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„Collective motion in a place filled with obstacles“

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Abstract

In recent years, models for collective behaviour have become a matter of special importance in the scientific world. Originally developed in mathematical physics, they are today also applied to describe emergence phenomena such as bird flocks. In the underlying master thesis a model for collective motion of two species, which will be referred to as swimmers and obstacles, is given and analyzed. It is sectioned in four parts: After a literature review, the dynamics of swimmers and obstacles without influence on each other are introduced. Then the coupled model shall be given and kinetic and macro equations are derived, which is the main goal of this work. To complete the thesis, the model is analyzed for its hyperbolicity.

Abstract

In der jüngsten Vergangenheit gewannen Modelle, die kollektives Verhalten beschreiben, in der Wissenschaft an Bedeutung. Ursprünglich stammen diese Modelle aus der mathematischen Physik, heutzutage werden sie auch angewandt, um kollektive Verhaltensphänomene zu beschreiben, wie man sie beispielsweise von Vogelschwärmen kennt. In der vorliegenden Masterarbeit wird ein Modell gegeben, das zwei Arten, sogenannte Schwimmer und Hindernisse, und deren Interaktion beschreibt und anschließend analysiert. Die Arbeit gliedert sich in vier Kapitel: Nach einer Literaturreview, werden die Dynamiken der Schwimmer und Hindernisse ohne wechselseitigen Einfluss aufeinander eingeführt. Anschließend wird das gekoppelte Modell aus Schwimmern und Hindernissen beschrieben und es erfolgt die Herleitung der kinetischen und makroskopischen Gleichungen, was den Kern der Arbeit bildet. Abschließend wird untersucht, ob es sich um ein hyperbolisches Modell handelt.

Contents

0 Acknoledgment

I would like to thank my advisor Sara Merino-Aceituno, who presented the field kinetic theory in her lecture at the mathematics department of the University of Vienna and later proposed this exciting topic for my master thesis. I am especially thankful for the countless hours of time she dedicated to this project and her the patience with which she answered my questions. Beyond that, I would like to thank my family, friends, roommates and fellow students for being as supportive as they possibly could be.

1 Introduction

Modelling the characteristics of collective motion gained a lot of attention in the scientific world recently. In order to understand the fascinating swarming behaviour as it occurs in bird flocks, fish suspensions, sperm dynamics, bacteria and many other areas, several models were given to describe the typical behaviour of self-propelled particles trying to align their movement with conspecifics. The behaviour phenomena of interest for bird flocks can be seen in this video[1] and for a little background on the science behind, there is for example a talk of Iain Couzin about his research on collective behaviour in this online presentation[2].

Agent-based models were initial position in modelling the emergence of collective motion. The goal was to investigate how order emerges on the basis of interaction of self-propelled particles. The mathematics behind comes originally from the studies of gas dynamics in (mathematical) physics. The interested reader can find further information on the application of kinetic theory in biology in [1] and explanation on mathematical details in [2].

1.1 Kinetic theory and the three levels of description

The models to describe the emergence phenomena are considered at different scales [2].
Microscopic models describe particle dynamics of the individual agents. These models contain all the information of a system and mostly they are deterministic, piece-wise deterministic Markov processes or stochastic differential equations, but also highly complex, and therefore, hard to analyse mathematically.
Kinetic equations are mathematical descriptions of the dynamics of large particle ensembles in terms of a phase space; usually integro-differential equations, which give the position and velocity (or orientation). Condoning the loss of information on the individual particle, we get a density of the particles in a given state. They can be obtained via taking the mean-field limit of the microscopic model. In general, it is hard to prove the correspondence between stochastic particle systems and kinetic equations and we must also mention that while mean-field limits are typically straightforward to compute formally, their rigorous proof tend to be very hard to obtain in general. But

[1] `www.youtube.com/watch?v=bb9ZTbYGRdc`
[2] `www.youtube.com/watch?v=6wy1CP-mMO8`

if the mean-field limit is true, kinetic equations are good approximations for dynamics with high particle density. But beware! One could get behaviour in the kinetic equation, which was not expected a priori at the particle level - even with rigorous mean-field limit.

Macroscopic models give a description with a finite number of position dependent quantities like mass density, the mean velocity or temperature. On this scale, the evolution of statistics of the distribution function is described by averaging. Intuitively one can imagine in some cases the derivation of macroscopic equations as speeding up time and zooming out in space at the same time such that we have many, many interactions between the particles and the distance between these interactions becomes small.

In the underlying thesis, a model for collective motion of self-propelled particles, the so-called swimmers, in a place filled with rod-shaped obstacles is given. The following sections consist of a literature review, then the particle model for the swimmers and obstacles is given. The main goal is to obtain kinetic and macroscopic equations and depict properties of the model.

2 Literature review

2.1 The Doi-Onsager model

The main references for this section are [3], [4], [5] and [6].

The Doi-Onsager model describes the behavior of rigid rod-like polymers in a fluid; in particular its rotation and translation. It combines two different, but related models: Firstly, it considers a single rod-shaped molecule and its distribution function which depends on position and orientation. The rods are assumed to not only interact with other molecules but also with the flow, where this flow is modelled via Brownian motion. Interactions between the molecules are modelled via a mean-field potential and a kinetic equation for the molecules is derived. Most of the existing literature focuses on models in which interactions occur through excluded-volume effects, even though effects of an external field could be included.

The second model is a free energy minimization problem that can be obtained via statistical mechanics techniques and was originally investigated by Onsager in 1949 and can be gleaned in [7].

2.1.1 Occuring phenomena

If the concentration of rods has reached a certain quantity, the particles want to align with others and form a nematic liquid crystal[3], i.e. they tend to be in a nematic phase. It has been shown by Onsager that this effect even occurs, if the spatial distribution of particles is uniform. For low concentrations, the rods are oriented randomly and give an isotropic distribution, meaning that the orientations are uniformly distributed and there is not a preferred direction. One special feature of the Doi-Onsager model is that it describes both phases, the isotropic and nematic phase.

2.1.2 The Doi model and Doi-Onsager

As already mentioned above, it is a model which can be obtained via mean-field kinetic theory and describes the properties of liquid crystal polymers in a solution. The agent based model tells us the position and orientation of a single rod-like molecule via a distribution function, while the interaction of molecules is modelled via a mean field potential. As a consequence, the model describes the interaction between the orientation of rod-like polymers on the microscopic scale and the macroscopic properties of the fluid in which these polymers are in.

The Doi-Onsager model is basically the Doi model in absence of flow. The Doi-Onsager model reduces to a non-local diffusion equation and reads

$$\frac{\partial f}{\partial t} = D_r \mathcal{R} \cdot (\mathcal{R} f + f \mathcal{R} U), \tag{2.1}$$

where $f = f(\theta)$ is the orientation distribution function of the particles. The position of the polymers is neglected since it has no notable effect in the model. D_r is the rotational diffusivity, which is proportional to the temperature of the solution and w.l.o.g. set to 1 and $\theta \in \mathbb{S}^2$, the two-dimensional sphere.
The differential operator on the sphere is $\mathcal{R} = \theta \times \frac{\partial}{\partial \theta}$. Note that $\mathcal{R} \cdot \mathcal{R}$ is the Laplace-Beltrami operator. This operator is a generalization of the Laplacian to a manifold. In $\mathbb{R}^2$ the operator is given by

$$\mathcal{R} \cdot \mathcal{R} f(\theta) = \frac{\partial^2}{\partial \theta^2} f(\theta)$$

[3]Polymers in a fluid correspond to liquid crystals.

In dimension 3 its connection to the standard Laplacian can be expressed as

$$\mathcal{R} \cdot \mathcal{R} f(\theta) = \nabla^2 f\left(\frac{\theta}{|\theta|}\right) = \nabla^2 f = \frac{\partial}{r\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\mathcal{R} \cdot \mathcal{R} f,$$

such that

$$\mathcal{R} \cdot \mathcal{R} f(\theta, \omega) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta^2}\frac{\partial^2 f}{\partial\omega^2},$$

where ω denotes the azimuth angle.

Remark 1. *Azimuth angle*
In a spherical coordinate system, the azimuth is an angular measure. Suppose you have a vector from a point of observation (origin) to a point of interest. Then you project this vector perpendicularly onto a plane of reference. The angle between the projected vector and a vector on the reference plane is called the azimuth or azimuthal angel.

Unfortunately, the notation of the operator on the sphere in arbitrary dimension is inconsistent, e.g. $\Delta\omega f$ or $\nabla^2_{\mathbb{S}^{n-1}f}$ are used too. U denotes the mean-field interaction potential and is of the form

$$U(\theta, [f]) = \int_{|\theta'|=1} K(\theta, \theta') f(\theta') d\theta'.$$

The interaction kernel $K(\theta, \theta')$ is a smooth, real-valued, symmetric function:

$$K(-\theta, \theta') = K(\theta, \theta'), \quad K(\theta, \theta') = K(\theta', \theta), \quad K(\theta, \theta') = K(A\theta, A\theta') \quad \forall A \in SO(3).$$

The symbol $SO(3)$ denotes the three-dimensional rotation group. In Dimension 2, one can parametrize $\mathbb{S}^1$ by the angle $\theta \in [0, 2\pi[$ and write the kernel as a convolution for a certain even function $K = K(\theta - \theta')$. This kernel satisfies $K(\theta) = K(\theta + \pi)$ and then, the mean-field interaction potential reads

$$U(\theta, [f]) = \int_0^{2\pi} K(\theta - \theta') f(\theta') d\theta'.$$

2.1.3 The kernel

Depending on the choice of this kernel, one gets different behaviour for the models. For example, Onsager considered the kernel $K(\theta, \theta') = \alpha|\theta \times \theta'| = |\sin\theta|$ in dimension 3, where α is a constant, measuring the potential intensity. This potential is suitable for describing the behaviour of excluded volume effects for rod-like polymers, when their concentration in the solution is not too high. Another choice is the so-called Maier-Saupe potential, given by $K(\theta, \theta') = |\theta \cdot \theta'|^2 = \sin^2 \theta$ in dimension 3, which turned out to be the more accurate potential for low-molecular weight nematics and allows direct computation of solutions, which is why it is preferred more often. There exist further analysis with the (asymmetric) dipole potential and other potentials too.

The Lyapunov functional for this model has a free energy defined by

$$A(f) = \int_{|x|=1} \left(f(t,x) \ln f(t,x) + \frac{1}{2} f(t,x) U(f) \right) dx.$$

It was originally proposed by Onsager to describe the equilibrium states, which correspond to the critical points of the functional, of rod-shaped polymer suspensions.

The following computations

$$\begin{aligned} \frac{\partial}{\partial t} \int_{|\theta|=1} f(t,\theta) d\theta &= \int_{|\theta|=1} \frac{\partial f}{\partial t}(t,\theta) d\theta \\ &= \int_{|\theta|=1} \mathcal{R} \cdot (\mathcal{R} f + f \mathcal{R} U) d\theta \\ &= - \int_{|\theta|=1} \mathcal{R}(1) \cdot (\mathcal{R} f + f \mathcal{R} U) d\theta \\ &\overset{i.b.p.}{=} 0, \end{aligned}$$

show that the integral of f is conserved, which is why normally equation 2.1 is solved together with a normalization condition $\int_\theta f(\theta) d\theta = 1$, where f is a probability distribution function. The Doi-Onsager model is sometimes also called Smoluchowski equation or non-linear Fokker-Planck equation.

2.1.4 Results for different potentials

1. The Onsager potential

   This potential is the most accurate one in terms of physics. For any positive integer k, the two-dimensional Doi-Onsager model with Onsager potential has at least one $\frac{\pi}{k}$ periodic non-constant stationary solution. It has been shown in [8] the eigenvalues and eigenvectors of the system can be completely characterized for a class of spherically symmetric kernels. The article also investigates the bifurcation diagram for the steady state problem, where the eigenvalues correspond to the bifurcation points. Furthermore Vollmer shows that the isotropic solution is the unique minimizer of the energy functional for low concentrations.

For the three-dimensional Onsager potential (and other potentials as well in 2D or 3D), it is not known if all equilibrium solutions exhibit axial symmetry.

2. The Maier-Saupe potential

> The Maier-Saupe theory of liquid crystals is perhaps the most successful microscopic theory proposed so far to explain the condensation of the nematic phase. Maier Saupe (1958)

All stationary solutions are axial-symmetric and there are estimates for the sharp characterization of the bifurcation regimes for the isotropic and nematic solutions, see also [9].

If the second moment of the Maier-Saupe potential is known, the whole potential is completely determined, while the Onsager potential depends on the whole orientation probability density function. In particular, the second moment of the Maier-Saupe potential specifies an equilibrium state completely, see also [10].

Two-dimensional vs. three-dimensional results: The authors of [9] were able to determine the structure of the stationary solutions for the Doi-Onsager model on the circle and on the sphere. It has been shown that the equilibrium solution to the Doi-Onsager equation with the Maier-Saupe potential exhibits axial symmetry in the two- and three-dimensional case.

2.2 The Vicsek model

Another important model for collective motion is the Vicsek[4] model, first introduced in [11]. It is a time-continuous system of stochastic differential equations with a gradient flow. It describes $N \in \mathbb{N}$ self-propelled particles with position $x_i \in \mathbb{R}^d$ and orientation $\omega_i \in \mathbb{S}^{d-1}$, where mostly authors consider dimension $d = 2$ or $d = 3$. We suppose each

[4] Tamas Vicsek

particle moves at constant speed v and tries to align its orientation with the orientation of its neighbours. Within the alignment process, certain mistakes or random changes in orientation happen, which is why a random noise term is also included in the model. The whole model reads

$$
\begin{aligned}
dx_i &= v\omega_i dt, \\
d\omega_i &= P_{\omega_i^\perp} \circ (\nu\bar{\omega}_i dt + \sqrt{2D}dB_t^i), \\
\bar{\omega}_i &= \frac{J_i}{|J_i|}, \quad J_i = \sum_{k=1}^{N} K(|x_i - x_k|)\omega_k,
\end{aligned}
$$

where K is a non-negative 'sensing function', which measures the influence of the agents on each other.

With $\nabla_\omega(\bar{\omega}_k \cdot \omega) = P_{\omega_k^\perp}\bar{\omega}_k$ we get

$$
d\omega_k = \nu P_{\omega_k^\perp} \circ \left(\bar{\omega}_k dt + \sqrt{2D}B_t^k\right),
$$

where $P_{\omega_k^\perp}$ denotes the orthogonal projection onto the normal plane and ensures that the norm of ω_k is 1. $(B_t^k)_{k=1,\dots,N}$ are independent Brownian motions in $\mathbb{R}^d$. The symbol '$\circ$' indicates the Stratonovich convention. The potential for particle k is given by

$$
V(\omega_k) = -\bar{\omega}_k \cdot \omega_k,
$$

which has a minimum at alignment $\omega_k = \bar{\omega}_k$, i.e. ω_k relaxes to $\bar{\omega}_k$ over time. The corresponding kinetic equation for the particle density $f = f(t, x, \omega)$ reads

$$
\begin{aligned}
&\partial_t f + \omega \cdot \nabla_x f + \nu\nabla_\omega \cdot (P_{\omega^\perp}\Omega_f f) = D\Delta_\omega f, \\
&\Omega_f = \frac{J_f}{|J_f|} \qquad J_f(t,x) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} K(|x-y|)\omega f(t,y,\omega)d\omega.
\end{aligned}
$$

With the Gibbs measure

$$
M_{\Omega_f} = \frac{e^{-V/D}}{Z} = \frac{e^{\nu(\Omega_f\cdot\omega)/D}}{Z},
$$

where Z is given by

$$
Z = \int_{\mathbb{R}^d} e^{\nu(\Omega_f\cdot\omega)/D}d\omega,
$$

one can rewrite the kinetic equation

$$\partial_t f + \omega \cdot \nabla_x f = D\nabla_\omega \cdot \left[M_{\Omega_f} \nabla_\omega \left(\frac{f}{M_{\Omega_f}} \right) \right],$$

where ∇_ω is the gradient on the sphere.

The Vicsek model exhibits a phase transition in which an unstructured movement of the agents becomes an ordered movement on the macroscopic scale if the particle density is sufficiently high.

2.3 The Cucker-Smale model

The Cucker-Smale[5] was first introduced in 2007, based on the preliminary version [12]. The authors have gotten their inspiration to derive this model mainly from the Vicsek model. Once formulated, further studies focused on the asymptotics, collision avoidance and other variants of the model, but also on kinetic and hydrodynamic limits for the individuals coupled with some classical models for hydrodynamics.

The Cucker-Smale model is for instance used to describe the dynamics of bird swarms. It is formulated in discrete and continuous time.

Let $i = 1, \dots, N$ be the number of individuals and x_i and v_i the position and velocity of individual i respectively. The model is then given by the following set of equations

$$\dot{x}_i(t) = v_i(t), \tag{2.2}$$

$$\dot{v}_i(t) = \frac{K}{N} \sum_{j=1}^{N} \psi(|x_i(t) - x_j(t)|)(v_j(t) - v_i(t)) \tag{2.3}$$

with initial condition $(x_i(0), v_i(0)) = (x_{i0}, v_{i0})$.
K denotes the coupling strength, i.e. how strong the interaction force between two agents is, and the so-called communication rate $\psi_{ij}(t) : \mathbb{R}^+ \to \mathbb{R}^+$ measures the interaction between the individuals and is usually assumed to be Lipschitz continuous and non-increasing.

A posible choice for the strengh ψ is $\frac{1}{(\sigma^2 + |x_i(t) - x_j(t)|^2)^\beta}$. where β is a constant that describes the rate of decay of the influence between the agents in the underlying space, see also [13].

[5] Felipe Cucker and Steve Smale

In their paper, the main goal of Cucker and Smale was to prove that under a certain communication rate between the agents, flocking happens. This means that the distance between the individuals remains constant and they move in the same direction: The population $(x_i(t), v_i(t))_{i=1}^N$ flocks if

$$\sup_{t\geq 0} |x_i(t) - x_j(t)| \leq \infty, \qquad \lim_{t\to\infty} |v_i(t) - v_j(t)| = 0 \qquad \forall i \geq 1, N \geq j$$

or equivalently if the mean velocity and center of mass are used:

$$\sup_{t\geq 0} |x_i(t) - \bar{x}(t)| \leq \infty, \qquad \lim_{t\to\infty} |v_i(t) - \bar{v}(t)| = 0 \qquad \forall 1 \leq i \leq N.$$

Suppose $(x_i(t), v_i(t))_{1\leq i\leq N}$ is the solution of 2.2 with initial condition $(x_i^0, v_i^0)_{1\leq i\leq N}$. Cucker and Smale managed to show that for the ψ from above, there is initial data that leads to either unconditional flocking for $\beta \in [0, \frac{1}{2}]$, meaning that there are positive x_a and a_b such that for all $t \geq 0$, $1 \leq i \leq N$

$$x_a \leq |x_i(t) - (\bar{t})| \leq x_A, \qquad |v_i(t) - \bar{v}| \leq |v_i^0 - \bar{v}| \exp\{-\psi_A t\}$$

or conditional flocking for $\beta > \frac{1}{2}$, which indicates that the conclusion from above still holds if $(x_i^0, v_i^0)_{1\leq i\leq N}$ satisfy

$$(1 + 2N\|x^0 - \bar{x}^0\|)^{\frac{1-2\beta}{2}} > \frac{3N(2N)^{\frac{3}{2}}}{K}\|v^0 - \bar{v}^0\| \left[\frac{1}{2\beta}^{\frac{1}{2\beta-1}} - \frac{1}{2\beta}^{\frac{1-2\beta}{2}} \right].$$

Then, all the individuals have the same velocity.

3 Collective motion in a place filled with obstacles

For a start, this section gives a particle model for the dynamics of two species, being referred to as swimmers and obstacles under the assumptions that swimmers and obstacles are of the same size. These swimmers and obstacles live in a fluid, which is assumed to be very viscous. One can think of sperm and fibers moving within seminal fluid.

We start with the dynamics of swimmers and obstacles separately and then the models are coupled and we try to obtain kinetic and macroscopic equations for the behaviour of the two species in the following.

3.1 Dynamics of the swimmers

For the particle dynamics of the swimmers, the Vicsek model provides the basis. Let the tuple $(x_i, \omega_i)_{i=1}^N$ be the position and orientation of swimmer i at time t, where $x_i \in \mathbb{R}^2$ and $\omega_i \in \mathbb{S}^1$, the one-dimensional sphere.

We assume that each swimmer moves at constant speed $v_0 > 0$ and tries to align its orientation with its neighbours with respect to a certain mean orientation $\bar{\omega}_i$, further specified by an alignment intensity $\nu > 0$. We model the alignment for swimmers as polar alignment, where the classical potential is $\omega \cdot \bar{\omega}$.

The Stochastic Differential Equation (SDE) Vicsek model for the individual behaviour of the swimmers on the micro level looks as follows

$$dx_i = v_0 \omega_i dt \quad v_0 > 0, \tag{3.1}$$

$$d\omega_i = P_{\omega_i^\perp} \circ \left(\nu \bar{\omega}_i dt + \sqrt{2D} dB_t^i \right) \qquad \nu > 0,\ D > 0. \tag{3.2}$$

The first equation describes how the position of one swimmer changes. The second equation describes the change of its orientation, which is influenced by two causes: alignment and noise. The mean orientation of swimmer i is given by the flux, which is normalized to ensure that we stay on the sphere

$$\bar{\omega_i} = \frac{J_i}{|J_i|}, \tag{3.3}$$

$$J_i = \frac{1}{N}\sum_{j=1}^{N} K(|x_i - x_j|)\omega_j. \tag{3.4}$$

The noise term is modelled via independent Brownian motion $(B_t^i)_{i=1}^N$. The symbol '$\circ$' denotes that the SDE has to be understood in the sense of Stratonovich. This is convenient, since we deal with SDE's on a manifold[6]. The projection operator $P_{\omega_i^\perp}$ on the orthogonal of ω_i guarantees that the norm of ω_i is set to 1 for all times, i.e. we do not leave the sphere.

For further analysis, we rewrite the second equation in the SDE model using the following Lemma.

Lemma 1. *Let $u, \omega \in \mathbb{S}^{d-1}$. It holds that*

$$P_{\omega^\perp}(u) = \nabla_\omega(u \cdot \omega).$$

Proof. To compute the gradient on the sphere, we first take $u \in \mathbb{S}^{d-1}$ fixed and independent of ω. We then take a path $\omega = \omega(\epsilon) \in \mathbb{S}^{d-1}$ such that $\omega(0) = \omega$ and $\frac{d}{d\epsilon}\omega(\epsilon)|_{\epsilon=0} = \delta_\omega$. Then

$$\begin{aligned}\nabla_\omega(u\cdot\omega)\delta_\omega &= \lim_{\epsilon\to 0}\frac{[u\cdot\omega(\epsilon)] - u\cdot\omega}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{u\cdot\omega + \epsilon\cdot\omega\cdot\delta_\omega + \mathcal{O}(\epsilon^2)}{\epsilon}\\ &= u\delta_\omega,\end{aligned}$$

where we have used the Taylor expansion $\omega(\epsilon) = \omega + \epsilon\delta_\omega + \mathcal{O}(\epsilon^2)$ in the second line. It follows that

$$\begin{aligned}\nabla_\omega(u\cdot\omega)\cdot\delta_\omega &= u\delta_\omega\\ \Rightarrow P_{T_\omega}(\nabla_\omega(u\cdot\omega)) &= P_{T_\omega}(u)\end{aligned}$$

and we obtain

$$\nabla_\omega(u\cdot\omega) = P_{T_\omega}(u)$$

which finishes the proof, since $\nabla_\omega(u\cdot\omega) \in T_\omega = \text{span}\{\omega^\perp\}$. □

[6]For a detailed formulation, see [14], especially section 1.2 and theorem 1.2.4

Now, the second equation in the SDE model can be written as

$$d\omega_i = \nu \nabla_{\omega_i} (\omega_i \cdot \bar{\omega}_i) dt + P_{\omega_i^\perp} \circ (\sqrt{2D} dB_t^i) \tag{3.5}$$

The first part on the right-hand side of the equation corresponds to alignment and is a gradient flow, the second part to noise. For $u \in \mathbb{S}^{d-1}$ fixed, we have

$$\frac{d\omega}{dt} = \nu \nabla_\omega (\omega \cdot u).$$

Given any initial point $\omega(0) = \omega_0 \in \mathbb{S}^{d-1}$, we try to maximize $V(\omega) := \omega \cdot u$ as fast as possible. We get

$$\arg\max_\omega V(\omega) = u.$$

This means that the swimmers want to relax towards the mean orientation $\bar{\omega}$ of its neighbours over time, where ν denotes the strength of relaxation.

3.1.1 Kinetic equation for swimmers

A rigorous derivation of the kinetic equation of the Vicsek model can be found in [15]. Let $f = f(t, x, \omega)$ be the (probability) distribution function of the swimmers where $f(t, x\omega) dx d\omega$ corresponds to the probability that we find a swimmer in $dx d\omega$ when we know the underlying dynamics of all swimmers. The kinetic equation for the swimmers is given by

$$\partial_t f + v_0 \omega \cdot \nabla_x f = \nabla_\omega \cdot \left[\nu (P_{\omega^\perp} \bar{\omega}_f) f + D \nabla_\omega f \right], \tag{3.6}$$

where ∇_ω denotes the divergence on the sphere. We write $M_{\bar{\omega}_f}$ in spherical coordinates while introducing a reference frame with $e_3 = \bar{\omega}_f$ and have

$$M_{\bar{\omega}_f}(\omega(\theta, \phi)) = \frac{1}{Z} \exp\left\{ \frac{1}{D} \cos\theta \right\}.$$

We then have

$$\begin{aligned}\nabla_\omega\left(\ln M_{\bar\omega_f}\right) &= \nabla_\omega(\ln\frac{1}{Z}\exp\left\{\frac{1}{D}\cos\theta\right\}\\ &= \frac{1}{D}\nabla_\omega(\cos\theta)\\ &= -\frac{1}{D}\nu(\cos\theta)\sin\theta e_\theta\\ &= \frac{1}{D}P_{\omega^\perp}\bar\omega_f.\end{aligned}$$

Within these computation, we have introduced the basis associated with the spherical coordinate system

$$(e_\theta, e_\phi) = ((\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), (-\sin\phi, \cos\phi, 0)).$$

Since

$$\begin{aligned}D\nabla_\omega\cdot\left[M_{\bar\omega_f}\nabla_\omega\left(\frac{f}{M_{\bar\omega_f}}\right)\right] &= D\nabla_\omega\cdot\left[\nabla_\omega f - f\nabla_\omega(\ln M_{\bar\omega_f})\right]\\ &= D\nabla_\omega f - \nabla_\omega\cdot(P_{\omega^\perp}\bar\omega_f f),\end{aligned}$$

the kinetic equation can be written in the equivalent form

$$\partial_t f + v_0\nabla_x\cdot(\omega f) = D\nabla_\omega\cdot\left[M_{\bar\omega_f}\nabla_\omega\left(\frac{f}{M_{\bar\omega_f}}\right)\right] \tag{3.7}$$

with

$$\begin{aligned}\bar\omega_f &= \frac{J_f}{|J_f|},\\ J_f(t,x) &= \int_{\mathbb{R}^2}\int_{\mathbb{S}} K(|x-z|)\omega f(z,\omega,t)dzd\omega.\end{aligned}$$

The Gibbs measure for the potential $V(\omega) = -\nu(\omega\cdot\bar\omega)$ is given by

$$M_{\bar\omega_f}(\omega) = \frac{\exp\left(\frac{\nu}{D}(\omega\cdot\bar\omega_f)\right)}{Z}$$

with a normalization constant Z such that $\int_{\mathbb{S}} M_{\bar\omega_f}(\omega)d\omega = 1$. Finally, we replace the right-hand side of kinetic equation for the swimmers with

$$\partial_t f + v_0\nabla_x\cdot(\omega f) = \left(\int_{\mathbb{S}^{d-1}} f d\omega\right)M_{\bar\omega_f} - f. \tag{3.8}$$

Now, there is a relaxation term on the right-hand side of the equation instead of the differential operator and both terms converge to the same type of equilibria $M = M(\omega)$.

Lemma 2. *Small note in order to explain the convergence to the same equilibria Let*

$$v = v(t),$$
$$\frac{dv}{dt} = a - v$$

and

$$w = v - a,$$
$$\frac{dw}{dt} = \frac{dv}{dt} = -w.$$

It follows that

$$w(t) = w(0)e^{-t} \qquad \textit{and} \qquad v(t) = a + (v(0) - a)e^{-t}$$

and we have that $\lim_{t\to\infty} v(t) = a$.

3.2 Dynamics of rod-shaped obstacles

Now, we give a model which aims to describe the behaviour of the obstacles over time. Let $(y_j, u_j)_{j=1}^M$ be the position and orientation of obstacle j at time t. Firstly, we assume that the center, denoted by y_j, of the obstacle does not move, but its orientation changes according to nematic alignment. Moreover we identify obstacle (y_j, u_j) as the same obstacle as $(y_j, -u_j)$ and use the Doi-Onsager model as a basis for the dynamics.

$$dy_j = 0, \tag{3.9}$$
$$du_j = P_{u_j^\perp} \circ \left(\sqrt{2D} dB_t^j\right) + \nabla_{u_j} U dt. \tag{3.10}$$

The first summand on the right-hand side of the second equation corresponds to Brownian noise, the second summand is a gradient flow, which describes the alignment.

For now, we assume that we have already found a method to compute the nematic mean $\bar{u}_j$ for each j. Then, we want to take U the maximal values whenever

$$u_j = \bar{u}_j \qquad \text{and} \qquad u_j = -\bar{u}_j.$$

The classical potential used for nematic alignment in the direction $\bar{u}$ is

$$U(u) = (u \cdot \bar{u})^2.$$

Remark 2. *Explanation of the classical potential for nematic alignment*
At the particle level, one has

$$Q = \sum_{j=1}^{M} u_j \otimes u_j.$$

Let v *be the leading eigenvector of* Q *with norm* 1 *and let* λ_1, λ_2 *be positive eigenvalues with* $\lambda_1 > \lambda_2$*. Then we have*

$$Qv = \lambda_1 v,$$
$$v^T Q v = \lambda_1 v^T v = \lambda_1 \|v\| = \lambda_1.$$

Since Q *is symmetric, it is diagonizable and we can write* $Q = R^T D R$ *with diagonal matrix* $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$*. We also have*

$$\max_{\omega \in \mathbb{S}} \{w^T Q w\} = \lambda_1$$

because

$$w^T Q w = w^T R^T D R w \overset{y=Rw}{=} y^T D y \overset{y_1^2+y_2^2=1}{=} \lambda_1 y_1^2 + \lambda_2 y_2^2$$

which is maximal when $y = (1, 0)$*. We have* $Qw = \lambda_1 w$ *and* $\arg\max\{w^T Q w\} = v$*. With* $w^T(u \otimes u)w = w^T(uu^T)w = w^T u(u \cdot w) = (u \cdot w)^2$ *we finally obtain*

$$w^T Q w = w^T \left(\sum_{j=1}^{M} u_j \otimes u_j \right) w = \sum_{j=1}^{M} (w \cdot u_j)^2.$$

3.2.1 Kinetic equation for rod-obstacles

Let $g = g(t, x, u)$ be the distribution of the obstacles. Again, we obtain the kinetic equation for the obstacles and get

$$\partial_t g = d \, \nabla_u \cdot \left[\bar{M}_{u_g} \nabla_u \left(\frac{g}{\bar{M}_{u_g}} \right) \right] \tag{3.11}$$

with the Gibbs measure

$$\bar{M}_{\bar{u}_g} = \frac{\exp\left(\frac{(u \cdot \bar{u}_g)^2}{d} \right)}{\bar{Z}}.$$

The constant $\bar{Z}$ is a normalizing constant. Following the same procedure as before, we consider the 'simplified' equation with a relaxation part on the right-hand side of the equation

$$\partial_t g = \left(\int g du \right) \bar{M}_{\bar{u}_g} - g.$$

The open question of the definition of the nematic mean $\bar{u}_g$ is discussed in the next chapter.

3.3 Coupling effects on swimmers and obstacles

The models for the swimmers and obstacles from the previous two subsections are combined in the next chapter. Different coupling approaches can be considered and investigated. Here we list a few options on how swimmers can influence obstacles and vice versa.

3.3.1 Swimmers impact on obstacles

- The position of an obstacle y_j can be affected in its position in various ways:
  - Repulsion by volume exclusion at the particle level
  - We can assume that swimmers create diffusion in the obstacle. $\Delta_x g$ with diffusion constant which depends on the concentration of swimmers.
- The orientation u_j can be affected
  - Obstacles could adopt the mean orientation of the swimmers.
  - Swimmers could have no influence at all.
  - There could be 'weighted influence', meaning that rods align not only with rods but also with swimmers, where the alignment with swimmers is weaker compared to the alignment with obstacles.

3.3.2 Obstacles effects on swimmers

- Change in position x_i
  - No influence
  - Repulsion by volume exclusion
- Change in orientation ω_i

- analogous to the changes of orientation in obstacles

- Alignment, which affects the orientation of both, obstacles and swimmers.

- Collisions affect the position of swimmers and obstacles respectively. First, consider a model, where only the position of swimmers changes and obstacles have a fixed position, i.e. center, which does not move in space.

- Non-symmetric interaction between particles and obstacles.

3.4 The coupling

We assume the coupling only has influence on the swimmers orientation ω, who want to align with other swimmers and also with obstacles. The position of obstacle y_j is not affected by the swimmers and the position x_i of a swimmer is also not affected by the obstacles. The orientation of the obstacles is also not affected by the swimmers. In other words: The obstacles lead the swimmers by influencing their orientation.

3.4.1 Particle level

At the particle level, we consider

$$d\omega_i = \nu \nabla_{\omega_i} V_{\Omega_i} dt + P_{\omega_i^\perp} \circ \left(\sqrt{2D} dB_t^i\right) + \nabla_{\omega_i} W_{\bar{u}(x_i)}. \tag{3.12}$$

Under the assumption that we are able to compute the nematic mean orientation $\bar{u}(x_i)$, we consider the classical potential for nematic alignment

$$W_u(\omega) = (\omega \cdot \bar{u})^2, \tag{3.13}$$

which we want to have a maximum at

$$\begin{cases} \bar{u} & \text{if } \omega \cdot \bar{u} > 0 \\ -\bar{u} & \text{if } \omega \cdot \bar{u} < 0 \end{cases}$$

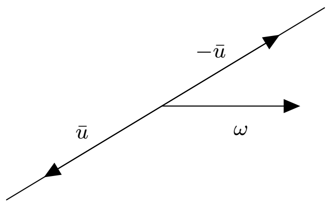

Figure 1: Nematic mean orientation

Remark 3. *The potential from above is the preferred choice, but we want to point to other potentials as well, e.g to the following potential, which is symmetric in* ω *and given by*

$$\tilde{W}(\omega) = (\omega \cdot u)\mathbb{1}_{\omega \cdot u > 0} - (\omega \cdot u)\mathbb{1}_{\omega \cdot u < 0}. \tag{3.14}$$

At the next stage we examine the Gibbs measure

$$\mathcal{Q} = \frac{e^{(\nu V_{\Omega_f} + W_{\bar{u}_g})/D}}{Z}.$$

It is an invariant stationary distribution normalized by $Z = \int_{\mathbb{R}^d} e^{(\nu V_{\Omega_f} + W_{\bar{u}_g})/D}$.

Definition 1. *Gibbs*[7] *measures are a huge class of distribution functions used for the analysis of data. Originally developed to do research in statistical mechanics, it has now found applications in machine learning, neuroscience, social network modeling and other areas of statistics. From [16], we learn that in mathematical terms, the Gibbs measure is an idealization of an equilibrium state of a physical system consisting of a large number of particles interacting with each other. It is a probability distribution of an underlying stochastic process parametrized by sites of a spatial lattice instead of time. Additionally it admits prescribed versions of conditional distributions w.r.t. configurations outside of finite regions. Further information can also be found in [17].*

3.4.2 Kinetic level

The next step in the derivation of the macroscopic model is to find a set of kinetic equations $f = f(t, x, \omega)$ and $g = g(t, y, u)$ for our particle model. They can be seen an intermediate stop between the microscopic and macroscopic model. The kinetic equations for the swimmers and obstacles read

$$\partial_t g = \left(\int_{\mathbb{S}^{d-1}} g \, du \right) \bar{M}_{\bar{u}_g} - g, \tag{3.15}$$

$$\partial_t f + v_0 \omega \cdot \nabla_x f = \left(\int_{\mathbb{S}^{d-1}} f d\omega \right) \mathcal{M}_{\Omega_f, \bar{u}_g} - f. \tag{3.16}$$

The first equation describes the distribution of obstacles. The second one the distribution of swimmers, where $\mathcal{M}_{\Omega_f, \bar{u}_g}$ expresses the alignment with obstacles and swimmers, $x_i \in \mathbb{R}^2, \omega \in \mathbb{S}$.

[7] Josiah Willard Gibbs (1839 - 1903)

3.5 Fixed distribution of obstacles - First part

We assume the obstacles have a fixed distribution. Then, the model reads

$$\partial_t g = 0, \tag{3.17}$$

$$\partial_t f + \omega \nabla_x f = \left(\int_{\mathbb{S}^{d-1}} f d\omega \right) \mathcal{M}_{\Omega_f, \bar{u}_g} - f. \tag{3.18}$$

The alignment is described by

$$\mathcal{M}_{\Omega_f, \bar{u}_g}(\omega) = e^{[(\omega \cdot \Omega) + (\omega \cdot \Omega)^2]/D}.$$

3.5.1 Macroscopic equations

Now, we derive a macroscopic model, which describes the evolution of statistics of the distribution functions f and g in the coupled system 3.17. Usually kinetic equations are easier to analyze since we do not have to deal with an equation per swimmer anymore but instead have an equation for the system, but contain less information than the particle model. In this particular scenario, the macroscopic equation for $\bar{u}_g$ is given by the initial value of g, so we start with obtaining the macroscopic equations for the particle density of the swimmers $\rho = \rho(t, x)$ and afterwards we derive the mean orientation $\Omega = \Omega(t, x)$ of the swimmers.

3.5.2 Rescaling

We scale time and space in 3.17 with the temporal and spatial macroscopic variables

$$t' = \epsilon t \quad x' = \epsilon x,$$

where $0 < \epsilon \ll 1$ is the scale parameter, since we are interested in large time and large space scales. With the rescaling we aim to zoom out of the space and speed up time. With $f^\epsilon(t', x'\omega) = f(t, x, \omega)$, the kinetic equation for the swimmers becomes

$$\epsilon \left(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon \right) = \left(\int_{\mathbb{S}^{d-1}} f^\epsilon d\omega \right) \mathcal{M}^\epsilon_{\Omega_f, \bar{u}_g} - f^\epsilon. \tag{3.19}$$

Since the left-hand side of 3.19 vanishes in the limit $\epsilon \to 0$, it indicates that the solution of the rescaled system f^ϵ should converge to an element in the kernel of the righ-hand side.

Proposition 1. *Considering the rescaled system 3.19 with the formal macroscopic limiting process. If $\epsilon \to 0$, it formally holds that for the total mass $\rho = \rho(t,x) \geq 0$ of f^0 and the director of the flux $\Omega(t,x)$ on the sphere*

$$f^\epsilon \to f^0 = \rho M_{\bar{u},\Omega},$$
$$\rho(t,x) = \int_{\omega \in \mathbb{S}^{d-1}} f^0(x,\omega,t) d\omega.$$

3.5.3 Consistency relation

To guarantee that the derivation of the macroscopic equations for the mean orientation and density work, we must make sure that the the following consistency relation, which gives a relation between the mean orientation of the swimmers and obstacles, holds

$$\Omega_f = \frac{\int \rho \omega f d\omega}{\left|\int \rho \omega f d\omega\right|}.$$

We keep in mind that $\int_{\mathbb{S}^{d-1}} \omega \bar{M}_{\bar{u}} d\omega = \int_{\mathbb{S}^{d-1}} \omega e^{(\omega\cdot\bar{u})^2} d\omega = 0$. The density function ρ cancels, since it does not depend on ω. We use the substitution

$$\begin{aligned}\omega &= P_\Omega(\omega) + P_{\Omega^\perp}(\omega)\\ &= (\omega\cdot\Omega)\Omega + (\omega\cdot\Omega^\perp)\Omega^\perp\\ &= \cos\theta\Omega + \sin\theta\Omega^\perp \quad \text{for } \theta \in [-\pi,\pi]\end{aligned}$$

for $\theta \in [-\pi,\pi]$. Then, we have

$$\int_{\mathbb{S}^{d-1}} \omega \mathcal{M}_{u,\Omega}(\omega) d\omega = \int_{\mathbb{S}^{d-1}} \omega e^{[(\omega\cdot\Omega)+\tilde{W}_u]/D} d\omega,$$

which is equivalent to

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \omega e^{[(\omega\cdot\Omega)+(\omega\cdot\Omega)^2]/D} d\omega\\ &= \int_{-\pi}^{\pi} (\cos\theta\Omega + \sin\theta\Omega^\perp) e^{[(\cos\theta+\tilde{W}_u(\omega)(\theta)]/D} d\theta\\ &= \left(\int_{-\pi}^{\pi} \cos\theta e^{[(\cos\theta+\tilde{W}_u(\omega)(\theta)]/D} d\theta\right)\Omega + \left(\int_{-\pi}^{\pi} \sin\theta e^{[(\cos\theta+\tilde{W}_u(\omega)(\theta)]/D} d\theta\right)\Omega^\perp\end{aligned}$$

The first summand does not cause any problems since it is multiplied by Ω, but we need the second summand to be zero, which imposes a condition in u. After substituting, we must solve

$$I(\phi) = \int_{-\pi}^{\pi} \sin\theta e^{\cos\theta} e^{\cos^2(\theta-\phi)} d\theta = 0,$$

where ϕ denotes the angle between Ω and u. The integral is 2π-periodic in θ and π-periodic in ϕ. Numerically, we find that the second integral is only zero if $\phi = \{0, \pm\frac{\pi}{2}, \pm\pi\}$, where $\pm\frac{\pi}{2}$ and $\pm\pi$ refer to the same obstacle orientation respectively. See also the appendix.

Remark 4. *In the limit $\epsilon \to 0$ we have*

$$(f^\epsilon, \Omega_{f^\epsilon}, u_{g^\epsilon}) \to (\rho\mathcal{M}_{u,\Omega}, \Omega, u)$$

and

$$\Omega_{f^\epsilon} = \frac{\int \omega f^\epsilon d\omega}{|\int \omega f^\epsilon d\omega|} \qquad \Omega = \frac{\int \omega \mathcal{M}_{\Omega,u} d\omega}{|\int \omega \mathcal{M}_{\Omega,u} d\omega|}.$$

The equation on the right-hand side indicates the relation between Ω and u. In our scenario the consistency relation is only true if $\Omega \perp u$ or $\Omega \parallel u$. Otherwise, the expression $f = \rho\mathcal{M}_{\Omega,u}$ does not make any sense. To obtain interesting dynamics, we will assume that $u = u(t,x)$ is not given. We will compute an equation for Ω and then, by the compatibility condition, one must have that u is either such that $\Omega \perp u$ or $\Omega \parallel u$. For this system, intuitively, one expects that $\Omega \parallel u$ would be the right solution but the stability of this solution is left for future work.

3.5.4 The particle density ρ

When we compute the formal limit $\epsilon \to 0$ and f^ϵ converges to f^0, then f^0 must be in the kernel of the right-hand side of 3.19. In particular, we have for any f belonging to the kernel that

$$\int_{\mathbb{S}^{d-1}} f d\omega = \rho.$$

In order to find the evolution for the density $\rho = \rho(t,x)$, the rescaled kinetic equation is integrated over $\mathbb{S}^{d-1}$ with respect to ω and divided by ϵ - analosoulsy to the procedure done in [18]:

$$\epsilon\left(\int_{\mathbb{S}^{d-1}}\partial_t f^\epsilon d\omega+\int_{\mathbb{S}^{d-1}}\omega\cdot\nabla_x f^\epsilon d\omega\right)=\int_{\mathbb{S}^{d-1}}\left[\left(\int_{\mathbb{S}^{d-1}}f^\epsilon d\omega\right)\mathcal{M}^\epsilon-f^\epsilon\right]d\omega=0$$
$$\epsilon\left(\partial_t\int_{\mathbb{S}^{d-1}}f^\epsilon d\omega+\nabla_x\cdot\left(\int_{\mathbb{S}^{d-1}}\omega f^\epsilon d\omega\right)\right)=0$$

Because of mass conservation, the right-hand side of the equation gives zero. In the formal limit $\epsilon\to0$ we have

$$\partial_t\int_{\mathbb{S}^{d-1}}\rho(t,x)\mathcal{M}_{\Omega,u}d\omega+\nabla_x\cdot\int_{\mathbb{S}^{d-1}}\omega\rho(t,x)\mathcal{M}_{\Omega,u}d\omega$$

and with the consistency relation

$$\partial_t\rho(t,x)\underbrace{\int_{\mathbb{S}^{d-1}}\mathcal{M}_{\Omega,u}d\omega}_{=1}+\nabla_x\cdot\left(\rho(t,x)\underbrace{\int_{\mathbb{S}^{d-1}}\omega\mathcal{M}_{\Omega,u}d\omega}_{=c_0\Omega}\right)=0.$$

Therefore we get

$$\partial_t\rho+\nabla_x\cdot(c_0\rho\Omega)=0$$

with

$$c_0=\left|\int_{\mathbb{S}^{d-1}}\omega\mathcal{M}_{\Omega,u}(\omega)d\omega\right|.$$

3.5.5 The orientation Ω

To obtain the macroscopic equation for the orientation $\Omega=\Omega(t,x)$ we multiply the rescaled kinetic equation by ω an integrate with respect to ω

$$\epsilon\left(\partial_t\int_{\mathbb{S}^{d-1}}\omega f^\epsilon d\omega+\int_{\mathbb{S}^{d-1}}\omega(\omega\nabla_x f^\epsilon)d\omega\right)=\int_{\mathbb{S}^{d-1}}\omega\left((\int_{\mathbb{S}^{d-1}}f^\epsilon d\omega)\mathcal{M}_{\Omega,u}(\omega)-f\right)d\omega$$

$$\iff\partial_t\int\omega f^\epsilon d\omega+\int_{\mathbb{S}^{d-1}}[(\omega\otimes\omega)\nabla_x]f^\epsilon d\omega=\frac{1}{\epsilon}\int\omega\underbrace{\left((\int_{\mathbb{S}^{d-1}}f^\epsilon d\omega)\mathcal{M}_{\Omega,u}(\omega)-f\right)}_{=:A(f)}d\omega.$$

The right-hand side of the equation does not necessarily give 0 in the limit $\epsilon\to0$, which is why we make use of the concept of General Collision Invariants (GCI). The idea is to relax the condition of convergence to zero on the right hand side of the equation.

The operator A is non-linear in f. According to [2], we want to define an operator $\bar{A}$, which is linear in f.

Take

$$\bar{A}(\Omega_{f^\epsilon}, f) = \mathcal{M}_{\Omega,u}(\omega) \int_{\mathbb{S}^{d-1}} f d\omega - f$$

and

$$\bar{A}^*(\Omega_0, \psi_{\Omega_0}) = \omega \cdot \beta = \int_{\mathbb{S}^{d-1}} \psi_{\Omega_0}(\omega) \mathcal{M}_{\Omega_0,u}(\omega) d\omega - \psi_{\Omega_0}(\omega)$$

for some $\beta \in \Omega_0^\perp$. We now try to find solutions of

$$\int_{\mathbb{S}^{d-1}} \psi_{\Omega_0}(\omega) \mathcal{M}_{\Omega_0,u}(\omega) d\omega - \psi_{\Omega_0}(\omega) = \omega \cdot \beta. \tag{3.20}$$

The concept of the GCI is used to obtain equation for the mean orientation Ω in kinetic theory generally. A formal definition reads

Definition 2. *A function $\psi_{\Omega_{f^\epsilon}}$ is a Generalised Collision Invariant associated with $\Omega_{f^\epsilon} \in \mathbb{S}^{d-1}$ of the operator A if*

$$\int_{\mathbb{S}^{d-1}} \bar{A}(\Omega_{f^\epsilon}, f) \psi_{\Omega_{f^\epsilon}} d\omega = 0$$

for all f such that

$$P_{\Omega_{f^\epsilon}^\perp} \left(\int_{\mathbb{S}^{d-1}} \omega f d\omega \right) = 0.$$

Lemma 3. *If we have that $\psi_{\Omega_{f^\epsilon}}$ is a GCI, then*

$$\frac{1}{\epsilon} \int_{\mathbb{S}^{d-1}} \psi_{\Omega_{f^\epsilon}} A(f^\epsilon) d\omega = 0.$$

Proof. With

$$P_{\Omega_{f^\epsilon}^\perp} \underbrace{\int_{\mathbb{S}^{d-1}} \omega f d\omega}_{= c\Omega_f^\epsilon} = 0$$

we can conclude that we indeed get (by the definition of the GCI)

$$\frac{1}{\epsilon} \int_{\mathbb{S}^{d-1}} \psi_{\Omega_{f^\epsilon}} A(f^\epsilon) d\omega = \frac{1}{\epsilon} \int_{\mathbb{S}^{d-1}} \psi_{\Omega_f^\epsilon}(\omega) \bar{A}(f^\epsilon, \Omega_{f^\epsilon}) d\omega = 0.$$

□

Inspired by Proposition 12.7 in the lecture notes [2] we suppose that

$$\int_{\mathbb{S}^{d-1}} \psi d\omega = 0$$

and substitute $\psi = -\beta \cdot \omega$ in 3.20. Assuming that the consistency relation holds, we get

$$-\beta \cdot \int_{\mathbb{S}^{d-1}} \omega \mathcal{M}_{\Omega_0,u}(\omega) + \beta \cdot \omega = \beta \cdot \omega$$
$$\int_{\mathbb{S}^{d-1}} \omega \mathcal{M}_{\Omega_0,u}(\omega) d\omega = c\Omega_0$$

with $\beta \cdot \Omega_0 = 0, \beta \in \Omega_0^\perp$.

Instead of ω, we now multiply our rescaled kinetic equation 3.19 with the GCI associated with Ω_0 to get zero on the right-hand side and use Lemma 12.1.6. of the same lecture notes.

Let $\psi_{\Omega_{f^\epsilon}} = \beta^\epsilon \cdot \omega$ with $\beta^\epsilon \in \Omega_f^\epsilon$

$$\int_{\mathbb{S}^{d-1}} \psi_{\Omega_{f^\epsilon}} \left(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon\right) d\omega = 0$$

In the limit $\epsilon \to 0$, we use the consistency relation and get for $\beta \in \Omega_0^\perp$

$$\int_{\mathbb{S}^{d-1}} \beta \cdot \omega \left(\partial_t(\rho \mathcal{M}_{\Omega,u}(\omega)) + \omega \cdot \nabla_x(\rho \mathcal{M}_{\Omega,u}(\omega))\right) d\omega = 0,$$

which is equivalent to

$$\beta \cdot \int_{\mathbb{S}^{d-1}} \omega \left(\partial_t(\rho \mathcal{M}_{\Omega,u}(\omega)) + \omega \cdot \nabla_x(\rho \mathcal{M}_{\Omega,u}(\omega))\right) d\omega = 0$$

or

$$P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega \left(\partial_t(\rho \mathcal{M}_{\Omega,u}(\omega)) + \omega \cdot \nabla_x(\rho \mathcal{M}_{\Omega,u}(\omega))\right) d\omega = 0.$$

We compute

$$P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega \partial_t \rho \mathcal{M}_{\Omega,u}(\omega) + \omega \rho \partial_t \mathcal{M}_{\Omega,u}(\omega) + (\omega \cdot \nabla_x \rho) \mathcal{M}_{\Omega,u}(\omega) \omega + \rho \omega (\omega \cdot \nabla_x \mathcal{M}_{\Omega,u}(\omega)) d\omega = 0.$$

1. Now, first we see that

$$P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}}\omega\partial_t\rho\mathcal{M}_{\Omega,u}(\omega)d\omega=\partial_t\rho P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}}\omega\mathcal{M}_{\Omega,u}(\omega)d\omega=\partial_t\rho P_{\Omega^\perp}\Omega=0.$$

Remark 5. *Small reminder:*

$$\mathcal{M}_{\Omega,u}(\omega)=\frac{e^{(\omega\cdot\Omega)+(\omega\cdot u)^2}}{Z}$$

where $u=u(x)$ only depends on space and $\Omega=\Omega(t,x)$ depends on time and space. The partial time derivative of $\mathcal{M}_{\Omega,u}(\omega)$ is then given by

$$\partial_t\mathcal{M}_{\Omega,u}(\omega)=\partial_t\frac{e^{(\omega\cdot\Omega)+(\omega\cdot u)^2}}{Z}=(\omega\cdot\partial_t\Omega)\mathcal{M}_{\Omega,u}(\omega)$$

2. We then compute

$$\begin{aligned}\rho P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}}\omega\partial_t\mathcal{M}_{\Omega,u}(\omega)d\omega&=\rho P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}}\omega(\omega\cdot\partial_t\Omega)\mathcal{M}_{\Omega,u}(\omega)d\omega\\&=\rho\left[P_{\Omega^\perp}\left(\int_{\mathbb{S}^{d-1}}\omega\otimes\omega\mathcal{M}_{\Omega,u}(\omega)d\omega\right)\partial_t\Omega\right].\end{aligned}$$

The integral

$$\int_{\mathbb{S}^{d-1}}\omega\otimes\omega e^{(\omega\cdot\Omega)+(\omega\cdot u)^2}d\omega.$$

is calculated with the substitution $\omega=\cos\theta\Omega+\sin\theta\Omega^\perp$. This leads to the four terms

$$A(u)\Omega\otimes\Omega+B(u)\Omega\otimes\Omega^\perp+C(u)\Omega^\perp\otimes\Omega+D(u)\Omega^\perp\otimes\Omega^\perp,$$

where

$$\begin{aligned}A(u)&=\int\cos^2\theta e^{\cos\theta+(\omega(\theta)\cdot u)^2}d\theta\\B(u)=C(u)&=\int\cos\theta\sin\theta e^{\cos\theta+(\omega(\theta)\cdot u)^2}d\theta\\D(u)&=\int\sin^2\theta e^{\cos\theta+(\omega(\theta)\cdot u)^2}d\theta.\end{aligned}$$

Remark 6. *Note that $\partial\Omega\in\Omega^\perp$ since Ω has norm 1 and*

$$0=\partial|\Omega|^2=\partial(\Omega\cdot\Omega)=2\Omega\cdot\partial\Omega.$$

The integral A vanishes because it holds that $\partial_t\Omega \perp \Omega$, which gives us

$$A(u)(\Omega \otimes \Omega)\partial_t\Omega = A(u)\Omega \underbrace{(\Omega \cdot \partial_t\Omega)}_{=0} = 0.$$

Both, B and C vanish since

$$B(u)(\Omega \otimes \Omega^\perp)\partial_t\Omega = B(u)\Omega(\Omega^\perp \cdot \partial_t\Omega)$$

and this gives us zero when we project on $\Omega^\perp$:

$$P_{\Omega^\perp}[B(u)(\Omega \otimes \Omega^\perp)\partial_t\Omega] = 0$$

and

$$C(u)(\Omega^\perp \otimes \Omega)\partial_t\Omega = C(u)\Omega^\perp(\Omega \cdot \partial_t\Omega) = 0.$$

For the last one, we obtain

$$D(u) = (\Omega^\perp \otimes \Omega^\perp)\partial_t\Omega = D(u)\Omega^\perp(\Omega^\perp \cdot \partial_t\Omega)$$

and

$$P_{\Omega^\perp}[D(u)(\Omega^\perp \otimes \Omega^\perp)\partial_t\Omega] = D(u)\Omega^\perp(\Omega^\perp \cdot \partial_t\Omega) = D(u)P_{\Omega^\perp}(\partial_t\Omega) = D(u)\partial_t\Omega.$$

So, now we have

$$\rho P_{\Omega^\perp}\left(\omega(\omega \cdot \partial_t\Omega)M_{\Omega,u}(\omega)d\omega\right) = \rho D(u,\Omega)\partial_t\Omega,$$

with D depending on u and Ω

$$D(u,\Omega) = \int_{-\pi}^{\pi} \sin^2\theta e^{\cos\theta} e^{(u\cdot(\cos\theta\Omega + \sin\theta\Omega^\perp))^2} d\theta.$$

3. Now we compute

$$P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega(\omega \cdot \nabla_x\rho)\mathcal{M}_{\Omega,u}(\omega)d\omega.$$

For the i-th component we have

$$\partial_{x_i}(\rho\mathcal{M}_{\Omega,u}(\omega)) = (\partial_{x_i}\rho)\mathcal{M}_{\Omega,u}(\omega) + \rho(\omega \cdot \partial_{x_i}\Omega)\mathcal{M}_{\Omega,u}(\omega)$$

and

$$\omega \cdot \nabla_x(\rho \mathcal{M}_{\Omega,u}(\omega)) = \sum_{i=1}^{N} (\omega_i \partial_{x_i} \rho) \mathcal{M}_{\Omega,u}(\omega) + \rho(\omega \cdot (\omega_i \partial_{x_i} \Omega)) \mathcal{M}_{\Omega,u}(\omega).$$

This gives us

$$P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega(\omega \cdot \nabla_x \rho) \mathcal{M}_{\Omega,u}(\omega) d\omega = P_{\Omega^\perp} \left(\int_{\mathbb{S}^{d-1}} \omega \otimes \omega \mathcal{M}_{\Omega,u}(\omega) d\omega \right) \nabla_x \rho$$

$$\begin{aligned}
&= \ P_{\Omega^\perp}(A(\omega)\Omega \otimes \Omega + B(u)\Omega \otimes \Omega^\perp + C(u)\Omega^\perp \otimes \Omega + D(u)\Omega^\perp \otimes \Omega^\perp)\nabla_x \rho \\
&= 0 + 0 + \Omega^\perp(\nabla_x \rho \cdot \Omega)C(u) + D(u)\Omega^\perp(\Omega^\perp \cdot \nabla_x \rho) \\
&= D(u=0) P_{\Omega^\perp} \nabla_x \rho \\
&= \left(\int_{-\pi}^{\pi} \sin^2 \theta e^{\cos\theta} d\theta \right) P_{\Omega^\perp} \nabla_x \rho \\
&=: c_2 P_{\Omega^\perp} \nabla_x \rho
\end{aligned}$$

since $C(u=0) = 0$ and $\Omega^\perp(\Omega^\perp \cdot \nabla_x \rho) = P_{\Omega^\perp} \nabla_x \rho$.

4. The last computation missing is the integral

$$P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \rho\omega(\omega \cdot \nabla_x \mathcal{M}_{\Omega,u}(\omega)) d\omega$$

with $\mathcal{M}_{\Omega,u}(\omega) = \frac{e^{\omega\cdot\Omega + (\omega\cdot u)^2}}{Z}$. Note that

$$\partial_{x_i} \mathcal{M}_{\Omega,u}(\omega) = [\omega \cdot \partial_{x_i}\Omega + 2(\omega \cdot u)(\omega \cdot \partial_{x_i} u)] \mathcal{M}_{\Omega,u}(\omega).$$

We first compute the integral

$$\begin{aligned}
&P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega\omega_i \partial_{x_i} M_{\Omega,u}(\omega) d\omega \\
&= P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega\omega_i [\omega \cdot \partial_{x_i}\Omega + 2(\omega \cdot u)(\omega \cdot \partial_{x_i} u)] \mathcal{M}_{\Omega,u}(\omega) d\omega \\
&= \underbrace{P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega\omega_i \omega \cdot \partial_{x_i}\Omega \mathcal{M}_{\Omega,u}(\omega) d\omega}_{=B^{(1)}} + \underbrace{2P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega\omega_i (\omega \cdot u)(\omega \cdot \partial_{x_i} u) \mathcal{M}_{\Omega,u}(\omega) d\omega}_{=B^{(2)}}.
\end{aligned}$$

In the j-th component one has

$$B_j^{(1)} = P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega_j \omega_i (\omega \cdot \partial_{x_i} \Omega) \mathcal{M}_{\Omega,u}(\omega) d\omega,$$
$$B_j^{(2)} = 2P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega_j \omega_i (\omega \cdot u)(\omega \cdot \partial_{x_i} u) \mathcal{M}_{\Omega,u}(\omega) d\omega.$$

We start with computing $B^{(1)}$. With the tensorial notation the integral can be written in the form

$$B^{(1)} = \left[P_{\Omega^\perp} \int_{\mathbb{S}^{d-1}} \omega \otimes \omega \otimes \omega \mathcal{M}_{\Omega,u}(\omega) d\omega : \nabla_x \Omega \right]_{2:3}.$$

In dimension 2, the integral is computed with the usual substitution

$$\omega = \cos\theta \Omega + \sin\theta \Omega^\perp$$
$$d\omega = d\theta,$$

where we get eight different combinations of tensors of Ω and $\Omega^\perp$. The coefficients $\cos^3\theta$ and $\cos^2\theta \sin\theta$ with Ω in the first component give 0 when we project on $\Omega^\perp$. The terms with $\Omega^\perp$ in the first component lead to the integrals[8]

$$P_{\Omega^\perp} \left[\int_{-\pi}^{\pi} \cos^2\theta \sin\theta M_\theta d\theta \ \Omega^\perp \otimes \Omega \otimes \Omega : \nabla_x \Omega \right]_{2:3}$$
$$P_{\Omega^\perp} \left[\int_{-\pi}^{\pi} \sin^3\theta e^{\cos\theta} d\theta \ \Omega^\perp \otimes \Omega^\perp \otimes \Omega^\perp : \nabla_x \Omega \right]_{2:3}$$
$$P_{\Omega^\perp} \left[\int_{-\pi}^{\pi} \sin^2\theta \cos\theta M_\theta d\theta \ \Omega^\perp \otimes \Omega^\perp \otimes \Omega : \nabla_x \Omega \right]_{2:3}$$
$$P_{\Omega^\perp} \left[\int_{-\pi}^{\pi} \cos\theta \sin^2\theta M_\theta d\theta \ \Omega^\perp \otimes \Omega \otimes \Omega^\perp : \nabla_x \Omega \right]_{2:3}.$$

The first two integrals vanish since the integrand is odd. Only the last two integrals survive and we obtain

[8] Note that the row of $\nabla_x \Omega$ is $\partial_{x_i} \Omega \in \Omega^\perp$.

$$
\begin{aligned}
&P_{\Omega^\perp}\left[\underbrace{\int_{-\pi}^{\pi}\cos\theta\sin^2\theta M_\theta d\theta}_{=\beta}\left(\Omega^\perp\otimes\Omega\otimes\Omega^\perp+\Omega^\perp\otimes\Omega^\perp\otimes\Omega\right):\nabla_x\Omega\right]_{2:3}\\
&=\beta P_{\Omega^\perp}\left[\ \Omega^\perp\otimes\Omega\otimes\Omega^\perp:\nabla_x\Omega\right]_{2:3}\\
&=\beta P_{\Omega^\perp}\left(\Omega^\perp\underbrace{[\Omega\otimes\Omega^\perp:\nabla_x\Omega]}_{\text{is a scalar}}\right)\\
&=\beta\Omega^\perp\left(\sum_i\Omega_i(\Omega^\perp\cdot\partial_{x_i}\Omega)\right)\\
&=\beta\Omega^\perp\big(\Omega^\perp\cdot\underbrace{(\sum_i\Omega_i\partial_{x_i}\Omega)}_{=(\Omega\cdot\nabla_x)\Omega)}\big)\\
&=\beta\Omega^\perp\left(\Omega^\perp\cdot[(\Omega\cdot\partial_x)\Omega]\right)=\beta P_{\Omega^\perp}\left(\Omega\cdot\nabla_x)\Omega\right)=\beta(\Omega\cdot\nabla_x)\Omega
\end{aligned}
$$

where we have used $(\Omega\cdot\nabla_x)\Omega\in\Omega^\perp$ in the last equation.

So

$$
B^{(1)}=\left(\int_{-\pi}^{\pi}\cos\theta\sin^2\theta M_\theta d\theta\right)(\Omega\cdot\nabla_x)\Omega=\beta[(\Omega\cdot\nabla_x)\Omega]
$$

For the j-th component of $B^{(2)}$ we have after using the same substitution as before

$$
\begin{aligned}
B_j^{(2)}&=P_{\Omega\perp}\int_{\mathbb{S}^{d-1}}\omega(\omega\cdot[(\omega\cdot\nabla_x)u])(\omega\cdot u)\mathcal{M}_{\Omega,u}(\omega)d\omega\\
&=\Omega^\perp\int_{-\pi}^{\pi}\sin\theta\left((\cos\theta\Omega+\sin\theta\Omega^\perp)\cdot\left[[(\cos\theta\Omega+\sin\theta\Omega^\perp)\cdot\nabla_x]u\right]\right)\\
&\left[\cos\theta(\Omega\cdot u)+\sin\theta(\Omega^\perp\cdot u)\right]M_\theta d\theta.
\end{aligned}
$$

Recall that $[v\cdot(v\cdot\nabla_x)u]=v\otimes v:\nabla_x u$ for any vector v. Since $|\Omega|=1$, we obtain the following, if u equals $\pm\Omega$

$$
=\Omega^\perp\left[\int_{-\pi}^{\pi}\sin\theta\cos\theta M_\theta(\pm1)d\theta\right]\left[(\cos\theta\Omega+\sin\theta\Omega^\perp)\otimes(\cos\theta\Omega+\sin\theta\Omega^\perp)d\theta\right]:\nabla_x u.
$$

The integrand $\sin\theta\cos\theta M_\theta$ is an odd function, so we conclude that we get zero, when multiplying with the tensors $\Omega\otimes\Omega$ and $\Omega^\perp\otimes\Omega^\perp$. The remaining terms are

$$\begin{aligned}
&= \Omega^\perp(\pm1)\underbrace{\left(\int_{-\pi}^{\pi}\sin^2\theta\cos^2\theta M_\theta d\theta\right)}_{=\gamma}\underbrace{\left(\Omega\otimes\Omega^\perp+\Omega^\perp\otimes\Omega\right):\nabla_x u}_{\Omega\cdot[(\Omega^\perp\cdot\nabla_x)u]=0\ \text{ if }\ u=\Omega}\\
&= (\pm1)\gamma\Omega^\perp(\Omega^\perp\cdot[(\Omega\cdot\nabla_x)u])\\
&= (\pm1)\gamma P_{\Omega^\perp}[(\Omega\cdot\nabla_x)u]\\
&= (\pm1)\gamma(\Omega\cdot\nabla_x)u\\
&= \gamma[(\Omega\cdot\nabla_x)\Omega].
\end{aligned}$$

We used $u=\pm\Omega$ in the last equation. When we put all the integrals together, we finally get the equation of the swimmers orientation Ω

$$\rho D(u,\Omega)\partial_t\Omega + D(0,\Omega)P_{\Omega^\perp}\nabla_x\rho + (\beta+\gamma)[(\Omega\cdot\nabla_x)\Omega] = 0$$

with

$$\begin{aligned}
D(u,\Omega) &= \int_{-\pi}^{\pi}\sin^2\theta e^{\cos\theta}e^{(u\cdot(\cos\theta\Omega+\sin\theta\Omega^\perp))^2}d\theta\\
D(0,\Omega) &= \int_{-\pi}^{\pi}\sin^2\theta e^{\cos\theta}d\theta\\
\beta &= \int_{-\pi}^{\pi}\cos\theta\sin^2\theta M_\theta d\theta\\
\gamma &= \int_{-\pi}^{\pi}\sin^2\theta\cos^2\theta M_\theta d\theta.
\end{aligned}$$

3.5.6 Summary and conclusion of the first part

Recapitulatory we get the system

$$\partial_t\rho + \nabla_x\cdot(c_0\rho\Omega) = 0 \tag{3.21}$$

$$c_1\partial_t\Omega + c_2P_{\Omega^\perp}\nabla_x\rho + c_3(\Omega\cdot\nabla_x)\Omega = 0 \tag{3.22}$$

for the swimmers density and orientation, where we have renamed the constants in the following way

$$\begin{aligned} c_0 &:= \left| \int_{\mathbb{S}^{d-1}} \omega \mathcal{M}_{\Omega,u}(\omega) d\omega \right| \\ c_1 &:= D(u, \Omega) \\ c_2 &:= D(0, \Omega) \\ c_3 &:= \beta + \gamma. \end{aligned}$$

Obtaining this system only works, if the consistency relation is true. We have seen that this is only the case if either $\Omega \perp u$ or $\Omega \parallel u$, i.e. if the swimmers follow this particular orientation from the beginning, which is a very strong restriction and therefore makes the model unsuitable. The second issue is that the equation of the mean orientation Ω is independent of u. The next section will resolve this inconvenience by considering another distribution.

3.6 Fixed distribution of obstacles - Second part

We again consider the kinetic model in 3.17:

$$\partial_t g = 0,$$
$$\partial_t f + \omega \cdot \nabla_x f = \left(\int_{\mathbb{S}^{d-1}} f(t,x,\omega) d\omega \right) \mathcal{M}_{\Omega_f, \bar{u}_g} - f.$$

But this time, we take the Gibbs measure adapted to the weighted distribution

$$\mathcal{M}_{\Omega_f, \bar{u}_g} = w_1 M_{\Omega_f} + w_2 \bar{M}_{\bar{u}_g}$$

with $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$, where $d, D > 0$ and the nematic direction $\bar{u}$, which is the leading eigenvector of the operator

$$Q_g = \int_{\mathbb{S}^1} (u \otimes u) g(u) du.$$ [9]

Note that we could have w_1 and w_2 being dependent on time and space: $w_1 = w_1(t,x)$ and $w_2 = w_2(t,x)$. The first term

$$M_{\Omega_f} = \frac{e^{\frac{\nu}{D} \omega \cdot \Omega_f}}{Z}$$

is a von-Mises distribution and $\bar{M}_{\bar{u}_g}$ is given by

$$\bar{M}_{\bar{u}_g} = \frac{e^{(\omega \cdot \bar{u}_g)^2 / d}}{Z}.$$

To prove that $\mathcal{M}_{\Omega_f, \bar{u}_g}$ is indeed a distribution on the sphere, we have to show that

$$\mathcal{M}_{\Omega_f, \bar{u}_g} \geq 0$$
$$\text{and} \int_{\mathbb{S}^{d-1}} \mathcal{M}_{\Omega_f, \bar{u}_g} d\omega = 1.$$

It is easy to see that $\mathcal{M}_{\Omega_f, \bar{u}_g}$ is always non-negative and the second condition is always true since each integral, M_{Ω_f} and $\bar{M}_{\bar{u}_g}$, is normalized and w_1 and w_2 are adding up to 1.

We want to do the same analysis as before and compute macro-equations.

[9] Note that $(u \otimes v)_{ij} = u_i v_j$, so Q_g is a 2×2 matrix.

3.6.1 Rescaling

We scale time and space with $t' = \epsilon t$ and $x' = \epsilon x$. The dynamics of the swimmers and obstacle distribution functions become

$$\partial_t g^\epsilon = 0, \tag{3.23}$$

$$\epsilon \left(\partial_t f^\epsilon + \omega \cdot \nabla_x f^\epsilon \right) = \left(\int_{\mathbb{S}^{d-1}} f^\epsilon d\omega \right) \mathcal{M}^\epsilon - f^\epsilon \quad (=: \mathcal{Q}(f^\epsilon)). \tag{3.24}$$

Formally one obtains

$$f^\epsilon \to f^0 = \rho(t,x) \mathcal{M}_{\Omega,u},$$

when taking the limit $\epsilon \to 0$. $\Omega = \Omega(t,x)$ corresponds to the mean orientation and $u = u(x)$, since u is fixed and therefore independent of time.

3.6.2 Consistency relation

We prove that the following consistency relation now holds for all ω (ρ already cancelled out)

$$\Omega_f = \frac{\int_{\mathbb{S}^{d-1}} \omega M_{\Omega_f}(\omega) d\omega + \int_{\mathbb{S}^{d-1}} \omega \bar{M}_{u_g}(\omega) d\omega}{\left| \int_{\mathbb{S}^{d-1}} \omega M_{\Omega_f}(\omega) d\omega + \int_{\mathbb{S}^{d-1}} \omega M_{u_g}(\omega) d\omega \right|}$$

In dimension 2, we have

$$\int_{\mathbb{S}^{d-1}} \omega M_{\Omega_f} d\omega = c_0 \Omega,$$

where c_0 and the normalization constant Z' are given by

$$c_0 = \int_{-\pi}^{\pi} \cos\theta \frac{e^{\cos\theta/d}}{Z'} d\theta, \qquad Z' = \int_{-\pi}^{\pi} e^{\cos\theta/d} d\theta.$$

Therefore we conclude

$$\int_{\mathbb{S}^{d-1}} \omega \bar{M}_{u_g} d\omega = \int_{\mathbb{S}^{d-1}} \underbrace{\omega e^{(\omega\cdot u)^2/D}}_{\text{odd}} d\omega = 0$$

and it is proven that the compatibility condition holds.

3.6.3 Equation for ρ

To determine the equation for the swimmer density we follow the same path as before, i.e. we integrate the rescaled kinetic equation over $\mathbb{S}^{d-1}$ and divide by ϵ.

$$\partial_t \rho^\epsilon + \nabla_x \cdot \int_{\mathbb{S}^{d-1}} \omega f^\epsilon d\omega = 0.$$

Now, let $\epsilon \to 0$

$$\partial_t \rho + \nabla_x \cdot \int_{\mathbb{S}^{d-1}} \omega \rho \mathcal{M}_{\Omega,u} d\omega = 0$$

Using the definition of $\mathcal{M}$, the integral can be written as

$$\int_{\mathbb{S}^{d-1}} \omega \rho \mathcal{M}_{\Omega,u} d\omega = \rho \left[\int_{\mathbb{S}^{d-1}} \omega w_1 M_\Omega d\omega + \int_{\mathbb{S}^{d-1}} \omega w_2 \bar{M}_u d\omega \right].$$

We make use of the consistency relation in the first summand, in the second integral we notice that ω is odd and $\bar{M}_u$ is even, so the product is even and the integral gives zero. Altogether we have

$$\int_{\mathbb{S}^{d-1}} \omega \rho \mathcal{M}_{\Omega,u} d\omega = \rho c_0 w_1 \Omega(t,x)$$

and therefore the equation for the swimmers density reads

$$\partial_t \rho + \nabla_x \cdot (c_0 w_1 \rho \Omega) = 0. \tag{3.25}$$

3.6.4 Equation for Ω

To obtain the equation for the mean orientation we multiply the rescaled kinetic equation in 3.23 by ω and integrate the expression over $\mathbb{S}^{d-1}$. Then we take the limit $\epsilon \to 0$ and make use of the consistency relation. The issue occuring is again that

$$\int_{\mathbb{S}^{d-1}} \omega \mathcal{Q}(f^\epsilon) \neq 0,$$

since the orientation is not a conserved quantity. So we need the GCI to relax the condition and consider certain functions which will give zero. We can use the same GCI as in the case before and follow the same prodecure

$$\psi_\Omega(\omega) = \beta \cdot \omega$$

with $\beta \in \Omega^\perp$. Now, we multiply the 3.23 by ψ_Ω and integrate with respect to ω and take the limit ϵ to 0:

$$\int_{\mathbb{S}^{d-1}} \left(\psi_{\Omega_{f^\epsilon}} \partial_t f^\epsilon + \psi_{\Omega_{f^\epsilon}} \omega \cdot \nabla_x f^\epsilon \right) d\omega = \int_{\mathbb{S}^{d-1}} \omega \psi_{\Omega_{f^\epsilon}} d\omega = 0.$$

As $\epsilon \to 0$, we have

$$\int_{\mathbb{S}^{d-1}} [(\beta\cdot\omega)\partial_t(\rho\mathcal{M}_{\Omega,u}) + (\beta\cdot\omega)(\omega\cdot\nabla_x)(\rho\mathcal{M}_{\Omega,u})]\, d\omega = 0$$

and for all $\beta \in \Omega^\perp$

$$P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}} [\omega\partial_t(\rho\mathcal{M}_{\Omega,u}) + \omega(\omega\cdot\nabla_x)(\rho\mathcal{M}_{\Omega,u})]\, d\omega = 0$$

Now, we split the integral in two parts.

1. The first term gives us

$$\begin{aligned}
&P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}} \omega\partial_t(\rho\mathcal{M}_{\Omega,u})d\omega \\
&= P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}} \omega\partial_t\rho M_\Omega d\omega + P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}} \omega\rho(w_1\omega\cdot\partial_t\Omega)M_\Omega d\omega \\
&= P_{\Omega^\perp}\left[\rho w_1\int_{\mathbb{S}^{d-1}} \omega(\omega\cdot\partial_t\Omega)M_\Omega d\omega\right] \\
&= P_{\Omega^\perp}\left[\rho w_1\left(\int_{\mathbb{S}^{d-1}} \omega\otimes\omega M_\Omega d\omega\right)\partial_t\Omega\right] \\
&= P_{\Omega^\perp}\left[\rho w_1\left(\int_{-\pi}^{\pi} (\cos\theta\Omega+\sin\theta\Omega^\perp)\otimes(\cos\theta\Omega+\sin\theta\Omega^\perp)e^{\cos\theta/d}d\theta\right)\partial_t\Omega\right] \\
&= P_{\Omega^\perp}\left[\rho w_1\left(\left(\underbrace{\int_{-\pi}^{\pi}\cos^2\theta e^{\cos\theta/d}d\theta}_{=c_1}\right)\Omega\otimes\Omega + \left(\underbrace{\int_{-\pi}^{\pi}\sin^2\theta e^{\cos\theta/d}d\theta}_{=c_2}\right)\Omega^\perp\otimes\Omega^\perp\right)\partial_t\Omega\right] \\
&= P_{\Omega^\perp}\left[\rho w_1(c_1\Omega\otimes\Omega + c_2\Omega^\perp\otimes\Omega^\perp)\partial_t\Omega\right] \\
&= w_1\rho c_2\partial_t\Omega.
\end{aligned}$$

   Note that in the last step, we have used that for all $v\in\mathbb{R}^2$ (so in particular for the identity id)

$$\begin{aligned}
P_{\Omega^\perp}\left[\left(\int_{\mathbb{S}^{d-1}}\omega\otimes\omega M_\Omega d\omega\right)v\right] &= c_1P_{\Omega^\perp}(\Omega(\Omega\cdot v)) + c_2P_{\Omega^\perp}(\Omega^\perp(\Omega^\perp\cdot v)) \\
&= 0 + c_2(\Omega^\perp\cdot v)\Omega^\perp \\
&= c_2P_{\Omega^\perp}v.
\end{aligned}$$

2. The second integral is computed in the following way

$$P_{\Omega^\perp}\int_{\mathbb{S}^{d-1}}\omega(\omega\cdot\nabla_x)(\rho\mathcal{M}_{\Omega,u})d\omega = P_{\Omega^\perp}\left[\int_{\mathbb{S}^{d-1}}\omega[(\omega\cdot\nabla_x)\rho]\mathcal{M}_{\Omega,u}d\omega + \rho\int_{\mathbb{S}^{d-1}}\omega(\omega\cdot\nabla_x)\mathcal{M}_{\Omega,u}d\omega\right]$$

Now we define

$$\mathcal{X}_1 := \int_{\mathbb{S}^{d-1}} \omega[(\omega \cdot \nabla_x)\rho]\mathcal{M}_{\Omega,u} d\omega = \left(\int_{\mathbb{S}^{d-1}} \omega \otimes \omega \mathcal{M}_{\Omega,u} d\omega\right) \nabla_x \rho,$$
$$\mathcal{X}_2 := \rho \int_{\mathbb{S}^{d-1}} \omega(\omega \cdot \nabla_x)\mathcal{M}_{\Omega,u} d\omega$$

and compute them separately.

(a) We start with $\mathcal{X}_1$.

$$\left(\int_{\mathbb{S}^{d-1}} \omega \otimes \omega \mathcal{M}_{\Omega,u} d\omega\right) \nabla_x \rho = \left(w_1 \int_{\mathbb{S}^{d-1}} \omega \otimes \omega M_\Omega d\omega + w_2 \int_{\mathbb{S}^{d-1}} \omega \otimes \omega \bar{M}_u d\omega\right) \nabla_x \rho.$$

When projected on $\Omega^\perp$, the first term of the sum gives us (analogous to the computations before)

$$w_1 \left(\int_{\mathbb{S}^{d-1}} \omega \otimes \omega M_\Omega d\omega\right) \nabla_x \rho = w_1 c_2 \nabla_x \rho.$$

The second part is computed with the substitution

$$\omega = \cos\alpha u + \sin\alpha u^\perp.$$

So, we have

$$\begin{aligned}
&w_2 \int_{\mathbb{S}^{d-1}} \omega \otimes \omega \bar{M}_u d\omega \\
&= w_2 \int_{\mathbb{S}^{d-1}} \omega \otimes \omega \frac{e^{(\omega\cdot u)^2}}{Z} d\omega \\
&= w_2 \int_{-\pi}^{\pi} \left[\cos^2\alpha(u \otimes u) + \sin^2\alpha(u^\perp \otimes u^\perp) + \cos\alpha \sin\alpha(u^\perp \otimes u + u \otimes u^\perp)\right] \underbrace{\frac{e^{\cos^2\alpha/D}}{Z}}_{=:m_\alpha} d\alpha \\
&= A(u \otimes u) + B(u^\perp \otimes u^\perp) + C(u^\perp \otimes u + u \otimes u^\perp),
\end{aligned}$$

where

$$A = \int_{-\pi}^{\pi} \cos^2\alpha \; m_\alpha d\alpha$$
$$B = \int_{-\pi}^{\pi} \sin^2\alpha \; m_\alpha d\alpha$$
$$C = \int_{-\pi}^{\pi} \cos\alpha \sin\alpha \; m_\alpha d\alpha.$$

In the last integral, we have an odd integrand, so $C = 0$.

Putting everything together we find

$$\begin{aligned} P_{\Omega^\perp}\mathcal{X}_1 &= P_{\Omega^\perp}\left[w_1 c_2 + A(u\otimes u) + B(u^\perp \otimes u^\perp)\right]\nabla_x\rho \\ &= P_{\Omega^\perp} w_1 c_2(\nabla_x\rho) + A P_{\Omega^\perp}P_u(\nabla_x\rho) + B P_{\Omega^\perp}P_{u^\perp}(\nabla_x\rho). \end{aligned}$$

The last step follows from

$$\begin{aligned} (u\otimes u)\nabla_x\rho = u(u\cdot\nabla_x\rho) = P_u(\nabla_x\rho) \\ (u^\perp\otimes u^\perp)\nabla_x\rho = u^\perp(u^\perp\cdot\nabla_x\rho) = P_{u^\perp}(\nabla_x\rho) \end{aligned}$$

Remark 7. *The relation between A and B is*

$$A = \int_{-\pi}^{\pi}\cos^2\alpha\ m_\alpha d\alpha = \int_{-\pi}^{\pi}(1-\sin^2\alpha)\ m_\alpha d\alpha = 1 - B$$

since $\int_{-\pi}^{\pi} m_\alpha d\alpha = \int_{\$} M_u d\omega = 1$. So they are not equal, but we observe that $A, B \le 1$ and $A + B = 1$.

(b) Now, we compute $P_{\Omega^\perp}\mathcal{X}_2$. Note that

$$\partial_{x_i}\mathcal{M}_{\Omega,u} = w_1(\omega\cdot\partial_{x_i}\Omega)M_\Omega + 2w_2(\omega\cdot u)(\omega\cdot\partial_{x_i}u)\bar{M}_u$$

and

$$\begin{aligned} (\omega\cdot\nabla_x)\mathcal{M}_{\Omega,u} &= \sum_{i=1}^{N}\omega_i\partial_{x_i}(\mathcal{M}_{\Omega,u}) \\ &= \sum_{i=1}^{N}\omega_i w_1(\omega\cdot\partial_{x_i}\Omega)M_\Omega + \sum_{i=1}^{N} 2\omega_i w_2(\omega\cdot u)(\omega\cdot\partial_{x_i}u)\bar{M}_u \\ &= \sum_{i=1}^{N} w_1(\omega\cdot(\omega_i\partial_{x_i}\Omega))M_\Omega + \sum_{i=1}^{N} 2w_2(\omega\cdot u)(\omega\cdot(\omega_i\partial_{x_i}\omega))\bar{M}_u \\ &= w_1\left(\omega\cdot\left(\sum_{i=1}^{N}\omega_i\partial_{x_i}\Omega\right)\right)M_\Omega + 2w_2(\omega\cdot u)\left(\omega\cdot\left(\sum_{i=1}^{N}\omega_i\partial_{x_i}u\right)\right)\bar{M}_u \\ &= w_1(\omega[(\omega\cdot\nabla_x)\Omega])M_\Omega + 2w_2(\omega\cdot u)(\omega\cdot[(\omega\cdot\nabla_x)u])\bar{M}_u. \end{aligned}$$

We can write $\mathcal{X}_2$ as

$$\mathcal{X}_2 = \rho \int_{\mathbb{S}^{d-1}} w_1 \omega (\omega \cdot [(\omega \cdot \nabla_x)\Omega]) M_\Omega d\omega + 2\rho \int_{\mathbb{S}^{d-1}} w_2 \omega (\omega \cdot u)[\omega \cdot ((\omega \cdot \nabla_x)u)] \bar{M_{\bar{u}}} d\omega$$
$$=: \mathcal{X}_2^{(1)} + \mathcal{X}_2^{(2)}$$

and compute the two integrals separately. The first one is computed analogously to the integrals $B^{(1)}$ and $B^{(2)}$ in part 1:

$$P_{\Omega^\perp} \mathcal{X}_2^{(1)} = \rho P_{\Omega^\perp} \left[\int_{\mathbb{S}^{d-1}} w_1 \omega (\omega \cdot [(\omega \cdot \nabla_x)\Omega]) M_\Omega d\omega \right]$$
$$= \rho w_1 P_{\Omega^\perp} \left[\int_{\mathbb{S}^{d-1}} \omega \otimes \omega \otimes \omega M_\Omega d\omega : \nabla_x \Omega \right]_{2:3}$$
$$= w_1 c_3 \rho (\Omega \cdot \nabla_x)\Omega$$

where the constant c_3 is given by

$$c_3 = \int_{-\pi}^{\pi} \sin^2\theta \cos\theta \frac{e^{\cos\theta/d}}{Z'} d\theta.$$

To compute $\mathcal{X}_2^{(2)}$, we use the same change of variables $\omega = \cos\alpha u + \sin\alpha u^\perp$ as before.

Remark 8. *Reminder Let p, v, u be vectors.*

$$(v \otimes p) : \nabla_x u = p \cdot ((v \cdot \nabla_x)u)$$
$$= p \cdot \left((\sum_i v_i \partial_i) u \right)$$
$$= \sum_j p_j (\sum_i v_i \partial_i u)_j$$
$$= \sum_{j,i} p_j v_i \partial_i u_j$$

$$v(v \otimes p : \nabla_x u) = [(v \otimes v \otimes p) : \nabla_x u]_{2:3}$$

$$\frac{\mathcal{X}_{2^{(2)}}}{2w_2} = \rho \int_{\mathbb{S}^{d-1}} \omega (\omega \cdot u) [\omega ((\omega \cdot \nabla_x)u)] \bar{M_{\bar{u}}} d\omega$$
$$= \rho u \cdot \left[\left[\int_{\mathbb{S}^{d-1}} \omega \otimes \omega \otimes \omega \otimes \omega \bar{M_{\bar{u}}} d\omega \right] : \nabla_x u \right]_{2:3}$$

$$= \rho \int_{-\pi}^{\pi} (\cos\alpha u + \sin\alpha u^{\perp}) \cos\alpha \left[(\cos\alpha u + \sin\alpha u^{\perp}) \cdot \left(\left((\cos\alpha u + \sin\alpha u^{\perp}) \cdot \nabla_x \right) u \right) m_\alpha \right] d\alpha$$

$$= \rho \int_{-\pi}^{\pi} (\cos\alpha u + \sin\alpha u^{\perp}) \cos\alpha \sin\alpha u^{\perp} \cdot \left((\cos\alpha u + \sin\alpha u^{\perp}) \cdot \nabla_x u \right) m_\alpha d\alpha$$

$$= \rho \int_{-\pi}^{\pi} (\cos\alpha u + \sin\alpha u^{\perp}) \cos\alpha \sin\alpha \left((\cos\alpha u + \sin\alpha u^{\perp}) \otimes u^{\perp} \right) : \nabla_x u \; m_\alpha d\alpha$$

With the remark from above we rewrite the integrand

$$= 2\rho w_2 \int_{-\pi}^{\pi} \cos\alpha^2 \sin\alpha u \cdot \left[\cos\alpha (u \otimes u^{\perp} : \nabla_x u) + \sin\alpha (u^{\perp} \otimes u^{\perp} : \nabla_x u) \right]$$
$$+ 2\rho w_2 \int_{-\pi}^{\pi} \sin\alpha^2 \cos\alpha u^{\perp} \cdot \left[\cos\alpha (u \otimes u^{\perp} : \nabla_x u) + \sin\alpha (u^{\perp} \otimes u^{\perp} : \nabla_x u) \right]$$

The terms with odd integrands $\cos\alpha^3 \sin\alpha m_\alpha$ and $\cos\alpha \sin\alpha^3 m_\alpha$ vanish when we integrate and the only summands remaining are

$$= 2\rho w_2 \left(\underbrace{\int_{-\pi}^{\pi} \cos\alpha^2 \sin^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha}_{=:\gamma_1} \right) \left[u (u^{\perp} \otimes u^{\perp} : \nabla_x u) + u^{\perp} (u \otimes u^{\perp} : \nabla_x u) \right]$$

We conclude

$$P_{\Omega^{\perp}} \mathcal{X}_2^{(2)} = 2\rho w_2 \gamma_1 \left((P_{\Omega^{\perp}} u)(u^{\perp} \otimes u^{\perp} : \nabla_x u) + (P_{\Omega^{\perp}} u^{\perp})(u \otimes u^{\perp} : \nabla_x u) \right).$$

Remark 9. *The term $P_{\Omega^{\perp}} \mathcal{X}_2^{(2)}$ describes a relaxation towards a weighted average of u and $u^{\perp}$.*
Recall that

$$\frac{d\Omega(t)}{dt} = P_{\Omega^{\perp}} u = \nabla_{\Omega} (\Omega \cdot u)$$

is a gradient flow, where the gradient is taken on the sphere.

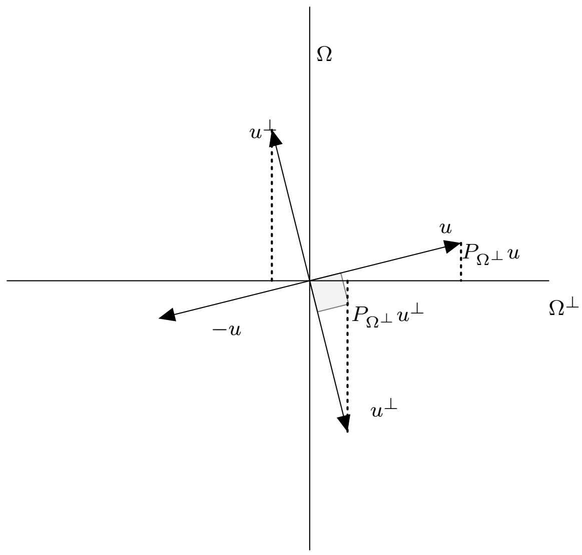

Figure 2: Ω and u

If we collect all our separately computed equations, we obtain the following model for the particle density and mean orientation

$$
\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (w_1 c_0 \rho \Omega) = 0, \\
&w_1 \rho c_2 \partial_t \Omega + w_1 \rho c_2 P_{\Omega^\perp}(\nabla_x \rho) + \rho A P_{\Omega^\perp} P_u(\nabla_x \rho) + \rho B P_{\Omega^\perp} P_{u^\perp}(\nabla_x \rho) + w_1 c_3 \rho (\Omega \cdot \nabla_x)\Omega \\
&+ 2\rho w_2 \gamma_1 \big(((P_{\Omega^\perp} u)(u^\perp \otimes u^\perp : \nabla_x u) + (P_{\Omega^\perp} u^\perp)(u \otimes u^\perp : \nabla_x u) \big) = 0.
\end{aligned}
$$

In 2D, the constants are given by

$$c_0 := \int_{-\pi}^{\pi} \cos\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \tag{3.26}$$

$$c_2 := \int_{-\pi}^{\pi} \sin^2\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \tag{3.27}$$

$$c_3 := \int_{-\pi}^{\pi} \cos\theta \sin^2\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \tag{3.28}$$

$$\gamma_1 := \int_{-\pi}^{\pi} \cos^2\alpha \sin^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha \tag{3.29}$$

$$A := \int_{-\pi}^{\pi} \cos^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha \tag{3.30}$$

$$B := \int_{-\pi}^{\pi} \sin^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha \tag{3.31}$$

4 Analysis of the coupled model

Now, that we have derived a macroscopic model for the swimmers and obstacles, we follow the approach given by Motsch and Navoret in [19] in this section. In the mentioned paper Motsch and Navoret give a procedure to solve the Vicsek model numerically. They note that it can be transformed from a non-conservative system into a conservative system, which allows to use standard hyperbolic theory for numerical simulations. In Prop. 3.1 they show that the macroscopic Vicsek model can be seen as the limit of a conservative hyperbolic model with a relaxation term. But the transformation into a conservative system comes with a downside; the conservative formulation of the Vicsek model is only equivalent to the non-conservative Vicsek model for smooth functions, meaning that analytically derived behaviour in the conservative system must not occur in the non-conservative system.

Here, as a very first step for a deeper theoretical and numerical analysis, the hyperbolicity of the model will be investigated.

4.1 Formulation and theoretical analysis of the model

The derived model for the density and mean orientation of the swimmers

$$\begin{aligned}
&\partial_t \rho + w_1 \nabla_x \cdot (c_0 \rho \Omega) = 0, \\
&w_1 \rho c_2 \partial_t \Omega + w_1 c_2 P_{\Omega^\perp}(\nabla_x \rho) + A P_{\Omega^\perp} P_u(\nabla_x \rho) + B P_{\Omega^\perp} P_{u^\perp}(\nabla_x \rho) + w_1 c_3 \rho (\Omega \cdot \nabla_x)\Omega \\
&+ 2\rho w_2 \gamma_1 \big(((P_{\Omega^\perp} u)(u^\perp \otimes u^\perp : \nabla_x u) + (P_{\Omega^\perp} u^\perp)(u \otimes u^\perp : \nabla_x u) \big) = 0
\end{aligned}$$

contains three summands in the second equation, which are non-conservative:

1. $A P_{\Omega^\perp} P_u(\nabla_x \rho)$
2. $B P_{\Omega^\perp} P_{u^\perp}(\nabla_x \rho)$
3. $2\rho w_2 \gamma_1 \big(((P_{\Omega^\perp} u)(u^\perp \otimes u^\perp : \nabla_x u) + (P_{\Omega^\perp} u^\perp)(u \otimes u^\perp : \nabla_x u) \big)$.

We try to reformulate the model into conservative form analogously to the procedure in [19], i.e we want something of the form

$$\partial_t h + \nabla_x \cdot (F(h)) = 0$$

for a certain function h.

Remark 10. *For the first two terms we compute*

$$\begin{aligned}\nabla_x \cdot (\rho P_{u^\perp}) &= \nabla_x \cdot (\rho(Id - u \otimes u)) \\ &= (\nabla_x \rho)(Id - u \otimes u) + \rho \nabla_x \cdot (Id - u \otimes u) \\ &= P_{u^\perp}(\nabla_x \rho) - \rho \nabla_x \cdot (u \otimes u) \\ &= P_{u^\perp}(\nabla_x \rho) - \rho[(\nabla_x \cdot u)u + (u \cdot \nabla_x)u].\end{aligned}$$

The last step follows from

$$\nabla_x \cdot (u \otimes u) = \partial_{x_i}(u_i u_j) = (\partial_{x_i} u_i)u_j + u_i \partial_{x_i} u_j = (\nabla_x \cdot u)u + (u \cdot \nabla_x)u$$

and analogously

$$\nabla_x \cdot (\rho P_u) = \nabla_x \cdot (\rho(u \otimes u)) - \rho[(\nabla_x \cdot u)u + (u \cdot \nabla_x)u].$$

Now, we have

$$\begin{aligned}BP_{\Omega^\perp} P_{u^\perp}(\nabla_x \rho) &= BP_{\Omega^\perp} \nabla_x \cdot (\rho(\mathrm{Id} - u \otimes u)) + BP_{\Omega^\perp} \rho \left((\nabla_x \cdot u)u + (u \cdot \nabla_x)u\right) \\ AP_{\Omega^\perp} P_u(\nabla_x \rho) &= AP_{\Omega^\perp} \nabla_x \cdot (\rho(u \otimes u)) - AP_{\Omega^\perp} \rho \left[(\nabla_x \cdot u)u + (u \cdot \nabla_x)u\right],\end{aligned}$$

where in each equation the first summand is conservative and the second summand is fixed.

Remark 11.

$$\nabla_x \cdot (\rho(Id - u \otimes u)) = \nabla_x \cdot (\rho Id) - \nabla_x \cdot (\rho(u \otimes u)) = \nabla_x \rho - \nabla_x \cdot (\rho u \otimes u)$$

With $A + B = 1$ (see remark 7), we combine the two terms

$$\begin{aligned}&AP_{\Omega^\perp} P_u(\nabla_x \rho) + BP_{\Omega^\perp} P_{u^\perp}(\nabla_x \rho) \\ &= BP_{\Omega^\perp} \nabla_x \rho + (A - B)P_{\Omega^\perp} \nabla_x \cdot (\rho u \otimes u) + (B - A)P_{\Omega^\perp} \rho \left[(\nabla_x \cdot u)u + (u \cdot \nabla_x u)u\right].\end{aligned}$$

In a next step, we simplify the system and assume u is constant. We then have that $\nabla_x u = 0$ and the equations are

$$\partial_t \rho + \nabla_x \cdot (w_1 c_0 \rho \Omega) = 0 \tag{4.1}$$

$$w_1 c_2 \rho \partial_t \Omega + w_1 c_2 P_{\Omega^\perp}(\nabla_x \rho) + P_{\Omega^\perp} B \nabla_x \rho \tag{4.2}$$

$$+ (A - B)P_{\Omega^\perp} \nabla_x \cdot (\rho u \otimes u) + w_1 c_3 \rho (\Omega \cdot \nabla_x)\Omega = 0 \tag{4.3}$$

Remark 12. *In 2D we have*

$$u^\perp \otimes u^\perp = \begin{pmatrix} u_2^2 & -u_2u_1 \\ -u_2u_1 & u_1^2 \end{pmatrix}$$

$$u \otimes u = \begin{pmatrix} u_1^2 & u_2u_1 \\ -u_2u_1 & u_2^2 \end{pmatrix}$$

and

$$u \otimes u^\perp = \begin{pmatrix} -u_1u_2 & u_1^2 \\ -u_2^2 & -u_2u_1 \end{pmatrix}$$

Therefore we get

$$\begin{aligned}(u \otimes u^\perp) : \nabla_x u &= (u \otimes u^\perp) \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_1} u_2 \\ \partial_{x_2} u_1 & \partial_{x_2} u_2 \end{pmatrix} \\ &= -u_1u_2\partial_{x_1}u_1 + u_1^2\partial_{x_1}u_2 - u_2^2\partial_{x_2}u_1 + u_2u_1\partial_{x_2}u_2\end{aligned}$$

where $\nabla_x(u \otimes u)$ *in dimension* 2 *is equal to*

$$(\nabla_x \cdot u)u + (u \cdot \nabla_x)u = (\partial_{x_1}u_1 + \partial_{x_2}u_2)(u_1, u_2) + (u_1\partial_{x_1}u_1 + u_2\partial_{x_2}u_1, u_1\partial_{x_1}u_2 + u_2\partial_{x_2}u_2).$$

Figure 3: Illustration of u and $u^\perp$

4.2 Parametrization and hyperbolicity

To analyse the system we parametrize the mean orientation Ω to polar coordinates and try to prove that the system is hyperbolic.

$$
\begin{aligned}
&\partial_t \rho + w_1 \nabla_x \cdot (c_0 \rho \Omega) = 0, \\
&w_1 \rho (c_2 \partial_t \Omega + c_3 (\Omega \cdot \nabla_x) \Omega) + (A - B)(\nabla_x \rho \cdot u)(u - (u \cdot \Omega)\Omega) \\
&+ (w_1 c_2 + B)(\nabla_x \rho - (\Omega \cdot \nabla_x \rho)\Omega) = 0
\end{aligned}
$$

Let $\phi = \phi(t,z)$, $z = (x,y) \in \mathbb{R}^2$ and $u = (u_1, u_2) \in \mathbb{R}^2$. Using polar coordinates $\Omega = (\cos\phi, \sin\phi)$, the two-dimensional system reads

$$
\begin{aligned}
&\partial_t \rho + w_1 (\partial_x (c_0 \rho \cos\phi) + \partial_y (\rho \sin\phi)) = 0, \\
&\partial_t \phi c_2 \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} + (c_3 \cos\phi \partial_x \phi + c_3 \sin\phi \partial_y \phi) \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \\
&+ \frac{(A-B)}{w_1 \rho} (\partial_x \rho u_1 + \partial_y \rho u_2)(-u_1 \sin\phi + u_2 \cos\phi) \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \\
&+ \frac{(w_1 c_2 + B)}{w_1 \rho} (-\sin\phi \partial_x \rho + \cos\phi \partial_y \rho) \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} = 0
\end{aligned}
$$

The second equation can only equal zero if

$$
\begin{aligned}
c_2 \partial_t \phi + (c_3 \cos\phi \partial_x \phi + c_3 \sin\phi \partial_y \phi) + \frac{(A-B)}{w_1 \rho} (\partial_x \rho u_1 + \partial_y \rho u_2)(-u_1 \sin\phi + u_2 \cos\phi) \\
+ \frac{(w_1 c_2 + B)}{w_1 \rho} (-\sin\phi \partial_x \rho + \cos\phi \partial_y \rho) = 0.
\end{aligned}
$$

So, finally we get

$$
\partial_t \rho + w_1 (\partial_x (c_0 \rho \cos\phi) + \partial_y (\rho \sin\phi)) = 0 \tag{4.4}
$$

$$
c_2 \partial_t \phi + (c_3 \cos\phi \partial_x \phi + c_3 \sin\phi \partial_y \phi) + \frac{(A-B)}{w_1 \rho} (\partial_x \rho u_1 + \partial_y \rho u_2)(-u_1 \sin\phi + u_2 \cos\phi) \tag{4.5}
$$

$$
+ \frac{(w_1 c_2 + B)}{w_1 \rho} (-\sin\phi \partial_x \rho + \cos\phi \partial_y \rho) = 0 \tag{4.6}
$$

Now, we look at the wave propagation in x-direction. This means we only consider the one-dimensional case and set the y parts to zero. Then, the system reads

$$\partial_t\rho + w_1\partial_x(c_0\rho\cos\phi) = 0$$
$$c_2\partial_t\phi + c_3\cos\phi\partial_x\phi + \frac{(A-B)}{w_1\rho}(\partial_x\rho u_1)(-u_1\sin\phi + u_2\cos\phi) + \frac{(w_1c_2+B)}{w_1\rho}(-\sin\phi\partial_x\rho) = 0.$$

We assume that $w_1, c_2 \neq 0$ and consider the equivalent form

$$\partial_t\rho + w_1\partial_x(c_0\rho\cos\phi) = 0$$
$$\partial_t\phi + \frac{c_3}{c_2}\cos\phi\partial_x\phi + \frac{(A-B)}{c_2w_1\rho}(\partial_x\rho u_1)(-u_1\sin\phi + u_2\cos\phi) + \frac{(w_1c_2+B)}{c_2w_1\rho}(-\sin\phi\partial_x\rho) = 0$$

which can also be written as

$$\partial_t\begin{pmatrix}\rho\\ \phi\end{pmatrix} + M\ \partial_x\begin{pmatrix}\rho\\ \phi\end{pmatrix} = 0 \tag{4.7}$$

with

$$M = M(\rho,\phi) = \begin{pmatrix} w_1c_0\cos\phi & -w_1c_0\rho\sin\phi \\ \frac{(A-B)[u_1u_2\cos\phi - u_1^2\sin\phi] - (w_1c_2+B)\sin\phi}{c_2w_1\rho} & \frac{c_3}{c_2}\cos\phi \end{pmatrix}.$$

The eigenvalues of M are

$$\lambda_{1,2} = \frac{w_1c_0 + \frac{c_3}{c_2}\cos\phi}{2} \pm \sqrt{D}$$

with

$$D = \frac{(w_1c_0 + \frac{c_3}{c_2})^2}{4}\cos^2\phi - \frac{c_0c_3w_1}{c_2}\cos^2\phi + \frac{(w_1c_2+B)c_0\sin^2\phi}{c_2}$$
$$- (A-B)\sin\phi\frac{c_0}{c_2}[u_1u_2\cos\phi - u_1^2\sin\phi]$$

$$= \frac{\cos^2\phi}{4}\left(w_1c_0 - \frac{c_3}{c_2}\right)^2 + \frac{(w_1c_2+B)c_0\sin^2\phi}{c_2}$$
$$- (A-B)\frac{c_0}{c_2}[u_1u_2\cos\phi\sin\phi - u_1^2\sin^2\phi].$$

We want to prove that D is always positive, which would mean that the matrix M has two, distinct real eigenvalues and the system is (strictly) hyperbolic. Recall from remark 7 that we have that $A+B=1$, $u=(u_1,u_2)$ is of norm 1 and $0 < w_1 < 1$.

Remark 13. *Our two-dimensional constants were*

$$\begin{aligned}
c_0 &:= \int_{-\pi}^{\pi} \cos\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \\
c_2 &:= \int_{-\pi}^{\pi} \sin^2\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \\
c_3 &:= \int_{-\pi}^{\pi} \cos\theta \sin^2\theta \frac{e^{\cos\theta/d}}{Z'} d\theta \\
A &:= \int_{-\pi}^{\pi} \cos^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha \\
B &:= \int_{-\pi}^{\pi} \sin^2\alpha \frac{e^{\cos^2\alpha/D}}{Z} d\alpha.
\end{aligned}$$

We first analyse $A - B$ numerically:

Theorem 1. *For all $d > 0$, it holds that $A - B \geq 0$*

$$A - B = \int_{-\pi}^{\pi} (\cos^2\phi - \sin^2\phi) \frac{e^{\cos^2\phi/d}}{Z'} d\phi,$$

where $Z = \int_{-\pi}^{\pi} e^{\cos^2\phi/d} d\phi$ and $A - B \to 0$ as $d \to \infty$.

Proof. For all d, integration by parts provides

$$\begin{aligned}
A - B &= \int_{-\pi}^{\pi} (\cos^2\phi - \sin^2\phi) \frac{e^{\cos^2\phi/d}}{Z} d\phi \\
&= \sin\phi\cos\phi \frac{e^{\cos^2\phi/d}}{Z}\bigg|_{-\pi}^{\pi} + 2\int_{-\pi}^{\pi} \sin^2\phi\cos^2\phi \frac{e^{\cos^2\phi/d}}{Z} d\phi \\
&= 2\int_{-\pi}^{\pi} \sin^2\phi\cos^2\phi \frac{e^{\cos^2\phi/d}}{Z} \\
&\geq 0.
\end{aligned}$$

□

In order to have a positive discriminant, it is now enough to show that

$$X := u_1 u_2 \cos\phi \sin\phi - u_1^2 \sin^2\phi \leq 0.$$

We rewrite X and define

$$Y := -2X = (u_1 \sin\phi - u_2 \cos\phi)^2 - u_2^2 + \sin^2\phi$$

Now, we consider Y as a function depending on u_2:

$$Y = Y(u_2) = \left(\pm\sqrt{1-u_2^2}\sin\phi - u_2\cos\phi \right)^2 - u_2^2 + \sin^2\phi.$$

For $\sin\phi = 0$, the function is $Y \equiv 0$. Now, let $\sin\phi \neq 0$. On the boundaries we have for $u_2 = \pm 1$ that $Y(-1) = Y(1) = 0$. Plotting the function Y and varying ϕ from $-\pi$ to π, we obtain that $Y \geq \frac{1}{4}$, or equivalently $X \leq \frac{1}{8}$.

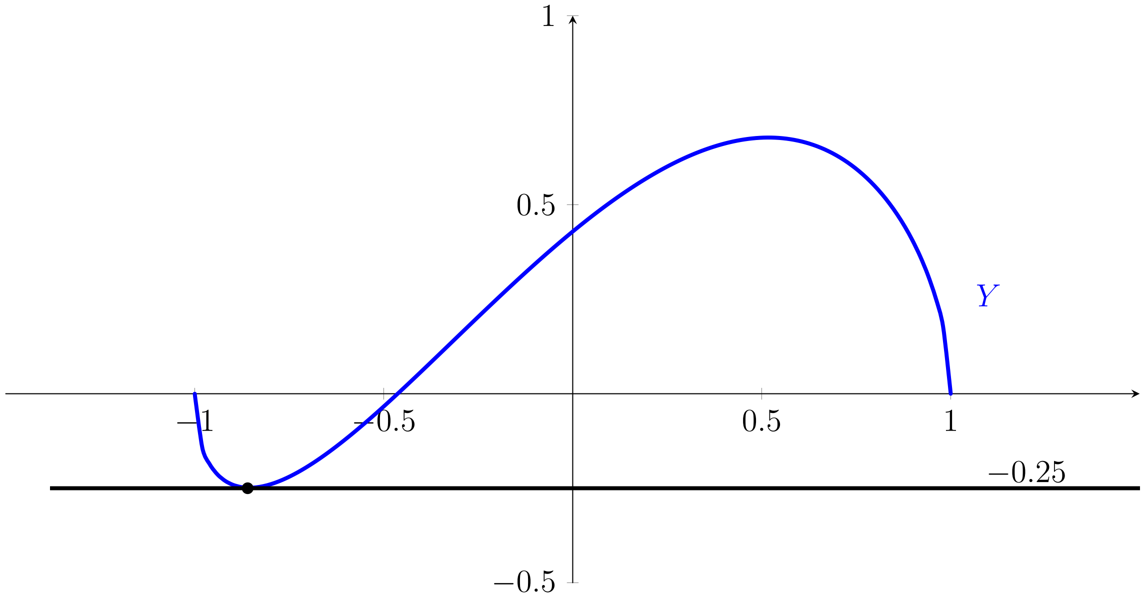

Figure 4: Plot of $Y(u_2)$

4.2.1 Non-hyperbolic example

Particular choices for the parameter lead to a non-hyperbolic system. Here in this section we give some examples to illustrate the issue. For instance, if we choose

$$d = D = 1 \quad w_1 = \frac{1}{2} \quad u_1 = \frac{1}{2} \text{ and } u_2 = \frac{\sqrt{3}}{2},$$

and compute the constants numerically, then, the discriminant $D = D(\phi, u_1, u_2)$ for $\phi \in [-\pi, \pi]$ is given by

$$D = 0,0000725\cos^2\phi + 0,66275\sin^2\phi - 0,10488\cos\phi\sin\phi.$$

Plotting this function shows that for $n \in \mathbb{Z}$, there are roots of four different forms

$$r_1 = 2(3.1416n - 1.4927) \quad r_2 = 2(3.1416n + 0.078135)$$
$$r_3 = 2(3.1416n - 1.5704) \quad r_4 = 2(3.1416n + 0.00034716).$$

Between those roots, the sign of D changes.

4.2.2 Hyperbolic example

For the following choice, one gets hyperbolicity:

$$d = D = \frac{1}{2} \quad w_1 = \frac{1}{2} \quad u_1 = 0.1 \text{ and } u_2 = \pm\sqrt{1 - 0.1^2}.$$

In this scenario, the discriminant D is always positive and therefore the matrix is diagonizable.

The full characterization of the hyperbolicity of the model is not known at this point. But with certain choices of the parameters, hyperbolic regions can be found. But: further analysis and possible numerical simulation hinge on the hyperbolicity of the model.

4.3 Relaxation limit of a conservative system

Assuming, we have found a hyperbolic region, we can approximate the geometric constraint by a term that relaxes towards $|\Omega| = 1$. The following proposition shows that our model for the swimmers and obstacles can be seen as the relaxation limit of a conservative, hyperbolic system with a relaxation term.

Proposition 2. *The relaxation model given by*

$$\begin{aligned}&\partial_t \rho^\eta + \nabla_x \cdot (w_1 c_0 \rho^\eta \Omega^\eta) = 0,\\ &\partial_t (w_1 c_2 \rho^\eta \Omega^\eta) + \nabla_x \cdot (w_1 c_3 \rho^\eta \Omega^\eta \otimes \Omega^\eta) + \nabla_x \cdot ((A-B)\rho^\eta u \otimes u) + w_1 c_2 \nabla_x \rho^\eta + B \nabla_x \rho^\eta \\ &= \frac{\rho^\eta}{\eta}(1 - |\Omega^\eta|^2)\Omega^\eta\end{aligned} \tag{4.8}$$

converges to 4.1 as $\eta \to 0$.

Proof. To prove the formal limit $\eta \to 0$ we follow the procedure in [19]. Recall that $|\Omega| = 1$. We take the second equation from 4.8 and compute the vector product with Ω. In the limit $\eta \to 0$ we obtain

$$\partial_t (w_1 \rho^0 \Omega^0) + \nabla_x \cdot (w_1 c_3 \rho^0 \Omega^0 \otimes \Omega^0) + \nabla_x \cdot \big((A-B)\rho^0 u \otimes u\big) + w_1 c_2 \nabla_x \rho^0 + B \nabla_x \rho^0 = \alpha \Omega^0$$

where we can specify α by taking the scalar product of the equation above with Ω^0

$$\begin{aligned}\alpha = \Omega^0 \cdot [&\partial_t (w_1 \rho^0)\Omega^0 + \rho^0 \partial_t (w_1 \Omega^0) + w_1 c_3 \nabla_x \rho^0 (\Omega^0 \otimes \Omega^0) + w_1 c_3 \rho (\nabla_x \cdot \Omega^0)\Omega^0 \\ &+ w_1 c_3 \rho (\Omega^0 \cdot \nabla_x)\Omega^0 + (A-B)(\nabla_x \rho^0) u \otimes u + w_1 c_2 \nabla_x \rho^0 + B \nabla_x \rho^0]\end{aligned}$$

Since $\nabla_x \cdot (\rho^0 \Omega^0 \otimes \Omega^0) = \nabla_x \cdot (\rho \Omega^0)\Omega^0 + \rho^0 (\Omega^0 \cdot \nabla_x)\Omega^0$ and since Ω^0 has norm 1, we get

$$\alpha = \partial_t (w_1 \rho^0) + w_1 c_3 \nabla_x \cdot (\rho^0 \Omega^0) + (A-B)(\nabla_x \rho^0 \cdot u)(u \cdot \Omega^0) + c_2 w_1 \Omega^0 \cdot \nabla_x \rho + B \Omega^0 \cdot \nabla_x \rho.$$

Now, we use $\partial_t w_1 \rho^0 = -\nabla_x \cdot (w_1 c_1 \rho^0 \Omega^0)$.

$$\alpha = \nabla_x \cdot (\rho^0 \Omega^0)[-w_1^2 c_1 + w_1 c_3] + (A-B)(\nabla_x \rho^0 \cdot u)(u \cdot \Omega^0) + c_2 w_1 \Omega^0 \cdot \nabla_x \rho^0 + B \Omega^0 \cdot \nabla_x \rho^0.$$

Back to equation 4.8, this gives us

$$\partial_t(w_1\rho^0\Omega^0) + \nabla_x \cdot (w_1c_3\rho^0\Omega^0 \otimes \Omega^0) + \nabla_x \cdot ((A-B)\rho^0\Omega^0 \otimes \Omega^0) + w_1c_2\nabla_x\rho^0 + B\nabla_x\rho^0$$
$$= \Omega^0 \left[\nabla_x \cdot (\rho^0\Omega^0)[-w_2c_1 + w_1c_3] + (A-B)(\nabla_x\rho^0 \cdot u)(u \cdot \Omega^0) + c_2w_1\Omega^0 \cdot \nabla_x\rho^0 + B\Omega^0 \cdot \nabla_x\rho\right]$$

which leads to

$$\begin{aligned}\Rightarrow \partial_t(w_1\rho^0\Omega^0) &+ w_1\rho^0\partial_t\Omega^0 + w_1c_3\nabla_x \cdot (\rho^0\Omega^0)\Omega^0 \\ &+ w_1c_3\rho^0(\Omega^0 \cdot \nabla_x)\Omega^0 + (\nabla_x\rho^0 \cdot u)u(A-B) + w_1c_2\nabla_x\rho^0 + B\nabla_x\rho^0 \\ &= \Omega^0\partial_t(w_1\rho^0) + w_1c_3\Omega^0(\nabla_x \cdot (\rho^0\Omega^0)) + (A-B)(\nabla_x\rho^0 \cdot u)(u \cdot \Omega^0)\Omega^0 \\ &\qquad + c_2w_1\Omega^0(\Omega^0 \cdot \nabla_x\rho^0) + B(\Omega \cdot \nabla_x\rho^0)\Omega^0.\end{aligned}$$

$$\begin{aligned}\Rightarrow w_1\rho^0\partial_t\Omega^0 &+ w_1c_3\rho^0(\Omega^0 \cdot \nabla_x)\Omega^0 + (\nabla_x\rho^0 \cdot u)(A-B)\underbrace{(u - (u \cdot \Omega^0)\Omega^0)}_{P_{\Omega^\perp}u} \\ &+ w_1c_2\underbrace{(\nabla_x\rho^0 - (\Omega^0 \cdot \nabla_x\rho^0)\Omega^0)}_{P_{\Omega^\perp}\nabla_x\rho^0} + B\underbrace{(\nabla_x\rho^0 - (\Omega^0 \cdot \nabla_x\rho^0)\Omega^0)}_{P_{\Omega^\perp}\nabla_x\rho^0} = 0,\end{aligned}$$

which finishes the proof. □

5 Conclusions

In this thesis, a macroscopic model for swimmers filled in a place with obstacles was given. For that matter, we started with a particle model, where the behaviour of each swimmer and obstacle was established and then kinetic and macroscopic equations were derived for the coupled model. The last section concludes with an investigation on the hyperbolicity of the model. It turned out that depending on the choice of the parameter values, hyperbolic and non-hyperbolic areas can be found. This issue complicates the use of standard theory to analyze the model numerically.

In this models, the obstacles were considered fixed, but it would be interesting to add dynamics to the obstacles

$$\partial_t g = \int_{\mathbb{S}^{d-1}} g\ du\ \bar{M}_{\bar{u}_g} - g,$$

$$\partial_t f + \omega \nabla_x f = \left(\int_{\mathbb{S}^{d-1}} f d\omega \right) \mathcal{M} - f$$

and effects in the coupling in both directions; swimmer influence swimmers and obstacles *and* vice versa. We also did not include dependencies on (ρ_s, ρ_o) on the weights w_1 and w_2.

In the last section, we have seen that for particular values of ϕ, we lose the hyperbolicity of the system. When instabilities arise, new patterns are formed. The stability of the linearized coupled system could be analyzed, assuming that ρ, Ω and u are constant (space-independent) solutions, following the procedure of section 5 in [20]. The behaviour of the system around small perturbations τ around a constant solution can be established and subsequently one would look for non-trivial wave-like solutions of the form

$$(\rho, \Omega, u) = (\bar{\rho}, \bar{\Omega}, \bar{u}) \exp\{i(k \cdot x - \alpha t)\}.$$

A Consistency relation the first coupling attempt

To solve the integral equation of the consistency relation in the first coupling part, the following code was used.

```
## Function to integrate the function f(x)
# within the interval [a,b] using the left box rule (summed rectangles).
#
# @param f f(x) function to be integrated
# @param a lower limit of the interval, scalar
# @param b upper limit of the interval, scalar
# @param nI number of partition intervals, integer
def rechteckInt(f,a,b,nI):

# Interpolation points are equidistantly distributed.
    dx = (b-a)/(nI)
    x = a
    I = 0 # initialize integral
    for i in range(0,nI,1):
        I = I+f(x)*dx
        x = x+dx

# Return Integral within [a,b]
    return I
## Function to perform the numeric integration of function f(x)
# in [a,b] with trapezoidal rule.
def trapezInt(f,a,b,nI):
# Interpolation points are equidistantly distributed.
    dx = (b-a)/(nI)
    x = a
    I = 0 # initialize integral
    for i in range(0,nI,1):
        I = I+(f(x)+f(x+dx))*dx/2.0
        x = x+dx
```

```
# Return integral
    return I
import time
# Define symbols
# Increment
st=0.01
x = sympy.Symbol('x')
# Define 0
Zero=0.1
y=sympy.Symbol('y')

## definition of f(x)
f = sympy.sin(x)*sympy.exp(sympy.cos(x))*sympy.exp((sympy.cos(x-y))**2)
print ('function')
display(f)

# Numerics start here
print ('NUMERIC SOLUTION')
before=time.time()
y = -numpy.pi
def ft(x):
    return sympy.sin(x)*sympy.exp(sympy.cos(x))*sympy.exp((sympy.cos(x-y))**2)
    IT1 = trapezkInt(ft,-numpy.pi,numpy.pi,100)
# loop
while (y<=numpy.pi):
    y=y+st
    def ft(x):
        return sympy.sin(x)*sympy.exp(sympy.cos(x))*sympy.exp((sympy.cos(x-y))**2)
    IT1 = rechteckInt(ft,-numpy.pi,numpy.pi,100)
    if abs(IT1)<Zero:
        print('y =', y)
        display(IT1)
        print ('---------------------')
```

```
# Print the function, values of and the corresponding solution of the integral and th
after=time.time()
Computationtime= after - before
print ('Computation time in sec:')
display(Computationtime)
```

function

$e^{\cos x}e^{\cos^2(x-y)}\sin x$

```
NUMERIC SOLUTION
y = -3.1315926535897933
0.0293008362135942
---------------------
y = -3.1215926535897935
0.0585896038167137
---------------------
y = -3.1115926535897938
0.0878542395868006
---------------------
y = -1.601592653589817
0.0884046751074984
---------------------
y = -1.591592653589817
0.0597183574328607
---------------------
y = -1.581592653589817
0.0310088717760839
---------------------
y = -1.571592653589817
0.002287356336762
---------------------
y = -1.561592653589817
0.0264350464679824
---------------------
```

```
y = -1.551592653589817
0.0551471939099414
---------------------
y = -1.541592653589817
0.0838379468563002
---------------------
y = -0.03159265358981578
0.0925120535514362
---------------------
y = -0.021592653589815776
0.0632523741270618
---------------------
y = -0.011592653589815776
0.033966642471436
---------------------
y = -0.0015926535898157755
0.00466692045178125
---------------------
y = 0.008407346410184225
0.0246347238188619
---------------------
y = 0.018407346410184225
0.0539262213692564
---------------------
y = 0.028407346410184227
0.0831955077581071
---------------------
y = 1.5384073464101853
0.0929705333479318
---------------------
y = 1.5484073464101853
0.0642889332725048
---------------------
```

```
y = 1.5584073464101853
0.0355823919344488
---------------------
y = 1.5684073464101853
0.00686204650107603
---------------------
y = 1.5784073464101853
0.0218609610206798
---------------------
y = 1.5884073464101853
0.0505754876906062
---------------------
y = 1.5984073464101853
0.0792703935417453
---------------------
y = 3.108407346410162
0.0971689009945928
---------------------
y = 3.118407346410162
0.0679144835867601
---------------------
y = 3.1284073464101616
0.038632093844468
---------------------
y = 3.1384073464101614
0.00933379214302369
---------------------
y = 3.148407346410161
0.0199683540386214
---------------------
Computation time in sec:
88.25006437301636
```

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