universität wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

"Continuity of phase transition for Bernoulli percolation on generalized slabs"

verfasst von / submitted by

Moritz Dober, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2021 / Vienna 2021

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	A 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Marcin Lis, PhD

Abstract

Duminil-Copin, Sidoravicius and Tassion proved in [5] that the phase transition for *Bernoulli bond percolation* on a slab $\mathbb{Z}^2 \times \{0, \ldots, k\}$ $(k \geq 0)$ proceeds continuously, which is to say that there exists almost surely no infinite cluster at the *critical density*, and remarked that the techniques would work equally well for any graph of the form $\mathbb{Z}^2 \times G$, where G is finite. The aspiration of this thesis is to carry out the proof in said general setup and provide justification for the required results in the preliminary sections.

In the introduction and Chapter 2, we briefly introduce Bernoulli bond percolation models on arbitrary graphs, state basic properties of percolation laws and prove the following two facts. Using the arguments of Burton and Keane [4], we show (almost sure) uniqueness of an infinite cluster in the *supercritical phase* for connected quasi-transitive graphs of subexponential growth. Subsequently, we verify that *k-dependent* models on the square lattice $\mathbb{Z}^2$ percolate if each edge is open with probability at least some $p_c(k) < 1$, by means of a Peierls argument involving the dual graph.

Chapter 3 consists of the proof of the main theorem divided into three steps and a subsequent corollary of the *gluing* technique therein.

Zusammenfassung

Duminil-Copin, Sidoravicius und Tassion haben in [5] gezeigt, dass der Phasenübergang für *Bernoulli-Kantenperkolation* auf einer Platte $\mathbb{Z}^2 \times \{0, \ldots, k\}$ ($k \geq 0$) stetig verläuft, was bedeutet, dass bei der *kritischen Dichte* fast sicher kein unendlicher Cluster existiert und merkten an, dass die Techniken genauso gut für jeden Graphen der Form $\mathbb{Z}^2 \times G$ funktionieren würden, wobei G endlich ist. Das Ziel dieser Arbeit ist es, den Beweis in besagtem allgemeinen Umstand auszuarbeiten und Rechtfertigung für die dazu verwendeten Resultate in den einleitenden Abschnitten zu liefern.

In der Einleitung und in Kapitel 2 geben wir eine kurze Einführung zu Bernoulli-Kantenperkolationsmodellen auf allgemeinen Graphen, erwähnen grundlegende Eigenschaften von Perkolationsverteilungen und beweisen die folgenden zwei Tatsachen. Unter der Verwendung der Argumente von Burton und Keane [4] zeigen wir die (fast sichere) Eindeutigkeit eines unendlichen Clusters in der *superkritischen Phase* für zusammenhängende, quasitransitive Graphen von subexponentiellem Wachstum. Anschließend verifizieren wir, dass *k-abhängige* Modelle auf dem quadratischen Gitter $\mathbb{Z}^2$ perkolieren, wenn jede Kante mit einer Wahrscheinlichkeit von zumindest einem gewissen $p_c(k) < 1$ offen ist, anhand eines Peierls Arguments, das den dualen Graphen involviert.

Kapitel 3 besteht aus dem Beweis des Hauptsatzes aufgeteilt in drei Abschnitte und einem anschließenden Korollar bezüglich der *Klebetechnik* darin.

Contents

Chapter 1

Introduction

Let G be a connected graph with vertex set V and set of edges E, and let $p \in [0, 1]$. We declare each edge $e \in E$ to be *open* with probability p, and *closed* with probability $1 - p$, independently of the states of other edges. Formally, let $(\omega_e)_{e \in E}$ be a sequence of independent and identically distributed random variables on $(\Omega, \mathfrak{F}, \mathbb{P}_p)$ with $\mathbb{P}_p[\omega_e = 1] = p = 1 - \mathbb{P}_p[\omega_e = 0]$, where $\omega_e = 1$ corresponds to the edge being open. One can take $\Omega = \{0, 1\}^E$ equipped with the σ-algebra $\mathfrak{F}$ generated by cylinder sets, $\mathbb{P}_p$ the p-Bernoulli measure and ω_e the canonical projections. Members of Ω are referred to as *configurations* and will be identified with the corresponding induced subgraphs consisting of all vertices in V and all open edges in E. The model is called the *Bernoulli (bond) percolation model* on G and was introduced by Broadbent and Hammersley in 1957 [3].

Define $\Theta_G(p)$ to be the $\mathbb{P}_p$-probability that there exists an infinite *cluster* (connected component) in the induced subgraph, and set

$$p_c = p_c(G) = \inf\{p \in [0, 1] : \Theta_G(p) > 0\},$$

which is called the *critical density*. If $\Theta_G(p) > 0$, then the model is said to *percolate* with respect to the parameter p.

Moreover, denote by $\theta_{G,v}(p)$ the $\mathbb{P}_p$-probability that the cluster of $v \in V$ in the induced subgraph is infinite. If G is countable, then $\Theta_G(p) > 0$ precisely if $\theta_{G,v}(p) > 0$ for all $v \in V$. As $p \mapsto \theta_{G,v}(p)$ can be seen to be the pointwise limit of a decreasing sequence of continuous non-decreasing functions, it is itself right-continuous on $[0, 1]$. In fact, if for any p, there exists at most one infinite cluster in the induced subgraph $\mathbb{P}_p$-a.s. (almost surely with respect to $\mathbb{P}_p$), then said convergence is uniform on compact subintervals of $(p_c, 1]$, and hence $\theta_{G,v}$ is continuous on $[0, 1] \setminus \{p_c\}$. For this class of graphs, continuity of $\theta_{G,v}$ is therefore equivalent to left-continuity at p_c, which in turn is equivalent to a.s.-absence of an infinite cluster at the critical value, whence this property is called *continuity of phase transition*.

The major questions in this field pertain to the behaviour of the model for p close to or at p_c. Only in rare cases is the critical density known explicitly; e.g., Kesten [11] proved that $p_c(\mathbb{Z}^2) = 1/2$. It has been established that there exists a.s. no infinite cluster at the critical value on $\mathbb{Z}^d$ for $d = 2$ (due to Kesten [11] and Harris [10]) and $d \geq 19$ (due to Hara and Slade [9]), and it is widely believed that the same holds true in the remaining dimensions. In fact, the unpublished proof by Hara and Slade has been extended quite recently by Fitzner and Hofstad [6] to apply to $d \geq 11$. The probably most famous and still open case remains in 3-dimensional space $\mathbb{Z}^3$. Nevertheless, continuity of phase transition is known since 1991 for half-spaces $\mathbb{Z}^{d-1} \times \mathbb{N}$ (and quarter-spaces, etc.) in all dimensions; see [2]. Another related result is that, for any d, the critical densities of slabs $\mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}$ converge to the critical density of the full-space $\mathbb{Z}^d$ as k tends to infinity. This has been proved by Grimmett and Marstrand [8].

In the last decade, there has been progress for quasi-planar graphs of the form $\mathbb{Z}^2 \times G$ where G is any finite graph. In 2014, Duminil-Copin, Sidoravicius and Tassion [5] published a proof for G being a string of finite length and noted that it may be adapted to work for arbitrary G. The aspiration of this thesis is to carry out the proof in said general setup. As the techniques rely on almost sure uniqueness of an infinite cluster in the *super-critical phase* [1], this fact will be proved preliminary for connected quasi-transitive graphs of subexponential growth using the arguments of Burton and Keane [4].

All figures were made with GeoGebra except Figure 2.1, which was made with Ipe.

Chapter 2

Definitions, notations and preliminaries

2.1 Graphs and properties of percolation laws

In this section, we mention some basic notions and notations concerning graphs and state basic properties of percolation laws.

2.1.1 Graphs

We will only consider *simple locally finite* graphs $G = (V, E)$, where E is a set consisting of two-element subsets of V, and where, for each $v \in V$, there exist only finitely many $e \in E$ with $v \in e$. Members of V are called vertices, members of E edges, and members of $e \in E$ endpoints of e. We say that an edge connects its endpoints. By abuse of notation, the vertex set V will frequently be referred to as the graph G itself. For a subset $V' \subseteq V$, we write $E(V')$ for the set of edges with both endpoints belonging to V'. A *path* is a finite sequence of vertices for which consecutive ones are *adjacent* to one another, meaning that they are connected by an edge. We will make use of further common terminology as *incident, initial, terminal, geodesic, self-avoiding, (maximum) degree* of a vertex (graph) and so on.

An *automorphism* on a graph $G = (V, E)$ is a bijection γ on the vertex set V that preserves adjacency, i.e. for all $u, v \in V$ we have that u is adjacent to v if and only if $\gamma(u)$ is adjacent to $\gamma(v)$. Then γ induces a bijection on the set of edges E, which we will also call γ. We denote by $Aut(G)$ the group of all automorphisms on G and write $H \cdot v := \{\gamma(v) : \gamma \in H\}$ for any $v \in V$ and $H \subseteq Aut(G)$.

Given two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, the *(Cartesian) graph product* $G_1 \times G_2$ is defined to be the graph with vertex set $V_1 \times V_2$ and edges between vertices $(u_1, u_2), (v_1, v_2)$ exactly when either $u_1 = v_1$ and u_2, v_2 are adjacent in G_2, or u_1, v_1 are adjacent in G_1 and $u_2 = v_2$. It is straightforward to check that one can lift an automorphism γ on G_1 to the product by setting $\overline{\gamma}((v_1, v_2)) = (\gamma(v_1), v_2)$, and analogously for automorphisms on G_2.

For a graph which is embedded in the plane $\mathbb{R}^2$ in such a way that distinct edges intersect only at common vertices, there exists a *dual graph* which is defined as follows. The vertex set of the dual graph is the set of faces of the embedding, and two of these faces are joined by an edge in the dual graph if they share a boundary edge. This ensures that the set of edges of the dual graph is in 1-1 correspondence with the set of edges of the original graph. As an example, the dual of the square lattice $\mathbb{Z}^2$ may be identified with $(1/2, 1/2) + \mathbb{Z}^2$, and hence is isomorphic to $\mathbb{Z}^2$.

2.1.2 Percolation laws

Let $G = (V, E)$ be a connected graph, $p \in [0, 1]$, and define $(\Omega, \mathfrak{F}, \mathbb{P}_p)$ as in the introduction. We call $\mathbb{P}_p$ the (nearest neighbour Bernoulli bond) percolation law on G with respect to the parameter p, and write $\mathbb{E}_p$ for the expectation operator regarding $\mathbb{P}_p$.

Remark 2.1.1 (Continuity of $\mathbb{P}_p$ for local events). Let $\mathcal{A} \in \mathfrak{F}$ be a *local* event, i.e. $\mathcal{A}$ depends on only finitely many edges. More formally, there exists $E' \subseteq E$ finite such that $\mathcal{A}$ belongs to the σ-algebra generated by ω_e, $e \in E'$, where ω_e is the projection on the e-coordinate. Then the map $p \mapsto \mathbb{P}_p[\mathcal{A}]$ is a polynomial in p, and hence continuous.

Lemma 2.1.1 (Invariance of $\mathbb{P}_p$ under automorphisms)**.** *Let $\mathcal{A} \in \mathfrak{F}$ and $\gamma \in Aut(G)$. Then $\gamma(\mathcal{A}) \in \mathfrak{F}$ and $\mathbb{P}_p[\gamma(\mathcal{A})] = \mathbb{P}_p[\mathcal{A}]$, where $\gamma(\mathcal{A}) = \{\gamma(\omega) : \omega \in \mathcal{A}\}$ and $\gamma(\omega)_e = \omega_{\gamma^{-1}(e)}$.*

Proof. The statement is immediate for cylinder events and generalizes to the whole product σ-algebra by a $\pi - \lambda$ argument. $\square$

Definition 2.1.1 (Increasing/decreasing events and functions)**.** Let $\leq$ be the componentwise order on Ω, i.e. $\omega \leq \omega'$ if and only if $\omega_e \leq \omega'_e$ for all edges $e \in E$. A function $f : \Omega \to \mathbb{R}$ is called *increasing* if $\omega \leq \omega'$ implies $f(\omega) \leq f(\omega')$, and *decreasing* if $-f$ is increasing. An event $\mathcal{A} \in \mathfrak{F}$ is called *increasing* (respectively, *decreasing*) if $\mathbb{1}_{\mathcal{A}}$ is increasing (respectively, decreasing).

For proofs of the following results, the reader may consult, e.g., Grimmett's book [7], pages 32-36 and 288-289.

Lemma 2.1.2 (Monotonicity in p)**.** *If $X : \Omega \to \mathbb{R}$ is an increasing random variable and $p \leq q$, then $\mathbb{E}_p[X] \leq \mathbb{E}_q[X]$, whenever these expectations are defined. This implies that, for an increasing event $\mathcal{A} \in \mathfrak{F}$ and $p \leq q$, $\mathbb{P}_p[\mathcal{A}] \leq \mathbb{P}_q[\mathcal{A}]$.*

Lemma 2.1.3 (Positive correlation - Harris-FKG inequality)**.** *Let $X, Y : \Omega \to \mathbb{R}$ be increasing random variables. It is the case that $\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y]$, whenever these expectations are defined. This implies that, for increasing events $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$, it holds that $\mathbb{P}_p[\mathcal{A} \cap \mathcal{B}] \geq \mathbb{P}_p[\mathcal{A}]\,\mathbb{P}_p[\mathcal{B}]$.*

Corollary 2.1.1 (Square-root trick)**.** *If $n \geq 1$ and $\mathcal{A}_1, \dots, \mathcal{A}_n$ are increasing events, then*

$$\max\{\mathbb{P}_p[\mathcal{A}_1], \dots, \mathbb{P}_p[\mathcal{A}_n]\} \geq 1 - (1 - \mathbb{P}_p[\mathcal{A}_1 \cup \dots \cup \mathcal{A}_n])^{1/n}.$$

2.2 Uniqueness of infinite cluster

In 1960, Harris proved in [10] that the critical density on $\mathbb{Z}^2$ is at least $1/2$. Using this fact together with self-duality of $\mathbb{Z}^2$, he obtained that in the super-critical phase there exists a.s. a unique infinite cluster. In 1987, Aizenman, Kesten and Newman [1] proved this fact on $\mathbb{Z}^d$ for general d. Two years later, Burton and Keane [4] published another proof which is much shorter and simpler and moreover works for any transitive amenable graph, and with a little more effort (namely Lemma 2.2.1) also for so-called quasi-transitive graphs of subexponential growth:

Definition 2.2.1. A graph G is called *quasi-transitive* if there are only finitely many orbits for the action of the automorphism group $Aut(G)$ on the vertex set, i.e. there exist $N \geq 1$ and vertices $v_1, \ldots, v_N$ such that $G = \bigcup_{i=1}^{N} Aut(G) \cdot v_i$.

Throughout this section, for a vertex v in some graph G and $n \geq 0$, we will write $\mathfrak{B}_n(v)$ for the ball of radius n and center v in G with respect to the graph distance, i.e. for the set of all vertices within graph distance n of v in G.

Definition 2.2.2. A graph G is said to have *subexponential growth* if for any vertex v of G, the balls with center v grow slower than any exponential, i.e.

$$\liminf_{n\to\infty} \sqrt[n]{|\mathfrak{B}_n(v)|} = 1.$$

Observe that, if $(a_n - a_{n-1})/a_n \geq \delta \in (0,1)$ eventually for a non-decreasing sequence (a_n) of non-negative integers, then $\liminf_{n\to\infty} \sqrt[n]{a_n} \geq (1-\delta)^{-1} > 1$. This fact will be used later to arrive at a contradiction to subexpoenntial growth.

Theorem 2.2.1. *Let G be a connected quasi-transitive graph of subexponential growth. For all $p \in [0,1]$, there exists either no infinite cluster $\mathbb{P}_p$-a.s. or exactly one infinite cluster $\mathbb{P}_p$-a.s..*

For the proof, we follow [7] where the statement is verified for $\mathbb{Z}^d$ using the Burton-Keane arguments and make necessary adjustments.

Lemma 2.2.1. *Let G be a connected quasi-transitive graph, $o, v \in G$ vertices. There exists $c > 0$ such that, for n large enough,*

$$\frac{|\mathfrak{B}_n(o) \cap Aut(G) \cdot v|}{|\mathfrak{B}_n(o)|} \geq c.$$

Proof. Let $G = \bigcup_{i=1}^{N} Aut(G) \cdot v_i$ and set $V_i := Aut(G) \cdot v_i$. Moreover, let $r_i := d(o, v_i)$ and $r := \max_{1\leq i\leq N} r_i$, where d denotes the graph distance in G. Then, for all $u \in G$, there exist $i \in \{1, \ldots, N\}$ and $\gamma \in Aut(G)$ such that $u = \gamma(v_i)$, and hence

$$d(\gamma(o), u) = d(\gamma(o), \gamma(v_i)) = d(o, v_i) = r_i \leq r,$$

i.e. $u \in \mathfrak{B}_r(\gamma(o))$. We deduce that $G = \bigcup_{\gamma\in Aut(G)} \mathfrak{B}_r(\gamma(o))$. Therefore, for $n \geq 1$, there exists $\Gamma_n \subseteq Aut(G)$ such that $\mathfrak{B}_n(o) \subseteq \bigcup_{\gamma\in\Gamma_n} \mathfrak{B}_r(\gamma(o))$. Assume w.l.o.g. that Γ_n does not contain superfluous elements. More precisely, assume that $\gamma(o) \neq \gamma'(o)$ for distinct $\gamma, \gamma' \in \Gamma_n$ and $\mathfrak{B}_r(\gamma(o)) \cap \mathfrak{B}_n(o) \neq \varnothing$ for all $\gamma \in \Gamma_n$. Then the triangle inequality implies $\bigcup_{\gamma \in \Gamma_n} \mathfrak{B}_r(\gamma(o)) \subseteq \mathfrak{B}_{n+2r}(o)$.

Since $|\mathfrak{B}_r(\gamma(o))| = |\mathfrak{B}_r(o)|$ for all $\gamma \in Aut(G)$, we have $|\mathfrak{B}_n(o)| \leq |\Gamma_n| \, |\mathfrak{B}_r(o)|$. Furthermore, we may construct $\tilde{\Gamma}_n \subseteq \Gamma_n$ with $\left|\tilde{\Gamma}_n\right| \geq \lceil |\Gamma_n| \,/\, |\mathfrak{B}_{2r}(o)| \rceil$ and such that the balls of radius r with center in $\tilde{\Gamma}_n \cdot o$ are pairwise disjoint. Indeed, suppose $\tilde{\Gamma}_n$ with $\left|\tilde{\Gamma}_n\right| < |\Gamma_n| \,/\, |\mathfrak{B}_{2r}(o)|$ satisfies the property just mentioned. Then

$$\left| \left\{ \gamma \in \Gamma_n : \gamma(o) \notin \bigcup_{\tilde{\gamma} \in \tilde{\Gamma}_n} \mathfrak{B}_{2r}(\tilde{\gamma}(o)) \right\} \right| \geq |\Gamma_n| - \left|\tilde{\Gamma}_n\right| |\mathfrak{B}_{2r}(o)| > 0,$$

so we can find $\gamma \in \Gamma_n \setminus \tilde{\Gamma}_n$ such that the balls of radius r and center in $(\tilde{\Gamma}_n \cup \{\gamma\}) \cdot o$ are still pairwise disjoint. Combining the two inequalities above, we get

$$\left|\tilde{\Gamma}_n\right| \geq \frac{|\mathfrak{B}_n(o)|}{|\mathfrak{B}_r(o)| \, |\mathfrak{B}_{2r}(o)|} =: c \, |\mathfrak{B}_n(o)| \, .$$

Together with $\mathfrak{B}_{n+2r}(o) \supseteq \bigcup_{\gamma \in \tilde{\Gamma}_n} \mathfrak{B}_r(\gamma(o))$ and $|\mathfrak{B}_r(\gamma(o)) \cap V_i| = |\mathfrak{B}_r(o) \cap V_i| =: \alpha_i$, this implies that eventually

$$|\mathfrak{B}_{n+2r}(o) \cap V_i| \geq \left| \bigcup_{\gamma \in \tilde{\Gamma}_n} \mathfrak{B}_r(\gamma(o)) \cap V_i \right| = \left|\tilde{\Gamma}_n\right| \alpha_i \geq c\alpha_i \, |\mathfrak{B}_n(o)| \, .$$

Let $\Delta = \Delta(G) < \infty$ be the maximum degree of G, then clearly $|\mathfrak{B}_{n+2r}(o)| \leq \Delta^{2r} \, |\mathfrak{B}_n(o)|$, and we finally obtain that

$$\frac{|\mathfrak{B}_{n+2r}(o) \cap V_i|}{|\mathfrak{B}_{n+2r}(o)|} = \underbrace{\frac{|\mathfrak{B}_{n+2r}(o) \cap V_i|}{|\mathfrak{B}_n(o)|}}_{\geq c\alpha_i} \underbrace{\frac{|\mathfrak{B}_n(o)|}{|\mathfrak{B}_{n+2r}(o)|}}_{\geq \Delta^{-2r}} \geq c\alpha_i \Delta^{-2r} > 0,$$

for n sufficiently large. This finishes the proof, for the V_i cover G and since, for each $v \in V_i$, we have $Aut(G) \cdot v = V_i$. $\square$

Lemma 2.2.2. *Let G be quasi-transitive and infinite. Events which are invariant under $Aut(G)$ are trivial, i.e. if $\mathcal{A} \in \mathfrak{F}$ and $\gamma(\mathcal{A}) = \mathcal{A}$ for all $\gamma \in Aut(G)$, then $\mathbb{P}_p[\mathcal{A}] \in \{0, 1\}$.*

Proof. The same proof as for translation invariant events on $\mathbb{Z}^d$ works. One only needs that all (iff one) orbits are infinite; see, e.g. [12], pages 238-239. $\square$

Fix G as in the theorem and infinite, and define N to be the number of infinite clusters in the induced subgraph of a configuration. Moreover, for a finite set of vertices $B \subseteq G$, let M_B be the number of infinite clusters intersecting B.

Proposition 2.2.1. *For any $p \in [0, 1]$, there exists $k = k(p) \in \{0, 1, \infty\}$ such that $\mathbb{P}_p$-a.s. $N = k$.*

Proof. Fix $p \in (0,1]$. By invariance of $\{N = k\}$, there exists $k = k(p) \geq 0$ such that $\mathbb{P}_p[N = k] = 1$. Suppose that $2 \leq k < \infty$ and fix $o \in G$.

Since $\mathfrak{B}_n := \mathfrak{B}_n(o)$ increases to G as n tends to infinity and $k < \infty$, we have that $\{M_{\mathfrak{B}_n} = N = k\}$ increases to $\{N = k\}$, and thus

$$\mathbb{P}_p[M_{\mathfrak{B}_n} = N] \geq \mathbb{P}_p[M_{\mathfrak{B}_n} = N = k] \nearrow \mathbb{P}_p[N = k] = 1 \quad \text{as } n \to \infty.$$

In particular, there exists $n \geq 1$ such that $\mathbb{P}_p[M_{\mathfrak{B}_n} = N] \geq 1/2$. As $\{M_{\mathfrak{B}_n} = N\}$ depends only on edges with at least one endpoint outside $\mathfrak{B}_n$, we obtain

$$\begin{aligned}
\mathbb{P}_p[N = 1] &\geq \mathbb{P}_p[\{M_{\mathfrak{B}_n} = N\} \cap \{\text{all edges in } \mathfrak{B}_n \text{ open}\}] \\
&= \mathbb{P}_p[M_{\mathfrak{B}_n} = N]\, \mathbb{P}_p[\text{all edges in } \mathfrak{B}_n \text{ open}] \\
&\geq \frac{1}{2} \cdot p^{|E(\mathfrak{B}_n)|} > 0,
\end{aligned}$$

contradicting $N = k \geq 2$ with probability 1. □

Definition 2.2.3. We call a vertex $v \in G$ a *trifurcation* if

- the open cluster of v is infinite,
- there exist precisely three open edges incident to v,
- closing all edges incident to v splits its open cluster into three disjoint infinite open clusters, which are denoted by $C_1(v), C_2(v), C_3(v)$, respectively.

Define $\mathcal{T}_v$ to be the event that v is a trifurcation.

Lemma 2.2.3. *Let $p \in [0,1]$. If $N \geq 3$ $\mathbb{P}_p$-a.s., then there exists a vertex $v_0 \in G$ such that $\mathbb{P}_p[\mathcal{T}_{v_0}] > 0$.*

Proof. Fix $p \in [0,1]$, $o \in G$, and assume $N \geq 3$ $\mathbb{P}_p$-a.s..

For a finite set of vertices $B \subseteq G$, we define $M_B(0)$ to be the number of infinite clusters intersecting B when all edges in B are set to be closed.

As before, for $n \geq 1$, set $\mathfrak{B}_n = \mathfrak{B}_n(o)$. Since $\{M_{\mathfrak{B}_n} \geq 3\}$ increases to $\{N \geq 3\}$ as $n \to \infty$, there exists $n \geq 1$ such that $\mathbb{P}_p[M_{\mathfrak{B}_n} \geq 3] \geq 1/2$. Since clearly $M_B(0) \geq M_B$, we have that

$$\begin{aligned}
1/2 &\leq \mathbb{P}_p[M_{\mathfrak{B}_n}(0) \geq 3] \\
&\leq \sum_{u,v,w \in \partial \mathfrak{B}_n} \mathbb{P}_p\Bigg[\underbrace{C_u \setminus \mathfrak{B}_n,\, C_v \setminus \mathfrak{B}_n,\, C_w \setminus \mathfrak{B}_n \text{ infinite, disjoint}}_{=:\kappa_{u,v,w}}\Bigg],
\end{aligned}$$

where $\partial \mathfrak{B}_n = \mathfrak{B}_n \setminus \mathfrak{B}_{n-1}$ and C_u, C_v, C_w are the clusters of u, v, w, respectively. We deduce that there exist deterministic $u, v, w \in \partial \mathfrak{B}_n$ with $\mathbb{P}_p[\kappa_{u,v,w}] > 0$.

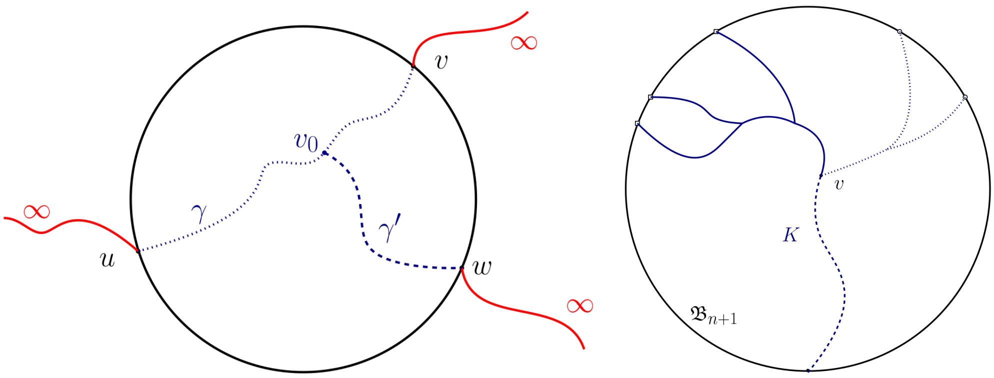


Figure 2.1: The construction of a trifurcation v_0 on the left, and the induced 3-partition of $\partial\mathfrak{B}_{n+1} \cap K$ by a trifurcation $v \in \mathfrak{B}_n$ on the right.

Now choose a shortest path γ connecting u to v within $\mathfrak{B}_n$ and a shortest path γ' connecting w to γ within $\mathfrak{B}_n$, and denote by v_0 the unique vertex at which γ and γ' intersect; see the left-hand side in Figure 2.1. Using independence, we finally obtain

$$\begin{aligned}\mathbb{P}_p\left[\mathcal{T}_{v_0}\right] &\geq \mathbb{P}_p\left[\kappa_{u,v,w} \cap \{\text{only } \gamma, \gamma' \text{ open in } \mathfrak{B}_n\}\right] \\ &= \mathbb{P}_p\left[\kappa_{u,v,w}\right] \mathbb{P}_p\left[\text{only } \gamma, \gamma' \text{ open in } \mathfrak{B}_n\right] > 0.\end{aligned}$$

□

Proof of Theorem 2.2.1. We only have to rule out the case $N = \infty$. Fix $p \in [0,1]$ and assume that $\mathbb{P}_p$-a.s. $N = \infty$. Then, by Lemma 2.2.3, there exists $v_0 \in G$ such that $\mathbb{P}_p\left[\mathcal{T}_{v_0}\right] > 0$. Take some arbitrary but fixed vertex o in G, and set $\mathfrak{B}_n = \mathfrak{B}_n(o)$. By Lemma 2.2.1, there exists $c > 0$ such that $|\mathfrak{B}_n \cap Aut(G) \cdot v_0| \geq c\,|\mathfrak{B}_n|$ for n large enough. Moreover, since $\mathbb{P}_p$ is invariant under automorphisms, for all $\gamma \in Aut(G)$ we have that $\mathbb{P}_p\left[\mathcal{T}_{\gamma(v_0)}\right] = \mathbb{P}_p\left[\gamma(\mathcal{T}_{v_0})\right] = \mathbb{P}_p\left[\mathcal{T}_{v_0}\right]$, and hence

$$\mathbb{E}_p\left[\sum_{v \in \mathfrak{B}_n} \mathbb{1}_{\mathcal{T}_v}\right] \geq |\mathfrak{B}_n \cap Aut(G) \cdot v_0|\,\mathbb{P}_p\left[\mathcal{T}_{v_0}\right] \geq c\,|\mathfrak{B}_n|\,\mathbb{P}_p\left[\mathcal{T}_{v_0}\right].$$

We aim to show that we can also bound this expectation from above by $c'\,|\partial\mathfrak{B}_n|$ for some $c' < \infty$ (where $\partial\mathfrak{B}_n = \mathfrak{B}_n \setminus \mathfrak{B}_{n-1}$ as before), which would contradict subexponential growth of G and finish the proof.

Let Y be a finite set with at least 3 members. We call a partition $\Pi = \{P_1, P_2, P_3\}$ of Y consisting of three non-empty sets a *3-partition*. Two 3-partitions $\Pi = \{P_1, P_2, P_3\}$ and $\Pi' = \{P_1', P_2', P_3'\}$ are said to be *compatible* if there exists an ordering of their elements such that $P_1 \supseteq P_2' \cup P_3'$ (this condition is symmetric in Π and Π'). A collection $\mathcal{P}$ of 3-partitions is called compatible if each pair of distinct members of $\mathcal{P}$ is compatible. We will use the following elementary lemma, whose proof is contained in the appendix (A).

Lemma 2.2.4. *If $\mathcal{P}$ is a compatible family of 3-partitions of Y, then $|\mathcal{P}| \leq |Y| - 2$.*

Now let $K \subseteq \mathfrak{B}_{n+1}$ be an open cluster. A trifurcation $v \in \mathfrak{B}_n \cap K$ induces a 3-partition $\Pi(v) = \{P_1, P_2, P_3\}$ of $\partial\mathfrak{B}_{n+1} \cap K$, namely $P_i = C_i(v) \cap \partial\mathfrak{B}_{n+1}$, $i = 1, 2, 3$; see the right-hand side of Figure 2.1. We clearly have

- $P_i \neq \varnothing$ for all i,
- P_i is a subset of a connected subgraph of $\mathfrak{B}_{n+1} \setminus \{v\}$,
- $P_i \not\leftrightarrow P_j$ in $\mathfrak{B}_{n+1} \setminus \{v\}$ for $i \neq j$.

We claim that, for two distinct trifurcations $v, v' \in \mathfrak{B}_n \cap K$, the 3-partitions $\Pi(v)$ and $\Pi(v')$ are compatible. Indeed, let $C_i = C_i(v), C'_j = C_j(v')$, $i, j = 1, 2, 3$, and let P_i, P'_j be the corresponding sets in the partitions, i.e. $P_i = C_i \cap \partial\mathfrak{B}_{n+1}$, $P'_j = C'_j \cap \partial\mathfrak{B}_{n+1}$. Since $C_1 \cup C_2 \cup C_3$ contains $K \setminus \{v\}$ and $v \neq v'$, there exists $i \in \{1, 2, 3\}$ with $v' \in C_i$, say $i = 1$. Now, as it is only possible for C'_j not to be contained in C_1 if it "leaves" through v (and $C'_j \cap C'_k = \varnothing$ for $j \neq k$), we deduce that two of C'_1, C'_2, C'_3 are contained in C_1, say $C'_2 \cup C'_3 \subseteq C_1$, and hence $P'_2 \cup P'_3 = (C'_2 \cup C'_3) \cap \partial\mathfrak{B}_{n+1} \subseteq C_1 \cap \partial\mathfrak{B}_{n+1} = P_1$.

The lemma gives that the number of trifurcations in $\mathfrak{B}_n \cap K$ is bounded from above by $|\partial\mathfrak{B}_{n+1} \cap K|$. Summing over all open clusters $K \subseteq \mathfrak{B}_{n+1}$, we arrive at

$$\sum_{v \in \mathfrak{B}_n} \mathbb{1}_{\mathcal{T}_v} = \sum_{K \subseteq \mathfrak{B}_{n+1}} \sum_{v \in \mathfrak{B}_n \cap K} \mathbb{1}_{\mathcal{T}_v} \leq \sum_{K \subseteq \mathfrak{B}_{n+1}} |\partial\mathfrak{B}_{n+1} \cap K| = |\partial\mathfrak{B}_{n+1}| \leq \Delta(G)\, |\partial\mathfrak{B}_n| \,,$$

where $\Delta(G)$ is the maximum degree of G, and taking expectations concludes the proof. □

2.3 k-dependent percolation

Let $\Omega = \{0,1\}^{E(\mathbb{Z}^2)}$, and let $\mathfrak{F}$ be the σ-algebra on Ω generated by cylinder sets. We equip the set of edges $E(\mathbb{Z}^2)$ with some metric $d : E(\mathbb{Z}^2) \times E(\mathbb{Z}^2) \to [0, \infty)$ that is invariant under $Aut(\mathbb{Z}^2)$, i.e. for all $e, e' \in E(\mathbb{Z}^2)$ and for any $\gamma \in Aut(\mathbb{Z}^2)$, we have $d(\gamma(e), \gamma(e')) = d(e, e')$. Furthermore, assume that balls of any radius (w.r.t. d) are finite. Later we will work with the following metric:

$$d_\infty(e, e') := \max_{z \in e, z' \in e'} \|z - z'\|_\infty \text{ for } e \neq e'.$$

Definition 2.3.1. Let $k \geq 0$. We say that a probability measure μ on $(\Omega, \mathfrak{F})$ is k-*dependent* (with respect to d) if, for all finite $E \subseteq E(\mathbb{Z}^2)$ with $\min_{e \neq e' \in E} d(e, e') > k$, we have that $\{\omega_e\}_{e \in E}$ are independent, where ω_e are the canonical projections.

Proposition 2.3.1. *For any $k \geq 0$, there exists $p_c = p_c(d, k) < 1$ such that, for all k-dependent probability measures μ with the property that $\mu(\omega_e = 1) \geq p_c$ for all $e \in E(\mathbb{Z}^2)$, the corresponding model percolates.*

Proof. We use a "Peierls argument". Let $p \in [0,1]$, let μ_p be a k-dependent probability measure with $\mu_p(\omega_e = 1) \geq p$ for all $e \in E(\mathbb{Z}^2)$, and define K to be the number of edges within d-distance k of some $e \in E(\mathbb{Z}^2)$. Notice that K does not depend on the choice of e by invariance of d under $Aut(\mathbb{Z}^2)$, and since $\mathbb{Z}^2$ is *arc-transitive*, meaning that $Aut(\mathbb{Z}^2)$ acts transitively on $E(\mathbb{Z}^2)$. Then, by the same argument as in the proof of Lemma 2.2.1, for any finite $E \subseteq E(\mathbb{Z}^2)$, there exist at least $\lceil |E| / K \rceil$ edges belonging to E that are all at d-distance greater than k from each other, and whose states are therefore independent.

Consider the dual graph of $\mathbb{Z}^2$, and declare each edge therein to be open if and only if the corresponding edge in $\mathbb{Z}^2$ is closed. Using the fact that the cluster C_0 of $0 \in \mathbb{Z}^2$ is finite precisely if there exists an open circuit surrounding 0 in the dual configuration, we get

$$\begin{aligned}
\mu_p(|C_0| < \infty) &= \mu_p\left(\bigcup_{n\geq 4}\{\exists \text{ dual open circuit of length } n \text{ surrounding } 0\}\right) \\
&\leq \sum_{n\geq 4}\sum_{\gamma\colon l(\gamma)=n} \mu_p(\gamma \text{ open}) \\
&\leq \sum_{n\geq 4} n4^n(1-p)^{\lceil n/K\rceil},
\end{aligned}$$

where the second sum in the second line is taken over all dual circuits γ of length n surrounding 0, and where we used the crude bound $n4^n$ for the number of such circuits. We finish the proof since the last sum even decreases to 0 as p increases to 1. $\square$

Chapter 3

Continuity of phase transition on generalized slabs

Let G be a finite connected graph and consider the *generalized slab* $\mathbb{S}_G = \mathbb{Z}^2 \times G$. The goal of this chapter is to prove the following theorem.

Theorem 3.0.1. *For any finite connected graph G, $\Theta_{\mathbb{S}_G}(p_c(\mathbb{S}_G)) = 0$.*

For the proof, we closely follow [5], where the statement is verified for the case of G being a string of arbitrary length. The methods and ideas rely on projections on $\mathbb{Z}^2$, whence they work equally well for arbitrary G. A way to think of the structure of $\mathbb{S}_G$ throughout the proof is the following. Imagine $|G|$ disjoint copies of $\mathbb{Z}^2$ on top of each other. We regard edges in $\{z\} \times G$ as "allowed jumps", meaning that there is an edge between vertices u and v in G if and only if we can jump between (z, u) and (z, v) in $\mathbb{S}_G$ for any $z \in \mathbb{Z}^2$.

We will now briefly outline the proof of the theorem, which essentially consists of three separate steps. We start by fixing some finite connected graph G and any p for which the model percolates with positive $\mathbb{P}_p$-probability. In Section 3.1, we construct a sequence $(\mathcal{A}_n)$ of local events which occur with high $\mathbb{P}_p$-probability, conditioned on what will be called the gluing lemma (Lemma 3.1.4). Section 3.2 consists of the so-called renormalization step: we define 4-dependent bond percolation laws on the coarse-grained lattices $4n\mathbb{Z}^2$ in such a way that an infinite cluster in the coarse-grained model induces one in the original one, and in which each edge is forced to be open if a translate of $\mathcal{A}_n$ occurs. By translation invariance of $\mathbb{P}_p$ and by the observations in Section 2.3 on k-dependent measures, the model percolates for $\mathbb{P}_p[\mathcal{A}_n]$ sufficiently close to 1, which may be achieved by increasing n due to step 1. Then, by continuity of $q \mapsto \mathbb{P}_q[\cdot]$ on local events, the model also percolates for q close enough to p, which allows us to conclude that $p > p_c$. The proof of the gluing lemma in Section 3.3 forms the final step.

We introduce some notation. For $R, S \subseteq B \subseteq \mathbb{R}^2$, define

$$\begin{aligned}
\overline{R} &= (R \cap \mathbb{Z}^2) \times G = \{(z, v) \in \mathbb{S}_G : z \in R\},\\
\{R \xleftrightarrow{B} S\} &= \{\text{there exists an open cluster in } \overline{B} \text{ that connects } \overline{R} \text{ and } \overline{S}\},\\
\{R \xleftrightarrow{!B!} S\} &= \{\text{there exists a unique open cluster in } \overline{B} \text{ that connects } \overline{R} \text{ and } \overline{S}\},\\
\{R \leftrightarrow \infty\} &= \{\overline{R} \text{ intersects an infinite cluster}\}.
\end{aligned}$$

Moreover, for $n \geq 1$, define

$$B_n = [-n, n]^2 \text{ and } \partial B_n = B_n \setminus B_{n-1}.$$

3.1 The finite-size criterion

From now on until the end of Section 3.2, fix some finite connected graph G and any $p \in [0, 1]$ with $\Theta_{\mathbb{S}_G}(p) > 0$.

Lemma 3.1.1. *There exists a sequence $(u_n)_{n\geq 1}$ of non-negative integers such that $u_n \leq n/3$ for all $n \geq 1$ and*

$$\lim_{n\to\infty} \mathbb{P}_p \left[B_{u_n} \xleftrightarrow{!B_n!} \partial B_n \right] = 1.$$

Proof. For $1 \leq k \leq n$, set

$$\mathcal{U}_{k,n} := \left\{ B_k \xleftrightarrow{!B_n!} \partial B_n \right\}.$$

By a.s. existence of a unique infinite cluster in $\mathbb{S}_G$, we have

$$\begin{aligned}
1 &= \mathbb{P}_p [\exists \text{ infinite cluster in } \mathbb{S}_G] \\
&= \lim_{k\to\infty} \mathbb{P}_p [B_k \leftrightarrow \infty] && \text{(by continuity of measure)}\\
&= \lim_{k\to\infty} \mathbb{P}_p [B_k \leftrightarrow \infty, \exists! \text{ infinite cluster}] && \text{(intersection with a.s.-event)}\\
&\leq \lim_{k\to\infty} \mathbb{P}_p \left[\{B_k \leftrightarrow \infty\} \cap \bigcup_{n\geq k} \mathcal{U}_{k,n} \right] && \text{(by inclusion)}\\
&= \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}_p [\{B_k \leftrightarrow \infty\} \cap \mathcal{U}_{k,n}] && \text{(by continuity of measure)}\\
&\leq \lim_{k\to\infty} \lim_{n\to\infty} \mathbb{P}_p [\mathcal{U}_{k,n}] && \text{(by inclusion).}
\end{aligned}$$

Therefore, for every $l \geq 1$, we can find $k_l \geq 1$ such that $\lim_{n\to\infty} \mathbb{P}_p [\mathcal{U}_{k_l,n}] > 1 - 1/(2l)$. Now, there exists $N_l \geq 1$ such that, for all $n \geq N_l$, we have $\mathbb{P}_p [\mathcal{U}_{k_l,n}] > 1 - 1/l$. We assume without loss of generality that $N_l \geq 3k_l$ and $N_l < N_{l+1}$ for all l.

Finally, for $N_l \leq n < N_{l+1}$, set $u_n := k_l$. Then by construction $u_n = k_l \leq N_l/3 \leq n/3$ and $\mathbb{P}_p [\mathcal{U}_{u_n,n}] = \mathbb{P}_p [\mathcal{U}_{k_l,n}] > 1 - 1/l$, which concludes the proof. ☐

For simplicity of notation, set $S_n := B_{u_n}$, and for $0 \le \alpha \le \beta \le n$, define the event

$$\mathcal{E}_n(\alpha, \beta) := \{S_n \xleftrightarrow{B_n} \{n\} \times [\alpha, \beta]\},$$

whose projection is illustrated in Figure 3.1.

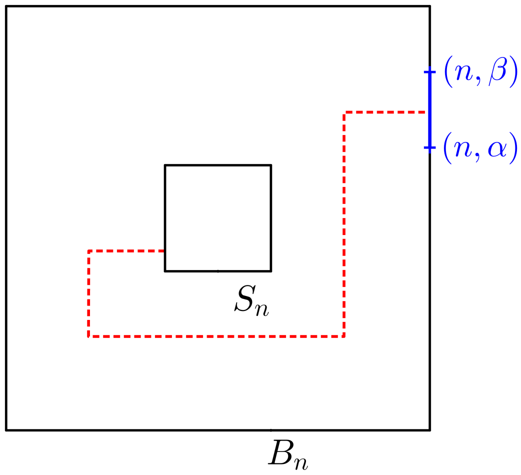


Figure 3.1: The projection of the event $\mathcal{E}_n(\alpha, \beta)$.

Lemma 3.1.2. *There exist sequences* $(\alpha_n)_{n\ge1}$, $(y_n)_{n\ge1}$ *with* $\alpha_n \in [0, n]$ *and* $y_n \in \{\alpha_n/4, 3\alpha_n/4\}$ *for all* n *such that*

$$\mathbb{P}_p[\mathcal{E}_n(\alpha_n, n)] \xrightarrow{n\to\infty} 1,$$
$$\mathbb{P}_p[\mathcal{E}_n(y_n - \alpha_n/4, y_n + \alpha_n/4)] \xrightarrow{n\to\infty} 1.$$

Proof. The proof essentially consists of multiple uses of the square-root trick and of Lemma 3.1.1.

We can decompose

$$\{S_n \xleftrightarrow{B_n} \partial B_n\} = \bigcup_{i=1}^{8} \overline{\gamma}_i(\mathcal{E}_n(0, n)),$$

where the γ_i are the symmetries of the square, but all these events have the same $\mathbb{P}_p$-probability by invariance. The square-root trick implies

$$\mathbb{P}_p[\mathcal{E}_n(0, n)] \ge 1 - \left(1 - \mathbb{P}_p\left[S_n \xleftrightarrow{B_n} \partial B_n\right]\right)^{1/8},$$

and hence, by Lemma 3.1.1, also $\mathbb{P}_p[\mathcal{E}_n(0, n)]$ tends to 1 as $n \to \infty$.

For $\alpha \in \{0, \dots, n-1\}$, we clearly have

$$\mathcal{E}_n(0, n) = \mathcal{E}_n(0, \alpha) \cup \mathcal{E}_n(\alpha + 1, n),$$

and we claim that, for n large enough,

- $\mathbb{P}_p\left[\mathcal{E}_n(0,0)\right] < \mathbb{P}_p\left[\mathcal{E}_n(1,n)\right]$,
- $\mathbb{P}_p\left[\mathcal{E}_n(0,n-1)\right] > \mathbb{P}_p\left[\mathcal{E}_n(n,n)\right]$.

Indeed, we have $\mathbb{P}_p\left[\mathcal{E}_n(0,0)^c\right] \geq \mathbb{P}_p\left[\text{edges in } \overline{(n,0)+B_1} \text{ closed}\right] =: c > 0$, but the square-root trick gives that

$$\max\{\underbrace{\mathbb{P}_p\left[\mathcal{E}_n(0,0)\right]}_{\leq 1-c}, \mathbb{P}_p\left[\mathcal{E}_n(1,n)\right]\} \xrightarrow{n\to\infty} 1,$$

whence $\mathbb{P}_p\left[\mathcal{E}_n(1,n)\right] \to 1$ as $n \to \infty$. In particular $\mathbb{P}_p\left[\mathcal{E}_n(1,n)\right] > 1-c \geq \mathbb{P}_p\left[\mathcal{E}_n(0,0)\right]$ for n large enough. The second part of the claim follows analogously.

Therefore, for n large enough, we can find $\alpha_n \in \{1,\dots,n-1\}$ such that

$$\mathbb{P}_p\left[\mathcal{E}_n(0,\alpha_n-1)\right] < \mathbb{P}_p\left[\mathcal{E}_n(\alpha_n,n)\right] \text{ and } \mathbb{P}_p\left[\mathcal{E}_n(0,\alpha_n)\right] \geq \mathbb{P}_p\left[\mathcal{E}_n(\alpha_n+1,n)\right].$$

Now the square-root trick, together with the inequalities above, implies that both $\mathbb{P}_p\left[\mathcal{E}_n(\alpha_n,n)\right]$ and $\mathbb{P}_p\left[\mathcal{E}_n(0,\alpha_n)\right]$ converge to 1 as $n \to \infty$.

Finally, since $\mathcal{E}_n(0,\alpha_n) = \mathcal{E}_n(0,\alpha_n/2) \cup \mathcal{E}_n(\alpha_n/2,\alpha_n)$, the square-root trick gives that, for each n, we can find $y_n \in \{\alpha_n/4, 3\alpha_n/4\}$ such that

$$\begin{aligned}\mathbb{P}_p\left[\mathcal{E}_n(y_n-\alpha_n/4, y_n+\alpha_n/4)\right] &= \max\left\{\mathbb{P}_p\left[\mathcal{E}_n(0,\alpha_n/2)\right], \mathbb{P}_p\left[\mathcal{E}_n(\alpha_n/2,\alpha_n)\right]\right\} \\ &\geq 1-(1-\mathbb{P}_p\left[\mathcal{E}_n(0,\alpha_n)\right])^{1/2} \xrightarrow{n\to\infty} 1,\end{aligned}$$

which finishes the proof. □

Lemma 3.1.3. *There exist infinitely many n such that $\alpha_{3n} \leq 4\alpha_n$.*

Proof. Suppose $\alpha_{3n} > 4\alpha_n$ for $n \geq N$. Then it follows by induction that $\alpha_{3^k N} > 4^k \alpha_N$ for $k \geq 1$, contradicting $\alpha_n \leq n$ for all n. □

For $n \geq 1$, define the following subsets of $\mathbb{Z}^2$, and see Figure 3.2 for an illustration:

$$\begin{aligned}B'_n &:= (2n, y_{3n}) + B_n, \\ S'_n &:= (2n, y_{3n}) + S_n, \\ Y_n^+ &:= \{3n\} \times [y_{3n}+\alpha_n, y_{3n}+n], \\ Y_n^- &:= \{3n\} \times [y_{3n}-n, y_{3n}-\alpha_n], \\ Z_n &:= \{3n\} \times [y_{3n}-\alpha_n, y_{3n}+\alpha_n].\end{aligned}$$

If $\alpha_{3n} \leq 4\alpha_n$, then $[y_{3n}-\alpha_n, y_{3n}+\alpha_n] \supseteq [y_{3n}-\alpha_{3n}/4, y_{3n}+\alpha_{3n}/4]$, and thus

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n\right] \geq \mathbb{P}_p\left[\mathcal{E}_{3n}(y_{3n}-\alpha_{3n}/4, y_{3n}+\alpha_{3n}/4)\right].$$

Together with Lemmata 3.1.2 and 3.1.3, this implies that

$$\limsup_{n\to\infty} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n\right] = 1. \tag{3.1}$$

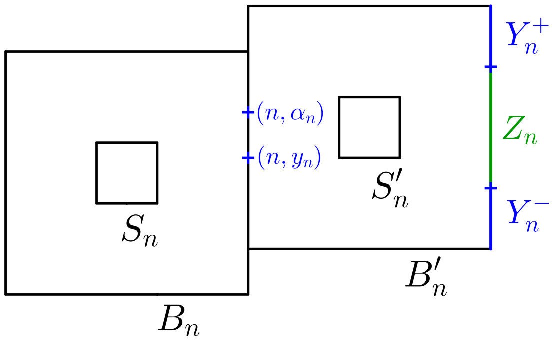


Figure 3.2: The regions $B'_n, S'_n, Y_n^+, Y_n^-, Z_n$.

Moreover, we have by reflection and translation invariance that

$$\mathbb{P}_p\left[S'_n \xleftrightarrow{B'_n} Y_n^-\right] = \mathbb{P}_p\left[S'_n \xleftrightarrow{B'_n} Y_n^+\right] = \mathbb{P}_p\left[\mathcal{E}_n(\alpha_n, n)\right],$$

and hence the Harris-FKG inequality gives that

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n\right] \mathbb{P}_p\left[\mathcal{E}_n(\alpha_n, n)\right]^2.$$

Applying $\limsup$ to the above inequality and using equation (3.1) and Lemma 3.1.2, we deduce that

$$\limsup_{n\to\infty} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] = 1. \tag{3.2}$$

The goal is to show that there exists a path from $\overline{S_{3n}}$ to $\overline{S'_n}$ with high probability. If G consists of a single vertex, then $\mathbb{S}_G \simeq \mathbb{Z}^2$ and the occurrence of the event in (3.2) already implies the existence of such a path, since in the plane "crossing paths necessarily intersect". See Figure 3.3. To proceed in the setting for general G, we make use of the following gluing lemma whose proof we postpone to Section 3.3.

Lemma 3.1.4 (Gluing lemma)**.** *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, p, G) > 0$ such that, for any n,*

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq 1 - \delta$$

implies

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}\cup B'_n} S'_n\right] \geq 1 - \varepsilon.$$

The gluing lemma 3.1.4 and equation (3.2) imply that

$$\limsup_{n\to\infty} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{4n}} S'_n\right] = 1. \tag{3.3}$$

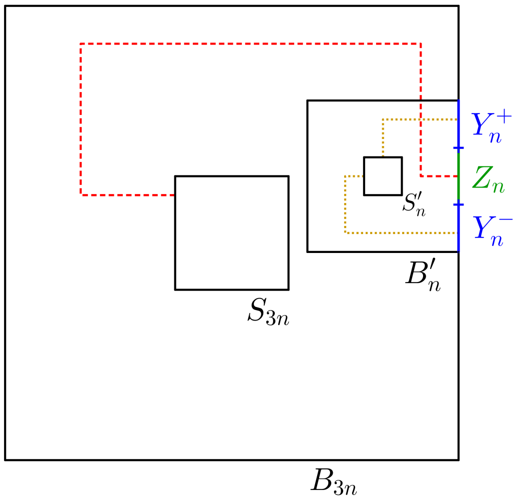


Figure 3.3: Paths from S_{3n} to Z_n, and from S'_n to Y_n^+ and Y_n^-, respectively.

Besides, we estimate

$$\begin{aligned}
&\mathbb{P}_p\left[S_{3n} \xleftrightarrow{(2n,0)+B_{6n}} (4n,0)+S_{3n}\right] \\
&\overset{(a)}{\geq} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{4n}} S'_n, S'_n \xleftrightarrow{(4n,0)+B_{4n}} (4n,0)+S_{3n}, S'_n \xleftrightarrow{!B'_n!} \partial B'_n\right] \\
&\overset{(b)}{\geq} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{4n}} S'_n, S'_n \xleftrightarrow{(4n,0)+B_{4n}} (4n,0)+S_{3n}\right] + \mathbb{P}_p\left[S'_n \xleftrightarrow{!B'_n!} \partial B'_n\right] - 1 \\
&\overset{(c)}{\geq} \underbrace{\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{4n}} S'_n\right]^2}_{\substack{\limsup(\ldots)=1 \\ \text{by } (3.3)}} + \underbrace{\mathbb{P}_p\left[S'_n \xleftrightarrow{!B'_n!} \partial B'_n\right]}_{\substack{\lim(\ldots)=1 \\ \text{by Lemma } 3.1.1}} - 1,
\end{aligned}$$

where (a) holds by inclusion of events (see Figure 3.4), (b) since in general $\mathbb{P}[\mathcal{A} \cap \mathcal{B}] \geq \mathbb{P}[\mathcal{A}] + \mathbb{P}[\mathcal{B}] - 1$ and (c) by the Harris-FKG inequality and invariance under reflection (across $\{2n\} \times \mathbb{Z}$). In summary, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{(2n,0)+B_{6n}} (4n,0)+S_{3n}\right] = 1. \tag{3.4}$$

3.2 Renormalization

Fix $n \in \mathbb{N}$ and define a percolation law on the coarse-grained lattice $4n\mathbb{Z}^2$ in the following way. We call an edge $\{z, z'\} \subseteq 4n\mathbb{Z}^2$ (with $\|z - z'\|_1 = 4n$) *good* if

- $z + S_{3n} \xleftrightarrow{R_n} z' + S_{3n}$ where $R_n = \frac{z+z'}{2} + B_{6n}$,

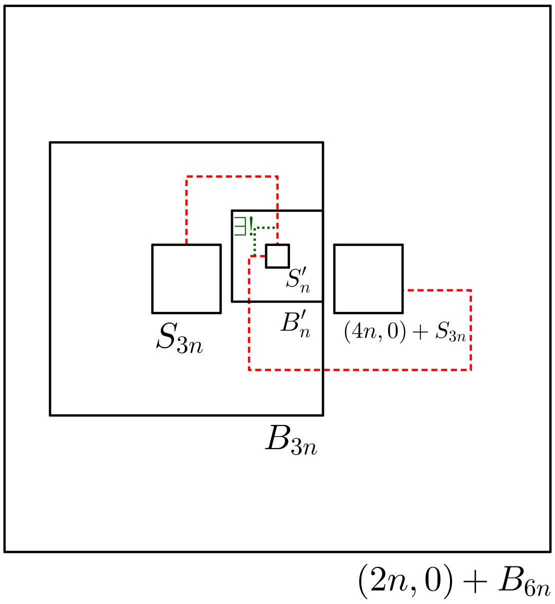


Figure 3.4: The projection of the event
$\{S_{3n} \xleftrightarrow{B_{4n}} S'_n, S'_n \xleftrightarrow{(4n,0)+B_{4n}} (4n,0)+S_{3n}, S'_n \xleftrightarrow{!B'_n!} \partial B'_n\}$.

- $z+S_{3n} \xleftrightarrow{!z+B_{3n}!} z+\partial B_{3n}$ and $z'+S_{3n} \xleftrightarrow{!z'+B_{3n}!} z'+\partial B_{3n}$.

Note that, with the notation from above, we have $z+B_{3n}, z'+B_{3n} \subseteq R_n$, and that a path from $z+S_{3n}$ to $z'+S_{3n}$ necessarily intersects $z+\partial B_{3n}$ and $z'+\partial B_{3n}$; see Figure 3.5. If e, e' are edges in $4n\mathbb{Z}^2$ as above with midpoints x, x', respectively, and $d_\infty(e,e') = \max_{z\in e, z'\in e'} \|z-z'\|_\infty > 16n$, then $\|x-x'\|_\infty > 12n$, and hence $x+B_{6n}$ and $x'+B_{6n}$ are disjoint. This implies that the set of good edges follows a percolation law which is 4-dependent with respect to d_∞ (if we identify $4n\mathbb{Z}^2$ with $\mathbb{Z}^2$), and by Proposition 2.3.1, there exists $\eta > 0$ such that whenever the probability to be good is at least $1-\eta$, the model percolates.

By invariance under translations and reflections, the $\mathbb{P}_p$-probability to be good is constant for any edge in $4n\mathbb{Z}^2$, and by the observations in Section 3.1, we can force it to be as close to 1 as we wish by increasing n. Indeed, since the $\mathbb{P}_p$-probabilities of both events in the second part of the definition of being good converge to 1 as n tends to infinity by Lemma 3.1.1, there exists $N \geq 1$ such that, for all $n \geq N$, the probabilities of these events exceed $1-\eta/3$. Now, by equation (3.4), we can find $n \geq N$ such that the $\mathbb{P}_p$-probability of the event in the first part is also greater than $1-\eta/3$. For this n, the event of being good is of $\mathbb{P}_p$-probability larger $1-\eta$.

Furthermore, the event that some fixed edge in $4n\mathbb{Z}^2$ is good is local, so (by continuity of $q \mapsto \mathbb{P}_q[\cdot]$ on local events, see Remark 2.1.1) we can find $q < p$ such that the $\mathbb{P}_q$-probability still exceeds $1-\eta$, and thus the model percolates with respect to $\mathbb{P}_q$.

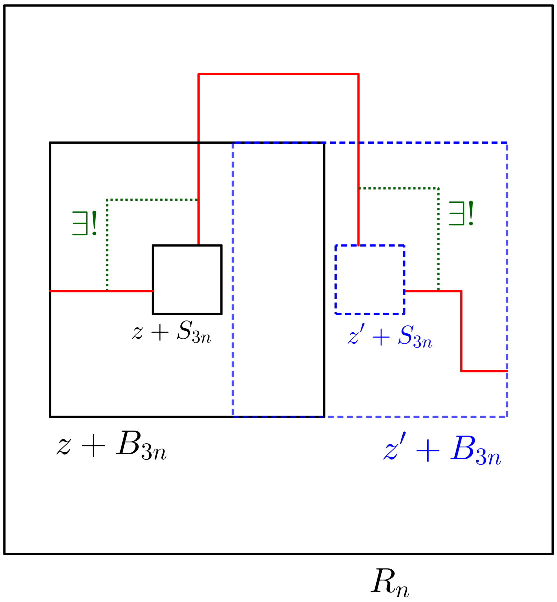


Figure 3.5: The projection of the event that the edge $\{z, z'\}$ is good.

By construction, the existence of an infinite path in the coarse-grained lattice implies the existence of one in the original lattice. We deduce that $q \geq p_c(\mathbb{S}_G)$, and since p with $\Theta_{\mathbb{S}_G}(p) > 0$ was arbitrary, we proved that $\Theta_{\mathbb{S}_G}(p_c(\mathbb{S}_G)) = 0$.

3.3 The gluing lemma

Fix $p \in (0, 1)$ and recall the gluing lemma:

Lemma 3.3.1. *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, p, G) > 0$ such that, for any n,*

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq 1 - \delta$$

implies

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\right] \geq 1 - \varepsilon.$$

If G consists of a single vertex, then the lemma holds for $\delta(\varepsilon) = \varepsilon$ simply by inclusion of events. Indeed, if a path from S_{3n} to Z_n in B_{3n} does not intersect S'_n itself already, then it has to intersect at least one of the paths from S'_n to Y_n^- or Y_n^+ in B'_n, and hence induce a path from S_{3n} to S'_n in $B_{3n} \cup B'_n$. From now on, we assume without further mention that $|G| \geq 2$.

The following lemma will be the main tool in proving the gluing lemma.

Lemma 3.3.2. *Let $s, t > 0$. Consider two local events $\mathcal{A}, \mathcal{B}$ depending only on edges in a finite set $F \subseteq E(\mathbb{S}_G)$. We identify configurations ω with their restrictions $\omega|_F := (\omega_e)_{e\in F}$, events $\mathcal{E}$ with $\{\omega|_F : \omega \in \mathcal{E}\}$, and $\mathbb{P}_p$ with the p-Bernoulli measure on $\{0,1\}^F$.*
Suppose there exists a map Φ from $\mathcal{A}$ to the set $\mathfrak{P}(\mathcal{B})$ of local subevents of $\mathcal{B}$ depending only on edges in F, satisfying the following properties:

1. *for all $\omega \in \mathcal{A}$, $|\Phi(\omega)| \geq t$,*
2. *for all $\omega' \in \mathcal{B}$, there exists a set $S = S(\omega') \subseteq F$ with $|S| \leq s$ such that*

$$\{\omega \in \mathcal{A} : \omega' \in \Phi(\omega)\} \subseteq \{\omega \in \mathcal{A} : \omega|_{S^c} = \omega'|_{S^c}\}.$$

Then

$$\mathbb{P}_p[\mathcal{A}] \leq \frac{(2/\min\{p, 1-p\})^s}{t} \mathbb{P}_p[\mathcal{B}].$$

Remark 3.3.1.

(i) The identifications in the lemma are justified because events depending only on edges in F are finite unions of (at most $2^{|F|}$) cylinder sets of the form $\mathcal{C}_F[\omega] := \{\tilde{\omega} : \tilde{\omega}_e = \omega_e \ \forall e \in F\}$ whose $\mathbb{P}_p$-measure coincides with the finite dimensional p-Bernoulli measure of the corresponding restrictions to F. The identifications implicitly require that, if the images of the map Φ are defined in terms of the whole configurations ω (rather than their restrictions $\omega|_F$), one has to ensure that every $\omega' \in \Phi(\omega)$ agrees with ω outside F, and that, if configurations agree on edges in F, then so do their images under Φ.

(ii) The lemma will be applied to show that a local event $\mathcal{A}$ has small $\mathbb{P}_p$-probability by finding a convenient $\mathcal{B}$ and constructing Φ as above in such a way that t is large or $\mathbb{P}_p[\mathcal{B}]$ small.

Proof. For $\omega \in \mathcal{A}$ and $\omega' \in \Phi(\omega)$, let $S = S(\omega')$ be as in property 2. Then we have

$$\mathbb{P}_p[\omega] = \prod_{e\in F} p^{\omega_e}(1-p)^{1-\omega_e} = \left(\prod_{e\in S^c} p^{\omega'_e}(1-p)^{1-\omega'_e}\right) \underbrace{\left(\prod_{e\in S} p^{\omega_e}(1-p)^{1-\omega_e}\right)}_{\leq \frac{\prod_{e\in S} p^{\omega'_e}(1-p)^{1-\omega'_e}}{\min\{p,1-p\}^{|S|}}}$$
$$\leq \frac{\mathbb{P}_p[\omega']}{\min\{p, 1-p\}^{|S|}} \leq \frac{\mathbb{P}_p[\omega']}{\min\{p, 1-p\}^s},$$

and hence

$$\frac{\mathbb{P}_p[\Phi(\omega)]}{\min\{p, 1-p\}^s} = \sum_{\omega'\in\Phi(\omega)} \frac{\mathbb{P}_p[\omega']}{\min\{p, 1-p\}^s} \geq |\Phi(\omega)|\, \mathbb{P}_p[\omega] \overset{1}{\geq} t\, \mathbb{P}_p[\omega].$$

Therefore, we get

$$\mathbb{P}_p[\mathcal{A}] = \sum_{\omega\in\mathcal{A}} \mathbb{P}_p[\omega] \le \frac{1}{t\min\{p,1-p\}^s} \sum_{\omega\in\mathcal{A}} \mathbb{P}_p[\Phi(\omega)]$$
$$= \frac{1}{t\min\{p,1-p\}^s} \sum_{\omega'\in\mathcal{B}} \left|\{\omega\in\mathcal{A} : \omega'\in\Phi(\omega)\}\right| \mathbb{P}_p[\omega'],$$

but, by property 2, we have that

$$\left|\{\omega\in\mathcal{A} : \omega'\in\Phi(\omega)\}\right| \le \left|\{\omega\in\mathcal{A} : \omega|_{S(\omega')^c} = \omega'|_{S(\omega')^c}\}\right| \le 2^{|S(\omega')|} \le 2^s.$$

Altogether, we obtain the desired inequality and finish the proof. □

Now, fix a relation $\prec$ on the set of edges in $\mathbb{S}_G$ that becomes an order when restricted to edges incident to any fixed vertex, and which is invariant under translations of $\mathbb{Z}^2$ (meaning $e \prec e'$ iff $\overline{\tau}(e) \prec \overline{\tau}(e')$ for all translations τ of $\mathbb{Z}^2$). It is easy to see that such a relation exists by considering an order on G and an order on edges incident to $0 \in \mathbb{Z}^2$. In addition, fix an arbitrary order $\ll$ on the set of vertices of $\mathbb{S}_G$.

We now define a lexicographic order on the set of self-avoiding paths in $\mathbb{S}_G$. For two paths $\gamma = (v_i)_{0\le i\le l}$, $\gamma' = (v'_i)_{0\le i\le l'}$, we set $\gamma < \gamma'$ if one of the following holds

- $v_0 \ll v'_0$,
- there exists $k < l \wedge l'$ such that $v_i = v'_i$ for $i \le k$ and $\{v_k, v_{k+1}\} \prec \{v'_k, v'_{k+1}\}$,
- $l < l'$ and $\gamma = (v'_i)_{0\le i\le l}$.

Definition 3.3.1. Fix n, let ω be a configuration with at least one path from $\overline{S_{3n}}$ to $\overline{Z_n}$ contained in $\overline{B_{3n}}$, and define $\gamma_{\min}(\omega)$ to be the minimal (with respect to $<$) such path in ω. Let $U(\omega)$ be the set of points $z \in B'_n$ satisfying

P1 $\overline{\{z\}} \cap \gamma_{\min}(\omega) \neq \varnothing$ (this implies $z \in B_{3n}$),

P2 $\overline{z + B_1}$ is connected to $\overline{S'_n}$ by an open path π contained in $\overline{B'_n}$, such that the distance between the projections of π and $\gamma_{\min}(\omega)$ onto $\mathbb{Z}^2$ is exactly 1.

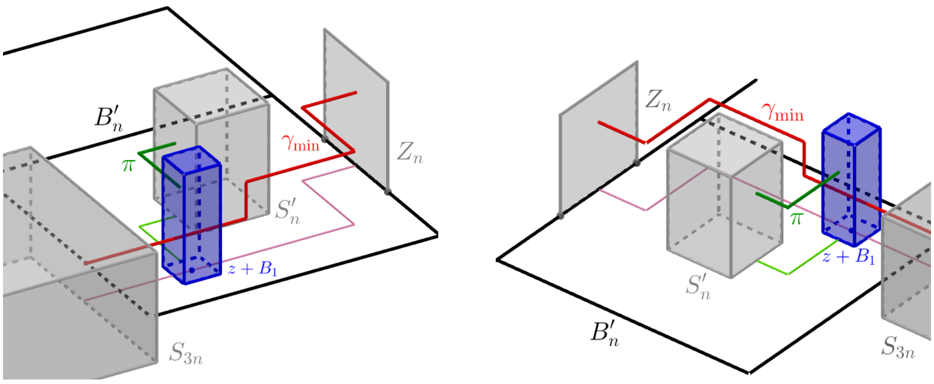


Figure 3.6: A configuration in $\{z \in U\}$ from two points of view.

We will now reformulate the gluing lemma so that we can use Lemma 3.3.2 to prove it. For this purpose, define the event

$$\mathcal{X}_n := \{S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\} \cap \{S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\}^c.$$

Then we have

$$\mathbb{P}_p\left[\{S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\}^c\right] \leq \mathbb{P}_p\left[\mathcal{X}_n\right] + \mathbb{P}_p\left[\{S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}^c\right].$$

Thus, in order to prove the gluing lemma, it suffices to show that (uniformly in n) the $\mathbb{P}_p$-probability of $\mathcal{X}_n$ is small whenever the $\mathbb{P}_p$-probability of $\{S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}$ is large:

Lemma 3.3.3 (Gluing lemma)**.** *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, p, G) > 0$ such that, for any n,*

$$\mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n}} Z_n, S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq 1 - \delta$$

implies

$$\mathbb{P}_p\left[\mathcal{X}_n\right] \leq \varepsilon.$$

Lemma 3.3.3 will be a consequence of the following two facts, in which $\mathcal{X}_n$ is decomposed into two parts where $|U|$ is either large or not.

If $\omega \in \mathcal{X}_n$ and $|U(\omega)|$ is not too large, then ω is "similar" to a configuration in $\{S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}^c$, in the sense that one needs to modify it only on edges incident to $\overline{U(\omega)}$ to turn it into a configuration ω' in said event and for which $U(\omega') = U(\omega)$, which "costs only a fixed finite price" according to Lemma 3.3.2. However, we already know from Lemma 3.1.2 that $\{S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}^c$ is arbitrarily unlikely when n is large.

Fact 1. *Let $\varepsilon > 0$ and $t > 0$. There exists $\delta = \delta(\varepsilon, t, p, G) > 0$ such that, for any n,*

$$\mathbb{P}_p\left[S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq 1 - \delta$$

implies $\mathbb{P}_p\left[\mathcal{X}_n \cap \{|U| < t\}\right] \leq \varepsilon$.

On the other hand, if $\omega \in \mathcal{X}_n$, then for any $z \in U(\omega)$, the witnessing paths for $\{S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}$ need to avoid $\gamma_{\min}(\omega)$ at $\overline{\{z\}}$ while their projections intersect in z. If $|U(\omega)|$ is large, this is intuitively unlikely by itself. One can create, for each $z \in U(\omega)$, a configuration $\omega^{(z)}$ that belongs to $\{S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\}$ and that disagrees with ω only locally around $\overline{\{z\}}$ while this z is also determined by knowledge of the modified configuration $\omega^{(z)}$ (which again "costs only a fixed finite price" according to Lemma 3.3.2).

Fact 2. *Let $\varepsilon > 0$. For t large enough, we have that, for any n,*

$$\mathbb{P}_p\left[\mathcal{X}_n \cap \{|U| \geq t\}\right] \leq \varepsilon\, \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\right] (\leq \varepsilon).$$

Suppose that Fact 1 and Fact 2 are true, and let $\varepsilon > 0$ be arbitrary. We first choose $t > 0$ such that the statement in Fact 2 holds for $\varepsilon/2$ and t. Now let $\delta > 0$ be such that the statement in Fact 1 holds for $\varepsilon/2, t, \delta$. If $\mathbb{P}_p\left[S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\right] \geq 1 - \delta$, then we have

$$\mathbb{P}_p[\mathcal{X}_n] = \underbrace{\mathbb{P}_p[\mathcal{X}_n \cap \{|U| < t\}]}_{\leq \varepsilon/2 \text{ by Fact } 1} + \underbrace{\mathbb{P}_p[\mathcal{X}_n \cap \{|U| \geq t\}]}_{\leq \varepsilon/2 \text{ by Fact } 2} \leq \varepsilon.$$

It remains to prove the two facts.

Proof of Fact 1. Fix $n \geq 1$. We will construct a map from $\mathcal{X}_n \cap \{|U| < t\}$ to $\{S'_n \xleftrightarrow{B'_n} Y_n^-, S'_n \xleftrightarrow{B'_n} Y_n^+\}^c$ so that we can apply Lemma 3.3.2.

Let $\omega \in \mathcal{X}_n$ with $|U(\omega)| < t$. Define ω' to be the configuration obtained from ω by closing, for each $z \in U(\omega)$, all edges $\{u, v\}$ such that $u \in \overline{\{z\}}$ and v is connected to $\overline{S'_n}$ by an open path in ω that is contained in B'_n. See Figure 3.7.

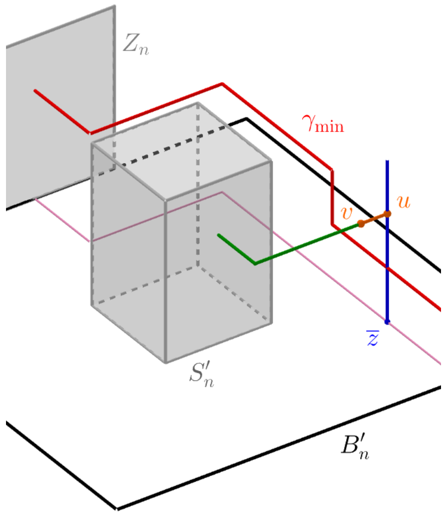


Figure 3.7: The kind of edges $\{u, v\}$ that are closed when defining ω' in the proof of Fact 1.

Then, in ω', there cannot exist both an open path from $\overline{S'_n}$ to $\overline{Y_n^-}$ and one from $\overline{S'_n}$ to $\overline{Y_n^+}$ contained in $\overline{B'_n}$. Indeed, suppose by way of contradiction that there exist such two paths. Then they are also open in ω, and since $\gamma_{\min}(\omega)$ does not intersect $\overline{S'_n}$ (because $\omega \in \mathcal{X}_n$), it must be the case that the $\mathbb{Z}^2$-projection of $\gamma_{\min}(\omega)$ intersects at least one of the projections of the two paths in some $z \in B'_n$. Choose z with this property in such a

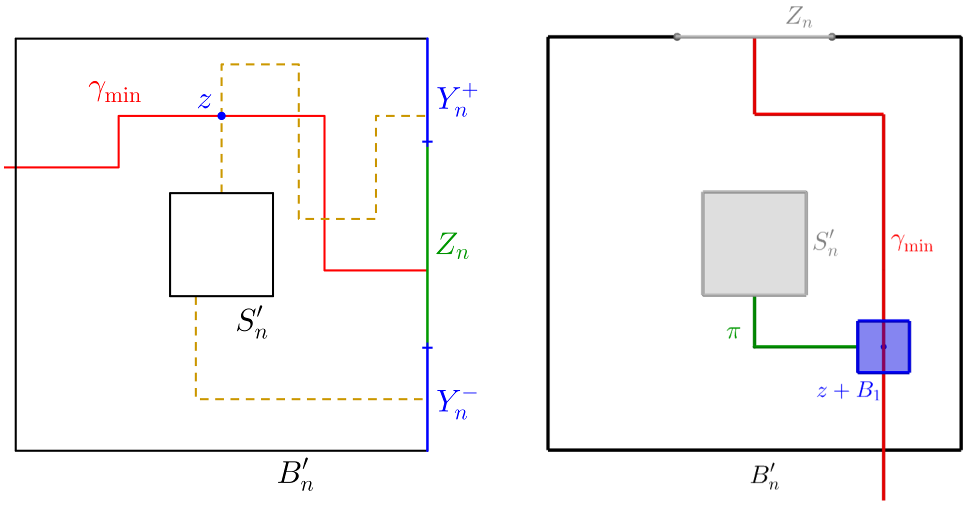


Figure 3.8: Concerning the proof of Fact 1: On the left side, projections of $\gamma_{\min}$ and two paths from $\overline{S_n'}$ to $\overline{Y_n^-}$ and $\overline{Y_n^+}$, respectively, and the choice of z. On the right side, projections of $\gamma_{\min}$ and π as in the definition of $U(\omega)$, which illustrates why no edge of π is closed when defining ω'.

way that when following the projection of the path from z on back to S_n', one does not traverse another z' with this property; depicted in the left-hand side of Figure 3.8. This choice ensures that $z \in U(\omega)$, and thus all edges incident to $\overline{\{z\}}$ that are contained in a path to $\overline{S_n'}$ were closed when defining ω'. In particular, one edge in one of the two paths is closed in ω', a contradiction.

Hence we constructed a map

$$\Phi : \mathcal{X}_n \cap \{|U| < t\} \to \{S_n' \overset{B_n'}{\longleftrightarrow} Y_n^-, S_n' \overset{B_n'}{\longleftrightarrow} Y_n^+\}^c$$
$$\omega \mapsto \omega'.$$

We claim that, if $\Phi(\omega) = \omega'$, then $U(\omega) = U(\omega')$. Indeed, let e be an edge which is open in ω but closed in ω'. Then e must be part of an open path in ω that intersects $\overline{S_n'}$ and is contained in $\overline{B_n'}$, and hence cannot be used by $\gamma_{\min}(\omega)$ (since $\omega \in \mathcal{X}_n$). This implies that $\gamma_{\min}(\omega)$ is open in ω', and thus, as $\omega' \leq \omega$ in the componentwise order, we conclude that $\gamma_{\min}(\omega) = \gamma_{\min}(\omega')$. Since condition **P1** in the definition of U depends only on $\gamma_{\min}$, we deduce that it is equivalent for ω and ω'. Now, fix $z \in B_n'$ that satisfies **P1** for ω and ω'. If z satisfies **P2** for ω', then clearly also for ω (since $\omega' \leq \omega$). On the other hand, suppose z satisfies **P2** for ω, i.e. there exists a path π in ω connecting $\overline{z + B_1}$ to $\overline{S_n'}$, such that the $\mathbb{Z}^2$-projections of π and $\gamma_{\min}(\omega)$ are at distance 1 from each other. See the right-hand side of Figure 3.8. As a consequence, π cannot use edges incident to $\overline{z'}$ for $z' \in U(\omega)$, so π is also open in ω', and hence z satisfies **P2** for ω'.

Now, if $\Phi(\omega) = \omega'$, then ω disagrees with ω' only on edges incident to $U(\omega) = U(\omega')$, and this set of edges is at most of size

$$(4+|G|-1)\left|\overline{U(\omega')}\right| = (3+|G|)\,|G|\,|U(\omega')| < ct,$$

where $c := (3+|G|)\,|G|$, because the maximum degree of $\mathbb{S}_G$ satisfies $\Delta(\mathbb{S}_G) = \Delta(\mathbb{Z}^2) + \Delta(G) \le 4 + |G| - 1$, and since $|U(\omega')| = |U(\omega)| < t$ by assumption.

We are now in the situation to apply Lemma 3.3.2 to obtain

$$\mathbb{P}_p\left[\mathcal{X}_n \cap \{|U| < t\}\right] \le (2/\min\{p, 1-p\})^{ct}\, \mathbb{P}_p\left[\{S'_n \overset{B'_n}{\longleftrightarrow} Y_n^-, S'_n \overset{B'_n}{\longleftrightarrow} Y_n^+\}^c\right],$$

which finishes the proof of Fact 1. □

Proof of Fact 2. For $R \ge 1$ and $v = (z,u) \in \mathbb{S}_G$, we write $\overline{B_R}(v)$ for $\overline{z + B_R}$ and $\partial\overline{B_R}(v)$ for $\overline{B_R}(v) \setminus \overline{B_{R-1}}(v)$. We claim that, for any $v \in \mathbb{S}_G$, for all three distinct neighbours w_1, w_2, w_3 of v and all distinct $w'_1, w'_2, w'_3 \in \partial\overline{B_2}(v)$, there exist three disjoint self-avoiding paths in $\overline{B_2}(v) \setminus \{v\}$ connecting w_i to w'_i, respectively. A rigorous proof of this claim is a bit lengthy due to some case distinctions and included in the appendix (Lemma B.0.1).

Fix $n \ge 1$, let $\omega \in \mathcal{X}_n$ with $|U(\omega)| \ge t$, and pick $z \in U(\omega)$. We identify z with some arbitrary but fixed vertex in $\overline{\{z\}}$. Construct the configuration $\omega^{(z)}$ as follows:

1. Choose three distinct vertices u, v, w adjacent to z such that $\{z, v\} \prec \{z, w\}$. We distinguish cases; see Figure 3.9.

   (a) First assume that $z + B_1$ does neither intersect Z_n nor S'_n. Define u' and v' to be, respectively, the first and last vertices of $\gamma_{\min}(\omega)$ that belong to $\partial\overline{B_2}(z)$. They exist and are distinct by property **P1** in the definition of $U(\omega)$, by the assumption above and since $\gamma_{\min}(\omega)$ is self-avoiding. Let π be a path witnessing property **P2** for z, and define w' to be the last vertex of π that belongs to $\partial\overline{B_2}(z)$. This is again possible by the assumption above. Then w' is connected to $\overline{S'_n}$ in $\overline{B'_n}$, and since $\omega \in \mathcal{X}_n$, we have that $w' \notin \{u', v'\}$.

   (b) If $z + B_1$ intersects Z_n, then define u' and w' as in (a), and v' to be a vertex on $\partial\overline{B_2}(z) \cap \overline{Z_n}$ distinct from w'.

   (c) If $z + B_1$ intersects S'_n, then define u', v' as in (a), and choose w' on $\partial\overline{B_2}(z) \cap \overline{S'_n}$ (necessarily distinct from u', v' since $\omega \in \mathcal{X}_n$).

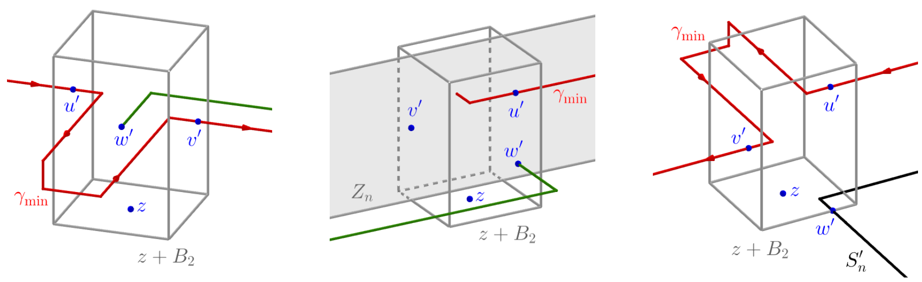


Figure 3.9: Choice of u', v', w' for all cases in 1.

2. Close all edges in $\overline{B_3}(z)$ except the edges incident to $\partial\overline{B_3}(z)$ that are used by $\gamma_{\min}(\omega)$ or π; see Figure 3.10.

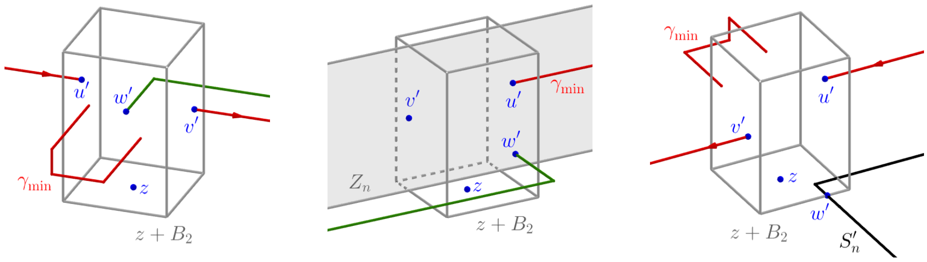


Figure 3.10: Closing edges according to step 2.

3. Open the edges $\{z, u\}, \{z, v\}, \{z, w\}$ and three disjoint self-avoiding paths $\gamma_u, \gamma_v, \gamma_w$ in $\overline{B_2}(z) \setminus \{z\}$ connecting u to u', v to v' and w to w', respectively; see Figure 3.11.

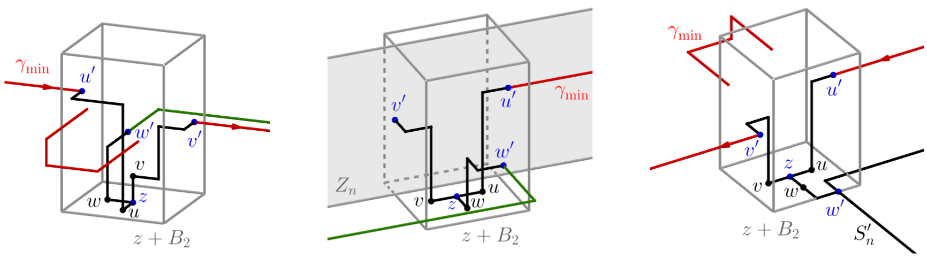


Figure 3.11: The configuration $\omega^{(z)}$ for all cases in 1.

Now, since it might be the case that $z + B_2 \not\subseteq B_{3n} \cap B_n'$, we have that $\gamma_u, \gamma_v, \gamma_w$ need not be contained in $\overline{B_{3n} \cap B_n'}$. If z is a corner of $B_{3n} \cap B_n'$, then it does not hold in general that, for given u, v, w, u', v', w' as above and in $\overline{B_{3n} \cap B_n'}$, one may find such paths in $(\overline{B_2}(z) \setminus \{z\}) \cap (\overline{B_{3n} \cap B_n'})$. One could argue that such three paths exist if one either carries out the construction above in $\overline{B_3}(z)$ instead of $\overline{B_2}(z)$, or if one chooses the relation $\prec$ and u, v, w in a suitable way. As the former method would increase the constant s in Lemma 3.3.2 (which we want to avoid due to a subsequent corollary of this proof), we continue with the latter one and provide justification in the appendix (Lemma B.0.2).

Then, by construction, $\omega^{(z)} \in \{S_{3n} \xleftrightarrow{B_{3n} \cup B_n'} S_n'\}$, and hence we defined a map

$$\begin{aligned} \Psi : \mathcal{X}_n \cap \{|U| \geq t\} &\to \mathfrak{P}(S_{3n} \xleftrightarrow{B_{3n} \cup B_n'} S_n') \\ \omega &\mapsto \{\omega^{(z)} : z \in U(\omega)\}. \end{aligned}$$

Let $\gamma_1 = \gamma_1(\omega)$ and $\gamma_2 = \gamma_2(\omega)$ be the parts of $\gamma_{\min}(\omega)$ up to u' and from v' on, respectively (γ_2 is possibly empty). Since γ_u and γ_v are chosen to be contained in $\overline{B_{3n}}$, it is clearly the case that $\tilde{\gamma} := (\gamma_1, \gamma_u^{-1}, z, \gamma_v, \gamma_2)$ (where γ_u^{-1} is the path γ_u in reversed order) is a self-avoiding path in $\overline{B_{3n}}$ connecting $\overline{S_{3n}}$ to $\overline{Z_n}$ that starts at the same vertex as $\gamma_{\min}(\omega)$ and is open in $\omega^{(z)}$. By the definition of the lexicographic order, as $\omega \geq \omega^{(z)}$ outside $\overline{B_2}(z)$, and since an open path in ω starting at $\overline{S_{3n}}$ and staying in $\overline{B_{3n}}$ can only enter $\overline{B_2}(z)$ through vertices used by $\gamma_{\min}(\omega)$ (not by π because $\omega \in \mathcal{X}_n$), we get that γ_1 is a prefix of $\gamma_{\min}(\omega^{(z)})$. By construction, the only possibility to proceed is to traverse γ_u^{-1}, and then $\{u, z\}$. Now, it might happen that starting from vertex z, there exists an open path to $\overline{Z_n}$ in $\omega^{(z)}$ using the edge $\{z, w\}$, but by assumption $\{z, v\} \prec \{z, w\}$. This means that $\gamma_{\min}(\omega^{(z)})$ uses $\{z, v\}$, then γ_v (again by lack of choice), and after that γ_2, since γ_2 is used by $\gamma_{\min}(\omega)$ from vertex v' on and again because $\omega \geq \omega^{(z)}$ outside $\overline{B_2}(z)$. We deduce that $\tilde{\gamma} = \gamma_{\min}(\omega^{(z)})$.

Since $\omega \in \mathcal{X}_n$, we have that no vertex on $\gamma_{\min}(\omega)$ is connected to $\overline{S'_n}$ in $\overline{B_{3n} \cup B'_n}$, and thus z is the only vertex on $\gamma_{\min}(\omega^{(z)})$ that is connected to $\overline{S'_n}$ in $\overline{B_{3n} \cup B'_n}$ without using an edge of $\gamma_{\min}(\omega^{(z)})$. This implies that $z \mapsto \omega^{(z)}$ is injective, and hence $|\Psi(\omega)| \geq |U(\omega)| \geq t$. By the same observation, if $\omega^{(z)} = \omega'^{(z')}$, then $z = z'$, and therefore ω and ω' agree on every edge with at least one endpoint outside $\overline{B_3}(z)$.

Altogether, the map Ψ satisfies the hypotheses of Lemma 3.3.2 for $s = \left|E(\overline{B_3})\right|$, and we get

$$\mathbb{P}_p\left[\mathcal{X}_n \cap \{|U| \geq t\}\right] \leq \frac{(2/\min\{p, 1-p\})^s}{t} \mathbb{P}_p\left[S_{3n} \xleftrightarrow{B_{3n} \cup B'_n} S'_n\right],$$

which concludes the proof of Fact 2. $\square$

Corollary 3.3.1. *There exists a constant $c = c(p, G) > 0$ such that the following holds. Let $\Lambda \subseteq \mathbb{Z}^2$ be a rectangle with side lengths at least 4. Let $R_i, S_i \subseteq \partial\Lambda$, $i = 1, 2$, be disjoint and such that any two paths in Λ connecting R_i to S_i, respectively, necessarily intersect. Assume that R_1, R_2, S_1, S_2 either contain at least 5 vertices each or that they are pairwise at sup norm distance at least 3 from each other. It is the case that*

$$\mathbb{P}_p\left[R_1 \overset{\Lambda}{\longleftrightarrow} R_2\right] \geq c \cdot \mathbb{P}_p\left[R_1 \overset{\Lambda}{\longleftrightarrow} S_1, R_2 \overset{\Lambda}{\longleftrightarrow} S_2\right].$$

The constant can be taken to be $c = (1 + c')^{-1}$, where

$$c' = \left(\frac{2}{\min\{p, 1-p\}}\right)^{|G|(49|G|+119)/2}.$$

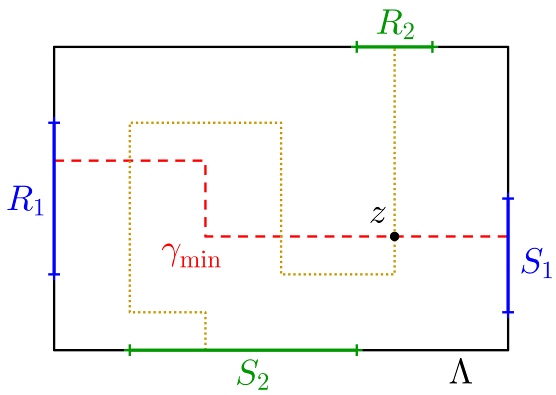


Figure 3.12: The projection of a typical situation in Corollary 3.3.1.

The proof is very similar to the proof of Fact 2.

Proof. Set $\mathcal{A} = \{R_1 \overset{\Lambda}{\longleftrightarrow} S_1, R_2 \overset{\Lambda}{\longleftrightarrow} S_2\} \cap \{R_1 \overset{\Lambda}{\longleftrightarrow} R_2\}^c$ and $\mathcal{B} = \{R_1 \overset{\Lambda}{\longleftrightarrow} R_2\}$. We intend to construct a map from $\mathcal{A}$ to $\mathcal{B}$ analogously to the proof of Fact 2.

Fix $\omega \in \mathcal{A}$, and let $\gamma_{\min}(\omega)$ be the minimal (w.r.t. $<$) open path from $\overline{R_1}$ to $\overline{S_1}$ within $\overline{\Lambda}$. As $\omega \in \mathcal{A}$, there exists $z \in \Lambda$ for which $\gamma_{\min}(\omega)$ intersects $\overline{\{z\}}$, and such that $\overline{\{z\}}$ is connected to $\overline{R_2}$ in $\overline{\Lambda}$ by an open path π whose $\mathbb{Z}^2$-projection intersects the projection of $\gamma_{\min}(\omega)$ only in z. Note that $z \notin R_1 \cup R_2$ since $\omega \in \mathcal{A}$. As before, we identify z with some arbitrary but fixed vertex in $\overline{\{z\}}$. Recall the definition of u', v', w' in the proof of Fact 2. In contrast to the situation there, it is possible that $\overline{B_2}(z)$ intersects the region $\overline{R_1}$ from where $\gamma_{\min}(\omega)$ starts. Since it is crucial to control the minimal path in the new configuration, we choose in this case u' to be the minimal (w.r.t. $\ll$) vertex in $\overline{B_2}(z) \cap \overline{R_1}$ if it is $\ll$-less than or equal to the initial vertex of $\gamma_{\min}(\omega)$, and otherwise to be the first vertex of $\gamma_{\min}(\omega)$ in $\partial\overline{B_2}(z)$. Under the assumption that R_i, S_i are of cardinality at least 5, one can always choose v' and w' in $\partial\overline{B_2}(z)$. Whereas under the assumption that R_i, S_i are at sup norm distance at least 3 apart, at most one of R_i, S_i may intersect $z + B_1$. If $z \in S_1$, then one can choose $v = v' = z$. Any of these cases may be put down to the setup of Lemma B.0.2 in the appendix, and hence existence of neighbours u, v, w with $\{z, v\} \prec \{z, w\}$ together with the corresponding paths follows. We now proceed as in steps 2,3 in the proof of Fact 2 and denote the resulting configuration by ω'. By construction and analogous arguments as before, we have that $\omega' \in \mathcal{B}$, and z is the only vertex on $\gamma_{\min}(\omega')$ that is connected to $\overline{R_2}$ in $\overline{\Lambda}$ without using an edge of $\gamma_{\min}(\omega')$. We deduce that $\omega \mapsto \omega'$ satisfies the assumptions of Lemma 3.3.2 for $t = 1$ and s being the number of edges in $[-3, 3]^2 \times G$. Counting these edges for G being the complete graph on $|G|$ vertices finishes the proof. $\square$

Appendix A

The proof of Lemma 2.2.4

Proof of Lemma 2.2.4. The proof proceeds by induction on $|Y|$.

If $|Y| = 3$, then there exists only one 3-partition. Assume that the statement holds for sets of cardinality at most n, and let $|Y| = n+1$. Let $\mathcal{P}$ be a compatible collection of 3-partitions of Y. Fix $y \in Y$ and set $Y' = Y \setminus \{y\}$.

Define $\mathcal{P}'$ to be the collection of all members in $\mathcal{P}$ all of whose parts differ from the singleton $\{y\}$. Then $\mathcal{P}'$ consists of 3-partitions of the form $\{P_1 \cup \{y\}, P_2, P_3\}$ for $P_i \subseteq Y'$ non-empty, and removing y from the corresponding parts results in a family of compatible 3-partitions of Y'. On the other hand, even though a 3-partition of Y' induces three 3-partitions of Y (by adding y to any part), any two of them are clearly not compatible. Thus the induction hypothesis implies that

$$|\mathcal{P}'| \leq |Y'| - 2 = |Y| - 3.$$

Therefore it suffices to show that $|\mathcal{P} \setminus \mathcal{P}'| \leq 1$, which is to say that two 3-partitions containing the singleton $\{y\}$ cannot be compatible. To this end, suppose conversely that $\Pi = \{\{y\}, P_2, P_3\}$ and $\Pi' = \{\{y\}, P_2'.P_3'\}$ are two compatible 3-partitions of Y. By definition, one part of Π contains two (distinct) parts of Π'. As $P_j' \neq \varnothing$ for $j = 2, 3$, and since $y \notin P_i$ for $i = 2, 3$, there exists $i \in \{2, 3\}$ such that

$$Y' = P_2' \cup P_3' \subseteq P_i \subsetneq Y',$$

a contradiction. $\square$

Appendix B

Constructing paths in small environments

This section is devoted to proving existence of the paths $\gamma_u, \gamma_v, \gamma_w$ in Fact 2 concerning the gluing lemma, and its corollary.

Lemma B.0.1. *Let $|G| \geq 2$. For any vertex $v \in \mathbb{S}_G$, for all three distinct neighbours w_1, w_2, w_3 of v and all distinct $w'_1, w'_2, w'_3 \in \partial\overline{B_2}(v)$, there exist three disjoint self-avoiding paths in $\overline{B_2}(v) \setminus \{v\}$ connecting w_i to w'_i, respectively.*

Before proving this, we introduce the following notation. For a path $p = (z_1, \dots, z_n)$ in $\mathbb{Z}^2$ and a vertex $u \in G$, we write $[p, u]$ for the path $((z_1, u), \dots, (z_n, u))$ in $\mathbb{S}_G$, which connects (z_1, u) to (z_n, u) by only changing the $\mathbb{Z}^2$-component according to p. Analogously, for a vertex $z \in \mathbb{Z}^2$ and a path p in G, we will write $[z, p]$ for the corresponding path in $\mathbb{S}_G$.

Proof. Let $v = (z, u)$, $w_i = (z_i, u_i)$, $w'_i = (z'_i, u'_i)$, $i = 1, 2, 3$, with $z, z_i, z'_i \in \mathbb{Z}^2$ and $u, u_i, u'_i \in G$. We first show the statement in a slightly different setup and require of w_1, w_2, w_3 only that z_1, z_2, z_3 are pairwise distinct and adjacent to z (we do not assume that w_i is adjacent to v).

Since G is connected, there exist self-avoiding paths γ_i, $i = 1, 2, 3$, in G connecting u_i to u'_i, respectively. We distinguish two cases:

Case 1: u'_1, u'_2, u'_3 not all equal; w.l.o.g., say $u'_1 \notin \{u'_2, u'_3\}$.

For $i = 1, 2, 3$, let $\overline{\gamma_i} = [z_i, \gamma_i]$ be the corresponding path in $\mathbb{S}_G$, which connects w_i to (z_i, u'_i). Let $\tilde{z_1} \in z + \partial B_2$ be adjacent to z_1, and let δ_1 be a path in $z + \partial B_2$ connecting $\tilde{z_1}$ to z'_1. Observe that, for any choice of z_i, z'_i, there exist disjoint self-avoiding paths δ_2, δ_3 in $(z + B_2) \setminus \{z, z_1\}$ connecting z_2 to z'_2 and z_3 to z'_3, respectively. For $i = 1, 2, 3$, let $\overline{\delta_i} = [\delta_i, u'_i]$ be the corresponding paths in $\mathbb{S}_G$, which let the G-component fixed. Then $p_1 = (\overline{\gamma_1}, (\tilde{z_1}, u'_1), \overline{\delta_1})$ and $p_i = (\overline{\gamma_i}, \overline{\delta_i})$, $i = 2, 3$, are paths as desired.

Case 2: $u'_1 = u'_2 = u'_3$.

Then we must have that $z'_1, z'_2, z'_3 \in z + \partial B_2$ are pairwise distinct. It is straightforward to check that, for any choice of z_i, z'_i, there is an ordering of $\{(z_i, z'_i) : i = 1, 2, 3\}$ such that there exist disjoint self-avoiding paths δ_2, δ_3 in $(z + B_2) \setminus \{z, z_1, z'_1\}$ connecting z_2 to z'_2 and z_3 to z'_3, respectively. Since $|G| \geq 2$, we can find $u'_4 \in G \setminus \{u'_1\}$. For $i = 2, 3$, let $\overline{\gamma_i} = [z_i, \gamma_i]$, $\overline{\delta_i} = [\delta_i, u'_i]$ and set $p_i = (\overline{\gamma_i}, \overline{\delta_i})$ as in case 1.

Moreover, let $\gamma_{1,1}, \gamma_{1,2}$ be self-avoiding paths in G connecting u_1 to u'_4 and u'_4 to u'_1, respectively, and as in case 1, let $\tilde{z_1} \in z + \partial B_2$ be adjacent to z_1 and δ_1 a path in $z + \partial B_2$ connecting $\tilde{z_1}$ to z'_1. If we now set $\overline{\delta_1} = [\delta_1, u'_4]$, $\overline{\gamma_{1,1}} = [z_1, \gamma_{1,1}]$ and $\overline{\gamma_{1,2}} = [z'_1, \gamma_{1,2}]$, then $p_1 = (\overline{\gamma_{1,1}}, (\tilde{z_1}, u'_4), \overline{\delta_1}, \overline{\gamma_{1,2}})$ is a self-avoiding path from w_1 to w'_1 disjoint from p_2, p_3.

Now, let w_1, w_2, w_3 be adjacent to v and keep the notation from above. Note that, if $z_i = z_j \neq z$, then $u_i = u_j = u$, and thus $i = j$.

If z_1 is not adjacent to z, then we must have $z_1 = z$ (since w_1 is adjacent to v). In this case, take $\tilde{z_1} \neq z_2, z_3$ adjacent to z, otherwise set $\tilde{z_1} = z_1$. If z_2 is not adjacent to z, then pick $\tilde{z_2} \neq \tilde{z_1}, z_3$ adjacent to z, otherwise set $\tilde{z_2} = z_2$. Finally, if z_3 is not adjacent to z, then choose $\tilde{z_3} \neq \tilde{z_1}, \tilde{z_2}$ adjacent to z, otherwise set $\tilde{z_3} = z_3$.

Now $\tilde{z_1}, \tilde{z_2}, \tilde{z_3}$ are pairwise distinct and adjacent to z. So, by the above, we get disjoint self-avoiding paths $\tilde{p_i}$, $i = 1, 2, 3$, connecting $(\tilde{z_i}, u_i)$ to w'_i, respectively, and by construction, these paths do not traverse vertices in $\overline{\{z\}}$. Then $p_i = ((z_i, u_i), (\tilde{z_i}, u_i), \tilde{p_i})$, $i = 1, 2, 3$, are as desired. $\square$

The next part is dedicated to the case when $z \in U(\omega)$ in the proof of Fact 2 is close to the boundary of $B_{3n} \cap B'_n$. We first restrict the choice of the relation $\prec$ on the set of edges in $\mathbb{S}_G$ and demand that change of entries in the $\mathbb{Z}^2$-component is $\prec$-less than in the G-component. More formally, we require for $z, z' \in \mathbb{Z}^2$ adjacent to one another and $u, u' \in G$ adjacent to one another that

$$\{(z, u), (z', u)\} \prec \{(z, u), (z, u')\}. \tag{B.1}$$

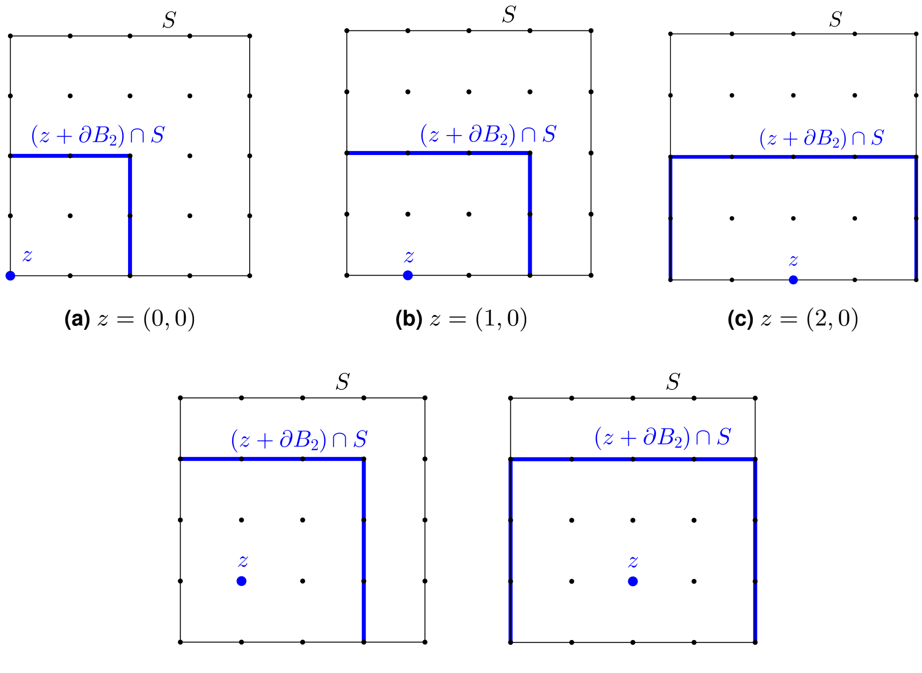
Lemma B.0.2. *Let $|G| \geq 2$, $R \subseteq \mathbb{Z}^2$ a rectangle with side-lengths at least 4, $v \in \overline{R}$, $w'_1, w'_2, w'_3 \in \partial \overline{B_2}(v) \cap \overline{R}$ such that the $\mathbb{Z}^2$-projections of w'_i and w'_3 are distinct for $i = 1, 2$. Then there exist three distinct neighbours w_1, w_2, w_3 of v in $\overline{R}$ with $\{v, w_2\} \prec \{v, w_3\}$ and three disjoint self-avoiding paths in $(\overline{B_2}(v) \setminus \{v\}) \cap \overline{R}$ connecting w_i to w'_i, respectively.*

Remark B.0.1. With the notation in the proof of Fact 2, $B_{3n} \cap B'_n$ corresponds to R, u' to w'_1, v' to w'_2 and w' to w'_3. The condition on the $\mathbb{Z}^2$-projections in the lemma is satisfied since the $\mathbb{Z}^2$-projection of the path π in **P2** in the definition of $U(\omega)$ is at distance 1 from the projection of $\gamma_{\min}$, and moreover u', v' are on $\gamma_{\min}$, whereas w' is on π.

Also note that this lemma does not fully cover the situation in Fact 2 as u' and v' might be in $\overline{B_{3n} \setminus B'_n}$, and w' might be in $\overline{B'_n \setminus B_{3n}}$. However, with the argument above about the $\mathbb{Z}^2$-projections in mind, these situations can be solved easily or may be put down to the setup of the lemma in such a way that the paths to u' and v' are contained in $\overline{B_{3n}}$ and the path to w' in $\overline{B'_n}$.

Regarding Corollary 3.3.1, the assumption about the $\mathbb{Z}^2$-projections is also fulfilled since z and the path π (in the proof of Corollary 3.3.1) are chosen in such a way that the projection of π intersects the projection of $\gamma_{\min}$ only in z.

Proof. We claim that it suffices to consider $G = \{0, 1\}$. To see this, let $v = (z, u)$, $z \in R$, $u \in G$. Since $|G| \geq 2$, there exists a neighbour u' of u in G. Observe that there exist three disjoint self-avoiding paths (possibly empty) in $(\overline{B_2}(v) \setminus \{v\}) \cap \overline{R}$ connecting w_i' to $((z + \partial B_2) \cap R) \times \{u, u'\}$, respectively, and such that their terminal vertices satisfy the conditions of the lemma. Now, consider $G' = \{u, u'\} \simeq \{0, 1\}$. Moreover, by translation invariance of $\prec$, we may assume w.l.o.g. that $R = S := [0, 4]^2$. Due to the assumption (B.1) on $\prec$ and since we do not assume anything about the order of edges in the same plane, i.e. of edges of the form $\{(z, u), (z', u)\}$ and $\{(z, u), (z'', u)\}$, it is enough to consider $v = (z, 0)$. Furthermore, by symmetry and the above arguments, we can restrict ourselves to the cases $z = (0, 0), (1, 0), (2, 0), (1, 1), (2, 1)$; see Figure B.1.



(a) $z = (0, 0)$ **(b)** $z = (1, 0)$ **(c)** $z = (2, 0)$

(d) $z = (1, 1)$ **(e)** $z = (2, 1)$

Figure B.1: Relevant possibilities for the position of $z \in S$.

As there is no prospect of a general construction for the three paths with the desired properties, and since even the case $z = (0, 0)$ includes $10 \cdot 8 \cdot 7 = 560$ possibilities for w_1', w_2', w_3', the remainder of this proof is rather a manual on how to work through them efficiently. By the assumption (B.1) on $\prec$, we can always choose $w_3 = (z, 1)$ and have no further restrictions on w_1, w_2, apart from being distinct from one another and from w_3. This allows us to neglect henceforth the order of w_1', w_2' as well. It is easy to see (but somewhat lengthy to write down) that the critical case $z = (0,0)$ implies the other ones. To this end, let $z = (0,0)$, and let $w'_i = (z'_i, u'_i)$ with $z'_i \in \partial B_2 \cap S$, $u'_i \in \{0,1\}$, $i = 1,2,3$. For the remainder of the proof, assume w.l.o.g. that two of the $\mathbb{Z}^2$-projections of w'_1, w'_2, w'_3 are on top of $\partial B_2 \cap S$, i.e. there exist $i \neq j$ such that $z'_i, z'_j \in [0,2] \times \{2\}$.

We first treat the case when the $\mathbb{Z}^2$-projections z'_1, z'_2, z'_3 are pairwise distinct. If $\{z'_1, z'_2, z'_3\} \neq \{(1,2),(2,2),(2,1)\}$, then there exist distinct $w_1, w_2 \neq w_3$ adjacent to v and three disjoint self-avoiding paths γ_i, $i = 1,2,3$, connecting w_i to $(z'_i, 0)$. Moreover, for each i, the path γ_i may be constructed in such a way that its $\mathbb{Z}^2$-projection does not traverse z'_j for $j \neq i$, which means that if $u'_i = 1$ and if γ_i does not traverse w'_i already, we can extend it to $\tilde{\gamma}_i = (\gamma_i, w'_i)$ and the resulting paths are still disjoint and self-avoiding. To verify this, one needs to check 18 possibilities. Indeed, ignoring their order, there are 6 possibilities to choose the z'_i (under the current assumptions), and as the order of w'_1, w'_2 is irrelevant, we only have to mark which one is z'_3. See Figure B.2 for a selection.

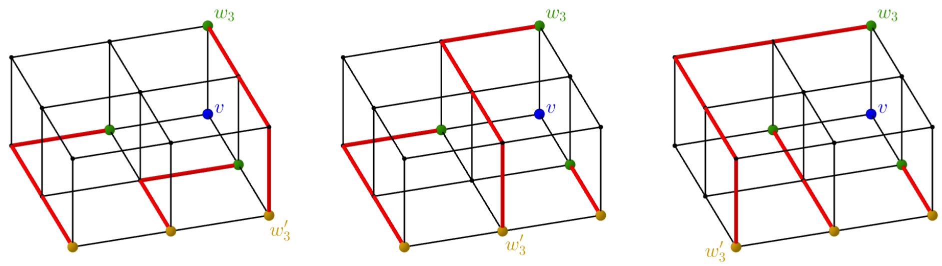


Figure B.2: A selection of paths when the $w'_i = (z'_i, 0)$ are pairwise distinct and not gathered around the corner.

Under the assumption that they are pairwise distinct, we are left with the case where the $\mathbb{Z}^2$-projections z'_1, z'_2, z'_3 are $(1,2),(2,2),(2,1)$ (in some order). This instance comprises 24 possibilities for the choice of w'_1, w'_2, w'_3, for each of which the existence of such three paths can be verified. Similarly as before, it suffices to consider the cases where $u'_3 = 0$ while making sure that no other path than γ_3 traverses $(z'_3, 1)$, which reduces it to 12 possibilities. See Figure B.3 for a selection.

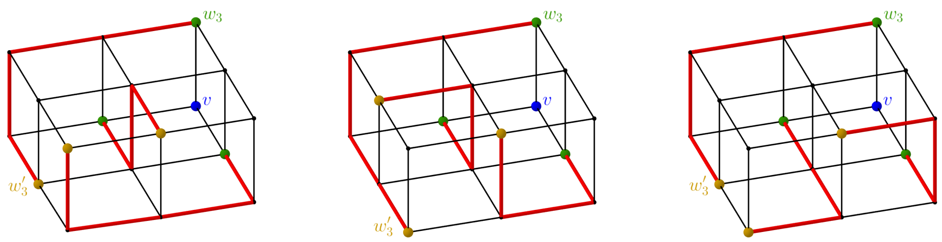


Figure B.3: A selection of paths when $\{z'_1, z'_2, z'_3\} = \{(1,2),(2,2),(2,1)\}$.

By hypothesis, z_1', z_2', z_3' not being pairwise distinct only occurs when w_1' is on top of w_2', or vice versa, which can happen in 24 ways, and it can be checked that three paths with the desired properties exist in all of them. Once again, one only needs to check the 12 cases where $u_3' = 0$ while ensuring that no other path than γ_3 traverses $(z_3', 1)$. See Figure B.4 for a collection.

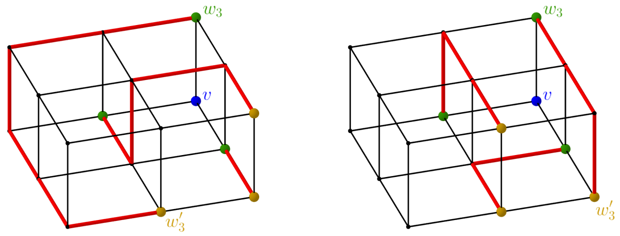


Figure B.4: A selection of paths when one of w_1', w_2' is on top of the other.

$\square$

Remark B.0.2. Since the only assumptions on G were to be finite and connected, the constructions of the paths in both proofs also work in $\mathbb{S}_T = \mathbb{Z}^2 \times T$ for some spanning tree T of G.

Bibliography

[1] M. Aizenman, H. Kesten, and C. M. Newman. “Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation”. In: *Communications in Mathematical Physics* 111.4 (1987), pp. 505–531.

[2] D. Barsky, G. Grimmett, and C. Newman. “Percolation in half-spaces: equality of critical densities and continuity of the percolation probability”. In: *Probability theory and related fields* 90.1 (1991), pp. 111–148.

[3] S. R. Broadbent and J. M. Hammersley. “Percolation processes: I. Crystals and mazes”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 53.3 (1957), pp. 629–641.

[4] R. M. Burton and M. Keane. “Density and uniqueness in percolation”. In: *Communications in Mathematical Physics* 121.3 (1989), pp. 501–505.

[5] H. Duminil-Copin, V. Sidoravicius, and V. Tassion. “Absence of infinite cluster for critical Bernoulli percolation on slabs”. In: *Communications on Pure and Applied Mathematics* 69.7 (2014), pp. 1397–1411.

[6] R. Fitzner and R. van der Hofstad. “Mean-field behavior for nearest-neighbor percolation in $d > 10$”. In: *Electronic Journal of Probability* 22 (2017), pp. 1–65.

[7] G. Grimmett. *Percolation.* 2nd ed. Grundlehren der mathematischen Wissenschaften; 321. Springer, 1999.

[8] G. Grimmett and J. Marstrand. “The supercritical phase of percolation is well behaved”. In: *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences* 430.1879 (1990), pp. 439–457.

[9] T. Hara and G. Slade. “Mean-Field Behaviour and the Lace Expansion”. In: *Probability and Phase Transition.* Ed. by G. Grimmett. Springer, Dordrecht, 1994, pp. 87–122.

[10] T. E. Harris. “A lower bound for the critical probability in a certain percolation process”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 56.1 (1960), pp. 13–20.

[11] H. Kesten. “The critical probability of bond percolation on the square lattice equals 1/2”. In: *Communications in mathematical physics* 74.1 (1980), pp. 41–59.

[12] R. Lyons and Y. Peres. *Probability on Trees and Networks.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.