^T\|_{2\to2}^2 &= \|\left(\alpha\beta^T - \beta\alpha^T\right)^* \left(\alpha\beta^T - \beta\alpha^T\right)\|_{2\to2} = \|\|\alpha\|^2\bar{\beta}\beta^T + \|\beta\|^2\bar{\alpha}\alpha^T\|_{2\to2} \\ &= \|\alpha\|^2\|\beta\|^2\|\mathrm{diag}(1, 1, 0, \dots, 0)\|_{2\to2} = \|\alpha\|^2\|\beta\|^2.\end{aligned}$$

The standard Euclidean scalar product on $\mathbb{C}^{m\times m}$ coincides with the Frobenius scalar product $(A|B) = \mathrm{tr}(A^*B)$. For a matrix $Z = \alpha\beta^T - \beta\alpha^T$ with orthogonal $\alpha, \beta \in \mathbb{C}^m$, the Frobenius norm works out to

$$\sqrt{\mathrm{tr}(Z^*Z)} = \sqrt{\mathrm{tr}(\|\alpha\|^2\bar{\beta}\beta^T + \|\beta\|^2\bar{\alpha}\alpha^T)} = \sqrt{2}\|\alpha\|\|\beta\|.$$

Therefore, the Frobenius norm and the matrix norm agree up to a constant on Σ, and $S = \Sigma \cap M_{II}^m = \Sigma \cap \sqrt{2}\mathbb{S}^{2m^2-1}$ is strongly pseudoconvex, as it is given by the intersection of a complex manifold with a strongly convex hypersurface.

The singular value decomposition expresses $Z \in M_{II}^m$ as $uv^T - vu^T + B$, where $\bar{u}$ and $\bar{v}$ are right singular vectors corresponding to the double singular value 1 satisfying $Z\bar{u} = -v$ and $Z\bar{v} = u$, and B satisfies $B\bar{u} = B\bar{v} = 0$, $u^*B = v^*B = 0$ and $\|B\| < 1$. By linearity, we have $B^T = -B$, implying that $B\bar{u} = B\bar{v} = 0$ and $u^*B = v^*B = 0$ are equivalent. In coordinates where $u = (1, 0, 0, \dots, 0)$ and $v = (0, 1, 0, \dots, 0)$, the conditions $B = -B^T$ and $B\bar{u} = B\bar{v} = 0$ simply mean that B is a skew symmetric matrix with the first two rows and columns empty. We conclude that the affine planes $\eta_{uv^T - vu^T} = uv^T - vu^T + \{B \in \mathbb{C}^{m\times m} : B = -B^T, B\bar{u} = B\bar{v} = 0\}$ for $uv^T - vu^T \in S$ provide a foliation of M_{II}^m by $\frac{(m-2)(m-3)}{2}$-dimensional complex manifolds, and that M_{II}^m, as an embedded piece of a vector bundle over S, is indeed a manifold.

The complex tangent space $T_{p'}^c S$ at $p' = ab^T - ba^T$ will be given by the complex vector space $\mathfrak{V} := \{a\beta^T - \beta a^T + \alpha b^T - b\alpha^T : \alpha, \beta \in \langle a, b\rangle^\perp \subset \mathbb{C}^m\}$. To show tangency, consider the complex curve $\gamma(t) = (a + t\alpha)(b + t\beta)^T - (b + t\beta)(a + t\alpha)^T$, with tangent vector $\gamma_t(0) = a\beta^T - \beta a^T + \alpha b^T - b\alpha^T$. It is contained in Σ and tangential to M_{II}^m, the latter because $\|\gamma(t)\|^2 = \|a + t\alpha\|^2\|b + t\beta\|^2 - |(a + t\alpha)^*(b + t\beta)|^2 = \|a\|^2\|b\|^2 + \mathcal{O}(|t|^2)$, hence $\gamma_t(0) \in T_{p'}^c S$. Since $\mathfrak{V}$ is isomorphic to $\langle a, b\rangle^\perp$ by the map $\gamma_t(0) \mapsto \big(\gamma_t(0)\bar{b}, -\gamma_t(0)\bar{a}\big)$, it has $2m - 4 = \dim_{CR} S$ dimensions, and $T_{p'}^c S = \mathfrak{V}$.

Given $B_0 \in T_{p'}\eta$, the map $B(Z) = (\mathbb{I}_m - ZZ^*)B_0(\mathbb{I}_m - Z^*Z)$ again provides a section of $T\eta$ along S, since for orthonormal $u, v \in \mathbb{C}^m$,

$$B(uv^T - vu^T) = (\mathbb{I}_m - uu^* - vv^*)B_0(\mathbb{I}_m - \bar{u}u^T - \bar{v}v^T) = -B(uv^T - vu^T)^T$$
$$B(uv^T - vu^T)\bar{u} = (\mathbb{I}_m - uu^* - vv^*)B_0(\bar{u} - \bar{u}) = 0, \text{ and } B(uv^T - vu^T)\bar{v} = 0.$$

Taking a CR vector $\bar{L}|_{p'} \in T_{p'}^{0,1} S$ with real part $\frac{1}{2}\left(a\beta^T - \beta a^T + \alpha b^T - b\alpha^T\right)$ and the curve $\gamma(t) = (a + t\alpha)(b + t\beta)^T - (b + t\beta)(a + t\alpha)^T$, we first obtain

$$\begin{aligned}
\gamma(t)\gamma(t)^* &= \|a + t\alpha\|^2(b + t\beta)(b + t\beta)^* + \|b + t\beta\|^2(a + t\alpha)(a + t\alpha)^* \\
&\quad - t\bar{t}(\beta^T\bar{\alpha})(a + t\alpha)(b + t\beta)^* - t\bar{t}(\alpha^T\bar{\beta})(b + t\beta)(a + t\alpha)^* \\
&= (b + t\beta)(b + t\beta)^* + (a + t\alpha)(a + t\alpha)^* + \mathcal{O}(|t|^2), \\
\gamma(t)^*\gamma(t) &= \overline{(b + t\beta)}(b + t\beta)^T + \overline{(a + t\alpha)}(a + t\alpha)^T + \mathcal{O}(|t|^2),
\end{aligned}$$

which simplifies the calculations for $R^S_{p'}$ significantly. We obtain

$$\begin{aligned}
\bar{L}|_{p'}B &= \frac{d}{d\bar{t}}\Big|_{t=0} B\circ\gamma(t) = \frac{d}{d\bar{t}}\Big|_{t=0} \left(\mathbb{I}_m - \gamma(t)\gamma(t)^*\right) B_0 \left(\mathbb{I}_m - \gamma(t)^*\gamma(t)\right) \\
&= \frac{d}{d\bar{t}}\Big|_{t=0} \Big(\big(\mathbb{I}_m - (b+t\beta)(b+t\beta)^* - (a+t\alpha)(a+t\alpha)^*\big) B_0 \\
&\cdot \big(\mathbb{I}_m - \overline{(b+t\beta)}(b+t\beta)^T - \overline{(a+t\alpha)}(a+t\alpha)^T)\big) + \mathcal{O}(|t|^2)\Big) \\
&= -b\beta^* B_0 - a\alpha^* B_0 - B_0\bar{\beta}b^T - B_0\bar{\alpha}a^T.
\end{aligned}$$

By the same calculations as in the proof of Lemma 10, we find that this already gives $R^S_{p'}(\bar{L}|_{p'}, B_0) = -b\beta^* B_0 - a\alpha^* B_0 - B_0\bar{\beta}b^T - B_0\bar{\alpha}a^T$, and that $\bar{L}|_{p'} \in \ker R^S_{p'}(\cdot, B_0)$ if and only if $\alpha, \beta \in \ker \bar{B}_0$. As a nonzero skew symmetric matrix, $\bar{B}_0$ has at least two nonzero singular values, hence $\operatorname{codim}_{\mathbb{C}} \ker \bar{B}_0 \geq 2$. Since $\bar{B}_0 a = \bar{B}_0 b = 0$, and $\alpha, \beta \in \langle a, b\rangle^{\perp}$, we obtain $\operatorname{codim}_{\mathbb{C}} \ker R^S_{p'}(\cdot, B_0) \geq 4$ and thus $\nu_{p'} = 2m - 8$. □

As in Proposition 2, there are counterexamples to regularity if M has exactly $2m-8$ positive Levi eigenvalues.

Example 9. *Let $\hat{S} \subset M^{m-2}_{II}$ be the strongly pseudoconvex hypersurface of antisymmetric $(m-2)\times(m-2)$ matrices of rank two and norm 1. It has $2m-8$ positive Levi eigenvalues. Given a C^k-smooth, but nowhere C^∞-smooth CR function ϕ on $\hat{S}$ strictly bounded by 1, the map $h : \hat{S} \to M^m_{II}$ given by*

$$h(Z) = \begin{pmatrix} Z & 0 & 0 \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{pmatrix}$$

is a C^k-smooth, but nowhere C^∞-smooth CR-transversal CR function.

3.2.3 Classical domains of the third kind

Domains of the third kind D^m_{III} are given by the sets of *symmetric* complex $m\times m$ matrices with norm less than 1. Equivalently,

$$D^m_{III} = \left\{ Z \in \mathbb{C}^{m\times m} : Z^T = Z, \mathbb{I}_m - Z^*Z > 0 \right\}.$$

Here the regularity result obtained from Corollary 2 only holds for $M \subset \mathbb{C}^m$.

Proposition 4. *Let $m \geq 2$ and $M \subset \mathbb{C}^m$ be a strongly pseudoconvex smooth hypersurface. Then every CR-transversal CR map h of regularity $C^{\frac{m(m+1)}{2}-m+1}$ from M into M_{III}^m is C^∞-smooth on a dense open subset of M.*

This is a consequence of the boundary orbit theorem for D_{III}^m, which tells us that every point $Z \in M_{III}^m$ may be mapped to aa^T for a unit vector $a \in \mathbb{C}^m$ by an ambient biholomorphism mapping M_{III}^m into itself. Almost completely analogously to the case of $M_I^{m,n}$, the following structural properties hold.

Lemma 12. *Let $a \in \mathbb{C}^m$ be a unit vector. Around $aa^T \in M_{III}^m$, the pseudoconvex hypersurface M_{III}^m is foliated by $\frac{m(m-1)}{2}$-dimensional complex (linear) manifolds. Its Levi form has exactly $m-1$ positive eigenvalues, and $\nu_{aa^T} = m-2$.*

Proof. As the intersection of the convex set of matrices of norm less than 1 with the linear subspace of symmetric matrices, D_{III}^m is convex, and thus M_{III}^m is pseudoconvex.

Let Σ be the m-dimensional complex manifold of symmetric matrices of rank 1. Near aa^T, it is parametrized by $z \mapsto zz^T$ for $z \in \mathbb{C}^m$ with $\Re(a^* z) > 0$. To check bijectivity, write $Z = \sigma uv^*$ for singular vectors $u, v \in \mathbb{C}^m$ and the nonzero singular value σ. Since u and v lie in the one-dimensional kernels of $ZZ^* - \sigma^2 \mathbb{I}_m = Z\bar{Z} - \sigma^2 \mathbb{I}_m$ and $Z^*Z - \sigma^2\mathbb{I}_m = \bar{Z}Z - \sigma^2 \mathbb{I}_m$, respectively, we infer by Cramer's rule that $\lambda u = \bar{v}$ for some $\lambda \in \mathbb{S}^1$. Letting $z := \sigma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \bar{v} = \sigma^{\frac{1}{2}} \lambda^{\frac{1}{2}} u$, we find that $Z = zz^T$. The only indeterminacy here - the choice of sign for the root $\lambda^{\frac{1}{2}}$ - is fixed by requiring $\Re(z^* a) > 0$.

The real hypersurface $S \subset \Sigma$ of rank one matrices with norm 1 is strongly pseudoconvex. Indeed, as $\|zz^T\|_{2\to 2} = \|z\|^2$, we have that $z \in \mathbb{S}^{2m-1}$ iff $zz^T \in S$, and the map $z \mapsto zz^T$ provides a holomorphic double cover of S by $\mathbb{S}^{2m-1}$, showing that $S \cong \mathbb{R}P^{2m-1}$.

The complex affine planes $\eta_{ww^T} := \{ww^T + B : B\bar{w} = 0, B^T = B\}$ for $w \in \mathbb{S}^{2m-1}$ provide a foliation of M_{III}^m near aa^T. As in the proof of Lemma 10, the singular value decomposition expresses $Z \in M_{III}^m$ as $uv^* + B$, where u, v are unit vectors (unique up to simultaneous multiplication by $\lambda \in \mathbb{S}^1$), and B satisfies $B^*u = Bv = 0$ and $\|B\| < 1$. Since as before, u and v lie in the one-dimensional kernels of $Z\bar{Z} - \mathbb{I}_m$ and $\bar{Z}Z - \mathbb{I}_m$, respectively, we may express $Z = ww^T + B$ for $w \in \mathbb{S}^{2m-1}$, implying $B^T = B$ by linearity. The condition $Bv = B^*u = 0$ simplifies to $B\bar{w} = 0$. In coordinates where $w = (1, 0, \ldots, 0)^T \in \mathbb{C}^m$, $B\bar{w} = 0$ just means that the first column is empty, a condition that is clearly linearly independent of $B^T = B$. Therefore, the space defined by $B^T = B$, $B\bar{w} = 0$ is a complex vector space of $\frac{m(m-1)}{2}$ dimensions for w near a.

Given $B_0 \in T_{aa^T}\eta$, we prove that $B(Z) = (\mathbb{I}_m - ZZ^*)B_0(\mathbb{I}_m - Z^*Z)$ provides a section of $T\eta$ along S. For $ww^T \in S$,

$$\begin{aligned}
B(ww^T)\bar{w} &= (\mathbb{I}_m - ww^*)B_0(\mathbb{I}_m - \bar{w}w^T)\bar{w} = (\mathbb{I}_m - ww^*)B_0(\bar{w} - \bar{w}) = 0, \\
B(ww^T)^T &= (\mathbb{I}_m - (\bar{w}w^T)^T)B_0^T(\mathbb{I}_m - (ww^*)^T) = B(ww^T) \text{ and} \\
B(aa^T) &= (\mathbb{I}_m - aa^*)B_0(\mathbb{I}_m - \bar{a}a^T) = B_0.
\end{aligned}$$

Consider a CR vector $\bar{L}|_{aa^T} \in T^{0,1}_{aa^T}S$. Complex tangent vectors $\alpha \in T^c_a\mathbb{S}^{2m-1}$ are characterized by $\alpha^* a = 0$. Since $z \mapsto zz^T$ is holomorphic and onto, we can just plug a suitable complex tangent $t \mapsto a + t\alpha$ into this map to obtain a curve $\gamma(t) = (a+t\alpha)(a+t\alpha)^T$ such that $\bar{L}|_{aa^T} = \gamma_* \frac{d}{dt}|_{t=0}$. Then, after rewriting $\gamma(t) = (a+t\alpha)(\bar{a}+\bar{t}\bar{\alpha})^*$, we obtain by the exact same calculation as in Lemma 10 that $R^S_{aa^T}(\bar{L}|_{aa^T}, B_0) = -a\alpha^* B_0 - B_0\bar{\alpha}a^T$. By multiplying from the right with $\bar{a}$, we find that $\bar{L}|_{aa^T} \in \ker R^S_{aa^T}(\cdot, B_0)$ if and only if $B_0\bar{\alpha} = 0$. Since $\bar{a} \in \ker B_0$, the codimension of the kernel of B_0 in $\langle \bar{a} \rangle^\perp$ equals the codimension of the full kernel of B_0, hence $\dim_{\mathbb{C}} \ker R^S_{aa^T}(\cdot, B_0) = m - 1 - \operatorname{codim}_{\mathbb{C}} \ker B_0 \leq m - 2$, implying $\nu_{aa^T} = m - 2$. $\square$

Here a counterexample for regularity of CR-transversal maps from source manifolds with less than $m-1$ positive Levi eigenvalues may be constructed in the exact same fashion as in the case of $M^{m,n}_I$. Let us instead consider a slightly different example map into M^m_{III}. It is unclear to the author whether it is biholomorphically equivalent to Example 2.2 given in [15].

Example 10. *Let ϕ be a nowhere smooth CR function of regularity C^k on $\mathbb{S}^{2m-3}$ strictly bounded by 1. Then the map $h : \mathbb{S}^{2m-3} \to M^m_{III}$ given by*

$$h(z) = \frac{1}{2}(z_1, \ldots, z_{m-1}, 1)^T(z_1, \ldots, z_{m-1}, 1) + \frac{\phi(z)}{2}(z_1, \ldots, z_{m-1}, -1)^T(z_1, \ldots, z_{m-1}, -1)$$

is a nowhere smooth CR-transversal CR embedding of regularity C^k.

Proof. We first consider the map $H : \mathbb{C}^{m-1}_z \times \mathbb{C}_w$ given by

$$H(z, w) = \frac{1}{2}(z_1, \ldots, z_{m-1}, 1)^T(z_1, \ldots, z_{m-1}, 1) + \frac{w}{2}(z_1, \ldots, z_{m-1}, -1)^T(z_1, \ldots, z_{m-1}, -1).$$

It is a holomorphic immersion on $\mathbb{C}^{m-1} \times \mathbb{B}^1 \subset \mathbb{C}^{m-1} \times \mathbb{C}$, since

$$\frac{\partial}{\partial z_j} H(z,w) = \frac{1+w}{2} \left(e_j \big(z_1, \ldots, z_{m-1}, \frac{1-w}{1+w}\big) + \big(z_1, \ldots, z_{m-1}, \frac{1-w}{1+w}\big) e_j^T \right),$$
$$\frac{\partial}{\partial w} H(z,w) = \frac{1}{2} (z_1, \ldots, z_{m-1}, -1)^T (z_1, \ldots, z_{m-1}, -1),$$

where e_j denotes the j^{th} standard unit vector. The matrices $\frac{\partial}{\partial z_j} H(z,w)$ for $1 \le j \le m-1$ are linearly independent, since their last columns - given by $\frac{1-w}{2} e_j$ - are. Observing that $\frac{\partial}{\partial z_j} H(z,w)_{m,m} = 0$, but $\frac{\partial}{\partial w} H(z,w)_{m,m} = \frac{1}{2} \neq 0$, we conclude that all partial derivatives of H are linearly independent, hence H is immersive. From the adapted singular value decomposition used in the proof of Lemma 12, we see that H maps $\mathbb{S}^{2m-3} \times \mathbb{B}^1$ injectively into M_{III}^m. Considering the graph map $\Phi : \mathbb{S}^{2m-3} \to \mathbb{S}^{2m-3} \times \mathbb{C}$, $\Phi(z) = (z, \phi(z))$, which clearly is a C^k, but nowhere C^∞-smooth CR embedding of $\mathbb{S}^{2m-3}$, we may write $h = H \circ \Phi$, showing that h is a C^k, but nowhere C^∞-smooth CR immersion of $\mathbb{S}^{2m-3}$ into M_{III}^m. Note that it is CR-transversal, since H is transversal to M_{III}^m, and Φ was CR-transversal. Since h is an injective immersion of the compact sphere, it is an embedding. $\square$

3.2.4 Classical domains of the fourth kind

Somewhat different from the first three series of classical symmetric domains, the models for these domains, denoted by D_{IV}^m for $m \geq 2$, are defined by simple quartic inequalities, first given in [3].

$$D_{IV}^m = \left\{ z \in \mathbb{C}^m : z^* z < 1, 1 + |z^T z|^2 - 2 z^* z > 0 \right\}.$$

The binding inequality is the second one. Indeed, a point $z \in \partial D_{IV}^m$ satisfying $z^* z = 1$ also satisfies $|z^T z| \leq 1$ by Cauchy's inequality, thus $1 + |z^T z|^2 - 2 z^* z \leq 0$. A low-dimensional toy image to have in mind is that of a lens-shaped region defined by $y^2 - \frac{1}{4}\left(1 - x^2\right)^2 < 0$, where we discard the unbounded region by requiring $x^2 + y^2 < 1$. The smooth part of the boundary of D_{IV}^m, which we will denote by M_{IV}^m, is given by those $z \in \mathbb{C}^m$ satisfying $1 + |z^T z|^2 - 2 z^* z = 0$ and $z^* z < 1$.

In fact, D_{IV}^m is biholomorphic to the tube domain over the light cone from Example 7. The tube domain over the future light cone is given by $\{(z_1, \ldots, z_{m-1}, z_m) \in \mathbb{C}^m : \Re(z_1)^2 + \cdots + \Re(z_{m-1})^2 < \Re(z_m)^2,\ \Re(z_m) > 0\}$. An explicit biholomorphism between the tube domain over the future light cone and D^m_{IV} is given in [15] as

$$(z_1, \dots, z_{m-1}, z_m) \mapsto \sqrt{2}i \left(2\frac{z_1}{\mathcal{F}(z+\mathbf{i})}, \dots, 2\frac{z_{m-1}}{\mathcal{F}(z+\mathbf{i})}, \frac{1+\mathcal{F}(z)}{\mathcal{F}(z+\mathbf{i})} \right),$$

where $\mathbf{i}$ denotes the vector $(0, \dots, 0, i) \in \mathbb{C}^m$ and where $\mathcal{F}(z) := z_m^2 - z_1^2 - \dots - z_{m-1}^2$ for any $z \in \mathbb{C}^m$.

Let us nevertheless reprove the regularity result for D^m_{IV} by computing the necessary quantities directly from Cartan's representation. As an example point in M^m_{IV} to base our calculations on, take $a := (\frac{1}{2}, \frac{i}{2}, 0, \dots, 0)^T$. Here, $a^T a = 0$ and $a^* a = \frac{1}{2}$. Contrary to the first three kinds of classical domains, M^m_{IV} will necessarily behave exactly like the tube over the light cone.

Proposition 5. *Let $m \geq 2$ and M be a minimal CR manifold. Then every CR map h of regularity C^{m-1} from M into M^m_{IV} which is of (real) rank ≥ 3 is C^∞-smooth on a dense open subset of M.*

This is an immediate consequence of the boundary orbit theorem for M^m_{IV}, which allows us to take any point in M^m_{IV} to $(\frac{1}{2}, \frac{i}{2}, 0, \dots, 0)$ by an ambient biholomorphism, and of Corollary 1. The relevant structural properties of M^m_{IV} do not differ at all from those of the tube over the light cone (Example 7).

Lemma 13. *Let $a \in \mathbb{C}^m$ be such that $a^T a = 0$ and $a^* a = \frac{1}{2}$. Around $a \in M^m_{IV}$, the pseudoconvex hypersurface M^m_{IV} is foliated by complex lines. Its Levi form has exactly $m-2$ positive eigenvalues, and $\nu_a = 0$.*

Proof. The complex quadric Σ defined by $z^T z = 0$ is a manifold where $z \neq 0$. Its intersection S with M^m_{IV} is given by $S = \{w \in \mathbb{C}^m, w^T w = 0, w^* w = \frac{1}{2}\}$. As it is the intersection of a complex manifold with the strongly pseudoconvex sphere given by $w^* w = \frac{1}{2}$, it is strongly pseudoconvex itself.

Near a point $w \in S$, the complex line given by $\eta_w(t) = w + t\bar{w}$ is contained in S. This is proven by straightforward calculation. Since $w^* w = \frac{1}{2}$ and $w^T w = 0$, we observe

$(w^* + \bar{t}w^T)(\bar{w} + \bar{t}w) = \bar{t}$ and similar cancellations, and arrive at

$$\begin{aligned}
&1 + |\eta_w^T(t)\eta_w(t)|^2 - 2\eta_w(t)^*\eta_w(t) \\
&= 1 + (w + t\bar{w})^*\overline{(w + t\bar{w})}(w + t\bar{w})^T(w + t\bar{w}) - 2(w + t\bar{w})^*(w + t\bar{w}) \\
&= 1 + (w^* + \bar{t}w^T)(\bar{w} + \bar{t}w)(w^T + t\bar{w}^T)(w + t\bar{w}) - 2(w^* + \bar{t}w^T)(w + t\bar{w}) \\
&= 1 + \bar{t}t - (1 + \bar{t}t) = 0.
\end{aligned}$$

It remains to calculate the tensor $R^S(\cdot, \bar{w})$ at $a \in S$, since the section $\bar{w}$ already spans $T\eta$ along S. A vector $v \in T_a^c S$ is characterized by $(a + tv)^T(a + tv) = \mathcal{O}(|t|^2)$ and $(a + tv)^*(a + tv) = \frac{1}{2} + \mathcal{O}(|t|^2)$, which is equivalent to $a^T v = a^* v = 0$. Take a CR vector $\bar{L}|_a \in T_a^{0,1}S$ with real part $\frac{1}{2}v$, and consider the holomorphic curve $\gamma(t) = a + tv$. Then $\bar{L}|_a \bar{w} = \frac{d}{d\bar{t}}|_{t=0}\overline{(a + tv)} = \bar{v}$, and we find that $\bar{v} \in T_a^\perp \eta_a = \langle \bar{a} \rangle^\perp$ already, since $a^* v = 0$. Therefore $R_a^S(\bar{L}|_a, \bar{a}) = \bar{v}$ only vanishes if v and thus $\bar{L}|_a$ vanish, implying $\nu_a = 0$. □

4 Appendix: The boundary orbit theorem

Here we prove the boundary orbit theorem for $D_I^{m,n}$ for the interested reader who wants to convince herself that it is true without having to get acquainted with the Lie algebra formalism used in [14], or indeed most sources on symmetric domains. The basics on the $SU(m, n)$ action on $D_I^{m,n}$, excluding the transitivity proofs, are sourced from Knapp's wonderfully accessible text [6].

Recall that for $1 \le m \le n$ the classical irreducible symmetric domain $D_I^{m,n}$ is given by those $m \times n$ matrices with norm less than one, equivalently, $D_I^{m,n} = \{Z \in \mathbb{C}^{m\times n} : \mathbb{I}_m - ZZ^* > 0\}$. The smooth piece of its boundary is given by those matrices $M_I^{m,n}$ with singular values $1 = \sigma_1 > \sigma_2 \ge \cdots \ge \sigma_m \ge 0$.

The matrix group $SU(m, n)$ is given by

$$SU(m, n) = \left\{ T \in \mathbb{C}^{(m+n)\times(m+n)} : T^* \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} T = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}, \det T = 1 \right\}.$$

Alternatively we can describe $SU(m, n)$ as the set of matrices of determinant one whose columns are orthonormal bases with respect to the sesquilinear form

$$\mathcal{F}_{m,n}(w, z) = \bar{w}_1 z_1 + \cdots + \bar{w}_m z_m - \bar{w}_{m+1} z_{m+1} - \cdots - \bar{w}_{m+n} z_{m+n}.$$

Write $T \in SU(m,n)$ as $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n,n}$, where $A^*A - C^*C = \mathbb{I}_m$, $A^*B - C^*D = 0$ and $B^*B - D^*D = -\mathbb{I}_n$. Then T acts on $D_I^{m,n}$ by the holomorphic map

$$T \cdot Z = (AZ + B)\,(CZ + D)^{-1}\,.$$

To convince ourselves that $CZ + D$ is invertible on $D_I^{m,n}$ and that $T \cdot Z \in D_I^{m,n}$, we first compute

$$\begin{aligned}
&(AZ+B)^*(AZ+B) - (CZ+D)^*(CZ+D) \\
&= \begin{pmatrix} Z \\ \mathbb{I}_n \end{pmatrix}^* \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Z \\ \mathbb{I}_n \end{pmatrix} \\
&= \begin{pmatrix} Z \\ \mathbb{I}_n \end{pmatrix}^* \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix} \begin{pmatrix} Z \\ \mathbb{I}_n \end{pmatrix} = Z^*Z - \mathbb{I}_n \leq 0.
\end{aligned}$$

For any $Z \in \overline{D_I^{m,n}}$, the form $(CZ+D)^*(CZ+D) = \mathbb{I}_n - Z^*Z + (AZ+B)^*(AZ+B)$ is thus positive semidefinite, and $\ker(CZ+D) \subseteq \ker(AZ+B)$. For $v \in \ker(CZ+D)$, we multiply $(CZ+D)v = 0$ and $(AZ+B)v = 0$ from the left by D^* and B^*, respectively, to obtain $0 = (B^*A - D^*C)Zv + (B^*B - D^*D)v = -\mathbb{I}_n v$, hence $(CZ+D)$ is invertible on $\overline{D_I^{m,n}}$. The rational function $T \mapsto T \cdot Z$ is therefore defined on a (T-dependent) open neighborhood of $\overline{D_I^{m,n}}$, which of course contains $M_I^{m,n}$.

To prove that $T \cdot Z \in D_I^{m,n}$ for any $Z \in D_I^{m,n}$, we compute

$$\begin{aligned}
&\mathbb{I}_n - (T\cdot Z)^*(T\cdot Z) = \mathbb{I}_n - \left((CZ+D)^{-1}\right)^* (AZ+B)^*(AZ+B)(CZ+D)^{-1} \\
&= -\left((CZ+D)^{-1}\right)^* \big((AZ+B)^*(AZ+B) - (CZ+D)^*(CZ+D)\big)(CZ+D)^{-1} \\
&= \left((CZ+D)^{-1}\right)^* (\mathbb{I}_n - Z^*Z)(CZ+D)^{-1} \geq 0.
\end{aligned}$$

The quadratic forms $\mathbb{I}_n - (T\cdot Z)^*(T\cdot Z)$ and $\mathbb{I}_n - Z^*Z$ are thus conjugate, implying that they have the same number of positive and zero eigenvalues, respectively. Therefore, $T\cdot Z$ and Z have the same number of eigenvalues less than as well as equal to 1, implying that $T \cdot Z \in D_I^{m,n}$ iff $Z \in D_I^{m,n}$ and similarly, that $T \cdot Z \in M_I^{m,n}$ iff $Z \in M_I^{m,n}$. We conclude that $Z \mapsto T \cdot Z$ is a biholomorphic map on $D_I^{m,n}$, extending to an open neighborhood of $\overline{D_I^{m,n}}$. Its holomorphic inverse is simply given by $T^{-1} \cdot Z$.

Lemma 14. *The group $SU(m,n)$ acts transitively on $D_I^{m,n}$ by biholomorphisms of $D_I^{m,n}$, and it acts transitively by ambient biholomorphisms on $M_I^{m,n}$.*

Proof. The singular value decomposition expresses $Z \in \mathbb{C}^{m\times n}$ as $U\Sigma V^*$ with $U \in U(m)$, $V \in U(n)$ and $\Sigma = \mathrm{diag}^{m,n}(\sigma_1, \dots, \sigma_m)$. To get rid of the determinants, let $\lambda = \det(U)\det(V)$ and note that $\lambda \in \mathbb{S}^1$. Now $K := \begin{pmatrix} \lambda^{-\frac{1}{2}}U & 0 \\ 0 & \lambda^{-\frac{1}{2}}V \end{pmatrix} \in SU(m,n)$ and $K \cdot \Sigma = \lambda^{-\frac{1}{2}}U\Sigma(\lambda^{-\frac{1}{2}}V)^{-1} = U\Sigma V^* = Z$. Thus, any matrix Z in $D_I^{m,n}$ or in $M_I^{m,n}$ may be mapped by an (ambient) biholomorphism to its singular value representation Σ.

We now show that $\Sigma \in D_I^{m,n}$ may be mapped to $\mathrm{diag}^{m,n}(0, \dots, 0)$ and $\Sigma \in M_I^{m,n}$ may be mapped to $\mathrm{diag}^{m,n}(1, 0, \dots, 0)$ by a suitable $T \in SU(m,n)$, which will complete the transitivity proof. First, consider $\Sigma \in D_I^{m,n}$ and let $\alpha_j = \mathrm{atanh}(\sigma_j)$, $s_j = \sinh(\alpha_j)$ and $c_j = \cosh(\alpha_j)$ for $j = 1, \dots, m$. It is easy to see that

$$T := \begin{pmatrix} \mathrm{diag}(c_1, \dots, c_m) & \mathrm{diag}(s_1, \dots, s_m) & 0 \\ \mathrm{diag}(s_1, \dots, s_m) & \mathrm{diag}(c_1, \dots, c_m) & 0 \\ 0 & 0 & \mathbb{I}_{n-m} \end{pmatrix}$$

lies in $SU(m,n)$, since its columns are evidently orthonormal with respect to $\mathcal{F}_{m,n}$, and $\det T = (c_1^2 - s_1^2)\dots(c_m^2 - s_m^2) = 1$. We check that $T \cdot \mathrm{diag}^{m,n}(0, \dots, 0) = \mathrm{diag}^{m,n}(s_1, \dots, s_m) \cdot \mathrm{diag}^{n,n}(c_1, \dots, c_m, 1 \dots, 1)^{-1} = \mathrm{diag}^{m,n}(\frac{s_1}{c_1}, \dots, \frac{s_m}{c_m}) = \Sigma$.

For $\Sigma = \mathrm{diag}^{m,n}(1, \sigma_2, \dots, \sigma_m) \in M_I^{m,n}$, we adapt $T \in SU(m,n)$ slightly to

$$T := \begin{pmatrix} \mathrm{diag}(1, c_2 \dots, c_m) & \mathrm{diag}(0, s_2, \dots, s_m) & 0 \\ \mathrm{diag}(0, s_2, \dots, s_m) & \mathrm{diag}(1, c_2, \dots, c_m) & 0 \\ 0 & 0 & \mathbb{I}_{n-m} \end{pmatrix}.$$

Denoting $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as earlier, and $E = \mathrm{diag}^{m,n}(1, 0, \dots, 0)$, we obtain that

$$\begin{aligned} AE + B &= \mathrm{diag}^{m,m}(1, c_2, \dots, c_m)\mathrm{diag}^{m,n}(1, 0, \dots, 0) + \mathrm{diag}^{m,n}(0, s_2, \dots, s_m) \\ &= \mathrm{diag}^{m,n}(1, s_2, \dots, s_m) \\ CE + D &= \mathrm{diag}^{n,m}(0, s_2, \dots, s_m)\mathrm{diag}^{m,n}(1, 0, \dots, 0) + \mathrm{diag}^{n,n}(1, c_2, \dots, c_m, 1, \dots, 1) \\ &= \mathrm{diag}^{n,n}(1, c_2, \dots, c_m, 1, \dots, 1), \end{aligned}$$

and thus $T \cdot \mathrm{diag}^{m,n}(1, 0, \dots, 0) = \mathrm{diag}^{m,n}(1, \sigma_2, \dots, \sigma_m) = \Sigma$. $\square$

5 Appendix: Rough differential geometry

It is a standard fact in differential geometry that preimages of points under submersions foliate the source manifold. More generally, by the *constant rank theorem* the same holds for preimages under maps of constant rank (cf. [9, p. 150]). Since the regularity problem neither allow us to work in the C^∞ setting nor to assume too much on the rank of the maps considered, it seems appropriate to present a self-contained proof of the rather general statement used in the proof of Corollary 1.

Lemma 15. *Let M, N be (second countable) C^l-manifolds and let $\Phi : M \to N$ be a C^l-smooth map. Then there exists a (countable) family of open sets $(O_j)_{j\in J}$ such that*

1. *the set $O := \bigcup_{j\in J} O_j$ is a dense open subset of M,*
2. *for any j, $\Phi(O_j) \subseteq N$ is a C^l-submanifold of N,*
3. *the restricted map $\Phi|_{O_j} : O_j \to \Phi(O_j)$ is a C^l-submersion,*
4. *the preimages $\eta_q := O_j \cap \Phi^{-1}(\Phi(q))$, $q \in O_j$ provide a C^l-foliation of O_j and*
5. *for each $q \in O_j$, $\ker D\Phi|_q = T_q\eta_q$.*

Proof. Denote $n = \dim N$, $m = \dim M$ and let $M_k = \{p \in M : \mathrm{rk}(D\Phi) \leq k\}$ for $0 \leq k \leq m$. In coordinates, $\mathrm{rk}(D\Phi) = k$ is equivalent to the existence of a non-vanishing $k \times k$ minor of $D\Phi$, which is an open condition, hence M_k is open. By construction, $\mathrm{rk}(D\Phi)$ is constant on each set $\tilde{M}_k := (M_k \setminus M_{k+1})^o$, $0 \leq k \leq m-1$ and on $\tilde{M}_m := M_m$. Furthermore, the sets $\tilde{M}_k$ together are dense in M by the following argument: For any given open set $O \subseteq M$, take the largest k such that $M_k \cap O \neq \varnothing$. Then $(M_k \setminus M_{k+1}) \cap O$ is a nonempty open set, hence $\tilde{M}_k \cap O \neq \varnothing$.

Consider $p \in \tilde{M}_k$. That $D\Phi$ is of rank k at p implies $\dim \Phi^* T^*_{\Phi(p)} N = k$. On a small neighborhood of $\Phi(p)$, choose coordinates $y_1, \ldots, y_k, \ldots, y_n$ such that the pullback forms $\Phi^* dy_1|_p, \ldots, \Phi^* dy_k|_p$ span $\Phi^* T^*_{\Phi(p)} N = k$. Since linear independence is an open condition, and $\mathrm{rk}(D\Phi) = k$ on a neighborhood of p, we can restrict to a smaller neighborhood U of $\Phi(p)$ where $\Phi^* dy_1, \ldots, \Phi^* dy_k$ span $\Phi^* T^* U$. Since $\Phi^* dy_1 = d(y_1 \circ \Phi)$, the functions $x_j := y_j \circ \Phi$ for $1 \leq j \leq k$ provide C^l-coordinates near p. We extend to a set of coordinates $x_1, \ldots, x_k, \ldots, x_m$ and observe that $dy^a(D\Phi \cdot \frac{\partial}{\partial x_b}) = (\Phi^* dy^a)(\frac{\partial}{\partial x_b}) = 0$ for

$1 \leq a \leq m$ and $k < b \leq m$, since $\Phi^* dy^a$ is a linear combination of $dx_1, \dots, dx_k$. Therefore Φ is constant on coordinate manifolds $(x_1, \dots, x_k, \cdot, \dots, \cdot)$. At the same time, Φ is immersive on the slice $(\cdot, \dots, \cdot, x_{k+1}, \dots, x_n)$, since $dy_a(D\Phi \cdot \frac{\partial}{\partial x_b}) = dx_a(\frac{\partial}{\partial x_b}) = \delta_{ab}$ for $1 \leq a, b \leq k$, which also shows that $\ker D\Phi|_q = \left\langle \frac{\partial}{\partial x_{k+1}}|_q, \dots, \frac{\partial}{\partial x_m}|_q \right\rangle$.

We conclude that, when restricted to a small enough neighborhood O_p of p, $\Phi(O_p)$ is the embedded image of $(\cdot, \dots, \cdot, x_{k+1}, \dots, x_n) \cap O_p$, and therefore a C^l-manifold of dimension k. The map Φ is thus a submersion into the C^l-submanifold $\Phi(O_p)$. Furthermore, the preimages of points $\Phi(q) \in \Phi(O_p)$ are precisely the coordinate manifolds $\eta_q := (x_1(q), \dots, x_k(q), \cdot, \dots, \cdot)$, which of course foliate O_p, and $\ker D\Phi|_q = T_q \eta_q$. If M is second countable, we may choose O_p from the countable topological basis to obtain a countable family of open sets $(O_j)_{j=1}^{\infty}$ as claimed. □

Since the leaves of a foliation locally can be equipped with a manifold structure, we can use Lemma 15 to study preimages of leaves of a foliation. It turns out that such preimages of leaves under a C^l-map *generically* foliate the domain as well.

Corollary 3. *Let M, N be (second countable) C^l-manifolds and let $\Phi : M \to N$ be a C^l-map. Suppose that N is foliated by submanifolds of dimension $k \in \mathbb{N}$, and denote by η_p the leaf through p. Then there exists a (countable) family of open sets $(O_j)_{j \in J}$, which together make up a dense open subset of M, such that the preimages $\Phi^{-1}(\eta_q) \cap O_j$, $q \in \Phi(O_j)$ are C^l-submanifolds of constant dimension (depending only on j), which foliate O_j. Furthermore, we have that $(D\Phi)^{-1}(T_{\Phi(q)}\eta_{\Phi(q)}) = T_q(\Phi^{-1}(\eta_{\Phi(q)}))$ for each $q \in O_j$.*

Proof. At each point $p \in N$ we can find coordinates $y_1, \dots, y_n$ on a neighborhood U of p such that the leaves of the foliation are given by the coordinate submanifolds $\{\cdot, \dots, \cdot, y_{k+1}, \dots, y_n\}$. On U, the foliation $\eta|_U$ can thus be equipped with a manifold structure itself, with coordinates given by $y_{k+1}, \dots, y_n$. The projection $\pi : U \to \eta|_U$, $\pi(p) = \eta_p$ is a map of the same regularity as η, hence we may apply Lemma 15 to the map $\pi \circ \Phi : \Phi^{-1}(U) \to \eta|_U$ to obtain a family of open sets $(O_j)_{j \in J}$ which together lie dense in $\Phi^{-1}(U)$, such that the preimages $\Phi^{-1} \circ \pi^{-1}(\eta_{\Phi(q)}) \cap O_j = \Phi^{-1}(\eta_{\Phi(q)}) \cap O_j$ for $q \in O_j$ foliate O_j. Finally, we obtain $(D\Phi)^{-1}(T_{\Phi(q)}\eta_{\Phi(q)}) = \ker D\pi|_{\Phi(q)} \circ D\Phi|_q = \ker D(\pi \circ \Phi)|_q = T_q(\Phi^{-1}(\eta_{\Phi(q)}))$ from part 5 of Lemma 15. □

References

[1] M. Salah Baouendi, Peter Ebenfelt, and Linda Preiss-Rothschild. *Real submanifolds in complex space and their mappings*. Princeton Math. Series 47. Princeton University Press, 1999.

[2] Shiferaw Berhanu and Ming Xiao. On the regularity of CR mappings between CR manifolds of hypersurface type. *Transactions of the American Mathematical Society*, 369, 11 2014.

[3] Elie Cartan. Sur les domaines bornés homogènes de l'espace de n variables complexes. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 11(1):116–162, 1935.

[4] A.S. Fedenko (originator). Hermitian symmetric space. *Encyclopedia of Mathematics*, URL: `http://encyclopediaofmath.org/index.php?title=Hermitian_symmetric_space&oldid=33416`. [Online; accessed 31-May-2020].

[5] Franc Forstnerič. Extending proper holomorphic mappings of positive codimension. *Inventiones Mathematicae*, 95:31–62, 1989.

[6] Anthony Knapp. *Bounded symmetric domains and holomorphic discrete series*. Ed. Boothby & Weiss, Marcel Dekker Inc., 1972.

[7] Bernhard Lamel and Nordine Mir. Convergence of formal CR mappings into strongly pseudoconvex Cauchy–Riemann manifolds. *Inventiones Mathematicae*, 210(3):963–985, 11 2017.

[8] Bernhard Lamel and Nordine Mir. On the $\mathbb{C}^\infty$ regularity of CR mappings of positive codimension. *Advances in Mathematics*, 335:696–734, 9 2018.

[9] John M. Lee. *Introduction to smooth manifolds*. Graduate Texts in Mathematics. Springer, New York, 2003.

[10] Ngaiming Mok and Sui Chung Ng. Germs of measure-preserving holomorphic maps from bounded symmetric domains to their cartesian products. *Journal für die reine und angewandte Mathematik*, 2012(669):47 – 73, 2012.

[11] R.M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Graduate Texts in Mathematics. Springer New York, 2013.

[12] E.M. Stein and R. Shakarchi. *Functional Analysis: Introduction to Further Topics in Analysis*. EBL-Schweitzer. Princeton University Press, 2011.

[13] E.B. Vinberg (originator). Symmetric domain. *Encyclopedia of Mathematics*, URL: `http://encyclopediaofmath.org/index.php?title=Symmetric_domain&oldid=16315`. [Online; accessed 27-May-2020].

[14] J. A. Wolf. *Fine structure of hermitian symmetric spaces*. Symmetric Spaces: Short Courses Presented at Washington University. Ed. Boothby & Weiss, Marcel Dekker Inc., 1972.

[15] Ming Xiao. Regularity of mappings into classical domains. *Submitted*, 2019.