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Abstract

Sobolev spaces play a key role in the proof of existence of solutions to linear symmetric hyperbolic systems and in the proof of local existence of solutions to non linear wave equations. Here we prove the existence and uniqueness of solutions to linear symmetric hyperbolic systems. We describe basic properties and prove important estimates of Sobolev spaces. By using the exact form of the duality we prove existence of solutions to linear symmetric hyperbolic systems. By the Sobolev embedding inequality we can connect the regularity of Sobolev spaces with classical differentiability. Consequently, we can obtain k times continuously differentiable or even smooth solutions.

Zussamenfassung

Sobolev-Räume spielen eine Schlüsselrolle sowohl beim Beweis der Existenz von Lösungen linearer symmetrischer hyperbolischer Systeme als auch beim Beweis der lokalen Existenz von Lösungen nichtlinearer Wellengleichungen. Hier zeigen wir die Existenz und Eindeutigkeit von Lösungen linearer symmetrischer hyperbolischer Systeme. Wir beschreiben grundlegende Eigenschaften und beweisen wichtige Abschätzungen in Sobolev-Räumen. Indem wir die exakte Form der Dualität benutzen zeigen wir die Existenz von Lösungen linearer symmetrischer hyperbolischer Systeme. Mithilfe der Sobolev-Einbettungsungleichung können wir die Regularität von Sobolev-Räumen mit der klassischen Differenzierbarkeit verbinden. Infolgedessen können wir k-mal stetig differenzierbare oder sogar glatte Lösungen erhalten.

1 Sobolev Spaces

1.1 Mollifiers

In this section we follow [Rin09, §5.1]. Mollifiers are a necessary tool to approximate functions in L^p by smooth functions with compact support. We denote by
$C_0^\infty(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) |\ \text{supp}(f) \text{ is compact}\}$ the set of smooth functions with compact support and by
$\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ the set of test functions on $\Omega \subset \mathbb{R}^n$ where Ω is open subset of $\mathbb{R}^n$. Finally, $\bar{B}_1(0)$ denotes the unit ball with center 0.
When we write $C_0^\infty(\mathbb{R}^n)$ it means we have functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Definition 1.1. A function $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(x) \geq 0$ is called a mollifier if

i) $\text{supp}(\phi) \subseteq \bar{B}_1(0)$

ii) $\int\limits_{\mathbb{R}^n} \phi(x)dx = 1.$

Assume that $u : \mathbb{R}^n \to \mathbb{R}^n$ is measurable, so it is integrable on any compact subset of $\mathbb{R}^n$. Now we define $(J_\varepsilon u)(x)$, for any $\varepsilon > 0$, by

$$(J_\varepsilon u)(x) = \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)u(y)dy,$$

where $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ is a scaled mollifier and $u \in L^1_{loc}(\mathbb{R}^n)$. From the definition it is clear that $J_\varepsilon u$ is a smooth function and since $\int\limits_{\mathbb{R}^n} \phi_\varepsilon(y)dy = 1$ we have

$$(J_\varepsilon u)(x) - u(x) = \int\limits_{\mathbb{R}^n} \phi_\varepsilon(y)[u(x-y) - u(x)]dy.$$

Consequently,

$$|(J_\varepsilon u)(x) - u(x)| \leq \sup_{|y|<\varepsilon} |u(x-y) - u(x)|.$$

If u is continuous then $J_\varepsilon u - u$ uniformly converges to zero on compact subsets of $\mathbb{R}^n$ as

$\varepsilon \to 0$. When $1 < p < \infty, u \in L^p(\mathbb{R}^n)$ and with q such that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
&|(J_\varepsilon u)(x)| \leq \int\limits_{\mathbb{R}^n} \phi_\varepsilon^{1/q}(x-y)\phi_\varepsilon^{1/p}(x-y)|u(y)|dy \leq \\
&\left(\int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)dy\right)^{1/q} \left(\int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)|u(y)|^p dy\right)^{1/p} = \\
&\left(\int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)|u(y)|^p dy\right)^{1/p}.
\end{aligned} \tag{1.1}$$

In the second step we used Hölder's inequality. So

$$|(J_\varepsilon u)(x)|^p \leq \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)|u(y)|^p dy.$$

Next we integrate both sides of this inequality and then use Fubini's theorem

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} |(J_\varepsilon u)(x)|^p dx \leq \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)|u(y)|^p dy dx = \\
&\int\limits_{\mathbb{R}^n} |u(y)|^p \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y) dx dy = \int\limits_{\mathbb{R}^n} |u(y)|^p dy.
\end{aligned} \tag{1.2}$$

Raising both sides of the inequality to the power $1/p$,

$$\left(\int\limits_{\mathbb{R}^n} |(J_\varepsilon u)(x)|^p dx\right)^{1/p} \leqslant \left(\int\limits_{\mathbb{R}^n} |u(y)|^p dy\right)^{1/p}.$$

We finally get

$$\|(J_\varepsilon u)\|_p \leq \|u\|_p. \tag{1.3}$$

(1.3) is also true for $p = 1$ and $p = \infty$. Similarly,

$$\|J_\varepsilon u - u\|_p \leqslant \left(\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \phi_\varepsilon(y)|u(x-y) - u(x)|^p dy dx\right)^{1/p}. \tag{1.4}$$

From the above inequality it is clear that if $u(.-y)$ converges to u in $L^p(\mathbb{R}^n)$ as $y \to 0$, then $J_\varepsilon u \to u$ in $L^p(\mathbb{R}^n)$. By Theorem 3.14 of [Rud87] continuous functions with compact support are dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. It follows that $\|u_n - u\|_p \to 0$ for certain $u_n \in C_0(\mathbb{R}^n)$, given $u \in L^p(\mathbb{R}^n)$ $1 \leq p < \infty$. Applying (1.3) to $u - u_n$ we obtain $\|J_\varepsilon u - J_\varepsilon u_n\|_p \leq \|u - u_n\|_p$. Hence $\|J_\varepsilon u - J_\varepsilon u_n\|_p \to 0$ when $n \to \infty$. Since u_n is a continuous function with compact support then by (1.4) we have $\|J_\varepsilon u_n - u_n\|_p \to 0$. By using the Minkowski inequality we obtain

$$\|J_\varepsilon u - u\|_p \le \|J_\varepsilon u - J_\varepsilon u_n\|_p + \|J_\varepsilon u_n - u_n\|_p + \|u_n - u\|_p.$$

Since the right hand side of this inequality converges to 0 when $\varepsilon \to 0$ and $n \to \infty$, then $J_\varepsilon u \to u$ in $L^p(\mathbb{R}^n)$. For proving the next lemma we will need this intermediate lemma:

Lemma 1.2. If $\Omega \subseteq \mathbb{R}^n$ is an open set then there exists an increasing sequence of sets $\{K_n\}_{n\ge 1}$ such that

1. K_n is compact
2. K_n are increasing: $K_n \subset \mathrm{int}K_{n+1}$
3. $\bigcup\limits_{n=1}^{\infty} K_n = \Omega$

Proof. Let us define $K_n = \{x \in \Omega : |x| \le n \text{ and } d(x, \partial\Omega) \ge \frac{1}{n}\}$, where $\partial\Omega$ is the boundary of Ω. From this definition it follows that K_n is bounded and closed, so K_n is compact. Let us prove that $K_n \subset \mathrm{int}K_{n+1}$. Suppose that $x \in K_n$ but $x \notin \mathrm{int}K_{n+1} \Rightarrow |x| \ge n+1 > n$ or $d(x, \partial\Omega) \le \frac{1}{n+1} < \frac{1}{n}$ so $x \notin K_n$ which is a contradiction to the assumption. Take $x \in \Omega$. As Ω is open, $\exists r > 0$ such that $B_r(x) \subset \Omega$. Choose n such that $r \ge \frac{1}{n}$ and $|x| \le n$ so $B_r(x) \supseteq B_{\frac{1}{n}}(x) \Longrightarrow d(x, \partial\Omega) \ge \frac{1}{n}$ so $x \in K_n$. □

Lemma 1.3. Let $u \in L^1_{loc}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is open. If

$$\int\limits_\Omega u\phi \, dx = 0$$

for every $\phi \in C_0^\infty(\Omega)$, then $u = 0$ a.e..

Proof. By assumption $u : \Omega \to \mathbb{R}$ is measurable so $u\chi_K$ is integrable for every compact subset $K \subseteq \Omega$. Here χ_K is the characteristic function of K. We define B_j to be the subset of Ω on which $|u(x)| \ge 1/j$, and from Lemma 1.2 there exist the increasing compact sets $\{K_l\}_{l\geqslant 1}$ such that $\Omega = \bigcup\limits_{l\ge 1} K_l$ and let $B_{j,l} = B_j \cap K_l$. We denote by

$$\nu_{j,l}(x) = \frac{u(x)}{|u(x)|}\chi_{B_{j,l}}(x),$$

then $J_\varepsilon \nu_{j,l} \in C_0^\infty(\Omega)$ for ε small enough. In fact there is an $\varepsilon_0 > 0$ and a corresponding compact subset $K_{l,0}$ of Ω such that $|J_\varepsilon \nu_{j,l}| \le \chi_{K_{l,0}}$ for all $\varepsilon \le \varepsilon_0$. Therefore $|(J_\varepsilon \nu_{j,l})u| \le |u|\chi_{K_{l,0}}$, and $|u|\chi_{K_{l,0}}$ is integrable. Choose any sequence $0 \le \varepsilon_i \le \varepsilon_0$ converging to zero. From the discussion preceding this lemma and the fact that $\nu_{j,l} \in L^p$ for any $1 \le p < \infty$, we conclude that $J_{\varepsilon_i}\nu_{j,l}$ converges to $\nu_{j,l}$ with respect to any L^p norm, for $1 \le p < \infty$. According to Theorem 3.12 of [Rud87] there is a subsequence ε_{i_k} such that $J_{\varepsilon_{i_k}}\nu_{j,l}(x)$

converges to $\nu_{j,l}(x)$ a.e.. Now we have that $J_{\varepsilon_{i_k}}\nu_{j,l}u$ is bounded by an integrable function and converges to $|u|\chi_{B_{j,l}}$ a.e., so by Lebesgue's dominated convergence theorem

$$\lim_{k\to\infty}\int_{\Omega}(J_{\varepsilon_{i_k}}\nu_{j,l})(x)u(x)dx = \int_{B_{j,l}}|u(x)|dx \geq \frac{1}{j}\mu(B_{j,l}).$$

By assumption of the lemma the left-hand side is zero, so we can conclude that the measure of the union of all the $B_{j,l}$ is zero. This union coincides with the set on which u is non-zero, so $u = 0$ a.e.. □

1.2 Weak differentiability, examples

In this section we follow [Rin09, §5.2]. We begin with some notations. A typical point $x \in \mathbb{R}^n$ is denoted by $x = (x^1, \dots, x^n)$. A multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is an n-tuple of non-negative integeres α_i. We denote by x^α the monomial $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$ and call it a monomial of order $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. By ∂^α is denoted the differential operator of order $|\alpha|$. If u is differentible and α is a multiindex then the differential of u with respect to α is

$$\partial^\alpha u = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}u,$$

where

$$\partial_i = \frac{\partial}{\partial x^i} \qquad i = 1, \dots, n.$$

If we write $\partial^\alpha u$ we understand that α is an n-dimensional multiindex and that we differentiate with respect to n variables and if we write ∂_i we assume i is an integer between 0 and n. Here we recall Leibniz rule

$$\partial^\alpha(uv)(x) = \binom{\alpha}{\beta}\partial^\beta u(x)\partial^{\alpha-\beta}v(x),$$

for functions u and v that are $|\alpha|$ times continuously differentiable near x, where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}.$$

When we write $H_s(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ it means we have functions from $\mathbb{R}^n$ to $\mathbb{C}$. But in general, when we do not write the function spaces range it means that functions are real valued.

Definition 1.4. A function $u \in L^1_{loc}(\mathbb{R}^n)$ is called k times weakly differentiable if for every multiindex α with $|\alpha| \leq k$ there is a function $u_\alpha \in L^1_{loc}(\mathbb{R}^n)$ such that

$$(-1)^{|\alpha|}\int_{\mathbb{R}^n}\partial^\alpha u\phi dx = (-1)^{|\alpha|}\int_{\mathbb{R}^n}u_\alpha\phi\ dx = \int_{\mathbb{R}^n}u\partial^\alpha\phi\ dx \tag{1.5}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n)$. The functions u_α are called the weak derivatives of u .

Remark 1.5. If eq. (1.5) holds for both v_α and u_α, then by Lemma 1.3 $u_\alpha = v_\alpha$ a.e.. Usually we shall write $\partial^\alpha u$ instead of u_α. Also we can replace $\mathbb{R}^n$ with any open subset Ω of $\mathbb{R}^n$.

Definition 1.6. The set of all k times weakly differentiable, complex valued functions defined on $\mathbb{R}^n$ such that all the weak derivatives are in $L^p(\mathbb{R}^n)$ $1 \leq p < \infty$, is denoted by $\mathcal{W}^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$. The set of equivalence classes of elements in $\mathcal{W}^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ we simply denote by $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$, where two elements are equivalent if the set on which they are different has measure zero. We define

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^p \, dx \right)^{1/p} \tag{1.6}$$

for an element u of $\mathcal{W}^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$.

Later we will prove that $\| \cdot \|_{W^{k,p}}$ is a norm on $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ spaces.

Remark 1.7. Analogously, we define $W^{k,p}(\mathbb{R}^n, \mathbb{R}^m)$ and $W^{k,p}(\Omega, \mathbb{C}^m)$ for any open subset Ω of $\mathbb{R}^n$.

Definition 1.8. Let the sequence $\{u_m\}_{m \geq 1}$ and $u \in W^{k,p}(\mathbb{R}^n)$. We say that $\{u_m\}_{m \geq 1}$ converges to u in $W^{k,p}(\mathbb{R}^n)$ and write $u_m \to u$ in $W^{k,p}(\mathbb{R}^n)$ if

$$\lim_{m \to \infty} \|u_m - u\|_{W^{k,p}(\mathbb{R}^n)} = 0.$$

Lemma 1.9. Assume $1 \leq p < \infty$ and let k be a non negative integer. Then $\| \cdot \|_{W^{k,p}}$ defined in (1.6) is a norm on $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ with respect to which $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ is a Banach space.

Proof. First we prove that $\| \cdot \|_{W^{k,p}}$ is a norm. From (1.6)

$$\|\lambda u\|_{W^{k,p}} = |\lambda| \|u\|_{W^{k,p}} \quad \forall \lambda \in \mathbb{R},$$

and

$$\|u\|_{W^{k,p}} = 0 \quad \text{ if and only if } \quad u = 0 \quad \text{a.e..}$$

We need to check that

$$\|u + v\|_{W^{k,p}} \leq \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$$

for $u, v \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$. If $p = 1$ then by (1.6) and by triangle inequality we obtain

$$\|u + v\|_{W^{k,1}} = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha (u + v)| \, dx \leq \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha u| \, dx + \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} |\partial^\alpha v| \, dx = \|u\|_{W^{k,1}} + \|v\|_{W^{k,1}}.$$

Let us assume $1 < p < \infty$, then

$$\|u+v\|_{W^{k,p}}^p = \sum_{|\alpha|\le k} \int\limits_{\mathbb{R}^n} |\partial^\alpha (u+v)|^p \; dx \le$$

$$\sum_{|\alpha|\le k} \int\limits_{\mathbb{R}^n} |\partial^\alpha u||\partial^\alpha (u+v)|^{p-1} \; dx + \sum_{|\alpha|\le k} \int\limits_{\mathbb{R}^n} |\partial^\alpha v||\partial^\alpha (u+v)|^{p-1} \; dx.$$

Note that if we let $q = p/(p-1)$ then we can use Hölder's inequality for counter measure:

$$\sum_{|\alpha|\le k} |\partial^\alpha u||\partial^\alpha (u+v)|^{p-1} \le \left(\sum_{|\alpha|\le k} |\partial^\alpha u|^p\right)^{1/p} \left(\sum_{|\alpha|\le k} |\partial^\alpha (u+v)|^{(p-1)\cdot q}\right)^{1/q}.$$

In particular

$$\sum_{|\alpha|\le k} \int\limits_{\mathbb{R}^n} |\partial^\alpha u||\partial^\alpha (u+v)|^{p-1} \; dx = \int\limits_{\mathbb{R}^n} \sum_{|\alpha|\le k} |\partial^\alpha u||\partial^\alpha (u+v)|^{p-1} \; dx \le$$

$$\int\limits_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} |\partial^\alpha u|^p\right)^{1/p} \left(\sum_{|\alpha|\le k} |\partial^\alpha (u+v)|^{(p-1)\cdot \frac{p}{p-1}}\right)^{1/q} dx.$$

Now let us apply Hölder's inequality for the Lebesgue measure:

$$\int\limits_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} |\partial^\alpha u|^p\right)^{1/p} \left(\sum_{|\alpha|\le k} |\partial^\alpha (u+v)|^p\right)^{1/q} dx \le$$

$$\left(\int\limits_{\mathbb{R}^n} \sum_{|\alpha|\le k} |\partial^\alpha u|^p dx\right)^{1/p} \left(\int\limits_{\mathbb{R}^n} \sum_{|\alpha|\le k} |\partial^\alpha (u+v)|^p dx\right)^{\frac{p-1}{p}} = \|u\|_{W^{k,p}} \|u+v\|_{W^{k,p}}^{p-1}.$$

We have a similar inequality for v:

$$\sum_{|\alpha|\le k} \int\limits_{\mathbb{R}^n} |\partial^\alpha v||\partial^\alpha (u+v)|^{p-1} \; dx \le \|v\|_{W^{k,p}} \|u+v\|_{W^{k,p}}^{p-1}.$$

Together these inequalities yield

$$\|u+v\|_{W^{k,p}}^p \le \|u\|_{W^{k,p}} \|u+v\|_{W^{k,p}}^{p-1} + \|v\|_{W^{k,p}} \|u+v\|_{W^{k,p}}^{p-1} =$$
$$\left(\|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}\right) \|u+v\|_{W^{k,p}}^{p-1}.$$

Finally we get $\|u+v\|_{W^{k,p}} \le \|u\|_{W^{k,p}} + \|v\|_{W^{k,p}}$. Next we need to prove that $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ is a Banach space. Assume that u_j is a Cauchy sequence in $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ then $\partial^\alpha u_j$ is a Cauchy sequence in $L^p(\mathbb{R}^n, \mathbb{C}^m)$. Since $L^p(\mathbb{R}^n, \mathbb{C}^m)$ is a Banach space then for every multiindex α with $|\alpha| \leq k$ there is a $u_\alpha \in L^p(\mathbb{R}^n, \mathbb{C}^m)$ such that $\partial^\alpha u_j \to u_\alpha$ in $L^p(\mathbb{R}^n, \mathbb{C}^m)$. Define $u := u_\alpha$ for $\alpha = 0$ then $\lim_{j\to\infty} u_j = u_{(0,\dots,0)}$ in $L^p(\mathbb{R}^n, \mathbb{C}^m)$. We need to prove that

$$u \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^m) \quad \text{and} \quad \partial^\alpha u = u_\alpha \quad (|\alpha| \leq k).$$

Fix $\phi \in C_0^\infty(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} u \partial^\alpha \phi dx = \lim_{j\to\infty} \int_{\mathbb{R}^n} u_j \partial^\alpha \phi dx =$$
$$\lim_{j\to\infty} (-1)^{|\alpha|} \int_{\mathbb{R}^n} \partial^\alpha u_j \phi dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_\alpha \phi dx.$$

We conclude that $u \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ and $\partial^\alpha u = u_\alpha$ for every $|\alpha| \leq k$. Finally

$$\|u - u_j\|^p_{W^{k,p}} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u - \partial^\alpha u_j|^p dx \to 0 \text{ when } j \to \infty.$$

So u_j converges to u in $W^{k,p}$ and u is weakly differentiable with weak derivatives $\partial^\alpha u = u_\alpha$. □

In case $p = 2$, $W^{k,2}$ is a Hilbert space.

Definition 1.10. Let $H^k(\mathbb{R}^n, \mathbb{C}^m) = W^{k,2}(\mathbb{R}^n, \mathbb{C}^m)$, so

$$H^k(\mathbb{R}^n, \mathbb{C}^m) = \{u : u \in L^2(\mathbb{R}^n, \mathbb{C}^m), \partial^\alpha u \in L^2(\mathbb{R}^n, \mathbb{C}^m) \quad \forall |\alpha| \leq k\},$$

where $\partial^\alpha u$ are the weak derivatives defined by (1.5).

Remark 1.11. Note that $H^k(\mathbb{R}^n, \mathbb{C}^m)$ is a complex Hilbert space with the inner product given by

$$(u, v) = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} \partial^\alpha u(x) \cdot \overline{\partial^\alpha v}(x) dx \quad \text{for} \quad u, v \in H^k(\mathbb{R}^n, \mathbb{C}^m), \tag{1.7}$$

and corresponding norm for this inner product is defined by

$$\|u\|_{H^k} = \left(\sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 \, dx \right)^{1/2}. \tag{1.8}$$

Let us give some examples of functions belonging to or not belonging to $H^k(\Omega)$ from [Bha12, page 182] .

Example 1.12. For $\Omega = (-1, 1)$ and $u(x) = |x| \quad \forall x \in (-1, 1)$. We show that $u \in H^1(-1, 1)$, but $u \notin H^2(-1, 1)$.

Proof. Let us compute $\int_{-1}^{1} |u(x)|^2 dx$:

$$\int_{-1}^{1} |u(x)|^2 dx = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < \infty \Rightarrow u(x) \in L^2(-1,1).$$

The first order weak derivative is

$$\frac{du}{dx} = g(x) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ -1 & \text{for } -1 < x < 0. \end{cases}$$

Consequently,

$$\int_{-1}^{1} |g(x)|^2 dx = \int_{-1}^{0} 1dx + \int_{0}^{1} 1dx = 2 < \infty$$

$\Longrightarrow \dfrac{du}{dx} \in L^2(-1,1)$. Thus $u, \dfrac{du}{dx} \in L^2(-1,1)$ from Definition 1.10 it follows that $u \in H^1(-1,1)$.
Now we need to compute the second order of weak derivative . For $\forall \phi \in \mathcal{D}(\mathbb{R})$

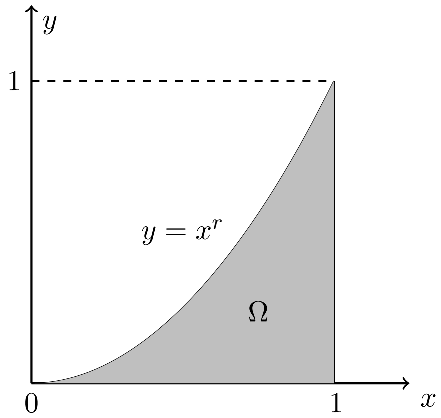
$$\begin{aligned} \left\langle \frac{d^2u}{dx^2}, \phi \right\rangle &= \left\langle \frac{d}{dx}\left(\frac{du}{dx}\right), \phi \right\rangle = \left\langle \frac{dg}{dx}, \phi \right\rangle = -\left\langle g, \frac{d\phi}{dx} \right\rangle = -\int_{-\infty}^{\infty} g(x)\frac{d\phi}{dx}(x)dx \\ &= \int_{-\infty}^{0} \frac{d\phi}{dx}dx - \int_{0}^{\infty} \frac{d\phi}{dx}dx = \phi\Big|_{-\infty}^{0} - \phi\Big|_{0}^{\infty} = 2\phi(0) = \langle 2\delta, \phi \rangle \Longrightarrow \end{aligned}$$

$\dfrac{d^2u}{dx^2} = 2\delta \in \mathcal{D}'(\mathbb{R})$.
Let us prove that the Dirac distribution is not integrable, $\delta \notin L^1_{loc}(-1,1)$.
Suppose to the contrary that is $\exists f \in L^1_{loc}(-1,1)$ with $\delta_{x_0} = f$.
Choose $\rho \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp}(\rho) \subseteq (-1,1)$, $\rho(0) = 1$ and we define
$\rho_l(x) := \rho(l(x - x_0)) \quad l \in \mathbb{N}$. Then $\operatorname{supp}(\rho_l) \subseteq (-1/l, 1/l)$, $\rho_l(x_0) = 1$. Now we have

$$\begin{aligned} 1 = \rho_l(x_0) = |\langle \delta_{x_0}, \rho_l \rangle| &= \Big| \int_{-1/l}^{1/l} f(x)\rho(l(x-x_0))dx \Big| \le \int_{-1/l}^{1/l} |f(x)||\rho(l(x-x_0))|dx \\ &\le ||\rho||_{L^\infty} \int_{-1/l}^{1/l} |f(x)|dx \to 0 \quad l \to \infty. \end{aligned}$$

This is a contradiction, so $\delta \notin L^1_{loc}(-1,1)$.
Since $L^2(-1,1) \subset L^1_{loc}(-1,1)$ and $2\delta \notin L^1_{loc}(-1,1) \Longrightarrow u \notin H^2(-1,1)$. □

Example 1.13. Let $\Omega = \{(x,y) : 0 < x < 1, 0 < y < x^r, r > 0\} \subset \mathbb{R}^2$ and $u(x,y) = x^\alpha, \quad \alpha \in \mathbb{R} \quad \forall (x,y) \in \Omega$. If $2\alpha + r > 1$ then $u \in H^1(\Omega)$.



Proof. Let us compute $\int_\Omega |u(x,y)|^2 dxdy$:

$$\int_\Omega |u(x,y)|^2 dxdy = \int_0^1 \left(\int_{y=0}^{y=x^r} x^{2\alpha} dy \right) dx = \int_0^1 x^{2\alpha} \int_{y=0}^{y=x^r} dydx$$
$$= \int_0^1 x^{2\alpha+r} dx < \infty \quad \text{if} \quad 2\alpha + r + 1 > 0.$$

Consequently, $u(x,y) \in L^2(\Omega)$ for $2\alpha + r > -1$.
Notice that u is a C^∞-function in Ω. Since the usual partial derivatives and weak derivatives of u will coincide in Ω then $\partial_x u = \alpha x^{\alpha-1}$, $\partial_y u = 0$ in Ω. It is clear that $\partial_y u \in L^2(\Omega)$ $\forall \alpha, r$ and

$$\int_\Omega |\partial_x u|^2 dxdy = \int_0^1 \left(\int_{y=0}^{y=x^r} \alpha^2 x^{2\alpha-2} dy \right) dx$$
$$= \alpha^2 \int_0^1 x^{2\alpha-2+r} dx < \infty \quad \text{if} \quad 2\alpha - 2 + r + 1 > 0.$$

Hence $\partial_x u \in L^2(\Omega)$ for $2\alpha + r > 1$. We have proved that $u, \partial_x u, \partial_y u \in L^2(\Omega)$ for $2\alpha + r > 1$. By Definition 1.10 $u \in H^1(\Omega)$ for $2\alpha + r > 1$. □

Example 1.14. If $\Omega = \mathbb{R}^2$ and $u(x_1, x_2)$ is given by

$$u(x_1,x_2) = \begin{cases} \ln|\ln r| & \text{for } 0 < r = (x_1^2 + x_2^2)^{\frac{1}{2}} < \dfrac{1}{e} \\ 0 & \text{for } \dfrac{1}{e} \le r < \infty, \end{cases}$$

then $u(x_1, x_2) \in H^1(\mathbb{R}^2)$.

Proof. Since $|u(x_1, x_2)| \to \infty$ when $r \to 0$, it follows that u is unbounded and discontinuous in $\mathbb{R}^2$.

First let us prove that $u \in L^2(\mathbb{R}^2)$:

$$\int_{\mathbb{R}^2} u(x_1,x_2)^2 dx_1 dx_2 = \int_{0<r<\frac{1}{e}} (\ln|\ln r|)^2 dx_1 dx_2 =$$
$$\int_0^{\frac{1}{e}} \int_0^{2\pi} (\ln|\ln r|)^2 r dr d\theta = 2\pi \int_0^{\frac{1}{e}} (\ln|\ln r|)^2 r dr.$$

Let us estimate $\int_0^{\frac{1}{e}} (\ln|\ln r|)^2 r dr$. For $0 < r < \dfrac{1}{e}$, we get $|\ln(r)| = -\ln r = \ln\dfrac{1}{r}$ and for $r = \dfrac{1}{e}, \ln\dfrac{1}{e} = -\ln e = -1$. So $\ln r$ monotonically increases from $-\infty$ to -1 when r increases from 0 to $\dfrac{1}{e}$. Hence, $0 < r < \dfrac{1}{e} \Longrightarrow e < \dfrac{1}{r} < \infty \Longrightarrow 1 < \ln\dfrac{1}{r} < \dfrac{1}{r} \Longrightarrow \ln|\ln r| = \ln\ln\frac{1}{r} < \ln\frac{1}{r} = -\ln r \Longrightarrow (\ln|\ln r|)^2 < (\ln r)^2$ for $0 < r < \dfrac{1}{e}$. Consequently,

$$\int_0^{\frac{1}{e}} (\ln|\ln r|)^2 r dr \le \int_0^{\frac{1}{e}} (\ln r)^2 r dr. \tag{1.9}$$

Let us compute $\int_0^{\frac{1}{e}} (\ln r)^2 r dr$ using partial integration

$$I = \int_0^{\frac{1}{e}} (\ln r)^2 d\frac{r^2}{2} = \frac{(\ln r)^2 r^2}{2}\Big|_{r\to 0+}^{r=\frac{1}{e}} - \int_0^{\frac{1}{e}} \frac{r^2}{2}\cdot\frac{2\ln r}{r} dr =$$
$$\frac{1}{2e^2} - \underbrace{\lim_{r\to 0+} \frac{(\ln r)^2 r^2}{2}}_{I_1} - \underbrace{\int_0^{\frac{1}{e}} r\ln r dr}_{I_2} = \tag{1.10}$$
$$\frac{1}{2e^2} - I_1 - I_2.$$

Now let us compute I_1 and I_2 :

$$I_1 = \lim_{r\to 0+} \frac{(\ln r)^2}{2r^{-2}} = \lim_{r\to 0+} \frac{2\ln r}{2r(-2r^{-3})} = \lim_{r\to 0+} \frac{\ln r}{-2r^{-2}} =$$
$$\lim_{r\to 0+} \frac{1}{-2r(-2r^{-3})} = \lim_{r\to 0+} \frac{r^2}{4} = 0,$$

and

$$I_2 = \int_0^{\frac{1}{e}} \ln r d\frac{r^2}{2} = \frac{r^2 \ln r}{2}\Big|_{r\to 0+}^{r=\frac{1}{e}} - \int_0^{\frac{1}{e}} \frac{r^2}{2}\frac{1}{r} dr =$$
$$-\frac{1}{2e^2} - \underbrace{\lim_{r\to 0+} \frac{r^2\ln r}{2}}_{I_3} - \frac{1}{4e^2},$$

$$I_3 = \lim_{r\to 0+} \frac{r^2\ln r}{2} = \lim_{r\to 0+} \frac{\ln r}{2r^{-2}} = \lim_{r\to 0+} \frac{1}{-4r\cdot r^{-3}} = \lim_{r\to 0+} -\frac{r^2}{4} = 0.$$

Hence $\quad I_2 = -\dfrac{1}{2e^2} - 0 - \dfrac{1}{4e^2} = -\dfrac{3}{4e^2}$.
Insert I_1 and I_2 in (1.10), then we obtain $I = \dfrac{1}{2e^2} - 0 + \dfrac{3}{4e^2} = \dfrac{5}{4e^2}$.
Finally by (1.9) we have that

$$\int_{\mathbb{R}^2} u(x_1, x_2)^2 dx_1 dx_2 = 2\pi \int_0^{\frac{1}{e}} (\ln|\ln r|)^2 r dr \leqslant \frac{2\pi \cdot 5}{4e^2} < \infty \Longrightarrow u \in L^2(\mathbb{R}^2).$$

Now we need to show that $\dfrac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^2), i = 1, 2,$ where $\dfrac{\partial u}{\partial x_i}$ is the weak derivatives. Let us denote by $\left[\dfrac{\partial u}{\partial x_i}(x)\right]$ the usual partial derivative of u with respect to $x_i \quad (i = 1, 2)$.
For $0 < r < \dfrac{1}{e}$

$$\begin{gathered}
\left[\frac{\partial u}{\partial x_i}(x)\right] = \left[\frac{d}{dr} \ln|\ln r|\right] \left[\frac{\partial r}{\partial x_i}\right] = \\
-\frac{1}{\ln r}\left[\frac{d}{dr}(-\ln r)\right] \cdot \frac{x_i}{r} = \frac{x_i}{r^2 \ln r}, \qquad i = 1, 2.
\end{gathered}$$

Since $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$ then $\left[\dfrac{\partial r}{\partial x_i}\right] = \dfrac{x_i}{r}$. From the definition of the function $u(x_1, x_2)$ it follows that $\left[\dfrac{\partial u}{\partial x_i}(x)\right] = 0$ for $\frac{1}{e} \leq r < \infty$. Thus

$$\left[\frac{\partial u}{\partial x_i}(x)\right] = \begin{cases} \dfrac{x_i}{r^2 \ln r}, & 0 < r < \dfrac{1}{e} \\ 0, & \dfrac{1}{e} \leq r < \infty. \end{cases} \tag{1.11}$$

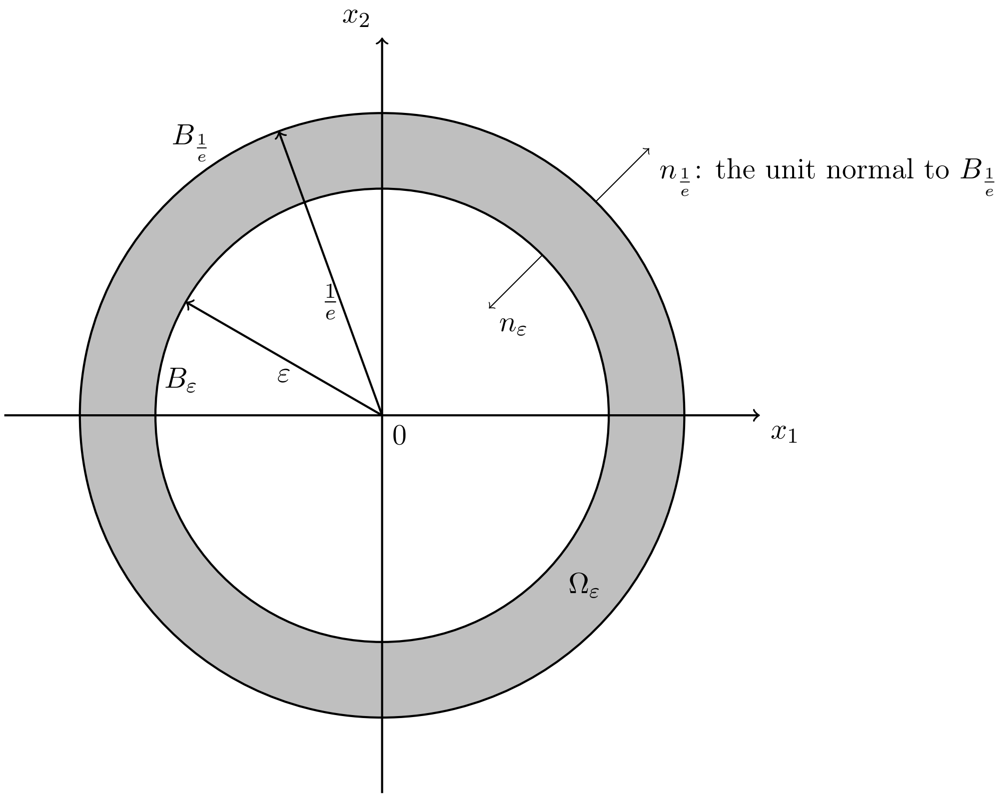
Let us prove that $\left[\dfrac{\partial u}{\partial x_i}(x)\right] \in L^2(\mathbb{R}^2), \quad i = 1, 2:$

$$\begin{gathered}
\int_{\mathbb{R}^2} \left[\frac{\partial u}{\partial x_i}(x)\right]^2 dx_1 dx_2 = \int_{0<r<\frac{1}{e}} \frac{x_i^2}{r^4 (\ln r)^2} dx_1 dx_2 \leq \\
\int_0^{\frac{1}{e}} \int_0^{2\pi} \frac{r^2}{r^4 (\ln r)^2} r dr d\theta = 2\pi \int_0^{\frac{1}{e}} \frac{dr}{r (\ln r)^2} = \\
2\pi \int_0^{\frac{1}{e}} \frac{d(\ln r)}{(\ln r)^2} = 2\pi \left[\frac{-1}{\ln r}\right]\Bigg|_0^{\frac{1}{e}} = 2\pi \cdot \frac{-1}{\ln(\frac{1}{e})} = 2\pi < \infty.
\end{gathered}$$

Let $\dfrac{\partial u}{\partial x_i} \quad i = 1, 2$ be the weak derivatives of u with respect to x_i. We will prove that the usual derivative and the weak derivative of u coincide. Then, $\forall \phi \in \mathcal{D}(\mathbb{R}^2)$,

$$\begin{aligned}\left\langle \frac{\partial u}{\partial x_i}, \phi \right\rangle = -\left\langle u, \frac{\partial \phi}{\partial x_i} \right\rangle = -\int_{\mathbb{R}^2} u(x) \frac{\partial \phi}{\partial x_i}(x) dx = -\int_{0<r<\frac{1}{e}} u(x) \frac{\partial \phi}{\partial x_i}(x) dx = \\ -\int_{0<r<\frac{1}{e}} \left[\frac{\partial (u\phi)}{\partial x_i} \right] dx + \int_{0<r<\frac{1}{e}} \left[\frac{\partial u}{\partial x_i}(x) \right] \phi(x) dx = \\ \underbrace{-\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \left[\frac{\partial (u\phi)}{\partial x_i} \right] dx}_{J_1} + \underbrace{\int_{\mathbb{R}^2} \left[\frac{\partial u}{\partial x_i}(x) \right] \phi(x) dx}_{J_2} = -J_1 + J_2,\end{aligned} \tag{1.12}$$

since $\left[\frac{\partial u}{\partial x_i}(x) \right] = 0$ for $r \geq \frac{1}{e}$. Hence, $\Omega_\varepsilon = \{x : x = (x_1, x_2), \varepsilon \leq r = (x_1^2 + x_2^2)^{\frac{1}{2}} \leq \frac{1}{e}\}$ is the closed annular circular domain enclosed by inner circle B_ε with radius ε and outer circle $B_{\frac{1}{e}}$ with radius $\frac{1}{e}$. The figure is given below.



Applying Green's theorem on the closed annular circular domain Ω_ε we get

$$\begin{aligned}J_1 = \lim_{\varepsilon \to 0} \int_{B_\varepsilon} u(x)\phi(x) n_i(B_\varepsilon) ds + \int_{B_{\frac{1}{e}}} u(x)\phi(x) n_i(B_{\frac{1}{e}}) ds = \\ \lim_{\varepsilon \to 0} \int_{B_\varepsilon} u(x)\phi(x) n_i(B_\varepsilon) ds,\end{aligned} \tag{1.13}$$

where $n_i(B_\varepsilon)$ (resp.$n_i(B_{\frac{1}{e}})$) is the i th component of the unit vector normal $\widehat{n}$ to B_ε (resp. $B_{\frac{1}{e}}$). From the definition of the function $u(x_1,x_2)$ it follows that $u=0$ on $B_{\frac{1}{e}}$ so the second term in (1.13) vanishes.

$$\begin{aligned}|J_1| &= \left|\int_{B_\varepsilon} u(x)\phi(x)n_i(B_\varepsilon)ds\right| = \left|\int_{B_\varepsilon} \ln|\ln\varepsilon|\phi(x)n_i(B_\varepsilon)ds\right| \\ &\leq \max|\phi(x)|\Big|\ln|\ln\varepsilon|\Big|2\pi\varepsilon \to 0,\end{aligned}$$

when $\varepsilon\to 0$. Since $\forall\varepsilon>0,\ \ln|\ln\varepsilon|<|\ln\varepsilon|$ and

$$\lim_{\varepsilon\to 0}\varepsilon|\ln\varepsilon| = \lim_{\varepsilon\to 0}\frac{-\ln\varepsilon}{\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0}\frac{-\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = 0$$

then $\varepsilon\ln|\ln\varepsilon|<\varepsilon|\ln\varepsilon|\to 0$ when $\varepsilon\to 0$. From the above discussion and from (1.13) it follows that $J_1=0$. Finally from (1.12) we have that $\forall\phi\in\mathcal{D}(\mathbb{R}^2)$,

$$\left\langle\frac{\partial u}{\partial x_i},\phi\right\rangle = \int_{\mathbb{R}^2}\left[\frac{\partial u}{\partial x_i}(x)\right]\phi(x)dx \Longrightarrow \frac{\partial u}{\partial x_i} = \left[\frac{\partial u}{\partial x_i}(x)\right]$$

with $\left[\frac{\partial u}{\partial x_i}(x)\right]$ defined by (1.11) for $i=1,2$. We have already proved that $\left[\frac{\partial u}{\partial x_i}(x)\right]\in L^2(\mathbb{R}^2)\Longrightarrow\frac{\partial u}{\partial x_i}(x)\in L^2(\mathbb{R}^2)$ for $i=1,2$, so $u(x_1,x_2)\in H^1(\mathbb{R}^2)$. □

Lemma 1.15. Let $1\leq p<\infty$ and let k be a non negative integer. Then the space $C_0^\infty(\mathbb{R}^n,\mathbb{C}^m)$ is dense in $W^{k,p}(\mathbb{R}^n,\mathbb{C}^m)$.

Proof. Assume that $u\in W^{k,p}(\mathbb{R}^n,\mathbb{C}^m)$ and $\phi\in C_0^\infty(\mathbb{R}^n)$ be such that $\phi(x)=1$ for $|x|\leq 1$. The existence such a ϕ is given in Proposition A.12 of [Rin09]. Let us show that $\phi_l u\in W^{k,p}(\mathbb{R}^n,\mathbb{C}^m)$, where $\phi_l(x)=\phi(x/l)$. Since $u\in W^{k,p}(\mathbb{R}^n,\mathbb{C}^m)$ it follows that $\partial^\alpha u\in L^p(\mathbb{R}^n,\mathbb{C}^m)$ for all $|\alpha|\leq k$. By the Leibniz formula

$$\partial^\alpha(\phi_l u) = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}\phi_l\partial^\beta u.$$

Since $\partial^{\alpha-\beta}\phi_l\in C_0^\infty(\mathbb{R}^n)$ and $\partial^\beta u\in L^p(\mathbb{R}^n,\mathbb{C}^m)$ then

$$\partial^{\alpha-\beta}\phi_l\partial^\beta u\in L^p(\mathbb{R}^n,\mathbb{C}^m):$$

$$\int\limits_{\mathbb{R}^n}|\partial^{\alpha-\beta}\phi_l\partial^\beta u|^p dx \leq \|\partial^{\alpha-\beta}\phi_l\|_\infty^p\|\partial^\beta u\|_p^p<\infty.$$

Consequently,

$$\partial^\alpha(\phi_l u)\in L^p(\mathbb{R}^n,\mathbb{C}^m)\text{ for all }|\alpha|\leq k.$$

We conclude that $\phi_l u \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$. To begin with we need to show that $\phi_l u$ converges to u in $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$:

$$\|\phi_l u - u\|^p_{W^{k,p}} = \sum_{|\alpha|\leq k} \int\limits_{\mathbb{R}^n} |\partial^\alpha(\phi_l u - u)|^p dx =$$

$$\sum_{|\alpha|\leq k} \int\limits_{\mathbb{R}^n} \left| \sum_{\beta<\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\phi_l \partial^\beta u + \phi_l \partial^\alpha u - \partial^\alpha u \right|^p dx.$$

In the above expression if $\beta < \alpha$, then $\partial^{\alpha-\beta}\phi_l\partial^\beta u$ converges to zero pointwise everywhere when $l \to \infty$ and it is bounded by a function in L^p. By Lebesgue's dominated convergence theorem $\partial^{\alpha-\beta}\phi_l\partial^\beta u$ convergence to zero in L^p. By definition $\phi_l(x) = 1$ for $|x| \leqslant l$, so $\phi_l u$ converges to u. Since $\phi_l u$ has compact support and $\phi_l u$ converges to u in $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ then we can assume that u also has compact support. Consequently $J_\varepsilon u$ is a smooth function with compact support. Let us show that

$$\partial^\alpha J_\varepsilon u(x) = \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)\partial^\alpha u(y)dy.$$

Since $\phi_\varepsilon(x-y) \in C_0^\infty(\mathbb{R}^n)$ and $\partial_x^\alpha \phi_\varepsilon(x-y) = (-1)^{|\alpha|}\partial_y^\alpha \phi_\varepsilon(x-y)$ we have that

$$\partial_x^\alpha J_\varepsilon u(x) = \int\limits_{\mathbb{R}^n} \partial_x^\alpha \phi_\varepsilon(x-y)u(y)dy = (-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \partial_y^\alpha \phi_\varepsilon(x-y)u(y)dy. \tag{1.14}$$

By the definition of the weak derivative and since $\phi_\varepsilon(x-y) \in C_0^\infty(\mathbb{R}^n)$ we get

$$(-1)^{|\alpha|} \int\limits_{\mathbb{R}^n} \partial_y^\alpha \phi_\varepsilon(x-y)u(y)dy = \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)\partial_y^\alpha u(y)dy. \tag{1.15}$$

So

$$\partial^\alpha J_\varepsilon u(x) = \int\limits_{\mathbb{R}^n} \phi_\varepsilon(x-y)\partial^\alpha u(y)dy.$$

Before the statement of Lemma 1.2 we proved that $J_\varepsilon u$ converges to u in L^p. Finally we conclude that all $u \in W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ can be approximated by $J_\varepsilon u \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$. So the space $C_0^\infty(\mathbb{R}^n, \mathbb{C}^m)$ is dense in $W^{k,p}(\mathbb{R}^n, \mathbb{C}^m)$ for $1 \leq p < \infty$. □

1.3 Schwartz space

In this section we follow lecture notes [HS09, §5.2].

Definition 1.16. We say that $\phi \in C^\infty(\mathbb{R}^n)$ is rapidly decreasing, if for all multiindices α, β :

$$q_{\alpha,\beta}(\phi) := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty.$$

The vector space of all rapidly decreasing functions on $\mathbb{R}^n$ we call Schwartz space and denote by $\mathcal{S}(\mathbb{R}^n)$.

Definition 1.17. We say that the sequence $(\phi_n)_{n\geqslant 1} \in \mathcal{S}(\mathbb{R}^n)$ converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$ if for all multiindices α, β :

$$q_{\alpha,\beta}(\phi_n - \phi) \to 0 \quad when \quad n \to \infty.$$

Also, we can define convergence in $\mathcal{S}(\mathbb{R}^n)$ by increasing sequence of semi-norms

$$Q_k(\phi) := \sum_{|\alpha|,|\beta|\leq k} q_{\alpha,\beta}(\phi), \quad k \in \mathbb{N}_0.$$

Theorem 1.18. *The space $\mathcal{S}(\mathbb{R}^n)$ has a metric $d : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ defined by*

$$d(\phi,\psi) := \sum_{k=0}^{\infty} 2^{-k} \frac{Q_k(\phi-\psi)}{1+Q_k(\phi-\psi)} \quad \phi,\psi \in \mathcal{S}(\mathbb{R}^n).$$

Proof. Let us prove that $d(\phi,\psi)$ is a metric. From the definition of $d(\phi,\psi)$ it is clear that $d(\phi,\psi) = 0 \Leftrightarrow \phi = \psi$ and $d(\phi,\psi) = d(\psi,\phi)$. It remains to show that for every $\rho \in \mathcal{S}(\mathbb{R}^n)$ $d(\phi,\psi) \leq d(\phi,\rho) + d(\rho,\psi)$. We will use that $f(x) = \dfrac{x}{1+x} : [0,\infty) \longrightarrow [0,\infty)$ is an increasing function and $Q_k(\phi-\psi) \leq Q_k(\phi-\rho) + Q_k(\rho-\phi)$ (as $Q_k(\phi)$ is a semi-norm). So

$$\frac{Q_k(\phi-\psi)}{1+Q_k(\phi-\psi)} \leq \frac{Q_k(\phi-\rho)+Q_k(\rho-\psi)}{1+Q_k(\phi-\rho)+Q_k(\rho-\psi)}.$$

Now we use the following simple fact that for every $a,b \geq 0$: $\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}$ and finally we get

$$\frac{Q_k(\phi-\rho)+Q_k(\rho-\psi)}{1+Q_k(\phi-\rho)+Q_k(\rho-\psi)} \leq \frac{Q_k(\phi-\rho)}{1+Q_k(\phi-\rho)} + \frac{Q_k(\rho-\psi)}{1+Q_k(\rho-\psi)}.$$

Multiplying both sides of the above inequality by 2^{-k} and summing for all k we get

$$\sum_{k=0}^{\infty} 2^{-k} \frac{Q_k(\phi-\psi)}{1+Q_k(\phi-\psi)} \leq \sum_{k=0}^{\infty} 2^{-k} \frac{Q_k(\phi-\rho)}{1+Q_k(\phi-\rho)} + \sum_{k=0}^{\infty} 2^{-k} \frac{Q_k(\rho-\psi)}{1+Q_k(\rho-\psi)}.$$

Now we have the desired result $d(\phi,\psi) \leq d(\phi,\rho)+d(\rho,\psi)$ and that $d(\phi,\psi)$ is a metric. □

Theorem 1.19. *Convergence with respect to the metric d is equivalent to S-convergence.*

Proof. Assume that the sequence $(\phi_n)_{n\geq 1}$ is converging to ϕ in $\mathcal{S}$. We need to show that $d(\phi_n,\phi) \longrightarrow 0$ as $n \longrightarrow \infty$.
Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\varepsilon > \dfrac{4}{2^{N+1}}$. Since the sequence $(\phi_n)_{n\geq 1}$ converges to ϕ in $\mathcal{S}$ there exists some $m_0 \in \mathbb{N}$ such that for all $n \geq m_0$ we have $Q_N(\phi_n - \phi) < \frac{\varepsilon}{4}$.

Thus we obtain for any $n \geq m_0$:

$$d(\phi_n, \phi) = \sum_{k=0}^{N} 2^{-k} \overbrace{\frac{Q_k(\phi_n - \phi)}{1 + Q_k(\phi_n - \phi)}}^{\leq Q_N(\phi_n - \phi)} + \sum_{k=N+1}^{\infty} 2^{-k} \overbrace{\frac{Q_k(\phi_n - \phi)}{1 + Q_k(\phi_n - \phi)}}^{\leq 1}$$
$$\leq Q_N(\phi_n - \phi) \sum_{k=0}^{N} 2^{-k} + \sum_{k=N+1}^{\infty} 2^{-k} = Q_N(\phi_n - \phi) \cdot 2(1 - 2^{-N-1}) + \frac{1}{2^{N+1}} \cdot 2$$
$$\leq \frac{\varepsilon}{4} \cdot 2(1-0) + \frac{\varepsilon}{4} \cdot 2 = \varepsilon.$$

So $d(\phi_n, \phi) \longrightarrow 0$ as $n \longrightarrow \infty$. And conversely, metric convergence implies S-convergence since $q_{\alpha,\beta}(\phi_n - \phi) \longrightarrow 0$ when $n \longrightarrow \infty$. □

Theorem 1.20. *$\mathcal{S}(\mathbb{R}^n)$ is a complete metric space.*

Proof. Let us prove that any Cauchy sequence $(\phi_n)_{n\geqslant 1} \in \mathcal{S}(\mathbb{R}^n)$ converges to some $\phi \in \mathcal{S}(\mathbb{R}^n)$. The space $L^\infty(\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_\infty$. Let us denote by $C_b(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. This space is also a Banach space with the norm $\|\cdot\|_\infty$. Suppose that $(\phi_j)_{j\geqslant 1}$ is a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$ then we obtain for every multiindices α, β, $x^\alpha D^\beta \phi_j$ is a Cauchy sequence in $C_b(\mathbb{R}^n)$. Thus $x^\alpha D^\beta \phi_j$ converges to some $\phi_{\alpha,\beta} \in C_b$. Let us put $\phi = \phi_{0,0}$. On the one hand $x^\alpha D^\beta \phi_j$ converges to $\phi_{\alpha,\beta}$ in C_b and on the other hand $x^\alpha D^\beta \phi_j$ converges point wise to $x^\alpha \phi_{0,\beta}$ so $\phi_{\alpha,\beta} = x^\alpha D^\beta \phi = x^\alpha \phi_{0,\beta}$.
Now we need to prove that $\phi_{\alpha,\beta} \in \mathcal{S}(\mathbb{R}^n)$. For arbitrary fixed $N \in \mathbb{N}_0$ and for any multiindices α, β :

$$\|x^\alpha D^\beta \phi\|_\infty \leq \|x^\alpha D^\beta \phi - x^\alpha D^\beta \phi_N\|_\infty + \|x^\alpha D^\beta \phi_N\|_\infty < \infty.$$

So $q_{\alpha,\beta}(\phi) < \infty$. It remains to prove that ϕ_j converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$. Since $x^\alpha D^\beta \phi_j$ is a Cauchy sequence then for any $\varepsilon > 0$ there exist N_0 such that

$$\|x^\alpha D^\beta \phi - x^\alpha D^\beta \phi_j\|_\infty = \lim_{n\to\infty} \|x^\alpha D^\beta \phi_n - x^\alpha D^\beta \phi_j\|_\infty < \varepsilon \quad (j > N_0).$$

Therefore $\phi_j \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$. □

Example 1.21. $\phi = e^{-a|x|^2}$, $Re(a) > 0$ is a smooth function but has not compact support so $e^{-a|x|^2} \notin \mathcal{D}(\mathbb{R}^n)$. This function is rapidly decreasing $\Longrightarrow e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n)$. Finally we have $e^{-a|x|^2} \in \mathcal{S}(\mathbb{R}^n) \setminus \mathcal{D}(\mathbb{R}^n)$.

1.4 Temperate distributions and Fourier transforms

In this section we follow lecture notes [HS09, §5.3 and §5.4].

Definition 1.22. A continuous linear functional $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a temperate distribution:

$$u(\phi_n) \to 0 \text{ in } \mathbb{C} \text{ when } \phi_n \to 0 \ (n \to \infty) \text{ in } \mathcal{S}(\mathbb{R}^n).$$

The set of all temperate distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.23. *Assume $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional. Then $u \in \mathcal{S}'(\mathbb{R}^n)$ if and only if $\exists C > 0$ $\exists N \in \mathbb{N}_0$ such that $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$*

$$|u(\phi)| \leq CQ_N(\phi) = C \sum_{|\alpha|,|\beta| \leq N} \|x^\alpha D^\beta \phi\|_\infty. \tag{1.16}$$

Proof. If $\phi_n \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ then by (1.16) $u(\phi_n) \to 0$. Hence $u \in \mathcal{S}'(\mathbb{R}^n)$. Now assume that u is continuous but (1.16) does not hold: $\forall N \in \mathbb{N} \quad \exists \phi_N \in \mathcal{S}(\mathbb{R}^n)$ such that

$$|u(\phi_N)| > NQ_N(\phi_N).$$

In particular, $\phi_N \neq 0$. Let us denote by $\psi_N = \phi_N/(NQ_N(\phi_N))$. This is a sequence in $\mathcal{S}(\mathbb{R}^n)$ with

$$q_{\alpha,\beta}(\psi_N) = \frac{q_{\alpha,\beta}(\phi_N)}{NQ_N(\phi_N)} = \frac{q_{\alpha,\beta}(\phi_N)}{N \cdot \sum\limits_{|\alpha|,|\beta| \leq N} q_{\alpha,\beta}(\phi_N)} < \frac{1}{N}$$

when $N \geq \max(|\alpha|, |\beta|)$. On the other hand $|u(\psi_N)| = \dfrac{|u(\phi_N)|}{NQ_N(\phi_N)} > 1$. We obtain $q_{\alpha,\beta}(\psi_N) \to 0$ but $|u(\psi_N)| > 1$. This is a contradiction to our assumption. □

A temperate distribution is a special case of a distribution, which is a continuous linear form on Schwartz space. We need the concept of a temperate distribution to take the Fourier transform.

Remark 1.24. *For* $1 \leq p \leq \infty \quad L^p(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n)$.

Proof. The cases $p = 1$ and $p = \infty$ are obvious. Let us prove theorem when $1 < p < \infty$. From Hölder's inequality we have, if $f \in L^p(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and with q such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$|\langle f, \phi \rangle| \leq \int |f\phi| \leq \|f\|_{L^p} \|\phi\|_{L^q}.$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \forall l \in \mathbb{N}, \quad |\phi(x)| \leq \dfrac{Q_l(\phi)}{(1+|x|)^l}$. From this inequality we have

$$\|\phi\|_{L^q}^q = \int_{\mathbb{R}^n} |\phi(x)|^q dx \leq Q_l^q(\phi) \int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{lq}}.$$

Choose sufficiently large l such that $lq > n$, hence $\int_{\mathbb{R}^n} \dfrac{dx}{(1+|x|)^{lq}}$ is finite. Then

$$|\langle f, \phi \rangle| \leq CQ_l(\phi).$$

From Theorem 1.23 it follows that $f \in \mathcal{S}'(\mathbb{R}^n)$. □

Remark 1.25. If $f \in C(\mathbb{R}^n)$ is of polynomial growth : $\exists C, M \geq 0$ such that

$$|f(x)| \leq C(1+|x|)^M \quad \forall x \in \mathbb{R}^n,$$

then $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. Let us estimate $|\langle f, \phi\rangle|$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$|\langle f, \phi\rangle| \leq \int_{\mathbb{R}^n} |f(x)||\phi(x)|dx \leq \int_{\mathbb{R}^n} C(1+|x|)^M \frac{Q_l(\phi)}{(1+|x|)^l}dx = CQ_l(\phi)\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{l-M}}.$$

If we choose $l > M+n$ then $\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^{l-M}} < \infty$ and that $|\langle f, \phi\rangle| \leq C_l Q_l(\phi)$. From Theorem 1.23 it follows that $f \in \mathcal{S}'(\mathbb{R}^n)$. □

There is a theorem connected with this remark [Fri82, Theorem 8.3.1], which states that every temperate distribution is a finite order derivative of a continuous function of polynomial growth on $\mathbb{R}^n$.

Definition 1.26. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a temperate distribution, then its Fourier transform $\hat{u} = \mathcal{F}u \in \mathcal{S}'(\mathbb{R}^n)$ is the temperate distribution defined by:

$$\langle \hat{u}, \phi\rangle = \langle u, \hat{\phi}\rangle \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n) \tag{1.17}$$

with

$$\hat{\phi}(\xi) = (\mathcal{F}\phi)(\xi) = \int\limits_{\mathbb{R}^n} \phi(x)e^{-i\xi\cdot x}dx, \quad \phi(x) = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \hat{\phi}(\xi)e^{i\xi\cdot x}d\xi.$$

Theorem 1.27. *The Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is linear and bijective , and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are sequentially continuous maps $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.*

Proof. From the above definition of Fourier transform it follows that $\mathcal{F}$ is linear and sequentially continuous. Let us prove injectivity. Assume that $\hat{u} = \mathcal{F}u = 0$, then $\forall \phi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \mathcal{F}u, \mathcal{F}^{-1}\phi\rangle = \langle u, \phi\rangle = 0 \Longrightarrow u = 0$.
Now we need to show surjectivity. First let us prove that $\forall \phi \in \mathcal{S}(\mathbb{R}^n) \quad \hat{\hat{\phi}} = (2\pi)^n\check{\phi}$. From the Fourier inverson formula we get

$$\phi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{\phi}(\xi)e^{i\xi\cdot x}d\xi = \int_{\mathbb{R}^n} ((2\pi)^{-n}\hat{\phi}(-\xi))e^{-i\xi\cdot x}d\xi = \mathcal{F}((2\pi)^{-n}\check{\hat{\phi}}).$$

Using the fact that $\forall \phi \in \mathcal{S}(\mathbb{R}^n) \quad \mathcal{F}(\check{\phi}) = (\mathcal{F}\phi)\check{}$ we obtain that

$$\check{\phi} = \mathcal{F}((2\pi)^{-n}\check{\check{\hat{\phi}}}) = (2\pi)^{-n}\mathcal{F}(\hat{\phi}) = (2\pi)^{-n}\hat{\hat{\phi}}.$$

Finally we have

$$\langle \hat{\hat{u}}, \phi\rangle = \langle \hat{u}, \hat{\phi}\rangle = \langle u, \hat{\hat{\phi}}\rangle = (2\pi)^n\langle u, \check{\phi}\rangle = (2\pi)^n\langle \check{u}, \phi\rangle \quad \text{where} \quad \check{\phi}(x) = \phi(-x).$$

Hence

$$\langle \hat{\hat{u}}, \phi\rangle = (2\pi)^n\langle \check{u}, \phi\rangle \quad \text{where} \quad \check{u}(\phi) = u(-\phi). \tag{1.18}$$

From (1.18) it follows that $u = \mathcal{F}(2\pi^{-n}\check{\hat{u}}) \Longrightarrow \mathcal{F}$ is surjective. Hence $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is bijective. Since $\mathcal{F}$ is sequentially continuous and bijective $\Rightarrow \mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is also sequentially continuous. □

Definition 1.28. The derivative of a temperate distribution u is defined by

$$\langle \partial^\alpha u, \phi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n). \tag{1.19}$$

Since $\phi \in \mathcal{S}(\mathbb{R}^n)$ then $\partial^\alpha \phi \in \mathcal{S}(\mathbb{R}^n)$. So $\langle u, \partial^\alpha \phi \rangle$ is well defined. From (1.19) follows that $\langle \partial^\alpha u, \phi \rangle$ is also well defined for $\forall \phi \in \mathcal{S}(\mathbb{R}^n)$ and $\partial^\alpha u$ is continuous on $\mathcal{S}(\mathbb{R}^n) \Longrightarrow \partial^\alpha u \in \mathcal{S}'(\mathbb{R}^n)$. Every temperate distribution is infinitely differentiable.

Theorem 1.29. *The map $\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, for all multiindices α is linear and continuous.*

Proof. Let us prove linearity. We need to prove that $\forall u, v \in \mathcal{S}'(\mathbb{R}^n)$ and $\forall a, b \in \mathbb{C} \quad \partial^\alpha(au + bv) = a\partial^\alpha u + b\partial^\alpha v$:

$$\begin{aligned}&\langle \partial^\alpha(au+bv), \phi \rangle = (-1)^{|\alpha|}\langle au+bv, \partial^\alpha \phi \rangle = (-1)^{|\alpha|}\langle au, \partial^\alpha \phi \rangle + (-1)^{|\alpha|}\langle bv, \partial^\alpha \phi \rangle \\ &= (-1)^{|\alpha|} a\langle u, \partial^\alpha \phi \rangle + (-1)^{|\alpha|} b\langle v, \partial^\alpha \phi \rangle = a\langle \partial^\alpha u, \phi \rangle + b\langle \partial^\alpha v, \phi \rangle = \langle a\partial^\alpha u + b\partial^\alpha v, \phi \rangle.\end{aligned}$$

Let us assume that $u_k \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ when $k \to \infty$. We need to show that for all multiindices α, $\partial^\alpha u_k \to \partial^\alpha u$ in $\mathcal{S}'(\mathbb{R}^n)$. By (1.19) it follows

$$\langle \partial^\alpha u_k - \partial^\alpha u, \phi \rangle = (-1)^{|\alpha|} \langle u_k - u, \partial^\alpha \phi \rangle.$$

Hence $|\langle \partial^\alpha u_k - \partial^\alpha u, \phi \rangle| = |\langle u_k - u, \partial^\alpha \phi \rangle| \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$. Since $u_k \to u$ in $\mathcal{S}'(\mathbb{R}^n)$, $k \to \infty$ it follows that $\partial^\alpha u_k \to \partial^\alpha u$ in $\mathcal{S}'(\mathbb{R}^n)$. So $\partial^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. □

Theorem 1.30. *If $f \in L^2(\mathbb{R}^n)$ then $\hat{f} \in L^2(\mathbb{R}^n)$.*

Proof. Recalling the Frechet-Riesz theorem [Wer05, Theorem V.3.6]:
There exists a unique $\nu \in L^2(\mathbb{R}^n)$ such that $\forall \phi \in L^2(\mathbb{R}^n)$:

$$\langle \hat{f}, \phi \rangle = \int_{\mathbb{R}^n} \phi(x)\overline{\nu}(x)dx.$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$ we obtain $\langle \hat{f}, \phi \rangle = \langle \overline{\nu}, \phi \rangle \Rightarrow \hat{f} = \overline{\nu}$ in $\mathcal{S}'(\mathbb{R}^n)$. Since $\overline{\nu} \in L^2(\mathbb{R}^n) \Rightarrow \hat{f} \in L^2(\mathbb{R}^n)$. □

Now we have collected enough knowledge and information to define Sobolev spaces. In the following section we introduce Sobolev spaces and their properties. These spaces are very useful in partial differential equations.

1.5 Sobolev Spaces

In this section we follow [Rin09, §5.2].
Let $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. We denote by $\lambda = \lambda(\xi) := (1+|\xi|^2)^{\frac{1}{2}}$, where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ and hence $\lambda^s = \lambda^s(\xi) = (1+|\xi|^2)^{\frac{s}{2}}$.

Definition 1.31. Let $s \in \mathbb{R}$. The Sobolev space $H_s(\mathbb{R}^n)$ is defined by

$$H_s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n), \lambda^s \hat{u} \in L^2(\mathbb{R}^n)\} \quad \text{where} \quad \hat{u} = (\mathcal{F}u)(\xi).$$

The space $H_s(\mathbb{R}^n)$ is called Sobolev space of order $s \in \mathbb{R}$ on $\mathbb{R}^n$.

Definition 1.32. For all $u, v \in H_s(\mathbb{R}^n)$, the inner product $(u,v)_{H_s(\mathbb{R}^n)}$ is defined by

$$(u,v)_{H_s(\mathbb{R}^n)} = (u,v)_s := \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \lambda^{2s}(\xi)\hat{u}(\xi)\bar{\hat{v}}(\xi)d\xi, \tag{1.20}$$

where $\hat{u} = \mathcal{F}u$, $\hat{v} = \mathcal{F}v$ and $\bar{\hat{v}}(\xi)$ is the complex conjugate of $\hat{v}(\xi)$.
The corresponding norm is

$$\|u\|_{H_s(\mathbb{R}^n)} = \|u\|_s = (u,u)_s^{\frac{1}{2}} = \left[\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \lambda^{2s}(\xi)|\hat{u}(\xi)|^2 d\xi\right]^{\frac{1}{2}}. \tag{1.21}$$

Note also that $\|u\|_{H_s(\mathbb{R}^n)} = (2\pi)^{\frac{-n}{2}} \|\lambda^s \hat{u}\|_{L^2(\mathbb{R}^n)}$.

Theorem 1.33. *$\forall s \in \mathbb{R}$, the space $H_s(\mathbb{R}^n)$, equipped with the inner product $(.,.)_s$ defined in (1.20) is a Hilbert space.*

Proof. For the proof we need to show that every Cauchy sequence in $H_s(\mathbb{R}^n)$ is convergent. Let (u_k) be any Cauchy sequence in $H_s(\mathbb{R}^n)$:

$$\|u_k - u_m\|_s^2 = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \lambda^{2s} |\hat{u}_k - \hat{u}_m|^2 d\xi \to 0 \quad \text{when} \quad k, m \to \infty.$$

On the other hand,

$$\int\limits_{\mathbb{R}^n} \lambda^{2s} |\hat{u}_k - \hat{u}_m|^2 d\xi = \|\lambda^s(\hat{u}_k - \hat{u}_m)\|_{L^2(\mathbb{R}^n)}^2 \to 0,$$

so $(\lambda^s \hat{u}_k)$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, which is a complete space $\Rightarrow \exists w \in L^2(\mathbb{R}^n)$ such that $\lambda^s \hat{u}_k \to w$ in $L^2(\mathbb{R}^n)$ when $k \to \infty$. From Remark 1.24 it follows that $w \in \mathcal{S}'(\mathbb{R}^n)$ $\Rightarrow \lambda^{-s} w \in \mathcal{S}'(\mathbb{R}^n)$. By Theorem 1.27 $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an isomorphism hence $\exists u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\mathcal{F}(u) = \hat{u} = \lambda^{-s} w \in \mathcal{S}'(\mathbb{R}^n)$. Since $u \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow \lambda^s \hat{u} \in \mathcal{S}'(\mathbb{R}^n)$. But $\lambda^s \hat{u} = w \in L^2(\mathbb{R}^n)$ which means that $u \in H_s(\mathbb{R}^n)$ and $\lambda^s \hat{u}_k \to \lambda^s \hat{u}$ in $L^2(\mathbb{R}^n)$ when $k \longrightarrow \infty$. Finally we have

$$\|u - u_k\|_s^2 = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \lambda^{2s} |\hat{u} - \hat{u}_k|^2 d\xi \to 0 \quad \text{when} \quad k \to \infty.$$

This shows that the Cauchy sequence (u_k) converges to $u \in H_s(\mathbb{R}^n)$, hence $H_s(\mathbb{R}^n)$ is complete space. $\square$

Proposition 1.34. For $s_1 \geq s_2$, $H_{s_1}(\mathbb{R}^n) \hookrightarrow H_{s_2}(\mathbb{R}^n)$ with

$$\|u\|_{s_2} \leq \|u\|_{s_1} \quad \forall u \in H_{s_1}(\mathbb{R}^n).$$

Proof. Let $u \in H_{s_1}(\mathbb{R}^n) \Rightarrow \int\limits_{\mathbb{R}^n} \lambda^{2s_1}(\xi)|\hat{u}(\xi)|^2 d\xi < \infty$.

Since $\lambda(\xi) = (1+|\xi|^2)^{\frac{1}{2}} \geq 1$ and $s_2 - s_1 \leq 0$ then $\lambda^{2(s_2-s_1)}(\xi) \leq 1$. Let us estimate $\|u\|_{s_2}$:

$$\begin{aligned}
\|u\|_{s_2} &= (2\pi)^{\frac{-n}{2}} \left(\int\limits_{\mathbb{R}^n} \lambda^{2s_2}(\xi)|\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = (2\pi)^{\frac{-n}{2}} \left(\int\limits_{\mathbb{R}^n} \lambda^{2(s_2-s_1)}(\xi)\lambda^{2s_1}(\xi)|\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\
&\leq (2\pi)^{\frac{-n}{2}} \left(\int\limits_{\mathbb{R}^n} \lambda^{2s_1}(\xi)|\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \|u\|_{s_1} < \infty.
\end{aligned}$$

Hence $u \in H_{s_2}(\mathbb{R}^n)$. We obtain $H_{s_1}(\mathbb{R}^n) \subset H_{s_2}(\mathbb{R}^n)$ with $\|u\|_{s_2} \leq \|u\|_{s_1} \Rightarrow H_{s_1}(\mathbb{R}^n) \hookrightarrow H_{s_2}(\mathbb{R}^n)$. $\square$

Remark 1.35. For $s \geq 0$, $H_s(\mathbb{R}^n) \hookrightarrow H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $\|u\|_{H^0} = \|u\|_{L^2}$.

For $s \geq 0$ we conclude that:

$$H_s(\mathbb{R}^n) = \{u : u \in L^2(\mathbb{R}^n), \lambda^s \hat{u} \in L^2(\mathbb{R}^n)\}$$

with $(u,v)_s$ and $\|.\|_s$ defined by eq. (1.20) and eq. (1.21) respectively.

Example 1.36. Let us show that the partial differential operator $-\Delta + k^2 : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is an isomorphism for all real $k \neq 0, \forall s \in \mathbb{R}$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the n-dimensional Laplace operator.
Set $A = -\Delta + k^2$. First we prove that for all $u \in H_{s+2}(\mathbb{R}^n)$, $Au = -\Delta u + k^2 u \in H_s(\mathbb{R}^n)$, where $k \in \mathbb{R} \setminus \{0\}, s \in \mathbb{R}$ and the map $A : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is continuous. Let $u \in H_{s+2}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Since $\mathcal{F}[-\Delta u + k^2 u] = (|\xi|^2 + k^2)\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ where $\hat{u} = \mathcal{F}(u)$, it follows that

$$\lambda^s \cdot \mathcal{F}[-\Delta u + k^2 u] = \lambda^s(|\xi|^2 + k^2) \cdot \hat{u}.$$

But

$$\lambda^s \cdot (|\xi|^2 + k^2) \leq \max\{1, k^2\}(1 + |\xi|^2) \cdot \lambda^s = C \cdot \lambda^{s+2}$$

with $C = \max\{1, k^2\} > 0$. Hence,

$$\lambda^s \cdot |\mathcal{F}[-\Delta u + k^2 u]| \leq C \cdot \lambda^{s+2} |\hat{u}|.$$

Since $u \in H_{s+2}(\mathbb{R}^n)$ then $\lambda^{s+2} \cdot \hat{u} \in L^2(\mathbb{R}^n) \Rightarrow \lambda^s \cdot \mathcal{F}[-\Delta u + k^2 u] \in L^2(\mathbb{R}^n) \Rightarrow -\Delta u + k^2 u \in H_s(\mathbb{R}^n)$. Moreover,

$$\| - \Delta u + k^2 u\|_s^2 = (2\pi)^{-n} \|\lambda^s \cdot \mathcal{F}[-\Delta u + k^2 u]\|_{L^2}^2 \leq (2\pi)^{-n} C^2 \|\lambda^{s+2} \hat{u}\|_{L^2}^2.$$

We obtain $\| -\Delta u + k^2 u\|_s \leq C\|u\|_{s+2}$ with $C > 0$. Hence $A : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is continuous.
Let us prove that $A : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is injective. For $u \in H_{s+2}(\mathbb{R}^n)$, $Au = -\Delta u + k^2 u = 0$ in $H_s(\mathbb{R}^n) \Rightarrow (|\xi|^2 + k^2)\hat{u} = 0$ in $\mathcal{S}'(\mathbb{R}^n)$. Since $|\xi|^2 + k^2 \neq 0 \Rightarrow \hat{u} = 0$ in $\mathcal{S}'(\mathbb{R}^n) \Rightarrow u = \mathcal{F}^{-1}\hat{u} = 0$ in $\mathcal{S}'(\mathbb{R}^n) \Rightarrow u = 0$ in $H_{s+2}(\mathbb{R}^n)$.
Now we need to prove that $A : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is surjective. Let us take any $f \in H_s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \Rightarrow \hat{f} \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow (|\xi|^2 + k^2)^{-1}\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$.
Set $v = (|\xi|^2 + k^2)^{-1}\hat{f} \in \mathcal{S}'(\mathbb{R}^n)$, then

$$f = \mathcal{F}^{-1}\hat{f} = \mathcal{F}^{-1}(|\xi|^2 + k^2)v = (-\Delta + k^2)\mathcal{F}^{-1}v.$$

Define $u = \mathcal{F}^{-1}v \in \mathcal{S}'(\mathbb{R}^n)$. Then $(-\Delta + k^2)u = f \in H_s(\mathbb{R}^n)$ and $(|\xi|^2 + k^2)\hat{u} = \hat{f}$. Let us estimate $\lambda^{s+2}\hat{u}$:

$$\lambda^{s+2}\hat{u} = \frac{\lambda^{s+2}}{|\xi|^2 + k^2}|\hat{f}| = \frac{\lambda^2}{|\xi|^2 + k^2}\lambda^s|\hat{f}| \leq C_1\lambda^s|\hat{f}|.$$

Since $|\xi|^2 + k^2 \geq \min\{1, k^2\}(1 + |\xi|^2)$ it follows that $\dfrac{1}{|\xi|^2 + k^2} \leq C_1 \dfrac{1}{1 + |\xi|^2}$ where $C_1 = \dfrac{1}{\min\{1, k^2\}} > 0$.
Hence

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \lambda^{2(s+2)}|\hat{u}|^2 d\xi \leq \frac{1}{(2\pi)^n}C_1^2 \int\limits_{\mathbb{R}^n} \lambda^{2s}|\hat{f}|^2 d\xi < \infty$$

then $u \in H_{s+2}(\mathbb{R}^n)$. Thus , $\forall f \in H_s(\mathbb{R}^n)\ \exists u \in H_{s+2}(\mathbb{R}^n)$ such that $Au = -\Delta u + k^2 u = f$ in $H_s(\mathbb{R}^n)$ it follows that $-\Delta + k^2$ is surjective from $H_{s+2}(\mathbb{R}^n)$ onto $H_s(\mathbb{R}^n)$. Hence $-\Delta + k^2 : H_{s+2}(\mathbb{R}^n) \longrightarrow H_s(\mathbb{R}^n)$ is a continuous, linear bijective map. Since $H_{s+2}(\mathbb{R}^n)$ and $H_s(\mathbb{R}^n)$ are Banach spaces it follows that the inverse map is also continuous.
So $-\Delta + k^2 : H_{s+2}(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is an isomorphism.

Definition 1.37. Let $u \in H_s(\mathbb{R}^n)$ and let t be a real number. We define a temperate distribution $(1 - \Delta)^t u$ whose Fourier transform is given by $(1 + |\xi|^2)^t \hat{u}(\xi)$.

Let us show that $(1 - \Delta)^t u$ is in $H_{s-2t}(\mathbb{R}^n)$ for $s, t \in \mathbb{R}$:

$$(1 + |\xi|^2)^{\frac{s-2t}{2}} \mathcal{F}\left[(1 - \Delta)^t u\right] = (1 + |\xi|^2)^{\frac{s-2t}{2}}(1 + |\xi|^2)^t \hat{u} = (1 + |\xi|^2)^{\frac{s}{2}}\hat{u} = \lambda^s \hat{u},$$

since $u \in H_s(\mathbb{R}^n)$ it follows that $(1 + |\xi|^2)^{\frac{s-2t}{2}} \mathcal{F}\left[(1 - \Delta)^t u\right] = \lambda^s \hat{u} \in L^2(\mathbb{R}^n)$. Hence $(1 - \Delta)^t u \in H_{s-2t}(\mathbb{R}^n)$. Let us prove that, for $t \in \mathbb{R}$

$$\|(1 - \Delta)^{t/2} u\|_{s-t} = \|u\|_s. \tag{1.22}$$

By the definition of the norm on $H_s(\mathbb{R}^n)$:

$$\|(1 - \Delta)^{t/2} u\|_{s-t}^2 = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \lambda^{2(s-t)}(\xi)|\mathcal{F}((1 - \Delta)^{t/2} u)|^2 d\xi =$$

$$\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} (1 + |\xi|^2)^{s-t}(1 + |\xi|^2)^t |\hat{u}|^2 d\xi = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \lambda^{2s}|\hat{u}|^2 d\xi = \|u\|_s^2.$$

From the above equation it follows that $(1-\Delta)^{t/2}$ is an homeomorphism from $H_s(\mathbb{R}^n)$ to $H_{s-t}(\mathbb{R}^n)$.

Remark 1.38. $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$ for any real s.

Proof. Assume that $u \in \mathcal{S}(\mathbb{R}^n)$. First let us show that $(1-\Delta)^t : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a homeomorphism:

$$(1-\Delta)^t u = \mathcal{F}^{-1}\left[(1+|\xi|^2)^t \hat{u}(\xi)\right].$$

Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a homeomorphism then $\hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$ and that $(1+|\xi|^2)^t \hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. So $\mathcal{F}^{-1}\left[(1+|\xi|^2)^t \hat{u}(\xi)\right] \in \mathcal{S}(\mathbb{R}^n)$. Consequently $(1-\Delta)^t : \mathcal{S} \to \mathcal{S}$ is a homeomorphism. By (1.22) we obtain that $(1-\Delta)^{-s/2}$ is an isometric map from $H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n, \mathbb{C})$ to $H_{0-(-s)}(\mathbb{R}^n) = H_s(\mathbb{R}^n)$. From Lemma 1.15 it follows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \mathbb{C})$. Since $(1-\Delta)^{-s/2}$ maps homeomorphically $\mathcal{S}(\mathbb{R}^n)$ into itself and $L^2(\mathbb{R}^n, \mathbb{C})$ into $H_s(\mathbb{R}^n)$ then $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$ for any real s. □

Let us mention that if $u \in \mathcal{S}(\mathbb{R}^n)$ and k is a positive integer, then $(1-\Delta)^k u$ can be interpreted in two ways. In one way, we interpret it as above and another way we interpret it as a differential operator acting on u, where Δ is the standard Laplacian. Let us prove the following lemma whose result is a very useful tool.

Lemma 1.39. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}\bar{\hat{g}} d\xi = \int_{\mathbb{R}^n} f\bar{g} dx. \tag{1.23}$$

Proof. First let us show that

$$\int_{\mathbb{R}^n} \hat{f} h dx = \int_{\mathbb{R}^n} f \hat{h} dx \quad \text{for} \quad f, h \in \mathcal{S}(\mathbb{R}^n). \tag{1.24}$$

From the definition of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$,

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{-i\xi\cdot x} d\xi.$$

Since $f, h \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ we can apply Fubini's theorem:

$$\int_{\mathbb{R}^n} \hat{f}(x)h(x)dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(\xi)e^{-i\xi\cdot x} d\xi\right) h(x)dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)h(x)e^{-i\xi\cdot x} d\xi dx =$$
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)h(x)e^{-i\xi\cdot x} dx d\xi = \int_{\mathbb{R}^n} f(\xi) \int_{\mathbb{R}^n} h(x)e^{-i\xi\cdot x} dx d\xi = \int_{\mathbb{R}^n} f(\xi)\hat{h}(\xi)d\xi.$$

So

$$\int_{\mathbb{R}^n} \hat{f} h dx = \int_{\mathbb{R}^n} f \hat{h} dx.$$

Take

$$h(x) = (2\pi)^{-n}\bar{\hat{g}}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \bar{g}(\xi) e^{i\xi\cdot x} d\xi,$$

then from the Fourier inversion formula we have $\hat{h} = \bar{g}$. By applying (1.24) we obtain

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi)\bar{\hat{g}}(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx. \tag{1.25}$$

$\square$

As a consequence of this Lemma we have the following Remark.

Remark 1.40. Let $u \in \mathcal{S}(\mathbb{R}^n)$ and $f = g = \partial^\alpha u$ in (1.23). Since

$$\widehat{\partial^\alpha u} = i^{|\alpha|}\xi^\alpha \hat{u}$$

according to (1.23) we have

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\partial^\alpha u(x)|^2 dx. \tag{1.26}$$

Remark 1.41. Any element g of the dual of $L^2(\mathbb{R}^n, \mathbb{C})$ is given by

$$g(\rho) = f[(1-\Delta)^{-s/2}\rho] \quad \text{where} \quad \rho \in L^2(\mathbb{R}^n, \mathbb{C}), \tag{1.27}$$

and f is in the dual of $H_s(\mathbb{R}^n)$.

Proof. Due to Theorem 6.16 of [Rud87] and identity (1.23), for any $g \in (L^2(\mathbb{R}^n, \mathbb{C}))'$ there is a $\chi \in L^2(\mathbb{R}^n, \mathbb{C})$ such that

$$g(\rho) = \int_{\mathbb{R}^n} \hat{\rho}\bar{\hat{\chi}} d\xi \quad \text{for all} \quad \rho \in L^2(\mathbb{R}^n, \mathbb{C}).$$

Let us define $\phi := (1-\Delta)^{s/2}\chi$ then $\hat{\phi} = (1+|\xi|^2)^{s/2}\hat{\chi}$. Since $\chi \in L^2(\mathbb{R}^n, \mathbb{C})$ then $\phi \in H_{-s}(\mathbb{R}^n)$. Now define $f \in (H_s(\mathbb{R}^n))'$ by

$$f(\psi) = \int_{\mathbb{R}^n} \hat{\psi}\bar{\hat{\phi}} d\xi = \int_{\mathbb{R}^n} \hat{\psi}(1+|\xi|^2)^{s/2}\bar{\hat{\chi}} d\xi \quad \text{for all} \quad \psi \in H_s(\mathbb{R}^n). \tag{1.28}$$

Taking $\rho = (1-\Delta)^{s/2}\psi \in L^2(\mathbb{R}^n, \mathbb{C})$ we get $\hat{\rho} = (1+|\xi|^2)^{s/2}\hat{\psi}$. From (1.28) we obtain

$$g(\rho) = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s/2}\hat{\psi}\bar{\hat{\chi}} d\xi = f(\psi) \quad \text{where} \quad \psi \in H_s(\mathbb{R}^n).$$

So $g(\rho) = f[(1-\Delta)^{-s/2}\rho] \quad \text{where} \quad \rho \in L^2(\mathbb{R}^n, \mathbb{C})$. $\square$

Let us analyze the relation between the spaces $H_k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n, \mathbb{C})$ for $k \in \mathbb{N}$.

Theorem 1.42. *For k non-negative integer the spaces $H_k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n, \mathbb{C})$ coincide:*

$$H_k(\mathbb{R}^n) \equiv H^k(\mathbb{R}^n, \mathbb{C})$$

and their norms are equivalent :
There are constants $C_{i,k} > 0, i = 1, 2,$ such that for all $u \in H_k(\mathbb{R}^n)$,

$$C_{1,k}\|u\|_k \leq \|u\|_{H^k} \leq C_{2,k}\|u\|_k$$

Proof. Recall the definitions of spaces $H_k(\mathbb{R}^n)$ and $H^k(\mathbb{R}^n, \mathbb{C})$

$$H_k(\mathbb{R}^n) = \{u : u \in L^2(\mathbb{R}^n), \lambda^k \hat{u} \in L^2(\mathbb{R}^n)\},$$
$$H^k(\mathbb{R}^n, \mathbb{C}) = \{u : u \in L^2(\mathbb{R}^n), \partial^\alpha u \in L^2(\mathbb{R}^n, \mathbb{C}) \quad \forall |\alpha| \leq k\}.$$

According to (1.26)

$$\int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \quad \text{for} \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Summing this equality for $|\alpha| \leq k$ we get

$$\sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx = \frac{1}{(2\pi)^n} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \quad \text{for} \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{1.29}$$

By Remark 1.38 $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_k(\mathbb{R}^n)$. Let us take $u \in H_k(\mathbb{R}^n)$ then

$$\|u\|_k^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 d\xi < \infty.$$

Let us mention that there are constant $c_{i,k} > 0, i = 1, 2$ such that

$$c_{1,k}(1 + |\xi|^2)^k \leq \sum_{|\alpha|\leq k} \xi^{2\alpha} \leq c_{2,k}(1 + |\xi|^2)^k. \tag{1.30}$$

Using this inequality and (1.29) we can estimate $\|u\|_{H^k}$:

$$\|u\|_{H^k}^2 = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx = \frac{1}{(2\pi)^n} \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi \leq$$
$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} c_{2,k}(1 + |\xi|^2)^k |\hat{u}(\xi)|^2 d\xi = c_{2,k}\|u\|_k^2 < \infty,$$

hence $u \in H^k(\mathbb{R}^n, \mathbb{C})$ and $\|u\|_{H^k} \leq \sqrt{c_{2,k}}\|u\|_k$. So we conclude that $H_k(\mathbb{R}^n) \subseteq H^k(\mathbb{R}^n, \mathbb{C})$. Let us prove the converse $H^k(\mathbb{R}^n, \mathbb{C}) \subseteq H_k(\mathbb{R}^n)$. By Lemma 1.15 $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n, \mathbb{C})$. Assume that $u \in H^k(\mathbb{R}^n, \mathbb{C})$ then

$$\|u\|_{H^k}^2 = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx < \infty. \tag{1.31}$$

Now we need to estimate $\|u\|_k$. Again we apply the left hand side of (1.30) and use (1.29)

$$\begin{aligned}\|u\|_k^2 = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^k|\hat{u}(\xi)|^2 d\xi \leq \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\sum_{|\alpha|\leq k} c_{1,k}^{-1}\xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi = \\ c_{1,k}^{-1}\frac{1}{(2\pi)^n}\sum_{|\alpha|\leq k}\int\limits_{\mathbb{R}^n}\xi^{2\alpha}|\hat{u}(\xi)|^2 d\xi = c_{1,k}^{-1}\sum_{|\alpha|\leq k}\int\limits_{\mathbb{R}^n}|\partial^\alpha u|^2 dx = c_{1,k}^{-1}\|u\|_{H^k}^2 < \infty,\end{aligned} \tag{1.32}$$

hence $u \in H_k(\mathbb{R}^n)$ and $\sqrt{c_{1,k}}\|u\|_k \leq \|u\|_{H^k}$. So we conclude that $H^k(\mathbb{R}^n,\mathbb{C}) \subseteq H_k(\mathbb{R}^n)$. Finally we get that $H^k(\mathbb{R}^n,\mathbb{C}) \equiv H_k(\mathbb{R}^n)$ and their norms are equivalent. There are positive constants $C_{i,k}$ $i = 1,2$ such that for all $u \in H_k(\mathbb{R}^n)$

$$C_{1,k}\|u\|_k \leq \|u\|_{H^k} \leq C_{2,k}\|u\|_k.$$

□

Lemma 1.43. If α is a multiindex and $s \in \mathbb{R}$ then

$$\|\partial^\alpha f\|_{s-|\alpha|} \leq C\|f\|_s \text{ for all } f \in \mathcal{S}(\mathbb{R}^n), \tag{1.33}$$

where C is a constant, which depends on α and s. Hence ∂^α is a bounded linear operator from $H_s(\mathbb{R}^n)$ to $H_{s-|\alpha|}(\mathbb{R}^n)$.

Proof. Due to Remark 1.38 $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$. Let us compute $\|\partial^\alpha f\|_{s-|\alpha|}$:

$$\|\partial^\alpha f\|_{s-|\alpha|}^2 = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^{s-|\alpha|}|i^{|\alpha|}\xi^\alpha\hat{f}(\xi)|^2 \leq C\|f\|_s^2.$$

Here we have used the inequality

$$|\xi^\alpha|^2(1+|\xi|^2)^{s-|\alpha|} \leq C(1+|\xi|^2)^s,$$

where C is a constant. Hence $\partial^\alpha f \in H_{s-|\alpha|}(\mathbb{R}^n)$. The operator $\partial^\alpha : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is bounded and linear. We can conclude that ∂^α can be extended to be a linear bounded operator from $H_s(\mathbb{R}^n)$ to $H_{s-|\alpha|}(\mathbb{R}^n)$. □

Lemma 1.44. Let $u, v \in H_s(\mathbb{R}^n)$ and let α be a multiindex with $|\alpha| \leq s$. Then u and v are in $W^{s,2}(\mathbb{R}^n)$ and

$$(u, \partial^\alpha v)_{L^2} = (-1)^{|\alpha|}(\partial^\alpha u, v)_{L^2}, \tag{1.34}$$

where

$$(u,v)_{L^2} = \int\limits_{\mathbb{R}^n} u\bar{v}dx.$$

Moreover, if $u, v \in H_s(\mathbb{R}^n)$, $s \geq 0$ and $t \leq s$ then $(1-\Delta)^{t/2}u, (1-\Delta)^{t/2}v \in L^2(\mathbb{R}^n,\mathbb{C})$ and

$$((1-\Delta)^{t/2}u, v)_{L^2} = (u, (1-\Delta)^{t/2}v)_{L^2}. \tag{1.35}$$

Proof. By Theorem 1.42 for any non negative integer s the spaces $H_s(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ coincide. Since $u, v \in H_s(\mathbb{R}^n)$ it follows that $\partial^\alpha v, \partial^\alpha u \in L^2(\mathbb{R}^n, \mathbb{C})$. Let us take $u, v \in \mathcal{S}(\mathbb{R}^n)$ and estimate $(u, \partial^\alpha v)_{L^2}$. From (1.23) we get:

$$(u, \partial^\alpha v)_{L^2} = \int\limits_{\mathbb{R}^n} u\overline{\partial^\alpha v}dx = (2\pi)^{-n}\int\limits_{\mathbb{R}^n} \hat{u}\overline{\widehat{\partial^\alpha v}}d\xi. \tag{1.36}$$

We need to compute $\overline{\widehat{\partial^\alpha v}}$:

$$\overline{\widehat{\partial^\alpha v}} = \overline{(i\xi)^\alpha \cdot \widehat{v}} = \overline{(i\xi)^\alpha} \cdot \overline{\widehat{v}} = (-i\xi)^\alpha \cdot \overline{\widehat{v}} = (-1)^{|\alpha|}(i\xi)^\alpha \overline{\widehat{v}}.$$

Inserting the above result in (1.36),we obtain

$$(u, \partial^\alpha v)_{L^2} = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{u}(-1)^{|\alpha|}(i\xi)^\alpha \overline{\widehat{v}}d\xi = (-1)^{|\alpha|}(2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{\partial^\alpha u}\overline{\widehat{v}}d\xi$$
$$= (-1)^{|\alpha|}\int_{\mathbb{R}^n} \partial^\alpha u(x)\overline{v}(x)dx = (-1)^{|\alpha|}(\partial^\alpha u, v)_{L^2}.$$

Let us prove that $(1-\Delta)^{t/2}u \in L^2(\mathbb{R}^n, \mathbb{C})$. Since $u \in H_s(\mathbb{R}^n)$ then $(1-\Delta)^{t/2}$ maps $H_s(\mathbb{R}^n)$ into $H_{s-t}(\mathbb{R}^n)$. By assumption $s - t \geq 0$ so $H_{s-t}(\mathbb{R}^n) \subseteq H_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, which means that $(1-\Delta)^{t/2}u \in L^2(\mathbb{R}^n)$. Now let us check that $((1-\Delta)^{t/2}u, v)_{L^2} = (u, (1-\Delta)^{t/2}v)_{L^2}$. From (1.23) we get for $u, v \in \mathcal{S}(\mathbb{R}^n)$:

$$((1-\Delta)^{t/2}u, v)_{L^2} = \int_{\mathbb{R}^n} ((1-\Delta)^{t/2}u\overline{v}dx = (2\pi)^{-n}\int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{t}{2}}\hat{u}\overline{\hat{v}}d\xi =$$
$$\int_{\mathbb{R}^n} u\overline{(1-\Delta)^{t/2}v}dx = (u, (1-\Delta)^{t/2}v)_{L^2}.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$, then all results follow in general. □

Lemma 1.45. Assume $u \in \mathcal{S}(\mathbb{R}^n)$ and assume $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ with all derivatives bounded. Then

$$\|\phi u\|_k \leq C\|u\|_k, \tag{1.37}$$

where C is a constant , which depends on k and the sup norm of up to $|k|$ derivatives of ϕ.

Proof. Due to Theorem 1.42 for any $k \geq 0$ integer, the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_k$ are equivalent. By assumption $|\partial^k\phi| \leq M$. We obtain

$$\|\phi u\|_k \leq C_1\|\phi u\|_{H^k} = C_1\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n} |\partial^\alpha(\phi u)|^2 dx\right)^{\frac{1}{2}} =$$

$$C_1\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n} |\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}\phi\partial^\beta u|^2 dx\right)^{\frac{1}{2}} \leq C_1\cdot M\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n} |\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^\beta u|^2 dx\right)^{\frac{1}{2}} \leq$$

$$\tilde{C}_1\cdot M\left(\sum_{|\alpha|\leq k}\int_{\mathbb{R}^n} |\partial^\alpha u|^2 dx\right)^{\frac{1}{2}} = \tilde{C}_1\cdot M\|u\|_{H^k} \leq C_2\cdot\tilde{C}_1\cdot M\|u\|_k = C\|u\|_k.$$

Let us prove (1.37) for negative k. First we need to show that for $u, v \in \mathcal{S}(\mathbb{R}^n)$

$$\|u\|_k = \sup_{v\in\mathcal{A}_k} \left| \int\limits_{\mathbb{R}^n} u\bar{v}dx \right|, \text{ where } \mathcal{A}_k = \{v \in \mathcal{S}(\mathbb{R}^n) : \|v\|_{-k} \le 1\}.$$

By (1.23) for $u, v \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\int\limits_{\mathbb{R}^n} u\bar{v}dx = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n} \hat{u}\bar{\hat{v}}d\xi = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^{k/2}\hat{u}(1+|\xi|^2)^{-k/2}\bar{\hat{v}}d\xi. \tag{1.38}$$

Let us apply the Cauchy-Schwarz inequality to the above equality:

$$\left|\int\limits_{\mathbb{R}^n} u\bar{v}dx\right| \le \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}|\hat{u}|(1+|\xi|^2)^{k/2}|\bar{\hat{v}}|(1+|\xi|^2)^{-k/2}d\xi \le$$

$$\left(\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}|\hat{u}|^2(1+|\xi|^2)^{k}d\xi\right)^{\frac{1}{2}}\left(\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}|\bar{\hat{v}}|^2(1+|\xi|^2)^{-k}d\xi\right)^{\frac{1}{2}} = \|u\|_k\|\bar{v}\|_{-k}.$$

Taking the supremum over $v \in \mathcal{A}_k$, in the above inequality we get:

$$\sup_{v\in\mathcal{A}_k}\left|\int\limits_{\mathbb{R}^n} u\bar{v}dx\right| \le \|u\|_k.$$

Now we can choose $v \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\hat{v}(\xi) = (1+|\xi|^2)^k\hat{u}(\xi)\|u\|_k^{-1},$$

where $\|u\|_k \ne 0$. Then,

$$\|v\|_{-k} = \left(\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^{-k}(1+|\xi|^2)^{2k}|\hat{u}(\xi)|^2\|u\|_k^{-2}d\xi\right)^{\frac{1}{2}} = \left(\|u\|_k^{-2}\cdot\|u\|_k^2\right)^{\frac{1}{2}} = 1.$$

Inserting this v in (1.38), we obtain

$$\int\limits_{\mathbb{R}^n} u\bar{v}dx = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\hat{u}(\xi)(1+|\xi|^2)^k\bar{\hat{u}}(\xi)\|u\|_k^{-1}d\xi = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}(1+|\xi|^2)^k|\hat{u}(\xi)|^2\|u\|_k^{-1}d\xi =$$
$$\|u\|_k^{-1}\cdot\|u\|_k^2 = \|u\|_k.$$

Consequently, for $u \ne 0$ we obtain

$$\sup_{v\in\mathcal{A}_k}\left|\int\limits_{\mathbb{R}^n} u\bar{v}dx\right| = \|u\|_k.$$

It is clear that this equality holds for $u = 0$. Let us compute the following:

$$\left|\int\limits_{\mathbb{R}^n} \phi u \bar{v} dx\right| = \left|\int\limits_{\mathbb{R}^n} u \overline{\bar{\phi} v} dx\right| = \left|\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \hat{u}(1+|\xi|^2)^{k/2} \overline{\widehat{\bar{\phi} v}}(1+|\xi|^2)^{-k/2} d\xi\right| \leq$$
$$\left(\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} |\hat{u}|^2 (1+|\xi|^2)^k d\xi\right)^{\frac{1}{2}} \left(\frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} |\widehat{\bar{\phi}\, v}|^2 (1+|\xi|^2)^{-k} d\xi\right)^{\frac{1}{2}} =$$
$$\|u\|_k \|\bar{\phi} v\|_{-k} \leq C\|u\|_k \|v\|_{-k}.$$

In the third step we have used the Cauchy-Schwarz inequality and inequality (1.37) for k non-negative in the last step. Let us take the supremum over $v \in \mathcal{A}_k$, then

$$\|\phi u\|_k = \sup_{v \in \mathcal{A}_k} \left|\int\limits_{\mathbb{R}^n} \phi u \bar{v} dx\right| \leq C\|u\|_k. \tag{1.39}$$

□

Corollary 1.46. Let $u \in \mathcal{S}(\mathbb{R}^n)$ and let $f \in C^\infty(\mathbb{R}^n, C)$ with all derivatives bounded. Assume m and l be non-negative integers, α be a multiindex with $|\alpha| \leq l + m$ then

$$\|f \partial^\alpha u\|_{-m} \leq C\|u\|_l,$$

where C depends on m, l and a bound of $\partial^\alpha f$ for $|\alpha| \leq m$.

Proof. According to Lemma 1.45 and Lemma 1.43

$$\|f \partial^\alpha u\|_{-m} \leq C_1 \|\partial^\alpha u\|_{-m} = C_1 \|\partial^\alpha u\|_{-m+|\alpha|-|\alpha|} \leq C_1 \cdot C_2 \|u\|_{-m+|\alpha|}.$$

Since $|\alpha| - m \leq l$ then by Proposition 1.34

$$\|u\|_{-m+|\alpha|} \leq \|u\|_l.$$

Finally we get

$$\|f \partial^\alpha u\|_{-m} \leq C\|u\|_l, \quad \text{where} \quad C = C_1 \cdot C_2.$$

□

Lemma 1.47. Let $s_1, s_2, s_3 \in \mathbb{R}$ with $s_1 < s_2 < s_3$ and let $u \in H_{s_3}(\mathbb{R}^n)$. If $a, b \in (0,1)$ are such that $a + b = 1$ and a is small enough then

$$\|u\|_{s_2} \leq \|u\|_{s_1}^a \cdot \|u\|_{s_3}^b. \tag{1.40}$$

Actually,

$$\|u\|_{s_2} \leq \|u\|_{s_1}^{\frac{s_3 - s_2}{s_3 - s_1}} \cdot \|u\|_{s_3}^{\frac{s_2 - s_1}{s_3 - s_1}}.$$

Proof. By our assumption $a, b \in (0,1)$ are such that $a+b=1$. Let us set $s_2 = ts_1 + (1-t)s_3$ then by Hölder's inequality we obtain

$$\|u\|_{s_2}^2 = \int\limits_{\mathbb{R}^n} (1+|\xi|^2)^{s_2} |\hat{u}(\xi)|^2 d\xi = \int\limits_{\mathbb{R}^n} [(1+|\xi|^2)^{s_1} |\hat{u}(\xi)|^2]^t [(1+|\xi|^2)^{s_3} |\hat{u}(\xi)|^2]^{1-t} d\xi \leq$$

$$\left(\int\limits_{\mathbb{R}^n} [(1+|\xi|^2)^{s_1} |\hat{u}(\xi)|^2]^{t \cdot \frac{1}{a}} d\xi \right)^a \cdot \left(\int\limits_{\mathbb{R}^n} [(1+|\xi|^2)^{s_3} |\hat{u}(\xi)|^2]^{(1-t) \cdot \frac{1}{b}} d\xi \right)^b =$$

$$\left(\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^{s_1} |\hat{u}(\xi)|^2 d\xi \right)^a \cdot \left(\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^{s_3} |\hat{u}(\xi)|^2 d\xi \right)^b = \|u\|_{s_1}^{2a} \cdot \|u\|_{s_3}^{2b}.$$

Here we take $a = t$ and $b = 1-t$ and the result follows

$$\|u\|_{s_2} \leq \|u\|_{s_1}^a \cdot \|u\|_{s_3}^b.$$

Since $t = \dfrac{s_3 - s_2}{s_3 - s_1}$ and $1 - t = \dfrac{s_2 - s_1}{s_3 - s_1}$ then we have

$$\|u\|_{s_2} \leq \|u\|_{s_1}^{\frac{s_3-s_2}{s_3-s_1}} \cdot \|u\|_{s_3}^{\frac{s_2-s_1}{s_3-s_1}}.$$

□

Let us consider situations where we can apply this lemma. Let $\{u_l\}_{l\geq 1}$ be a bounded sequence in $H_{s_3}(\mathbb{R}^n)$ and let $\{u_l\}_{l\geq 1}$ is a Cauchy sequence in $H_{s_1}(\mathbb{R}^n)$ with $s_1 < s_3$. By (1.40) we can prove that $\{u_l\}_{l\geq 1}$ is a Cauchy sequence with respect to any norm $\|\cdot\|_{s_2}$ such that $s_1 < s_2 < s_3$.

$$\|u_l - u_m\|_{s_2} \leq \|u_l - u_m\|_{s_1}^a \cdot \|u_l - u_m\|_{s_3}^b. \tag{1.41}$$

The sequence $\{u_l\}_{l\geq 1}$ is bounded in $H_{s_3}(\mathbb{R}^n)$ so that $\|u_l - u_m\|_{s_3}^b \leq M$. Since $\{u_l\}_{l\geq 1}$ is a Cauchy sequence in $H_{s_1}(\mathbb{R}^n)$ then $\forall \varepsilon > 0 \quad \exists\, N$ such that for $l, m > N$ $\|u_l - u_m\|_{s_1} < \left(\dfrac{\varepsilon}{M}\right)^{\frac{1}{a}}$. So

$$\|u_l - u_m\|_{s_2} < \varepsilon.$$

Due to Fatou's lemma the limit u is in $H_{s_3}(\mathbb{R}^n)$:

$$\|u\|_{s_3}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^{s_3} |\hat{u}|^2 d\xi \leq \liminf_{l\to\infty} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s_3} |\hat{u}_l|^2 d\xi < \infty.$$

Let us mention that the sequence $\{u_l\}_{l\geq 1}$ converges to u with respect to the weak topology on $H_{s_3}(\mathbb{R}^n)$. If f is an element of the dual of $H_{s_3}(\mathbb{R}^n)$ then there is a $\phi \in H_{-s_3}(\mathbb{R}^n)$ such that

$$f(v) = \int\limits_{\mathbb{R}^n} \hat{v} \bar{\hat{\phi}} d\xi$$

for all $v \in H_{s_3}(\mathbb{R}^n)$. Let the sequence $\{\phi_m\}_{m\geq 1} \in \mathcal{S}(\mathbb{R}^n)$ converges to ϕ with respect to the norm $\|\cdot\|_{-s_3}$. Then

$$f(u_l) - f(u) = \int_{\mathbb{R}^n} \hat{u}_l(\hat{\bar{\phi}} - \hat{\bar{\phi}}_m)d\xi + \int_{\mathbb{R}^n} (\hat{u}_l - \hat{u})\hat{\bar{\phi}}_m d\xi + \int_{\mathbb{R}^n} \hat{u}(\hat{\bar{\phi}}_m - \hat{\bar{\phi}})d\xi.$$

Due to (1.23) we have

$$f(u_l) - f(u) = (2\pi)^n \left[\int\limits_{\mathbb{R}^n} u_l(\bar{\phi} - \bar{\phi}_m)dx + \int\limits_{\mathbb{R}^n} (u_l - u)\bar{\phi}_m dx + \int\limits_{\mathbb{R}^n} u(\bar{\phi}_m - \bar{\phi})dx\right].$$

For given $\varepsilon > 0$, we fix m, independently of l, so that $\int\limits_{\mathbb{R}^n} |\bar{\phi} - \bar{\phi}_m|dx < \dfrac{\varepsilon}{3C \cdot (2\pi)^n}$ where $C = \max\{|u_l|, |u|, |\bar{\phi}_m|\}$. For this fixed m, we can choose l large enough such that $\int\limits_{\mathbb{R}^n} |u_l - u|dx < \dfrac{\varepsilon}{3C \cdot (2\pi)^n}$. Then

$$|f(u_l) - f(u)| \leq (2\pi)^n \left[\int\limits_{\mathbb{R}^n} |u_l||\bar{\phi} - \bar{\phi}_m|dx + \int\limits_{\mathbb{R}^n} |u_l - u||\bar{\phi}_m|dx + \int\limits_{\mathbb{R}^n} |u||\bar{\phi}_m - \bar{\phi}|dx\right] <$$
$$(2\pi)^n \left[\frac{\varepsilon \cdot C}{3C \cdot (2\pi)^n} + \frac{\varepsilon \cdot C}{3C \cdot (2\pi)^n} + \frac{\varepsilon \cdot C}{3C \cdot (2\pi)^n}\right] = \varepsilon$$

Consequently the sequence $\{u_l\}_{l\geq 1}$ converges to u with respect to the weak topology on $H_{s_3}(\mathbb{R}^n)$.

1.6 Dualities

In this section we introduce the duality of Sobolev spaces and follow [Rin09, §5.4] and lecture note [HS09]. We start with the definition of bilinear form $\langle \cdot, \cdot \rangle_{H_{-s}, H_s}$.

Remark 1.48. Assume $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and refer to φ as a regular temperate distribution. Then we obtain

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}^n} \varphi(x)\psi(x)dx = (\psi, \bar{\varphi})_{L^2} = (2\pi)^{-n}(\hat{\psi}, \hat{\bar{\varphi}})_{L^2} = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\psi}(\xi)\bar{\hat{\bar{\varphi}}}(\xi)d\xi =$$
$$(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\psi}(\xi)\hat{\varphi}(-\xi)d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-s/2}\hat{\psi}(\xi)(1 + |\xi|^2)^{s/2}\hat{\varphi}(-\xi)d\xi,$$

using the identity (1.23) and the fact that $\bar{\hat{\bar{\varphi}}}(\xi) = \hat{\varphi}(-\xi)$. Let us prove the last identity:

$$\hat{\bar{\varphi}}(\xi) = \int_{\mathbb{R}^n} \bar{\varphi}(x)e^{-i\xi\cdot x}dx.$$

Consequently,

$$\bar{\hat{\bar{\varphi}}}(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-i(-\xi)\cdot x}dx = \hat{\varphi}(-\xi).$$

According to the Cauchy-Schwarz inequality:

$$|\langle \varphi, \psi \rangle| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |(1+|\xi|^2)^{-s/2}\hat{\psi}(\xi)(1+|\xi|^2)^{s/2}\hat{\varphi}(-\xi)|d\xi \leq$$
$$(2\pi)^{-n} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s}|\hat{\psi}(\xi)|^2 d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s}|\hat{\varphi}(\xi)|^2 d\xi \right)^{1/2} = \|\psi\|_{-s}\|\varphi\|_s. \tag{1.42}$$

Due to Remark 1.38 the Schwarz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_s(\mathbb{R}^n)$ for all s. Consequently we can extend the map

$$(\varphi, \psi) \longmapsto \langle \varphi, \psi \rangle$$

uniquely to a bilinear map $H_{-s}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \longrightarrow \mathbb{C}$, which we write as

$$(u, v) \longmapsto \langle u, v \rangle := (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{u}(\xi)\hat{v}(-\xi)d\xi \quad \text{for} \quad u \in H_{-s}(\mathbb{R}^n), v \in H_s(\mathbb{R}^n). \tag{1.43}$$

By (1.42) we obtain

$$|\langle u, v \rangle_{H_{-s}, H_s}| \leq \|u\|_{-s}\|v\|_s. \tag{1.44}$$

So $\langle u, v \rangle_{H_{-s}, H_s} : H_{-s}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is a continuous bilinear map.

Theorem 1.49. *The bilinear form $\langle \cdot, \cdot \rangle_{H_{-s}, H_s}$ of (1.43) produces an isometric isomorphism*

$$H_{-s}(\mathbb{R}^n) \to (H_s(\mathbb{R}^n))^*,$$

where $(H_s(\mathbb{R}^n))^*$ is the topological dual of $H_s(\mathbb{R}^n)$. So $H_{-s}(\mathbb{R}^n)$ consists precisely of the linear and continuous forms on $H_s(\mathbb{R}^n)$.

Proof. Let us fix $u \in H_{-s}(\mathbb{R}^n)$ and define $\varphi_u : v \to \varphi_u(v) = \langle u, v \rangle_{H_{-s}, H_s}$. Then $v \mapsto \varphi_u(v)$ is a continuous and linear map on $H_s(\mathbb{R}^n)$ with $|\varphi_u(v)| \leq \|u\|_{-s}\|v\|_s$.
We will show that the map $u \to \varphi_u$ is an isometric isomorphism $H_{-s} \to (H_s)^*$. First let us prove isometry . For this we need to show that there exists some $v \in H_s(\mathbb{R}^n)$, $\|v\|_s = 1$ with $|\varphi_u(v)| = \|u\|_{-s}$.
Let us set $v_0 = \mathcal{F}^{-1}((1+|\xi|^2)^{-s}\bar{\hat{u}}(-\xi))$ and $v := \dfrac{v_0}{\|v_0\|_s}$. From the definition of v_0 it follows that $v_0 \in \mathcal{S}'(\mathbb{R}^n)$ and

$$(1+|\xi|^2)^{s/2}\hat{v_0} = (1+|\xi|^2)^{-s/2}\bar{\hat{u}}(-\xi) \in L^2(\mathbb{R}^n, \mathbb{C}).$$

Consequently $v_0 \in H_s(\mathbb{R}^n)$ and

$$\|v_0\|_s = \left((2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{s}(1+|\xi|^2)^{-2s}|\hat{u}(\xi)|^2 d\xi \right)^{1/2} =$$
$$\left((2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s}|\hat{u}(\xi)|^2 d\xi \right)^{1/2} = \|u\|_{-s}.$$

Let us compute $\langle u, v_0\rangle_{H_{-s},H_s}$:

$$\langle u, v_0\rangle_{H_{-s},H_s} = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{u}(\xi)\hat{v}_0(-\xi)d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{u}(\xi)(1+|\xi|^2)^{-s}\bar{\hat{u}}(\xi)d\xi =$$
$$(2\pi)^{-n}\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2(1+|\xi|^2)^{-s}d\xi = \|u\|_{-s}^2.$$

Since $\|v\|_s = \left\|\frac{v_0}{\|v_0\|_s}\right\|_s = 1$ and $\langle u, v_0\rangle_{H_{-s},H_s} = \|u\|_{-s}^2$ then

$$|\varphi_u(v)| = |\langle u, v\rangle_{H_{-s},H_s}| = |\langle u, \frac{v_0}{\|v_0\|_s}\rangle_{H_{-s},H_s}| = \frac{|\langle u, v_0\rangle_{H_{-s},H_s}|}{\|v_0\|_s} = \frac{\|u\|_{-s}^2}{\|u\|_{-s}} = \|u\|_{-s}.$$

It follows that $|\varphi_u(v)| = \|u\|_{-s}$. From this isometry, injectivity follows. It remains to show surjectivity. Let $u' \in (H_s)^*$, then by Riesz-Fréchet representation theorem $\exists\omega \in H_s(\mathbb{R}^n)$ with

$$u'(v) = (v,\omega)_s = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s\hat{v}(\xi)\bar{\hat{\omega}}(\xi)d\xi.$$

Now set $u := \mathcal{F}^{-1}((1+|\xi|^2)^s\bar{\hat{\omega}}(-\xi))$. Let us prove that $u \in H_{-s}(\mathbb{R}^n)$ and $u'(v) = \langle u, v\rangle_{H_{-s},H_s} = \varphi_u(v)$ for all $v \in H_s(\mathbb{R}^n)$. Since $\omega \in H_s(\mathbb{R}^n)$ and

$$(1+|\xi|^2)^{-s/2}\hat{u} = (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^s\bar{\hat{\omega}}(-\xi) = (1+|\xi|^2)^{s/2}\bar{\hat{\omega}}(-\xi) \in L^2(\mathbb{R}^n,\mathbb{C})$$

then $u \in H_{-s}(\mathbb{R}^n)$. For all $v \in H_s(\mathbb{R}^n)$ we have

$$u'(v) = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s\hat{v}(\xi)\bar{\hat{\omega}}(\xi)d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}(1+|\xi|^2)^s\hat{v}(-\xi)\bar{\hat{\omega}}(-\xi)d\xi =$$
$$(2\pi)^{-n}\int_{\mathbb{R}^n}\hat{u}(\xi)\hat{v}(-\xi)d\xi = \langle u, v\rangle_{H_{-s},H_s}.$$

Consequently the map $\langle u, v\rangle_{H_{-s},H_s} : H_{-s}(\mathbb{R}^n) \times H_s(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an isometric isomorphism. $\square$

Since $L^2(\mathbb{R}^n,\mathbb{C})$ and $H_s(\mathbb{R}^n)$ are isomorphic and $L^2(\mathbb{R}^n,\mathbb{C})$ is separable, also $H_s(\mathbb{R}^n)$ is separable. Assume

$$u \in L^p\{[0,T], H_s(\mathbb{R}^n,\mathbb{C}^m)\}, \quad v \in L^q\{[0,T], H_{-s}(\mathbb{R}^n,\mathbb{C}^m)\},$$

where $\frac{1}{p}+\frac{1}{q} = 1$.
Since $(1-\Delta)^{\frac{s}{2}} : H_s(\mathbb{R}^n,\mathbb{C}^m) \to L^2(\mathbb{R}^n,\mathbb{C}^m)$ and $(1-\Delta)^{\frac{-s}{2}} : H_{-s}(\mathbb{R}^n,\mathbb{C}^m) \to L^2(\mathbb{R}^n,\mathbb{C}^m)$ then

$$(1-\Delta)^{\frac{s}{2}}u \in L^p\{[0,T], L^2(\mathbb{R}^n,\mathbb{C}^m)\}, \quad (1-\Delta)^{\frac{-s}{2}}v \in L^q\{[0,T], L^2(\mathbb{R}^n,\mathbb{C}^m)\}.$$

We define

$$\langle u, v\rangle = \int_0^T ((1-\Delta)^{\frac{s}{2}}u, (1-\Delta)^{\frac{-s}{2}}v)_{L^2}dt, \tag{1.45}$$

since the integrand is measurable and bounded by an integrable function. Let us show that (1.45) does not depend on s. Let $r > s$ and

$$u \in L^p\{[0,T], H_r(\mathbb{R}^n, \mathbb{C}^m)\}$$

$$v \in L^q\{[0,T], H_{-r}(\mathbb{R}^n, \mathbb{C}^m)\}.$$

Then by (1.35) we obtain

$$\int_0^T ((1-\Delta)^{\frac{r}{2}} u, (1-\Delta)^{\frac{-r}{2}} v)_{L^2} dt = \int_0^T ((1-\Delta)^{\frac{r-s}{2}} (1-\Delta)^{\frac{s}{2}} u, (1-\Delta)^{\frac{-r}{2}} v)_{L^2} dt =$$
$$\int_0^T ((1-\Delta)^{\frac{s}{2}} u, (1-\Delta)^{\frac{-s}{2}} v)_{L^2} dt.$$

We conclude that (1.45) does not depend on s, until the right hand side is defined. Let us mention that for non-negative integer k and multiindex $|\alpha| \leq k$, if

$$u \in L^p\{[0,T], H_k(\mathbb{R}^n, \mathbb{C}^m)\}, \quad v \in L^q\{[0,T], H_k(\mathbb{R}^n, \mathbb{C}^m)\},$$

then by Lemma 1.43

$$\partial^\alpha u \in L^p\{[0,T], H_0(\mathbb{R}^n, \mathbb{C}^m)\}, \quad \partial^\alpha v \in L^q\{[0,T], H_0(\mathbb{R}^n, \mathbb{C}^m)\}.$$

Due to (1.34)

$$\langle \partial^\alpha u, v\rangle = \int_0^T ((1-\Delta)^{\frac{s}{2}} \partial^\alpha u, (1-\Delta)^{\frac{-s}{2}} v)_{L^2} dt = \int_0^T (\partial^\alpha u, v)_{L^2} dt =$$
$$\int_0^T (-1)^{|\alpha|} (u, \partial^\alpha v)_{L^2} dt = (-1)^{|\alpha|} \int_0^T ((1-\Delta)^{\frac{s}{2}} u, (1-\Delta)^{\frac{-s}{2}} \partial^\alpha v)_{L^2} dt =$$
$$(-1)^{|\alpha|} \langle u, \partial^\alpha v\rangle.$$

Proposition 1.50. Assume

$$X_s = L^1\{[0,T], H_s(\mathbb{R}^n, \mathbb{C}^m)\}$$
$$Y_{-s} = L^\infty\{[0,T], H_{-s}(\mathbb{R}^n, \mathbb{C}^m)\}.$$

Then for any $F \in X_s^*$ there exists $y \in Y_{-s}$ such that

$$F(x) = \langle x, y\rangle$$

for all $x \in X_s$ and $\|y\|_{Y_{-s}} = \|F\|_{X_s^*}$.

Proof. For a given $F \in X_s^*$, define $G \in X_0^*$ by

$$G(x_0) = F[(1-\Delta)^{\frac{-s}{2}} x_0]. \tag{1.46}$$

From this definition it follows that $\|G\|_{X_0^*} = \|F\|_{X_s^*}$. Since $X_0 = L^1\{[0,T], L^2(\mathbb{R}^n, \mathbb{C}^m)\}$, according to Proposition 3.6 of [Rin09] we obtain a $y_0 \in Y_0 = L^\infty\{[0,T], L^2(\mathbb{R}^n, \mathbb{C}^m)\}$, such that

$$G(x_0) = \langle x_0, y_0\rangle \quad \text{and} \quad \|y_0\|_{Y_0} = \|F\|_{X_s^*}$$

for all $x_0 \in X_0$. Now set $y = (1-\Delta)^{\frac{s}{2}} y_0$. Then from (1.46) and from (1.35), we obtain:

$$F(x) = G[(1-\Delta)^{\frac{s}{2}} x] = \langle (1-\Delta)^{\frac{s}{2}} x, (1-\Delta)^{\frac{-s}{2}} y\rangle = \langle x, y\rangle \quad \text{for} \quad x \in X_s.$$

Since $y = (1-\Delta)^{\frac{s}{2}} y_0 \in Y_{-s}$, then from (1.22) we obtain

$$\|y\|_{Y_{-s}} = \|y_0\|_{Y_0}. \tag{1.47}$$

So $\|F\|_{X_s^*} = \|y_0\|_{Y_0} = \|y\|_{Y_{-s}}$, as claimed. $\square$

2 Sobolev Embedding

2.1 Young's inequality, Sobolev embedding

In this section we follow [Rin09, §6.1 and §6.2].

Theorem 2.1. *Assume p, q be positive real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

for all non-negative a, b. This inequality is called Young's inequality.

Proof. Let us note that if either a or b are zero, then the inequality holds. So let $a, b > 0$. Dividing both sides of (2.1) by b^q we get:

$$\frac{a}{b^{q-1}} \leq \frac{1}{p} \cdot \frac{a^p}{b^q} + \frac{1}{q}.$$

Let us set $t = a/b^{q-1}$. Since $\dfrac{a^p}{b^q} = \dfrac{a^p}{b^{p(q-1)}} = t^p$ then the above inequality will be equivalent to

$$t \leq \frac{1}{q} + \frac{t^p}{p}. \tag{2.2}$$

So inequalities (2.1) and (2.2) are equivalent. Hence we need to show that (2.2) holds. Let us note that the function

$$\frac{t^{-1}}{q} + \frac{t^{p-1}}{p}$$

tends to infinity when $t \to \infty$ and also when $t \to 0$. Let us show that it has a unique minimum at $t = 1$. We need to differentiate $\dfrac{t^{-1}}{q} + \dfrac{t^{p-1}}{p}$ and solve the following equation:

$$\begin{aligned} -\frac{1}{q}t^{-2} + \frac{p-1}{p}t^{p-2} &= 0 \\ t^{-2}\left(-\frac{1}{q} + \frac{p-1}{p}t^p\right) &= 0. \end{aligned}$$

Since $t \neq 0$ then

$$-\frac{1}{q} + \frac{p-1}{p} t^p = 0$$
$$t^p = \frac{p}{q(p-1)} = 1$$
$$t = 1.$$

So the function $\dfrac{t^{-1}}{q} + \dfrac{t^{p-1}}{p}$ has a unique minimum at $t = 1:$

$$\frac{t^{-1}}{q} + \frac{t^{p-1}}{p} \geq \frac{1}{q} + \frac{1}{p} = 1.$$

Consequently,

$$\frac{1}{q} + \frac{t^p}{p} \geq t.$$

So we obtain (2.1). □

We can generalize the Young's inequality in this way.

Theorem 2.2. *Let $p_1, \dots, p_k$ be positive numbers such that*

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = 1. \tag{2.3}$$

If $a_1, \dots, a_k$ are non-negative numbers then

$$a_1 \cdots a_k \leq \frac{a_1^{p_1}}{p_1} + \dots + \frac{a_k^{p_k}}{p_k}. \tag{2.4}$$

Proof. Let us prove (2.4) by induction. It holds for $k = 2$. Assume that it holds for some $k \geq 2$. We need to prove that it holds for $k+1$. Let $p_1, \dots, p_{k+1}$ satisfy

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} + \frac{1}{p_{k+1}} = 1. \tag{2.5}$$

Let us set $r_i = p_i$ for $i = 1, \dots, k-1$ and

$$r_k = \frac{p_k p_{k+1}}{p_k + p_{k+1}}.$$

Then $r_1, \dots, r_k$ are positive numbers which satisfy (2.3). Consequently,

$$a_1 \dots a_{k+1} \leq \frac{a_1^{p_1}}{p_1} + \dots + \frac{a_{k-1}^{p_{k-1}}}{p_{k-1}} + \frac{(a_k a_{k+1})^{r_k}}{r_k}.$$

Let us set $p = p_k/r_k$ and $q = p_{k+1}/r_k$, then apply (2.1). We obtain that

$$(a_k a_{k+1})^{r_k} \leq \frac{a_k^{r_k p}}{p} + \frac{a_{k+1}^{r_k q}}{q} = r_k \left(\frac{a_k^{p_k}}{p_k} + \frac{a_{k+1}^{p_{k+1}}}{p_{k+1}} \right).$$

So we obtain (2.4) for $k+1$. □

Lemma 2.3. Assume $p_1, \dots, p_k$ be positive numbers with

$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = 1.$$

If $u_i \in L^{p_i}(\mathbb{R}^n)$ for $i = 1, \dots, k$ then $u_1 \dots u_k \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |u_1 \dots u_k| dx \leq \|u_1\|_{p_1} \dots \|u_k\|_{p_k}. \tag{2.6}$$

Proof. Let us note that if $\|u_i\|_{p_i} = 0$ for some i then both sides of (2.6) are equal to zero. So we can assume that $\|u_i\|_{p_i} > 0$. Let us set $v_i = u_i / \|u_i\|_{p_i}$ then $\|v_i\|_{p_i} = 1$. According to (2.4) we have

$$\int_{\mathbb{R}^n} |v_1 \dots v_k| dx \leq \int_{\mathbb{R}^n} \left(\frac{|v_1|^{p_1}}{p_1} + \dots + \frac{|v_k|^{p_k}}{p_k} \right) dx =$$
$$\frac{1}{p_1} \int_{\mathbb{R}^n} |v_1|^{p_1} dx + \dots + \frac{1}{p_k} \int_{\mathbb{R}^n} |v_k|^{p_k} dx = \frac{1}{p_1} \|v_1\|_{p_1}^{p_1} + \dots + \frac{1}{p_k} \|v_k\|_{p_k}^{p_k} =$$
$$\frac{1}{p_1} + \dots + \frac{1}{p_k} = 1.$$

Consequently

$$\int_{\mathbb{R}^n} \left| \frac{u_1}{\|u_1\|_{p_1}} \dots \frac{u_k}{\|u_k\|_{p_k}} \right| dx \leq 1.$$

Multiplying both sides of this inequality by $\|u_1\|_{p_1} \dots \|u_k\|_{p_k}$ we get (2.6). □

Corollary 2.4. Assume $p_1, \dots, p_k$ be positive numbers such that the equality (2.3) holds. If $u_i \in L^{2p_i}(\mathbb{R}^n)$ for $i = 1, \dots, k$ then $u_1 \dots u_k \in L^2(\mathbb{R}^n)$ and

$$\|u_1 \dots u_k\|_2 \leq \|u_1\|_{2p_1} \dots \|u_k\|_{2p_k}. \tag{2.7}$$

Proof. Let us apply the above Lemma to $u_1^2, \dots, u_k^2$ then

$$\int_{\mathbb{R}^n} |u_1^2 \dots u_k^2| dx \leq \|u_1^2\|_{p_1} \cdots \|u_k^2\|_{p_k}.$$

Taking the square root, we obtain:

$$\left(\int_{\mathbb{R}^n} |u_1 \dots u_k|^2 dx \right)^{1/2} \leq \left(\int_{\mathbb{R}^n} |u_1|^{2p_1} dx \right)^{1/2p_1} \cdots \left(\int_{\mathbb{R}^n} |u_k|^{2p_k} dx \right)^{1/2p_k}.$$

So

$$\|u_1 \dots u_k\|_2 \leq \|u_1\|_{2p_1} \dots \|u_k\|_{2p_k}.$$

□

Definition 2.5. We denote by $C_b^k(\mathbb{R}^n, \mathbb{C})$ the space which consists of all the C^k functions whose derivatives up to order k are bounded. We define the norm

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} = \sum_{|\alpha|=0}^{k} \|\partial^\alpha f(x)\|_{C_b(\mathbb{R}^n,\mathbb{C})} = \sum_{|\alpha|=0}^{k} \sup |\partial^\alpha f(x)|.$$

Theorem 2.6 (Sobolev embedding). *Assume k be a non negative integer and let $s > k + \dfrac{n}{2}$. Then for all $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_s, \tag{2.8}$$

where C is a constant depending on k, n and s.

Proof. First let us prove the case $k = 0$. Due to the Fourier inversion formula for $\mathcal{S}(\mathbb{R}^n)$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&|f(x)| = |(2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\xi\cdot x} d\xi| \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi = \\
&(2\pi)^{-n} \int_{\mathbb{R}^n} (1+|\xi|^2)^{-s/2} (1+|\xi|^2)^{s/2} |\hat{f}(\xi)| d\xi \leq \\
&(2\pi)^{-n} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s} |\hat{f}(\xi)|^2 d\xi\right)^{1/2} = \\
&(2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{-s} d\xi\right)^{1/2} \|f\|_s.
\end{aligned}$$

If $s > n/2$ then $(1+|\xi|^2)^{-s}$ is integrable and we obtain (2.8). If $s - |\alpha| > n/2$ then $(1+|\xi|^2)^{-s+|\alpha|}$ is integrable for any multiindex α. With similar assumptions for $\partial^\alpha f$ we get

$$\|\partial^\alpha f\|_{C_b(\mathbb{R}^n,\mathbb{C})} \leq C\|\partial^\alpha f\|_{s-|\alpha|}.$$

By Lemma 1.43 we have

$$\|\partial^\alpha f\|_{C_b(\mathbb{R}^n,\mathbb{C})} \leq C\|\partial^\alpha f\|_{s-|\alpha|} \leq C\|f\|_s.$$

Summing these inequalities for all α such that $|\alpha| \leq k$ we get

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_s.$$

□

Due to (2.8) we can associate the elements of $H_s(\mathbb{R}^n)$ with elements of $C_b^k(\mathbb{R}^n, \mathbb{C})$ for $s > k + n/2$. Let us prove this. If $\phi_l \to u$ in $H_s(\mathbb{R}^n)$, $\phi_l \in \mathcal{S}(\mathbb{R}^n)$, then by (2.8)

$$\|\phi_n - \phi_m\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C\|\phi_n - \phi_m\|_s.$$

So ϕ_l is a Cauchy sequence in $C_b^k(\mathbb{R}^n,\mathbb{C})$. By Theorem 3.12 of [Rud87] there is a subsequence of ϕ_l which converges to u a.e.. So we conclude that u is a function in $C_b^k(\mathbb{R}^n,\mathbb{C})$. Due to Theorem 1.42 for any non negative integer k the space $H_k(\mathbb{R}^n)$ coincides with $H^k(\mathbb{R}^n,\mathbb{C})$ and the norms are equivalent. So we conclude that for $l > n/2+k$

$$\|f\|_{C_b^k(\mathbb{R}^n,\mathbb{C})} \leq C\|f\|_{H^l}.$$

Lemma 2.7. Assume $\Omega \subset \mathbb{R}^n$ be open and let $u \in L^2_{loc}(\Omega)$ is l times weakly differentiable. Then $u \in C^k(\Omega)$ for $l > k + n/2$.

Proof. Since $u \in L^2_{loc}(\Omega)$ and it is l times weakly differentiable then the weak derivatives are in $L^2_{loc}(\Omega)$. In the proof of Lemma 1.15 we show that if $u \in W^{l,2}(\Omega) = H^l(\Omega)$ and $\phi \in C_0^\infty(\Omega)$ then $\phi u \in W^{l,2}(\Omega) = H^l(\Omega)$. From the above observations it follows that ϕu is a C^k function. By Proposition A.12 of [Rin09], for any compact subset $K \subseteq \Omega$, there is a $\phi \in C_0^\infty(\Omega)$ such that $\phi(x) = 1$ for $x \in K$. Consequently $u \in C^k(\Omega)$. □

2.2 Gagliardo-Nirenberg inequalities

In this section we will follow [Rin09, §6.3]. We will prove some inequalities of Gagliardo, Nirenberg and Moser. Let Y be a real vector space with inner product $\langle\cdot,\cdot\rangle$, where the norm is induced by an inner product $|y|_Y^2 = \langle y, y\rangle$ for all $y \in Y$. In the following lemmas we will always use such a vector space Y. Let us denote by $B(\mathbb{R}^n, Y)$ the bounded linear transformations from $\mathbb{R}^n$ to Y. Note that $B(\mathbb{R}^n, Y)$ is a real vector space with a norm.

Definition 2.8. We say that $f : \mathbb{R}^n \longrightarrow Y$ is differentiable at $x \in \mathbb{R}^n$ if there exists $T \in B(\mathbb{R}^n, Y)$ such that

$$\lim_{h\to 0} \frac{|f(x+h) - f(x) - Th|_Y}{\|h\|_2} = 0. \tag{2.9}$$

We call T the derivative of f at x and denote it by $(Df)(x)$. If f is differentiable at every $x \in \mathbb{R}^n$ then we get a map $Df : \mathbb{R}^n \longrightarrow B(\mathbb{R}^n, Y)$. If the map $Df : \mathbb{R}^n \longrightarrow B(\mathbb{R}^n, Y)$ is continuous we say that f is continuously differentiable. We denote the kth derivative by $D^k f$.

In this section we are interested in $C_0^\infty(\mathbb{R}^n, Y)$. Let $f \in C_0^\infty(\mathbb{R}^n, Y)$, and define

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|_Y^p dx\right)^{1/p}, \quad \|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|_Y$$

for $1 \leq p < \infty$. Also,

$$\|D^l f\|_p = \left(\sum_{|\alpha|=l} \int_{\mathbb{R}^n} |(\partial^\alpha f)(x)|_Y^p dx\right)^{1/p}, \quad \|D^l f\|_\infty = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|=l} |(\partial^\alpha f)(x)|_Y. \tag{2.10}$$

Assume $f, g : \mathbb{R}^n \longrightarrow Y$ are differentiable at $x \in \mathbb{R}^n$ and let $\phi(x) = \langle f(x), g(x)\rangle$. Then ϕ is differentiable at x and

$$[(D\phi)(x)]h = \langle [(Df)(x)]h, g(x)\rangle + \langle f(x), [(Dg)(x)]h\rangle. \tag{2.11}$$

Let e_j be the vector in $\mathbb{R}^n$ whose jth component is 1 and all other components are zero. Then the partial derivative $\partial_j f$ is the function whose value at x is given by $(Df)(x)e_j$. So

$$(\partial_j f)(x) = [(Df)(x)]e_j. \tag{2.12}$$

Consequently,

$$\partial_j \phi(x) = \langle \partial_j f(x), g(x)\rangle + \langle f(x), \partial_j g(x)\rangle. \tag{2.13}$$

Let us note that if $f : \mathbb{R}^n \longrightarrow Y$ is smooth, then all the partial derivatives exist and are also smooth.

Lemma 2.9. Assume $1 \leq j \leq n$ and assume that $k, r \in \mathbb{R}$ with $1 \leq r \leq k$. Then for all $f \in C_0^\infty(\mathbb{R}^n, Y)$,

$$\|\partial_j f\|_{2k/r}^2 \leq C\|f\|_{2k/(r-1)}\|\partial_j^2 f\|_{2k/(r+1)}, \tag{2.14}$$

where a constant C depends on an upper bound on k.

Proof. Let $2 \leq q \in \mathbb{R}$ and let us define ϕ_j by

$$\phi_j(x) = \langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j f(x)\rangle^{\frac{q-2}{2}},$$

where the last factor is 1 if $q = 2$. From our assumption on $f(x)$ it follows that this function has compact support. We need to show that $\phi_j(x)$ is continuously differentiable. According to observations which we made in the beginning of this section, the expressions $\langle f(x), \partial_j f(x)\rangle$ and $\langle \partial_j f(x), \partial_j f(x)\rangle$ are smooth functions with compact support. If $q = 2$ then ϕ_j is smooth, so assume that $q > 2$. If $(\partial_j f)(\xi) \neq 0$, then we conclude that ϕ_j is smooth in a neighborhood of ξ. Let us assume that ξ is such that $(\partial_j f)(\xi) = 0$. Denote by $\psi_j(x) = \langle \partial_j f(x), \partial_j f(x)\rangle$, then ψ_j is smooth and for all $1 \leq k \leq n$:

$$\partial_k \psi_j(x) = \langle \partial_k \partial_j f(x), \partial_j f(x)\rangle + \langle \partial_j f(x), \partial_k \partial_j f(x)\rangle.$$

So $\psi_j(\xi) = \partial_k \psi_j(\xi) = 0$ and we conclude that

$$\psi_j(x) = O(|x - \xi|^2).$$

Applying the Cauchy-Schwarz inequality to $\phi_j(x)$ we get:

$$\begin{aligned} |\phi_j(x)| =& |\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j f(x)\rangle^{\frac{q-2}{2}}| \leq |f(x)|_Y\,[\psi_j(x)]^{1/2}\,[\psi_j(x)]^{\frac{q-2}{2}} = \\ & |f(x)|_Y\,[\psi_j(x)]^{\frac{q-1}{2}} = O(|x-\xi|^{q-1}). \end{aligned}$$

So we conclude that ϕ_j is differentiable in a neighborhood of ξ and that the derivative is zero. If $(\partial_j f)(x) \neq 0$ we can differentiate ϕ_j with respect to the kth variable:

$$\begin{aligned} (\partial_k \phi_j)(x) = & \langle \partial_k f(x), \partial_j f(x)\rangle\,[\psi_j(x)]^{\frac{q-2}{2}} + \langle f(x), (\partial_k \partial_j f)(x)\rangle\,[\psi_j(x)]^{\frac{q-2}{2}} + \\ & (q-2)\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), (\partial_k \partial_j f)(x)\rangle\,[\psi_j(x)]^{\frac{q-4}{2}}. \end{aligned} \tag{2.15}$$

From the above observations we conclude that if ξ is such that $\partial_j f(\xi) \neq 0$, then $\phi_j(x)$ is continuously differentiable at ξ. If ξ is such that $\partial_j f(\xi) = 0$, then $D\phi_j(\xi) = 0$. Let us show that $D\phi_j(x)$ is continuous at ξ. Let $\partial_j f(x_l) \neq 0$ and $x_l \to \xi$ with $\partial_j f(\xi) = 0$, then from (2.15) we get $\partial_k \phi_j(x_l) \to \partial_k \phi_j(\xi) = 0$. Hence ϕ_j is continuously differentiable. Let us compute $(\partial_j \phi_j)(x)$:

$$\begin{aligned}(\partial_j\phi_j)(x) = \langle \partial_j f(x), \partial_j f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} + \langle f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} + \\ (q-2)\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-4}{2}}.\end{aligned} \tag{2.16}$$

Integrating (2.16) over $\mathbb{R}^n$ and using the triangle inequality we get:

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} \langle \partial_j f(x), \partial_j f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} dx\Big| \leq \Big|\int_{\mathbb{R}^n} (\partial_j\phi_j)(x)dx\Big| + \\ &\Big|\int_{\mathbb{R}^n} \langle f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} + (q-2)\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-4}{2}} dx\Big|.\end{aligned} \tag{2.17}$$

Since $\int_{\mathbb{R}^n} (\partial_j\phi_j)(x)dx = 0$ then

$$\begin{aligned}\int_{\mathbb{R}^n} |\partial_j f(x)|_Y^q dx \leq \int_{\mathbb{R}^n} |\langle f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} + \\ (q-2)\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-4}{2}} |dx.\end{aligned} \tag{2.18}$$

Let us denote

$$I = \langle f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-2}{2}} + (q-2)\langle f(x), \partial_j f(x)\rangle\langle \partial_j f(x), \partial_j^2 f(x)\rangle \left[\psi_j(x)\right]^{\frac{q-4}{2}}.$$

We need to estimate $|I|$. Applying the Cauchy-Schwarz inequality we get:

$$\begin{aligned}&|I| \leq |f(x)|_Y|\partial_j^2 f(x)|_Y|\partial_j f(x)|_Y^{q-2} + \\ &(q-2)|f(x)|_Y|\partial_j f(x)|_Y|\partial_j f(x)|_Y|\partial_j^2 f(x)|_Y|\partial_j f(x)|_Y^{q-4} = \\ &(q-1)|f(x)|_Y|\partial_j^2 f(x)|_Y|\partial_j f(x)|_Y^{q-2}.\end{aligned}$$

Inserting this result into (2.18)

$$\int_{\mathbb{R}^n} |\partial_j f(x)|_Y^q dx \leq (q-1)\int_{\mathbb{R}^n} |f(x)|_Y|\partial_j^2 f(x)|_Y|\partial_j f(x)|_Y^{q-2}dx. \tag{2.19}$$

For $q = 2$, if we interpret $|\partial_j f(x)|_Y^{q-2}$ as 1, then

$$\int_{\mathbb{R}^n} |\partial_j f(x)|_Y^q dx \leq \int_{\mathbb{R}^n} |f(x)|_Y|\partial_j^2 f(x)|_Y dx \leq \|f\|_{2k/(r-1)}\|\partial_j^2 f\|_{2k/(r+1)}.$$

In the last step we have used Hölder's inequality, where $k = r \geq 1$. By assumption, $1 \leq r < k$, so we set

$$q = \frac{2k}{r},\ q_1 = \frac{2k}{r-1},\ q_2 = \frac{2k}{r+1},\ q_3 = \frac{q}{q-2}.$$

Hence $q > 2$. Since $1/q_1 + 1/q_2 + 1/q_3 = 1$ then we can apply Hölder's inequality to (2.19). So we obtain

$$\int_{\mathbb{R}^n} |\partial_j f(x)|_Y^q dx \le (q-1)\|f\|_{2k/(r-1)}\|\partial_j^2 f\|_{2k/(r+1)}\|\partial_j f\|_q^{q-2}.$$

Consequently,

$$\|\partial_j f\|_{2k/r}^2 \le C\|f\|_{2k/(r-1)}\|\partial_j^2 f\|_{2k/r+1}.$$

Let us also mention that if $r = 1$ then $2k/(r-1)$ is interpreted as ∞, and the constant only depends on an upper bound on k. □

Lemma 2.10. Assume $1 \le j, l, i \in \mathbb{Z}$ and assume that $k, r \in \mathbb{R}$ with $j \le r \le k+1-i$ and $l \ge j$. Then for all $\phi \in C_0^\infty(\mathbb{R}^n, Y)$,

$$\|D^l\phi\|_{2k/r} \le C\left[\|D^{l-j}\phi\|_{2k/(r-j)} + \|D^{l+i}\phi\|_{2k/(r+i)}\right], \tag{2.20}$$

where a constant C depends on n and an upper bound on k and $l+i$.

We interpret $2k/(r-j)$ as ∞ when $r = j$.

Proof. Let us apply (2.14), then we have

$$\|D^l\phi\|_{2k/r}^2 \le C\|D^{l-1}\phi\|_{2k/(r-1)}\|D^{l+1}\phi\|_{2k/(r+1)}, \tag{2.21}$$

for $l \ge 1$ and $1 \le r \le k$. Recall that

$$ab \le \frac{1}{2}(\varepsilon a + \varepsilon^{-1}b)^2$$

for all non-negative a, b and $\varepsilon > 0$. Apply this inequality to (2.21):

$$\|D^l\phi\|_{2k/r} \le C\left[\varepsilon\|D^{l-1}\phi\|_{2k/(r-1)} + \varepsilon^{-1}\|D^{l+1}\phi\|_{2k/(r+1)}\right].$$

This is inequality (2.20) in the case $i = j = 1$. Let us prove that

$$\|D^l\phi\|_{2k/r} \le C\left[\varepsilon\|D^{l-j}\phi\|_{2k/(r-j)} + C(\varepsilon)\|D^{l+i}\phi\|_{2k/(r+i)}\right], \tag{2.22}$$

where r, k, j, l, i satisfy the conditions of the lemma, also the condition that $j, i \le \gamma$. We prove this inequality by induction. From the observation above we know that it is true for $\gamma = 1$. Assume that inequality (2.22) works for γ and let us prove it for $\gamma + 1$. We need to show that we can increase j to $j+1$. Assume that we have conditions of the lemma with j replaced by $j+1$ and $1 \le i, j \le \gamma$. Let us apply the induction hypothesis to $r' = r - j, k' = k, l' = l - j, i' = j$ and $j' = 1$, then we obtain

$$\|D^{l-j}\phi\|_{2k/(r-j)} \le C\left[\varepsilon_1\|D^{l-j-1}\phi\|_{2k/(r-j-1)} + C(\varepsilon_1)\|D^l\phi\|_{2k/r}\right].$$

Inserting this inequality into (2.22) we obtain:

$$\|D^l\phi\|_{2k/r} \le C\left[\varepsilon C\left[\varepsilon_1\|D^{l-j-1}\phi\|_{2k/(r-j-1)} + C(\varepsilon_1)\|D^l\phi\|_{2k/r}\right] + C(\varepsilon)\|D^{l+i}\phi\|_{2k/(r+i)}\right]. \tag{2.23}$$

So $\|D^l\phi\|_{2k/r}$ appears also on the right hand side. Let us fix ε_1 and assume that ε is small enough, so that the coefficient of $\|D^l\phi\|_{2k/r}$ can be smaller than $1/2$. Then, we can move it over to the left hand side and obtain (2.22) for $j+1$

$$\|D^l\phi\|_{2k/r} \le C\left[\varepsilon\|D^{l-j-1}\phi\|_{2k/(r-j-1)} + C(\varepsilon)\|D^{l+i}\phi\|_{2k/(r+i)}\right] \tag{2.24}$$

Hence (2.22) holds for all r, k, j, l, i, which satisfy the conditions of the lemma and $i \le \gamma,\ j \le \gamma+1$. Now let us assume that conditions of the lemma are satisfied with i replaced by $i+1$ and that $1 \le i \le \gamma$ and $j \le \gamma+1$. Let us apply the induction hypothesis with $r' = r+i,\ k' = k,\ j' = i,\ l' = l+i$ and $i' = 1$:

$$\|D^{l+i}\phi\|_{2k/(r+i)} \le C\left[\varepsilon_2\|D^l\phi\|_{2k/r} + C(\varepsilon_2)\|D^{l+i+1}\phi\|_{2k/(r+i+1)}\right]. \tag{2.25}$$

Inserting this result into (2.22) and using a similar argument as above, we obtain the induction hypothesis for $i+1$. Consequently we have the induction hypothesis with γ replaced by $\gamma+1$. □

For $j = l$ and $r + i = k$, as a consequence of this lemma we have

$$\|D^l\phi\|_{2k/r} \le C\left[\|\phi\|_{2k/(r-l)} + \|D^{l+k-r}\phi\|_2\right] \tag{2.26}$$

for all $\phi \in C_0^\infty(\mathbb{R}^n, Y)$, $l \in \mathbb{N}$, $k, r \in \mathbb{R}$ such that $l \le r$ and $k - r \in \mathbb{N}$.

Lemma 2.11. Assume l, μ and i be non-negative integers with $l \le \max\{\mu, i\}$ and assume $q, \varrho, \rho \in [1, \infty]$. Let us set

$$\alpha = \frac{n}{q} - \frac{n}{\varrho} + \mu - l, \quad \beta = -\frac{n}{q} + \frac{n}{p} - i + l \tag{2.27}$$

and assume that α and β are non zero. If $0 < C_1, C_2 \in \mathbb{R}$ are constants such that the inequality

$$\|D^l\phi\|_q \le C_1\|D^\mu\phi\|_\varrho + C_2\|D^i\phi\|_\rho$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^n, Y)$, then α and β have the same sign and

$$\|D^l\phi\|_q \le (C_1 + C_2)\|D^\mu\phi\|_\varrho^{\beta/(\alpha+\beta)}\|D^i\phi\|_\rho^{\alpha/(\alpha+\beta)} \quad \text{for all} \quad \phi \in C_0^\infty(\mathbb{R}^n, Y).$$

If $q = \infty$, then we interpret n/q as 0 and similarly for n/ϱ and n/ρ.

Proof. If we set $Q = \|D^l\phi\|_q$, $R = \|D^\mu\phi\|_\varrho$ and $P = \|D^i\phi\|_\rho$ then the assumed inequality is transformed to

$$Q \le C_1 R + C_2 P.$$

Let us replace $\phi(x)$ by $\phi(sx)$, where $0 < s \in \mathbb{R}$. Then from (2.10) we get:

$$\|D^l\phi(sx)\|_q = s^{l-\frac{n}{q}}\|D^l\phi(x)\|_q, \quad \|D^\mu\phi(sx)\|_\varrho = s^{\mu-\frac{n}{\varrho}}\|D^\mu\phi(x)\|_\varrho$$

and

$$\|D^i\phi(sx)\|_\rho = s^{i-\frac{n}{\rho}}\|D^i\phi(x)\|_\rho.$$

Consequently

$$s^{l-n/q}Q \le C_1 s^{\mu-n/\varrho}R + C_2 s^{i-n/\rho}P.$$

Multiplying both sides of the above inequality by $s^{n/q-l}$ we get:

$$Q \le C_1 s^{\mu-n/\varrho-l+n/q}R + C_2 s^{i-n/\rho-l+n/q}P.$$

So

$$Q \le C_1 s^\alpha R + C_2 s^{-\beta}P. \tag{2.28}$$

Assume that α and β have different sign. If s tends to zero or ∞ then we can conclude that $Q = 0$. Hence $\|D^l\phi\|_q = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^n, Y)$. Since this is false, we conclude that α and β have the same sign. If P or R is zero then ϕ is zero. Hence the inequality holds. Let us set $s = (P/R)^{1/(\alpha+\beta)}$, where P and R are non zero. Inserting s in (2.28) we have:

$$\begin{aligned} Q \le & C_1 \left(\frac{P}{R}\right)^{\frac{\alpha}{\alpha+\beta}} R + C_2 \left(\frac{P}{R}\right)^{\frac{-\beta}{\alpha+\beta}} P = \\ & C_1 P^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}} + C_2 P^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}} = \\ & (C_1 + C_2) P^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}. \end{aligned}$$

Consequently,

$$\|D^l\phi\|_q \le (C_1 + C_2)\|D^i\phi\|_\rho^{\alpha/(\alpha+\beta)}\|D^\mu\phi\|_\varrho^{\beta/(\alpha+\beta)}.$$

□

Corollary 2.12. Assume $1 \le l \in \mathbb{Z}$ and let $k, r \in \mathbb{R}$ be such that $l \le r$ and $k - r \in \mathbb{N}$. Then for all $\phi \in C_0^\infty(\mathbb{R}^n, Y)$

$$\|D^l\phi\|_{2k/r} \le C\|\phi\|_{2k/(r-l)}^{(k-r)/(k+l-r)}\|D^{k+l-r}\phi\|_2^{l/(k+l-r)}, \tag{2.29}$$

where C is a constant.

Proof. Since the corollary conditions satisfy Lemma 2.11 then we can apply it to (2.26). And that the result of this corollary follows. Note that from the assumptions of Lemma 2.11 and (2.26) we conclude that:

$$q = \frac{2k}{r},\ \varrho = \frac{2k}{r-l},\ \mu = 0,\ i = k+l-r,\ \rho = 2. \tag{2.30}$$

Inserting this in (2.27) we obtain

$$\begin{aligned}
\alpha &= n\cdot\frac{r}{2k} - n\cdot\frac{(r-l)}{2k} + 0 - l = \frac{nr - nr + nl - 2kl}{2k} = \frac{l(n-2k)}{2k} \\
\beta &= -n\cdot\frac{r}{2k} + \frac{n}{2} - (k+l-r) + l = \frac{-n(r-k)}{2k} + (r-k) = (r-k)\frac{(2k-n)}{2k} \\
\alpha+\beta &= \frac{(l+k-r)(n-2k)}{2k}.
\end{aligned} \tag{2.31}$$

So

$$\begin{aligned}
\frac{\alpha}{\alpha+\beta} &= \frac{l}{k+l-r} \\
\frac{\beta}{\alpha+\beta} &= \frac{k-r}{k+l-r}.
\end{aligned}$$

Applying Lemma 2.11 and using the above results we obtain

$$\|D^l\phi\|_{2k/r} \le C\|\phi\|_{2k/(r-l)}^{(k-r)/(k+l-r)}\|D^{k+l-r}\phi\|_2^{l/(k+l-r)}.$$

Let us prove the case when $\alpha = \beta = 0$, where α and β are given in (2.27). Since $l \ge 1$ and $k-r$ is positive then $\alpha = \beta = 0$ if and only if $n = 2k$. The constant C in (2.26) only depends on an upper bound on n, on k and on $l+k-r$. Assume that $n = 2k$, and set $k_\varepsilon = k+\varepsilon$ and $r_\varepsilon = r+\varepsilon$ for $\varepsilon \in (0,1)$. We can apply the Corrolary for $l, k_\varepsilon, r_\varepsilon$, and we can choose constant which is independent of ε. Since $k_\varepsilon - r_\varepsilon = k - r$, so it remains to prove that

$$\lim_{t\to t_0}\|\phi\|_t = \|\phi\|_{t_0},$$

for smooth function ϕ with compact support and $1 < t_0 \le \infty$.
Since $\|\phi\|_t = (\int\limits_{\mathbb{R}^n}(|\phi(x)|_Y^t dx)^{1/t}$ and the norm $|\phi(x)|_Y$ defines a non negative real-valued continuous function with compact support. Then we only need to prove that $\lim\limits_{t\to t_0}\|\psi\|_t = \|\psi\|_{t_0}$, where ψ is a continuous real-valued function with compact support. First consider the case $1 < t_0 < \infty$. Since $\psi(x)$ is a continuous real-valued function with compact support then $|\psi|^t$ converges to $|\psi|^{t_0}$ everywhere and is bounded by an integrable function. By Lebesgue's dominated convergence theorem, $\|\psi\|_t^t \to \|\psi\|_{t_0}^{t_0}$. So we conclude that $\|\psi\|_t \to \|\psi\|_{t_0}$. Now let us prove the case $t_0 = \infty$. Let $\psi = 0$ outside of a compact set K. Consequently, for $1 < t < \infty$,

$$\|\psi\|_t = \left(\int\limits_K |\psi(x)|^t dx\right)^{1/t} \le \|\psi\|_\infty\,[\mu(K)]^{1/t}.$$

So,

$$\limsup_{t\to\infty} \|\psi\|_t \le \|\psi\|_\infty.$$

Let us denote by $A_\alpha = \{x \in \mathbb{R}^n : \ |\psi(x)| \ge \alpha\}$, for any real number $\alpha > 0$. Then

$$\alpha\, [\mu(A_\alpha)]^{1/t} \le \left(\int\limits_{A_\alpha} |\psi(x)|^t dx \right)^{1/t} = \|\psi\|_t.$$

If $\mu(A_\alpha) > 0$, we conclude that

$$\alpha \le \liminf_{t\to\infty} \|\psi\|_t.$$

Let $\|\psi\|_\infty = \alpha_0$ then

$$\|\psi\|_\infty \le \liminf_{t\to\infty} \|\psi\|_t \quad \text{on} \quad A_{\alpha_0}.$$

By combining the above results we have:

$$\|\psi\|_\infty \le \liminf_{t\to\infty} \|\psi\|_t \le \limsup_{t\to\infty} \|\psi\|_t \le \|\psi\|_\infty.$$

Hence, $\lim\limits_{t\to\infty} \|\psi\|_t = \|\psi\|_\infty$. □

Corollary 2.13. Let k, l be positive integers such that $k \ge l+1$. We interpret $2k/(r-l)$ as ∞ when $r = l$. According to the above corollary there is a constant C such that

$$\|D^l \phi\|_{2k/l} \le C \|\phi\|_\infty^{1-l/k} \|D^k \phi\|_2^{l/k} \text{ for all } \phi \in C_0^\infty(\mathbb{R}^n, Y). \tag{2.32}$$

Lemma 2.14. Let $\phi_1, \dots, \phi_l \in C_0^\infty(\mathbb{R}^n, Y)$ and let $\alpha_1, \dots, \alpha_l$ be multiindices with $\sum\limits_{i=1}^{l} |\alpha_i| = k$. Then

$$\|\partial^{\alpha_1}\phi_1 \dots \partial^{\alpha_l}\phi_l\|_2 \le C \sum_{i=1}^{l} \|D^k \phi_i\|_2 \prod_{j\ne i} \|\phi_j\|_\infty. \tag{2.33}$$

Proof. Let us set $k_i = |\alpha_i|$ and $p_i = k/k_i$ then $1/p_1 + \dots + 1/p_l = 1$. By applying (2.7) we obtain

$$\|\partial^{\alpha_1}\phi_1 \dots \partial^{\alpha_l}\phi_l\|_2 \le \|\partial^{\alpha_1}\phi_1\|_{2k/k_1} \cdots \|\partial^{\alpha_l}\phi_l\|_{2k/k_l}. \tag{2.34}$$

If only one $k_i \ne 0$, then (2.33) is true. So let us assume that $k_i \le k-1$. By applying the inequality (2.32) with $Y = \mathbb{R}$ we obtain that

$$\|\partial^{\alpha_1}\phi_1 \dots \partial^{\alpha_l}\phi_l\|_2 \le C \|\phi_1\|_\infty^{1-k_1/k} \|D^k\phi_1\|_2^{k_1/k} \cdots \|\phi_l\|_\infty^{1-k_l/k} \|D^k \phi_l\|_2^{k_l/k}. \tag{2.35}$$

Since $1 - k_i/k = \sum\limits_{j\ne i} k_j/k$ then we can write the factors in l groups of the form

$$\left(\|D^k \phi_i\|_2 \prod_{j \ne i} \|\phi_j\|_\infty \right)^{k_i/k}. \tag{2.36}$$

Let us apply the inequality (2.4):

$$\left(\|D^k\phi_1\|_2\prod_{j\neq 1}\|\phi_j\|_\infty\right)^{k_1/k}\cdots\left(\|D^k\phi_l\|_2\prod_{j\neq l}\|\phi_j\|_\infty\right)^{k_l/k}\leq$$

$$\left(\|D^k\phi_1\|_2\prod_{j\neq 1}\|\phi_j\|_\infty\right)^{\frac{k_1}{k}\cdot\frac{k}{k_1}}\cdot\frac{k_1}{k}+\cdots+\left(\|D^k\phi_l\|_2\prod_{j\neq l}\|\phi_j\|_\infty\right)^{\frac{k_l}{k}\cdot\frac{k}{k_l}}\cdot\frac{k_l}{k}=$$

$$\|D^k\phi_1\|_2\prod_{j\neq 1}\|\phi_j\|_\infty\cdot\frac{k_1}{k}+\cdots+\|D^k\phi_l\|_2\prod_{j\neq l}\|\phi_j\|_\infty\cdot\frac{k_l}{k}\leq$$

$$\sum_{i=1}^{l}\|D^k\phi_i\|_2\prod_{j\neq i}\|\phi_j\|_\infty.$$

Inserting this in (2.35) we get

$$\|\partial^{\alpha_1}\phi_1\ldots\partial^{\alpha_l}\phi_l\|_2\leq C\sum_{i=1}^{l}\|D^k\phi_i\|_2\prod_{j\neq i}\|\phi_j\|_\infty.$$

□

Let us show that (2.33) also holds when $\phi_1,\ldots,\phi_l\in H^k(\mathbb{R}^n)$, are such that $\|\phi_i\|_\infty<\infty$ for $i=1,\ldots,l$. By Lemma 1.15 there is a sequence $\phi_{i,m}\in C_0^\infty(\mathbb{R}^n)$ converging to ϕ_i for $i=1,\ldots,l$. From the proof of Lemma 1.15 and inequality (1.3) for $p=\infty$ it follows that $\|\phi_{i,m}\|_\infty\leq\|\phi_i\|_\infty$. According to Theorem 3.12 of [Rud87] we can choose subsequences of $\phi_{i,m}$ such that $\partial^{\alpha_i}\phi_{i,m}$ converges to $\partial^{\alpha_i}\phi_i$ almost everywhere. By applying Fatou's lemma 1.28 of [Rud87] we get the inequality (2.33) for $\phi_1,\ldots,\phi_l\in H^k(\mathbb{R}^n)$.

3 Symmetric hyperbolic systems

In this chapter we follow [Rin09, Chapter 7]. Let us recall the definition of absolutely continuous functions.

Definition 3.1. A complex function f, defined on an interval I is called absolutely continuous (AC) if $\forall\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \varepsilon$$

for any n and any disjoint intervals $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in I whose lengths satisfy

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta.$$

Lemma 3.2. (Grönwall's lemma) Let $f \in L^\infty([T_0, T])$, $k \in L^1([T_0, T])$ be non-negative functions and let G be an increasing, non-negative function on $[T_0, T]$, where $T_0 \in \mathbb{R}, T > T_0$. If

$$f(t) \leq G(t) + \int_{T_0}^{t} k(s) f(s) ds \quad \text{for all } t \in [T_0, T] \tag{3.1}$$

then

$$f(t) \leq G(t) \exp\left(\int_{T_0}^{t} k(s) ds \right) \quad \text{for all } t \in [T_0, T].$$

Proof. Since G is an increasing function it will suffice to prove the statement for $t = T$. Therefore, we may suppose that $G = G(T)$ is constant. Let us extend k and f to the entire real line by setting them zero outside the interval $[T_0, T]$. Let us denote

$$F(t) = G + \int_{T_0}^{t} k(s) f(s) ds.$$

F is differentiable almost everywhere and $F' = kf$ by Theorem 7.11 of [Rud87]. Also F is a real-valued continuous and increasing function defined on $[T_0, T]$. Due to Theorem 7.18(c) of [Rud87] F is absolutely continuous (AC) on $[T_0, T]$. Similarly, $\int_{T_0}^{t} k(s) ds$ is absolutely continuous (AC) on $[T_0, T]$. Hence

$$g(t) = F(t) \exp\left(-\int_{T_0}^{t} k(s)ds\right)$$

is absolutely continuous. By Theorem 7.18 (a) of [Rud87] $g(t)$ is differentiable a.e. and the derivative is

$$g' = kf \exp\left(-\int_{T_0}^{t} k(s)ds\right) - kF \exp\left(-\int_{T_0}^{t} k(s)ds\right) =$$
$$k(f-F) \exp\left(-\int_{T_0}^{t} k(s)ds\right).$$

From (3.1) it follows that $g' \leq 0$. Since g is a real-valued and absolutely continuous function on $[T_0, T]$ then by Theorem 7.20 of [Rud87]

$$g(t) - g(T_0) = \int_{T_0}^{t} g'(x)dx \quad T_0 \leq t \leq T.$$

Consequently,

$$g(t) - g(T_0) \leq 0$$
$$g(t) \leq g(T_0) = G.$$

Hence,

$$F(t) \exp\left(-\int_{T_0}^{t} k(s)ds\right) \leq G,$$
$$F(t) \leq G \exp\left(\int_{T_0}^{t} k(s)ds\right).$$

From (3.1) it follows that $f(t) \leq F(t)$. So

$$f(t) \leq G(t) \exp\left(\int_{T_0}^{t} k(s)ds\right).$$

$\square$

3.1 Energy inequalities

To define symmetric hyperbolic systems we need to recall some definitions.

Definition 3.3. An $N \times N$ square matrix A is called symmetric if it is equal to its transpose $A = A^T$:

$$a_{i,j} = a_{j,i} \quad \text{for every} \quad 1 \leq i, j \leq N.$$

Definition 3.4. An $N \times N$ symmetric real matrix A is called positive definite if

$$x^T A x > 0 \quad \text{for all} \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Let us define linear symmetric hyperbolic systems. These are equations of the following form :

$$\sum_{\mu=0}^{n} A^\mu \partial_\mu u + Bu = f \tag{3.2}$$

$$u(0, \cdot) = u_0,$$

where $A^\mu, \quad \mu = 0, 1, \cdots, n$ and B are $N \times N$ real matrix-valued smooth functions on $\Omega \subseteq \mathbb{R}^{n+1}$. The derivatives of $A^\mu, \quad \mu = 0, 1, \cdots, n$ and B are supposed to be bounded. Here f and u are $\mathbb{R}^N$-valued functions defined on Ω, and u_0 is a smooth function on $\mathbb{R}^n$. By using Einstein summation convention we can rewrite (3.2) as

$$\begin{aligned} & A^\mu \partial_\mu u + Bu = f \\ & u(0, \cdot) = u_0. \end{aligned} \tag{3.3}$$

We say that (3.3) is a symmetric hyperbolic system if $A^\mu, \quad \mu = 0, 1, \cdots, n$ are symmetric. We suppose that A^0 is positive definite with a uniform positive lower bound. Hence there is a real constant $c_0 > 0$ such that $A^0 \geq c_0$. This means that for every $x \in \mathbb{R}^n$ $x^T A^0 x \geq c_0 x^T I x = c_0 |x|^2$ where I is the identity matrix. Let us denote by $L = A^\mu \partial_\mu + B$. So (3.3) can be rewritten as

$$Lu = f \tag{3.4}$$

$$u(0, \cdot) = u_0.$$

Let u be a smooth solution to (3.3) on $S_T = [0, T] \times \mathbb{R}^n$ and suppose that $u, \partial_t u$ satisfy uniform Schwartz bounds. This means that for every α and β multiindices there is a constant $C_{\alpha,\beta}$ such that

$$|x^\alpha| [|\partial^\beta u| + |\partial^\beta \partial_t u|](t, x) \leq C_{\alpha,\beta} \tag{3.5}$$

on S_T. Then by (3.3), f also satisfies uniform Schwartz bounds. Let us analyze the basic energy inequality. Let

$$E = E(t) = \frac{1}{2} \int_{\mathbb{R}^n} u^T A^0 u \, dx,$$

where u^T is a transpose of u. Let us differentiate E with respect of t :

$$\partial_t E = \partial_t \left[\frac{1}{2} \int_{\mathbb{R}^n} u^T A^0 u \, dx \right] = \frac{1}{2} \int_{\mathbb{R}^n} \partial_t (u^T A^0 u) dx =$$
$$\frac{1}{2} \int_{\mathbb{R}^n} [\partial_t u^T A^0 u + u^T ((\partial_t A^0) u + A^0 \partial_t u)] dx =$$
$$\frac{1}{2} \int_{\mathbb{R}^n} [\partial_t u^T A^0 u + u^T (\partial_t A^0) u + u^T A^0 \partial_t u] dx.$$

Since A^0 is a symmetric matrix, $a_{ij} = a_{ji}$ for $i, j = 1, \cdots, N$, then

$$\partial_t u^T A^0 u = u^T A^0 \partial_t u.$$

So

$$\partial_t E = \int_{\mathbb{R}^n} \left[\frac{1}{2} u^T (\partial_t A^0) u + u^T A^0 \partial_t u \right] dx.$$

By equation (3.3) we have that

$$A^0 \partial_t u + A^i \partial_i u + Bu = f.$$

Consequently we can rewrite the second term on the right hand side as

$$\int_{\mathbb{R}^n} u^T A^0 \partial_t u \, dx = \int_{\mathbb{R}^n} \left[u^T f - u^T A^i \partial_i u - u^T B u \right] dx.$$

We need to compute $u^T A^i \partial_i u$. Let us begin with

$$\partial_i (u^T A^i u) = \partial_i u^T A^i u + u^T [(\partial_i A^i) u + A^i \partial_i u].$$

Since A^i is a symmetric matrix then

$$\partial_i u^T A^i u = u^T A^i \partial_i u.$$

Consequently,

$$\partial_i (u^T A^i u) = 2 u^T A^i \partial_i u + u^T (\partial_i A^i) u$$
$$u^T A^i \partial_i u = \frac{1}{2} \left[\partial_i (u^T A^i u) - u^T (\partial_i A^i) u \right].$$

Integrating both sides of the above equation we obtain

$$\int_{\mathbb{R}^n} u^T A^i \partial_i u \, dx = \frac{1}{2} \int_{\mathbb{R}^n} \left[\partial_i (u^T A^i u) - u^T (\partial_i A^i) u \right] dx = -\frac{1}{2} \int_{\mathbb{R}^n} u^T (\partial_i A^i) u \, dx,$$

where we have used the fact that $\int\limits_{\mathbb{R}^n} \partial_i(u^T A^i u)dx = 0$.
Summing the above equations we have that

$$
\begin{aligned}
\partial_t E = \int\limits_{\mathbb{R}^n} \left[\frac{1}{2}u^T(\partial_t A^0)u + u^T f + \frac{1}{2}u^T(\partial_i A^i)u - u^T B u\right] dx = \\
\int\limits_{\mathbb{R}^n} u^T \left(\frac{1}{2}\partial_t A^0 + \frac{1}{2}\partial_i A^i - B\right) u\, dx + \int\limits_{\mathbb{R}^n} u^T f\, dx.
\end{aligned}
\tag{3.6}
$$

By our assumption, B and the derivatives of A, have an upper bound hence

$$
|u^T \left(\frac{1}{2}\partial_t A^0 + \frac{1}{2}\partial_i A^i - B\right) u| \leq M|u|^2. \tag{3.7}
$$

By our assumption A_0 has a uniform positive lower bound so

$$
u^T A_0 u \geq C_0 |u|^2. \tag{3.8}
$$

The first term on the right hand side of (3.6) is bounded by CE:

$$
\int\limits_{\mathbb{R}^n} |u^T \left(\frac{1}{2}\partial_t A^0 + \frac{1}{2}\partial_i A^i - B\right) u|\, dx \leq \int\limits_{\mathbb{R}^n} M|u|^2\, dx \leq \frac{M}{C_0}\int\limits_{\mathbb{R}^n} u^T A^0 u\, dx \leq CE.
$$

Let us estimate $\int\limits_{\mathbb{R}^n} |u^T f|\, dx$ by using the Cauchy-Schwarz inequality:

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} |u^T f|\, dx \leq \left(\int\limits_{\mathbb{R}^n} |u|^2\, dx\right)^{1/2} \left(\int\limits_{\mathbb{R}^n} |f(t,\cdot)|^2 dx\right)^{1/2} \leq \\
\left(\frac{1}{C_0}\int\limits_{\mathbb{R}^n} u^T A^0 u\, dx\right)^{1/2} \cdot \|f(t,\cdot)\|_2 = CE^{1/2}\|f(t,\cdot)\|_2.
\end{aligned}
$$

So we have

$$
\partial_t E \leq\ CE + CE^{1/2}\|f(t,\cdot)\|_2, \tag{3.9}
$$

where C is a constant whose value may change from line to line. Let us take $E_\varepsilon = E + \varepsilon$ for $\varepsilon > 0$. Since (3.9) holds for $E_\varepsilon > 0$ then we can divide by $\sqrt{E_\varepsilon}$:

$$
\frac{\partial_t E_\varepsilon}{E_\varepsilon^{1/2}} \leq CE_\varepsilon^{1/2} + C\|f(t,\cdot)\|_2.
$$

Integrating the above inequality over $[0,t]$ we get:

$$
\int\limits_0^t \frac{\partial_t E_\varepsilon}{E_\varepsilon^{1/2}} ds \leq \int\limits_0^t CE_\varepsilon^{1/2}(s)ds + \int\limits_0^t C\|f(s,\cdot)\|_2 ds
$$

$$E_\varepsilon^{1/2}(t) \le E_\varepsilon^{1/2}(0) + C\int_0^t E_\varepsilon^{1/2}(s)ds + C\int_0^t \|f(s,\cdot)\|_2 ds. \tag{3.10}$$

Now we apply Grönwall's lemma, Lemma 3.2, where

$$G(t) = E_\varepsilon^{1/2}(0) + C\int_0^t \|f(s,\cdot)\|_2 ds$$

and

$$k(t) = C.$$

So

$$E_\varepsilon^{1/2}(t) \le \left(E_\varepsilon^{1/2}(0) + C\int_0^t \|f(s,\cdot)\|_2 ds\right) e^{Ct}.$$

If $\varepsilon \to 0$ then

$$E^{1/2}(t) \le \left(E^{1/2}(0) + C\int_0^t \|f(s,\cdot)\|_2 ds\right) e^{Ct}.$$

From this inequality follows the uniqueness of solutions to (3.3). Let u_1 and u_2 be two different solutions for (3.3). Let us consider the energy of $u_1 - u_2$. Since $u_1(0,\cdot) = u_2(0,\cdot) = u_0$ then $E^{1/2}(0) = 0$. The second term of the right hand side of the inequality is also 0. Since $E^{1/2}(t) \le 0$ it follows that $u_1(t,\cdot) = u_2(t,\cdot)$ on S_T.

Lemma 3.5. Let (3.3) have a solution and satisfy the conditions mentioned at the beginning of this section. Let us define

$$E_k[u] = \frac{1}{2}\sum_{|\alpha|\le k}\int_{\mathbb{R}^n} (\partial^\alpha u)^T A^0 \partial^\alpha u\, dx. \tag{3.11}$$

Then

$$\partial_t E_k \le C E_k + C E_k^{1/2}\|f\|_{H^k}, \tag{3.12}$$

where the constants depend on the bounds on A^μ and B.

Proof. We need to estimate E_k. Since

$$E[u] = \frac{1}{2}\int_{\mathbb{R}^n} u^T A^0 u\, dx$$

then

$$E_k[u] = \sum_{|\alpha|\le k} E[\partial^\alpha u].$$

Hence for $k = 0$ we obtain $E = E_0[u]$. Let us differentiate the above equation with respect of t :

$$\partial_t E_k[u] = \sum_{|\alpha|\le k} \partial_t E[\partial^\alpha u]. \tag{3.13}$$

We recall that

$$L\partial^\alpha u = \partial^\alpha L u + [L, \partial^\alpha]u = \partial^\alpha f + [L, \partial^\alpha]u.$$

From (3.9) it follows that (3.12) holds for $k = 0$. Consequently,

$$\begin{aligned} &\partial_t E[\partial^\alpha u] \le CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|L\partial^\alpha u\|_2 = \\ &CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|\partial^\alpha f + [L, \partial^\alpha]u\|_2. \end{aligned} \tag{3.14}$$

Let us compute $[L, \partial^\alpha]u$:

$$\begin{aligned} &[L, \partial^\alpha]u = A^\mu \partial_\mu \partial^\alpha u + B\partial^\alpha u - \partial^\alpha(A^\mu \partial_\mu u + Bu) = \\ &A^\mu \partial_\mu \partial^\alpha u + B\partial^\alpha u - [A^\mu \partial^\alpha \partial_\mu u + (\partial^\alpha A^\mu)\partial_\mu u + (\partial^\alpha B)u + B\partial^\alpha u] = \\ &-(\partial^\alpha A^\mu)\partial_\mu u - (\partial^\alpha B)u. \end{aligned} \tag{3.15}$$

Since all derivatives of A^μ and B are bounded then from the above expression we obtain

$$|[L, \partial^\alpha]u| \le C\left(\sum_{|\mu|=1} |\partial_\mu u| + |u|\right). \tag{3.16}$$

From (3.8) it follows that

$$|u|^2 \le Cu^T A^0 u$$

and

$$|\partial_\mu u|^2 \le C(\partial_\mu u)^T A^0 \partial_\mu u.$$

Consequently,

$$\begin{aligned} |[L, \partial^\alpha]u|^2 \le C\left(\sum_{|\mu|\le 1} |\partial_\mu u|^2 + |u|^2\right) \le \\ C\left(\sum_{|\mu|\le 1} (\partial_\mu u)^T A^0 \partial_\mu u + u^T A^0 u\right). \end{aligned}$$

Integrating both sides of this inequality over $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} |[L, \partial^\alpha]u|^2 dx \le C \int_{\mathbb{R}^n} \left(\sum_{|\mu|\le 1} (\partial_\mu u)^T A^0 \partial_\mu u + u^T A^0 u\right) dx.$$

So,

$$\|[L, \partial^\alpha]u\|_2 \le C\ E_1^{1/2}[u] \le C\ E_k^{1/2}[u],$$

where $k \geq 1$. Let us insert this result in (3.14). To not burden our inequalities with many constants we will use the same notation C for all of them.

$$\begin{aligned}
&\partial_t E[\partial^\alpha u] \leq CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\left(\|\partial^\alpha f + [L,\partial^\alpha]u\|_2\right) \leq \\
&CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|\partial^\alpha f\|_2 + CE^{1/2}[\partial^\alpha u]\|[L,\partial^\alpha]u\|_2 \leq \\
&CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|f\|_{H^k} + CE^{1/2}[\partial^\alpha u]E_k^{1/2}[u] \leq \\
&CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|f\|_{H^k} + CE^{1/2}[\partial^\alpha u]E_k^{1/2}[u].
\end{aligned}$$

In the second step we have used the Minkowski inequality. In the third step we have used the fact $\|\partial^\alpha f\|_2 \leq \|f\|_{H^k}$, when $|\alpha| \leq k$. This follows from the definition of the $\|\cdot\|_2$ norm and (1.8)

$$\|\partial^\alpha f\|_2 \leq \|f\|_{H^{|\alpha|}} \leq \|f\|_{H^k}.$$

Hence we obtain

$$\partial_t E[\partial^\alpha u] \leq CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|f\|_{H^k} + CE^{1/2}[\partial^\alpha u]E_k^{1/2}[u].$$

Inserting this result in (3.13) we get

$$\begin{aligned}
\partial_t E_k[u] = \sum_{|\alpha|\leq k} \partial_t E[\partial^\alpha u] &\leq \sum_{|\alpha|\leq k}\left(CE[\partial^\alpha u] + CE^{1/2}[\partial^\alpha u]\|f\|_{H^k} + CE^{1/2}[\partial^\alpha u]E_k^{1/2}[u]\right) \\
&\leq CE_k[u] + CE_k^{1/2}[u]\|f\|_{H^k} + CE_k[u] \leq CE_k[u] + CE_k^{1/2}[u]\|f\|_{H^k}.
\end{aligned}$$

□

Corollary 3.6. Let (3.3) have a solution and satisfy the conditions mentioned at the beginning of this section then

$$E_k^{1/2}(t) \leq C\left[E_k^{1/2}(0) + \int_0^t \|f(s,\cdot)\|_{H^k} ds\right] \text{ for } t \in [0,T], \tag{3.17}$$

where the constant C depends on k, the bounds on A^μ and B and on T.

Proof. Let us define $\Lambda_k = e^{-Ct}E_k + \varepsilon$ for $\varepsilon > 0$, here C is the first constant from (3.12). We need to estimate $\partial_t \Lambda_k$:

$$\begin{aligned}
&\partial_t \Lambda_k = \partial_t\left[e^{-Ct}E_k + \varepsilon\right] = -Ce^{-Ct}E_k + e^{-Ct}\partial_t E_k \leq \\
&-Ce^{-Ct}E_k + e^{-Ct}\left[CE_k + CE_k^{1/2}\|f\|_{H^k}\right] = Ce^{-Ct}E_k^{1/2}\|f\|_{H^k} \leq \\
&Ce^{-Ct/2}\left[e^{-Ct}E_k + \varepsilon\right]^{1/2}\|f\|_{H^k} = Ce^{-Ct/2}\Lambda_k^{1/2}\|f\|_{H^k}.
\end{aligned} \tag{3.18}$$

Above in the first inequality, we have used (3.12). Since we assumed that C depends on T it follows that $e^{-Ct/2}$ can be estimated by a constant. Let us rewrite the inequality

$$\partial_t \Lambda_k \leq C\Lambda_k^{1/2}\|f\|_{H^k}.$$

Since $\Lambda_k \geq \varepsilon > 0$, we can divide both sides of the above inequality by $\Lambda_k^{1/2}$:

$$\frac{\partial_t \Lambda_k}{\Lambda_k^{1/2}} \leq C\|f\|_{H^k}. \tag{3.19}$$

Let us integrate this inequality over $[0, t]$:

$$\int_0^t \frac{\partial_t \Lambda_k}{\Lambda_k^{1/2}} ds \leq \int_0^t C\|f\|_{H^k} ds.$$

Consequently

$$\Lambda_k^{1/2}(t) \leq \Lambda_k^{1/2}(0) + C\int_0^t \|f(s,\cdot)\|_{H^k} ds.$$

When $\varepsilon \to 0$ the difference between Λ_k and E_k tends to a constant factor. Inserting in the above inequality $\Lambda_k = CE_k$ we get the final result

$$E_k^{1/2}(t) \leq C\left[E_k^{1/2}(0) + \int\limits_0^t \|f(s,\cdot)\|_{H^k} ds\right].$$

□

Let us mention that the above estimates also hold for $\mathbb{C}^N$-valued solutions to (3.3). The real and imaginary parts of $\mathbb{C}^N$-valued solutions can be considered as two $\mathbb{R}^N$-valued solutions with suitable right hand sides. Let us define

$$E_k[u] = E_k[\operatorname{Re} u] + E_k[\operatorname{Im} u]$$

then we obtain (3.12) and (3.17) for $\mathbb{C}^N$-valued solutions.
The following Lemma will help us to prove existence of solutions to symmetric hyperbolic systems.

Lemma 3.7. Let u be a solution of (3.3), under the assumptions made at the beginning of this section. Then for $t \in [0, T]$ and for any $k \in \mathbb{Z}$ we obtain

$$\|u(t,\cdot)\|_k \leq C\left[\|u(0,\cdot)\|_k + \int_0^t \|f(s,\cdot)\|_k ds\right] \tag{3.20}$$

where the constant C depends on k, the bounds on A^μ, B and on T.

Proof. First we prove (3.20) for k a non-negative integer. Let us apply (3.17). By Theorem 1.42 the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_k$ are equivalent for k non-negative integer. Hence

$$E_k^{1/2}(t) \leq C_1\left[E_k^{1/2}(0) + \int\limits_0^t \|f(s,\cdot)\|_k ds\right].$$

Let us estimate $E_k^{1/2}$. From the equation (3.11) it follows that

$$E_k = \frac{1}{2}\sum_{|\alpha|\le k}\int\limits_{\mathbb{R}^n}(\partial^\alpha u)^T A^0 \partial^\alpha u\, dx \ge \frac{1}{2}\sum_{|\alpha|\le k}\int\limits_{\mathbb{R}^n} c_0|\partial^\alpha u|^2 dx$$
$$\ge C_0^2\|u\|_{H^k}^2.$$

In the first step we use the fact that A^0 has a uniform positive lower bound c_0. So

$$E_k^{\frac{1}{2}} \ge C_0\|u\|_{H^k} \ge C_2\|u\|_k. \tag{3.21}$$

Above in the second step we use the equivalency of the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_k$. Choosing the constant C_3 big enough we can assume that $E_k^{1/2}(0) \le C_3\|u(0,\cdot)\|_k$. Combining these results we get

$$\|u(t,\cdot)\|_k \le \frac{E_k(t)^{1/2}}{C_2} \le \frac{C_1}{C_2}\left[E_k^{1/2}(0) + \int_0^t \|f(s,\cdot)\|_k ds\right] \le$$
$$\frac{C_1}{C_2}\left[C_3\|u(0,\cdot)\|_k + \int_0^t \|f(s,\cdot)\|_k ds\right] \le C\left[\|u(0,\cdot)\|_k + \int_0^t \|f(s,\cdot)\|_k ds\right].$$

We need to prove (3.20) for k a negative integer. For $t\in[0,T]$ we define

$$g(t,\cdot) = (1-\Delta)^k u(t,\cdot). \tag{3.22}$$

From the conditions of the lemma it follows that $g(t,\cdot)$ also satisfies uniform Schwartz bounds. Let us compute $\|g(t,\cdot)\|_{-k}$:

$$\|g(t,\cdot)\|_{-k} = \left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\mathcal{F}[(1-\Delta)^k u]|^2(1+|\xi|^2)^{-k}d\xi\right]^{1/2} =$$
$$\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{2k}|\hat{u}|^2(1+|\xi|^2)^{-k}d\xi\right]^{1/2} =$$
$$\left[\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{k}|\hat{u}|^2 d\xi\right]^{1/2} = \|u(t,\cdot)\|_k.$$

Here we have used the Fourier transform definition of $(1-\Delta)^k u$, that is $\mathcal{F}[(1-\Delta)^k u] = (1+|\xi|^2)^k\hat{u}$. Let us estimate $\|u(t,\cdot)\|_k$:

$$\begin{aligned}
&\|u(t,\cdot)\|_k = \|g(t,\cdot)\|_{-k} \le C E_{-k}^{1/2}[g](t) \le \\
&C\left[E_{-k}^{1/2}[g](0) + \int_0^t \|Lg(s,\cdot)\|_{-k}\, ds\right] \le \\
&C\left[\|g(0,\cdot)\|_{-k} + \int_0^t \|Lg(s,\cdot)\|_{-k}\, ds\right] = \\
&C\left[\|u(0,\cdot)\|_{k} + \int_0^t \|Lg(s,\cdot)\|_{-k}\, ds\right].
\end{aligned} \tag{3.23}$$

In the second step we have used (3.21) and in the third step we have used (3.17). We need to estimate the last term. Recall that

$$[L,(1-\Delta)^{-k}]g = L(1-\Delta)^{-k}g - (1-\Delta)^{-k}Lg =$$
$$Lu - (1-\Delta)^{-k}Lg = f - (1-\Delta)^{-k}Lg.$$

Consequently,

$$(1-\Delta)^{-k}Lg = f - [L,(1-\Delta)^{-k}]g.$$

We have that $\|\lambda^k(\xi)\hat{u}\|_2 = (2\pi)^{n/2}\|u\|_k$. Applying the Minkowski inequality to the above expression for the norm $\|\cdot\|_2$ we obtain

$$\|(1-\Delta)^{-k}Lg(t,\cdot)\|_k \le \|f(t,\cdot)\|_k + \|[L,(1-\Delta)^{-k}]g(t,\cdot)\|_k. \tag{3.24}$$

By (1.22) we have that $\|(1-\Delta)^{-k}Lg(t,\cdot)\|_k = \|Lg(t,\cdot)\|_{-k}$. Let us rewrite the above inequality

$$\|Lg(t,\cdot)\|_{-k} \le \|f(t,\cdot)\|_k + \|[L,(1-\Delta)^{-k}]g(t,\cdot)\|_k.$$

We need to estimate $\|[L,(1-\Delta)^{-k}]g(t,\cdot)\|_k$:

$$\|[L,(1-\Delta)^{-k}]g(t,\cdot)\|_k = \|[A^\mu\partial_\mu + A^0\partial_t + B,(1-\Delta)^{-k}]g(t,\cdot)\|_k \le$$
$$\|[A^\mu\partial_\mu,(1-\Delta)^{-k}]g(t,\cdot)\|_k + \|[A^0\partial_t,(1-\Delta)^{-k}]g(t,\cdot)\|_k + \|[B,(1-\Delta)^{-k}]g(t,\cdot)\|_k.$$

Let us estimate each term of the right hand side of the above inequality. For $1 \le \mu \le n$ we obtain

$$\begin{aligned}
&\|[A^\mu\partial_\mu,(1-\Delta)^{-k}]g\|_k = \|(1-\Delta)^k([A^\mu\partial_\mu,(1-\Delta)^{-k}]g)\|_{-k} = \\
&\|(1-\Delta)^k(A^\mu\partial_\mu(1-\Delta)^{-k}g) - (1-\Delta)^k(1-\Delta)^{-k}A^\mu\partial_\mu g\|_{-k} = \\
&\|(1-\Delta)^k(A^\mu\partial_\mu(1-\Delta)^{-k}g) - A^\mu\partial_\mu g\|_{-k} = \\
&\|A^\mu\partial_\mu g - A^\mu\partial_\mu g + P(\partial)(\partial_\mu(1-\Delta)^{-k}g)\|_{-k} = \\
&\|P(\partial)(\partial_\mu(1-\Delta)^{-k}g)\|_{-k} \le C\|(1-\Delta)^{-k}g\|_k = C\|g\|_{-k}.
\end{aligned} \tag{3.25}$$

The first equality follows from (1.22). Since the linear partial derivative operator $P(\partial)$ has the order less than or equal to $2k-1$ then $P(\partial)\partial_\mu$ will have order less than or equal to $2k$. From Lemma 1.43 the last inequality follows.
With similar steps we obtain an estimate for $\|[A^0\partial_t,(1-\Delta)^{-k}]g\|_k$:

$$\begin{aligned}
&\|[A^0\partial_t,(1-\Delta)^{-k}]g\|_k = \|P(\partial)(\partial_t(1-\Delta)^{-k}g)\|_{-k} \le \\
&C\|\partial_t(1-\Delta)^{-k}g\|_{k-1} = C\|\partial_t g\|_{-k-1}.
\end{aligned} \tag{3.26}$$

Since the linear partial derivative operator $P(\partial)$ has order less than or equal to $2k-1$ then by Lemma 1.43 we obtain the first inequality. It remains to estimate the last term

$$\|[B,(1-\Delta)^{-k}]g\|_k \le C\|(1-\Delta)^{-k}g\|_k = C\|g\|_{-k}. \tag{3.27}$$

Combining the above results we obtain

$$\|[L, (1-\Delta)^{-k}]g(t,\cdot)\|_k \le C\|g(t,\cdot)\|_{-k} + C\|\partial_t g(t,\cdot)\|_{-k-1}.$$

It remains to estimate $\|\partial_t g(t,\cdot)\|_{-k-1}$. From (1.22) it follows that

$$\|\partial_t g(t,\cdot)\|_{-k-1} = \|(1-\Delta)^{-k}\partial_t g\|_{k-1}. \tag{3.28}$$

To compute $\|(1-\Delta)^{-k}\partial_t g\|_{k-1}$ we need to define the operator L_0 :

$$L_0 u = (A^0)^{-1}Lu.$$

With this notation we have

$$\begin{aligned} &[L_0, (1-\Delta)^{-k}]g = L_0(1-\Delta)^{-k}g - (1-\Delta)^{-k}L_0 g = \\ &L_0 u - (1-\Delta)^{-k}L_0 g = (A^0)^{-1}f - (1-\Delta)^{-k}L_0 g. \end{aligned} \tag{3.29}$$

We can rewrite the last term as

$$(1-\Delta)^{-k}L_0 g = (1-\Delta)^{-k}(L_0 - \partial_t)g + (1-\Delta)^{-k}\partial_t g.$$

Inserting this expression to (3.29) we get

$$[L_0, (1-\Delta)^{-k}]g = (A^0)^{-1}f - \left[(1-\Delta)^{-k}(L_0 - \partial_t)g + (1-\Delta)^{-k}\partial_t g\right].$$

So

$$(1-\Delta)^{-k}\partial_t g = (A^0)^{-1}f - (1-\Delta)^{-k}(L_0 - \partial_t)g - [L_0, (1-\Delta)^{-k}]g.$$

Let us estimate $(1-\Delta)^{-k}\partial_t g$ in H_{k-1}.

$$\begin{aligned} &\|(1-\Delta)^{-k}\partial_t g\|_{k-1} \le \|(A^0)^{-1}f\|_{k-1} + \|(1-\Delta)^{-k}(L_0 - \partial_t)g\|_{k-1} \\ &+ \|[L_0, (1-\Delta)^{-k}]g\|_{k-1}. \end{aligned} \tag{3.30}$$

We need to estimate each term of this sum. Due to Lemma 1.45

$$\|(A^0)^{-1}f\|_{k-1} \le C\|f\|_{k-1}. \tag{3.31}$$

Let us estimate the second and the third terms on the right hand side of (3.30).

$$\begin{aligned} &L_0 = (A^0)^{-1}A^0\partial_t + (A^0)^{-1}A^i\partial_i + (A^0)^{-1}B = \partial_t + B^i\partial_i + B^0 \\ &L_0 - \partial_t = B^i\partial_i + B^0, \end{aligned}$$

where we denote $(A^0)^{-1}A^i = B^i$ and $(A^0)^{-1}B = B^0$.

$$\begin{aligned} &\|(1-\Delta)^{-k}(L_0 - \partial_t)g\|_{k-1} = \|(L_0 - \partial_t)g\|_{-k-1} = \|B^i\partial_i g + B^0 g\|_{-k-1} \le \\ &\|B^i\partial_i g\|_{-k-1} + \|B^0 g\|_{-k-1} \le C\|g\|_{-k} + C\|g\|_{-k-1} \le C\|g\|_{-k}. \end{aligned} \tag{3.32}$$

In the second inequality we have used Lemma 1.43 and Lemma 1.46.
We need to compute the last term of (3.30)

$$\|[L_0,(1-\Delta)^{-k}]g\|_{k-1} = \|[\partial_t + B^i\partial_i + B^0,(1-\Delta)^{-k}]g\|_{k-1} \leq$$
$$\|[\partial_t,(1-\Delta)^{-k}]g\|_{k-1} + \|[B^i\partial_i,(1-\Delta)^{-k}]g\|_{k-1} + \|[B^0,(1-\Delta)^{-k}]g\|_{k-1}.$$

Let us note that $\|[\partial_t,(1-\Delta)^{-k}]g\|_{k-1} = 0$. With similar steps like in (3.25) we obtain:

$$\|[B^i\partial_i,(1-\Delta)^{-k}]g\|_{k-1} \leq C\|g\|_{-k-1}.$$

$$\|[B^0,(1-\Delta)^{-k}]g\|_{k-1} \leq C\|(1-\Delta)^{-k}g\|_{k-1} = C\|g\|_{-k-1} \leq C\|g\|_{-k}.$$

We conclude that

$$\|[L_0,(1-\Delta)^{-k}]g\|_{k-1} \leq C\|g\|_{-k}. \tag{3.33}$$

Inserting (3.31), (3.32) and (3.33) in (3.30) we obtain

$$\|\partial_t g(t,\cdot)\|_{-k-1} \leq C\left[\|f(t,\cdot)\|_{k-1} + \|g(t,\cdot)\|_{-k} + \|g(t,\cdot)\|_{-k}\right] \leq$$
$$C\left[\|u(t,\cdot)\|_k + \|f(t,\cdot)\|_k\right].$$

Finally we can estimate $\|u(t,\cdot)\|_k$:

$$\|u(t,\cdot)\|_k \leq C\left[\|u(0,\cdot)\|_k + \int_0^t \left[\|u(s,\cdot)\|_k + \|f(s,\cdot)\|_k\right]ds\right].$$

Since the conditions of Grönwall's lemma are satisfied we can apply Lemma 3.2:

$$\|u(t,\cdot)\|_k \leq C\left[\|u(0,\cdot)\|_k + \int_0^t \|f(s,\cdot)\|_k ds\right]e^{Ct} \leq$$
$$C\left[\|u(0,\cdot)\|_k + \int_0^t \|f(s,\cdot)\|_k ds\right].$$

□

Corollary 3.8. Let u be a solution of (3.3)under the assumptions made at the beginning of this section. Then for $t \in [0,T]$ and for $k \in \mathbb{Z}$ we obtain

$$\|u(t,\cdot)\|_k \leq C\left[\|u(T,\cdot)\|_k + \int_t^T \|f(s,\cdot)\|_k ds\right], \tag{3.34}$$

where the constant C depends on k, the bounds on A^μ, B and on T.

Proof. Let us define the operator L_1 :

$$L_1 = -A^0(T-t,x)(\partial_t u)(t,x) + A^i(T-t,x)(\partial_i u)(t,x) + B(T-t,x)u(t,x).$$

and denote

$$v(t,x) = u(T-t,x).$$

The operators $-L_1$ and L are of the same type. Hence we can apply Lemma 3.7 and obtain

$$\|v(t,\cdot)\|_k \leq C\left[\|v(0,\cdot)\|_k + \int_0^t \|L_1 v(s,\cdot)\|_k ds\right],$$

for all $k \in \mathbb{Z}$ and all $t \in [0,T]$. Since $(L_1 v)(t,x) = (Lu)(T-t,x)$ we can rewrite the above inequality as

$$\|u(t,\cdot)\|_k \leq C\left[\|u(T,\cdot)\|_k + \int_t^T \|Lu(s,\cdot)\|_k ds\right].$$

□

Now we are ready to prove uniqueness and existence of solutions to (3.3).

3.2 Uniqueness and Existence

In this last section we follow [Rin09, §7.3 and §7.4].

Theorem 3.9 (**Uniqueness**). *Let A^μ and B be maps from $\mathbb{R}^{n+1}$ into the space of real-valued $N \times N$ matrices with A^μ, $\mu = 0,1,\ldots,n$ symmetric in C^1 and B in C^0. Suppose that for any compact interval $[T_1,T_2]$, A^0 is positive definite on $[T_1,T_2]\times\mathbb{R}^n$ with a constant positive lower bound and that all the matrices A^μ are bounded on $[T_1,T_2]\times\mathbb{R}^n$. We assume that $f:\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^N$ is continuous. Let u_1 and u_2 be two C^1-solutions to (3.3) defined on $(a,b)\times\mathbb{R}^n$ where $a<0$ and $b>0$, with corresponding initial data u_{01} and u_{02}. Assume that $T_1 \leq 0, T_2 \geq 0$ and let $[T_1,T_2]$ be a compact subinterval of (a,b). If $u_{01}(x) = u_{02}(x)$ for $x \in B_r(x_0)$ then there is an $s_0 > 0$ depending on the lower bound on A^0 and the upper bound on A^i in $[T_1,T_2]$ such that*

$$u_1(t,x) = u_2(t,x) \quad \textit{for} \quad (t,x) \in C_{x_0,r,s_0,T_1,T_2}, \tag{3.35}$$

where

$$C_{x_0,r,s_0,T_1,T_2} = C = \{(t,x)\in[T_1,T_2]\times\mathbb{R}^n : |t| < r/s_0, \quad x \in B_{r-s_0 t}(x_0)\}. \tag{3.36}$$

Moreover, if u is a C^1 solution to (3.3) on $[T_1,T_2]\times\mathbb{R}^n, u_0(x) = 0$ for $x\in B_r(x_0)$ and $f(t,x) = 0$ for $(t,x)\in C$, then $u(t,x) = 0$ for $(t,x)\in C$.

Proof. First we prove the second statement of the theorem, then by setting $u = u_1 - u_2$ the first statement follows. Let us denote

$$D = C_{x_0,r,s_0,0,T_2}.$$

Let us compute the following expression

$$\partial_\alpha\left[e^{-kt}u^T A^\alpha u\right],$$

where k is a constant and u^T is the transpose of u. Here we use notation $\partial_0 = \partial_t$:

$$
\begin{aligned}
&\partial_\alpha \left[e^{-kt}u^T A^\alpha u\right] = -ke^{-kt}u^T A^0 u + e^{-kt}\partial_\alpha \left[u^T A^\alpha u\right] = \\
&- ke^{-kt}u^T A^0 u + e^{-kt} \left[\partial_\alpha(u^T)A^\alpha u + u^T(\partial_\alpha(A^\alpha)u + A^\alpha \partial_\alpha u)\right] = \\
&- ke^{-kt}u^T A^0 u + e^{-kt} \left[2u^T A^\alpha \partial_\alpha u + u^T \partial_\alpha(A^\alpha)u\right] = \\
&- ke^{-kt}u^T A^0 u + e^{-kt} \left[2u^T f - 2u^T Bu + u^T \partial_\alpha(A^\alpha)u\right] = \\
&e^{-kt}u^T \left[-kA^0 - 2B + \partial_\alpha A^\alpha\right] u + 2e^{-kt}u^T f.
\end{aligned}
$$

In the third equality we used that $\partial_\alpha(u^T)A^\alpha u = u^T A^\alpha \partial_\alpha u$, since the A^α are symmetric matrices. In the forth equality we used the equality (3.3), that is $A^\alpha \partial_\alpha u = f - Bu$. Let us integrate both sides of the above equality over D :

$$
\int_D \partial_\alpha \left[e^{-kt}u^T A^\alpha u\right] dx = \int_D e^{-kt}u^T \left[-kA^0 - 2B + \partial_\alpha A^\alpha\right] u + 2e^{-kt}u^T f dx. \tag{3.37}
$$

Let us apply Stokes' theorem for the ordinary Euclidean metric on $\mathbb{R}^{n+1}$ [Rin09, eq.(10.3)] to the left hand side of the above equality

$$
\int_D \partial_\alpha \left[e^{-kt}u^T A^\alpha u\right] dx = \int_{\partial D} (e^{-kt}u^T A^\alpha u)N_\alpha d\sigma, \tag{3.38}
$$

where $N_\alpha, \quad \alpha = 0, 1, \ldots, n$ is the unit outward normal to ∂D, and $d\sigma$ is its surface element. Let us denote by B the bottom of the cone D and by H the hull of D, then $\partial D = B \cup H$. Let us rewrite the right hand side of (3.38)

$$
\int_{\partial D} (e^{-kt}u^T A^\alpha u)N_\alpha d\sigma = \int_B (e^{-kt}u^T A^\alpha u)N_\alpha d\sigma + \int_H (e^{-kt}u^T A^\alpha u)N_\alpha d\sigma.
$$

The integral over B in the above equality is 0 since $u(0, x) = 0$. For s_0 big enough N_0 is much bigger than $N_i, \quad i = 1, 2 \ldots, n$. Since $N_0 \cdot A^0$ dominates $\sum_{i=1}^{n} N_i \cdot A^i$ and is positive definite then $(e^{-kt}u^T A^\alpha u)N_\alpha$ is positive definite on H. Let us fix such an s_0, then the right hand side of (3.38) is non negative. By the conditions of the theorem f is zero in D. By assumption A^0 has a positive uniform lower bound and B, and the $\partial_\alpha A^\alpha$ have a uniform upper bound on D. Hence for some positive constant M and by choosing k large enough we obtain

$$
\int_D e^{-kt}u^T \left[-kA^0 - 2B + \partial_\alpha(A^\alpha)\right] udx \leq -M \int_D e^{-kt}u^T udx. \tag{3.39}
$$

Since in (3.37) the left-hand side is non-negative and the right hand side is non-positive then both sides of the equality have to be zero. So u is zero in D. The claim in the negative times follows by reversing time. $\square$

From the theorem we conclude the following Remark.

Remark 3.10. If the initial data coincide then $u_1(t,x) = u_2(t,x)$ in their domain. Let us suppose that u_0 has compact support and for any compact interval $[T_1, T_2]$, there is a compact set K such that $f(t,x) = 0$ for $t \in [T_1, T_2]$ and $x \notin K$, then there is a compact set K_1 such that $u(t,x) = 0$ for $t \in [T_1, T_2]$ and $x \notin K_1$.

Theorem 3.11 (**Existence**). *In the initial value problem (3.3), suppose that $u_0 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$, $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$. Let the real-valued $N \times N$ matrices A^μ, $\mu = 0, 1, \ldots, n$ and B be smooth functions on $\mathbb{R}^{n+1}$, with all derivatives bounded. Let A^μ be symmetric and A^0 positive definite with a uniform positive lower bound. Then there exists a unique solution $u \in C^\infty([0,T) \times \mathbb{R}^n, \mathbb{R}^N)$ to (3.3) and a compact set $K \subset \mathbb{R}^n$ such that $u(t,x) = 0$ for $t \in [0,T]$, $T > 0$ and $x \notin K$.*

Proof. Recall that

$$L = A^\mu \partial_\mu + B.$$

Let us denote

$$\begin{aligned} L^* = -A^0\partial_t u - A^i \partial_i u + (-\partial_t A^0 - \partial_i A^i + B^T)u = \\ -\partial_t(A^0 u) - \partial_i(A^i u) + B^T u. \end{aligned} \tag{3.40}$$

Then $-L^*$ and L are operators of the same type. Hence we can apply Corollary 3.8 to the operator L^*

$$\|\phi(t,\cdot)\|_{-k} \leq C \int_t^T \|(L^*\phi)(s,\cdot)\|_{-k} ds \tag{3.41}$$

for every $\phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ such that $\phi(t,x) = 0$ for all $t \geq T$. Let us note that if $\psi, \phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$, $\psi(t,x) = \phi(t,x) = 0$ for $t \geq T$ and $L^*\phi = L^*\psi$ then from (3.41)

$$\phi(t,\cdot) = \psi(t,\cdot) \text{ for all } t \in [0,T].$$

Let us define for such a ϕ and for $f \in L^1\{[0,T], H_k(\mathbb{R}^n, \mathbb{C}^N)\}$

$$F(L^*\phi) = \langle f, \phi \rangle = \int_0^T (\phi(t), f(t))_{L^2} dt. \tag{3.42}$$

From the above observations and since ϕ is a smooth function it follows that F is a well defined functional. Assume that $k \geq 0$. From (1.44) it follows that

$$|F(L^*\phi)| = |\langle f, \phi \rangle| \leq \|f\|_k \|\phi\|_{-k}.$$

Inserting in the above inequality (3.41) we obtain

$$|F(L^*\phi)| \leq C \int_0^T \|(L^*\phi)(s,\cdot)\|_{-k} ds. \tag{3.43}$$

Consequently we can consider $L^*\phi$ as an element of

$$X = L^1\{[0,T], H_{-k}(\mathbb{R}^n, \mathbb{C}^N)\}.$$

Let us denote by M the following subspace of X:

$$M = \operatorname{span}\{L^*\phi :\ \phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N) \text{ such that } \phi(t,x) = 0 \text{ for } t \geq T\} \subseteq X.$$

So F is a bounded linear functional on M. Applying the Hahn-Banach theorem (Theorem 5.16 [Rud87]), we can extend F to a bounded linear functional on X. The norm of the extension and the norm of the functional restricted to M coincide. According to Proposition 1.50 there is a $u \in L^\infty\{[0,T], H_k(\mathbb{R}^n, \mathbb{C}^N)\}$ such that

$$F(L^*\phi) = \int_0^T (\phi(t), f(t))_{L^2} dt = \int_0^T (L^*\phi(t), u(t))_{L^2} dt.$$

First we prove the theorem for $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ such that $f(t,\cdot) = 0$ for all $t \leq 0$. Hence we can extend u to $L^\infty\{[-\infty,T], H_k(\mathbb{R}^n, \mathbb{C}^N)\}$ by setting it 0 for $t < 0$. So

$$\int_{-\infty}^T (\phi(t), f(t))_{L^2} dt = \int_{-\infty}^T (L^*\phi(t), u(t))_{L^2} dt \tag{3.44}$$

$\forall \phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N)$ such that $\phi(t,\cdot) = 0$ for all $t \geq T$. According to Lemma A.5 of [Rin09] there exists a $U \in L^2_{\text{loc}}[(-\infty,T) \times \mathbb{R}^n, \mathbb{C}^N]$ which is k times weakly differentiable with respect to x such that

$$\int_{-\infty}^T \int_{\mathbb{R}^n} \phi \cdot \bar{f} dx dt = \int_{-\infty}^T \int_{\mathbb{R}^n} L^*\phi \cdot \bar{U} dx dt. \tag{3.45}$$

Assume inductively that for $j + |\alpha| \leq k$ and $j \leq l \leq k-1$ there is a function $U_{j,\alpha} \in L^2_{\text{loc}}[(-\infty,T) \times \mathbb{R}^n, \mathbb{C}^N]$ such that

$$\int_{-\infty}^T \int_{\mathbb{R}^n} \phi \cdot \bar{U}_{j,\alpha} dx dt = (-1)^{j+|\alpha|} \int_{-\infty}^T \int_{\mathbb{R}^n} \partial_t^j \partial^\alpha \phi \cdot \bar{U} dx dt \quad \forall\, \phi \in C_0^\infty((-\infty,T) \times \mathbb{R}^n, \mathbb{C}^N).$$

For brevity we write $\partial_t^j \partial^\alpha U = U_{j,\alpha}$. The case $l = 0$ follows from (3.45). Let us mention that any $\psi \in C_0^\infty((-\infty,T) \times \mathbb{R}^n, \mathbb{C}^N)$ we can write as $\psi = A^0\phi$. Let us set

$$g = (A^0)^{-1}[f - A^i \partial_i U - BU] \tag{3.46}$$

and rewrite (3.45) as

$$\int_{-\infty}^T \int_{\mathbb{R}^n} \psi \cdot \bar{g} dx dt = -\int_{-\infty}^T \int_{\mathbb{R}^n} \partial_t \psi \cdot \bar{U} dx dt. \tag{3.47}$$

From the induction assumption the weak derivatives $\partial^\alpha \partial_t^j g$ exists and is in $L^2_{\mathrm{loc}}[(-\infty, T)\times \mathbb{R}^n, \mathbb{C}^N]$ for $|\alpha| + j \leq k-1$ and any non-negative integer j with $j \leq l$. Let us insert $\partial^\alpha \partial_t^l \psi$ in (3.47) instead of ψ. Hence we get that the induction holds for $l+1$. So U is k times weakly differentiable with respect to (t,x) in $(-\infty, T)\times \mathbb{R}^n$. From Lemma 2.7 it follows that for k large enough U is continuously differentiable. From the definition of L^* in (3.40), it follows that L^* is the adjoint operator of L. Due to (3.45) and the standard properties of adjoint operators we have

$$\langle \phi, f\rangle = \langle L^*\phi, U\rangle = \langle \phi, (L^*)^* U\rangle = \langle \phi, LU\rangle \quad \forall \phi \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{C}^N). \tag{3.48}$$

Hence $LU = f$ and $U = 0$ for $t \leq 0$. It remains to show that U is smooth. Here for each k we obtain its corresponding U. Due to uniqueness we can assume that the solutions coincide when they are in C^1. Since the solutions U coincide for k large enough then the smoothness of the solution U follows.
Let us now prove the existence theorem in the general case $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$. Let us define

$$f_\varepsilon(t,x) = \eta(t/\varepsilon) f(t,x). \tag{3.49}$$

Here $\eta \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\eta(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ 1, & \text{for } t \geq 1 \\ [0,1]\,, & \text{otherwise.} \end{cases}$$

We have already proved that there is a smooth solution u_ε to the equation $Lu_\varepsilon = f_\varepsilon$ such that $u_\varepsilon(t,x) = 0$ for all $t \leq 0$. Due to Theorem 3.9 there is a compact set K such that

$$u_\varepsilon(t,x) = 0 \text{ for } x \notin K,\ t \leq T \text{ and } \ \forall \varepsilon > 0. \tag{3.50}$$

Applying to (3.20) we obtain

$$\|(u_{\varepsilon_1} - u_{\varepsilon_2})(t,\cdot)\|_k \leq C \int_0^t |\eta(s/\varepsilon_1) - \eta(s/\varepsilon_2)| \|f(s,\cdot)\|_k ds.$$

Consequently the Cauchy sequence $u_\varepsilon(t,\cdot)$ converges in any H^k, which is a Banach space. So $u_\varepsilon(t,\cdot)$ converges to u in H^k. By (2.8) $u_\varepsilon(t,\cdot)$ converges to u in any C^k-norm when $\varepsilon \longrightarrow 0$. Applying this to (3.3) we get convergence of any number of time derivatives. Hence we obtain a smooth solution u on $(0,T)\times\mathbb{R}^n$. We need to extend the solution to $t = 0$. Let us define $u(0,\cdot) = 0$. The higher time derivatives at 0 we get recursively from (3.3). We need to prove that $\partial_t u$ and its higher time derivatives are continuous when $t \to 0+$. Applying (3.20) for $u_\varepsilon(t,x)$ we obtain

$$\|u_\varepsilon(t,\cdot)\|_k \leq C \int_0^t \|f_\varepsilon(s,\cdot)\|_k ds \leq C \int_0^t \|f(s,\cdot)\|_k ds.$$

The right hand side of the above inequality is independent of ε, hence this inequality holds for u. Applying Lemma 2.7 for k large enough we get the above inequality for any C^k-norm in particular for C^0-norm. We conclude that $u(t,\cdot)$ converges to 0 when $t \to 0+$, hence $u(t,\cdot)$ is continuous at 0. From (3.3) we get

$$\partial_t u(0,\cdot) = (A^0)^{-1}[f - A^i \partial_i u(0,\cdot)].$$

We conclude that $\partial_t u(t,\cdot)$ converges to its limit in any C^k-norm. With similar steps, using the above equality, we obtain the continuity at 0 for all higher time derivatives of u. So the equation (3.3) for $u_0 = 0$ and for $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ has a smooth solution on $[0,T) \times \mathbb{R}^n$. For $u_0 \neq 0$ we replace u with $u - u_0\psi$, where $\psi \in C_0^\infty(\mathbb{R})$ and $\psi(t) = 1$ for $t \in [-1, T+1]$. Consequently for $f \in C_0^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ and by assumption that $u_0 \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$ we obtain a smooth solution. □

Corollary 3.12. In the initial value problem (3.3), suppose that $u_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, $f \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$. The real-valued $N \times N$ matrices A^μ, B are supposed to be smooth functions on $\mathbb{R}^{n+1}$. Let A^μ be symmetric and for any compact interval $[T_1, T_2]$ there are constants $a_0, b_0 > 0$ such that $A^0 \geq a_0$ and $\|A^\mu\| \leq b_0$, $\mu = 0, 1, \ldots, n$ on $[T_1, T_2] \times \mathbb{R}^n$. Then there is a unique solution $u \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R}^N)$ to (3.3).

Proof. First we construct the solution on $[T_1, T_2] \times \mathbb{R}^n$ where $T \in (0, \infty)$. Then we extend the solution to $\mathbb{R}$. Let us assume that s_0 be the same as in Theorem 3.9. For $[T_1, T_2] = [0, T]$ and $r \geq T s_0 + 1$, we define

$$C_r = \{(t,x) \in [0,T] \times \mathbb{R}^n : x \in B_{r - s_0 t}(0)\}. \tag{3.51}$$

Assume that $\psi_r \in C_0^\infty(\mathbb{R}^{n+1})$ such that

$$\psi_r(t,x) = \begin{cases} 1, & (t,x) \in C_{2r+2s_0T} \\ [0,1], & \text{otherwise.} \end{cases}$$

Let $\phi_r \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\phi_r(x) = \begin{cases} 1, & x \in B_r(0) \\ 0, & x \notin B_{2r}(0). \end{cases}$$

Denoting by

$$A_r^0 = \psi_r A^0 + (1 - \psi_r) A^0(0,0), \qquad A_r^i = \psi_r A^i, \qquad B_r = \psi_r B, \qquad u_{0_r} = \phi_r u_0,$$

and $f_r(t,x) = \psi_r(t,x)\phi_r(x) f(t,x)$. The equation

$$\begin{aligned} &A_r^\mu \partial_\mu u + B_r u = f_r, \\ &u(0,\cdot) = u_{0_r} \end{aligned} \tag{3.52}$$

has a smooth solution u_r by Theorem 3.11. Since s_0 depends on a lower bound on A^0 and an upper bound on A^i then s_0 depends on a lower bound on A_r^0 and an upper bound on A^i_r. Applying Theorem 3.9 with this s_0 to (3.52) we obtain that $u_r(t,x) = 0$ when $t \in [0,T]$ and $x \notin B_{2r+s_0t}(0)$. Hence we conclude when $u_r(t,x) \neq 0$ and $t \in [0,T]$ then

$$A^\mu_r(t,x) = A^\mu(t,x) \qquad \text{and} \qquad B_r(t,x) = B(t,x). \tag{3.53}$$

So u_r is a solution to the equation

$$\begin{aligned} &A^\mu \partial_\mu u + Bu = f_r, \\ &u(0,\cdot) = u_{0_r} \end{aligned} \tag{3.54}$$

on $[0,T) \times \mathbb{R}^n$. Then u_r is a solution to (3.3) in region C_r. Let us prove uniqueness of the solution. Consider two different solutions u_{r_1} and u_{r_2} where $r_1 < r_2$. From uniqueness it follows that $u_{r_1} = u_{r_2}$ on C_{r_1}. Assume that $(t,x) \in [0,T) \times \mathbb{R}^n$. Let us choose r such that $(t,x) \in C_r$ and define $u(t,x) = u_r(t,x)$. We have shown that the solution does not depend on r. So we obtain a smooth solution to the (3.3) on $[0,T) \times \mathbb{R}^n$. It remains to extend the solution on $[0,T) \times \mathbb{R}^n$ to $\mathbb{R}^{n+1}$. By using the uniqueness argument we define the solution for arbitrary T. The claim in the opposite time direction follows by reversing time. □

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