MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

"Towards a descriptive version of Ramsey's Theorem"

verfasst von / submitted by

Fabian Ebner, BSc

angestrebter akademischer Grad / in partial fulfillment of the requirements for the degree of

Master of Science (MSc)

Wien, 2020 / Vienna 2020

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Benjamin Miller, PhD

Abstract

This thesis explores aspects of Ramsey theory in the descriptive set-theoretic context. The motivating question was: When does a Borel function from a countable Borel equivalence relation E to 2 admit an E-complete homogeneous Borel set? This thesis mostly focuses on two related questions: When is the underlying space a countable union of homogeneous Borel sets? In the new context, what is the relevance of the usual method of constructing counterexamples to infinitary Ramsey statements by comparing two linear orders?

First, we look at the Hausdorff condensation of linear orders and give two proofs of the fact that ω_1 is a strict upper bound for the supremum of the Hausdorff ranks of certain definable sets of scattered linear orders. Then, we deal with continuous embeddability in the class $\Gamma_{\mathcal{G}^2}$ of pairs of analytic directed graphs on a Polish space whose joint Borel chromatic number is uncountable. Many results from a paper by Miller and Lecomte are generalized to pairs of analytic directed graphs, most importantly the basis and anti-basis results. Finally, we look at the class $\Gamma_{\mathcal{F}^*}$ of Borel functions $f\colon E \setminus \Delta(X) \to 2$, where E is a non-smooth Borel equivalence relation on a Polish space X, with the property that every Borel set B for which $f \restriction ((E \setminus \Delta(X)) \restriction B)$ is constant is E-smooth. There is a natural example f_0 of such a function, which turns out to be minimal among the functions in $\Gamma_{\mathcal{F}^*}$ whose domain comes from a countable equivalence relation.

Zusammenfassung

In dieser Masterarbeit erkunden wir Aspekte der Ramsey-Theorie im Kontext der Deskriptiven Mengenlehre. Die motivierende Fragestellung lautete: Wann gibt es für eine borelsche Funktion von einer abzählbaren borelschen Äquivalenzrelation E nach 2 eine E-vollständige homogene Borel-Menge? Wir beschäftigen uns hauptsächlich mit zwei verwandten Fragen: Wann kann der darunterliegende Raum als abzählbare Vereinigung von homogenen Borel-Mengen geschrieben werden? Welche Relevanz hat die übliche Methode, um Gegenbeispiele für Ramsey-Aussagen im Unendlichen zu konstruieren, i.e. das Vergleichen von zwei linearen Ordnungen, im neuen Kontext?

Zu Beginn untersuchen wir die Hausdorff-Kondensation von linearen Ordnungen und geben zwei Beweise dafür, dass ω_1 eine strikte obere Schranke für das Supremum der Hausdorff-Ränge von bestimmten definierbaren Mengen von zerstreuten linearen Ordnungen ist. Anschließend untersuchen wir die stetige Einbettbarkeit in der Klasse $\Gamma_{\mathcal{G}^2}$ von Paaren analytischer gerichteter Graphen auf polnischen Räumen, deren gemeinsame borelsche chromatische Zahl überabzählbar ist. Viele Resultate einer Arbeit von Miller und Lecomte werden zu Paaren von analytischen gerichteten Graphen verallgemeinert, insbesondere die Basis- und Anti-Basis-Resultate. Abschließend untersuchen wir die Klasse $\Gamma_{\mathcal{F}^*}$ von borelschen Funktionen $f\colon E\setminus\Delta(X) \to 2$, wobei E eine nicht-glatte borelsche Äquivalenzrelation auf einem polnischen Raum ist und jede Borel-Menge B, für welche $f \restriction ((E \setminus \Delta(X)) \restriction B)$ konstant ist, E-glatt ist. Es gibt ein natürliches Beispiel f_0 einer solchen Funktion, die minimal unter jenen Funktionen in $\Gamma_{\mathcal{F}^*}$ ist, deren Domäne von einer abzählbaren Äquivalenzrelation kommt.

Introduction

The original motivation for this thesis was to investigate to what extent there is an analog of Ramsey's Theorem in the descriptive set-theoretic context. For each binary relation R on a set X, function $f\colon R \to 2$ and $k < 2$, we call a set $Y \subseteq X$ *f-homogeneous (with value k)* if $f \restriction (R \restriction Y)$ is constant (with value k). For each equivalence realtion E on a set X, a subset of X is *E-complete* if it intersects every E-class.

Question 1. *Given a countable Borel equivalence relation E on a Polish space, under what circumstances does a Borel function $f\colon E \to 2$ admit an E-complete f-homogeneous Borel set?*

A *graph* on a set X is a symmetric irreflexive subset of $X \times X$, a *directed graph* on X is an irreflexive subset of $X \times X$ and an *oriented graph* on X is an anti-symmetric directed graph on X. For each directed graph G on a set X, let $G^{\pm 1} = \{(x, y) \in X \times X \mid (x, y) \in G \text{ or } (y, x) \in G\}$.

Although Question 1 served as a starting point, this thesis mostly deals with the following two related questions:

Question 2. *Given an analytic directed graph G on a Polish space X and a Borel function $f\colon G \to 2$, under what circumstances is X a countable union of f-homogeneous Borel sets?*

Question 3. *Does the usual method of building counterexamples to infinitary Ramsey statements, i.e. by comparing two linear orders, also work in the descriptive set-theoretic context, and to what extent are such counterexamples canonical?*

For each set X, let $\Delta(X)$ denote the *diagonal* $\{(x, x) \in X^2 \mid x \in X\}$ on X. For each linear order R on a set X, let $\operatorname{dom}(R) = X$, $<_R = R \setminus \Delta(X)$, $(x, y)_R = \{z \in X \mid x <_R z <_R y\}$ and $[x, y)_R = \{z \in X \mid x \mathrel{R} z <_R y\}$. A linear order R on a set X is *dense* if $\forall x, y \in X\ (x <_R y \implies (x, y)_R \neq \emptyset)$, and a set $Y \subseteq X$ is *R-convex* if $\forall x, y \in Y\ (x, y)_R \subseteq Y$.

Given a linear order R on a set X and an equivalence relation E on X whose classes are R-convex, let R/E denote the linear order on X/E given by

$$[x]_E \mathrel{R/E} [y]_E \iff x \mathrel{R} y,$$

and let E'_R denote the superequivalence relation of E given by

$$x \mathrel{E'_R} y \iff |([x]_E, [y]_E)_{R/E} \cup ([y]_E, [x]_E)_{R/E}| < \aleph_0.$$

Note that the classes of E'_R are R-convex. Recursively define an increasing sequence of equivalence relations whose classes are R-convex by setting $E^0_R =$

$\Delta(X)$, $E_R^{\alpha+1} = (E_R^\alpha)'_R$ for all ordinals α and $E_R^\lambda = \bigcup_{\beta<\lambda} E_R^\beta$ for all limit ordinals λ. The *Hausdorff rank $\rho_H(R)$ of R* is the least ordinal α with the property that $E_R^\alpha = E_R^{\alpha+1}$.

Let $\leq_\mathbb{Q}$ denote the usual order on $\mathbb{Q}$. A linear order R is *scattered* if there is no embedding of $\leq_\mathbb{Q}$ into R. For each equivalence relation E, we abuse language by saying that a partial order $R \subseteq E$ is an *assignment of linear orders to the classes of E* if the restriction of R to each E-class is a linear order. A *Borel space* $(X, \mathcal{S})$ is a set X together with a σ-algebra $\mathcal{S}$ on X, and $(X, \mathcal{S})$ is *standard* if there is a Polish topology on X whose Borel sets are exactly the elements of $\mathcal{S}$.

Section 1 deals with the Hausdorff condensation of linear orders and relates to Question 3. We give two different proofs of the following well-known result:

Theorem 1.8. *Suppose that E is a countable Borel equivalence relation on a standard Borel space X and R is a Borel assignment of scattered linear orders to the classes of E. Then* $\sup_{x\in X} \rho_H(R \restriction [x]_E) < \omega_1$.

For each directed graph G on a set X, a set $Y \subseteq X$ is *G-dependent* if $\exists y, y' \in Y\ (y, y') \in G$. For each sequence $(G_i)_{i\in I}$ of directed graphs on a set X, a set $Y \subseteq X$ is *$(G_i)_{i\in I}$-dependent* if $\forall i \in I\ Y$ is G_i-dependent, and *$(G_i)_{i\in I}$-independent* if it is not $(G_i)_{i\in I}$-dependent, and a function $c\colon X \to Z$ is a *coloring* of $(G_i)_{i\in I}$ if $\forall z \in Z\ c^{-1}(\{z\})$ is $(G_i)_{i\in I}$-independent. For each sequence $(G_i)_{i\in I}$ of analytic directed graphs on a Polish space X and class Γ, define $\chi_\Gamma((G_i)_{i\in I})$ to be the least cardinal κ for which there is a Polish space Y and a Γ-measurable coloring $c\colon X \to Y$ of $(G_i)_{i\in I}$ such that $|c[X]| = \kappa$. When Γ is the class of all Borel subsets of X, we use $\chi_B((G_i)_{i\in I})$ to denote the *Borel chromatic number* of $(G_i)_{i\in I}$, and when Γ is the class of all subsets of X with the property of Baire, we use $\chi_{BP}((G_i)_{i\in I})$ to denote the respective chromatic number.

For all sequences $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ of binary relations on sets X and Y, a map $\pi\colon X \to Y$ is a *homomorphism* from $(R_i)_{i\in I}$ to $(S_i)_{i\in I}$ if $(x, x') \in R_i \implies (\pi(x), \pi(x')) \in S_i$ for each $x, x' \in X$ and $i \in I$, a *reduction* of $(R_i)_{i\in I}$ to $(S_i)_{i\in I}$ if $(x, x') \in R_i \iff (\pi(x), \pi(x')) \in S_i$ for each $x, x' \in X$ and $i \in I$, and an *embedding* of $(R_i)_{i\in I}$ into $(S_i)_{i\in I}$ if it is an injective reduction of $(R_i)_{i\in I}$ to $(S_i)_{i\in I}$. For each pair of functions $f\colon G \to 2$ and $g\colon H \to 2$, where G and H are directed graphs on the sets X and Y, we call $\pi\colon X \to Y$ an *embedding* of f into g if π is an embedding of $(f^{-1}(\{k\}))_{k<2}$ into $(g^{-1}(\{k\}))_{k<2}$. Note that every embedding of f into g is an embedding of G into H.

A *quasi-order* on a set X is a reflexive transitive binary relation on X. For each quasi-order $\sqsubseteq$ on a set X, a set $A \subseteq X$ is an *$\sqsubseteq$-antichain* if $\forall a, b \in$

A ($a \neq b \implies a \not\sqsubseteq b$), and a *strong* $\sqsubseteq$*-antichain* if $\forall a, b \in A$ ($a \neq b \implies \forall x \in X$ ($x \not\sqsubseteq a$ or $x \not\sqsubseteq b$)).

Let $\Gamma_{\mathcal{G}} = \{G \mid G$ is an analytic directed graph on a Polish space X such that $\chi_B(G) > \aleph_0\}$, $\Gamma_{\mathcal{G}^2} = \{(G_k)_{k<2} \mid G_0$ and G_1 are analytic directed graphs on a Polish space X such that $\chi_B((G_k)_{k<2}) > \aleph_0\}$ and $\Gamma_{\mathcal{F}} = \{f \colon G \to 2 \mid G$ is an analytic directed graph on a Polish space X and f is a Borel function such that $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$. As it should cause no confusion, we use $\sqsubseteq_c$ to denote the quasi-order of continuous embeddability on all three of these classes.

In Section 2, we deal with Question 2 by looking at continuous embeddability in the classes $\Gamma_{\mathcal{G}^2}$ and $\Gamma_{\mathcal{F}}$. Many results from [8] by Miller and Lecomte are generalized to pairs of directed analytic graphs, most importantly the basis and anti-basis results stated in the following paragraphs.

Theorem 2.36. *There is a continuum-sized strong* $\sqsubseteq_c$*-antichain of minimal-under-*$\sqsubseteq_c$ *pairs of graphs in* $\Gamma_{\mathcal{G}^2}$*. In particular, any basis for* $\Gamma_{\mathcal{G}^2}$ *with respect to* $\sqsubseteq_c$ *is at least continuum-sized.*

Let $\mathrm{par} \colon \mathbb{N} \to 2$ be the unique map satisfying $\mathrm{par}(n) \equiv n \pmod 2$ for each $n \in \mathbb{N}$. For each pair $S \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ and $k < 2$, let S^k be given by $S^k(i) = \{\mathbf{s} \in S(i) \mid \mathrm{par}(|\mathbf{s}(0)|) = k\}$ for each $i < 2$, let G^S be the directed graph on $2^{\mathbb{N}}$ given by $G^S = \{(\mathbf{s}(j) \frown (|j-i|) \frown c)_{j<2} \mid i \in 2, \mathbf{s} \in S(i), c \in 2^{\mathbb{N}}\}$, and for each $n > 0$, let G^S_n be the finite approximation of G^S on 2^n given by $G^S_n = \{(\mathbf{s}(j) \frown (|j-i|) \frown t)_{j<2} \mid i \in 2, \mathbf{s} \in S(i) \cap (2^{<n} \times 2^{<n}), t \in 2^{n-(|\mathbf{s}(0)|+1)}\}$.

Let $\sqsubseteq$ and $\sqsubset$ denote extension and strict extension on both $\mathbb{N}^{\leq\mathbb{N}}$ and $2^{\leq\mathbb{N}}$. Fix a sequence $s_n \in 2^n$ for each $n \in \mathbb{N}$ such that $\{s_{2n+k} \mid n \in \mathbb{N}\}$ is dense in $2^{<\mathbb{N}}$ for each $k < 2$, and let $\mathbb{S}_0 = (\{(s_n, s_n) \mid n \in \mathbb{N}\}, \emptyset)$. We call $S \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ *dense* if $\forall r \in 2^{<\mathbb{N}} \exists \mathbf{s} \in S(0) \forall j < 2\ r \sqsubseteq \mathbf{s}(j)$ and *strongly dense* if $S(0) \supseteq \mathbb{S}_0(0)$. We call $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ *strongly dense* if $\mathbf{S}(k)(0) \supseteq \mathbb{S}_0^k(0)$ for each $k < 2$.

A subset B of a Polish space X is $\aleph_0$*-universally Baire* if for every Polish space Y and Borel function $\pi \colon X \to Y$, the set $\pi^{-1}(B)$ has the property of Baire.

Theorem 2.31. *Suppose that* $\Gamma = \{f \colon G \to 2 \mid G$ *is an analytic graph on a Polish space which admits an* $\aleph_0$*-universally Baire measurable reduction to a locally countable analytic graph on a Polish space, as well as an* $\aleph_0$*-universally Baire measurable reduction to an analytic acyclic graph on a Polish space, and* f *is a symmetric Borel function such that* $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$*. Then the set* $\{f_0\}$*, where* $f_0 \colon (G^{\mathbb{S}_0})^{\pm 1} \to 2$ *is given by* $f_0(\mathbf{x}) = k \iff \mathbf{x} \in (G^{\mathbb{S}_0^k})^{\pm 1}$ *for each* $k < 2$*, is a one-element basis for* $\sqsubseteq_c \restriction \Gamma$*.*

Theorem 2.32. *Suppose that $\Gamma = \{f \colon G \to 2 \mid G$ is an analytic oriented graph on a Polish space which admits an $\aleph_0$-universally Baire measurable reduction to a locally countable analytic directed graph on a Polish space, as well as an $\aleph_0$-universally Baire measurable reduction to an analytic directed graph H on a Polish space for which $H^{\pm 1}$ is an acylic graph, and f is a Borel function such that $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$. Then the set $\{f_0\}$, where $f_0 \colon G^{\mathbb{S}_0} \to 2$ is given by $f_0(\mathbf{x}) = k \iff \mathbf{x} \in G^{\mathbb{S}_0^k}$ for each $k < 2$, is a one-element basis for $\sqsubseteq_c \restriction \Gamma$.*

We let $\leq_{lex}$ denote the lexicographic ordering on $2^{\mathbb{N}}$ as well the lexicographic ordering on 2^n for each $n \in \mathbb{N}$. An *aligned function* on $2^{<\mathbb{N}}$ is a function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ given by $f(s) = \bigoplus_{n<|s|} \mathbf{u}_n^f(s(n))$, where $\mathbf{u}_n^f \in (2^{k_n^f})^2$ for some positive natural number k_n^f for each $n \in N$, and where the empty concatenation denotes the empty sequence. For each aligned function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$, let $f_\infty \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be given by $f_\infty(c) = \bigcup_{n\in\mathbb{N}} f(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$. An aligned function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is *order-preserving* if $\forall c, d \in 2^{\mathbb{N}}$ $(c \leq_{lex} d \implies f_\infty(c) \leq_{lex} f_\infty(d))$, or equivalently, if $\forall n \in \mathbb{N}$ $u_n^f(0) \leq_{lex} u_n^f(1)$, and is *order-reversing* if $\forall c, d \in 2^{\mathbb{N}}$ $(c \leq_{lex} d \implies f_\infty(c) \geq_{lex} f_\infty(d))$, or equivalently, if $\forall n \in \mathbb{N}$ $u_n^f(0) \geq_{lex} u_n^f(1)$. An aligned function is *monotonic* if it is either order-preserving or order-reversing. For pairs $S, T \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$, an *aligned embedding* of S into T is an aligned function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ for which $f \restriction 2^n$ is an embedding of G_n^S into $G^T_{k_0^f+\ldots+k_{n-1}^f}$ for each $n > 0$. We use $\sqsubseteq_a$ to denote the quasi-order of monotonic aligned embeddability on the set of dense pairs in $\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$. As a consequence of Proposition 2.8, for each $S, T \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$, we call an embedding of G^S into G^T an (*order-preserving*, *order-reversing* or *monotonic*) *aligned embedding* if it is of the form f_∞ for some (order-preserving, order-reversing or monotonic) aligned embedding $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ of S into T.

Theorem 2.27. *(cf. [8, Theorem 3.10]) Suppose that G_0 and G_1 are analytic directed graphs on a Polish space X such that there is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to a pair of locally countable analytic directed graphs on a Polish space, $\mathcal{T}$ is a finite subset of $(\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$, and $\pi^{\mathbf{T}}$ is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$. Then exactly one of the following holds:*

(1) $\chi_B((G_k)_{k<2}) \leq \aleph_0$.

(2) There is a strongly dense pair $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ and a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G_k)_{k<2}$ such that $\pi^{\mathbf{T}} \circ \pi$ is an aligned embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$.

A Borel equivalence relation E on a standard Borel space X is *smooth* if it is Borel reducible to equality on a standard Borel space, and a Borel set $B \subseteq X$ is *E-smooth* if $E \upharpoonright B$ is smooth. In particular, if E is a non-smooth Borel equivalence relation and B is an E-complete Borel set, then B is E-non-smooth. Let $\mathbb{E}_0$ be the non-smooth countable Borel equivalence relation on $2^{\mathbb{N}}$ given by $c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m > n \ c(m) = d(m)$ and let $\mathbb{R}_0 \subseteq \mathbb{E}_0$ denote the Borel relation on $2^{\mathbb{N}}$ given by $c \mathbin{\mathbb{R}_0} d \iff (c = d$ or $\exists n \in \mathbb{N}$ $(c(n) < d(n)$ and $\forall m > n \ c(m) = d(m)))$.

Section 3 is related to Questions 1 and 3, and here we look at continuous embeddability on the class $\Gamma_{\mathcal{F}^*}$ of Borel functions $f \colon E \setminus \Delta(X) \to 2$, where E is a non-smooth Borel equivalence relation on a Polish space X, with the property that every f-homogeneous Borel set is E-smooth. There is a natural example $f_0 \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ of such a function given by $f_0(\mathbf{c}) = 0 \iff (\mathbf{c} \in {\leq_{lex}} \iff \mathbf{c} \in \mathbb{R}_0)$, and the following hold:

Theorem 3.9. *Suppose that Γ is the class of symmetric Borel functions $f \colon E \setminus \Delta(X) \to 2$ in $\Gamma_{\mathcal{F}^*}$ of the form $f(\mathbf{x}) = 0 \iff (\mathbf{x} \in R \iff \mathbf{x} \in S)$, where R and S are Borel assignments of linear orders to the classes of E. Then $\{f_0\}$ is a one-element basis for $\sqsubseteq_c \upharpoonright \Gamma$.*

Theorem 3.11. *Suppose that $\Gamma = \{f \colon E \setminus \Delta(X) \to 2 \mid f \in \Gamma_{\mathcal{F}^*}$ and E is a countable Borel equivalence relation$\}$. Then f_0 is minimal with respect to $\sqsubseteq_c \upharpoonright \Gamma$.*

For each $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$, we let $\sim S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ denote the pair given by $\sim S(i) = (\bigcup_{n \in \mathbb{N}} 2^n \times 2^n) \setminus S(i)$ for each $i < 2$. Note that $G^{\sim S} = \mathbb{E}_0 \setminus (\Delta(2^{\mathbb{N}}) \cup G^S)$.

We also show that the functions from $\mathbb{E}_0$ to 2 that are fully determined by a function from $\bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ to 2 form a basis for a large subclass of $\Gamma_{\mathcal{F}^*}$. In fact, one can show that functions generated by strongly dense pairs which satisfy a certain technical condition (see the following definition and theorem) form such a basis. For each pair $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$, we say that an aligned embedding $g \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is *S-homogeneous* if the pair given by $T \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ given by $T(i) = \{(g(\mathbf{s}(j)) \frown (\mathbf{u}_n^g(|j - i|) \upharpoonright k))_{j<2} \mid n \in \mathbb{N}, \mathbf{s} \in 2^n \times 2^n, k$ is minimal such that $\mathbf{u}_n^g(0)(m) = \mathbf{u}_n^g(1)(m)$ for each $k < m < k_n^g$ and $\mathbf{u}_n^g(0)(k) = i\}$ for each $i < 2$ is such that either $T(i) \subseteq S(i)$ for each $i < 2$ or $T(i) \subseteq \sim S(i)$ for each $i < 2$. Note that g is S-homogeneous if and only if $g_\infty[2^{\mathbb{N}}]$ is f-homogeneous for the function $f \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ given by $f(\mathbf{c}) = 0 \iff \mathbf{c} \in G^S$.

Theorem 3.14. *Suppose that $\Gamma = \{f \colon E \setminus \Delta(X) \to 2 \mid E$ is a non-smooth Borel equivalence relation on a Polish space X and f is a Borel function such* *that there is an $\aleph_0$-universally Baire measurable reduction of $(f^{-1}(\{k\}))_{k<2}$ to a pair of locally countable analytic directed graphs on a Polish space, and there is no E-non-smooth f-homogeneous Borel subset of $X\}$. Then the set $\{(G^S, G^{\sim S}) \mid S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ is a strongly dense pair such that there is no S-homogeneous aligned embedding$\}$ is a basis for $\sqsubseteq_c \restriction \Gamma$.*

1 Hausdorff condensation

Proposition 1.1. *Suppose that R is a linear order on a set X and $f\colon \mathbb{Q} \to X$ is an embedding of $\leq_{\mathbb{Q}}$ into R. Then for each ordinal α, the map $f_\alpha\colon \mathbb{Q} \to X/E_R^\alpha$ given by $f_\alpha(q) = [f(q)]_{E_R^\alpha}$ is an embedding of $\leq_{\mathbb{Q}}$ into R/E_R^α.*

Proof. For each ordinal α, the fact that the quotient map from X to X/E_R^α is a homomorphism from R to R/E_R^α ensures that f_α is a homomorphism from $\leq_{\mathbb{Q}}$ to R/E_R^α, so the fact that $\leq_{\mathbb{Q}}$ is a linear order and R/E_R^α is a partial order ensures that it is sufficient to show that f_α is injective. Since $f_0 = f$ and the least ordinal β for which two elements are E_R^β-related is never a limit ordinal, it is sufficient to show that if f_α is an embedding of $\leq_{\mathbb{Q}}$ into R/E_R^α, then $f_{\alpha+1}$ is injective. To see this, note that if f_α is an embedding of $\leq_{\mathbb{Q}}$ into R/E_R^α and $q_0 <_{\mathbb{Q}} q_1$, then $(f_\alpha(q_0), f_\alpha(q_1))_{R/E_R^\alpha}$ is infinite, thus $[f(q_0)]_{E_R^{\alpha+1}} \neq [f(q_1)]_{E_R^{\alpha+1}}$. ☐

Proposition 1.2 (Hausdorff [4])**.** *Suppose that R is a linear order on a set X. Then $E_R^{\rho_H(R)} = X \times X$ if and only if R is scattered.*

Proof. To see ($\Longrightarrow$), it is sufficient to show that if R is not scattered, then $E_R^{\rho_H(R)} \neq X \times X$, which is a direct consequence of Proposition 1.1.

To see ($\Longleftarrow$), it is sufficient to show that if $E_R^{\rho_H(R)} \neq X \times X$, then R is not scattered. To see this, note that if C and D are $E_R^{\rho_H(R)}$-classes with $C <_{R/E_R^{\rho_H(R)}} D$, then the definition of $\rho_H(R)$ ensures that the open interval $(C, D)_{R/E_R^{\rho_H(R)}}$ is infinite, so $R/E_R^{\rho_H(R)}$ is a dense linear order. In particular, if $E_R^{\rho_H(R)} \neq X \times X$, then $R/E_R^{\rho_H(R)}$ is a non-trivial dense linear order, and therefore is not scattered. ☐

Let $\mathrm{LO}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N}^2)$ be the set of linear orders on $\mathbb{N}$.

Proposition 1.3. *The Hausdorff rank of each $R \in \mathrm{LO}(\mathbb{N})$ is countable.*

Proof. It is sufficient to note that $(E_R^\alpha)_{\alpha<\omega_1}$ is an increasing sequence of equivalence relations on $\mathbb{N}$ and $\mathbb{N}$ is countable. ☐

It is straightforward to check that $\mathrm{LO}(\mathbb{N})$ is closed, hence a Polish space. Note that the map from $\mathcal{P}(\mathbb{N}^2) \times \mathrm{LO}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N}^2)$ given by $(E, R) \mapsto E'_R$ if E is an equivalence relation whose classes are R-convex and $(E, R) \mapsto \emptyset$ otherwise is Borel, so a straightforward induction ensures that for each countable ordinal α, the map from $\mathrm{LO}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N}^2)$ given by $R \mapsto E_R^\alpha$ is Borel.

For each tree T on a set X, let $\mathrm{Prune}(T)$ be the set of all elements of T with a proper extension in T. Let $\mathrm{Prune}^0(T) = T$, $\mathrm{Prune}^{\alpha+1}(T) = \mathrm{Prune}(\mathrm{Prune}^\alpha(T))$ for all ordinals α and $\mathrm{Prune}^\lambda(T) = \bigcap_{\alpha<\lambda} \mathrm{Prune}^\alpha(T)$ for all limit ordinals λ. Let the *pruning rank* $\rho_P(T)$ of T be the least ordinal α for which $\mathrm{Prune}^\alpha(T) = \mathrm{Prune}^{\alpha+1}(T)$, and for each $t \in T$, let the *pruning rank* $\rho_P^T(t)$ of t within T be the maximal ordinal α for which $t \in \mathrm{Prune}^\alpha(T)$ and ∞ if no such ordinal exists. For each $x \in X$, let $(x) \frown T$ denote the tree $\{(x) \frown t \mid t \in T\}$.

Proposition 1.4. *Suppose that $A \subseteq \mathrm{LO}(\mathbb{N})$ is an analytic set of scattered linear orders on $\mathbb{N}$. Then $\sup_{R\in A} \rho_H(R) < \omega_1$.*

Proof. For an injective enumeration $Q = (q_n)_{n\in\mathbb{N}}$ of a subset of $\mathbb{Q}$ and a linear order R on a subset of $\mathbb{N}$, define the tree $T^Q(R)$ on $\mathbb{N}$ of *attempts* at embedding $\leq_\mathbb{Q} \restriction Q[\mathbb{N}]$ into R by

$$t \in T^Q(R) \iff \forall n, m < |t| \ (q_n \leq_\mathbb{Q} q_m \iff t(n) \ R \ t(m)).$$

Lemma 1.5. *Suppose that R is a scattered linear order on a subset of $\mathbb{N}$ for which $\rho_H(R) \geq \lambda + 2n$ for some limit ordinal λ and natural number n, and $Q = (q_k)_{k\in\mathbb{N}}$ is an injective enumeration of a subset of $\mathbb{Q}$. Then $\rho_P(T^Q(R)) \geq \lambda + n$.*

Proof. We proceed by induction on $\lambda + 2n$. To see the base case where $\lambda = \omega$ and $n = 0$, it is sufficient to note that the domain of R is infinite, so for each $l \in \mathbb{N}$, there is an embedding of $\leq_\mathbb{Q} \restriction \{q_k \mid k < l\}$ into R, thus $\rho_P(T^Q(R)) \geq \omega$.

To see the limit case, suppose that $\rho_H(R) \geq \lambda$ for some limit ordinal $\lambda > \omega$. Fix an increasing sequence $(\mu_m)_{m\in\mathbb{N}}$ of limit ordinals such that $(\mu_m + m)_{m\in\mathbb{N}}$ is cofinal in λ, and note that the induction hypothesis ensures that $\rho_P(T^Q(R)) \geq \mu_m + m$ for every $m \in \mathbb{N}$, thus $\rho_P(T^Q(R)) \geq \lambda$.

To see the successor case, suppose that $\rho_H(R) \geq \lambda + 2n$ for some limit ordinal λ and $n > 0$. Let $\alpha = \lambda + 2(n-1)$, and note that there are infinitely many E_R^α-classes, since otherwise $\rho_H(R) \leq \alpha + 1$. Also note that for (R/E_R^α)-adjacent E_R^α-classes C and D, at least one of $\rho_H(R \restriction C)$ and $\rho_H(R \restriction D)$ is equal to α, since otherwise C and D are contained in the same E_R^α-class. Therefore, there are E_R^α-classes C_0 and C_1 and a natural number m such that $\rho_H(R \restriction C_0) = \rho_H(R \restriction C_1) = \alpha$ and $C_0 <_{R/E_R^\alpha} [m]_{E_R^\alpha} <_{R/E_R^\alpha} C_1$.

Let Q_0 and Q_1 be the unique subsequences of Q such that for all $q \in Q[\mathbb{N}]$:

(i) $q \in Q_0[\mathbb{N}] \iff q <_\mathbb{Q} q_0$, and

(ii) $q \in Q_1[\mathbb{N}] \iff q_0 <_\mathbb{Q} q$.

Let $f\colon \mathbb{N} \to 2 \times \mathbb{N}$ be the unique bijection such that $q_{k+1} = Q_{f(k)(0)}(f(k)(1))$ for all $k \in \mathbb{N}$. Let $T = \{s \in \mathbb{N}^{<\mathbb{N}} \mid (m) \frown s \in T^Q(R)\}$ and $T_i = T^{Q_i}(R \restriction C_i)$ for each $i < 2$.

[9, Proposition 1.1.7] ensures that there is a homomorphism φ from $\sqsubset \restriction T_j$ to $\sqsubset \restriction T_{1-j}$ for some $j < 2$. Without loss of generality, we can assume that $j = 0$. Let $\pi\colon \prod_{i<2} T_i \to T$ be the map given by

$$\pi(s_0, s_1) = (s_{f(k)(0)}(f(k)(1)))_{k<\min(|s_0|,|s_1|)}$$

and note that if $s_i \sqsubset t_i$ for $i < 2$, then $\pi(s_0, s_1) \sqsubset \pi(t_0, t_1)$. Therefore, the map $\psi\colon T_0 \to T$ given by

$$\psi(s_0) = \pi(s_0, \varphi(s_0))$$

is a homomorpism from $\sqsubset \restriction T_0$ to $\sqsubset \restriction T$, thus [9, Proposition 1.1.7] ensures that $\rho_P^T(\emptyset) \geq \rho_P^{T_0}(\emptyset)$. The induction hypothesis ensures that $\rho_P(T_0) \geq \lambda+(n-1)$, so the fact that $\rho_P^{T^Q(R)}(\emptyset) \geq \rho_P^T(\emptyset)+1$ ensures that $\rho_P^{T^Q(R)}(\emptyset) \geq \lambda+(n-1)$, thus $\rho_P(T^Q(R)) \geq \lambda + n$. ❒

Fix an injective enumeration $Q = (q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. The relation $S \subseteq A^2$ given by $R_0 \; S \; R_1 \iff \rho_P(T^Q(R_0)) < \rho_P(T^Q(R_1))$ is clearly well-founded. Note that if R is scattered, then $\rho_P^{T^Q(R)}(\emptyset)$ is countable, thus [9, Proposition 1.1.7] implies that if R_0 and R_1 are scattered, then $\rho_P(T^Q(R_0)) < \rho_P(T^Q(R_1))$ if and only if there is a homomorphism from $\sqsubset \restriction (0) \frown T^Q(R_0)$ to $\sqsubset \restriction T^Q(R_1)$. As T^Q is Borel and A is analytic, S is analytic. The Kunen-Martin Theorem (see [9, Theorem 1.4.31]) ensures that $\sup_{R\in A} \rho_P(T^Q(R)) < \omega_1$, and Lemma 1.5 ensures that $\rho_H(R) < \rho_P(T^Q(R)) + \omega$ for all $R \in A$, thus $\sup_{R\in A} \rho_H(R) < \omega_1$. □

One can also give a more direct proof of Proposition 1.4 using the following lemma:

Lemma 1.6. *Suppose that R is a linear order on a set X, L is a linear order with minimal element 0_L and $\{F^l \mid l \in \mathrm{dom}(L)\}$ is a set of equivalence relations on X whose classes are R-convex such that $F^{0_L} = \Delta(X)$ and $\forall l >_L 0_L \; F^l = \bigcup_{k<_L l}(F^k)'_R$. Then*

(1) $F^l = E_R^{\alpha_l}$ for all l in the well-founded part of L, where α_l is the unique ordinal for which $L \restriction [0_L, l)_L \cong \leq \restriction \alpha_l$, and

(2) $F^l \supseteq E_R^{\rho_H(R)}$ for all l in the ill-founded part of L.

Proof. By the definition of $E_R^{\alpha_l}$, (1) holds.

To see (2), it is sufficient to show that if α is an ordinal and F^l is a superequivalence relation of E_R^α for all l in the ill-founded part of L, then F^l is a superequivalence relation of $E_R^{\alpha+1}$ for all l in the ill-founded part of L. To see this, it is sufficient to note that if l is in the ill-founded part of L, then there is a $k <_L l$ in the ill-founded part of L, and since $E_R^\alpha \subseteq F^k$, it follows that $E_R^{\alpha+1} \subseteq F^l$. $\square$

Alternative proof of Proposition 1.4. Define $S \subseteq A^2$ by $R_0\ S\ R_1$ if and only if there is a triple $(L, (F_0^l)_{l\in\mathrm{dom}(L)}, (F_1^l)_{l\in\mathrm{dom}(L)}) \in \mathcal{P}(\mathbb{N}^2) \times \mathcal{P}(\mathbb{N}^2)^{\mathrm{dom}(L)} \times \mathcal{P}(\mathbb{N}^2)^{\mathrm{dom}(L)}$ such that:

(i) L is a linear order on a subset of $\mathbb{N}$ with minimal element 0,

(ii) $\forall i < 2 \forall l \in \mathrm{dom}(L)\ F_i^l$ is an equivalence relation on $\mathbb{N}$ whose classes are R_i-convex,

(iii) $\forall i < 2\ F_i^0 = \Delta(\mathbb{N})$,

(iv) $\forall i < 2 \forall l >_L 0\ F_i^l = \bigcup_{k<_L l} (F_i^k)'_{R_i}$ and

(v) $\exists k \in \mathrm{dom}(L)\ F_0^k = \mathbb{N} \times \mathbb{N} \neq F_1^k$.

Notice that (i)–(v) are Borel conditions, thus S is analytic.

Lemma 1.7. *Suppose that $R_0, R_1 \in \mathrm{LO}(\mathbb{N})$ are scattered. Then $\rho_H(R_0) < \rho_H(R_1)$ if and only if $R_0\ S\ R_1$.*

Proof. To see ($\Longrightarrow$), suppose that $\rho_H(R_0) < \rho_H(R_1)$. Let L be a linear order on a subset of $\mathbb{N}$ with minimal element 0 such that $L \cong\ \leq \restriction (\rho_H(R_0) + 1)$. For all $i < 2$ and $l \in \mathrm{dom}(L)$, set $F_i^l = E_{R_i}^{\alpha_l}$, where α_l is the unique ordinal for which $L \restriction [0, l)_L \cong\ \leq \restriction \alpha_l$. Then the triple $(L, (F_0^l)_{l\in\mathrm{dom}(L)}, (F_1^l)_{l\in\mathrm{dom}(L)})$ witnesses that R_0 and R_1 are S-related.

To see ($\Longleftarrow$), suppose that the triple $(L, (F_0^l)_{l\in\mathrm{dom}(L)}, (F_1^l)_{l\in\mathrm{dom}(L)})$ witnesses that $R_0\ S\ R_1$, and fix $k \in \mathrm{dom}(L)$ such that $F_0^k = \mathbb{N} \times \mathbb{N} \neq F_1^k$. Since the triple satisfies (i)–(iv), Lemma 1.6 implies that every such k is in the well-founded part of L and also that $F_i^k = E_{R_i}^{\alpha_k}$ for each $i \in 2$, where α_k is the unique ordinal for which $L \restriction [0, k)_L \cong\ \leq \restriction \alpha_k$. Therefore $\rho_H(R_0) \leq \alpha_k < \rho_H(R_1)$. $\square$

Lemma 1.7 implies that S is well-founded, so an application of the Kunen-Martin Theorem yields a countable upper bound on the rank of S. Lemma 1.7 also implies that any upper bound on the rank of S is an upper bound for $\{\rho_H(R) \mid R \in A\}$, completing the proof. $\square$

Theorem 1.8. *Suppose that E is a countable Borel equivalence relation on a standard Borel space X and R is a Borel assignment of scattered linear orders to the classes of E. Then* $\sup_{x \in X} \rho_H(R \restriction [x]_E) < \omega_1$.

Proof. Applying Proposition 1.4 to the analytic set $A \subseteq \mathcal{P}(\mathbb{N}^2)$ given by

$$a \in A \iff \exists x \in X \; R \restriction [x]_E \cong a$$

yields the desired bound. $\square$

2 Continuous embeddability of pairs of directed graphs of uncountable Borel chromatic number

Proposition 2.1. *Suppose that $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ is dense and $B \subseteq 2^{\mathbb{N}}$ is non-meager and has the property of Baire. Then B is G^S-dependent.*

Proof. Let B be a non-meager set with the property of Baire. [9, Proposition 1.7.4] ensures that there is an $r \in 2^{<\mathbb{N}}$ such that $B \cap \mathcal{N}_r$ is comeager in $\mathcal{N}_r$. The fact that S is dense ensures that there is an $\mathbf{s} \in S(0)$ such that $r \sqsubseteq \mathbf{s}(j)$ for each $j < 2$. Let $\varphi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map given by $\varphi(\mathbf{s}(j) \frown (j) \frown c) = \mathbf{s}(1-j) \frown (1-j) \frown c$ for each $j < 2$ and $c \in 2^{\mathbb{N}}$, and $\varphi(t \frown c) = t \frown c$ for each $t \in 2^{<\mathbb{N}} \setminus \{\mathbf{s}(j) \frown (j) \mid j < 2\}$ and $c \in 2^{\mathbb{N}}$. The fact that φ is a homeomorphism ensures that $B \cap \varphi^{-1}(B) \cap \mathcal{N}_r$ is comeager in $\mathcal{N}_r$. It remains to note that if $x \in B \cap \varphi^{-1}(B) \cap \mathcal{N}_r$, then $(x, \varphi(x)) \in G^S \restriction B$, thus B is G^S-dependent. □

Proposition 2.2. *Suppose that $B \subseteq 2^{\mathbb{N}}$ is non-meager and has the property of Baire, and $\mathbf{S} \in (\mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2)^2$ is such that $\mathbf{S}(k)$ is dense for each $k < 2$. Then $\chi_{BP}((G^{\mathbf{S}(k)} \restriction B)_{k<2}) > \aleph_0$.*

Proof. Assume, towards a contradiction, that the coloring $c \colon 2^{\mathbb{N}} \to \mathbb{N}$ witnesses that $\chi_{BP}((G^{\mathbf{S}(k)} \restriction B)_{k<2}) \leq \aleph_0$. Since c is Baire measurable, there is an $n \in \mathbb{N}$ such that $B \cap c^{-1}(\{n\})$ is non-meager. Since $\mathbf{S}(k)$ is dense for each $k < 2$, two applications of Proposition 2.1 yield that $B \cap c^{-1}(\{n\})$ is $G^{\mathbf{S}(k)}$-dependent for each $k < 2$, a contradiction. □

Proposition 2.3. *[8, Proposition 1.2] The directed graph $G^{\mathbb{S}_0}$ is an oriented treeing of $\mathbb{E}_0$.*

Proposition 2.4. *[8, Proposition 1.3] Suppose that $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ is strongly dense and G^S is an acyclic graph. Then $S(i) = \mathbb{S}_0(0)$ for each $i < 2$.*

Proposition 2.5. *[8, Proposition 1.4] Suppose that $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ is strongly dense and G^S is an oriented graph such that $(G^S)^{\pm 1}$ is an acyclic graph. Then $S = \mathbb{S}_0$.*

Proposition 2.6. *Suppose that $\mathbf{S} \in (\mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2)^2$ is strongly dense such that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint graphs and $G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}$ is an acyclic graph. Then $\mathbf{S}(k)(i) = \mathbb{S}_0^k(0)$ for each $k, i < 2$.*

Proof. As $(\mathbf{S}(0)(i) \cup \mathbf{S}(1)(i))_{i<2}$ is strongly dense and $G^{\mathbf{S}(0)\cup\mathbf{S}(1)}$ is an acyclic graph, Proposition 2.4 ensures that $\mathbf{S}(0)(0) \cup \mathbf{S}(1)(0) = \mathbb{S}_0(0)$, and the fact that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint ensures that $\mathbf{S}(0)(0)$ and $\mathbf{S}(1)(0)$ are disjoint. Therefore, the fact that $\mathbf{S}$ is strongly dense ensures that $\mathbf{S}(k)(0) = \mathbb{S}_0^k(0)$ for each $k < 2$. Finally, the fact that $G^{\mathbf{S}(k)}$ is symmetric ensures that if $(s,t) \in \mathbf{S}(k)(0)$, then $(t,s) \in \mathbf{S}(k)(1)$, thus $\mathbf{S}(k)(i) = \mathbb{S}_0^k(0)$ for each $k, i < 2$. □

Proposition 2.7. *Suppose that $\mathbf{S} \in \left(\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2\right)^2$ is strongly dense such that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint, $G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}$ is an oriented graph and $\left(G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}\right)^{\pm 1}$ is an acyclic graph. Then $\mathbf{S}(k) = \mathbb{S}_0^k$ for each $k < 2$.*

Proof. Since $(\mathbf{S}(0)(i) \cup \mathbf{S}(1)(i))_{i<2}$ is strongly dense, $G^{\mathbf{S}(0)\cup\mathbf{S}(1)}$ is an oriented graph and $\left(G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}\right)^{\pm 1}$ is an acyclic graph, Proposition 2.5 ensures that $\mathbf{S}(0)(0) \cup \mathbf{S}(1)(0) = \mathbb{S}_0(0)$ and $\mathbf{S}(k)(1) = \emptyset = \mathbb{S}_0^k(1)$ for each $k < 2$, and the fact that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint ensures that $\mathbf{S}(0)(0)$ and $\mathbf{S}(1)(0)$ are disjoint. Therefore, the fact that $\mathbf{S}$ is strongly dense ensures that $\mathbf{S}(k)(0) = \mathbb{S}_0^k(0)$ for each $k < 2$. □

Proposition 2.8. *[8, Proposition 1.6] Suppose that $S, T \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ and $f\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is an aligned embedding of S into T. Then f_∞ is a continuous embedding of G^S into G^T.*

For each directed graph G on a set X, let E_G denote the equivalence relation on X generated by G, and for each $x \in X$, let $G_x = \{y \in X \mid (x,y) \in G\}$ and $G^x = \{y \in X \mid (y,x) \in G\}$.

Proposition 2.9. *(cf. [8, Proposition 2.2]) Suppose that X and Y are Polish spaces, G_0 and G_1 are locally countable Borel directed graphs on X such that $\chi_B((G_k)_{k<2}) > \aleph_0$, H_0 and H_1 are directed graphs on Y and $\pi\colon X \to Y$ is a Borel reduction of $(G_k)_{k<2}$ to $(H_k)_{k<2}$. Then there is a Borel set $B \subseteq X$ such that $\chi_B((G_k \restriction B)_{k<2}) > \aleph_0$ and $\pi \restriction B$ is injective.*

Proof. Let $G = G_0 \cup G_1$, $H = H_0 \cup H_1$ and $X' = \{x \in X \mid G_x \cup G^x \neq \emptyset\}$, and note that $\chi_B((G_k \restriction X')_{k<2}) > \aleph_0$. The fact that G_0 and G_1 are locally countable and the Lusin-Novikov Uniformization Theorem (see, for example, [5, Theorem 18.10]) ensure that X' is Borel.

Lemma 2.10. *The map $\pi \restriction X'$ is countable-to-one.*

Proof. The fact that G_0 and G_1 are locally countable ensures that it is sufficient to show that

$$\forall x_0, x_1 \in X' \; (\pi(x_0) = \pi(x_1) \implies x_0 \; E_G \; x_1).$$

Towards this end, suppose that $x_0, x_1 \in X'$ are such that $\pi(x_0) = \pi(x_1)$. Fix $x_2 \in G_{x_0} \cup G^{x_0}$ and note that $\pi(x_2) \in H_{\pi(x_0)} \cup H^{\pi(x_0)} = H_{\pi(x_1)} \cup H^{\pi(x_1)}$, so $x_2 \in \bigcap_{i<2}(G_{x_i} \cup G^{x_i})$, thus $x_0 \mathrel{E_G} x_2 \mathrel{E_G} x_1$. ☐

Lemma 2.10 ensures that we may apply the Lusin-Novikov Uniformization Theorem to get Borel sets $X_n \subseteq X'$ such that $X' = \bigcup_{n\in\mathbb{N}} X_n$ and $\pi \restriction X_n$ is injective for each $n \in \mathbb{N}$. Fix $m \in \mathbb{N}$ such that $\chi_B((G_k \restriction X_m)_{k<2}) > \aleph_0$ and let $B = X_m$. ☐

Proposition 2.11. *Suppose that $(G_i)_{i\in I}$ is a sequence of analytic directed graphs on a Polish space X and $\mathcal{B}$ is a countable Borel partition of X such that $\chi_B((G_i \restriction B)_{i\in I}) \leq \aleph_0$ for each $B \in \mathcal{B}$. Then $\chi_B((G_i)_{i\in I}) \leq \aleph_0$.*

Proof. For each $B \in \mathcal{B}$, fix a Borel coloring $c_B \colon B \to \mathbb{N}$ witnessing that $\chi_B((G_i \restriction B)_{i\in I}) \leq \aleph_0$. Let $c \colon X \to \mathcal{B} \times \mathbb{N}$ be given by $c(x) = (B, c_B(x)) \iff x \in B$. For each $B \in \mathcal{B}$ and $n \in \mathbb{N}$, the fact that $C_{B,n} = c^{-1}(\{(B, n)\}) = {c_B}^{-1}(\{n\})$ is $(G_i \restriction B)_{i\in I}$-independent and the fact that $C_{B,n} \subseteq B$ ensure that $C_{B,n}$ is $(G_i)_{i\in I}$-independent, thus c witnesses that $\chi_B((G_i)_{i\in I}) \leq \aleph_0$. ☐

Theorem 2.12. *(cf. [8, Theorem 2.4]) Suppose that G_0 and G_1 are locally countable Borel directed graphs on a Polish space X, $\mathcal{T}$ is a finite subset of $(\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ and $\pi^{\mathbf{T}}$ is a Borel reduction of $(G_k)_{k<2}$ to $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$. Then exactly one of the following holds:*

1. *$\chi_B((G_k)_{k<2}) \leq \aleph_0$.*
2. *There is a strongly dense pair $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ and a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G_k)_{k<2}$ such that $\pi^{\mathbf{T}} \circ \pi$ is an aligned embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$.*

Proof. Proposition 2.2 and the fact that colorings can be pulled back through homomorphisms ensure that conditions (1) and (2) are mutually exclusive, thus it is sufficient to show that $\neg(1) \implies (2)$. Towards this end, suppose that $\chi_B((G_k)_{k<2}) > \aleph_0$. By repeatedly applying Proposition 2.9, we may assume that $\pi^{\mathbf{T}}$ is injective for each $\mathbf{T} \in \mathcal{T}$.

By the Feldman-Moore Theorem (see [3, Theorem 1]), there is a countable group Γ of Borel automorphisms of X such that $E_{G_0 \cup G_1} = \bigcup_{\gamma\in\Gamma} \mathrm{graph}(\gamma)$. Fix an increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of finite symmetric neighborhoods of 1_Γ such that $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. Let F_n denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathrel{F_n} d$ if and only if $c(m) = d(m)$ for all $m \geq n$.

By standard change of topology results (see, for example, [5, Chapter 13]), we may assume that X is a zero-dimensional Polish space, Γ acts on X by homeomorphisms, and the sets $\{x \in X \mid (\gamma_j \cdot x)_{j<2} \in G_k\}$, $\{x \in X \mid$

$\pi^{\mathbf{T}}(x)\ F_n\ \pi^{\mathbf{T}}(\gamma_0 \cdot x)\}$ and $\{x \in X \mid s \sqsubseteq \pi^{\mathbf{T}}(x)\}$ are clopen for each $\gamma_0, \gamma_1 \in \Gamma$, $k < 2$, $\mathbf{T} \in \mathcal{T}$, $n \in \mathbb{N}$ and $s \in 2^{<\mathbb{N}}$.

We will recursively define clopen subsets U_n of X, $\mathbf{S}_n \in (\mathcal{P}(2^n \times 2^n)^2)^2$, $\gamma_n \in \Gamma$, $k_n \in \mathbb{N}$ and $\pi^{\mathbf{T}}_n \colon 2^n \to 2^{k_0+\cdots+k_{n-1}}$ for each $\mathbf{T} \in \mathcal{T}$ and $n \in \mathbb{N}$. For each sequence $s \in 2^{<\mathbb{N}}$ and $k < 2$, let $\gamma_s \colon X \to X$ be the Borel automorphism given by $\gamma_s = \gamma_0^{s(0)} \cdots \gamma_n^{s(n)}$ and G^s_k be the Borel directed graph on U_n given by

$$G^s_k = \{\mathbf{x} \in U_n \times U_n \mid (\gamma_s \cdot \mathbf{x}(j))_{j<2} \in G_k\}.$$

Let $U_0 = X$ and $\pi^{\mathbf{T}}_0(\emptyset) = \emptyset$ for each $\mathbf{T} \in \mathcal{T}$. By construction, the sequences $(U_m, (\pi^{\mathbf{T}}_m)_{\mathbf{T}\in\mathcal{T}})_{m\leq n}$ and $(\gamma_m, \mathbf{S}_m, k_m)_{m<n}$ will satisfy the following conditions:

(i) $\forall m \leq n\ \chi_B((G^s_k)_{k<2, s\in 2^n}) > \aleph_0$.

(ii) $\forall m < n \forall k < 2\ (s_m, s_m) \in \mathbf{S}_m(\mathrm{par}(m))(0))$.

(iii) $\forall m < n\ U_{m+1} \subseteq U_m \cap {\gamma_m}^{-1}(U_m)$.

(iv) $\forall m < n \forall x \in U_{m+1} \forall \mathbf{s} \in 2^m \times 2^m \forall i, k < 2$
$(\mathbf{s} \in \mathbf{S}_m(k)(i) \iff (\gamma_{\mathbf{s}(j)} \gamma_m^{|j-i|} \cdot x)_{j<2} \in G_k)$.

(v) $\forall m < n \forall x \in U_{m+1} \forall \mathbf{T} \in \mathcal{T}\ \pi^{\mathbf{T}}(x)\ F_{k_0+\cdots+k_m}\ \pi^{\mathbf{T}}(\gamma_m \cdot x)$.

(vi) $\forall m \leq n \forall x \in U_m \forall s \in 2^m \forall \mathbf{T} \in \mathcal{T}\ \pi^{\mathbf{T}}_m(s) \sqsubseteq \pi^{\mathbf{T}}(\gamma_s \cdot x)$.

(vii) $\forall m < n \forall s, t \in 2^m \forall \gamma \in \Gamma_m\ \gamma\gamma_s[U_{m+1}] \cap \gamma_t\gamma_m[U_{m+1}] = \emptyset$.

(viii) $\forall m < n \forall s \in 2^{m+1}\ \ \mathrm{diam}(\gamma_s[U_{m+1}]) \leq 1/(m+1)$.

Granting that we have already found such sequences $(U_m, (\pi^{\mathbf{T}}_m)_{\mathbf{T}\in\mathcal{T}})_{m\leq n}$ and $(\gamma_m, \mathbf{S}_m, k_m)_{m<n}$, let $\mathbb{P}_n$ be the set of tuples p of the form $(\gamma_p, \mathbf{S}_p, k_p, (\pi^{\mathbf{T}}_p)_{\mathbf{T}\in\mathcal{T}})$, where $\gamma_p \in \Gamma$, $\mathbf{S}_p \in (\mathcal{P}(2^n \times 2^n)^2)^2$ is such that $(s_n, s_n) \in \mathbf{S}_p(\mathrm{par}(n))(0)$, $k_p \in \mathbb{N}$ and $\pi^{\mathbf{T}}_p \colon 2^{n+1} \to 2^{k_0+\cdots+k_{n-1}+k_p}$ for each $\mathbf{T} \in \mathcal{T}$. For each $p \in \mathbb{P}_n$, let U_p be the open set of $x \in X$ which satisfy the following (open) conditions:

(iii') $x \in U_n \cap {\gamma_p}^{-1}(U_n)$.

(iv') $\forall \mathbf{s} \in 2^n \times 2^n \forall i, k < 2\ (\mathbf{s} \in \mathbf{S}_p(k)(i) \iff (\gamma_{\mathbf{s}(j)} \gamma_p^{|j-i|} \cdot x)_{j<2} \in G_k)$.

(v') $\forall \mathbf{T} \in \mathcal{T}\ \pi^{\mathbf{T}}(x)\ F_{k_0+\cdots+k_{n-1}+k_p}\ \pi^{\mathbf{T}}(\gamma_p \cdot x)$.

(vi') $\forall s \in 2^n \forall i < 2 \forall \mathbf{T} \in \mathcal{T}\ \pi^{\mathbf{T}}_p(s \frown (i)) \sqsubseteq \pi^{\mathbf{T}}(\gamma_s \gamma_p^i \cdot x)$.

(vii') $\forall s, t \in 2^n \forall \gamma \in \Gamma_n\ \gamma_p \cdot x \neq {\gamma_t}^{-1}\gamma\gamma_s \cdot x$.

For each $p \in \mathbb{P}_n$, $s \in 2^n$ and $i, k < 2$, let $G_k^{p,s \frown (i)}$ be the Borel directed graph on U_p given by

$$G_k^{p,s \frown (i)} = \{\mathbf{x} \in U_p \times U_p \mid (\gamma_s \gamma_p^i \cdot \mathbf{x}(j))_{j<2} \in G_k\}.$$

Lemma 2.13. *There is a $p \in \mathbb{P}_n$ such that $\chi_B((G_k^{p,s})_{k<2,s\in 2^{n+1}}) > \aleph_0$.*

Proof. Suppose, towards a contradiction, that for each $p \in \mathbb{P}_n$, there are $(G_k^{p,s})_{k<2,s\in 2^{n+1}}$-independent Borel sets $B_{p,m}$ for $m \in \mathbb{N}$ such that $U_p = \bigcup_{m\in\mathbb{N}} B_{p,m}$. For each $p \in \mathbb{P}_n$ and $m \in \mathbb{N}$, fix $i_{p,m} < 2$ such that $\gamma_p^{i_{p,m}}[B_{p,m}]$ is $(G_k^s)_{k<2,s\in 2^n}$-independent. Let

$$U = U_n \setminus \bigcup_{p\in\mathbb{P}_n, m\in\mathbb{N}} \gamma_p^{i_{p,m}}[B_{p,m}]$$

and note that $\chi_B((G_k^s \restriction U)_{k<2,s\in 2^n}) > \aleph_0$. Let K be the Borel graph on U given by

$$K = \{(x, y) \in U \times U \mid \exists s, t \in 2^n \exists \gamma \in \Gamma_n \; {\gamma_t}^{-1} \gamma \gamma_s \cdot x = y\}.$$

Since K has bounded vertex degree, [7, Proposition 4.5] ensures that there is an $m \in \mathbb{N}$ and a Borel coloring $c \colon U \to m$ of K. Since $\{c^{-1}(\{l\}) \mid l \in m\}$ is a finite Borel partition of U, Proposition 2.11 ensures that there is a K-independent Borel set $U' \subseteq U$ such that $\chi_B((G_k^s \restriction U')_{k<2,s\in 2^n}) > \aleph_0$.

Fix $\mathbf{x} \in U' \times U'$ such that $(\gamma_{s_n} \cdot \mathbf{x}(j))_{j<2} \in G_{\mathrm{par}(n)}$. We will show that conditions (iii') through (vii') hold for $\mathbf{x}(0)$. The definition of Γ ensures that there is a $\gamma_p \in \Gamma$ such that $\gamma_p \cdot \mathbf{x}(0) = \mathbf{x}(1)$, thus condition (iii') holds. Let $\mathbf{S}_p \in (\mathcal{P}(2^n \times 2^n)^2)^2$ be given by

$$\mathbf{S}_p(k)(i) = \{s \in 2^n \times 2^n \mid (\gamma_{s(j)} \gamma_p^{|j-i|} \cdot \mathbf{x}(j))_{j<2} \in G_k\},$$

for each $k < 2$, thus $(s_n, s_n) \in \mathbf{S}_p(\mathrm{par}(n))(0)$ and condition (iv') holds. For each $\mathbf{T} \in \mathcal{T}$, the fact that $\pi^{\mathbf{T}}$ is a reduction of $(G_k)_{k<2}$ to $(G^{\mathbf{T}(k)})_{k<2}$ and the fact that $\mathbf{x}(0) \; E_{G_0 \cup G_1} \; \mathbf{x}(1)$ ensure that there is a $k_{\mathbf{T}} \in \mathbb{N}$ such that $\pi^{\mathbf{T}}(\mathbf{x}(0)) \; F_{k_{\mathbf{T}}} \; \pi^{\mathbf{T}}(\mathbf{x}(1))$, thus there is a $k_p \in \mathbb{N}$ large enough, so that condition (v') holds. For each $\mathbf{T} \in \mathcal{T}$, let $\pi_p^{\mathbf{T}} \colon 2^{n+1} \to 2^{k_0+\cdots+k_{n-1}+k_p}$ be given by $\pi_p^{\mathbf{T}}(s \frown (i)) = \pi^{\mathbf{T}}(\gamma_s \gamma_p^i \cdot \mathbf{x}(0)) \restriction (k_0 + \cdots k_{n-1} + k_p)$, thus condition (vi') holds. The fact that U' is K-independent ensures that condition (vii') holds. It follows that the tuple $p = (\gamma_p, \mathbf{S}_p, k_p, (\pi_p^{\mathbf{T}})_{\mathbf{T}\in\mathcal{T}}) \in \mathbb{P}_n$, thus there is an $m \in \mathbb{N}$ such that $\{\mathbf{x}(0), \mathbf{x}(1)\} \cap \gamma_p^{i_{p,m}}[B_{p,m}] \neq \emptyset$, contradicting the fact that $\mathbf{x}(0), \mathbf{x}(1) \in U' \subseteq U$. $\square$

Lemma 2.13 ensures that there is a $p \in \mathbb{P}_n$ such that $\chi_B((G_k^{p,s})_{k<2,s\in 2^{n+1}}) > \aleph_0$. Let $\gamma_n = \gamma_p$, $\mathbf{S}_n = \mathbf{S}_p$, $k_n = k_p$ and $\pi_{n+1}^{\mathbf{T}} = \pi_p^{\mathbf{T}}$ for each $\mathbf{T} \in \mathcal{T}$. The fact that X is zero-dimensional and condition (vii') ensure that there is a countable clopen partition $\mathcal{V}$ of U_p such that the following conditions hold:

(vii") $\forall V \in \mathcal{V} \forall s, t \in 2^n \forall \gamma \in \Gamma_n\ \gamma\gamma_s[V] \cap \gamma_t\gamma_n[V] = \emptyset$.

(viii") $\forall V \in \mathcal{V} \forall s \in 2^{n+1}\ \operatorname{diam}(\gamma_s[V]) \leq 1/(n+1)$.

By Proposition 2.11, there is a $V \in \mathcal{V}$ such that $\chi_B((G_k^{p,s} \upharpoonright V)_{k<2,s\in 2^{n+1}}) > \aleph_0$. Let $U_{n+1} = V$. Conditions (iii')–(vi') and (vii")–(viii") and the fact that $\chi_B((G_k^s)_{k<2,s\in 2^{n+1}}) = \chi_B((G_k^{p,s} \upharpoonright V)_{k<2,s\in 2^{n+1}})$ ensure that conditions (i)–(viii) hold at stage $n+1$. This completes the recursive construction.

Let $\pi \colon 2^{\mathbb{N}} \to X$ be given by

$$\{\pi(c)\} = \bigcap_{n\in\mathbb{N}} \gamma_{c\upharpoonright n}\left[U_n\right],$$

and note that conditions (iii),(vii) and (viii) ensure that π is well-defined and a continuous injection.

Lemma 2.14. *Suppose that $n \in \mathbb{N}$, $s \in 2^n$ and $c \in 2^{\mathbb{N}}$. Then $\pi(s \frown c) = \gamma_s \cdot \pi((0)^n \frown c)$.*

Proof. Note that

$$\begin{aligned}
\{\pi(s \frown c)\} &= \bigcap_{m\geq n} \gamma_{(s\frown c)\upharpoonright m}\left[U_m\right] \\
&= \bigcap_{m\geq 0} \gamma_s\gamma_{(0)^n\frown c\upharpoonright m}\left[U_{m+n}\right] \\
&= \gamma_s\left[\bigcap_{m\geq 0} \gamma_{(0)^n\frown c\upharpoonright m}\left[U_{m+n}\right]\right] \\
&= \gamma_s\left[\bigcap_{m\geq n} \gamma_{((0)^n\frown c)\upharpoonright m}\left[U_m\right]\right] \\
&= \{\gamma_s \cdot \pi((0)^n \frown c)\},
\end{aligned}$$

thus $\pi(s \frown c) = \gamma_s \cdot \pi((0)^n \frown c)$. ◻

Let $\mathbf{S} \in \left(\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2\right)^2$ be given by $\mathbf{S}(k)(i) = \bigcup_{n\in\mathbb{N}} \mathbf{S}_n(k)(i)$ for each $i, k < 2$, and note that $\mathbf{S}$ is strongly dense.

Lemma 2.15. *Suppose that $n \in \mathbb{N}$, $\mathbf{s} \in 2^n \times 2^n$, $i < 2$, $d \in 2^{\mathbb{N}}$, and $\mathbf{c} \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is given by $\mathbf{c}(j) = \mathbf{s}(j) \frown (|j - i|) \frown d$ for all $j < 2$. Then $\mathbf{c} \in G^{\mathbf{S}(k)} \iff (\pi \times \pi)(\mathbf{c}) \in G_k$ for each $k < 2$.*

Proof. Lemma 2.14 ensures that $\pi(\mathbf{c}(j)) = \gamma_{\mathbf{s}(j) \frown (|j-i|)} \cdot \pi((0)^{n+1} \frown d)$ for each $j < 2$, and since $\pi((0)^{n+1} \frown d) \in U_{n+1}$, condition (iv) ensures that $\mathbf{c} \in G^{\mathbf{S}(k)} \iff \mathbf{s} \in \mathbf{S}_m(k)(i) \iff (\pi \times \pi)(\mathbf{c}) \in G_k$ for each $k < 2$. ❒

Lemma 2.16. *Suppose that $\mathbf{c} \notin \mathbb{E}_0$. Then $(\pi \times \pi)(\mathbf{c}) \notin E_{G_0 \cup G_1}$.*

Proof. To see that $\mathbf{c} \notin \mathbb{E}_0$ implies $(\pi \times \pi)(\mathbf{c}) \notin E_{G_0 \cup G_1}$, it is sufficient to show that if $n \in \mathbb{N}$ and $\mathbf{c}(0)(n) \neq \mathbf{c}(1)(n)$, then there is no $\gamma \in \Gamma_n$ such that $\gamma \cdot \pi(\mathbf{c}(0)) = \pi(\mathbf{c}(1))$. To see this, suppose, towards a contradiction, that $n \in \mathbb{N}$ is such that $\mathbf{c}(0)(n) \neq \mathbf{c}(1)(n)$, and $\gamma \in \Gamma_n$ is such that $\gamma \cdot \pi(\mathbf{c}(0)) = \pi(\mathbf{c}(1))$. Fix $\mathbf{s} \in 2^n \times 2^n$, $i < 2$ and $\mathbf{d} \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $\mathbf{c}(j) = \mathbf{s}(j) \frown (|j - i|) \frown \mathbf{d}(j)$ for all $j < 2$. Since Γ_n is symmetric, we may assume that $i = 0$. Lemma 2.14 ensures that $\pi(\mathbf{c}(j)) = \gamma_{\mathbf{s}(j)} \gamma_n^j \cdot \pi((0)^{n+1} \frown \mathbf{d}(j))$ for each $j < 2$, so the fact that $\pi((0)^{n+1} \frown \mathbf{d}(j)) \in U_{n+1}$ for each $j < 2$ ensures that $\pi(\mathbf{c}(1)) \in \gamma\gamma_{\mathbf{s}(0)}[U_{n+1}] \cap \gamma_{\mathbf{s}(1)}\gamma_n[U_{n+1}]$, which contradicts condition (vii). ❒

Lemma 2.15 and Lemma 2.16 ensure that π is an embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G_k)_{k<2}$. It remains to show that $\pi^{\mathbf{T}} \circ \pi$ is an aligned embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$. For the rest of the proof, we may assume that $\mathcal{T} \neq \emptyset$, as otherwise, there is nothing left to show.

Lemma 2.17. *Suppose that $\mathbf{T} \in \mathcal{T}$, $n \in \mathbb{N}$ and $c \in 2^{\mathbb{N}}$. Then $\pi_n^{\mathbf{T}}(c \restriction n) \sqsubseteq (\pi^{\mathbf{T}} \circ \pi)(c)$.*

Proof. It is sufficient to note that if $x \in U_n$ is such that $\pi(c) = \gamma_{c \restriction n} \cdot x$, then condition (vi) ensures that $\pi_n^{\mathbf{T}}(c \restriction n) \sqsubseteq \pi^{\mathbf{T}}(\gamma_{c \restriction n} \cdot x)$. ❒

Lemma 2.17 ensures that if $\mathbf{T} \in \mathcal{T}$ and $n \in \mathbb{N}$, then $\pi_n^{\mathbf{T}}(s_n) \sqsubseteq \pi_{n+1}^{\mathbf{T}}(s_n \frown (j))$ for each $j < 2$. In particular, it follows that for each $n \in \mathbb{N}$, there is a unique pair $\mathbf{u}_n \in 2^{k_n} \times 2^{k_n}$ such that $\pi_{n+1}^{\mathbf{T}}(s_n \frown (j)) = \pi_n^{\mathbf{T}}(s_n) \frown \mathbf{u}_n(j)$ for each $j < 2$.

Lemma 2.18. *Suppose that $n \in \mathbb{N}$. Then $\mathbf{u}_n(0) \neq \mathbf{u}_n(1)$.*

Proof. Fix $\mathbf{T} \in \mathcal{T}$. The fact that π and $\pi^{\mathbf{T}}$ are injective ensures that $((\pi^{\mathbf{T}} \circ \pi)(s_n \frown (j) \frown (0)^{\mathbb{N}}))_{j<2}$ is injective. Lemma 2.17 and condition (v) ensure that there is a $c \in 2^{\mathbb{N}}$ such that $(\pi^{\mathbf{T}} \circ \pi)(s_n \frown (j) \frown (0)^{\mathbb{N}}) = \pi_n^{\mathbf{T}}(s_n) \frown \mathbf{u}_n(j) \frown c$ for each $j < 2$. It follows that $\mathbf{u}_n(0) \neq \mathbf{u}_n(1)$. ❒

Lemma 2.19. *Suppose that $n \in \mathbb{N}$, $s \in 2^n$, $j < 2$ and $\mathbf{T} \in \mathcal{T}$. Then $\pi_{n+1}^{\mathbf{T}}(s \frown (j)) = \pi_n^{\mathbf{T}}(s) \frown \mathbf{u}_n(j)$.*

Proof. Since $\pi^{\mathbf{T}}_{n+1}(s_n \frown (j)) = \pi^{\mathbf{T}}_n(s_n) \frown \mathbf{u}_n(j)$, Lemma 2.14 and Lemma 2.17 ensure that

$$\pi^{\mathbf{T}}_n(s) \sqsubseteq \pi^{\mathbf{T}}_{n+1}(s \frown (j)) \sqsubseteq (\pi^{\mathbf{T}} \circ \pi)(s \frown (j) \frown (0)^{\mathbb{N}}) = \pi^{\mathbf{T}}(\gamma_s \gamma_n^j \cdot \pi((0)^{\mathbb{N}})).$$

Since condition (v) ensures that $\pi^{\mathbf{T}}(\gamma_s \gamma_n^j \cdot \pi((0)^{\mathbb{N}}))\ F_{k_0+\cdots k_{n-1}}\ \pi^{\mathbf{T}}(\gamma_{s_n} \gamma_n^j \cdot \pi((0)^{\mathbb{N}}))$, it follows that $\pi^{\mathbf{T}}_{n+1}(s \frown (j)) = \pi^{\mathbf{T}}_n(s) \frown \mathbf{u}_n(j)$. ☐

Let $\mathbf{T} \in \mathcal{T}$, $n > 0$, $\mathbf{s} \in 2^n \times 2^n$ and $k < 2$, and note that

$$\begin{aligned} \mathbf{s} \in G_n^{\mathbf{S}(k)} &\iff (\mathbf{s}(j) \frown (0)^{\mathbb{N}})_{j<2} \in G^{\mathbf{S}(k)} \\ &\iff (\pi(\mathbf{s}(j) \frown (0)^{\mathbb{N}}))_{j<2} \in G_k \\ &\iff ((\pi^{\mathbf{T}} \circ \pi)(\mathbf{s}(j) \frown (0)^{\mathbb{N}}))_{j<2} \in G^{\mathbf{T}(k)} \\ &\iff (\pi^{\mathbf{T}}_n(\mathbf{s}(j)))_{j<2} \in G^{\mathbf{T}(k)}_{k_0+\cdots k_{n-1}}, \end{aligned}$$

thus $\pi^{\mathbf{T}} \circ \pi$ is an aligned embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G^{\mathbf{T}(k)})_{k<2}$, which completes the proof. ☐

Remark 2.20. *The directed graphs $G_0 = <_{lex} \cap \mathbb{E}_0$ and $G_1 = >_{lex} \cap \mathbb{E}_0$ on $2^{\mathbb{N}}$ satisfy $\chi_B((G_k)_{k<2}) > \aleph_0$ and, together with the set of reductions $\mathcal{T} = \{\mathrm{id}_{2^{\mathbb{N}}}\}$, show that requiring each $\pi^{\mathbf{T}} \circ \pi$ to be a monotonic aligned embedding in the conclusion of Theorem 2.12 is not possible.*

In order to generalize Theorem 2.12, we first need to generalize the $\mathbb{G}_0$-dichotomy and a few technical results to pairs of analytic directed graphs.

Theorem 2.21. *(cf. [9, Theorem 2.2.1]) Suppose that G_0 and G_1 are analytic directed graphs on a Hausdorff space X. Then exactly one of the following holds:*

(1) $\chi_B((G_k)_{k<2}) \leq \aleph_0$.

(2) There is a continuous homomorphism $\pi\colon 2^{\mathbb{N}} \to X$ from $(G^{\mathbb{S}_0^k})_{k<2}$ to $(G_k)_{k<2}$.

Proof. [9, Proposition 1.4.8] ensures that there is a continuous surjection $\varphi_{G_i}\colon \mathbb{N}^{\mathbb{N}} \to G_i$ for each $i < 2$, and [9, Propositions 1.4.1, 1.4.4 and 1.4.8] ensure that there is a continuous function $\varphi_X\colon \mathbb{N}^{\mathbb{N}} \to X$ for which $\varphi_X[\mathbb{N}^{\mathbb{N}}]$ is the union of the left and right projections of $G_0 \cup G_1$ onto X.

We will recursively define a decreasing sequence $(B^\alpha)_{\alpha<\omega_1}$ of Borel subsets of X such that $\chi_B((G_k \restriction \sim B^\alpha)_{k<2}) \leq \aleph_0$ for each $\alpha < \omega_1$. Let $B^0 = X$ and for each limit ordinal $\lambda < \omega_1$, let $B^\lambda = \bigcap_{\alpha<\lambda} B^\alpha$. To describe the construction of $B^{\alpha+1}$ from B^α, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \varphi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}, \varphi^a \colon 2^{n^a} \to \mathbb{N}^{n^a}$ and $\psi^a_n \colon 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for each $n < n^a$. A *one-step extension* of an approximation a is an approximation b for which the following hold:

(i) $n^b = n^a + 1$.

(ii) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}$ $(s \sqsubseteq t \implies \varphi^a(s) \sqsubseteq \varphi^b(t))$.

(iii) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}$ $(s \sqsubseteq t \implies \psi^a_n(s) \sqsubseteq \psi^b_n(t))$.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \varphi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}, \varphi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^\mathbb{N}$ and $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \mathbb{N}^\mathbb{N}$ for each $n < n^\gamma$, and

$$(\varphi_{G_{\mathrm{par}(n)}} \circ \psi^\gamma_n)(t) = ((\varphi_X \circ \varphi^\gamma)(s_n \frown (j) \frown t))_{j<2}$$

for each $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. A configuration γ is *compatible* with an approximation a if the following conditions hold:

(i) $n^a = n^\gamma$.

(ii) $\forall t \in 2^{n^a}$ $\varphi^a(t) \sqsubseteq \varphi^\gamma(t)$.

(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}$ $\psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

A configuration is *compatible* with a set $Y \subseteq X$ if $(\varphi_X \circ \varphi^\gamma)[2^{n^\gamma}] \subseteq Y$. An approximation a is Y*-terminal* if no configuration is compatible with a one-step extension of a and with Y. Let $A(a, Y)$ denote the set of points of the form $(\varphi_X \circ \varphi^\gamma)(s_{n^a})$, where γ varies over all configurations compatible with both a and Y.

Lemma 2.22. *Suppose that $Y \subseteq X$, a is a Y-terminal approximation. Then $A(a, Y)$ is $G_{\mathrm{par}(n^a)}$-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, which are compatible with both a and Y, such that $((\varphi_X \circ \varphi^{\gamma_j})(s_{n^a}))_{j<2} \in G_{\mathrm{par}(n^a)}$. Fix $d \in \mathbb{N}^\mathbb{N}$ such that $\varphi_{G_{\mathrm{par}(n^a)}}(d) = ((\varphi_X \circ \varphi^{\gamma_j})(s_{n^a}))_{j<2}$ and let γ be the configuration given by $n^\gamma = n^a + 1, \varphi^\gamma(t \frown (j)) = \varphi^{\gamma_j}(t)$ for each $j < 2$ and $t \in 2^{n^a}$, $\psi^\gamma_n(t \frown (j)) = \psi^{\gamma_j}_n(t)$ for each $j < 2, n < n^a$ and $t \in 2^{n^a-(n+1)}$, and $\psi^\gamma_{n^a}(\emptyset) = d$. It follows that γ is compatible with a one-step extension of a, which contradicts the fact that a is Y-terminal. ◻

For each B^α-terminal approximation a, [9, Proposition 2.2.15] ensures that there is a $G_{\mathrm{par}(n^a)}$-independent Borel set $B(a, B^\alpha) \supseteq A(a, B^\alpha)$. For each $\alpha < \omega_1$, let $\mathcal{A}_{B^\alpha}$ denote the set of all B^α-terminal approximations, which is countable, and let $B^{\alpha+1} = B^\alpha \setminus \bigcup_{a \in \mathcal{A}_{B^\alpha}} B(a, B^\alpha)$. It follows that B^α is Borel for each $\alpha < \omega_1$.

Lemma 2.23. *Suppose that $\alpha < \omega_1$ and a is an approximation which is not $B^{\alpha+1}$-terminal. Then there is a one-step extension of a which is not B^{α}-terminal.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with both b and $B^{\alpha+1}$. Then $(\varphi_X \circ \varphi^{\gamma})(s_{n^b}) \in B^{\alpha+1}$, so $A(b, B^{\alpha}) \cap B^{\alpha+1} \neq \emptyset$, thus b is not B^{α}-terminal. □

Fix $\alpha < \omega_1$ such that $\mathcal{A}_{B^\alpha} = \mathcal{A}_{B^{\alpha+1}}$, and let a_0 be the unique approximation for which $n^{a_0} = 0$. Note that $A(a_0, Y) = Y$ for each $Y \subseteq X$, so if a_0 is B^{α}-terminal, then $B^{\alpha+1} = \emptyset$, which, together with the fact that $B(a, B^{\beta})$ is $G_{\mathrm{par}(n^a)}$-independent for each $\beta \leq \alpha$ and $a \in \mathcal{A}_{B^\beta}$, implies that $\chi_B((G_k)_{k<2}) \leq \aleph_0$.

Otherwise, if a_0 is not B^{α}-terminal, by recursively applying Lemma 2.23, we construct for each $n \in \mathbb{N}$, a one-step extension a_{n+1} of a_n, which is not B^{α}-terminal. Define $\varphi, \psi_n \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $\varphi(c) = \bigcup_{n\in\mathbb{N}} \varphi^{a_n}(c \restriction n)$ and $\psi_n(c) = \bigcup_{m>n} \psi_n^{a_m}(c \restriction (m - (n+1)))$ for each $n \in \mathbb{N}$. Clearly, these functions are continuous.

It remains to show that the function $\pi = \varphi_X \circ \varphi$ is a homomorphism from $(G^{\mathbb{S}_0^k})_{k<2}$ to $(G_k)_{k<2}$. To see this, it is sufficient to show that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then

$$(\varphi_{G_{\mathrm{par}(n)}} \circ \psi_n)(c) = ((\varphi_X \circ \varphi)(s_n \frown (j) \frown c))_{j<2}.$$

And to see this, it is sufficient to show that if U is an open neighborhood of $((\varphi_X \circ \varphi)(s_n \frown (j) \frown c))_{j<2}$ and V is an open neighborhood of $(\varphi_{G_{\mathrm{par}(n)}} \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Fix $m > n$ such that $\prod_{j<2} \varphi_X(\mathcal{N}_{\varphi^{a_m}(s_n \frown (j) \frown s)}) \subseteq U$ and $\varphi_{G_{\mathrm{par}(n)}}(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \restriction m - (n+1)$. The fact that a_m is not B^{α}-terminal ensures that there is a configuration γ which is compatible with a_m. It follows that $((\varphi_X \circ \varphi^{\gamma})(s_n \frown (j) \frown s))_{j<2} \in U$ and $(\varphi_{G_{\mathrm{par}(n)}} \circ \psi_n^{\gamma})(s) \in V$, thus $U \cap V \neq \emptyset$, which completes the proof. □

Proposition 2.24. *(cf. [8, Proposition 3.7]) Suppose that G_0 and G_1 are locally countable analytic directed graphs on a Polish space X such that $\chi_B((G_k)_{k<2}) > \aleph_0$. Then there is a Borel set $B \subseteq X$ such that $G_k \restriction B$ is Borel for each $k < 2$ and $\chi_B((G_k \restriction B)_{k<2}) > \aleph_0$.*

Proof. By Theorem 2.21, there is a continuous homomorphism $\pi \colon 2^{\mathbb{N}} \to X$ from $(G^{\mathbb{S}_0^k})_{k<2}$ to $(G_k)_{k<2}$, and for each $k < 2$, an application of [8, Proposition 3.5] to π and G_k yields a Borel set $B_k \subseteq X$ such that $\pi^{-1}(B_k)$ is comeager in $2^{\mathbb{N}}$ and $G_k \cap (B_k \times X)$ is Borel. Let $B = \bigcap_{k<2} B_k$, and note that $G_k \restriction B$ is Borel for each $k < 2$. Since $\pi^{-1}(B) = \bigcap_{k<2} \pi^{-1}(B_k)$ is comeager in $2^{\mathbb{N}}$, Proposition 2.2 ensures that $\chi_B((G^{\mathbb{S}_0^k} \restriction \pi^{-1}(B))_{k<2}) > \aleph_0$, thus $\chi_B((G_k \restriction B)_{k<2}) > \aleph_0$. □

Proposition 2.25. *(cf. [8, Proposition 3.8]) Suppose that X and Y are Polish spaces, G_0 and G_1 are analytic directed graphs on X, $A \subseteq X$ is analytic, $\chi_B((G_k \upharpoonright A)_{k<2}) > \aleph_0$, and $\varphi\colon A \to Y$ is $\aleph_0$-universally Baire measurable. Then there is a Borel set $B \subseteq X$ such that $B \subseteq A$, $\varphi \upharpoonright B$ is Borel and $\chi_B((G_k \upharpoonright B)_{k<2}) > \aleph_0$.*

Proof. Since $\chi_B((G_k \upharpoonright A)_{k<2}) > \aleph_0$, Theorem 2.21 ensures that there is a continuous homomorphism $\pi\colon 2^{\mathbb{N}} \to X$ from $(G^{\mathbb{S}_0^k})_{k<2}$ to $(G_k \upharpoonright A)_{k<2}$. It follows that $\pi[2^{\mathbb{N}}] \subseteq A$, so [8, Proposition 3.3] ensures that there is a Borel set $B \subseteq X$ such that $\varphi \upharpoonright B$ is Borel and $\pi^{-1}(B)$ is comeager. Therefore, Proposition 2.2 ensures that $\chi_B((G^{\mathbb{S}_0^k} \upharpoonright \pi^{-1}(B))_{k<2}) > \aleph_0$, thus $\chi_B((G_k \upharpoonright B)_{k<2}) > \aleph_0$. $\square$

Proposition 2.26. *(cf. [8, Proposition 3.9]) Suppose that X and Y are Polish spaces, G_0 and G_1 are analytic directed graphs on X, H_0 and H_1 are analytic directed graphs on Y, $\chi_B((G_k)_{k<2}) > \aleph_0$, and $\pi\colon X \to Y$ is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to $(H_k)_{k<2}$. Then there is a Borel set $B \subseteq Y$ such that $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$ and $(H_k \upharpoonright B)_{k<2}$ admits a Borel embedding into $(G_k)_{k<2}$.*

Proof. Proposition 2.25 ensures that there is a Borel set $B_X \subseteq X$ such that $\chi_B((G_k \upharpoonright B_X)_{k<2}) > \aleph_0$ and $\pi \upharpoonright B_X$ is Borel, thus $\chi_B((H_k \upharpoonright \pi[B_X])_{k<2}) > \aleph_0$. The Jankov-von Neumann Uniformization Theorem (see, for example, [5, Theorem 18.1]) ensures that there is a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable function $\varphi\colon \pi[B_X] \to B_X$ such that

$$\forall y \in \pi\,[B_X]\;\; \pi(\varphi(y)) = y.$$

It follows that φ is an embedding of $(H_k \upharpoonright \pi[B_X])_{k<2}$ into $(G_k)_{k<2}$. By [5, Theorem 21.6], φ is $\aleph_0$-universally Baire measurable, thus Proposition 2.25 ensures that there is a Borel set $B \subseteq \pi[B_X]$ such that $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$ and $\varphi \upharpoonright B$ is Borel. It follows that $\varphi \upharpoonright B$ is a Borel embedding of $(H_k \upharpoonright B)_{k<2}$ into $(G_k)_{k<2}$. $\square$

Theorem 2.27. *(cf. [8, Theorem 3.10]) Suppose that G_0 and G_1 are analytic directed graphs on a Polish space X such that there is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to a pair of locally countable analytic directed graphs on a Polish space, $\mathcal{T}$ is a finite subset of $(\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$, and $\pi^{\mathbf{T}}$ is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$. Then exactly one of the following holds:*

(1) $\chi_B((G_k)_{k<2}) \leq \aleph_0$.

(2) *There is a strongly dense pair* $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ *and a continuous embedding* $\pi \colon 2^{\mathbb{N}} \to X$ *of* $(G^{\mathbf{S}(k)})_{k<2}$ *into* $(G_k)_{k<2}$ *such that* $\pi^{\mathbf{T}} \circ \pi$ *is an aligned embedding of* $(G^{\mathbf{S}(k)})_{k<2}$ *into* $(G^{\mathbf{T}(k)})_{k<2}$ *for each* $\mathbf{T} \in \mathcal{T}$.

Proof. Proposition 2.2 and the fact that colorings can be pulled back through homomorphisms ensure that conditions (1) and (2) are mutually exclusive, thus it is sufficient to show that $\neg(1) \implies (2)$.

Towards this end, suppose that $\chi_B((G_k)_{k<2}) > \aleph_0$. By Proposition 2.26 there are locally countable analytic directed graphs H_0 and H_1 on a Polish space Y for which $\chi_B((H_k)_{k<2}) > \aleph_0$, as well as a Borel embedding φ of $(H_k)_{k<2}$ into $(G_k)_{k<2}$. By Proposition 2.24, there is a Borel set $B' \subseteq Y$ such that $\chi_B((H_k \upharpoonright B')_{k<2}) > \aleph_0$ and $H_k \upharpoonright B'$ is Borel for each $k < 2$. Since $\pi^{\mathbf{T}} \circ (\varphi \upharpoonright B')$ is $\aleph_0$-universally Baire measurable for each $\mathbf{T} \in \mathcal{T}$, $|\mathcal{T}|$-many applications of Proposition 2.25 ensure that there is a Borel set $B \subseteq B'$ such that $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$ and $\pi^{\mathbf{T}} \circ (\varphi \upharpoonright B)$ is Borel for each $\mathbf{T} \in \mathcal{T}$.

By standard change of topology results, there is a Polish topology τ on B which is compatible with the Borel structure on B and for which $\varphi \upharpoonright B$ is continuous. Since $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$, Theorem 2.12 ensures that there is a strongly dense pair $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ and a continuous embedding $\psi \colon 2^{\mathbb{N}} \to (B, \tau)$ of $(G^{\mathbf{S}(k)})_{k<2}$ into $(H_k \upharpoonright B)_{k<2}$ such that $\pi^{\mathbf{T}} \circ (\varphi \upharpoonright B) \circ \psi$ is an aligned embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G^{\mathbf{T}(k)})_{k<2}$ for each $\mathbf{T} \in \mathcal{T}$. It follows that the function $\pi = (\varphi \upharpoonright B) \circ \psi$ is as desired. □

Corollary 2.28. *Suppose that* $\Gamma = \{(G_k)_{k<2} \mid G_0$ *and* G_1 *are analytic directed graphs on a Polish space such that* $\chi_B((G_k)_{k<2}) > \aleph_0$, *and there is an* $\aleph_0$*-universally Baire measurable reduction of* $(G_k)_{k<2}$ *to a pair of locally countable analytic directed graphs on a Polish space*$\}$. *Then the set* $\{(G^{\mathbf{S}(k)})_{k<2} \mid \mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$ *is strongly dense*$\}$ *is a basis for* $\sqsubseteq_c \upharpoonright \Gamma$.

Proof. This follows directly from Theorem 2.27. □

Corollary 2.29. *Suppose that* G_0 *and* G_1 *are disjoint analytic graphs on a Polish space* X *such that* $\chi_B((G_k)_{k<2}) > \aleph_0$, *and there is an* $\aleph_0$*-universally Baire measurable reduction of* $(G_k)_{k<2}$ *to a pair of locally countable analytic graphs on a Polish space, as well as an* $\aleph_0$*-universally Baire measurable reduction of* $(G_k)_{k<2}$ *to a pair of analytic graphs* $(H_k)_{k<2}$ *on a Polish space* Y *for which* $H_0 \cup H_1$ *is an acyclic graph. Then there is a continuous embedding of* $((G^{\mathbb{S}_0^k})^{\pm 1})_{k<2}$ *into* $(G_k)_{k<2}$.

Proof. An application of Proposition 2.26 yields a Borel set $B \subseteq Y$ such that $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$ and a Borel embedding $\varphi \colon B \to X$ of $(H_k \upharpoonright B)_{k<2}$ into $(G_k)_{k<2}$. By standard change of topology results, there is a Polish topology

τ on B which is compatible with the Borel structure on B and for which φ is continuous. It follows that there is an $\aleph_0$-universally Baire measurable reduction of $(H_k \upharpoonright B)_{k<2}$ to a pair of locally countable analytic graphs on a Polish space, and since $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$, Theorem 2.27 yields a continuous embedding $\pi\colon 2^{\mathbb{N}} \to (B, \tau)$ of $(G^{\mathbf{S}(k)})_{k<2}$ into $(H_k \upharpoonright B)_{k<2}$ for a strongly dense pair $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$. Note that $\varphi \circ \pi$ is a continuous embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G_k)_{k<2}$. The fact that G_0 and G_1 are disjoint ensures that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint, and the fact that $H_0 \cup H_1$ is an acyclic graph ensures that that $G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}$ is an acyclic graph. Therefore, Proposition 2.6 ensures that $\mathbf{S}(k)(i) = \mathbb{S}_0^k(0)$ for each $k, i < 2$, thus $G^{\mathbf{S}(k)} = (G^{\mathbb{S}_0^k})^{\pm 1}$ for each $k < 2$. □

Corollary 2.30. *Suppose that G_0 and G_1 are disjoint analytic directed graphs on a Polish space X such that $G_0 \cup G_1$ is an oriented graph and $\chi_B((G_k)_{k<2}) > \aleph_0$, and there is an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to a pair of locally countable analytic directed graphs on a Polish space, as well as an $\aleph_0$-universally Baire measurable reduction of $(G_k)_{k<2}$ to a pair of analytic directed graphs $(H_k)_{k<2}$ on a Polish space Y for which $(H_0 \cup H_1)^{\pm 1}$ is an acyclic graph. Then there is a continuous embedding of $(G^{\mathbb{S}_0^k})_{k<2}$ into $(G_k)_{k<2}$.*

Proof. An application of Proposition 2.26 yields a Borel set $B \subseteq Y$ such that $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$ and a Borel embedding $\varphi\colon B \to X$ of $(H_k \upharpoonright B)_{k<2}$ into $(G_k)_{k<2}$. By standard change of topology results, there is a Polish topology τ on B which is compatible with the Borel structure on B and for which φ is continuous. It follows that there is an $\aleph_0$-universally Baire measurable reduction of $(H_k \upharpoonright B)_{k<2}$ to a pair of locally countable analytic directed graphs on a Polish space, and since $\chi_B((H_k \upharpoonright B)_{k<2}) > \aleph_0$, Theorem 2.27 yields a continuous embedding $\pi\colon 2^{\mathbb{N}} \to (B, \tau)$ of $(G^{\mathbf{S}(k)})_{k<2}$ into $(H_k \upharpoonright B)_{k<2}$ for a strongly dense pair $\mathbf{S} \in (\mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2)^2$. Note that $\varphi \circ \pi$ is a continuous embedding of $(G^{\mathbf{S}(k)})_{k<2}$ into $(G_k)_{k<2}$. The fact that G_0 and G_1 are disjoint ensures that $G^{\mathbf{S}(0)}$ and $G^{\mathbf{S}(1)}$ are disjoint, the fact that $G_0 \cup G_1$ is an oriented graph ensures that $G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)}$ is an oriented graph, and the fact that $(H_0 \cup H_1)^{\pm 1}$ is an acyclic graph ensures that $(G^{\mathbf{S}(0)} \cup G^{\mathbf{S}(1)})^{\pm 1}$ is an acyclic graph. Therefore, Proposition 2.7 ensures that $\mathbf{S}(k)(i) = \mathbb{S}_0^k(i)$ for each $k, i < 2$. □

Theorem 2.31. *Suppose that $\Gamma = \{f\colon G \to 2 \mid G$ is an analytic graph on a Polish space which admits an $\aleph_0$-universally Baire measurable reduction to a locally countable analytic graph on a Polish space, as well as an $\aleph_0$-universally Baire measurable reduction to an analytic acyclic graph on a Polish space, and*

f is a symmetric Borel function such that $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$. Then the set $\{f_0\}$, where $f_0 \colon (G^{\mathbb{S}_0})^{\pm 1} \to 2$ is given by $f_0(\mathbf{x}) = k \iff \mathbf{x} \in (G^{\mathbb{S}_0^k})^{\pm 1}$ for each $k < 2$, is a one-element basis for $\sqsubseteq_c \restriction \Gamma$.

Proof. This follows directly from Corollary 2.29 and Propositions 2.2 and 2.3. □

Theorem 2.32. *Suppose that $\Gamma = \{f \colon G \to 2 \mid G$ is an analytic oriented graph on a Polish space which admits an $\aleph_0$-universally Baire measurable reduction to a locally countable analytic directed graph on a Polish space, as well as an $\aleph_0$-universally Baire measurable reduction to an analytic directed graph H on a Polish space for which $H^{\pm 1}$ is an acylic graph, and f is a Borel function such that $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$. Then the set $\{f_0\}$, where $f_0 \colon G^{\mathbb{S}_0} \to 2$ is given by $f_0(\mathbf{x}) = k \iff \mathbf{x} \in G^{\mathbb{S}_0^k}$ for each $k < 2$, is a one-element basis for $\sqsubseteq_c \restriction \Gamma$.*

Proof. This follows directly from Corollary 2.30 and Propositions 2.2 and 2.3. □

Now we turn our attention to anti-basis results.

Proposition 2.33. *Suppose that $T \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ is a pair minimal under $\sqsubseteq_a$ such that $\chi_B(G^T, G^{\sim T}) > \aleph_0$. Then $(G^T, G^{\sim T})$ is minimal under $\sqsubseteq_c$.*

Proof. Suppose that G_0 and G_1 are analytic directed graphs on a Polish space X for which $\chi_B((G_k)_{k<2}) > \aleph_0$, and $\varphi \colon X \to 2^{\mathbb{N}}$ is a continuous embedding of $(G_k)_{k<2}$ into $(G^T, G^{\sim T})$. To see that $(G^T, G^{\sim T})$ is minimal under $\sqsubseteq_c$, it is sufficient to show that $(G^T, G^{\sim T}) \sqsubseteq_c (G_k)_{k<2}$.

Towards this end, note that $\chi_B(G_0) > \aleph_0$, thus [8, Theorem 3.10] ensures that there is a strongly dense $S \in \mathcal{P}(\bigcup_{n \in \mathbb{N}} 2^n \times 2^n)^2$ and a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of G^S into G_0 such that $\varphi \circ \pi$ is a monotonic aligned embedding of G^S into G^T. The fact that every monotonic aligned embedding is a reduction of $\mathbb{E}_0$ to $\mathbb{E}_0$ ensures that $\varphi \circ \pi$ is a reduction of $G^{\sim S}$ to $G^{\sim T}$, which, together with the fact that φ is a reduction of G_1 to $G^{\sim T}$, ensures that π is a reduction of $G^{\sim S}$ to G_1. It follows that π is a continuous embedding of $(G^S, G^{\sim S})$ into $(G_k)_{k<2}$.

The minimality of T under $\sqsubseteq_a$ ensures that there is a monotonic aligned embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of G^T into G^S. The fact that every monotonic aligned embedding is a reduction of $\mathbb{E}_0$ to $\mathbb{E}_0$ ensures that ψ is a continuous embedding of $(G^T, G^{\sim T})$ into $(G^S, G^{\sim S})$, thus $\pi \circ \psi$ is a continuous embedding of $(G^T, G^{\sim T})$ into $(G_k)_{k<2}$. □

For each $A \in \mathcal{P}(2^{<\mathbb{N}})^2$, let $S^A \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ be given by $S^A(i) = \{(s,s) \mid s \in A(i)\}$ for each $i < 2$. A *nicely aligned function* on $2^{<\mathbb{N}}$ is a function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ given by $f(s) = \bigoplus_{n<|s|}(u_n^f \frown |s(n) - c^f(n)|)$, where $c^f \in 2^{\mathbb{N}}$ and $u_n^f \in 2^{k_n^f}$ for some natural number k_n^f for each $n \in \mathbb{N}$, and where the empty concatenation denotes the empty sequence. Note that a nicely aligned function f is an aligned function, and f is order-preserving if $c^f = (0)^{\mathbb{N}}$ and order-reversing if $c^f = (1)^{\mathbb{N}}$. We say that f is an *aligned embedding* of A into B if it is an aligned embedding of S^A into S^B, or equivalently, if $\forall i < 2 \forall s \in 2^{<\mathbb{N}}$ $(s \in A \iff f(s) \in B)$. We call $A \in \mathcal{P}(2^{<\mathbb{N}})^2$ *dense* if S^A is dense. As it should cause no confusion, we let $\sqsubseteq_a$ also denote the quasi-order of monotonic aligned embeddability on the set of dense pairs in $\mathcal{P}(2^{<\mathbb{N}})^2$.

For each $c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}$, let $A_c \in \mathcal{P}(2^{<\mathbb{N}})^2$ be given by $A_c(0) = A_c(1) = \{s \in 2^{<\mathbb{N}} \mid \exists m \in \mathrm{supp}(c)\ |\mathrm{supp}(s)| \equiv 2^{2m} \pmod{2^{2m+1}}\}$.

Proposition 2.34. *[8, Proposition 6.15] The set $\{A_c \mid c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}\}$ is a continuum-sized strong $\sqsubseteq_a$-antichain of minimal-under-$\sqsubseteq_a$ dense pairs in $\mathcal{P}(2^{<\mathbb{N}})^2$.*

Proposition 2.35. *Suppose that $c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}$. Then $\chi_B(G^{S^{A_c}}, G^{\sim S^{A_c}}) > \aleph_0$.*

Proof. Since Proposition 2.34 ensures that S^{A_c} is dense, Proposition 2.2 ensures that it is sufficient to show that $\sim S^{A_c}$ is dense. Towards this end, suppose that $r \in 2^{<\mathbb{N}}$, and fix an $s \sqsupseteq r$ such that $|\mathrm{supp}(s)| \equiv 2 \pmod 4$. It follows that $(s,s) \notin S^{A_c}(0)$, thus $\sim S^{A_c}$ is dense. $\square$

Theorem 2.36. *There is a continuum-sized strong $\sqsubseteq_c$-antichain of minimal-under-$\sqsubseteq_c$ pairs of graphs in $\Gamma_{\mathcal{G}^2}$. In particular, any basis for $\Gamma_{\mathcal{G}^2}$ with respect to $\sqsubseteq_c$ is at least continuum-sized.*

Proof. [8, Theorem 5.5, Proposition 5.14] and Proposition 2.34 ensure that $\{G^{S^{A_c}} \mid c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}\}$ is a strong $\sqsubseteq_c$-antichain, thus Proposition 2.35 ensures that $\mathcal{A} = \{(G^{S^{A_c}}, G^{\sim S^{A_c}}) \mid c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}\}$ is a strong $\sqsubseteq_c$-antichain. Finally, Proposition 2.33 and Proposition 2.34 ensure that each element of $\mathcal{A}$ is minimal under $\sqsubseteq_c$. $\square$

Let $\Gamma_{\mathbb{E}_0} = \{f \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2 \mid f \text{ is Borel}, \chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0\}$.

Theorem 2.37. *There is a continuum-sized strong $\sqsubseteq_c$-antichain of minimal-under-$\sqsubseteq_c$ functions in $\Gamma_{\mathbb{E}_0}$. In particular, any basis for $\Gamma_{\mathbb{E}_0}$ with respect to $\sqsubseteq_c \restriction \Gamma_{\mathbb{E}_0}$ and any basis for $\Gamma_{\mathcal{F}}$ with respect to $\sqsubseteq_c$ is at least continuum-sized.*

Proof. For each $c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}$, let $f_c \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be given by $f_c(\mathbf{d}) = 0 \iff \mathbf{d} \in G^{S^{A_c}}$. [8, Theorem 5.5, Proposition 5.14] and Proposition 2.34 ensure that $\{G^{S^{A_c}} \mid c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}\}$ is a strong $\sqsubseteq_c$-antichain, thus Proposition 2.35 ensures that $\mathcal{A} = \{f_c \mid c \in 2^{\mathbb{N}} \setminus \{(0)^{\mathbb{N}}\}\}$ is a strong $\sqsubseteq_c$-antichain, and in particular $\mathcal{A} \subseteq \Gamma_{\mathbb{E}_0}$. Finally, Proposition 2.33 and Proposition 2.34 ensure that each element of $\mathcal{A}$ is minimal under $\sqsubseteq_c$. $\square$

Let $A_0 \in \mathcal{P}(2^{<\mathbb{N}})^2$ be given by $A_0(0) = A_0(1) = \{s \in 2^{<\mathbb{N}} \mid \exists m \in \operatorname{supp}(s)\ |\operatorname{supp}(s)| \equiv 2^{2m} \pmod{2^{2m+1}}\}$.

Proposition 2.38. *Suppose that $A \in \mathcal{P}(2^{<\mathbb{N}})^2$ and f is an aligned embedding of A into A_0. Then $\chi_B(G^{S^A}, G^{\sim S^A}) > \aleph_0$.*

Proof. [8, Proposition 7.1] ensures that S^A is dense, so Proposition 2.2 ensures that it is sufficient to show that $\sim S^A$ is dense. Towards this end, suppose that $r \in 2^{<\mathbb{N}}$, and note that the properties of nicely aligned functions ensure that there is a sequence $s \in 2^4$ such that $|f(r \frown s)| \equiv 2 \pmod 4$. It follows that $f(r \frown s) \notin A_0$, so $r \frown s \notin A$, thus $\sim S^A$ is dense. $\square$

Theorem 2.39. *Suppose that $A \in \mathcal{P}(2^{<\mathbb{N}})^2$ and there is a monotonic aligned embedding of A into A_0. Then there is a continuum-sized strong $\sqsubseteq_c$-antichain $\mathcal{A} \subseteq \Gamma_{\mathcal{G}^2}$ such that $\mathbf{G} \sqsubseteq_c (G^{S^A}, G^{\sim S^A})$ for each $\mathbf{G} \in \mathcal{A}$.*

Proof. By [8, Proposition 7.8], there is a continuum-sized strong $\sqsubseteq_a$-antichain $\mathcal{B}$ of dense pairs in $\mathcal{P}(2^{<\mathbb{N}})^2$ such that $B \sqsubseteq_a A$ for each $B \in \mathcal{B}$. Let $\mathcal{A} = \{(G^{S^B}, G^{\sim S^B}) \mid B \in \mathcal{B}\}$. The fact that every monotonic aligned embedding is a reduction of $\mathbb{E}_0$ to $\mathbb{E}_0$ ensures that $\mathbf{G} \sqsubseteq_c (G^{S^A}, G^{\sim S^A})$ for each $\mathbf{G} \in \mathcal{A}$. [8, Theorem 5.5 and Proposition 5.14] ensure that $\{G^{S^B} \mid B \in \mathcal{B}\}$ is a strong $\sqsubseteq_c$-antichain, thus Proposition 2.38 and the fact that $B \sqsubseteq_a A_0$ for each $B \in \mathcal{B}$ ensure that $\mathcal{A}$ is a strong $\sqsubseteq_c$-antichain. $\square$

For each $A \in \mathcal{P}(2^{<\mathbb{N}})^2$, let $f_A \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be given by $f_A(\mathbf{c}) = 0 \iff \mathbf{c} \in G^{S^A}$. In particular, note that $\{f_A \mid A \in \mathcal{P}(2^{<\mathbb{N}})^2\} \subseteq \Gamma_{\mathbb{E}_0}$.

Theorem 2.40. *Suppose that $A \in \mathcal{P}(2^{<\mathbb{N}})^2$ and there is a monotonic aligned embedding of A into A_0. Then there is a continuum-sized strong $\sqsubseteq_c$-antichain $\mathcal{A} \subseteq \Gamma_{\mathbb{E}_0}$ such that $f \sqsubseteq_c f_A$ for each $f \in \mathcal{A}$.*

Proof. By [8, Proposition 7.8], there is a continuum-sized strong $\sqsubseteq_a$-antichain $\mathcal{B}$ of dense pairs in $\mathcal{P}(2^{<\mathbb{N}})^2$ such that $B \sqsubseteq_a A$ for each $B \in \mathcal{B}$. Let $\mathcal{A} = \{f_B \mid B \in \mathcal{B}\}$. The fact that every monotonic aligned embedding is a reduction of $\mathbb{E}_0$ to $\mathbb{E}_0$ ensures that $f \sqsubseteq_c f_A$ for each $f \in \mathcal{A}$. [8, Theorem 5.5 and Proposition 5.14] ensure that $\{G^{S^B} \mid B \in \mathcal{B}\}$ is a strong $\sqsubseteq_c$-antichain, thus Proposition 2.38 and the fact that $B \sqsubseteq_a A_0$ for each $B \in \mathcal{B}$ ensure that $\mathcal{A}$ is a strong $\sqsubseteq_c$-antichain. $\square$

3 Borel functions without homogeneous E-non-smooth Borel sets

Proposition 3.1. *Suppose that G is a directed graph on a Polish space X and $f\colon G \to 2$ is a Borel function. Then the following are equivalent:*

(1) There is no sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and B_n is f-homogeneous for each $n \in \mathbb{N}$.

(2) $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0$.

Proof. This directly follows from the fact that for each $k < 2$, a subset of X is $f^{-1}(\{k\})$-independent if and only if it is f-homogeneous with value $1-k$. $\square$

Remark 3.2. *Proposition 3.1 ensures that the class of functions $\{f\colon G \to 2 \mid G$ is an analytic directed graph on a Polish space X and f is a Borel function for which there is no sequence $(B_n)_{n\in\mathbb{N}}$ of Borel $B_n \subseteq X$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and B_n is f-homogeneous for each $n \in \mathbb{N}\}$ is the same as $\Gamma_{\mathcal{F}}$.*

Proposition 3.3. *Suppose that E is a non-smooth Borel equivalence relation on a Polish space X and $f\colon E \setminus \Delta(X) \to 2$ is a Borel function for which there is no E-non-smooth f-homogeneous Borel subset of X. Then $\chi_B((f^{-1}(\{k\}))_{k<2}) > \aleph_0$.*

Proof. This follows from Proposition 3.1 and the fact that if $X = \bigcup_{n\in\mathbb{N}} B_n$ for Borel sets $B_n \subseteq X$, then there is an $m \in \mathbb{N}$ such that B_m is E-non-smooth. $\square$

Let $\Gamma_{\mathcal{F}^*}$ be the class of functions $\{f\colon E \setminus \Delta(X) \to 2 \mid E$ is a non-smooth Borel equivalence relation on a Polish space X and f is a symmetric Borel function such that there is no E-non-smooth f-homogeneous Borel subset of $X\}$, and note that Proposition 3.3 ensures that $\Gamma_{\mathcal{F}^*} \subseteq \Gamma_{\mathcal{F}}$. The next result ensures that $\Gamma_{\mathcal{F}^*}$ is non-empty.

For each binary relation $R \subseteq X \times Y$, the *flip of* R is the relation R^{-1} on $Y \times X$ given by $y \mathrel{R^{-1}} x \iff x \mathrel{R} y$.

Proposition 3.4. *Suppose that E is a non-smooth Borel equivalence relation on a Polish space X, R is a Borel linear order on X, and S is a Borel assignment of scattered linear orders to the classes of E. Then the function $f\colon E \setminus \Delta(X) \to 2$ given by $f(\mathbf{x}) = 0 \iff (\mathbf{x} \in R \iff \mathbf{x} \in S)$ is in $\Gamma_{\mathcal{F}^*}$.*

Proof. It is sufficient to show that if $B \subseteq X$ is an f-homogeneous Borel set, then $E \restriction B$ is smooth. The fact that B is f-homogeneous ensures that $(R \cap E) \restriction B \in \{S \restriction B, S^{-1} \restriction B\}$, so $(R \cap E) \restriction B$ is a Borel assignment of scattered linear orders to the classes of $E \restriction B$, thus [2, Proposition 2.9] ensures that $E \restriction B$ is smooth. $\square$

Proposition 3.5. *For each $c \in 2^{\mathbb{N}}$, $\mathbb{R}_0 \upharpoonright [c]_{\mathbb{E}_0}$ is a scattered linear order. More precisely, the order type of $\mathbb{R}_0 \upharpoonright [c]_{\mathbb{E}_0}$ is $\mathbb{N}$ if $c \in [(0)^{\mathbb{N}}]_{\mathbb{E}_0}$, $-\mathbb{N}$ if $c \in [(1)^{\mathbb{N}}]_{\mathbb{E}_0}$ and $\mathbb{Z}$ otherwise.*

Proof. Suppose that $g \colon 2^{\mathbb{N}} \setminus \{(1)^{\mathbb{N}}\} \to 2^{\mathbb{N}}$ is the function given by $g((1)^n \frown (0) \frown c) = (0)^n \frown (1) \frown c$ for each $n \in \mathbb{N}$ and $c \in 2^{\mathbb{N}}$. It is straightforward to check that $c \mathrel{\mathbb{R}_0} d \iff \exists n \in \mathbb{N}\ g^n(c) = d$, and that this implies the conclusion of the proposition. $\square$

We let $f_0 \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be the symmetric Borel function given by $f_0(\mathbf{c}) = 0 \iff (\mathbf{c} \in {\leq_{lex}} \iff \mathbf{c} \in \mathbb{R}_0)$, and note that Proposition 3.4 ensures that $f_0 \in \Gamma_{\mathcal{F}^*}$.

Proposition 3.6. *Suppose that $E \subseteq \mathbb{E}_0$ is a non-smooth countable Borel equivalence relation. Then there is an order-preserving aligned embedding $g_\infty \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(\mathbb{E}_0, \mathbb{R}_0)$ into $(E, \mathbb{R}_0)$.*

Proof. Note that if $h' \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is an aligned embedding with respect to Conley's notion (see [2, page 3]), then there is an aligned embedding with respect to our notion $h \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that $h_\infty(c) = \bigcup_{n \in \mathbb{N}} h'(c \upharpoonright n)$ for each $c \in 2^{\mathbb{N}}$, thus [2, Proposition 2.1] ensures that there is an order-preserving aligned embedding $h_\infty \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ into E. Let $g \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ be the aligned embedding given by $\mathbf{u}^g_n(j) = \mathbf{u}^h_{2n}(j) \frown \mathbf{u}^h_{2n+1}(|i_n - j|)$ for each $j < 2$ and $n \in \mathbb{N}$, where $i_n < 2$ is unique such that $(\mathbf{u}^h_{2n+1}(|i_n - j|) \frown (0)^\infty)_{j<2} \in \mathbb{R}_0$ for each $n \in \mathbb{N}$, and note that g_∞ is as desired. $\square$

Proposition 3.7. *There is a continuous embedding of f_0 into $1 - f_0$.*

Proof. The map $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\pi(c)(2n) = 1 - \pi(c)(2n+1) = c(n)$ for all $n \in \mathbb{N}$ and $c \in 2^{\mathbb{N}}$ is a continuous embedding of $(\leq_{lex}, \mathbb{R}_0)$ into $(\leq_{lex}, \mathbb{R}_0^{-1})$, thus π is also a continuous embedding of f_0 into $1 - f_0$. $\square$

Proposition 3.8. *Suppose that R and S are Borel assignments of linear orders to the classes of $\mathbb{E}_0$ and $f \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ is the symmetric Borel function given by $f(\mathbf{c}) = 0 \iff (\mathbf{c} \in R \iff \mathbf{c} \in S)$. Then at least one of the following holds:*

1. *There is an $\mathbb{E}_0$-non-smooth f-homogeneous Borel set.*
2. *There is a continuous embedding of f_0 into f.*

Proof. An application of [2, Theorem 2.12] yields an $\mathbb{E}_0$-non-smooth compact set $K \subseteq 2^{\mathbb{N}}$ such that $R \upharpoonright K \in \{({\leq_{lex}} \cap \mathbb{E}_0) \upharpoonright K, ({\geq_{lex}} \cap \mathbb{E}_0) \upharpoonright K, \mathbb{R}_0 \upharpoonright K, \mathbb{R}_0^{-1} \upharpoonright K\}$. Proposition 3.6 ensures that there is an order-preserving aligned embedding $\pi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(\mathbb{E}_0, \mathbb{R}_0)$ into $(\mathbb{E}_0 \upharpoonright K, \mathbb{R}_0)$. Let R' and S' be the pullbacks of R and S under π, and note that S' is a Borel assignment of linear orders to the classes of $\mathbb{E}_0$. The fact that π is an embedding of $(\mathbb{E}_0, \leq_{lex}, \mathbb{R}_0)$ into $(\mathbb{E}_0 \upharpoonright K, \leq_{lex}, \mathbb{R}_0)$ ensures that $R' \in \{\leq_{lex} \cap \mathbb{E}_0, \geq_{lex} \cap \mathbb{E}_0, \mathbb{R}_0, \mathbb{R}_0^{-1}\}$.

A second application of [2, Theorem 2.12] yields an $\mathbb{E}_0$-non-smooth compact set $K' \subseteq 2^{\mathbb{N}}$ such that $S' \upharpoonright K' \in \{(\leq_{lex} \cap \mathbb{E}_0) \upharpoonright K', (\geq_{lex} \cap \mathbb{E}_0) \upharpoonright K', \mathbb{R}_0 \upharpoonright K', \mathbb{R}_0^{-1} \upharpoonright K'\}$. Proposition 3.6 ensures that there is an order-preserving aligned embedding $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(\mathbb{E}_0, \mathbb{R}_0)$ into $(\mathbb{E}_0 \upharpoonright K', \mathbb{R}_0)$. Let R'' and S'' be the pullbacks of R' and S' under ψ. The fact that ψ is an embedding of $(\mathbb{E}_0, \leq_{lex}, \mathbb{R}_0)$ into $(\mathbb{E}_0 \upharpoonright K', \leq_{lex}, \mathbb{R}_0)$ ensures that $R'', S'' \in \{\leq_{lex} \cap \mathbb{E}_0, \geq_{lex} \cap \mathbb{E}_0, \mathbb{R}_0, \mathbb{R}_0^{-1}\}$. Let $f''\colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be the symmetric Borel function given by $f''(\mathbf{c}) = 0 \iff (\mathbf{c} \in R'' \iff \mathbf{c} \in S'')$, and note that $\pi \circ \psi$ is a continuous embedding of f'' into f.

The fact that $R'', S'' \in \{\leq_{lex} \cap \mathbb{E}_0, \geq_{lex} \cap \mathbb{E}_0, \mathbb{R}_0, \mathbb{R}_0^{-1}\}$ ensures that $f'' \in \{0, 1, f_0, 1 - f_0\}$. If f'' is constant, then $(\pi \circ \psi)[2^{\mathbb{N}}]$ is an $\mathbb{E}_0$-non-smooth f-homogeneous Borel set, thus condition (1) holds, and if f'' is not constant, then Proposition 3.7 ensures that there is a continuous embedding $\varphi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of f_0 into f'', so $\pi \circ \psi \circ \varphi$ is a continuous embedding of f_0 into f, thus condition (2) holds. $\square$

Theorem 3.9. *Suppose that Γ is the class of symmetric Borel functions $f\colon E \setminus \Delta(X) \to 2$ in $\Gamma_{\mathcal{F}^*}$ of the form $f(\mathbf{x}) = 0 \iff (\mathbf{x} \in R \iff \mathbf{x} \in S)$, where R and S are Borel assignments of linear orders to the classes of E. Then $\{f_0\}$ is a one-element basis for $\sqsubseteq_c \upharpoonright \Gamma$.*

Proof. Fix $f \in \Gamma$ and let R and S be the Borel assignments of linear orders to the classes of E that define it. Since E is a non-smooth Borel equivalence relation, [6, Theorem 1.1] ensures that there is a continuous embedding $\pi\colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E. Let R' and S' be the pullbacks of R and S under π, and note that R' and S' are Borel assignments of linear orders to the classes of $\mathbb{E}_0$. Let $f'\colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be the symmetric Borel function given by $f'(\mathbf{c}) = 0 \iff (\mathbf{c} \in R' \iff \mathbf{c} \in S')$, and note that π is a continuous embedding of f' into f.

Note that if $B \subseteq 2^{\mathbb{N}}$ is an $\mathbb{E}_0$-non-smooth f'-homogeneous Borel set, then $\pi[B]$ is an E-non-smooth f-homogeneous Borel set, thus the fact that $f \in \Gamma_{\mathcal{F}^*}$ ensures that there is no $\mathbb{E}_0$-non-smooth f'-homogeneous Borel set. Therefore, Proposition 3.8 ensures that there is a continuous embedding $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of f_0 into f', thus $\pi \circ \psi$ is a continuous embedding of f_0 into f. $\square$

Proposition 3.10. *Suppose that E is a non-smooth countable Borel equivalence relation on a Polish space X, $f\colon E \setminus \Delta(X) \to 2$ is Borel, and $\varphi\colon X \to 2^{\mathbb{N}}$*

is a Borel reduction of $(f^{-1}(\{k\}))_{k<2}$ *to* $({f_0}^{-1}(\{k\}))_{k<2}$. *Then there is a continuous embedding* $\pi\colon 2^{\mathbb{N}} \to X$ *of* f_0 *into* f.

Proof. By replacing X with $X \setminus \{x \in X \mid |[x]_E| = 1\}$ if necessary, we may assume without loss of generality that every E-class consists of at least two elements. The fact that φ is a reduction of $E \setminus \Delta(X)$ to $\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}})$ then ensures that φ is injective, thus φ is a reduction of E to $\mathbb{E}_0$.

By the Lusin-Novikov Uniformization Theorem, there is a Borel function $\psi\colon \varphi[X] \to X$ such that

$$\forall y \in \varphi[X] \;\; \varphi(\psi(y)) = y,$$

thus the fact that φ is a reduction of $(f^{-1}(\{k\}))_{k<2}$ to $({f_0}^{-1}(\{k\}))_{k<2}$ ensures that ψ is an embedding of $({f_0}^{-1}(\{k\}) \restriction \varphi[X]))_{k<2}$ into $(f^{-1}(\{k\}))_{k<2}$. The fact that φ is a reduction of E to $\mathbb{E}_0$ ensures that $\varphi[X]$ is an $\mathbb{E}_0$-non-smooth Borel set, thus Proposition 3.6 yields an order-preserving aligned embedding $g_\infty\colon 2^{\mathbb{N}} \to \varphi[X]$ of $(\mathbb{E}_0, \mathbb{R}_0)$ into $(\mathbb{E}_0 \restriction \varphi[X], \mathbb{R}_0 \restriction \varphi[X])$. By [9, Proposition 1.7.5], there is a dense G_δ set $C \subseteq 2^{\mathbb{N}}$ such that $(\psi \circ g_\infty) \restriction C$ is continuous, and since every dense G_δ set in $2^{\mathbb{N}}$ is comeager, [10, Proposition 12.7] ensures that C is $\mathbb{E}_0$-non-smooth. A second application of Proposition 3.6 yields an order-preserving aligned embedding $h_\infty\colon 2^{\mathbb{N}} \to C$ of $(\mathbb{E}_0, \mathbb{R}_0)$ into $(\mathbb{E}_0 \restriction C, \mathbb{R}_0 \restriction C)$. It remains to note that $g_\infty \circ h_\infty$ is a continuous embedding of $({f_0}^{-1}(\{k\}))_{k<2}$ into $({f_0}^{-1}(\{k\}) \restriction \varphi[X])_{k<2}$, thus it follows that $\pi = \psi \circ g_\infty \circ h_\infty$ is a continuous embedding of f_0 into f. □

Theorem 3.11. *Suppose that* $\Gamma = \{f\colon E \setminus \Delta(X) \to 2 \mid f \in \Gamma_{\mathcal{F}^*}$ *and* E *is a countable Borel equivalence relation*$\}$. *Then* f_0 *is minimal with respect to* $\sqsubseteq_c \restriction \Gamma$.

Proof. This follows directly from Proposition 3.10. □

Question 3.12. *Is the set* $\{f_0\}$ *a one-element basis for* $\sqsubseteq_c \restriction \Gamma_{\mathcal{F}^*}$*?*

Proposition 3.13. *Suppose that* E *and* F *are Borel equivalence relations on a Polish space* X, E *is non-smooth and* $f\colon E \setminus \Delta(X) \to 2$ *is the Borel function given by* $f(\mathbf{x}) = 0 \iff \mathbf{x} \in F$. *Then there is an* E*-non-smooth* f*-homogeneous Borel set.*

Proof. The Kanovei-Zapletal Canonization Theorem (see, for example, [1, Theorem 8]) ensures that there is an E-non-smooth Borel set B such that $F \restriction B \in \{\Delta(B), E \restriction B, B \times B\}$. It remains to note that B is f-homogeneous with value 1 in the first case, and B is f-homogeneous with value 0 in the latter two cases. □

Theorem 3.14. *Suppose that* $\Gamma = \{f \colon E \setminus \Delta(X) \to 2 \mid E$ *is a non-smooth Borel equivalence relation on a Polish space* X *and* f *is a Borel function such that there is an* $\aleph_0$*-universally Baire measurable reduction of* $(f^{-1}(\{k\}))_{k<2}$ *to a pair of locally countable analytic directed graphs on a Polish space, and there is no* E*-non-smooth* f*-homogeneous Borel subset of* $X\}$*. Then the set* $\{(G^S, G^{\sim S}) \mid S \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ *is a strongly dense pair such that there is no* S*-homogeneous aligned embedding*$\}$ *is a basis for* $\sqsubseteq_c \restriction \Gamma$.

Proof. Fix $f \colon E \setminus \Delta(X) \to 2$ in Γ, and note that since E is a non-smooth Borel equivalence relation, [6, Theorem 1.1] ensures that there is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E. Let $f' \colon \mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}) \to 2$ be the Borel function given by $f'(\mathbf{c}) = f((\pi \times \pi)(\mathbf{c}))$, and note that π is an embedding of f' into f. If $B \subseteq 2^{\mathbb{N}}$ is an $\mathbb{E}_0$-non-smooth f'-homogeneous Borel set, then $\pi[B]$ is an E-non-smooth f-homogeneous Borel set, thus, without loss of generality, we may assume that $f = f'$.

By Proposition 3.3 and [8, Theorem 3.10], there is a strongly dense pair $S \in \mathcal{P}(\bigcup_{n\in\mathbb{N}} 2^n \times 2^n)^2$ and a continuous embedding $\varphi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of G^S into $f^{-1}(\{0\})$ such that $\mathrm{id}_{2^{\mathbb{N}}} \circ \varphi$ is a monotonic aligned embedding of G^S into $f^{-1}(\{0\})$. The fact that every aligned embedding is a reduction of $\mathbb{E}_0$ to $\mathbb{E}_0$ ensures that φ is an embedding of $(G^S, G^{\sim S})$ into $(f^{-1}(\{k\}))_{k<2}$, and together with [5, Corollary 15.2], it ensures that if $g \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is an aligned embedding, then $(\varphi \circ g_\infty)[2^{\mathbb{N}}]$ is an $\mathbb{E}_0$-non-smooth Borel set. The fact that if g is S-homogeneous, then $(\varphi \circ g_\infty)[2^{\mathbb{N}}]$ is f-homogeneous and the fact that there is no f-homogeneous $\mathbb{E}_0$-non-smooth Borel set ensure that there is no S-homogeneous aligned embedding, thus S is as desired. □

We finish with a proposition that is complementary to a special case of [2, Proposition 2.9].

For each $c, d \in 2^{\mathbb{N}}$, let $c \wedge d$ denote $c \restriction n$ for the maximal $n \in \mathbb{N}$ satisfying $c \restriction n = d \restriction n$. For each linear order R on a set X, $x, y \in X$ are *R-adjacent* if $x \neq y$ and $\neg \exists z \in X$ $(x <_R z <_R y$ or $y <_R z <_R x)$, and x is an *R-endpoint* if x is R-minimal or R-maximal.

Proposition 3.15. *Suppose that* E *is a countable Borel equivalence relation on a Polish space* X *and* R *is a Borel linear order on* X *which admits a Borel reduction* $\varphi \colon X \to 2^{\mathbb{N}}$ *to* $\leq_{lex}$*. Then* $B = \{x \in X \mid R \restriction [x]_E$ *is not a dense linear order without endpoints*$\}$ *is an* E*-smooth Borel set.*

Proof. The fact that φ is a reduction of a linear order ensures that φ is injective, thus [5, Corollary 15.2] ensures that $\varphi[B]$ is Borel and the inverse map $\varphi^{-1} \colon \varphi[B] \to B$ is Borel. It follows that the relation $F = \{(\varphi(x), \varphi(y)) \mid (x, y) \in E\} \cup \Delta(2^{\mathbb{N}})$ is a Borel equivalence relation on $2^{\mathbb{N}}$. Let $C = \{c \in 2^{\mathbb{N}} \mid$

$\leq_{lex} \restriction [c]_F$ is not a dense linear order without endpoints$\}$, and note that the fact that (R, E) is the pullback of $(\leq_{lex}, F)$ under φ and the fact that $[\varphi(x)]_F \subseteq \varphi[B]$ for each $x \in X$ ensure that $\varphi[B] \subseteq C$. It follows that if C is an F-smooth Borel set, then B is an E-smooth Borel set, thus, without loss of generality, we may assume that $X = 2^{\mathbb{N}}$ and $R = \leq_{lex}$.

Let $B' = \{c \in 2^{\mathbb{N}} \mid \leq_{lex} \restriction [c]_E$ has an endpoint$\}$ and $B'' = \{c \in 2^{\mathbb{N}} \setminus B' \mid [c]_E$ contains $(\leq_{lex} \restriction [c]_E)$-adjacent elements$\}$, and note that since E is countable, the Lusin-Novikov Uniformization Theorem ensures that B' and B'' are Borel. Also note that $(B' \times B'') \cap E = \emptyset$. The fact that a linear order without endpoints is not dense if and only if there are adjacent elements ensures that $B = B' \cup B''$, thus B is Borel. Let $T' = \{c \in B' \mid c$ is the $\leq_{lex}$-least $(\leq_{lex} \restriction [c]_E)$-endpoint$\}$, and note that T' is a Borel transversal of $E \restriction B'$. To construct a Borel transversal of $E \restriction B''$, we require the following lemma:

Lemma 3.16. *Suppose that $C \subseteq 2^{\mathbb{N}}$, $d, e \in C$ are $(\leq_{lex} \restriction C)$-adjacent, $d', e' \in C$ are $(\leq_{lex} \restriction C)$-adjacent and $d \wedge e = d' \wedge e'$. Then $\{d', e'\} = \{d, e\}$.*

Proof. Without loss of generality, we may assume that $d \leq_{lex} d'$, $(d \wedge e) \frown (0) \sqsubset d, d'$ and $(d \wedge e) \frown (1) \sqsubset e, e'$. Note that $d \leq_{lex} d' \leq_{lex} e$, thus the fact that d and e are $(\leq_{lex} \restriction C)$-adjacent ensures that $d = d'$, and also note that at least one of the following holds:

(1) $d \leq_{lex} e' \leq_{lex} e$.

(2) $d = d' \leq_{lex} e \leq_{lex} e'$.

If (1) holds, then the fact that d and e are $(\leq_{lex} \restriction C)$-adjacent ensures that $e = e'$, and if (2) holds, then the fact that d' and e' are $(\leq_{lex} \restriction C)$-adjacent ensures that $e = e'$, completing the proof of the lemma. $\square$

Fix a well-order $\leq_w$ of $2^{<\mathbb{N}}$. Let $\sigma \colon B'' \to 2^{<\mathbb{N}}$ be the function sending c to the $\leq_w$-minimal $s \in 2^{<\mathbb{N}}$ for which there are $(\leq_{lex} \restriction [c]_E)$-adjacent $d, e \in [c]_E$ with $d \wedge e = s$, and note that the definition of B'' ensures that φ is well-defined and Borel. Let $T'' = \{d \in B'' \mid \exists e \in [d]_E$ $(d, e$ are $(\leq_{lex} \restriction [d]_E)$-adjacent and $d \wedge e = \sigma(d))$ and $\sigma(d) \frown (0) \sqsubset d\}$, and note that T'' is Borel. For each $c \in B''$, the fact that there are $(\leq_{lex} \restriction [c]_E)$-adjacent $d, e \in [c]_E$ ensures that $T'' \cap [c]_E \neq \emptyset$, and if $d', e' \in [c]_E$ are $(\leq_{lex} \restriction [c]_E)$-adjacent and $d' \wedge e' = d \wedge e$, then Lemma 3.16 ensures that $\{d', e'\} = \{d, e\}$, thus $|T'' \cap [c]_E| = 1$. It follows that T'' is a Borel transversal of $E \restriction B''$, and the fact that $(B' \times B'') \cap E \neq \emptyset$ ensures that $T = T' \cup T''$ is a Borel transversal of $E \restriction B$. Therefore, the map $\pi \colon B' \to T$ with $\mathrm{graph}(\pi) = \{(c, t) \mid c \in B, t \in [c]_E \cap T\}$ is a Borel reduction of $E \restriction B$ to equality on T, thus B is E-smooth. $\square$

References

[1] A classical proof of the Kanovei-Zapletal canonization. *Contemporary Mathematics*, 533:281–286, 2011.

[2] Clinton T. Conley. Canonizing relations on nonsmooth sets. `http://www.math.cornell.edu/~clintonc/onlinepapers/canonize.pdf`.

[3] Jacob Feldman and Calvin C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras I. *Transactions of the American Mathematical Society*, pages 289–324, 1977.

[4] Felix Hausdorff. Grundzüge einer Theorie der geordneten Mengen. *Mathematische Annalen*, 65:435–505, 1908.

[5] Alexander S. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.

[6] Alexander S. Kechris, Leo A. Harrington, and Alain Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. *Journal of the American Mathematical Society*, vol. 3, no. 4:903–928, 1990.

[7] Alexander S. Kechris, Slawomir Solecki, and Stevo Todorcevic. Borel chromatic numbers. *Advances in Mathematics*, 141:1–44, 1999.

[8] Dominique Lecomte and Benjamin D. Miller. Basis theorems for non-potentially closed sets and graphs of uncountable Borel chromatic number. *Journal of Mathematical Logic*, 8, 2007.

[9] Benjamin D. Miller. An introduction to classical descriptive set theory. `https://homepage.univie.ac.at/benjamin.miller/descriptivesettheory.pdf`. Lecture notes.

[10] Benjamin D. Miller. Measure theory and countable Borel equivalence relations. `https://homepage.univie.ac.at/benjamin.miller/measuretheory.pdf`. Lecture notes.