# MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

„Cofinitary Groups"

verfasst von / submitted by

Maximilian Vötsch, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna 2020

| | |
|---|---|
| Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet: | UA 066 821 |
| Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet: | Masterstudium Mathematik |
| Betreut von / Supervisor: | Vera Fischer, Privatdoz. PhD |
| Mitbetreut von / Co-Supervisor: | |

## Abstract

The topic of this thesis is cofinitary groups, which are special subgroups of the infinite permutation group $S_\omega$. We will begin by giving an overview of the algebraic properties of cofinitary groups. We will survey the algebraic properties of cofinitary groups, where the main results give us bounds on the size of cofinitary groups based on their orbit structure. We will then examine how to construct cofinitary groups using inverse limits and automorphisms of Boolean algebras. We then begin looking at maximal cofinitary groups and their possible sizes as well as the combinatorial characteristic $\mathfrak{a}_g$. In chapter 4 we will use forcing to show that there are infinitely many, non-isomorphic, maximal cofinitary groups, by constructing a group with $n$ infinite and $m$ finite orbits, for any tuple $(n, m) \in \mathbb{N}_{>0} \times \mathbb{N}$. In chapter 5, we use forcing constructions to show the existence of a maximal cofinitary group into which every countable group embeds. Finally, we show that we can tightly control the possible sizes of cofinitary groups in a model by adapting a novel proof from the theory of maximal almost disjoint families.

## Abriss

Das Thema dieser Arbeit sind kofinitäre Gruppen, eine spezielle Klasse an Untergruppen der unendlichen Permutationsgruppen. Wir beginnen mit einer Übersicht der algebraischen Resultate für diese Gruppen. Die wichtigsten Resultate in diesem Kapitel sind strukturelle Einschränkungen der Kardinalität von kofinitären Gruppen durch ihre Orbitstruktur. In weiterer Folge betrachten wir Konstruktionen von kofinitären Gruppen mittels projektiver Limits und Automorphismen von Boolschen Algebren. Der Rest der Thesis befasst sich mit maximalen kofinitären Gruppen, wobei wir zuerst die möglichen Größen, sowie die kombinatorische Charakteristik $\mathfrak{a}_g$ betrachten. In Kapitel 4 werden wir Forcing verwenden, um zu jedem Tupel $(n, m) \in \mathbb{N}_{>0} \times \mathbb{N}$ eine maximale kofinitäre Gruppe zu finden welche $n$ unendliche und $m$ endliche Orbits aufweist, wodurch wir unendlich viele nicht isomorphe Gruppen konstruieren können. In Kapitel 5 konstruieren wir mittels Forcing eine maximale kofinitäre Gruppe in welche wir alle abzählbar unendlichen Gruppen einbetten können. Im letzten Kapitel zeigen wir eine Konstruktion, welche uns die möglichen Größen von maximalen kofinitären Gruppen in unserem Modell steuern lässt.

# Acknowledgments

I would like to thank Vera Fischer for providing a great environment to write this thesis and for the support I have received from her throughout the process.

I would also like to thank my parents who have enabled me to finish my studies with their support and have been providing me with an environment that not many students get to enjoy.

Furthermore, I would like to thank Paul for helping me create the illustration in Chapter 3 and I would like to thank Peter Cameron for the helpful correspondence, along with Alexandra Elbakyan for making scientific research accessible to everyone.

Lastly, I would like to thank Jasmin for her love and patience even when I stayed up into the wee hours of the morning working on my manuscript, as well as Pan and Lucy for the mountains of cat hair they have left between the pages of my drafts.

# Contents

# 1 Introduction

One of the fundamental objects studied in algebra is the family of permutation groups, with broad-reaching results such as Cayley's Theorem, which asserts that all finite groups embed into a subgroup of some finite permutation group. Similar results exist for infinite permutation groups, as we will see in Chapter 5. These types of groups also play a large role in model theory, as automorphisms of structures are defined by permutations of the universe.

The theory of infinite permutation groups is a vast topic, and an interested reader might want to consult [6] to gain some insight into the general theory and problems that exist in the field.

We will be examining a special class of subgroups of infinite permutation groups which have found a "new home" in set theory, as they closely relate to an object of interest in that field, maximal almost disjoint (mad) families. The particular subgroups of $S_\omega$, which we are interested in, are called cofinitary groups.

**Definition 1.1** (Almost Disjointness)**.** Two sets $A$, $B$ are called almost disjoint if $|A \cap B| < \omega$, i.e. they have finite intersection.

Let $\mathcal{A}$ be a set of infinite subsets of the natural numbers. Then we call $\mathcal{A}$ an almost disjoint family if all sets $A, B \in \mathcal{A}$ are pairwise almost disjoint.

If, additionally, for any infinite set $C \subset \omega$ we have

$$C \in \mathcal{A} \text{ or } \exists D \in \mathcal{A}\ |C \cap D| = \omega,$$

then we call $\mathcal{A}$ a maximal almost disjoint (mad) family. Furthermore the minimal size of an infinite mad family of subsets of the natural numbers is denoted by $\mathfrak{a}$.

Analogously, we can apply this concept to bijective functions of the natural numbers:

**Definition 1.2** (Cofinitary Permutation)**.** A permutation $\sigma \in S_\omega$ is called cofinitary if it has only finitely many fixed points or is the identity permutation.

Finally, we may define the central object of interest of this text:

**Definition 1.3** (Cofinitary Group)**.** A subgroup $G \leq S_\omega$ is called a cofinitary group if all $\sigma \in G$ are cofinitary.

Similarly to mad families, we may define a notion of maximality as follows: Let $G$ be a cofinitary group. Then $G$ is maximal if for any $\sigma \in S_\omega$ we have

$$\text{either } \sigma \in G \text{ or } \langle G, \sigma \rangle \text{ is not cofinitary.}$$

We can also characterize cofinitary groups in terms of almost disjointness, where two functions are said to be almost disjoint if they are almost disjoint as sets.

**Lemma 1.4.** *Let $G$ be a subgroup of $S_\omega$. Then $G$ is cofinitary iff $G$ is an almost disjoint family of sets.*

*Proof.* Any two permutations $\sigma, \rho$ which agree on infinitely many points would give us a permutation $\sigma^-1\rho$ which has infinitely many fixed points and any non-cofinitary permutation would have infinite intersection with the identity permutation, hence we have shown the equivalence. $\square$

From a set theoreticians point of view, we note that this definition naturally generalizes, for uncountable cardinal numbers $\kappa$, to groups of permutations of $\kappa$ with strictly less than $\kappa$-many fixed points. A treatment of maximal cofinitary groups of uncountable degree can be found in [10].

To aid intuition, let us consider a simple example of a cofinitary group before moving on:

**Example 1.5.** (i). The group $\langle f \rangle$, where $f \in Sym(\mathbb{N})$ is given by

$$f(x) \coloneqq \begin{cases} x+2 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = 1, \\ x-2 & \text{otherwise,} \end{cases}$$

is a countable cofinitary group and $\langle f \rangle \cong (\mathbb{Z}, +)$.

(ii). The element $g \in Sym(\mathbb{N})$ defined as $g = (123)(45)(67)(89)\dots$ can not be an element of a cofinitary group, as $g \circ g \neq id$ has infinitely many fixed points, even though $g$ itself is cofinitary.

This example illustrates the main difficulty of constructing these groups, which is the fact that we have to guarantee that all non-trivial words of elements will only have finitely many fixed points.

This is conceptually similar to the word problem, which is known to be undecidable for finitely presentable groups by Novikov [26]. The word problem can be solved for a group if we find an algorithm to determine whether an arbitrary word made up of group elements is equivalent to the identity.

# 2 Preliminaries

We will now establish some of the notation and conventions which we will use for the remainder of this thesis. Alongside these fundamental definitions, we will state some fundamental theorems.

For indices, we generally use lowercase Latin letters when indexing over the natural numbers and lowercase Greek letters when indexing in the transfinite case.

## 2.1 Model Theory

We will be using model-theoretic concepts all throughout this thesis, as there seems to be a strong connection between the theory of permutation groups and model theory. Forcing also relies on some model-theoretic arguments for some of the most central theorems of the technique.

All our structures will be in some language $\mathcal{L}$ which is a triple $(C, F, R)$ where $C$ is a set of constant symbols, $F$ is a set of function symbols, and $R$ is a set of relation symbols. A set $M$, along with interpretations of the symbols in $\mathcal{L}$, is called a structure. We say a set $T$ of $\mathcal{L}$-sentences is a theory, and we call it consistent if we can not derive a contradiction from the sentences in $T$. An example of a theory would be $\mathsf{PA}$, the axioms of Peano arithmetic. We call a structure $\mathfrak{M}$ a model of $T$ if all sentences of $T$ hold in $\mathfrak{M}$. Note that a model only exists if the theory is consistent, as models need to be free of logical contradictions.

One theorem that we will be using a lot throughout this thesis, even though those uses often are implicit will be the theorem of Löwenheim-Skolem:

**Theorem 2.1.** *Let $\mathfrak{B}$ be an $\mathcal{L}$-structure and let $B$ be its universe. Furthermore, let $S \subseteq B$ and let $\kappa$ be an infinite cardinal.*

*(i). If $max(|S|, |\mathcal{L}|) \leq \kappa \leq |B|$ then $\mathfrak{B}$ has an elementary substructure of size $\kappa$ containing $S$.*

*(ii). If $\omega \leq max(|B|, |\mathcal{L}|) \leq \kappa$ then there exists an elementary extension of $\mathfrak{B}$ of cardinality $\kappa$.*

Another concept that will appear is that of types:

**Definition 2.2** (Type)**.** Let $\mathfrak{A}$ be an $\mathcal{L}$-structure and let $B \subseteq A$. A set $p(x)$ of $\mathcal{L}$-formulas is called a type over $B$ if it is maximally finitely satisfiable in $\mathfrak{A}$. This means that for any finite subset of $q(x) \subseteq p(x)$, there is some element $a \in A$ such that $a$ satisfies all formulas in $q(x)$.

We say a type $p(x)$ is realized in $\mathfrak{A}$ if there is an element $a \in A$ such that $a$ satisfies all the formulas in $p(x)$. If this is not the case we say that the structure $\mathfrak{A}$ omits the type.

We say a structure $\mathfrak{M}$ is $\omega$-homogeneous if any isomorphism of finite substructures can be extended to an automorphism of $\mathfrak{M}$.

Finally we require one last theorem that will be used for constructions later in the thesis:

**Definition 2.3** (Skeleton)**.** For a language $\mathcal{L}$ the skeleton $\mathcal{K}$ of an $\mathcal{L}$-structure $\mathfrak{M}$ is the class of all finitely-generated $\mathcal{L}$-structures which are isomorphic to substructures of $\mathfrak{M}$. We say the structure $\mathfrak{M}$ is $\mathcal{K}$-saturated if its skeleton is $\mathcal{K}$ and for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and for all embeddings $f : \mathfrak{A} \to \mathfrak{M}$ and $g : \mathfrak{A} \to \mathfrak{B}$ there is an embedding $h : \mathfrak{B} \to \mathfrak{M}$ with $f = h \circ g$.

One important property of $\mathcal{K}$-saturated structures is that they are isomorphic.

**Theorem 2.4.** *Let $\mathcal{L}$ be a countable language and let $\mathcal{K}$ be a countable class of finitely-generated $\mathcal{L}$-structures. There is a countable $\mathcal{K}$-saturated $\mathcal{L}$-structure $\mathfrak{M}$ if and only if*

*(i). $\mathcal{K}$ is downward closed, i.e. if $\mathfrak{A} \in \mathcal{K}$, then all elements of the skeleton of $\mathfrak{A}$ belong to $\mathcal{K}$.*

*(ii). Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$. Then there is some $\mathfrak{D} \in \mathcal{K}$ and embeddings of $\mathfrak{A}$ and $\mathfrak{B}$ into $\mathfrak{D}$.*

*(iii). Let $\mathfrak{B}, \mathfrak{C} \in \mathcal{K}$ such that they have a common substructure $\mathfrak{A}$ that embeds into $\mathfrak{B}$ and $\mathfrak{C}$ via $e_1$ and $e_2$ respectively. Then there is some $\mathfrak{D} \in \mathcal{K}$ and embeddings $f : \mathfrak{B} \to \mathfrak{D}$ and $g : \mathfrak{C} \to \mathfrak{D}$, such that $f \circ e_1 = g \circ e_2$.*

*We call this $\mathfrak{M}$ the Fraïssé limit of $\mathcal{K}$.*

The third property of the above theorem is called the "amalgamation property" and one might replace it with the so-called "strong amalgamation property", which stipulates that

$$im(f(\mathfrak{B})) \cap im(g(\mathfrak{C})) = im(f(e_1(\mathfrak{A}))) \quad (= im(g(e_2(\mathfrak{A})))).$$

For a more thorough introduction to model theory, as well as the proofs to the theorems mentioned above, we would recommend either [29] or [23].

## 2.2 Set Theory

All of the set theoretic proofs in this thesis will be using the axioms of ZFC (Zermelo-Fränkel-Choice) with additional axioms specified as necessary. Our set theoretic language will be that of $(\emptyset, \emptyset, \{\in\})$ with the usual interpretation. All other symbols (subsets, intersections, ...) are definable in terms of this language, and we merely see them as a form of "syntactic sugar" to make proofs readable to the working mathematician.

Generally, we will follow the notational conventions laid out by [21], which is also one of the main references used for set theoretic questions. Another frequently recommended textbook about set theory is [15].

One important idea is that we can always treat maps $f : A \to B$ as a special subset of $A \times B$, in which elements of $A$ may only appear in at most one pair. If the function is only partially defined on $A$ we will often write $f : A \rightharpoonup B$. $dom(f)$ and $ran(f)$ are the domain and range of the map $f$ respectively.

When discussing cardinalities, as a convention we will use $\omega$ in place of $\aleph_0$ and $\mathfrak{c}$ instead of $2^\omega$ or $2^{\aleph_0}$ to denote the size of the continuum. Should other cardinal numbers appear, then we will either define their meaning explicitly or stick to standard $\aleph_\alpha$ notation indexed via ordinal numbers.

Let $X$ be a set and let $\kappa$ be a cardinal number. Some commonly used shorthand notation throughout the thesis will be $\mathcal{P}(X)$ to denote the power set operation, $X^\kappa$ to denote sequences of length $\kappa$ formed with elements of $X$ and $[X]^\kappa$ as the set of all $\kappa$ sized subsets of $X$. If $|X| < \kappa$ we take this set to be empty. Furthermore we define

$$[X]^{<\kappa} := \bigcup_{\alpha<\kappa} [X]^\alpha,$$

the set of all less than $\kappa$ sized subsets of $X$.

Most proofs from the fourth chapter onwards will be utilizing forcing as a proof technique. Forcing is a powerful machinery used to construct models of ZFC in which we can guarantee the existence of certain sets. Any reader that is not familiar with forcing is urged to familiarize themselves with the concept in order to be able to follow the logic of the proofs. The standard texts for this are once again [21] and [15]. Another recommended introductory text, that is somewhat less technical, is [25].

Finally we will state one theorem that will be used frequently in later sections:

**Theorem 2.5** ($\Delta$-System Lemma). *Let $\kappa$ be any infinite cardinal and let $\lambda > \kappa$ be a regular cardinal such that*

$$\forall \alpha < \lambda (|\alpha^{<\kappa}| < \lambda).$$

*Then for any family of sets $\mathcal{A}$ with $|\mathcal{A}| \geq \lambda$ and $\forall x \in \mathcal{A}(|x| < \kappa)$ there is $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| = \lambda$ and there exists a fixed set $r$, called the root, such that for any $a, b \in \mathcal{B}$ we have $a \cap b = r$.*

A proof of this theorem can be found in [21].

## 2.3 Group Theory

As we will sometimes (implicitly) treat the groups we work with as a model theoretic structure we need a language of groups. The one we will be using is

$$\mathcal{L}_G \coloneqq (\{\underline{1}\}, \{\underline{\cdot}, \underline{\phantom{x}}^{-1}\}, \emptyset),$$

where the usual interpretations are used. The theory of groups includes the usual axioms of neutral and inverse elements as well as associativity.

Most groups we will be working with in the later sections will be of infinite cardinality, thus we can't rely on a lot of classical results to aid us in classification, as they mostly apply to finite groups.

First let us remind ourselves of a central definition that will appear a lot all throughout the text:

**Definition 2.6** (Group action)**.** Let $G$ be a group and let $S$ be a set. A group action is a function $\mu : G \times S \to S$ such that the following conditions hold:

(i). $\mu(g, \mu(h, s)) = \mu(gh, s)$ for all $g, h \in G$, $s \in S$;

(ii). $\mu(1, s) = s$ for all $s \in S$.

We will not be distinguishing left and right group actions as these definitions are essentially the same for our purpose.

We can define an equivalence relation $\sim$ for any group action, in the following way

$$s \sim t \iff \exists g \in G \quad g(s) = t,$$

where $s, t \in S$. The equivalence classes of this relationship are called the orbits of the action $\mu$. If there is only one equivalence class we call the action transitive.

Further, let us recall that the stabilizer of a point $s \in S$ is defined as $G_s \coloneqq \{g \in G \mid g(s) = s\}$, i.e. the set of permutations with fixed point $s$. We note that $G_s$ is a subgroup of $G$.

**Definition 2.7** (Free group)**.** Let $A$ be a set of symbols. We define the free group on $A$ to be the group with the presenation

$$F(A) \coloneqq \langle A \mid \emptyset \rangle.$$

The elements of this group are reduced words made up of letters from the alphabet $A$. In the case of free groups we call the cardinality of the base set the rank of the group. Any free group has the universal property that a function

$f : A \to G$ from the base set into some group $G$ extends uniquely to a group homomorphism $F : F(A) \to G$.

If we let $G$ and $H$ be groups, then we call $G * H$ the free product of groups which is defined as

$$G * H := \langle G \cup H \mid R_G \cup R_H \rangle,$$

where $R_G$ denotes the set of relations of $G$.

If we now assume we have an action $\mu$ of some group $G$ and we let $w := g_1 g_2 \dots g_n$ be a word in the group, then we can evaluate the action of $\mu(w, x)$ step by step due to the associativity of the group action. Our convention will be that the evaluation of $\mu(w, x)$, also written as $w(x)$ when no confusion about the action can arise, will be done from right to left, i.e. the first element we evaluate will be $x_n := \mu(g_n, x)$, then $\mu(g_{n-1}, x_n)$ and so on.

# 3 The Algebraic Perspective

Before focusing on the class of maximal cofinitary permutation groups, which requires a lot of set theoretic machinery, we will take an excursion into the classical treatment of these groups.

The study of cofinitary groups arose naturally after research was conducted by Wielandt [32], and subsequently Neumann [24], on the structure of finitary groups, permutation groups on infinite sets whose elements all have finite support. As opposed to the cofinitary groups, maximality of this class of groups is trivial, as the group that contains all permutations with infinitely many fixed points is also finitary, since $|supp(f \circ g)| \leq |supp(f)| + |supp(g)|$, where $supp(f) :=$ $\{n \mid f(n) \neq n\}$.

Considering some of the results presented later, there seems to be little hope of finding a theorem for classifying them in full generality.

## 3.1 Permutation Groups

In this section we will review a few of the definitions from the theory of permutation groups which we will use a lot throughout the rest of the chapter. For a more in depth treatment of the theory of permutation groups, see [6] or [8].

Let $G$ be a permutation group defined on a set $S$. Then we call the cardinality of $S$ the degree of $G$. The action of $G$ on $S$ which we obtain by applying a permutation $g$ as a bijective function on $S$ is called the natural action of $G$.

A permutation which only exchanges two elements and leaves all others in place is called a transposition. If the set $S$ is finite, we can define the sign of a permutation $\sigma$ to be

$$sgn(\sigma) := \begin{cases} 1 & \text{if } \sigma \text{ can be written as a product of an even} \\ & \text{number of transpositions,} \\ -1 & \text{otherwise} \end{cases}$$

The alternating group $A_n \leq S_n$ is then defined as the group containing all permutations with positive sign.

In the theory of permutation groups, there are a number of group actions with special properties that can provide us additional means to aid in classification.

We call a group $G$ semiregular, if no permutation other than the identity has a fixed point or equivalently, the stabilizer $G_s$ is trivial for all $s \in S$. If the group $G$ also acts transitively, we call the group regular.

If $G$ is a permutation group with a regular normal subgroup $N \trianglelefteq G$, then we can identify the set $S$ with $N$ by fixing $s \in S$ and then using the bijection

$$
\begin{aligned}
f\colon N &\longrightarrow S \\
n &\longmapsto t := n(s)
\end{aligned}
$$

Additionally we note that the above map also induces an isomorphism between the action of $G_s$ on $S$ and the action of $G_s$ on $N$ via conjugation. First note that the action of $G_s$ on $N$ is closed, so we always stay inside $N$. Now let $n \in N$ be such that $t = n(s)$, then

$$(g^{-1}n_1 g)(s) = g^{-1}(t)$$

and we see that by regularity of $N$ we get a uniquely determined element $n_1$ for each $n$, the one mapping $s$ to $g^{-1}(t)$.

We know that $N \cap G_s$ will always be trivial, so if we take $G_1$ and note that $G = NG_1$ (if $g(1) = k$, then $n \in N$ such that $n(1) = k$ gives us $n^{-1}g \in G_1$, which yields a unique solution to the equation $g = nx$ for $x \in G_1$), then we see that $G$ is the semidirect product of $N$ and $G_1$.

Let $k \in \omega$, we say that $G$ is $k$-transitive on $S$ if $G$ acts transitively on $S^k$, the space of $k$-tuples under the componentwise action. If $G$ is $k$-transitive and for every pair of tuples $(a, b)$ there is a unique $g \in G$ such that $g$ maps $a$ to $b$ then we say $G$ is sharply $k$-transitive. As an example, the finite symmetric group $S_n$ is both sharply $n$ and $n - 1$ transitive.

It is a theorem that for $k \geq 4$ the only sharply $k$-transitive groups are either the symmetric groups $S_k$ or $S_{k+1}$, the alternating group $A_{k+2}$ and the Mathieu groups $M_{11}$ for $k = 4$ and $M_{12}$ for $k = 5$. Thus all the sharply $k$-transitive cofinitary groups are either isomorphic to these or have $k < 4$. Those interested in a proof of this theorem should consult [30] or [33].

Let $G$ be a group acting on a set $S$ and let $\sim$ be an equivalence relation defined on $S \times S$. We say $\sim$ is $G$-invariant if for all $s, t \in S$ and all $g \in G$ we have

$$s \sim t \iff g(s) \sim g(t).$$

Any action admits two trivial $G$-invariant equivalence relations, equality, i.e. $s \sim t \iff s = t$, and the universal relation where $s \sim t$ for all $s, t \in S$.

A group $G$ acting on $S$ is said to be primitive if these are the only possible equivalence relations on $S$ which are $G$-invariant.

Lastly, we need one more definition that will allow us to more precisely characterize the groups we work with.

**Definition 3.1** (Type)**.** For a permutation group $G$ we call the set

$$typ(G) := \{n \in \omega \mid \exists \sigma \in G \setminus \{id\}\ |fix(\sigma)| = n\}$$

the type of $G$. If $max(typ(G))$ exists, then we say that the type of $G$ is bounded.

Note that any semiregular group will always be of type 0. Note that this specific type is not the concept introduced before, but it could be defined as a model theoretic type in a language of group theory that allows for group actions.

**Lemma 3.2.** *Let $G \leq S_\omega$. Then there exists a relational structure $M$ on the universe $\omega$ such that*

*(i). $G \leq Aut(M)$,*

*(ii). $G$ and $Aut(M)$ have the same orbits in $\omega^n$ for all $n \in \omega$.*

*Proof.* For each $n \in \omega$ let us decompose $\omega^n$ into orbits under the action of $G$, in total there are countably many, so let us fix an enumeration as $O_1, O_2, \ldots$ Now associate a relation symbol $R_i$ to each orbit $O_i$ such that for a tuple $x$

$$R_i(x) \iff x \in \omega^n \text{ and } x \in O_i.$$

The Lemma directly follows from the construction of the structure $M := (\emptyset, \emptyset, (R_i)_{i\in\omega})$. □

*Remark* 1. This relational structure is called the "canonical relational structure". Note that there may be many more non-isomorphic structures for which $G \leq Aut(M)$ holds.

## 3.2 Residually Finite Groups

One class of groups that often appear when studying cofinitary permutation groups are the residually finite groups, also known as the "finitely appproximable" groups.

**Definition 3.3** (Residually Finite Group)**.** A group $G$ is said to be residually finite if for each $g \in G \setminus \{1\}$ there is a homomorphism $\phi\colon G \to H$ to a finite group $H$ with $\phi(g) \neq 1$.

We note that any finite group is trivially residually finite via the identity homomorphism. Some other examples are the finitely generated nilpotent groups or finitely generated linear groups (which is a famous result by Mal'cev [22]) , along with the free groups, which we want to quickly examine in more detail. The proof of the following proposition appears in [7].

**Proposition 3.4.** *Let $G$ be a free group of finite rank $n$. Then $G$ is residually finite.*

*Proof.* Let $x_1, \dots, x_n$ be the generators of $G$ and let $w$ be a reduced word in $G$. Write $w = x_{i_1}^{e_1} x_{i_2}^{e_2} \dots x_{i_m}^{e_m}$ where each $x_{i_k}$ is a generator and $e_j = \pm 1$. We will construct a homomorphism $\phi \colon G \to S_{m+1}$ as follows. If $e_j = 1$ we let $\phi(x_{i_j})$ be such that it maps $k \mapsto k+1$ and if $e_j = -1$ then it maps $k+1 \mapsto k$. These restrictions will impart certain conditions on the permutations we may map our generators to, but as long as we make choices in accordance to them we obtain $\phi(w)(1) = m+1$. Note that it may happen that a generator does not appear in $w$ and we can thus freely choose any element $f$ of $S_{m+1}$ such that $\phi(w) = f$. $\square$

## 3.3 Cofinitary Groups

As a unified structural theory for general cofinitary groups currently seems outside of our grasp, we will consider subclasses of cofinitary groups that share some common structure. Often the structure we consider is that of the orbits of the natural action. This observation, along with all the results in this section, is due to Cameron [5]. Those cofinitary groups where all orbits are finite are particularly nice to work with and have some unifying features.

Before we begin with said examination, we present some elementary facts:

**Proposition 3.5.** *Let $G$ be a cofinitary group.*

*(i). Any subgroup $H \leq G$ is cofinitary.*

*(ii). If $G$ is cofinitary and its action on $S$ has an infinite orbit $O$, then it must act faithfully on $O$.*

*(iii). If $G$ acts cofinitarily on $S_1$ and $S_2$ then it also acts cofinitarily on $S_1 \cup S_2$ and $S_1 \times S_2$.*

Now let us begin showing some non-trivial results on cofinitary groups.

**Proposition 3.6.** *Let $G$ be a group. The following conditions are equivalent.*

*(i). $G$ is isomorphic to a permutation group of countable degree with finite orbits.*

*(ii). $G$ is isomorphic to a cofinitary permutation group of countable degree with all orbits finite.*

*(iii). $G$ has a countable family of subgroups of finite index with trivial intersection.*

*(iv). $G$ is a product of countably many finite groups.*

*Proof.* To see that (i) implies (ii), we begin by enumerating the orbits of $G$ as $O_1, O_2, \ldots$. Now define $\Delta_1 \coloneqq O_1$ and inductively let $\Delta_i$ be a $G$-orbit in $\Delta_{i-1} \times O_i$. Next, let $\Delta'_i$ be the regular representation of the transitive constituent $G^{\Delta_i}$ (the transitive permutation group on $\Delta_i$ induced by $G$), which is always finite. Note that for any non-trivial element $g \in G$ there exists an $i \in \omega$ such that $g$ acts non-trivially on $O_i$. This tells us that it also acts fixed point freely on $\Delta'_j$ for all $j \geq i$, thus assuring every permutation in $G$ is cofinitary on $\bigcup_i \Delta'_i$.

Assume $G$ is a cofinitary permutation group with all orbits finite, then we can express it as a subgroup of the Cartesian product of its transitive constituents. Let $G_1, G_2, \ldots$ be an enumeration of these constituents and define homomorphisms $\pi_i \colon G \to G_i$ as the natural projections on the $i$th coordinate of the cartesian product. Let $H_i \coloneqq ker(\phi_i)$ then $H_i \leq G$ and $[G : H_i] = |G_i| < \omega$. The family $(H_i)_{i\in\omega}$ now satisfies (iii).

To see (iii) implies (iv), we first recall that any subgroup of finite index contains a normal subgroup of finite index, by the property of the given family, we know that for every non-trivial element $x$ of $G$ there is $N_x \leq G$ with $x \notin N_x$ such that for the quotient map $\phi_{N_x} \colon G \to G/N_x$ we get $x \notin ker(\phi_N)$. We then see that we can embed $G$ into

$$K \coloneqq \prod_{x\in G\setminus\{id\}} G/N_x,$$

via the injective group homomorphism

$$\begin{aligned} \phi \colon G &\longrightarrow K \\ x &\longmapsto (\phi_{N_{x_1}}(x), \phi_{N_{x_2}}(x), \ldots) \end{aligned}$$

The product $K$ is isomorphic to a countable product of groups, as we need only countably many finite groups that we obtain from taking quotients.

Finally, to see that (iv) implies (i), recall that every finite group is isomorphic to a subgroup of a finite permutation group. Let $(G_i)_{i\in\mathbb{N}}$ be the countable family of finite groups and let $(m_i)_{i\in\mathbb{N}}$ be the size of the symmetric group $S_{m_i}$ such that there exists $H_i \leq S_{m_i}$ such that $G_i \cong H$. Then we can define $G = \prod G_i \cong \prod H_i \leq \prod S_{m_i}$. $\square$

Comparing (iii) to the definition of residually finite groups immediately yields the following corollary.

**Corollary 3.7.** *Any countable residually finite group is isomorphic to a cofinitary group with finite orbits.*

Analyzing this particular class of groups, we come to see that it is closed under countable direct products, which is a trivial consequence of (iv) in the above proposition.

Using this fact we get another corollary.

**Corollary 3.8.** *The free group $F_{\mathfrak{c}}$ of rank $2^\omega$ is isomorphic to a cofinitary group.*

*Proof.* First, let us recall that the set $2^\omega$ consists of infinite sequences of 1s and 0s. Let us define a homomorphism $\phi\colon F_{\mathfrak{c}} \to F_2^\omega$ where $F_2^\omega$ is a countable direct product of free groups on two generators. We note that this infinite product has uncountably many elements.

Let $w = r_1^{e_1} \dots r_n^{e_n}$ be a word in $F_{\mathfrak{c}}$ and consider each $r_i$ as an infinite sequence of 1s and 0s where $r_i(k)$ denotes the $k$th element of the sequence. Then we define

$$\phi(w) := (r_1(1)^{e_1} r_2(1)^{e_2} \dots r_n(1)^{e_n}, r_1(2)^{e_1} r_2(2)^{e_2} \dots r_n(2)^{e_n}, \dots),$$

which gives us an embedding of the set of words in $F_{\mathfrak{c}}$ into $F_2^\omega$. We also immediately see that this map is a isomorphism, telling us that $F_{\mathfrak{c}}$ is in fact cofinitary. □

Finally, using this next lemma, we are able to fully classify the cofinitary permutation groups which admit only finite orbits.

**Lemma 3.9.** *Let $G$ be a cofinitary group which has infinitely many finite orbits of size $n$. Then $|G| = n$.*

*Proof.* We will show this by contradiction.

We begin by fixing $n! + 1$ distinct elements in $G$, say $g_1, \dots, g_{n!+1}$, then we know that there are $n!$ possible permutations they can induce on an orbit of size $n$. By the pigeonhole principle, there must be at least one permutation induced on infinitely many orbits by $g_1$.

Now, consider only those orbits and see that $g_2$ must induce one permutation of the $n$ elements of these orbits infinitely often. Continuing iteratively, we obtain an infinite set of orbits on which each element $g_k$ induces the same permutation of elements.

As there can be only $n!$ many of those, at least two elements $g_i$ and $g_j$ must induce the same permutation on those orbits, thus $g_i g_j^{-1}$ would have infinitely many fixed points, a contradiction.

To see that $|G| = n$ we note that the elements of $G$ need to act regularly on all but finitely many of these orbits. This allows us to conclude that $|G| = n$ as elements are uniquely determined by the regular action on these orbits. □

This lets us obtain the following corollary:

**Corollary 3.10.** *Let $G$ be a cofinitary group with all orbits finite. Then it is either of countable degree or finite.*

As the last part of this section, we will be looking at some of the results concerning the normal subgroups of cofinitary groups. In classical group theory, gaining an understanding of the normal subgroups of a group makes it easier to understand possible homomorphisms into other groups, as well as the quotient groups. In infinite group theory, this becomes rather difficult, as indicated by the oftentimes very complicated automorphism groups of infinite permutation groups.

The Schreier-Ulam Theorem [28] indicates that no cofinitary group can be a normal subgroup of $S_\omega$, as the only two nontrivial subgroups of this group are $\bigcup_{n\in\omega} S_n$ and $\bigcup_{n\in\omega} A_n$.

Note, once again, that the result depends on the existence of finite orbits, which seem to aid greatly in obtaining elementary results.

This next result is once again presented in [5].

**Proposition 3.11.** *Let $G$ be an infinite, transitive cofinitary group and let $N \trianglelefteq G$ be a normal subgroup. If $N$ has a finite orbit, then it is semiregular and $G/N$ acts as a cofinitary group on the set of orbits of $N$.*

*Proof.* Assume $N$ has two orbits of different size $O_1$ and $O_2$ and let wlog $|O_1| < |O_2|$, then by transitivity of $G$ there exists an element $g \in G$ and elements $y \in O_1$ and $x_1 \in O_2$ such that $g(y) = x_1$. As $|O_1| < |O_2|$ there must be an $x_2 \in O_2$, such that $g(x_2) \notin O_1$. Finally, as $N$ is transitive on its orbits there exists an $f \in N$, such that $f(x_1) = x_2$. This gives us

$$N \ni (g^{-1}fg)(y) = g^{-1}(f(g(y))) = g^{-1}(x_2) \notin O_1.$$

Together with Lemma 3.9 this tells us that $N$ is finite and acts regularly on all but finitely many orbits. By a similar argument as above we get that $N$ must be a semiregular group. Now let $K$ be the kernel of the action of $G$ on the orbit set $\mathcal{O}$ of $N$. Then $N \leq K$. As $K$ is semiregular $|K| \leq |N|$, thus $N = K$. thus $G/N$ acts faithfully on $\mathcal{O}$.

It remains to show that $G/N$ acts cofinitarily. Indeed, let $g \in G$ fix infinitely many orbits in $O$. By the pigeonhole principle there must be one permutation of the set $1, \ldots, n$ that occurs infinitely often. $N$ must also act the same way on infinitely many of these orbits, so there is an $h \in N$ such that $gh^{-1}$ fixes an element in all of these orbits, which tells us that $gh^{-1} = id$, so $g \in N$. $\square$

Using this result and the fact that a cofinitary group always acts faithfully on infinite orbits, allows us to classify the actions of normal subgroups as follows:

**Corollary 3.12.** *Let $G$ be as above and let $N \trianglelefteq G$. Then $N$ acts faithfully on each orbit.*

Assuming primitivity of our cofinitary group will yield another structural result, for which we need the next definition.

**Definition 3.13** (Frobenius Group)**.** A group $G$ is said to be a Frobenius group if it is transitive and of type $\{0, 1\}$.

**Proposition 3.14.** *Suppose $G$ is an infinite, primitive, transitive cofinitary group and let $N \trianglelefteq G$ be a non-trivial abelian normal subgroup of $G$. Then one of the two following cases holds:*

*(i). $G$ is a Frobenius group,*

*(ii). $N$ is an elementary abelian $p$-group and $G$ is a semidirect product of $N$ with an irreducible cofinitary linear group of infinite dimension over $\mathbb{F}_p$.*

*Remark* 2. This result draws an explicit connection to the notion of cofinitary linear groups, which are subgroups of $GL(V)$ (where $V$ is some vector space) where every element has finite dimensional fixed point space.

*Proof.* A normal subgroup induces an equivalence relation through its orbits, so we know that if $G$ is primitive and transitive, $N$ must also be primitive. As any cofinitary group acts faithfully on infinite orbits, we get that $N$ must be regular.

This allows us to identify $N$ with the set of elements permuted by $N$ so that $N$ acts by right multiplication and $G_1$ acts by conjugation on this set. Since $G$ is primitive, $N$ has no non-trivial proper $G_1$-invariant subgroup.

Suppose now that $G$ is not a Frobenius group, then there is some $g \in G_1$ with non-trivial centraliser in $N$.

Assume $N$ has an element of finite order, and let $p$ be a prime dividing $|C_N(g)|$, then the elements of order dividing $p$ in $N$ form a characteristic subgroup, which must be all of $N$, and thus an elementary abelian $p$-group of infinite dimension.

Otherwise $N$ is torsion-free and so there is a non-trivial element $n \in N$ such that for a non-trivial $h \in H$ these elements commute, which would mean that $h$ also commutes with all powers of $n$, contradicting cofinitarity.

We know that $G \leq S_\omega$ is a subgroup of $GL(V)$, the infinite linear group over any vector space $V$ and thus trivially a subgroup of $AGL(V)$, the affine linear group. These groups decompose as $G = V \rtimes G_0$ where $V$ is the additive group of the vector space and $G_0$ is a linear group on $V$. In the case of $G$ primitive $G_0$ must be irreducible on $V$. □

Note that the stated proof differs from the one given in [5] which as pointed out to me in personal communication contained a minor flaw. Peter Cameron further stated that the result still holds when we do not ask for $G$ to be cofinitary, with the minor alteration that both $N$ and $H$ will be defined as linear over the rational numbers.

## 3.4 Topology of Cofinitary Groups

In this section we will examine how we can turn a cofinitary group into a topological group. This section is based on the definitions and results given in [4], in particular Chapters 2.3 and 2.4. There are still a number of open questions regarding the topological properties of cofinitary groups, especially the question if any maximal cofinitary groups are closed. In a recent paper Horowitz and Shelah showed the existence of a Borel maximal cofinitary group, assuming ZF [14].

Before we get into this, let us recall the basic definition.

**Definition 3.15** (Topological group)**.** Let $G$ be a group, we say that $G$ is a topological group if $G$ is a topological space and both the group law $\cdot\colon G \times G \to G$ and taking inverses ${}^{-1}\colon G \to G$ are continuous functions under the topology on $G$.

For a symmetric group of countable degree acting on the set $S$ and any of its subgroups, we can define a natural topology via pointwise convergence. We may assume $S$ to be $\omega$ without loss of generality. A sequence of permutations $f_n$ converges to a limit $f$ if for all $i \in \omega$ there is an $N \in \omega$, such that $f_n(i) = f(i)$ for all $n > N$.

To see that under this notion of convergence we have a topological group, let $\lim_{n\to\omega} g_n = g$ and $\lim_{n\to\omega} f_n = f$, then both $\lim_{n\to\omega} f_n^{-1} = f^{-1}$ and $\lim_{n\to\omega} f_n g_n = fg$. As this is very easy to show we will not give an explicit proof and leave it as an exercise to the reader.

In fact we can define a metric on the symmetric group that will induce this topology of pointwise convergence. For any $c \in (0, 1)$ we can define

$$d_c(g,h) := \begin{cases} 0 & \text{if} \quad g = h, \\ c^{-i} & \text{if} \quad g(n) = h(n) \quad \text{for} \quad n < i \quad \text{but} \quad g(i) \neq h(i). \end{cases}$$

This metric is a very intuitive notion, as it measures the length of the initial segment that two functions (interpreted as sequences, where the $n$th element is given by $f(n)$) agree on. Note that this topology is not complete. We can let $g_n := (012\ldots n-1)$, which is Cauchy in $S_\omega$, but the limit of $g_n$ is not in $S_\omega$ as 0 is not in the domain of $\lim_{n\to\omega} g_n$.

We can modify the metric to be

$$d'(g,h) \coloneqq max(d_c(g,h), d_c(g^{-1}, h^{-1})),$$

which defines the same topology but is complete.

**Proposition 3.16.** *Let $G \leq S_\omega$. Then $G$ is closed if and only if $G = Aut(M)$ for some first order structure $M$ on $\omega$.*

*Proof.* Let $g_n \to g$ be a sequence in $G$ and suppose $G$ is closed. Let $M$ be the canonical relational structure of $G$. Let us further suppose $\tilde{g} \in Aut(M)$ and let $\bar{a}$ be a tuple of elements of $\omega$. There is some $g' \in G$ such that $g'(\bar{a}) = \tilde{g}(\bar{a})$ by Lemma 3.2. We iteratively construct a sequence $g_n$ by choosing $g_n = g'$ for the tuple $\bar{a} = (0, \dots, n-1)$, this sequence converges to $\tilde{g}$ and since $G$ is closed we know that $\tilde{g} \in G$.

For the other implication we can assume wlog that $M$ is a purely relational structure.

Now suppose $G = Aut(M)$, and let $g_n \to g$ be a sequence in $G$. Let $a \in M$. Then there exists $n \in \omega$ such that $g_n(a) = g(a)$. Let $\bar{a}$ now be a tuple in $M$ satisfying a relation $R$. Note that as $g_n$ is an automorphism we know that there is some $\bar{n}$ such that $(g_{\bar{n}}(a_1), \dots, g_{\bar{n}}(a_n)) = (g(a_1), \dots, g(a_n)) \eqqcolon g(\bar{a})$ and thus $R(\bar{a})$ implies $R(g(\bar{a}))$. Thus $g$ is an automorphism of $M$ and so $G$ is closed. □

**Corollary 3.17.** *Let $M$ be a countably infinite first order structure $M$ then either $|Aut(M)| \leq \aleph_0$ or $|Aut(M)| = 2^{\aleph_0}$. The first case is true if and only if the stabiliser of some tuple is the identity.*

*Proof.* Let us assume there is some tuple whose stabiliser is the identity. This implies that $G$ must be a discrete group, and as such $G$ must be countable.

If not, then the identity and thus every point must be a limit point, which gives the other case. □

Similar results exist that help us understand other important topological subgroups of $S_\omega$:

**Proposition 3.18.** *Let $G \leq S_\omega$.*

*(i). $G$ is open if and only if it contains the stabilizer of a finite tuple in $S_\omega$.*

*(ii). $G$ is discrete if and only if there is a finite tuple whose stabilizer in $G$ is the identity.*

*(iii). $G$ is compact if and only if it is closed and all orbits are finite.*

*(iv). $G$ is locally compact if and only if it is closed and there is a finite tuple such that all the orbits of its stabilizer are finite.*

*Proof.* All but the last two statements have been shown previously. Before we show the second to last one, let us note that the last one is a trivial consequence of it.

Let us assume that there exists an infinite orbit $O$. Let $a \in O$ and define $X_b \coloneqq \{g \in G \mid g(a) = b\}$. These point stabilizers form an open cover of $G$. We note that any finite subset of $X \coloneqq \{X_b \mid b \in O\}$ will not form a cover of $G$. As $S_\omega$ is Hausdorff we see that the closedness is a necessary condition as well.

Now, assume $G$ is closed and has finitely many orbits. We enumerate the orbits as $O_1, O_2, \ldots$ Towards a contradiction we may assume that there is a cover of $G$ that is infinite and admits no finite subcover.

Let $g \restriction O_i$ be the restriction of $g \in G$ to the finite permutation group $S_{O_i}$ in the natural way. Assume that for a fixed $i$, for all $h \in S_{O_i}$ the induced cover of the set

$$G_h \coloneqq \{g \in G \mid g \restriction O_i = h\}$$

has a finite subcover. This is clearly absurd, as this would contradict our assumption. Thus for all $i \in \omega$ there is at least one $h_i \in S_{O_i}$, such that $G_{h_i}$ has no finite subcover. Let the sequence $(\hat{h}_i)_{i \in \omega}$ denote these elements. As the group is closed, we know that the limit

$$\bar{g} \coloneqq \bigcup_{i \in \omega} G_{\hat{h}_i}$$

must lie in $G$. Thus $\bar{g}$ must lie in some member of the cover, say $S$.

As $S$ is open there exists some $m$, such that

$$\bigcup_{i=1}^{m} G_{\hat{h}_i} \subseteq S$$

a contradiction. □

To end this section, we'll just state one more fact about the closure of permutation groups with finite orbits inside of $S_\omega$, namely that the closure of $G \leq S_\omega$, $\bar{G}$ is the inverse limit of the inverse system $G_i = G/N_i$ where $N_i$ is the normal subgroup fixing $O_1 \cup O_2 \cup \cdots \cup O_i$, with the morphisms taken to be the canonical projections from $G_i$ into $G_j$ for $i > j$.

### 3.5 Constructions

Before we begin introducing the methods of forcing to construct cofinitary groups, we will examine some of the classical techniques that can be used to obtain them.

Over the years many different ways of constructing permutation groups with few fixed points have been discovered, with the papers of Koppelberg [20] and

Cameron [5]. outlining a multitude of possible approaches. Of those, we will consider two exemplary ones, the first one for its simplicity and the second one for its interesting results.

### 3.5.1 Constructions using Inverse Limits

The results of this section are due to [20].

**Definition 3.19** (Inverse Limit)**.** Let $(I, \leq)$ be a directed poset and let $(A_i)_{i \in I}$ be a family of groups. Let $f_{ij} : A_j \to A_i$ be a homomorphism for all $i \leq j$ with the properties

(i). $f_{ii}$ is the identity homomorphism,

(ii). $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

We call this type of object an inverse system and define its inverse limit to be

$$\varprojlim_{i \in I} A_i := \left\{ \vec{a} \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j) \quad \text{for all } i \leq j \text{ in } I \right\}.$$

Let $\lambda$ be a limit ordinal and let $(G_\iota)_{\iota \in \lambda}$ and $(\phi_{\iota,\kappa})_{\kappa \leq \iota < \lambda}$ be an inverse system of groups. We then let $G := \varprojlim_{\iota < \lambda} G_\iota$. be the inverse limit of the system and define $X := \bigcup_{\iota < \lambda} G_\iota$ to be the disjoint union of the $G_\iota$. We let $g \in G$ act on $X$ in the following way:

If $x \in G_\alpha$, then $g(x) = g_\alpha(x)$ where $g_\alpha$ is the $\alpha$th element in the tuple that makes up $g$. This allows us to view $G$ as a subgroup of $S_X$.

The set of fixed points of any $g \in G \setminus \{1\}$ can not be of size $\lambda$ by our definition, as otherwise all $g_\gamma$ for $\gamma < \lambda$ would be the identity due to it being an inverse limit, a contradiction. In particular, if $\lambda = \omega$ any element that is not the identity can only have finitely many fixed points.

This construction is called a “tree-like” one by Koppelberg, due to its utilization of set theoretic trees to obtain an inverse system, many of which allow for the construction of a cofinitary group. For more information about the theory of trees refer to the second chapter of [21].

Now let us consider some concrete examples:

**Example 3.20.** Let $\lambda = \omega$, and let all the $G_\iota$ be finite groups with strictly increasing cardinalities. This will result in $|X| = \omega$ and $|G| = 2^\omega$. Depending on the individual properties of the $G_\iota$ we can influence the properties of $G$, for example we can let all $G_\iota$ be abelian, which will result in the abelian group with $2^\omega$ many generators.

If we venture outside of our usual realm of groups of countable rank, we are able to experiment with all sorts of possible cardinal numbers with different properties. For example, one might want to use a limit cardinal of countable cofinality and let the sizes of the groups in the inverse system be dictated by a cofinal sequence. The resulting rank and cardinality of the group are then dictated by König's Theorem.

Finally, let us examine an example based around a specific class of trees:

**Definition 3.21** ($\kappa$-Kurepa Tree)**.** Let $\kappa$ be a cardinal number. We call a tree $(T, <)$ a $\kappa$-Kurepa tree if it has at least $\kappa^+$ many branches of length $\kappa$ and levels of size less than $\kappa$.

Now let $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{F}| \geq \kappa^+$ and for any $\alpha < \kappa$ the following holds

$$| \{\alpha \cap F \mid F \in \mathcal{F}\} | < \kappa.$$

We call this $\mathcal{F}$ a $\kappa$-Kurepa family.

*Remark* 3. Note that a $\kappa$-Kurepa family does not depend on the ordering of $\kappa$ allowing us to use any arbitrary unordered set in its stead. Further, the existence of $\kappa$-Kurepa families is equivalent to the existence of $\kappa$-Kurepa trees. For a proof of this, see Chapter 2 Theorem 5.18 of [21].

In a similar way we can define a $\kappa$-Kurepa group and show that its existence is equivalent to that of $\kappa$-Kurepa families and trees.

**Definition 3.22** ($\kappa$-Kurepa Group)**.** Let $\kappa$ be an infinite cardinal and let $G \leq S_\kappa$ and $|G| \geq \kappa^+$. We call $G$ a $\kappa$-Kurepa group if $G$ is cofinitary and for any $\alpha < \kappa$ the following holds

$$|\{g{\restriction}\alpha \mid g \in G\}| < \kappa.$$

**Theorem 3.23.** *A $\kappa$-Kurepa group exists if and only if a $\kappa$-Kurepa tree exists.*

*Proof.* Let $G$ be a $\kappa$-Kurepa group. Then we see immediately that it is a $\kappa$-Kurepa family of subsets of $\kappa \times \kappa$.

Conversely, let $(T, <)$ be a $\kappa$-Kurepa tree. We let $G_\alpha$ be the free abelian group generated by the elements of level $\alpha$ of the tree. For $\beta \geq \alpha$ we obtain a surjective homomorphism of groups,

$$\begin{aligned} \phi\colon G_\beta &\longrightarrow G_\alpha \\ h &\longmapsto g \end{aligned}$$

by mapping the generator $h$ of $G_\beta$ to the unique generator $g$ of $G_\alpha$ for which $g < h$ holds. Taking the inverse limit of this system gives us a $\kappa$-Kurepa group. ∎

### 3.5.2 Constructions via Automorphisms of Structures

Another viable way of constructing cofinitary groups is via the use of automorphisms of certain structures. Two examples we will examine in this section will be automorphisms of Boolean algebras and automorphisms of relational structures. For the two constructions presented in this chapter, as well as additional results, see [20] and [5] respectively. For a more comprehensive examination of the connections of permutation groups and model theory see [19].

The difficulty in the construction of cofinitary groups from automorphisms of Boolean algebras is in finding those automorphisms with few fixed points. The following result due to Koppelberg shows us that they exist for certain Boolean algebras.

**Proposition 3.24.** *Let $\mathcal{B}$ be a free product of pairwise isomorphic Boolean algebras $(\mathcal{B}_i)_{i\in I}$ and let $\phi : I \to I$ be a permutation of the indices. The automorphism $g$ of $\mathcal{B}$ induced by $\phi$ will have*

$$fix(g) \subseteq \bigcup \{\mathcal{B}_j \mid j \text{ lies in a finite orbit of } \phi\} .$$

Thus there exist large cofinitary groups of automorphisms of Boolean algebras, as we can extend the group generated by $g$ to a cofinitary group of arbitrary size as we will see in the next chapter.

The next two results will use the construction via Fraïssé limits as illustrated in the introductory chapter on model theory.

**Proposition 3.25.** *Let $M$ be a countable $\omega$-homogeneous structure whose skeleton has the strong amalgamation property. There exists a cofinitary dense subgroup of $Aut(M)$ which is free of countable rank.*

*Proof.* We begin by enumerating all possible pairs of tuples of distinct elements of the same type. Thus we will get a list of the following form

$$\{((a_{11}, \ldots, a_{1n}), (b_{11}, \ldots b_{1n})), ((a_{21}, \ldots, a_{2m}), (b_{21}, \ldots b_{2m})), \ldots\} ,$$

where $tp(a_{ki}) = tp(b_{kj})$ for all $i, j, k \in \omega$. We further enumerate all elements of $M$ as $x_1, x_2, \ldots$ allowing us to identify $x_i$ with $i \in \omega$.

Now let us construct our group iteratively. At stage $2n$ we add a new partial permutation $f_n$ mapping the first member of the $n$th pair to the second one. At stage $2n+1$ we extend each previously constructed permutation and their inverses in such a way that they end up as partial isomorphisms for all elements up to $n$. Particularly, if $f_k^{\pm 1}(m)$ is not defined for some $m \leq n$, then we choose it to be an element $l$ where $l > n$ and $l$ does not appear any of the other permutations constructed up until that point.

If we take the limit of this construction, we obtain a countable set of permutations $(f_i)_{i\in\omega}$ which all define automorphisms of $M$. If we now consider the group $F := \langle f_i \rangle_{i\in\omega}$, then it is dense in $Aut(M)$ as it has the same orbits.

To see that the group is cofinitary, let $w = f_{i_1}^{n_1} \dots f_{i_l}^{n_l}$ be an arbitrary cyclically reduced word. It is sufficient to consider these words as conjugation preserves fixed points. Assume $x$ is a fixed point of $w$, such that it does not arise due to a point that appears in the pair used to construct $f_{i_k}$ in an even step. Without loss of generality we may assume that the evaluation of $w$ does not yield $x$ more than once, i.e. $f_{i_1}^{n_1} \dots f_{i_j}^{n_j}(x) \neq x$ for $j < l$.

Considering cyclic permutations $w' = f_{j_1}^{m_1} \dots f_{j_k}^{m_k}$ of $w$, we will find one where the fixed point $x'$ corresponding to $x$ has the following property

$$x' > f_{j_1}^{m_1}(x') \text{ and } x' > f_{j_k}^{-m_k}(x').$$

By construction there is at most one choice of $y$ and $f_j^m$ such that $y$ comes before $x$ in our enumeration and $f_j^m(y) = x$, where $x$ and $y$ are not members of the $j$th pair. Thus we have that $f_{j_1}^{m_1} = f_{j_1}^{-m_k}$, contradicting our assumption of $w$ being cyclically reduced.

Thus any fixed point has to arise via the elements of the pairs of tuples used to construct the $f_i$, yielding only finitely many fixed points for any non-trivial word. □

There exist many other interesting groups that can be constructed similarly to the one above using a Fraïssé type construction, one particular example is that of a transitive discrete unbounded cofinitary group of countable degree.

## 3.6 Maximal cofinitary groups

As opposed to the finitary permutation groups, which have a single maximal group that contains all other finitary permutation groups as subgroups, cofinitary groups admit no unique maximal group. A standard argument invoking Zorn's Lemma at least guarantees us the existence of these groups, making their study feasible.

One of the central questions of interest when it comes to maximal cofinitary groups is their size. Using forcing methods we can find models with all manner of differently sized maximal cofinitary groups. Particularly interesting is the minimal size. Analogously to mad families we define the following:

**Definition 3.26.** Inside a model $M$ let $\mathfrak{a}_g$ denote the minimal cardinality of a maximal cofinitary group.

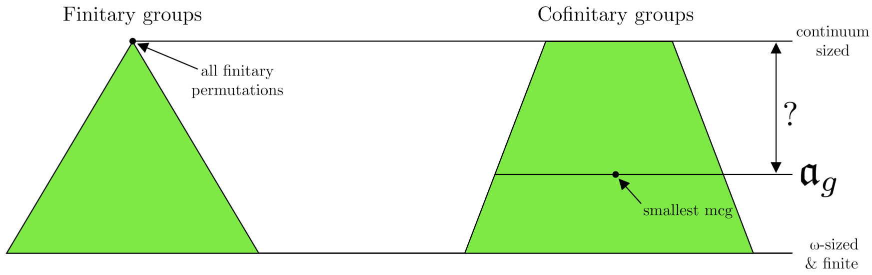


Figure 1: A comparison between the hierarchy of finitary and cofinitary groups of countable degree.

There exist some ZFC-provable inequalities for $\mathfrak{a}_g$, as we will discuss in the remainder of this section. A closely related cardinal characteristic is $\mathfrak{a}_f$, the minimal size of a maximal almost disjoint family of functions defined on a countable set. We can use simple diagonalization to show that $\mathfrak{a}_f$ is at least uncountable.

**Lemma 3.27.** $\mathfrak{a}_f > \omega$

*Proof.* Assume $A$ is a countable almost disjoint family of functions. We enumerate all the functions in $A$ and define $f : \omega \to \omega$ by taking $f(n) \neq f_k(n)$ for all $k < n$. Now $A \cup \{f\}$ will be almost disjoint, contradicting maximality. □

In the case of $\mathfrak{a}_g$ the same holds, however the argument is much more technical. We can't simply construct an element using a diagonalization argument, which stems from the fact that we need to guarantee that all elements of the free product of the countable group and the new function remain cofinitary.

**Theorem 3.28.** $\mathfrak{a}_g > \omega$

*Proof.* The proof we will show here is due to Adeleke [1], for an alternative one see [31]. We will show that for any countable, cofinitary groups $G$ and $H$ there exists a permutation $\sigma$ such that $\langle G, \sigma H \sigma^{-1} \rangle$ is a cofinitary group. The theorem then follows immediately.

We begin by enumerating all words of the following form:

$$w_i(y) = (yh_{i1}y^{-1})g_{i1}(yh_{i2}y^{-1})g_{i2} \dots (yh_{in(i)}y^{-1})g_{in(i)},$$

where we take $y$ to be a placeholder variable, $h_{ij} \in H \setminus \{id\}$ and $g_{ij} \in G \setminus \{id\}$. It is sufficient to use these words, as all non-trivial words in the free product of $G$ and $yHy^{-1}$ are either of this form or a conjugate.

Our goal is to iteratively construct a permutation $\sigma$ such that all $w_i(\sigma)$ have only finitely many fixed points. For this we construct an ascending sequence of finite partial functions $y_1 \subseteq y_2 \subseteq \dots$ such that

(i). $\{0, 1, \dots, i-1\} \subseteq dom(y_i)$ and $\{0, 1, \dots, i\} \subseteq ran(y_i)$,

(ii). For any $x \in dom(y_i) \setminus dom(y_{i-1})$ we have that the word $w_j(y_i)$ does not have $x$ as a fixed point for $j \in [1, i]$.

(iii). Each $y_i$ is a map of the form $x_{i1} \mapsto x_{i2} \mapsto \dots \mapsto x_{il(i)}$

Let us begin by picking two variables $a, b$ and define $y_1(a, b)$ to be

$$a \mapsto 0 \mapsto b \mapsto 1.$$

We need to check whether or not we can find concrete values for $a$ and $b$ such that the second condition holds.

For this we consider the equation $w_1(y_1(a, b))(a) = a$, $w_1(y_1(a, b))(b) = b$ and $w_1(y_1(a, b))(0) = 0$.

All of these equations give us certain necessary conditions on the pair $(a, b)$, namely $g_{1n(1)}(0) \neq a, b$, $g_{1n(1)}(a) \neq a, b$, $g_{1n(1)}(b) \neq a, b$ and $h_{11}(b) \notin \{0, b, 1\}$, for it to satisfy property (ii). As is evident, there are infinitely many pairs $(a, b)$ that satisfy these conditions.

Note that the solution sets of these conditions are either finite or form a non intersecting curve in the discrete space $\omega^2$.

Now let us construct $y_{n+1}$ assuming $y_n$ is known. Let $S = \omega \setminus dom(y_s)$ and denote the $k$th element of $S$ by $s_k$. Once again we want to find a pair $(a, b)$ that can take values from $S \setminus \{s_1\}$. We now define

$$y_{n+1} \coloneqq y_n \cup \{(a, x_{n1}), (x_{n,l(n)}, b), (b, s_1)\},$$

which clearly is a partial function extending $y_n$ satisfying properties (i) and (iii).

To see property (ii), we once again consider a set of equations. Begin by noting that $dom(y_{n+1}) \setminus dom(y_n) = \{a, b, x_{n,l(n)}\}$. Thus for each $w_i(y)$ with $1 \leq i \leq n+1$ we get three equations that restrict our choice of $(a, b)$, leading to similar restrictions on our pair as above, leaving us with infinitely many choices still.

However, we will still have to make sure that our choice of $(a, b)$ adds no new fixed points to words $w_j(y)$ and elements of $D_j \coloneqq dom(y_n) \setminus dom(y_{j-1})$. The case we need to consider in particular are those $x \in D_j$ and $w_j(y)$ where $w_j(y_{n+1})(x)$ is defined but $w_j(y_n)(x)$ is not as otherwise the induction hypothesis guarantees us that $x$ is not a fixed point.

If $w_j(y_{n+1})(x)$ becomes defined, then it must be due to one of the components of the pair $(a, b)$ appearing in its evaluation. Thus for each $w_j$ and $x$ we get conditions similar to the ones above that give us a finite or 1 dimensional, meaning at least one of the canonical projections $\pi_1$ or $\pi_2$ is injective, solution set as discussed above that we can eliminate from the space $S^2 \coloneqq S \times S$. Eliminating a finite amount of these lines from $S^2$ will still leave infinitely many choices for $(a, b)$ and thus we are done.

Now, taking the limit $\sigma = \bigcup_{i \in \omega} y_i$, we get a permutation of the naturals with a single cycle and the property that the number of fixed points of each $w_i(\sigma)$ is bounded via its index $i$. $\square$

Another absolute bound we have established previously in Corollary 3.8 is that $\mathfrak{a}_g \leq \mathfrak{c}$, as we have constructed a cofinitary group of size continuum, thus there must be a maximal cofinitary group containing it that is also of size continuum. Another lower bound of $\mathfrak{a}_g$ is the cardinal invariant $non(\mathcal{M})$ [3].

Besides these inequalities, there are some consistency results, such as $\mathfrak{a} < \mathfrak{a}_g$ being consistent with ZFC [3]. A model of $\mathfrak{a} < \mathfrak{a}_g$ would be the Random Real Model, in which $\omega_1 = \mathfrak{a} < non(\mathcal{M}) = \mathfrak{c}$, see [2] for details.

# 4 The Isomorphism Type of Maximal Cofinitary Groups

After our study of cofinitary groups in the classical sense of group theory, this chapter is dedicated to developing the theory of forcing on cofinitary groups, motivated by their relation to mad families.

In particular we will find that there are at least countably many non-isomorphic maximal cofinitary groups, by being able to construct groups with an arbitrary number of orbits.

The notation and basic results in Section 4.2 on forcing of maximal cofinitary groups are from Fischer [11] and the results of Sections 4.1 and 4.3 about the isomorphism classes are due to Kastermans [17].

## 4.1 An Upper Bound on Orbits

Before we begin going through the motions that will allow us to use forcing, we will use an algebraic argument to gain a first, motivating result for the study of isomorphism classes of maximal cofinitary groups.

**Theorem 4.1.** *The natural action of a maximal cofinitary group can not have infinitely many orbits.*

*Proof.* Towards a contradiction we assume that $G$ is a maximal cofinitary group whose set of orbits $\mathcal{O}$ under the natural action is of cardinality $\omega$.

Without loss of generality we can assume that this group has no orbits of size 1, as there can only be finitely many of these and thus they can be ignored in our construction.

We will now construct a cofinitary permutation $f \notin G$ and then show that $\langle G, f\rangle$ is a cofinitary group contradicting maximality.

First, let us fix an enumeration of the orbits of $G$ as $O_1, O_2, \dots$ acting on $\omega$ denoted by $(O_i)_{i\in\omega}$. We now define $f$ recursively via an ascending sequence of partial functions $(f_i)_{i\in\omega}$ with $f_j \subseteq f_k$ for $j \leq k$. We begin by defining $f_0 \coloneqq \emptyset$. Assuming $f_n$ has been defined, we can define

$$k \coloneqq min\left(\omega \setminus (dom(f_n) \cap ran(f_n))\right).$$

We also define $m \coloneqq min(O_j)$ where

$$j \coloneqq min\left(\{j \in \omega \mid O_j \cap (dom(f_n) \cup ran(f_n)) = \emptyset \text{ and } k \notin O_j\}\right).$$

Finally we define $f_{n+1} \coloneqq f_n \cup \{(k, m)\}$ if $k \notin dom(f_n)$ and $f_{n+1} \coloneqq f_n \cup \{(m, k)\}$ otherwise and set $f \coloneqq \bigcup_{i=0}^{\infty} f_i$.

Now let us check that $f$ is a bijective function on $\omega$. By construction we see that our function is total, as any number $n$ will appear in both the domain and range of the partial function $f_{2n}$ and furthermore it can only appear once in the range and once in the domain.

Furthermore, this construction guarantees that $fix(f) = \emptyset$ and as such $f$ is a cofinitary permutation. It is also obvious that $f \notin G$ due to its effect on the orbits of $G$.

It now remains to show that $\langle G, f \rangle$ is a cofinitary group. For this we consider the free product $G * F(\{f\})$ and let $w \in G * F(\{f\})$ and show that the evaluation of any such word will only have finitely many fixed points.

We will show this via a graph theoretic argument on the orbit graph of our group action.

**Definition 4.2** (Orbit Graph)**.** For a group $G$ acting on a set $S$ inducing the orbits $(O_i)_{i \in I}$ as well as a function $f \colon S \to S$, we define the ($G$-)orbit graph of $f$ to be an undirected graph $T = (V, E)$ where $V := \{O_i \mid i \in I\}$ and

$$(O_j, O_k) \in E \iff \exists m \in O_j \exists n \in O_k \; f(m) = n.$$

*Remark* 4. If $(O_j, O_k)$ is induced by a unique pair $(m, n)$ then we will refer to the edge via $(m, n)$ instead.

Inspecting the orbit graph of our permutation $f$, we notice the following:

**Claim 4.3.** *The orbit graph of $f$ contains no non-trivial circuits. In other words, the orbit graph of $f$ is an infinite tree.*

*Proof of Claim.* Assume that this is false, then there must exist a circuit of length $n > 1$ that we can write as $O_1 O_2 \cdots O_n O_{n+1}$ where $O_{n+1} = O_1$. This means there are edges $(O_i, O_{i+1}) \in E$ that form this circuit. Each edge has an associated pair of elements $(k_i, l_i) \in O_i \times O_{i+1}$ such that $f(k_i) = l_i$.

Since the circuit is of finite length, there must be some $m \in \omega$ such that the $G$-orbit graph of $f_m$ includes the circuit, but the one of $f_{m-1}$ does not.

Thus there exists a unique pair $(k, l) \in f_m \setminus f_{m-1}$ that is used to complete the circuit, connecting the orbits $O_s$ and $O_{s+1}$ for some $s \leq n$. However, both of these orbits are already path connected in the orbit graph of $f_{m-1}$ which leads to a contradiction, as both

$$O_s \cap (dom(f_{m-1}) \cup ran(f_{m-1})) \neq \emptyset,$$

and

$$O_{s+1} \cap (dom(f_{m-1}) \cup ran(f_{m-1})) \neq \emptyset,$$

which means the pair $(k, l)$ could not have been selected in the construction of $f_m$. $\square$

We can now, as mentioned before, consider reduced words $w \in G * F(\{f\})$ and the evaluation of their action on the orbit tree. In fact, as such elements are of the form

$$w = g_0 f^{k_0} g_1 f^{k_1} \dots g_{l-1} f^{k_{l-1}} g_l,$$

with $k_i \neq 0$ for all $i < l$. We will only observe a change between vertices in the graph when evaluating the element $f$, as elements from $G$ remain in their orbits.

Now suppose that $w$ has infinitely many fixed points in $G * F(\{f\})$ and take $n \in \omega$ to be an arbitrary fixed point of $w$ and consider the path $p(w, n) = (O_i)_{i<l}$ of orbits that we pass through when evaluating $w(n)$. Necessarily for $n$ to be a fixed point we have $n \in O_1, O_l$ which means $O_l = O_1$. Thus the path $p(w, n)$ has to be a circuit, but since the orbit graph is a tree, we must backtrack all the steps taken away from $O_1$ eventually.

Let $O_m$ be the orbit occurring in $p(w, n)$ that has maximal distance from $O_1$. If there are multiple such orbits, let $O_m$ be the one where $m$ is minimal among them. We know that there must be a pair $(k, l) \in f$ that occurs in the evaluation of $w$ and causes us to pass from $O_{m-1}$ to $O_m$. The next step in our path will be from $O_m$ back to $O_{m-1}$ and by construction of $f$ this step has to occur via the same pair $(k, l)$. As $w$ is reduced, we know that we have to evaluate an element $g' \in G$ before we are able to go back via the edge $(l, k)$, but this means that $g'$ must have a fixed point at $l$.

Thus for every fixed point $n$ of $w$ we find that there must be a corresponding fixed point in one of the elements of $G$ occurring in $w$. As there are infinitely many fixed points but finitely many such elements one of them must have infinitely many fixed points by the pigeonhole principle, call it $g_j$.

As all the $g \in G$ and $f$ are bijective functions, we know that any initial segment of $w$ will also be a bijective function, thus we know that $w' := g_0 f^{k_0} \dots f^{k_{j-1}} g_j : \omega \to \omega$ will have $w'(n) \neq w'(m)$ for $n \neq m$ and thus we find that each fixed point of $w$ corresponds to a different fixed point of $g_j$, meaning it must have infinitely many. Hence $G$ can not be cofinitary, a contradiction. $\square$

## 4.2 The Basics of Forcing Cofinitary Groups

Having obtained an upper bound on the number of the orbits of a maximal cofinitary group's action, we now begin introducing a manner of basic notions that will allow us to construct cofinitary groups via forcing, eventually letting us construct groups with an arbitrary number of orbits.

**Definition 4.4.** Let $A$ be a set and let $\widehat{W}_A \subseteq W_A$ be the subset of words such that for $w \in \widehat{W}_A$ we have either $w = a^n$ for some $a \in A$, $n \neq 0$ or $w = a_1 v a_2$ with $a_1, a_2 \in A$ and $a_1 \neq a_2$, i.e. the set of cyclically reduced words made up of letters from $A$.

Note that any $w \in W_A$ can be written as some $w = u^{-1} w' u$ with $u \in W_A$ and $w' \in \widehat{W}_A$, which means that if we consider $A$ to be a set of permutations, then the cycle structure of $w$ is determined via a word $w' \in \widehat{W}_A$.

As a matter of notational convenience, for $f\colon S \to S$ we let

$$fix(f) := \{s \in S \mid f(s) = s\}$$

be the set of fixed points of a function.

**Definition 4.5** (Cofinitary Representation)**.** Let $G$ be a group and let $\rho\colon G \to S_\omega$ be a homomorphism of groups. We call $\rho$ a cofinitary representation of $G$, if

$$\forall g \in G\ |fix(\rho(g))| < \omega.$$

If $B$ is a set, we say the map $f\colon B \to S_\omega$ induces a cofinitary representation, if the induced homomorphism of the free group $\phi\colon F(B) \to S_\omega$ is a cofinitary representation of $F(B)$.

**Definition 4.6** (Evaluations)**.** Let $A$ be a set, let $s \subseteq A \times \omega \times \omega$ and let $a \in A$ and define

$$s_a := \{(n, m) \mid (a, n, m) \in s\}.$$

Furthermore, for a word $w \in W_A$ we define the relation $e_w[s] \subseteq \omega \times \omega$ recursively as follows.

If $w = a$ for some $a \in A$ then $(n, m) \in e_w[s]$ if $(n, m) \in s_a$ and if $w = a^i v$ for some $v \in W_A$ and $i = 1, -1$ without cancellation, then

$$(n, m) \in e_w[s] \iff \exists k\ (k, m) \in e_{a^i}[s] \wedge (n, k) \in e_v[s].$$

If, furthermore, $s_a$ is a finite injective partial function for all $a \in A$, then so is $e_w[s]$ and we call it the evaluation of $w$ on $s$.

Lastly, we define

$$oc_A((s, F)) := \{a \in A \mid a \in dom(s)\} \cup \{a \in A \mid \exists w \in F\ a \in w\}.$$

If $s$ is as above with the additional condition of every $s_a$ being a partial function or empty, then the evaluation $e_w[s]$ of a word $w$ corresponds to a partial function $\omega \to \omega$ and we write $e_w[s] \downarrow$ if $n \in dom(e_w[s])$ and $e_w[s] \uparrow$ otherwise.

**Definition 4.7** (Evaluations 2)**.** For disjoint sets $A, B$, a function $f\colon B \to S_\omega$, a word $w \in W_{A\cup B}$ and $s \subseteq A \times \omega \times \omega$, we define

$$e_w[s, f] \coloneqq e_w[s \cup \{ (b, k, l) \mid (f(b))(k) = l \}].$$

All notions concerning $e_w$ defined before apply equally to this extended notion.

*Remark* 5. Let $A$, $B$, $w$, $s$ and $f$ be as in the above definition. Then for $u, v \in W_{A\cup B}$ such that $w = uv$ without cancellation it holds that $n \in dom(e_w[s, f])$ if and only if $n \in dom(e_v[s, f])$ and $e_v[s, f](n) \in dom(e_u[s, f])$.

Moreover, for $w \in \hat{W}_{A\cup B}$ we have that

$$n = e_w[s, f](n) \iff e_v[s, f](n) = e_{vu}[s, f](e_v[s, f]).$$

Thus $e_w[s, f]$ and $e_{vu}[s, f]$ have the same number of fixed points.

**Definition 4.8.** Let $A$ and $B$ be disjoint sets and $f : B \to S_\omega$ a function such that the induced homomorphism $\rho : F(B) \to S_\omega$ is a cofinitary representation. Then we define the poset $\mathbb{Q}_{A,\rho}$ as follows:

(i). The conditions of $\mathbb{Q}_{A,\rho}$ are pairs $(s, W)$ such that $s \in [A \times \omega \times \omega]^{<\omega}$ and $s_a$ is a partial finite injective function for every $a \in A$ and $W \subseteq \widehat{W}_{A\cup B}$ is finite.

(ii). For two conditions $(s_1, W_1) \leq (s_2, W_2)$ iff $s_1 \supseteq s_2$, $W_1 \supseteq W_2$ and for every $n \in \omega$ and $w \in W_2$, if $e_w[s_1, \rho](n) = n$ then already $e_w[s_2, \rho](n) \downarrow$ and $e_w[s_2, \rho](n) = n$, i.e. the extension adds no new fixed points to the evaluation.

As usual, we want to know whether our forcing poset fulfills any of the chain conditions, thus providing us with information about the cardinals of a generic extension constructed via this poset.

**Proposition 4.9.** *$\mathbb{Q}_{A,\rho}$ has the countable chain condition (c.c.c.).*

*Proof.* Assuming $|A| > \aleph_0$ we show this by contradiction, otherwise there are at most countably many possible elements for the first component of the tuples in $\mathbb{Q}_{A,\rho}$ as

$$|A \times \omega \times \omega|^{<\omega} = |\omega|^{<\omega} = \omega$$

and any two tuples that agree on the first component are trivially compatible.

Let $C$ be a set of conditions with $|C| > \omega$. We will now use the $\Delta$-System Lemma to show there must be some compatible conditions in $\mathcal{C}$.

We first apply the lemma to the set

$$\Delta_1 \coloneqq \{s \mid (s, W) \in C\},$$

obtaining some uncountable subset $\Delta_1'$ of it along with finite $t \subset A \times \omega \times \omega$ such that $s_1 \cap s_2 = t$ for any $s_1, s_2 \in \Delta_1$. Similarly we obtain finite sets $A_1, A_2$ as roots of $\Delta$-systems $\Delta_2'$ and $\Delta_3'$ for the sets

$$\Delta_2 := \{oc_A(W) \mid \exists p \in \Delta_1 \ (p, W) \in C\}$$

and

$$\Delta_3 := \{dom(p) \cup oc_A(W) \mid \exists p \in \Delta_1 \ (p, W) \in C\}$$

respectively.

We note that $dom(t)$ and $A_1$ are subsets of $A_2$ as

$$\begin{aligned} A_2 &= (dom(s_1) \cup oc_A(W_1)) \cap (dom(s_2) \cup oc_A(W_2)) \\ &= (dom(s_1) \cap dom(s_2)) \cup \cdots \cup (oc_A(W_1) \cap oc_A(W_2)) = t \cup \cdots \cup A_1. \end{aligned}$$

Next, we define

$$\Delta_4 := \{s \in \Delta_1 \mid s \cap (A_2 \times \omega \times \omega) = t\}.$$

We see that $\Delta_4$ is also uncountable, as $s \cap (A_2 \times \omega \times \omega) \supset t$.

Finally define

$$\Delta_5 := \{(s, W) \in C \mid s \in \Delta_4, oc_A(W) \in \Delta_2' \text{ and } (dom(s) \cup oc_A(W)) \in \Delta_3'\}$$

and note that this set is also uncountable.

Let $(s, W_s), (u, W_u) \in \Delta_5$ then we have $(s \cup u, W_s \cup W_u) \in \mathbb{Q}_{A,\rho}$ and

$$s \cap (oc_A(W_u) \times \omega \times \omega) \subseteq t$$

as $dom(s) \cap oc_A(W_u) \subseteq A_2$.

Thus for $w \in W_u$ we get that $e_w(s \cup u, \rho)(n) = n$ is equivalent to

$$e_w(t \cup u, \rho)(n) = e_w(u, \rho)(n) = n$$

and thus

$$(s \cup u, W_s \cup W_u) \leq (u, W_u).$$

Note that since $s$ and $u$ were arbitrary and union is symmetric we are done. $\square$

*Remark* 6. In fact, this proof establishes the stronger property of $\mathbb{Q}_{A,\rho}$ having the $(\aleph_1-)$*Knaster property.*

Before we can begin using this poset for forcing, we need to check that it behaves the way we want it to.

**Definition 4.10** (Generic Representation)**.** Let $\mathcal{G}$ be a $\mathbb{Q}_{A,\rho}$-generic filter over a family of dense sets $\mathcal{F}$. We define $\rho_{\mathcal{G}}\colon A \cup B \to S_\infty$ as

$$\rho_{\mathcal{G}}(x) := \begin{cases} \rho(x) & \text{if } x \in B, \\ \bigcup \left\{ s_x \mid \exists F \subset \hat{W}_{A\cup B}\ (s,F) \in \mathcal{G} \right\} & \text{if } x \in A. \end{cases}$$

From this definition it is not apparent whether

$$\bigcup \left\{ s_x \mid \exists F \subseteq \widehat{W}_{A\cup B}\ (s,F) \in \mathcal{G} \right\}$$

actually defines a cofinitary permutation. We will now introduce a Lemma that will establish that fact and aid us in the proof of the main theorem of this section. This result is due to [12].

**Lemma 4.11** (Domain and Range Extension Lemma)**.** *Let $A$ and $B$ be disjoint sets and $\rho\colon B \to S_\omega$ a function inducing a cofinitary representation. Then*

*(i). For any $(s,F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $n \in \omega$ such that $n \notin dom(s_a)$ there exist cofinitely many $m \in \omega$ such that $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.*

*(ii). For any $(s,F) \in \mathbb{Q}_{A,\rho}$, $a \in A$ and $n \in \omega$ such that $n \notin ran(s_a)$ there exist cofinitely many $m \in \omega$ such that $(s \cup \{(a,m,n)\}, F) \leq (s,F)$.*

Before we will prove this Lemma let us introduce a helper definition and another helpful Proposition.

**Definition 4.12** ($a$-Good Word)**.** Let $A$ and $B$ be disjoint sets, $a \in A$, $j \in \omega \setminus \{0\}$ and $w \in W_{A\cup B}$. We call $w$ an a-good word of rank j if it is of the form

$$w = a^{\alpha_1} v_1 a^{\alpha_2} v_2 \dots a^{\alpha_j} v_j,$$

where $v_i \in W_{A\setminus\{a\}\cup B}$ for all $i \leq j$ and $\alpha_i \neq 0$.

Using this definition we will now show a slightly stronger statement than the above Lemmas for $a$-good words.

**Proposition 4.13.** *Let $A$ be a set, $s \in [A \times \omega \times \omega]^{<\omega}$ such that every $s_a$ is a partial injective finite function, let $a \in A$ and let $w \in W_{A\cup B}$ be a-good. For any $n \in \omega \setminus dom(s_a)$ and any finite $C \subseteq \omega$ there are cofinitely many $m \in \omega$ such that*

$$\forall l \in \omega\ e_w[s \cup \{(a,n,m)\}, \rho](l) \in C \iff e_w[s,\rho](l)\downarrow \text{ and } e_w[s,\rho](l) \in C$$

*Proof.* We will show this via induction over the rank of $w$. If the rank is 1 and $w$ is a-good, it must be of the form $w = a^{\alpha_1} v_1$.

First assume that $\alpha_1 > 0$. We pick $m \in \omega \setminus (C \cup dom(s_a))$. Assume

$$e_w[s \cup \{(a, n, m)\}, \rho](l) \in C$$

and $e_w[s, \rho](l) \uparrow$. This would mean that there is some $1 \le i \le \alpha_1$ such that

$$e_{a^i v_1}[s \cup \{(a, n, m)\}, \rho](l) = m \text{ but } m \notin dom(s_a)$$

and so $e_w[s \cup \{(a, n, m)\}, \rho](l) \uparrow$. Thus $i = \alpha_1$ and

$$e_w[s \cup \{(a, n, m)\}, \rho](l) = m \notin C,$$

contradicting our assumption. The other direction of the equivalence is always true.

Now let $\alpha_1 < 0$. We select

$$m \in \omega \setminus \bigcup_{i=-1}^{\alpha_1} ran(e_{a^i u_1}[s, \rho]).$$

Assume $e_w[s \cup \{(a, n, m)\}, \rho](l) \in C$ and $e_w[s, \rho](l) \uparrow$. This means that there is a $\alpha_1 \le i \le -1$ minimal in magnitude such that

$$e_{a^i u_1}[s \cup \{(a, n, m)\}, \rho](l) = n.$$

Thus $e_{a^i u_1}[s, \rho](l) \uparrow$ and $e_{a^{i+1} u_1}[s, \rho](l) \downarrow$, contradicting our choice of $m$.

Assume we have shown our proposition up to rank $j - 1$. Our word of rank $j$ is of the form $w = a^{\alpha_1} u_1 \widehat{w}$, where $\widehat{w}$ is $a$-good of rank $j - 1$. We define

$$C' \coloneqq e_{a^{\alpha_1} u_1}[s, \rho]^{-1}(C),$$

and use the induction hypothesis to get a cofinite set $S_1 \subseteq \omega$ using the proposition with $\widehat{w}$ and $C'$. Using the hypothesis again, this time for $a^{\alpha_1} u_1$ and $C$ we get another cofinite set $S_2$.

Consider now $m \in S_1 \cap S_2$ and assume $e_w[s \cup \{(a, n, m)\}, \rho](l) \in C$. This tells us that $e_{\widehat{w}}[s \cup \{(a, n, m)\}, \rho](l) \in C'$ and thus $e_{\widehat{w}}[s, \rho](l) \in C'$. As such,

$$e_{a^{\alpha_1} u_1}[s \cup \{(a, n, m)\}, \rho](e_{\widehat{w}}[s, \rho](l)) \in C$$

and by definition we get

$$e_{a^{\alpha_1} u_1}[s, \rho](e_{\widehat{w}}[s, \rho](l)) = e_w[s, \rho](l) \in C.$$

$\square$

*Proof of 4.11.* Clearly it is sufficient to show that either of these statements holds for arbitrary singleton sets $F = \{w\}$ as in general, $F$ is finite and the intersection of finitely many cofinite sets is still cofinite.

(i). First, we may assume that $a \in w$, as otherwise we are already done. In case $w$ is a-good, the statement follows directly from Proposition 4.13.

Otherwise $w$ will be of the form $w = uva^\alpha$ where $u \in W_{A\setminus\{a\}\cup B}$, $v \in W_{A\cup B}$ a-good and $\alpha \in \mathbb{Z}$. Let $\widehat{w} := va^\alpha u$, which is also a-good.

By the previous proposition, we know that if we fix $n \in \omega \setminus dom(s_a)$, and set $C := fix(s_a)$, then we will get a cofinite set $\widehat{C}$ such that for all $m \in \widehat{C}$ we have $(s \cup \{(a,n,m)\}, \{\widehat{w}\}) \leq (s, \{\widehat{w}\})$.

We will now show that these $m$ also fulfill the relation

$$(s \cup \{(a,n,m)\}, \{w\}) \leq (s, \{w\}).$$

To check, pick $l \in fix(e_{\widehat{w}}[s \cup \{(a,n,m)\}, \rho])$, by Remark 5 this gives us that

$$e_{va^\alpha}[s \cup \{(a,n,m)\}, \rho](l) \in fix(e_w[s \cup \{(a,n,m)\}, \rho])$$

and as $\widehat{w}$ is a-Good, we know that $l \in fix(e_{\widehat{w}}[s,\rho])$ and as such

$$e_{va^\alpha}[s,\rho](l) \in fix(e_w[s,\rho]).$$

(ii). Let us fix $(s, \{w\}) \in \mathbb{Q}_{A,\rho}$ and $a \in A$. Substituting $a \mapsto a^{-1}$ in $w$, we get a new word $w'$. Now we define

$$s' := S \cup \{\, (b,n,m) \mid (b,n,m) \in s \wedge b \neq a \,\}$$

i.e. we use the map $s$ but invert the function defined by $s_a$. Now we can use the previous case to find a cofinite set $\widehat{C}$, such that for $m \notin dom(s'_a) = ran(s_a)$ we get that for $n \in \widehat{C}$ we have $(s' \cup \{(a,m,n)\}, \{\widehat{w}\}) \leq (\widehat{s}, \{\widehat{w}\})$, which is equivalent to $(s \cup \{(a,n,m)\}, \{w\}) \leq (s, \{w\})$.

□

**Corollary 4.14.** *Let $A$ and $B$ be sets, let $w \in W_{A\cup B}$ and let $A_0 := oc_A(w) \subset A$ be the set of letters of $A$ occuring in $w$. Furthermore let $C, D \subseteq \omega$ be finite sets and let $(s, F) \in \mathbb{Q}_{A,\rho}$. Then there exists a finite $t \subseteq A_0 \times \omega \times \omega$ such that $(t \cup s, F) \leq (s, F)$ and $dom(e_w[s \cup t, \rho]) \supseteq C$ and $ran(e_w[s \cup t, \rho]) \supseteq D$.*

*Proof.* Applying Lemma 4.11 repeatedly for the sets $C$ and $D$ and elements from $A_0$ we get a descending chain of conditions that after a finite number of applications of the Lemma fulfills all the properties we ask for. $t$ may simply be taken to be the union of all the pairs added during the construction of the chain. $\square$

Using this Lemma, we can show another fact that establishes that the previously defined extension $\rho_G$ is a sensible choice.

**Lemma 4.15.** *For all $w \in \widehat{W}_{A\cup B}$ we have that*

$$(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n) = m$$

*for some $n, m \in \omega$ implies that $e_w[s, \rho](n) \downarrow$ and $e_w[s, \rho](n) = m$.*

*Proof.* We will show this via induction on the number of appearances of letters from $A$ in $w$. If there are none, then we are already done, as we get that $w \in \widehat{W}_B$, meaning that $\rho$ fully defines the behavior of $\rho_{\mathcal{G}}$ with respect to $w$.

Assuming we have shown the statement for words with at most $k$ letters from $A$, we now consider a word $w \in \widehat{W}_{A\cup B}$ with $k + 1$ letters from $A$.

Assume towards a contradiction, that $e_w[s, \rho](n) \uparrow$ and

$$(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n) = m.$$

Thus, we can find an element $a \in A$ and words $u, v \in W_{A\cup B}$ such that $w = ua^{\pm 1}v$ and $e_v[s, \rho](n) \downarrow$ while $e_{a^{\pm 1}v}[s, \rho](n) \uparrow$. Furthermore we can write $w = w_0 w_1$ where $w_0$ does not contain $a$ and $w_1$ is a-good.

From Lemma 4.11, we know that there must exist some set of tuples $\bar{s} \subseteq \{a\} \times \omega \times \omega$ such that $(s \cup \bar{s}, F) \leq (s, F)$ and $e_{w_1}[s \cup \bar{s}, \rho](n) \downarrow$. We chose $\bar{s}$ such that

$$\bar{n} := e_{w_1}[s \cup \bar{s}, \rho](n) \neq e_{w_0}^{-1}[s, \rho](m)$$

if $e_{w_0}^{-1}[s, \rho](m)$ is defined. Using that

$$(s, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_w[\rho_G](n) = m$$

and

$$(s \cup \bar{s}, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_{w_1}[\rho_G](n) = \bar{n},$$

we get that

$$(s \cup \bar{s}, F) \Vdash_{\mathbb{Q}_{A,\rho}} e_{w_0}[\rho_G](\bar{n}) = m$$

and as $w_0$ contains at most $k$ elements from $A$, the induction hypothesis yields $e_{w_0}[s \cup \bar{s}, \rho](\bar{n}) = m$ and since there is no occurence of $a$ in $w_0$ we get that $e_{w_0}[s, \rho](\bar{n}) = m$, a contradiction. $\square$

The next definition and lemma are due to Kastermans and Zhang [18].

**Definition 4.16** (Hitable Function)**.** Let $G \leq S_\omega$ and let $f\colon \omega \rightharpoonup \omega$ be a partial, infinite function. We call $f$ hitable with respect to $G$ if the free product $G * \langle f \rangle$ does not contain any words with infinitely many fixed points other than those that evaluate as the identity.

Note that for this to be the case $f$ must be injective and may only have finitely many fixed points.

**Lemma 4.17** (Hitting Lemma)**.** *Let $A$ and $B$ be disjoint sets and let $\rho\colon B \to S_\omega$ be a function inducing a cofinitary representation. Furthermore, let $f\colon \omega \rightharpoonup \omega$ be a hitable function with respect to $im(\rho) \leq S_\omega$. Then for any $(s, F) \in \mathbb{Q}_{A,\rho}$ and $a \in A$ there exists $n \in dom(f)$, $n \notin dom(s_a)$ such that $(s \cup \{(a, n, f(n))\}, F) \leq (s, F)$.*

*Proof.* We begin by showing this for $F = \{w\}$ where $w$ is a reduced word. If $w$ does not contain $a$, then any tuple $(a, n, f(n))$ where $n \notin dom(s_a)$ will suffice and as $dom(s_a)$ is finite we are done.

Let us assume

$$e_w[(s \setminus s_a) \cup \{\, (a, n, f(n)) \mid n \in ran(f) \,\}, \rho] \cong id,$$

where defined, then $a$ must occur at least twice in $w$. If $a$ were to occur only once, then we can find a cyclic permutation of $w$ of the form $a^{\pm 1}w'$ which would contradict the fact that $im(\rho_G) * \langle f \rangle$ is cofinitary. Thus $w$ must contain either the pattern $a^{\pm 2}$ or a subword of the form $a^{\pm 1}w'a^{\pm 1}$ with $a^{\pm 1} \notin w'$. We define $f'$ to be the subset of $f$ that does not contain any of the fixed points of $f$ or any of the (finitely many) pairs used in both $a^{\pm 1}$ when evaluating the pattern $a^{\pm 1}w'a^{\pm 1}$.

We now define $f''$ iteratively. Let $f''_0 \coloneqq \emptyset$ and let $f'_0 \coloneqq f'$. Assuming $f''_n$ and $f'_n$ have been defined we define

$$f''_{n+1} \coloneqq f''_n \cup \{(m, n)\},$$

where $m = min(dom(f''_n))$. If we consider the possible evaluations of $a^{\pm 1}w'a^{\pm 1}$ and let the pair used in place of one of the $a$ be $(m, n)$, then there are at most two other pairs $p_1, p_2 \in f'_n$ that will be used for the evaluation in the other $a$. We then let

$$f'_{n+1} \coloneqq f'_n \setminus \{(m, n), p_1, p_2\}.$$

Finally, we let $f'' \coloneqq \bigcup_{n \in \omega} f''_n$, which is a partial, infinite function. Using this $f''$ we get that

$$e_w[(s \setminus s_a) \cup \{\, (a, n, f''(n)) \mid n \in ran(f) \,\}, \rho]$$

is nowhere defined.

In the case where

$$e_w[(s \setminus s_a) \cup \{\, (a, n, f(n)) \mid n \in ran(f) \,\}, \rho] \not\cong id,$$

we simply remove one of the pairs of $f$ that is used in an occurence of $a$ for each fixed point of its evaluation to get $f''$.

As $s_a$ is finite, there is only a finite number of pairs $(m, n) \in f''$ such that $m \in dom(s_a)$ or $n \in ran(s_a)$. Removing these still leaves us with infinitely many candidate pairs, we call this set $\hat{f}$ and define $\hat{s} := s \cup \{(a, m, n) | (m, n) \in \hat{f}\}$

Now we consider the fixed points of $e_w[\hat{s}, \rho]$, which, by definition, can only be finitely many. For every $n \in fix(e_w[\hat{s}, \rho]) \setminus fix(e_w[s, \rho])$ there must be some $(c, d) \in \hat{f}$ that the iterative evaluation of $w$ might contain (along with a pair from $s_a$). Thus, if we remove at most $|s_a|$ pairs from $\hat{f}$ for each fixed point, we can eliminate all new fixed points obtained by adding $\hat{f}$ to $s$, leaving us with an infinite set of candidate pairs. If $F$ is not a singleton set we must consider all the evaluations of the words in $F$ and remove all pairs from $f$ that can give rise to fixed points by repeatedly using the two steps used to construct $f''$ from $f$. After having done this for each word the rest of the proof works the same. □

## 4.3 A Lower Bound on the Number of Isomorphism Classes of Maximal Cofinitary Groups

Having established some forcing machinery, we may now begin proving the main theorem of this section. As with any forcing argument, we will begin by defining the sets our $\mathbb{Q}_{A,\rho}$-generic filter will intersect.

**Definition 4.18.** Let $A$ be a set and let $\rho \colon B \to S_\omega$ be a function inducing a cofinitary representation. Let $a \in A$, $n \in \omega$ and let $w \in \hat{W}_{A \cup B}$ then we define the following sets:

- $D_{a,n} := \{(s, F) \in \mathbb{Q}_{A,\rho} \mid n \in dom(s_a)\}$,
- $R_{a,n} := \{(s, F) \in \mathbb{Q}_{A,\rho} \mid n \in ran(s_a)\}$,
- $W_w := \{(s, F) \in \mathbb{Q}_{A,\rho} \mid w \in F\}$.
- Let $T \in [\omega]^\omega$ then we define

  $$T_{a,n} := \{(s, F) \in \mathbb{Q}_{A,\rho} \mid \exists k \geq n \; k \in dom(s_a) \cap T \text{ and } s_a(k) \in T\}.$$

- Let $f \colon S \rightharpoonup S$ be hitable with respect to the cofinitary group $\langle \rho(B) \rangle$. Then define

$$F_{a,n} \coloneqq \{(s,F) \in \mathbb{Q}_{A,\rho} \mid \exists k \geq n \; k \in dom(s_a) \text{ and } s_a(k) = f(k)\}.$$

**Proposition 4.19.** *These posets are dense subsets of $\mathbb{Q}_{A,\rho}$ for any choice of $n \in \omega$, $a \in A$ and $w \in \widehat{W}_{A\cup B}$.*

*Proof.* Let $(s,F)$ be arbitrary in $\mathbb{Q}_{A,\rho}$.

- If $n \in dom(s_a)$, then $(s,F)$ is also contained in $D_{a,n}$ and we're done.

  Otherwise we find cofinitely many good extensions $(s \cup \{(a,n,m)\}, F) \in D_{a,n}$ with respect to $w$ for all $w \in F$ by Lemma 4.11.

  As $F$ is finite, we take the finite intersection of the sets of possible tuples for each word, yielding a cofinite set $S$ of candidates.

  Thus we can pick an arbitrary triple $(a,n,m) \in S$ and will find that $(s \cup \{(a,n,m)\}, F) \in D_{a,n}$ and $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.

- Similarly, for $(s,F)$ with $n \notin ran(s)$ we find an extension $(s \cup \{(a,m,n)\}, F) \in R_{a,n}$ such that $(s \cup \{(a,m,n)\}, F) \leq (s,F)$ using Lemma 4.11, arguing the same way as above.

- We can trivially extend $(s,F)$ to $(s, F \cup \{w\})$, which lies in $W_w$.

- We can use 4.11 with $n \in T \setminus dom(s_a)$ to find a cofinite set that after intersecting with $T$ yields infinitely many pairs $(n,m) \in T$ such that $(s \cup \{(a,n,m)\}, F) \in T_{a,n}$ and $(s \cup \{(a,n,m)\}, F) \leq (s,F)$.

- The density of this set follows in a straightforward manner from Lemma 4.17, as it directly provides an unbounded set of possible extensions.

$\square$

*Remark* 7. When considering the families of all such sets, $\mathcal{D} \coloneqq \{D_{a,n} \mid a \in A, n \in \omega\}$ with $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{F}$ defined analogously, which are of size $max(\omega, |A|)$ as their elements are indexed over $A \times \omega$. The family $\mathcal{W}$ is indexed over the elements in $\hat{W}_{A\cup B}$, and as such has cardinality $|\widehat{W}_{A\cup B}| = |A \cup B|^{<\omega} = max(|A \cup B|, \omega) = max(|A| + |B|, \omega) \leq \mathfrak{c}$.

Now we will prove one final proposition before we finally show how we can construct maximal cofinitary groups with arbitrary orbit structure.

**Proposition 4.20.** *Let $A = \{a\}$ be a singleton set, let $B$ be a set with $|B| < \mathfrak{c}$ and $a \notin B$. Furthermore let $\rho \colon B \to S_\omega$ be a function inducing a cofinitary representation of $F(B)$.*

*Assuming the existence of a $\mathbb{Q}_{A,\rho}$-generic filter, the following are true:*

*(i). We can find a cofinitary extension $\rho_{\mathcal{G}}\colon A \cup B \to S_\omega$ that extends $\rho$ such that $\rho_{\mathcal{G}} \restriction B = \rho$ and $im(\rho_{\mathcal{G}}) \cong \rho(F(B)) * (\mathbb{Z}, +)$.*

*(ii). Let $T \in [\omega]^\omega$ be infinite. Then we can find a cofinitary extension $\rho_{\mathcal{G}}\colon A \cup B \to S_\infty$ that extends $\rho$ such that $\rho_{\mathcal{G}} \restriction B = \rho$, $im(\rho_{\mathcal{G}}) \cong \rho(F(B)) * (\mathbb{Z}, +)$ and $|(T \times T) \cap \rho_{\mathcal{G}}(a)| = \omega$.*

*(iii). Let $f\colon \omega \rightharpoonup \omega$ be hitable with respect to $\rho(F(B))$.*

*Then we can find a cofinitary extension $\rho_{\mathcal{G}}\colon A \cup B \to S_\infty$ that extends $\rho$ such that $\rho_{\mathcal{G}} \restriction B = \rho$, $im(\rho_{\mathcal{G}}) \cong \rho(F(B)) * (\mathbb{Z}, +)$ and $|f \cap \rho_{\mathcal{G}}(a)| = \omega$.*

*Proof.* (i). For this construction we consider the collections of sets $(D_{a,n})_{n\in\omega}$, $(R_{a,n})_{n\in\omega}$ and $(W_w)_{w\in\widehat{W}_{A\cup B}}$, whose elements we have shown to be dense. Let $\mathcal{G}$ be a $\mathbb{Q}_{A,\rho}$-generic filter, such that for all $n \in \omega$ and $w \in \widehat{W}_{A\cup B}$ we have $\mathcal{G} \cap D_{a,n} \neq \emptyset$, $\mathcal{G} \cap R_{a,n} \neq \emptyset$ and $\mathcal{G} \cap W_w \neq \emptyset$.

Examining the generic representation $\rho_{\mathcal{G}}$ as defined above, we notice immediately that $a$ maps to an element of $S_\omega$ due to the intersection with the dense sets given, which force it to be a total bijective function.

It remains to show that $\rho_{\mathcal{G}}$ induces a cofinitary representation of $F(A\cup B)$. To see this, we take any $w \in W_{A\cup B}$ and find $\widehat{w} \in \widehat{W}_{A\cup B}$, $u \in W_{A\cup B}$ such that $w = u^{-1}\widehat{w}u$.

As $W_{\widehat{w}}$ is dense, there must be some $(s, F) \in W_{\widehat{w}}$ such that $(s, F) \in \mathcal{G}$. Let $m \in \omega$ be a fixed point of $e_{\widehat{w}}[\rho_{\mathcal{G}}]$, then there must be a condition $(t, E) \in \mathcal{G}$ with

$$(t, E) \Vdash_{\mathbb{Q}_{A,\rho}} e_{\widehat{w}}[\rho_{\mathcal{G}}](m) = m,$$

with $(t, E) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ and we get $e_{\widehat{w}}[t, \rho](m) = m$, which implies $e_{\widehat{w}}[s, \rho](m) = m$. This means that $fix(e_{\widehat{w}}[\rho_{\mathcal{G}}]) = fix(e_{\widehat{w}}[s, \rho])$, which is finite.

To check that $e_w[\rho_{\mathcal{G}}]$ has at most finitely many fixed points we can see from the definiton that this set is just

$$e_{u^{-1}}[\rho_{\mathcal{G}}](fix(e_{\hat{w}}[\rho_{\mathcal{G}}])),$$

which is finite yet again.

The structure of the new group as a free product follows trivially from adding a single element to the set $B$, extending $\rho$ and considering the new free group's structure.

(ii). The arguments of this and the next item are very much the same, except that we add other, previously defined dense sets that our filter has to have non-empty intersections with. For this construction we include the family $T_{a,n}$ of dense sets.

The families $(D_{a,n})_{n\in\omega}$, $(R_{a,n})_{n\in\omega}$ and $(W_w)_{w\in\widehat{W}_{A\cup B}}$ guarantee us the same properties as before, while the non-empty intersection with all the $(T_{a,n})_{n\in\omega}$ guarantees us the property $|\rho_{\mathcal{G}}(a)\cap T\times T|=\omega$.

(iii). For this construction we add the family $(F_{a,n})_{n\in\omega}$ to the collection of dense sets that our $\mathbb{Q}_{A,\rho}$-generic filter has to have non-empty intersections with. With the other properties as in (i), the intersection with $F_{a,n}$ guarantees us that $|\rho_{\mathcal{G}}(a)\cap f|=\omega$.

□

*Remark* 8. This proposition is an alternative way of proving that $\mathfrak{a}_g>\omega$ as in the case where $B$ is countable we know that we can construct a generic filter explicitly. For larger cardinalities of $A$ or $B$ we will need to use Martin's Axiom (MA), which states that a generic filter exists for any collection of dense sets with cardinality less that $\mathfrak{c}$. In some sense this axiom can be thought of as a generalization of $CH$.

We can now utilize this proposition when we construct a maximal cofinitary group of arbitrary orbit structure.

**Theorem 4.21.** *Let $(m,n)\in\omega\times\omega\setminus\{0\}$. Then, assuming Martin's Axiom, there exists a maximal cofinitary group such that its natural action has $m$ finite and $n$ infinite orbits.*

*Proof.* Begin by fixing a tuple $(m,n)\in\omega\times\omega\setminus\{0\}$. To construct a cofinite group with $n$ infinite and $m$ finite orbits, we first fix an arbitrary partition of

$$\omega=\bigcup_{i=1}^{n}O_i\cup\bigcup_{j=1}^{m}\overline{O}_j,$$

where all $O_i$ are infinite and $\overline{O}_j$ are finite.

Now we will construct sequences of generators

$$g_i\coloneqq\{g_{i,\alpha}\in Sym(O_i)|\alpha<\mathfrak{c}\},$$

and

$$\bar{g}_j\coloneqq\{g_{j,\alpha}\in Sym(\bar{O}_j)|\alpha<\mathfrak{c}\},$$

such that $\langle g_i\rangle$ is transitive and $\langle g_i\rangle\cong F(g_i)$. For $\langle\bar{g}_j\rangle$ we simply ask for transitivity on $\overline{O}_j$.

Assuming we have constructed these sequences up to some $\alpha\in\mathfrak{c}$, we define $G_{i,\alpha}=\langle g_i\rangle_{i<\alpha}$, $\overline{G}_{i,\alpha}=\langle\bar{g}_j\rangle_{i<\alpha}$ and $G_\alpha\coloneqq\langle g_\beta\rangle_{\beta<\alpha}$, which are defined as

$$g_\beta(x) = \begin{cases} g_{i,\beta}(x) & \text{if } x \in O_i, \\ \bar{g}_{j,\beta}(x) & \text{if } x \in \overline{O}_j. \end{cases}$$

As $\overline{O}_j$ is finite we may simply take $g_{j,0} \coloneqq \sigma$ where is any cyclic permutation of $\overline{O}_j$. Further, we set $g_{j,\alpha} \coloneqq g_{j,0}$. This guarantees us transitivity of $\langle \bar{g}_j \rangle$ on $\overline{O}_j$, which is all we ask for in this case.

For infinite orbits we define the permutation $g_{i,0}$ to be $\sigma_i \circ f \circ \sigma_i^{-1}$ where $\sigma_i$ is the order preserving bijection from $\omega$ onto $O_i$ and

$$f(x) \coloneqq \begin{cases} x+2 & \text{if } x \text{ is even}, \\ 0 & \text{if } x = 1, \\ x-2 & \text{otherwise}. \end{cases}$$

This $f$ generates a countable cofinitary group on $\omega$ isomorphic to $(\mathbb{Z}, +)$. And as such, the $g_{i,0}$ do the same on $O_i$.

Next we fix an enumeration of the elements of $S_\omega$ as $(f_\alpha)_{(\alpha\in\mathfrak{c})}$. Then we proceed recursively, at step $\alpha$ we check if $\langle f_\alpha, G_\alpha\rangle$ is cofinitary. If this is not the case, we use utilize construction (i) from Proposition 4.20 with $B$ such that $|B| = |G_{i,\alpha}|$ and $\rho\colon B \to S_\infty$ as a cofinitary representation of $G_{i,\alpha}$. The existence of the necessary filter is guaranteed by MA, as mentioned in the previous remark. Doing this for all orbits $O_i$ we can define $g_{i,\alpha} = \rho_{\mathcal{G}}(a)$.

In the case of $\langle f_\alpha, G_\alpha\rangle$ being cofinitary, we must have at least one tuple $(i, j)$ such that $f_\alpha \cap O_i \times O_j$ is infinite.

For the case $i = j$ we get that $f_\alpha \restriction O_i \times O_i$ is a hitable function with respect to $G_{i,\alpha}$, and as such we can use construction (iii) (with parameters as in the previous paragraph) from Proposition 4.20 to define $g_{i,\alpha}$ and the first construction to obtain all $g_{k,\alpha}$ for $k \neq i$.

For the case $i \neq j$, we first use construction (ii) of Proposition 4.20 with

$$T = \sigma_j^{-1} ran(f_\alpha \cap O_i \times O_j),$$

to construct $g_{j,\alpha}$. Next, consider a partial function $h\colon O_i \rightharpoonup O_i$ defined as

$$h \coloneqq (f_\alpha \cap O_i \times O_j)^{-1} \circ g_{j,\alpha} \circ (f_\alpha \cap O_i \times O_j),$$

which is infinite as $g_{j,\alpha}$ is a total bijective function on $O_j$.

If $h$ is hitable with respect to $G_{i,\alpha}$ we can again use the third construction from Proposition 4.20, otherwise we simply resort to the first one to define $g_{i,\alpha}$. For all other $k \in \omega \setminus \{i, j\}$ we use the first construction to get $g_{k,\alpha}$.

Finally we need to check whether $G_\mathfrak{c} \coloneqq \langle g_\alpha \rangle_{\alpha<\mathfrak{c}}$ fulfills our requirements.

Our group has the required orbit structure as adding an element preserves the orbits by construction.

The fact that each $G_\alpha$ is cofinitary is immediately clear by construction, as we always guarantee that $\langle G_\alpha, g_{\alpha+1}\rangle$ is cofinitary by construction and the case of $\alpha$ being a limit ordinal being trivial.

Finally, we need to show that $G_\mathfrak{c}$ is maximal. Arguing by contradiction, assume that there is some $f \in S_\omega$ such that $\langle f, G_\mathfrak{c}\rangle$ is cofinitary. But since our construction ranges over all $f \in S_\omega$, $f = f_\alpha$ for some $\alpha < \mathfrak{c}$, thus at step $\alpha$ in our construction we would have constructed a $g_{i,\alpha}$ such that $g_{i,\alpha} \cap f_\alpha$ is infinite or such that for a $g_{j,\alpha}$ we have that $f_\alpha^{-1} g_{j,\alpha} f_\alpha$ is not cofinitary or such that $g_{i,\alpha} \cap f_\alpha^{-1} g_{j,\alpha} f_\alpha$ is infinite. Thus we get that either $f \in G_{\alpha+1}$ or $\langle f, G_{\alpha+1}\rangle$ is not cofinitary. $\square$

# 5 A Universal Maximal Cofinitary Group

In this section we will show that there exist cofinitary groups into which every countable group can be embedded, this result was first shown in [31] and then proven in a different manner in [16], which is what Section 5.2 is based on.

## 5.1 More Basics of Forcing Cofinitary Groups

The results of this section, which is a continuation of Section 4.2, are once again from the paper by Fischer [11].

We begin by showing a generalization of (i) from Proposition 4.20, namely that our forcing notion from the previous section can be used with an arbitrary set $A$ to add $|A|$-many elements to our cofinitary group.

**Proposition 5.1.** *Let $A$ and $B$ be sets, let $\rho\colon B \to S_\omega$ induce a cofinitary representation and let $\mathcal{G}$ be a $\mathbb{Q}_{A,\rho}$-generic filter. Then $\rho_{\mathcal{G}}\colon A \cup B \to S_\infty$ induces a cofinitary representation $\hat{\rho}_{\mathcal{G}}\colon F(A \cup B) \to S_\infty$. Furthermore $\rho_{\mathcal{G}}{\restriction}B = \rho$ and $\hat{\rho}_{\mathcal{G}}{\restriction}B = \hat{\rho}$.*

*Proof.* The proof of this statement is the same as the one from (i) of Proposition 4.20, with the one change being that our collections of dense sets are now indexed over $A$ as well as $\omega$. Everything else in the proof still holds the way it was stated, since we never used the fact that $A$ was a singleton set. $\square$

*Remark* 9. By choosing to include $(W_w)_{\widehat{W}_{A\cup B}}$ in our family of dense sets, we guarantee that there will be now relations that impede on the freeness of the newly added elements, as any non-trivial word $w \in \hat{W}_{A\cup B}$ under $\hat{\rho}_{\mathcal{G}}$ can have at most finitely many fixed points and as such will not map to the identity, meaning $\hat{\rho}_{\mathcal{G}}(A \cup B) = \hat{\rho}(B) * F(A)$.

This result shows us that the image of $\rho_{\mathcal{G}}$ will be a cofinitary group, but we still need to show that if we choose $A$ to be large enough, the group will not only be cofinitary, but also maximal.

**Definition 5.2** (Complete embedding)**.** Let $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ be posets and let $\mathbb{Q} \subseteq \mathbb{P}$. Then $\mathbb{Q}$ is completely contained in $\mathbb{P}$ if

(i). For any $q, q' \in \mathbb{Q}$ such that $q \leq_{\mathbb{Q}} q'$ we have $q \leq_{\mathbb{P}} q'$,

(ii). For all $q, q' \in \mathbb{Q}$ such that $q \perp_{\mathbb{Q}} q'$ we have $q \perp_{\mathbb{P}} q'$,

(iii). All maximal antichains in $\mathbb{Q}$ are maximal in $\mathbb{P}$.

Alternatively, this third condition may be stated equivalently as:

(iii') For all $p \in \mathbb{P}$, there is some $q \in \mathbb{Q}$, such that for all $q' \in \mathbb{Q}$ with $q' \leq_{\mathbb{Q}} q$ we have $q' ||_{\mathbb{P}} p$.

*Remark* 10. To see that this definition is equivalent, consider that (iii) tells us that for any $p \in \mathbb{P}$ and an antichain $A$ in $\mathbb{Q}$ there is at least one element $r \in A$ that is compatible with $p$, as otherwise the antichain would not be maximal in $\mathbb{P}$, pick the element that extends both $p$ and $r$ as $q$ in the second definition.

To see that (iii') and (iii) are equivalent, let us assume (iii') holds and let $A$ be an antichain in $\mathbb{Q}$. Assume $p \perp_{\mathbb{P}} q$ for all $q \in A$. Then we know that there is some element $r$ that fulfills (iii') for $p$ and some element $s \in A$ such that $r||_{\mathbb{Q}}s$. Thus we can find a common extension $r'$ of both $r$ and $s$, which has $r'||_{\mathbb{P}}p$ and is a common extension of both $s$ and $p$, contradicting the assumption $p \perp_{\mathbb{P}} q$ for all $q \in A$.

Assume (iii') is false, then there is some $p \in \mathbb{P}$ such that for all $q \in \mathbb{Q}$ there is a $q' \leq q$ such that $q' \perp_{\mathbb{P}} p$. Let $D \coloneqq \{q \in \mathbb{Q} \mid q \perp p\}$, then this set is dense and thus there is a maximal antichain $A \subseteq D$ in $\mathbb{Q}$. This antichain is not maximal in $\mathbb{P}$ as $p \perp q$ for all $q \in D$.

**Definition 5.3** (Restriction of Poset)**.** Let $A_0 \subseteq A$. Then for a condition $p = (s, F) \in \mathbb{Q}_{A,\rho}$, we write $s{\restriction}A_0$ for $s \cap A_0 \times \omega \times \omega$. Furthermore, we write $p{\restriction}A_0$ for $(s{\restriction}A_0, F)$. We call this the *restriction* of $p$ to $A_0$.

Furthemore we write $p|{\restriction}A_0$ for $(s{\restriction}A_0, F \cap \hat{W}_{A_0 \cup B})$. This is called the *strong restriction* of $p$ to $A_0$. Note that $p|{\restriction}A_0 \in \mathbb{Q}_{A_0,\rho}$, while $p{\restriction}A_0$ is generally not.

**Lemma 5.4.** *Let $A_0 \subseteq A$. Then $\mathbb{Q}_{A_0,\rho}$ is completely contained in $\mathbb{Q}_{A,\rho}$.*

*Proof.* If $A_0 = A$ or $\emptyset$ there is nothing to show, so we assume that $A_0$ is a proper subset of $A$ and define $A_1 \coloneqq A \setminus A_0$. Let $p = (s, F) \in \mathbb{Q}_{A_0,\rho}$ be a condition. Then we immediately see that for $p \in \mathbb{Q}_{A,\rho}$ and a condition $q = (t, E) \in \mathbb{Q}_{A_0,\rho}$ such that $q \leq p$ in $\mathbb{Q}_{A_0,\rho}$ we immediately have $q \leq_{\mathbb{Q}_{A,\rho}} p$.

Furthermore, we see that for $p, q \in \mathbb{Q}_{A_0,\rho}$ we get

$$q \perp_{\mathbb{Q}_{A_0,\rho}} p \iff q \perp_{\mathbb{Q}_{A,\rho}} p,$$

as the incompatibility is due to an element contained within $A_0$.

Thus it remains to show that one of the equivalent third conditions from Definition 5.2 holds.

**Claim 5.5.** *For all $(s, F) \in \mathbb{Q}_{A,\rho}$ there exists a $t$ such that $s{\restriction}A_0 \subset t \subset A_0 \times \omega \times \omega$ where for any $a \in A_0$ $t_a$ is a partial injective finite function and if $(r, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t, F \cap \widehat{W}_{A_0 \cup B})$, then $(s \cup r, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$.*

*Proof of claim.* Let $w_j \in F \setminus \widehat{W}_{A_0 \cup B}$. This means that each $w_j$ is of the form

$$u_{k+1} v_k u_k v_{k-1} u_{k-1} \dots v_1 u_1$$

with $u_i \in W_{A_0 \cup B}$ and $v_i \in W_{A_1 \cup B}$ for all $1 \leq i \leq k+1$, where all words except for $u_1$ and $u_{k+1}$ must be non empty and each $v_i$ starts and ends with a letter from $A_1$.

Now we can use Corollary 4.14 to inductively construct an element $t \subset A_0 \times \omega \times \omega$. To do so, we repeatedly apply it for each of the words $(u_i)_{1 \leq i \leq k+1}$ and the condition $(s, F)$ yielding us a $t'_i \subseteq A_0 \times \omega \times \omega$ with $s \restriction A_0 \subseteq t'$ and $dom(e_{u_i}[s \cup t'_i, \rho]) \supseteq ran(e_{v_i}[s, \rho])$ and $ran(e_{u_i}[s \cup t'_i, \rho]) \supseteq dom(e_{v_{i+1}}[s, \rho])$ and $(s \cup t, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ where $t = \bigcup_{1 \leq i \leq k+1} t'_i$.

Now let $(r, E) \in \mathbb{Q}_{A_0,\rho}$ such that

$$(r, E) \leq_{\mathbb{Q}_{A_0,\rho}} (t, F \cap W_{A_0 \cup B}).$$

To see that $(s \cup r, F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ fix $w \in F$ and let $n \in \omega$ be a fixed point of $e_w[s \cup r, \rho]$. If $w \in W_{A_0 \cup B}$ then we are done as $(r, E) \leq (t, F \cap W_{A_0 \cup B}$. Otherwise, for $w \in F \setminus \widehat{W}_{A_0 \cup B}$ we know that by construction of $t$ that if $e_w[s \cup r, \rho](n)\downarrow$ for some $n \in \omega$, then we already have $e_w[s \cup t, \rho](n)\downarrow$. As $(s \cup t, F) \leq (s, F)$ we know that $e_w[s, \rho](n)\downarrow$ and we are done. $\square$

It remains to show that for all pars of conditions $(s, F)$, $(r, E)$ as above we also have that $(s \cup r, E) \leq (r, E)$. For this, assume $e_w[s \cup r, \rho](n) = n$ for some $n \in \omega$. As $r \supset t \supset s \restriction A_0$ and $w \in \widehat{W}_{A_0 \cup B}$ we see that the evaluation of $s \cup r$ must be the same as the evaluation of $r$ thus $e_w[r, \rho](n) = n$.

Thus we get $(s \cup r, E \cup F) \leq_{\mathbb{Q}_{A,\rho}} (s, F)$ and $(s \cup r, E \cup F) \leq_{\mathbb{Q}_{A,\rho}} (r, E)$. $\square$

**Lemma 5.6.** *Let $A \coloneqq A_0 \cup A_1$ where $A_0 \cap A_1 = \emptyset$, let $(t, E) \in \mathbb{Q}_{A_0,\rho}$ and suppose*

$$(t, E) \Vdash_{\mathbb{Q}_{A_0,\rho}} (s_1, F_1) \leq_{\mathbb{Q}_{A_1,\rho_{\mathcal{G}}}} (s_2, F_2)$$

*then*

$$(t \cup s_1, F_1) \leq_{\mathbb{Q}_{A,\rho}} (t \cup s_2, F_2).$$

*Proof.* By assumption we already get $t \cup s_1 \supset t \cup s_2$, $F_1 \supset F_2$. Next, choose $w \in F_2$ and let $n \in \omega$ be any fixed point of $e_w[t \cup s_1, \rho](n) = n$. Now, let $\mathcal{G}$ be $\mathbb{Q}_{A_0,\rho}$-generic and let $(t, E) \in \mathcal{G}$, this means that $e_w[s_1, \rho_{\mathcal{G}}](n) = n$ and by our hypothesis, we get $e_w[s_2, \rho_{\mathcal{G}}](n) = n$. Using Lemma 4.15 and the fact that $w \in \widehat{W}_{A_1 \cup B}$, we get $e_w[t \cup s_2, \rho](n) = n$. $\square$

**Lemma 5.7.** *Suppose $\mathcal{G}$ is $\mathbb{Q}_{A,\rho}$-generic over $V$ and let $A \coloneqq A_0 \dot{\cup} A_1$ such that $A_0, A_1 \neq \emptyset$ and $A_0 \cap A_1 = \emptyset$. Then $\mathcal{H} \coloneqq \mathcal{G} \cap \mathbb{Q}_{A_0,\rho}$ is $\mathbb{Q}_{A_0,\rho}$-generic over $V$ and*

$$\mathcal{K} \coloneqq \{p \restriction A_1 \mid p \in \mathcal{G}\},$$

*is $\mathbb{Q}_{A_1,\rho}$-generic over $V[\mathcal{H}]$. Also $\rho_{\mathcal{G}} = (\rho_{\mathcal{H}})_{\mathcal{K}}$.*

*Proof.* We know that $\mathbb{Q}_{A_0,\rho}$ is completely contained in $\mathbb{Q}_{A,\rho}$ by Lemma 5.4, as such we know that for any maximal antichain $C$ of elements in $\mathbb{Q}_{A_0,\rho}$, $C$ is also a maximal antichain in $\mathbb{Q}_{A,\rho}$. As such we know that $\mathcal{G} \cap C = S$ where $S \subset \mathbb{Q}_{A_0,\rho}$ and thus

$$\mathcal{H} \cap C = (\mathcal{G} \cap \mathbb{Q}_{A_0,\rho}) \cap C = (\mathcal{G} \cap C) \cap \mathbb{Q}_{A_0,\rho} = S \cap \mathbb{Q}_{A_0,\rho} = S.$$

Finally, to show that $\mathcal{K}$ is $\mathbb{Q}_{A_1,\rho_\mathcal{H}}$-generic in $V[\mathcal{H}]$, we consider a dense set $D \subseteq \mathbb{Q}_{A_1,\rho_\mathcal{H}}$ with $D \in V[\mathcal{H}]$. Next we define

$$D' := \left\{p \in \mathbb{Q}_{A,\rho} \mid p|\restriction A_0 \Vdash_{\mathbb{Q}_{A_0,\rho}} p\restriction A_1 \in \dot{D}\right\}.$$

As $D$ is dense there must be a condition $p$ in $\mathcal{H}$ such that

$$p \Vdash_{\mathbb{Q}_{A_0,\rho}} \text{“}D \text{ is dense”}.$$

Now let $(s,F) = q \leq_{\mathbb{Q}_{A,\rho}} p$. Then by Claim 5.5 we find $q' \leq_{\mathbb{Q}_{A_0,\rho}} q|\restriction A_0$ such that if $q_1 \leq_{\mathbb{Q}_{A_0,\rho}} q'$ then $q_1 \parallel_{\mathbb{Q}_{A,\rho}} q$.

Now, as $D$ is dense, we can also find a condition $r = (s', F') \in \mathbb{Q}_{A_1,\rho}$ and a condition $(t,E) \leq_{\mathbb{Q}_{A_0,\rho}} p$ such that

$$(t,E) \Vdash_{\mathbb{Q}_{A_0,\rho}} \dot{r} \in \dot{D} \wedge \dot{r} \leq_{\mathbb{Q}_{A_1,\rho}} \dot{q}|\restriction A_1.$$

Using Lemma 5.6, we now get that $(t \cup s', F') \leq_{\mathbb{Q}_{A,\rho}} (t \cup s\restriction A_1, F)$ and thus $(t \cup s', F' \cup E) \leq_{\mathbb{Q}_{A,\rho}} (s,F)$.

We see that $(t \cup s', F' \cup E) \in D'$ allowing us to conclude that $D'$ is dense below $p$. Since $p \in \mathcal{G}$ we know that there is some $p' \in D' \cap \mathcal{G}$ and in $V[\mathcal{H}]$ we obtain that $p'\restriction A_1 \in D$ and thus $\mathcal{K} \cap D \neq \emptyset$.

Finally, to see that $\rho_\mathcal{G} = (\rho_\mathcal{H})_\mathcal{K}$ we first understand that they must agree on $B$, as it stays the same in our extensions. For an element $a_1 \in A_1$, we then see that

$$\begin{aligned}
\rho_\mathcal{G}(a_1) &= \bigcup\{s_{a_1} \mid \exists F \subset \cap W_{A\cup B}\ (s,F) \in \mathcal{G}\} \\
&= \bigcup\{(s\restriction A_1)_{a_1} \mid \exists F \subset \cap W_{A\cup B}\ (s,F) \in \mathcal{G}\} \\
&= \bigcup\{(s_{a_1} \mid \exists F \subset \cap W_{A\cup B}\ (s,F) \in \mathcal{K}\} \\
&= (\rho_\mathcal{H})_\mathcal{K}(a_1)
\end{aligned}$$

Lastly, consider an element $a_0 \in A_0$:

$$\begin{aligned}\rho_{\mathcal{G}}(a_0) &= \bigcup\{s_{a_1} \mid \exists F \subset \cap W_{A\cup B}\ (s, F) \in \mathcal{G}\} \\ &= \bigcup\{s_{a_0} \mid \exists F \subseteq \hat{W}_{A_0\cup B}\ (s, F) \in \mathcal{H}\} \\ &= (\rho_{\mathcal{H}})_{\mathcal{K}}(a_0)\end{aligned}$$

The second equality is due to the property of filters, if some condition $(s, F) \in \mathcal{G}$ forces some property of $s_{a_0}$, then we find a condition $(s', F') \in \mathbb{Q}_{A_0,\rho} \cap \mathcal{G}$ with $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (s', F')$ defined by $F' = F \cap \hat{W}_{A_0\cup B}$ and $s' = s{\restriction}A_0$. $\square$

**Lemma 5.8.** *Let $B$ be a set and suppose $\rho\colon B \to S_\infty$ induces a non-trivial cofinitary representation. Let $b \in B$ such that $\rho(b) \neq 1$, let $(s, F) \in \mathbb{Q}_{A,\rho\restriction B\setminus\{b\}}$ and let $a \in A$. Then there is some $N \in \omega$, such that for all $n \geq N$ we have $(s \cup \{(a, n, \rho(b)(n))\}, F) \leq_{\mathbb{Q}_{A,\rho\restriction B\setminus\{b\}}} (s, F)$*

*Proof.* We first begin by enumerating the words of $F$ in which $a$ occurs, as all others don't concern us for our statement. Denote them by $w_1, \ldots, w_l$. Any word $w_i$ is of the form

$$u_{i,j_i} a^{k_{i,j_i}} u_{i,j_i-1} \ldots u_{i,1} a^{k_{i,1}} u_{i,0},$$

where the $u_{i,l_i} \in W_{A\setminus\{a\}\cup B\setminus\{b\}}$ and non empty except possibly the ones at indices 0 and $j_i$.

Next we use Lemma 4.11 to ensure that for any $u_{i,l}$ with $dom(e_{u_{i,l}}[s, \rho])$ and $ran(e_{u_{i,l}}[s, \rho])$ finite we have

$$dom(e_{a^{k_{i,l+1}}}[s, \rho]) \supset ran(e_{u_{i,l}}[s, \rho]),$$

and

$$ran(e_{a^{k_{i,l}}}[s, \rho]) \supset dom(e_{u_{i,l}}[s, \rho]).$$

For each $w_1, \ldots w_l$ let $\bar{w}_i$ be the word where every instance of $a$ has been replaced by $b$. As $\rho$ induces a cofinitary representation we know that the evaluation $e_{\bar{w}_i}[s, \rho]$ will always have at most finitely many fixed points, even if it is totally defined.

Let $\bar{w}_{i,l}$ be the subword of $\bar{w}_i$ that begins with the word $u_{i,l}$ and define

$$\begin{aligned}N_i \coloneqq max\big(\{e_v[s, \rho](n) \;\big|\; e_{\bar{w}_i}[s, \rho](n) = n \text{ and } v = b^{sign(k_{i,l})p}\bar{w}_{i,l} \text{ and} \\ 0 \leq p \leq |k_{i,m}| \text{ and } 0 \leq m \leq j_i\}\big),\end{aligned}$$

for the $i$ where $e_{\bar{w}_i}[s, \rho]$ is totally defined.

Now pick $N \in \omega$ such that

$$N \geq max(\{N_i \mid i < l\}, dom(s_a), \{n \mid \rho(b)(n) \in ran(s_a)\}).$$

For any $n \geq N$ and $\bar{w}_i$ where $e_{\bar{w}_i}[s, \rho]$ is not fully defined we have

$$dom(e_{w_i}[s, \rho]) = dom(e_{w_i}[s \cup \{(a, n, \rho(b)(n)\}, \rho],$$

due to the stipulations on range and domain above. If now $e_{\bar{w}_i}[s, \rho]$ is fully defined, then for all $n \geq N$ we have

$$e_{w_i}[s \cup \{(a, n, \rho(b)(n)\}, \rho](k) = k \implies e_{w_i}[s, \rho](k) = k.$$

□

**Theorem 5.9.** *Let $A$ and $B$ be sets and let $\rho\colon B \to S_\omega$ induce a cofinitary representation. If $|A| > \aleph_0$, and $\mathcal{G}$ is a $\mathbb{Q}_{A,\rho}$-generic filter over $V$, then $im(\rho_{\mathcal{G}})$ is a maximal cofinitary group in $V[G]$ of cardinality $|A \cup B|$.*

*Proof.* Towards a contradiction assume that $im(\rho_{\mathcal{G}})$ is not a maximal cofinitary group. Thus there must be a permutation $f \in S_\omega$ such that $f \notin im(\rho_{\mathcal{G}})$ and $\langle im(\rho_{\mathcal{G}}), f\rangle$ is a cofinitary group. We can thus extend the domain of $\rho_{\mathcal{G}}$ with a single element $x$ and define $\dot{\rho}_{\mathcal{G}}\colon A \cup B \cup \{x\} \to S_\omega$ such that $\dot{\rho}_{\mathcal{G}}(x) = f$ and $\dot{\rho}_{\mathcal{G}}\restriction(A \cup B) = \rho_{\mathcal{G}}$.

As $f \in V[\mathcal{G}]$ there is a name $\dot{f}$ for $f$. As $f$ is countable, there is an at most countable set $A_0 \subset A$ such that $\dot{f}$ is a $\mathbb{Q}_{A_0,\rho}$-name. Thus $f \in V[\mathcal{H}]$ for $\mathcal{H} \coloneqq \mathcal{G} \cap \mathbb{Q}_{A_0,\rho}$. Now we define $A_1 \coloneqq A \setminus A_0$ and $\mathcal{K} \coloneqq \{p\restriction A_1 \mid p \in \mathcal{G}\}$ Next define

$$D_{f,N} \coloneqq \{(s, F) \in \mathbb{Q}_{A_1,\rho_{\mathcal{H}}} \mid \exists n \geq N \ s_a(n) = f(n)\}.$$

For every $N \in \omega$ and $a \in A_1$ this set is dense by Proposition 4.19. In $V[\mathcal{H}][\mathcal{K}]$ we have that $(\rho_{\mathcal{H}})_{\mathcal{K}}(a)(n) = f(n)$ for all $a \in A_1$ and infinitely many $n \in \omega$. By Lemma 5.8 we have $(\rho_{\mathcal{H}})_{\mathcal{K}} = \rho_{\mathcal{G}}$ which contradicts that $\dot{\rho}_{\mathcal{G}}$ induces a cofinitary representation. □

## 5.2 Constructing a Universal Cofinitary Group

Our first hurdle in constructing this universal group is wether or not we can even represent the groups we want to embed as subgroups of $S_\omega$ at all, which we have already shown for a few special classes of groups in Section 3.

**Definition 5.10.** A group $G$ is said to have cofinitary action if there exists a group homomorphism $\rho\colon G \to S_\omega$ which admits a cofinitary representation.

At this point we do not know whether all countable groups even have a cofinitary action. This fact will be established first in this section before finally constructing a group into which all countable groups can be embedded.

Before we can start with the Lemma that will establish this, we need to alter the forcing notion that we have been using so far to accommodate us.

**Definition 5.11.** Let $G$ be a countable group and let $f\colon G \to G$ be the identity function of $G$ as a set. We then let $\hat{f}\colon F(G) \to G$ be the group homomorphism obtained via the universal property of the free group.

Let $A$ be a set of the same cardinality as $G$. Then $\mathbb{Q}^G_{A,\rho}$ is the forcing notion defined as:

(i). The conditions of $\mathbb{Q}^G_{A,\rho}$ are pairs $(s, F)$ where $s \subseteq A \times \omega \times \omega$ is finite and $s_a$ is a partial finite injective function for every $a \in A$ and $W \subseteq \widehat{W}_{A \cup B}$ is finite. Furthermore for every word $w \in ker(\hat{f}) \subset W_A$ we require $e_w[s, \rho] \cong id$ wherever it is defined.

(ii). For two conditions $(s_1, W_1) \leq (s_2, W_2)$ if $s_1 \supseteq s_2$, $W_1 \supseteq W_2$ and for every $n \in \omega$ and $w \in W_2$, if $e_w[s_1, \rho](n) = n$ then already $e_w[s_2, \rho](n)\!\downarrow$ and $e_w[s_2, \rho](n) = n$.

*Remark* 11. Note that $\mathbb{Q}^G_{A,\rho} \subseteq \mathbb{Q}_{A,\rho}$ and thus $\mathbb{Q}^G_{A,\rho}$ inherits the countable chain condition.

This restriction of the poset now allows us to force relations in the group $\rho_{\mathcal{G}}(A)$, by allowing us to have certain words be the identity when evaluated. As a subset of $\mathbb{Q}_{A,\rho}$ all the universal statements about $\mathbb{Q}_{A,\rho}$ hold for our new notion $\mathbb{Q}^G_{A,\rho}$ as well, particularly Lemma 4.11 and Proposition 5.1.

The relations that define our group $G$ also play another role by providing us with a way of refining a condition.

**Definition 5.12** (Applying Relations)**.** Let $(s, F) \in \mathbb{Q}^G_{A,\rho}$ then we say $t \in [A \times \omega \times \omega]^{<\omega}$, where every $t_a$ is a partial injective function, is obtained from $s$ by applying relations if

$$(a, n, m) \in t \iff \exists w \in W_A \; aw \in ker(\hat{f}) \text{ and } e_w[s, \rho](m) = n.$$

Note that a $t$ obtained by applying relations is not necessarily an element of $\mathbb{Q}^G_{A,\rho}$ as it may be infinite. To avoid this, we can stipulate that the $a$ appearing in the first coordinate of $t$ may only be ones that appear in $s$ along with possibly finitely more from a set $A' \subseteq A$.

We call this $A'$-applying relations.

**Lemma 5.13.** *Let $(s, F) \in \mathbb{Q}^G_{A,\rho}$, $\bar{A} \subseteq A$ finite and let $t$ be obtained from $s$ by $\bar{A}$-applying relations. Then* *(i).* $s \subseteq t$,

*(ii).* *$t$ is constant under $\bar{A}$-applying relations,*

*(iii).* $(t, F) \in \mathbb{Q}^G_{A,\rho}$,

*(iv).* $(t, F) \leq (s, F)$.

*Proof.* (i). Using $a^{-1}$ in place of $w$, this is clear by definition.

(ii). Let $q$ be the element obtained from $t$ by $\bar{A}$-applying relations. Towards a contradiction we assume $q \setminus t \neq \emptyset$, thus there is an element $(a, n, m) \in q \setminus t$ and a word $w \in W_A$ such that for $a \in dom(t) = dom(s) \cup \bar{A}$ we have $aw \cong id$.

Let $n \in \omega$ be arbitrary and assume $e_w[s, \rho](n) = m$. Then the pair $(a, m, n)$ would have been added when applying relations to $s$ already.

As such, the only case for $q \setminus t$ to not be empty is for $l \in \omega$ such that $e_w[s, \rho](l)\uparrow$ but $e_w[t, \rho](l)\downarrow$.

This means there is some element $a' \in dom(s) \cup \bar{A}$ appearing in $w$ which appears in the first coordinate of a tuple added while $\bar{A}$-applying relations to $s$, so we can write $w = ua'v$. Let $(a', j, k) \in t \setminus s$ be that pair.

By definition we know that for this pair to be added, there must be a word $w'$ such that $a'w' \in ker(\hat{\rho})$ and $e'_w[s, \rho](k) = j$. As $a'w' \cong 1$ when $w'$ is defined this means we can substitute $a'$ for $(w')^{-1}$ in $w$. Repeating this for all tuples which were added when $\bar{A}$-applying relations to $s$ we obtain a new word $\bar{w}$ which has the same properties as $w$ but $e_{\bar{w}}[s, \rho](l)\downarrow$, thus $(a', j, k) \in t$.

(iii). As both $s$ and $\bar{A}$ are finite, there are only finitely many pairs that can be added when $\bar{A}$-applying relations. Thus $t \in [A \times \omega \times \omega]^{<\omega}$.

Let $w \in ker(\hat{f})$. Then $e_w[s, \rho] \cong id$ where it is defined. By the construction from the previous point, we see that $e_w[t, \rho] \cong id$ as well. Thus $(t, F) \in \mathbb{Q}^G_{A,\rho}$

(iv). Let $n \in \omega$ and $w \in F$ such that $e_w[t, \rho](n) = n$. As we have shown above, we must have $e_w[s, \rho](n)\downarrow$ which implies $(t, F) \leq (s, F)$.

□

Now we can begin using forcing arguments to construct the groups we want.

**Theorem 5.14.** *Let $H$ be a cofinitary group with cofinitary representation $\rho$ and let $G$ be an at most countable group. Then there exists a set of cofinitary permutations $F \subseteq S_\omega$ such that $\langle F \rangle \cong G$. In particular the group we obtain is $H \times G \cong H \times \langle F \rangle \leq S_\infty$ and $H \times \langle F \rangle$ is a cofinitary group.*

*Proof.* We will use a forcing argument to show this. Let us first show that the sets $D_{a,n}$, $R_{a,n}$ and $W_w$ defined in Definition 4.18 are also dense with respect to $\mathbb{Q}^G_{A,\rho}$.

We begin by enumerating $A$ and we write $A_n$ for the set containing the first $n$ elements of this sequence.

Let us fix some $a \in A$, $n \in \omega$ and $(s, F) \in \mathbb{Q}^G_{A,\rho}$. We let $t \in \mathbb{Q}^G_{A,\rho}$ be obtained from $s$ by $A_n$-applying relations to $s$. If $n \in dom(t_a)$, then we are done by Lemma 5.13. If this is not the case, then we can use Lemma 4.11 to find an extension $(r, F) \in D_{a,n}$ such that $(r, F) \leq_{\mathbb{Q}_{A,\rho}} (t, F)$.

It remains to show that there is an $r$ such that $(r, F) \in \mathbb{Q}^G_{A,\rho}$. Towards a contradiction, assume there is some $w \in ker(\hat{f})$ such that $e_w[r, \rho] \not\cong id$. Let us now pick the shortest such $w$. Now there must be some $k \in \omega$ such that $e_w[r, \rho](k) \neq k$. The previous Lemma tells us that applying relations can not cause this, so we must have that the pair $(a, n, m)$ which was added via our application of Lemma 4.11 must be used in the evaluation $e_w[r, \rho](k)$. As there are cofinitely many possible choices for $r$, we can simply choose $m$ large enough so that this case is avoided, as only finitely many choices for $m$ will lead to $e_w[r, \rho](k) \neq k$.

The argument for the density of $R_{a,n}$ follows analogously and $W_w$ is trivially a dense set.

We can now find a $\mathbb{Q}^G_{A,\rho}$-generic filter that has non empty intersection with all of the dense sets defined above. Using 5.1 we get a cofinitary representation induced by $\rho_{\mathcal{G}}$.

We define $F := \rho(A)$. From our construction we know that every $a \in A$ maps to a cofinitary permutation. Furthermore we see that by our construction, we get that every word $w \in W_A$ such that $w \cong id$ we have $\hat{\rho}_{\mathcal{G}}(w) = id$. Thus $\rho(A) \cong F(A)/W_{G,id} \cong G$. □

Lastly we will just need to show a simple result that allows us to use CH for our proof.

**Lemma 5.15.** *There are $2^\omega$ many countable groups up to isomorphism.*

*Proof.* Each group law can be thought of as a function $f : \omega \times \omega \to \omega$ which we know there are at most continuum many.

To see there are at least continuum many consider that for any subset of the primes we can form the direct product of the cyclic groups of the orders of the primes, obtaining continuum many non-isomorphic countable groups. □

With these results, we can now finally show the main result of this section.

**Theorem 5.16.** *Assuming ZFC+CH, there is a maximal cofinitary group into which every countable group embeds.*

*Proof.* We begin by enumerating all countable groups. By CH, we know there are $\omega_1$ many and thus we enumerate them as $(G_\alpha)_{\alpha<\omega_1}$. We do the same with all permutatinos in $S_\omega$ and get a sequence $(g_\alpha)_{\alpha<\omega_1}$.

Now we use Theorem 5.14 to adjoin one group after the other to $G_0$ yielding us a universal cofinitary group $U$. After the step where we adjoin group $G_\alpha$ we also check whether $g_\alpha$ is part of our group, if it is we are done. If $G_\alpha * \langle g_\alpha \rangle$ is cofinitary, we can use construction (iii) from Proposition 4.20 to construct an element $f$ which we add to $G_\alpha$.

Once we have constructed $G_{\omega_1}$ it will be maximal and all countable groups will embed into it. □

Finally, we will see that this construction does not necessarily stipulate an assumption of CH on our part, but can also be done by assuming MA. Our main theorem then becomes:

**Theorem 5.17.** *Martin's Axiom implies the existance of a maximal cofinitary group into which every countable group can be embedded.*

*Proof.* The proof of this Theorem proceeds exactly as above, with the one change being the fact that the $H$ we use in in Theorem 5.14 is no longer countable, which does not change the statement of it. The transfinite induction goes through as stated and we use MA to obtain the necessary generic filter for each step. □

# 6 The Spectrum of Maximal Cofinitary Groups

In this section we will discuss the possible sizes of maximal cofinitary groups. We will find that there are models in which we can control the spectrum of maximal cofinitary groups very tightly. For this we will start with models of ZFC + GCH and then construct generic extensions using an alteration of our familiar poset.

**Definition 6.1** (Spectrum)**.** Let $V$ be a model of ZFC and GCH and let $\mathcal{S}$ be the class of all sets in $V$ that fulfill some property. The spectrum of $\mathcal{S}$ is the class of all possible sizes of such structures,

$$C(\mathcal{S}) := \{|S| \mid S \in \mathcal{S}\}.$$

Both $\mathcal{S}$ and $C(\mathcal{S})$ may also be sets, depending on the model and the nature of $\mathcal{S}$.

**Example 6.2.** (i). Let $V$ be any model of ZFC. Then the spectrum $C(fin)$ of the class of finite sets $fin$ is $\omega$.

(ii). If the size of objects in the class $\mathcal{S}$ is linked to the continuum, then the spectrum of this class changes depending on the model, while the spectrum of some classes such as $fin$ in universal. For example if we consider $C(mcg)$ the spectrum of maximal cofinitary groups then it must be the singleton set of $\omega_1$ for models of CH, and the set $2^\omega$ for models of MA, but this need not be $\omega_1$ in this case.

We will now work in a model $V$ of ZFC + GCH. Let $C(\omega)$ be a closed set of cardinals with the following properties:

(i). $min(C(\omega)) = \omega^+$,

(ii). for all $\mu \in C(\omega)$, if $cof(\mu) = \omega$, then $\mu^+ \in C(\omega)$,

(iii). if $|C(\omega)| \geq \omega^+$, then the interval $[\omega^+, |C(\omega)|] \subseteq C(\omega)$.

We call a set $C(\omega)$ satisfying these properties a potential spectrum.

Note that for a cardinal $\kappa$ there are many possible sets $C(\kappa)$.

## 6.1 The Existence Result

We will now show that there is a ccc forcing notion $\mathbb{P}$ such that in the $\mathbb{P}$-generic extension of $V$ the spectrum of maximal cofinitary groups coincides with the set $C(\omega)$. The proof of the theorem follows the one given in [10], which was itself a generalization of the initial result presented by Brendle, Spinas and Zhang [3].

**Example 6.3.** A model for which $C(\omega) = \{\omega_1\}$ will have only maximal cofinitary groups of size $\omega_1$. In Section 3 we showed that there will always be maximal cofinitary groups of size continuum by Zorn's Lemma, thus we must have $\mathfrak{c} = \omega_1$.

**Definition 6.4.** Let $\xi$ be a cardinal and let $I_\xi \coloneqq \{(\eta, \xi) \mid \eta < \xi\}$ be the set of ordinals less than $\xi$. Let $\mathbb{Q}_{I_\xi,\rho}$ be the forcing notion defined like before, but instead of an abstract index set $A$, we now index over the set of tuples $I_\xi$. A $\mathbb{Q}_{I_\xi,\rho}$-generic extension of $V$ will contain a maximal cofinitary group of cardinality $\xi$ by Proposition 5.1.

Furthermore, let

$$\mathbb{P} \coloneqq \prod_{\xi \in C(\omega)} \mathbb{Q}_{I_\xi,\rho},$$

such that every element $p \in \mathbb{P}$ has at most finitely many non-empty sets in its $|C(\omega)|$-many coordinates.

We say that $s \leq_{\mathbb{P}} t$ if $s_\eta \leq_{\mathbb{Q}_{I_\eta,\rho}} t_\eta$ for all $\eta \in C(\omega)$.

For an element $p \in \mathbb{P}$ we write

$$supp(p) \coloneqq \{\xi \in C(\omega) \mid p_\xi \neq \emptyset\},$$

which we call the support of $p$.

The fact that $\mathbb{P}$ is ccc follows immediately from the next lemma.

**Lemma 6.5.** *Let $\mathbb{Q}$ be a product of $\omega^+$-Knaster posets $(\mathbb{Q}_i)_{i\in I}$ with finite supports. Then $\mathbb{Q}$ is also $\omega^+$-Knaster.*

*Proof.* Let $A \subseteq \mathbb{Q}$ be a set of conditions with $|A| = \omega^+$. Assume that there are some $p, q \in \mathbb{Q}$ such that $p \perp_{\mathbb{Q}} q$.

For any $i \in I$ such that $|A_i \cap \mathbb{Q}_i| \geq \omega^+$ we can use the fact that $\mathbb{Q}_i$ is $\omega^+$-Knaster and obtain $B_i \subset A_i$ with $|B_i| = \omega^+$ and for all $p, q \in B_i$ we have $p \parallel_{\mathbb{Q}_i} q$. We then restrict the $i$th coordinate of $A$ to elements from $B_i$ and get a new set $A'$ which is of size at least $\omega^+$ and such that all elements are compatible on the $i$th coordinate.

For $i \in I$ where $|A_i \cap \mathbb{Q}_i| < \omega^+$ we will, by regularity of $\omega$, be able to find a compatible subset $B_i$ such that the restriction of $A$ to $B_i$ on the $i$th coordinate $A'$ will still be of cardinality $\omega^+$.

Applying either of these steps for each $i \in I$ will yield a set $B$ of size $\omega^+$ where all elements are compatible. $\square$

*Remark* 12. Note that a product of ccc posets $\mathbb{Q}_i$ will not necessarily be ccc itself.

Knowing that $\mathbb{P}$ will preserve all cardinals in our extension $V[\mathcal{G}]$, we can now show the existence part of the section's main theorem.

**Lemma 6.6.** *For every $\xi \in C(\omega)$ there exists a maximal cofinitary group of size $\xi$ in the extension $V[\mathcal{G}]$ where $\mathcal{G}$ is $\mathbb{P}$-generic.*

*Proof.* We know that for each $\xi \in C(\omega)$ we adjoin a maximal cofinitary group of size $\xi$ via the poset $\mathbb{Q}_{I_\xi,\rho}$, as products of dense sets will be dense these groups will be exist in the $\mathbb{P}$-generic extension $V[\mathcal{G}]$. However we still need to show that all these groups will still be maximal.

Let us fix a $\psi \in C(\omega)$ and towards a contradiction assume that $G_\psi$ is not a maximal cofinitary group in $V^{\mathbb{P}}$. This means that there must be some $f \in S_\omega$ and a $\mathbb{P}$-name for it along with a condition $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \langle G_\psi, \dot{f} \rangle \text{ is cofinitary.}$$

We know that $f$ has a nice name and as $\mathbb{P}$ is ccc we know there are $\omega$ many antichains $(A_i)_{i\in\omega}$ such that for every $p \in A_n$ there is $k \in \omega$ such that $p \Vdash_{\mathbb{P}} \dot{f}(n) = k$.

Next we will aim to define a poset $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$, where we define $\bar{\mathbb{P}} := \prod_{\xi \in C'} \mathbb{Q}_{\xi,\rho}$ with finite supports and

$$C' := \left( \left( \bigcup_{i\in\omega, b\in A_i} supp(b) \right) \cup supp(p) \right) \setminus \{\psi\}.$$

Note that this set is at most countable. We let $\bar{\mathbb{Q}} := \mathbb{Q}_{A_\psi,\rho}$ where

$$A_\psi := \left( \bigcup_{i\in\omega, b\in A_i} oc_{I_\psi}(b(\psi)) \right) \cup oc_{I_\psi}(p(\psi)),$$

which is also a countable set.

By Lemma 5.4 we note that $\mathbb{Q}_{A_\psi,\rho}$ is completely contained in $\mathbb{Q}_{A_\psi,\rho}$. Also note that $p$ is a $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$-condition and similarly all the $b \in A_i$ for $i \in \omega$, meaning that $\dot{f}$ is a $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$-name. Thus

$$p \Vdash_{\bar{\mathbb{P}}\times\bar{\mathbb{Q}}} \langle G_\psi, \dot{f} \rangle \text{ is cofinitary.}$$

Now let $\mathcal{G}'$ be $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$-generic and let $p \in \mathcal{G}'$. We note that by 5.8 we have

$$\Vdash_{\mathbb{Q}_{I_\psi \setminus A_\psi, \rho_{A_\psi}}} \langle im(\rho_{A_\psi \mathcal{H}}) \rangle \text{ is a maximal cofinitary group of size } \psi.$$

We see that

$$(\bar{\mathbb{P}} \times \mathbb{Q}_{A_\psi}) * \mathbb{Q}_{I_\psi \setminus B_\psi} = \bar{\mathbb{P}} \times (\mathbb{Q}_{A_\psi} * \mathbb{Q}_{I_\psi \setminus A_p si}) = \bar{\mathbb{P}} \times \mathbb{Q}_{I_\xi},$$

where $*$ denotes an iterated forcing step and so

$$p \Vdash_{\bar{\mathbb{P}} \times \mathbb{Q}_{I_\psi}} \langle G_\psi, \dot{f} \rangle \text{ is cofinitary,}$$

which is a contradiction. □

## 6.2 The Nonexistence Result

The proof of the main lemma of this section is an adaption of the proof given in [9], which concerns itself with mad families, to our setting of maximal cofinitary groups.

It remains to show that for any cardinal number $\lambda \notin C(\omega)$ there exist no maximal cofinitary groups of size $\lambda$ in our model $V[\mathcal{G}]$, which is what we will show in the remainder of this chapter. Once again $V$ is a model of ZFC and GCH.

**Lemma 6.7.** *Let $\lambda$ be a cardinal number such that $\lambda \notin C(\omega)$ and suppose $\mathcal{G}$ is $\mathbb{P}$-generic. Then there is no maximal cofinitary group of size $\lambda$ in $V[\mathcal{G}]$.*

*Proof.* Fix $\lambda \notin C(\omega)$. Towards a contradiction, suppose that in our model $V[\mathcal{G}]$ there is a maximal cofinitary group $G_\lambda = \{g_\alpha\}_{\alpha<\lambda}$ of size $\lambda$. First, let us define

$$\mu := max\left(\{\xi \in C(\omega) \mid \xi < \lambda\}\right),$$

as the largest cardinal in $C(\omega)$ less than $\kappa$.

By definition of $C(\omega)$ we get $\mu \geq cof(\mu) \geq \omega_1$ and by GCH we get $\mu^\kappa = \mu$. Also note that $\mu \geq |[C(K)]^\kappa|$.

Next let us define some helper notions.

**Definition 6.8.** (i). Let $\dot{f}$ be a $\mathbb{P}$-name for a cofinitary permutation. Then we can assume that $\dot{f}$ is a nice name, so we can find $\omega$-many maximal antichains $(A_i)_{i\in\omega}$ with the property that $A_n$ decides $\dot{f}(n)$. We then define

$$\Delta_{\dot{f}} := \bigcup_{i\in\omega} A_i,$$

as the set of conditions involved in $\dot{f}$.

(ii). Let $\dot{f}$ be a $\mathbb{P}$-name for a cofinitary permutation and let $\Delta_{\dot{f}}$ be the set of conditions involved in $\dot{f}$. Then

$$J_{\dot{f}} := \bigcup_{p\in\Delta_{\dot{f}}, \xi\in supp(p)} dom(p(\xi)),$$

where $dom(p(\xi)) = dom((s, F)) = dom(s)$ by an abuse of notation. We call $J_{\dot{f}}$ the support of $\dot{f}$.

(iii). Any countable set $J'$ with $J_{\dot{f}} \subseteq J' \subseteq I := \bigcup_{\xi \in C(\omega)} I_\xi$ is referred to as a support of $\dot{f}$.

For each $\dot{g}_\alpha \in G_\lambda$ let $J_\alpha$ be a support of $\dot{g}_\alpha$. We define the set

$$K^* := \bigcup \{I_\xi \mid \xi \in C(\omega) \text{ and } \xi \leq \mu\},$$

and

$$S := K^* \cup \bigcup \{J_\alpha \mid \alpha \in C(\omega)\}.$$

**Definition 6.9.** Now let $K$ be a set such that $K^* \subseteq K \subseteq S$ and $|K| = \mu$ and let $\dot{f}$ be a $\mathbb{P}$-name for a cofinitary permutation. Then:

(i). A support $J$ for $\dot{f}$ is said to be a $K$-support if whenever $J \cap (I_\gamma \setminus K) \neq \emptyset$ then $|J \cap (I_\gamma \setminus K)| = \omega$.

(ii). A $K$-support $J$ of $\dot{f}$ is said to be $K$-standard if $J \cap K = J \cap S$.

If $\dot{f}$ is a $\mathbb{P}$-name for a cofinitary permutation and $K$ is as above, then $\dot{f}$ has a $K$-support. Furthermore, any support $J$ of $\dot{f}$ can be made into a $K$-support.

To see this, consider that $J$ is countable, so we can add countably many tuples of the form $(\eta, \gamma)$ for any $\gamma$ fulfilling the condition in $(i)$ above.

With $K$ as in the above definition, let $G(K)$ be the group of all permutations of the index set $I = \bigcup_{\xi \in C(\omega)} I_\xi$ such that any element $g \in G$ is the identity on $K$ and the orbits of the action of $G(K)$ are the individual $I_\xi$.

Each $g \in G(K)$ defines an automorphism $\phi_g$ of $\mathbb{P}$ if for a $p \in \mathbb{P}$ we let $\phi_g(p)$ be a condition with the same support as $p$ and for every tuple $(a, m, n) \in p(\xi)$ we let $\phi_g((a, m, n)) = (\phi_g(a), m, n) \in \phi_g(p(\xi))$.

The fact that $\phi_g$ is an automorphism is easily seen as it merely permutes the labels of the elements of the components $\mathbb{Q}_{I_\xi, \rho}$ of $\mathbb{P}$ and as such also preserves the relation $\leq_{\mathbb{P}}$ and antichains of $\mathbb{P}$.

As a consequence of this any $K$-support $J$ remains a $K$-support under the action of $g \in G(K)$.

For any $K$-support $J$ we can define the following set,

$$\overline{J} := \{\gamma \mid J \cap (I_\gamma \setminus K) \neq \emptyset\}.$$

In our case the set $\bar{J}$ is of size at most $\omega$.

**Lemma 6.10.** *For $K$-supports $J_0$ and $J_1$ we have that there is a $g \in G(K)$ such that $g(J_0) = J_1$ if and only if $J_0 \cap K = J_1 \cap K$ and $\overline{J_0} = \overline{J_1}$.*

*Proof.* If there is $g \in G(K)$ such that $g(J_0) = J_1$, then we immediately get that

$$J_0 \cap K = g(J_0 \cap K) = g(J_0) \cap K = J_1 \cap K,$$

and the second condition follows from $g$ having the orbits $I_\xi$.

To see the other direction, note that if we do not have $J_0 \cap K = J_1 \cap K$ then we can not have $g \restriction K \cong id_K$ and if we don't have $\overline{J_0} = \overline{J_1}$ then a function taking $J_0$ to $J_1$ could not have the orbits $I_\xi$ $\square$

For a fixed $K$ we get that there are at most $\mu$-many orbits under the action of $G(K)$ on the sets of $K$-supports due to the fact that $|[K]^\omega| = \mu$, i.e. there are $\mu$-many choices for static sets of $K$ under the action of $G(K)$, and $|[C(\omega)]^\omega| \leq \mu^\omega = \mu$, which is the number of possible choices of index sets that are non-isomorphic.

We also know that any orbit contains a $K$-standard support and thus we find that as there are at most $\omega^\omega = \omega^+$-many different names for cofinitary permutations with the same support we find that there are at most $\mu$-many names for cofinitary permutations with $K$-standard supports.

Now if $\dot{f}$ is a $\mathbb{P}$-name for a cofinitary permutation, the fact that $\mathbb{P}$ is ccc guarantees us the existence of a set $B(\dot{f}) \in [\lambda]^\omega \cap V$ such that

$$\Vdash_{\mathbb{P}} \exists \alpha \in \check{B}(\dot{f}) \ |\dot{g}_\alpha \cap \dot{f}| = \omega.$$

**Definition 6.11.** Let $K \subset S$ such that $|K| = \mu$ and $K^* \subset K$. Let

$$B(K) \coloneqq \bigcup \{B(\dot{x}) \mid \dot{x} \text{ is a } \mathbb{P}\text{-name for a cofinitary permutation with a } K\text{-standard support}\}.$$

By the above observation on the number of names of $K$-standard supports, $|B(K)| = \mu$.

Now we construct recursive sequences of sets as follows:

Let $K_0 \coloneqq K^*$ and let $M_0 \coloneqq \emptyset$. Now define $M_1 \coloneqq B(K^*)$ and let

$$K_1 \coloneqq K_0 \cup \bigcup \{J_\alpha \mid \alpha \in M_0\}.$$

Assuming $K_\delta$ has been defined we define $M_{\delta+1} \coloneqq B(K_\delta)$ and

$$K_{\delta+1} \coloneqq K_\delta \cup \bigcup \{J_\alpha \mid \alpha \in M_{\delta+1}\}.$$

If $\delta$ is a limit, then let $K_\delta \coloneqq \bigcup_{\eta<\delta} K_\eta$ and let $M_\delta \coloneqq \bigcup_{\eta<\delta} M_\eta$. Finally, let $K \coloneqq \bigcup_{\eta<\omega^+} K_\eta$ and let $M \coloneqq \bigcup_{\eta<\omega^+} M_\eta$. By construction $M$ is of size $\mu$.

There is an $\alpha \in \lambda \setminus M$ and let us consider $\dot{f} = \dot{g}_\alpha$. Let $J$ be a support for $\dot{f}$ and by definition of $K$ there must be some $K_\gamma$ such that $J \cap K_\gamma = J \cap K$. We may assume that $J$ is a $K_\gamma$-support and thus there is a $g \in G(K_\gamma)$ such that $g(J)$ is a $K_\gamma$-standard support.

As $g(J)$ is $K_\gamma$-standard, we have that $g(J) \cap K_\gamma = g(J) \cap S$ and thus we get $g(J) \cap (K_{\gamma+1} \setminus K_\gamma) = \emptyset$. We can thus find $h \in G(K_{\gamma+1})$ with $h \restriction J = g$. This means that $g(\dot{f}) = h(\dot{f})$ and as $g(J)$ is $K_\gamma$-standard we note that $B(g(\dot{f})) \subseteq M_{\gamma+1}$ and $\bigcup_{\delta \in M_{\gamma+1}} J_\delta \subseteq K_{\gamma+1}$. By definition of $B(g(\dot{f}))$ we get

$$\Vdash_{\mathbb{P}} \exists \delta \in \check{M}_{\gamma+1} \ |g(\dot{f}) \cap \dot{g}_\delta| = \omega.$$

Next we use the fact that $h(\dot{f}) = g(\dot{f})$ and $h(\dot{g}_\delta) = \dot{g}_\delta$ as $J_\delta \subseteq K_{\gamma+1}$ to obtain

$$\Vdash_{\mathbb{P}} \exists \delta \in \check{M}_{\gamma+1} \ |h(\dot{f}) \cap h(\dot{g}_\delta)| = \omega.$$

Finally we get

$$\Vdash_{\mathbb{P}} \exists \delta \in \check{M}_{\gamma+1} \ |\dot{f} \cap \dot{g}_\delta| = \omega,$$

from the fact that $h$ is an automorphism of $\mathbb{P}$. As $\dot{f} = \dot{g}_\alpha$ this is a contradiction. $\square$

Finally, this yields the main theorem of this section:

**Theorem 6.12.** *Let $C(\omega)$ be a potential spectrum. Then, assuming GCH, we can find a generic extension, which preserves cardinals, in which $C(mcg) = C(\omega)$.*

# 7 Open Questions

Finally, here are some (to my knowledge) open questions about cofinitary groups:

- Is it consistent with $ZFC$ that $\mathfrak{a}_f \neq \mathfrak{a}_g$?
- Are there closed maximal cofinitary groups?
- Are all closed cofinitary groups of countable degree locally compact?
- How many non-isomorphic maximal cofinitary groups of a fixed cardinality are there?
- Which uncountable groups admit a cofinitary representation?

# 8 References


[1] SA Adeleke. “Embeddings of infinite permutation groups in sharp, highly transitive, and homogeneous groups”. In: *Proceedings of the Edinburgh Mathematical Society* 31.2 (1988), pp. 169–178.

[2] Andreas Blass. “Combinatorial cardinal characteristics of the continuum”. In: *Handbook of set theory*. Springer, 2010, pp. 395–489.

[3] Jörg Brendle, Otmar Spinas, and Yi Zhang. “Uniformity of the meager ideal and maximal cofinitary groups”. In: *Journal of Algebra* 232.1 (2000), pp. 209–225.

[4] Peter J Cameron. “Oligomorphic permutation groups”. In: *Perspectives in Mathematical Sciences II: Pure Mathematics*. World Scientific, 2009, pp. 37–61.

[5] Peter J. Cameron. “Cofinitary Permutation Groups”. In: *Bulletin of the London Mathematical Society* 28.2 (Mar. 1996), pp. 113–140. eprint: `http://oup.prod.sis.lan/blms/article-pdf/28/2/113/1011007/28-2-113.pdf`. URL: `https://dx.doi.org/10.1112/blms/28.2.113`.

[6] Peter J. Cameron. *Permutation Groups*. London Mathematical Society Student Texts. Cambridge University Press, 1999.

[7] Daniel E Cohen and Daniel Ellis Cohen. *Combinatorial group theory: a topological approach*. Vol. 14. Cambridge University Press, 1989.

[8] John D Dixon and Brian Mortimer. *Permutation groups*. Vol. 163. Springer Science & Business Media, 1996.

[9] Arthur Fischer, Vera Fischer, and Sy David Friedman. “Global Spectra”.

[10] Vera Fischer. “Maximal Cofinitary Groups Revisited”. In: *Mathematical Logic Quarterly* 61.4-5 (2015), pp. 367–379.

[11] Vera Fischer and Asger Törnquist. “Template iterations and maximal cofinitary groups”. In: *Fundamenta Mathematicae* 230 (Oct. 2013).

[12] Su Gao and Yi Zhang. “Definable sets of generators in maximal cofinitary groups”. In: *Advances in Mathematics* 217 (Jan. 2008), pp. 814–832.

[13] Lorenz J Halbeisen. *Combinatorial set theory*. Springer, 2012.

[14] Haim Horowitz and Saharon Shelah. “A Borel maximal cofinitary group”. In: *arXiv preprint arXiv:1610.01344* (2016).

[15] Thomas Jech. *Set theory*. Springer Science & Business Media, 2013.

[16] Bart Kastermans. *Cofinitary Groups and Other Almost Disjoint Families of Reals*. 2009. arXiv: `0910.0279`.

[17] Bart Kastermans. “Isomorphism types of maximal cofinitary groups”. In: *Bull. Symbolic Logic* 15.3 (Sept. 2009), pp. 300–319. URL: https://doi.org/10.2178/bsl/1246453976.

[18] Bart Kastermans and Yi Zhang. “Cardinal Invariants Related to Permutation Groups”. In: *Annals of Pure and Applied Logic* 143 (Nov. 2006), pp. 139–146.

[19] Richard W Kaye and Dugald Macpherson. *Automorphisms of first-order structures*. Oxford University Press, 1994.

[20] Sabine Koppelberg. “Groups of permutations with few fixed points”. In: *algebra universalis* 17.1 (Dec. 1983), pp. 50–64. URL: https://doi.org/10.1007/BF01194513.

[21] Kenneth Kunen. *Set theory an introduction to independence proofs*. Vol. 102. Elsevier, 2014.

[22] Anatoliı I Mal’cev. “On the faithful representation of infinite groups by matrices”. In: *Mat. Sb* 8.50 (1940), pp. 405–422.

[23] David Marker. *Model theory: an introduction*. Vol. 217. Springer Science & Business Media, 2006.

[24] Peter M Neumann. “The structure of finitary permutation groups”. In: *Archiv der Mathematik* 27.1 (1976), pp. 3–17.

[25] Weaver Nik. *Forcing for mathematicians*. World Scientific, 2014.

[26] Petr Sergeevich Novikov. “Algorithmic Unsolvability of the Word Problem in Group Theory”. In: (1958).

[27] Ralf Schindler. *Set theory: exploring independence and truth*. Springer, 2014.

[28] Józef Schreier and Stanisław Ulam. “Über die Permutationsgruppe der natürlichen Zahlenfolge”. In: *Studia Math* 4.1 (1933), pp. 134–141.

[29] Katrin Tent and Martin Ziegler. *A course in model theory*. Vol. 40. Cambridge University Press, 2012.

[30] Jacques Tits. “Généralisations des groupes projectifs basées sur leurs propriétés de transitivité”. In: *Mémoires de la Classe des Sciences* 27 (1952).

[31] J.K. Truss. “Embeddings of infinite permutation groups”. In: *Proceedings of Groups - St. Andrews 1985*. Ed. by E. F. Robertson and Colin MatthewEditors Campbell. London Mathematical Society Lecture Note Series. Cambridge University Press, 1987, pp. 335–351.

[32] Helmut Wielandt and Adolf Mader. *Unendliche Permutationsgruppen.* Universität Tübingen, 1960.

[33] Mitsuo Yoshizawa. “On infinite four-transitive permutation groups”. In: *Journal of the London Mathematical Society* 2.3 (1979), pp. 437–438.