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"Generalisations of Small Cancellation: The `RSym` Algorithm on Hyperbolic One-Relator Groups"

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Abstract

This thesis considers several conditions on group presentations which yield hyperbolicity of a group. A classic example from small cancellation theory is the following: A group presentation $\langle \mathcal{S} | \mathcal{R} \rangle$ satisfies the $C'(1/6)$ condition if for any subword of a relator R that occurs in two different elements of the set of cyclically reduced conjugates of elements of $\mathcal{R} \cup \mathcal{R}^{-1}$, the length is strictly smaller than $1/6\,|R|$. Since the development of classical small cancellation theory in the 1960's, there have been published many generalisations and variations of those conditions from which we present two recent ones. First, we consider a condition for one-relator groups by Blufstein and Minian and then an algorithmic approach called `RSym` by Holt et al.

A conjecture by Louder and Wilton says that one-relator groups with a relator of primitivity rank 3 are hyperbolic. The primitivity rank of a word w in the alphabet $\mathcal{S}$ is the smallest rank that a subgroup H of the free group $F(\mathcal{S})$ can have such that w is a non-primitive element of H.

We use the `RSym` algorithm to analyse this conjecture for small examples, i.e. groups with relators of length up to 15. Furthermore, we examine if the `RSym` algorithm succeeds on groups which are hyperbolic by Blufstein and Minian.

Zusammenfassung

Diese Masterarbeit behandelt einige Voraussetzungen an Gruppenpräsentationen, aus denen folgt, dass eine Gruppe hyperbolisch ist. Ein klassisches Beispiel aus der Small Cancellation Theorie is das Folgende: Eine Gruppe mit Präsentation $\langle \mathcal{S} | \mathcal{R} \rangle$ erfüllt die $C'(1/6)$ Voraussetzung, wenn für jedes Unterwort eines Relators R, das in zwei verschiedenen Elementen der Menge aller zyklisch reduzierten konjugierten Elemente von $\mathcal{R} \cup \mathcal{R}^{-1}$ vorkommt, die Länge echt kleiner ist als $1/6\,|R|$. Seit der Entwicklung der klassischen Small Cancellation Theorie in den 1960ern wurden viele Verallgemeinerungen und Variationen dieser Voraussetzungen veröffentlicht von denen wir zwei kürzlich erschienene vorstellen werden. Zuerst betrachten wir eine Voraussetzung für Gruppen mit genau einem Relator von Blufstein und Minian und dann den sogenannten `RSym` Algorithmus von Holt et al.

Eine Vermutung von Louder und Wilton besagt, dass Gruppen mit genau einem Relator, wobei der Relator einen Primitivrang von 3 hat, hyperbolisch sind. Der Primitivrang eines Wortes w im Alphabet $\mathcal{S}$ ist der kleinste Rang den eine Untergruppe der freien Gruppe $F(\mathcal{S})$ haben kann, sodass w ein nichtprimitives Element von H ist.

Wir nutzen den `RSym` Algorithmus um diese Vermutung für kleine Beispiele, also Relatoren bis zu Länge 15, zu analysieren. Weiterhin untersuchen wir, ob der `RSym` Algorithmus für Gruppen erfolgreich ist, die nach Blufstein und Minian hyperbolisch sind.

Contents

1 Introduction

This thesis analyses different conditions for hyperbolicity of a group with the main focus on the algorithmical approach of the `RSym` procedure, introduced by Holt et al. in [HLN+19].

We begin in Chapter 2 by reproducing preliminiary definitions and results about van Kampen diagrams, hyperbolic groups and the word problem.

Chapter 3 is devoted to small cancellation theory. The main focus is on the classical small cancellation conditions as they are defined in [LS77, Chapter V], but we also present a recently published condition by Blufstein and Minian in [BM19]. This condition is a weakening of the classical $C'(1/4) - T(4)$ condition for one-relator groups.

In Chapter 4 we present a new type of group presentation introduced in [HLN+19] called *pregroup presentation*, so called *coloured van Kampen diagrams* over those presentations and the `RSym`, `RSym`$^+$ and `VerifySolver` algorithms along with some important results of the paper. Pregroup presentations give the advantage that we can ignore the failing of small cancellation conditions on short relators. We view the group G presented by a pregroup presentation $\mathcal{P}$ as a quotient of a universal group $U(P)$. This universal group is generated by the elements of the pregroup and its relators are the relators of G of length 3. Then $G \cong U(P)/\langle\langle \hat{\mathcal{R}} \rangle\rangle$, where $\langle\langle \hat{\mathcal{R}} \rangle\rangle$ denotes the normal closure of the relators of length greater than 3. Rimlinger proved in [Rim87] that a finitely generated group G is virtually free if and only if G is the universal group $U(P)$ of a finite pregroup P. Therefore, any group that is a quotient of a virtually free group by finitely many additional relators has a finite pregroup presentation. In coloured van Kampen diagrams over those pregroup presentations, a face that is labelled by a relator of length 3, i.e. by a relator of $U(P)$ is coloured red and a face that is labelled by a relator of longer length is coloured green. The ***RSym* *scheme*** is a curvature distribution scheme over coloured van Kampen diagrams with certain properties. If the curvature of any non-boundary green face of a diagram can be bounded above by a negative number $-\varepsilon$, then `RSym` is said to succeed on the diagram. If `RSym` succeeds on every diagram over a pregroup presentation $\mathcal{P}$, then the group presented by $\mathcal{P}$ is hyperbolic.

A slight modification of `RSym` is the `RSym`$^+$ algorithm, where faces with negative curvature can transfer some of it to neighbouring faces. Here, it is often useful to increase the dual distance to the boundary of the faces for which the curvature needs to be bounded as above. One says that `RSym`$^+$ (resp. `RSym`) *succeeds* for a diagram *at level* d if it succeeds for all internal faces that are of dual distance at least $d + 1$ from the external face. If `RSym`$^+$ (resp. `RSym`) succeeds at level d and another condition regarding the relators of length 3 is satisfied, then the group presented by $\mathcal{P}$ is hyperbolic.

Our first result is the following lemma which proves to be very helpful, when working with one-relator groups for example.

Lemma. *Let G be a group with a pregroup presentation $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ where $V_P = \emptyset$ and assume that **RSym** fails on exactly one decomposition of a relator $R \in \hat{\mathcal{R}}$ with total curvature 0. If, in addition, there is one step which appears only once in this decomposition, then **RSym**$^+$ succeeds at level 2.*

Additionally, `RSym` can be used to efficiently solve the word problem of a group if its pregroup presentation satisfies a certain condition. The `VerifySolver` algorithm, which is also introduced in [HLN+19], checks if a given pregroup presentation satisfies this condition.

Holt et al. have implemented functions to the computer algebra systems GAP and MAGMA that try to verify that `RSym` succeeds on a given pregroup presentation at level 1. We make use of those in the last chapter.

In Chapter 5 we analyse in which cases `RSym` and `VerifySolver` succeed on groups that satisfy other conditions for hyperbolicity. In [HLN+19] Holt et al. show that `RSym` generalises the classical small cancellation conditions $C'(1/6)$ and $C'(1/4) - T(4)$ and that `VerifySolver` also succeeds on presentations that satisfy those conditions. We present the proof and afterwards focus on one-relator groups.

First, we consider relators of primitivity rank 3. The primitivity rank of an element w of a free group F denotes the smallest rank that a subgroup K of F can have such that w is non-primitive in K. There is a conjecture by Louder and Wilton that one-relator groups with relators of primitivity rank at least 3 are hyperbolic (see [LW18]).

We analyse if `RSym` can be used to prove the conjecture for small relators and if an implementation of `RSym`$^+$ could give additional insight. To this end, we prove the following result using the GAP functions:

Theorem. *Let $G = \langle \mathcal{S} | R \rangle$ be a one-relator group with R of primitivity rank 3. Then there exists a pregroup presentation $\mathcal{P}$ of G such that:*

(i) If R is of length less or equal to 10, then `RSym` succeeds on $\mathcal{P}$.

(ii) If R is of length less or equal to 12, then `RSym`$^+$ succeeds on $\mathcal{P}$ at level 2.

Note that hyperbolicity of one-relator groups with relators of primitivity rank 3 and of length up to 11 also follows from [IS98]. In this paper Ivanov and Schupp consider relators that contain a letter a for which either the sum of occurrences of a and a^{-1} is not greater than 3 or the sum of occurrences of a is greater or equal to 4 and the subwords in between are pairwise different. This includes all relators of length up to 11 and primitivity rank 3. Then they describe exactly which form such a relator can take such that the corresponding one-relator group is not hyperbolic and prove that all other groups are hyperbolic. It can be shown that all relators with such a letter a that are *not* hyperbolic by Ivanov and Schupp have primitivity rank 2 [CH20]. However, there are words of length 12 and primitivity rank 3 for which there exists no such a, so our theorem also yields hyperbolicity for groups that are not already covered by the theorems of Ivanov and Schupp. We refer the reader to a related paper [OS19] by Olshanskii and Sapir for a strengthening of the results of Ivanov and Schupp.

Finally, we consider one-relator groups that satisfy the Blufstein-Minian condition (BM-condition). We prove the following theorem:

Theorem. *For one-relator groups with relators of length up to 12 that satisfy the BM-condition, there exists a pregroup presentation on which `RSym`$^+$ succeeds at level 2.*

Furthermore, using the GAP functions, we prove the following:

Theorem. *Let $G = \langle \mathcal{S} | R \rangle$ be a one-relator group of rank 3 that satisfies the BM-condition. Then there exists a pregroup presentation $\mathcal{P}$ of G such that:*

(i) If R is of length less or equal to 10 or of length 12, then `RSym` *succeeds on $\mathcal{P}$.*

(ii) If R is of length less or equal to 14 and contains all 3 letters of $\mathcal{S}$, then `RSym` *succeeds on $\mathcal{P}$.*

(iii) If R is of length 14, then `RSym`$^+$ *succeds on $\mathcal{P}$ at level 2.*

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2 Preliminaries

We begin by recalling some basic terminology of group theory. Let G be a group with presentation $\mathcal{Q} := \langle \mathcal{S} | \mathcal{R} \rangle$, where $\mathcal{S}$ is a set of elements called *generators* or *letters*. A *word* is a written product of those generators and their inverses. When we refer to the elements of $\mathcal{S}$ as letters, we sometimes call $\mathcal{S}$ the *alphabet*. Consider a word $w = x_1 x_2 \cdots x_{|w|}$, where $x_i \in \mathcal{S} \cup \mathcal{S}^{-1}$ for each $i \in \{1, \ldots, |w|\}$. Then a *subword* of w is a word of the form $x_i \cdots x_j$, where $1 \leq i \leq j \leq |w|$. The set $\mathcal{R}$ consists of words called *relators*. If a word w is an element of $\mathcal{R}$, then $w = 1$ in G. Even though a group can have many different presentations, we sometimes write $G = \langle \mathcal{S} | \mathcal{R} \rangle$.

Definition 2.1. We say that a word w in an alphabet $\mathcal{S}$ is *reduced* if no subword is of the form xx^{-1} for all $x \in \mathcal{S} \cup \mathcal{S}^{-1}$. We say that w is *cyclically reduced* if it is reduced and the first letter is not the inverse of the last letter of w.

Let $G = \langle \mathcal{S} | \mathcal{R} \rangle$, where $\mathcal{R}$ is a set of cyclically reduced relators of G. Let $F(\mathcal{S})$ be the free group on $\mathcal{S}$. Then the group G is defined by the quotient $F(\mathcal{S})/\langle\langle \mathcal{R} \rangle\rangle$, where $\langle\langle \mathcal{R} \rangle\rangle$ is the normal closure of $\mathcal{R}$ in $F(\mathcal{S})$. We have that an element $w \in \langle\langle \mathcal{R} \rangle\rangle$ if and only if in the free group $F(\mathcal{S})$, w is a product of conjugates of elements of $\mathcal{R} \cup \mathcal{R}^{-1}$. The elements of $\langle\langle \mathcal{R} \rangle\rangle$ are called *relations*.

A *van Kampen diagram* (see Section 2.1) is a diagram in the Euclidean plane, which contains all the essential information about such a product. Van Kampen diagrams play a crucial role in small cancellation theory and we work with them throughout this thesis. They give rise to a combinatorial definition of *hyperbolic groups*, which we present in Section 2.2. A famous problem posed by Dehn in 1911 that is still very relevant today is the *word problem* that asks if a given word is trivial in G. In Section 2.3 we present a proof that this problem is solvable for hyperbolic groups.

2.1 Van Kampen Diagrams

Van Kampen diagrams were introduced 1933 by van Kampen in [vK33] in order to examine which words represent the identity element in a finitely presented group. According to [LS77, Chapter V, p. 236] they did not get a lot of attention for thirty years until in 1966, Lyndon independently arrived at the idea of cancellation diagrams and Weinbaum rediscovered van Kampen's paper and its use for small cancellation theory at the same time.

Let $G = \langle \mathcal{S} | \mathcal{R} \rangle$ be as above.

Definition 2.2. A *van Kampen diagram* D for the word w over the presentation $\langle \mathcal{S} | \mathcal{R} \rangle$ is a finite, connected, oriented, labelled planar graph with a fixed embedding into the plane. Furthermore, each edge is labelled by an element of $\mathcal{S} \cup \mathcal{S}^{-1}$, such that:

(i) The boundary of D (denoted by ∂D) is labelled by w;

(ii) The boundary of any bounded region of $\mathbb{R}^2 \setminus D$, i.e. any bounded complementary component, is labelled by a cyclically reduced element of $\mathcal{R}$.

The boundary words are obtained by reading the labels on the edges, where an element on an edge is given a ± 1 exponent, depending on the orientation of the edge. We say that ∂D is labelled by w if there exists a vertex on ∂D, starting from which one can read w or w^{-1} depending on the direction.

We call the bounded regions of $\mathbb{R}^2 \setminus D$ *faces*. A non-trivial path of maximal length that is common to two adjacent faces of D is called a *consolidated edge*.

The following result is one of the central tools in geometric small cancellation theory. A proof can be found in [LS77, Chapter V.1].

Theorem 2.3. *Let G be a group given by the presentation $\langle \mathcal{S} | \mathcal{R} \rangle$. An element $w \in F(\mathcal{S})$ is in $\langle\langle \mathcal{R} \rangle\rangle$, i.e. satisfies $w = 1$ in G, if and only if there exists a van Kampen diagram with boundary label w.*

2.2 Definition of Hyperbolic Groups

Hyperbolicity of a metric space is a large-scale geometric property that was introduced by Gromov in [Gro83]. A group G is called *hyperbolic* if one (and, hence, any) of its Cayley graphs with respect to finite generating sets equipped with the word metric is a hyperbolic space. There are multiple ways to formally state the definition of a hyperbolic group. In this thesis, we work with the combinatorial definition using van Kampen diagrams.

Definition 2.4. A van Kampen diagram over a presentation $\langle \mathcal{S} | \mathcal{R} \rangle$ is called *reduced* if no two adjacent faces are labelled by $w_1 w_2$ and $w_2^{-1} w_1^{-1}$ with a common consolidated edge labelled by w_1 and w_1^{-1} for some subwords $w_1, w_2 \in F(\mathcal{S})$.

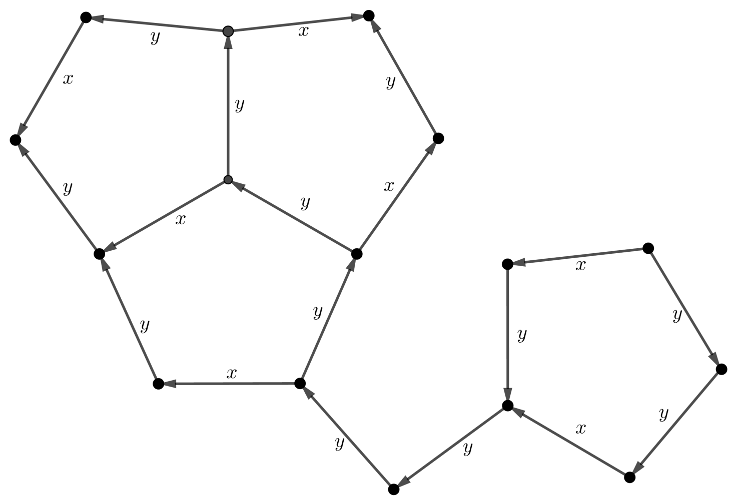

Figure 1: A reduced van Kampen diagram for $w = yxy^2x^{-1}y^{-1}xy^{-1}x^{-1}y^{-4}x^{-1}y^2xy$ over the presentation $\langle x, y | xyx^{-1}y^{-2} \rangle$.

Definition 2.5. Given a group G with presentation $\langle \mathcal{S} | \mathcal{R} \rangle$. Let $w \in F(\mathcal{S})$ such that $w = 1$ in G. Then we denote by $\mathrm{Area}(w)$ the minimal number of faces in a reduced van Kampen diagram with boundary word w.

Definition 2.6. Given a presentation $\langle \mathcal{S} | \mathcal{R} \rangle$, its *Dehn function* is the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = \max\{\mathrm{Area}(w) | w \in F(\mathcal{S}), w = 1 \text{ in } G, |w| = n\}.$$

Note that two distinct presentations of a group G can have two different Dehn functions. There are however properties of those Dehn functions that are invariant, as is indicated by the following lemma, which is a well-known result. A proof can be found for example in [Bri93].

Lemma 2.7. *Let f and g be the Dehn functions of two distinct presentations for the group G. Then there exist constants $a, b, c, d \in \mathbb{N}$ such that*

$$f(n) \leq a \cdot g(bn + c) + d \textit{ for all } n \in \mathbb{N}.$$

Definition 2.8. A finitely presented group G is called *hyperbolic* if for any presentation of G, the Dehn function f is bounded above by a linear function $\hat{f}$, i.e. $f(n) \leq \hat{f}(n)$ for all $n \in \mathbb{N}$.

Remark 2.9. From Lemma 2.7, we conclude that if we find one presentation of G whose Dehn function can be bounded above by a linear function, then this is possible for the Dehn functions of all finite presentations. Therefore, hyperbolicity of a group is not dependent on the choice of presentation.

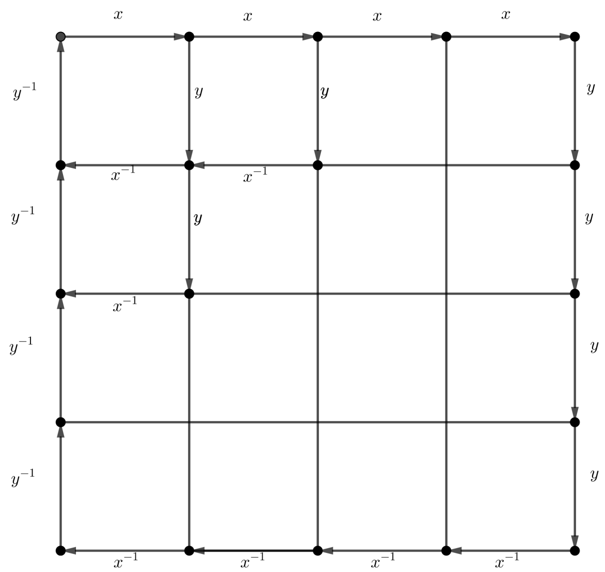

Figure 2: A reduced van Kampen diagram for $[x^4, y^4]$ in $\mathbb{Z}^2$.

Example 2.10. (i) Finitely generated free groups are hyperbolic since in the free presentation the only word $w = 1$ is the trivial word and $\mathrm{Area}(1) = 0$.

(ii) Every finite group is hyperbolic. For any finite group G there exists a finite presentation where the generating set consists of all elements of G and the set of relators consists of all multiplication rules $ab = c$. Consider a word $w = 1$ in G. If $|w| = 3$, then w is a relator of G and, hence, $\mathrm{Area}(w) = 1$. Now let $|w| > 3$, hence $w = abv$, where $a, b, \in G$ and v a word. If $ab = c$ in G for some c, then w is equivalent to $abc^{-1}cv = cv$, where cv is shorter than w. We repeat this process until we arrive at a word of length 3 which is a relator. Hence, we have $\mathrm{Area}(w) \leq |w|$.

(iii) The free abelian group $\mathbb{Z}^2$ is not hyperbolic because $\mathrm{Area}([x^n, y^n]) = n^2$ (see Figure 2).

2.3 The Word Problem

The word problem was posed by Dehn in 1911 and is still one of the central problems in combinatorial group theory.

Definition 2.11. Let $G = \langle \mathcal{S} | \mathcal{R} \rangle$ be a finitely presented group. The *word problem* for G is the problem of deciding algorithmically whether or not a given element $w \in F(\mathcal{S})$ represents the trivial element in G.

In general the word problem is proven to be unsolvable. One could approach a solution by listing all of the finite products of conjugates of elements of $\mathcal{R} \cup \mathcal{R}^{-1}$ since $\mathcal{S}$ and $\mathcal{R}$ are countable. If $w = 1$ in G, this would eventually lead to a word that is equal to w and we can give the definite answer that w represents 1 in G. On the other hand, if $w \neq 1$, this algorithm would continue forever and we would never get a definite answer. However, there are certain groups for which the word problem is solvable and one example of those is hyperbolic groups.

Theorem 2.12. *Hyperbolic groups have solvable word problem.*

Proof. Let G be a hyperbolic group with finite presentation $\langle \mathcal{S} | \mathcal{R} \rangle$ and let w be a word in $F(\mathcal{S})$. Since $\mathcal{R}$ is finite, there are only finitely many van Kampen diagrams of a given area with boundary length $|w|$. Since the area of a word of a given length is bounded above for hyperbolic groups, we can search all of the diagrams with a boundary word of length $|w|$ in finite time and if none of them has a boundary word that is equal to w, we can give the definite answer that w is not trivial in G. $\square$

Note that we do not use the fact that the area of a diagram is bounded by a *linear* function. In fact, the proof is valid for all groups whose Dehn function can be bounded above by an arbitrary computable function. Even though hyperbolic groups are our main interest in this thesis, we state the more general result for completeness.

Theorem 2.13. *Groups whose Dehn function can be bounded above by a computable function have solvable word problem.*

When Dehn first posed the word problem, he provided algorithms that solved it for fundamental groups of closed orientable two-dimensional manifolds (see Example 3.5 for a presentation). In these algorithms he used the fact that, when multiplying two cyclic conjugates of the relator R, there was very little cancellation [LS77, Chapter V]. This idea laid the foundation for the *small cancellation theory*, which we introduce in the next chapter. It furthermore gave rise to *Dehn's algorithm*, which solves the word problem for hyperbolic groups more efficiently than the procedure in the proof of Theorem 2.12 and gives a *Dehn presentation* for those groups.

3 Small Cancellation Theory

Small cancellation theory gives conditions on the relators in a group presentation that lead to strong results about the properties of the group. Amongst those results is the fact that a finitely presented group, which satisfies certain small cancellation conditions, is hyperbolic and that the word problem can be solved by Dehn's algorithm for those groups. The theory was developed by Lyndon, Greendlinger and others in the 1960's. In the first two sections of this chapter we present aspects of classical small cancellation theory. In Section 3.1 we define small cancellation conditions under which a group is known to be hyperbolic. In Section 3.2 we show that under a certain condition, the word problem of a group is solvable. Finally, in Section 3.3 we present the definition of a recently developed condition for one-relator groups by Blufstein and Minian.

3.1 Classical Small Cancellation Conditions

The definitions and results presented in this section are mostly from [LS77, Chapter V], which is the standard work for this topic and to which we refer the reader for a complete presentation of the field. We again work in our standard setting: Let G be a group with presentation $\langle \mathcal{S} | \mathcal{R} \rangle$, where $\mathcal{S}$ is the set of generators and $\mathcal{R}$ a set of cyclically reduced relators. Furthermore, let $\mathcal{S}$ be finite and let $\mathcal{R}$ be finite and *symmetrized*, i.e. closed under inversion and cyclic permutation.

Definition 3.1. Suppose in the set of relators $\mathcal{R}$, there are two elements R_1, R_2 such that $R_1 = bc_1$ and $R_2 = bc_2$, with $b, c_1, c_2 \in F(\mathcal{S})$ subwords. Then b is called a *piece*.

Definition 3.2. We say that a presentation satisfies the $C'(\lambda)$ *condition* if for any relator $R \in \mathcal{R}$, where $R = bc$ with b a piece, we have

$$|b| < \lambda \, |R| \, ,$$

where λ is a positive real number.

We say that a presentation satisfies the $C(p)$ *condition* if no element of $\mathcal{R}$ consists of less than p pieces.

Remark 3.3. $C'(1/p)$ implies $C(p+1)$ for $p \in \mathbb{N}$.

One can slightly alter the definition of a van Kampen diagram given in Section 2.1 to a *consolidated van Kampen diagram*, where the vertices of degree 2 are suppressed and the incident edges to such a vertex are joined to one edge and labelled by the product of the corresponding generators. The interior consolidated edges are then labelled by pieces. In the following we freely pass between van Kampen diagrams and consolidated van Kampen diagrams without further comment.

Definition 3.4. A finite presentation $\mathcal{Q}$ satisfies the $T(q)$ *condition* if in every reduced consolidated van Kampen diagram over $\mathcal{Q}$ all internal vertices have degree at least q.

Example 3.5. Let $G = \langle a_1, b_1, \ldots, a_g, b_g | R := a_1^{-1} b_1^{-1} a_1 b_1 \cdots a_g^{-1} b_g^{-1} a_g b_g \rangle$ be the fundamental group of a closed orientable surface of genus g. The only non-trivial pieces of R are single letters, so for any piece p, we have $|p| = 1 < 1/(4g-1)\,|R|$. Therefore, G satisfies the $C'(1/(4g-1))$ and $C(4g)$ conditions.

Furthermore, G satisfies $T(4)$. For any $i \in \{1, \ldots, g\}$, call the edges labelled by $a_i^{\pm 1}$ *a-edges* and the edges labelled by $b_i^{\pm 1}$ *b-edges*. Since an a-edge is always followed and preceded by a b-edge and a b-edge is always followed and preceded by an a-edge, the degree of an internal vertex in any diagram over the presentation of G is at least 4.

The following theorem is a consequence of Greendlinger's Lemma, a proof of which can be found in [LS77, Chapter V.4].

Theorem 3.6. *If a finite presentation of G satisfies the $C'(1/6)$ or $C'(1/4) - T(4)$ condition, then G is hyperbolic.*

Another famous result is the following theorem:

Theorem 3.7 ([GS90, Corollary 4.1]). *If a finite presentation of G satisfies the $C(p)-T(q)$ condition, with $1/p+1/q<1/2$, then G is hyperbolic.*

From Theorem 3.6 together with Example 3.5 we deduce the following corollary:

Corollary 3.8. *The fundamental group G of a closed orientable surface of genus $g \geq 2$ is hyperbolic.*

Remark 3.9. For genus $g=1$ the fundamental group G is the free abelian group of rank 2 which is not hyperbolic, see Example 2.10 (iii).

Also, note that this example shows that the statement of Theorem 3.7 does not hold for $1/p+1/q=1/2$ in general. See [IS98] and [OS19] for more information on when a $C(p)-T(q)$ diagram with $1/p+1/q=1/2$ is hyperbolic.

3.2 Dehn's Algorithm

As we have seen in the last section, groups with a $C'(1/6)$ presentation are hyperbolic and hence have solvable word problem. In this section we describe an algorithm that solves it efficiently. Dehn's algorithm solves the word problem for any group G that has a presentation where each word that is trivial in G contains more than half of a relator.

Let G be a group with finite presentation $\langle \mathcal{S} | \mathcal{R} \rangle$, where $\mathcal{R}$ is a symmetrized set of cyclically reduced relators. Furthermore, let all freely reduced non-trivial words in $F(\mathcal{S})$ that are trivial in G contain more than half of some element of $\mathcal{R}$.

Algorithm 3.10 (Dehn's Algorithm). Let w be a non-trivial word in $F(\mathcal{S})$.

1. List all relators $R \in \mathcal{R}$ with $|R| < 2\,|w|$.
2. Look for a relator $R = ct$ with $|t| < |c|$ in this list, such that we can factorize $w = bcd$.
3. If no such R exists, the element w is non-trivial in G. Otherwise, we can write $w = bt^{-1}d$ and $bt^{-1}d$ is of shorter length than bcd.
4. Repeat this process until you either get that $w=1$ or that w is non-trivial in G.

Remark 3.11. Note that Step 1 of Dehn's algorithm can be executed in finite time since $\mathcal{R}$ and the set of all words in $F(\mathcal{S})$ shorter than $2\,|w|$ are finite.

The algorithm gives rise to a group presentation of the following form:

Definition 3.12. A *Dehn presentation* of a group G is of the form $\langle \mathcal{S} | c_1t_1, \ldots, c_nt_n \rangle$, where $c_i = t_i^{-1}$ in G, $|t_i| < |c_i|$ for all $i \in \{1, \ldots, n\}$ and any word w which is trivial in G contains at least one of the c_i as a subword.

The following theorem is a well-known fact. A proof can be found for example in [BH99, Chapter III.Γ].

Theorem 3.13. *A group is hyperbolic if and only if it has a Dehn presentation.*

In the rest of the section we present a proof of the fact that the word problem of a group G with a $C'(1/6)$ presentation is solvable by Dehn's algorithm, which yields that G is hyperbolic by Theorem 3.13.

Theorem 3.14. *If a finite presentation $\mathcal{Q}$ satisfies the $C'(1/6)$ condition, then Dehn's algorithm solves the word problem in $\mathcal{Q}$.*

We are following the proof of Theorem 3.14 in [BRS07, III]. First, we need the following lemma:

Lemma 3.15 ([BRS07, III])**.** *Let D be a van Kampen diagram with at least 2 faces, such that the boundary of all internal faces has ≥ 6 edges and all vertices have degree ≥ 3. For $i = 1, \dots, 5$, let b_i be the number of faces whose boundary intersects ∂D in exactly one connected segment having exactly i internal edges, forming a connected part of their internal boundary. Then*

$$3b_1 + 2b_2 + b_3 \geq 6.$$

Proof of Theorem 3.14. Let G be a group with finite presentation $\mathcal{Q} = \langle \mathcal{S} | \mathcal{R} \rangle$ that satisfies the $C'(1/6)$ condition. We need to show that any cyclically reduced word w that is trivial in G contains at least half of an element of $\mathcal{R}$. Let D be a van Kampen diagram for such a w over $\mathcal{Q}$. Then D either consists of a single topological disc, or there are multiple extremal discs which intersect the rest of the diagram in just one vertex (see Figure 1 in Chapter 2.1 for example).

If D consists of a single topological disc, we may assume that the disc consists of more than a single face since otherwise the claim is proven. Therefore, we can apply Lemma 3.15 since $\mathcal{Q}$ satisfies the $C(7)$ condition and we can suppress vertices of degree 2. It follows from the lemma that there are at least 2 faces F_1, F_2 in D whose internal boundary consists of at most 3 pieces. Note that since the presentation satisfies the $C'(1/6)$ condition, for any three pieces a, b, c of a relator $R \in \mathcal{R}$, we have that $|a| + |b| + |c| < 1/2\,|R|$. Therefore, F_1 and F_2 meet the boundary of D in segments of length greater than half of their boundary. We conclude that w contains more than half of the relators labelling the boundaries of F_1 and F_2.

Now consider the case that D contains at least two extremal discs. Again we assume that none of those discs consists of a single face since otherwise the claim is proven. Then, for each disc Lemma 3.15 holds. Fix one of the discs and its faces F_1, F_2 with the properties as above. Since there is only one vertex joining the disc and the diagram, for at least one of the faces F_i, this vertex is not in the interior of the boundary component $\partial D \cap \partial F_i$. Therefore, w contains more than half of the boundary label of F_i. $\square$

Remark 3.16. Note that the $C'(1/5)$ condition would not be sufficient to satisfy the condition of Dehn's algorithm since the sum of the length of three pieces could be equal to half of the relator, as can be seen from the hexagonal tesselation of the euclidean plane.

3.3 The Blufstein-Minian Condition

For one-relator groups, Blufstein and Minian developed a condition which is weaker than $C'(1/6)$ and $C'(1/4) - T(4)$ in [BM19]. It allows a diagram to contain *tripods*, i.e. vertices of degree 3, if they satisfy the *(T') condition.* They prove that this condition ensures hyperbolicity by defining *strictly systolic angled complexes* and showing that groups that act geometrically on such complexes are hyperbolic.

Definition 3.17. A presentation $\mathcal{Q} = \langle \mathcal{S} | R \rangle$ of a one relator-group satisfies the *(T') condition* if the following is true for every reduced van Kampen diagram over $\mathcal{Q}$: For each interior vertex of degree 3, whose incident consolidated edges are labelled by the words w_1,w_2 and w_3, we have

$$|w_1| + |w_2| + |w_3| < \frac{|R|}{2}.$$

Remark 3.18. Note that a presentation of a one-relator group that satisfies the $C'(1/6)$ condition also satisfies the (T') condition since the sum of the length of three pieces is strictly smaller than $1/6\,|R| + 1/6\,|R| + 1/6\,|R| = 1/2\,|R|$.

Theorem 3.19 ([BM19])**.** *If a presentation of a one-relator group G satisfies the $C'(1/4)$ and the (T') condition, then G is hyperbolic.*

Definition 3.20. We say that a presentation satisfies the *Blufstein-Minian condition* or *BM-condition* if the assumptions of Theorem 3.19 are satisfied.

Example 3.21. Consider the group G with presentation

$$\mathcal{Q} = \langle x, y | R := xy^{-1}x^{-3}y^{-1}x^{-1}y^2xy^{-1} \rangle.$$

The pieces of the relator R are $x^{\pm 1}, y^{\pm 1}, xy^{-1}, x^{-2}, (x^{-1}y^{-1})^{\pm 1}$, $(x^{-1}y)^{\pm 1}$ and $y^{-1}x^{-1}$. Since R is of length 11 and there exist pieces of length 2, the presentation Q does not satisfy $C'(1/6)$. Furthermore, $C'(1/4) - T(4)$ is not satisfied since there exist diagrams with vertices of degree 3. One example of such a diagram is shown in Figure 3. However, we can prove that the group is hyperbolic using the Blufstein-Minian condition. Since the maximal piece length is 2, we see that $\mathcal{Q}$ satisfies the $C'(1/4)$ condition. Furthermore, the maximal tripod length is 5, which is smaller than $|R|/2$. In order to show this, we only have to check that vertices between two pieces of length 2 are not incident with a third edge labelled by a piece of length 2. Since each length 2 piece can be found exactly at two positions of the relator, there is always only one possibility how the third edge of a tripod can be labelled. It is straightforward to check that this is always a single letter. Four of those vertices are shown in Figure 3. The only other possible tripod-vertex between two length 2 pieces is between the pieces $y^{-1}x^{-1}$ and x^{-2}. Here the third edge of the tripod can only be labelled by $y^{\pm 1}$ depending on the orientation. Therefore, the BM-condition is satisfied and G is hyperbolic.

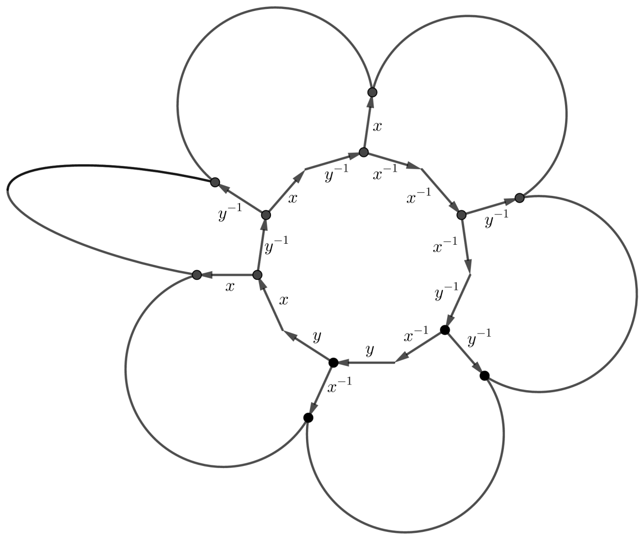

Figure 3: A van Kampen diagram over the group $\langle x, y | xy^{-1}x^{-3}y^{-1}x^{-1}y^2xy^{-1} \rangle$.

4 The `RSym` Procedure

In 2019, Holt et al. introduced a new algorithmic approach to prove hyperbolicity of a group in [HLN+19]. They make use of *pregroup presentations* (see Section 4.1) and define a *curvature distribution scheme* called `RSym` (see Section 4.3) on *coloured van Kampen diagrams* (see Section 4.2) over those pregroup presentations. The `RSym` procedure is said to *succeed* on a pregroup presentation $\mathcal{P}$ if the curvature of every internal non-boundary face with boundary length larger than 3 can be bounded above by a negative constant in every diagram. If `RSym` succeeds on $\mathcal{P}$, then the group presented by $\mathcal{P}$ is hyperbolic. A small extension of `RSym`, called `RSym`$^+$ is presented in Section 4.4. In this algorithm a face with negative curvature can give some of it to neighbouring green faces. Especially when working with the `RSym`$^+$ algorithm, it is often useful to increase the dual distance of the faces that need to have negative curvature, to the boundary. One says that `RSym`$^+$ *succeeds* on a diagram *at level* d if it succeeds on all internal faces that are of dual distance of at least $d + 1$ from the external face. If `RSym`$^+$ succeeds at level d and another condition regarding the relators of length 3 is satisfied, then the group presented by $\mathcal{P}$ is hyperbolic. In Section 4.5 we explain in which case one can use a Dehn algorithm to solve the word problem for groups on which `RSym` succeeds and in Section 4.6 we describe how Holt et al. implemented a function that verifies that `RSym` succeeds on a pregroup presentation and how we use this function in the next chapter. The definitions and results in this chapter are from [HLN+19] unless stated otherwise.

4.1 Pregroups and Pregroup Presentations

In [HLN+19], Holt et al. define a new kind of group presentation based on pregroups, which were introduced by Stallings in [Sta71]. When working with pregroup presentations, we can view a group G as a quotient of a virtually free group (see Definition 4.1) $U(P)$ rather than just a free group as in classical small cancellation theory. This enables us to ignore the failure of small cancellation on relators of length 3 since they are in the set of relations of $U(P)$.

Definition 4.1. Let φ be a property. We say that a group G is *virtually* φ if there exists a subgroup $H \leq G$ of finite index which has property φ.

Definition 4.2. A *pregroup* is a set P , together with a partial multiplication $(x, y) \to xy$ which is defined for $(x, y) \in D(P) \subseteq P \times P$ and an involution $\sigma : x \to x^\sigma$, such that the following axioms hold:

(i) There exists a distinguished element 1 in P such that for each $x \in P$, we have $(1, x), (x, 1) \in D(P)$ and $1x = x1 = x$;

(ii) $(x, x^\sigma), (x^\sigma, x) \in D(P)$ and $xx^\sigma = x^\sigma x = 1$ for all $x \in P$;

(iii) If $(x, y) \in D(P)$, then $(y^\sigma, x^\sigma) \in D(P)$ and $(xy)^\sigma = y^\sigma x^\sigma$ for all $x, y \in P$;

(iv) If $(x, y), (y, z) \in D(P)$, then $(xy, z) \in D(P)$ if and only if $(x, yz) \in D(P)$ for all $x, y, z \in P$. In this case $(xy)z = x(yz)$;

(v) If $(x, y), (y, z), (z, t) \in D(P)$, then at least one of $(xy, z), (yz, t) \in D(P)$ for all $x, y, z, t \in P$.

When working with words over P, we write $[xy]$ to denote that the letters x, y are to be multiplied.

Definition 4.3. We write X^σ to denote the set $X := P\backslash\{1\}$ equipped with the involution σ and let $F_P(X^\sigma) := \langle X | xx^\sigma : x \in X\rangle$ be the free product of copies of $\mathbb{Z}$ and copies of $\mathbb{Z}_2$.

We then let $V_P := \{xy[xy]^\sigma : x, y \in X, (x, y) \in D(P), x \neq y^\sigma\}$ be the set of all length 3 relators over X and define the *universal group* as follows:

$$U(P) := \langle X | \{xx^\sigma : x \in X\} \cup V_P\rangle = F_P(X^\sigma)/\langle\langle V_P\rangle\rangle,$$

where $\langle\langle V_P\rangle\rangle$ denotes the normal closure of V_P in $F_P(X^\sigma)$.

Definition 4.4. Let $w = y_1 \cdots y_n \in F(X)$ be a word. We say that w is *σ-reduced* if it contains no consecutive pair $y_i y_i^\sigma$ of letters. We say that w is *cyclically σ-reduced* if it is σ-reduced and $y_n \neq y_1^\sigma$.

Furthermore, we say that w is $P-$reduced if there is no pair $(y_i, y_{i+1}) \in D(P)$. The word w is *cyclically P-reduced* if it is P-reduced and $(y_n, y_1) \notin D(P)$.

The following equivalence relation was defined by Stallings and enables us to show that the group $U(P)$ has solvable word problem.

Definition 4.5. Let $v = v_1 \cdots v_n \in F(X)$ be P-reduced and $w = w_1 \cdots w_m$ any word in $F(X)$. We say that v is an *interleave* of w, denoted by $v \approx w$, if $n = m$ and there exist elements $s_0 = 1, s_1, \ldots, s_{n-1}, s_n = 1 \in P$, such that $(s_{i-1}^{\sigma}, v_i), (v_i, s_i), (\left[s_{i-1}^{\sigma} v_i\right], s_i) \in D(P)$ and $w_i = \left[s_{i-1}^{\sigma} v_i s_i\right]$ for all $i \in \{1, \ldots, n\}$.

Theorem 4.6 ([Sta71]). *For a pregroup P, two words u and v represent the same element in $U(P)$ if and only if $u \approx v$.*

Proposition 4.7. *Let P be a finite pregroup. Then the word problem in $U(P)$ is solvable by a Dehn algorithm.*

Proof. From Theorem 4.6 follows that the only P-reduced word representing 1 in $U(P)$ is the empty word, so the word problem can be solved by reducing words using the products in $D(P)$. $\square$

Definition 4.8. Let P be a pregroup with X and σ as before. Let $\hat{\mathcal{R}} \in F(X)$ be a set of cyclically P-reduced words. Then we define a group presentation $\mathcal{P}$ on the set X as follows:

$$\mathcal{P} = \langle X^{\sigma} | V_P | \hat{\mathcal{R}} \rangle := \langle X | \{ x x^{\sigma} : x \in X \} \cup V_P \cup \hat{\mathcal{R}} \rangle.$$

$\mathcal{P}$ is called a *pregroup presentation.*

In the following we want to show that there exists a large class of groups for which we can find pregroup presentations. The following theorem was proved by Rimlinger in [Rim87].

Theorem 4.9. *A finitely generated group G is virtually free if and only if G is the universal group $U(P)$ of a finite pregroup P.*

Holt et al. deduce the following result as an immediate corollary of Theorem 4.9.

Corollary 4.10. *If a group G has a pregroup presentation $\mathcal{P} = \langle X^{\sigma} | V_P | \hat{\mathcal{R}} \rangle$, then*

$$G \cong U(P) / \langle\langle \hat{\mathcal{R}} \rangle\rangle,$$

where $\langle\langle \hat{\mathcal{R}} \rangle\rangle$ denotes the normal closure of $\hat{\mathcal{R}}$ in $U(P)$. Any group that is a quotient of a virtually free group by finitely many additional relators has a finite pregroup presentation.

Example 4.11. Let $G = \langle x, y, z | y^3, (zy)^4, z^2 y^{-1} x y z^{-1} x \rangle$. First note that this presentation of G does not satisfy any of the classical small cancellation conditions for hyperbolicity we have presented in Chapter 3. In the course of this chapter we explain how to show that it is hyperbolic in a different way. We define the pregroup $P = \{1, x, y, z, X, Y, Z\}$ with products $X = x^{\sigma}, Y = y^{\sigma}, Z = z^{\sigma}, Y^2 = y, y^2 = Y$. It can be easily checked that the axioms (i)-(v) from Definition 4.2 hold. Let X^{σ} be like in Definition 4.3. Then

$F_P(X^\sigma) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ is the free group on x, y, z and $V_P = \{yyy, YYY\}$. The universal group is then defined as

$$U(P) = F_P(X^\sigma)/\langle\langle V_P \rangle\rangle = \langle x, y, z | yyy \rangle$$

and we get that the pregroup presentation

$$\mathcal{P} = \left\langle x, y, z, X, Y, Z | xX, yX, zZ, yyy, (zy)^4, z^2 YxyZx \right\rangle$$

is a presentation for G.

4.2 Coloured Diagrams and Curvature Distribution Schemes

The `RSym` procedure works over coloured van Kampen diagrams, a generalisation of van Kampen diagrams, which are defined in this section. Furthermore, we define curvature distribution schemes over those diagrams, an example of which is the `RSym` scheme, which is presented in the next section.

Definition 4.12. A *coloured (van Kampen) diagram* Γ over the pregroup presentation $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ is a van Kampen diagram with edges labelled by elements of X^σ and faces labelled by elements of $V_P \cup \hat{\mathcal{R}}^\pm$, where $\hat{\mathcal{R}}^\pm$ denotes the set of elements of $\hat{\mathcal{R}}$ and its inverses. The faces labelled by an element of V_P are coloured red and the faces labelled by an element of $\hat{\mathcal{R}}^\pm$ are coloured green. We view the complement of the diagram as another green face, called *external face*. For a vertex $v \in \Gamma$ we define $\delta_G(v)$ to be the number of green faces that are incident with v and $\delta_R(v)$ the number of red faces incident with v.

Note that the red faces are triangles.

Definition 4.13. The *coloured area* of a coloured diagram Γ is an ordered pair $(a, b) \in \mathbb{N} \times \mathbb{N}$, where a is the number of internal green faces and b is the number of red triangles. Let Δ be a coloured diagram with coloured area (c, d). We say that the coloured area of Γ is less than or equal to the coloured area of Δ if $a < c$ or if $a = c$ and $b \leq d$.

Definition 4.14. A diagram is called semi-σ-reduced if no two distinct adjacent faces are labelled by $w_1 w_2$ and $w_2^{-1} w_1^{-1}$ for some relator $w_1 w_2 \in V_P \cup \hat{\mathcal{R}}^\pm$ and have a common consolidated edge labelled by w_1 and w_1^{-1}. It is σ-reduced if the same also holds for a single face adjacent to itself.

Definition 4.15. A coloured diagram is *semi-P-reduced* if no two distinct adjacent green faces are labelled by $w_1 w_2$ and $w_3^{-1} w_1^{-1}$ and have a common consolidated edge labelled by w_1 and w_1^{-1}, where w_2 and w_3 are equal in $U(P)$.

Definition 4.16. We say that a coloured diagram is *green-rich* if $\delta_G(v) \geq 2$ for all $v \in \Gamma$, i.e. each vertex is adjacent to at least 2 green faces, including the external face.

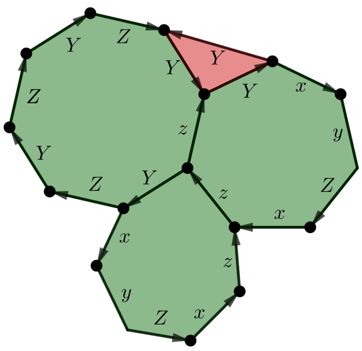

Figure 4: A σ-reduced, semi-P-reduced, green-rich coloured van Kampen diagram for $w = (ZY)^2 ZyxyZxZXzYX$ over the pregroup presentation $\langle x, y, z, X, Y, Z | xX, yX, zZ, yyy, (zy)^4, z^2YxyZx \rangle$.

Definition 4.17. A *red blob* in a coloured diagram Γ is a nonempty subset B of the set of red triangles in Γ, such that any nonempty proper subset C of B has at least one edge in common with $B \setminus C$.

Remark 4.18. Note that in a green rich diagram every red blob is simply connected. It then follows from [HLN+19, Proposition 4.12] that in a green rich diagram of minimal coloured area the boundary word of every simply connected red blob has no proper subword equal to 1 in $U(P)$.

Next, we generalise Definition 4.5 to *cyclic interleaves*, to define another pregroup presentation where the set $\hat{\mathcal{R}}$ is replaced by all cyclic interleaves of elements of $\hat{\mathcal{R}}$. This enables us to state a version of the classical van Kampen Lemma (Theorem 2.3) for pregroup presentations.

Definition 4.19. Let $v = v_1 \cdots v_n \in F(X)$ be cyclically P-reduced and $w = w_1 \cdots w_m$ any word. We say that v is a *cyclic interleave* of w, denoted by $v \approx^C w$, if $n = m$ and there exist elements $s_0, s_1, \ldots, s_{n-1}, s_n = s_0 \in P$ such that $(s_{i-1}^\sigma, v_i), (v_i, s_i), ([s_{i-1}^\sigma v_i], s_i) \in D(P)$ and $w_i = [s_{i-1}^\sigma v_i s_i]$ for all $i \in \{1, \ldots, n\}$.

Theorem 4.20 ([HLN+19])**.** *$\approx^C$ is an equivalence relation on the set of all cyclically P-reduced words.*

Definition 4.21. Let w be a cyclically P-reduced word. Then we define

$$\mathcal{I}(w) := \{v \in F(X) | v \approx^C w\}.$$

Furthermore, we define $\mathcal{I}(\hat{\mathcal{R}}) = \cup_{R \in \hat{\mathcal{R}}} \mathcal{I}(R)$ and a pregroup presentation

$$\mathcal{I}(\mathcal{P}) := \langle X^\sigma | V_P | \mathcal{I}(\hat{\mathcal{R}}) \rangle.$$

Theorem 4.22 ([HLN+19])**.** *The normal closure of $V_P \cup \hat{\mathcal{R}}$ is equal to the normal closure of $V_P \cup \mathcal{I}(\hat{\mathcal{R}})$. Hence $\mathcal{P}$ and $\mathcal{I}(\mathcal{P})$ define the same group G.*

Theorem 4.23 ([HLN+19])**.** *Let G be a group with pregroup presentation $\langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$. A cyclically P-reduced word w is trivial in G if and only if some cyclic interleave of w is the boundary of a coloured van Kampen diagram over $\langle X^\sigma | V_P | \mathcal{I}(\hat{\mathcal{R}}) \rangle$.*

Definition 4.24. Let Γ be a coloured van Kampen diagram over the pregroup presentation $\mathcal{P}$ with vertex set $V(\Gamma)$, edge set $E(\Gamma)$, and set of internal faces $F(\Gamma)$. A *curvature distribution* is a function $\rho : V(\Gamma) \cup E(\Gamma) \cup F(\Gamma) \to \mathbb{R}$ such that

$$\sum_{x \in V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)} \rho(x) = 1.$$

Definition 4.25. Let $\mathcal{K}$ be a set of coloured diagrams over $\mathcal{I}(P)$. A *curvature distribution scheme on* $\mathcal{K}$ is a map $\Psi : \mathcal{K} \to \{\rho_\Gamma : \Gamma \in \mathcal{K}\}$ that associates a curvature distribution to every diagram in $\mathcal{K}$.

4.3 The `RSym` Scheme

We are now ready to define the `RSym` scheme, which is an example of a curvature distribution scheme and explain how it can be used to prove that a group is hyperbolic. We first define the set $\mathcal{D}$ of diagrams on which `RSym` is applicable.

Definition 4.26. Let $\mathcal{P}$ be a pregroup presentation. Then $\mathcal{D}$ denotes the set of all coloured diagrams Γ over $\mathcal{I}(\mathcal{P})$ with the following properties:

(i) the boundary word of Γ is cyclically P-reduced;

(ii) the diagram Γ is σ-reduced and semi-P-reduced;

(iii) Γ is green-rich;

(iv) no proper subword of the boundary word of a simply connected red blob in Γ is equal to 1 in $U(P)$.

Algorithm 4.27 (`RSym`)**.** For any diagram $\Gamma \in \mathcal{D}$ do:

1. In the beginning, every vertex, red triangle and internal green face has curvature $+1$ and every edge has curvature -1.

2. Any edge is adjacent to two faces (possibly one of them is the external face). It gives curvature $-1/2$ to any adjacent red triangle and, if it is adjacent to a green face, gives curvature $-1/2$ to its end vertex obtaining the orientation from the green face in question.

3. Each vertex divides its curvature equally amongst its incident internal green faces (counted with multiplicity). If there are none, the curvature remains on the vertex.

4. Each red blob sums the curvature of its triangles to get the blob curvature and distributes it equally amongst its adjacent internal green faces.

5. Return the curvature distribution.

Remark 4.28. In the `RSym` scheme, fix a curvature distribution ρ_Γ on a diagram Γ. By Euler's formula we have

$$\sum_{x \in V(\Gamma) \cup E(\Gamma) \cup F(\Gamma)} \rho_\Gamma(x) = |E| \cdot (-1) + (1 + |E|) \cdot (+1) = 1,$$

so `RSym` is indeed a curvature distribution scheme.

Definition 4.29. A V^σ*-letter* is a letter $x \in X$ such that either $x^\sigma = x$ or x is a letter of a relator in V_P.

Definition 4.30. We say that `RSym` *succeeds on a diagram* $\Gamma \in \mathcal{D}$ if there exists a constant $\varepsilon > 0$, such that the curvature of every internal non-boundary green face is $\leq -\varepsilon$. We say that it *succeeds on a pregroup presentation* $\mathcal{P}$ if `RSym` succeeds for every $\Gamma \in \mathcal{D}$.

The following theorem is a central result from [HLN$^+$19].

Theorem 4.31. *If* `RSym` *succeeds on a pregroup presentation $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ for some $\varepsilon > 0$, where no $R \in \hat{\mathcal{R}}$ is of length 1 or 2 and no two distinct conjugates of relators $R, S \in \mathcal{I}(\hat{\mathcal{R}})^\pm$ have a common prefix consisting of all but one letter of R or S, then its Dehn function is bounded above by*

$$f(n) = n\left(6 + r + \frac{3+r}{2\varepsilon}\right) - \frac{3+r}{\varepsilon},$$

where r is the maximum length of a relator in $\hat{\mathcal{R}}$. In particular, the group presented by $\mathcal{P}$ is hyperbolic.

The following lemmas from [HLN$^+$19] can be helpful for proving that `RSym` succeeds on a given diagram.

Lemma 4.32. *Let v be a vertex of a diagram $\Gamma \in \mathcal{D}$ and let x be the number of times it is incident with the external face. We have*

(i) The vertex v gives curvature

$$\frac{2 - \delta_G(v)}{2(\delta_G(v) - x)}$$

to each incident internal green face if $x \neq \delta_G(v)$.

(ii) If $x \geq 2$, then v gives curvature at most $-1/2$ to each incident internal green face. Otherwise the curvature values given to an incident internal green face are as in Table 1.

Table 1: Curvature values χ_{int} and $\chi_{boundary}$ of an internal and a boundary vertex in a diagram $\Gamma \in \mathcal{D}$.

$\delta_G(v)$	χ_{int}	$\chi_{boundary}$
2	0	0
3	$-1/6$	$-1/4$
4	$-1/4$	$-1/3$
5	$-3/10$	$-3/8$
6	$-1/3$	$-2/5$
≥ 7	$\leq -5/14$	$\leq -5/12$

Table 2: Curvature values χ of a red blob B in a diagram $\Gamma \in \mathcal{D}$.

$\lvert\partial(B)\rvert$	$\lvert\partial(B) \cap \partial(\Gamma)\rvert$	χ
3	0	$-1/6$
3	1	$-1/4$
4	0	$-1/4$
4	1	$-1/3$
5	0	$-3/10$
6	0	$-1/3$

Proof. Initially, the vertex curvature is $+1$. Thus, the curvature the vertex can distribute is $1-1/2\cdot\delta_G(v)$. This curvature is split equally among $\delta_G(v)-x$ faces, so Part (i) follows. Part (ii) can be computed using the formula from Part (i). $\square$

Lemma 4.33. *Let B be a red blob in a diagram $\Gamma \in \mathcal{D}$ and f an internal green face adjacent to B. If B is simply connected, then $|\partial(B)|,|\partial(B) \cap \partial(\Gamma)|$ and the curvature values that B gives to f are as in Table 2.*

Proof. Let B be a red blob with boundary length l and area t. Since B is simply connected and Γ is green rich, we have that every vertex of B lies on ∂B. Furthermore, $t = l - 2$ holds: For two triangles attached to each other, we have $l = 4$ and $t = 2$, so the formula is true. Now assume that we have a red blob consisting of n red triangles with length l and area t, such that $t = l - 2$. Then, since every vertex of B lies on ∂B, attaching another triangle increases l and t by 1 each, so the claim follows by induction.

After Step 2 of algorithm 4.27, the curvature of each triangle is $-1/2$, so, in Step 4 the red blob gives curvature

$$\frac{-t}{2\,|\partial(B) \setminus \partial(\Gamma)|} = \frac{-l+2}{2(l - |\partial(B) \cap \partial(\Gamma)|)}$$

from which the values in Table 2 can be computed. $\square$

The following definition can be thought of as a generalisation of a piece in the classical small cancellation theory.

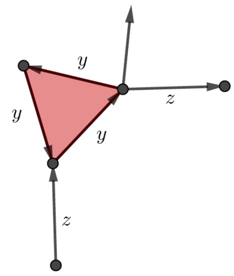

Figure 5: A red triangle adjacent to a face labelled by R_1 in Example 4.36.

Definition 4.34. Fix a decomposition of a relator $R = w_1 w_2 \cdots w_k$ into words labelling the consolidated edges $e_1, e_2, \ldots, e_k$ of an internal face labelled by R. A *step* is either a single subword w_i if the faces on the other sides of e_i and e_{i+1} are both green or two consecutive subwords $w_i w_{i+1}$ if e_{i+1} is adjacent to a red blob.

Fix a word $w = x_1 \cdots x_{|w|}$ such that $R = w^k$ with k maximal among such expressions for R. A *location* on R is an ordered triple (i, a, b), where $i = \{1, \ldots, |w|\}$, $a = x_{i-1}$ (or $x_{|w|}$ if $i = 1$) and $b = x_i$.

A *place* on R is a location (i, a, b) for which there exists a σ-reduced diagram Γ with a face f labelled by R, such that there is a vertex between a and b of degree at least 3. A place is *green* if the face meeting f at b is green and *red* otherwise.

Remark 4.35. Note that the maximal curvature of a step is $-1/6$ since either the end vertex of a step is adjacent to at least 3 green faces or the step contains an edge that is adjacent to a red blob, which gives curvature at most $-1/6$ as well.

Example 4.36. `RSym` succeeds on the pregroup presentation

$$\mathcal{P} = \left\langle x, y, z, X, Y, Z | xX, yX, zZ, yyy, (zy)^4, z^2 YxyZx \right\rangle$$

from Example 4.11 with $\varepsilon = 1/6$. To see this, consider each relator in $\hat{\mathcal{R}} = \{R_1 := (zy)^4, R_2 := z^2 YxyZx\}$ individually. A consolidated edge between an internal non-boundary face labelled by R_1 and another green face can only be labelled by z or y since neither zy or yz nor YZ or ZY are subwords of R_2. If R_1 were adjacent to itself, the diagram would not be σ-reduced. Hence, a consolidated edge is of length 1. If the edge is labelled by z and the following edge is adjacent to a red triangle labelled by yyy, then the step curvature is at most $-1/3$. This is because the end vertex v of the step zy has green valency $\delta_G(v) \geq 3$ since neither Yz nor Zy is a subword of any relator, so the red triangle and the end vertex of the step give at most $-1/6$ of curvature respectively (see Figure 5). If the edge labelled by y is a consolidated edge of two green faces, then we have two edges whose end vertices give curvature at most $-1/6$ respectively, so the total curvature of a face labelled by R_1 is at most $1 - 1/6 \cdot 8 = -1/3 < -1/6$.

Next, consider an internal non-boundary face f labelled by R_2, as shown in Figure 6. If any of the boundary edges of f labelled by y or Y were adjacent to a red triangle,

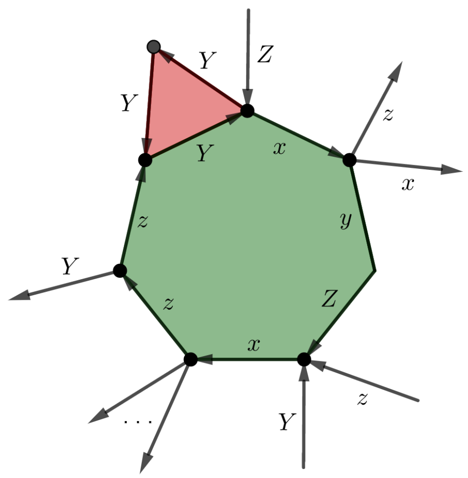

Figure 6: The decomposition of R_2 in Example 4.36.

this triangle would give curvature at most $-1/6$ and the vertices at the beginning and end of the subword $(zY)^{\pm 1}$ would give curvature $-1/6$ each. Therefore, we have at least 5 vertices in the decomposition of R_2 that give curvature at most $-1/6$, so only one of the edges labelled by y or Y can be adjacent to a red triangle since otherwise the curvature of the face would already be negative. The other edge is then part of a green step $(zY)^{\pm 1}$ whose incident vertices both give curvature at most $-1/4$. Hence, if there is one red triangle adjacent to f, the curvature is bounded by $1 - 1/2 - 4/6 = -1/6$ and if f is not adjacent to a red triangle, the curvature is bounded by $1 - 4/4 - 1/6 = -1/6$.

Remark 4.37. See [Cha20] for an application of the `RSym` algorithm to the Fibonacci groups $\langle x_1, x_2, \ldots, x_n | x_i x_{i+1} = x_{i+2}$ for $i = 1, \ldots, n\rangle$ with subscripts taken mod n, which yields that they are hyperbolic for n odd and $n \geq 11$.

4.4 `RSym`$^+$ at Level 2

`RSym`$^+$ is an extension of the `RSym` algorithm by an additional step that allows faces with more than enough negative curvature to give some of it to their neighbouring green faces.

In this thesis we give examples of multiple presentations on which `RSym` does not succeed, but `RSym`$^+$ does, if one increases the dual distance of non-boundary faces, which need to have negative curvature, to the boundary.

Algorithm 4.38 (`RSym`$^+$)**.** For any diagram $\Gamma \in \mathcal{D}$ do:

1.-4. See Algorithm 4.27.

5. Each face that has curvature $\chi \leq -\varepsilon$ can give a total of $\chi + \varepsilon$ of curvature to any adjacent green faces with curvature greater than $-\varepsilon$.

6. Return the curvature distribution.

Definition 4.39. For $d \geq 1$, we say that $\texttt{RSym}^+$ (resp. `RSym`) succeeds on a presentation P *on level* d if we can bound the total curvature above by $-\varepsilon$ for all faces that are of dual distance at least $d+1$ from the external face.

Theorem 4.40 ([HLN+19])**.** *Assume that* $\texttt{RSym}^+$ *(resp.* `RSym`*) succeeds at level* d *on a presentation* $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ *of* G*, where no* $R \in \hat{\mathcal{R}}$ *is of length* 1 *or* 2 *and no two distinct conjugates of relators* $R, S \in \mathcal{I}(\hat{\mathcal{R}})^{\pm}$ *have a common prefix consisting of all but one letter of* R *or* S*. If additionally no* V^σ*-letter is trivial in* G*, then its Dehn function is bounded above by*

$$f(n) = n\left((3+r)\frac{(r-1)^d - 1}{r-2}\left(1 + \frac{1}{\varepsilon}\right) + 3\right) - \frac{3+r}{\varepsilon},$$

where r *is the maximum length of a relator in* $\hat{\mathcal{R}}$*. In particular,* G *is hyperbolic.*

We now give an example to illustrate the usefulness of $\texttt{RSym}^+$ and then prove a lemma which becomes very helpful in Chapter 5, where we analyse `RSym` over one-relator groups. When working with decompositions of relators, we use the following notation:

Definition 4.41. Let $R = x_1 \cdots x_n$ be a relator in a pregroup presentation. When describing a decomposition of R, we let

$$\xrightarrow{x_1 \cdots x_k} (\chi_1) \xrightarrow{x_{k+1} \cdots x_{k+t}}$$

denote that $x_1 \cdots x_k$ and $x_{k+1} \cdots x_{k+t}$ are steps which are seperated by a vertex that gives curvature χ_1. A complete decomposition of R is denoted by

$$\xrightarrow{x_1 \cdots x_k} (\chi_1) \xrightarrow{x_{k+1} \cdots x_{k+t}} (\chi_2) \xrightarrow{\cdots} \cdots \xrightarrow{x_{n-i} \cdots x_n} (\chi_s),$$

where (χ_s) denotes the vertex between the last and the first step.

Example 4.42. Let $G = \langle x, y, z | z^{-2}y^{-1}z^{-2}y^{-1}x^{-2}y^{-2}z^{-1}yx^2 \rangle$. Then

$$\mathcal{P} = \langle x, y, z, X, Y, Z | xX, yY, zZ, Z^2YZ^2YX^2Y^2Zyx^2 \rangle,$$

is a pregroup presentation for G. We show that `RSym` does not succeed on $\mathcal{P}$, but $\texttt{RSym}^+$ succeeds on $\mathcal{P}$ at level 2. The only decomposition of a face f on which `RSym` fails is the following:

$$\xrightarrow{Z^2Y} (-1/6) \xrightarrow{Z^2Y} (-1/6) \xrightarrow{X^2Y} (-1/6) \xrightarrow{YZ} (-3/10) \xrightarrow{yx^2} (-1/6). \tag{4.1}$$

At level 1, the internal non-boundary face f has total curvature $1 - 4/6 - 3/10 = 1/30$ and the adjacent (boundary) faces also have non-negative curvature. To analyse $\texttt{RSym}^+$

at level 2, consider the step $p := YZ$ of the failing decomposition. Since G is a one-relator group, the face $\hat{f}$ that is on the other side of the edge labelled by p is also a green face, labelled by the same relator. At level 2, the face $\hat{f}$ has to be non-boundary as well, so the decomposition of $\hat{f}$ with maximal curvature is

$$\xrightarrow{Z^2} (-1/3) \xrightarrow{YZ} (-1/4) \xrightarrow{ZY} (-1/6) \xrightarrow{X^2Y} (-1/6) \xrightarrow{YZ} (-3/10) \xrightarrow{yx^2} (-1/6),$$

which results in a total curvature of $-17/60$. Now, if we chose ε small enough, the face $\hat{f}$ can give curvature $-\varepsilon - 1/30$ to f, so that the curvature of both faces is small enough. Since Decomposition (4.1) is the only decomposition on which `RSym` fails, all other non-boundary faces in the diagram have curvature less than $-\varepsilon$, so `RSym`$^+$ succeeds at level 2. There is no V^σ-letter in $\mathcal{P}$, so G is hyperbolic.

Lemma 4.43. *Let G be a group with a pregroup presentation $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ where $V_P = \emptyset$ and assume that `RSym` fails on exactly one decomposition of a relator $R \in \hat{\mathcal{R}}$ with total curvature* 0*. If, in addition, there is one step which appears only once in this decomposition, then `RSym`$^+$ succeeds at level 2.*

Proof. Since $V_P = \emptyset$, all of the faces of the failing diagram are green. Consider the step p which appears only once in the failing decomposition and let the face on the other side of p be called f. Note that if f is also labelled by R, the edge that is labelled by p cannot correspond to the same location of the relator as in the original face since otherwise the diagram would not be σ-reduced. Therefore, the face f has negative curvature $-x$ for some value x because its boundary word is either a relator different than R on which `RSym` succeeds or its a decomposition of R other than the failing one since p appears only once in the failing decomposition. Say that p can be found n-times in the relator. Then there can be at most n bad faces, i.e. faces with curvature 0, adjacent to f. In Step 5 of the `RSym`$^+$ procedure, f donates $-(x - \varepsilon)/n$ of curvature to each adjacent bad face. Since each bad face has at least one face like f as its neighbour, we conclude that, if we choose ε small enough, `RSym`$^+$ succeeds at level 2. $\square$

4.5 `RSym` and the Word Problem

If a pregroup presentation $\mathcal{P}$ on which `RSym` succeeds satisfies the following extra condition, it can be used to solve the word problem.

Definition 4.44. `RSym` *verifies a solver* for $\mathcal{I}(\mathcal{P})$ if, for any green boundary face f in any $\Gamma \in \mathcal{D}$ with positive total curvature, the removal of f shortens $\partial(\Gamma)$.

In this section we present an algorithm introduced in [HLN+19] that checks, if `RSym` verifies a solver for $\mathcal{I}(\mathcal{P})$ in the case where $\mathcal{I}(\hat{\mathcal{R}}) = \hat{\mathcal{R}}$. If the algorithm succeeds, we can use a standard Dehn algorithm in order to solve the word problem. Note that there is a large class of groups that have a pregroup presentation for which $\mathcal{I}(\hat{\mathcal{R}}) = \hat{\mathcal{R}}$, for example all quotients of free products of free and finite groups. This is the case for all of the groups in this thesis, so we present the algorithm only for this case. We refer the reader to [HLN+19, Chapter 8] for an algorithm that solves the word problem in the case $\mathcal{I}(\hat{\mathcal{R}}) \neq \hat{\mathcal{R}}$, which is different than a standard Dehn algorithm.

The algorithm `VerifySolver` considers up to three steps from any green place and up to four steps from any red place of any relator and computes, if the total curvature after those steps is ≤ 0. If it is not, `VerifySolver` checks if the length of those steps accounts for less than half of the length of the relator. Note that it suffices to check up to three steps from any starting green place or up to four steps from any starting red place since any boundary vertex and any boundary triangle give curvature at most $-1/4$ and internal steps give at most $-1/6$ of curvature. Therefore, the total curvature of a boundary face would be less or equal to 0 if the internal boundary consisted of more than three or four steps respectively.

Algorithm 4.45 (`VerifySolver`). Given a pregroup presentation $\mathcal{P} = \langle X^\sigma | V_P | \hat{\mathcal{R}} \rangle$ on which `RSym` succeeds, for each $R \in \hat{\mathcal{R}}$ and each place P_S on R do the following:

1. If P_S is red, compute the maximum curvature χ_0 that a boundary red blob and the corresponding end vertex of the step give to the face. Let $\phi_0 = 1 + \chi_0$ and consider each possible following step P_1. If P_S is green, let $\phi_0 = 3/4$.

2. Consider all possible sequences of up to three steps from P_S (green case) or P_1 (red case) and for each step at position i of the sequence, compute the maximal curvature χ_i and let $\phi_i = \phi_{i-1} - \chi_i$.

3. If for some $i \in \{1, 2, 3\}$, we have that $\phi_i > 0$ and the length of the subword labelling the steps up to step i is $\geq |R|/2$, return fail.

Return true.

Example 4.46. `VerifySolver` does not succeed for the group with pregroup presentation

$$\mathcal{P} = \left\langle x, y, z, X, Y, Z | xX, yX, zZ, yyy, R_1 = (zy)^4, R_2 = z^2YxyZx \right\rangle$$

from Example 4.11. Consider a boundary face f labelled by R_2. Then, in the decomposition of f shown in Figure 7, the face has total curvature $1/12$ since the red triangle gives curvature $-1/4$, the internal vertices give curvature $-1/6$ and $-1/4$ and the end vertex gives curvature $-1/4$. The subword labelling the internal boundary $YxyZ$ is of length $4 > |R_2|/2$ and therefore `VerifySolver` does not succeed.

Remark 4.47. Note that all boundary faces in Example 4.46 for which `VerifySolver` fails are adjacent to a boundary red triangle since otherwise the total curvature would be at most 0. Furthermore, the edge adjacent to this red triangle is either followed by an edge labelled by x or Z if the red triangle is in the beginning, or it is preceded by an edge labelled by z or x if it is located at the end of the internal boundary. Therefore, the failing of `VerifySolver` is always due to the existence of the relator $(zy)^4$. If we would consider those faces over the group $G = \langle x, y, z | yyy, R_2 \rangle$, the curvature of those faces would be at most 0. Therefore, the group

$$G = \langle x, y, z | yyy, zy^{-1}xyz^{-1}x \rangle$$

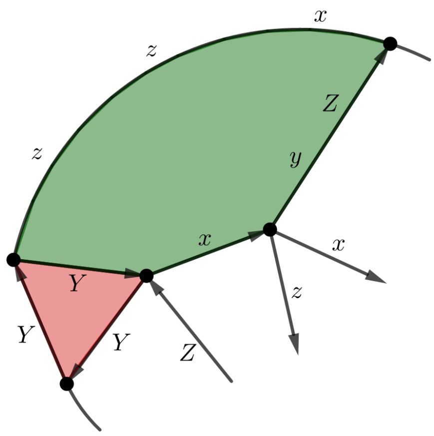

Figure 7: The decomposition of R_2 as a boundary face in Example 4.46.

is an example of a group for which we can find a pregroup presentation on which `RSym` and `VerifySolver` succeed.

4.6 Implementation and Usage of `RSym`

Holt et al. implemented a function in the computer algebra systems GAP and MAGMA called `IsHyperbolic` that searches to verify that `RSym` succeeds on a given pregroup presentation for which $\mathcal{I}(\hat{\mathcal{R}}) = \hat{\mathcal{R}}$. It either returns true if `RSym` succeeds or it returns fail along with additional information on the diagram where `RSym` might fail. The authors point out that the function `IsHyperbolic` might return fail even if `RSym` succeeds on a presentation, in which case it would output a diagram that does not exist. Furthermore, the `RSym`$^+$ algorithm and algorithms at level d are not implemented at the moment. Therefore, if `IsHyperbolic` returns fail for a pregroup presentation, one can either try to show by hand that the diagram(s) given in the output do(es) not exist or that `RSym` or `RSym`$^+$ succeed at a higher level d and that no V^σ-letter is trivial in the group.

The GAP package where `IsHyperbolic` is implemented is called `walrus`. In the current version of `walrus` (4.10.2), the function `IsHyperbolic` tries possible decompositions of relators until it finds one on which `RSym` fails. Then, the computation stops and returns fail along with the information about this decomposition. When trying to show that `RSym`$^+$ succceeds at level d on this group, it can be helpful to get information about *all* of the decompositions where `RSym` might fail.

We therefore made some small alterations to the `walrus` code for our own use, so that `IsHyperbolic` does not stop when it finds a failing decomposition, but continues to try out all of the possibilities and in the end returns all of the possible decompositions where

`RSym` might fail.

5 `RSym` in Relation to Other Conditions for Hyperbolicity

In this chapter we examine how `RSym` behaves on groups that satisfy other conditions for hyperbolicity, starting with small cancellation conditions in Section 5.1. In Section 5.2 we analyse under which conditions `RSym` succeeds on one-relator-groups that have a short relator (i.e. of length up to 15) of primitivity rank 3 and if `RSym` succeeds on one-relator groups that satisfy the Blufstein-Minian condition.

Note that a group with presentation $\mathcal{Q} = \langle \mathcal{S} | \mathcal{R} \rangle$ where $|R| > 3$ for all relators $R \in \mathcal{R}$ has a canonical pregroup presentation where the only products of the pregroup are between the elements of $\mathcal{S}$ and their inverses, $V_P = \emptyset$ and $\hat{\mathcal{R}} = \mathcal{R}$. We sometimes say that `RSym` succeeds on $\mathcal{Q}$, meaning that the algorithm succeeds on the corresponding canonical pregroup presentation.

5.1 `RSym` on Small Cancellation Groups

The following theorem is a result from [HLN+19].

Theorem 5.1. *Let $\mathcal{Q}$ be a presentation for the group G, satisfying $C(p)-T(q)$ for some $(p, q) \in \{(7, 3), (5, 4), (4, 5)\}$. Then `RSym` succeeds on $\mathcal{Q}$, so G is hyperbolic.*

Proof. Since $\mathcal{Q}$ satisfies the $C(p)$ condition, all faces have at least $7, 5$ or 4 edges, so V_P is empty and there are no red faces. Fix a diagram Γ over $\mathcal{Q}$ and let f be a non-boundary face of Γ. Using Lemma 4.32, we find that any vertex v of f that is not on $\partial\Gamma$ gives curvature

$$\frac{2-\delta(v)}{2\delta(v)} = \frac{1}{\delta(v)} - \frac{1}{2} \leq \frac{1}{q} - \frac{1}{2}$$

to f since all vertices have degree at least q. If v is on $\partial\Gamma$, it gives curvature at most $-1/3$ since f is not a boundary face and hence $\delta(v) \geq 4$. Since $-1/3$ is smaller than $1/q - 1/2$ for all q, we can assume that all vertices are internal and arrive at the following upper bound on the curvature $\kappa(f)$ of f:

$$\kappa(f) \leq 1 - p\left(\frac{1}{q} - \frac{1}{2}\right) \leq -1/6$$

for any $(p, q) \in \{(7, 3), (5, 4), (4, 5)\}$. □

The fact that $C'(1/6)$ implies $C(7) - T(3)$ and $C'(1/4) - T(4)$ implies $C(5) - T(4)$ gives the following corollary:

Corollary 5.2. *Let $\mathcal{Q}$ be a group presentation satisfying $C'(1/6)$ or $C'(1/4) - T(4)$. Then `RSym` succeeds on $\mathcal{Q}$.*

Another result of Holt et al. is the following:

Theorem 5.3. *Let $\mathcal{Q}$ be a group presentation satisfying $C'(1/6)$ or $C'(1/4) - T(4)$. Then `VerifySolver` succeeds on $\mathcal{Q}$.*

Proof. The internal boundary of a boundary face f labelled by a relator R with positive curvature in a diagram over a $C'(1/6)$ presentation consists of at most three steps since the starting and end vertices give curvature at most $-1/4$ each and an internal vertex gives curvature at most $-1/6$. Therefore, the length of the internal boundary is less than $3/6 \cdot |R|$. In a $C'(1/4) - T(4)$ diagram the internal boundary of f consists of at most two steps since any internal vertex gives curvature at most $-1/4$. Again the length of the internal boundary is less than half of $|R|$ because of the $C'(1/4)$ condition. □

5.2 RSym on One-Relator Groups

In this section we analyse one-relator groups of rank 3 using computational methods in parts. We start with groups that have relators of primitivity rank 3 and then focus on groups that are hyperbolic by the Blufstein-Minian condition.

5.2.1 RSym and primitivity rank 3

Louder and Wilton conjecure that one-relator groups with relators of primitivity rank 3 are hyperbolic (see [LW18]). In this section we want to analyse to what extent one can use `RSym` to confirm this conjecture. In subsequent work [CH20] we used these and additional methods to show that every rank 3 one-relator group with a relator of primitivity rank 3 and of length at most 17 is hyperbolic.

The definition of the primitivity rank of a relator was introduced by Puder in [Pud14].

Definition 5.4. An element w of a free group F is called primitive if it is contained in a free generating set of F.

It is well known that a primitive word of a free group F is also primitive in every subgroup of F containing it. The other direction is not true in general. If a word w is non-primitive in F, it could be either primitive or non-primitive in a given subgroup of F. The *primitivity rank* of a word w is the smallest rank that a subgroup K of F can have, such that w is non-primitive in K.

Definition 5.5. Let F be a free group and $w \in F$ a non-primitive element. The *primitivity rank* of w is

$$\pi(w) := \min\{\text{rank}(K) | w \in K \leq F \text{ and } w \text{ not primitive in } K\}.$$

In this section, we are working with a list[1] of rank 3 one-relator groups with relators of primitivity rank 3 of length up to 15. First, we run the `IsHyperbolic` function of the `walrus` package in GAP on all of these groups with $\varepsilon = 1/100$. The results can be found in Table 3. They give rise to the following theorem:

[1] The list was provided by C. Cashen, one of the supervisors of this thesis.

Table 3: Results of `IsHyperbolic` with $\varepsilon = 1/100$ on rank 3 one-relator groups with a relator R of primitivity rank 3.

$\lvert R\rvert$	number of groups[2]	number of fails
≤ 10	637	0
11	$3,115$	4
12	$16,806$	1
13	$96,040$	37
14	$582,475$	98
15	$2,099,313$	671

Theorem 5.6. *Let $G = \langle \mathcal{S} | R \rangle$ be a one-relator group with R of primitivity rank* 3*. Then there exists a pregroup presentation $\mathcal{P}$ of G such that:*

(i) If R is of length less or equal to 10*, then **RSym** succeeds on $\mathcal{P}$.*

(ii) If R is of length less or equal to 12*, then **RSym**$^+$ succeeds on $\mathcal{P}$ at level 2.*

Proof. The first statement is a purely computational result (see Table 3).

In order to prove the second statement, we consider the 5 groups for which `IsHyperbolic` returns fail. We have 4 such groups with relators of length 11. In a pregroup presentation with generators x, y, z and X, Y, Z where $x^\sigma = X$, $y^\sigma = Y$ and $z^\sigma = Z$, the relators of those four groups are: $ZYXZxzXZxzy$, $ZYZXZxZXZxy$, $ZYZXYxyZYZy$ and $ZYZXzxyZYZy$. For each group, there exists only one failing decomposition of the respective relator, namely:

$$\begin{aligned} &\xrightarrow{XZxz} (-1/6) \xrightarrow{XZxz} (-1/6) \xrightarrow{y} (-1/4) \xrightarrow{Z} (-1/4) \xrightarrow{Y} (-1/6), \\ &\xrightarrow{ZXZx} (-1/6) \xrightarrow{ZXZx} (-1/6) \xrightarrow{y} (-1/4) \xrightarrow{Z} (-1/4) \xrightarrow{Y} (-1/6), \\ &\xrightarrow{yZYZ} (-1/6) \xrightarrow{yZYZ} (-1/6) \xrightarrow{X} (-1/4) \xrightarrow{Y} (-1/4) \xrightarrow{x} (-1/6) \\ \text{and } &\xrightarrow{yZYZ} (-1/6) \xrightarrow{yZYZ} (-1/6) \xrightarrow{X} (-1/4) \xrightarrow{z} (-1/4) \xrightarrow{x} (-1/6). \end{aligned}$$

In each of these decompositions the total curvature equals 0 and there exists a step that does not come up more than once. Using Lemma 4.43, we conclude that `RSym`$^+$ succeeds at level 2 for those four groups.

Now we repeat the process for the failing group with a relator of length 12 which has the following presentation:

$$\langle x, y, z, X, Y, Z | xX, yX, zZ, Z^2Y^2Z^2Y^2X^2zx \rangle.$$

[2]The groups in the list are $\mathrm{Aut}(F(\mathcal{S}))$ classes of cyclic subgroups, so they are not necessarily distinct isomorphism classes of groups.

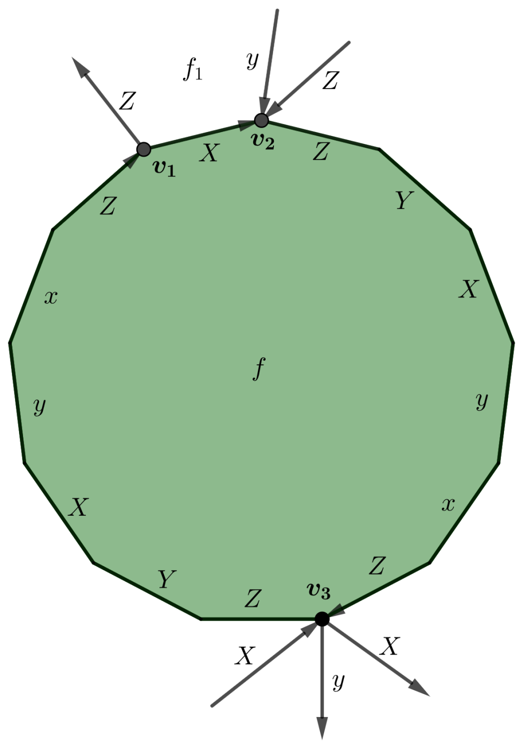

Figure 8: An internal face in a diagram over the presentation $\langle x, y, z, X, Y, Z | xX, yY, zZ, (ZYXyxZ)^2X \rangle$ on which `RSym` fails.

Again there is only one decomposition on which `RSym` fails, namely

$$\xrightarrow{Z^2Y^2} (-1/6) \xrightarrow{Z^2Y^2} (-1/6) \xrightarrow{X} (-1/6) \xrightarrow{X} (-1/6) \xrightarrow{z} (-1/6) \xrightarrow{x} (-1/6),$$

which yields total curvature 0, so Lemma 4.43 applies and the claim is proven. $\square$

For groups with a relator R of length 13, we get 37 groups on which `IsHyperbolic` returns fail. Lemma 4.43 applies for 20 of them, so we end up with 17 groups that need to be investigated further. In the following we give an example of one of those groups and show how we can prove that `RSym`$^+$ succeeds at level 2.

Example 5.7. A one-relator group with $|R| = 13$ for which `RSym` fails is the group

$$G = \langle x, y, z | (z^{-1}y^{-1}x^{-1}yxz^{-1})^2x^{-1} \rangle$$

with pregroup presentation

$$\mathcal{P} = \langle x, y, z, X, Y, Z | xX, yY, zZ, R := (ZYXyxZ)^2X \rangle.$$

There are 12 different ways to decompose R such that `RSym` fails on a non-boundary face labelled by R, so Lemma 4.43 is not applicable. However, we argue that `RSym`$^+$ still succeeds at level 2. One example of a diagram where `RSym` fails is shown in Figure 8.

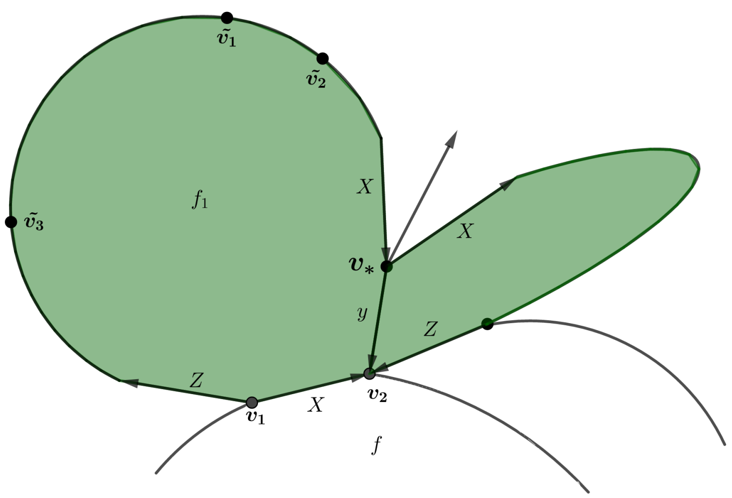

Figure 9: The decomposition of f_1 if v_2 has degree 4.

First note that the consolidated edge labelled by X between the faces labelled by f and f_1 in the figure is a step in every decomposition of R since neither ZX nor XZ can be a step of R. Furthermore, the three vertices v_1, v_2 and v_3 in the diagram in Figure 8 appear in every decomposition of R and the degree shown in this diagram is minimal over all diagrams. Hence, the total curvature of $17/60$ is maximal for f. Consider the face f_1 in Figure 8. We compute an upper bound on the curvature of f_1, when f_1 is non-boundary. Note that when f and f_1 are both non-boundary faces and v_2 is a boundary vertex, then v_2 has valency at least 5. We initially assume that v_2 is an internal vertex.

The first case to consider is that the valency of v_2 is 4 as in Figure 8. By $\tilde{v_1}, \tilde{v_2}, \tilde{v_3}$ we denote the three vertices on the boundary of f_1 that correspond to the location of v_1, v_2, v_3 in the decomposition labelling f. They give curvature at most $-1/6 - 1/4 - 3/10 = -43/60$ since the degrees of v_1, v_2, v_3 are minimal. The face f_1 has two additional vertices v_1 and v_2 that give curvature $-1/6$ and $-1/4$. Furthermore, since v_2 has valency 4, there is a third additional vertex v_* (see Figure 9), which gives curvature at most $-1/4$ since neither X^2 nor x^2 is a subword of R. Therefore, we can bound the total curvature $\kappa(f_1)$ of f_1 by

$$\kappa(f_1) \leq 1 - 43/60 - 1/6 - 2/4 = -23/60.$$

Hence, if we choose ε small enough, f_1 has enough negative curvature to donate $-17/60-\varepsilon$ of it to f.

The next question to consider is what happens, when the degree of v_2 is 5 since then vertex v_* might not exist. In this case the total curvature of f would be $1-1/6-6/10 = 7/30$. The faces would be arranged like in Figure 10 and we see that there is a new vertex $\tilde{v_*}$ that also gives curvature at most $-1/4$. Again f_1 has enough negative curvature for itself and f.

If v_2 is internal with degree larger or equal to 6 or v_2 is a boundary vertex with degree

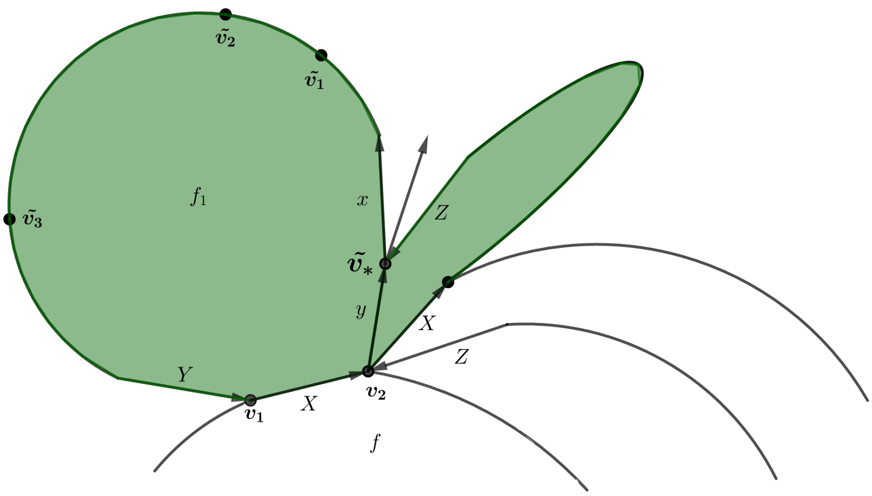

Figure 10: The decomposition of f_1 if v_2 has degree 5.

larger or equal to 5, the total curvature of f is $\leq 1 - 1/3 - 1/6 - 3/10 = 1/5$ and the total curvature of f_1 is $\leq 1 - 1/3 - 1/6 - 3/10 - 1/4 - 1/6 = -13/60$ which is sufficient for f_1 and f.

In all of these cases, any additional bad face, i.e. face with non-negative curvature, that might be adjacent to f_1 via a consolidated edge labelled by X would produce 2 more vertices on the boundary of f_1 giving curvature at most $-1/6$ and $-1/4$, which could be donated right back to the bad face in question. Therefore, f_1 can provide enough negative curvature to any adjacent bad face, which concludes the argument that `RSym`$^+$ succeeds at level 2 since any face on which `RSym` fails has a neighbouring face like f_1.

This leaves us with 16 groups that need to be investigated in the same manner. While doing this for some randomly picked groups, we were not able to find an example where `RSym`$^+$ does not succeed at level 2 for relators of length ≥ 13. Since the number of groups to analyse grows larger as the length of the relator gets longer, it is not feasible to perform a procedure like in Example 5.7 one by one for each group by hand. Instead, implementing an algorithm that tries to verify that `RSym`$^+$ succeeds at level 2 on a given pregroup presentation might give more insight.

Next, we consider if `VerifySolver` succeeds on the groups on which `RSym` does. The following example shows that this is not the case in general.

Example 5.8. Consider the group G given by the pregroup presentation

$$\mathcal{P} = \langle x, y, z, X, Y, Z | xX, yZ, zZ, R := Z^4 Y X^2 yx \rangle.$$

The only possible steps longer than 1 are Z^2 and Z^3. Therefore, any decomposition of R consists of at least 7 steps. Since the maximal curvature of a step is $-1/6$, we see that the total curvature of an internal face is at most $-1/6$, so `RSym` succeeds on $\mathcal{P}$.

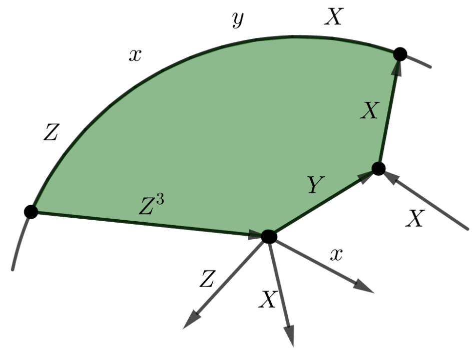

Figure 11: The decomposition of R as a boundary face in Example 5.8.

Table 4: Results of `IsHyperbolic` with $\varepsilon = 1/100$ on rank 3 one-relator groups with relators R that satisfy the BM-condition.

$\lvert R\rvert$	number of groups	number of fails
≤ 10	606	0
11	$2,973$	11
12	$14,961$	0
13	$95,085$	67
14	$564,718$	13

A boundary face with curvature $1/30 > 0$ whose internal boundary is labelled by a subword of R of length $5 > \lvert R\rvert/2$ is shown in Figure 11. Hence, `VerifySolver` does not succeed on $\mathcal{P}$.

5.2.2 `RSym` and the Blufstein-Minian condition

In this section, we analyse if `RSym` succeeds on groups that satisfy the Blufstein-Minian condition (BM-condition). For this purpose, we are working with a list of shortest representatives of $\mathrm{Aut}(F)$ equivalence classes of cyclic subgroups of F of rank 3 with a relator of length up to 14. We have checked computationally which words of our list satisfy the BM-condition and then ran the function `IsHyperbolic` on them. The computational results can be found in Table 4. Before analysing the computational results, we state a more general observation that is true for one-relator groups of all ranks and which can be proved by hand.

Theorem 5.9. *For one-relator groups with relators of length up to 12 that satisfy the BM-condition, there exists a pregroup presentation on which `RSym`$^+$ succeeds at level 2.*

Proof. Consider a one-relator group G with presentation $\mathcal{Q} = \langle \mathcal{S} | R \rangle$ that satisfies the BM-condition. If $|R| < 7$, then there exists no tripod in any diagram over $\mathcal{Q}$ since the minimum length of a tripod is 3. Hence, the presentation satisfies the $C'(1/4) - T(4)$ condition, so `RSym` succeeds on $\mathcal{Q}$ by Corollary 5.2.

For $|R| = 7$ or 8, the maximal length of a step is 1 since $\mathcal{Q}$ satisfies the $C'(1/4)$ condition. Hence, $\mathcal{Q}$ satisfies the $C'(1/6)$ condition and `RSym` succeeds on $\mathcal{Q}$ by Corollary 5.2.

For $|R| = 9$, the maximal length of a step is 2 and the maximal length of a tripod is 4. Consider a face f labelled by a decomposition of R that consists of only five steps. Then four of these steps are of length 2, so there are 3 vertices that cannot have degree 3, hence the maximum curvature of f is $-1/12$. Let f be labelled by a decomposition of R that consist of 6 steps and assume that `RSym` does not succeed on $\mathcal{Q}$ because of this decomposition. Then there are 6 vertices of degree 3 on the boundary of f and the total curvature is 0. Because of the (T') condition, in the decomposition there are three steps of length 1 and three steps of length 2 and the steps of length 1 and 2 are alternating. Consider an edge labelled by a step p of length 1. It is preceded and followed by an edge of length 2 because its start and end vertex have degree three and a tripod has maximal length 4. Furthermore, the third edge in the tripod is also labelled by a word of length 1. Hence, the decomposition of the face $\hat{f}$, by which we denote the face on the other side of the consolidated edge labelled by p, contains at least three consecutive steps of length 1. Therefore, if $\hat{f}$ is non-boundary, it has total curvature at most $-1/6$. Since $\hat{f}$ is only adjacent to a finite number of bad faces (i.e. faces with curvature 0), in Step 5 of the $\texttt{RSym}^+$ algorithm, it can give enough of its curvature to all of them if we chose ε small enough. Therefore, $\texttt{RSym}^+$ succeeds at level 2 on $\mathcal{Q}$ in this case.

For $|R| = 10$, the maximal step length and maximal tripod length remain the same. If a decomposition consists of 5 steps, all of them are of length 2, so there can be no tripods, which results in a total curvature of at most $-1/4$. If it consists of 6 steps, we obtain at least two vertices that are in between two edges labelled by words of length 2 that give curvature at most $-1/4$ each because of the maximal tripod length. Hence, `RSym` succeeds on $\mathcal{Q}$.

For $|R| = 11$, we have a maximal step length of 2 and a maximal tripod length of 5. With the same argument as for $|R| = 9$, one can show that $\texttt{RSym}^+$ succeeds on $\mathcal{Q}$ at level 2.

For $|R| = 12$, `RSym` might not succeed on a decomposition of R with 6 vertices of degree 3 where each step is of length 2. Since the maximal tripod length is 5, all adjacent faces have curvature less than 0 in this case because their boundary contains an edge of length 1. Hence, $\texttt{RSym}^+$ again succeeds on $\mathcal{Q}$ at level 2. $\square$

By hand, we were able to show that in general for relators of length 9 and 12, $\texttt{RSym}^+$ succeeds at level 2. In fact, when investigating one-relator groups of rank 3 using the `walrus` package in GAP, we obtain that `RSym` already succeeds on all relators of length 9 and 12 at level 1. Therefore, the decompositions of length 6 that we construct in the proof of Theorem 5.9, where `RSym` might fail, do not appear for groups of rank 3. Another observation is that all of the groups on which `IsHyperbolic` fails have relators that are words in only 2 letters. From the computational results we deduce the following

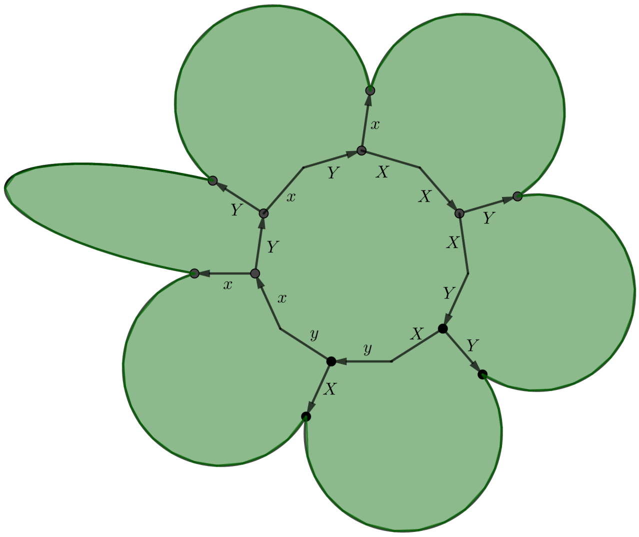

Figure 12: A diagram over the pregroup presentation $\langle x, y, X, Y | xX, yY, xYX^3YXy^2xY \rangle$ on which RSym fails.

theorem:

Theorem 5.10. *Let $G = \langle \mathcal{S} | R \rangle$ be a one-relator group of rank 3 that satisfies the BM-condition. Then there exists a pregroup presentation $\mathcal{P}$ of G such that:*

*(i) If R is of length less or equal to 10 or of length 12, then **RSym** succeeds on $\mathcal{P}$.*

*(ii) If R is of length less or equal to 14 and contains all 3 letters of S, then **RSym** succeeds on $\mathcal{P}$.*

Example 5.11. An example of a one-relator group, which is hyperbolic by the Blufstein-Minian condition, but where $\mathtt{RSym}^+$ does not succeed on the corresponding pregroup presentation at level 1 is the group

$$G =< x, y | xy^{-1}x^{-3}y^{-1}x^{-1}y^2xy^{-1} > .$$

In Example 3.21 we have already shown that G satisfies the BM-condition. The diagram where $\mathtt{RSym}^+$ fails at level 1 is shown in Figure 12. The non-boundary face has total curvature 0 and the adjacent faces have a total curvature of 1/6 respectively. However, this is the only decomposition of the relator where RSym fails, so $\mathtt{RSym}^+$ succeeds at level 2 by Lemma 4.43.

When analysing the groups with relators of length 13 on which RSym fails, we see that there are 52 groups for which Lemma 4.43 applies, so we end up with 15 groups that need to be investigated further. Again, we were not able to find an example where $\mathtt{RSym}^+$ does not succeed at level 2.

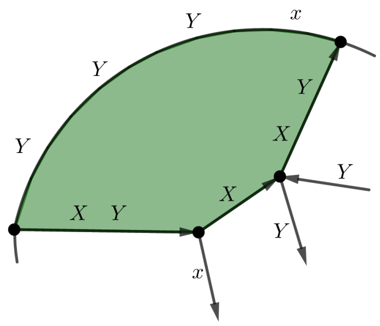

Figure 13: The decomposition of R as a boundary face in Example 5.14.

For relators of length 14, there are 9 groups for which Lemma 4.43 is applicable. In a pregroup presentation with generators x, y, z and X, Y, Z where $x^\sigma = X$, $y^\sigma = Y$ and $z^\sigma = Z$, the relators of the remaining four groups are: $Z^2YZ^2YzY^2ZY^2Zy$, $Z^3Y^2ZYZ^2yZyzY$, $Z^2YZ^2YzyZY^2zy^2$ and $zY^2z^3Y^2Z^3YZY$. Analysing those presentations in a similar manner as in Example 5.7 yields that `RSym`$^+$ also succeeds on them at level 2.

Proposition 5.12. *For one-relator groups of rank* 3 *with relators of length up to* 14 *that satisfy the BM-condition, there exists a pregroup presentation on which* `RSym`$^+$ *succeds at level* 2.

Finally, we want to investigate if `VerifySolver` succeeds on the groups on which `RSym` does. Since we have seen in the proof of Theorem 5.9 that a one-relator group $G = \langle \mathcal{S}|R\rangle$ with $|R| \leq 8$ satisfying the BM-condition has either a $C'(1/6)$ or a $C'(1/4) - T(4)$ presentation, the following result follows immediately from Theorem 5.3:

Proposition 5.13. *For one-relator groups with presentation* $\langle \mathcal{S}|R\rangle$ *with* $|R| \leq 8$ *that satisfy the BM-condition,* `VerifySolver` *succeeds on the corresponding canonical pregroup presentation* $\mathcal{P}$.

On presentations of groups with relators of length ≥ 9, `VerifySolver` does not succeed in general, as is shown by the following example:

Example 5.14. Consider the group given by the presentation

$$\mathcal{P} = \langle x, y, X, Y | xX, yY, R := Y^3XYX^2Yx\rangle.$$

The only possible steps of R longer than 1 are Y^2 and XY. In order to prove that the (T') condition is verified, it suffices to check if a vertex between edges labelled by Y^2 and XY can have degree 3. This is not the case since the edge following Y^2 needs to be labelled by Y and the edge preceding XY needs to be labelled by X. Since the other possible steps of length 2 are always preceded and followed by a step of length 1, the maximal tripod length is 4. Furthermore, $\mathcal{P}$ satisfies the $C'(1/4)$ condition, hence, it satisfies the BM-condition. Since $|R| = 9$, `RSym` succeeds on $\mathcal{P}$ by Theorem 5.10. However, `VerifySolver` does not succeed on $\mathcal{P}$ because a boundary face of the form shown in Figure 13 has total curvature $1/12 > 0$ and the internal boundary is labelled by a subword of R of length $5 > |R|/2$.

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