



# DIPLOMARBEIT / DIPLOMA THESIS

Titel der Diplomarbeit / Title of the Diploma Thesis

„The Schwarz–Christoffel Mapping and some related  
applications “

verfasst von / submitted by

Dr. Paolo Raimo

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Magister der Naturwissenschaften (Mag. rer. nat.)

Wien, 2021 / Vienna, 2021

Studienkennzahl lt. Studienblatt / A 190 406 884

degree programme code as it appears on  
the student record sheet:

Studienrichtung lt. Studienblatt / Lehramtsstudium

degree programme as it appears on UF Mathematik  
UF Informatik und Informatikmanagement

the student record sheet:

Betreut von / Supervisor:

Univ Prof. Bernhard Lamel, PhD

# Contents

1	Introduction.....	4
1.1	Complex analysis .....	4
1.2	The Mapping Problem .....	6
1.2.1	Some other conformal mapping plotted examples.....	9
1.2.2	Phase plot: visualizing of complex functions.....	10
1.3	Conformal transformations.....	11
1.4	Moebius transformation .....	13
1.5	The History of Schwarz-Christoffel Mapping.....	14
1.6	Last developments: numerical methods .....	16
1.6.1	Crowding.....	17
1.6.2	Conformal mappings onto multiple connected domains.....	18
2	Schwarz-Christoffel transformation .....	19
2.1	Introduction.....	19
2.2	Schwarz-Christoffel theorem.....	20
2.2.1	Theorem formulation .....	20
2.3	Schwarz-Christoffel mapping from the unit disk.....	22
2.3.1	Special case for regular polygons .....	22
2.4	Schwarz-Christoffel mapping from a strip.....	24
3	Example of computer assisted solutions.....	24
3.1	Introduction to MATLAB SC Toolbox: notation on angles.....	24
3.2	Example: upper half-plane mapped to a strip.....	25
3.2.1	Integration.....	26
3.2.2	MATLAB power expressions and explicit solution of indefinite integrals.....	27
3.3	Example 2: unit disk mapped to a L-shaped region .....	28
3.3.1	Crowding on an L-shaped region.....	29
4	Physics applications to 2D fluid flow and 2D electrostatics .....	30
4.1	Two-dimensional free-streamline flows.....	31
4.2	Image of a two-dimensional fluid flow turning a corner.....	32

4.2.1	Fluid flow over a step .....	34
4.2.2	Fluid flow around a slit .....	35
4.3	Electrostatic fields representation using SC mapping.....	36
5	Beautiful Math (aesthetic pattern based on SC transformation).....	37
5.1	Implementation Details.....	38
5.2	Gallery of Polygonal Patterns .....	39
6	Bibliography.....	40
7	Index of figures .....	43
8	Index of tables .....	44
9	Abstract .....	45
10	Kurzzusammenfassung.....	45

# 1 Introduction

## 1.1 Complex analysis

Following books have been used for the basic definitions:

- Complex variables and applications (Brown & Churchill, 2009)
- Visual complex analysis (Needham, 1998)
- Schwartz Christoffel Mapping (Driscoll & Trefethen, 2002)
- Complex Analysis and Conformal Mapping (Olver, 2017)

### Definition 1.1. Complex numbers.

A complex number has the form  $z=a+ib$ ,  $a, b \in \mathbb{R}$ , where:

- the imaginary unit is defined as  $i=\sqrt{-1}$
- $a$  is the real part of  $z$ ,  $a=Re(z)$ ,
- $b$  is the imaginary part of  $z$ ,  $b=Im(z)$ .
- The set of all complex numbers is denoted by  $\mathbb{C}$ .

### Definition 1.2. Arguments of complex number.

Let  $r$  and  $\theta$  ( $r, \theta \in \mathbb{R}$ ) be polar coordinates of the point  $a, b$  ( $(a, b) \in \mathbb{R}^2$ ) that corresponds to a non-zero complex number  $z=a+ib$ .

Since  $a=r \cdot \cos\theta$  and  $b=r \cdot \sin\theta$ , the number  $z$  can be written in polar form as

$$z= r \cdot (\cos\theta + i\sin\theta) = r e^{i\theta}$$

where  $r$  is a positive real number called the modulus of  $z$  (written also as  $|z|$ ), representing the distance of  $z$  from the origin  $0$ .

The phase or argument (written as  $argz$ )  $\theta$  is a real number, is defined up to an integer multiple of  $2\pi$ , represents the oriented angle between the real axis and the ray from the origin to  $z$ .  $\theta$  is measured in radians.

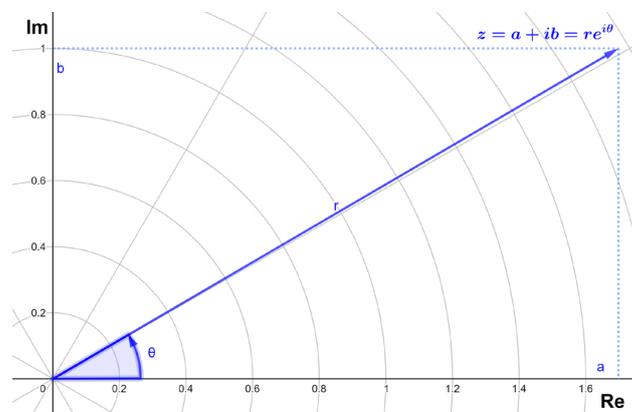


Figure 1: Cartesian and polar representation of a complex number.

**Theorem 1.1. Law of arguments of products.**

Let  $b, c \in \mathbb{C}$ . Then  $\arg (b \cdot c) = \arg b + \arg c$ . (1)

**Theorem 1.2. Law of arguments of power.**

Let  $b \in \mathbb{C}, c \in \mathbb{R}$ . Then  $\arg b^c = c \cdot \arg b$ . (2)

**Definition 1.3 Complex Function**

A complex function is a function  $f$  whose domain and range are subsets of the set  $\mathbb{C}$  of complex numbers.

The function  $f$  of the complex variable  $z$  is then a rule that assigns to each value  $z$  in a set  $D$ , one and only one complex value  $w$ :  $w=f(z)$ , where  $w$  is called the image of  $z$  under  $f$ .

The set  $D$  is the *definition domain of  $f$* , the set of all images  $R = \{w = f(z): z \in D\}$  is called the *range of  $f$* .

Any such complex function  $f(z)$ , with  $z \in D$ , can be uniquely written as a complex combination:

$$f(z) = f(x + i y) = u(x, y) + i v(x, y) = w$$

of two real functions, each depending on the two real variables  $x, y$ :

its real part  $u(x, y) = \text{Re} (f(z)) = \text{Re} (w)$  and its imaginary part  $v(x, y) = \text{Im} (f(z)) = \text{Im} (w)$ .

The term “complex analysis” refers to the calculus of complex-valued functions  $f(z)$  depending on a single complex variable  $z$ .

Complex differentiable functions (differently from what happens in the real domain  $\mathbb{R}$ ) are necessarily analytic, meaning that they can be represented by convergent power series, and hence are infinitely differentiable.

A complex function  $f(z)$  that is analytic in every point of a certain region  $R$  is said to be analytic in  $R$ .

The points of non-analyticity are called *singular points*.

If the function  $f(z)$  is analytic on the whole complex plane, then  $f(z)$  is called *entire*.

Analytic functions have a wide range of physical and mathematical applications, and most of the results named in this work apply only to analytic functions.

**Definition 1.4 Polygons**

For the rest of this document, a (generalized) **polygon**  $\Gamma$  is defined by a collection of vertices  $w_1, \dots, w_n$  and real interior angles  $\alpha_1, \dots, \alpha_n$ . We define for indexing purposes:  $w_{n+1} = w_1$  and  $w_0 = w_n$ . The vertices, which lie in the extended complex plane  $\mathbb{C} \cup \{\infty\}$ , are given in anticlockwise order with respect to the interior of the polygon.

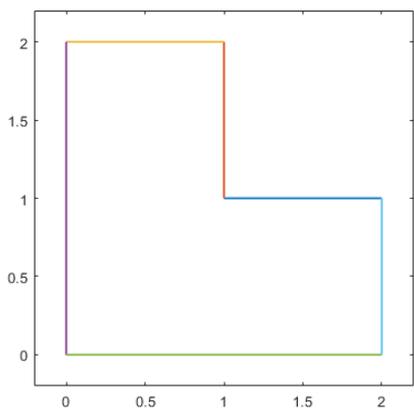


Figure 2: A generalized polygon drawn in MATLAB with SC Toolbox

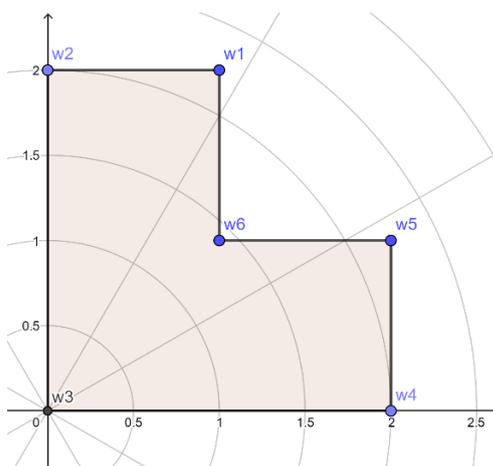


Figure 3: A generalized polygon drawn in GeoGebra

## 1.2 The Mapping Problem

Mapping between two copies of the complex plane, show very important geometrical aspects of complex functions.

A mapping  $f$  from a set  $D$  in the complex  $z$  plane to a set  $D'$  in the complex  $w$  plane is denoted by:

$$w = f(z), z \in D;$$

i.e.,  $f$  takes a point from  $D$  in the  $z$ -plane into a unique point  $w$ , in the  $w$ -plane ( $D$  is the domain set,  $D'$  the target set or range).

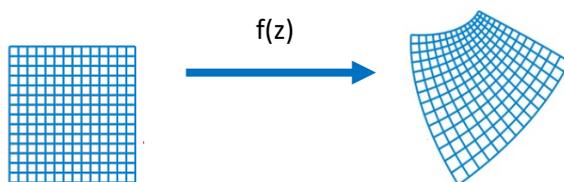


Figure 4: Conformal mapping

### Definition 1.5 Conformal mapping.

Let  $w = f(z)$  be a complex mapping defined in a domain  $D$  and let  $z_0$  be a point in  $D$ . Then we can say that  $w = f(z)$  is *conformal* at  $z_0$ , if it preserves oriented angles between curves through  $z_0$ .

In particular for every pair of smooth oriented curves  $C_1$  and  $C_2$  in  $D$  intersecting at  $z_0$ , the angle between  $C_1$  and  $C_2$  at  $z_0$  is equal to the angle between the image curves  $C'_1$  and  $C'_2$  at  $f(z_0)$  in both magnitude and sense.

The transformation  $w = f(z)$  is then said to be a “*conformal transformation*”, or a *conformal mapping*, when it is conformal at each point in  $D$ .

The transformation  $w = f(z)$  is conformal if and only if it is holomorphic and its derivative  $f'(z)$  is everywhere non-zero on  $D$ .

To classify the definitions of the different mapping in relation on how angles are eventually preserved, we can say that:

- a mapping is called **conformal** (or angle-preserving, or biholomorphic) if it preserves local oriented angles between curves with respect to their magnitude and orientation
- If every angle is mapped to an angle of same magnitude but opposite sign (orientation), the mapping is called **anticonformal**
- a mapping preserving the magnitude of angles, but without the information whether the orientation is preserved or not, is called an **isogonal mapping** (Brown & Churchill, 2009, p. 241).

To summarize, conformal maps preserve both angles and the shapes of infinitesimally small figures, but not their size or curvature.

As visible in Figure 5, in the transformation  $w = z^a$ ,  $a \in \mathbb{R}$ , ( $a > 1$ ) we see an angle preserving dilatation and an anticlockwise rotation of the grid in the original square. The right side of the square rotates 45 degrees for each 0.5 increase of the exponent parameter  $a$ .

The sequence of rotations (ignoring the related dilatations) in Figure 5 could be perceived like the opening a handheld fan.

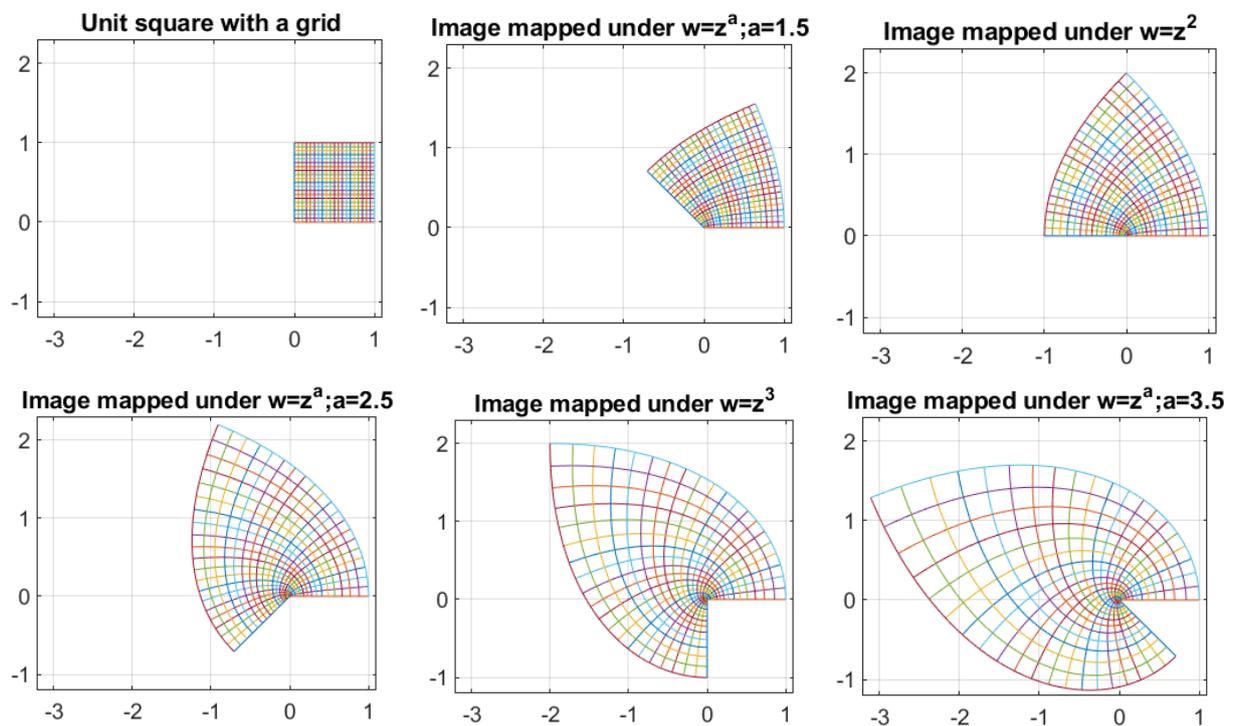


Figure 5: Conformal maps preserve angles and the shapes of infinitesimally small figures. Unit square is in the  $z$  plane,  $w = z^a$ ,  $a=1.5, 2, 2.5, 3, 3.5$ . Implemented with MATLAB & Chebfun

Similarly, for the same transformation  $w = z^a$ ,  $a \in \mathbb{R}$ , ( $a < 1$ ) we see that the dilation (as a contraction here) of the original square and a clockwise rotation of the grid in the original square. The real half axis  $\text{Re}(z) > 0$  is mapped to himself. Angles in the figure are preserved, as made visible looking at the grid.

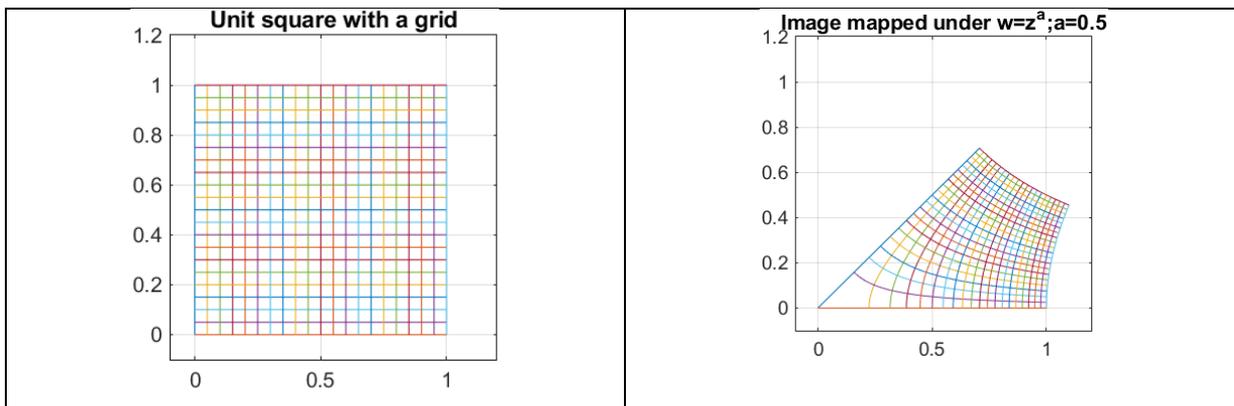


Figure 6: Conformal maps preserve both angles and the shapes of infinitesimally small figures. Left: unit square in the  $z$  Plane; right:  $w = z^{0.5} = \sqrt{z}$ . Implemented with MATLAB & Chebfun

Conformal maps can be defined between domains in higher-dimensional Euclidean spaces, and more generally on a Riemannian or semi-Riemannian manifold.

An analytic function is conformal at any point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic. Conformal mapping is extremely important in complex analysis, as well as in many areas of physics and engineering. For additional figures describing conformal transformation see also (Needham, 1998, p. 190)).

**Theorem 1.3. Riemann’s Theorem:** There exists a conformal one to one transformation from any simple connected region in the complex plane (other than the plane itself) to the unit disk. It is uniquely determined up to the choice of three points in and their images. (Swan & Bruss, n.d.)

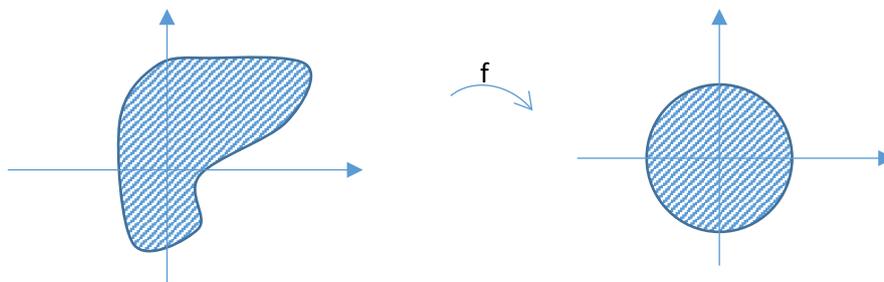


Figure 7: Conformal Mapping and Riemann Theorem

The Riemann mapping theorem guarantees that if  $D$  is any simply connected domain in the plane (other than the entire plane itself), then there exists a one-to-one conformal mapping  $w=f(z)$  that maps  $D$  onto the unit disk  $|w| < 1$ .

As a result of that, and because the inverse of a one-to-one conformal transformation is again a conformal transformation, any two simply connected regions of the plane are conformally equivalent.

The two regions can then be mapped one to the other by a one to one conformal transformation. (Hendriks, 2009, p. 6)

### 1.2.1 Some other conformal mapping plotted examples

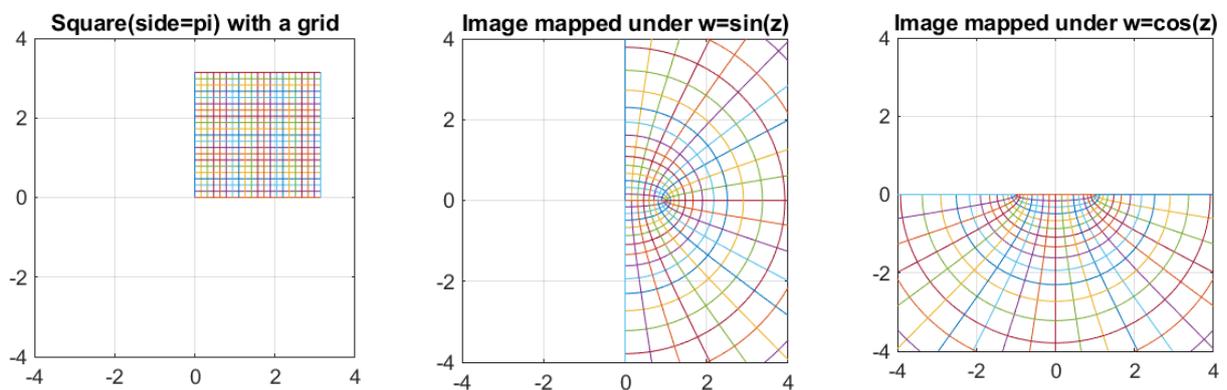
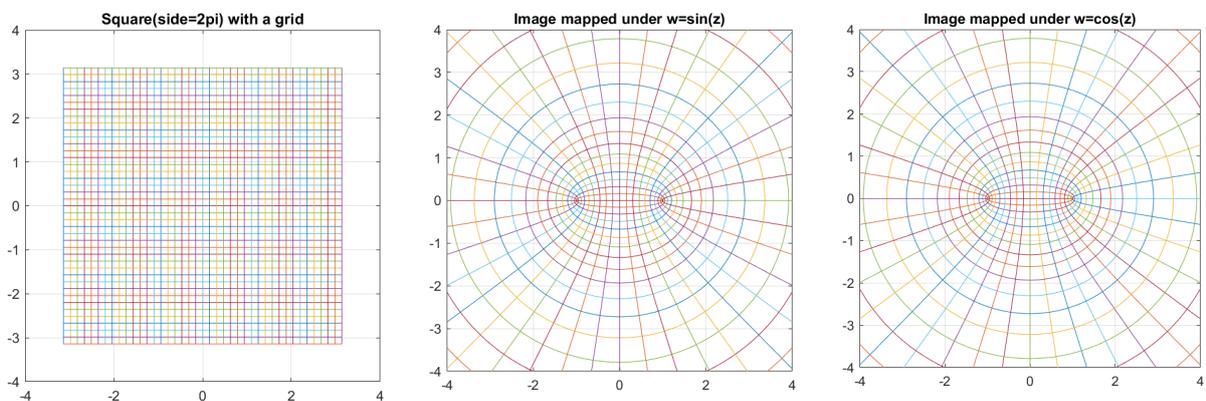


Figure 8: Square (side= $\pi$ ) in  $z$  plane Q1 and related mapping under  $w=\sin(z)$  and  $w=\cos(z)$



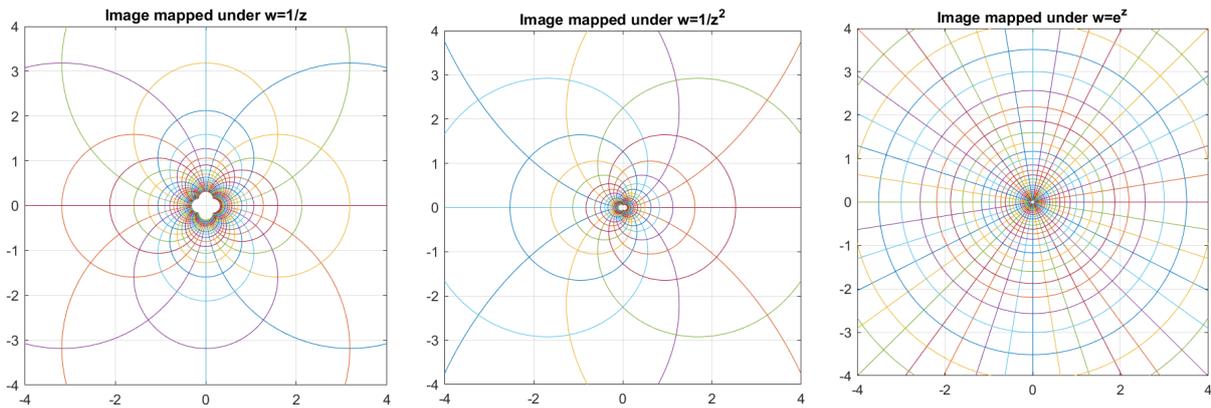


Figure 9: Square (side=2) centered in the origin and related mapping:  $w=\sin(z)$ ,  $w=\cos(z)$ ,  $w=1/z$ ,  $w=1/z^2$ ,  $w=e^z$ .

### 1.2.2 Phase plot: visualizing of complex functions.

Following information refer mostly to Chebfun MATLAB SW Guide (Driscoll et al., 2014), Wegert work on visualization of phase portraits (Wegert, 2012) and the two MATLAB related libraries "Complex Function Explorer" and "Phase Plots of Complex Functions" (Wegert, n.d.).

Phase portraits are a technique for visualizing complex analytic functions of a single complex variable. They rely on the visual encoding of complex numbers through their phase (or argument).

In a standard phase plot red indicates a phase of zero and the colors go from red to yellow, green, cyan, blue, magenta as the phase increases every 60 degrees (in the interval  $[0; 2\pi]$ ), in anticlockwise order. This generates for  $f(z) = z$  something like this:

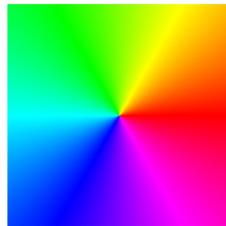


Figure 10: Two-dimensional color scheme used for phase plot, according to traditional color wheel.

It is important to say even if phase plot can nearly identify analytic functions, this is not valid in general and it is possible to find a non-analytic function sharing the same phase plot with a certain analytic function (Wegert, 2012, p. 37). For this reason, we refer just to analytic functions and their properties.

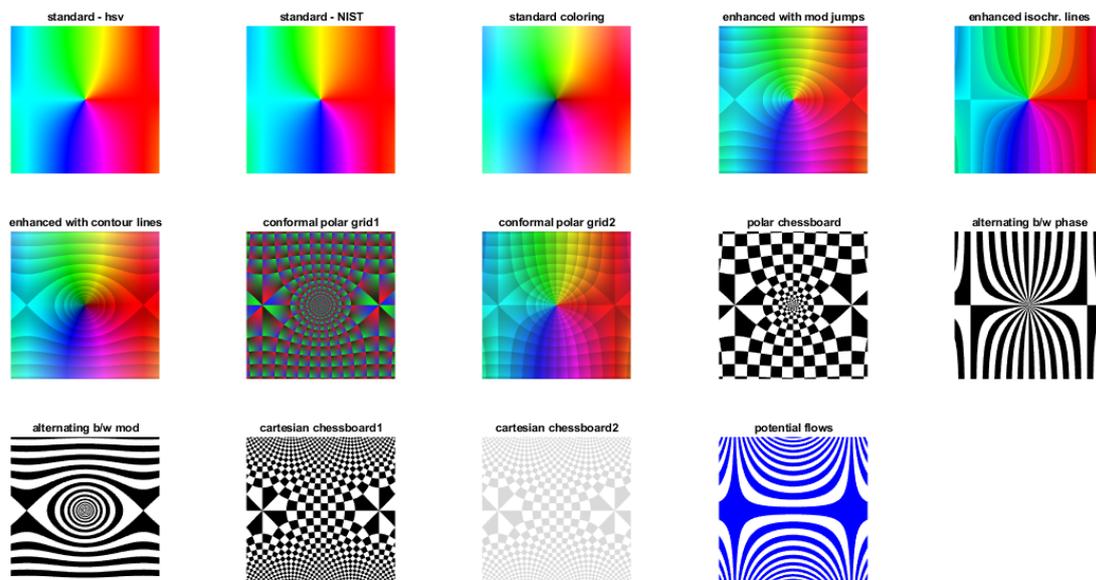


Figure 11: Phase plot construction for the same function  $f=\sin(z)$  using different representation options (MATLAB and Visual Complex Functions library) (Wegert, n.d.).

The SW Mathematica offers a similar capability on phase portrait, without a specific package for realizing so many different phase plot pictures. An example of Color-wheel phase plot picture in Mathematica we can be checked in Figure 12.

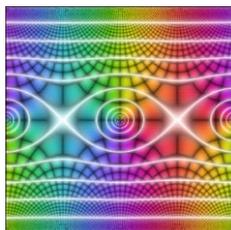


Figure 12:  $f(z)=\sin(z)$  (plotted with Mathematica)

`ComplexPlot[Sin[z], {z, -Pi - Pi I, Pi + Pi I}, ColorFunction -> "CyclicReImLogAbs", Frame -> False]`

### 1.3 Conformal transformations

The Riemann mapping theorem (formulated in Riemann's PhD thesis, 1851) ensures that every simply connected domain<sup>1</sup> can be conformally mapped onto the unit disk. This theorem guarantees the existence of such mappings, but it tells us nothing about how to explicitly construct the mapping.

The search for the mapping formulas is what still needs to be addressed.

Example of standard conformal transformations:

1. Translation:  $w = z + a; a \in \mathbb{C}$

---

<sup>1</sup> Analogous representation was obtained later by Lars Ahlfors to multiply connected domain.

2. Rotation:  $w = e^{i\theta} \cdot z$
3. Rotating dilatation:  $w = b \cdot z$ ,  $b \in \mathbb{C}$ ;  $b \neq 0$
4. Non-rotating dilatation, (expansion, homogeneous dilatation):  $w = b \cdot z$ ,  $b \in \mathbb{R}$ ;  $b > 0$  (if  $b < 1$  it is a contraction, if  $b > 1$  an enlargement)
5. Linear transformation  $w = a + b \cdot z$ ;  $a, b \in \mathbb{C}$ . It is in effect a composition of a translation, a rotation, and a dilatation
6. Inversion.  $w = 1/z$
7. Composition of the above (it is equivalent to just one transformation)
8. Möbius transformation (or linear-fractional transformation):

$$w = \frac{az + b}{cz + d} \quad \text{with } (ad - bc) \neq 0$$

It can be proven, that Möbius transformation is a combination of two translations, a rotation, a dilatation and an inversion (Needham, 1998, pp. 122–124). See more details in paragraph 1.4.

The conformal transformation 1 to 8 outlined above are visible in Figure 13 , we use as starting image the unit square placed in the first quadrant.

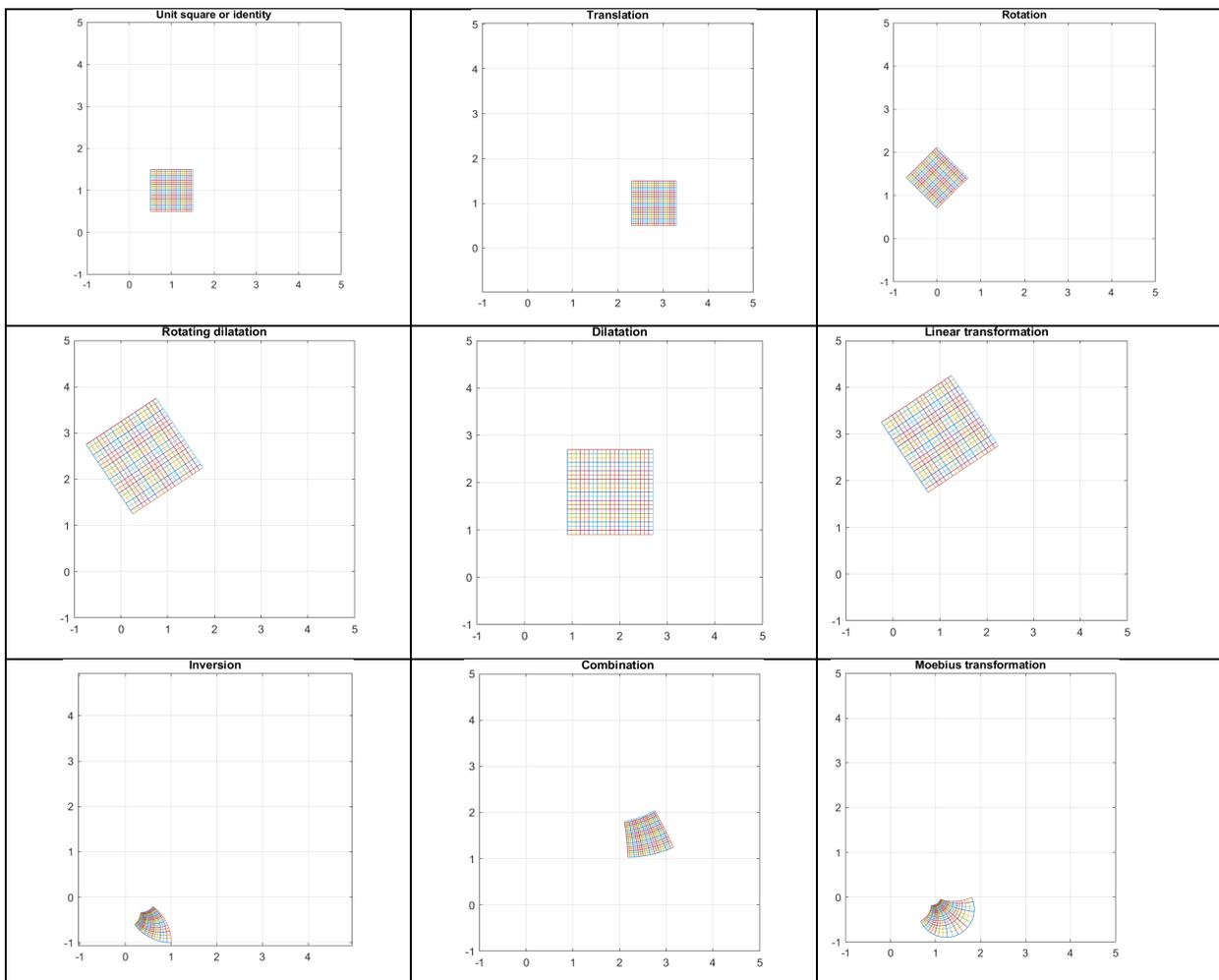


Figure 13: Unit square and conformal transformation 1 to 8

9. **Schwarz-Christoffel transformation** represent a group of conformal mappings onto polygonal domains.

The historical development of the Schwarz-Christoffel mapping is outlined in chapter 1.5 and a detailed description is in chapter 2.

Conformal transformations have many applications in physics, (e.g.: in fluid flow, mechanics, electrostatics and elasticity) because many physical properties remain invariant after such transformations (see chapter 4) and this helps simplifying and solving such problems.

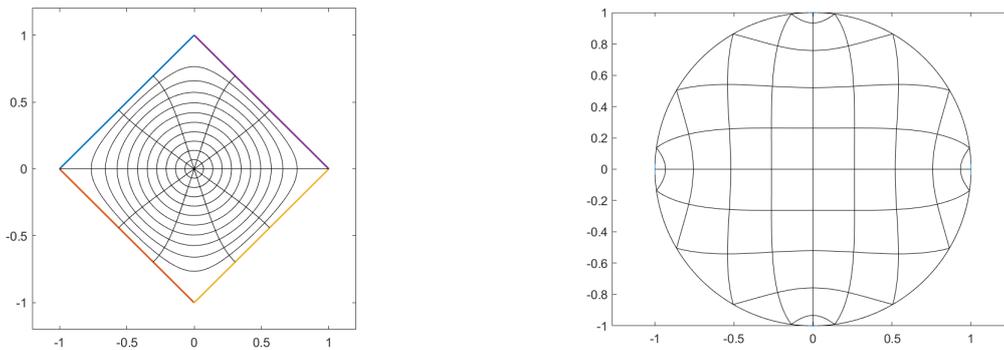


Figure 14: Example of Conformal Mapping using MATLAB

## 1.4 Moebius transformation

We describe here some properties of the Moebius transformation that will be used as reference in the next chapter.

We already said that a Moebius transformation has the form:

$$w = f(z) = \frac{az + b}{cz + d} \quad \text{with } (ad - bc) \neq 0 \quad (3)$$

We can outline following properties (Iske, 2020, pp. 63–67):

- $f$  maps the extended complex plane  $\mathbb{C} \cup \{\infty\}$ , onto itself.
- $f$  maps the class of circled and lines (generalized circle or "circline") to itself.
- $f$  is conformal at every point except its pole.
- the inverse transformation

$$z = f^{-1}(w) = \frac{dw - b}{-cw + a} \quad \text{with } (ad - bc) \neq 0 \quad (4)$$

is also a Möbius transformation.

- A Möbius transformation is uniquely determined by three (different) points  $z_i, i = 1, 2, 3$ .

**Example 1.4.1:** Finding a Möbius transformation that maps the upper half-plane  $\{\text{Im}(z) > 0\}; z \in \mathbb{C}$  to the inside of the unit-disk  $E \{ |w| \leq 1 \}; w \in \mathbb{C}$ , such that a given point  $z_1=i$  is mapped to  $w=0$  and  $z_2= \infty$  to  $w=-1$ .

1.  $z_1=i$  is mapped to  $w=0$

$$0 = \frac{a(i) + b}{c(i) + d}; b = -ia$$

2.  $z_2= \infty$  is mapped to  $w=-1 \Rightarrow c= -a$
3. we can choose one of the parameters: e.g.,  $a=1 \Rightarrow$  we obtain  $a=1; b=-i; c=-1$
4. We need to choose  $d$  in order to map the upper half-plane to the inside of the disk:  $d=-i$  ensures that.

$$w = f(z) = \frac{z - i}{-z - i}$$

The real axis is mapped to the circle line.

The function we just found can be rewritten as  $w= -c(z)$ , where  $c(z)$  is called the *Caley transformation* and is also mapping the upper half-plane to the unit disk:

$$c(z) = \frac{z - i}{z + i}$$

**Example 1.4.2:**

According to the Riemann Theorem for each transformation  $w=T(z)$  from the upper half-plane  $\text{Im}(z)>0$  to the unit disk there is an *inverse transformation*  $z=T^{-1}(w)$  from the unit disk to the upper half-plane.

If  $w= T(z)= (1 - i w)/(1+i w)$  then the inverse transformation is  $z= T^{-1}(w) = (iz - i)/(z + 1)$ .

Both transformations are Möbius transformations.

This example preludes to the Schwarz-Christoffel mapping from the unit disk, outlined later in chapter 2.3.

Now we introduce the Schwarz-Christoffel formula for constructing a conformal mapping from the upper half-plane onto a region  $G$  bounded by a polygonal curve.

## 1.5 The History of Schwarz-Christoffel Mapping

The idea of conformal mapping was first proposed by Carl Gauss in the 1820s.

In 1851, Bernhard Riemann presented the famous Riemann mapping theorem in his doctoral dissertation, in which he proved that a bijective conformal mapping exists between any two simply connected regions.

In complex analysis, a Schwarz-Christoffel mapping is *historically* a conformal transformation of the upper half-plane onto the interior of a simple polygon. It was discovered independently by **Hermann Schwarz** in 1869 and **Elwin Christoffel** in 1867. Therefore, the mapping onto a simply connected polygonal domain is known as the *Schwarz-Christoffel transformation*.

Elwin Bruno Christoffel was born in Montjoie Aachen, Germany on 10.11.1829.

He studied at the University of Berlin, one of his teachers was Peter Dirichlet. His work on the Schwarz-Christoffel transformations were published in various papers between 1867 and 1870.

Hermann Amandus Schwarz was born in Hermsdorf, (Germany, now Poland) on 25.01.1843. He studied at the Technical University of Berlin, starting with chemistry. Later two of his teachers, Weierstrass and Kummer convinced him to study Mathematics. Schwarz's published his work on this subject in two papers in 1869.



Figure 15: Schwarz's 1869 conformal map of a square onto a disk, reproduced from "Gesammelte Mathematische Abhandlungen (Schwarz, 1972)

For more information on their life the reader can be referred to St. Andrews Scotland university biographies website: <http://www-history.mcs.st-andrews.ac.uk/BiogIndex.html>.

There are three important points to consider:

1. Most of the domains can be approximated by polygonal domains
2. The mapping formulas are from standard domains like the upper half-plane\_onto polygonal domains, but very often the mapping is needed in the other direction, from the more complex domain (the polygonal) to the standard one.

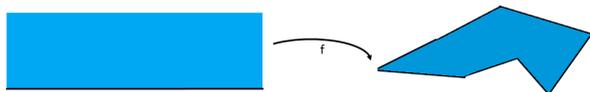


Figure 16: Schwarz-Christoffel transformation: the upper half-plane is mapped onto a polygon bounded domain.

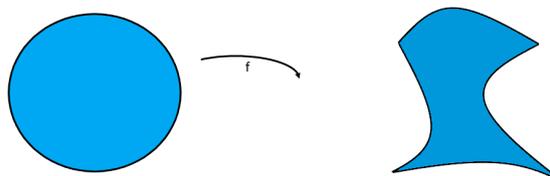


Figure 17: Circular arc polygon domain mapping: the unit disk is mapped onto a polygon bounded domain, having arcs as sides.

3. The problem of choosing the correct parameters. The mapping formula contains the preimages of the corners of the polygon and the related interior angle in these corners. The form of the image domain depends on these corner and angles. This problem is normally described as the "parameter problem" of the Schwarz-Christoffel transformation.

Christoffel and Schwarz expanded this topic to different forms of the mapping formula by changing some of the properties of the image domains. This includes mappings onto the exterior of a polygon, domains with curved boundary parts and domains bounded by circular arc polygon.

While they have discovered many mapping formulas, the parameter problem restricted them to just a number of simple examples.

The Schwarz-Christoffel mapping received a rising interest again in the second half of the last century. The invention of the computer and the availability of always more performing calculation power, the availability of new dedicated software packages, the development of new algorithms, made the possibility to solve the parameter problem step by step easier.

## **1.6 Last developments: numerical methods**

The possibility to solve the mapping as a computer aided problem caused the publication of many articles about the different aspects of the Schwarz-Christoffel transformation numerics.

Even if the Schwarz Christoffel formula has been known since 1867-1869 it has not been used a lot, because of the complication of solving it without a computer.

The prevertices are normally impossible to be calculated per hand from the polygon, and the locations of the prevertices affect the side lengths (but not the angles) of the image polygon in a highly nonlinear way.

End of last century, a number of software packages regarding the Schwarz-Christoffel transformation become available. Some tools were written in Fortran or in other programming languages, some other are based on commercial existing SW like MATLAB (see Driscoll *SC Toolbox* (Driscoll, 1996) (Driscoll, 1994) (Driscoll, 2005) and related articles) or Mathematica.

They relate on the assumption that the potential users work most in universities and have already anyway the commercial "core" software needed to use these programs or tools. The tools based on these commercial SW are then free to use.

There is a huge number of small Mathematica or MATLAB programs or apps capable to show properties of Conformal mapping or SC mapping, but most of them share a common issue: the two mathematical platforms evolve very quickly, sometimes old commands are deprecated or after a "reasonable" time become invalid. They have every year a major release and some minor deliveries during the year.

This is not an issue for those people who want to write a brand-new piece of software, but it makes already developed software after some years potentially not usable anymore. In the best-case scenario, the user receives lot of warnings, even if this do not compromise the result. It is very difficult to reuse the existing software if the original author does not maintain it year after year.

Wolfram Mathematica offers on their website also some generic programs not needing a license (named *demonstrations* or *handbooks*) and the related software code. There are some available “Mathematica *demonstrations*” written for conformal or even SC mapping, they are also officially maintained by Wolfram Mathematica. For a graphical demonstration on Conformal mapping, you can e.g. look at Wolfram Mathematica website <https://demonstrations.wolfram.com/ACatalogOfConformalMappings/> (Rangel-Mondragon, 2011).

For the numeric aspect of conformal mappings onto circular arc polygon domains please refer to Bjørstad and Grosse (Bjørstad & Grosse, 1987, pp. 19–32).

A more general and detailed discussion of the numeric is available in Trefethen and Driscoll (Driscoll & Trefethen, 2002) or in "Numerical Approximation of the Schwarz-Christoffel Mapping" (§14.4.1 in *Handbook of Complex Variables*. (Krantz, 2012, pp. 175–179)).

A good overview of different numerical methods for conformal mapping can be found in Luteberget (Luteberget, 2010, pp. 36–55).

### 1.6.1 Crowding

Crowding (Hendriks, 2009, pp. 20–21) is a form of ill-conditioning that causes trouble in virtually all numerical methods for conformal mapping. The situation can be exemplified by the map from the disk to a rectangle with side length respectively  $a$  and 1, as in Figure 18. The figure has been produced and calculated during this work with MATLAB SC Toolbox, inspired from Driscoll and Trefethen book (Driscoll & Trefethen, 2002).

The angles at which the curves meet at the origin are the same as in the disk (because of conformal mapping). As the aspect ratio “ $a$ ” grows, the angles between some pairs of these curves become exponentially small.

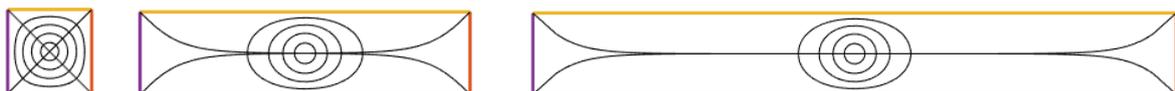


Figure 18: Crowding explained through a map from the disc to a rectangle with a different side length ratio  $a$ , for  $a=1$ ,  $a=4$ ,  $a=8$

This indicates that small changes in a point in the disk such as those induced by roundoff error on a computer can be amplified enormously by the map.

According to Driscoll It is quite difficult to distinguish adjacent prevertices in standard double precision arithmetic if the “ $a$ ” ratio is larger than about 23, this is a sort of floating-point arithmetic limits. Crowding occurs whenever the target region has areas that are relatively long and thin (elongated regions). There are some workarounds to reduce this issue, but we will not discuss them here (Driscoll, 2005). For a simple example on how crowding can be reduced using a different algorithm see chapter 3.3.1.

### 1.6.2 Conformal mappings onto multiple connected domains

The bigger progress in this area in the last 20 years was the extension of the mapping formula to a multiply connected polygonal domains.

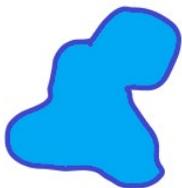


Figure 19: Simply connected domain



Figure 20: Connected domain, not simply

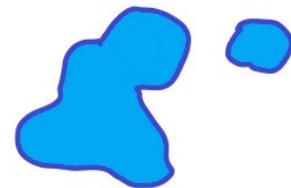


Figure 21: Not connected domain

The solutions of these problems can only be calculated with a computer, that is why the development of numeric was essential for getting solutions to this problem.

A formula for conformal mappings onto doubly connected polygonal domains were discovered from Komatu in 1945, but a Software package to perform the calculation (named DSCPACK) was just developed by Hu in 1998 (Hu, 1998).

De Lillo and Pfaltzgraff proposed first an alternative approach to the mappings onto doubly connected domains (DeLillo & Pfaltzgraff, 1998) and then later a solution for so named Multiply Connected Schwarz-Christoffel domains (MCSC) (DeLillo et al., 2004) (Figure 22 and Figure 23). De Lillo’s formula for MCSC has a similar structure to the original SCT for single domain for the unbounded case and a much complex one for the bounded one.

We will not cover multiple connected domains in this work, but for any interest see also De Lillo’s article(DeLillo & Kropf, 2011) describing numerical computation for multiple connected domains.

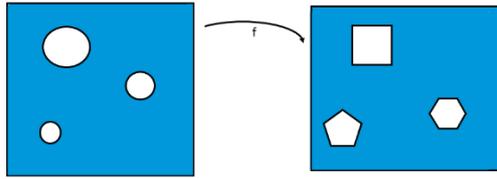


Figure 22: MCSC map: map onto a multiply connected polygonal domain (Bauer, 2015, pp. 1–3).

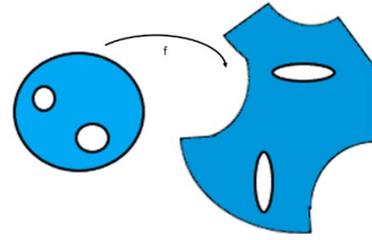


Figure 23: map onto a multiply connected arc-polygon domain (Bauer, 2015, pp. 1–3)

## 2 Schwarz-Christoffel transformation

As already introduced in chapter 1, the classical Schwarz-Christoffel transformation is a *conformal mapping* from the upper half-plane to a polygon.

### 2.1 Introduction

The idea behind the Schwarz–Christoffel (SC) transformation is that a conformal transformation  $f$  may have a derivative  $f'$  that can be expressed as (Driscoll & Trefethen, 2002, p. 1)

$$f' = \prod f_k \quad (5)$$

for certain functions  $f_k$ .

A big number of conformal maps variations can fit into this basic formulation.

In effect this is valid for all conformal transformations whose analytic forms are known as Schwarz–Christoffel maps.

From a geometric point of view, the formulation in (5) means:

$$\arg f' = \sum \arg f_k \quad (6)$$

In the classical transformation, each  $\arg f_k$  is designed to be a step function, so the resulting  $\arg f'$  is piecewise constant with specific jumps with the result that  $f$  maps the real axis to a polygon.

Let  $P$  be the region in the complex plane  $\mathbb{C}$  bounded by a polygon  $\Gamma$  with vertices  $w_1, \dots, w_n$ , given in anticlockwise order, and real interior angles  $\alpha_1, \dots, \alpha_n$  ( $P$  is the interior of the polygon  $\Gamma$ ).

$P$  is assumed for now to be bounded and without cusps or slits. The coefficients are  $\alpha_k \in (0, 2)$  for each  $k$ .

Let  $f$  be a conformal map of the upper half-plane  $\mathbf{H}^+$  onto  $P$  and let  $z_k = f^{-1}(w_k)$  be the  $k$ th prevertex.

We take  $z_n = \infty$ , without loss of generality. If infinity is not already a prevertex, we can simply introduce its image (which lies on  $\Gamma$ ) as a new vertex with interior angle  $\pi$ . The other prevertices  $z_1, \dots, z_{n-1}$  lie on the real axis and are then real.

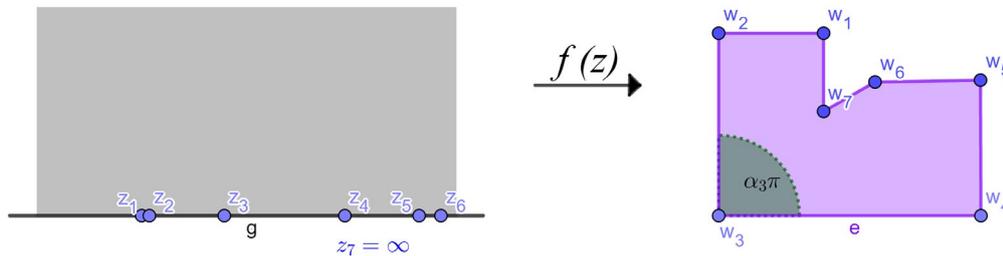


Figure 24: Notational convention for SCT

As with all conformal maps, the main effort is in getting the boundary right.

By the *Schwarz reflection principle* (also known as *Schwarz's symmetric principle* (Needham, 1998, p. 254)) (Krantz, 2012, p. 95),  $f$  can be analytically continued across the segment  $(z_k, z_{k+1})$ . In particular,  $f'$  exists on this segment, and we see that  $\arg f'$  must be constant there.

The proof of the Schwarz–Christoffel mapping fundamental theorem is in next chapter 2.2.

## 2.2 Schwarz-Christoffel theorem

The proof follows mainly the two books of Mathews (Mathews & Howell, 2012, p. 370) and Driscoll (Driscoll & Trefethen, 2002, p. 3) we have already referenced in this document.

We analyze in this chapter the construction of a one-to-one conformal mapping from the upper half-plane  $\text{Im}(z) > 0$  onto a domain  $P$  in the  $w$  plane where the boundary consists of straight-line segments. Many applications involving conformal mappings require the solutions of the “Schwarz-Christoffel Parameter Problem”, that is the problem of determining the prevertices of a Schwarz-Christoffel mapping.

We want to find a function  $w = f(z)$  with the property expressed below in (3) (Chapter 2.2.1)

Three of the prevertices  $z_j$ , including if the case the already fixed  $z_n = \infty$ , may be chosen arbitrarily.

The remaining  $n-3$  prevertices are then obtained solving a non-linear equations system.

### 2.2.1 Theorem formulation

#### Theorem Schwarz-Christoffel

Let  $P$  be the interior of a polygon  $\Gamma$  in the  $w$  plane with vertices  $w_1, w_2, \dots, w_n$  and interior angles  $\alpha_1, \dots, \alpha_n$  specified in the positive (anticlockwise) order.

There exists a one-to-one conformal mapping  $w = f(z)$  from the upper half-plane  $\text{Im}(z) > 0$  onto  $P$  satisfying the boundary conditions:

$$w_k = f(x_k) \text{ for } k = 1, 2, \dots, n-1 \text{ and} \quad (7)$$

$$w_n = f(\infty), \text{ where } x_1 < x_2 < \dots < x_{n-1} < \infty$$

The derivative  $f'(z)$  is

$$f'(z) = A(z - x_1)^{(\alpha_1 - 1)}(z - x_2)^{(\alpha_2 - 1)} \dots (z - x_{n-1})^{(\alpha_{n-1} - 1)} \quad (8)$$

and the function  $f(z)$  can be expressed as an indefinite integral:

$$f(z) = A + C \int^z \prod_{k=1}^{n-1} (\zeta - x_k)^{\alpha_{k-1}} d\zeta \quad (9)$$

where  $A$  and  $C$  are suitably chosen complex constants (determining orientation, size, and position of the polygon  $\Gamma$ ).

In this special case, mapping from upper half-plane, all prevertices on the real axis, therefore  $x_i \in \mathbb{R} \cup \{\infty\}$ .

As already stated above, three of the points  $\{x_i\}$  may be chosen arbitrarily. If we make the decision to take  $x_n = \infty$ , then just two of the remaining  $\{x_i\}$  can be chosen.

#### Theorem Proof:

For simplicity, we just consider the case where all prevertices are finite and the product ranges over the indices 1 to  $n$ . By the Schwarz reflection principle, the mapping function  $f$  can be analytically continued across the segment  $(x_k, x_{k+1})$  of the real axis, so that  $f'(z)$  exists there.

We also see that  $\arg f'(z)$  must be constant on these segments, and that  $\arg f'(z)$  must jump at each  $x_k$ ,

$$\lim_{z \rightarrow x_k^+} \arg f'(z) - \lim_{z \rightarrow x_k^-} \arg f'(z) = (1 - \alpha_k)\pi = \beta_k \quad (10)$$

$\beta_k$  is the turning angle at  $w_k$  (or the exterior angle at vertex  $k$ ).

The amount of rotation is in effect determined by  $\arg f'(x)$ . The function  $f_k = (z - x_k)^{-\beta_k}$  is analytic in the upper half-plane  $\mathbf{H}^+$  (or  $\text{Im}(z) > 0$ ), has the specific jump on  $z = x_k$  and has  $\arg f_k$  constant on the real axis.

For all these reasons, now we can write a conformal map to a polygon as:

$$f'(z) = C \prod_{k=1}^{n-1} f_k(z) \quad (11)$$

By summing up the previous formula for  $f'(z)$  we obtain the SC Formula for a half-plane

$$f(z) = A + C \int^z \prod_{k=1}^{n-1} (\zeta - x_k)^{\alpha_{k-1}} d\zeta$$

(same formula outlined in (9),  $A$  and  $C$  are suitably chosen complex constants).

Formula (9) gives a representation for  $f$  in terms of an indefinite integral.

Also, the integral will involve a multivalued function, and a specific branch must be selected to fit the boundary values specified in the problem. Table 1 is useful for solving problems with 2 or 3 prevertices and angles that are always a multiple of  $\pi/2$  (exponent are then  $\pm\frac{1}{2}, \pm 1, \dots$ ).

Table 1: Indefinite Integrals

function	integral
$\frac{1}{(z^2 - 1)^{1/2}}$	$i \operatorname{Arcsin}(z) = \log(z + (z^2 - 1)^{1/2}) - i\frac{\pi}{2}$
$\frac{1}{(z^2 + 1)}$	$\operatorname{Arctan}(z)$
$\frac{1}{(-z^2 + 1)^{1/2}}$	$\operatorname{Arcsin}(z)$
$\frac{1}{z(z^2 - 1)^{1/2}}$	$-\operatorname{Arcsin}\left(\frac{1}{z}\right)$
$\frac{1}{z(z + 1)^{1/2}}$	$-2 \operatorname{Arctanh}[(z + 1)^{1/2}]$
$(1 - z^2)^{1/2}$	$\frac{1}{2}z(1 - z^2)^{1/2} + \operatorname{Arcsin} z$

## 2.3 Schwarz-Christoffel mapping from the unit disk

As important as the previous formula, is the one applying to the conformal map from the unit disk  $E$ .

Let  $P$  be the interior of a polygon  $\Gamma$  in the  $w$  plane with vertices  $w_1, w_2, \dots, w_n$  and interior angles  $\alpha_1, \dots, \alpha_n$  specified in the positive (anticlockwise) order.

Let  $f$  be any conformal map from the unit disk  $E$  to  $P$ .

Then:

$$f(z) = A + C \int^z \prod_{k=1}^n \left(1 - \frac{\zeta}{z_k}\right)^{\alpha_k - 1} d\zeta \quad (12)$$

$A$  and  $C$  are suitably chosen complex constants,  $w_k = f(z_k)$  for  $k = 1, \dots, n$ .

This variation can be derived from the basic SC principle; the only substantive difference between this formulation and the one mapping from the half upper plan, is that the product runs *over all  $n$  prevertices*. The form of the integrand appears to be slightly different, but in fact it is a constant multiple of the original form. The reason for the change is so that the branch cuts of the integrand will point away from the origin if the principal branch of the logarithm is used to compute them.

### 2.3.1 Special case for regular polygons

Just a note for a special case. If the polygon  $\Gamma$  is a regular one, then all angles have the same size. If  $n$  is the number of vertices, all exterior angles are  $2\pi/n$  big.

The formula can be then rewritten as:

$$f(z) = A + C \int^z \prod_{k=1}^n \left(1 - \frac{\zeta}{z_k}\right)^{2/n} d\zeta \quad (13)$$

$A$  and  $C$  are suitably chosen complex constants,  $w_k = f(z_k)$  for  $k = 1, \dots, n$ .

To show this special case we can write a small program that realizes the mapping for each regular polygon with  $n$  vertices ( $n \geq 3$ ). For details see also (Driscoll & Trefethen, 2002, pp. 16–20).

```
%function disctonregpoly(n) % n >=3;
b(1:n,1) = -2/n;
for j=1:n
    v(j,1) = exp((j*2*pi*i)/n);
end;
ang(1:n,1) = (1 + b(1:n,1));
p = polygon(v,ang);
fmap = diskmap(p);
fmap = center(fmap,0);
plot(fmap, 20, 20)
```

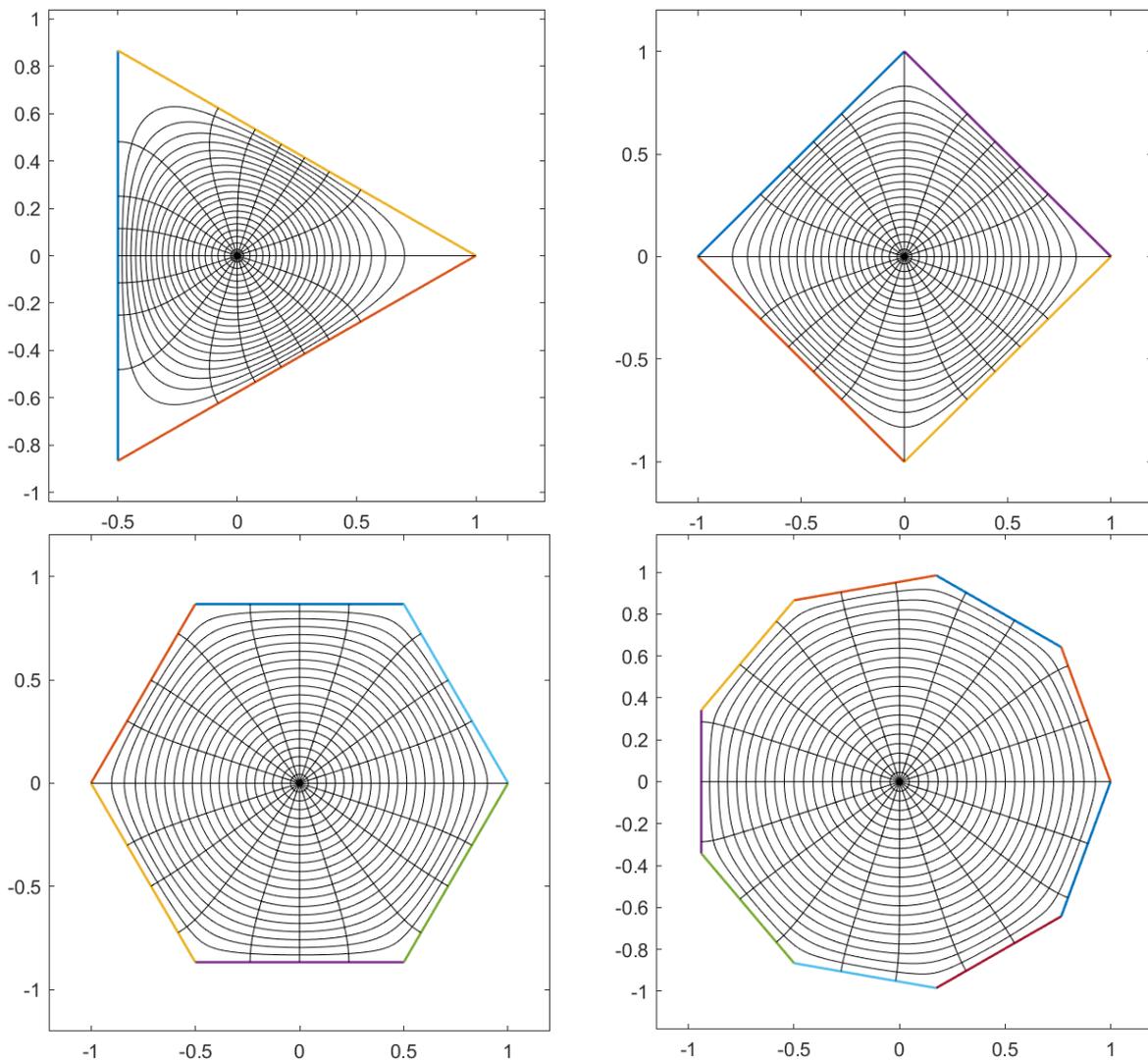


Figure 25: SC mapping from unit disc to a  $n$ -regular polygon for  $n = 3, 4, 6, 9$

## 2.4 Schwarz-Christoffel mapping from a strip

The infinite strip  $S = \{0 < \text{Im } z < 1\}$  is very useful as a canonical domain.

The first reason is because many applications involve an infinite channel. Secondly, because  $S$  is essentially the logarithm of the upper half-plane, crowded prevertex on the real line may become well separated on the  $S$  boundary and therefore easier to treat numerically. It was implemented and advocated as a method for reducing crowding (see chapter 1.6.1).

Under this map, the two ends of the channel always have identical “divergence angles.”

For the Schwartz-Christoffel formula for a strip see please the reference in literature below. (Driscoll & Trefethen, 2002, p. 46)

Because the two ends of the strip are preassigned, only one degree of freedom remains in the map: sideways translation of the strip, which can be fixed by placing a designated prevertex at zero.

There are then just  $(n-1)$  real conditions necessary to calculate the remaining prevertices, where  $n$  is the number of vertices, once the channel ends have been excluded.

We will use this mapping in chapter 4.3, to show the application to capacity electric field.

## 3 Example of computer assisted solutions.

The Software programs and versions used for all following examples and for all the pictures present in this work are:

- MATLAB R2020b with SC Toolbox 2.4.1, Chebfun, “Phase plot of complex functions”
- Wolfram Mathematica 12.0
- GeoGebra Classic 6 (just for drawing)

### 3.1 Introduction to MATLAB SC Toolbox: notation on angles

You can find in literature different expressions for the SC transformation and relative formulation, using either the interior or the exterior angles. We decided to use the interior angles  $\alpha_i$  firstly because it more common, but mainly to align to the documentation of SC Toolbox.

The sum of all exterior angles is  $2\pi$ . Another way is to check if angles have been calculated correctly is  $(n - \sum_{i=1}^n \alpha_i) = 2$  or  $\sum_{i=1}^n (\alpha_i) = n - 2$ , where  $n$  is the number of vertices in the polygon.

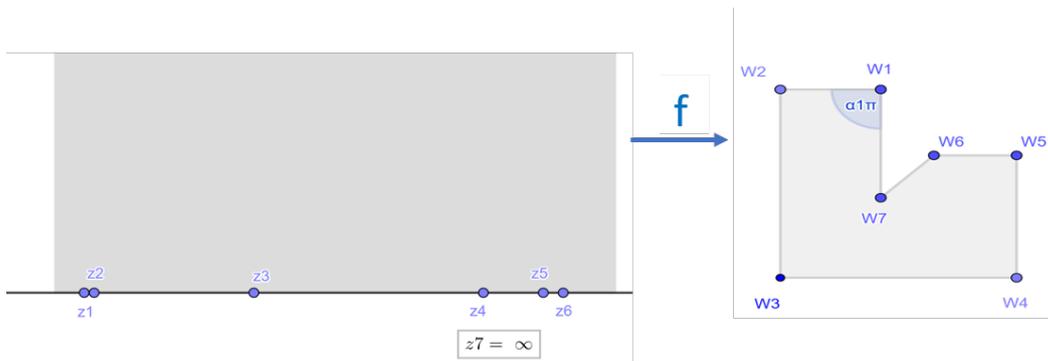
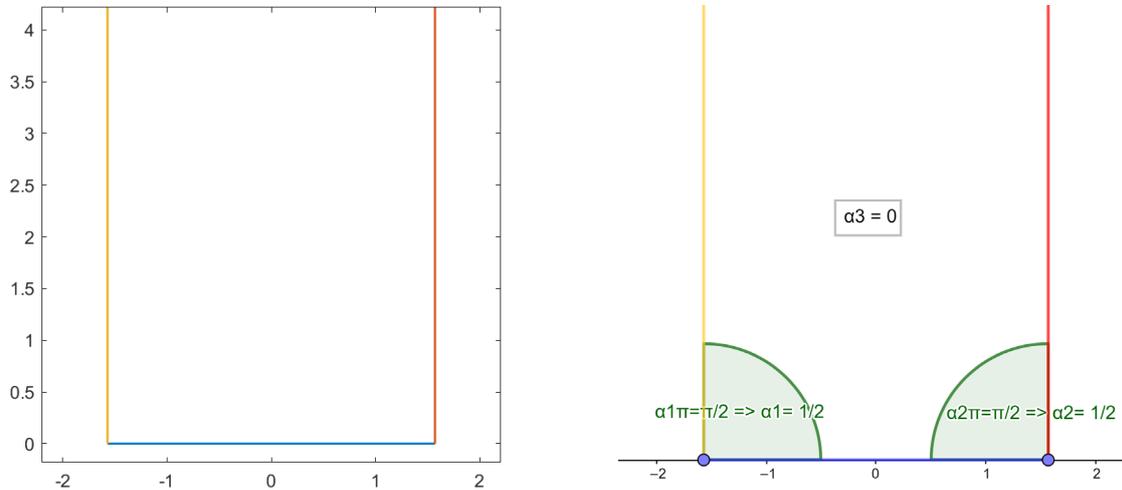


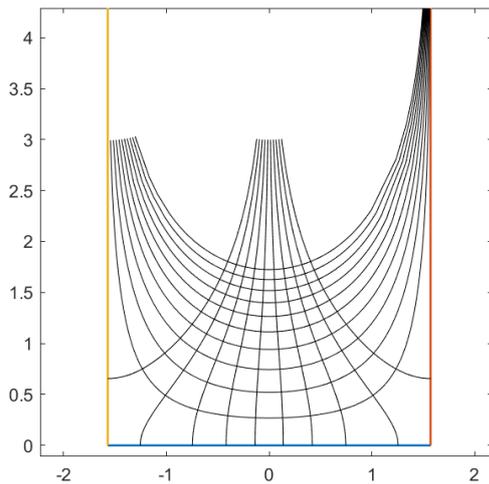
Figure 26: Prevertices  $z_i$  and vertices  $w_j$ ; angles according to SC Toolbox: standard. If the transformation is from the extended half-plane, we can name the  $z_i$  as  $x_i$ , to make it clear they have all a value in  $\mathbb{R}$

We give now two examples taken from Matthews (page 373-380) and Driscoll but always using Driscoll notation for the angles). Using this notation for all internal angles we have for all polygon with slits  $\alpha_{\text{slit}} = 2$ ; for vertex  $w_i$  at infinity ( $-2 \leq \alpha_i \leq 0$ ).

### 3.2 Example: upper half-plane mapped to a strip

Example 1 (Mathews & Howell, 2012, p. 373): function mapping the upper half-plane  $\text{Im}(z) > 0$  to the semi-infinite strip





MATLAB answer: commands in blue; output in black

```
>> q = polygon([-pi/2,pi/2,i*Inf],[1/2,1/2,0]);
```

```
>> plot (q)
```

```
>> f = hplmap(q)
```

hplmap object:

vertex	alpha	prevertex
-1.57080 + 0.00000i	0.50000	-1.000000000000e+00
1.57080 + 0.00000i	0.50000	1.000000000000e+00
0.00000 + Inf	0.00000	Inf

```
c = 6.123234e-17 + 1i
```

```
Apparent accuracy is 1.71e-11
```

```
>> plot (f)
```

Figure 27: mapping the upper half-plane  $\text{Im}(z) > 0$  to the semi-infinite strip

This Calculates the 3 prevertices in SC Toolbox: :  $z_1 = -1$  ;  $z_2 = +1$ ;  $z_3 = z_n = \infty$

The angles are  $\alpha_1 = \alpha_2 = 1/2$ ;  $\alpha_3 = 0$

The integral of  $f'(z) = \frac{-i}{(z^2-1)^{1/2}}$  can be calculated with a mathematical program like MATLAB or

Mathematica and we know the analytic solution is  $f(z) = \text{Arcsin } z = -i \log(z + (z^2 - 1)^{1/2}) - \frac{\pi}{2}$

MATLAB answer: commands in blue; output in black

```
>> syms z
```

```
>> expr = -i/(z^2-1)^(1/2);
```

```
>> F = int(expr)
```

```
F = -log(z + (z^2 - 1)^(1/2))*1i
```

That is the right result, but in exp/log form.

### 3.2.1 Integration

SC Toolbox in MATLAB or Mathematica perform for the SC mapping calculation a numeric integration.

Let's see if it is possible to get, at least in some special easier cases, an explicit expression of the definite integral that solve the  $f(z)$  performing the mapping,

We can try to integrate in MATLAB reusing the already calculated prevertices and  $\alpha$  parameters.

We obtain first as solution of (9), the definite integral  $f(z)$  where A and B still need to be calculated.

**$f(z) = B + A G(z)$ .** The function  $G(z)$  is the result of the MATLAB program.

```
v = [-pi/2,pi/2,i*Inf];
ang = [1/2,1/2,0];
n = size(v);
index = n(2);
p = polygon(v,ang);
plot (p);
fmap = hplmap(p)
prev = [prevertex(fmap)]
ang = [angle(p)]
syms z;
fder = 1;
for j = 1:(index-1)
    fderk = (z- prev(j))^(ang(j)-1)
    fder = fder * fderk;
end
fder
G = int(fder)
```

MATLAB calculates  $G(z) = i \operatorname{Arcsin}(z)$ . A and B still need to be calculated with an equation system, using the conditions coming from the image values  $f(x_i) = w_i$ .

$$\begin{cases} f(z) = A i \operatorname{Arcsin}(z) + B \\ f(-1) = -\pi/2 \\ f(1) = \pi/2 \end{cases} \longrightarrow \begin{cases} -\frac{\pi}{2} = A \frac{-i\pi}{2} + B \\ \frac{\pi}{2} = A \frac{i\pi}{2} + B \end{cases} \longrightarrow \begin{cases} \pi = Ai\pi - 2B \\ \pi = Ai\pi + 2B \end{cases} \text{ that is solved}$$

for  $B=0$  and  $A=-i$

That means  $f(z) = \operatorname{Arcsin} z$  is the solution.

Please read also next chapter, to get more information on how the code showed above not always gives a compact solution for the definite integral.

### 3.2.2 MATLAB power expressions and explicit solution of indefinite integrals

A manual analytic solution with lot of vertices become complex and additionally, if the angles are not multiple of  $\pi/2$ , is nearly impossible. In this case we need to use a computer and find a numeric solution of the problem. All Software tool we have used “just” perform a numeric integration to find the prevertices of the SC transformation, without giving an explicit description of the function  $f : H^+ \rightarrow P$  that maps e.g. the half-plane to a certain region  $P$ . This is true even if an analytic solution would be possible. In general terms it is possible to try to ask a certain software (e.g., MATLAB) to find the explicit analytic primitive function, even if the answer is not always written as we would expect.

To be precise, MATLAB has to be explicitly asked to recognize that several different binomial term with same power should be multiplied, to get a shorter more compact form (e.g.):

$$\frac{1}{(z+1)^a} \frac{1}{(z-1)^a} = \frac{1}{(z^2-1)^a}$$

The consequence is that the solution of the integral is not as compact as it could be.

This can be done with the “simplify” command, using the ‘steps’ option.

MATLAB command	MATLAB answer
<code>syms z; % just defining the variable</code>	
<code>simplify((z+1)*(z-1))</code>	$z^2 - 1$
<code>simplify((z+1)^-1*(z-1)^-1)</code>	$1/(z^2 - 1)$
<code>simplify((z+1)^-2*(z-1)^-2)</code>	$1/((z-1)^2*(z+1)^2)$
<code>simplify((z+1)^-2*(z-1)^-2,'Steps',10)</code>	$1/((z-1)^2*(z+1)^2)$
<code>simplify((z+1)^-2*(z-1)^-2,'Steps',57)</code>	$1/(z^2 - 1)^2$

Of course, this is not always needed, but it is good for teaching, to show how in simple cases you get to the same solution, manually or with a MATLAB program.

In this case the automation of the integral procedure must get a manual human step where the function to be integrated is simplified.

MATLAB has various commands like “*simplify*” (algebraic simplification), “*rewrite*” (rewrite expression in terms of another function) and “*combine*” (combine terms of identical algebraic structure) to change the structure of a function or a mathematical expression. Other command that can be used are “*collect*”, “*expand*”, “*numden*”, “*simplifyFraction*”.

Solving simple examples can be noted that if we are looking to find not only the prevertices, but also the explicit function solving the required SC conformal mapping, the MATLAB commands “*simplify*” and “*rewrite*” can be used.

However, to write a program capable to compact every (simple) solution is not easy, it is enough to know that the numeric calculated solution is correct.

### 3.3 Example 2: unit disk mapped to a L-shaped region

For the unit disk mapping we present just a computer aided solution to the parameter problem (finding the prevertices  $z_i$ ) (Driscoll, 1994).

MATLAB SC Toolbox: [commands in blue](#); output in black

```
>> p = polygon ([i -1+i -1-i 1-i 1 0])
```

p = polygon object:

Vertex	Angle/pi
0.0000 + 1.0000i	0.5000
-1.0000 + 1.0000i	0.5000
-1.0000 - 1.0000i	0.5000
1.0000 - 1.0000i	0.5000
1.0000 + 0.0000i	0.5000
0.0000 + 0.0000i	1.5000

```
>> plot (p)
```

```
>> f = diskmap (p)
```

diskmap object:

vertex	alpha	prevertex	arg/pi
0.00000 + 1.00000i	0.50000	0.98974 + 0.14286i	0.045628948204
-1.00000 + 1.00000i	0.50000	0.98811 + 0.15378i	0.049144854229
-1.00000 - 1.00000i	0.50000	0.95325 + 0.30217i	0.097710854028
1.00000 - 1.00000i	0.50000	-1.00000 + 0.00000i	1.000000000000
1.00000 + 0.00000i	0.50000	-0.00000 - 1.00000i	1.500000000000
0.00000 + 0.00000i	1.50000	1.00000 + 0.00000i	2.000000000000

```
c = -0.48783135 + 0.29499692i
```

Conformal center at 0.4955 - 0.5829i

Apparent accuracy is 6.31e-08

```
>> plot (f)
```

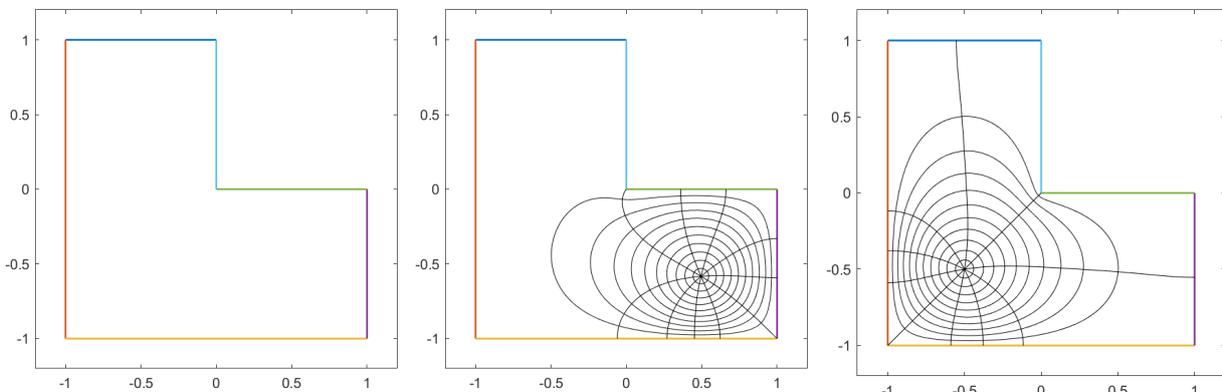


Figure 28:L-shaped region; unit disk mapped to a L-shaped region; same with map centered in (-0.5 -0.5i)

What is showed in Figure 28 are the images of ten evenly spaced circles centered at the origin and ten evenly spaced radii in the unit disk. Some sectors of the new grid are greatly distorted than others. The intersections of radii and circles preserve the orthogonality (conformal mapping). The 3<sup>rd</sup> figure on the right is obtained from the 2<sup>nd</sup> changing later the conformal center to  $(-0.5, -0.5 i)$  and re-plotting the map. In this case, of course, the prevertices are in general complex numbers.

SC Toolbox can also map the disc to the exterior polygon as in Figure 29, plotted for the already calculated mapping.

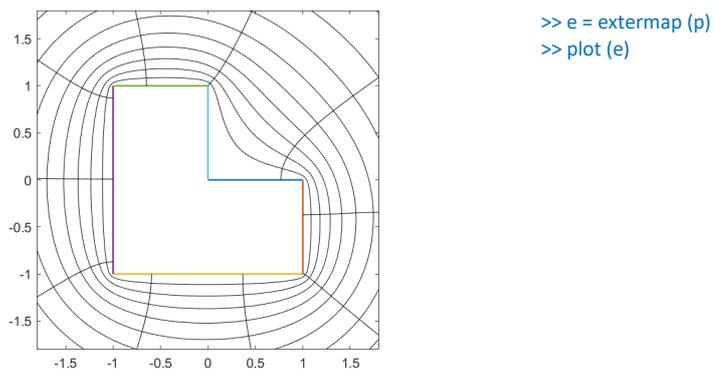


Figure 29: unit disk mapped to the exterior of a L-shaped region.

### 3.3.1 Crowding on an L-shaped region

Crowding has been introduced in paragraph 1.6.1.

If we check the effect of crowding on an L-shaped region we can see that the numeric computation can even fail if crowding is relevant. To avoid that we need a better and possibly more resource consuming algorithm.

In MATLAB SC Tool this is solved using a different algorithm launched through the `crdiskmap(f)` command instead of standard `diskmap(f)` as it is shown in Figure 32 and Figure 33.

The `crdiskmap` use a specific algorithm that may add trivial vertices to the original polygon. This can increase the calculation time if the region has many elongations and therefore many additional vertices.

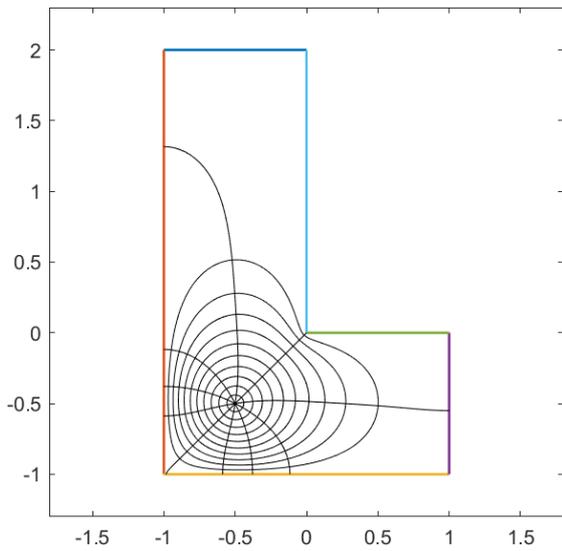


Figure 30: unit disk mapped to a L-shaped region: longer side  $a=2$

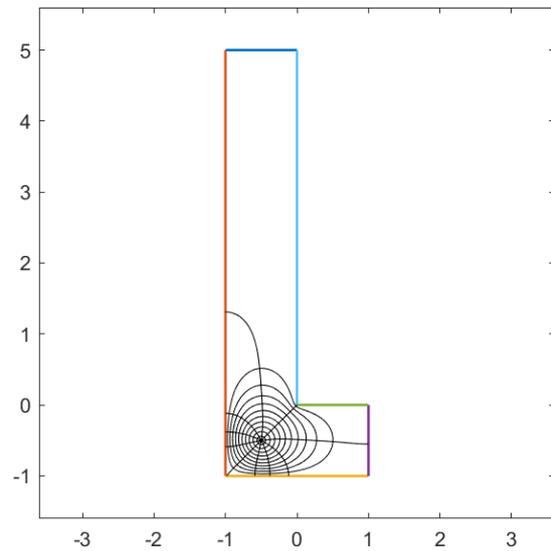


Figure 31: unit disk mapped to a L-shaped region: longer side  $a=5$

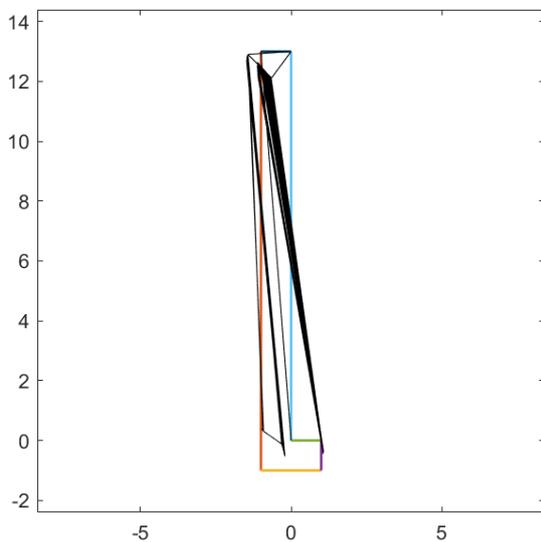


Figure 32: unit disk mapped to a L-shaped region: longer side  $a=13$ . Standard algorithm is failing.

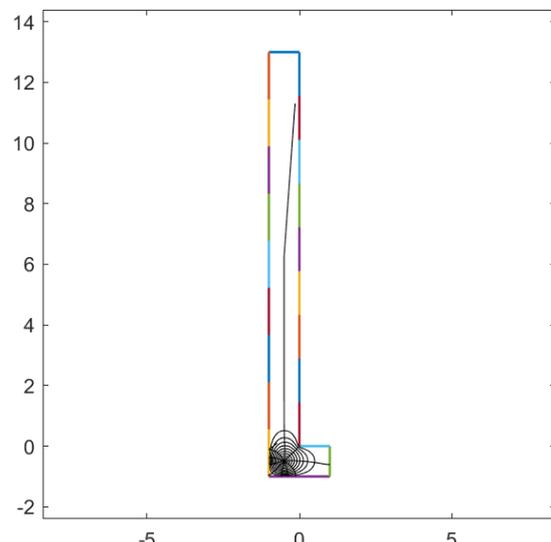


Figure 33: unit disk mapped to a L-shaped region: longer side  $a=13$ . Solved with modified algorithm `crdiskmap(f)`.

## 4 Physics applications to 2D fluid flow and 2D electrostatics

In this section we show that the SC idea can be applied to the classical fluid mechanics problem of ideal free-streamline flows (Brown & Churchill, 2009, p. 382) (Mathews & Howell, 2012, pp. 349–360).

We said already in paragraph 1.2 that the transformation  $w = f(z)$  is conformal if and only if it is holomorphic and its derivative  $f'(z)$  is everywhere non-zero on  $D$ .

We consider therefore here only complex functions  $f(x + iy) = u(x, y) + iv(x, y)$  that have the property to be holomorphic functions.

## 4.1 Two-dimensional free-streamline flows

We consider here just *holomorphic functions*, defined on a *simply connected* region.

If the domain of flow  $D_w$  in the  $uv$  plane is the image of a domain  $D_z$  under a transformation

$$w = f(z) = u(x, y) + iv(x, y), f(z) \text{ holomorphic in } D_z$$

we can consider his antiderivative

$$F(f(z)) = F(w) = \phi(u, v) + i \psi(u, v)$$

For the class of holomorphic functions is true that  $F(w)$  is harmonic in the Domain  $D_w$  defined by the transformation  $w=f(z)$ . A demonstration of that is present in most complex analysis books (e.g. (Needham, 1998, pp. 511-538)). The two functions  $\phi(u, v)$  and  $\psi(u, v)$  are both harmonic as well.

The functions  $\phi[u(x, y), v(x, y)]$  and  $\psi[u(x, y), v(x, y)]$  are harmonic in  $D_z$ .

The function  $F(w) = \phi(u, v) + i \psi(u, v)$  is named **complex potential (of the flow) in the  $(u,v)$  plane** and the composite function  $F[f(z)] = \phi[u(x, y), v(x, y)] + i\psi[u(x, y), v(x, y)]$  is the complex potential in the  $(x,y)$  plane.

We can write for the derivative of  $F(z)$ :

$$F'(z) = \phi_x(x, y) + i\psi_x(x, y) = \phi_x(x, y) - i\phi_y(x, y) = f(z)$$

(because of the Cauchy–Riemann equations (Brown & Churchill, 2009, p. 385))

We can also define:

- The velocity as  $V = \overline{F'(z)}$
- The module of the velocity is  $|V| = |F'(z)|$
- The function  $\psi(x, y)$  is the stream function. The level curves  $\psi(x, y) = \text{constant}$ , are called the streamlines (same for  $\psi(u,v)$  ).
- A streamline or natural boundary  $\psi(u, v) = \text{constant}$  in the  $(u,v)$  plane corresponds to a streamline or natural boundary  $\psi[u(x, y), v(x, y)] = \text{constant}$  in the  $(x,y)$  plane.
- The function  $\phi(x, y)$  is called the velocity potential. The level curves  $\phi(x, y) = \text{constant}$ , are called equipotentials or iso-potential (same for  $\phi(u,v)$  ).
- The equipotential curves are orthogonal to the streamline curves (Mathews & Howell, 2012, p. 310) in both planes.

The streamline defining a natural boundary works as a wall for the for the fluid flow.

We consider only the two-dimensional problem and assume to have a steady-state flow (the fluid flow is unchanging in time).

The flow motion is assumed to be the same in all horizontal planes parallel to the  $z$  plane, the velocity being parallel to that plane and constant everywhere in time.

Now we are considering just irrotational, incompressible fluids (Brown & Churchill, 2009, pp. 383–385) (Mathews & Howell, 2012, pp. 89–92). Additionally, we can also say that holomorphic functions in simply connected closed domains behaves like the functions describing conservative fields in physic (the integral over a closed path is zero) and are capable to model fluid dynamics, electrostatic or electromagnetic field functions.

The definitions above can be ported to describe different physical phenomena, in **Error! Reference source not found**. Table 2 there is an overview on how the name of the curves  $\phi$  and  $\psi$  changes accordingly (Mathews & Howell, 2012, p. 329).

Table 2: Interpretations for Level curves

Physical phenomenon	Level curves	
	$\phi(x, y) = \text{constant}$	$\psi(x, y) = \text{constant}$
<b>Fluid flow</b>	<b>Equipotentials</b>	<b>Streamlines</b>
<b>Electrostatics</b>	<b>Equipotential curves</b>	<b>Flux lines</b>
Heat flow	Isothermals	Heat flow lines
Gravitational field	Gravitational potential	Lines of force
Magnetism	Potential	Lines of force
Diffusion	Concentration	Lines of flow
Elasticity	Strain function	Stress lines
Current flow	Potential	Lines of flow

## 4.2 Image of a two-dimensional fluid flow turning a corner.

In the next example, where we try to find the image of a (two-dimensional) fluid flow under a conformal transformation.

Consider a flow in the first quadrant  $x > 0, y > 0$  that comes in downward parallel to the  $y$  axis but is forced to turn a corner near the origin, similar to what is shown in Figure 34.

To determine the flow, we look for a transformation that maps the first quadrant on the half upper plane like  $w = f(z) = z^2$ :

$$w = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$$

The Q1 boundary of the  $z$  plane is mapped to the real axis in the  $w$  plane, that is the boundary of the upper half-plane  $\text{Im}(w) > 0$ .

The *complex potential* for a uniform flow to the right in the upper half of the  $w$  plane is  $F = Aw$ , where  $A$  is a positive real constant.

Using now the definitions from the previous paragraph 4.1:

- *complex potential in the Q1 quadrant:*  $F = Aw = Az^2 = Ax^2 - Ay^2 + i2Axy = \phi(x, y) + i\psi(x, y)$
- velocity potential:  $\phi(x, y) = Ax^2 - Ay^2$
- stream function for the flow:  $\psi(x, y) = 2Axy$
- the streamlines are the branches of the hyperbolas  $2Axy = \text{constant}$ .
- the fluid velocity is  $V = \overline{F'(z)} = \overline{2Az} = 2A(x - iy)$ .
- the module of the velocity  $|V| = 2A\sqrt{x^2 + y^2}$ , decreasing as far as the  $(x, y)$  point near the origin

This example can be formulated also mapping in the other direction from the upper half  $z$  plane to the first quadrant of the  $w$  plane, in this case we can use the SC transformation (9) .

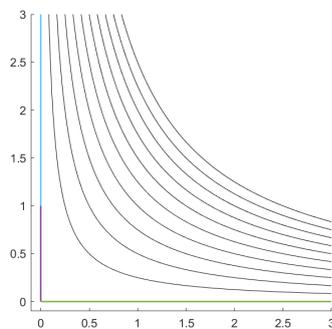


Figure 34: Streamline of a fluid flow in Q1.

```
%fluid flow streamlines
v= [i,0,Inf];
ang = [1,1/2,-1/2];
p = polygon(v,ang);
axis([-0.1 3 -0.1 3]), hold on
plot (p);
fmap = hplmap(p);
plot(fmap,0,1*(1:10))
```

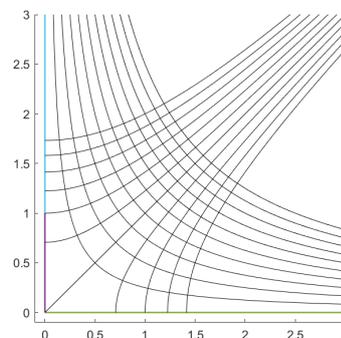


Figure 35: Streamline & iso-potential of a fluid flow in Q1.

```
%fluid flow streamlines % &
iso-potential
v= [i,0,Inf];
ang = [1,1/2,-1/2];
p = polygon(v,ang);
axis([-0.1 3 -0.1 3]), hold on
plot (p);
fmap = hplmap(p);
plot(fmap,1*(-5:5),1*(1:10))
```

Figure 34 is obtained as Schwarz-Christoffel mapping from the upper half-plane to the first quadrant, the boundary polygon is expressed as a triangle with a vertex to infinite, a vertex in  $(0, 0)$  and a trivial vertex in  $(0, i)$ . Sometimes it is needed to add trivial vertices to the polygon.

This approach allows us to find the *streamlines* for the fluid flow in a domain of the  $w$  plane bounded by straight line segments. Figure 35: Streamline & iso-potential of a fluid flow in Q1. Figure 35 is the same with addition of the iso-potential curves: the mapping clearly preserves the orthogonality of streamlines and iso-potentials.

The transformation from the upper half-plane to the first quadrant is of the type  $w = f(z) = z^{1/2}$ , this can be calculated explicitly with the SC transformation, where the exponent  $1/2$  is direct consequence of the right angle in the polygon.

Using MATLAB, we get as SC solution for this mapping:

$$f(z) = A + C \cdot [2 \cdot (z - 1)^{1/2}]$$

that is also the same solution we find resolving manually the Integral in (9).

### 4.2.1 Fluid flow over a step

We consider in this chapter only SC mapping from the half-plane  $\text{Im}(z) > 0$ .

We want to show in the next chapter that the Schwarz-Christoffel mapping allow us to find the *streamlines* for flows in the  $w$  plane bounded by straight line segments.

We consider here  $z = t + iy$ .

The flow in the upper half-plane  $\text{Im}(z) > 0$  is seen as decomposed in streamlines with constant  $y = c$ .

The image of the streamline  $-\infty < t < \infty$  will be a in the  $w$  plane a streamline given by the parametric equations

$$\begin{cases} u = u(t, c) \\ v = v(t, c) \end{cases} \quad -\infty < t < \infty$$

and will be oriented in the anticlockwise (positive) sense.

The streamline  $u = u(t, 0), v = v(t, 0)$  is considered to be a boundary wall for the for the fluid flow.

Consider the conformal mapping:

$$w = f(z) = \frac{1}{\pi} [ (z-1)^{1/2} + \text{Log}(z + (z^2-1)^{1/2}) ]$$

which is obtained by using the Schwarz-Christoffel formula, to map the upper half-plane  $\text{Im}(z) > 0$  onto the domain in the  $w$  plane (**Error! Reference source not found.**) that lies above the boundary.

The boundary consists of the rays  $u \leq 0, v = 1$  and  $u \geq 0, v = 0$  and the segment  $u = 0, -1 \leq v \leq 0$

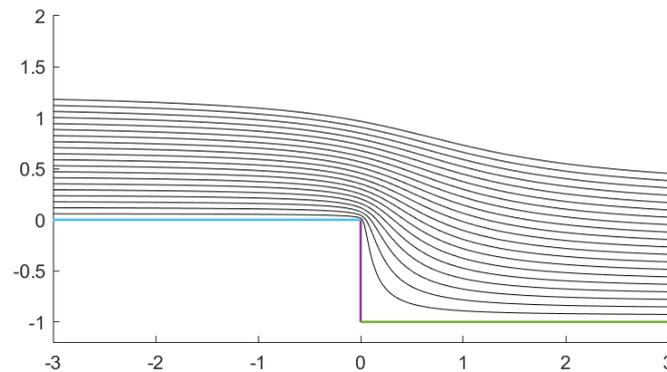


Figure 36: Domain in the  $w$  plane (Flow over a step)

The image of horizontal streamlines in the  $z$  plane are curves in the  $w$  plane given by the parametric equation:

$$w = f(t + ic) = \frac{1}{\pi} (t^2 - c^2 - 1 + i2ct)^{1/2} + \frac{1}{\pi} \text{Log}[t + ic + (t^2 - c^2 - 1 + i2ct)^{1/2} ], \text{ for } -\infty < t < \infty$$

The new flow is that of a step in the bed of a deep stream and is illustrated in Figure 36.

We use MATLAB SC Toolbox for the graphical part (Figure 36), where just the boundary and twenty streamlines are showed, hiding the radial iso-potential curves.

```
v = [i, 0, Inf];
ang = [3/2, 1/2, -1];
n = size(v);
```

```

index = n(2);
p = polygon(v,ang);
axis([-3 3 -0.2 2.5]), hold on
plot (p);
fmap = hplmap(p);
plot (fmap, 0, 20)

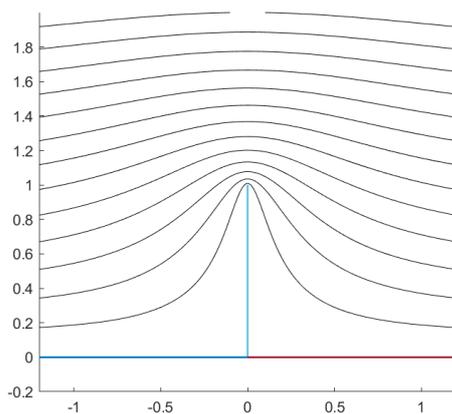
```

#### 4.2.2 Fluid flow around a slit

We try to graphically show the fluid flow around a slit. This can be achieved with one conformal transformation  $w=f(z)$ , or with a two-steps process(Mathews & Howell, 2012, p. 299). The function  $Z=(z^2-1)$  maps the upper half-plane  $\text{Im}(z)>0$  onto the  $Z$ -plane slit along the ray  $Y = 0, X = -1$ . Then the function  $w = Z^{1/2}$  maps the slit plane onto the slit half-plane, as shown in Figure 37.

Figure 37: The composite transformation  $w = f(z) = (z^2-1)^{1/2}$  and the intermediate steps  $Z=(z^2-1)$  and  $w= Z^{1/2}$

Defining the slit and looking for the function  $w=f(z)$  with SC Toolbox:



```

v= [0 , i, 0, Inf];
ang = [1/2, 2 ,1/2, -1];
n = size(v);
index = n(2);
p = polygon(v,ang);
axis([-1.2 1.2 -0.2 2]), hold on
plot (p);
fmap = hplmap(p);
plot (fmap, 0, 30)
prev = [prevertex(fmap)];
ang = [angle(p)];
syms z;
fder = 1;
for j = 1:(index-1)
    fderk = (z- prev(j))^(ang(j)-1)
    fder = fder * fderk;
end
fder
G = int(fder,z,'IgnoreAnalyticConstraints',true)

```

Figure 38: MATLAB SC Toolbox Fluid flow around a slit

The Answer from the MATLAB program (using 'IgnoreAnalyticConstraints' option) is:

```

fder = z/((z - 1)^(1/2)*(z + 1)^(1/2))
G = (z^2 - 1)^(1/2)

```

We get as MATLAB answer  $G(z) = \sqrt{(z^2 - 1)}$ , with in general  $f(z) = A \cdot G(z) + B$ , see below.

$$\begin{cases} f(z) = A \cdot G(z) + B \\ f(-1) = A \cdot 0 + B = 0 \\ F(0) = A \cdot i = i \end{cases} \quad \longrightarrow \quad \begin{cases} f(z) = G(z) \\ B = 0 \\ A = 1 \end{cases}$$

As  $B=0$  and  $A=1$ , the final answer is  $f(z) = \sqrt{(z^2 - 1)}$

### 4.3 Electrostatic fields representation using SC mapping

As we already discussed, conformal mapping  $f \rightarrow w=f(z)$  is an analytical function maps the complex plane  $Z$  into the complex plane  $W$ .

In chapter 4 we have described some properties of holomorphic functions and introduced the idea of complex potential. Electrical field is also a conservative field and can be described using holomorphic functions.

We refer now to  $F$ ,  $\psi$  and  $\phi$  in general, independently of the plane  $(x,y)$  or  $(u,v)$ .

We can reuse the definitions outlined in chapter 4:

- The function  $F = \phi + i \psi$  is named complex potential
- The function  $\psi$  is named flux function. The level curves  $\psi = constant$ , are called iso-flux.
- A iso-flux or natural boundary  $\psi = constant$  in the  $(u,v)$  plane corresponds to a iso-flux or natural boundary  $\psi = constant$  in the  $(x,y)$  plane.
- The function  $\phi$  is called the velocity potential. The level curves  $\phi = constant$ , are called equipotentials or iso-potential.
- The equipotential curves are orthogonal to the iso-flux curves in both planes.

The two examples below represent electrostatic fields in high voltage applications (Rogowski electrodes, designed to minimize the electric field on their ends) (*Rogowski Electrodes --QuickField FEA Software*, n.d., p. 1).

The electric field is shown in the  $w$  plane. In this chapter we use the SC transformation from a strip to a polygon that has been drafty discussed in chapter 2.4. The electrodes, have a potential of  $+a$  and  $-a$ .

We can try to represent Rogowski electrodes using again MATLAB SC Toolbox.

In many cases, it is desirable to produce uniform electric fields between a set of electrodes. However, simple parallel plate electrodes are finite in size and therefore have electric field enhancement at the edges of the plates (Figure 40).

Rogowski also studied the case of finite plate above an infinite ground plane, shown in Figure 39.

In these 2 figures is very clear how the SC transformation works: the potential and the flux are represented by iso-potential and iso-flux lines in the  $W$  plane (Costa, 1999, pp. 23–24). The orthogonality of iso-potential and iso-flux lines is preserved through the conformal transformation.

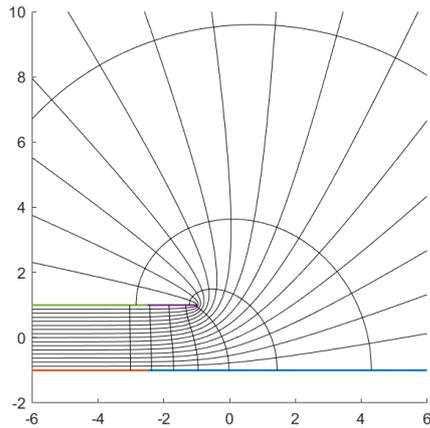


Figure 39: Flow and potential lines associated with finite plate above infinite plate

```
%function condensator
w = [Inf,1000-i,-2.5-i,Inf,-1+i,-2.5+i];
ang = [0,2,1,-2,2,1];
p = polygon(w,ang);
f1 = stripmap(p,[1 4]);
axis([-6 6 -2 10]), hold on
plot(f1,15,15)
```

MATLAB SC Toolbox code (Figure 39), realized for this work

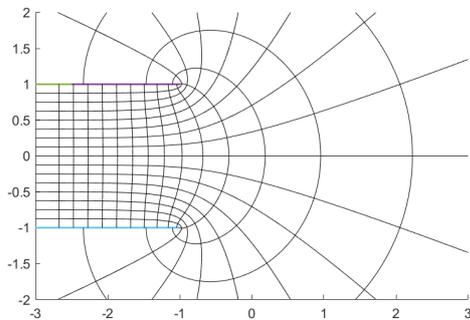


Figure 40: Flow and potential lines of electric field: Rogowski electrode profile

```
%function condensator
w = [Inf,-1-i,-2.5-i,Inf,-1+i,-2.5+i];
ang = [0,2,1,-2,2,1];
p = polygon(w,ang);
f1 = stripmap(p,[1 4]);
axis([-3 3 -2 2]), hold on
plot(f1,15,15)
```

MATLAB SC Toolbox code (Figure 40), realized for this work

## 5 Beautiful Math (aesthetic pattern based on SC transformation)

This final section is inspired from two very interesting articles written by Peichang Ouyang and other colleagues. (Ouyang et al., 2015) (Ouyang & Chung, 2014)

The triangle is a basic shape in any geometrical space. However, a hyperbolic triangle could not exist in Euclidean space because its angle sum is less than  $\pi$ . This simple example suggests that hyperbolic and Euclidean spaces are in theory “inconsistent”.

For planar figures, conformal mapping is a communication method between these two mathematical worlds. Conformal mapping can “translate” beautiful Euclidean art into another world, with different rules. This method of transforming Euclidean symmetric patterns into hyperbolic ones is also named “hyperbolization” of Euclidean ornaments.

As an example, we take the Poincaré hyperbolic disk, a two-dimensional space having hyperbolic geometry defined on.

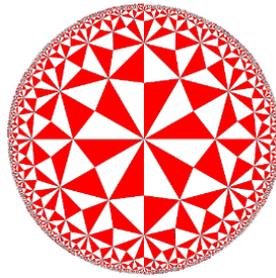


Figure 41: Poincaré disk realized with Mathematica. The aesthetic patterns in the figure below can have arbitrary resolution without distortion.

The Schwarz-Christoffel mapping provides the numerical method for conformal mappings. This will in effect allow to transform hyperbolic patterns, from a disc e.g., into a regular  $n$ -sided polygon. This is similar to what the Dutch artist M.C. Escher solved in his art pictures.

The modern solution of the problem is to use Schwarz-Christoffel mapping, that can perfectly preserve the pattern structure.

The applications of Schwarz-Christoffel mapping in arts are still on going and have still to be fully discovered. Using computer numeric integration and very modern software it is possible to introduce a method to create aesthetic patterns transforming e.g., Escher's "Circle Limit I" to a "Square Limit".



Figure 42: Escher's "Circle Limits I" and same picture conformally transformed with Schwarz-Christoffel mapping to a new "Square Limits"(Fong, 2019)

## 5.1 Implementation Details

The numerical realization of Schwarz-Christoffel mapping has been refined for half a century. For the following picture the authors have used MATLAB SC Toolbox , the same tool mentioned above in this text many times.

The authors have just applied three functions of SC Toolbox (polygon, diskmap, evalinv).

These functions were called in the order described by the following steps:

1.  $p = \text{polygon}(v)$ , where  $v = [v_1, v_2, \dots, v_n]$  are the polygon vertices
2.  $f = \text{diskmap}(p)$ , which yields the desired Schwarz-Christoffel mapping  $f$
3.  $d = \text{evalinv}(f, w)$ , which yields the inverse of  $f$  at  $w$ .

Let  $w(i, j)$  be the element in row  $i$  and column  $j$  of matrix  $w$ . Then the corresponding conformal point in the unit disc  $D$  is  $w(i, j) = \text{evalinv}(f, w(i, j))$ . Steps 1 through 3 outlined above realize numerically the conformal mapping  $f$  between  $w$  and  $D$ .

Figure 43 shows an example that conformally transforms a disc tiling into a regular pentagon. Thus, once we have a disc pattern, we can transform it into a specific polygon region and obtain a polygonal aesthetic pattern. (For the creation of disc patterns with the symmetry of triangle groups, see the “Beautiful Math, Part 3” article)(Ouyang & Chung, 2014). Because the inverse mapping method ( $d = evalinv(f, w)$  command) with SC Toolbox consumes lot of memory resources, the polygonal aesthetic patterns are indeed difficult to obtain. The transformation below takes nearly 30 minutes of CPU time on a professional university computer and the picture has just 2000x2000 pixel.

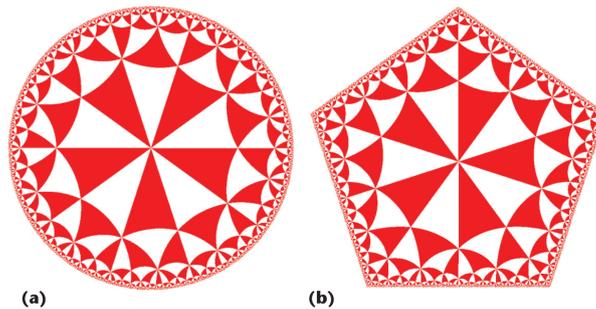


Figure 43: (a) Poincare disk and (b) same picture converted to a pentagon.

## 5.2 Gallery of Polygonal Patterns

More complex pattern obtained by Ouyang and Chung and presented in their articles (Ouyang et al., 2015) (Ouyang & Chung, 2014), an example of these patterns is copied in Figure 44. This is to show how mathematics allow people to express relations between numbers as a sort of “modern art”.

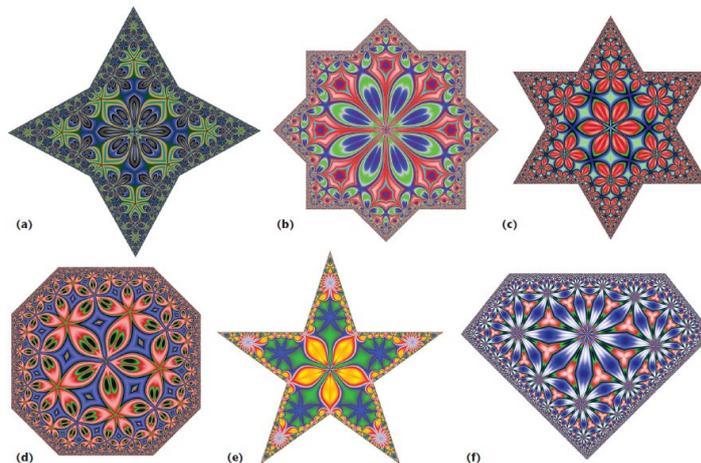


Figure 44: Specially designed polygons. (a) Cross, (b) overlapping squares, (c) overlapping triangles, (d) non-regular octagon, (e) star, (f) diamond

## 6 Bibliography

- Bjørstad, P., & Grosse, E. (1987). Conformal mapping of circular arc polygons. *SIAM Journal on Scientific and Statistical Computing*, 8(1), 19–32.
- Brown, J. W., & Churchill, R. V. (2009). *Complex variables and applications*. Boston: McGraw-Hill Higher Education,.
- Costa, S. (1999). *Applicazione delle trasformazioni di Schwarz-Christoffel per la progettazione assistita dal calcolatore di dispositivi elettromeccanici*. [https://www.matematicamente.it/tesi/Costa-trasformazioni\\_schwarz\\_christoffel.pdf](https://www.matematicamente.it/tesi/Costa-trasformazioni_schwarz_christoffel.pdf)
- DeLillo, T. K., Elcrat, A. R., & Pfaltzgraff, J. A. (2004). Schwarz-Christoffel mapping of multiply connected domains. *Journal d'Analyse Mathématique*, 94(1), 17–47.
- DeLillo, T. K., & Kropf, E. H. (2011). Numerical computation of the Schwarz–Christoffel transformation for multiply connected domains. *SIAM Journal on Scientific Computing*, 33(3), 1369–1394.
- DeLillo, T. K., & Pfaltzgraff, J. A. (1998). Numerical Conformal Mapping Methods for Simply and Doubly Connected Regions. *SIAM Journal on Scientific Computing; Philadelphia*, 19(1), 17. <http://dx-doi-org.uaccess.univie.ac.at/10.1137/S1064827596303545>
- Driscoll, T. A. (1994). *Schwarz-Christoffel Toolbox User's Guide*. Cornell University.
- Driscoll, T. A. (1996). Algorithm 756: A MATLAB toolbox for Schwarz-Christoffel mapping. *ACM Transactions on Mathematical Software (TOMS)*, 22(2), 168–186. <http://dl.acm.org/citation.cfm?id=229475>
- Driscoll, T. A. (2005). Algorithm 843: Improvements to the Schwarz-Christoffel toolbox for MATLAB. *ACM Transactions on Mathematical Software (TOMS)*, 31(2), 239–251.
- Driscoll, T. A., Hale, N., & Trefethen, L. N. (2014). *Chebfun guide*. Pafnuty Publications, Oxford.
- Driscoll, T. A., & Trefethen, L. N. (2002). *Schwarz-Christoffel mapping*. Cambridge University Press.
- Fong, C. (2019). Analytical Methods for Squaring the Disc. *ArXiv:1509.06344 [Math]*. <http://arxiv.org/abs/1509.06344>
- Hendriks, W. H. (2009). *The Schwarz-Christoffel transformation and elliptic functions*.

- Hu, C. (1998). Algorithm 785: A software package for computing Schwarz-Christoffel conformal transformation for doubly connected polygonal regions. *ACM Transactions on Mathematical Software (TOMS)*, 24(3), 317–333.
- Iske, A. (2020). *Komplexe Funktionen für Studierende der Ingenieurwissenschaften*. Technische Universität Hamburg. <https://www.math.uni-hamburg.de/teaching/export/tuhh/cm/kf/20/vorlesungen/kf.pdf>
- Krantz, S. G. (2012). *Handbook of complex variables*. Springer Science & Business Media.
- Luteberget, B. S. (2010). *Numerical approximation of conformal mappings* [Master's Thesis]. Institutt for matematiske fag.
- Mathews, J., & Howell, R. (2012). *Complex analysis for mathematics and engineering*. Jones & Bartlett Publishers.
- Needham, T. (1998). *Visual complex analysis*. Oxford University Press.
- Olver, P. J. (2017). *Complex analysis and conformal mapping*. University of Minnesota.
- Ouyang, P., & Chung, K. (2014). Beautiful math, part 3: Hyperbolic aesthetic patterns based on conformal mappings. *IEEE Computer Graphics and Applications*, 34(2), 72–79.
- Ouyang, P., Ding, F., & Wang, X. (2015). Beautiful Math, Part 4: Polygonal Aesthetic Patterns Based on the Schwarz-Christoffel Mapping. *IEEE Computer Graphics and Applications*, 35(4), 22–25.
- Rangel-Mondragon, J. (2011, March 7). *A Catalog of Conformal Mappings*. <http://demonstrations.wolfram.com/ACatalogOfConformalMappings/>
- Rogowski electrodes—QuickField FEA Software. (n.d.). Retrieved February 23, 2020, from [https://quickfield.com/advanced/rogowski\\_electrodes.htm](https://quickfield.com/advanced/rogowski_electrodes.htm)
- Schwarz, H. A. (1972). *Gesammelte mathematische abhandlungen* (Vol. 260). American Mathematical Soc.
- Swan, Y. C., & Bruss, F. T. (n.d.). *Schwarz-Christoffel Transformation and Applications in Applied Probability*.

Wegert, E. (n.d.). *Complex Function Explorer—File Exchange—MATLAB Central*. Retrieved January 18, 2021, from <https://www.mathworks.com/matlabcentral/fileexchange/44375-phase-plots-of-complex-functions>

Wegert, E. (2012). *Visual complex functions: An introduction with phase portraits*. Springer Science & Business Media.

## 7 Index of figures

Figure 1: Cartesian and polar representation of a complex number.....	4
Figure 2: A generalized polygon drawn in MATLAB with SC Toolbox .....	6
Figure 3: A generalized polygon drawn in GeoGebra.....	6
Figure 4: Conformal mapping.....	6
Figure 5: Conformal maps preserve angles and the shapes of infinitesimally small figures. Unit square is in the $z$ plane, $w = z^a$ , $a=1.5, 2, 2.5, 3, 3.5$ . Implemented with MATLAB & Chebfun .....	7
Figure 6: Conformal maps preserve both angles and the shapes of infinitesimally small figures. Left: unit square in the $z$ Plane; right: $w = z^{0.5} = \sqrt{z}$ . Implemented with MATLAB & Chebfun .....	8
Figure 7: Conformal Mapping and Riemann Theorem.....	8
Figure 8: Square (side=1) in $z$ plane Q1 and related mapping under $w=\sin(z)$ and $w=\cos(z)$ .....	9
Figure 9: Square (side=2) centered in the origin and related mapping: $w=\sin(z)$ , $w=\cos(z)$ , $w=1/z$ , $w=1/z^2$ , $w=e^z$ .....	10
Figure 10: Two-dimensional color scheme used for phase plot, according to traditional color wheel...	10
Figure 11: Phase plot construction for the same function $f=\sin(z)$ using different representation options (MATLAB and Visual Complex Functions library) (Wegert, n.d.).....	11
Figure 12: $f(z)=\sin(z)$ (plotted with Mathematica) .....	11
Figure 13: Unit square and conformal transformation 1 to 8.....	12
Figure 14: Example of Conformal Mapping using MATLAB.....	13
Figure 15: Schwarz's 1869 conformal map of a square onto a disk, reproduced from "Gesammelte Mathematische Abhandlungen (Schwarz, 1972) .....	15
Figure 16: Schwarz-Christoffel transformation: the upper half-plane is mapped onto a polygon bounded domain.....	15
Figure 17: Circular arc polygon domain mapping: the unit disk is mapped onto a polygon bounded domain, having arcs as sides. ....	15
Figure 18: Crowding explained through a map from the disc to a rectangle with a different side length ratio $a$ , for $a= 1, a=4, a=8$ .....	17
Figure 19: Simply connected domain .....	18
Figure 20: Connected domain, not simply .....	18
Figure 21: Not connected domain.....	18
Figure 22: MCSC map: map onto a multiply connected polygonal domain (Bauer, 2015, pp. 1–3). .....	19
Figure 23: map onto a multiply connected arc-polygon domain (Bauer, 2015, pp. 1–3) .....	19
Figure 24: Notational convention for SCT .....	20
Figure 25: SC mapping from unit disc to a $n$ -regular polygon for $n = 3, 4, 6, 9$ .....	23

Figure 26: Prevertices  $z_i$  and vertices  $w_j$  ; angles according to SC Toolbox: standard. If the transformation is from the extended half-plane, we can name the  $z_i$  as  $x_i$ , to make it clear they have all a value in  $\mathbb{R}$  .. 25

Figure 27: mapping the upper half-plane  $\text{Im}(z) > 0$  to the semi-infinite strip..... 26

Figure 28:L-shaped region; unit disk mapped to a L-shaped region; same with map centered in  $(-0.5 -0.5i)$  ..... 28

Figure 29: unit disk mapped to the exterior of a L-shaped region..... 29

Figure 30: unit disk mapped to a L-shaped region: longer side  $a=2$  ..... 30

Figure 31: unit disk mapped to a L-shaped region: longer side  $a=5$  ..... 30

Figure 32: unit disk mapped to a L-shaped region: longer side  $a=13$ . Standard algorithm is failing. .... 30

Figure 33: unit disk mapped to a L-shaped region: longer side  $a=13$ . Solved with modified algorithm `crdiskmap(f)`..... 30

Figure 34: Streamline of a fluid flow in  $Q_1$ ..... 33

Figure 35: Streamline & iso-potential of a fluid flow in  $Q_1$ ..... 33

Figure 36: Domain in the  $w$  plane (Flow over a step)..... 34

Figure 37: The composite transformation  $w = f(z) = (z^2-1)^{1/2}$ , and the intermediate steps  $Z = (z^2-1)$  and  $w = Z^{1/2}$  ..... 35

Figure 38: MATLAB SC Toolbox Fluid flow around a slit..... 35

Figure 39: Flow and potential lines associated with finite plate above infinite plate ..... 37

Figure 40: Flow and potential lines of electric field: Rogowski electrode profile ..... 37

Figure 41: Poincare disk realized with Mathematica. The aesthetic patterns in the figure below can have arbitrary resolution without distortion. .... 38

Figure 42: Escher's "Circle Limits I" and same picture conformally transformed with Schwarz-Christoffel mapping to a new "Square Limits"(Fong, 2019)..... 38

Figure 43: (a) Poincare disk and (b) same picture converted to a pentagon. .... 39

Figure 44: Specially designed polygons. (a) Cross, (b) overlapping squares, (c) overlapping triangles, (d) non-regular octagon, (e) star, (f) diamond..... 39

## 8 Index of tables

Table 1: Indefinite Integrals..... 22

Table 2: Interpretations for Level curves ..... 32

## 9 Abstract

This work describes the Conformal Mapping of domains by holomorphic functions. It gives in particular an overview on the Schwarz-Christoffel transformation and related developments.

A very quick historic information on the development of the SC Transformation from the original discovery (Hermann Schwarz in 1869; Elwin Christoffel in 1867), current finding and results, enabled by the using of complex numeric algorithms and computers, are given as well.

The Schwarz-Christoffel Transformation enables in many application fields the search for complex solutions in complex geometries if a simple solution in the canonic domain already exists.

Between the very large number of applications that can use the Schwarz-Christoffel transformation and his mathematical developments, we have chosen some simple examples related to fluid dynamics and electrostatic. The solutions are represented graphically using a computer.

The graphic representation is obtained in MATLAB or Mathematica (using special tools or libraries) and the software code written (or adapted) for drawing these images has been provided as well.

## 10 Kurzzusammenfassung

Diese Arbeit beschreibt die Konformen Abbildungen von Domänen mittels holomorpher Funktionen, insbesondere beschäftigt sich mit der Schwarz-Christoffel Abbildung.

Es wird eine kurze historische Darstellung gemacht und anschließend werden die wichtigsten Sätze dargestellt.

Die Schwarz-Christoffel Abbildung erlaubt in viele Anwendungsgebiete, bei denen eine Lösung für einfache Domänen existiert, die Suche nach komplexeren Lösungen in komplexeren Geometrien.

Es würden einfache Beispiele aus der Fluidodynamik und Elektrostatik ausgewählt, gerechnet und graphisch dargestellt. Die Berechnung und die graphische Darstellung basieren auf MATLAB oder Mathematica, mit der Hilfe von ausgewählte „Tools“ oder „Libraries“.

Die Abbildungen und die Software Programme, die geschrieben oder adaptiert worden sind, werden ausführlich in dieser Arbeit dargestellt.