DISSERTATION / DOCTORAL THESIS

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“Moduli spaces of Calabi-Yau compactifications

and

special regions therein“

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Contents

Abstract

The focus of this thesis lies on the study of Calabi-Yau moduli spaces of string compactifications. For this purpose results of supersymmetric localisation in an $N = 2$ supersymmetric gauge theory in two dimensions, called gauged linear sigma model (GLSM), are used. This approach allows to examine the moduli space in regions which do not have an obvious geometric interpretation. Different regions correspond to different vacuum configurations of the GLSM, termed phases. In these phases the low energy physics of the GLSM is often described by models which exhibit a so-called hybrid or pseudo-hybrid behaviour. The analysis of these areas in the moduli space is not only of interest for physics, but also for mathematics, especially for the branch of enumerative geometry.

We will first review concepts from string theory and string compactifications. Then we will introduce extended supersymmetry in two dimension and define GLSMs. We summarize the supersymmetric localisation results relevant for this thesis. Abelian one-parameter GLSMs are discussed in detail, because they play the main role in this thesis. We will use the introduced concepts to evaluate the sphere partition function in various phases of the abelian one-parameter GLSMs. Thereby we find a general structure, valid in all phases of the abelian one-parameter GLSMs. We will describe the building blocks entering the general form of the sphere partition function and draw connections to results known in physics and mathematics. The GLSM and supersymmetric localisation also provide a way to study questions related to the swampland program. After discussing swampland conjectures relevant for this thesis, we show that the refined swampland distance conjecture holds for Kaluza-Klein states in pseudo-hybrid phases of abelian one-parameter GLSMs. In the last part of the thesis we propose a relation between the swampland distance conjecture and the de Sitter conjectures. This relation allows to motivate a lower bound on a parameter appearing in the swampland distance conjecture.

Zusammenfassung

Der Fokus dieser Doktorarbeit liegt in der Untersuchung von Parameterräumen von Calabi-Yau Kompaktifizierungen. Um diese Räume zu studieren, werden Resultate aus der supersymmetrischen Lokalisierung von $N = 2$ supersymmetrischen Eichtheorien in zwei Dimensionen verwendet. Im Speziellen wurden geeichte lineare Sigmamodelle ("Gauged Linear Sigma Models"-GLSMs) studiert. Der Zugang über GLSMs erlaubt die Untersuchung der Parameterräume in Regionen, wo keine offensichtliche geometrische Interpretation vorhanden ist. Unterschiedliche Regionen in den Parameterräumen korrespondieren zu unterschiedlichen Vakuum-Konfigurationen des GLSMs. Diese Konfigurationen werden Phasen genannt. Oft sind in solchen Phasen die Niedrigenergiebeschreibungen des GLSM durch sogenannt Hybrid- oder Pseudohybridmodelle gegeben. Die Untersuchung dieser Regionen ist nicht nur vom physikalischem Standpunkt interessant, sondern auch vom mathematischen, vor allem für das Teilgebiet der enumerativen Geometrie.

Die Arbeit beginnt mit einer Wiederholung der Grundlagen der Stringtheorie und der Kompaktifizierung. Nach dieser Einführung wird die erweiterte Supersymmetrie in zwei Dimensionen erläutert und GLSMs werden definiert. Die für diese Arbeit relevanten Resultate der supersymmetrischen Lokalisierung werden zusammengefasst. Der Klasse der abelschen Einparameter-GLSMs ist ein eigenes Kapitel gewidmet, da auf diesen das Hauptaugenmerk der Doktorarbeit liegt. Die eingeführten Grundlagen werden dann dazu verwendet, um die sogenannt Sphere Partition Function in den verschiedenen Phasen der abelschen Einparameter-GLSMs auszuwerten. Es wird gezeigt, dass die Sphere Partition Function eine Struktur annimmt welche in allen Phasen gültig ist. Diese spezielle Form besteht aus einzelnen Teilbeiträgen, welche genauer beschrieben werden und in Verbindung mit Resultaten aus der Mathematik gebracht werden. Der Zugang über GLSMs und supersymmetrische Lokalisierung erlaubt es auch Fragen im Rahmen des Swampland Programmes zu untersuchen. Dazu werden die Swampland Vermutungen (Conjectures) diskutiert, welche für die Arbeit von Bedeutung sind. Danach wird gezeigt, dass die Refined Swampland Distance Conjecture für Kaluza-Klein Zustände in pseudohybrid Phasen der abelschen ein-Parameter GLSMs erfüllt ist. Im letzten Teil der Doktorarbeit wird ein möglicher Zusammenhang zwischen der Swampland Distance Conjecture und der de Sitter Conjecture motiviert. Dieser Zusam-

menhang kann verwendet werden, um eine untere Grenze für einen Parameter in der Swampland Distance Conjecture zu erhalten.

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Chapter 1

Introduction

String theory was first invented to describe phenomena related to strong interactions, but was soon found to be a valuable candidate for an unification of quantum field theories of particle physics and gravity. In order to achieve a consistent quantisation, superstring theory requires 10 spacetime dimensions. To extract 4 dimensional physics, a concept called compactification is applied. In compactification the extra dimensions are "curled up" on an internal space. These internal spaces have to have certain properties dictated by current observable physics and expected extensions thereof. Candidates for such internal spaces are Calabi-Yau manifolds. Nowadays experiments do not reach sufficient energy scales to resolve the presence of these internal dimensions. There is not just a single choice for the internal space and the set of possible spaces is called the string landscape. In view of this obstacle, the question if string theory describes our universe is still open and highly debated. An interesting approach to those phenomena is given by the swampland program. This approach uses string theory constructions to derive properties which an effective theory should have in order to have an ultra-violet completion to a theory of quantum gravity.

However, string theory is much more than a candidate for unification, especially regarding the connections to mathematics. A prime example is that it lead to the discovery of mirror symmetry. Today there are many definitions of mirror symmetry (see e.g. [1, 2, 3, 4]), but maybe the best known is the formulation in terms of a duality between type IIA string theory on a Calabi-Yau threefold and type IIB string theory on the mirror Calabi-Yau threefold. Mirror symmetry also provided insights into enumerative geometry which were set off by the work [5].

In light of the insights it has delivered, string theory is a valuable tool to study mathematical questions. This thesis studies such a topic at the interface of mathematics and physics: the Calabi-Yau moduli spaces.

These moduli spaces arise in the process of compactification and parameterize the families of worldsheet conformal field theories (CFTs). Different loci in the moduli spaces correspond to a different worldsheet CFT realizations. The best studied regions in these moduli spaces are loci where one has a geometric intuition of the spacetime physics. At these loci of the moduli space tools from algebraic geometry can typically be applied. Further, the low-energy physics is given in terms of a non-linear sigma model. At these loci the parameters of the CFT correspond to the moduli of the Calabi-Yau, which parameterize the size and the shape of the manifold. In order to move away from these loci one has to take into account instanton corrections and these corrections make an analysis hard. In addition, the concrete CFT realization at a specific locus is often not known.

A tool for studying a certain subset of the moduli space, namely the stringy Kähler moduli space $\mathcal{M}_K$, is given by the gauged linear sigma model (GLSM) [6]. The GLSM provides a common UV description for the CFTs encountered in $\mathcal{M}_K$. In the GLSM the parameters are realized as certain coupling parameters. Different values of these parameters result in different low-energy configurations of the GLSM, called phases. The low energy effective theory given in a phase corresponds to the worldsheet CFT in this locus of the moduli space. Correlators and partition functions of the GLSM correspond to geometric quantities on the moduli space. The difficulty of instanton-corrections in the calculation of these quantities is overcome by techniques from supersymmetric localisation in the GLSM. One important result from these techniques, which takes a main role in this thesis, is the so called sphere partition function [7, 8].

One main result of this thesis is, that we were able to show for the class of one-parameter abelian and certain two parameter abelian GLSMs that the sphere partition function evaluated in a specific phase takes a specific form valid in certain limiting regions. A preprint of these results is given by

[9] : D. Erkinger and J. Knapp, *Sphere partition function of Calabi-Yau GLSMs*, `2008.03089`

and these are further discussed in Chapter 6.

Many of the swampland conjectures are motivated in the geometric regimes. The GLSM and the sphere partition function provide a way to study swampland conjectures away from the geometric regimes. In this thesis we studied the refined swampland distance conjecture in so called pseudo-hybrid phases of certain one-parameter GLSMs. The outcome of this analysis is published in

[10] : D. Erkinger and J. Knapp, *Refined swampland distance conjecture and exotic hybrid Calabi-Yaus*, *JHEP* **07** (2019) 029 [`1905.05225`]

and more details can be found in Chapter 8.

In the course of this thesis we further draw a connection between the swampland distance conjecture and the de Sitter conjectures. This was done by considering various examples of string compactifications and no-go theorems in classical de Sitter supergravity solutions. The results appeared in

[11] : D. Andriot, N. Cribiori and D. Erkinger, *The web of swampland conjectures and the TCC bound*, *JHEP* **07** (2020) 162 [2004.00030]

and are summarized in Chapter 9.

1.1 Outline of this Thesis

In Chapter 2 we introduce basic concepts of string theory. The focus is to show the necessity of extra dimensions, due to the presence of the so called conformal anomaly. In Chapter 3 we describe the concept of compactification and introduce Calabi-Yau spaces and their parameter spaces. We discuss extended supersymmetry in 2 dimensions in Chapter 4 and introduce the GLSM. We further provide some background on supersymmetric localisation. Chapter 5 is devoted to abelian one-parameter GLSMs. These models are the most studied objects in this thesis. In Chapter 6 we show that the sphere partition function of abelian one-parameter GLSMs has a universal structure in the different phases of the GLSMs. After the introduction of some swampland conjectures in Chapter 7 the sphere partition function in GLSMs with a pseudo-hybrid phase is used in Chapter 8 to show that the refined swampland distance conjecture holds for Kaluza-Klein states in these models. Chapter 9 motivates on an example basis a relation between the swampland distance conjecture and the de Sitter conjectures. Thereby a lower bound on the order one parameter appearing in the swampland distance conjecture is produced.

Chapter 2

Basics of String Theory

In this chapter we want to introduce the basic concepts of string theory. The focus lies not on the spectrum of the theory, but on worldsheet features and the reason why we encounter extra dimensions in string theory. We will briefly talk about the bosonic string and move then on to superstring theory. Afterwards we perform the path integral which gives us the gauged fixed superstring action. We introduce certain concepts of superconformal field theory, which will be used to see that only in certain dimensions the superstring is anomaly free. Most of the following material is taken from [12] and [13]. Our discussion will be focused on key concepts and we will not give detailed calculations. These can be found in the mentioned literature. We will stick with the notation of [12] so the interested reader can easily look up further details.

2.1 The Classic Superstring

The basic idea of string theory is to replace the point particle by an extended one-dimensional object, the string. As a consequence the worldline of a particle becomes the worldsheet Σ of a string. Similar to the case of the particle, where the action is given by the length of the worldline in d dimensional Minkowski space, the action of the string is given by the area of Σ

$$\begin{aligned} S_{NG} &= -T \int_\Sigma \mathrm{d}A, \\ &= -T \int_\Sigma \mathrm{d}^2\sigma \left(-\det \left(\frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \right) \right)^{\frac{1}{2}}, \end{aligned} \tag{2.1}$$

where $\sigma^\alpha = (\tau, \sigma)$ are coordinates on the worldsheet, with $\tau_i < \tau < \tau_f$ and $0 \leq \sigma \leq l$. The functions $X^\mu(\tau, \sigma)$, $\mu = 0, 1, \ldots, d-1$ are embedding the worldsheet into d dimensional Minkowski space, with Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$, which is usually called spacetime. The expression

$$\frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}, \tag{2.2}$$

is the induced, or pulled-back, metric on the worldsheet from the embedding space. The subscript in S_{NG} stands for Nambu-Goto who studied this action first. T is called the string tension and is of mass dimension two $[T] = [Mass]^2$. Related to the tension is the parameter α':

$$\alpha' = \frac{1}{2\pi T}, \tag{2.3}$$

with $[\alpha'] = [Length]^2$. Further derived quantities are the string length scale:

$$l_s = 2\pi\sqrt{\alpha'}, \tag{2.4}$$

and the string mass scale:

$$M_s = \left(\alpha'\right)^{-\frac{1}{2}}. \tag{2.5}$$

In string theory we have two possibilities regarding the boundary behaviour of the worldsheet. We can either study worldsheets with boundaries, which gives open strings or worldsheets with no boundaries which results in closed strings. We will focus on the closed strings, because open strings play no role in this thesis. For closed strings we have:

$$X^\mu(\tau, \sigma + l) = X^\mu(\tau, \sigma). \tag{2.6}$$

A classically equivalent action to S_{NG} is given by the Polyakov action:

$$S_P = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.7}$$

In this action an additional field $h_{\alpha\beta}(\tau, \sigma)$ is introduced, with $h = \det(h_{\alpha\beta})$. $h_{\alpha\beta}(\tau, \sigma)$ is a metric on the worldsheet with signature $(-, +)$. S_P is the action of d massless bosons coupled to gravity in two dimensions. The local supersymmetric extension of (2.7) is given by:

$$\begin{aligned} S = -\frac{1}{8\pi}\int \mathrm{d}^2\sigma e \Bigg(&\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \\ &-i\overline{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu \left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\overline{\chi}_\beta \psi_\mu\right)\Bigg), \end{aligned} \tag{2.8}$$

where the ψ^μ are the superpartners of the X^μ. Further ρ^β are the 2-dimensional Dirac matrices in curved space and χ_α is the gravitino.

Symmetries of the Action

The action (2.8) has a huge number of local symmetries. Next we state these symmetries and use the convention that not stated fields are invariant under the respective transformation. We begin with local supersymmetry:

$$\sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu, \qquad \delta_\epsilon \psi^\mu = \frac{1}{2}\rho^\alpha\left(\sqrt{\frac{2}{\alpha'}}\partial_\alpha X^\mu - \frac{i}{2}\overline{\chi}_\alpha\psi^\mu\right)\epsilon,$$
$$\delta_\epsilon e^a_\alpha = \frac{i}{2}\bar{\epsilon}\rho^a\chi_\alpha, \qquad \delta_\epsilon\chi_\alpha = 2D_\alpha\epsilon, \tag{2.9}$$

with $\epsilon = \epsilon(\tau,\sigma)$ a Majorana spinor and D_α is a covariant derivative with torsion. The action is also invariant under Weyl transformations:

$$\delta_\Lambda\psi^\mu = -\frac{1}{2}\Lambda\psi^\mu, \qquad \delta_\Lambda e^a_\alpha = \Lambda e^a_\alpha, \qquad \delta_\Lambda\chi_\alpha = \frac{1}{2}\Lambda\chi_\alpha, \tag{2.10}$$

with $\Lambda = \Lambda(\tau,\sigma)$. There is also invariance with respect to super-Weyl transformations:

$$\delta_\eta\chi_\alpha = \rho_\alpha\eta, \tag{2.11}$$

with $\eta = \eta(\tau,\sigma)$ a Majorana spinor. A further symmetry is local Lorentz invariance in two dimensions given by

$$\delta_l\psi^\mu = -\frac{1}{2}l\overline{\rho}\psi^\mu, \qquad \delta_l e^a_\alpha = l\varepsilon^a_b e^b_\alpha, \qquad \delta_l\chi_\alpha = -\frac{1}{2}l\overline{\rho}\chi_\alpha, \tag{2.12}$$

with $l = l(\tau,\sigma)$ as parameter. Of course reparameterizations are also part of the symmetries:

$$\delta_\xi X^\mu = -\xi^\beta\partial_\beta X^\mu, \qquad \delta_\xi\psi^\mu = -\xi^\beta\partial_\beta\psi^\mu, \tag{2.13}$$
$$\delta_\xi e^a_\alpha = -\xi^\beta\partial_\beta e^a_\alpha - e^a_\beta\partial_\alpha\xi^\beta, \qquad \delta_\xi\chi_\alpha = -\xi^\beta\partial_\beta\chi_\alpha - \chi_\beta\partial_\alpha\xi^\beta, \tag{2.14}$$

and $\xi = \xi(\tau,\sigma)$. These local symmetries are accompanied by global spacetime Poincaré transformations.

Superconformal Gauge

It is always possible to choose locally the so-called superconformal gauge:

$$e^a_\alpha = e^\phi\delta^a_\alpha, \qquad \chi_\alpha = \rho_\alpha\lambda, \tag{2.15}$$

by application of a supersymmetry transformation, a reparameterization and a local Lorentz transformations. λ is a spinor. Classically one can completely gauge away the above degrees of freedom by application of a Weyl and super-Weyl rescaling, but these symmetries are anomalous in the quantum theory and can only be restored under certain conditions as we will discuss in Section 2.3. Another question is if the superconformal gauge can be reached globally. If yes there exists a globally defined spinor ϵ and a vector field ξ^α with

$$\begin{aligned}(\Pi\epsilon)_\alpha &= \frac{1}{2}\rho^\beta\rho_\alpha D_\beta\epsilon = \tau_\alpha,\\ (P\xi)_{\alpha\beta} &= \nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - \nabla_\gamma\xi^\gamma h_{\alpha\beta} = t_{\alpha\beta},\end{aligned} \tag{2.16}$$

where $t_{\alpha\beta}$ is an arbitrary tensor and τ_α a spin 3/2 spinor, with the properties:

$$\rho^\alpha \tau_\alpha = 0, \qquad t_{\alpha\beta} = t_{\beta\alpha}, \qquad t^\beta_\beta = 0. \tag{2.17}$$

These conditions are equivalent to the absence of zero modes of the adjoint operators $\Pi^\dagger$ and $P^\dagger$, with respect to the metric on the space of infinitesimal deformations. The zero modes are called moduli and supermoduli:

$$\# \text{ moduli } = \dim \ker P^\dagger, \tag{2.18}$$

$$\# \text{ supermoduli } = \dim \ker \Pi^\dagger. \tag{2.19}$$

The zero modes of P and Π are called conformal Killing vectors and conformal Killing spinors. They are a sign that the gauge fixing is not complete.

Equations of Motion and Conserved Charges

We will work in the superconformal gauge (2.15), where the action simplifies to

$$S = -\frac{1}{8\pi} \int \mathrm{d}^2\sigma \left(\frac{2}{\alpha'} \partial_\alpha X^\mu \partial^\alpha X_\mu + 2i\overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right). \tag{2.20}$$

This action is still invariant under local reparameterizations and supersymmetry transformations with parameters

$$P\xi = 0, \qquad \Pi\epsilon = 0, \tag{2.21}$$

where P and Π are given in (2.16). The equations of motion are given by

$$\partial_\alpha \partial^\alpha X^\mu = 0, \tag{2.22}$$

$$\rho^\alpha \partial_\alpha \psi^\mu = 0. \tag{2.23}$$

Although we have fixed e^a_α and χ_α, we still need to take into account their equations of motion, evaluated in the chosen gauge, as constraints. The variation of the action with respect to e^a_α gives the energy-momentum tensor:

$$T_{\alpha\beta} = \frac{2\pi}{e} \frac{\delta S}{\delta e^\beta_a} e_{\alpha a}. \tag{2.24}$$

By varying the action with respect to χ_α we get the supercurrent:

$$T_{F\alpha} = \frac{2\pi}{e} \frac{\delta S}{i\delta \overline{\chi}^\alpha}, \tag{2.25}$$

where F is just an additional label. The constraints are given by:

$$T_{\alpha\beta} = 0, \qquad T_{F\alpha} = 0. \tag{2.26}$$

They also fulfil

$$T^{\alpha}_{\alpha} = 0, \qquad \rho^{\alpha} T_{F\alpha} = 0. \tag{2.27}$$

It is convenient to change worldsheet coordinates to light-cone coordinates given by:

$$\sigma^{\pm} = \tau \pm \sigma, \tag{2.28}$$

with

$$\partial_+ = \frac{\partial}{\partial \sigma^+} = \frac{1}{2}\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \sigma}\right), \qquad \partial_- = \frac{\partial}{\partial \sigma^-} = \frac{1}{2}\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \sigma}\right). \tag{2.29}$$

The equations of motion simplify to

$$\partial_+ \partial_- X^{\mu} = 0, \tag{2.30}$$
$$\partial_- \psi^{\mu}_+ = \partial_+ \psi^{\mu}_- = 0. \tag{2.31}$$

Spinor components are denoted by $\pm$. We see that we can split the X^{μ} into a left- and right-moving part

$$X^{\mu}(\sigma^{\pm}) = X^{\mu}_L(\sigma^+) + X^{\mu}_R(\sigma^-), \tag{2.32}$$

and also

$$\psi^{\mu}_+ = \psi^{\mu}_+(\sigma^+), \qquad \psi^{\mu}_- = \psi^{\mu}_-(\sigma^-). \tag{2.33}$$

To solve the above equations we must specify boundary conditions. We are interested in the closed string and so we additionally demand

$$X^{\mu}(\tau, \sigma) = X^{\mu}(\tau, \sigma + l). \tag{2.34}$$

The fermions allow for periodic and anti-periodic boundary conditions, which can be imposed independently on both spinor components:

$$\psi_+(\tau, \sigma) = \pm \psi^{\mu}_+(\tau, \sigma + l), \qquad \psi_-(\tau, \sigma) = \pm \psi^{\mu}_-(\tau, \sigma + l). \tag{2.35}$$

Periodic boundary conditions are called Ramond (R) boundary conditions and anti-periodic ones are called Neveu-Schwarz (NS). The action (2.20) takes the following form in light-cone coordinates:

$$S = \frac{1}{2\pi} \int \mathrm{d}^2\sigma \left(\frac{2}{\alpha'} \partial_+ X \cdot \partial_- X + i \left(\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_- \right) \right). \tag{2.36}$$

Let us also study (2.24) and (2.25) in light-cone coordinates. One finds for the energy-momentum tensor:

$$\begin{aligned} T_{++} &= -\frac{1}{\alpha'} \partial_+ X^{\mu} \partial_+ X_{\mu} - \frac{i}{2} \psi^{\mu}_+ \partial_+ \psi_{+\mu}, \\ T_{--} &= -\frac{1}{\alpha'} \partial_- X^{\mu} \partial_- X_{\mu} - \frac{i}{2} \psi^{\mu}_- \partial_- \psi_{-\mu}, \\ T_{+-} &= T_{-+} = 0. \end{aligned} \tag{2.37}$$

For the supercurrent, as consequence of (2.27), only two of the four components are non-vanishing and these are denoted by:

$$T_{F\pm} = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi^{\mu}_{\pm}\partial_{\pm}X_{\mu}. \tag{2.38}$$

In the coordinates $\sigma^{\pm}$ the conservation of the energy-momentum current and the supercurrent is given by:

$$\partial_- T_{++} = \partial_+ T_{--} = 0, \tag{2.39}$$
$$\partial_- T_{F+} = \partial_+ T_{F-} = 0. \tag{2.40}$$

We see the appearance of an infinite number of conserved charges, because (super)currents of the form

$$f(\sigma^{\pm})T_{\pm\pm}, \qquad \epsilon^{\pm}(\sigma^{\pm})T_{F\pm}, \tag{2.41}$$

are also conserved. By choosing a basis of functions respecting the periodic boundary conditions of the system we get the following conserved (super)charges at $\tau = 0$:

$$L_n = -\frac{l}{4\pi^2}\int_0^l \mathrm{d}\sigma^{-\frac{2\pi}{l}in\sigma}T_{--}(\sigma), \tag{2.42}$$
$$G_r = -\frac{1}{\pi}\sqrt{\frac{l}{2\pi}}\int_0^l \mathrm{d}\sigma e^{-\frac{2\pi}{l}ir\sigma}T_{F-}(\sigma). \tag{2.43}$$

The Dirac brackets[1] of these charges give the centerless super-Virasoro algebra:

$$\begin{aligned}
\{L_m, L_n\} &= -i(m-n)L_{m+n}, \\
\{L_m, G_r\} &= -i\left(\frac{1}{2}m - r\right)G_{m+r}, \\
\{G_r, G_s\} &= -2iL_{r+s}.
\end{aligned} \tag{2.44}$$

We only gave the right-moving part, but there is also a similar left-moving part, which can be obtained by simply replacing the oscillators by the bared oscillators. The algebra (2.44) is modified in the process of quantization. This gives rise to the so-called Weyl anomaly. This is best understood by discussing super-conformal field theories, which will be done in Section 2.3, but before we will use the Faddeev-Popov method to obtain the gauge fixed action.

[1]It is necessary to use Dirac brackets, because second class constraints appear.

2.2 Quantization of the Superstring

We will right away discuss path integral quantization. Our goal is to obtain the gauge fixed action. There are other ways to quantize the superstring, namely canonical quantization and light-cone quantization. As usual in the process of canonical quantization one replaces Poisson-brackets by commutators. In addition, the quantum analogues of the constraints (2.26) have to be imposed on the spectrum. In light-cone quantization one chooses a specific gauge, in which the constraints are automatically fulfilled. For details on these approaches see [12].

Path Integral Quantization

We start from the path integral and use the Faddeev-Popov method to gauge fix the action. We start from the path integral of the form

$$Z = \int \mathcal{D}h\mathcal{D}\chi\mathcal{D}X\mathcal{D}\Psi e^{iS}, \tag{2.45}$$

where S is given by (2.8). The above expression should be taken with a grain of salt, because it is plagued by an overcounting of gauge equivalent functions. We do not want to go into details, but heuristically the procedure works as follows. The general idea is to consider a fixed configuration $\hat{h}_{\alpha\beta}$ and $\hat{\chi}_\alpha$ and integrate over the gauge parameters and divide by the volume of the gauge group. In this process a change of variables occurs and as for ordinary integrals determinants of Jacobians appear. These determinants can be written as functional integrals over ghost-fields. After this change of variables the integral over the symmetry parameters factors out[2]

and can be taken out of the path integral. For the superstring one gets the following ghost action

$$\begin{aligned} S_{gh} = -\frac{i}{2\pi}\int \mathrm{d}\sigma^2\sqrt{-\hat{h}}\Big\{ & b^{\alpha\beta}\hat{\nabla}_\alpha c_\beta + \overline{\beta}^\alpha\hat{\nabla}_\alpha\gamma \\ & - i\overline{\hat{\chi}}_\alpha\left[c^\beta\hat{\nabla}_\beta\beta^\alpha + \frac{3}{2}\beta^\beta\hat{\nabla}^\alpha c_\beta - \frac{i}{4}b^{\alpha\beta}\rho_\beta\gamma\right]\Big\}. \end{aligned} \tag{2.46}$$

$\hat{\nabla}_\alpha$ is the torsion-free connection with respect to $\hat{h}_{\alpha\beta}$ and in the superconformal gauge the part proportional to $\overline{\hat{\chi}}_\alpha$ is absent. $b_{\alpha,\beta}$, c_α are anti-commuting spin 2 and spin 1 fields and β_α, γ are commuting spin 3/2 and spin 1/2 fields respectively. $b_{\alpha\beta}$ is symmetric traceless and β_α is ρ traceless. The ghost system (2.46) is also a superconformal field theory and we will discuss in the next section the ghost contribution to the super-Virasoro algebra and anomaly cancelation in the critical dimension.

[2]This is true as long as we do not encounter anomalies. See [12, 13] for comments on this issue.

2.3 $N = 1$ Superconformal Symmetry

In the previous sections we discussed the fermionic string and the corresponding ghost fields. All of these theories are theories with a $N = 1$ superconformal symmetry. We will call such theories $N = 1$ SCFTs and discuss some general aspects of these theories. We consider a theory in the complex plane, with (anti-)holomorphic coordinate z ($\bar{z}$). We are interested in a supersymmetric theory and this is most easily described in superspace. We are considering $N = 1$ superspace and extend the bosonic coordinate z by a Grassmann one θ:

$$\mathbf{z} = (z, \theta) \quad \left(\bar{\mathbf{z}} = (\bar{z}, \bar{\theta})\right), \tag{2.47}$$

with the property $\theta^2 \left(\bar{\theta}^2\right) = 0$. In Section 4.1 we will introduce $N = 2$ superspace. The Grassmann property results in a finite Taylor series for superfields:

$$\Phi(\mathbf{z}, \bar{\mathbf{z}}) = \phi_0(z, \bar{z}) + \theta\phi_1(z, \bar{z}) + \overline{\theta\phi}_1(z, \bar{z}) + \theta\bar{\theta}\phi_2(z, \bar{z}). \tag{2.48}$$

We introduce super-derivatives

$$D = \partial_\theta + \theta\partial_z, \qquad \overline{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}. \tag{2.49}$$

From now one we focus on the holomorphic part ($\mathbf{z} = (z, \theta)$). The anti-holomorphic part follows by a similar discussion. If we impose $\overline{D}\Phi = 0$ on a general superfield (2.48) we obtain a chiral superfield

$$\Phi(\mathbf{z}) = \phi_0(z) + \theta\phi_1(z). \tag{2.50}$$

Superconformal transformations in superspace are given by the following transformations:

$$\mathbf{z} = (z, \theta) \to \mathbf{z}' = (z'(z, \theta), \theta'(z, \theta)), \tag{2.51}$$

such that

$$D = \left(D\theta'\right) D'. \tag{2.52}$$

The transformation behaviour (2.52) restricts the possible transformations and we refer to [12] were the form of the superconformal transformations is discussed in more detail[3].

The counterpart of conformal chiral primary fields are holomorphic superconformal primary fields, which transform under a superconfromal transformation by:

$$\Phi(\mathbf{z}) = \left(D\theta'\right)^{2h} \Phi'(\mathbf{z}'). \tag{2.53}$$

[3]See also [14] where the form of superconformal transformations is obtained by starting from a supersymmetric line element.

In the next step we introduce an infinite set of generators, which encode the infinitesimal superconformal transformations on the component fields (2.50):

$$\begin{aligned} \delta_\xi \phi(z) &= -\left[T_\zeta, \phi(z)\right], \\ \delta_\epsilon \phi(z) &= -\left[T_{F_\epsilon}, \phi(z)\right], \end{aligned} \tag{2.54}$$

where $\xi(z)$ parameterises conformal transformations and $\epsilon(z)$ supersymmetry transformations. The generators are given by

$$\begin{aligned} T_\xi &= \oint \frac{\mathrm{d}z}{2\pi i} \xi(z) T(z), \\ T_{F_\epsilon} &= \oint \frac{\mathrm{d}z}{2\pi i} \epsilon(z) T_F(z), \end{aligned} \tag{2.55}$$

where $T(z)$ is the conserved current associated to conformal transformations (energy-momentum tensor) and T_F is the conserved current under supersymmetry transformations (supercurrent). We next consider the quantum theory and we use operator product expansions (OPEs) to study the behaviour of the quantum superconformal theory. Of central interest are the following OPEs, which encode the superconformal algebra. In our case we only have one supercurrent and in this case we encounter the $N = 1$ superconformal algebra:

$$\begin{aligned} T(z)T(w) &= \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \\ T(z)T_F(w) &= \frac{\frac{3}{2}T_F(w)}{(z-w)^2} + \frac{\partial T_F(w)}{z-w} + \ldots, \\ T_F(z)T_F(w) &= \frac{\frac{c}{6}}{(z-w)^3} + \frac{\frac{1}{2}T(w)}{(z-w)} + \cdots, \end{aligned} \tag{2.56}$$

where c is the central charge. We expand the conserved currents in modes:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n, & L_n &= \oint \frac{\mathrm{d}z}{2\pi i} z^{n+1} T(z), \\ T_F(z) &= \frac{1}{2} \sum_{r\in\mathbb{Z}+a} z^{-\frac{3}{2}-r} G_r, & G_r &= 2 \oint \frac{\mathrm{d}z}{2\pi i} T_F(z) z^{r+\frac{1}{2}}. \end{aligned} \tag{2.57}$$

The generators fulfil the following Hermiticity relations:

$$L_n^\dagger = L_{-n}, \qquad\qquad G_r^\dagger = G_{-r}. \tag{2.58}$$

In (2.57) the parameter a distinguishes between R ($a = 0$) and NS ($a = \frac{1}{2}$) sector. Let us mention that on the complex plane the situation of periodic (NS sector) and anti-periodic (R sector) fermions is reversed compared to the situation on the cylinder in Section 2.1. We can now use (2.56) to write down the algebra of the modes, which results in the $N = 1$ super-Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n}, \\
[L_m, G_r] &= \left(\frac{1}{2}m - r\right)G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s}.
\end{aligned} \tag{2.59}$$

The Super-Virasoro Algebra of the Superstring

In (2.44) we obtained the Poisson brackets of the classical conserved charges in the superstring. In the previous section we gave in (2.59) the quantum result in a general $N = 1$ superconformal theory. Let us now comment on the central charge of the superstring.

In addition to the fields X^μ and ψ^μ we also have the ghost fields from the gauge fixing (2.46). To obtain c in these theories we would have to calculate the OPE of the energy-momentum tensor $T(z)$ with itself. We skip this and refer to [12] for details. The results for the central charges are

	X^μ, ψ^μ	b, c	β, γ
c	$\frac{3}{2}d$	-26	11

, (2.60)

where d is the spacetime dimension. We can now combine the results of (2.60) and find that for a vanishing central charge we need

$$d = 10. \tag{2.61}$$

A discussion why the vanishing of the central charges implies the absence of the Weyl anomaly can be found in [13, 12] and [15]. Let us also mention that not all contributions to $c = 15$, which cancel the contribution from the ghost systems, must be a free field theory. If we are interested in describing a theory with a 4-dimensional spacetime, we get a contribution of $c = 6$ from the X^μ, ψ^μ system and have a leftover $c = 9$ which can be described by a different superconformal theory. We will comment on this in the next section.

Chapter 3

String Compactifications and Calabi-Yau Manifolds

In Section 2.3 we saw that anomaly freedom of superstring theory requires to study a superconformal field theory with central charge $c = 15$. In order to make contact with currently observable physics in four spacetime dimensions we split this into an external theory with $c = 6$ and an internal theory with $c = 9$. The choice of the internal theory influences the physics observable in the external theory. The process of obtaining four dimensional physics out of string theory is called compactification.

The choice of the external theory is more or less fixed, because we want a theory which describes four dimensional Minkowski space. This can be achieved by using the free theory of $X^\mu, \psi^\mu, \quad \mu = 0, \ldots, 3$.

In the choice of the internal theory there is no such principle which would single out a specific theory. Only certain requirements on the observable spacetime theory reduce the set of possible theories. A requirement is $\mathcal{N} = 1$ supersymmetry. It was show in e.g. [16] that this requires $N = 2$ worldsheet supersymmetry[1].

A detailed discussion on the relation between properties of the spacetime theory and the internal conformal field theory can be found in [12].

A possible choice is to take for the $c = 9$ theory the same free theory as for the $c = 6$ part. This gives a theory in 10 dimensional Minkowski spacetime. The six extra dimensions are taken to be a compact space, with typical length scale which is small compared to current observable limits. The internal space is restricted by consistency conditions, which lead to Calabi-Yau manifolds. We will give more details on this procedure below.

It is not necessary to choose a compactification which has an obvious geometric interpretation as in the Calabi-Yau setup. A possibility would be $N = 2$ supersymmetric Landau-Ginzburg models, which have at a first glance no geometrical interpretation. However we will see in Section 5.2 that the

[1] We will denote worlsheet supersymmetry by N and target space supersymmetry by $\mathcal{N}$.

Landau-Ginzburg and Calabi-Yau setting occure as low energy descriptions of one model.

A more detailed discussion of the situation can be found in section 1.3 of [17].

3.1 $N = 2$ Superconformal Theories

Let us here discuss the features of a theory with $N = 2$ superconformal symmetry in 2 dimensions. As mentioned, the internal CFT needs this symmetry in order to obtain spacetime supersymmetry, but let us mention that in contrast to the $N = 1$ superconformal symmetry, discussed in Section 2.3, this symmetry is not the remnant of a local symmetry. Similar to Section 2.3 we call such theories $N = 2$ SCFTs. Our discussion follows [12, 18, 17, 19]. Remember that two dimensional theories have a holomorphic and an anti-holomorphic sector. The subsequently described structure can appear in either of these sectors. Therefore it is more convenient to denote such theories by $N = (2,2),(0,2),(2,0)$, depending in which sector we find the $N = 2$ superconformal symmetry. We focus on the holomorphic part and denote the anti-holomorphic quantities by bared operators.

An $N = 2$ SCFT has, compared to an $N = 1$ SCFT, a second supercurrent T_F^- and a conserved $U(1)$ current $J(z)$. We will focus on an algebraic approach. Extended superspace will be introduced in Section 4.1. Let us start from the operator product expansions of the conserved currents:

$$
\begin{aligned}
T(z)T(w) &= \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots, \\
T(z)T_F^{\pm}(w) &= \frac{\frac{3}{2}T_F^{\pm}(w)}{(z-w)^2} + \frac{\partial T_F^{\pm}(w)}{z-w} + \dots, \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \dots, \\
J(z)J(w) &= \frac{\frac{c}{3}}{(z-w)^2} + \dots, \\
J(z)T_F^{\pm}(w) &= \pm\frac{T_F^{\pm}(w)}{(z-w)} + \dots, \\
T_F^{+}(z)T_F^{-}(w) &= \frac{\frac{2c}{3}}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} + \dots, \\
T_F^{\pm}(z)T_F^{\pm}(w) &= \text{finite}.
\end{aligned}
\tag{3.1}
$$

An $N = 1$ superconformal sub-algebra is spanned by T and $T_F = \frac{1}{2\sqrt{2}}\left(T_F^+ + T_F^-\right)$. We can again perform a mode expansion and obtain the $N = 2$ super-Virasoro algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m+n}, \\
\left[L_m, G^{\pm}_{n\pm a}\right] &= \left(\frac{1}{2}m - n \mp a\right) G^{\pm}_{m+n+\pm a}, \\
[L_m, J_n] &= -nJ_{m+n}, \\
[J_m, J_n] &= \frac{c}{3} m\delta_{m+n}, \\
\left[J_m, G^{\pm}_{n\pm a}\right] &= \pm G^{\pm}_{m+n\pm a}, \\
\left\{G^{+}_{m+a}, G^{-}_{n-a}\right\} &= 2L_{m+n} + (m-n+2a)J_{m+n} \\
&\quad + \frac{c}{3}\left[(m+a)^2 - \frac{1}{4}\right]\delta_{m+n}, \\
\left\{G^{+}_{m+a}, G^{+}_{n+a}\right\} &= \left\{G^{-}_{m-a}, G^{-}_{n-a}\right\} = 0,
\end{aligned} \tag{3.2}$$

where J_n are the modes of $J(z)$ and $G^{\pm}_n$ are the modes of $T^{\pm}(z)$. The modes fulfil the following Hermiticity conditions

$$L^{\dagger}_n = L_{-n}, \qquad \left(G^{+}_{n+a}\right)^{\dagger} = G^{-n}_{-n-a}, \qquad J^{\dagger}_n = J_{-n}. \tag{3.3}$$

The real parameter a encodes the boundary condition on fermions:

$$a \in \begin{cases} \mathbb{Z} & \text{R-sector, anti-periodic,} \\ \mathbb{Z} + \frac{1}{2} & \text{NS-sector, periodic,} \end{cases} \tag{3.4}$$

and lies in the range $0 \leq a < 1$, as algebras for a and $a+1$ are isomorphic. We see from (3.2) that L_0 and J_0 form the maximal commuting subalgebra and so each state $|\phi\rangle$ in the Hilbert space is characterised by two quantum numbers (h, q):

$$L_0|\phi\rangle = h|\phi\rangle, \qquad J_0|\phi\rangle = q|\phi\rangle. \tag{3.5}$$

To construct a highest weight representation we further separate the generators into raising and lowering operators. The lowering operators are given by

$$L_n, J_m, G^{\pm}_r, \quad n, m, r > 0. \tag{3.6}$$

A highest weight state $|\phi\rangle$ has the property

$$\begin{aligned}
G^{\pm}_r|\phi\rangle = L_m|\phi\rangle = J_n|\phi\rangle = 0 \quad n, m, r > 0. \\
L_0|\phi\rangle = h_\phi|\phi\rangle, \quad J_0|\phi\rangle = q_\phi|\phi\rangle.
\end{aligned} \tag{3.7}$$

By the operator-state correspondence to any highest weight state (3.7) corresponds a so-called primary field $\phi^{(h,q)}$. If we focus on the NS sector and require unitarity of the representation we find that:

$$\begin{aligned}0 \leq \left|G^{\mp}_{-1/2}|\phi\rangle\right|^2 + \left|G^{\pm}_{1/2}|\phi\rangle\right|^2 &= \langle\phi| \left\{G^{\mp}_{-1/2}, G^{\pm}_{1/2}\right\} |\phi\rangle, \\ &= 2\left(h \pm \frac{1}{2}q\right) \langle\phi|\phi\rangle, \\ \Rightarrow h &\geq \frac{1}{2}|q|. \end{aligned} \tag{3.8}$$

Of particular interest are states which saturate the bound (3.8). The corresponding fields are called:

$$h = \begin{cases} \frac{q}{2} & \phi^{(\frac{q}{2},q)} \text{ chiral-primary,} \\ -\frac{q}{2} & \phi^{(-\frac{q}{2},q)} \text{ anti-chiral-primary.} \end{cases} \tag{3.9}$$

For a chiral-primary state $|\phi\rangle$ we further find

$$\begin{aligned}0 \leq \left|G^{+}_{-3/2}|\phi\rangle\right|^2 + \left|G^{-}_{3/2}|\phi\rangle\right|^2 &= \langle\phi| \left\{G^{+}_{-3/2}, G^{-}_{3/2}\right\} |\phi\rangle, \\ &= \langle\phi| \left(2L_0 - 3J_0 + \frac{2}{3}c\right) |\phi\rangle, \\ &= \left(-4h + \frac{2}{3}c\right) \langle\phi|\phi\rangle, \\ \Rightarrow h &\leq \frac{c}{6}. \end{aligned} \tag{3.10}$$

We can deduce from (3.8) that a chiral primary must fulfil

$$G^{+}_{-1/2}|\phi\rangle = G^{-}_{1/2}|\phi\rangle = 0. \tag{3.11}$$

In a similar way, from (3.8) if follows that an anti-chiral primary satisfies

$$G^{-}_{-1/2}|\phi\rangle = G^{+}_{1/2}|\phi\rangle = 0. \tag{3.12}$$

The conditions (3.11), (3.12) for the anti-holomorphic part follow simply by the replacement $G \to \overline{G}$. Observe that $G^{+}_{-1/2}$ and $G^{+}_{-1/2}$ give globally defined supercharges:

$$G^{\pm}_{-1/2} = \oint \frac{\mathrm{d}z}{2\pi i} T^{\pm}_{F}(z). \tag{3.13}$$

If we now consider a state $|\phi\rangle$ with (h, q) in the R-sector we find:

$$\begin{aligned}0 \leq \left|G^{+}_{0}|\phi\rangle\right|^2 + \left|G^{-}_{0}|\phi\rangle\right|^2 &= \langle\phi| \left\{G^{+}_{0}, G^{-}_{0}\right\} |\phi\rangle, \\ &= 2\left(h - \frac{c}{24}\right) |\phi\rangle, \\ \Rightarrow h &\geq \frac{c}{24}. \end{aligned} \tag{3.14}$$

Chiral Rings

If one studies the operator product of two chiral primary fields one finds that it only contains regular terms. This fact allows the point-wise multiplication of chiral primaries with the product operation given by:

$$(\phi_i \cdot \phi_j)(w) \equiv \lim_{z \to w} \phi_i(z)\phi_j(w) = \sum_k C_{ij}^k \phi_k(w), \tag{3.15}$$

where one the righthand-side only chiral primaries appear. In this way chiral primaries generate a ring. A similar ring structure, with multiplication given by (3.15), is found for the anti-chiral primaries. For an $N = (2, 2)$ theory four ring structures can be formed:

$$(c, c), \quad (a, c), \quad (c, a), \quad (a, a), \tag{3.16}$$

with c = chiral and a = anti-chiral. Two of the rings are always (charge) conjugate to each other: (c, c) to (a, a) and (a, c) to (c, a). A detailed analysis of the ring structure can be found in [20].

Spectral Flow

In the $N = 2$ super-Virasoro algebra (3.2) the additional parameter a appears. Different as define different algebras, but these algebras are all isomorphic. This can be shown by replacing $a = \eta + \frac{1}{2}$ in (3.2) and with the definitions:

$$\begin{aligned} L_n^\eta &= U_\eta L_n U_\eta^{-1} = L_n + \eta J_n + \frac{c}{6}\eta^2 \delta_n, \\ G_{n\pm a}^{\eta\pm} &= U_\eta G_{n+a}^{\pm} U_\eta^{-1} = G_{n\pm(a+\eta)}^{\pm}, \\ J_n^\eta &= U_\eta J_n U_\eta^{-1} = J_n + \frac{c}{3}\eta\delta_n, \end{aligned} \tag{3.17}$$

where we introduced a one-parameter isomorphism U_η relating the different algebras. The operator U_η allows us to obtain the transformed states:

$$|\phi_\eta\rangle = U_\eta |\phi\rangle. \tag{3.18}$$

The operator U_η is called spectral flow operator. From (3.2) we see that η does not modify the operators L_n and J_n and therefore we can act with them on the states in the new algebra. Of particular interest are:

$$L_0|\phi_\eta\rangle = h_\eta|\phi_\eta\rangle, \qquad\qquad J_0|\phi_\eta\rangle = q_\eta|\phi_\eta\rangle. \tag{3.19}$$

On the other hand we can also act with the transformed operators (3.17):

$$\begin{aligned} h|\phi_\eta\rangle &= U_\eta L_0|\phi\rangle = L_0^\eta|\phi_\eta\rangle \\ &= \left(L_0 + \eta J_0 + \frac{c}{6}\eta^2\right)|\phi_\eta\rangle = \left(h_\eta + \eta q_\eta + \frac{c}{6}\eta^2\right)|\phi_\eta\rangle, \end{aligned} \tag{3.20}$$

$$q|\phi_\eta\rangle = U_\eta J_0|\phi\rangle = J_0^\eta|\phi_\eta\rangle = \left(J_0 + \frac{c}{3}\eta\right)|\phi_\eta\rangle = \left(q_\eta + \frac{c}{3}\eta\right)|\phi_\eta\rangle, \tag{3.21}$$

where h and q are the conformal weight and $U(1)$ charge of the state before the spectral flow. It follows that:

$$\begin{aligned} h_\eta &= h - \eta q + \frac{c}{6}\eta^2, \\ q_\eta &= q - \frac{c}{3}\eta. \end{aligned} \tag{3.22}$$

Let us next study how chiral primary states in the NS sector behave under spectral flow by $\eta = \frac{1}{2}$. We clearly end up in the R sector under such a flow. We find by using (3.22):

$$\left|\frac{q}{2}, q\right\rangle_{\mathrm{NS}} \xrightarrow{\eta=\frac{1}{2}} \left|h_{\frac{1}{2}} = \frac{c}{24}, q_{\frac{1}{2}} = q - \frac{c}{6}\right\rangle_{\mathrm{R}}, \tag{3.23}$$

and by considering (3.14) we find that we get a ground state in the R sector. We see that there is a one-to-one correspondence between NS sector chiral primaries and ground states in the R sector. In a similar way we can show that there is a one-to-one correspondence between anti-chiral primaries and R ground states (flow by $\eta = -\frac{1}{2}$).

Next we consider spectral flow by $\eta = 1$. We again start in the NS sector and we end up in the NS sector. If we consider the behaviour of a chiral primary field, we find

$$\left|\frac{q}{2}, q\right\rangle_{\mathrm{NS}} \xrightarrow{\eta=1} \left|h_1 = -\frac{q_1}{2}, q_1 = q - \frac{c}{3}\right\rangle_{\mathrm{NS}}, \tag{3.24}$$

and see that we end up with an anti-chiral primary. Of particular interest is the behaviour of the NS vacuum $|0,0\rangle_{\mathrm{NS}}$ under the flow:

$$|0,0\rangle_{\mathrm{NS}} \xrightarrow{\eta=1} |\frac{c}{6}, -\frac{c}{3}\rangle_{\mathrm{NS}}. \tag{3.25}$$

This is the charge conjugate of the unique state in the chiral ring saturating (3.10). We see that such a chiral primary state with $h = \frac{c}{6}$ must always exist in the theory.

Let us mention that we only discussed spectral flow in one sector. Of course it is possible to simultaneously apply a flow in both sectors. The first appearance of spectral flow was in [21].

Deformations

Let us consider operators of holomorphic/anti-holomorphic conformal weight $(h, \overline{h})$. There are three kinds of possible deformations:

1. $h + \overline{h} > 2$, Irrelevant operators: A deformation by such an operator will not have an effect on the theory at its infrared fix point.
2. $h + \overline{h} < 2$, Relevant operators: Such an operator has direct influence at the infrared behaviour.

3. $h+\overline{h}=2$, Marginal operators: These operators can be used to deform a conformal theory to a nearby conformal theory. The central charge stays the same.

We focus on deformations of the third kind, in particular to operators with $(h,\overline{h})=(1,1)$. In fact we need to focus on $(1,1)$ operators which are still $(1,1)$ after the deformation of the theory by any of the $(1,1)$ operators. These operators are called truly marginal. We can construct such operators in the following way. We start from ϕ in the (c,c) ring with $h=\overline{h}=1/2$ and $q=\overline{q}=1$ and define

$$\hat{\phi}=\oint \mathrm{d}z T_F^-(z)\phi(w,\overline{w}). \tag{3.26}$$

The operator $\hat{\phi}$ has $h=1,\overline{h}=1/2$ and $q=0,\overline{q}=1$. Next we introduce

$$\Phi_{(1,1)}(w,\overline{w})=\oint \mathrm{d}\overline{z}\overline{T}_F^-(\overline{z})\hat{\phi}(w,\overline{w}), \tag{3.27}$$

with conformal weight $h=1,\overline{h}=1$ and $q=\overline{q}=0$. In a similar way we can start from an operator ϕ in the (a,c) ring with $h=\overline{h}=1/2$ and $-q=\overline{q}=1$ and construct

$$\hat{\phi}(w,\overline{w})=\oint \mathrm{d}\overline{z}\overline{T}_F^-(\overline{z})\phi(w,\overline{w}), \tag{3.28}$$

$$\Phi_{-1,1}=\oint \mathrm{d}z T_F^+(z)\hat{\phi}(w,\overline{w}), \tag{3.29}$$

with $\Phi_{-1,1}$ of conformal weight $h=\overline{h}=1$ and $q=\overline{q}=0$. Both of the operators $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$ are truly marginal as shown in [22, 23].

These truly marginal operators span the moduli space of the superconformal theory. In this moduli space each point corresponds to a superconformal theory and by the truly marginal operators we can deform one theory to another. A metric on the moduli space is given by the Zamolodchikov metric [24]. The metric is block diagonal in the deformations[2] $\Phi_{(1,1)}$ and $\Phi_{(-1,1)}$. So locally the moduli space is a product of the space of deformations of type $\Phi_{(1,1)}$ and of the space of deformations of type $\Phi_{(-1,1)}$.

3.2 Compactifications and Strings on curved Backgrounds

This section is based on [15, 12]. We talked previously about the possibility of compactifying the superstring on a target space with structure

$$\mathcal{M}_4\times K_6, \tag{3.30}$$

[2] A discussion from a sigma model perspective can be found in e.g. [25].

where $\mathcal{M}_4$ is 4 dimensional Minkowski space and K_6 is a compact six-dimensional space. The main consistency requirement is that we do not break conformal invariance. To study this requirement we must consider string theory on a curved background. Up to now we have considered string theory on Minkowski space. The extension to a non-trivial background is given by starting from (2.7) and replacing $\eta_{\mu\nu}$ by $G_{\mu\nu}(X)$

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu. \tag{3.31}$$

In contrast to (2.7), (3.31) is an interacting theory and called non-linear sigma model. Let us also comment more on the form of (3.31). The massless excitations of the string contain a graviton field and if one expands $G_{\mu\nu}(X)$ around fluctuations

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + g_{\mu\nu}(X), \tag{3.32}$$

the non-trivial background can be interpreted as a coherent-state of gravitons. An expansion around a classical string solution reveals that one can trust a perturbative analysis of (3.31) as long as

$$\frac{\sqrt{\alpha'}}{L} = \frac{l_s}{L} \ll 1, \tag{3.33}$$

where L is a typical length scale of the background geometry. The spectrum of the string contains further massless fields, namely the dilaton ϕ and an anti-symmetric tensorfield the B-field. The string can also be coupled to these fields, but for our discussion it is sufficient to focus on (3.31).

The action (3.31) is classically conformally invariant. In the process of quantization and renormalization the coupling constants of our theory become dependent on the scale μ. This breaks conformal invariance unless the β-functions of the couplings vanish. In our theory the coupling is given by $G_{\mu\nu}$ and it can be shown [15, 13] that at one loop order:

$$\beta_{\mu\nu}(G) = \alpha' \mathcal{R}_{\mu\nu} + \dots . \tag{3.34}$$

We see that conformal invariance requires a Ricci-flat target space.

A further requirement on the compactification space K_6 is, that it has covariant constant spinors. This condition ensures, that some supersymmetry is preserved in 4 dimensions. The requirement of covariant constant spinors restricts the allowed holonomy $\mathcal{H}$ of the manifold. Of particular interest for phenomenology are manifolds with $\mathcal{H} = SU(3)$. This can be seen by considering the compactification of the low energy effective actions of the superstring as done for example in [12]. Ricci-flatness ensures that the holonomy $\mathcal{H} \subseteq SU(N)$.

In conclusion the above requirements of Ricci flatness and $SU(3)$ holonomy lead to Calabi-Yau manifolds, which will be discussed in the next section.

3.3 Calabi-Yau Manifolds

We follow [17, 12, 26]. Calabi-Yau (CY) manifolds M of complex dimension n are Kähler manifolds with vanishing first Chern class $c_1(M) = 0$. Kähler manifolds are manifolds which have a Hermitian metric with a closed Kähler form J. By Yau's Theorem [27] vanishing of the first Chern class is equivalent to the vanishing of the Ricci-tensor $\mathcal{R}$. It can be shown that vanishing of the Ricci-tensor implies that the holonomy group of the manifold is a subgroup of $SU(n)$. As in the literature, we will take CY manifolds to have holonomy group exactly $SU(n)$. A further consequence of the vanishing of the first Chern class is, that the canonical line bundle $K(M) = \bigwedge^n T^{*1,0}M$ is topologically trivial. This implies the existence of a nowhere vanishing unique global section. For $K(M)$ this section is given by a globally defined nowhere vanishing holomorphic n-form on M: Ω. It is possible to show that Ω is unique. Next we study the cohomology groups of CY manifolds as these encode topological properties of the manifold. We start be recalling the relation between the deRahm cohomology groups $H^k(M, \mathbb{C})$ and the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(M)$:

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}(M). \tag{3.35}$$

The Hodge numbers $h^{p,q}$ are given by

$$h^{p,q}(M) = \dim_{\mathbb{C}} \left(H^{p,q}_{\bar{\partial}}(\mathcal{M}) \right). \tag{3.36}$$

These numbers are related to the Betti-Numbers $b_r = \dim \left(H^k(M) \right)$ on a Kähler manifold by:

$$b_r = \sum_{p+q=r} h^{p,q}. \tag{3.37}$$

The Hodge numbers are usually represented in a Hodge diamond. The Hodge numbers of a manifold fulfil certain relations, like Hodge duality and complex conjugation, which are reflected in the form of the Hodge diamond. For more details on these relations see e.g. [12]. We will provide the Hodge diamond for CY manifolds with $n = 3$, so called CY threefolds (3-folds). This is the right dimension for superstring theory compactifications to 4 dimensions. In these manifolds we find:

$$\begin{array}{ccccccc}
 & & & h^{0,0}=1 & & & \\
 & & h^{1,0}=0 & & h^{0,1}=0 & & \\
 & h^{2,0}=0 & & h^{1,1} & & h^{0,2}=0 & \\
h^{3,0}=1 & & h^{2,1} & & h^{1,2}=h^{2,1} & & h^{0,3}=1 \\
 & h^{3,1}=0 & & h^{2,2}=h^{1,1} & & h^{1,3}=0 & \\
 & & h^{3,2}=0 & & h^{2,3}=0 & & \\
 & & & h^{3,3}=1 & & &
\end{array} \tag{3.38}$$

The only independent Hodge numbers of a CY 3-fold are $h^{1,1}$ and $h^{2,1}$. These will be important in the next section when we consider the moduli space. Further we can conclude from (3.38) and (3.37) that the Euler number of a CY 3-fold is

$$\chi(M) = 2\left(h^{1,1}(M) - h^{1,2}(M)\right). \tag{3.39}$$

Moduli Space of Calabi-Yau Manifolds

Next we ask ourselves if it is possible to deform the Ricci tensor such that the CY stays Ricci-flat. In our discussion we follow [28, 12]. We can formalize our question in the following way: Let g be a Ricci-flat metric on M and consider $g_{\mu\nu} + \delta g_{\mu\nu}$. The question is now what infinitesimal deformations δg fulfil:

$$R_{\mu\nu}(g) = 0 \quad \Rightarrow \quad R_{\mu\nu}\left(g + \delta g\right) = 0. \tag{3.40}$$

Of course every diffeomorphism of g has this property. Therefore these have to be excluded. This can be achieved by a procedure, similar to gauge fixing. We set:

$$\nabla^{\mu}\delta g_{\mu\nu} = 0. \tag{3.41}$$

In this gauge the expansion of the righthand side of (3.40) to first order in δg results in:

$$\nabla^{\rho}\nabla_{\rho}\delta g_{\mu\nu} + 2R_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma}\delta g_{\rho\sigma} = 0. \tag{3.42}$$

This equation is called Lichnerowicz equation. On a Kähler manifold (M, J) the components of mixed type $\delta g_{\mu\bar{\nu}}$ and of pure type $\delta g_{\mu\nu}, \delta g_{\bar{\mu}\bar{\nu}}$ separately solve (3.42). The variation of mixed type can be interpreted as components of a real $(1,1)$-form:

$$i\delta g_{\mu\bar{\nu}} \mathrm{d}x^{\mu} \wedge \mathrm{d}\bar{x}^{\bar{\nu}}. \tag{3.43}$$

This $(1,1)$ form is harmonic for solutions of (3.42). Also we can associate a complex $(2,1)$ form to the components of pure type

$$\Omega_{\kappa\lambda}{}^{\bar{\nu}}\delta g_{\bar{\mu}\bar{\nu}} \mathrm{d}x^{\kappa} \wedge \mathrm{d}x^{\lambda} \wedge \mathrm{d}\bar{x}^{\bar{\mu}}, \tag{3.44}$$

where Ω is the holomorphic $(3,0)$ form introduced in Section 3.3. This is again a harmonic form on solutions of (3.42).

The above findings can be summarized in the statement that there is a one-to-one correspondence between solutions of (3.42) and elements of $H^{(1,1)}(M)$ and $H^{(2,1)}(M)$. The Kähler form J of M is itself a closed $(1,1)$-form:

$$J = ig_{\mu\bar{\nu}} \mathrm{d}x^{\mu} \wedge \mathrm{d}\bar{x}^{\bar{\nu}}, \tag{3.45}$$

and it is clear that deformations of mixed type correspond to deformations of the Kähler class. We can expand the $\delta g_{\mu\bar{\nu}}$ in a basis of real harmonic $(1,1)$ forms:

$$\delta g_{\mu\bar{\nu}} = \sum_{\alpha=1}^{h^{1,1}} \zeta^\alpha b^\alpha_{\mu\bar{\nu}}, \quad \zeta^\alpha \in \mathbb{R}, \tag{3.46}$$

where $h^{1,1} = \dim\left(H^{(1,1)}_{\bar{\partial}}(M)\right)$. The parameters ζ^α are called Kähler moduli. It is important to note that the moduli have to be chosen such that the deformed metric stays positive definite. This is equivalent to the conditions:

$$\int_C J > 0, \qquad \int_S J^2 > 0, \qquad \int_M J^3 > 0, \tag{3.47}$$

for all curves C and surfaces S on M. The parameters fulfilling (3.47) form a subset of $\mathbb{R}^{h^{1,1}}$ called Kähler cone. The pure type deformations can be expanded into a basis of harmonic $(2,1)$ forms:

$$\Omega_{ijk}\delta g^k_{\bar{l}} = \sum_{\alpha=1}^{h^{2,1}} z^\alpha b^\alpha_{ij\bar{l}}, \quad z^\alpha \in \mathbb{C}, \tag{3.48}$$

with $h^{(2,1)} = \dim\left(H^{(2,1)}(M)\right)$ The parameters z^α are called complex structure moduli and as the name suggest correspond to deformations of the complex structure[3].

In total there are $h^{1,1} + 2h^{2,1}$ real deformation parameters. These parameters form the moduli space $\mathcal{M}$ of the CY. Up to now we have not taken into account string theory. In string theory we have an additional massless antisymmetric tensor field B. Excitations of the B-field correspond to harmonic two-forms on the CY manifold. This allows to combine the B-filed excitations with the Kähler deformations:

$$(i\delta g_{\mu\bar{\nu}} + \delta B_{\mu\bar{\nu}})\,\mathrm{d}x^\mu \wedge \mathrm{d}\bar{x}^{\bar{\nu}} = \sum_{\alpha=1}^{h^{1,1}} t^\alpha b^\alpha, \quad t^\alpha \in \mathbb{C}. \tag{3.49}$$

Nevertheless the imaginary part of the t^α must still fulfil (3.47). This process is called complexification of the Kähler cone.

As discussed in [28] the metric on $\mathcal{M}$ is locally block diagonal and therefore the moduli space looks locally like

$$\mathcal{M} = \mathcal{M}_{CS} \times \mathcal{M}_{KS}, \tag{3.50}$$

where $\mathcal{M}_{CS}$ is the space of complex structure deformations and $\mathcal{M}_{KS}$ is the space of Kähler deformations. Both spaces are special Kähler, which means

[3] See [12] for an explanation of this correspondence.

that their Kähler potentials

$$K_{CS} = -\log \int_M \Omega \wedge \overline{\Omega}, \tag{3.51}$$

$$K_{KS} = -\log \kappa(J, J, J), \tag{3.52}$$

can be obtained from pre-potentials. Pre-potentials are certain holomorphic functions on the moduli space. More details on them and their geometry can be found in [28, 29]. In (3.52) we introduced

$$\kappa(\rho, \sigma, \tau) = \int_M \rho \wedge \sigma \wedge \tau, \tag{3.53}$$

with ρ, σ and τ in $H^{(1,1)}(M)$. Further, (3.52) is just the classical expression and the Kähler potential on $\mathcal{M}_{KS}$ receives perturbative and non-perturbative corrections in α'. This will be evident in Chapter 6.

Strings on Calabi-Yau Manifolds

Let us come back to the physical description of a string on a Calabi-Yau manifold. As mentioned in Section 3.2, a string propagating on a non-trivial background is given by a 2 dimensional non-linear sigma model and in the case of the superstring by the supersymmetric version thereof[4].

We will follow [30, 31] to discuss some aspects of the supersymmetric non-linear sigma model (NLSM). In general terms, a NLSM describes maps Φ from a Riemann surface Σ to a target space M:

$$\Phi : \Sigma \to M. \tag{3.54}$$

To obtain an $N = (2, 2)$ supersymmetric NLSM, M must be a Kähler manifold. By a similar reasoning as in Section 3.2 for a CY target space the model has $N = (2, 2)$ superconformal invariance, which gives the required $N = (2, 2)$ superconformal symmetry on the worldsheet. The action is given by, with notation as in [30]:

$$\begin{aligned} S = \frac{1}{4\pi} \int_\Sigma \mathrm{d}^2 z \, \Big(& g_{i\bar{\jmath}} \left(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^{\bar{\jmath}} + \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^{\bar{\jmath}} \right) + B_{i\bar{\jmath}} \left(\partial_z \phi^i \partial_{\bar{z}} \bar{\phi}^{\bar{\jmath}} - \partial_{\bar{z}} \phi^i \partial_z \bar{\phi}^{\bar{\jmath}} \right) \\ & + i g_{i\bar{\jmath}} \psi^i D_z \psi^{\bar{\jmath}} + i g_{i\bar{\jmath}} \chi^i D_{\bar{z}} \chi^{\bar{\jmath}} + R_{i\bar{k}j\bar{l}} \psi^i \psi^{\bar{k}} \chi^j \chi^{\bar{l}} \Big) . \end{aligned} \tag{3.55}$$

The explicit action can be derived from the results in Section 4.1 by using superspace techniques. The scalar fields ϕ^i are complex coordinates of the map Φ to M (3.54) and $\bar{\phi}^{\bar{\imath}} = \overline{\phi^i}$ are the complex conjugated ones. The fermionic

[4] In order to describe string theory we would need to couple the non-linear sigma model to two dimensional gravity. However we can discuss certain aspects without taking gravity into account.

degrees of freedom are given by ψ and χ, which are left-moving and right-moving respectively. The fermions can be seen as sections of the following bundles:

$$\begin{aligned}
\psi^i &\in \Gamma\left(K^{1/2}\otimes \phi^* T^{(1,0)}M\right), \quad \psi^{\bar\imath} \in \Gamma\left(K^{1/2}\otimes \phi^* T^{(0,1)}M\right),\\
\chi^i &\in \Gamma\left(\overline{K}^{1/2}\otimes \phi^* T^{(1,0)}M\right), \quad \chi^{\bar\imath} \in \Gamma\left(\overline{K}^{1/2}\otimes \phi^* T^{(0,1)}M\right),
\end{aligned} \tag{3.56}$$

where K is the canonical bundle (holomorphic cotangent bundle) on Σ and $\phi^* T^{(-,-)}M$ is the pullback of the (anti)-holomorphic tangentbundle on M. In (3.55) D is a covariant derivative on the respective bundle (3.56) and R is the Riemann tensor of M. We also introduced a B-field on M. The action (3.55) is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta\phi^i &= i\epsilon^-\psi^i + i\overline{\epsilon}^-\chi^i, & \delta\overline{\phi}^{\bar\imath} &= i\epsilon^+\psi^{\bar\imath} + i\overline{\epsilon}^+\chi^{\bar\imath},\\
\delta\psi^i &= -\epsilon^+\partial\phi^i - i\overline{\epsilon}^-\chi^j\Gamma^i_{jk}\psi^k, & \delta\psi^{\bar\imath} &= -\epsilon^-\partial\overline{\phi}^{\bar\imath} - i\overline{\epsilon}^+\chi^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar k}\psi^{\bar k},\\
\delta\chi^i &= -\overline{\epsilon}^+\overline{\partial}\phi^i - i\epsilon^-\psi^j\Gamma^i_{jk}\chi^k, & \delta\chi^{\bar\imath} &= -\overline{\epsilon}^-\overline{\partial}\overline{\phi}^{\bar\imath} - i\epsilon^+\psi^{\bar\jmath}\Gamma^{\bar\imath}_{\bar\jmath\bar k}\chi^{\bar k},
\end{aligned} \tag{3.57}$$

with $\epsilon^\pm$ are sections of $K^{-1/2}$ and $\overline{\epsilon}^\pm$ are sections of $\overline{K}^{-1/2}$. In (3.57) Γ is the Levi-Cevita connection on M.

Next we want to study the geometric interpretation of the (c, c) and (a, c) rings. This is best done in the topological version of the NLSM. In the topological version we have a Grassmann scalar symmetry operator Q. The operator Q is nilpotent and physical sates correspond to the cohomology classes of Q. In the literature the operator Q is called BRST operator. Further, in a topological theory the energy momentum tensor T is Q exact:

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}. \tag{3.58}$$

For our purposes we want to build an operator which annihilates states in the $(c, c)/(a, c)$ ring. We can construct such operators if we look back at (3.13):

$$\begin{aligned}
Q_A &= G^-_{-1/2} + \overline{G}^+_{-1/2}, \quad (a, c),\\
Q_B &= G^+_{-1/2} + \overline{G}^+_{-1/2}, \quad (c, c).
\end{aligned} \tag{3.59}$$

Both operators are nilpotent, but they are fermionic and not scalar. Also it can be shown that the energy momentum tensor is not $Q_{A/B}$ exact. To obtain a topological theory a so-called topological twist has to be performed. In order to twist the theory, the energy momentum tensor has to be modified. In our case there are two possibilities called A-twist and B-twist:

$$\left.\begin{aligned} T &\to T + \frac{1}{2}\partial J,\\ \overline{T} &\to \overline{T} - \frac{1}{2}\overline{\partial J}, \end{aligned}\right\}\text{A-twist},\qquad \left.\begin{aligned} T &\to T - \frac{1}{2}\partial J,\\ \overline{T} &\to \overline{T} - \frac{1}{2}\overline{\partial J}, \end{aligned}\right\}\text{B-twist}. \tag{3.60}$$

If we apply the A-twist, the operator Q_A becomes scalar. This can be derived from (3.1). The physical states are given by the (a, c) states. In the B-twisted model the BRST operator is Q_B and physical states are given by (c, c) states. After the twist the topological NLSM is referred to as A-model or B-model respectively. A second motivation for the topological twist is, that we now have a globally defined BRST operator on an arbitrary Riemann surface. This is important when gravity is coupled to the NLSM.

A-Model

The A-twist (3.60) affects the bundle structure (3.56) in the following way

$$\begin{aligned} &\psi^i \in \Gamma\left(\phi^* T^{(1,0)} M\right), &&\psi^{\bar{\imath}}_z \in \Gamma\left(K \otimes \phi^* T^{(0,1)} M\right), \\ &\chi^i_{\bar{z}} \in \Gamma\left(\overline{K} \otimes \phi^* T^{(1,0)M}\right), &&\chi^{\bar{\imath}} \in \Gamma\left(\phi^* T^{(0,1)} M\right). \end{aligned} \tag{3.61}$$

The fermions $\psi^i, \chi^{\bar{\imath}}$ became worldheet scalars. The transformation of the fields under the operator Q_A is obtained by setting $\epsilon^- = \overline{\epsilon}^+ = \epsilon$ and $\epsilon^+ = \overline{\epsilon}^- = 0$ in (3.57). The so obtained transformations lead to the identifications:

$$\psi^i \leftrightarrow \mathrm{d}\phi^i, \qquad \chi^{\bar{\imath}} \leftrightarrow \mathrm{d}\overline{\phi}^{\bar{\imath}}, \qquad Q_A \leftrightarrow \mathrm{d} = \partial + \overline{\partial}, \tag{3.62}$$

where d is the de Rham operator on M. As further discussed in [30, 31] the operators constructed out of worldsheet scalars are in one-to-one correspondence with the de Rham cohomology of M:

$$\begin{array}{c} w(\phi)_{i_1,\ldots,i_p,\bar{\imath}_1,\ldots,\bar{\imath}_q} \psi^{i_1} \cdots \psi^{i_p} \chi^{\bar{\imath}_1} \cdots \chi^{\bar{\imath}_q} \\ \updownarrow \\ w(\phi)_{i_1,\ldots,i_p,\bar{\imath}_1,\ldots,\bar{\imath}_q} \mathrm{d}\phi^{i_1} \cdots \mathrm{d}\phi^{i_p} \mathrm{d}\overline{\phi}^{\bar{\imath}_1} \cdots \mathrm{d}\overline{\phi}^{\bar{\imath}_q} \end{array}, \tag{3.63}$$

with w totally anti-symmetric. The action can be rewritten into

$$S = i \int_\Sigma \{Q_A, \mathcal{D}\} - 2\pi \int_\Sigma \phi^* \left(B + iJ\right), \tag{3.64}$$

where J is the Kähler form on M and

$$\mathcal{D} = 2\pi g_{i\bar{\jmath}} \left(\psi^{\bar{\jmath}} \overline{\partial} \phi^i + \partial \overline{\phi}^{\bar{\jmath}} \chi^i\right). \tag{3.65}$$

Observe that the second term in (3.64) depends only on the class $\beta = \phi_*(\Sigma) \in H_2(M, \mathbb{Z})$. We introduce:

$$t_\beta = \int_\Sigma \phi^* \left(B + iJ\right), \quad \phi_*(\Sigma) \in \beta. \tag{3.66}$$

As argued for example in [31] the partition function:

$$Z = \sum_{\beta \in H_2} e^{-2\pi i t_\beta} \int_{\phi_*(\Sigma) \in \beta} \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\chi e^{-i \int \{Q_A, \mathcal{D}\}}, \tag{3.67}$$

localises[5] to maps ϕ with $\mathcal{D} = 0$.

These are holomorphic maps:

$$\overline{\partial}\phi = 0. \tag{3.68}$$

From a physics standpoint the sum over the holomorphic maps corresponds to a sum over different worldsheet instanton sectors.

B-Model

The effect of the B-twist (3.60) is the following shift in the bundles (3.56):

$$\begin{aligned} \psi_z^i &\in \Gamma\left(K \otimes \phi^* T^{(1,0)} M\right), \quad \psi^{\bar{\imath}} \in \Gamma\left(\phi^* T^{(0,1)} M\right), \\ \chi_{\bar{z}}^i &\in \Gamma\left(\overline{K} \otimes \phi^* T^{(1,0)} M\right), \quad \chi^{\bar{\imath}} \in \Gamma\left(\phi^* T^{(0,1)} M\right). \end{aligned} \tag{3.69}$$

In the study of the B-model it is convenient to group the worldsheet scalars $\psi^{\bar{\imath}}, \chi^{\bar{\imath}}$ into

$$\eta^{\bar{\imath}} = \psi^{\bar{\imath}} + \chi^{\bar{\imath}}, \qquad \theta_j = g_{\bar{\imath}j}\left(\psi^{\bar{\imath}} - \chi^{\bar{\imath}}\right). \tag{3.70}$$

In addition the one-form ρ^i is introduced with $(1,0)$ component given by ψ_z^i and $(0,1)$ component $\chi_{\bar{z}}^i$. The relevant cohomology is the one of the operator Q_B which can be studied by setting $\epsilon^+ = \overline{\epsilon}^+ = \epsilon$ and $\epsilon^- = \overline{\epsilon}^- = 0$ in (3.57). The supersymmetry transformations suggest the identifications:

$$\eta^{\bar{\imath}} \leftrightarrow \mathrm{d}\overline{\phi}^{\bar{\imath}}, \qquad \theta_i \leftrightarrow \frac{\partial}{\partial \phi^i}, \qquad Q_B \leftrightarrow \overline{\partial}. \tag{3.71}$$

The scalar operators of the theory are in one-to-one correspondence [31] with $(0,q)$ forms on M with values in $\bigwedge^p T^{(1,0)} M$:

$$\begin{array}{c} w(\phi)^{i_1,\ldots,i_p}_{\bar{\imath}_1,\ldots,\bar{\imath}_q} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_q} \theta_{i_1} \cdots \theta_{i_p} \\ \updownarrow \\ w(\phi)^{i_1,\ldots,i_p}_{\bar{\imath}_1,\ldots,\bar{\imath}_q} \mathrm{d}\overline{\phi}^{\bar{\imath}_1} \cdots \mathrm{d}\overline{\phi}^{\bar{\imath}_q} \dfrac{\partial}{\partial \phi^{i_1}} \cdots \dfrac{\partial}{\partial \phi^{i_p}} \end{array}, \tag{3.72}$$

with w anti-symmetric in i and $\bar{\imath}$. These $(0,q)$ forms can be related to $(3-p,q)$ forms by using the holomorphic $(3,0)$ form Ω of the CY. If we introduce

$$\mathcal{D} = g_{j\bar{k}}\left(\rho_z^j \overline{\partial \phi}^{\bar{k}} + \rho_{\bar{z}}^j \partial \overline{\phi}^{\bar{k}}\right), \tag{3.73}$$

$$U = \int_\Sigma \left(-\theta_j D\rho^j - \frac{i}{2} R_{j\bar{\jmath}k\bar{k}} \rho^j \wedge \rho^k \eta^{\bar{\jmath}} \theta_l g^{l\bar{k}}\right), \tag{3.74}$$

[5]The localisation argument for supersymmetric field theories will be discussed in Section 4.3.

the action can be written in the form:

$$S = i \int_{\Sigma} \{Q_B, \mathcal{D}\} + U. \tag{3.75}$$

Similar to the A-model it can be argued that the path integral and correlation functions localize to (see Section 4.3 and [31]) the loci where $\mathcal{D} = 0$. For the B-model these loci are given by constant maps:

$$\partial \bar{\phi}^{\bar{k}} = \overline{\partial} \bar{\phi}^{\bar{k}} = 0. \tag{3.76}$$

The worldsheet gets mapped to a point in M and in the B-model we do not get instanton corrections.

Comparison of A- and B-Model.

The first point we want to mention is that the A-model (3.64) only depends on the Kähler structure of the target space and does not get influenced by the complex structure of M. For the B-model the situation is reversed. This is also consistent if we consider deformations of the theory. As discussed in Section 3.1, truly marginal deformations can be constructed from operators in the $(c, c)/(a, c)$ ring with $U(1)$ charges $(1, 1)/(-1, 1)$ respectively. In the A-model the operators with $U(1)$ charges $(-1, 1)$ correspond to elements of $H^{1,1}(M)$, which coincide, as discussed in Section 3.3, with deformations of the Kähler structure of M. The same argument is possible in the B-model, where the operators of charge $(1, 1)$ correspond to $(0, 1)$ forms with values in $T^{(1,0)}$. As mentioned, these forms correspond to $(2, 1)$ forms via the unique holomorphic $(3, 0)$ form Ω. These forms parameterise the deformations of the complex structure of M. The final point we want to make concerns the partition function. The A-model partition function localises to holomorphic maps and the path integral can be rewritten into an integral over the moduli space of such maps. Further we encountered instanton corrections. An interesting aspect of the partition function in the A-model is that it encodes enumerative invariants, so-called Gromov-Witten invariants, which describe certain intersection numbers in the target space M. These invariants are of interest for mathematicians as well. Unfortunately the partition function in the A-model is hard to compute. One way to calculate the partition function uses results from supersymmetric localisation in the gauged linear sigma model. We will come back to such techniques in Chapter 6. Another way is given by mirror symmetry which uses that in the B-model the partition function localises to constant maps and the remaining integral is relatively easy to compute. We comment on this in Section 3.4.

3.4 Mirror Symmetry

Nowadays there are various definitions of mirror symmetry. An overview of the developments can be found in the introductory chapter of [3]. We will focus on the following definition, namely that the operator algebra of the A-model on a CY M is isomorphic to the B-model operator algebra on the CY $\widetilde{M}$. M and $\widetilde{M}$ are called mirror pair. For CY threefolds this results in

$$h^{p,q}(M) = h^{3-p,q}(\widetilde{M}). \tag{3.77}$$

As we saw in Section 3.3, the A-model depends on the Kähler structure and the B-Model on the complex structure. It follows that mirror symmetry must map the moduli space of Kähler structures of M to the moduli space of complex structure of $\widetilde{M}$. This map is called mirror map, because mirror CY threefolds $M, \widetilde{M}$ have a “mirrored” Hodge diamond (3.38):

$$h^{1,1}_M \leftrightarrow h^{2,1}_{\widetilde{M}} \qquad\qquad h^{2,1}_M \leftrightarrow h^{1,1}_{\widetilde{M}}. \tag{3.78}$$

This definition of mirror symmetry goes back to the following observation. Form the standpoint of the $N = 2$ superconformal symmetry it is just convention which ring we call (c, c) or (a, c) as it is just a relative sign of the $U(1)$ charges. This point was first raised in [23] and further elaborated on in [20]. They used the fact that the number $h^{p,q}$ of (c, c)-ring elements with $U(1)_{L/R}$ charges (p, q) coincides with the Hodge numbers of a CY manifold M. By spectral flow it follows that the number of (a, c) elements with charge $(-p, q)$ is given by $h^{\frac{c}{3}-p,q}$. They concluded that there must exist a CY manifold $\widetilde{M}$ with $\dim\left(H^{(p,q)}\left(\widetilde{M}\right)\right) = h^{3-p,q}$. Since this observation various algorithms have been developed to construct mirror manifolds, see e.g. [32, 33].

Chapter 4

The GLSM and Supersymmetric Localisation

We start this chapter by first formulating $N = (2, 2)$ supersymmetry in terms of transformations on superspace. The superspace formalism allows us to write down supersymmetric Lagrangians in a straightforward manner. Afterwards we will discuss the gauged linear sigma model (GLSM). We describe the basic idea behind the GLSM and discuss renormalization and quantum effects in this model. We introduce the basics of supersymmetric localisation and give results from this technique related to the GLSM.

4.1 $N = (2, 2)$ Supersymmetry in 2 Dimensions

In this section we follow [3] and we will use the same notation as in this reference.

Superspace

We start from a 2-dimensional Minkowski-space with coordinates x^0, x^1 and metric

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

The starting point of the superspace formalism is to extend the coordinates by fermionic coordinates. We are interested in $N = (2, 2)$ supersymmetry and for that purpose we need 4 additional fermionic coordinates

$$\theta^{\pm}, \overline{\theta}^{\pm}. \tag{4.2}$$

These fermionic coordinates are complex and related by

$$\left(\theta^{\pm}\right)^* = \overline{\theta}^{\pm}. \tag{4.3}$$

By $\pm$ we denote the spin/chirality under 2-dimensional Lorentz transformations. In detail these transformations read

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \tag{4.4}$$

$$\theta^{\pm} \to e^{\pm\frac{\gamma}{2}}\theta^{\pm}, \quad \overline{\theta}^{\pm} \to e^{\pm\frac{\gamma}{2}}\overline{\theta}^{\pm}, \tag{4.5}$$

where γ is the rapidity. The fermionic coordinates mutually anti-commute and their fermionic nature renders them nilpotent. To summarize the above discussion we repeat the coordinates of the superspace:

$$x^0, x^1, \theta^{\pm}, \overline{\theta}^{\pm}. \tag{4.6}$$

Functions on superspace are called superfields and can be Taylor expanded in terms of $\theta^{\pm}, \overline{\theta}^{\pm}$:

$$\begin{aligned} \mathcal{F}(x^0, x^1, \theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-) = f(x^0, x^1) &+ \theta^+ f_+(x^0, x^1) \\ &+ \theta^- f_-(x^0, x^1) + \overline{\theta}^+ f'_+(x^0, x^1) \\ &+ \overline{\theta}^- f'_-(x^0, x^1) + \theta^+\theta^- f_{+-}(x^0, x^1) + \ldots \end{aligned} \tag{4.7}$$

The nilpotency of the fermionic coordinates lets us conclude, that there can be at most $2^4 = 16$ components. A superfield $\mathcal{F}$ is called bosonic (fermionic) if $[\theta^{\alpha}, \mathcal{F}] = 0(\{\theta^{\alpha}, \mathcal{F}\} = 0)$. In the superspace formalism supersymmetry transformations are encoded in differential operators, which act on the superfields. To write down these operators we first introduce:

$$x^{\pm} = x^0 \pm x^1, \qquad \partial_{\pm} = \frac{\partial}{\partial x^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{4.8}$$

The operators on superspace are given by

$$\begin{gathered} \mathcal{Q}_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + i\overline{\theta}^{\pm}\partial_{\pm}, \quad \overline{\mathcal{Q}}_{\pm} = -\frac{\partial}{\partial\overline{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm}, \\ \left\{\mathcal{Q}_{\pm}, \overline{\mathcal{Q}}_{\pm}\right\} = -2i\partial_{\pm}. \end{gathered} \tag{4.9}$$

The following operators are useful in writing down Lagrangians and provide a way to reduce the components of a superfield, as we will discuss later:

$$\begin{gathered} D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - i\overline{\theta}^{\pm}\partial_{\pm}, \quad \overline{D}_{\pm} = -\frac{\partial}{\partial\overline{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm} \\ \{D_{\pm}, D_{\pm}\} = 2i\partial_{\pm} \end{gathered} \tag{4.10}$$

The operators (4.10) anti-commute with the operators (4.9). Further operations on superfields are the vector and axial R-rotation given by

$$\begin{aligned} e^{i\alpha F_V} &: \mathcal{F}(x^{\mu}, \theta^{\pm}, \overline{\theta}^{\pm}) \mapsto e^{i\alpha q_V}\mathcal{F}(x^{\mu}, e^{-i\alpha}\theta^{\pm}, e^{i\alpha}\overline{\theta}^{\pm}), \\ e^{i\beta F_A} &: \mathcal{F}(x^{\mu}, \theta^{\pm}, \overline{\theta}^{\pm}) \mapsto e^{i\beta q_A}\mathcal{F}(x^{\mu}, e^{\mp i\beta}\theta^{\pm}, e^{\pm i\beta}\overline{\theta}^{\pm}), \end{aligned} \tag{4.11}$$

with q_V, q_A called vector or axial R-charge, respectively. These R-symmetries can be combined into left- and right moving R-symmetries:

$$\begin{aligned} J_L &= \frac{1}{2}\left(F_V - F_A\right), \\ J_R &= \frac{1}{2}\left(F_V + F_A\right). \end{aligned} \tag{4.12}$$

From (4.11) the transformations for the component fields of $\mathcal{F}$ can be read off. To reduce the component fields of a general superfield (4.7) the operators (4.10) can be used to impose constraints. The first example for such a constraint is given by

$$\overline{D}_\pm \Phi = 0. \tag{4.13}$$

A superfield Φ which fulfills (4.10) is called chiral superfield. To construct the component expansion of a chiral superfield we introduce new coordinates:

$$y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm, \qquad \widetilde{\theta}^\pm = \theta^\pm, \qquad \widetilde{\overline{\theta}}^\pm = \overline{\theta}^\pm. \tag{4.14}$$

The derivatives transform in the following way:

$$\frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial y^\pm}, \quad \frac{\partial}{\partial \theta^\pm} = -i\widetilde{\overline{\theta}}^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \widetilde{\theta}^\pm}, \quad \frac{\partial}{\partial \overline{\theta}^\pm} = i\widetilde{\theta}^\pm \frac{\partial}{\partial y^\pm} + \frac{\partial}{\partial \widetilde{\overline{\theta}}^\pm}. \tag{4.15}$$

The previous result let us conclude, that in the new coordinates (4.14)

$$\overline{D}_\pm = -\frac{\partial}{\partial \widetilde{\overline{\theta}}^\pm}. \tag{4.16}$$

This leads to a simplification of the constraint (4.13) and we can write down a chiral superfield in components

$$\Phi(x^\mu, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm). \tag{4.17}$$

We also see from (4.13) that a sum and a product of chiral fields is again chiral. The complex conjugate of a chiral field $\overline{\Phi}$ is called an anti-chiral field and fulfils

$$D_\pm \overline{\Phi} = 0. \tag{4.18}$$

The second type of constrained superfield we introduce is the twisted-chiral superfield U. A twisted-chiral superfield obeys:

$$\overline{D}_+ U = D_- U = 0. \tag{4.19}$$

As in the case of the chiral field, the expansion in component fields of U is done by first introducing new coordinates:

$$\widetilde{y}^\pm = x^\pm \mp i\theta^\pm\overline{\theta}^\pm, \qquad \widetilde{\theta}^\pm = \theta^\pm, \qquad \widetilde{\overline{\theta}}^\pm = \overline{\theta}^\pm. \tag{4.20}$$

This choice of coordinates gives the following simplifications:

$$\overline{D}_+ = -\frac{\partial}{\partial \widetilde{\overline{\theta}}^+}, \qquad D_- = \frac{\partial}{\partial \widetilde{\theta}^-}. \tag{4.21}$$

The above results lead to the following expansion of U:

$$U(x^\mu, \theta^\pm, \overline{\theta}^\pm) = u(\widetilde{y}^\pm) + \theta^+ \overline{\chi}_+(\widetilde{y}^\pm) + \overline{\theta}^- \chi_-(\widetilde{y}^\pm) + \theta^+ \overline{\theta}^- E(\widetilde{y}^\pm). \tag{4.22}$$

The complex conjugate $\overline{U}$ is called twisted anti-chiral superfield and satisfies

$$D_+ \overline{U} = \overline{D}_- \overline{U} = 0. \tag{4.23}$$

We now write down supersymmetric actions. These actions are invariant under the transformation

$$\delta = \varepsilon_+ \mathcal{Q}_- - \varepsilon_- \mathcal{Q}_+ - \overline{\varepsilon}_+ \overline{\mathcal{Q}}_- + \overline{\varepsilon}_- \overline{\mathcal{Q}}_+. \tag{4.24}$$

Given an arbitrary differentiable function K of some superfields $\mathcal{F}_i$ one can show that the following functional is invariant under (4.24):

$$\int \mathrm{d}^2 x \mathrm{d}^4 \theta K(\mathcal{F}_i) = \int \mathrm{d}^2 x \mathrm{d}\theta^+ \mathrm{d}\theta^- \mathrm{d}\overline{\theta}^- \mathrm{d}\overline{\theta}^+ K(\mathcal{F}_i). \tag{4.25}$$

The invariance of (4.25) follows from the representation of the supersymmetry operators (4.9) and remembering that the integral over $\mathrm{d}^4\theta$ picks out the prefactor of $\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$. A term of the form (4.25) is called D-term. For chiral superfields Φ_i we can write down the following supersymmetric functional

$$\int \mathrm{d}^2 x \mathrm{d}^2 \theta W(\Phi_i) = \int \mathrm{d}^2 x \mathrm{d}\theta^- \mathrm{d}\theta^+ W(\Phi_i)\Big|_{\overline{\theta}^\pm = 0}. \tag{4.26}$$

$W(\Phi_i)$ is holomorphic in the Φ_is. A functional of the form (4.26) is called F-term. In the case of twisted-chiral fields U_i the following functional is invariant under (4.24):

$$\int \mathrm{d}^2 x \mathrm{d}^2 \widetilde{\theta} \widetilde{W}(U_i) = \int \mathrm{d}^2 x \mathrm{d}\overline{\theta}^- \mathrm{d}\theta^+ \widetilde{W}(U_i)\Big|_{\overline{\theta}^+ = \theta^- = 0}, \tag{4.27}$$

and called twisted F-term.

Quantum Field Theory Aspects

Let us now study a $N = (2,2)$ supersymmetric quantum field theory. We start from a classical $N = (2,2)$ supersymmetric, Poincaré invariant field theory. In such a theory there a four conserved supercharges:

$$Q_+, Q_-, \overline{Q}_+, \overline{Q}_-, \tag{4.28}$$

and from energy, momentum and angular momentum invariance we get the Hamiltonian, the momentum and the angular momentum charges:

$$H, P, M. \tag{4.29}$$

We assume that the theory is invariant under both vector and axial R-symmetries and denote the conserved charges by:

$$F_V, F_A. \tag{4.30}$$

In the quantum theory the above conserved charges correspond to operators which generate the respective symmetry transformations. The supercharges (4.28) generate the supersymmetry transformations δ by:

$$\delta\mathcal{O} = \left[\hat{\delta}, \mathcal{O}\right], \tag{4.31}$$

with

$$\hat{\delta} := i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\overline{\epsilon}_+ \overline{Q}_- + i\overline{\epsilon}_- \overline{Q}_+, \tag{4.32}$$

and $\mathcal{O}$ is an operator on the Hilbert space of the quantum theory. Note that $\overline{Q}_\pm = Q_\pm^\dagger$. We assume that all symmetries of the classical action are non-anomalous. In that case we have the following operator algebra:

$$\begin{aligned}
Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_+^2 &= 0, \\
\left\{Q_\pm, \overline{Q}_\pm\right\} &= H \pm P, \\
\left\{\overline{Q}_+, \overline{Q}_-\right\} &= \{Q_+, Q_-\} = 0, \\
\left\{Q_-, \overline{Q}_+\right\} &= \left\{Q_+, \overline{Q}_-\right\} = 0, \\
[iM, Q_\pm] = \mp Q_\pm, &\quad \left[iM, \overline{Q}_\pm\right] = \mp\overline{Q}_\pm, \\
[iF_V, Q_\pm] = -iQ_\pm, &\quad \left[iF_V, \overline{Q}_\pm\right] = i\overline{Q}_\pm, \\
[iF_A, Q_\pm] = \mp iQ_\pm, &\quad \left[iF_A, \overline{Q}_\pm\right] = \pm i\overline{Q}_\pm.
\end{aligned} \tag{4.33}$$

The extension of the above algebra by central charges is discussed in [3]. Superfields furnish a representation of the algebra (4.33). The components of a chiral superfield $(\phi, \psi_\pm, F)$, see (4.17), span a representation called chiral multiplet. The lowest component ϕ fulfils

$$\left[\overline{Q}_\pm, \phi\right] = 0. \tag{4.34}$$

On the other hand, if we have an operator ϕ, which obeys (4.34) a chiral multiplet can be constructed by:

$$\psi_\pm := [iQ_\pm, \phi], \qquad\qquad F := \{Q_+, [Q_-, \phi]\}. \tag{4.35}$$

The components of the twisted-chiral superfield (4.22) give a twisted-chiral multiplet. The lowest component satisfies

$$\left[\overline{Q}_+, v\right] = \left[Q_-, v\right] = 0. \tag{4.36}$$

A twisted-chiral multiplet can be constructed starting from an operator which fulfils (4.36):

$$\begin{aligned} \overline{\chi}_+ := \left[iQ_+, v\right], \quad \chi_- := -\left[i\overline{Q}_-, v\right], \\ E := -\left\{Q_+, \left[\overline{Q}_-, v\right]\right\}. \end{aligned} \tag{4.37}$$

A further interesting aspect is, that the algebra (4.33) has an outer automorphism:

$$Q_- \leftrightarrow \overline{Q}_-, \qquad\qquad F_V \leftrightarrow F_A. \tag{4.38}$$

The remaining operators are unchanged. Two $N = (2, 2)$ supersymmetric field theories which are equivalent as quantum field theories and for which the isomorphism of the Hilbert spaces transforms the operators as in (4.38) are called mirror. The isomorphism exchanges chiral multiplets with twisted chiral multiplets and an unbroken (broken) axial R-symmetry gets exchanged with an unbroken (broken) vector R-symmetry. However it is pure convention which operator is called Q_- or $\overline{Q}_-$. We follow the convention of [3] and set the holomorphic variables of a non-linear sigma model or Landau-Ginzburg model (see Section 4.1) as lowest components of a chiral superfield. This convention could also be exchanged and we could view the variables as lowest component of a twisted-chiral superfield.

Non-Linear Sigma Models and Landau-Ginzburg Models

Let us consider n chiral mulitplets $\Phi^i \quad i = 1, \ldots, n$ and a real function $K\left(\Phi^i, \overline{\Phi}^{\bar{\imath}}\right)$. In order to construct a Lagrangian we further assume that

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K\left(\Phi^i, \overline{\Phi}^{\bar{\imath}}\right), \tag{4.39}$$

is positive definite. A supersymmetric Lagrangian density is given by:

$$\mathcal{L}_{kin} = \int \mathrm{d}^4\theta K\left(\Phi^i, \overline{\Phi}^{\bar{\imath}}\right) \tag{4.40}$$

The positive definite property guarantees a non-degenerate kinetic term for the component fields. In addition this property allows us to view (4.39) as Kähler metric on $\mathbb{C}^n = \{(z^1, \ldots, z^n)\}$:

$$\mathrm{d}s^2 = g_{i\bar{\jmath}} \mathrm{d}z^i \mathrm{d}\bar{z}^{\bar{\jmath}}. \tag{4.41}$$

The Levi-Civita connection on $T\mathbb{C}^n$ is given by:

$$\Gamma^i_{jk} = g^{i\bar{\jmath}} \partial_j g_{k\bar{\jmath}} \tag{4.42}$$

The component action of (4.40) is covariant under holomorphic coordinate changes up to equations of motion. A further invariance of the action is given by the following transformation:

$$K\left(\Phi^i, \overline{\Phi}^{\bar{\imath}}\right) \mapsto K\left(\Phi^i, \overline{\Phi}^{\bar{\imath}}\right) + f\left(\Phi^i\right) + \overline{f}\left(\overline{\Phi}^{\bar{\imath}}\right), \tag{4.43}$$

where $f\left(\Phi^i\right)$ is holomorphic in the Φ^is. This transformation does not change the metric (4.39) and has the form of a Kähler-transformation known from complex manifolds. The above results show that the given construction can be applied for each coordinate patch of a Kähler manifold M and then the different patches can be glued together. In this setting (4.40) gives an action for maps from the worldsheet Σ to M:

$$\phi : \Sigma \to M, \tag{4.44}$$

where $\phi = (\phi^i)$ are the lowest components of the Φ^is. The fermions are spinors with values in the pull-back of the tangent bundle $\phi^* TM$:

$$\begin{aligned} \psi^i_\pm &\in \Gamma\left(\Sigma, \phi^* TM^{(1,0)} \otimes S_\pm\right), \\ \overline{\psi}^{\bar{\imath}}_\pm &\in \Gamma\left(\Sigma, \phi^* TM^{(0,1)} \otimes S_\pm\right), \end{aligned} \tag{4.45}$$

where $S_\pm$ are the positive and negative spinor bundles on the worldsheet. These bundles are the square root of the (anit-)canonical bundle K (see (3.56)). The above construction gives the supersymmetric non-linear sigma model on a Kähler manifold M with metric g, but this is not a global formulation and only valid patchwise. Supersymmetry must be checked patch by patch. The Lagrangian for the non-linear sigma model in components was already given in Section 3.3.

Next we consider the following F-term:

$$\mathcal{L}_W = \frac{1}{2}\left(\int \mathrm{d}^2\theta W\left(\Phi^i\right) + c.c.\right), \tag{4.46}$$

where $c.c.$ stands for complex conjugate. $W\left(\Phi^i\right)$ is a holomorphic function of $\Phi^i, \quad i = 1, \ldots, n$ and called superpotential. W is holomorphic on M in the case of a sigma model and so can only be non-trivial if M is non-compact. The total Lagrangian density is given by:

$$\mathcal{L} = \int \mathrm{d}^4\theta K\left(\Phi^i, \overline{\Phi}^{\bar{\jmath}}\right) + \frac{1}{2}\left(\int \mathrm{d}^2\theta W\left(\Phi^i\right) + c.c.\right). \tag{4.47}$$

We will call a model with a non-trivial potential Landau-Ginzburg model and use the name non-linear sigma model for models without a potential.

Let us also consider the vector and axial R-symmetry of (4.47). We will give only the main points and refer to [3] for a more detailed discussion. The action on the chiral fields is given by (4.11) and can be interpreted as the action of the group $U(1)_V \times U(1)_A$. We treat first the classical (non-quantum) case. The $U(1)_A$ symmetry can always be made a symmetry classically (e.g. by assigning axial R-charge 0 to all fields). In order for $U(1)_V$ to be a symmetry classically the superpotential $W\left(\Phi^i\right)$, see (4.46), must be quasi-homogeneous such that

$$W\left(\lambda^{q_i}\Phi^i\right) = \lambda^2 W\left(\Phi^i\right), \tag{4.48}$$

and also the Kähler potential (4.40) must be invariant up to a Kähler transformation (4.43). It is also possible that not all $U(1)$s are preserved and only smaller subgroups are a symmetry. In the quantum theory the $U(1)_V$ symmetry remains a symmetry. The $U(1)_A$ symmetry is in general anomalous and gets broken to a smaller subgroup. In the case $c_1(M) = 0$ the $U(1)_A$ symmetry is unbroken, where $c_1(M)$ is the first Chern class of the tangent bundle of M.

Supersymmetric Gauge Theories

We focus here on the simplest case of a $U(1)$ gauge group and give only some details on the non-Abelian case. In Section 4.2 we will talk again about the non-Abelian case, but with applications to the gauged linear sigma model in mind. We can motivate the construction of supersymmetric gauge theories by starting with the following simple Lagrangian:

$$L = \int \mathrm{d}^4\theta \overline{\Phi}\Phi, \tag{4.49}$$

where Φ is a chiral superfield. It is obvious that (4.49) is invariant under a constant phase rotation

$$\Phi \to e^{-i\alpha}\Phi. \tag{4.50}$$

In order to gauge this symmetry we first replace the constant α by a chiral field $A(x^\mu, \theta^\pm, \overline{\theta}^\pm)$. Next we apply

$$\Phi \to e^{iA}\Phi, \tag{4.51}$$

to

$$\overline{\Phi}\Phi \to \overline{\Phi}e^{-i(\overline{A}-A)}\Phi, \tag{4.52}$$

and see that (4.49) is not invariant. In order to fix this we use the same method as in the gauging of non-supersymmetric theories: We introduce an additional superfield $V(x^\mu, \theta^\pm, \overline{\theta}^\pm)$. V is a real superfield and transforms by:

$$V \to V + i(\overline{A} - A). \tag{4.53}$$

The gauge invariant extension of (4.49) is given by:

$$L = \int \mathrm{d}^4\theta \overline{\Phi} e^V \Phi, \tag{4.54}$$

which is invariant under (4.51) and (4.53). A field $V(x^\mu, \theta^\pm, \overline{\theta}^\pm)$ with transformation behaviour (4.53) is called vector superfield. This field has the following component expansion

$$\begin{aligned} V(x^\mu, \theta^\pm, \overline{\theta}^\pm) = {} & \theta^-\overline{\theta}^+ (v_0 - v_1) + \theta^+\overline{\theta}^+ (v_0 + v_1) \\ & - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ & i\theta^-\theta^+ \left(\overline{\theta}^-\overline{\theta}_- + \overline{\theta}^+\overline{\lambda}_+\right) \\ & + i\overline{\theta}^+\overline{\theta}^- \left(\theta^-\lambda_- + \theta^+\lambda_+\right) \\ & \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D. \end{aligned} \tag{4.55}$$

The expansion (4.55) is only valid in a special gauge called Wess-Zumino gauge. The components v_0, v_1 define a one-form field, σ is a complex scalar field, $\lambda_\pm$, $\overline{\lambda}_\pm$ give a Dirac fermion and D is a real scalar field. In the Wess-Zumino gauge the following residual gauge symmetry remains:

$$v_\mu(x) \to v_\mu(x) - \partial_\mu \alpha(x), \tag{4.56}$$

with the other fields invariant. This transformation does not spoil the expansion (4.55). A supersymmetry transformation is implemented by:

$$\delta := \epsilon_+ \mathcal{Q}_- - \epsilon_- \mathcal{Q}_+ - \overline{\epsilon}_+ \overline{\mathcal{Q}}_- + \overline{\epsilon}_- \overline{\mathcal{Q}}_+. \tag{4.57}$$

The operators $\mathcal{Q}_\pm$, $\overline{\mathcal{Q}}_\pm$ were introduced in (4.9). In order to stay in Wess-Zumino gauge a supersymmetry transformation must be augmented by a gauge transformation. We refer to [3] where the transformations of the superfield components are given. A further superfield we introduce is:

$$\Sigma = \overline{D}_+ D_- V, \tag{4.58}$$

which is invariant under gauge transformations $V \to V + i(\overline{A} - A)$. The operators $\overline{D}_+$ and D_- are given in (4.10). Σ is a twisted chiral superfield (see (4.19)). In components (4.58) reads

$$\begin{aligned} \Sigma(\widetilde{y}) = {} & \sigma(\widetilde{y}) + i\theta^+\lambda_+(\widetilde{y}) - i\overline{\theta}^-\lambda_-(\widetilde{y}) \\ & \theta^+\overline{\theta}^- \left[D(\widetilde{y}) - iv_{01}(\widetilde{y})\right], \end{aligned} \tag{4.59}$$

with $\widetilde{y}^\pm$ introduced in (4.20) and

$$v_{01} = \partial_0 v_1 - \partial_1 v_0. \tag{4.60}$$

v_{01} is the field strength of v_μ and Σ is called super-field strength of V. Let us also comment on the necessary modifications in the case of a non-abelian gauge symmetry G . The vector field V takes now values in the Lie algebra $\mathfrak{g}$ of G:

$$V = V^a t^a_{\mathcal{R}}, \tag{4.61}$$

where $t^a_{\mathcal{R}}$ are the generators of $\mathfrak{g}$ in the representation $\mathcal{R}$. The transformation of V and Φ have to be modified, compared to the abelian case (4.53), (4.51):

$$\begin{aligned} e^V &\to e^{i\overline{A}} e^V e^{-iA}, \\ \Phi \to e^{iA}\Phi \quad \overline{\Phi} &\to \overline{\Phi} e^{-i\overline{A}}. \end{aligned} \tag{4.62}$$

where $A = A^a t^a_{\mathcal{R}}$ is a chiral superfield. The Lagrangian given in (4.54) is invariant under (4.62). It remains to adjust the superfield strength Σ (4.58). In the non-abelian case we have:

$$\begin{aligned} \Sigma &= \frac{1}{2}\left\{\overline{\mathcal{D}}_+, \mathcal{D}_-\right\}, \\ \mathcal{D}_\pm = e^{-V} D_\pm e^V \quad \overline{\mathcal{D}}_\pm &= e^V \overline{D}_\pm e^{-V}. \end{aligned} \tag{4.63}$$

The superfield strength Σ is still a twisted chiral superfield.

4.2 Gauged Linear Sigma Models

In this section we will write down the Lagrangian for the gauged linear sigma model, or GLSM for short. The GLSM is a supersymmetric model. This model resembles some features seen in the supersymmetric non-linear sigma models and the Landau-Ginzburg models discussed in Section 4.1, but has an additional gauge theory. The subsequent discussion mostly follows [3, 6, 34]. The GLSM was first introduced in [6] in order to give a field theoretic realization of the Landau-Ginzburg/Calabi-Yau correspondence, which we will discuss in Section 5.2.

Abelian Models

The GLSMs of most significance in this thesis are those with a gauge group $U(1)^k = \prod_{a=1}^k U(1)_a$ and N chiral matter fields $\Phi_i, \quad i = 1, \ldots, N$. The field Φ_i carries charge Q_{ia} under $U(1)_a$:

$$\Phi_i \to e^{iQ_{ia}A_a}\Phi_i. \tag{4.64}$$

A general Lagrangian is given by:

$$\begin{aligned} L = & \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i e^{Q_{ia}V_a} \Phi_i - \sum_{a,b=1}^{k} \frac{1}{2e_{a,b}^2} \overline{\Sigma}_a \Sigma_b \right) \\ + & \left(\int \mathrm{d}^2\theta W(\Phi_i) + c.c. \right) \\ + & \frac{1}{2} \left(\int \mathrm{d}^2\widetilde{\theta} \widetilde{W}(\Sigma_a) + c.c \right). \end{aligned} \tag{4.65}$$

The first line in (4.65) contains the kinetic terms of Φ_i and V_a. In the second line a superpotential (F-term) is added (see (4.26)) and in the last line a twisted F-term is shown (see (4.27)). In order to preserve the gauge symmetry the superpotential $W(\Phi_i)$ in (4.65) must be gauge invariant. To preserve $U(1)_A \times U(1)_V$ R-symmetry $W(\Phi_i)$ must be quasi-homogeneous and $\widetilde{W}(\Sigma_a)$ must be linear (see the discussion in Section 4.1). We set:

$$\widetilde{W}(\Sigma_a) = \sum_{a=1}^{k} (-t_a \Sigma_a), \tag{4.66}$$

with $t_a = \zeta_a - i\theta_a$. The term (4.66) is called Fayet-Iliopoulos θ-term (see [6]). The auxiliary field D_a of the vector superfield V_a (4.55) and F_i of the chiral superfield Φ_i (4.17) can be eliminated from (4.65). After this elimination the following potential for the scalar fields can be read off from (4.65):

$$\begin{aligned} U = & \sum_{a=1}^{k} \sum_{i=1}^{N} |Q_{ia}\sigma_a|^2 |\phi_i|^2 \\ & + \sum_{i,j=1}^{N} \sum_{a,b=1}^{k} \frac{\left(e^{a,b}\right)^2}{2} \left(Q_{ia} |\phi_i|^2 - \zeta_a \right) \left(Q_{jb} |\phi_j|^2 - \zeta_b \right) \\ & + \sum_{i=1}^{N} \left| \frac{\partial W}{\partial \phi_i} \right|^2. \end{aligned} \tag{4.67}$$

$\left(e^{a,b}\right)^2$ is the inverse matrix of $\frac{1}{e_{a,b}^2}$. (4.67) determines the structure of the vacuum manifold, which we will study in greater detail in Section 4.2.

General Terminology and Non-Abelian Models

In Section 4.2 we considered the special case of a $U(1)^k$ gauge symmetry. Here we reformulate the above statements in more general terms, which also apply to theories with non-abelian gauge symmetry. A further advantage of the following terminology is to make the connection to the mathematics literature (see e.g. [35]) more apparent. The subsequent discussion follows [34].

To construct a GLSM we need to choose:

- A gauge group G, assumed to be a compact Lie group
- Matter representations $\rho_{\mathcal{V}_i}$ with representation spaces $\mathcal{V}_i$
- A superpotential W
- A twisted superpotential $\widetilde{W}$

We assume that the homomorphisms $\rho_{\mathcal{V}_i}$:

$$\rho_{\mathcal{V}_i} : G \to GL(\mathcal{V}_i), \tag{4.68}$$

give a faithful complex representation. The chiral superfields $\Phi_i \quad i = 1, \ldots, N$ take values in $\mathcal{V}_i$. The scalar component of Φ_i is denoted by ϕ_i. The superpotential W is a G-invariant polynomial in the Φ_is. We denote the total space of the chiral fields by $\mathcal{V} = \bigoplus_{i=1}^{N} \mathcal{V}_i$. In addition we have a vector superfield V, taking values in $\mathfrak{g}_{\mathbb{C}}$. The scalar component of V is denoted by σ. $\mathfrak{g}_{\mathbb{C}}$ is the complexified Lie-algebra of G. We denote the corresponding superfield strength by Σ. The twisted chiral superpotential $\widetilde{W}$ is G-invariant and polynomial in the Σs. It is possible to preserve the vector and axial $U(1)_V \times U(1)_A$ R-symmetries with integral charges under the following conditions: $U(1)_V$ is preserved when it is possible to assign R-charges to Φ_i such that $W(\Phi)$ has R-charge 2. The vector R-charge operators $R_{\mathcal{V}_i}$ are elements of $\mathrm{End}(\mathcal{V}_i)$. Charge integrality is given if $e^{i\pi R_{\mathcal{V}_i}} = \rho_{\mathcal{V}_i}(J)$ for some $J \in G$. For $U(1)_A$ to exist classically σ must have axial R-charge 2 and $\widetilde{W}(\sigma)$ must be linear. $U(1)_A$ is anomaly free under the Calabi-Yau condition:

$$\rho_{\mathcal{V}} : G \to SL(\mathcal{V}). \tag{4.69}$$

The twisted superpotential is given by:

$$\widetilde{W}(\sigma) = -t(\sigma), \tag{4.70}$$

with $t \in \mathfrak{g}^*_{\mathbb{C}}$, $t = \zeta - i\theta$. ζ and θ are the FI-θ parameters introduced in Section 4.2. These terms are only possible if the gauge group has $U(1)$ subgroups. Next we write down the scalar potential and we focus on gauge groups of the following form:

$$G = U(1)^k \times \widetilde{G}, \tag{4.71}$$

where $\widetilde{G}$ is a simple Lie group with Lie algebra $\widetilde{\mathfrak{g}}$. Similar to the abelian models (Section 4.2) the auxiliary F_i and D fields can be replaced by their equations of motion. After these fields have been integrated out the potential for the scalar fields reads:

$$\begin{aligned} U = & \frac{1}{8e^2}\operatorname{Tr}\left([\sigma,\overline{\sigma}]^2\right) + \frac{1}{2}\sum_{i=1}^{N}\left(\phi_i^\dagger\left\{\sigma_{\mathcal{V}_i},\overline{\sigma}_{\mathcal{V}_i}\right\}\phi_i\right) \\ & + \frac{e_a^2}{2}\sum_{a=1}^{k}\left(\sum_{i=1}^{N}Q_{i,a}|\phi_i|^2 - \zeta_a\right)^2 + \frac{g^2}{2}\sum_{k=1}^{\dim(\widetilde{\mathfrak{g}})}\left(\sum_{i=1}^{N}\phi_i^\dagger t_{\mathcal{V}_i}^k\phi_i\right)^2, \\ & + \sum_{i=1}^{N}\left|\frac{\partial W(\phi)}{\partial\phi_i}\right|^2. \end{aligned} \tag{4.72}$$

where $\sigma_{\mathcal{V}_i} = \sigma_a t^a_{\mathcal{V}_i}$, $t^a_{\mathcal{V}_i}$ are the generators of $\widetilde{\mathfrak{g}}$ in the representation $\rho_{\mathcal{V}_i}$ and g is the coupling constant of the gauge field. We want to highlight the following terms of (4.72):

$$\left.\begin{aligned} \left(\sum_{i=1}^{N}Q_{i,a}|\phi_i|^2 - \zeta_a\right) &= \mu_a(\phi) - \zeta_a, \\ \left(\sum_{i=1}^{N}\overline{\phi}_i t^k_{\mathcal{V}_i}\phi_i\right) &= \mu_k(\phi), \end{aligned}\right\} \quad \text{D-terms,} \\ \frac{\partial W(\phi)}{\partial\phi_i} \quad \text{F-terms.} \tag{4.73}$$

We rewrote the D-terms in such a way to emphasise their interpretation as moment maps:

$$\mu_a : \mathcal{V} \to i\mathfrak{g}^*, \tag{4.74}$$

on the symplectic quotient which determines the vacuum manifold. These terms will be important in the study of the structure of the vacuum manifold in Section 4.2.

Let us close this section by making the connection with the abelian model with gauge group $U(1)^k$ discussed in Section 4.2. We set[1]

$$(e^{a,b})^2 = \delta^{a,b}(e_a)^2, \tag{4.75}$$

which gives for the potential (4.67):

$$\begin{aligned} U = & \sum_{a=1}^{k}\sum_{i=1}^{N}\left|Q_{ia}\sigma_a\right|^2|\phi_i|^2 \\ & + \sum_{a=1}^{k}\frac{(e_a)^2}{2}\left(\sum_{i=1}^{N}Q_{ia}\,|\phi_i|^2 - \zeta_a\right)^2 \\ & + \sum_{i=1}^{N}\left|\frac{\partial W}{\partial\phi_i}\right|^2. \end{aligned} \tag{4.76}$$

[1] We focus on this specific case, because all models studied in this thesis are of this kind.

We now compare (4.76) and (4.72). If we look at the first term in the first line of (4.72) we do not find a matching term in (4.76). This is expected, because we have an abelian gauge group and so all commutators vanish. The second term in (4.72) corresponds to the first line of (4.76). The remaining terms in (4.76) can be identified with:

$$\sum_{i=1}^{N}\left(Q_{ia}\left|\phi_i\right|^2\right)-\zeta_a \quad \text{D-terms,} \tag{4.77}$$

$$\frac{\partial W}{\partial \phi_i} \quad \text{F-terms.} \tag{4.78}$$

Structure of the Vacuum Manifold and Phases

In this section we analyse the structure of the vacuum manifold of a given GLSM. Our discussion follows [34, 3]. To obtain a Lorentz invariant vacuum the vacuum expectation values (VEVs) of spinors and vector fields have to be zero, but scalar fields can develop a non-zero VEV. In a general GLSM the scalar fields have a non-zero potential U (4.72). U is manifestly positive and therefore a vacuum is obtained, if U vanishes. Due to the structure of U (see (4.72)) each term of U has to vanish on its own. Therefore we first focus on the D- and F-term equations (4.73). The crucial feature of the D-term is that, depending on the value of ζ, certain ϕ_i are forced to be non-zero to get a vanishing D-term.

These non-zero values break the gauge group. We see from the first term of U that σ must take values in the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. It follows from the second term in the first line of (4.72) that the σ components corresponding to broken generators ($t^a_{\mathcal{V}_i}\phi_i \neq 0$) have to vanish. The symmetry breaking patterns divide the FI-θ parameter space into different chambers. These chambers are called phases of the GLSM. Different phases show different low energy physics.

We next analyse a phase in which the gauge group is broken to a finite subgroup. All σ components must be zero. The continuous part of the gauge group is completely Higgsed. In such a phase the low energy physics can be analysed classically. The space of D-term solutions modulo gauge group actions is the symplectic quotient:

$$\mu^{-1}\left(\zeta\right)/G, \tag{4.79}$$

and is either a smooth manifold or an orbifold. The symplectic quotient is equivalently described by a complex quotient:

$$\mu^{-1}\left(\zeta\right)/G \simeq \left(\mathcal{V}-F_\zeta\right)/G_{\mathbb{C}}, \tag{4.80}$$

see also [6] where this aspect is discussed in more detail. $F_\zeta \subset \mathcal{V}$ is called deleted set and corresponds to the $\phi \in \mathcal{V}$ whose gauge orbit does not intersect

$\mu^{-1}(\zeta)$. The superpotential W gives a holomorphic function W_ζ on (4.80). Classical vacua form the critical locus of W_ζ:

$$\mathrm{d}W^{-1}(0) \cap \mu^{-1}(\zeta)/G = \mathrm{Crit}(W_\zeta). \tag{4.81}$$

If all modes transverse to $\mathrm{Crit}(W_\zeta)$ are massive the low energy physics is given by a non-linear sigma-model with target space $\mathrm{Crit}(W_\zeta)$ after the massive modes have been integrated out. The target space is in that case a CY manifold or orbifold. If W_ζ has an isolated critical point the low energy description is given by a Landau-Ginzburg orbifold model. If none of the above holds $\left(\mu^{-1}(\zeta)/G, W_\zeta\right)$ describes a hybrid phase.

Between two phases some of the solutions ϕ_i to the D- and F-term equations (4.73) can leave a continuous subgroup unbroken. If this happens σ can take arbitrary values in the Cartan subalgebra of the unbroken gauge group. These non-compact flat directions in the effective target space are called Coulomb branch. As mentioned in [34] if some of the gauge group is broken it would be more precise to call such a direction mixed Coulomb-Higgs branch. To get the exact location of the Coulomb branch quantum effects have to be taken into account. We will study this situation in more detail in Section 4.2.

It is also possible that in a phase some solutions of (4.73) leave a continuous group unbroken and the values of σ for the unbroken gauge group are still constrained. These phases are strongly coupled and the classical analysis is not applicable. Such phases were analysed in [36] and a duality between a strongly coupled phase of one model and a weakly coupled of another model was given in [37]. We refer to these for further details on strongly coupled phases.

Quantum Corrections and Renormalization

In the analysis of the various phases of a gauged linear sigma model we are interested in the low energy physics. Therefore we have to take into account the renormalization of the parameters in the theory. We focus on the case of an abelian gauge group $U(1)^k = \prod_{a=1}^k U(1)_a$, because models of this kind will play a main role in this thesis. The Lagrangian is given in (4.65) and the twisted superpotential has the form (4.66). In the models of interest the only parameters which receive quantum corrections are the FI-parameters ζ_a. The corrections are not higher than one-loop order. The other parameters do not receive corrections. This is a consequence of supersymmetric non-renormalization theorems (see e.g. [38]). A detailed calculation is given in [3] and we will only state the outcome of the analysis. The crucial result is that the ζ_a values do not get renormalized if:

$$\sum_{i=1}^{N} Q_{ia} = 0 \quad \forall a. \tag{4.82}$$

This is the Calabi-Yau condition introduced in (4.69) to obtain an anomaly free $U(1)_A$ symmetry. It follows that if the ζ_a values are not renormalized also $U(1)_A$ is non-anomalous. In the following discussion we assume that ζ_a is not renormalized.

Let us come back to the analysis of the Coulomb branches between the different phases. We first focus on the non-Abelian case. Such branches appear for certain FI-θ parameters t_a values. At such values the first term in the first line of (4.72) breaks the gauge theory to a maximal torus T of G. σ can take arbitrary large values in $\mathfrak{t}_{\mathbb{C}} = \mathrm{Lie}(T)_{\mathbb{C}}$. The matter fields charged under T are heavy as one can see by looking at the second term in the first line of (4.72). To obtain the effective theory these heavy fields have to be integrated out. In that process an effective twisted superpotential for σ is obtained (see [34]):

$$\widetilde{W}_{eff}(\sigma) = \widetilde{W}(\sigma) + \pi i \sum_{\alpha>0} \langle \alpha, \sigma \rangle - \sum_{Q} \langle Q, \sigma \rangle \left(\log \langle Q, \sigma \rangle - 1 \right). \tag{4.83}$$

The sums run over the positive roots α of $\mathfrak{g}$ and over the weights Q of the representations $\mathcal{V}_i, \quad i = 1, \ldots, N$. From (4.83) the effective FI-θ parameter $t_{eff,a}(\sigma)$ can be obtained by:

$$t_{eff,a}(\sigma) = -\frac{\partial \widetilde{W}_{eff}(\sigma)}{\partial \sigma_a}. \tag{4.84}$$

The parameters (4.84) enter into the effective potential [34, 6, 39]:

$$U_{eff} = \min_{n \in P} \frac{e_{eff}^2}{2} \sum_a \left| t_{eff,a}(\sigma) + 2\pi i n \right|^2, \tag{4.85}$$

where P is the weight lattice of T. From (4.85) we see that the vacuum is at

$$t_{eff,a}(\sigma) \equiv 0 \quad \text{modulo } 2\pi i P \quad \forall a. \tag{4.86}$$

The Coulomb branch is then parameterized by (4.86). For the discussion of mixed Coulomb-Higgs-branches we refer the interested reader to [34].

Let us close this section by specializing (4.83) to the abelian model with $U(1)^k$ symmetry discussed previously. In this case we do not have positive roots and the weights are given by the charges Q_{ia}. This results in

$$\widetilde{W}_{eff}(\sigma) = -\sum_{a=1}^{k} t_a \sigma_a - \sum_{a=1}^{k} \sum_{i=1}^{N} Q_{ia} \sigma_a \left(\log \left(\sum_{b=1}^{k} Q_{ib} \sigma_b \right) - 1 \right). \tag{4.87}$$

We next use (4.84) to calculate the effective FI-θ parameters:

$$t_{eff,a}(\sigma) = t_a + \sum_{i=1}^{N} Q_{ia} \log \left(\sum_{b=1}^{k} Q_{ib} \sigma_b \right). \tag{4.88}$$

The Coulomb branch location is then parameterized by:

$$e^{-t_a} = \prod_{i=1}^{N} \left(\sum_{b=1}^{k} Q_{ib} \sigma_b \right)^{Q_{ia}} \qquad a = 1, \ldots, k. \tag{4.89}$$

4.3 Supersymmetric Localisation

In this section we will first outline the procedure to study supersymmetry on curved backgrounds. Afterwards we state the localisation argument, which makes it possible to get fully quantum corrected results in supersymmetric theories. We close the section by providing localisation results in GLSMs. An extensive review on supersymmetric localisation in quantum field theory is given in [40]. In our discussion we follow certain contributions of [40].

Supersymmetry in Curved Backgrounds

We follow [41, 42], where additional details and references to the original literature can be found. We start from a supersymmetric field theory with Lagrangian $\mathcal{L}$ in a flat spacetime, with metric η. We denote the infinitesimal supersymmetry transformations by δ. These transformations generate the supersymmetry algebra. Invariance of the action under δ requires:

$$\delta\mathcal{L} = \partial_\mu \left(\ldots \right)^\mu . \tag{4.90}$$

To define the above theory on a curved manifold with metric $g_{\mu\nu}$ two paths can be followed. The first is a trial and error procedure. In a first step we make the replacement:

$$\begin{aligned} \eta_{\mu\nu} &\to g_{\mu\nu}, \\ \partial_\mu &\to \nabla_\mu, \end{aligned} \tag{4.91}$$

where ∇_μ is a covariant derivative with respect to $g_{\mu\nu}$. Invariance under supersymmetry is given when

$$\delta\mathcal{L} = \nabla_\mu \left(\ldots \right)^\mu , \tag{4.92}$$

but this is in general not the case. It is necessary to amend the original transformation δ and $\mathcal{L}$ with additional terms to achieve (4.92). The drawback of the above procedure is that it has to be done on a case by case basis. Further it is not guaranteed that the above process leads to a closed supersymmetry algebra.

A second procedure was outlined in [43]. The supersymmetric theory is first coupled to supergravity and afterwards the rigid limit is taken. In this limit the supergravity gets non-dynamical and the metric $g_{\mu\nu}$ is set to a fixed background metric. In the rigid limit the equations of motion for the auxiliary fields in the supergravity multiplet do not need to be imposed. However we require Poincaré invariance and a supersymmetric background. This sets the fermionic fields in the supergravity multiplet and their supersymmetric variation δ to zero:

$$\delta \mathcal{F}_i = 0, \tag{4.93}$$

where by $\mathcal{F}_i$ we collectively denote the fermionic components of the supergravity multiplet. The left hand side of (4.93) is an expression in the bosonic components of the multiplet and the spinor parameter of the supersymmetric transformation δ. Supersymmetric backgrounds correspond to solutions of (4.93). Equations on the spinorial parameter are often referred to as generalized Killing spinor equations in this context.

Supersymmetric Localisation Argument

The main resources for this section are [44, 42, 45]. Supersymmetric localisation formulas can be seen as an extension of the localisation formulas known in mathematics like the Atiyah-Bott-Berline-Vergne localisation formula. This relation is discussed in [42], where also the main ideas behind the Atiyah-Bott formula are explained. We focus on a supersymmetric theory with Lagrangian $\mathcal{L}$. Let $\mathcal{Q}$ be a generator of a fermionic symmetry of the theory, which squares to a bosonic generator $\mathcal{B}$:

$$\mathcal{Q}^2 = \mathcal{B}. \tag{4.94}$$

$\mathcal{B}$ can be a generator of a linear combination of any bosonic symmetry of our theory. The Euclidean path integral is given by:

$$\mathcal{Z} = \int \mathcal{D}X e^{-S[X]}, \tag{4.95}$$

where X stands collectively for the fields in the theory. We deform (4.95) by:

$$\mathcal{Z}_t = \int \mathcal{D}X e^{-S[X]-t\mathcal{Q}V[X]}., \tag{4.96}$$

with

$$\mathcal{Q}^2 V = \mathcal{B}V = 0. \tag{4.97}$$

This does not alter the path integral result, because (4.96) is independent of t:

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{Z}_t &= -\int \mathcal{D}X (\mathcal{Q}V[X]) e^{-S[X]-t\mathcal{Q}V[X]}, \\ &= -\int \mathcal{D}X \mathcal{Q}\left(V[X]) e^{-S[X]-t\mathcal{Q}V[X]}\right), \\ &= 0. \end{aligned} \tag{4.98}$$

In the last line of (4.98) we integrated by parts in field space and assumed that the integral converges sufficiently fast so that boundary terms can be neglected. The result (4.98) makes it possible to calculate the path integral in the limit $t \to \infty$:

$$\int \mathcal{D}X e^{-S[X]} = \lim_{t\to\infty} \int \mathcal{D}X e^{-S[X]-t\mathcal{Q}V[X]} \tag{4.99}$$

We assume that $\mathcal{Q}V[X]$ is positive semi-definite and so (4.99) localises to the saddle points of $\mathcal{Q}V[X]$. The term

$$\mathcal{Q}V[X], \tag{4.100}$$

is called localisation action. Next we expand (4.99) around a saddle point X_0 of (4.100):

$$X = X_0 + \frac{1}{\sqrt{t}}\widetilde{X}, \tag{4.101}$$

where the t-prefactor was chosen such that the kinetic term is canonically normalized. We next expand the exponent of (4.99) around (4.101):

$$S[X_0] + \frac{1}{2}\iint \left.\frac{\delta^2\left(\mathcal{Q}V\right)}{\delta X^2}\right|_{X=X_0} \widetilde{X}^2 + \cdots . \tag{4.102}$$

In the limit $t \to \infty$ we can neglect the higher terms of (4.102), because these are weighted by negative powers of t. The fluctuations $\widetilde{X}$ can be integrated out and the path integral (4.95) results in:

$$\mathcal{Z} = \int_{\mathcal{V}_{loc}} \mathcal{D}X_0 e^{-S[X_0]} \frac{1}{\mathrm{SDet}\left.\left(\frac{\delta^2(\mathcal{Q}V)}{\delta X^2}\right)\right|_{X=X_0}}, \tag{4.103}$$

where $\mathcal{V}_{loc}$ is the localisation locus and SDet is the superdeterminant. The remaining integration is a path-integral over a lower dimensional field theory. The most favourable case is, if the remaining integral is over a zero-dimensional field theory, which gives an ordinary integral.

Let us mention that (4.103) also holds if we insert a BPS operator $\mathcal{O}$:

$$\langle\mathcal{O}\rangle = \int_{\mathcal{V}_{loc}} \mathcal{D}X_0 \mathcal{O}(X_0) e^{-S[X_0]} \frac{1}{\mathrm{SDet}\left.\left(\frac{\delta^2(\mathcal{Q}V)}{\delta X^2}\right)\right|_{X=X_0}}, \tag{4.104}$$

with

$$\mathcal{Q}\mathcal{O} = 0. \tag{4.105}$$

We want to emphasise that the result of (4.103) does not depend on the chosen localisation action (4.100), because we deformed the path-integral by a $\mathcal{Q}$-exact term. This follows from a similar argument as in (4.98). The localisation locus $\mathcal{V}_{loc}$ can be shown to be always a BPS configuration:

$$\mathcal{Q}\mathcal{F}_i = 0, \tag{4.106}$$

where $\mathcal{F}_i$ stands collectively for the fermionic fields. If we would integrate over a non-trivial $\mathcal{Q}$ orbit the path integral would vanish over the non-commuting coordinates of the orbit[2].

On the other hand we showed above that only the saddle points of the localisation action contribute in the limit $t \to \infty$. We can conclude that only $\mathcal{Q}$- invariant saddle points give a non-zero path integral. We see that, the path integral is taken over the intersection of saddle points with $\mathcal{Q}$-invariant configurations. The remaining integral is then over the moduli space of (4.106).

4.4 Sphere Partition Function of the GLSM

In this section we will present the results from supersymmetric localisation for the partition function in a GLSM on a sphere S^2 (see [7, 8]) and therefore called sphere partition function. Let us also mention that the GLSM was further localised on a hemisphere D^2 ([46, 47, 48]), whereby in [46] also the localisation on an annulus was performed. A further result is the elliptic genus obtained in [49, 50], which is given by supersymmetric localisation on a torus T^2. We refer to the literature for details on these results.

As mentioned in Section 4.3, in the process of supersymmetric localisation one has to choose a supercharge and a localisation action and those choices are not unique. Although in the end they must give the same result. For the GLSM on S^2 two possible ways were analysed. In both approaches the chosen supercharge for the localisation is the same and they differ by the applied localisation action. These two different choices are called Coulomb- and Higgs-branch localisation. We do not give the details of these computations and just state the result for Z_{S^2} for the Coulomb branch localisation:

$$\begin{aligned} Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}} \int \left(\prod_j \frac{\mathrm{d}\sigma_j}{2\pi} \right) & Z_{\text{class}}\left(\sigma, \mathfrak{m}\right) Z_{\text{gauge}}\left(\sigma, \mathfrak{m}\right) \\ & \cdot \prod_{\Phi_i} Z_{\Phi_i}\left(\sigma, \mathfrak{m}\right), \end{aligned} \tag{4.107}$$

where we have chosen the notation in accordance with [7]. The rightmost product in (4.107) runs over the different matter fields and the different components

[2]This argument was first given in [31].

read

$$Z_{\text{gauge}} = (-1)^{\frac{1}{2}\sum_{\alpha>0}\alpha(\mathfrak{m})} \prod_{\alpha>0} \left(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \right), \tag{4.108}$$

$$Z_{\Phi_i} = \prod_{w\in\rho_{\mathcal{V}_i}} \frac{\Gamma\left(\frac{R[\Phi_i]}{2} - iw(\sigma) - \frac{w(\mathfrak{m})}{2}\right)}{\Gamma\left(1 - \frac{R[\Phi_i]}{2} + iw(\sigma) - \frac{w(\mathfrak{m})}{2}\right)}, \tag{4.109}$$

$$Z_{\text{class}} = e^{-4\pi i\zeta\,\mathrm{Tr}(\sigma) - i\theta\,\mathrm{Tr}(\mathfrak{m})}, \tag{4.110}$$

where α are the roots of the Lie algebra $\mathfrak{g}$ and $\alpha > 0$ restricts the sum to the positive ones. In (4.107) $|\mathcal{W}|$ denotes the order of the Weyl-group of G. By w we denote a weight in the representation $\rho_{\mathcal{V}_i}$. In the process of localisation, σ which is the real part of the vector multiplet scalar, is forced to take values in the Cartan subalgebra of $\mathfrak{g}$. Also $\mathfrak{m}$ lies in the Cartan subalgebra and parameterises the gauge flux on S^2. The gauge flux is GNO quantized [51], which means for any representation $\rho_{\mathcal{V}_i}$ and for any weight w of this representation:

$$w\,(\mathfrak{m}) \in \mathbb{Z}. \tag{4.111}$$

$R[\Phi_i]$ gives the vector R-charge of the chiral superfield Φ_i. The alternating sign in (4.108) was not given in [7, 8], but later found in [46, 48]. A more general result compared to (4.107) can be found in [7, 8], for example we neglected twisted mass terms, but the displayed form is the most relevant for this thesis.

Chapter 5

Abelian one Parameter Models

In this chapter we describe a certain class of GLSMs, namely models with a $G = U(1)$ gauge group which fulfil the Calabi-Yau condition (see Section 4.2). The discussion follows [6, 10, 9].

The models of interest have the following field content[1]

$$\begin{array}{c|c|c|c|c|c|c} & p_1 & p_{2_1,\ldots,2_k} & x_{1,\ldots,5-n-j+k} & x_{\alpha_1,\ldots,\alpha_n} & x_{\beta_1,\ldots,\beta_j} & \text{FI} \\ \hline U(1) & -d_1 & -d_2 & 1 & \alpha & \beta & \zeta \\ \hline U(1)_V & 2-2d_1q & 2-2d_2q & 2q & 2\alpha q & 2\beta q & \end{array} \tag{5.1}$$

where

$$0 \leq k \leq 3, \qquad 0 \leq n \leq 2, \qquad 0 \leq j \leq 2. \tag{5.2}$$

In addition we have:

$$d_1, d_2, \alpha, \beta \in \mathbb{Z}_{\geq 0}. \tag{5.3}$$

In all models the Calabi-Yau condition is fulfilled:

$$5 + k - n - j + \alpha n + j\beta = d_1 + kd_2. \tag{5.4}$$

The $U(1)_V$ charges are positive and so it follows that

$$0 \leq q \leq \frac{1}{\max[d_1, d_2]}. \tag{5.5}$$

The superpotential takes the form

$$W = p_1 G_{d_1}(x_n) + \sum_{i=1}^{k} p_{2_i} G_{i,d_2}(x_n), \tag{5.6}$$

model-data					IR-description	
label	α^n	β^j	d_1	d_2^k	$\zeta \gg 0$	$\zeta \ll 0$
			F-type			
F1	-	-	5	-	$\mathbb{P}_{1^5}[5]$	LG orbifold
F2	-	2	6	-	$\mathbb{P}_{1^4,2}[6]$	LG orbifold
F3	-	4	8	-	$\mathbb{P}_{1^4,4}[8]$	LG orbifold
F4	2	5	10	-	$\mathbb{P}_{1^3,2,5}[10]$	LG orbifold
F5	-	2	4	3	$\mathbb{P}_{1^5,2}[4,3]$	Pseudo-Hybrid
F6	2^2	3	6	4	$\mathbb{P}_{1^3,2^2,3}[6,4]$	Pseudo-Hybrid
F7	4	6	12	2	$\mathbb{P}_{1^4,4,6}[12,2]$	Pseudo-Hybrid
			C-type			
C1	-	-	4	2	$\mathbb{P}_{1^6}[4,2]$	Pseudo-Hybrid
C2	-	3	6	2	$\mathbb{P}_{1^5,3}[6,2]$	Pseudo-Hybrid
C3	-	-	3	2^2	$\mathbb{P}_{1^7}[3,2,2]$	Pseudo-Hybrid
			K-type			
K1	-	-	3	3	$\mathbb{P}_{1^6}[3,3]$	Hybrid
K2	-	2^2	4	4	$\mathbb{P}_{1^4,2^2}[4,4]$	Hybrid
K3	2^2	3^2	6	6	$\mathbb{P}_{1^2,2^2,3^2}[6,6]$	Hybrid
			M-type			
M1	-	-	2	2^3	$\mathbb{P}_{1^8}[2,2,2,2]$	Non-linear σ

Table 5.1: Model data of one-parameter abelian GLSMs.

with $G_{d_1}(G_{i,d_2})$ a weighted homogeneous polynomial of degree $d_1(d_2)$ in the x-fields. In total there are 14 different abelian-one parameter models, with parameter values as given in Table 5.1.

Depending on whether $\zeta \lesseqgtr 0$ we encounter a different low energy description, called phases (see Section 4.2). We can identify ζ with the real part of the Kähler parameter of a Calabi-Yau manifold. We denote the Kähler moduli space by $\mathcal{M}_K$. In the models $\mathcal{M}_K$ decomposes into two chambers/phases. A the boundary between these two phases a Coulomb branch appears. The location of the Coulomb branch is determined by the zeros of (4.88):

$$t = -\sum_{i=1}^{N} Q_i \log\left(Q_i \sigma\right). \tag{5.7}$$

[1]In this section an abuse of notation is taking place, we denote the superfields and their scalar components by the same lowercase letter, which interpretation is valid should be obvious from the context.

For the models of interest we find singular points which are located at:

$$(\zeta, \theta) = \left(\frac{1}{2\pi} \log \left(\frac{d_1^{d_1} d_2^{k d_2}}{\alpha^{n\alpha} \beta^{j\beta}} \right), (d_1 + k d_2)\pi + 2\pi\mathbb{Z} \right). \tag{5.8}$$

Observe that it is a matter of convention if a factor of $\frac{1}{2\pi}$ appears in front of the ζ value in (5.8). In this thesis we use the convention $t = 2\pi\zeta - i\theta$, which gives the prefactor. In addition to the singular points each phase contains a limiting point, which is either at a finite or infinite distance in $\mathcal{M}_K$. The nature of these special points can be encoded into the local exponents $\mathbf{a} = (a_1, a_2, a_3, a_4), \quad a_i \in \mathbb{Q}$ at each point. The local exponents can be obtained from the Picard-Fuchs operator $\mathcal{L}(z)$ associated to the mirror Calabi-Yau of interest. z is a local coordinate in the complex structure moduli space of the mirror. Nevertheless it is meaningful to associate a Picard-Fuchs operator to the phases of a GLSM, because the sphere and hemisphere partition function of the GLSM fulfil GKZ and Picard-Fuchs equations [52, 53, 54]. The $\mathbf{a}$ are determined by the solutions of the indicial equation of the Picard-Fuchs differential operator at the respective special point. The different labels correspond to different monodromy behaviours of the solutions of the Picard-Fuchs equation around the singular points. These Picard-Fuchs operators can also be found in [55]. In Table 5.2 we give a list of possible local exponents for the one-parameter models.

The assignment of the different points in $\mathcal{M}_K$ to the labels F, C, K and M as been chosen in accordance with [55, 56]. The point at $\zeta \gg 0$ is always a M-point and this phase is called geometric phase. For $\zeta \ll 0$ we encounter M-, F-, K- and C-type limiting points. The finite distance points in the moduli space are given by C- and F-type points. C-type points occur at the boundary of the phases, but can also be encountered as limiting points in a phase. In the latter case they have been studied in [57, 58], where they where named pseudo-hybrids (see Section 5.4). At F-type points the low energy description is given in terms of Landau-Ginzburg orbifold theories, which are describe in Section 5.2, but also pseudo-hybrids are possible for these points. K-type points have low energy descriptions in terms of so-called hybrid models (see Section 5.3), which are Landau-Ginzburg orbifolds fibred over a base manifold.

To study the low energy configuration in a phase we need to solve the D- and F-term equations (4.73). If we insert the characteristics of the models (5.1) in these equations we find:

$$\begin{aligned} -d_1|p_1|^2 - d_2 \sum_{i=1}^{k} |p_{2_i}|^2 + \sum_{i=1}^{5-n-j+k} |x_i|^2 \\ + \alpha \sum_{i=1}^{n} |x_{\alpha_i}|^2 + \beta \sum_{i=1}^{j} |x_{\beta_i}|^2 = \zeta, \qquad \text{D-term}, \end{aligned} \tag{5.9}$$

type	a	distance on $\mathcal{M}$	description
F	(a,b,c,d)	finite	Landau-Ginzburg, pseudo-hybrid
C	(a,b,b,c)	finite	(mirror of) conifold, pseudo-hybrid
K	(a,a,b,b)	infinite	hybrid
M	(a,a,a,a)	infinite	geometric

Table 5.2: F-, C-, K-, and M-points of one-parameter models.

and

$$\left.\begin{aligned} G_{d_1}(x_n) = 0, \quad G_{i,d_2}(x_n) = 0, \quad i = 1, \dots, k \\ G'_l(x_n) = p_1 \frac{\partial G_{d_1}}{\partial x_l} + \sum_{i=1}^{k} p_{2_k} \frac{\partial G_{i,d_2}}{\partial x_l} = 0 \end{aligned}\right\} \text{F-terms}, \tag{5.10}$$

with

$$l \in \{1, \dots, 5-n-j+k, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_j\}. \tag{5.11}$$

We also assume that the $G'_k(x_n)$ are transverse in the following sense:

$$G'_l(x_n) = 0 \quad \forall l \iff x_n = 0 \quad \forall n. \tag{5.12}$$

We subsequently perform an analysis along the lines of Section 4.2 to describe the different possible solutions to (5.9) and (5.10). In Table 5.1 we give an overview of the low energy description in the respective phase.

Certain subsets of these abelian one parameter models have been studied in [59, 60] and a full classification has been given in [61] (see also [56] for the full list of models).

5.1 Geometric Phase

In this case we see from (5.9) that not all x_n can simultaneously vanish. This gives the deleted set:

$$F_{\zeta \gg 0} = \{x_i = 0 \quad \forall i\}. \tag{5.13}$$

As consequence of the F-term equations $G'_l(x_n)$ (5.10) and (5.12) in a vacuum configuration:

$$p_1 = p_{2_i} = 0 \quad \forall i. \tag{5.14}$$

Next we need to take into account the first line of the F-term equations in (5.10), which gives the constraints:

$$G_{d_1}(x_n) = G_{i,d_2}(x_n) = 0, \quad i = 1, \dots, k. \tag{5.15}$$

We reinsert (5.14) into the D-term (5.9):

$$\sum_{i=1}^{5-n-j+k} |x_i|^2 + \alpha \sum_{i=1}^{n} |x_{\alpha_i}|^2 + \beta \sum_{i=1}^{j} |x_{\beta_i}|^2 = \zeta, \tag{5.16}$$

and see that this is the defining equation of a sphere $S^{2(5+k)-1}$. If we take into account the $U(1)$ action and (5.15), we can conclude that the vacuum configuration is a complete intersection in weighted projective space:

$$\mathbb{P}^{5+k-1}_{1^{5+k-n-j}\alpha^n\beta^j}[d_1, \underbrace{d_2, \ldots, d_2}_{k\text{-times}}], \tag{5.17}$$

where the dimension is denoted by superscript and the weights by subscript. The weighted homogeneous degree of the defining equations is given by the number in the bracket. If we expand around the vacuum we see that the gauge group gets completely broken. All modes transverse to the vacuum are massive. The low energy effective theory is given by a non-linear sigma model with target (5.17). Due to (5.4) the target space is a complete intersection Calabi-Yau.

5.2 Landau Ginzburg Phase

Such phases are realized in the models F1, F2, F3 and F4. In all of these models we only have a single p-field, namely p_1. Becasue ζ is negative we see from (5.9) that p_1 cannot vanish and so the deleted set is

$$F_{\zeta \ll 0} = \{p_1 = 0\}. \tag{5.18}$$

The second line of the F-term equations (5.10) and the transversality condition (5.12) forces the x_is to vanish. As consequence $G_{d_1}(x_n) = 0$ has to be satisfied. If we now set the x_is to zero in (5.9), we see that p_{d_1} gets a VEV:

$$|p_{d_1}| = \sqrt{\frac{|\zeta|}{d_1}}. \tag{5.19}$$

We can apply a gauge transformation to set the phase of (5.19) to zero and obtain a unique vacuum configuration. The choice of a vacuum breaks the $U(1)$ symmetry to $\mathbb{Z}_{d_1}$. An expansion around the vacuum reveals that the x_i are massless, as long as, the fields appear with exponents ≥ 3 in G_{d_1}. The massless fields interact via an effective superpotential, which is obtained by setting p_{d_1} to its VEV in (5.6). This effective superpotential has a degenerate critical point at the origin. A model with such a superpotential and a $\mathbb{Z}_{d_1}$ symmetry is called Landau-Ginzburg orbifold model.

Landau-Ginzburg/Calabi Yau Correspondence

This correspondence is the observation that certain $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg orbifold models and certain $\mathcal{N} = 2$ supersymmetric non-linear sigma models with Calabi-Yau targetspace flow to isomorphic conformal field theories in the infrared. This goes back to the work [62, 63], in which a relation between minimal models and Calabi-Yau manifolds was described, and further arguments in favour of the correspondence were given in [64, 65, 66, 20]. The work of Witten [6] showed that the two theories are really two phases of an underlying GLSM[2]. This can be seen, by combining the results from Section 5.1 and Section 5.2. This correspondence was also extended to hybrid models (see [68, 69] and Chapter 6).

5.3 Hybrid Phases

Hybrid phases bear some similarities to Landau-Ginzburg orbifold phases. However in sharp contrast the vacuum manifold is no longer a single point, as we will see in the subsequent analysis.

K-Type Hybrid Phase

From Table 5.1 we can read off that there are 3 models of this kind: K1, K2 and K3. In these models $k = 1$ and $d_1 = d_{2_1}$. We start our analysis by looking at the D-term (5.9) in these models. If we take into account that $\zeta \ll 0$, we see that the deleted set is:

$$F_{\zeta \ll 0} = \{p_1 = p_{2_1} = 0\}. \tag{5.20}$$

In a vacuum configuration the F-terms (5.10) require $x_i = 0, \quad \forall i$. This result can be reinserted into (5.9) and gives

$$|p_1|^2 + |p_{2_1}|^2 = \frac{|\zeta|}{d_1}. \tag{5.21}$$

By the $U(1)$ action this describes a $\mathbb{P}^1_{d_1 d_1}$. The choice of a specific vacuum breaks the $U(1)$ gauge symmetry to $\mathbb{Z}_{d_1}$. By expanding around a chosen vacuum we see that the x_i are massless. We get an effective superpotential and the low energy dynamics is described in terms of a Landau-Ginzburg orbifold. Observe that we now have a Landau-Ginzburg orbifold model for every point of $\mathbb{P}^1_{d_1 d_1}$. Therefore the Landau-Ginzburg orbifold is actually fibred over $\mathbb{P}^1_{d_1 d_1}$.

M-Type Hybrid Phase

For the abelian one parameter models we only have one instance, in which such a phase is realized in the $\zeta \ll 0$ phase namely in the M1 model (Table

[2]See [67] were a similar result was obtained by using mirror symmetry.

(5.1)). The deleted set is

$$F_{\zeta \ll 0} = \{p_1 = p_{2_1} = p_{2_2} = p_{2_3} = 0\}. \tag{5.22}$$

and the D-term equation for a vacuum configuration reads:

$$|p_1|^2 + |p_{2_1}|^2 + |p_{2_2}|^2 + |p_{2_3}|^2 = \frac{|\zeta|}{2}. \tag{5.23}$$

In this case the vacuum manifold is a $\mathbb{P}^3_{2222}$. The non-minimal charges of the p-fields require a special treatment. This was done in [70], where it was found that the Calabi-Yaus related to the two phases of this model are not birational. It was shown that the $\zeta \ll 0$ phase is the non-commutative resolution of a singular branched double cover over $\mathbb{P}^3$ with branching locus $\det A = 0$. A originates from a remodelling of the superpotential W. In this model a generic superpotential W is given by

$$W = p_1 G_2(x_n) + \sum_{i=1}^{3} p_{2_i} G_{i,2}(x_n), \tag{5.24}$$

where $G_2(x_n)$ and the $G_{i,2}(x_n)$s are quadratic polynomials in the x_i. W can be rewritten into

$$W = \sum_{i=1}^{8} x_i A^{ij}(p) x_j, \tag{5.25}$$

where A is a symmetric matrix with linear entries in the p_is.

Although this phase has many features similar to a geometric phase, we will see in Chapter 6 that this phase is more hybrid like from a sphere partition function standpoint. Also the state space of the low energy theory is hybrid like [68].

5.4 Pseudo-Hybrid Phases

Pseudo-hybrid phases are the last possibility of phases we encounter for $\zeta \ll 0$. The characteristic feature of these models is that the phase decomposes into several components, or stated otherwise the vacuum equations of the GLSM allow for multiple solutions. These components show different symmetry breaking patterns. This characteristic makes a unique R-charge assignment in the IR-theory impossible. The low energy description for pseudo-hybrid models is currently not fully understood. In [57] arguments where given that the associated conformal field theories are singular. From Table 5.1 we see that in the C-type and for certain F-type models we encounter a pseudo-hybrid behaviour.

F-Type

There are 3 F-type examples with a pseudo-hybrid phase, F5, F6 and F7 (see Table 5.1) . These models have been previously studied in [60, 71, 72, 73, 56]. From the D-term equation (5.9) we deduce that the deleted set is

$$F_{\zeta\ll 0} = \{p_1 = p_{2_1} = 0\}, \tag{5.26}$$

and vanishing of (5.10) requires $x_i = 0$. We see that a generic vacuum is given for $(p_1, p_{2_1}) \in \mathbb{P}^1_{d_1 d_{2_1}}$ and the gauge group is completely broken. But at the point $(p_1, p_{2_1}) = (1, 0)$ there is an unbroken $\mathbb{Z}_{d_1}$ and the low energy description is given in terms of a Landau-Ginzburg orbifold model with R-charge assignment $q = \frac{1}{d_1}$. A second Landau-Ginzburg orbifold description with $\mathbb{Z}_{d_{2_i}}$ is encountered at the point $(p_1, p_{2_1}) = (0, 1)$.

C-Type

We see from Table 5.1 that there are 3 instances of this phase. In contrast to the pseudo hybrid F-type models the C-Type models will also have vacuum branches which are not point like and where a subgroup of G is unbroken. As these models play a prominent role in Chapter 8, we will discuss each realization separately.

C3

This model was analysed in [57] and we repeat the analysis from the GLSM viewpoint here. By the D-term equation (5.9) the deleted set is

$$F_{\zeta\ll 0} = \{p_1 = p_{2_1} = p_{2_2} = 0\} \tag{5.27}$$

and it follows from the F-terms (5.10) that $x_1 = \ldots = x_7 = 0$. The classical vacuum is a $\mathbb{P}^2_{322}$. The next step is to consider fluctuations of the x_i and we see that for a generic point in $\mathbb{P}^2_{322}$ we do not find proper vacua. In such cases the gauge group is completely broken. Due to quadratic terms in the superpotential the x_i are all massive and they give a zero contribution to the central charge. It follows that these vacua are not Calabi-Yau. The R-symmetry is completely broken, because the non-zero VEVs of the p_i require that they have zero R-charge. In this case it is not possible to find a R-charge assignment for the x_i fields such that the Landau-Ginzburg potential has R-charge two.

The interesting feature is that there are two special points where the vacuum manifold has a different behaviour. First at the point $(p_1, p_{2_1}, p_{2_2}) = (1, 0, 0)$ the gauge group gets broken to a $\mathbb{Z}_3$. We recover a Landau-Ginzburg orbifold with superpotential $W_{LG} = G_{d_1}$ in $\mathbb{C}^7/\mathbb{Z}_3$. The superpotential does not contain quadratic terms and all x_i are massless. It is possible to preserve the R-symmetry, by assigning the charge $\frac{2}{3}$ to all x_i. This choice of R-charge assignment implies that we set $q = \frac{1}{3}$ in (5.1). The central charge of this Landau-Ginzburg model is $\hat{c} = \frac{7}{3}$. This implies that we do not get a superconformal field theory of Calabi-Yau type and hints that this Landau-Ginzburg model alone cannot describe the theory at low energies.

Another vacuum branch is given by the curve $\mathcal{C} = (0, p_{2_1}, p_{2_2})$. The preserved symmetry group is a $\mathbb{Z}_2$. The R-symmetry is preserved by the assignment of R-charge 1 (set $q = \frac{1}{2}$) to the x_is. The $\mathbb{Z}_2$ Landau-Ginzburg orbifold fibred over $\mathcal{C}$ has a quadratic superpotential and at a first glance we would expect that we only have massive degrees of freedom. But it is possible to write the superpotential in the form

$$W = \sum_{i,j=1}^{7} x_i A^{ij}(p) x_j, \tag{5.28}$$

where A^{ij} is a generic 7×7 matrix. The matrix A has entries linear in $p_{2_1,2_2}$. In the case when the rank of $A(p)$ drops (i.e. $\det A(p) = 0$) there will be massless degrees of freedom. In [74, 70] situations similar to the above described where discussed.

We saw that in the $\zeta \ll 0$ phase we found two branches where the gauge symmetry is broken to a $\mathbb{Z}_2$ and a $\mathbb{Z}_3$ respectively, whereby the former corresponds to a hybrid-type and the latter to a Landau-Ginzburg orbifold.

C1

The zero of the scalar potential is given by $(p_1, p_{2_1}) \in \mathbb{P}^1_{42}$ and $x_i = 0$. A generic point (p_1, p_{2_1}) does not lead to a well-defined vacuum, but a the point $(p_1, p_{2_1}) = (1, 0)$ we find a Landau-Ginzburg orbifold with $W_{LG} = G_4$ in $\mathbb{C}^6/\mathbb{Z}_4$. To obtain the R-charge assignment we set $q = \frac{1}{4}$ in the Table (5.1). The central charge of the IR CFT is given by $\hat{c} = 3$. Another branch is located at $p_{2_1} \neq 0$. At this branch a $\mathbb{Z}_2$ is unbroken and the R-charges are given by setting $q = \frac{1}{2}$. This theory is massive. Again it is tempting to argue that this massive theory does not contribute in the IR, but we see some effects of this branch in the sphere partition function (see Section 6.3).

C2

The zeros of the classical potential are at $(p_1, p_{2_1}) \in \mathbb{P}^1_{62}$ and $x_i = 0$. At $(p_1, p_{2_1}) = (1, 0)$ we find a Landau-Ginzburg orbifold model in $\mathbb{C}^7/\mathbb{Z}_6$ and potential $W_{LG} = G_6$. The x_6 field is massive and the R-charge assignment is given by $q = \frac{1}{6}$. The central charge of the IR CFT is $\hat{c} = \frac{10}{3}$, which is higher than the value for the Calabi-Yau case. A second branch sits at $p_{2_1} \neq 0$. In this branch a $\mathbb{Z}_2$ is unbroken and the R-charges can be read off from Table (5.1) by setting $q = \frac{1}{2}$.

Chapter 6

The Structure of the Sphere Partition Function

In Section 4.4 we have introduced the sphere partition function of the gauged linear sigma model and described its dependence on the moduli of the underlying Calabi-Yau manifold. In this chapter we want to make the connection to known structures on the Calabi-Yau moduli space and conjecture a universal form of the sphere partition function, whereby universal we mean valid throughout the moduli space.

In [75] it was shown that the sphere partition function gives the fully quantum corrected exponentiated Kähler potential on the moduli space. This was checked by mirror symmetry in the geometric regime. The results of [76, 77, 78] suggest that this relation to the Kähler potential is valid away from the geometric regime. In view of these results and the common UV origin in terms of the GLSM it is natural to expect a general structure of the sphere partition function.

We begin by introducing universal structures on the moduli space and finish by matching the conjectured form for a class of abelian one parameter models and certain abelian two parameter models. For that purpose we evaluate the sphere partition function in the different phases of the respective GLSM and identify the structures introduced before. This chapter is based on the author's work [9].

6.1 Universal Structures on Calabi-Yau Moduli Spaces

In this section we will introduce two structures which appear on the moduli space of Calabi-Yau compactifications. The first is tt^* geometry as first studied in [79] and Givental's I and J functions (e.g. [80]).

tt^* Geometry

We will introduce the basic aspects of tt^* geometry and follow in our discussion the papers [79, 81, 30, 3] and [82]. In contrast to the original paper [79] we focus right away to $N = (2,2)$ supersymmetric conformal theories (SCFT) with central charge c. We introduced the $N = 2$ SCFT algebra in Section 3.1. We focus on the NS-NS sector and on the R-R sector which we will call simply NS- and Ramond-sectors subsequently. These SCFTs have four supercharges, $Q_\pm, \overline{Q}_\pm$, which can be combined into four nilpotent operators:

$$\begin{aligned} Q_A &= \overline{Q}_+ + Q_-, \\ Q_B &= \overline{Q}_- + \overline{Q}_+, \end{aligned} \tag{6.1}$$

and their complex conjugates. In the NS-sector the operator $Q_A\,(Q_B)$ annihilates states in the $(a,c)((c,c))$-ring (see Section 3.1). In a similar manner $Q_A^\dagger\left(Q_B^\dagger\right)$ annihilates states in the $(c,a)((a,a))$-ring. The Ramond ground states are defined by the states which are annihilated by an operator (6.1) and the complex conjugate thereof. The operators (6.1) are not defined on a general Riemann surface due to their spinorial origin. To make the discussion valid on a generic Riemann surface a topological twist has to be performed. For the twist there are different possibilities and each possibility results in a different operator in (6.1), or their complex conjugate, becoming scalar. The twisting procedure was discussed for non-linear sigma models with a CY target in Section 3.3. This choice also singles out a ring structure, which provides the physical operators. In the following we will assume that we have taken one choice. This theory is subsequently called topological theory and the complex conjugate twisted theory will by called anti-topological. We discussed in Section 3.1 that a certain subset of these rings can be used to construct perturbations of the theory and we will denote the parameters describing these deformations by $t^i, \overline{t}^i$. The ring structure is encoded in the structure constants (see Section 3.1):

$$\phi_i \phi_j = C_{ij}^k \phi_k, \tag{6.2}$$

where $C_{ik}^l\left(\overline{C}_{ij}^k\right)$ depends on $t^i\left(\overline{t}^i\right)$. As shown in Section 3.1 the different sectors are related by spectral flow. By spectral flow we can get a Ramond ground state from a state in the NS-sector. In the NS-sector we can identify a unique vacuum state $|0\rangle_{\mathrm{NS}}$, namely the state with lowest energy and left-right-charge $(q_L, q_R) = (0,0)$. Ground states in the Ramond sector can then by obtained by acting with a ring element ϕ_i on the NS vacuum: $\phi_i|0\rangle_{\mathrm{NS}}$, and afterwards performing a spectral flow. We will denote the obtained state in the Ramond sector by $|i\rangle$. Its now possible to realize the ring structure (6.2) directly on the ground states by

$$\phi_i|j\rangle = \phi_i\phi_j|0\rangle = C_{ij}^k \phi_k|0\rangle = C_{ij}^k|k\rangle. \tag{6.3}$$

where $|0\rangle$ is the ground state in the Ramond sector obtained under spectral flow from the NS sector vacuum state. It is important to note, that we introduced the relation between the states using spectral flow as defined in a $N=(2,2)$ superconformal theory. We refer to [79] for a detailed discussion of the case with only $N=(2,2)$ supersymmetry available.

Let us now consider a change of the parameters $t^i, \bar{t}^i$. Such a change results in a variation of the ground states in the full Hilbert space of the theory. Note that the full Hilbert space is unchanged. The ground states can be viewed as sections $|i(t,\bar{t})\rangle$ of the ground state bundle $\mathcal{V}$ over the parameter space $\mathcal{M}$ spanned by $t^i, \bar{t}^i$. We introduce a covariant derivative with a connection, such that non-orthogonal variations of the ground states are projected out. The defining property is

$$\langle a(t,\bar{t})|D_i|b(t,\bar{t})\rangle = \langle a(t,\bar{t})|\frac{\partial}{\partial t^i} - A_i|b(t,\bar{t})\rangle \equiv 0, \tag{6.4}$$

$$\langle a(t,\bar{t})|\overline{D}_i|b(t,\bar{t})\rangle = \langle a(t,\bar{t})|\frac{\partial}{\partial \bar{t}^i} - \overline{A}_i|b(t,\bar{t})\rangle \equiv 0. \tag{6.5}$$

It follows

$$A_{iab} = \langle a(t,\bar{t})|\frac{\partial}{\partial t^i}|b(t,\bar{t})\rangle, \qquad \overline{A}_{iab} = \langle a(t,\bar{t})|\frac{\partial}{\partial \bar{t}^i}|b(t,\bar{t})\rangle. \tag{6.6}$$

The above discussion is also valid for the anti-topological theory, with the appropriate notational adjustments. We denote the ground states in the anti-topologically twisted theory by $|\bar{b}(t,\bar{t})\rangle$ and the singled out ground state by $|\bar{0}\rangle$. But these are the same Ramond ground states as before and therefore the conjugate twisted theory provides only a different basis of these states. Therefore they must be related to the basis given by the twisted theory:

$$|a\rangle = M_a^{\bar{b}}|\bar{b}\rangle. \tag{6.7}$$

By the CPT operation, which changes $|a\rangle$ to $|\bar{a}\rangle$, it follows that

$$MM^* = \mathbf{1}. \tag{6.8}$$

The topologically twisted theory provides a holomorphic basis in which

$$(\overline{A}_i)_k^l = 0, \tag{6.9}$$

and similarly the anti-topological twist results in an anti-holomorphic basis with $(A_i)_{\bar{k}}^{\bar{l}} = 0$. The paring on the ground state bundle is either given by the pairing of the Ramond ground states:

$$\eta_{ij} = \langle j|i\rangle, \tag{6.10}$$

or by

$$g_{i\bar{j}} = \langle \bar{j}|i\rangle, \tag{6.11}$$

which gives a Hermitian metric. A crucial observation in [79] was, that the connection on $\mathcal{V}$, introduced above, fulfils the so called tt^*- equations:

$$[D_i, D_j] = 0, \qquad \left[\overline{D}_i, \overline{D}_j\right] = 0, \tag{6.12}$$

$$[D_i, C_j] = [D_j, C_i]\,, \qquad \left[\overline{D}_i, \overline{C}_j\right] = \left[\overline{D}_j, \overline{C}_i\right], \tag{6.13}$$

$$\left[D_i, \overline{D}_j\right] = -\left[C_i, \overline{C}_j\right], \tag{6.14}$$

where $C_j = (C_j)_l^k$ are the structure constants introduced in eqn. 6.2. From the tt^*- equations it follows that

$$\nabla_i = D_i - C_i, \tag{6.15}$$

and $\nabla_{\bar{\imath}}$ have vanishing curvature on $\mathcal{V}$.

The flatness of the connection allows to identify the fibres of $\mathcal{V}$ with a fixed fibre V at a chosen point by parallel transport. Chose V to be the vector space of ground states. $\nabla_i, \nabla_{\bar{\imath}}$ reduce to the ordinary derivatives $\frac{\partial}{\partial t^i}, \frac{\partial}{\partial \bar{t}^i}$ in this setup. CPT provides a real structure on $\mathcal{V}$, by declaring CPT invariant states as real.

We now focus on a subring generated by operators of conformal dimension $(\frac{1}{2}, \frac{1}{2})$. This ring is called deformation ring in [30] and [82]. The generators of the ring correspond to marginal deformations of the SCFT and in Section 3.1 we gave a procedure on how to construct these deformations. We will denote this ring by $\mathcal{H}^{def}$. We assume that we have m such elements. A basis of $\mathcal{H}^{def}$ is given by

$$\left\{|0\rangle, |a_1\rangle, |a_2\rangle, \ldots, |a_m\rangle, |a^1\rangle, |a^2\rangle, \ldots |a^m\rangle, |\Omega\rangle\right\}, \tag{6.16}$$

$|a^i\rangle$ are the dual states with respect to (6.10) and $|\Omega\rangle$ originates from the unique state in the NS sector with conformal dimension $(\frac{c}{6}, \frac{c}{6})$ (see Section 3.1). We see that $\mathcal{H}^{def}$ has dimension $2m + 2$. In the case of a SCFT with $c = 9$ the bundle V decomposes into

$$V = \mathcal{L} \oplus (\mathcal{TM} \otimes \mathcal{L}) \oplus \overline{(\mathcal{TM} \otimes \mathcal{L})} \oplus \overline{\mathcal{L}}. \tag{6.17}$$

$\mathcal{L}$ is the line bundle corresponding to the state $|0\rangle$. The fibres of $(\mathcal{TM} \otimes \mathcal{M})$ are spanned by the $|\alpha_i\rangle$, where $\mathcal{TM}$ is the holomorphic tangent space of $\mathcal{M}$. The conjugate bundles are spanned by the states:

$$|a_{\bar{\imath}}\rangle = g_{\bar{\imath}k}|a^k\rangle, \qquad |\overline{0}\rangle = g_{\overline{0}0}|0\rangle, \tag{6.18}$$

$g_{\bar{\jmath}i}$ is given in (6.11). By restricting the indices i, j to the marginal deformations we can introduce the metric

$$G_{i\bar{\jmath}} = \frac{g_{i\bar{\jmath}}}{\langle\overline{0}|0\rangle}. \tag{6.19}$$

G is the Zamolodchikov metric [24] as shown in [79]. Further by the tt^* - equations it follows that

$$G_{i\bar{j}} = -\partial_i \partial_j \log\langle\overline{0}|0\rangle, \tag{6.20}$$

and so

$$\langle\overline{0}|0\rangle = e^{-K(t,\bar{t})}, \tag{6.21}$$

$K(t,\bar{t})$ is the Kähler potential of $G_{i\bar{j}}$.

We have now all the ingredients to make the connection to the sphere partition function of the GLSM. In [75] it was conjectured and tested by examples, that

$$Z_{S^2}^{phase}(\mathsf{t},\bar{\mathsf{t}}) = e^{-K(\mathsf{t},\bar{\mathsf{t}})} = \langle\overline{0}|0\rangle, \tag{6.22}$$

where the last equality follows from tt^*- geometry and *phase* means evaluation of the sphere partition function in a certain phase of the GLSM. The authors of [78] verified the conjecture by using tt^*-geometry arguments. Observe that the t are the FI-theta parameters of the GLSM (6.71) and not the flat coordinates t, related to the marginal deformations. To extract enumerative invariants from the result of the GLSM we first need to change the coordinates from t to t. The procedure on how to extract this coordinate change in geometric and Landau-Ginzburg phases was given in [75] and [82]. The change coincides with the mirror map.

Givental's I and J Function

Let us first mention that our discussion will be heuristic, because the background material to fully understand the approach is vast and for our approach we only need certain aspects. This section follows the approach and notation of [83, 84]. See also [85, 86].

On of the main aspects of Givental's approach is that the genus zero invariants [1] are encoded in certain subspaces $\mathcal{L}_\bullet$. These subspaces are Lagrangian cones in a symplectic vector space $(\mathcal{V}_\bullet, \Omega_\bullet)$, where $\Omega_\bullet$ denotes the symplectic form.

The vector space $\mathcal{V}_\bullet$ consists of Laurent series with values in the state space $H_\bullet$:

$$\mathcal{V}_\bullet = H_\bullet \otimes \mathbb{C}((z^{-1})). \tag{6.23}$$

Observe that we did not explicitly state the nature of the state space $H_\bullet$, because it depends on the theory of interest. In Gromov-Witten (GW) theory

[1] We are here not very specify about the nature and definition of the term invariant. The exact nature of the invariants is highly dependent on the theory we study (e.g. Gromov-Witten invariants, FJRW invariants)

it would correspond to a subset of the cohomology of the Calabi-Yau hypersurface X and in FJRW theory it would be given by the FJRW theory state space and in hybrid theory by the appropriate state space of this theory. We denote this by $\bullet$.

We can introduce a basis of $H_\bullet$: $\Phi_0, \ldots, \Phi_k$ and $\Phi_0 = \mathbf{1}_\bullet$ is the unique identity element of the respective theory. The dual basis is denoted by $\Phi^0, \ldots, \Phi^k$. The symplectic form is given by

$$\Omega_\bullet(f_1, f_2) = \mathrm{Res}_{z=0} \left\langle f_1(-z), f_2(z) \right\rangle_\bullet , \tag{6.24}$$

where $\langle \cdot, \cdot \rangle_\bullet$ is a pairing on the state space $H_\bullet$.

The invariants of these theories describe certain intersection numbers in a moduli space of genus g curves. The attributes of the moduli space, the curves it parameterizes and the intersection numbers depend on the theory. We will not give details on the construction of the invariants and we denote the invariants of these theories collectively by

$$\langle \tau_{a_1}(\Phi_{i_1}), \ldots, \tau_{a_n}(\Phi_{i_n}) \rangle^\bullet_{g,n,\delta} , \tag{6.25}$$

g gives the genus of the curve and n the number of insertions. δ is a label which need not be present in all possible theories. For example in FJRW theory all $\delta > 0$ invariants are set to zero and in GW theory it labels the homology class $\delta \in H_2(X, \mathbb{Z})$. The genus g invariants can be collected in a generating function

$$F^g_\bullet = \sum_{\substack{a_1,\ldots,a_n \\ h_1,\ldots,h_n}} \sum_{\delta \geq 0} \langle \tau_{a_1}(\Phi_{h_1}), \ldots, \tau_{a_n}(\Phi_{h_n}) \rangle^\bullet_{g,n,\delta} \frac{t^{h_1}_{a_1} \cdots t^{h_n}_{a_n}}{n!}, \tag{6.26}$$

$t^{h_i}_{a_i}$ are formal variables associated to $\tau_{a_i}(\Phi_{h_i})$. As mentioned in the beginning the invariants are encoded in a subspace $\mathcal{L}_\bullet \subset \mathcal{V}_\bullet$. This subspace is constructed out of (6.26). Every point of $\mathcal{L}_\bullet$ can be written in the form

$$\begin{aligned} -z\Phi_0 &+ \sum_{\substack{0 \leq h \leq k \\ a \geq 0}} t^h_a \Phi_h z^a \\ &+ \sum_{\substack{n \geq 0 \\ \delta \geq 0}} \sum_{\substack{0 \leq h_1,\ldots,h_n \leq k \\ a_1,\ldots,a_n \geq 0}} \sum_{\substack{0 \leq \epsilon \leq k \\ l \geq 0}} \frac{t^{h_1}_{a_1} \cdots t^{h_n}_{a_n}}{n!(-z)^{l+1}} \\ &\cdot \langle \tau_{a_1}(\Phi_{h_1}), \ldots, \tau_{a_n}(\Phi_{h_n}), \tau_l(\Phi_\epsilon) \rangle^\bullet_{g,n+1,\delta} \, \Phi^\epsilon. \end{aligned} \tag{6.27}$$

It is possible to focus on points with $a = a_i = 0$:

$$\begin{aligned} J_\bullet(t, z) = -z\Phi_0 &+ \sum_{0 \leq h \leq k} t^h_0 \Phi_h \\ &+ \sum_{\substack{n \geq 0 \\ \delta \geq 0}} \sum_{0 \leq h_1,\ldots,h_n \leq k} \sum_{\substack{0 \leq \epsilon \leq k \\ l \geq 0}} \frac{t^{h_1}_0 \cdots t^{h_n}_0}{n!(-z)^{l+1}} \\ &\cdot \langle \tau_0(\Phi_{h_1}), \ldots, \tau_0(\Phi_{h_n}), \tau_l(\Phi_\epsilon) \rangle^\bullet_{g,n+1,\delta} \, \Phi^\epsilon. \end{aligned} \tag{6.28}$$

because the other points of $\mathcal{L}_\bullet$ follow uniquely from these points (see e.g. [87] for details on this relation). The J-function is then given by

$$t = \sum_{h=0}^{k} t_0^h \Phi_h \mapsto J_\bullet(t, z), \tag{6.29}$$

which maps form the state space $H_\bullet$ to $\mathcal{V}_\bullet$. The $J(t, z)$ function can be obtained from a function usually called I-function. For example in the setting of GW theory for a Calabi-Yau threefold X the I-function is a solution of the Picard-Fuchs equation of the mirror $X^\vee$.

The relation between I and J is given by the mirror map. See also the discussion in the Section 3.4. The Landau-Ginzburg/Calabi-Yau correspondence (see Section 5.2) can be described in this setting by a symplectic transformation

$$\mathbb{U}_{\text{LG-CY}} : \mathcal{V}_{\text{FJRW}} \to \mathcal{V}_{\text{GW}}, \tag{6.30}$$

with $\mathbb{U}_{\text{LG-CY}}(\mathcal{L}_{\text{FJRW}}) = \mathcal{L}_{\text{GW}}$. We again refer to [83] for more details.

6.2 Universal Structure of the Z_{S^2} in Phases of the GLSM

In [9] the following form of the Z_{S^2} in a phase which is a Landau-Ginzburg orbifold, with orbifold group G, fibred over the manifold B was conjectured:

$$\begin{aligned} Z_{S^2}^{phase}(\mathsf{t}, \bar{\mathsf{t}}) &= C \sum_{\delta \in G} \int_B (-1)^{\text{Gr}} \frac{\hat{\Gamma}_\delta(H)}{\hat{\Gamma}_\delta^*(H)} I_\delta(u(\mathsf{t}), H) I_\delta(\overline{u}(\bar{\mathsf{t}}), H), \\ &= \langle \overline{I}, I \rangle. \end{aligned} \tag{6.31}$$

The sum δ runs over certain twisted sectors in the orbifold group G called narrow sectors, which will be described in detail below. Gr is the eigenvalue of a grading operator on the narrow state space and by abuse of notation Gr in the above integral stands for the eigenvalue of a state in the respective sector. $\hat{\Gamma}_\delta, \hat{\Gamma}_\delta^*$ denote the component of the Gamma class and its conjugate in the sector δ. $I_\delta(u(\mathsf{t}), H)$ represents the component of Givental's I-function in the sector δ. By H we denote the generator of the Kähler cone in $H^2(B)$. C is a normalization constant.

Let us also explain the meaning of the second line in eqn. 6.31. For this purpose we expand the I function in a basis e_r of a subspace of the physical state space. The subspace will be denoted by $\mathcal{H}^{def}$. We will clarify the nature of this subspace below, when we discuss specific phases and justify the label def, which is similar to the label introduced in Section 6.1 for the subring generated by the marginal deformation operators. We expand $|I\rangle$ in the following form:

$$|I\rangle = \sum_r I_r e_r. \tag{6.32}$$

In order to expand $\langle \overline{I}|$ in a basis we need a notion of complex conjugation. The results of [88] in geometric phases suggest the following definition

$$\langle \overline{I}| = \sum_r \overline{I}_r e_r^*, \qquad \overline{I}(\overline{u}) = (-1)^{\mathrm{Gr}} \frac{\hat{\Gamma}}{\hat{\Gamma}^*} I(\overline{u}), \tag{6.33}$$

where e_r^* is the dual of e_r such that $\langle e_{r'}^*, e_r\rangle = c \cdot \delta_{r,r'}$. c is a normalization constant. This definition is also suggested by the results in the phases studied below. It seams natural to identify the paring $\langle \cdot, \cdot \rangle$ with the pairing given in (6.10). A comparison of (6.33) and of (6.11) suggest to interpret (6.33) as implementing the action of the matrix M (6.7). Further the basis expansion allows us to write the sphere partition function in another form namely

$$Z_{S^2}^{phase} = \overline{I} M I, \tag{6.34}$$

where I and $\overline{I}$ are interpreted as $\dim \mathcal{H}^{def}$ column and row vectors respectively. The $\dim \mathcal{H}^{def} \times \dim \mathcal{H}^{def}$ matrix M represents the action of $(-1)^{\mathrm{Gr}} \frac{\hat{\Gamma}}{\hat{\Gamma}^*}$. The decomposition of the I function in a basis of $\mathcal{H}^{def}$ allows the extraction of the coordinate transformation to flat coordinates. For that purpose one focuses on the component of the I-function in the direction of the unique ground state $|0\rangle$, denoted by I_0 and the components in the direction of the marginal deformations, denoted by I_j. The flat coordinates are given by

$$t_j(u) = \frac{I_j}{I_0}, \tag{6.35}$$

and the J-function follows from

$$J(t(u)) = \frac{I}{I_0}. \tag{6.36}$$

This transformation is realized in the sphere partition function by a change of normalization

$$\begin{aligned}\tilde{Z}_{S^2}^{phases}(t, \overline{t}) &= C \sum_{\delta \in G} \int_B (-1)^{\mathrm{Gr}} \frac{\hat{\Gamma}_\delta(H)}{\hat{\Gamma}_\delta^*(H)} \frac{I_\delta(u(t), H) I_\delta(\overline{u}(\overline{t}), H)}{I_0(u(t)) \overline{I}_0(\overline{u}(t))}, \\ &= \langle \overline{J}, J \rangle. \end{aligned} \tag{6.37}$$

Landau-Ginzburg Orbifold Phases and FJRW Theory

In Landau-Ginzburg orbifold models the conjectured form of the sphere partition function can be explicitly tested, because for certain examples expression for the I-function and Gamma class are known from FJRW theory ([89, 90]). The authors of [82] showed how to obtain this quantities directly form the Landau-Ginzburg data. We will follow their discussion and refer to their paper for more details.

We consider a Landau-Ginzburg model with an orbifold group G. The model comes with N fields x_i and with a holomorphic quasi-homogeneous G invariant superpotential W. The potential has the property $\mathrm{d}W^{-1}(0) = \{0\}$. The left-$R$-charge q_i of the x_is is chosen such that that W has left-R-charge 1: $W(\lambda^{q_i}x_i) = \lambda W(x_i)$. The vector R-charge of W is 2. Let W be of (quasi-homogenous) degree d. It follows that there exists an $\mathbb{Z}_d$ orbifold action $\langle J\rangle$, with $J = \left(e^{2\pi i q_1}, \ldots, e^{2\pi i q_N}\right)$. Our focus lies on models with $G = \langle J\rangle$, but the following statements are valid in a more general setup (see [82]). As shown in [91, 92] the state space consists of γ-twisted sectors

$$\mathcal{H} = \sum_{\gamma\in G} \mathcal{H}_\gamma. \tag{6.38}$$

The different $\mathcal{H}_\gamma$ are spanned by fields that satisfy untwisted boundary conditions in the γ-twisted sectors. In our case of interest we have $\gamma = J^l \quad (l = 0, \ldots, d-1)$. In this setup the untwisted boundary conditions read $x^i(e^{2\pi i}z) = e^{2\pi i q_i l}x_i(z) \quad q_i l \in \mathbb{Z}$. These fields serve as building blocks for G-invariant states. Among the constructed states of $\mathcal{H}$ we can single out the ground-states $|0\rangle_\gamma^{(c,c)}$, $|0\rangle_\gamma^{(a,c)}$ and $|0\rangle_\gamma^{R}$ of the (c,c)-, (a,c)-ring and the Ramond sector respectively. The spectral flow operation provides an isomorphism between these states [20]:

$$\mathcal{U}_{\left(-\frac{1}{2},-\frac{1}{2}\right)}|0\rangle_\gamma^{(c,c)} = |0\rangle_\gamma^R, \qquad \mathcal{U}_{(-1,0)}|0\rangle_\gamma^{(c,c)} = |0\rangle_{\gamma J}^{(a,c)}. \tag{6.39}$$

$\mathcal{U}_{(r,\bar{r})}$ is the spectral flow operator with R-charges $(\widehat{c}r, \widehat{c}\overline{r})$ and $\widehat{c} = \sum_{i=1}^N (1 - 2q_i)$. The states in the (c,c) ring can be expressed in terms of G-invariant monomials of the Jacobi ring of $W_\gamma = W|_{\mathrm{Fix}\,\gamma}$. The other states can be obtained via spectral flow. The eigenvalue of the vacuum states under the action of the generators $F_{L/R}$ of the left and right moving R-symmetries give their left and right R charges $(q, \overline{q})$:

$$F_L|0\rangle_\gamma = \left(\mathrm{age}(\gamma) - \frac{N}{2} + \sum_{j:\ell q_j\in\mathbb{Z}} q_j + \frac{\widehat{c}}{2}\right)|0\rangle_\gamma \tag{6.40}$$

$$F_R|0\rangle_\gamma = \left(-\mathrm{age}(\gamma) + \frac{N}{2} - n_\gamma + \sum_{j:\ell q_j\in\mathbb{Z}} q_j + \frac{\widehat{c}}{2}\right)|0\rangle_\gamma, \tag{6.41}$$

with

$$\mathrm{age}(\gamma) = \sum_j q_j, \qquad n_\gamma = \dim(\mathrm{Fix}(\gamma)). \tag{6.42}$$

We can further subdivide the sectors of the (c,c)-ring (and their images under spectral flow) into narrow and broad. Narrow sectors contain only the vacuum as a ground state. We label the one dimensional narrow-sectors by ϕ_δ, with $\delta \in G$. The paring on the (c,c)-ring is defined by

$$\langle \phi_\delta, \phi_{\delta'} \rangle = \frac{1}{|G|}\delta_{\delta,\delta'^{-1}}. \tag{6.43}$$

By the use of spectral flow (6.39) we can get the pairing in the (a,c) ring. The definition of the
I-function and Gamma class requires to take into account marginal deformations. For our cases of interest the marginal deformations correspond to elements of the (a,c) ring with left/right R-charges $(-1,1)$. The information of the marginal deformations can be encoded into a $h \times (h+N)$ matrix q, given a h dimensional space of marginal deformations. q can be calculated from the defining data of the Landau-Ginzburg orbifold [82]. In our case we start from a GLSM with gauge group $U(1)^h$ and a Landau-Ginzburg phase, the matrix q can be read off from the GLSM data. We start from the matrix C of GLSM gauge charges and split it into blocks $\mathsf{C} = (L,S)$, where L is a $h \times h$ matrix containing the charges of the fields with non-zero VEV in the Landau-Ginzburg phase. The matrix q is then given by $q = L^{-1}\mathsf{C}$. The matrix L and q can also be obtained without a GLSM description, see [82]. The matrix q is used to define the I-function and the Gamma class. The Gamma class and the I-function are expandable in terms of basis elements $e_\delta^{(a,c)}$ of the (a,c) ring, but the labelling of the different components in terms of FJRW-theory is more convenient. The FJRW labelling is closer to the labelling of the (c,c)-ring and we can relate the different labelling conventions by

$$e_{J\delta}^{(a,c)} = e_\delta^{(c,c)} = e_{\delta^{-1}}, \tag{6.44}$$

where the last equality is the FJRW basis. We chose the label for our cases to be given by e_l, because in our examples $\delta = J^l, \quad l = 0,\ldots,d-1$. In [82] the I-function for Landau-Ginzburg orbifolds is defined by

$$\begin{aligned} I_\ell(u) = -\sum_{\substack{k_1,\ldots,k_h \geq 0 \\ k' \equiv \ell \mod d}} &\frac{u^k}{\prod_{a=1}^h \Gamma(k_a+1)} \\ &\cdot \prod_{j=1}^N \frac{(-1)^{\langle -\sum_{a=1}^h k_a q_{a,h+j} + q_j \rangle}\Gamma(\langle \sum_{a=1}^h k_a q_{a,h+j} - q_j\rangle)}{\Gamma(1+\sum_{a=1}^h k_a q_{a,h+j} - q_j)}, \end{aligned} \tag{6.45}$$

where $\langle x \rangle = x - \lfloor x \rfloor$ and $u^k = \prod_i u_i^{k_i}$. The integers k_i have periodicities encoded in the matrix L associated to the action of the orbifold group G:

$$k \sim k + L^T m \quad \forall m \in \mathbb{Z}^h. \tag{6.46}$$

The matrix L encodes the embedding of the Landau-Ginzburg orbifold group into the GLSM gauge group and by this one can relate different values of k to different sectors labelled by l. A systematic way to obtain this relation was provided in [82]. The Landau-Ginzburg I-function can now be written as:

$$I_{LG}(u) = \sum_{\delta \in G} I_\delta(u) e_\delta^{(a,c)}. \tag{6.47}$$

The information used to write down the Gamma class is also encoded in the q matrix. The Gamma class is given by [82]:

$$\widehat{\Gamma}_{LG} e_\gamma^{(a,c)} = \widehat{\Gamma}_\gamma e_\gamma^{(a,c)} \qquad \widehat{\Gamma}_\delta = \prod_{j=1}^{N} \Gamma\left(1 - \left\langle \sum_{a=1}^{h} k_a q_{a,h+j} - q_j \right\rangle\right). \tag{6.48}$$

One sees that the Gamma class acts diagonally on $\mathcal{H}^{(a,c)}$. Observe that $\widehat{\Gamma}_l = \widehat{\Gamma}_{\delta^{-1}J}$. The conjugate expression is given by

$$\widehat{\Gamma}^*_{LG} e_\gamma^{(a,c)} = \widehat{\Gamma}^*_\gamma e_\gamma^{(a,c)} \qquad \widehat{\Gamma}^*_\delta = \prod_{j=1}^{N} \Gamma\left(\left\langle \sum_{a=1}^{h} k_a q_{a,h+j} - q_j \right\rangle\right). \tag{6.49}$$

We also introduce

$$\mathrm{Gr} = \sum_{j=1}^{N} \left\langle - \sum_{a=1}^{h} k_a q_{a,h+j} + q_j \right\rangle, \tag{6.50}$$

which coincides with the eigenvalues of the grading operator defined on the FJRW state space. The sphere partition function is then given by

$$Z_{S^2}^{LG}(\mathsf{t}, \bar{\mathsf{t}}) = \frac{1}{|G|} \sum_\delta (-1)^{\mathrm{Gr}} \frac{\widehat{\Gamma}_\delta}{\widehat{\Gamma}^*_\delta} I_\delta(u(\mathsf{t})) I_\delta(\overline{u}(\bar{\mathsf{t}})) = \langle \overline{I}_{LG}(\overline{u}(\bar{\mathsf{t}})), I_{LG}(u(\mathsf{t})) \rangle, \tag{6.51}$$

with pairing (6.43) and

$$\langle \overline{I}_{LG}(\overline{u}(\bar{\mathsf{t}})| = \sum_\delta (-1)^{\mathrm{Gr}} \frac{\widehat{\Gamma}_\delta}{\widehat{\Gamma}^*_\delta} I_\delta(\overline{u}(\bar{\mathsf{t}})) e_{\delta^{-1}}. \tag{6.52}$$

To determine the J-function and the flat coordinates t_a, we start from the element I_0 (associated to the basis element $e_0^{(a,c)}$). This element is the unique element with left/right R-charges $(q, \overline{q}) = (0, 0)$. Next we identify the elements I_{δ_a} $(a = 1, \ldots, h)$ of charges $(q, \overline{q}) = (-1, 1)$ corresponding to the marginal deformations. Then the flat coordinates are

$$t_a = \frac{I_{\delta_a}}{I_0}. \tag{6.53}$$

The definition of the J-function then reads

$$J_{LG}(t) = \frac{I_{LG}(u(t))}{I_0(u(t))}. \tag{6.54}$$

Geometric Phases

In order to evaluate (6.31) in geometric phases, we can turn to the literature, because these phases are well studied. The relation between the I-function and the sphere partition function in geometric phases of abelian and non-abelian GLSM has been noted in [93, 94, 95, 54, 96, 97].

We focus on nef complete intersection Calabi-Yaus in smooth toric varieties, for which a general expression for the I-function has been given in [98, 1]. Our discussion follows [1], where also the result for the degree 8 two-parameter example has been discussed (see Section 6.4). We start from a smooth toric variety X_Σ, defined by a fan Σ. By $\mathcal{L}_1, \ldots, \mathcal{L}_l$ we denote the line bundles of X_Σ generated by the global sections. To the toric variety X_Σ we associate an $(N-)$lattice polytope Δ^*. Next we consider a smooth Calabi-Yau complete intersection $X \subset X_\Sigma$, defined by a the global section of $\mathcal{V} = \oplus_{i=1}^{l} \mathcal{L}_i$. Let D_ρ be the divisor associated to the one dimensional cone $\rho \in \Sigma(1)$ of Σ. By an abuse of notation we denote the cohomology class of a divisor D_ρ by $D_\rho \in H^2(X_\Sigma)$. We use an integral basis $H_1, \ldots, H_h$ of $H^2(X_\Sigma, \mathbb{Z})$, which lies in the closure of the Kähler cone. We introduce $\delta = \sum_{i=1}^{h} t_i H_i$. Let $\beta \in H_2(X_\Sigma, \mathbb{Z})$ and we define $\mathcal{L}_i(\beta) = \int_\beta c_1(\mathcal{L}_i)$ and $D_\rho(\beta) = \int_\beta D_\rho$. Then the I-function I_X is given by

$$
\begin{aligned}
I_X(u, H) = \prod_i u_i^{H_i} \sum_{\beta \in M(X_\Sigma)} \prod_{i=1}^{h} u_i^{\int_\beta H_i} \Bigg(& \frac{\prod_{i=1}^{\ell} \prod_{m=-\infty}^{\mathcal{L}_i(\beta)} (c_1(\mathcal{L}_i) - m)}{\prod_{i=1}^{\ell} \prod_{m=-\infty}^{0} (c_1(\mathcal{L}_i) - m)} \\
& \cdot \frac{\prod_\rho \prod_{m=-\infty}^{0} (D_\rho - m)}{\prod_\rho \prod_{m=-\infty}^{D_\rho(\beta)} (D_\rho - m)} \Bigg)
\end{aligned}
\tag{6.55}
$$

where $M(X_\Sigma)$ is the Mori cone. If we have a GLSM description, the generators of the Mori cone can be obtained from the row vectors of the matrix of GLSM charges C. The column vectors span the secondary fan of X_Σ. The components of I_X can be obtained by an expansion in $H_1, \ldots, H_h$. The Gamma class of X and its conjugate can be written as

$$
\widehat{\Gamma}_X(H) = \frac{\prod_\rho \Gamma(1 - D_\rho)}{\prod_{i=1}^{\ell} \Gamma(1 - c_1(\mathcal{L}_i))}, \qquad \widehat{\Gamma}_X^*(H) = \frac{\prod_\rho \Gamma(1 + D_\rho)}{\prod_{i=1}^{\ell} \Gamma(1 + c_1(\mathcal{L}_i))} \tag{6.56}
$$

where H collectively denotes $H_1, \ldots, H_h$. Note that in the literature the definition of $\widehat{\Gamma}_X(H)$ and $\widehat{\Gamma}_X^*(H)$ might be exchanged. We follow the convention of [46]. The Gamma class is invertible as one can see by expansion into a power series in H. Therefore it is reasonable to write down expressions like $\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$. The relevant pairing $\langle \cdot, \cdot \rangle$ is given by the Mukai pairing [99, 88]:

$$
\langle \alpha, \beta \rangle = \int_X \alpha^\vee \wedge \beta, \tag{6.57}
$$

where $\alpha, \beta \in H^{even}(X, \mathbb{C})$. In the Calabi-Yau case $\alpha^\vee = (-1)^{\mathrm{Gr}}\alpha$. The grading operator Gr acts as follows on $H^{even}(X, \mathbb{C})$:

$$\mathrm{Gr}\alpha = k\,\alpha, \qquad \text{for} \qquad \alpha \in H^{2k}(X, \mathbb{C}). \tag{6.58}$$

This coincides with the definition in [100]. We next focus on the cohomology of the Calabi-Yau, which descends from the cohomology of the ambient space X_σ and neglect the cohomology associated to divisors on X without a counterpart in the ambient geometry (primitive cohomology). This restriction replicates the restriction to the narrow sectors seen in the Landau-Ginzburg setting. We can now adjust (6.31) to the geometric setting and find

$$Z^{geom}_{S^2}(\mathsf{t}, \bar{\mathsf{t}}) = \int_X \frac{\widehat{\Gamma}_X(H)}{\widehat{\Gamma}^*_X(H)} I_X(u(\mathsf{t}), H) I_X(\overline{u}(\bar{\mathsf{t}}), H) = \langle \overline{I}_X, I_X \rangle \tag{6.59}$$

The pairing is evaluated using the intersection ring of X.

The authors of [88] recognized the relation of the Gamma class to perturbative corrections and also the quotient $\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$ has been first observed by the authors. They linked the quotient to complex conjugation via K-theory arguments. The main point relies on the isomorphism between $H^{even}(X, \mathbb{C})$ and $K_{hol}(X) \otimes \mathbb{C}$, where $K_{hol}(X)$ is holomorphic K-theory [101]. This isomorphism involves the Gamma class [102, 103, 104, 105]:

$$\mu : [\mathcal{E}] \mapsto \mathrm{ch}(\mathcal{E}) \wedge \widehat{\Gamma}_X. \tag{6.60}$$

It was then argued, that complex conjugation for $w \in H^{even}(X, \mathbb{C})$ is given by the following procedure:

$$w \mapsto \mathrm{ch}^{-1}\left(\frac{w}{\widehat{\Gamma}_X}\right) \mapsto \mathrm{ch}^{-1}\left(\frac{\overline{w}}{\widehat{\Gamma}^*_X}\right) \mapsto \overline{w}\,\frac{\widehat{\Gamma}_X}{\widehat{\Gamma}^*_X}, \tag{6.61}$$

where the map in the middle is complex conjugation in $K_{hol}(X)$.

Observe that in our approach via the sphere partition function we have some ambiguity in identifying the pairing and complex conjugation operation. This comes form the fact that the grading operator Gr that acts on the state space appears twice: once in the definition of the Mukai pairing and once in $(-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$ in (6.33)). As consequence, the signs coming from $(-1)^{\mathrm{Gr}}$ cancel and we could, at least from the point of view of the sphere partition function, use a pairing $\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$ instead of the Mukai pairing and define complex conjugation via $\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$ instead of (6.33).

To specify the flat coordinates and the J-function we focus on distinguished components in the cohomology, namely: $\mathcal{H}^{def}_{marg} = H^2(X, \mathbb{C})$ and $H^0(X, \mathbb{C})$. The flat coordinates are defined by the corresponding components I_i ($i = 1, \ldots, h$) and I_0 of the I-function:

$$t_i(u) = \frac{I_i}{I_0}, \tag{6.62}$$

and the J-function is defined by

$$J_X(t) = \frac{I_X(u(t))}{I_0(u(t))}. \tag{6.63}$$

Hybrid Phases

We also tested our proposal (6.31) in hybrid phases. The low energy description is given by models which are fibrations of Landau-Ginzburg orbifolds over a base manifold B. Such models have been studied in e.g. [106, 107, 108]. For a certain class of one-parameter hybrids a mathematical description has been given in [109, 110, 111, 69, 112] by generalising FJRW theory. By our sphere partition function approach we recover results from mathematics for the I-functions and the Gamma class in the one-parameter case and conjecture expressions for the two-parameter examples discussed below. In (6.31) we integrate over a non Calabi-Yau base manifold B. In the discussed examples we often used the following identities between the characteristic classes of a algebraic variety B:

$$\mathrm{Td}(B) = e^{\frac{c_1(B)}{2}}\widehat{A}(B) = e^{\frac{c_1(B)}{2}}\widehat{\Gamma}_B\widehat{\Gamma}^*_B, \tag{6.64}$$

where Td is the Todd class, c_1 is the first Chern class, $\widehat{A}$ is the A-roof genus, and $\widehat{\Gamma}$ is the Gamma class to bring the sphere partition function into the form of (6.31). Similar to the geometric phase our approach does not allow to identify the correct definition of the pairing. By following the arguments of [99, 88] we can interpret the integral over B as an artefact of the Mukai pairing. This requires to modify the definition of $\alpha^\vee$ in (6.57) to be $\alpha^\vee = (-1)^{\mathrm{Gr}}e^{\frac{c_1(B)}{2}}\alpha$. The results from the sphere partition function would then further suggest that $(-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$ in the conjugation operation (6.33) would have to be modified to $(-1)^{\mathrm{Gr}}e^{-\frac{c_1(B)}{2}}\frac{\widehat{\Gamma}}{\widehat{\Gamma}^*}$.

Pseudo-Hybrid Phases

In pseudo-hybrid phases we do not have a unique R-charge assignment. This is related to the fact that no enumerative problem, like FJRW, is known for these phases, but a general state space isomorphism between general hybrid models and CY complete intersections has been given in [112]. The corresponding CFTs are singular. Some properties of the low-energy physics have been studied in [57] and it is also possible to evaluate the sphere partition function in such a phase. The vacuum equations in these models have several components, as one can see from the D- and F-term equations and this is also visible in the structure of the sphere partition function. The different components of the sphere partition function show a factorizations similar to (6.31) as one can see in the examples discussed below.

6.3 Abelian One-Parameter Models

The abelian GLSMs with gauge group $\mathsf{G} = U(1)$ and one Kähler parameter, studied in Chapter 5, provide a class of models in which we can test the proposed formula (6.31).

Evaluation of the Sphere Partition Function

In the models listed in Table 5.1 the sphere partition function takes the following form, after the shift $\sigma \to -iq + \sigma$:

$$Z_{S^2} = \frac{e^{-4\pi\zeta q}}{2\pi} \sum_{m\in\mathbb{Z}} \int_{-\infty+iq}^{\infty+iq} \mathrm{d}\sigma Z_{p_1} Z_{p_2}^k Z_1^{5+k-n-j} Z_\alpha^n Z_\beta^j \\ \cdot e^{(-2\pi\zeta - i\theta)\left(i\sigma+\frac{m}{2}\right)} e^{(-2\pi\zeta+i\theta)\left(i\sigma-\frac{m}{2}\right)}, \quad (6.65)$$

with

$$\begin{aligned} Z_{p_1} &= \frac{\Gamma\left(\frac{1}{2}(m+2i\sigma)d_1+1\right)}{\Gamma\left(\frac{1}{2}(m-2i\sigma)d_1\right)}, & Z_{p_2} &= \frac{\Gamma\left(\frac{1}{2}(m+2i\sigma)d_2+1\right)}{\Gamma\left(\frac{1}{2}(m-2i\sigma)d_2\right)}, \\ Z_1 &= \frac{\Gamma\left(-\frac{m}{2}-i\sigma\right)}{\Gamma\left(-\frac{m}{2}+i\sigma+1\right)}, & Z_\alpha &= \frac{\Gamma\left(-\frac{1}{2}\alpha(m+2i\sigma)\right)}{\Gamma\left(i\sigma\alpha-\frac{m\alpha}{2}+1\right)}, \\ Z_\beta &= \frac{\Gamma\left(-\frac{1}{2}\beta(m+2i\sigma)\right)}{\Gamma\left(i\sigma\beta-\frac{m\beta}{2}+1\right)}. \end{aligned} \quad (6.66)$$

The result of the evaluation of the sphere partition function depends on the phase. In the following we give only an overview and refer for details to the Appendix A. The steps in the evaluation of the sphere partition function are :

1. Apply the residue theorem to rewrite (6.65) as sum over poles. The contributing poles depend on the phase.
2. Center the integration around the location of the poles by the variable transformation
$$\sigma \to \varepsilon + const \quad (6.67)$$
so that the poles are now at $\varepsilon = 0$.
3. The next step is to simplify the sums over the magnetic charge lattice (parametrized by m) and the sum over the different poles.
4. The integrand is simplified by application of the identity
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}. \quad (6.68)$$

By applying the above steps we can bring the sphere partition function (6.65) in the form

$$Z_{S^2} = \sum_i Z_{S^2,i} \tag{6.69}$$

where

$$Z_{S^2,i} = -\frac{1}{2\pi} \sum_{finite} (-1)^{sgn} \oint \mathrm{d}\varepsilon \mathcal{Z}_{i,sing}(\varepsilon) |\mathcal{Z}_{i,reg}(\mathsf{t},\varepsilon)|^2. \tag{6.70}$$

$\mathcal{Z}_{i,sing}$ contains all the singular terms and so has poles and $\mathcal{Z}_{i,reg}$ is regular in the sense that it has no poles. We also introduced:

$$\mathsf{t} = 2\pi\zeta - i\theta. \tag{6.71}$$

The form (6.70) is valid in all phases. Let us give the following definitions

$$\begin{aligned} \gcd(\beta,\alpha) &= \kappa_1, \qquad \frac{\alpha}{\kappa_1} = \tau_\alpha, \qquad \frac{\beta}{\kappa_1} = \tau_\beta, \\ \gcd(d_1,d_2) &= \kappa_2, \qquad \frac{d_1}{\kappa_2} = \tau_{d_1}, \qquad \frac{d_2}{\kappa_2} = \tau_{d_2}. \end{aligned} \tag{6.72}$$

which we will use later to write down results for certain subsets of models. Subsequently we will discuss the explicit form of the contributing terms.

$\zeta \gg 0$ Phase

In the large volume phase the poles of Z_1, Z_α and Z_β give a contribution, but, as shown in the Appendix A, it is sufficient to sum only over the poles of Z_β. After application of the above steps we arrive at:

$$Z_{S^2}^{\zeta\gg 0} = -\frac{1}{2\pi} \oint_0 \mathrm{d}\varepsilon \mathcal{Z}_{1,sing}(\varepsilon) |\mathcal{Z}_{1,reg}(\varepsilon,\mathsf{t})|^2, \tag{6.73}$$

with

$$\begin{aligned} \mathcal{Z}_{1,reg}(\varepsilon) = \sum_{a=0}^{\infty} &(-1)^{a(5+k-n-j+\alpha n+j\beta)} e^{-\mathsf{t}(i\varepsilon+a+q)} \\ &\cdot \frac{\Gamma\left(ad_1+i\varepsilon d_1+1\right)}{\Gamma\left(a+i\varepsilon+1\right)^{5+k-n-j}\Gamma\left(a\alpha+i\varepsilon\alpha+1\right)^n} \\ &\cdot \frac{\Gamma\left(ad_2+i\varepsilon d_2+1\right)^k}{\Gamma(a\beta+i\varepsilon\beta+1)^j}, \end{aligned} \tag{6.74}$$

and

$$\mathcal{Z}_{1,sing}(\varepsilon) = \frac{\pi^4 \sin\left(\pi\left(i\varepsilon d_1\right)\right)\sin\left(\pi\left(i\varepsilon d_2\right)\right)^k}{\sin\left(\pi\left(i\varepsilon\right)\right)^{5+k-n-j}\sin\left(\pi\left(i\varepsilon\alpha\right)\right)^n\sin\left(\pi\left(i\varepsilon\beta\right)\right)^j}. \tag{6.75}$$

	F1				F2				F3				F4			
δ	1	2	3	4	1	2	4	5	1	3	5	7	1	3	7	9
pole order	1				1				1				1			

	K1		K2		K3		M1
δ	1	2	1	3	1	5	1
pole order	2		2		2		4

Table 6.1: Pole order and contributing sectors for Landau-Ginzburg and hybrid models.

$\zeta \ll 0$ **Phase**

We found that in this phase the sphere partition function is given by two contributions

$$Z_{S^2}^{\zeta\ll 0} = Z_{S^2,1}^{\zeta\ll 0} + Z_{S^2,2}^{\zeta\ll 0}, \tag{6.76}$$

where the first contribution comes from the poles of Z_{p_1} and the second term picks the remaining poles of Z_{p_2} which have not been accounted for in the first part. The second part only contributes in pseudo-hybrid models. For the next steps we focus on Landau-Ginzburg and hybrid phases. In these phases $d_1 = d_2$ and therefore we can further simplify $Z_{S^2,1}^{\zeta\ll 0}$. A characteristic of these phases is that $Z_{S^2,1}^{\zeta\ll 0}$ splits into a sum of different contributions. We label these contributions by δ, with $0 < \delta < d_1, \delta \in \mathbb{Z}_{>0}$. The dependence on δ is such that a contribution, labelled by δ, are zero unless:

$$\left\langle \frac{\delta}{d_1} \right\rangle \neq 0, \qquad \left\langle \alpha \frac{\delta}{d_1} \right\rangle \neq 0, \qquad \left\langle \alpha \frac{\beta}{d_1} \right\rangle \neq 0. \tag{6.77}$$

The set of non-zero contributing δ values, will be called *narrow*. This is in accordance with the *narrow* sectors introduced in Section 6.2, which also fulfil (6.77). The contributing sectors and the pole orders are given in Table 6.1.

As shown in Appendix A $Z_{S^2,1}^{\zeta\ll 0}$ takes the form

$$Z_{S^2,1}^{\zeta\ll 0} = \frac{1}{2\pi i d_1} \sum_{\delta\in narrow} \oint_0 \mathrm{d}\varepsilon \frac{(-1)^{\mathrm{Gr}}}{\varepsilon^{k+1}} \frac{\hat{\Gamma}_\delta(\varepsilon)}{\hat{\Gamma}^*_\delta(\varepsilon)} |I_\delta^{\zeta\ll 0}(\mathtt{t},\varepsilon)|^2, \tag{6.78}$$

with

$$
\begin{aligned}
I_\delta^{\zeta\ll 0}(\mathsf{t},\varepsilon) = &\sum_{a=0}^{\infty} e^{\mathsf{t}(\frac{\varepsilon}{d_1}+a+\frac{\delta}{d_1}-q)}(-1)^{a(5+k-n-j+\alpha n+j\beta)}\\
&\cdot\frac{\Gamma\left(1+\varepsilon\right)^{k+1}\Gamma\left(a+\frac{\varepsilon}{d_1}+\frac{\delta}{d_1}\right)^{5+k-n-j}}{\Gamma\left(\frac{\varepsilon}{d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j}\Gamma\left(\alpha\frac{\varepsilon}{d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^{n}}\\
&\cdot\frac{\Gamma\left(a\alpha+\alpha\frac{\varepsilon}{d_1}+\frac{\alpha}{d_1}\delta\right)^{n}\Gamma\left(a\beta+\beta\frac{\varepsilon}{d_1}+\frac{\beta}{d_1}\delta\right)^{j}}{\Gamma\left(\beta\frac{\varepsilon}{d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^{j}\Gamma\left(\delta+ad_1+\varepsilon\right)^{k+1}},
\end{aligned}
\tag{6.79}
$$

and

$$
(-1)^{\mathrm{Gr}} = (-1)^{\delta(k+1)}(-1)^{(5+k-n-j)\left\lfloor\frac{\delta}{d_1}\right\rfloor}(-1)^{n\left\lfloor\alpha\frac{\delta}{d_1}\right\rfloor}(-1)^{j\left\lfloor\beta\frac{\delta}{d_1}\right\rfloor}. \tag{6.80}
$$

$\widehat{\Gamma}_\delta(\epsilon)$ is given by

$$
\begin{aligned}
\hat{\Gamma}_\delta(\varepsilon) = \Gamma\left(1-\varepsilon\right)^{k+1}\Gamma\left(\frac{\varepsilon}{d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j}&\\
\cdot\Gamma\left(\alpha\frac{\varepsilon}{d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^{n}\Gamma\left(\beta\frac{\varepsilon}{d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^{j}&.
\end{aligned}
\tag{6.81}
$$

and

$$
\widehat{\Gamma}^*_\delta(\epsilon) = \widehat{\Gamma}_{d-\delta}(-\epsilon). \tag{6.82}
$$

In the following we will focus on specific phases and show that (6.79), (6.81) and (6.82) match the expression known from FJRW theory in Landau-Ginzburg and hybrid models. Let us also give the following expression, which we will use in subsequent sections:

$$
\gamma_\delta(H) = (-1)^{\mathrm{Gr}}\frac{\hat{\Gamma}_\delta(H)}{\hat{\Gamma}^*_\delta(H)}. \tag{6.83}
$$

Landau-Ginzburg Phases

For the abelian one parameter models the models F1, F2, F3 and F4 in Table 5.1 have a Landau-Ginzburg phase (see Section 5.2). We first extract the q matrix from the GLSM data by dividing the GLSM charge vectors by the charge of the p-field

$$
q = \begin{pmatrix} 1 & -\frac{1}{d_1} & -\frac{1}{d_1} & -\frac{1}{d_1} & -\frac{\alpha}{d_1} & -\frac{\beta}{d_1}\end{pmatrix}. \tag{6.84}
$$

The matrix q allows us now to calculate the $\hat{\Gamma}$-function and the I-functions from the results given in Section 6.2. We insert (6.84) into (6.45), (6.48), (6.49) and (6.50) as given in Section 6.2. We find for (6.48):

$$\begin{aligned}\widehat{\Gamma}_\delta = \Gamma\left(1-\left\langle -\frac{k}{d}-\frac{1}{d}\right\rangle\right)^3 \Gamma\left(1-\left\langle -\frac{k\alpha}{d}-\frac{\alpha}{d}\right\rangle\right) \\ \cdot\Gamma\left(1-\left\langle -\frac{k\beta}{d}-\frac{\beta}{d}\right\rangle\right).\end{aligned} \tag{6.85}$$

and application of (6.84) in (6.47) gives

$$\begin{aligned}I_{LG}(u) = -\sum_{k\geq 0}\frac{u^k}{\Gamma(k+1)}(-1)^{3\left\langle\frac{k+1}{d}\right\rangle+\left\langle\alpha\frac{k+1}{d}\right\rangle+\left\langle\beta\frac{k+1}{d}\right\rangle} \\ \cdot\frac{\Gamma\left(\left\langle-\frac{k}{d}-\frac{1}{d}\right\rangle\right)^3\Gamma\left(\left\langle-\frac{k\alpha}{d}-\frac{\alpha}{d}\right\rangle\right)\Gamma\left(\left\langle-\frac{k\beta}{d}-\frac{\beta}{d}\right\rangle\right)}{\Gamma\left(1-\frac{k}{d}-\frac{1}{d}\right)^3\Gamma\left(1-\frac{k\alpha}{d}-\frac{\alpha}{d}\right)\Gamma\left(1-\frac{k\beta}{d}-\frac{\beta}{d}\right)}.\end{aligned} \tag{6.86}$$

A shift of the k summation and a simplification of the fractional parts gives

$$\widehat{\Gamma}_\delta = \Gamma\left(\left\langle\frac{k}{d}\right\rangle\right)^3\Gamma\left(\left\langle\frac{k\alpha}{d}\right\rangle\right)\Gamma\left(\left\langle\frac{k\beta}{d}\right\rangle\right), \tag{6.87}$$

$$\begin{aligned}I_{LG}(u) = -\sum_{\delta=1}^{d-1}\sum_{n\geq 0}\frac{u^{dn+\delta-1}}{\Gamma(dn+\delta)}\frac{(-1)^{3\left\langle\frac{\delta}{d}\right\rangle+\left\langle\frac{\alpha\delta}{d}\right\rangle+\left\langle\frac{\beta\delta}{d}\right\rangle}\Gamma\left(1-\left\langle\frac{\delta}{d}\right\rangle\right)^3}{\Gamma\left(1-n-\frac{\delta}{d}\right)^3\Gamma\left(1-\alpha n-\frac{\alpha\delta}{d}\right)} \\ \cdot\frac{\Gamma\left(1-\left\langle\frac{\alpha\delta}{d}\right\rangle\right)\Gamma\left(1-\left\langle\frac{\beta\delta}{d}\right\rangle\right)}{\Gamma\left(1-\beta n-\frac{\beta\delta}{d}\right)}\end{aligned} \tag{6.88}$$

The identity

$$3\left\langle\frac{\delta}{d}\right\rangle+\left\langle\frac{\alpha\delta}{d}\right\rangle+\left\langle\frac{\beta\delta}{d}\right\rangle = \delta - 3\left\lfloor\frac{\delta}{d}\right\rfloor-\left\lfloor\frac{\alpha\delta}{d}\right\rfloor-\left\lfloor\frac{\beta\delta}{d}\right\rfloor, \tag{6.89}$$

and the Gamma function reflexion formula (6.68) can be used to obtain

$$\begin{aligned}I_{LG}(u) &= -\sum_{\delta=1}^{d-1}\sum_{n\geq 0}(-1)^\delta(-1)^{dn}\frac{u^{dn+\delta-1}}{\Gamma(dn+\delta)} \\ &\quad\cdot\frac{\Gamma\left(n+\frac{\delta}{d}\right)^3\Gamma\left(\alpha n+\frac{\alpha\delta}{d}\right)\Gamma\left(\beta n+\frac{\beta\delta}{d}\right)}{\Gamma\left(\left\langle\frac{\delta}{d}\right\rangle\right)^3\Gamma\left(\left\langle\frac{\alpha\delta}{d}\right\rangle\right)}, \\ &= \sum_{\delta=1}^{d-1}I_\delta(u).\end{aligned} \tag{6.90}$$

In a similar way we obtain the expressions for (6.48) and (6.50):

$$\mathrm{Gr} = \delta - \left(3\left\lfloor\frac{\delta}{d}\right\rfloor + \left\lfloor\alpha\frac{\delta}{d}\right\rfloor + \left\lfloor\beta\frac{\delta}{d}\right\rfloor\right), \tag{6.91}$$

$$\widehat{\Gamma}^*_\delta = \Gamma\left(\left\langle\frac{d-k}{d}\right\rangle\right)^3 \Gamma\left(\left\langle\alpha\frac{d-k}{d}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{d-k}{d}\right\rangle\right). \tag{6.92}$$

The above results can now be inserted into the postulated formula (6.51):

$$\begin{aligned} Z^{LG}_{S^2} &= \sum_{\delta,\delta'}(-1)^{\delta+3\lfloor\frac{\delta}{d}\rfloor+\lfloor\alpha\frac{\delta}{d}\rfloor+\lfloor\beta\frac{\delta}{d}\rfloor} I_\delta(\overline{u}(\overline{t}))I_{\delta'}(u(t))\left\langle e_{\delta^{-1}}, e_{\delta'}\right\rangle \\ &\quad\cdot\frac{\Gamma\left(\left\langle\frac{\delta}{d}\right\rangle\right)^3\Gamma\left(\left\langle\frac{\delta\alpha}{d}\right\rangle\right)\Gamma\left(\left\langle\frac{\delta\beta}{d}\right\rangle\right)}{\Gamma\left(\left\langle\frac{d-\delta}{d}\right\rangle\right)^3\Gamma\left(\left\langle\alpha\frac{d-\delta}{d}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{d-\delta}{d}\right\rangle\right)} \\ &= \frac{1}{d}\sum_{\delta}(-1)^{\delta+3\lfloor\frac{\delta}{d}\rfloor+\lfloor\alpha\frac{\delta}{d}\rfloor+\lfloor\beta\frac{\delta}{d}\rfloor} I_\delta(\overline{u}(\overline{t}))I_\delta(u(t)) \\ &\quad\cdot\frac{\Gamma\left(\left\langle\frac{\delta}{d}\right\rangle\right)^3\Gamma\left(\left\langle\frac{\delta\alpha}{d}\right\rangle\right)\Gamma\left(\left\langle\frac{\delta\beta}{d}\right\rangle\right)}{\Gamma\left(\left\langle\frac{d-\delta}{d}\right\rangle\right)^3\Gamma\left(\left\langle\alpha\frac{d-\delta}{d}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{d-\delta}{d}\right\rangle\right)}. \end{aligned} \tag{6.93}$$

We evaluated the pairing, as given in (6.43).

To test the result (6.93) we evaluate (6.78) further, now focusing on Landau-Ginzburg phases. In Landau-Ginzburg phases we have only first order poles, as one can see from Table 6.1 and therefore we can evaluate (6.78) in a straight forward way to get:

$$Z^{\zeta\ll 0}_{S^2} = \frac{1}{d_1}\sum_{\delta\in narrow}(-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}_\delta(0)}{\widehat{\Gamma}^*_\delta(0)}\left|I^{\zeta\ll 0}_\delta(\mathtt{t},0)\right|^2. \tag{6.94}$$

The next important observation is, that the given δ values in Table 6.1 coincide with the narrow sectors introduced in Section 6.2.
Evaluation of (6.81) and (6.82) gives:

$$\widehat{\Gamma}_\delta(0) = \Gamma\left(\left\langle\frac{\delta}{d_1}\right\rangle\right)^3\Gamma\left(\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{\delta}{d_1}\right\rangle\right), \tag{6.95}$$

$$\widehat{\Gamma}^*_\delta(0) = \Gamma\left(\left\langle\frac{d_1-\delta}{d_1}\right\rangle\right)^3\Gamma\left(\left\langle\alpha\frac{d_1-\delta}{d_1}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{d_1-\delta}{d_1}\right\rangle\right), \tag{6.96}$$

and (6.79) results into

$$\begin{aligned} I^{\zeta\ll 0}_\delta(\mathtt{t},0) &= \sum_{a=0}^{\infty}\frac{e^{\mathtt{t}(a+\frac{\delta}{d_1}-q)}(-1)^{a(3+\alpha+\beta)}}{\Gamma\left(\left\langle\frac{\delta}{d_1}\right\rangle\right)^3\Gamma\left(\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)\Gamma\left(\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)} \\ &\quad\cdot\frac{\Gamma\left(a+\frac{\delta}{d_1}\right)^3\Gamma\left(a\alpha+\frac{\alpha}{d_1}\delta\right)\Gamma\left(a\beta+\frac{\beta}{d_1}\delta\right)}{\Gamma\left(\delta+ad_1\right)}. \end{aligned} \tag{6.97}$$

We can now compare (6.93) with the GLSM calculation (6.94) and find a match.

The final task we perform is to read of the matrix M introduced in (6.34), therefore we expand (6.94) in terms of δ and get

$$M = \begin{pmatrix} \frac{\gamma_{\delta_1}(0)}{d_1} & 0 & 0 & 0 \\ 0 & \frac{\gamma_{\delta_2}(0)}{d_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{d_1\gamma_{\delta_2}(0)} & 0 \\ 0 & 0 & 0 & -\frac{1}{d_1\gamma_{\delta_1}(0)} \end{pmatrix}. \tag{6.98}$$

γ_δ was introduced in (6.83).

Geometry

For an analysis of the vacuum structure see Section 5.1. In the $\zeta \gg 0$ phase we can follow the steps outlined in [46] in the context of the hemisphere partition function to find a match with the expression introduced in Section 6.3. We start from (6.73) and after a variable transformation we introduce

$$\widehat{\Gamma}(H) = \frac{\Gamma\left(1-\frac{H}{2\pi i}\right)^{5-n-j+k}\Gamma\left(1-\alpha\frac{H}{2\pi i}\right)^{n}\Gamma\left(1-\beta\frac{H}{2\pi i}\right)^{j}}{\Gamma\left(1-d_1\frac{H}{2\pi i}\right)\Gamma\left(1-d_2\frac{H}{2\pi i}\right)^{k}}. \tag{6.99}$$

The conjugate $\hat{\Gamma}^*$ is obtained from $\hat{\Gamma}$ by the replacement $i \to -i$. The next identification we make is

$$\begin{aligned} I^{\zeta\gg 0}(\mathtt{t},H) &= \widehat{\Gamma}(H)^*\mathcal{Z}_{1,reg}\left(\frac{-H}{2\pi}\right) \\ &= \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^{5-n-j+k}\Gamma\left(1+\alpha\frac{H}{2\pi i}\right)^{n}\Gamma\left(1+\beta\frac{H}{2\pi i}\right)^{j}}{\Gamma\left(1+d_1\frac{H}{2\pi i}\right)\Gamma\left(1+d_2\frac{H}{2\pi i}\right)^{k}} \\ &\quad\cdot\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)}u(\mathtt{t})^{\left(\frac{H}{2\pi i}+a+q\right)} \\ &\quad\cdot\frac{\Gamma\left(1+ad_1+d_1\frac{H}{2\pi i}\right)}{\Gamma\left(1+a+\frac{H}{2\pi i}\right)^{5+k-n-j}\Gamma\left(1+a\alpha+\alpha\frac{H}{2\pi i}\right)^{n}} \\ &\quad\cdot\frac{\Gamma\left(1+ad_2+d_2\frac{H}{2\pi i}\right)^{k}}{\Gamma\left(1+a\beta+\beta\frac{H}{2\pi i}\right)^{j}}, \end{aligned} \tag{6.100}$$

with $u(\mathtt{t}) = e^{-\mathtt{t}}$ and $\mathcal{Z}_{1,reg}$ was given in (6.74). Eqn. (6.73) can, by the above results, be rewritten into

$$Z_{S^2}^{\zeta\gg 0} = (2\pi i)^3\frac{d_2^k d_1}{\alpha^n\beta^j}\oint_0\frac{\mathrm{d}H}{2\pi i}\frac{1}{H^4}\frac{\widehat{\Gamma}(H)}{\widehat{\Gamma}^*(H)}I^{\zeta\gg 0}(u(\mathtt{t}),H)I^{\zeta\gg 0}(\overline{u}(\bar{\mathtt{t}}),H). \tag{6.101}$$

The next step is to use the fact that the one-parameter abelian models in the phase $\zeta \gg 0$ are described by a non-linear sigma model with Calabi-Yau target space X of the form (5.17). We identify H with the hyperplane class of the ambient projective space X_Σ and lift the integration in (6.101) into a geometric setting. For that purpose we need the top Chern class of the normal bundle ξ of X. We state the total Chern class of ξ:

$$c(\xi) = (1 + d_1 H)(1 + d_2 H)^k. \tag{6.102}$$

In our models of interest the normal bundle χ has rank $k+1$ and the top Chern class ξ evaluates to:

$$c_{k+1}(\xi) = d_1 d_2^k H^{k+1}. \tag{6.103}$$

The top Chern class of the normal bundle can be used to pull back integrations on X to the embedding space:

$$\begin{aligned} \int_X g(H) &= \int_{X_\Sigma} c_{k+1}(\xi) \wedge g(H) \\ &= \frac{d_1 d_2^k}{3!} \frac{\partial^3}{\partial H^3} g(H)|_{H=0} = d_1 d_2^k \oint \frac{\mathrm{d}z}{2\pi i} \frac{1}{z^4} g(z). \end{aligned} \tag{6.104}$$

We can now rewrite (6.101) by applying (6.104) into

$$Z_{S^2}^{\zeta \gg 0} = \frac{(2\pi i)^3}{\alpha^n \beta^j} \int_X \frac{\widehat{\Gamma}_X(H)}{\widehat{\Gamma}^*_X(H)} I^{\zeta \gg 0}(u(\mathfrak{t}), H) I^{\zeta \gg 0}(\overline{u}(\overline{\mathfrak{t}}), H). \tag{6.105}$$

We see that the above result matches the form of the proposed formula in geometric regimes (6.59). Next we expand the components in the integrant (6.104) in powers of H and extract the H^3 component to read of the matrix M introduced in (6.34):

$$\frac{M}{8\pi^3} = \begin{pmatrix} \frac{\chi(X)\zeta(3)}{4\pi^3} & 0 & 0 & -i\kappa \\ 0 & 0 & -i\kappa & 0 \\ 0 & -i\kappa & 0 & 0 \\ -i\kappa & 0 & 0 & 0 \end{pmatrix}, \tag{6.106}$$

with $\kappa = \frac{d_1 d_2^k}{\alpha^n \beta^j}$ the triple intersection number and $\chi(X)$ the Euler number of the Calabi-Yau (X). The normalization by $8\pi^3$ is chosen to obtain a canonical normalized $\zeta(3)$ term, see also [75] and [113].

K-Type Hybrid Models

We focus on the models K1, K2 and K3 (see Table 5.1, Section 5.3) which describe a Landau-Ginzburg orbifold, with orbifold groups $G = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$

respectively, fibred over $\mathbb{P}^1$ in the phase $\zeta \ll 0$. After setting $k = 1$ and a variable transformation (6.78) can be brought into the following form:

$$Z_{S^2,1}^{\zeta\ll 0} = \frac{2\pi i}{d_1} \sum_{\delta \in Narrow} \oint \frac{\mathrm{d}H}{2\pi i} \frac{1}{H^2} (-1)^{\mathrm{Gr}} \frac{\Gamma_\delta(H)}{\Gamma^*_\delta(H)} I_\delta^{\zeta\ll 0}(\mathsf{t}, H) I_\delta^{\zeta\ll 0}(\bar{\mathsf{t}}, H), \quad (6.107)$$

with

$$\begin{aligned}\Gamma_\delta(H) = {}& \Gamma\left(1 - \frac{H}{2\pi i}\right)^2 \Gamma\left(\frac{H}{2\pi i d_1} + \left\langle \frac{\delta}{d_1} \right\rangle\right)^{6-n-j} \\ & \cdot \Gamma\left(\alpha \frac{H}{2\pi i d_1} + \left\langle \alpha \frac{\delta}{d_1} \right\rangle\right)^n \Gamma\left(\beta \frac{H}{2\pi i d_1} + \left\langle \beta \frac{\delta}{d_1} \right\rangle\right)^j,\end{aligned} \quad (6.108)$$

$$\begin{aligned}\Gamma^*_\delta(H) = {}& \Gamma\left(1 + \frac{H}{2\pi i}\right)^2 \Gamma\left(-\frac{H}{2\pi i d_1} + \left\langle \frac{d_1 - \delta}{d_1} \right\rangle\right)^{6-n-j} \\ & \cdot \Gamma\left(-\alpha \frac{H}{2\pi i d_1} + \left\langle \alpha \frac{d_1 - \delta}{d_1} \right\rangle\right)^n \\ & \cdot \Gamma\left(-\beta \frac{H}{2\pi i d_1} + \left\langle \beta \frac{d_1 - \delta}{d_1} \right\rangle\right)^j,\end{aligned} \quad (6.109)$$

and

$$\begin{aligned}\mathcal{I}_\delta^{\zeta\ll 0}(\mathsf{t}, H) = {}& \sum_{a=0}^{\infty} e^{\mathsf{t}\left(\frac{H}{2\pi i d_1} + a + \frac{\delta}{d_1} - q\right)} (-1)^{a(6-n-j+\alpha n + j\beta)} \\ & \cdot \frac{\Gamma\left(1 + \frac{H}{2\pi i}\right)^2 \Gamma\left(a + \frac{H}{2\pi i d_1} + \frac{\delta}{d_1}\right)^{6-n-j}}{\Gamma\left(\frac{H}{2\pi i d_1} + \left\langle \frac{\delta}{d_1} \right\rangle\right)^{6-n-j} \Gamma\left(\alpha \frac{H}{2\pi i d_1} + \left\langle \alpha \frac{\delta}{d_1} \right\rangle\right)^n} \\ & \cdot \frac{\Gamma\left(a\alpha + \alpha \frac{H}{2\pi i d_1} + \frac{\alpha}{d_1}\delta\right)^n \Gamma\left(a\beta + \beta \frac{H}{2\pi i d_1} + \frac{\beta}{d_1}\delta\right)^j}{\Gamma\left(\beta \frac{H}{2\pi i d_1} + \left\langle \beta \frac{\delta}{d_1} \right\rangle\right)^j \Gamma\left(\delta + a d_1 + \frac{H}{2\pi i}\right)^2}\end{aligned} \quad (6.110)$$

As in the geometric phase we want to rewrite the residue integral into an integral over the vacuum manifold, which is for the K-type hybrids a $\mathbb{P}^1$. We apply (6.104) and get

$$Z_{S^2,1}^{\zeta\ll 0} = \frac{2\pi i}{d_1} \sum_{\delta \in Narrow} \int_{\mathbb{P}^1} (-1)^{\mathrm{Gr}} \frac{\Gamma_\delta(H)}{\Gamma^*_\delta(H)} I_\delta^{\zeta\ll 0}(\mathsf{t}, H) I_\delta^{\zeta\ll 0}(\bar{\mathsf{t}}, H). \quad (6.111)$$

Similar to the models discussed before we next extract the matrix M (see (6.34)) and as expected the procedure is a mixture of the Landau-Ginzburg orbifold phases and the geometric phase. We first expand all δ components and perform a series expansion in H. To get the matrix M we read of the coefficients of H^1. We find:

$$M = \begin{pmatrix} -\frac{\nu}{d_1^2}\gamma_{\delta_1}(0) & 2\pi i\frac{1}{d_1}\gamma_{\delta_1}(0) & 0 & 0 \\ 2\pi i\frac{1}{d_1}\gamma_{\delta_1}(0) & 0 & 0 & 0 \\ 0 & 0 & -\frac{\nu}{d_1^2}\frac{1}{\gamma_{\delta_1}(0)} & 2\pi i\frac{1}{d_1}\frac{1}{\gamma_{\delta_1}(0)} \\ 0 & 0 & 2\pi i\frac{1}{d_1}\frac{1}{\gamma_{\delta_1}(0)} & 0 \end{pmatrix}. \tag{6.112}$$

with

$$\begin{array}{c|c|c|c} & \text{K1} & \text{K2} & \text{K3} \\ \hline \nu & \log 3^{18} & \log 2^{40} & \log\left(2^{32}3^{18}\right) \end{array}. \tag{6.113}$$

We can match our results to known results in mathematics and therefore focus on the K1 model. This model was studied in [68, 69]. In [68], using techniques from FJRW theory, the following I function was found

$$\begin{aligned} I_{hyb} = z &\sum_{\substack{d>0 \\ d\not\equiv -1 \mod 3}} e^{\left(d+1+\frac{H^{(d+1)}}{z}\right)t} z^{-6\langle\frac{d}{3}\rangle} \\ &\cdot \frac{\Gamma\left(\frac{H^{(d+1)}}{3z}+\frac{d}{3}+\frac{1}{3}\right)^6}{\Gamma\left(\frac{H^{(d+1)}}{3z}+\langle\frac{d}{3}\rangle+\frac{1}{3}\right)^6} \frac{\Gamma\left(\frac{H^{(d+1)}}{z}+1\right)^2}{\Gamma\left(\frac{H^{(d+1)}}{z}+d+1\right)^2}, \end{aligned} \tag{6.114}$$

with t not being the flat coordinate. We will give the relation between t and $\mathtt{t}$ subsequently. The sum can by simplified by the replacement $d = 3n+\delta$, with $\delta = 0, 1$. After this replacement one sees that the $\langle\cdot\rangle$ operation can be dropped. By noting that $H^{(3n+\delta)} = H^{(\delta)}$, because the exponent $(3n+\delta)$ is defined modulo 3 and shifting the δ summation we can write

$$\begin{aligned} I_{hyb} = z &\sum_{\delta=1}^{2}\sum_{n=0}^{\infty} e^{\left(3n+\delta+\frac{H^{(\delta)}}{z}\right)t} z^{-2(\delta-1)} \\ &\cdot \frac{\Gamma\left(\frac{H^{(\delta)}}{3z}+\frac{\delta}{3}+n\right)^6}{\Gamma\left(\frac{H^{(\delta)}}{3z}+\frac{\delta}{3}\right)^6} \frac{\Gamma\left(\frac{H^{(\delta)}}{z}+1\right)^2}{\Gamma\left(\frac{H^{(\delta)}}{z}+3n+\delta\right)^2}. \end{aligned} \tag{6.115}$$

We give the result (6.110) for the K1 model:

$$\begin{aligned} I_{\delta}^{\zeta\ll 0}(\mathtt{t}, H) = &\frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^2}{\Gamma\left(\frac{H}{3\cdot 2\pi i}+\frac{\delta}{3}\right)^6} \\ &\cdot \sum_{a=0}^{\infty} e^{\mathtt{t}\left(\frac{H}{3\cdot 2\pi i}+a+\frac{\delta}{3}-q\right)}(-1)^{6a}\frac{\Gamma^6\left(a+\frac{H}{3\cdot 2\pi i}+\frac{\delta}{3}\right)}{\Gamma^2\left(\delta+3a+\frac{H}{2\pi i}\right)}, \end{aligned} \tag{6.116}$$

and see that we can find a match with (6.115) if we identify

$$q = 0, \qquad H^{(\delta)} = \frac{H}{2\pi i}, \qquad z = 1, \qquad e^{3t} = e^{\mathsf{t}}. \tag{6.117}$$

Observe that the superscript of $H^{(\delta)}$ in (6.115) labels the sector of the narrow state space. From the sphere partition function we cannot directly extract this label, because the paring is partially evaluated. Also the choice of $z = 1$ solely by comparison with the sphere partition function is not unique also $z = -1$ would be possible, because also under this identification the sphere partition function would stay invariant. A further analysis of the J function and the enumerative invariants would be necessary to fix z unambiguously.

M-Type Model

This model was also analysed in [114], where the sphere partition function and Gromov-Witten invariants have been computed. Despite similar features to a geometric phase the evaluation of the sphere partition function is more along the lines of the K-type hybrids. The vacuum manifold is a $\mathbb{P}^3$ (see Section 5.3) and (6.78) can be brought into the following form

$$Z^{\zeta\ll 0}_{S^2,1} = \frac{(2\pi i)^3}{2}\int_{\mathbb{P}^3} (-1)^{\mathrm{Gr}} \frac{\Gamma_1(H)}{\Gamma_1^*(H)} |I_1^{\zeta\ll 0}(\mathsf{t}, H)|^2, \tag{6.118}$$

with (6.79):

$$\begin{aligned} I_1^{\zeta\ll 0}(\mathsf{t}, H) = &\frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^4}{\Gamma\left(\frac{H}{2\cdot 2\pi i}+\frac{1}{2}\right)^8} \\ &\cdot \sum_{a=0}^{\infty} e^{\mathsf{t}\left(\frac{H}{2\cdot 2\pi i}+a+\frac{1}{2}-q\right)} (-1)^{8a} \frac{\Gamma\left(a+\frac{H}{2\cdot 2\pi i}+\frac{1}{2}\right)^8}{\Gamma\left(1+2a+\frac{H}{2\pi i}\right)^4}, \end{aligned} \tag{6.119}$$

and (6.81), (6.82) are given by:

$$\Gamma_1(H) = \Gamma\left(1 - \frac{H}{2\pi i}\right)^4 \Gamma\left(\frac{1}{2} + \frac{H}{2\cdot 2\pi i}\right)^8, \tag{6.120}$$

$$\Gamma_1^*(H) = \Gamma\left(1 + \frac{H}{2\pi i}\right)^4 \Gamma\left(\frac{1}{2} - \frac{H}{2\cdot 2\pi i}\right)^8. \tag{6.121}$$

We can extract the M matrix as in the K-type hybrids and obtain

$$M = \begin{pmatrix} -\frac{\tau^3}{12} - \zeta(3) & i\pi\frac{\tau^2}{2} & 2\pi^2\tau & -4i\pi^3 \\ i\pi\frac{\tau^2}{2} & 2\pi^2\tau & -4i\pi^3 & 0 \\ 2\pi^2\tau & -4i\pi^3 & 0 & 0 \\ -4i\pi^3 & 0 & 0 & 0 \end{pmatrix}, \tag{6.122}$$

with

$$\tau = \log 2^{16}. \tag{6.123}$$

A comparison of (6.122) with the result for the M matrix in geometric phases (6.106) shows different structures, although both results correspond to a point of maximal unipotent monodromy.

In mathematics the model was also studied in [68] along the lines of FJRW theory. The following I-function was given

$$I_{hyb}(t) = \sum_{\substack{d>0 \\ d\not\equiv -1 \mod 2}} \frac{ze^{(d+1+\frac{H^{(d+1)}}{z})t}}{2^{8\lfloor \frac{d}{2} \rfloor}} \frac{\prod_{\substack{1\leq b\leq d \\ b\equiv d+1 \mod 2}} \left(H^{(d+1)}+bz\right)^8}{\prod_{1\leq b\leq d} \left(H^{(d+1)}+bz\right)^4}. \tag{6.124}$$

We rewrite the above result into the form

$$I_{hyb}(t) = \frac{\Gamma\left(1+\frac{H^{(1)}}{z}\right)^4}{\Gamma\left(\frac{1}{2}+\frac{H^{(1)}}{2z}\right)^8} \sum_{n=0}^{\infty} ze^{(2n+1+\frac{H^{(1)}}{z})t} \frac{\Gamma\left(\frac{1}{2}+\frac{H^{(1)}}{2z}+n\right)^8}{\Gamma\left(1+\frac{H^{(1)}}{z}+2n\right)^4}, \tag{6.125}$$

by application of the identity

$$z^l \frac{\Gamma\left(1+\frac{x}{z}+l\right)}{\Gamma\left(1+\frac{x}{z}\right)} = \prod_{k=1}^{l} (x+kz). \tag{6.126}$$

Similar to the K-type hybrids we can match the above results with (6.119) by identifying:

$$q=0, \qquad H^{(1)} = \frac{H}{2\pi i}, \qquad z=1, \qquad e^{2t} = e^{\mathsf{t}}. \tag{6.127}$$

Pseudo-Hybrid-Models

This phases of the GLSM were analysed in Section 5.4. The pseudo-hybrid phase of these models have also been studied in [10].

As seen in Table 5.1 we need to analyse certain F-type models and all of the C-type models. In C-type models the sphere partition function decomposes into components with first order pole and second order pole contributions, where in the F-type models we only have first order pole contributions.

F-Type Models

The pseudo-hybrid phase in these models has two different Landau-Ginzburg orbifold components, with orbifold groups $\mathbb{Z}_{d_1}$ and $\mathbb{Z}_{d_2}$ (see also [10], Section 5.4). As expected for Landau-Ginzburg phases we only encounter first order poles and a split into various twisted sectors. Evaluation of the sphere partition function results into

$$
\begin{aligned}
Z_{S^2}^{\zeta\ll 0} &= \frac{1}{d_1}\sum_{\delta=1}^{d_1-1}(-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}_\delta(0)}{\widehat{\Gamma}^*_\delta(0)}I_\delta(\mathsf{t},0)I_\delta(\bar{\mathsf{t}},0)\\
&+\frac{1}{d_2}\sum_{\delta=1}^{\tau_{d_2}-1}\sum_{\gamma=0}^{\kappa_2-1}(-1)^{\widetilde{\mathrm{Gr}}}\frac{\widetilde{\widehat{\Gamma}}_\delta(0)}{\widetilde{\widehat{\Gamma}}^*_\delta(0)}\widetilde{I}_{\delta,\gamma}(\mathsf{t},0)\widetilde{I}_{\delta,\gamma}(\bar{\mathsf{t}},0).
\end{aligned}
\tag{6.128}
$$

The above parameters are given in (6.72). We used

$$
\begin{aligned}
\widehat{\Gamma}_\delta(0) = \Gamma&\left(\left\langle\tau_{d_2}\frac{\tau_{d_1}-\delta}{\tau_{d_1}}\right\rangle\right)^k\Gamma\left(\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j}\\
&\cdot\Gamma\left(\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^n\Gamma\left(\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^j,
\end{aligned}
\tag{6.129}
$$

$$
\begin{aligned}
(-1)^{\mathrm{Gr}} = (-1)^\delta&(-1)^{k\left\lfloor\tau_{d_2}\frac{\delta}{\tau_{d_1}}\right\rfloor}(-1)^{(5+k-n-j)\left\lfloor\frac{\delta}{d_1}\right\rfloor}\\
&\cdot(-1)^{n\left\lfloor\alpha\frac{\delta}{d_1}\right\rfloor}(-1)^{j\left\lfloor\beta\frac{\delta}{d_1}\right\rfloor},
\end{aligned}
\tag{6.130}
$$

and

$$
\begin{aligned}
\widetilde{\widehat{\Gamma}}_\delta(0) = \Gamma&\left(\left\langle\tau_{d_1}\frac{\tau_{d_2}-\delta}{\tau_{d_2}}\right\rangle\right)\Gamma\left(\left\langle\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{6-n-j}\\
&\cdot\Gamma\left(\left\langle\alpha\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^n\Gamma\left(\left\langle\beta\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^j,
\end{aligned}
\tag{6.131}
$$

$$
\begin{aligned}
(-1)^{\widetilde{\mathrm{Gr}}} = (-1)^\delta&(-1)^{\gamma(\tau_{d_2}+\tau_{d_1})}(-1)^{\left\lfloor\frac{d_2}{d_1}\delta\right\rfloor}(-1)^{(6-n-j)\left\lfloor\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rfloor}\\
&\cdot(-1)^{n\left\lfloor\alpha\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rfloor}(-1)^{j\left\lfloor\beta\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rfloor},
\end{aligned}
\tag{6.132}
$$

where γ is introduced to rewrite the sum over the poles. The I-functions are given by:

$$
\begin{aligned}
I_\delta(\mathsf{t},0) = &\frac{\Gamma\left(\left\langle\frac{\tau_{d_2}}{\tau_{d_1}}\delta\right\rangle\right)^k\Gamma\left(\left\langle\tau_{d_2}\frac{\tau_{d_1}-\delta}{\tau_{d_1}}\right\rangle\right)^k}{\widehat{\Gamma}_\delta(0)}\\
&\cdot\sum_{a=0}^{\infty}e^{\mathsf{t}(a+\frac{\delta}{d_1}-q)}(-1)^{a(5+k-n-j+\alpha n+j\beta)}\\
&\cdot\frac{\Gamma\left(a+\frac{\delta}{d_1}\right)^{5+k-n-j}\Gamma\left(a\alpha+\frac{\alpha}{d_1}\delta\right)^n\Gamma\left(a\beta+\frac{\beta}{d_1}\delta\right)^j}{\Gamma\left(\delta+ad_1\right)\Gamma\left(ad_2+\frac{\tau_{d_2}}{\tau_{d_1}}\delta\right)^k},
\end{aligned}
\tag{6.133}
$$

and

$$\widetilde{I}_\delta(\mathsf{t},0) = \frac{\Gamma\left(\left\langle \tau_{d_1}\frac{\delta}{\tau_{d_2}}\right\rangle\right)\Gamma\left(\left\langle \tau_{d_1}\frac{\tau_{d_2}-\delta}{\tau_{d_2}}\right\rangle\right)}{\widehat{\widetilde{\Gamma}}_\delta(0)} \cdot \sum_{a=0}^{\infty}(-1)^{a(6-n-j+\alpha n+j\beta)}e^{\mathsf{t}(a+\frac{\tau_{d_2}\gamma+\delta}{d_2}-q)} \cdot \frac{\Gamma\left(a+\frac{\tau_{d_2}\gamma+\delta}{d_2}\right)^{6-n-j}\Gamma\left(a\alpha+\alpha\frac{\tau_{d_2}\gamma+\delta}{d_2}\right)^n}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}\delta+d_1a+\tau_{d_1}\gamma\right)} \cdot \frac{\Gamma\left(a\beta+\beta\frac{\tau_{d_2}\gamma+\delta}{d_2}\right)^j}{\Gamma\left(\delta+d_2a+\tau_{d_2}\gamma\right)} \tag{6.134}$$

We can compare (6.128) to the results in the Landau-Ginzburg phases (6.94)) and see that the structure is similar to the Landau-Ginzburg case, except that we now have two contributions. We can also compare (6.129) and (6.131), which would provide valid candidates for an identification as Gamma classes, and see that they have an extra term in contrast to the results in the Landau-Ginzburg phases (see (6.95) and (6.96)). The structure of the I-function is more along the lines of hybrid models (6.110). In the F7 model the second contribution is absent, which results from the fact that the greatest common divisor of the p-field charges coincides with the charge of the second p-field (see also Appendix A.3). This is expected because one of the Landau-Ginzburg model components is massive.

C-Type Models

In contrast to the F-type pseudo-hybrids the C-type models have a base manifold B of non-zero dimension. In all C-type models we see a contribution with a one-dimensional B and a Landau-Ginzburg component. We first discuss the C1 and C2 models and afterwards the C3 model, because the different components of the sphere partition function arise in different ways in these models. Again these models have been studied in [10] and have been summarized in Section 5.4, where more details can be found.

C1 and C2 In these models, by a similar reason as for the F7 model, the contribution $Z^{\zeta\ll 0}_{S^2,2} = 0$ (see (6.76)). But in contrast to the previously studied models $Z^{\zeta\ll 0}_{S^2,1}$ now splits into a contribution with first order poles and one with a second order pole. The second order poles are encountered for $\delta = \tau_{d_1}$ and we can write $Z^{\zeta\ll 0}_{S^2,1}$ in the following form

$$
\begin{aligned}
Z^{\zeta\ll 0}_{S^2,1} = {} & \frac{1}{d_1}\sum_{\delta}\Bigg|_{\delta\neq\tau_{d_1}} (-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}_\delta(0)}{\widehat{\Gamma}^*_\delta(0)} I_\delta(\mathsf{t},0)I_\delta(\bar{\mathsf{t}},0) \\
& + \frac{2\pi i}{d_2}\oint\frac{\mathrm{d}\varepsilon}{2\pi i}\frac{(-1)^{\widetilde{\mathrm{Gr}}}}{\varepsilon^2}\frac{\widetilde{\widehat{\Gamma}}(\varepsilon)}{\widetilde{\widehat{\Gamma}}^*(\varepsilon)}\widetilde{I}(\mathsf{t},\varepsilon)\widetilde{I}(\bar{\mathsf{t}},\varepsilon).
\end{aligned}
\tag{6.135}
$$

The quantities $(-1)^{\mathrm{Gr}}$, the $\widehat{\Gamma}_\delta(0), \widehat{\Gamma}^*_\delta(0)$ functions and $I_\delta(\mathsf{t},0)$ resemble the F-type results (see (6.130), (6.129) and (6.133) respectively). In the second line we introduced:

$$
\begin{aligned}
\widetilde{\widehat{\Gamma}}(\varepsilon) = {} & \Gamma\left(1-\frac{\varepsilon}{2\pi i}\right)\Gamma\left(1-\frac{\tau_{d_2}}{\tau_{d_1}}\frac{\varepsilon}{2\pi i}\right)\Gamma\left(\frac{\varepsilon}{2\pi i d_1}+\left\langle\frac{1}{k_2}\right\rangle\right)^{6-n-j} \\
& \cdot\Gamma\left(\alpha\frac{\varepsilon}{2\pi i d_1}+\left\langle\alpha\frac{1}{k_2}\right\rangle\right)^{n}\Gamma\left(\beta\frac{\varepsilon}{2\pi i d_1}+\left\langle\beta\frac{1}{k_2}\right\rangle\right)^{j},
\end{aligned}
\tag{6.136}
$$

$$
(-1)^{\widetilde{\mathrm{Gr}}} = (-1)^{\tau_{d_1}}(-1)^{\tau_{d_2}}(-1)^{6-n-j\left\lfloor\frac{1}{k_2}\right\rfloor}(-1)^{n\left\lfloor\frac{\alpha}{k_2}\right\rfloor}(-1)^{j\left\lfloor\frac{\beta}{k_2}\right\rfloor}, \tag{6.137}
$$

where one can obtain the conjugate expressions by using (6.82) and

$$
\begin{aligned}
\widetilde{I}(\mathsf{t},\varepsilon) = {} & \frac{\Gamma\left(1-\frac{\varepsilon}{2\pi i}\right)\Gamma\left(1+\frac{\varepsilon}{2\pi i}\right)\Gamma\left(1-\frac{\tau_{d_2}}{\tau_{d_1}}\frac{\varepsilon}{2\pi i}\right)\Gamma\left(1+\frac{\tau_{d_2}}{\tau_{d_1}}\frac{\varepsilon}{2\pi i}\right)}{\widetilde{\widehat{\Gamma}}(\varepsilon)} \\
& \cdot\sum_{a=0}^{\infty} e^{\mathsf{t}\left(\frac{\varepsilon}{2\pi i d_1}+a+\frac{1}{\kappa_2}-q\right)}(-1)^{a(6-n-j+\alpha n+j\beta)} \\
& \cdot\frac{\Gamma\left(a+\frac{\varepsilon}{2\pi i d_1}+\frac{1}{\kappa_2}\right)^{6-n-j}\Gamma\left(a\alpha+\alpha\frac{\varepsilon}{2\pi i d_1}+\frac{\alpha}{\kappa_2}\right)^{n}}{\Gamma\left(\tau_{d_2}+ad_1+\frac{\varepsilon}{2\pi i}\right)} \\
& \cdot\frac{\Gamma\left(a\beta+\beta\frac{\varepsilon}{2\pi i d_1}+\frac{\beta}{\kappa_2}\right)^{j}}{\Gamma\left(ad_2+\tau_{d_2}\frac{\varepsilon}{2\pi i\tau_{d_1}}+\tau_{d_1}\right)}.
\end{aligned}
\tag{6.138}
$$

The first line in (6.135) is similar to the result obtained in the Landau-Ginzburg case (6.94) and the second line matches the results found in the hybrid models (6.111) .

C3 This is the only model of the class of interest in which $Z^{\zeta\ll 0}_{S^2,2}\neq 0$ and has a second order pole. $Z^{\zeta\ll 0}_{S^2,1}$ has only first order poles and so we can write the sphere partition function in the following way

$$
\begin{aligned}
Z_{S^2}^{\zeta \ll 0} = & \frac{1}{d_1} \sum_{\delta} (-1)^{\mathrm{Gr}} \frac{\widehat{\Gamma}_{\delta}(0)}{\widehat{\Gamma}^*_{\delta}(0)} I_{\delta}(\mathsf{t}, 0) I_{\delta}(\bar{\mathsf{t}}, 0) \\
& + \frac{2\pi i}{d_2} \oint \frac{\mathrm{d}\varepsilon}{2\pi i} \frac{(-1)^{\widetilde{\mathrm{Gr}}}}{\varepsilon^2} \frac{\widetilde{\widehat{\Gamma}}(\varepsilon)}{\widetilde{\widehat{\Gamma}}^*(\varepsilon)} \widetilde{I}(\mathsf{t}, \varepsilon) I(\bar{\mathsf{t}}, \varepsilon).
\end{aligned}
\tag{6.139}
$$

In the above expression the $(-1)^{\mathrm{Gr}}$, the $\widehat{\Gamma}_{\delta}(0)$, $\widehat{\Gamma}^*_{\delta}(0)$ functions and $I_{\delta}(\mathsf{t}, 0)$ are as in the C1 and C2 model given by the F-type expressions (6.130), (6.129), and (6.133), respectively. For the second contribution we introduce

$$
\begin{aligned}
\widetilde{\widehat{\Gamma}}(\varepsilon) = & \Gamma\left(1 - \frac{\varepsilon}{2\pi i}\right)^2 \Gamma\left(-\tau_{d_1} \frac{\varepsilon}{2\pi i \tau_{d_2}} + \left\langle \tau_{d_1} \frac{\tau_{d_2} - 1}{\tau_{d_2}} \right\rangle\right) \\
& \cdot \Gamma\left(\frac{\varepsilon}{2\pi i d_2} + \left\langle \frac{1}{d_2} \right\rangle\right)^{7-n-j} \Gamma\left(\alpha \frac{\varepsilon}{2\pi i d_2} + \left\langle \frac{\alpha}{d_2} \right\rangle\right)^n \\
& \cdot \Gamma\left(\beta \frac{\varepsilon}{2\pi i d_2} + \left\langle \frac{\beta}{d_2} \right\rangle\right)^j,
\end{aligned}
\tag{6.140}
$$

$$
(-1)^{\widetilde{\mathrm{Gr}}} = (-1)^{\left\lfloor \frac{d_2}{d_1} \right\rfloor} (-1)^{(7-n-j)\left\lfloor \frac{1}{d_2} \right\rfloor} (-1)^{n\left\lfloor \frac{\alpha}{d_2} \right\rfloor} (-1)^{j\left\lfloor \frac{\beta}{d_2} \right\rfloor},
\tag{6.141}
$$

and

$$
\begin{aligned}
\widetilde{I}(\varepsilon, \mathsf{t}) = & \frac{\Gamma\left(1 - \frac{\varepsilon}{2\pi i}\right)^2 \Gamma\left(1 + \frac{\varepsilon}{2\pi i}\right)^2 \Gamma\left(-\tau_{d_1} \frac{\varepsilon}{2\pi i \tau_{d_2}} + \left\langle \tau_{d_1} \frac{\tau_{d_2}-1}{\tau_{d_2}} \right\rangle\right)}{\widetilde{\widehat{\Gamma}}(\varepsilon)} \\
& \cdot \Gamma\left(\tau_{d_1} \frac{\varepsilon}{2\pi i \tau_{d_2}} + \left\langle \tau_{d_1} \frac{1}{\tau_{d_2}} \right\rangle\right) \\
& \cdot \sum_{a=0}^{\infty} (-1)^{a(7-n-j+\alpha n+j\beta)} e^{\mathsf{t}\left(\frac{\varepsilon}{2\pi i d_2} + a + \frac{1}{d_2} - q\right)} \\
& \cdot \frac{\Gamma\left(a + \frac{\varepsilon}{2\pi i d_2} + \frac{1}{d_2}\right)^{7-n-j} \Gamma\left(a\alpha + \alpha \frac{\varepsilon}{2\pi i d_2} + \frac{\alpha}{d_2}\right)^n}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}} + d_1 a + \tau_{d_1} \frac{\varepsilon}{2\pi i \tau_{d_2}}\right)} \\
& \cdot \frac{\Gamma\left(a\beta + \beta \frac{\varepsilon}{2\pi i d_2} + \frac{\beta}{d_2}\right)^j}{\Gamma\left(1 + d_2 a + \frac{\varepsilon}{2\pi i}\right)^2}
\end{aligned}
\tag{6.142}
$$

As in the previous pseudo-hybrid models we see in (6.139) a part which resembles the result known from the Landau-Ginzburg phases and a part which looks similar to the result in hybrid phases.

6.4 Two-Parameter Examples

Next we show that the proposed formula also holds in examples with two Kähler parameters. We study the model with gauge group $\mathsf{G} = U(1)^2$ and field content

$$\begin{array}{c|cc|ccccc|c} & p & x_6 & x_3 & x_4 & x_5 & x_1 & x_2 & \text{FI} \\ \hline U(1)_1 & -4 & 1 & 1 & 1 & 1 & 0 & 0 & \zeta_1 \\ U(1)_2 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & \zeta_2 \\ \hline U(1)_V & 2-8q_1 & 2q_1-4q_2 & 2q_1 & 2q_1 & 2q_1 & 2q_2 & 2q_2, & \end{array} \tag{6.143}$$

and $0 \leq q_1 \leq \frac{1}{4}$ and $0 \leq q_2 \leq \frac{1}{8}$. In this model the superpotential has the form $W = pG_{(4,0)}(x_1, \dots, x_6)$. This model is one of the standard examples for the two parameter case (see [115],[116]).

The sphere partition function is given by

$$Z_{S^2} = \frac{1}{(2\pi)^2} \sum_{m \in \mathbb{Z}^2} \int_{-\infty}^{\infty} \mathrm{d}^2\sigma Z_p Z_6 Z_5^3 Z_1^2 e^{-4\pi i(\zeta_1\sigma_1+\zeta_2\sigma_2)-i(\theta_1 m_1+\theta_2 m_2)}, \tag{6.144}$$

where

$$\begin{aligned} Z_p &= \frac{\Gamma\left(4(i\sigma_1 - q_1 + \frac{1}{4}) + \frac{4m_1}{2}\right)}{\Gamma\left(1 - 4(i\sigma_1 - q_1 + \frac{1}{4}) + \frac{4m_1}{2}\right)}, \\ Z_6 &= \frac{\Gamma\left(-\left((i\sigma_1 - q_1) - 2(i\sigma_2 - q_2)\right) - \frac{m_1 - 2m_2}{2}\right)}{\Gamma\left(1 + \left((i\sigma_1 - q_1) - 2(i\sigma_2 - q_2)\right) - \frac{m_1-2m_2}{2}\right)}, \\ Z_5 &= \frac{\Gamma\left(-(i\sigma_1 - q_1) - \frac{m_1}{2}\right)}{\Gamma\left(1 + (i\sigma_1 - q_1) - \frac{m_1}{2}\right)}, \\ Z_1 &= \frac{\Gamma\left(-(i\sigma_2 - q_2) - \frac{m_2}{2}\right)}{\Gamma\left(1 + (i\sigma_2 - q_2) - \frac{m_2}{2}\right)}. \end{aligned} \tag{6.145}$$

The phase structure of this model is a follows

- **I**: Geometric phase ($\zeta_1 \gg 0, \zeta_2 \gg 0$): hypersurface $G_{(4,0)}(x_1, \dots, x_6) = 0$ in the toric ambient space defined by the $U(1)^2$-charges of $x_1, \dots, x_6$

- **II**: Orbifold phase ($2\zeta_1 + \zeta_2 \gg 0, \zeta_2 \ll 0$): this is a singular hypersurface $G_{(4,0)}(x_1, \dots, x_5, 1)$ in the ambient space spanned by the charges of $x_1, \dots, x_5$ under $2U(1)_1 + U(1)_2$

- **III**: Landau-Ginzburg orbifold phase ($2\zeta_1 + \zeta_2 \ll 0, \zeta_2 \ll 0$): Orbifold group $G = \mathbb{Z}_8$ and $W_{LG} = G_{(4,0)}(x_1, \dots, x_5, 1)$

- **IV**: Hybrid phase ($\zeta_1 \ll 0, \zeta_2 \gg 0$): this is a fibration of a Landau-Ginzburg orbifold with $G = \mathbb{Z}_4$ over $B = \mathbb{P}^1$.

We will focus on the Landau-Ginzburg, geometric and hybrid phase in the following. Similar to the one-parameter phases the goal is to use the residue theorem to evaluate the sphere partition function. Note that aspects from supersymmetric localisation for this model have also been studied in [113, 49, 117]. In the evaluation of the sphere partition function we follow the approach given in [118]. More details on the approach and certain steps in the evaluation are given in Appendix B.

Phase I: Geometric phase

Aspects related to the sphere partition function of this phase were also studied in [113]. As discussed in Appendix B in the geometric phase we can focus on the poles coming from Z_1 and Z_5. We next choose $q_1 = q_2 = 0$ in order to get R-charges compatible with the non-linear sigma model. The transformations given in Appendix B.1 lead to:

$$
\begin{aligned}
Z_{S^2}^{geom} = \frac{1}{(2\pi)^2} \sum_{n_5, n_1, n'_5, n'_1 \geq 0} \oint \mathrm{d}^2\varepsilon Z_p Z_6 Z_5^3 Z_1^2 e^{-4\pi(\zeta_1\varepsilon_1 + \zeta_2\varepsilon_2)}, \\
\cdot e^{(-2\pi\zeta_1 - i\theta_1)n_5 + (-2\pi\zeta_2 - i\theta_2)n_1} e^{(-2\pi\zeta_1 + i\theta_1)n'_5 + (-2\pi\zeta_2 + i\theta_2)n'_1}
\end{aligned}
\tag{6.146}
$$

where

$$
\begin{aligned}
Z_p &= \frac{\Gamma\left(1 + 4n_5 + 4\varepsilon_1\right)}{\Gamma\left(-4n'_5 - 4\varepsilon_1\right)} \\
Z_6 &= \frac{\Gamma\left(-n_5 + 2n_1 - \varepsilon_1 + 2\varepsilon_2\right)}{\Gamma\left(1 + n'_5 - 2n'_1 + \varepsilon_1 - 2\varepsilon_2\right)} \\
Z_5 &= \frac{\Gamma\left(-n_5 - \varepsilon_1\right)}{\Gamma\left(1 + n'_5 + \varepsilon_1\right)} \\
Z_1 &= \frac{\Gamma\left(-n_1 - \varepsilon_2\right)}{\Gamma\left(1 + n'_1 + \varepsilon_2\right)}.
\end{aligned}
\tag{6.147}
$$

Similar to the one-parameter geometric phases we want to rewrite the integral into expression known from geometry and therefore we define $\varepsilon_i = \frac{H_i}{2\pi i}$ $(i = 1, 2)$ with $H_i \in H^2(X, \mathbb{C})$. We use the identity (6.68) multiple times and introduce

$$
\begin{aligned}
I_X(\mathsf{t}, H) = \sum_{n_5, n_1 \geq 0} & e^{-\mathsf{t}_1 n_5} e^{-\mathsf{t}_2 n_1} e^{-\mathsf{t}_1 \frac{H_1}{2\pi i}} e^{-\mathsf{t}_2 \frac{H_2}{2\pi i}} \\
& \cdot \frac{\Gamma\left(1 + \frac{H_1}{2\pi i}\right)^3 \Gamma\left(1 + \frac{H_2}{2\pi i}\right)^2 \Gamma\left(1 + 4n_5 + 4\frac{H_1}{2\pi i}\right)}{\Gamma\left(1 + 4\frac{H_1}{2\pi i}\right) \Gamma\left(1 + n_5 + \frac{H_1}{2\pi i}\right)^3 \Gamma\left(1 + n_1 + \frac{H_2}{2\pi i}\right)^2} \\
& \cdot \frac{\Gamma\left(1 + \frac{H_1}{2\pi i} - 2\frac{H_2}{2\pi i}\right)}{\Gamma\left(1 + n_5 - 2n_1 + \frac{H_1}{2\pi i} - 2\frac{H_2}{2\pi i}\right)}
\end{aligned}
\tag{6.148}
$$

and write the sphere partition function in the following form:

$$Z_{S^2}^{geom} = -\frac{(2\pi i)^5}{(2\pi)^2} \oint \frac{d^2H}{(2\pi i)^2} \frac{4H_1}{H_1^3 H_2^2 (H_1 - 2H_2)} \frac{\widehat{\Gamma}}{\widehat{\Gamma}^*} I_X(\mathsf{t}, H) I_X(\bar{\mathsf{t}}, H), \quad (6.149)$$

with

$$\widehat{\Gamma} = \frac{\Gamma\left(1 - \frac{H_1}{2\pi i}\right)^3 \Gamma\left(1 - \frac{H_2}{2\pi i}\right)^2 \Gamma\left(1 - \frac{H_1}{2\pi i} + 2\frac{H_2}{2\pi i}\right)}{\Gamma\left(1 - \frac{4H_1}{2\pi i}\right)}, \quad (6.150)$$

the Gamma-class of the CY hypersurface X. The final step is to rewrite the integration as integration over the Calabi-Yau X and therefore we first consider a powers series of the quantity $h(H_1, H_2) = \sum_{i,j\leq 0} a_{i,j} H_1^i H_2^j$. An integral of this quantity over X can be written in the following form

$$\begin{aligned} \int_X h(H_1, H_2) = 8a_{3,0} + 4a_{2,1} &= \int_{X_\Sigma} (4H_1) h(H_1, H_2) \\ &= \oint_0 \frac{d^2H}{(2\pi i)^2} \left[\frac{8}{H_1^4 H_2} + \frac{4}{H_1^3 H_2^2}\right] h(H_1, H_2), \end{aligned} \quad (6.151)$$

where the result follows by taking into account the non-zero triple intersection numbers of X:

$$H_1^3 = 8, \qquad H_1^2 H_2 = 4. \quad (6.152)$$

To transform the sphere partition function into such a form, we first remember the transformation formula for multivariate residues (see [119]). Let $\{f_1(z_i), \ldots, f_n(z_i)\}$ and $\{g_1(z_i), \ldots, g_n(z_i)\}$ be holomorphic functions in the n variables $z_1, \ldots, z_n$ satisfying

$$g_k(z_i) = T_{kj} f_j(z_i), \quad (6.153)$$

where T is a holomorphic matrix. Then

$$\mathrm{Res}\left(\frac{h(z_i)\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n}{f_1(z_i) \cdot \ldots \cdot f_n(z_i)}\right) = \mathrm{Res}\left(\det T \frac{h(z_i)\mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n}{g_1(z_i) \cdot \ldots \cdot g_n(z_i)}\right). \quad (6.154)$$

Applied to our case we find

$$\begin{pmatrix} H_2^2 \\ H_1^5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 4H_1^3 & H_1 + 2H_2 \end{pmatrix}}_{T} \begin{pmatrix} H_2^2 \\ H_1^3 (H_1 - 2H_2) \end{pmatrix} \quad (6.155)$$

and consequently

$$\det T = H_1 + 2H_2. \quad (6.156)$$

By the above result we can bring the sphere partition function into the form

$$\begin{aligned} Z_{S^2}^{geom} &= -\frac{(2\pi i)^5}{(2\pi)^2} \oint \left[\frac{8}{H_1^4 H_2} + \frac{4}{H_1^3 H_2^2}\right] \frac{\widehat{\Gamma}}{\widehat{\Gamma}^*} I(\mathsf{t}) I(\bar{\mathsf{t}}) \\ &= (2\pi i)^3 \int_X \frac{\widehat{\Gamma}}{\widehat{\Gamma}^*} I(\mathsf{t}) I(\bar{\mathsf{t}}). \end{aligned} \quad (6.157)$$

We can write $Z^{geom}_{S^2}$ in the following form

$$\frac{Z_{S^2}}{8\pi^3} = \bar{\mathbf{I}}^{\mathsf{T}} M \mathbf{I}, \tag{6.158}$$

with

$$M = \begin{pmatrix} -\frac{168\zeta(3)}{4\pi^3} & 0 & 0 & 0 & 0 & -4i \\ 0 & 0 & 0 & 0 & -4i & 0 \\ 0 & 0 & 0 & -4i & -8i & 0 \\ 0 & 0 & -4i & 0 & 0 & 0 \\ 0 & -4i & -8i & 0 & 0 & 0 \\ -4i & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.159}$$

and

$$\mathbf{I} = \begin{pmatrix} I^{(0,0)} \\ I^{(0,1)} \\ I^{(1,0)} \\ I^{(1,1)} \\ I^{(2,0)} \\ I^{(2,1)} + 2I^{(3,0)} \end{pmatrix}, \qquad \bar{\mathbf{I}} = \begin{pmatrix} \bar{I}^{(0,0)} \\ \vdots \end{pmatrix}. \tag{6.160}$$

The $I^{(i,j)}$ are the coefficients of $H_1^i H_2^j$ in the expansion of the I-function with respect to H_1, H_2. We see that The I-function and the Gamma class match with (6.55) and (6.56). Another consistency check is given by acting with the Picard-Fuchs operators [116]

$$\begin{aligned} \mathcal{L}_1 &= \theta_1^2(\theta_1 - 2\theta_2) - 4z_1(4\theta_1 + 3)(4\theta_1 + 2)(4\theta_1 + 1) \\ \mathcal{L}_2 &= \theta_2^2 - z_2(2\theta_2 - \theta_1 + 1)(2\theta_1 - \theta_1), \end{aligned} \tag{6.161}$$

where $z_i = e^{-\mathsf{t}_i}$ and $\theta_i = z_i \frac{\partial_i}{\partial z_i}$ on the components of the I-function appearing in (6.159). As expected the components are annihilated by the operators $\mathcal{L}_1, \mathcal{L}_2$.

Phase III: Landau-Ginzburg Phase

Note that in [82] the hemisphere partition function of this phase was studied. In this phase the gauge group gets broken to $G = \mathbb{Z}_8$ and typical for Landau-Ginzburg orbifold models we encounter different sectors in the state space, which we will label by $\gamma \in \{0, \ldots, 8\}$. The sectors $\gamma = 0, 4$ are broad. Subsequently we show how the narrow sectors, now labelled by δ, arise in the evaluation of the sphere partition function. But first we need to determine the location of the poles. This is done in Appendix B, where we show that only the poles coming from Z_p and Z_6 contribute. The necessary transformations applied to (6.144) are given in Appendix B.1.

As outlined in Appendix B.1 the sums appearing in the sphere partition function need to be simplified. This is not straight forward and we will apply a two step procedure. First we introduce

$$a = n_p + 4n_6 + 8m_2, \quad c = n_p + 4n_6, \quad b = 4m_1 + n_p, \quad d = n_p. \tag{6.162}$$

We must be careful, because the new summation variables need to fulfil the following constraints

$$a - c \in 8\mathbb{Z}, \quad b - d \in 4\mathbb{Z}, \quad c - d \in 4\mathbb{Z}_{\geq 0}, \quad a - b \in 4\mathbb{Z}_{\geq 0}. \tag{6.163}$$

These equations can be obtained by inserting the definitions (6.162) and taking into account that $n_p, n_6, m_1, m_2 \in \mathbb{Z}$. With the constraints in mind we introduce in a second step the following summation variables

$$\begin{aligned} a &= 8l + \delta_1 \quad & c &= 8k + \delta_1 \quad & \delta_1 &= 0, 1, \ldots, 7, \\ b &= 4p + \delta_2 \quad & d &= 4q + \delta_2 \quad & \delta_2 &= 0, 1, \ldots, 3. \end{aligned} \tag{6.164}$$

To fulfil the constraints (6.163) we restrict to the following δ_1, δ_2 combinations:

δ_1	0	1	2	3	4	5	6	7
δ_2	0	1	2	3	0	1	2	3
$\kappa = \delta_1 - \delta_2$	0	0	0	0	4	4	4	4.

(6.165)

This restriction on the δ values suggests to set $\delta_1 = \delta_2 + \kappa$. We will now set $\delta_2 \equiv \delta$ and write the sphere partition function in the following form:

$$\begin{aligned} Z_{S^2}^{LG} = &-\frac{1}{8(2\pi i)^2} \sum_{\kappa \in \{0,4\}} \left(\sum_{\delta=0}^{3} \oint_{(0,0)} \mathrm{d}^2\varepsilon \frac{1}{\pi^3} \right. \\ &\cdot \frac{\sin\left(\pi\left(\frac{\delta+1}{4} + \frac{\varepsilon_1}{4}\right)\right)^3 \sin\left(\pi\left(\frac{\delta+1+\kappa}{8} + \frac{\varepsilon_1+4\varepsilon_2}{8}\right)\right)^2}{\sin(\pi\varepsilon_1)\sin\left(\pi\left(\frac{\kappa}{4} + \varepsilon_2\right)\right)} \\ &\cdot \left| e^{\mathrm{t}_1\frac{\varepsilon_1}{4}} e^{\mathrm{t}_2\frac{\varepsilon_1+4\varepsilon_2}{8}} \sum_{l=0}^{\infty} \sum_{p=0}^{2l+\frac{\kappa}{4}} (-1)^p e^{\frac{\mathrm{t}_1}{4}(4p+\delta)} e_2^{\frac{\mathrm{t}_2}{8}(8l+\delta+\kappa)} \right. \\ &\left.\left. \cdot \frac{\Gamma\left(p + \frac{\delta+1}{4} + \frac{\varepsilon_1}{4}\right)^3 \Gamma\left(l + \frac{\delta+1+\kappa}{8} + \frac{\varepsilon_1+4\varepsilon_2}{8}\right)^2}{\Gamma(1+4p+\delta+\varepsilon_1)\Gamma\left(1+2l-p+\frac{\kappa}{4}+\varepsilon_2\right)} \right|^2 \right). \end{aligned} \tag{6.166}$$

As expected only first order poles appear. $\delta = 3$ gives no contribution in accordance with the expectation, because it would correspond to a broad sector. Next we evaluate the integral and apply the transformations $\kappa \to 4\kappa$, and $\delta \to \delta - 1$. We introduce

$$\begin{aligned} (-1)^{\mathrm{Gr}_\kappa} &= (-1)^{\delta}(-1)^{\kappa}(-1)^{3\left\lfloor\frac{\delta}{4}\right\rfloor}(-1)^{2\left\lfloor\frac{\delta+4\kappa}{8}\right\rfloor}, \\ \widehat{\Gamma}_{\delta,\kappa}(0) &= \Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{\delta+4\kappa}{8}\right\rangle\right)^2, \end{aligned} \tag{6.167}$$

and define the following I function

$$I_{\delta,\kappa}(\mathtt{t}_1,\mathtt{t}_2,0) = \sum_{l=0}^{\infty}\sum_{p=0}^{\infty} \frac{(-1)^p e^{\frac{\mathtt{t}_1}{4}(4p+\delta-1)} e^{\frac{\mathtt{t}_2}{8}(8l+\delta-1+4\kappa)}}{\Gamma\left(\langle\frac{\delta}{4}\rangle\right)^3 \Gamma\left(\langle\frac{\delta+4\kappa}{8}\rangle\right)^2} \cdot \frac{\Gamma\left(p+\frac{\delta}{4}\right)^3 \Gamma\left(l+\frac{\delta+4\kappa}{8}\right)^2}{\Gamma(4p+\delta)\Gamma(1+2l-p+\kappa)}, \tag{6.168}$$

to write $Z_{S^2}^{LG}$ in the following way:

$$Z_{S^2}^{LG} = \frac{1}{8}\sum_{\delta=1}^{3}\left((-1)^{\mathrm{Gr}_0}\frac{\widehat{\Gamma}_{\delta,0}(0)}{\widehat{\Gamma}^*_{\delta,0}(0)} I_{\delta,0}(\mathtt{t}_1,\mathtt{t}_2,0) I_{\delta,0}(\bar{\mathtt{t}}_1,\bar{\mathtt{t}}_2,0) + (-1)^{\mathrm{Gr}_1}\frac{\widehat{\Gamma}_{\delta,1}(0)}{\widehat{\Gamma}^*_{\delta,1}(0)} I_{\delta,1}(\mathtt{t}_1,\mathtt{t}_2,0) I_{\delta,1}(\bar{\mathtt{t}}_1.\bar{\mathtt{t}}_2,0)\right). \tag{6.169}$$

From the previous expression we can read off the M matrix

$$M = \begin{pmatrix} \frac{\gamma_1(0)}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\gamma_2(0)}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma_3(0)}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{8\gamma_3(0)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{8\gamma_2(0)} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{8\gamma_1(0)} \end{pmatrix}, \tag{6.170}$$

where γ_δ is given by (6.83) with (6.167) at $\kappa = 0$ inserted. (6.167) at $\kappa = 1$ is not needed in the expression for M. Let us now match the result for the Landau-Ginzburg phase to the proposal given in (6.51). We first introduce the q-matrix

$$q = \begin{pmatrix} 1 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{6.171}$$

and use the result of [82] that (6.47) can be rewritten in the following form:

$$I_{LG}(u) = \sum_{r=1}^{3}\left[\frac{1}{\Gamma\left(\frac{r}{4}\right)^3\Gamma\left(\frac{r}{8}\right)^2}\widehat{\varpi}_r^{ev} e_r + \frac{1}{\Gamma\left(\frac{r}{4}\right)^3\Gamma\left(\frac{r}{8}+\frac{1}{2}\right)^2}\widehat{\varpi}_r^{od} e_{r+4}\right], \tag{6.172}$$

with

$$\begin{aligned}\widehat{\varpi}_r^{ev} = (-1)^{r+1} &\sum_{n\in 2\mathbb{Z}_{\geq 0}} \frac{\Gamma\left(n+\frac{r}{4}\right)^4}{\Gamma\left(4n+r\right)} \left(-2^{12}\psi^4\right)^{n+\frac{r-1}{4}} \\ &\cdot \sum_m \frac{\Gamma\left(m+\frac{n}{2}+\frac{r}{8}\right)^2}{\Gamma\left(n+\frac{r}{4}\right)\Gamma\left(2m+1\right)} \left(2\phi\right)^{2m} \\ + (-1)^r &\sum_{n\in 2\mathbb{Z}_{\geq 0}+1} \frac{\Gamma\left(n+\frac{r}{4}\right)^4}{\Gamma\left(4n+r\right)} \left(-2^{12}\psi^4\right)^{n+\frac{r-1}{4}} \\ &\cdot \sum_m \frac{\Gamma\left(m+\frac{n}{2}+\frac{r}{8}+\frac{1}{2}\right)^2}{\Gamma\left(n+\frac{r}{4}\right)\Gamma\left(2m+2\right)} \left(2\phi\right)^{2m+1},\end{aligned} \tag{6.173}$$

and

$$\begin{aligned}\widehat{\varpi}_r^{odd} = (-1)^{r+1} &\sum_{n\in 2\mathbb{Z}_{\geq 0}+1} \frac{\Gamma\left(n+\frac{r}{4}\right)^4}{\Gamma\left(4n+r\right)} \left(-2^{12}\psi^4\right)^{n+\frac{r-1}{4}} \\ &\cdot \sum_m \frac{\Gamma\left(m+\frac{n}{2}+\frac{r}{8}\right)^2}{\Gamma\left(n+\frac{r}{4}\right)\Gamma\left(2m+1\right)} \left(2\phi\right)^{2m} \\ + (-1)^r &\sum_{n\in 2\mathbb{Z}_{\geq 0}} \frac{\Gamma\left(n+\frac{r}{4}\right)^4}{\Gamma\left(4n+r\right)} \left(-2^{12}\psi^4\right)^{n+\frac{r-1}{4}} \\ &\cdot \sum_m \frac{\Gamma\left(m+\frac{n}{2}+\frac{r}{8}+\frac{1}{2}\right)^2}{\Gamma\left(n+\frac{r}{4}\right)\Gamma\left(2m+2\right)} \left(2\phi\right)^{2m+1}.\end{aligned} \tag{6.174}$$

The next step is given by the following transformations

$$\begin{aligned}&(6.173)\begin{cases} k = m+\frac{n}{2} & n\in 2\mathbb{Z} \\ k = m+\frac{n+1}{2} & n\in 2\mathbb{Z}+1 \end{cases}, \\ &(6.174)\begin{cases} k = m+\frac{n}{2} & n\in 2\mathbb{Z} \\ k = m+\frac{n-1}{2} & n\in 2\mathbb{Z}+1 \end{cases},\end{aligned} \tag{6.175}$$

which do not affect the limits of the summation, because we applied an integer shift. We identify

$$e^{\mathtt{t}_1} = -2^{11}\psi^4\phi^{-1} \tag{6.176}$$

$$e^{\mathtt{t}_2} = 2^2\phi^2, \tag{6.177}$$

and rewrite (6.172) into

$$I_{LG}(u) = \sum_{\delta=1}^{3}\Big[(-1)^{\delta+1}e_\delta I_{\delta,0}(\mathtt{t}_1,\mathtt{t}_2) + (-1)^{\delta}e_{\delta+4}I_{\delta,1}(\mathtt{t}_1,\mathtt{t}_2)\Big], \tag{6.178}$$

where (6.168) was inserted. The Gamma function (6.48) in this model is given by:

$$\widehat{\Gamma}_{\delta} = \Gamma\left(1 - \left\langle -\frac{k_1+1}{4}\right\rangle\right)^3 \Gamma\left(1 - \left\langle -\frac{k_1+1}{8} - \frac{k_2}{2}\right\rangle\right)^2. \tag{6.179}$$

After the reparameterization

$$k_1 = 4n + r - 1 \qquad r = 1, \ldots, 4 \qquad k_2 = 2m + s \qquad s = 0, 1, \tag{6.180}$$

given in [82] we get

$$\widehat{\Gamma}_{\delta} = \Gamma\left(1 - \left\langle -\frac{r}{4}\right\rangle\right)^3 \Gamma\left(1 - \left\langle -\frac{n+s}{2} - \frac{r}{8}\right\rangle\right)^2, \tag{6.181}$$

where we dropped integer shifts from $\langle \cdot \rangle$. We will now split the above formula into two contributions with either $n + s \in 2\mathbb{Z}$ or not:

$$\widehat{\Gamma}_{\delta} = \begin{cases} \Gamma\left(1 - \left\langle -\frac{r}{4}\right\rangle\right)^3 \Gamma\left(1 - \left\langle -\frac{r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z} \\ \Gamma\left(1 - \left\langle -\frac{r}{4}\right\rangle\right)^3 \Gamma\left(1 - \left\langle -\frac{1}{2} - \frac{r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z}+1 \end{cases}. \tag{6.182}$$

Our interest lies in the narrow sector, in which

$$\left\langle \frac{r}{4}\right\rangle \neq 0, \qquad \left\langle \frac{r}{8}\right\rangle \neq 0, \qquad \left\langle \frac{r}{2} + \frac{r}{8}\right\rangle \neq 0, \tag{6.183}$$

and it follows that

$$\widehat{\Gamma}_{\delta} = \begin{cases} \Gamma\left(\left\langle \frac{r}{4}\right\rangle\right)^3 \Gamma\left(\left\langle \frac{r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z} \\ \Gamma\left(\left\langle \frac{r}{4}\right\rangle\right)^3 \Gamma\left(\left\langle \frac{4+r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z}+1 \end{cases}. \tag{6.184}$$

Eqn. (6.49) can be rewritten by the same steps which result in

$$\widehat{\Gamma}^*_{\delta} = \begin{cases} \Gamma\left(\left\langle \frac{4-r}{4}\right\rangle\right)^3 \Gamma\left(\left\langle \frac{8-r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z} \\ \Gamma\left(\left\langle \frac{4-r}{4}\right\rangle\right)^3 \Gamma\left(\left\langle \frac{4-r}{8}\right\rangle\right)^2 & n+s \in 2\mathbb{Z}+1 \end{cases}. \tag{6.185}$$

The grading operator (6.50) is given by

$$\mathrm{Gr} = \begin{cases} r - 3\left\lfloor \frac{r}{4}\right\rfloor - 2\left\lfloor \frac{r}{8}\right\rfloor & n+s \in 2\mathbb{Z} \\ r + 1 - 3\left\lfloor \frac{r}{4}\right\rfloor - 2\left\lfloor \frac{4+r}{8}\right\rfloor & n+s \in 2\mathbb{Z}+1 \end{cases}. \tag{6.186}$$

We insert (6.178), (6.184), (6.185) and (6.186) into (6.51) and find

$$\begin{aligned} Z_{S^2}^{LG} = \sum_{\delta,\delta'=1}^{3} \Bigg(& (-1)^{\delta-3\lfloor\frac{\delta}{4}\rfloor-2\lfloor\frac{\delta}{8}\rfloor} \frac{\Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{\delta}{8}\right\rangle\right)^2}{\Gamma\left(\left\langle\frac{4-\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{8-\delta}{8}\right\rangle\right)^2} \\ & \cdot (-1)^{\delta+1} I_{\delta,0}(\bar{t_1},\bar{t_2}) \left\langle e_{\delta^{-1}}\right| \\ & + (-1)^{\delta+1-3\lfloor\frac{\delta}{4}\rfloor-2\lfloor\frac{4+\delta}{8}\rfloor} \frac{\Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{4+\delta}{8}\right\rangle\right)^2}{\Gamma\left(\left\langle\frac{4-\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{4-\delta}{8}\right\rangle\right)^2} \\ & \cdot (-1)^{\delta} I_{\delta,1}(\bar{t_1},\bar{t_2}) \left\langle e_{(\delta+4)^{-1}}\right| \Bigg) \\ & \cdot \Big((-1)^{\delta'+1} I_{\delta',0}(t_1,t_2) \left|e_{\delta'}\right\rangle + (-1)^{\delta'} I_{\delta',1}(t_1,t_2) \left|e_{(\delta'+4)}\right\rangle \Big). \end{aligned} \tag{6.187}$$

If we evaluate the pairing (6.43) we find for the proposed form of the sphere partition function:

$$\begin{aligned} Z_{S^2}^{LG} &= \frac{1}{8}\sum_{\delta=1}^{3} \Bigg((-1)^{\delta-3\lfloor\frac{\delta}{4}\rfloor-2\lfloor\frac{\delta}{8}\rfloor} \frac{\Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{\delta}{8}\right\rangle\right)^2}{\Gamma\left(\left\langle\frac{4-\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{8-\delta}{8}\right\rangle\right)^2} I_{\delta,0}(\bar{t_1},\bar{t_2}) I_{\delta,0}(t_1,t_2) \\ &\quad + (-1)^{\delta+1-3\lfloor\frac{\delta}{4}\rfloor-2\lfloor\frac{4+\delta}{8}\rfloor} \frac{\Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{4+\delta}{8}\right\rangle\right)^2}{\Gamma\left(\left\langle\frac{4-\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{4-\delta}{8}\right\rangle\right)^2} I_{\delta,1}(\bar{t_1},\bar{t_2}) I_{\delta,1}(t_1,t_2) \Bigg) \\ &= \frac{1}{8}\sum_{\kappa=0}^{1}\sum_{\delta=1}^{3} \Bigg((-1)^{\delta+\kappa-3\lfloor\frac{\delta}{4}\rfloor-2\lfloor\frac{4\kappa+\delta}{8}\rfloor} \frac{\Gamma\left(\left\langle\frac{\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{4\kappa+\delta}{8}\right\rangle\right)^2}{\Gamma\left(\left\langle\frac{4-\delta}{4}\right\rangle\right)^3 \Gamma\left(\left\langle\frac{8-4\kappa-\delta}{8}\right\rangle\right)^2} \\ &\quad \cdot I_{\delta,\kappa}(\bar{t_1},\bar{t_2}) I_{\delta,\kappa}(t_1,t_2) \Bigg). \end{aligned} \tag{6.188}$$

We see that (6.188) coincides with the GLSM result (6.169).

Phase IV: Hybrid Phase

The final phase we will consider is the hybrid phase. We start by analysing the phase structure. The D-terms are given by

$$\begin{aligned} -4|p|^2 + |x_6|^2 + \sum_{i=3}^{5} |x_i|^2 &= \zeta_1 \\ -2|x_6|^2 + |x_1|^2 + |x_2| &= \zeta_2. \end{aligned} \tag{6.189}$$

and for $\zeta_1 \ll 0, \zeta_2 \gg 0$ we find for the vacuum:

$$p = \sqrt{-\frac{\zeta_1}{4}}, \qquad |x_1|^2 + |x_2|^2 = \zeta_2. \tag{6.190}$$

The non zero vacuum values break $U(1)_2$ completely and $U(1)_1$ is broken to $\mathbb{Z}_4$. The vacuum manifold is a $\mathbb{P}^1$ and the low energy description is given by a $\mathbb{Z}_4$ Landau-Ginzburg orbifold fibred over $\mathbb{P}^1$. In this phase we choose $q_1 = \frac{1}{4}, q_2 = 0$. A discussion which poles contribute can again be found in Appendix B. The upshot is that in this phase only the poles of the contributions Z_1 and Z_p have to be taken into account. This is also expected, because these are the contributions corresponding to the fields with a non-zero VEV in this phase. After the transformations given in Appendix B.1 the sphere partition function becomes

$$
\begin{aligned}
Z_{S^2} = &-\frac{1}{4(2\pi)^2} \sum_{n_i, n_i'=0}^{\infty} \oint \mathrm{d}^2\varepsilon e^{\pi\zeta_1\varepsilon_1} e^{-4\pi\zeta_2\varepsilon_2} \frac{\Gamma\left(-n_p - \varepsilon_1\right)}{\Gamma\left(1 + n_p' + \varepsilon_1\right)} \\
&\cdot \frac{\Gamma\left(\frac{1}{4} + \frac{n_p}{4} + \frac{\varepsilon_1}{4} + 2n_1 + 2\varepsilon_2\right)}{\Gamma\left(1 - \frac{1}{4} - \frac{n_p'}{4} - \frac{\varepsilon_1}{4} - 2n_1' - 2\varepsilon_2\right)} \\
&\cdot \left[\frac{\Gamma\left(\frac{1}{4} + \frac{n_p}{4} + \frac{\varepsilon_1}{4}\right)}{\Gamma\left(1 - \frac{1}{4} - \frac{n_p'}{4} - \frac{\varepsilon_1}{4}\right)}\right]^3 \left[\frac{\Gamma\left(-n_1 - \varepsilon_2\right)}{\Gamma\left(1 + n_1' + \varepsilon_2\right)}\right]^2 \\
&\cdot e^{\frac{2\pi\zeta_1 + i\theta_1}{4} n_p} e^{\frac{2\pi\zeta_1 - i\theta_1}{4} n_p'} e^{-(2\pi\zeta_2 + i\theta_2) n_1} e^{-(2\pi\zeta_2 - i\theta_2) n_1'}.
\end{aligned}
\tag{6.191}
$$

We see, in the ϵ_1 contribution only first order poles appear and so this integral can be easily evaluated:

$$
\begin{aligned}
Z_{S^2} = &-\frac{2\pi i}{4(2\pi)^2} \sum_{a,b,n_1,n_1'} \sum_{\delta=1}^{4} \oint \mathrm{d}\varepsilon_2 (-1)^{\delta} \frac{1}{\pi^2} \frac{\sin\pi\left(\frac{\delta}{4} + 2\varepsilon_2\right) \sin^3 \pi\frac{\delta}{4}}{\sin^2 \pi\varepsilon_2} \\
&\cdot \frac{\Gamma\left(a + \frac{\delta}{4} + 2n_1 + 2\varepsilon_2\right) \Gamma\left(b + \frac{\delta}{4} + 2n_1' + 2\varepsilon_2\right)}{\Gamma\left(4a + \delta\right) \Gamma\left(4b + \delta\right) \Gamma\left(1 + n_1 + \varepsilon_2\right)^2} \\
&\cdot \frac{\Gamma\left(a + \frac{\delta}{4}\right)^3 \Gamma\left(b + \frac{\delta}{4}\right)^3}{\Gamma\left(1 + n_1' + \varepsilon_2\right)^2} e^{-(2\pi\zeta_2 + i\theta_2) n_1} e^{-(2\pi\zeta_2 - i\theta_2) n_1'} e^{-4\pi\zeta_2\varepsilon_2} \\
&\cdot e^{\frac{2\pi\zeta_1 + i\theta_1}{4}(4a + \delta - 1)} e^{\frac{2\pi\zeta_1 - i\theta_1}{4}(4b + \delta - 1)},
\end{aligned}
\tag{6.192}
$$

with

$$
n_p + 1 = 4a + \delta, \quad n_p' + 1 = 4b + \delta, \qquad a, b \in \mathbb{Z}_{\geq 0}, \quad \delta = 1, 2, 3, 4. \tag{6.193}
$$

We set $\epsilon_2 = \frac{H}{2\pi i}$ and note that the contribution for $\delta = 4$ vanishes, which leads to the hypothesis that $\delta = 4$ might correspond to a broad sector. After further manipulations the integral can be brought into the following form

$$
\begin{aligned}
Z_{S^2} = &\frac{2\pi i}{4} \sum_{\delta=1}^{3} \int_{\mathbb{P}^1} (-1)^{\delta} \frac{\Gamma\left(\frac{\delta}{4} + \frac{H}{\pi i}\right) \Gamma\left(\frac{\delta}{4}\right)^3 \Gamma\left(1 - \frac{H}{2\pi i}\right)^2}{\Gamma\left(1 - \frac{\delta}{4} - \frac{H}{\pi i}\right) \Gamma\left(1 - \frac{\delta}{4}\right)^3 \Gamma\left(1 + \frac{H}{2\pi i}\right)^2} \\
&\cdot I_{\delta}(\mathsf{t}_1, \mathsf{t}_2, H) I_{\delta}(\bar{\mathsf{t}}_1, \bar{\mathsf{t}}_2, H),
\end{aligned}
\tag{6.194}
$$

with

$$
\begin{aligned}
I_\delta(\mathtt{t}_1,\mathtt{t}_2,H) = {} & \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^2}{\Gamma\left(\frac{\delta}{4}+\frac{H}{\pi i}\right)\Gamma\left(\frac{\delta}{4}\right)^3} e^{-\mathtt{t}_2\frac{H}{2\pi i}} \sum_{a,n\geq 0} e^{\frac{\mathtt{t}_1}{4}(4a+\delta-1)} e^{-\mathtt{t}_2 n} \\
& \cdot \frac{\Gamma\left(a+\frac{\delta}{4}+2n+2\frac{H}{2\pi i}\right)\Gamma\left(a+\frac{\delta}{4}\right)^3}{\Gamma(4a+\delta)\,\Gamma\left(1+n+\frac{H}{2\pi i}\right)^2}
\end{aligned}
\tag{6.195}
$$

The structure of (6.194) suggests the following definitions

$$
\begin{aligned}
\widehat{\Gamma}_\delta(H) &= \Gamma\left(\frac{\delta}{4}+\frac{H}{\pi i}\right)\Gamma\left(\frac{\delta}{4}\right)^3\Gamma\left(1-\frac{H}{2\pi i}\right)^2 \\
\widehat{\Gamma}^*_\delta(H) &= \Gamma\left(1-\frac{\delta}{4}-\frac{H}{\pi i}\right)\Gamma\left(1-\frac{\delta}{4}\right)^3\Gamma\left(1+\frac{H}{2\pi i}\right)^2 .
\end{aligned}
\tag{6.196}
$$

The factor of $(-1)^\delta$ can be identified with $(-1)^{\mathrm{Gr}}$. Again we extract the M matrix:

$$
M = \begin{pmatrix}
\gamma_1(0)\log 2^3 & -\frac{i\pi}{2}\gamma_1(0) & 0 & 0 & 0 & 0 \\
-\frac{i\pi}{2}\gamma_1(0) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma_2(0)\log 2^2 & -\frac{i\pi}{2}\gamma_2(0) & 0 & 0 \\
0 & 0 & -\frac{i\pi}{2}\gamma_2(0) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\gamma_1(0)}\log 2^3 & -\frac{i\pi}{2}\frac{1}{\gamma_1(0)} \\
0 & 0 & 0 & 0 & -\frac{i\pi}{2}\frac{1}{\gamma_1(0)} & 0
\end{pmatrix},
\tag{6.197}
$$

where $\gamma_\delta(0)$ follows from (6.83) with (6.196) inserted. Also in this phase we can test our proposed I-function by application of the Picard-Fuchs system (6.161) onto the I components, but first we need to transform the equation to local coordinates by the transformation

$$
y_1 = z_1^{-\frac{1}{4}}, \qquad y_2 = z_2. \tag{6.198}
$$

The Picard-Fuchs operators now read

$$
\begin{aligned}
\mathcal{L}_1 &= 4(\theta_1-1)(\theta_1-2)(\theta_1-3) - \frac{y_1^4}{64}\theta_1^2(\theta_1+8\theta_2) \\
\mathcal{L}_2 &= \theta_2^2 - \frac{y_2}{16}(\theta_1+8\theta_2)(\theta_1+8\theta_2+4).
\end{aligned}
\tag{6.199}
$$

We identify

$$
e^{-\mathtt{t}_1} = y_1^{-4}, \qquad e^{-\mathtt{t}_2} = y_2, \tag{6.200}
$$

and expand the I-function into a power series in H. The I-function encodes six periods, where the H^0 coefficients provide three power series $\varrho_{0,\delta}$, $\quad \delta = 1,2,3$. The remaining periods are given by the coefficients of H^1: $\varrho_{1,\delta}$, which involve logarithms in y_2. These periods are all annihilated by the Picard-Fuchs system.

A further Hybrid Example

In all the above discussed hybrid models we have a base manifold of the from $\mathbb{P}^1$. We will now consider a model with a hybrid phase, which has a different base manifold. Again we consider a GLSM with gauge group $U(1)^2$ and the following field content

$$\begin{array}{c|cc|ccccc|c} & p & x_6 & x_4 & x_5 & x_1 & x_2 & x_3 & \text{FI} \\ \hline U(1)_1 & -6 & 1 & 2 & 3 & 0 & 0 & 0 & \zeta_1 \\ U(1)_2 & 0 & -3 & 0 & 0 & 1 & 1 & 1 & \zeta_2 \\ \hline U(1)_V & 2-12q_1 & 2q_1-6q_2 & 4q_1 & 6q_1 & 2q_2 & 2q_2 & 2q_2, & \end{array} \tag{6.201}$$

with superpotential $W = pG_{6,0}(x_1,\dots,x_6)$. The hybrid phase is at $\zeta_1 \ll 0, \zeta_2 \gg 0$. By considering the D- and F-term equations we find for the vacuum equations

$$p = \sqrt{-\frac{\zeta_1}{6}}, \qquad |x_1|^2 + |x_2|^2 + |x_3|^2 = \zeta_2 \tag{6.202}$$

and see the appearance of a $\mathbb{P}^2$ vacuum manifold. The gauge group $U(1)_1$ gets broken to $\mathbb{Z}_6$ and $U(1)_2$ is completely broken. The low energy description is given in terms of a $\mathbb{Z}_6$ Landau-Ginzburg orbifold fibred over $\mathbb{P}^2$. We can evaluate the sphere partition function along the lines of the previous example and the contributing poles come from the Z_1 and Z_p contributions. In this phase a consistent choice of the R-charge is given by $q_1 = \frac{1}{6}, q_2 = 0$. Some steps in the process of rewriting the integral as integral over the residues are given in Appendix B.2. We find that one integration variable gives only first order poles and we can easily evaluate that integration. In the end we find the following expression:

$$\begin{aligned} Z_{S^2}^{hyb}(\mathtt{t}_1,\mathtt{t}_2) = \frac{1}{6}\frac{(2\pi i)^4}{(2\pi)^2}\sum_{\delta\in\{1,5\}}\oint \mathrm{d}\frac{H}{2\pi i}\frac{(-1)^{\mathrm{Gr}}}{H^3}\frac{\widehat{\Gamma}_\delta(H)}{\widehat{\Gamma}^*_\delta(H)} \\ \cdot |I_\delta^{HYB}(H,\mathtt{t}_1,\mathtt{t}_2)|^2, \end{aligned} \tag{6.203}$$

with

$$\begin{aligned} I_\delta^{HYB}(H,\mathtt{t}_1,\mathtt{t}_2) = \sum_{a,b\geq 0}(-1)^b \frac{e^{\mathtt{t}_1(a+\frac{\delta-1}{6})}e^{-\mathtt{t}_2(b+\frac{H}{2\pi i})}\Gamma(1+\frac{H}{2\pi i})^3}{\Gamma\left(\langle\frac{\delta}{6}\rangle+3\frac{H}{2\pi i}\right)\Gamma\left(\langle 2\frac{\delta}{6}\rangle\right)\Gamma\left(\langle 3\frac{\delta}{6}\rangle\right)} \\ \cdot\frac{\Gamma\left(\frac{\delta}{6}+a+3b+3\frac{H}{2\pi i}\right)\Gamma\left(\frac{\delta}{3}+2a\right)\Gamma\left(\frac{\delta}{2}+3a\right)}{\Gamma\left(\delta+6a\right)\Gamma\left(1+b+\frac{H}{2\pi i}\right)^3}, \end{aligned} \tag{6.204}$$

and

$$\begin{aligned} &(-1)^{\mathrm{Gr}} = (-1)^{\lfloor\frac{\delta}{6}\rfloor+\lfloor 2\frac{\delta}{6}\rfloor+\lfloor 3\frac{\delta}{6}\rfloor+\delta}, \\ &\widehat{\Gamma}_\delta(H) = \Gamma\left(1-\frac{H}{2\pi i}\right)^3\Gamma\left(\left\langle\frac{\delta}{6}\right\rangle+3\frac{H}{2\pi i}\right)\Gamma\left(\left\langle 2\frac{\delta}{6}\right\rangle\right)\Gamma\left(\left\langle 3\frac{\delta}{6}\right\rangle\right), \\ &\widehat{\Gamma}^*_\delta(H) = \widehat{\Gamma}_{6-\delta}(-H). \end{aligned} \tag{6.205}$$

By a series expansion in H and writing out all δ components we can read off the following M matrix

$$M = \begin{pmatrix} -\frac{3}{4}\gamma_1(0)\left(\nu^2+\eta\right) & -\gamma_1(0)i\pi\nu & \frac{2}{3}\pi^2\gamma_1(0) & 0 & 0 & 0 \\ -\gamma_1(0)i\pi\nu & \frac{2}{3}\pi^2\gamma_1(0) & 0 & 0 & 0 & 0 \\ \frac{2}{3}\pi^2\gamma_1(0) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4}\frac{1}{\gamma_1(0)}\left(\nu^2-\eta\right) & \frac{1}{\gamma_1(0)}i\pi\nu & -\frac{2}{3}\pi^2\frac{1}{\gamma_1(0)} \\ 0 & 0 & 0 & \frac{1}{\gamma_1(0)}i\pi\nu & -\frac{2}{3}\pi^2\frac{1}{\gamma_1(0)} & 0 \\ 0 & 0 & 0 & -\frac{2}{3}\pi^2\frac{1}{\gamma_1(0)} & 0 & 0 \end{pmatrix}, \tag{6.206}$$

with

$$\nu = 4\log 2 + 3\log 3 \qquad\qquad \eta = \psi_1(\frac{1}{6}) - \psi_1(\frac{5}{6}). \tag{6.207}$$

$\psi_1(x)$ is the polygamma function and $\gamma_1(0)$ is given in (6.83), with operators taken from (6.205).

Chapter 7

A Selection of Swampland Conjectures

In this chapter we will review the swampland conjectures which played a key role in the papers [10] and [11]. These papers will be discussed in subsequent chapters. The swampland program is an approach to classify effective field theories. The main idea is to split effective field theories into two groups [120]:

1. Effective theories which look semi-classically consistent but which cannot be embedded into a quantum gravity theory in the UV (the *Swampland*).

2. Effective theories which can be coupled to quantum gravity in the UV (the *Landscape*).

The boundary between these two classes is defined by several swampland criteria. These are properties which an effective field theory must have in order to have a consistent completion in the UV to a quantum gravity theory. Up to now most of the proposed criteria are conjectures and many are motivated from string theory constructions. In the following we will focus on the conjectures important for our discussion. We will follow the review [121] and the lecture notes [122].

7.1 (Refined) Swampland Distance Conjecture

This conjecture was first given in [123] and further refined in [124, 125]. In the following discussion we have in mind a theory coupled to gravity. This theory has a moduli space, which is parameterized by expectation values of some scalar fields. These scalar fields have no potential. The first statement of the conjecture is that given an arbitrary point in the scalar moduli space, there exists another point at an arbitrarily large geodesic distance $\mathcal{D}$. The conjecture further states that by approaching $\mathcal{D} \to \infty$ an infinite tower of states appears with mass-scale

$$m(\varphi) \approx m_i e^{-\lambda \mathcal{D}} \quad \text{for} \quad \mathcal{D} \to \infty, \quad \lambda \sim \mathcal{O}(1), \tag{7.1}$$

where all the quantities are given in Planckian units. $m_i = m(\varphi_i)$ is the initial mass-scale. The assessment $\lambda \sim \mathcal{O}(1)$ is part of the refined distance conjecture, which says that the tower appears for super-Planckian distances $\mathcal{D}$.

7.2 de Sitter Conjectures

In [126] the difficulties in constructing de Sitter-like vacua in string theory motivated the proposal of a conjecture, which places de Sitter-like vacua in the swampland. The conjecture applies to a d-dimensional theory, $d > 2$, of scalar fields φ^i coupled to gravity, of the form

$$\mathcal{S} = \int \mathrm{d}^d x \sqrt{|g_d|} \left(\frac{M_p^2}{2} \mathcal{R}_d - \frac{1}{2} g_{ij} \partial_\mu \varphi^i \partial^\mu \varphi^j - V(\varphi) \right) , \tag{7.2}$$

with M_p the Planck mass, μ the d-dimensional indices, g_{ij} the field space metric and $V(\varphi)$ the scalar potential. In order for (7.2) to be a low energy effective theory of string theory the potential $V(\varphi)$ must obey the following condition [126]:

$$|\nabla V| \geq c\, V \;, \quad c \sim \mathcal{O}(1) \;, \tag{7.3}$$

where $|\nabla V| = \sqrt{g^{ij} \partial_i V \partial_j V}$, where the statement is given in Planckian units. This conjecture excludes de-Sitter extrema as one can see by looking at an extremum of the potential $\nabla V = 0$. In this case (7.2) gives:

$$\mathcal{R}_d = \frac{V}{2d(d-2)}. \tag{7.4}$$

From (7.3) the absence of de Sitter solutions follows. Away from extrema the conjecture restricts the slope of the potential, which is of consequence for cosmological models. Further studies of the implications of the proposal lead to the refinement of the conjecture in [127] (see [11] for further references), because it was argued that the original conjecture was too strong. In [127] it was argued that (7.3) is only valid in asymptotic regions of field space. In [128] the Trans-Planckian Censorship conjecture (TCC) was put forward and used to provide a physical motivation for the refined de Sitter conjecture and a bound on the value c. This was done by using physical arguments regarding quantum fluctuations. The analysis of [128] resulted in the following conditions on the scalar potential:

$$0 < V(\varphi) < V_0\, e^{-c_0\, \mathcal{D}} \quad \Rightarrow \quad \left\langle \frac{|V'|}{V} \right\rangle_{\mathcal{D} \to \infty} \geq c_0 = \frac{2}{\sqrt{(d-1)(d-2)}}. \tag{7.5}$$

V_0, φ_i are constants. $\mathcal{D}(P,Q)$, is the geodesic distance in field space between two arbitrary points P, Q in field space. For a canonically normalised field φ this can be written in the form $\mathcal{D} = |\varphi - \varphi_i|$. The ratio $|V'|/V$ is averaged over an interval in field space and is considered here in the limit of large distances. Here the conjecture is considered for a single field, but it was extended in [128] to the multi-field case. From (7.5) a lower bound for the value of c in (7.3), can be read off and for $d = 4$ it follows:

$$\text{TCC bound:} \qquad c \geq \sqrt{\frac{2}{3}} \approx 0,8165. \tag{7.6}$$

Chapter 8

The Refined Swampland Distance Conjecture in Exotic Calabi Yaus

This chapter is based on the paper [10]. We will use GLSM techniques to test the RSDC (see Section 7.1). This conjecture was tested before in [129, 130]. We will extend the discussion to more exotic Calabi-Yau phases.

The scalar moduli space, mentioned in the definition of the RSDC (see Section 7.1), which can be probed by GLSM techniques is the Kähler moduli space $\mathcal{M}_K$ of the Calabi-Yau. In order to test the RSDC we have to calculate the length of geodesics in $\mathcal{M}_K$. This is a delicate task, because the Kähler moduli space decomposes into different chambers and paths in the moduli space will cross the boundaries between these chambers. This requires to calculate geodesics in different chambers and to match them at the boundary. If the predictions of the RSDC are true, then all these components of the geodesic have to be of order $\mathcal{O}(1)$ in Planckian units.

To start our calculations we need to obtain a metric on $\mathcal{M}_K$. The Kähler moduli space is a Kähler manifold and so the metric can be obtained from a Kähler potential $K(t,\bar{t})$:

$$g_{t\bar{t}} = \partial_t \partial_{\bar{t}} K(t,\bar{t}). \tag{8.1}$$

We will us the sphere partition function to calculate the Kähler potential (see also Chapter 6). Given the metric we can then calculate the distance between two generic points p_1, p_2 in $\mathcal{M}_K$ via

$$\Theta(p_1,p_2) = \int_{\tau_1(p_1)}^{\tau_2(p_2)} d\tau \sqrt{2 g_{t\bar{t}}(\tau) \frac{\partial t}{\partial \tau} \frac{\partial \bar{t}}{\partial \tau}}, \tag{8.2}$$

where τ is an affine parameter. The geodesics can then be obtained by solving the geodesic equations:

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{8.3}$$

Due to quantum corrections we can solve (8.3) only numerically. In our case we consider models with $\dim_{\mathbb{C}} \mathcal{M}_K = 1$. We rewrite (8.3) into a radial and an angular variable $x(\tau) = (r(\tau), \varphi(\tau))$ and the geodesic equations become:

$$\ddot{\varphi} = \frac{1}{2} g^{\varphi\varphi} \partial_\varphi g_{rr} \dot{r}^2 - g^{\varphi\varphi} \partial_r g_{\varphi\varphi} \dot{r} \dot{\varphi} - \frac{1}{2} g^{\varphi\varphi} \partial_\varphi g_{\varphi\varphi} \dot{\varphi}^2, \tag{8.4}$$

$$\ddot{r} = -\frac{1}{2} g^{rr} \partial_r g_{rr} \dot{r}^2 - g^{rr} \partial_\varphi g_{rr} \dot{r} \dot{\varphi} + \frac{1}{2} g^{rr} \partial_r g_{\varphi\varphi} \dot{\varphi}^2. \tag{8.5}$$

We solve the geodesic equations and then evaluate (8.2) on the obtained solution to get the length of the geodesic. We were able to solve the geodesic equations numerically up to order $\mathcal{O}(r^{50})$. In our calculations we start from a finite-distance point in a chamber of $\mathcal{M}_K$ and cross the boundary to approach a point at infinite distance. In the second chamber the exponential drop-off becomes significant after a path of length $\approx \Theta_\lambda$ has been traversed, which signalises the appearance of a tower of light states after that distance. Let Θ_0 be the proper distance from the starting point to the chamber boundary, then we can characterize the full length of the geodesic by [129]

$$\Theta_c = \Theta_0 + \Theta_\lambda = \Theta_0 + \frac{1}{\lambda}. \tag{8.6}$$

By the RSDC this quantity should be of order $\mathcal{O}(1)$. Subsequently we will test this assertion in examples.

Different to [10] we corrected a missing factor of $\sqrt{2}$ in the results. The origin of this factor is explained in Section 9.2). Further we choose a different naming convention for the models of interest compared to [10]. The naming is chosen in accordance with Chapter 5.

8.1 Models with $\dim \mathcal{M}_K = 1$

We have analysed these models in Chapter 5 and will here repeat features which are important for our subsequent analysis. K-type points are at infinite distance in the moduli space and in [56] it was argued that the tower of massless states, as predicted by the SDC, is given in terms of D-brane states. The M-points are at infinite distance and for the one-parameter models the RSDC has been studied in [129] for geodesics from a Landau-Ginzburg point to a M-point. Therefore the focus of our analysis lies on geodesics from C-points and F-points not studied in [129] to M points. We will discuss all three models C-types (see Table 5.1 and Table 8.1). The C3-type was previously studied in [57] from a GLSM standpoint. We also study the F5 model which does not correspond to a standard Landau-Ginzburg model. The last example we will consider is a C-type pseudo hybrid phase in a non-abelian GLSM, which was first given in [131].

label	$\zeta \gg 0$	$h^{1,1}$	$h^{2,1}$	$\mathbf{a}$ at $\zeta \ll 0$
C3	$\mathbb{P}^6[3,2,2]$	1	73	$\left(\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{2}{3}\right)$
C1	$\mathbb{P}^5[4,2]$	1	89	$\left(\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right)$
C2	$\mathbb{P}^5_{1^5 3}[6,2]$	1	129	$\left(\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{5}{6}\right)$

Table 8.1: C-type pseudo-hybrids from $U(1)$ GLSMs.

C-Type Pseudo-Hybrid Examples from abelian GLSMs

We analysed the vacuum structures of these models in Section 5.4. and repeat certain characteristics in Table 8.1 (see also [56]). We focus here on the location of the singularity and the structure of the sphere partition function in these models.

C3-Model

In our calculations of the geodesic we will cross the phase boundary and therefore we need to calculate the location of the singular points at the boundary. These are located at (5.8):

$$e^{-t} = -\frac{1}{432} \quad \Rightarrow \quad \zeta = \frac{1}{2\pi}\log 2^4 3^3, \quad \theta = \pi \mod 2\pi. \tag{8.7}$$

To calculate the metric we first need to calculate the sphere partition function in the respective phase. The general steps for this procedure have been outlined in Section 6.3. We will simply give the results and refer to the original paper [10] and Appendix A for more details. In the $\zeta \gg 0$ phase only the poles in the x-field contributions have to be accounted for and we find

$$Z_{S^2}^{\zeta\gg 0} = -(z\bar{z})^q \operatorname{Res}_{\tau=0}\left((z\bar{z})^\tau \pi^4 \frac{\sin(2\pi\tau)^2 \sin(3\pi\tau)}{\sin(\pi\tau)^7} f[\tau,z]\right), \tag{8.8}$$

where

$$f[\tau,z] = \left|\sum_{a=0}^{\infty}(-z)^a \frac{\Gamma(1+2a+2\tau)^2\,\Gamma(1+3a+3\tau)}{\Gamma(1+a+\tau)}\right|^2, \tag{8.9}$$

and

$$z =^{-2\pi\zeta+i\theta} \equiv e^{-t}. \tag{8.10}$$

In the $\zeta \ll 0$ phase we find that the poles coming from the p_1 and p_2 contribution have to be taken into account. This was also observed in Section 6.3. The sphere partition function is given by

$$Z_{S^2}^{\zeta\ll 0} = Z_{S^2,Z_{p_1}}^{\zeta\ll 0} + Z_{S^2,Z_{p_2}}^{\zeta\ll 0}, \tag{8.11}$$

where

$$\begin{aligned} Z^{\zeta\ll 0}_{S^2,Z_{p_1}} = \sum_{\delta=0}^{2} \pi^{-4}(-1)^{\delta} (z\bar{z})^{q-\frac{1+\delta}{3}} \operatorname{Res}_{\tau=0} \Bigg((z\bar{z})^{\tau} \\ \cdot \frac{\sin\left(\pi\left(\frac{2-\delta}{3}+\tau\right)\right)^7}{\sin(3\pi\tau)\sin\left(\pi\left(\frac{1-2\delta}{3}+2\tau\right)\right)^2} f_1[\tau,z,\delta] \Bigg), \end{aligned} \tag{8.12}$$

with

$$f_1[\tau,z,\delta] = \left| \sum_{a=0}^{\infty} (-z)^{-a} \frac{\Gamma\left(\frac{1+\delta}{3}+a-\tau\right)^7}{\Gamma(1+3a+\delta-3\tau)\,\Gamma\left(\frac{2+2\delta}{3}+2a-2\tau\right)^2} \right|^2 . \tag{8.13}$$

$Z^{\zeta\ll 0}_{S^2,Z_{p_2}}$ is given by

$$\begin{aligned} Z^{r\ll 0}_{S^2,Z_{p_2}} = (z\bar{z})^{q-\frac{1}{2}} \operatorname{Res}_{\tau=0} \Bigg(\pi^{-4} (z\bar{z})^{\tau} \\ \cdot \left(\frac{\sin\left(\pi\left(\frac{1}{2}-\tau\right)\right)^7}{\sin\left(\pi\left(-\frac{1}{2}+3\tau\right)\right)\sin(2\tau\pi)^2} f_2[\tau,z,0] \right) \end{aligned} \tag{8.14}$$

with

$$f_2[\tau,z,\delta] = \left| \sum_{a=0}^{\infty} (-z)^{a} \frac{\Gamma\left(\frac{1+\delta}{2}+a-\tau\right)^7}{\Gamma(1+2a+\delta-2\tau)^2\,\Gamma\left(\frac{3+3\delta}{2}+3a-3\tau\right)} \right|^2 . \tag{8.15}$$

C1-Model

The position of the singular points is at (5.8):

$$e^{-t} = \frac{1}{1024} \quad \to \quad \zeta = \frac{1}{2\pi}\log 2^{10}, \quad \theta = 0 \mod 2\pi. \tag{8.16}$$

Again the evaluation of the sphere partition function follows the steps outlined in the previous chapters and we simply give the results. In the $\zeta \gg 0$ phase we find

$$Z^{r\gg 0}_{S^2} = -(z\bar{z})^q \pi^4 \operatorname{Res}_{\tau=0} \left(\frac{\sin 4\pi\tau \sin 2\pi\tau}{(\sin\pi\tau)^6} (z\bar{z})^{\tau} \left|f[z,\tau]\right|^2 \right), \tag{8.17}$$

with

$$f[z,\tau] = \sum_{n=0}^{\infty} z^n \frac{\Gamma(1+4n+4\tau)\,\Gamma(1+2n+2\tau)}{\Gamma(1+n+\tau)^6}. \tag{8.18}$$

In the phase $\zeta \ll 0$ we find two contributions, which is similar to the C3 behaviour:

$$Z^{\zeta\ll 0}_{S^2} = Z^{\zeta\ll 0}_{S^2,Z_{p_1}} + Z^{\zeta\ll 0}_{S^2,Z_{p_2}}. \tag{8.19}$$

In contrast to the C3-model we need to take care of a possible over-counting of poles. This issue was analysed in detail in the appendix of [10] and Appendix A. The upshot of the analysis is that the poles of the p_2 contribution result in

$$Z^{\zeta\ll 0}_{S^2,Z_{p_2}} = (z\bar{z})^{q-\frac{1}{2}} \operatorname{Res}_{\tau=0}\left(-\pi^{-4}\frac{\left(\sin\left(\left(\frac{1}{2}+\tau\right)\pi\right)\right)^6}{\sin 4\pi\tau \sin 2\pi\tau}(z\bar{z})^{\tau}\left|\tilde{f}_1[z,\tau]\right|^2\right), \quad (8.20)$$

with

$$\tilde{f}_1[z,\tau] = \sum_{a=0}^{\infty} z^{-a}\frac{\Gamma\left(a+\frac{1}{2}-\tau\right)^6}{\Gamma\left(4a+2-4\tau\right)\Gamma\left(2a+1-2\tau\right)}. \quad (8.21)$$

The poles of the p_1 contribution give

$$\begin{aligned} Z^{\zeta\ll 0}_{S^2,Z_{p_1}} = & \sum_{\delta=0,2}(z\bar{z})^{q-\frac{\delta+1}{4}}(-1)^{\delta}\operatorname{Res}_{\tau=0}\Bigg(\pi^{-4}(z\bar{z})^{\tau} \\ & \cdot\frac{\sin\left(\pi\left(\frac{3-\delta}{4}+\tau\right)\right)^6}{\sin 4\pi\tau\sin\left(\pi\left(\frac{1-\delta}{2}+2\tau\right)\right)}\left|\tilde{f}_2[z,\tau,\delta]\right|^2\Bigg), \end{aligned} \quad (8.22)$$

with

$$\tilde{f}_2[z,\tau,\delta] = \sum_{a=0}^{\infty} z^{-a}\frac{\Gamma\left(a+\frac{\delta+1}{4}-\tau\right)^6}{\Gamma\left(1+4a+\delta-4\tau\right)\Gamma\left(2a+\frac{\delta+1}{2}-2\tau\right)}. \quad (8.23)$$

C2-Model

The singular point is at

$$e^{-t} = \frac{1}{6912} \quad \rightarrow \quad \zeta = \frac{1}{2\pi}\log 3^3 2^8, \quad \theta = 0 \mod 2\pi. \quad (8.24)$$

In the $\zeta \gg 0$ phase the sphere partition function evaluates to

$$Z^{\zeta\gg 0}_{S^2} = -(z\bar{z})^{q}\operatorname{Res}_{\tau=0}\left(\pi^4(z\bar{z})^{\tau}\frac{\sin 6\pi\tau\sin 2\pi\tau}{\left(\sin\pi\tau\right)^5\sin 3\pi\tau}\left|f_1[z,\tau]\right|^2\right), \quad (8.25)$$

with

$$f_1[z,\tau] = \sum_{a=0}^{\infty} z^{a}\frac{\Gamma\left(1+6a+6\tau\right)\Gamma\left(1+2a+2\tau\right)}{\Gamma\left(1+a+\tau\right)^5\Gamma\left(1+3a+3\tau\right)}. \quad (8.26)$$

In the $\zeta \ll 0$ phase we see again two contributions:

$$\begin{aligned} Z^{\zeta\ll 0}_{S^2,Z_{p_2}} = & (z\bar{z})^{q-\frac{1}{2}}\operatorname{Res}_{\tau=0}\Bigg(\pi^{-4}(z\bar{z})^{\tau} \\ & \cdot\frac{\left(\sin\left(\pi\left(\frac{1}{2}+\tau\right)\right)\right)^5\sin\left(\pi\left(-\frac{1}{2}+3\tau\right)\right)}{\sin 6\pi\tau\sin 2\pi\tau}\left|\tilde{f}_1[z,\tau]\right|^2\Bigg), \end{aligned} \quad (8.27)$$

label	$\zeta \gg 0$: $\mathbb{P}^5_{w_i}[d_1, d_2]$	$h^{1,1}$	$h^{2,1}$	$\mathbf{a}$ at $\zeta \ll 0$	μ (sing. at $e^{-t} = \mu^{-1}$)
F5	$\mathbb{P}^5_{1^5 2}[4,3]$	1	79	$\left(\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}\right)$	$-2^6 3^3$
F6	$\mathbb{P}^5_{1^3 2^2 3}[6,4]$	1	79	$\left(\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}\right)$	$2^{10} 3^3$
F7	$\mathbb{P}^5_{1^4 4,6}[12,2]$	1	243	$\left(\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\right)$	$2^8 3^3$

Table 8.2: F-type pseudo-hybrids from $U(1)$ GLSMs.

with

$$\tilde{f}_1[z,\tau] = \sum_{a=0}^{\infty} z^{-a} \frac{\Gamma\left(\frac{1}{2}+a-\tau\right)^5 \Gamma\left(\frac{3}{2}+3a-3\tau\right)}{\Gamma\left(3+6a-6\tau\right)\Gamma\left(1+2a-2\tau\right)}, \tag{8.28}$$

and

$$\begin{aligned} Z^{\zeta\ll 0}_{S^2,Z_{p_1}} = & \sum_{\delta\in\{0,1,3,4\}} (z\bar{z})^{q-\frac{1+\delta}{6}} (-1)^{\delta} \operatorname{Res}_{\tau=0} \Bigg(\pi^{-4} (z\bar{z})^{\tau} \\ & \cdot \frac{\left(\sin\left(\pi\left(\frac{5-\delta}{6}+\tau\right)\right)\right)^5 \sin\left(\pi\left(\frac{1-\delta}{2}+3\tau\right)\right)}{\sin 6\pi\tau \sin\left(\pi\left(\frac{1+\delta}{3}-2\tau\right)\right)} \left|\tilde{f}_2[z,\tau,\delta]\right|^2 \Bigg), \end{aligned} \tag{8.29}$$

with

$$\tilde{f}_2[z,\tau,\delta] = \sum_{a=0}^{\infty} z^{-a} \frac{\Gamma\left(\frac{\delta+1}{6}+a-\tau\right)^5 \Gamma\left(\frac{1+\delta}{2}+3a-3\tau\right)}{\Gamma\left(\delta+1+6a-6\tau\right)\Gamma\left(\frac{1+\delta}{3}+2a-2\tau\right)}. \tag{8.30}$$

A pole cancelation takes place for $\delta \in \{0, 4\}$.

F-type Examples

We studied the F-type pseudo hybrids models previously in Section 5.4. The relevant data for our analysis of these models is given in Table 8.2.

The position of the singular points can be read off from Table 8.2. Due to the similar nature of the pseudo hybrid F-type models and the C-type models we only compute the sphere partition function for the F5 model.

In the $\zeta \ll 0$ phase the sphere partition function reads

$$Z^{\zeta\gg 0}_{S^2} = -(z\bar{z})^q \operatorname{Res}_{\tau=0} \left((z\bar{z})^{\tau} \pi^4 \frac{\sin\left(4\pi\tau\right)\sin\left(3\pi\tau\right)}{\sin\left(\pi\tau\right)^5 \sin\left(2\pi\tau\right)} \left|f[z,\tau,0]\right|^2 \right), \tag{8.31}$$

with

$$f[z,\tau] = \sum_{a=0}^{\infty} (-1)^a z^a \frac{\Gamma\left(4a+4\tau+1\right)\Gamma\left(3a+3\tau+1\right)}{\Gamma\left(a+\tau+1\right)^5 \Gamma\left(2a+2\tau+1\right)}. \tag{8.32}$$

In the $\zeta \ll 0$ phase we have again two contributions, but only first order poles occur. This is in contrast to the C-type models. We find

$$Z^{\zeta\ll 0}_{S^2,Z_{p_1}} = \frac{(z\bar{z})^{q-\frac{3}{4}} \left(\left|f_1[z,0,0]\right|^2 \sqrt{z\bar{z}} - \left|f_1[z,0,2]\right|^2 \right)}{16\pi^5}, \tag{8.33}$$

with

$$f_1[z,0,0] = \sum_{a=0}^{\infty}(-z)^{-a}\frac{\Gamma\left(a+\frac{1}{4}\right)^5\Gamma\left(2a+\frac{1}{2}\right)}{\Gamma\left(4a+1\right)\Gamma\left(3a+\frac{3}{4}\right)}. \tag{8.34}$$

$$f_1[z,0,2] = \sum_{a=0}^{\infty}(-z)^{-a}\frac{\Gamma\left(a+\frac{3}{4}\right)^5\Gamma\left(2a+\frac{3}{2}\right)}{\Gamma\left(4a+3\right)\Gamma\left(3a+\frac{9}{4}\right)}, \tag{8.35}$$

and

$$Z_{S^2,Z_{p_2}}^{\zeta\ll 0} = -\frac{3\sqrt{3}\,(z\bar{z})^{q-\frac{2}{3}}\left(|f_2[z,0,0]|^2\sqrt[3]{z\bar{z}} - |f_2[z,0,1]|^2\right)}{32\pi^5}, \tag{8.36}$$

with

$$f_2[z,0,0] = \sum_{a=0}^{\infty}(-z)^{-a}\frac{\Gamma\left(a+\frac{1}{3}\right)^5\Gamma\left(2a+\frac{2}{3}\right)}{\Gamma\left(4a+\frac{4}{3}\right)\Gamma\left(3a+1\right)}, \tag{8.37}$$

$$f_2[z,0,1] = \sum_{a=0}^{\infty}(-z)^{-a}\frac{\Gamma\left(a+\frac{2}{3}\right)^5\Gamma\left(2a+\frac{4}{3}\right)}{\Gamma\left(4a+\frac{8}{3}\right)\Gamma\left(3a+2\right)}. \tag{8.38}$$

For more details on the calculation and a discussion about the pole cancelation see [10] and Appendix A.

8.2 Non-Abelian Example with a Pseudo-Hybrid Phase

Due to the abelian nature of the previous GLSMs, these models can be realized in terms of toric geometry. Nevertheless the applied methods can also be used in non-abelian GLSMs and C-type pseudo-hybrid phases also occur for non-abelian models. A first example of a one-parameter model with a pseudo-hybrid phase was given in [131]. Additional examples with $\dim\mathcal{M}_K = 1$ where found in [132].

The non-abelian nature of the gauge group makes an analysis challenging. Such models have in general a strongly coupled phase, where in the low-energy effective theory a continuous subgroup of the GLSM gauge group is unbroken. It is possible to calculate the sphere partition function for such phases, but the result is not absolutely convergent and resummation techniques have to be applied. This restricts practical calculations to the lowest orders in the expansion. Also another difficulty is related to the fact that all known one-parameter non-abelian GLSMs have more than one singularity at the phase boundary. It is not clear if one should choose the metric for $\zeta \gg 0$ or $\zeta \ll 0$ between the singular points. Both show poor convergence properties there. In addition the expression for the sphere partition function is more complicated which hampers the numerical calculations. A higher efficiency in the calculations can be obtained by a good choice of coordinates on $\mathcal{M}_K$. In [129] it was demonstrated that for abelian examples a good choice is a coordinate ψ, with the property that the singular point is at $\psi = 1$. In the case of multiple singular points at the phase boundary there is no obvious choice. The difficulties outlined above, restricted our analysis to a specific region in the moduli space and we could not analyse geodesics crossing the phase boundaries. We focused on the weakly coupled pseudo-hybrid phase, where we were able to perform explicit calculations.

We studied a GLSM with $U(2)$ gauge group which has been discussed in [131]. The field content reads

$$\begin{array}{c|cc|cc} \phi & p^1,\dots,p^5 & p^6,p^7 & x_1,x_2 & x_3,\dots,x_5 \\ \hline U(2) & \det^{-1} & \det^{-2} & \det\otimes\Box & \Box \\ R & 4q & 8q & 1-6q & 1-2q \end{array}, \tag{8.39}$$

where $\Box$ refers to the fundamental representation and det is the determinantal representation of $U(2)$. The superpotential is

$$W = \sum_{i,j=1}^{5} A^{ij}(p)[x_i x_j], \tag{8.40}$$

where $[x_i x_j] = \varepsilon_{ab} x_i^a x_j^b$ $(a,b = 1,2)$. The structure of the antisymmetric 5×5 matrix $A(p)$ is restricted by gauge invariance to the following form: The first 2×2 block is cubic in $p^1,\dots,p^5$ and bilinear in $(p^{1,\dots,5}, p^{6,7})$ and so transforms in the $\det^{-3}$ representation. The lower 3×3 block has to be linear in $p^{1,\dots,5}$ and transforms in the $\det^{-1}$ representation. The off-diagonal entries transfrom by $\det^{-2}$ and must therefore be quadratic in $p^{1,\dots,5}$ and linear in $p^{6,7}$.

In the phase $\zeta \ll 0$ the limiting point is of M-type. This is a strongly coupled phase and a $SU(2)$ subgroup of the gauge group remains unbroken. The geometry of the vacuum manifold is a smooth Pfaffian Calabi-Yau in weighted $\mathbb{P}^7$ [133]. The Calabi-Yau is characterised by the $\operatorname{rank} A(p) = 2$ locus. For $\zeta \gg 0$ we encounter a pseudo-hybrid phase of type C. The limiting point is at finite distance in the moduli space. The model has two singular points at

$$e^{-t_\pm} = (540 \pm 312\sqrt{3}). \tag{8.41}$$

These singularities are not at the same theta angle: $\theta_+ = 0$ and $\theta_- = \pi$ $(\mathrm{mod}\, 2\pi)$. In our discussion we calculate the length Θ_0 of geodesics starting at the pseudo-hybrid point and ending at the ζ-value of the nearest singular point. In [131] the sphere partition function of this model has already been computed. For our analysis we just need the sphere partition function in the phase $\zeta \gg 0$. We can bring the sphere partition function into the following form

$$
\begin{aligned}
Z_{S^2} = -\frac{1}{8\pi} \int_{\gamma+i\mathbb{R}^2} & \mathrm{d}\tau_1 \wedge \mathrm{d}\tau_2, (Z_1)^5 (Z_2)^2 (Z_3)^2 (Z_4)^2 \\
& \cdot (Z_5)^3 (Z_6)^3 Z_G Z_{\text{classical}}
\end{aligned}
\tag{8.42}
$$

with $\tau_i = -q - i\sigma_i$ $(i = 1, 2)$ and $\gamma = -q(1,1)^T$. The different contributions read

$$
Z_1 = Z_{p^{1,\ldots,5}} = \frac{\Gamma\left(-\tau_1 - \tau_2 + \frac{1}{2}(m_1 + m_2)\right)}{\Gamma(1 + \tau_1 + \tau_2 + \frac{1}{2}(m_1 + m_2))}, \tag{8.43}
$$

$$
Z_2 = Z_{p^{6,7}} = \frac{\Gamma\left(-2\tau_1 - 2\tau_2 + (m_1 + m_2)\right)}{\Gamma(1 + 2\tau_1 + 2\tau_2 + (m_1 + m_2))}, \tag{8.44}
$$

$$
\begin{aligned}
Z_3 Z_4 = Z^1_{x_{1,2}} Z^2_{x_{1,2}} = & \frac{\Gamma\left(\frac{1}{2} + 2\tau_1 + \tau_2 - \frac{1}{2}(2m_1 + m_2)\right)}{\Gamma(\frac{1}{2} - 2\tau_1 - \tau_2 - \frac{1}{2}(2m_1 + m_2))} \\
& \cdot \frac{\Gamma\left(\frac{1}{2} + \tau_1 + 2\tau_2 - \frac{1}{2}(m_1 + 2m_2)\right)}{\Gamma(\frac{1}{2} - \tau_1 - 2\tau_2 - \frac{1}{2}(m_1 + 2m_2))},
\end{aligned}
\tag{8.45}
$$

$$
Z_5 Z_6 = Z^1_{x_{3,4}} Z^2_{x_{3,4}} = \frac{\Gamma\left(\frac{1}{2} + \tau_1 - \frac{1}{2}m_1\right)}{\Gamma\left(\frac{1}{2} - \tau_1 - \frac{1}{2}m_1\right)} \frac{\Gamma\left(\frac{1}{2} + \tau_2 - \frac{1}{2}m_2\right)}{\Gamma\left(\frac{1}{2} - \tau_2 - \frac{1}{2}m_2\right)}, \tag{8.46}
$$

$$
Z_G = (-1)^{m_1 - m_2} \left(\frac{1}{4}(m_1 - m_2)^2 - (\tau_1 - \tau_2)^2\right), \tag{8.47}
$$

$$
Z_{\text{Classical}} = e^{8\pi\zeta q} e^{4\pi r(\tau_1+\tau_2) - i\theta(m_1+m_2)}. \tag{8.48}
$$

The evaluation of the sphere partition function in the pseudo-hybrid phase results in

$$
Z^{\zeta \gg 0}_{S^2} = Z^{\zeta \gg 0}_{S^2,1} + 2 Z^{\zeta \gg 0}_{S^2,2}, \tag{8.49}
$$

with

$$
\begin{aligned}
Z^{\zeta \gg 0}_{S^2,1} = - \frac{9\sqrt{3}\,(z\bar{z})^{\frac{1}{3} - 2q}}{256\pi^7} & \left(\sqrt[3]{z\bar{z}}\,\partial^2_{x_2} \left|f_1[z, x_1, x_2, 1]\right|^2 |_{(0,0)} \right. \\
& \left. -\partial^2_{x_2} \left|f_1[z, x_1, x_2, 0]\right|^2 |_{(0,0)}\right),
\end{aligned}
\tag{8.50}
$$

and

$$
\begin{aligned}
f_1[z, x_1, x_2, \delta] = & \sum_{l=0}^{\infty} (-z)^l \sum_{b=0}^{3l+\delta} (-1)^b \left(-2b + \delta + 3l - x_1 + x_2\right) \\
& \cdot \frac{\Gamma\left(l + \frac{\delta+1}{3} - x_1 - x_2\right)^5 \Gamma\left(2l + \frac{2}{3}(\delta + 1) - 2x_1 - 2x_2\right)^2}{\Gamma\left(-b + 3l + \delta - 2x_1 - x_2 + 1\right)^2 \Gamma\left(b - x_1 - 2x_2 + 1\right)^2} \\
& \cdot \frac{\Gamma\left(-b + l + \frac{\delta}{3} + x_2 + \frac{1}{3}\right)^3}{\Gamma\left(-b + 2l + \frac{2+2\delta}{3} - x_1\right)^3}.
\end{aligned}
\tag{8.51}
$$

$Z_{S^2,2}^{\zeta\gg 0}$ is given by

$$Z_{S^2,2}^{\zeta\gg 0} = \frac{1}{8}\left(z\bar{z}\right)^{\frac{1}{2}-2q}\Big(3\left|f_2[z,0,0]\right|^2\log\left(z\bar{z}\right) \\ -4\partial_{x_2}\left|f_2[z,x_1,x_2]\right|^2|_{(0,0)} + \partial_{x_1}\left|f_2[z,x_1,x_2]\right|^2|_{(0,0)}\Big)\,, \tag{8.52}$$

where

$$f_2[z,x_1,x_2] = \sum_{a=0}^{\infty} z^a \sum_{b=0}^{2a}\left(-2b+a-x_1+x_2-\frac{1}{2}\right) \\ \cdot\frac{\Gamma\left(a-x_1-x_2+\frac{1}{2}\right)^5}{\Gamma\left(-b+2a-2x_1-x_2+1\right)^2\Gamma\left(b+a-x_1-2x_2+\frac{3}{2}\right)^2} \\ \cdot\frac{\Gamma\left(2a-2x_1-2x_2+1\right)^2}{\Gamma\left(-b+a-x_1+\frac{1}{2}\right)^3\Gamma\left(b-x_2+1\right)^3}, \tag{8.53}$$

The main challenge in computing the sphere partition function is the appearance of multi-dimensional residue. Procedures for the calculation of such integrals have been given in e.g. [134, 135, 118, 136]. Further details on the calculation can be found in the appendix of [10]. In Appendix B we discuss multidimensional residues for an abelian model. Extraction of the leading behaviour of the sphere partition function results in [131]:

$$Z_{S^2}^{\zeta\gg 0} = \frac{\Gamma\left(\frac{1}{3}\right)^{10}\left(z\bar{z}\right)^{-2q+\frac{1}{3}}}{2\sqrt{3}\pi\Gamma\left(\frac{2}{3}\right)^8} - \left(z\bar{z}\right)^{-2q+\frac{1}{2}}\left(-3\log\left(z\bar{z}\right)\right. \\ \left.+36+8\log(4)\right)+\dots \tag{8.54}$$

The displayed behaviour is expected for a pseudo-hybrid phase.

8.3 Testing the Refined Swampland Distance Conjecture

We apply the results from Section 8.1 to calculate the length of geodesics.

C-Type Pseudo-Hybrid Examples from abelian GLSMs

C3

The sphere partition function of this model was discussed in Section 8.1 and for the numerical calculations we make the following variable transformation

$$z = -\frac{1}{2^4 3^3 \psi^7}. \tag{8.55}$$

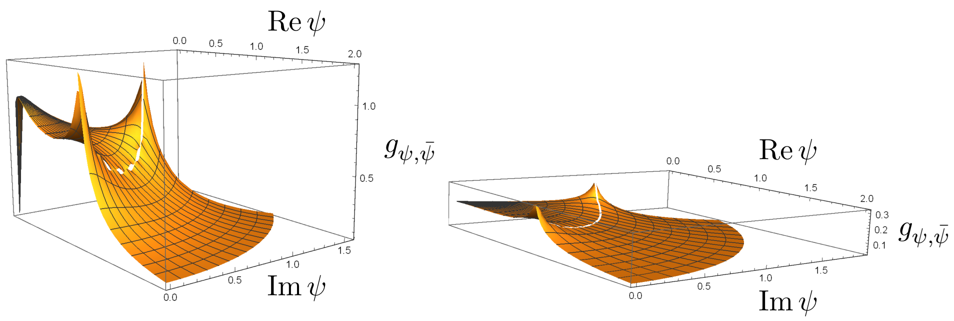

Figure 8.1: Metrics for C3 (left) and the quintic (right).

In the next step we switch to polar coordinates:

$$\begin{aligned} \psi = re^{i\varphi} &= -\frac{1}{(2^4 3^3)^{1/7}} z^{-1/7} \qquad \Leftrightarrow \\ r &= \frac{1}{(2^4 3^3)^{1/7}} e^{\frac{2\pi\zeta}{7}}, \varphi = -\frac{\theta+\pi}{7}. \end{aligned} \tag{8.56}$$

Note that we have chosen a parameterization of $\mathcal{M}_K$ in terms of the *classical* Kähler parameter $z = e^{-t}$ with $t = 2\pi\zeta - i\theta$. This parameterization is the natural one from a GLSM perspective. The coordinate transformation, applied above, places the pseudo-hybrid point at $\psi = 0$. The singular points at the phase boundary are at $(r, \varphi) = (1, 0 \mod \frac{2\pi}{7})$. Near the limiting points of the phases the metric displays the following leading behaviour:

$$\begin{aligned} g_{\psi\bar{\psi}}^{\zeta \ll 0} &= -\frac{2^8 7^3 \sqrt{3}\pi^7}{3^3 \Gamma\left(\frac{1}{6}\right)^4 \Gamma\left(\frac{1}{3}\right)^{10}} r^{1/3} \log(r) + \ldots, \\ g_{\psi\bar{\psi}}^{\zeta \gg 0} &= \frac{3}{4r^2 \log(r)^2} + \ldots. \end{aligned} \tag{8.57}$$

A plot of the metric is given in Figure 8.1 in which we additionally displayed the metric of the quintic.

In both models the limiting point in the $\zeta \ll 0$ phase is at finite distance. However the logarithm leads to a different behaviour near the limiting point and resembles the behaviour near the large volume point. Further approaching the pseudo-hybrid point the polynomial behaviour wins over the logarithmic dependence and renders the distance finite. This behaviour is best visible if we plot the metric for fixed φ values as in Figure 8.2. Near the phase boundary we only find a divergent behaviour if we directly approach the singularity (solid line).

In Figure 8.2 we also see at which r value the φ dependence sets in.

Our goal is to calculate geodesics which start near the pseudo-hybrid point, cross the phase boundary and approach the limiting point in the large volume

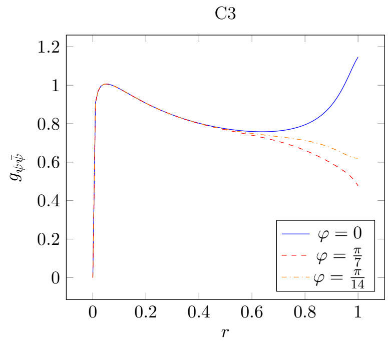

Figure 8.2: Metric for constant φ-values in the pseudo-hybrid phase of C3.

phase. In the following we will denote the distance in the $\zeta \ll 0$ phase by Θ_0. Θ_0 is given by the geodesic distance to the phase boundary. To approximate the distance behaviour in the $\zeta \gg 0$ phase we follow the approach given in [129]. The idea is to look at the leading behaviour of the metric in the $\zeta \gg 0$ phase and calculate the distance Θ for a path with fixed φ. This results in

$$\Theta \approx \frac{1}{\lambda} \log(|\log(|z|)|) + \frac{\alpha_1}{\log(|z|)^3} + \alpha_0. \tag{8.58}$$

If we change the coordinate from z to ψ we find:

$$\Theta \approx \frac{1}{\lambda} \log(\frac{1}{2\pi} \log(2^4 3^3 r^7)) + \frac{\alpha_1}{\left(\frac{1}{2\pi} \log(2^4 3^3 r^7)\right)^3} + \alpha_0. \tag{8.59}$$

In the next step we calculate geodesics and their lengths for different starting values of φ. Due to numerical issues we cannot directly start from $r = 0$ and therefore start from $r = 10^{-6}$. We approximate the additional distance by a straight line and find that it is negligible. In order to extract the quantities introduced in (8.6) we fit the leading behaviour of the distance (8.59) to our obtained results. The results are given in Table 8.3. Due to the symmetry of the metric around $\varphi = \frac{\pi}{7}$ (see Figure 8.1) we calculate geodesics for the range $0 \leq \varphi \leq \frac{\pi}{7}$ (In principle φ can take values up to $\frac{2\pi}{7}$). We want to mention that for small φ values the geodesics are rather short in the large volume phase. Therefore they are not very useful for testing the conjecture. Nevertheless they do not display any behaviour which would justify an exclusion for the study in the small radius regime, which is the focus of our studies, and therefore we keep them. Let us display the mean values for the fitted parameters

$$\Theta_0 \approx 1,2639, \qquad \lambda^{-1} \approx 1,3587, \qquad \Theta_c \approx 2,6226. \tag{8.60}$$

The results agree with the RSDC. If we look at the results for the quintic given in [129], we see that our results for Θ_0 are a factor 2 larger.

$\varphi(0)\frac{70}{\pi}$	α_0	α_1	λ^{-1}	Θ_0	Θ_c
1	$1,4408$	$-0,0934$	$0,9827$	$1,3029$	$2,2855$
2	$1,2365$	$0,1037$	$1,6631$	$1,2947$	$2,9578$
3	$1,1884$	$0,1346$	$1,6232$	$1,2823$	$2,9055$
4	$1,1855$	$0,1198$	$1,4062$	$1,2703$	$2,6765$
5	$1,17$	$0,127$	$1,4304$	$1,2621$	$2,6925$
6	$1,1564$	$0,1322$	$1,3992$	$1,2547$	$2,654$
7	$1,1534$	$0,1259$	$1,3055$	$1,2476$	$2,5531$
8	$1,1578$	$0,1179$	$1,2928$	$1,2439$	$2,5366$
9	$1,1549$	$0,1175$	$1,266$	$1,2412$	$2,5072$
10	$1,1574$	$0,1119$	$1,2182$	$1,2394$	$2,4577$

Table 8.3: Length parameters for C3.

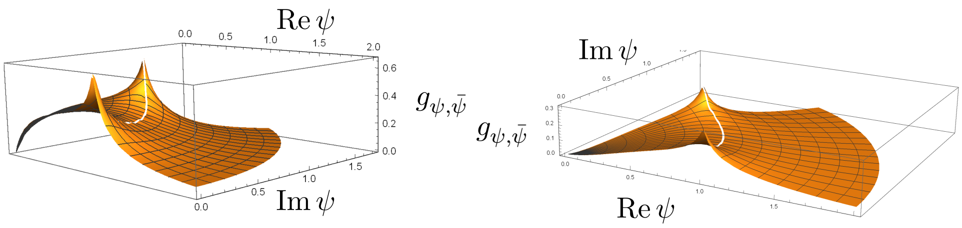

Figure 8.3: Metrics for C1 (left) and C2 (right).

C1

The results of Section 8.1 imply

$$z = \frac{1}{1024\psi^6}, \tag{8.61}$$

and in angular coordinates

$$\psi = re^{i\varphi} = \frac{1}{(2^{10})^{1/6}} z^{-1/6} \quad \Leftrightarrow \quad r = \frac{1}{(2^{10})^{1/6}} e^{\frac{2\pi\zeta}{6}}, \varphi = -\frac{\theta}{6}. \tag{8.62}$$

The singularity is at $(r, \varphi) = (1, 0 \mod \frac{2\pi}{6})$. We expand the metric near the limiting points and find

$$g_{\psi\bar{\psi}}^{\zeta \ll 0} = -\frac{2^7 3^3 \pi^6}{\Gamma\left(\frac{1}{4}\right)^{12}} r\log(r) + \ldots, \qquad g_{\psi\bar{\psi}}^{\zeta \gg 0} = \frac{3}{4r^2\log(r)^2} + \ldots. \tag{8.63}$$

The metric is visible in Figure 8.3.

In Figure 8.4 we can read off at which value of r the angle dependency sets in.

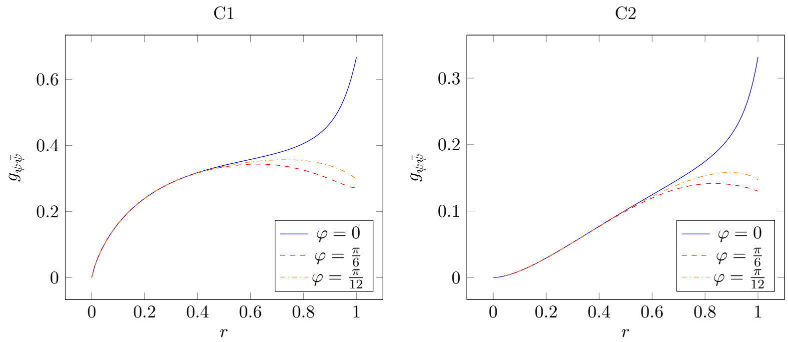

Figure 8.4: Metric for constant φ values in the pseudo-hybrid phase for C1 (left) and C2 (right).

$\varphi(0)\frac{60}{\pi}$	α_0	α_1	λ^{-1}	Θ_0	Θ_c
1	0,7749	−0,1167	1,0363	0,7898	1,8261
2	0,4833	0,1831	1,6336	0,7808	2,4144
3	0,5242	0,1486	1,3767	0,7708	2,1476
4	0,4866	0,1832	1,4202	0,7633	2,1836
5	0,4704	0,1975	1,4027	0,7561	2,1588
6	0,4774	0,1915	1,3164	0,7498	2,0662
7	0,4697	0,1946	1,325	0,7454	2,0704
8	0,4795	0,185	1,262	0,7417	2,0037
9	0,4706	0,1933	1,2729	0,7397	2,0126
10	0,4699	0,1937	1,2749	0,7393	2,0142

Table 8.4: Length parameters for C1.

In the large volume phase the asymptotic behaviour is given by (8.58) and adapted to our coordinates it reads:

$$\Theta \approx \frac{1}{\lambda}\log(\frac{1}{2\pi}\log(2^{10}r^6)) + \frac{\alpha_1}{\left(\frac{1}{2\pi}\log(2^{10}r^6)\right)^3} + \alpha_0. \tag{8.64}$$

The results of the fitting procedure are given in Table 8.4.

In this model we again used the symmetry of the metric to focus on the region $\varphi \leq \frac{\pi}{6}$. For the same reasons as in the C3 case we started from $r = 10^{-6}$. The mean values of the fitted parameters are

$$\Theta_0 \approx 0,7577, \qquad \lambda^{-1} \approx 1,3321, \qquad \Theta_c \approx 2,0898. \tag{8.65}$$

We see that they are in the bounds of the RSDC.

$\varphi(0)\frac{60}{\pi}$	α_0	α_1	λ^{-1}	Θ_0	Θ_c
1	$0,1596$	$-0,1831$	$0,9593$	$0,4216$	$1,3808$
2	$-0,2426$	$0,401$	$1,5029$	$0,4152$	$1,9181$
3	$-0,146$	$0,323$	$1,2836$	$0,409$	$1,6926$
4	$-0,2228$	$0,4285$	$1,3833$	$0,4042$	$1,7875$
5	$-0,2102$	$0,4345$	$1,3251$	$0,3993$	$1,7244$
6	$-0,2049$	$0,4402$	$1,2931$	$0,3956$	$1,6886$
7	$-0,2164$	$0,4567$	$1,3003$	$0,3926$	$1,6929$
8	$-0,2071$	$0,4659$	$1,259$	$0,3903$	$1,6493$
9	$-0,2239$	$0,4822$	$1,2876$	$0,3891$	$1,6768$
10	$-0,2069$	$0,4687$	$1,2506$	$0,3885$	$1,6391$

Table 8.5: Length parameters for C2.

C2

From Section 8.1 we read off the position of the singularity and introduce

$$z = \frac{1}{6912\psi^6} \tag{8.66}$$

and

$$\psi = re^{i\varphi} = \frac{1}{(2^8 3^3)^{1/6}} z^{-1/6} \quad \Leftrightarrow \quad r = \frac{1}{(2^8 3^3)^{1/6}} e^{\frac{2\pi\zeta}{6}}, \varphi = -\frac{\theta}{6}. \tag{8.67}$$

This parameterization places the singularity at $(r, \varphi) = (1, 0 \mod \frac{2\pi}{6})$. The leading behaviour of the metric near the limiting points reads

$$g_{\psi\bar{\psi}}^{\zeta \ll 0} = -\frac{2^5 3^5 \sqrt{3}\pi^3 \Gamma\left(\frac{5}{3}\right)^2}{\Gamma\left(\frac{1}{6}\right)^8} r^2 \log(r) + \ldots, \quad g_{\psi\bar{\psi}}^{\zeta \gg 0} = \frac{3}{4r^2 \log(r)^2} + \ldots. \tag{8.68}$$

We plotted the metric in Figure 8.3. In the $\zeta \ll 0$ phase we calculate the length of the geodesic numerically and in the $\zeta \gg 0$ phase we fit the result against the asymptotic behaviour. The starting point is again at $r = 10^{-6}$ and in the chosen parameterization the asymptotic behaviour reads

$$\Theta \approx \frac{1}{\lambda} \log(\frac{1}{2\pi} \log(2^8 3^3 r^6)) + \frac{\alpha_1}{\left(\frac{1}{2\pi} \log(2^8 3^3 r^6)\right)^3} + \alpha_0. \tag{8.69}$$

The mean values of the fitted parameters are

$$\Theta_0 \approx 0,4005 \qquad \lambda^{-1} \approx 1,2845 \qquad \Theta_c \approx 1,685, \tag{8.70}$$

and the single results are given in Table 8.5. We see that the results are in agreement with the RSDC.

Comparing C3, C1 and C2

We see from the results for Θ_0 (8.60),(8.65), (8.70), that the values differ. The variations of the results can be explained by studying the asymptotic expansion of the metric in these phases[1]: (8.57), (8.63), (8.68).

In all three models the logarithmic behaviour is suppressed by the polynomial contribution if we approach $r = 0$. In the C3 model we only have a suppression by a factor of $r^{1/3}$, which is weaker then in the C1 and C2 model and explains why we obtain the longest distance Θ_0 in the C3 model. The dominance of the logarithmic behaviour in the C3 model in comparison to the C1 and C2 model is visible in the Figures 8.3 and 8.1. Figures 8.2 and 8.4, in which the metric was plotted for various φ values, give an even better impression of this behaviour.

Let us also compare our results to the results for the one parameter hypersurface examples in [129]. As the asymptotic behaviour of the metrics in Landau-Ginzburg phases is different to the pseudo-hybrid phases we find bigger values for Θ_0 in our models. The biggest deviation is seen in the C3 model, but this is expected as we discussed in the previous paragraph. We see that the specifics of the phase influence the value of Θ_0. This could be used to make more precise statements about the order one parameter in (7.1).

F-Type Example

Now we study geodesic in the model introduced in Section 8.1. We chose the following parameterization

$$z = -\frac{1}{2^6 3^3 \psi^6}, \tag{8.71}$$

and so

$$r = \frac{1}{(2^6 3^3)^{1/6}} e^{\frac{2\pi\zeta}{6}}, \qquad \varphi = -\frac{\theta + \pi}{6}. \tag{8.72}$$

The leading order behaviour of the metric in the $\zeta \ll 0$ phase is given by

$$g_{\psi\bar{\psi}}^{\zeta\ll 0} = \frac{3^3 \pi \Gamma\left(\frac{1}{3}\right)^6 \Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^{10}} \frac{1}{r} + \dots, \tag{8.73}$$

and although the metric is singular at $r = 0$ the distance remains finite (this can be seen by integrating $\sqrt{g_{\psi\bar{\psi}}^{\zeta\ll 0}}$ using (8.2). A plot of the metric is given in Figure 8.5, where we display the metric for certain fixed φ values in Figure 8.6.

The leading behaviour at the limiting point in the $\zeta \gg 0$ phase is similar to the previous models.

[1] The leading behaviour of the metric has also been studied in [56]. We match our results to the results of [56] in the appendix of [10]

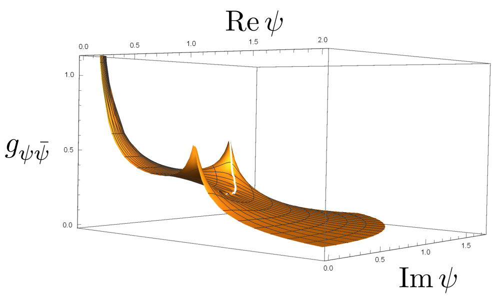

Figure 8.5: Metric plot of F5.

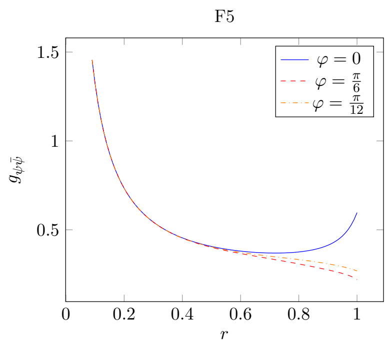

Figure 8.6: Metric for constant φ-values in the pseudo-hybrid phase of F5.

The symmetry of the metric in φ around $\varphi = \frac{\pi}{6}$ allows us to restrict the analysis of geodesic to $0 \leq \varphi \leq \frac{\pi}{6}$. We start from $r = 10^{-6}$. The reason for this starting point is the same as in the previous models, but in contrast to the cases above we add a distance of $7 \cdot 10^{-5}$ to the result for[2] Θ_0.

The results for Θ_0 and the fitted parameters of the asymptotic behaviour in the large volume phase are given in Table 8.6. The mean values are

$$\Theta_0 \approx 1,1719, \quad \Theta_\lambda \approx 1,3072, \quad \Theta_c \approx 2,479. \tag{8.74}$$

Let us compare these results to the models studied in the previous section. In the F-type model we see that the Θ_0 distance is larger as compared to the C1 and C2 model, but the longest distance is still found in the C3 model. If we look at Figure 8.7 we can gain an intuition why this behaviour is the case. The F-type model metric diverges at $r = 0$, but soon drops below the value of the C3 model.
We finally remark that the results are in accordance with the RSDC.

[2] This distance can be calculated by using (8.2) and considering a path from $r = 0$ to the starting point, with fixed φ value.

$\varphi(0)\frac{60}{\pi}$	α_0	α_1	λ^{-1}	Θ_0	Θ_c
1	$1,1082$	$-0,1444$	$1,0416$	$1,1999$	$2,2415$
2	$0,8185$	$0,1879$	$1,5171$	$1,1911$	$2,7082$
3	$0,8683$	$0,1486$	$1,3166$	$1,1831$	$2,4997$
4	$0,8017$	$0,2175$	$1,428$	$1,1769$	$2,6049$
5	$0,8139$	$0,2126$	$1,3342$	$1,1702$	$2,5044$
6	$0,8041$	$0,2216$	$1,3353$	$1,1657$	$2,501$
7	$0,7973$	$0,2301$	$1,3199$	$1,1614$	$2,4813$
8	$0,8072$	$0,2246$	$1,2661$	$1,1583$	$2,4244$
9	$0,7921$	$0,236$	$1,3048$	$1,1567$	$2,4615$
10	$0,8186$	$0,2172$	$1,208$	$1,1552$	$2,3632$

Table 8.6: Length parameters for F5.

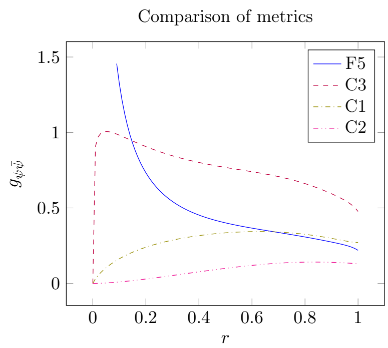

Figure 8.7: Metric for the central φ-value in the $\zeta \ll 0$-phases of C3, C1, C2 and F5.

Non-Abelian Example with a Pseudo-Hybrid Phase

We now apply the results given in Section 8.2 to compute Θ_0 for geodesics starting at the pseudo-hybrid point and ending at the ζ-value of the nearest singular point. From (8.41), we calculate

$$\zeta_+ = \frac{1}{2\pi} \log \frac{1}{540 + 312\sqrt{3}} \approx -6.99 \tag{8.75}$$

$$\zeta_- = \frac{1}{2\pi} \log \frac{1}{|540 - 312\sqrt{3}|} \approx 0.92, \tag{8.76}$$

and we conclude that t_- is the closest singularity to the pseudo-hybrid phase. We introduce a coordinate ψ, such that the pseudo-hybrid point is at $\psi = 0$ and the nearest singularity is at $\psi = 1$. The analysis of [133] suggests to set

$$z = (540 - 312\sqrt{3})\psi^7. \tag{8.77}$$

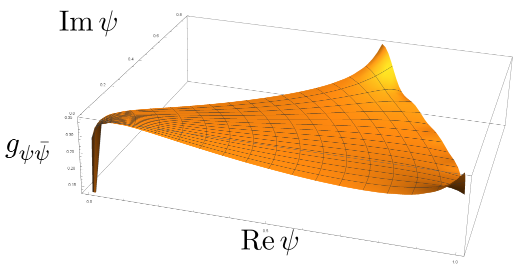

Figure 8.8: Metric for the non-abelian model from the pseudo-point ($r = 0$) to the nearest singularity ($r = 1$).

$\varphi(0)\frac{70}{\pi}$	1	2	3	4	5
Θ_0	0,7646	0,7603	0,7557	0,7523	0,7493
$\varphi(0)\frac{70}{\pi}$	6	7	7	9	10
Θ_0	0,7464	0,7443	0,7429	0,7419	0,7415

Table 8.7: Length parameters for the non-abelian model.

We define polar coordinates in the following way

$$\psi = re^{i\varphi}, \qquad r = \frac{e^{-\frac{2\pi}{7}\zeta}}{|540 - 312\sqrt{3}|^{\frac{1}{7}}} \qquad \varphi = \frac{\theta + \pi}{7}. \tag{8.78}$$

In these coordinates the nearest singularity is at $(r, \varphi) = (1, 0 \mod 2\pi)$. The leading behaviour of the metric near the pseudo-hybrid point is

$$\begin{aligned} g_{\psi\bar{\psi}} &= \frac{2^8 7^3 \left(2 - \sqrt{3}\right) \pi^8}{3^3 \Gamma\left(\frac{1}{6}\right)^2 \Gamma\left(\frac{1}{3}\right)^{14}} r^{\frac{1}{3}} \left(-\log(r) + \frac{1}{42} \log\left(\frac{2^{16}}{\left(540 - 312\sqrt{3}\right)^6}\right)\right) + \dots \\ &= -\alpha \frac{7^3 \Gamma\left(\frac{5}{6}\right)^9}{2^{\frac{8}{3}} \pi^{\frac{9}{2}}} r^{\frac{1}{3}} \log(r) + \dots . \end{aligned} \tag{8.79}$$

We introduced

$$\alpha = \left(\frac{2\sqrt{3} - 3}{2^{\frac{2}{3}}}\right) \approx 0,292, \tag{8.80}$$

and observe a similar leading behaviour as in the C3 model, except for the prefactor α. This similarity can be used as an additional check. We expect that the calculated distances are by a factor of $\sqrt{\alpha}$ smaller compared to the C3 results.

The metric is depicted in Figure 8.8. The metric is symmetric around $\varphi = \frac{\pi}{7}$. Figure 8.9 shows at which point the φ dependency of the metric sets in. In Table 8.7 we give the numerical results for the distances for various starting values of φ.

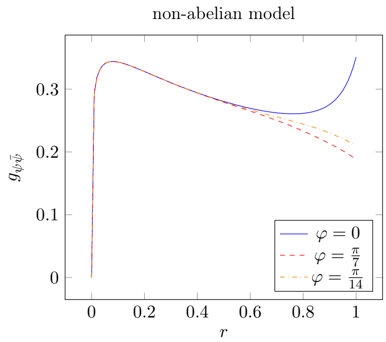

Figure 8.9: Metric constant φ-values of the non-abelian model.

The calculation of the mean value results in

$$\Theta_0 \approx 0,7499. \tag{8.81}$$

This is in agreement with the RSDC. Next we compare the results to the C3-model and find

$$\frac{\Theta_0}{\sqrt{\alpha}} \approx 1,2711. \tag{8.82}$$

As expected this lies close to the value of $\Theta_0^{C3} \approx 1,2639$.

Θ_0 can also be approximated by computing the integral (8.2) for the leading term of the metric. This term is independent of φ. The distance integral results in $\Theta_0 \approx 0,6747$, which is smaller than (8.81). This deviation can be traced back to the fact that we are integrating up to the boundary of the convergence radius at $r = 1$. Near $r = 1$ the subleading terms give larger contributions. If we cut off the integrate, say at $r = 0.9$ the leading term is a good approximation.

Chapter 9

The Web of Swampland Conjectures and the TCC bound

In this chapter we discuss the results obtained in [11]. The goal of the paper was to check the TCC bound (7.6) for ten de-Sitter no-go theorems and draw a connection to the swampland distance conjecture. Further a lower bound on the parameter λ, which appears in the swampland distance conjecture (Section 7.1) was proposed.

We will first recap the test of the TCC bound for the de-Sitter no-go theorems. Afterwards we summarize results found in the literature for the swampland distance conjecture. In the end of the chapter we discuss certain relations between different swampland conjectures. We close the discussion by proposing a generalization of the distance conjecture and a relation between the generalized distance conjecture and the de-Sitter conjecture.

In the discussion our focus lies on the aspects mainly studied by the author of this thesis and therefore some aspects of [11] will only be briefly summarized.

9.1 No-Go Theorems on classical de Sitter and c Values

De Sitter string backgrounds are backgrounds in which the 10d space-time splits into a 4d de Sitter manifold and a 6d compact manifold. The focus of the analysis lies on classical string backgrounds. These backgrounds are solutions of the 10d supergravity theory in a regime with low energy (α' corrections can be neglected) and small string coupling g_s. In [11] de Sitter solutions of 10d type II supergravities with D_p-branes and orientifold O_p-planes were considered. De Sitter solutions are difficult to construct and several no-go theorems have been established. The goal is to check the TCC bound (7.6) for ten no-go theorems. For this purpose the no-go theorems have to be rewritten into a condition on the 4d scalar potential V. The no-go theorems studied can always be brought into the form:

$$a\,V + \sum_{i=1}^{3} b_i\,\partial_{\varphi^i} V \leq 0\ , \quad a > 0, \tag{9.1}$$

where φ_i are scaler fields. (9.1) is true if certain assumptions are made. The assumptions are exactly the respective no-go theorems. It is possible to extract a value for c from (9.1). For that purpose (9.1) is rewriting into a form similar to (7.3):

$$(9.1) \to cV + \partial_{\varphi_2} V \leq 0. \tag{9.2}$$

φ_2 is a linear combination of the scalar fields entering (9.1). We skip the details of the analysis and refer to section 2 of [11]. In this section a table of the no-go theorems discussed is given and a detailed description of the procedure to turn the no-go theorems into the form (9.1) is laid out. The outcome of the analysis is that:

$$c \geq \sqrt{\frac{2}{3}}. \tag{9.3}$$

We see in (9.3) that the c-values of the no-go theorems analysed fulfil the TCC bound (7.6). This is noteworthy, because the values were obtained by a purely classical approach, with no quantum gravity arguments used. Let us next comment on the asymptotic behaviour of the potential V if we send φ_2 (9.2) to infinity. As show in section 4.3 of [11] for the no-go theorems studied the potential takes the following asymptotic form:

$$V = V_i e^{-c\mathcal{D}} \quad \mathcal{D} \to \infty, \tag{9.4}$$

with $\mathcal{D} = |\phi_2 - \phi_i|$ and V_i, ϕ_i are constants. c conincides with the value obtained from (9.2). The behaviour seen in (9.4) provides evidence for the proposed relation between the distance conjecture and the de Sitter conjectures as formulated in Section 9.3.

9.2 Distance Conjecture and λ Values

In this section we want to report on examples in the literature where different λ values for the refined swampland distance conjecture have been calculated. We harmonise conventions on the distance definition and compare the obtained λ values. A test of the distance conjecture requires the identification of an infinite tower of states with the described behaviour. The identification of an appropriate tower is not straightforward and often not only one tower of states becomes massless. In [137, 129, 138, 139, 140, 56, 141, 142, 10, 143, 144] such towers of states were studied. In [145, 146, 147, 148, 149, 150, 151, 152] the relation between such towers and the weak gravity conjecture was investigated.

After a tower which becomes massless has been identified it remains to check, if the tower shows the behaviour as claimed by the distance conjecture. If a valid tower was found λ can be estimated. In the subsequent discussion we analyse the results of papers which were focused on Kaluza-Klein states [129, 10] or brane states [137, 142, 140, 138, 56, 144]. This analysis allows us to give an estimate for λ in various infinite distance directions in the moduli space and for different states.

Let us start our discussion by first clarifying the notion of distance $\mathcal{D}(P,Q)$ from a point P to Q on a (field) space/manifold. In real coordinates x^i we express the metric by $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ and the distance along a curve γ from P to Q is given by

$$\mathcal{D}(P,Q) = \int_\gamma \sqrt{g_{ij}\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial s}}\,\mathrm{d}s, \tag{9.5}$$

where s is an affine parameter along γ. Next we consider a complex manifold of real dimension $2r$ and introduce complex holomorphic coordinates z^α and anti-holomorphic coordinates $\overline{z^\alpha} = \bar{z}^{\bar{\alpha}}$, $\alpha = 1,\ldots,r$. The metric can be expanded in this coordinates in the following way (see e.g. appendix A.2 of [153]):

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = 2g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha\mathrm{d}\bar{z}^{\bar{\beta}} + g_{\alpha\beta}\mathrm{d}z^\alpha\mathrm{d}z^\beta + g_{\bar{\alpha}\bar{\beta}}\mathrm{d}\bar{z}^{\bar{\alpha}}\mathrm{d}\bar{z}^{\bar{\beta}}. \tag{9.6}$$

In the following we consider a Hermitian metric. A Hermitian metric fulfils $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$. This property simplifies the line element to $\mathrm{d}s^2 = 2g_{\alpha\bar{\beta}}\mathrm{d}z^\alpha\mathrm{d}\bar{z}^{\bar{\beta}}$ and the distance (9.5) then becomes

$$\mathcal{D}(P,Q) = \int_\gamma \sqrt{2\,g_{\alpha\bar{\beta}}\frac{\partial z^\alpha}{\partial s}\frac{\partial \bar{z}^{\bar{\beta}}}{\partial s}}\mathrm{d}s. \tag{9.7}$$

The factor $\sqrt{2}$ appearing in the above definition was left out in [129, 10, 56, 121]. We will rescale the λ values obtained in these works by $\sqrt{2}$ subsequently.

Kaluza-Klein States

In [129, 10] trajectories in Kähler moduli space of CY hypersurfaces and complete intersections were studied and λ values for a tower of Kaluza-Klein states were obtained. In [129] these result were calculated by mirror symmetry and in [10] a GLSM approach was used. The method of [10] is described in Chapter 8 of this thesis. The values are in the range

$$0.6013 \le \lambda \le \sqrt{2}. \tag{9.8}$$

As discussed above, we rescaled the original results by $\sqrt{2}$. In Table 9.1 we give an overview of the individual λ values and in addition display the references and report further details of the different models. If a range of λ values is given for a certain model, then these values correspond to different field space directions or different phases in which λ has been calculated. In contrast to the original work, were $\Theta_\lambda = \frac{1}{\lambda}$ was given, we directly give the λ value (divided by $\sqrt{2}$).
The values for $\mathbb{P}^5[3,3], \mathbb{P}^5_{1^4 2^2}[4,4]$ and $\mathbb{P}^5_{1^2 2^2 3^2}[6,6]$ are new results and have not been calculated before. In the $h^{1,1} = 2$ model the calculations of geodesics is a computationally intense task. Therefore the calculations in [129] were done with an asymptotic expansion of the Kähler potential in the various phases. The geodesics were studied deep in a phase. In Table 9.1 we simply state the results obtained in different phases and refer to sections 5.1.3, 5.1.4 of [129] for more details.

Brane States

In contrast to the previous Section 9.2 we will now study brane states and their relation to the distance conjecture. In this section we focus on the complex structure moduli space $\mathcal{M}_{cs}$ of type IIB string theory compactified on a Calabi-Yau threefold CY_3 (see e.g.[28] for a review). $\mathcal{M}_{cs}$ is in general not smooth and has singularities. A theorem by Schmid [154] was used in [137] to study the singular points in $\mathcal{M}_{cs}$ through the monodromy behaviour around the singular points. The theorem allows to derive an asymptotic form for the Kähler potential K near the singular points in $\mathcal{M}_{cs}$:

$$e^{-K} = P(\operatorname{Im} t) + \mathcal{O}(e^{2\pi i t}), \tag{9.9}$$

where t is a coordinate on $\mathcal{M}_{cs}$ such that the singular point is at $t \to i\infty$. P is a polynomial in the imaginary part of t. The Weil-Peterson metric for the one parameter case then takes the following asymptotic form

$$g_{t\bar{t}} = \frac{1}{4}\frac{n}{(\operatorname{Im} t)^2} + \frac{\#}{(\operatorname{Im} t)^3} + \cdots + \mathcal{O}(e^{2\pi i t}). \tag{9.10}$$

n is called the nilpotency index and reflects the nature of the singular point. In addition n is the degree of the polynomial $P(\operatorname{Im} t)$. For more details we refer to [137]. The universal structure of (9.10) allows to extract the leading behaviour of the distance approaching a singularity:

$$\mathcal{D}(P,Q) = \int_P^Q \sqrt{2 g_{t\bar{t}}}\, |\mathrm{d}t| \;\approx\; \sqrt{\frac{n}{2}}\, \ln \frac{\operatorname{Im} t|_Q}{\operatorname{Im} t|_P} \;\to\; \infty. \tag{9.11}$$

The authors of [137] identified BPS states in $\mathcal{N} = 2$ supergravity originating from wrapped D_3-branes as possible candidate states to become massless. Their approach allowed them to study the asymptotic behaviour of the mass

Tower states	Setting	Reference	λ
Kaluza–Klein states	$\mathbb{P}^4[5]$, $h^{1,1}=1$	Table 3 of [129]	$0.7168 \leq \lambda \leq 0.8168$
	$\mathbb{P}^4_{1^4 2}[6]$, $h^{1,1}=1$	Table 4 of [129]	$0.7389 \leq \lambda \leq 0.8165$
	$\mathbb{P}^4_{1^4 4}[8]$, $h^{1,1}=1$	Table 5 of [129]	$0.7579 \leq \lambda \leq 0.8175$
	$\mathbb{P}^4_{1^3 25}[10]$, $h^{1,1}=1$	Table 6 of [129]	$0.7451 \leq \lambda \leq 0.8175$
	$\mathbb{P}^6_{1^7}[3,2,2]$, $h^{1,1}=1$	Table 4 of [10]	$0.6013 \leq \lambda \leq 1.0177$
	$\mathbb{P}^5_{1^6}[4,2]$, $h^{1,1}=1$	Table 5 of [10]	$0.6121 \leq \lambda \leq 0.9650$
	$\mathbb{P}^5_{1^5 3}[6,2]$, $h^{1,1}=1$	Table 6 of [10]	$0.6654 \leq \lambda \leq 1.0425$
	$\mathbb{P}^5_{1^5 2}[4,3]$, $h^{1,1}=1$	Table 7 of [10]	$0.6591 \leq \lambda \leq 0.9600$
	$\mathbb{P}^5[33]$, $h^{1,1}=1$	new	$0.7834 \leq \lambda \leq 0.8837$
	$\mathbb{P}^5_{1^4 2^2}[4,4]$, $h^{1,1}=1$	new	$0.7835 \leq \lambda \leq 0.9182$
	$\mathbb{P}^5_{1^2 2^2 3^2}[6,6]$, $h^{1,1}=1$	new	$0.7828 \leq \lambda \leq 0.9447$
	$\mathbb{P}^4_{1^2 2^3}[8]$, $h^{1,1}=2$, Hybrid-Orbifold	Section 5.1.4 of [129]	$\sqrt{\frac{2}{3}}$
	$\mathbb{P}^4_{1^2 2^3}[8]$, $h^{1,1}=2$, Hybrid-$\mathbb{P}^1$	Section 5.1.4 of [129]	$\sqrt{2}$
	$\mathbb{P}^4_{1^2 2^2 6}[12]$, $h^{1,1}=2$, Hybrid-Orbifold	Section 5.2 of [129]	$\sqrt{\frac{2}{3}}$
	$\mathbb{P}^4_{1^2 2^2 6}[12]$, $h^{1,1}=2$, Hybrid-$\mathbb{P}^1$	Section 5.2 of [129]	$\sqrt{2}$
	$\mathbb{P}^4_{1^3 69}[18]$, $h^{1,1}=2$, Hybrid-Orbifold	Section 5.3 of [129]	$\sqrt{\frac{2}{3}}$
	$\mathbb{P}^4_{1^3 69}[18]$, $h^{1,1}=2$, Hybrid-$\mathbb{P}^2$	Section 5.3 of [129]	1
Brane states	CY_3, type IIB, $n=1$	[137]	$\frac{1}{2}\sqrt{2}$
	CY_3, type IIB, $n=2$	[137]	1
	CY_3, type IIB, $n=3$	[137]	$\frac{1}{2}\sqrt{\frac{2}{3}}$
	$\mathbb{P}^5[c,c]$, large complex struct. point	(6.5) of [56]	$\frac{1}{2}\sqrt{\frac{2}{3}}$
	$\mathbb{P}^5[c,c]$, small complex struct. point	(6.22) of [56]	$\frac{1}{2}\sqrt{2}$
	CY_3 with orientifold, type IIB	Section 4.2 of [144]	$\frac{1}{2}\sqrt{6}$, $\frac{1}{2}\sqrt{\frac{2}{3}}$

Table 9.1: Values of λ in the distance conjecture 7.1 that were obtained in the literature, as well as a few new ones. The states of the tower becoming massless are either Kaluza–Klein states or brane states. For each value or range of values, we give the setting in which it was obtained and the reference in the literature. More details can be found in the main text.

of these states. For BPS states the mass is given by $M_{\mathbf{q}} = Z_{\mathbf{q}}$, where $Z_{\mathbf{q}}$ is the central charge and $\mathbf{q}$ the charge vector. In [137] the following asymptotic behaviour was obtained

$$\frac{M_{\mathbf{q}}(Q)}{M_{\mathbf{q}}(P)} \simeq \frac{(\mathrm{Im}\, t)^s|_P}{(\mathrm{Im}\, t)^s|_Q} \simeq e^{-\lambda\ \mathcal{D}(P,Q)}\ , \tag{9.12}$$

where the second equality is obtained from (9.11), with

$$\lambda = s\sqrt{\frac{2}{n}} \quad s = \begin{cases} 1 & \text{if} \quad n \quad \text{mod } 2 = 0 \\ \frac{1}{2} & \text{if} \quad n \quad \text{mod } 2 \neq 0 \end{cases}. \tag{9.13}$$

The value of n is bounded by the complex dimension of the CY and in the case of CY threefolds it can take values in $n = \{1, 2, 3\}$, which leads to the following λ values

$$\lambda = \left\{ \frac{1}{2}\sqrt{2}\ , \quad 1\ , \quad \frac{1}{2}\sqrt{\frac{2}{3}} \right\}, \tag{9.14}$$

as also given in Table 9.1.

These general results were reproduced in [56] for concrete one parameter examples. All the studied models in [56] have an infinite distance point in the large complex structure point (LCS) and therefore we can focus on the result for models of the type $\mathbb{P}^5[c, c]$, which have an additional infinite distance point at the small complex structure point (SCS). For all other models the small complex structure point is at finite distance. In [56] the following λ values:

$$\mathbb{P}^5[c, c]: \qquad \lambda_{LCS} = \frac{1}{\sqrt{6}}\ , \qquad \lambda_{SCS} = \frac{1}{\sqrt{2}}\ , \tag{9.15}$$

for D_3-branes at the SCS point in the $\mathbb{P}^5[3, 3]$ model and mirror D_0-D_2 bound states at the LCS point were found. The behaviour of the D_0 branes near the LCS point is less dominant with $\lambda = \sqrt{3/2}$.

Having obtained massless states it remains to show that an infinite tower of such states exits. The existence of such towers for different singularity types was studied in [137] and [138] in a systematic way. Infinite distances and massless states in the Kähler moduli space where studied in [140]. An analysis of possible states in type IIA/IIB orientifold compactifications was done in [142] and [144]. By studying O_3/O_7 on CY_3 in type IIB the following λ values where obtained in [144] for D_3-branes: $\lambda = 1/\sqrt{6}$ and $\lambda = \sqrt{3/2}$. The results for the λ values in the different models are summarized in Table 9.1.

Comments and λ Values

As one can see by studying Table 9.1 all λ values lie in the range

$$\lambda \ \geq \ \frac{1}{2}\sqrt{\frac{2}{3}} \ \approx \ \frac{1}{2}0.8165 \ \approx \ 0.4082. \tag{9.16}$$

Observe that the general analysis of [137] on CY_3 in type IIB with $n = 3$ gives precisely this value. Note that the value of (9.16) is exactly half of the TCC bound (7.6). This result is very interesting, because regarding the framework in which the λ values have been obtained, there is a priory no reason to expect a relation to the TCC bound. In [11] further matching patterns in the λ values and the c values for the no-go theorems where observed. A possible relation between these setups will be discussed in the subsequent section.

9.3 The Web of Conjectures

We saw, that both order one parameters λ and c have a lower bound, see (9.16) and (9.3) respectively. The bound on c is consistent with the TCC bound (7.6) and the bound on λ is given by one-half of the TCC value. In [11] similar patterns in the c values for the no-go theorems and the λ values (see Table 9.1) were observed. Further in both conjectures an exponential behaviour can be identified, see (7.1) and (9.4). These results can be interpreted as a relation in disguise between the two conjectures. The idea of a relation has been considered before and we will summarize the various suggestions given in the literature bellow. Afterwards we will present a generalization of the distance conjecture and a new conjecture relating the de Sitter conjecture and the distance conjecture.

Relating Conjectures in the Literature

An interesting aspect of the various swampland conjectures is, that they are related. The rigour at which these connections are established varies and for a more detailed account on possible relations we refer to [121]. For the de Sitter swampland conjecture arguments for relations to the distance conjecture and the weak gravity conjecture have been given before (see e.g. [127]) and with the results of the previous sections in mind we want to make the connection more precise.

We will start by looking at the settings in which these conjectures are established. The distance conjecture is mostly checked in Calabi-Yau compactifications and in contrast the de Sitter conjecture is typically verified in compactifications on manifolds with fluxes and sources in order to generate a non-trivial 4d potential V and a cosmological constant Λ. As one can see both setups are rather different. Nevertheless the conjectures should hold in both of them. Due to the different frameworks it is not clear if a relation between the two conjectures can be given in any framework or even in a single setup. A possibility is to view the relation between the conjectures as a map or duality between two different settings or theories. Arguments in that direction were given in [127]. If we view the conjectures in a single setup one could compare the mass $m(\varphi)$ entering the distance conjecture to the mass of φ related to $\partial_\varphi^2 V$ for a scalar potential $V(\varphi)$. These masses are related to different states and a relation between them is not evident. In the following arguments will be given to rather relate m to V itself.

A guiding principle in relating the conjectures are the observed exponential behaviours. An exponential behaviour typical appears in large field limits. As argued in [127] these limits correspond to parametrically controlled regimes in string theory and typically in such regimes a dual description is available. This hints to view one conjecture valid on one side of the duality, which is then mapped on the other side of the duality to a different conjecture. The mapping is done by identifying the exponential behaviour of some quantity. Similar ideas were used in [127] to identify in the large field limit the asymptotic form of the scalar potential V with the exponential behaviour of m. Along these arguments we deduce the possibility of having the following relation:

$$\frac{m}{m_i} \approx \left|\frac{V}{V_i}\right|^{\alpha} \approx e^{-\lambda \mathcal{D}} \quad \text{when } \mathcal{D} \to \infty \ , \tag{9.17}$$

with some constants m_i, V_i, α and $\mathcal{D}$ the geodesic field distance. In the case of the de Sitter conjecture (Section 7.2) with exponential behaviour the following equality can be read off

$$\lambda = \alpha \cdot c. \tag{9.18}$$

The next swampland conjecture we want to consider is the scalar weak gravity conjecture [155]. For a single scalar field φ with mass m the conjecture can be written in the form

$$(\partial_\varphi m)^2 \geq m^2 \ . \tag{9.19}$$

In the case when the equality is saturated an exponential behaviour is obtained. We want to mention the similarity to the de Sitter conjecture (7.3). This will be important in the following discussion. The scalar weak gravity conjecture was generalized in [156] and named strong scalar weak gravity conjecture. In the proposed extension an inequality similar to (9.19) is applied to the self-interactions of a single scalar field φ given by a potential $V(\varphi)$. The basic idea is to replace m^2 in (9.19) by the second derivative V''. The extension of the conjecture as given in [156] reads

$$2\frac{(V''')^2}{V''} - V'''' \geq V'' \quad \leftrightarrow \quad (V'')^2 \left(\frac{1}{V''}\right)'' \geq V''. \tag{9.20}$$

In the saturation case one again recovers an exponential behaviour

$$\left(\frac{1}{V''}\right)'' - \frac{1}{V''} = 0 \leftrightarrow V'' = \left(Ae^{\varphi} + Be^{-\varphi}\right)^{-1} , \tag{9.21}$$

where A, B are constants of integration. We can rewrite (9.20) and exchange V'' by m^2 and recover a distance conjecture like behaviour

$$m^2 \left(\frac{1}{m^2}\right)'' \geq 1. \tag{9.22}$$

A further generalization was put forward in the talk [157]. It was proposed to replace m by the scalar potential of a de Sitter conjecture:

$$V = m^{2\gamma}, \tag{9.23}$$

whereby γ was not specified. This is the same relation as the first part of (9.17).

Let us also look at a conjecture put forward in [158], in which the distance conjecture was studied in an anti-de Sitter space-time. It was argued that in the limit of a vanishing cosmological constant $\Lambda \to 0$, a tower of states of mass-scale m becomes light:

$$m \sim |\Lambda|^{\alpha} \quad \text{for } \Lambda \to 0, \tag{9.24}$$

with $\alpha \geq \frac{1}{2}$. As argued by the authors of [158] in a supersymmetric vacuum $\alpha = \frac{1}{2}$ (strong AdS distance conjecture). Further a relation to the de Sitter conjecture adjusted to AdS space was discussed. Extending the conjecture to quasi-de Sitter or de Sitter space-times by replacing Λ by V in (9.24) the relation (9.17) is recovered in an asymptotic limit. Observe that in [158] the tower of states and the cosmological constant Λ are in the same setup and we argued for a map between different setups.

For references regarding further relations between the conjectures we refer to [11].

Proposals

The arguments given in Section 9.3, the lower bounds on λ and c (see (9.16) and (9.3)) and the appearing exponential behaviours, see (7.1) and (9.4), lead to the subsequent conjectural statements.

Distance Conjecture

The goal is to relax the original distance conjecture (see Section 7.1) to a weaker more general statement. The statement is inspired by the bound on λ (9.16) and by the map to the potential V of the de Sitter conjecture. The following statement, inspired by the structure of the TCC (7.5), was proposed in [11]:

Consider a field space (e.g. a scalar moduli space) appearing in a d-dimensional low energy effective theory of a quantum gravity and a geodesic distance $\mathcal{D}$ in this field space. Whenever $\mathcal{D} \to \infty$, a tower of states becomes light, with a typical mass scale m such that in Planckian units

$$0 < m \leq m_0 \, e^{-\lambda_0 \mathcal{D}}, \tag{9.25}$$

with positive constants m_0, λ_0, where

$$\lambda_0 = \frac{1}{\sqrt{(d-1)(d-2)}}. \tag{9.26}$$

Focusing on the cases of a single canonically normalized field φ, the distance can be written $\mathcal{D} = |\varphi - \varphi_i|$, where $\varphi > \varphi_i$ is assumed. We next calculate the average of the ratio $\frac{\partial_\phi m}{m}$ along a path in the moduli space:

$$-\left\langle \frac{m'}{m} \right\rangle = -\frac{1}{\varphi - \varphi_i} \int_{\varphi_i}^{\varphi} \mathrm{d}\tilde{\varphi} \frac{\partial_{\tilde{\varphi}} m}{m} = -\frac{\ln \frac{m(\varphi)}{m(\varphi_i)}}{\varphi - \varphi_i} \geq -\frac{\ln \frac{m_0}{m(\varphi_i)}}{\varphi - \varphi_i} + \lambda_0, \tag{9.27}$$

where in the last step (9.25) was used. Equation (9.27) further implies:

$$\left\langle \frac{|m'|}{m} \right\rangle \geq \left| \left\langle \frac{m'}{m} \right\rangle \right| \geq \frac{|\ln \frac{m_0}{m(\varphi_i)}|}{\mathcal{D}} + \lambda_0, \tag{9.28}$$

and in a large distance regime we can conclude

$$(9.25) \Rightarrow \left\langle \frac{|m'|}{m} \right\rangle_{\mathcal{D}\to\infty} \geq \lambda_0. \tag{9.29}$$

The previous derivation is modelled along the lines of the TCC (7.5) given in [128]. The natural extension to the multi-field case is by replacing $m' = \partial_\varphi m$ by ∇m. The standard distance conjecture (7.1) is a special case of (9.25). In this case the mass-scale has an exponential behaviour in the large distance limit $\mathcal{D} \to \infty$ and from (9.25) or (9.29) one can obtain the following constraint:

$$m = m_i \, e^{-\lambda \mathcal{D}} \quad \Rightarrow \quad \lambda \geq \lambda_0. \tag{9.30}$$

(9.25) can be interpreted to allow an exponential behaviour of m in various directions, but these exponentials are subdominant compared to the saturation case of the inequality. This viewpoint leads to the bound

$$\lambda \geq \frac{1}{2}\sqrt{\frac{2}{3}} \quad \text{in } d = 4, \tag{9.31}$$

which was verified in all the examples of Table 9.1 and in general for wrapped D_3 brane states in[1] [137].

The mentioned examples motivate the value of λ_0 (9.26) in $d = 4$ to be half of the TCC value (7.5): $\lambda_0 = \frac{1}{2} c_0$.

Let us mention, that different to the TCC (7.5), where the inequality on the potential V was motivated by physical arguments, our approach does not provide such arguments for (9.25).

Scalar Weak Gravity Conjecture

Let us repeat the result given in (9.29)

$$\left\langle \frac{|\partial_\varphi m|}{m} \right\rangle_{\mathcal{D}\to\infty} \geq \lambda_0, \tag{9.32}$$

[1]In the meantime further evidence for the bound where found in [159] and [160] by studying the distance conjecture in relation to the weak gravity conjecture.

which resembles the form of a scalar weak gravity conjecture. Therefore the scalar weak gravity conjecture inequality (9.19) can be extended to the more general form (9.32). This makes the relation to the distance conjecture, as discussed above, more suggestive. The formulation in terms of (9.32) allows for behaviours different than exponential form. In the case where m is an exponential (9.32) gives

$$(\partial_\varphi m)^2 \geq \lambda_0^2 m^2. \tag{9.33}$$

The extension to the multi-field case is given by the replacement $(\partial_\varphi m)^2 \to (\nabla_\varphi m)^2$. If we compare (9.33) to the original scalar weak gravity conjecture (9.19) we see the additional parameter λ_0. The modified version (9.33) is valid for all cases where the original conjecture holds, because $\lambda_0 < 1$ in $d = 4$. The modified version puts forward the possibility of other cases, in which the lower bound λ_0 is obtained.
The proposed form (9.32) also suggest to modify extensions of the scalar weak gravity conjecture and we refer to [11] for further comments.

Relating the Distance and de Sitter Conjectures

In the Section 9.3 we discussed a possible relation between the distance and de Sitter conjectures. We further highlighted a link between m and V. The weaker version of the distance conjecture (9.25) makes the link, by looking at the TCC (7.5), even more prominent.
The similar results for λ and c in the studied examples and the relation to the TCC bound further suggest a possible link. In the examples studied before the conjectures act on different compactification setups. This and the previous points lead us to following proposal [11]:

We conjecture the existence of a map between two compactification setups to d dimensions where, in each of them, a direction φ_k in field space is selected: (setup$_1$, φ_1) $\leftrightarrow$ (setup$_2$, φ_2). In the first setup, the generalized distance conjecture (9.25) *with mass m applies, and in the second one the de Sitter conjecture in TCC form* (7.5) *applies. We denote generically by $\mathcal{D}$ the field space geodesic distance along $\varphi_{k=1,2}$; it can on both sides be arbitrarily large. The proposed map is then*

$$\frac{m}{m_i} \simeq \left|\frac{V}{V_i}\right|^{\frac{1}{2}} \quad \text{for } \mathcal{D} \to \infty \ , \tag{9.34}$$

for some constants m_i, V_i. The symbol $\simeq$ is understood as an equality of the two functions m, V up to the exchange $\varphi_1 \leftrightarrow \varphi_2$.

If both m and V show an exponential behaviour in large field distances, both sides of (9.34) are equal to $e^{-\lambda\mathcal{D}}$. In that case (9.17) can be matched by setting $\alpha = \frac{1}{2}$ and further

$$\lambda = \frac{1}{2}c \ \geq \ \lambda_0 = \frac{1}{2}c_0. \tag{9.35}$$

For further comments on the asymptotic (exponential) form of V see section 4.3 of [11]. In the case of a single field it follows from (9.34):

$$\frac{|m'|}{m_i} \simeq \frac{1}{2}\frac{|V'|}{V_i}, \tag{9.36}$$

in a large distance limit. (9.36) generalizes (9.35) to non-exponential behaviour.

Let us close the discussion by mentioning that, similar to the proposal given for the distance conjecture, a fundamental principle underlying the proposed map is at the moment unclear to us.

Chapter 10

Summary / Outlook

In this thesis we were able to show that the sphere partition function of all one-parameter abelian and certain two-parameter abelian GLSMs has the same structure in certain phases of the GLSMs. An extension of our study would be to consider non-abelian GLSMs and check if a similar structure appears in the different phases of the respective GLSM. Promising results in that direction have been given in e.g [93, 94, 95, 97], in which the appearance of the I function in the sphere partition function of certain non-abelian GLSMs in phases with a geometric interpretation was shown.

Our results for the hybrid models could provide a way to explicitly extract FJRW invariants, along the lines of [75] where Gromov-Witten invariants were extracted. The FJRW invariants, together with the mirror map, have been defined in [68], but no invariants have been calculated. In view of our conjectural I- and J- functions for the two parameter hybrid phases studied in this thesis, it would be interesting to check if they are compatible with FJRW theory. Furthermore it could be insightful to reconcile the results from physics for hybrid models e.g. [106, 108] with the results known in mathematics. Although the focus of this thesis was on the sphere partition function similar techniques can be applied to the hemisphere partition function of the GLSM [46, 47, 48]. This could provide insights into brane states for hybrid models. A related analysis was done for geometric and Landau-Ginzburg phases in [82]. Some results for brane states in hybrid models were given in [69] and it would be interesting to see how these fit into the setting of the GLSM and supersymmetric localisation.

We also showed that the refined swampland distance conjecture holds for pseudo-hybrid phases in the one-parameter abelian GLSMs and the pseudo-hybrid phase of a non-abelian GLSM. Nevertheless a further study of non-abelian GLSMs could provide further insights for this conjecture, because these models provide a way to study CYs which are not complete intersections. Another possible direction would be to study candidates for massless states in infinite distance regions by GLSM techniques. Some preliminary results for

K-type hybrid models have been given in [56] .

Further it would be interesting to see how GLSMs techniques relate to results obtain from asymptotic Hodge theory. For example how the structure of the sphere partition function fits into the classification of singularities given in [137, 138] or the results found for the weak gravity conjecture in [160, 159].

In this thesis we also provided a conjecture for a lower bound on the order one parameter appearing in the refined swampland distance conjecture and outlined a possible relation between the swampland distance conjecture and the de Sitter conjectures. In order to strengthen these conjectures a better understanding of them away from the geometric regimes is needed. Especially for the conjectured lower bound a deeper knowledge of the possible tower of massless states has to be established. However it is nice that these directions for further research overlap with the ones identified above treatable by GLSM techniques.

Appendix A

Evaluation of the Sphere Partition Function - One Parameter Abelian

In this appendix we give further details on the evaluation of the sphere partition function for the models introduced in Chapter 5. In this class of models we perform a single variable residue calculation. For the two parameter models we need a more elaborate procedure, which we outline in Appendix B.

A.1 Form of the Sphere Partition Function

Our starting point is the sphere partition function as given in (6.65):

$$
\begin{aligned}
Z_{S^2} = \frac{(z\bar{z})^q}{2\pi} \sum_{m\in\mathbb{Z}} \int_{-\infty+iq}^{\infty+iq} \mathrm{d}\sigma Z_{p_1}(\sigma,m) Z_{p_2}^k(\sigma,m) Z_1^{5+k-n-j}(\sigma,m) \\
\cdot Z_\alpha^n(\sigma,m) Z_\beta^j(\sigma,m) \bar{z}^{i\sigma+\frac{m}{2}} z^{i\sigma-\frac{m}{2}}
\end{aligned}
\tag{A.1}
$$

with the following contributions:

$$\begin{aligned}
Z_{p_1} &= \frac{\Gamma\left(\frac{1}{2}(m+2i\sigma)d_1+1\right)}{\Gamma\left(\frac{1}{2}(m-2i\sigma)d_1\right)}, \\
Z_{p_2} &= \frac{\Gamma\left(\frac{1}{2}(m+2i\sigma)d_2+1\right)}{\Gamma\left(\frac{1}{2}(m-2i\sigma)d_2\right)}, \\
Z_1 &= \frac{\Gamma\left(-\frac{m}{2}-i\sigma\right)}{\Gamma\left(-\frac{m}{2}+i\sigma+1\right)}, \\
Z_\alpha &= \frac{\Gamma\left(-\frac{1}{2}\alpha(m+2i\sigma)\right)}{\Gamma\left(i\sigma\alpha-\frac{m\alpha}{2}+1\right)}, \\
Z_\beta &= \frac{\Gamma\left(-\frac{1}{2}\beta(m+2i\sigma)\right)}{\Gamma\left(i\sigma\beta-\frac{m\beta}{2}+1\right)}.
\end{aligned} \tag{A.2}$$

For later convenience we introduced:

$$z = e^{-2\pi\zeta+i\theta}, \qquad \bar{z} = e^{-2\pi\zeta-i\theta}. \tag{A.3}$$

These are related to $\mathtt{t}$ (6.71) by:

$$z = e^{-\mathtt{t}}, \qquad \bar{z} = e^{-\bar{\mathtt{t}}}. \tag{A.4}$$

Our goal is to evaluate (A.1) by a residue calculation. For this purpose we need to calculate the positions of the poles and choose a contour such that the integral is convergent.

A.2 Situation of the Poles and Contour of Integration

We see that for large ζ values the integrand is dominated by

$$e^{-4\pi i\zeta\sigma} = e^{-4\pi i\zeta\,\mathrm{Re}(\sigma)}e^{4\pi\zeta\,\mathrm{Im}(\sigma)},$$

and it follows that we have to close the contour such that it encloses:

- the positive imaginary axis for $\zeta \ll 0$,
- the negative imaginary axis for $\zeta \gg 0$.

The next task is to determine the positions of the poles. The Γ-function has poles whenever its argument is a negative integer. In the integral poles can be cancelled by the Γ-functions in the denominators. From (A.2) we find that the poles are at:

$$\begin{aligned}
Z_{p_1}:\quad \sigma &= \frac{i\,(d_1 m + 2n_1 + 2)}{2d_1} && n_1 \geq \max[0, -d_1 m] \\
Z_{p_2}:\quad \sigma &= \frac{i\,(d_2 m + 2n_2 + 2)}{2d_2} && n_2 \geq \max[0, -d_2 m] \\
Z_1:\quad \sigma &= \frac{1}{2} i\,(m - 2n_3) && n_3 \geq \max[0, m] \\
Z_\alpha:\quad \sigma &= \frac{i\,(\alpha m - 2n_\alpha)}{2\alpha} && n_\alpha \geq \max[0, \alpha m] \\
Z_\beta:\quad \sigma &= \frac{i\,(\beta m - 2n_\beta)}{2\beta} && n_\beta \geq \max[0, \beta m].
\end{aligned} \tag{A.5}$$

To calculate the positions of the poles we use the following formula:

$$\max(x, y) = \frac{x + y + |x - y|}{2}. \tag{A.6}$$

We find for the positions of the poles of Z_{p_1}:

$$\begin{aligned}
\sigma &= \frac{i\,(d_1 m + 2n_1 + 2)}{2d_1} \\
&\geq i\frac{(d_1|m| + 2)}{2d_1} \\
&\geq i\frac{|m|}{2} + i\frac{1}{d_1} \\
&\geq i\frac{1}{d_1}.
\end{aligned} \tag{A.7}$$

We see that the poles of Z_{p_1} lie along the positive imaginary axis. The same conclusion follows for Z_{p_2}, because it is the same situation except that d_1 is replaced by d_2. Next we study the poles of Z_α:

$$\begin{aligned}
\sigma &= \frac{i\,(\alpha m - 2n_4)}{2\alpha}, \\
&\leq i\frac{(\alpha m - \alpha m - \alpha|m|)}{2\alpha}, \\
&\leq -i\frac{|m|}{2}.
\end{aligned} \tag{A.8}$$

We conclude that the contributions of Z_α lie along the negative imaginary axis. This also holds for Z_1 and Z_β.

A.3 Double Counting of Poles

Next we study when poles coincide. The quantities given in (6.72) will be repeatedly used in this analysis and therefore we repeat them here for convenience:

$$\begin{aligned} \gcd(\beta,\alpha) &= \kappa_1, \quad \frac{\alpha}{\kappa_1} = \tau_\alpha, \quad \frac{\beta}{\kappa_1} = \tau_\beta, \\ \gcd(d_1,d_2) &= \kappa_2, \quad \frac{d_1}{\kappa_2} = \tau_{d_1}, \quad \frac{d_2}{\kappa_2} = \tau_{d_2}. \end{aligned} \tag{A.9}$$

We begin by studying possible intersections for the poles on the negative imaginary axis ($\zeta \gg 0$ phase). We study the intersections of poles of Z_α and Z_β:

$$\frac{i\,(\alpha m - 2n_\alpha)}{2\alpha} = \frac{i\,(\beta m - 2n_\beta)}{2\beta} \Rightarrow n_\beta = \tau_\beta \frac{n_\alpha}{\tau_\alpha}. \tag{A.10}$$

Next we use the restrictions on the Z_α poles given in (A.5) and find:

$$\begin{aligned} n_\beta \geq \frac{\tau_\beta}{\tau_\alpha}\max[0,\alpha m] &= \frac{\tau_\beta}{\tau_\alpha}\frac{\alpha m + \alpha|m|}{2} \\ &= \frac{\beta m + |\beta m|}{2} \\ &= \max[0,\beta m]. \end{aligned} \tag{A.11}$$

In the above calculation we used formula (A.6). We see that the restrictions on the Z_β poles given in (A.5) is fulfilled and so:

$$Z_\alpha \subset Z_\beta \quad \Leftrightarrow \quad n_\alpha = \tau_\alpha k \quad k \in \mathbb{Z}_{\geq 0}. \tag{A.12}$$

As similar calculation with Z_β and Z_1 shows that if one sums over the poles of Z_β one gets all poles of Z_1. We can conclude that if we sum over all Z_β poles we need in addition a summation over all Z_α poles where the n_α are of the form:

$$n_\alpha = \tau_\alpha n + \delta \qquad\qquad \delta = 1,\ldots,\tau_\alpha - 1 \quad n \in \mathbb{N}_{\geq 0}. \tag{A.13}$$

At first, one might wonder if one hits poles of Z_1 by doing the extra summation over (A.13), but we know that in order for a Z_α pole to be a pole of Z_1 we must have:

$$n_1 = \frac{n_\alpha}{\alpha}, \quad n_1 \in \mathbb{Z}_{\geq 0}. \tag{A.14}$$

This is never an integer for (A.13). For the poles on the positive imaginary axis ($\zeta \ll 0$ phase) the poles of Z_{p_1} and Z_{p_2} could possibly coincide. We calculate:

$$\begin{aligned} \frac{i\,(d_2 m + 2n_2 + 2)}{2d_2} &= \frac{i\,(d_1 m + 2n_1 + 2)}{2d_1} \Rightarrow \\ n_1 &= \frac{\tau_{d_1}}{\tau_{d_2}}\,(n_2+1) - 1. \end{aligned} \tag{A.15}$$

Let us next check the constraint on the poles of Z_{p_1}:

$$\begin{aligned} n_1 &\geq \frac{\tau_{d_1}}{\tau_{d_2}}\,(\max[0, -d_2 m] + 1) - 1, \\ &\geq \frac{\tau_{d_1}}{\tau_{d_2}}\left(\frac{-d_2 m + d_2|m|}{2} + 1\right) - 1, \\ &\geq \max[0, -d_1 m] + \frac{d_1}{d_2} - 1, \\ &> \max[0, -d_1 m] - 1, \\ &\geq \max[0, -d_1 m]. \end{aligned} \tag{A.16}$$

Hence, we recover the constraint on poles of Z_{p_1} given in (A.5). By the above results we found for the poles:

$$Z_{p_2} \subset Z_{p_1} \quad \Leftrightarrow \quad n_2 + 1 = \tau_{d_2} k \quad k \in \mathbb{Z}_{>0}. \tag{A.17}$$

As a result we can sum over the poles of Z_{p_1} and in addition we need to sum over the poles of Z_{p_2} of the form:

$$n_2 + 1 = \tau_{d_2} k + \delta, \qquad \delta = 1, \ldots \tau_{d_2} - 1, \quad k \in \mathbb{Z}_{\geq 0}, \tag{A.18}$$

which can be rewritten into:

$$n_2 = \tau_{d_2} k + \delta, \qquad \delta = 0, \ldots, \tau_{d_2} - 2, \quad k \in \mathbb{Z}_{\geq 0}. \tag{A.19}$$

Let us also remark, that we have models where we only have a Z_1 contribution. Nevertheless our discussion is applicable by simply setting $\beta = 1$ and $j = 0$ in the following. Of course, the sum is then over the Z_1 poles.

A.4 $\zeta \ll 0$ Contributions

We first sum over the poles of Z_{p_1} and afterwards we focus on the remaining poles of Z_{p_2} given by (A.19).

Z_{p_1} Contribution

We perform the shift:

$$\sigma \to \varepsilon + \frac{i\,(d_1 m + 2n_1 + 2)}{2d_1}, \tag{A.20}$$

and write the integral as a sum over residue integrals:

$$\begin{aligned} Z_{1,S^2}^{\zeta\ll 0} = \frac{(z\bar{z})^q}{2\pi} \sum_{n_1=0}^{\infty} \sum_{-\frac{n_1}{d_1}\leq m} \oint \mathrm{d}\varepsilon Z_{p_1}(\varepsilon, n_1, m) Z_{p_2}^k(\varepsilon, n_1, m) \\ \cdot Z_1^{5+k-n-j}(\varepsilon, n_1, m) Z_\alpha^n(\varepsilon, n_1, m) Z_\beta^j(\varepsilon, n_1, m) \\ \cdot \bar{z}^{i\varepsilon - \frac{n_1}{d_1} - \frac{1}{d_1}} z^{i\varepsilon - \frac{n_1}{d_1} - \frac{1}{d_1} - m}. \end{aligned} \tag{A.21}$$

To simplify the sum over m we introduce:

$$l = d_1 m + n_1, \tag{A.22}$$

and replace the sum over m by:

$$\sum_{-\frac{n_1}{d_1}\leq m} \Rightarrow \sum_{l=0}^{\infty}. \tag{A.23}$$

We need to be careful when using this transformation because m is an integer and therefore:

$$m = \frac{l-n_1}{d_1} \Rightarrow l = d_1 b + \delta, \quad n_1 = d_1 a + \delta, \tag{A.24}$$

with $\delta = 0, \dots, d_1 - 1$ and $a, b \in \mathbb{N}_{\geq 0}$. We find for the integral

$$\begin{aligned} Z_{1,S^2}^{\zeta\ll 0} = \frac{1}{2\pi} \sum_{\delta=0}^{d_1-1} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \oint_0 \mathrm{d}\varepsilon Z_{p_1}(\varepsilon, a, b, m) Z_{p_2}^k(\varepsilon, a, b, m) \\ \cdot Z_1^{5+k-n-j}(\varepsilon, a, b, m) Z_\alpha^n(\varepsilon, a, b, m) Z_\beta^j(\varepsilon, a, b, m) \\ \cdot \bar{z}^{i\varepsilon - a - \frac{\delta+1}{d_1} + q} z^{i\varepsilon - b - \frac{\delta+1}{d_1} + q}. \end{aligned} \tag{A.25}$$

After the transformation (A.20) and the shift of the summation variables the integral contributions can be brought into the following forms:

$$\begin{aligned}
Z_{p_1}(\varepsilon,a,b,m) &= (-1)^{\delta}(-1)^{ad_1}\frac{\pi}{\sin\pi\left(i\varepsilon d_1\right)} \\
&\cdot\frac{1}{\Gamma\left(\delta+bd_1-i\varepsilon d_1+1\right)\Gamma\left(\delta+ad_1-i\varepsilon d_1+1\right)}, \\
Z_{p_2}(\varepsilon,a,b,m) &= (-1)^{ad_2}\frac{\pi}{\sin\pi\left(-i\varepsilon d_2+\frac{d_2}{d_1}(\delta+1)\right)} \\
&\cdot\frac{1}{\Gamma\left(ad_2-i\varepsilon d_2+\frac{d_2}{d_1}(\delta+1)\right)} \\
&\cdot\frac{1}{\Gamma\left(bd_2-i\varepsilon d_2+\frac{d_2}{d_1}(\delta+1)\right)}, \\
Z_1(\varepsilon,a,b,m) &= (-1)^{b}\Gamma\left(a-i\varepsilon+\frac{\delta+1}{d_1}\right)\Gamma\left(b-i\varepsilon+\frac{\delta+1}{d_1}\right) \\
&\cdot\frac{\sin\pi\left(-i\varepsilon+\frac{\delta+1}{d_1}\right)}{\pi}, \\
Z_{\alpha}(\varepsilon,a,b,m) &= (-1)^{\alpha b}\frac{\sin\pi\left(-i\varepsilon\alpha+\frac{\alpha}{d_1}(\delta+1)\right)}{\pi} \\
&\cdot\Gamma\left(a\alpha-i\varepsilon\alpha+\frac{\alpha}{d_1}(\delta+1)\right) \\
&\cdot\Gamma\left(b\alpha-i\varepsilon\alpha+\frac{\alpha}{d_1}(\delta+1)\right), \\
Z_{\beta}(\varepsilon,a,b,m) &= (-1)^{\beta b}\frac{\sin\pi\left(-i\varepsilon\beta+\frac{\beta}{d_1}(\delta+1)\right)}{\pi} \\
&\cdot\Gamma\left(a\beta-i\varepsilon\beta+\frac{\beta}{d_1}(\delta+1)\right) \\
&\cdot\Gamma\left(b\beta-i\varepsilon\beta+\frac{\beta}{d_1}(\delta+1)\right).
\end{aligned} \tag{A.26}$$

We next observe that by the Calabi-Yau condition (5.4) we can write

$$(-1)^{ad_1}(-1)^{akd_2} = (-1)^{(5+k-n-j+\alpha n+j\beta)a}. \tag{A.27}$$

This will be useful to write the integral in a compact form. We introduce:

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{1,reg}(\varepsilon,z,\delta) = &\sum_{a=0}^{\infty} z^{i\varepsilon-a-\frac{\delta+1}{d_1}+q}(-1)^{a(5+k-n-j+\alpha n+j\beta)} \\ &\cdot\Gamma\left(a-i\varepsilon+\frac{\delta+1}{d_1}\right)^{5+k-n-j} \\ &\cdot\frac{\Gamma\left(a\alpha-i\varepsilon\alpha+\frac{\alpha}{d_1}(\delta+1)\right)^{n}}{\Gamma\left(\delta+ad_1-i\varepsilon d_1+1\right)} \\ &\cdot\frac{\Gamma\left(a\beta-i\varepsilon\beta+\frac{\beta}{d_1}(\delta+1)\right)^{j}}{\Gamma\left(ad_2-i\varepsilon d_2+\frac{d_2}{d_1}(\delta+1)\right)^{k}},\end{aligned} \tag{A.28}$$

and

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{1,sing}(\varepsilon,\delta) = &\frac{1}{\pi^4}\sin\left(\pi\left(-i\varepsilon+\frac{\delta+1}{d_1}\right)\right)^{5+k-n-j} \\ &\cdot\frac{\sin\left(\pi\left(-i\varepsilon\alpha+\frac{\alpha}{d_1}(\delta+1)\right)\right)^{n}}{\sin\pi\left(i\varepsilon d_1\right)} \\ &\cdot\frac{\sin\left(\pi\left(-i\varepsilon\beta+\frac{\beta}{d_1}(\delta+1)\right)\right)^{j}}{\sin\left(\pi\left(-i\varepsilon d_2+\frac{d_2}{d_1}(\delta+1)\right)\right)^{k}}\end{aligned} \tag{A.29}$$

The integral can now be written in a compact way:

$$Z^{\zeta\ll 0}_{1,S^2} = \frac{1}{2\pi}\sum_{\delta=0}^{d_1-1}\oint d\varepsilon(-1)^{\delta}\mathcal{Z}^{\zeta\ll 0}_{1,sing}(\varepsilon,\delta)|\mathcal{Z}^{\zeta\ll 0}_{1,reg}(\varepsilon,z,\delta)|^2. \tag{A.30}$$

We see that it is convenient to make the following shift

$$\delta+1\to\delta, \tag{A.31}$$

to get:

$$Z^{\zeta\ll 0}_{1,S^2} = -\frac{1}{2\pi}\sum_{\delta=1}^{d_1}\oint d\varepsilon(-1)^{\delta}\mathcal{Z}^{\zeta\ll 0}_{1,sing}(\varepsilon,\delta)|\mathcal{Z}^{\zeta\ll 0}_{1,reg}(\varepsilon,z,\delta)|^2, \tag{A.32}$$

with

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{1,reg}(\varepsilon,z,\delta)=&\sum_{a=0}^{\infty} z^{i\varepsilon-a-\frac{\delta}{d_1}+q}(-1)^{a(5+k-n-j+\alpha n+j\beta)}\\ &\cdot\Gamma\left(a-i\varepsilon+\frac{\delta}{d_1}\right)^{5+k-n-j}\\ &\cdot\frac{\Gamma\left(a\alpha-i\varepsilon\alpha+\frac{\alpha}{d_1}\delta\right)^{n}\Gamma\left(a\beta-i\varepsilon\beta+\frac{\beta}{d_1}\delta\right)^{j}}{\Gamma\left(\delta+ad_1-i\varepsilon d_1\right)\Gamma\left(ad_2-i\varepsilon d_2+\frac{d_2}{d_1}\delta\right)^{k}},\end{aligned}\tag{A.33}$$

and

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{1,sing}(\varepsilon,\delta)=&\frac{1}{\pi^4}\frac{\sin\left(\pi\left(-i\varepsilon+\frac{\delta}{d_1}\right)\right)^{5+k-n-j}\sin\left(\pi\left(-i\varepsilon\alpha+\frac{\alpha}{d_1}\delta\right)\right)^{n}}{\sin\pi\left(i\varepsilon d_1\right)}\\ &\cdot\frac{\sin\left(\pi\left(-i\varepsilon\beta+\frac{\beta}{d_1}\delta\right)\right)^{j}}{\sin\left(\pi\left(-i\varepsilon d_2+\frac{d_2}{d_1}\delta\right)\right)^{k}}.\end{aligned}\tag{A.34}$$

What we can immediately conclude is that $\delta=d_1$ will never give a pole, because this pole gets cancelled. Therefore we can write

$$Z^{\zeta\ll 0}_{1,S^2}=-\frac{1}{2\pi}\sum_{\delta=1}^{d_1-1}\oint d\varepsilon(-1)^{\delta}\mathcal{Z}^{\zeta\ll 0}_{1,sing}(\varepsilon,\delta)|\mathcal{Z}^{\zeta\ll 0}_{1,reg}(\varepsilon,z,\delta)|^2.\tag{A.35}$$

Before we further manipulate the integral we introduce the fractional part $\langle x\rangle$ of x:

$$\langle x\rangle = x-\lfloor x\rfloor.\tag{A.36}$$

$\lfloor x\rfloor$ is the floor operator which gives the minimum integer n with $x\geq n$. One can show that:

$$\left\langle \alpha-\alpha\frac{k}{d}\right\rangle = 1-\left\langle\alpha\frac{k}{d}\right\rangle \quad \text{if } \left\langle\alpha\frac{k}{d}\right\rangle\neq 0,\tag{A.37}$$

with $\alpha,k,d\in\mathbb{N}$. This allows the following manipulations, for $\alpha,\beta,k,d\in\mathbb{N}$:

$$\begin{aligned}\sin\left(\pi\left(i\beta\sigma+\alpha\frac{k}{d}\right)\right)&=\sin\left(\pi\left(i\beta\sigma+\left\langle\alpha\frac{k}{d}\right\rangle+\left\lfloor\alpha\frac{k}{d}\right\rfloor\right)\right),\\ &=(-1)^{\lfloor\alpha\frac{k}{d}\rfloor}\sin\left(\pi\left(i\beta\sigma+\left\langle\alpha\frac{k}{d}\right\rangle\right)\right),\\ &=\frac{(-1)^{\lfloor\alpha\frac{k}{d}\rfloor}\pi}{\Gamma\left(i\beta\sigma+\langle\alpha\frac{k}{d}\rangle\right)\Gamma\left(1-i\beta\sigma-\langle\alpha\frac{k}{d}\rangle\right)},\\ &=\frac{(-1)^{\lfloor\alpha\frac{k}{d}\rfloor}\pi}{\Gamma\left(i\beta\sigma+\langle\alpha\frac{k}{d}\rangle\right)\Gamma\left(-i\beta\sigma+\langle\alpha\frac{d-k}{d}\rangle\right)},\end{aligned}\tag{A.38}$$

where the last line follows from (A.37). A crucial observation is that (A.34) only gives poles for δs such that:

$$\left\langle \frac{\delta}{d_1} \right\rangle \neq 0, \qquad \left\langle \alpha\frac{\delta}{d_1} \right\rangle \neq 0, \qquad \left\langle \alpha\frac{\beta}{d_1} \right\rangle \neq 0. \tag{A.39}$$

We will call such δ values narrow and the set of all such values narrow sector. This notation is in accordance with the terminology introduced in 6.2 in FJRW theory. In the narrow sectors we can use (A.37) and (A.38) to rewrite (A.34). We apply the variable transformation:

$$\varepsilon \to \frac{i\varepsilon}{d_1}, \tag{A.40}$$

which gives an overall sign due to the change of orientation. It is then possible to write (A.35) in the form:

$$\begin{aligned} Z^{\zeta\ll 0}_{1,S^2} = &\frac{1}{2\pi i d_1}\left(\frac{d_1}{d_2}\right)^k \sum_{\substack{\delta\in\text{narrow}\\ \frac{\tau_{d_2}}{\tau_{d_1}}\delta\in\mathbb{Z}}} \oint_0 d\varepsilon \frac{(-1)^{\text{Gr}}}{\varepsilon^{k+1}} \frac{\Gamma_\delta(\varepsilon)}{\Gamma^*_\delta(\varepsilon)} |I^{\zeta\ll 0}_\delta(z,\varepsilon)|^2, \\ &+ \frac{1}{d_1} \sum_{\substack{\delta\in\text{narrow}\\ \frac{\tau_{d_2}}{\tau_{d_1}}\delta\notin\mathbb{Z}}} (-1)^{\widetilde{\text{Gr}}} \frac{\widetilde{\Gamma}_\delta(0)}{\widetilde{\Gamma}^*_\delta(0)} |\widetilde{I}^{\zeta\ll 0}_\delta(z,0)|^2, \end{aligned} \tag{A.41}$$

with

$$\begin{aligned} I^{\zeta\ll 0}_\delta(z,\varepsilon) = &\sum_{a=0}^{\infty} z^{\left(-\frac{\varepsilon}{d_1}-a-\frac{\delta}{d_1}+q\right)} (-1)^{a(5+k-n-j+\alpha n+j\beta)} \\ &\cdot \frac{\Gamma\left(1+\varepsilon\right)\Gamma\left(1+\frac{\tau_{d_2}}{\tau_{d_1}}\varepsilon\right)^k}{\Gamma\left(\frac{\varepsilon}{d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j}\Gamma\left(\alpha\frac{\varepsilon}{d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^n} \\ &\cdot \frac{\Gamma\left(a+\frac{\varepsilon}{d_1}+\frac{\delta}{d_1}\right)^{5+k-n-j}}{\Gamma\left(\beta\frac{\varepsilon}{d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^j} \\ &\cdot \frac{\Gamma\left(a\alpha+\alpha\frac{\varepsilon}{d_1}+\frac{\alpha}{d_1}\delta\right)^n \Gamma^j\left(a\beta+\beta\frac{\varepsilon}{d_1}+\frac{\beta}{d_1}\delta\right)}{\Gamma\left(\delta+ad_1+\varepsilon\right)\Gamma\left(\frac{\tau_{d_2}}{\tau_{d_1}}\delta+ad_2+\frac{\tau_{d_2}}{\tau_{d_1}}\varepsilon\right)^k}, \end{aligned} \tag{A.42}$$

and

$$(-1)^{\text{Gr}} = (-1)^{\delta\left(\frac{\tau_{d_2}}{\tau_{d_1}}k+1\right)} (-1)^{(5+k-n-j)\left\lfloor\frac{\delta}{d_1}\right\rfloor} (-1)^{n\left\lfloor\alpha\frac{\delta}{d_1}\right\rfloor} (-1)^{j\left\lfloor\beta\frac{\delta}{d_1}\right\rfloor}. \tag{A.43}$$

We also introduced:

$$\begin{aligned}\Gamma_\delta(\varepsilon) = \Gamma\left(1-\varepsilon\right)\Gamma\left(1-\frac{\tau_{d_2}}{\tau_{d_1}}\varepsilon\right)^k \Gamma\left(\frac{\varepsilon}{d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j} \\ \cdot\Gamma\left(\alpha\frac{\varepsilon}{d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^n \Gamma\left(\beta\frac{\varepsilon}{d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^j,\end{aligned} \tag{A.44}$$

and

$$\begin{aligned}\Gamma^*_\delta(\varepsilon) = \Gamma\left(1+\varepsilon\right)\Gamma\left(1+\frac{\tau_{d_2}}{\tau_{d_1}}\varepsilon\right)^k \Gamma\left(-\frac{\varepsilon}{d_1}+\left\langle\frac{d_1-\delta}{d_1}\right\rangle\right)^{5+k-n-j} \\ \cdot\Gamma\left(-\alpha\frac{\varepsilon}{d_1}+\left\langle\alpha\frac{d_1-\delta}{d_1}\right\rangle\right)^n \Gamma\left(-\beta\frac{\varepsilon}{d_1}+\left\langle\beta\frac{d_1-\delta}{d_1}\right\rangle\right)^j.\end{aligned} \tag{A.45}$$

In the second line in (A.41) we used the fact that for $\frac{\tau_{d_2}}{\tau_{d_1}}\delta \notin \mathbb{Z}$ we only get a first order pole. We introduced:

$$\begin{aligned}\widetilde{I}^{\zeta\ll 0}_\delta(z,0) = &\sum_{a=0}^{\infty} z^{\left(-a-\frac{\delta}{d_1}+q\right)}(-1)^{a(5+k-n-j+\alpha n+j\beta)} \\ &\cdot\frac{\Gamma\left(\left\langle\tau_{d_2}\frac{\delta}{\tau_{d_1}}\right\rangle\right)^k}{\Gamma\left(\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j}\Gamma\left(\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^n\Gamma\left(\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^j} \\ &\cdot\frac{\Gamma^{5+k-n-j}\left(a+\frac{\delta}{d_1}\right)\Gamma^n\left(a\alpha+\frac{\alpha}{d_1}\delta\right)\Gamma^j\left(a\beta+\frac{\beta}{d_1}\delta\right)}{\Gamma\left(\delta+ad_1\right)\Gamma\left(\frac{\tau_{d_2}}{\tau_{d_1}}\delta+ad_2\right)^k},\end{aligned} \tag{A.46}$$

$$(-1)^{\widetilde{\mathrm{Gr}}} = (-1)^\delta(-1)^{\left\lfloor\frac{\tau_{d_2}}{\tau_{d_1}}\delta\right\rfloor k}(-1)^{(5+k-n-j)\left\lfloor\frac{\delta}{d_1}\right\rfloor}(-1)^{n\left\lfloor\alpha\frac{\delta}{d_1}\right\rfloor}(-1)^{j\left\lfloor\beta\frac{\delta}{d_1}\right\rfloor}, \tag{A.47}$$

$$\begin{aligned}\widetilde{\Gamma}_\delta(0) = \Gamma\left(\left\langle\tau_{d_2}\frac{\tau_{d_1}-\delta}{\tau_{d_1}}\right\rangle\right)^k\Gamma\left(\left\langle\frac{\delta}{d_1}\right\rangle\right)^{5+k-n-j} \\ \cdot\Gamma\left(\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^n\Gamma\left(\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^j,\end{aligned} \tag{A.48}$$

and

$$\begin{aligned}\widetilde{\Gamma}^*_\delta(0) = \Gamma\left(\left\langle\tau_{d_2}\frac{\delta}{\tau_{d_1}}\right\rangle\right)^k\Gamma\left(\left\langle\frac{d_1-\delta}{d_1}\right\rangle\right)^{5+k-n-j} \\ \cdot\Gamma\left(\left\langle\alpha\frac{d_1-\delta}{d_1}\right\rangle\right)^n\Gamma\left(\left\langle\beta\frac{d_1-\delta}{d_1}\right\rangle\right)^j.\end{aligned} \tag{A.49}$$

Z_{p_2} Contribution

Here we must sum over the poles we missed previously. These are of the form (A.19). Again we start by shifting the integration variable by

$$\sigma \to \varepsilon + \frac{i\left(d_2 m + 2n_2 + 2\right)}{2d_2}. \tag{A.50}$$

After replacing n_2 by (A.19), using now s as a summation variable, we get for the integral:

$$\begin{aligned} Z_{2,S^2}^{\zeta\ll 0} = \frac{(z\bar{z})^q}{2\pi} \sum_{\delta=0}^{\tau_{d_2}-2} \sum_{s=0}^{\infty} \sum_{-\frac{\tau_{d_2}}{d_2}s-\frac{\delta}{d_2}\leq m} \oint \mathrm{d}\varepsilon Z_{p_1}(\varepsilon,s,\delta,m) Z_{p_2}^k(\varepsilon,s,\delta,m) \\ \cdot Z_1^{5+k-n-j}(\varepsilon,s,\delta,m) Z_\alpha^n(\varepsilon,s,\delta,m) Z_\beta^j(\varepsilon,s,\delta,m) \\ \cdot \bar{z}^{i\varepsilon-\frac{s}{\kappa_2}-\frac{\delta+1}{d_2}} z^{i\varepsilon-\frac{s}{\kappa_2}-\frac{\delta+1}{d_2}-m}. \end{aligned} \tag{A.51}$$

The next step is to simplify the sum over m. Note that $\frac{\delta}{d_2}$ is never integer and so:

$$m \geq -\frac{\tau_{d_2}}{d_2}s - \frac{\delta}{d_2} \Rightarrow m \geq -\frac{\tau_{d_2}}{d_2}s = -\frac{s}{\kappa_2}. \tag{A.52}$$

Because m needs to be integer we introduce:

$$l = \kappa_2 m + s \Rightarrow m = \frac{l-s}{\kappa_2} \Rightarrow l = \kappa_2 b + \gamma \quad s = \kappa_2 a + \gamma, \tag{A.53}$$

with $\gamma = 0, 1, \ldots, \kappa_2 - 1$. After the above transformations, the integral reads:

$$\begin{aligned} Z_{2,S^2}^{\zeta\ll 0} = \frac{(z\bar{z})^q}{2\pi} \sum_{\delta=0}^{\tau_{d_2}-2} \sum_{\gamma=0}^{\kappa_2-1} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \oint \mathrm{d}\varepsilon Z_{p_1}(\varepsilon,a,b,\gamma,\delta,m) \\ \cdot Z_{p_2}^k(\varepsilon,a,b,\gamma,\delta,m) Z_1^{5+k-n-j}(\varepsilon,a,b,\gamma,\delta,m) \\ \cdot Z_\alpha^n(\varepsilon,a,b,\gamma,\delta,m) Z_\beta^j(\varepsilon,a,b,\gamma,\delta,m) \bar{z}^{i\varepsilon-a-\frac{\gamma}{\kappa_2}-\frac{\delta+1}{d_2}} \\ \cdot z^{i\varepsilon-b-\frac{\gamma}{\kappa_2}-\frac{\delta+1}{d_2}}. \end{aligned} \tag{A.54}$$

We can transform the contributions under the integral to:

$$\begin{aligned}
Z_{p_1}(\varepsilon,a,b,\gamma,\delta,m) &= \frac{\pi(-1)^{d_1 a}(-1)^{\tau_{d_1}\gamma}}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}(\delta+1)+d_1 b-i\varepsilon d_1+\tau_{d_1}\gamma\right)} \\
&\quad\cdot\frac{1}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}(\delta+1)+d_1 a-i\varepsilon d_1+\tau_{d_1}\gamma\right)} \\
&\quad\cdot\frac{1}{\sin\pi\left(\frac{\tau_{d_1}}{\tau_{d_2}}(\delta+1)-i\varepsilon d_1\right)}, \\
Z_{p_2}(\varepsilon,a,b,\gamma,\delta,m) &= \frac{\pi(-1)^{\delta}(-1)^{d_2 a}(-1)^{\tau_{d_2}\gamma}}{\Gamma\left(\delta+d_2 b-i\varepsilon d_2+\tau_{d_2}\gamma+1\right)} \\
&\quad\cdot\frac{1}{\Gamma\left(\delta+d_2 a-i\varepsilon d_2+\tau_{d_2}\gamma+1\right)\sin\pi i\varepsilon d_2},
\end{aligned} \tag{A.55}$$

and

$$\begin{aligned}
Z_1(\varepsilon,a,b,\gamma,\delta,m) &= (-1)^{b}\frac{\Gamma\left(a-i\varepsilon+\frac{\delta+1}{d_2}+\frac{\gamma}{\kappa_2}\right)}{\pi} \\
&\quad\cdot\Gamma\left(b-i\varepsilon+\frac{\delta+1}{d_2}+\frac{\gamma}{\kappa_2}\right) \\
&\quad\cdot\sin\pi\left(-i\varepsilon+\frac{\delta+1}{d_2}+\frac{\gamma}{\kappa_2}\right), \\
Z_\alpha(\varepsilon,a,b,\gamma,\delta,m) &= (-1)^{\alpha b}\Gamma\left(a\alpha-i\varepsilon\alpha+\frac{\alpha}{d_2}(\delta+1)+\frac{\gamma\alpha}{\kappa_2}\right) \\
&\quad\cdot\Gamma\left(b\alpha-i\varepsilon\alpha+\frac{\alpha}{d_2}(\delta+1)+\frac{\gamma\alpha}{\kappa_2}\right) \\
&\quad\cdot\frac{\sin\pi\left(-i\varepsilon\alpha+\frac{\alpha}{d_2}(\delta+1)+\frac{\gamma\alpha}{\kappa_2}\right)}{\pi}, \\
Z_\beta(\varepsilon,a,b,\gamma,\delta,m) &= (-1)^{\beta b}\Gamma\left(a\beta-i\varepsilon\beta+\frac{\beta}{d_2}(\delta+1)+\frac{\gamma\beta}{\kappa_2}\right) \\
&\quad\cdot\Gamma\left(b\beta-i\varepsilon\beta+\frac{\beta}{d_2}(\delta+1)+\frac{\gamma\beta}{\kappa_2}\right) \\
&\quad\cdot\frac{\sin\pi\left(-i\varepsilon\beta+\frac{\beta}{d_2}(\delta+1)+\frac{\gamma\beta}{\kappa_2}\right)}{\pi}.
\end{aligned} \tag{A.56}$$

As in the case of the Z_{p_1} contribution we can shift $\delta + 1 \to \delta$ and introduce

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{2,reg}(\varepsilon, z, \delta, \gamma) = &\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)} z^{i\varepsilon - a - \frac{\gamma}{\kappa_2} - \frac{\delta}{d_2} + q} \\ &\cdot \Gamma\left(a - i\varepsilon + \frac{\delta}{d_2} + \frac{\gamma}{\kappa_2}\right)^{5+k-n-j} \\ &\cdot \frac{\Gamma\left(a\alpha - i\varepsilon\alpha + \frac{\alpha}{d_2}\delta + \frac{\gamma\alpha}{\kappa_2}\right)^n}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}\delta + d_1 a - i\varepsilon d_1 + \tau_{d_1}\gamma\right)} \\ &\cdot \frac{\Gamma\left(a\beta - i\varepsilon\beta + \frac{\beta}{d_2}\delta + \frac{\gamma\beta}{\kappa_2}\right)^j}{\Gamma\left(\delta + d_2 a - i\varepsilon d_2 + \tau_{d_2}\gamma\right)^k},\end{aligned} \tag{A.57}$$

and

$$\begin{aligned}\mathcal{Z}^{\zeta\ll 0}_{2,sing}(\varepsilon, \delta, \gamma) = &\frac{1}{\pi^4}\sin\left(\pi\left(-i\varepsilon + \frac{\delta}{d_2} + \frac{\gamma}{\kappa_2}\right)\right)^{5+k-n-j} \\ &\cdot \frac{\sin\left(\pi\left(-i\varepsilon\alpha + \frac{\alpha}{d_2}\delta + \frac{\gamma\alpha}{\kappa_2}\right)\right)^n}{\sin\left(\pi\left(\frac{\tau_{d_1}}{\tau_{d_2}}\delta - i\varepsilon d_1\right)\right)} \\ &\cdot \frac{\sin\left(\pi\left(-i\varepsilon\beta + \frac{\beta}{d_2}\delta + \frac{\gamma\beta}{\kappa_2}\right)\right)^j}{\sin\left(\pi i\varepsilon d_2\right)^k}.\end{aligned} \tag{A.58}$$

The integral can now be written in the following form:

$$\begin{aligned}Z^{\zeta\ll 0}_{2,S^2} = &-\frac{1}{2\pi}\sum_{\delta=1}^{\tau_{d_2}-1}\sum_{\gamma=0}^{\kappa_2-1}(-1)^{k\delta}(-1)^{\tau_{d_1}\gamma}(-1)^{k\tau_{d_2}\gamma} \\ &\cdot \oint \mathrm{d}\varepsilon\, \mathcal{Z}^{\zeta\ll 0}_{2,sing}(\varepsilon, \delta, \gamma)|\mathcal{Z}^{\zeta\ll 0}_{2,reg}(\varepsilon, z, \delta, \gamma)|^2.\end{aligned} \tag{A.59}$$

The next step is to study the pole structure of (A.58). It is evident that potential poles can only come from the second factor in the denominator. We also find, similar to the Z_{p_1} case, that in the case of a non-vanishing contribution:

$$\begin{gathered}\left\langle \frac{\tau_{d_1}}{\tau_{d_2}}\delta \right\rangle \neq 0, \\ \left\langle \frac{\delta}{d_2} + \frac{\gamma}{\kappa_2} \right\rangle \neq 0, \quad \left\langle \alpha\left(\frac{\delta}{d_2} + \frac{\gamma}{\kappa_2}\right) \right\rangle \neq 0, \quad \left\langle \beta\left(\frac{\delta}{d_2} + \frac{\gamma}{\kappa_2}\right) \right\rangle \neq 0,\end{gathered} \tag{A.60}$$

such values we again call narrow. By this observation we can apply (A.38) and with the transformation:

$$\varepsilon \to \frac{i\epsilon}{d_2}, \tag{A.61}$$

and we can bring (A.59) in the form:

$$Z_{2,S^2}^{\zeta\ll 0} = \frac{(-1)^{k+1}}{2\pi i d_2} \sum_{\delta,\gamma\in\text{narrow}} \oint \mathrm{d}\varepsilon \frac{(-1)^{\mathrm{Gr}_{\delta,\gamma}}}{\varepsilon^k} \frac{\Gamma_{\delta,\gamma}(\varepsilon)}{\Gamma^*_{\delta,\gamma}(\varepsilon)} |I_{\delta,\gamma}^{\zeta\ll 0}(z,\varepsilon)|^2, \tag{A.62}$$

with:

$$\begin{aligned}(-1)^{\mathrm{Gr}_{\delta,\gamma}} = {} & (-1)^{k\delta}(-1)^{\tau_{d_1}\gamma}(-1)^{k\tau_{d_2}\gamma}(-1)^{\left\lfloor \frac{\tau_{d_1}}{\tau_{d_2}}\delta\right\rfloor} \\ & \cdot (-1)^{(5+k-n-j)\left\lfloor \frac{\delta}{d_2}+\frac{\gamma}{\kappa_2}\right\rfloor} \\ & \cdot (-1)^{n\left\lfloor \alpha\left(\frac{\delta}{d_2}+\frac{\gamma}{\kappa_2}\right)\right\rfloor}(-1)^{j\left\lfloor \beta\left(\frac{\delta}{d_2}+\frac{\gamma}{\kappa_2}\right)\right\rfloor},\end{aligned} \tag{A.63}$$

$$\begin{aligned}\Gamma_{\delta,\gamma}(\varepsilon) = {} & \Gamma(1-\varepsilon)^k\,\Gamma\left(-\frac{\tau_{d_1}}{\tau_{d_2}}\varepsilon + \left\langle \tau_{d_1}\frac{\tau_{d_2}-\delta}{\tau_{d_2}}\right\rangle\right) \\ & \cdot \Gamma\left(\frac{\varepsilon}{d_2} + \left\langle \frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{5+k-n-j} \\ & \cdot \Gamma\left(\alpha\frac{\varepsilon}{d_2} + \left\langle \alpha\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{n}\Gamma\left(\beta\frac{\varepsilon}{d_2} + \left\langle \beta\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{j}, \\ \Gamma^*_{\delta,\gamma}(\varepsilon) = {} & \Gamma(1+\varepsilon)^k\,\Gamma\left(-\frac{\tau_{d_1}}{\tau_{d_2}}\varepsilon + \left\langle \tau_{d_1}\frac{\delta}{\tau_{d_2}}\right\rangle\right) \\ & \cdot \Gamma\left(-\frac{\varepsilon}{d_2} + \left\langle \frac{d_2-\delta-\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{5+k-n-j} \\ & \cdot \Gamma\left(-\alpha\frac{\varepsilon}{d_2} + \left\langle \alpha\frac{d_2-\delta-\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{n} \\ & \cdot \Gamma\left(-\beta\frac{\varepsilon}{d_2} + \left\langle \beta\frac{d_2-\delta-\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{j},\end{aligned} \tag{A.64}$$

and

$$
\begin{aligned}
I_{\delta,\gamma}^{\zeta\ll 0}(z,\varepsilon) = & \frac{\Gamma(1+\varepsilon)^k}{\Gamma\left(\frac{\varepsilon}{d_2}+\left\langle\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^{5+k-n-j}\Gamma\left(\alpha\frac{\varepsilon}{d_2}+\left\langle\alpha\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^n} \\
& \cdot\frac{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}\varepsilon+\left\langle\tau_{d_1}\frac{\delta}{\tau_{d_2}}\right\rangle\right)}{\Gamma\left(\beta\frac{\varepsilon}{d_2}+\left\langle\beta\frac{\delta+\tau_{d_2}\gamma}{d_2}\right\rangle\right)^j} \\
& \cdot\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)}z^{-\frac{\varepsilon}{d_2}-a-\frac{\gamma}{\kappa_2}-\frac{\delta}{d_2}+q} \\
& \cdot\frac{\Gamma\left(a+\frac{\varepsilon}{d_2}+\frac{\delta}{d_2}+\frac{\gamma}{\kappa_2}\right)^{5+k-n-j}}{\Gamma\left(\frac{\tau_{d_1}}{\tau_{d_2}}\delta+d_1a+\frac{\tau_{d_2}}{\tau_{d_2}}\varepsilon+\tau_{d_1}\gamma\right)} \\
& \cdot\frac{\Gamma\left(a\alpha+\alpha\frac{\varepsilon}{d_2}+\frac{\alpha}{d_2}\delta+\frac{\gamma\alpha}{\kappa_2}\right)^n\Gamma\left(a\beta+\beta\frac{\varepsilon}{d_2}+\frac{\beta}{d_2}\delta+\frac{\gamma\beta}{\kappa_2}\right)^j}{\Gamma\left(\delta+d_2a+\varepsilon+\tau_{d_2}\gamma\right)^k},
\end{aligned}
\tag{A.65}
$$

A.5 $\zeta \gg 0$ Contributions

In the $\zeta \gg 0$ phase our strategy is to first sum over the poles of Z_β and afterwards over the remaining poles of Z_α (A.13).

Z_β Contribution

We start by performing the shift:

$$
\sigma \to \varepsilon + \frac{i\left(\beta m - 2n_\beta\right)}{2\beta}, \tag{A.66}
$$

and write (A.1) as sum over residues:

$$
\begin{aligned}
Z_{1,S^2}^{\zeta\gg 0} = & -\frac{(z\bar{z})^q}{2\pi}\sum_{n_\beta=0}^{\infty}\sum_{m\leq\frac{n_\beta}{\beta}}\oint \mathrm{d}\varepsilon Z_{p_1}(\varepsilon,n_\beta,m)Z_{p_2}^k(\varepsilon,n_\beta,m) \\
& \cdot Z_1^{5+k-n-j}(\varepsilon,n_\beta,m)Z_\alpha^n(\varepsilon,n_\beta,m)Z_\beta^j(\varepsilon,n_\beta,m) \\
& \cdot \bar{z}^{i\varepsilon+\frac{n_\beta}{\beta}}z^{i\varepsilon+\frac{n_\beta}{\beta}-m}.
\end{aligned}
\tag{A.67}
$$

The overall minus sign is introduced because we close the path clockwise. The sum over m can by simplified by the following transformation:

$$
l = n_\beta - \beta m \geq 0 \quad m = \frac{n_\beta - l}{\beta}. \tag{A.68}
$$

We know that m must be integer and therefore we set:

$$n_\beta = \beta a + \delta \quad l = \beta b + \delta \quad \delta = 0, 1, \dots \beta - 1. \tag{A.69}$$

By the above transformations we can bring (A.67) into the form:

$$\begin{aligned} Z_{1,S^2}^{\zeta \gg 0} = & -\frac{(z\bar{z})^q}{2\pi} \sum_{\delta=0}^{\beta-1} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \oint \mathrm{d}\varepsilon Z_{p_1}(\varepsilon, a, b, \delta) Z_{p_2}^k(\varepsilon, a, b, \delta) \\ & \cdot Z_1^{5+k-n-j}(\varepsilon, a, b, \delta) Z_\alpha^n(\varepsilon, a, b, \delta) Z_\beta^j(\varepsilon, a, b, \delta) \\ & \cdot \bar{z}^{i\varepsilon + a + \frac{\delta}{\beta}} z^{i\varepsilon + b + \frac{\delta}{\beta}}. \end{aligned} \tag{A.70}$$

The integral contributions are given by:

$$\begin{aligned} Z_{p_1}(\varepsilon, a, b, \delta) &= -(-1)^{bd_1} \frac{\sin\left(\pi\left(i\varepsilon d_1 + \frac{\delta d_1}{\beta}\right)\right)}{\pi} \\ &\cdot \Gamma\left(a d_1 + \frac{\delta d_1}{\beta} + i\varepsilon d_1 + 1\right) \\ &\cdot \Gamma\left(b d_1 + \frac{\delta d_1}{\beta} + i\varepsilon d_1 + 1\right), \\ Z_{p_2}(\varepsilon, a, b, \delta) &= -(-1)^{bd_2} \frac{\sin\left(\pi\left(i\varepsilon d_2 + \frac{\delta d_2}{\beta}\right)\right)}{\pi} \\ &\cdot \Gamma\left(a d_2 + \frac{\delta d_2}{\beta} + i\varepsilon d_2 + 1\right) \\ &\cdot \Gamma\left(b d_2 + \frac{\delta d_2}{\beta} + i\varepsilon d_2 + 1\right), \\ Z_1(\varepsilon, a, b, \delta) &= -(-1)^a \frac{\pi}{\sin\left(\pi\left(i\varepsilon - \frac{\delta}{\beta}\right)\right)} \\ &\cdot \frac{1}{\Gamma\left(a + \frac{\delta}{\beta} + i\varepsilon + 1\right)\Gamma\left(b + \frac{\delta}{\beta} + i\varepsilon + 1\right)}, \\ Z_\alpha(\varepsilon, a, b, \delta) &= -(-1)^{a\alpha} \frac{\pi}{\sin\left(\pi\left(i\varepsilon\alpha - \frac{\delta\alpha}{\beta}\right)\right)} \\ &\cdot \frac{1}{\Gamma\left(a\alpha + \frac{\delta\alpha}{\beta} + i\varepsilon\alpha + 1\right)\Gamma\left(b\alpha + \frac{\delta\alpha}{\beta} + i\varepsilon\alpha + 1\right)}, \\ Z_\beta(\varepsilon, a, b, \delta) &= -(-1)^{a\beta}(-1)^\delta \frac{\pi}{\sin\left(\pi\left(i\varepsilon\beta\right)\right)} \\ &\cdot \frac{1}{\Gamma(a\beta + i\varepsilon\beta + \delta + 1)\Gamma(b\beta + i\varepsilon\beta + \delta + 1)}. \end{aligned} \tag{A.71}$$

We introduce the following quantities:

$$\mathcal{Z}^{\zeta\gg 0}_{1,reg}(\varepsilon,z,\delta)=\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)}z^{i\varepsilon+a+\frac{\delta}{\beta}+q}$$
$$\cdot\frac{\Gamma\left(ad_1+\frac{\delta d_1}{\beta}+i\varepsilon d_1+1\right)}{\Gamma\left(a+\frac{\delta}{\beta}+i\varepsilon+1\right)^{5+k-n-j}\Gamma\left(a\alpha+\frac{\delta\alpha}{\beta}+i\varepsilon\alpha+1\right)^{n}}$$
$$\cdot\frac{\Gamma\left(ad_2+\frac{\delta d_2}{\beta}+i\varepsilon d_2+1\right)^{k}}{\Gamma\left(a\beta+i\varepsilon\beta+\delta+1\right)^{j}}, \tag{A.72}$$

$$\mathcal{Z}^{\zeta\gg 0}_{1,sing}(\varepsilon,\delta)=\frac{\pi^4\sin\left(\pi\left(i\varepsilon d_1+\frac{\delta d_1}{\beta}\right)\right)}{\sin\left(\pi\left(i\varepsilon-\frac{\delta}{\beta}\right)\right)^{5+k-n-j}\sin\left(\pi\left(i\varepsilon\alpha-\frac{\delta\tau_\alpha}{\tau_\beta}\right)\right)^{n}}$$
$$\cdot\frac{\sin\left(\pi\left(i\varepsilon d_2+\frac{\delta d_2}{\beta}\right)\right)^{k}}{\sin\left(\pi\left(i\varepsilon\beta\right)\right)^{j}} \tag{A.73}$$

and write the integral as

$$Z^{\zeta\gg 0}_{1,S^2}=-\frac{1}{2\pi}\sum_{\delta=0}^{\beta-1}(-1)^{j\delta}\oint_0 \mathrm{d}\varepsilon\,\mathcal{Z}^{\zeta\gg 0}_{1,sing}(\varepsilon,\delta)|\mathcal{Z}^{\zeta\gg 0}_{1,reg}(\varepsilon,z,\delta)|^2. \tag{A.74}$$

We want to remark that the minus signs in (A.71) add up to one:

$$(-1)^{5+k-n-j+n+j+k+1}=(-1)^{6+2k}=1. \tag{A.75}$$

The next task is to study for which parameters we can hit a pole of (A.73). The outcome of this analysis is, that we only get poles if $\delta=0$. In that case we can further simplify (A.74) to

$$Z^{\zeta\gg 0}_{1,S^2}=-\frac{1}{2\pi}\oint_0 \mathrm{d}\varepsilon\,\mathcal{Z}^{\zeta\gg 0}_{1,sing}(\varepsilon)|\mathcal{Z}^{\zeta\gg 0}_{1,reg}(\varepsilon,z)|^2, \tag{A.76}$$

with

$$\mathcal{Z}^{\zeta\gg 0}_{1,reg}(\varepsilon,z)=\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)}z^{i\varepsilon+a+q}$$
$$\cdot\frac{\Gamma\left(ad_1+i\varepsilon d_1+1\right)}{\Gamma\left(a+i\varepsilon+1\right)^{5+k-n-j}\Gamma\left(a\alpha+i\varepsilon\alpha+1\right)^{n}}$$
$$\cdot\frac{\Gamma\left(ad_2+i\varepsilon d_2+1\right)^{k}}{\Gamma(a\beta+i\varepsilon\beta+1)^{j}} \tag{A.77}$$

and

$$\mathcal{Z}_{1,sing}^{\zeta\gg0}(\varepsilon) = \frac{\pi^4 \sin(\pi i\varepsilon d_1) \sin(\pi i\varepsilon d_2)^k}{\sin(\pi i\varepsilon)^{5+k-n-j} \sin(\pi i\varepsilon\alpha)^n \sin(\pi i\varepsilon\beta)^j}. \tag{A.78}$$

We can rewrite (A.78) into the form:

$$\begin{aligned}\mathcal{Z}_{1,sing}^{\zeta\gg0}(\varepsilon) = {}& \frac{d_1 d_2^k}{\alpha^n \beta^j \varepsilon^4} \frac{\Gamma(1+i\varepsilon)^{5+k-n-j}\,\Gamma(1-i\varepsilon)^{5+k-n-j}}{\Gamma(1+i\varepsilon d_1)\,\Gamma(1-i\varepsilon d_1)} \\ & \cdot \frac{\Gamma(1+i\varepsilon\alpha)^n\,\Gamma(1-i\varepsilon\alpha)^n\,\Gamma(1+i\varepsilon\beta)^j\,\Gamma(1-i\varepsilon\beta)^j}{\Gamma(1+i\varepsilon d_2)^k\,\Gamma(1-i\varepsilon d_2)^k}.\end{aligned} \tag{A.79}$$

Next we make the following transformation:

$$\varepsilon \to -\frac{\varepsilon}{2\pi}, \tag{A.80}$$

and introduce

$$\begin{aligned}\Gamma(\varepsilon) = {}& \frac{\Gamma\left(1-\frac{\varepsilon}{2\pi i}\right)^{5+k-n-j}\Gamma\left(1-\frac{\varepsilon}{2\pi i}\right)^{5+k-n-j}}{\Gamma\left(1-d_1\frac{\varepsilon}{2\pi i}\right)} \\ & \cdot \frac{\Gamma\left(1-\alpha\frac{\varepsilon}{2\pi i}\right)^n\Gamma\left(1-\beta\frac{\varepsilon}{2\pi i}\right)^j}{\Gamma\left(1-d_2\frac{\varepsilon}{2\pi i}\right)^k},\end{aligned} \tag{A.81}$$

and

$$\begin{aligned}I^{\zeta\gg0}(\varepsilon,z) = {}& \sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)} z^{i\varepsilon+a+q} \\ & \cdot \frac{\Gamma\left(1+\frac{\varepsilon}{2\pi i}\right)^{5+k-n-j}\Gamma\left(1+\frac{\varepsilon}{2\pi i}\right)^{5+k-n-j}\Gamma\left(1+\alpha\frac{\varepsilon}{2\pi i}\right)^n}{\Gamma\left(1+d_1\frac{\varepsilon}{2\pi i}\right)} \\ & \cdot \frac{\Gamma\left(1+\beta\frac{\varepsilon}{2\pi i}\right)^j}{\Gamma\left(1+d_2\frac{\varepsilon}{2\pi i}\right)^k} \\ & \cdot \frac{\Gamma\left(ad_1+d_1\frac{\varepsilon}{2\pi i}+1\right)}{\Gamma\left(a+\frac{\varepsilon}{2\pi i}+1\right)^{5+k-n-j}\Gamma\left(a\alpha+\alpha\frac{\varepsilon}{2\pi i}+1\right)^n} \\ & \cdot \frac{\Gamma\left(ad_2+d_2\frac{\varepsilon}{2\pi i}+1\right)^k}{\Gamma\left(a\beta+\beta\frac{\varepsilon}{2\pi i}+1\right)^j}.\end{aligned} \tag{A.82}$$

By using the above quantities we can bring the integral in the following form:

$$Z_{1,S^2}^{\zeta\gg0} = -(2\pi)^2 \frac{d_1 d_2^k}{\alpha^n\beta^j} \oint_0 \mathrm{d}\varepsilon \frac{1}{\varepsilon^4} \frac{\Gamma(\varepsilon)}{\Gamma^*(\varepsilon)} |I^{\zeta\gg0}(\varepsilon,z)|^2, \tag{A.83}$$

with

$$\Gamma^*(\varepsilon) = \Gamma(-\varepsilon). \tag{A.84}$$

Z_α Contribution

As previously discussed, it is not enough to sum only over the poles of Z_β. We miss the Z_α poles of the form (A.13). We will now take into account these remaining poles and again begin by shifting the integration variable by

$$\sigma \to \sigma + \frac{i\,(\alpha m - 2n_4)}{2\alpha}. \tag{A.85}$$

Afterwards we replace n_4 by (A.13). These transformations bring (A.1) into the form:

$$\begin{aligned} Z_{2,S^2}^{\zeta\gg 0} = &-\frac{(z\bar{z})^q}{2\pi}\sum_{\delta=1}^{\tau_\alpha-1}\sum_{s=0}^{\infty}\sum_{m\leq\frac{s}{\kappa_1}+\frac{\delta}{\alpha}}\oint \mathrm{d}\sigma Z_{p_1}(\sigma,s,\delta,m)Z_{p_2}^k(\sigma,s,\delta,m) \\ &\cdot Z_1^{5+k-n-j}(\sigma,s,\delta,m)Z_\alpha^n(\sigma,s,\delta,m)Z_\beta^j(\sigma,s,\delta,m) \\ &\cdot \bar{z}^{i\sigma+\frac{s}{\kappa_1}+\frac{\delta}{\alpha}}z^{i\sigma+\frac{s}{\kappa_1}+\frac{\delta}{\alpha}-m}. \end{aligned} \tag{A.86}$$

The overall minus sign is due the orientation of the path. The next step is to simplify the sum over m. We see that $\frac{\delta}{\alpha}$ can never be an integer and therefore it is valid to set

$$m \leq \frac{s}{\kappa_1}. \tag{A.87}$$

We introduce:

$$l = s - \kappa_1 m \quad \to m = \frac{s-l}{\kappa_1}. \tag{A.88}$$

Because m needs to be integer it follows that:

$$s = \kappa_{\alpha\beta}a + \gamma, \quad l = \kappa_1 b + \gamma, \quad \gamma = 0, \ldots, \kappa_1 - 1. \tag{A.89}$$

We can now write the integral as

$$\begin{aligned} Z_{2,S^2}^{\zeta\gg 0} = &-\frac{(z\bar{z})^q}{2\pi}\sum_{\delta=1}^{\tau_\alpha-1}\sum_{\gamma=0}^{\kappa_1-1}\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\oint \mathrm{d}\sigma Z_{p_1}(\sigma,a,b,\delta,\gamma)Z_{p_2}^k(\sigma,a,b,\delta,\gamma) \\ &\cdot Z_1^{5+k-n-j}(\sigma,a,b,\delta,\gamma)Z_\alpha^n(\sigma,a,b,\delta,\gamma)Z_\beta^j(\sigma,a,b,\delta,\gamma) \\ &\cdot \bar{z}^{i\sigma+a+\frac{\gamma}{\kappa_1}+\frac{\delta}{\alpha}}z^{i\sigma+b+\frac{\gamma}{\kappa_1}+\frac{\delta}{\alpha}}, \end{aligned} \tag{A.90}$$

with the following contributions

$$\begin{aligned}
Z_{p_1}(\sigma,a,b,\delta,\gamma) &= -(-1)^{bd_1}\frac{\sin\left(\pi\left(\frac{\delta d_1}{\alpha}+i\sigma d_1+\frac{\gamma d_1}{\kappa_1}\right)\right)}{\pi}\\
&\cdot\Gamma\left(ad_1+\frac{\delta d_1}{\alpha}+i\sigma d_1+\frac{\gamma d_1}{\kappa_1}+1\right)\\
&\cdot\Gamma\left(bd_1+\frac{\delta d_1}{\alpha}+i\sigma d_1+\frac{\gamma d_1}{\kappa_1}+1\right),\\
Z_{p_2}(\sigma,a,b,\delta,\gamma) &= -(-1)^{bd_2}\frac{\sin\left(\pi\left(\frac{\delta d_2}{\alpha}+i\sigma d_2+\frac{\gamma d_2}{\kappa_1}\right)\right)}{\pi}\\
&\cdot\Gamma\left(ad_2+\frac{\delta d_2}{\alpha}+i\sigma d_2+\frac{\gamma d_2}{\kappa_1}+1\right)\\
&\cdot\Gamma\left(bd_2+\frac{\delta d_2}{\alpha}+i\sigma d_2+\frac{\gamma d_2}{\kappa_1}+1\right),\\
Z_1(\sigma,a,b,\delta,\gamma) &= -(-1)^{a}\frac{\pi}{\sin\left(\pi\left(\frac{\delta}{\alpha}+i\sigma+\frac{\gamma}{\kappa_1}\right)\right)}\\
&\cdot\frac{1}{\Gamma\left(a+\frac{\delta}{\alpha}+i\sigma+\frac{\gamma}{\kappa_1}+1\right)}\\
&\cdot\frac{1}{\Gamma\left(b+\frac{\delta}{\alpha}+i\sigma+\frac{\gamma}{\kappa_1}+1\right)},\\
Z_{\alpha}(\sigma,a,b,\delta,\gamma) &= -(-1)^{a\alpha}(-1)^{\delta}(-1)^{\gamma\tau_\alpha}\frac{\pi}{\sin\left(\pi\left(i\sigma\alpha\right)\right)}\\
&\cdot\frac{1}{\Gamma\left(a\alpha+i\sigma\alpha+\delta+\gamma\tau_\alpha+1\right)}\\
&\cdot\frac{1}{\Gamma\left(b\alpha+i\sigma\alpha+\delta+\gamma\tau_\alpha+1\right)},\\
Z_{\beta}(\sigma,a,b,\delta,\gamma) &= -(-1)^{a\beta}(-1)^{\gamma\tau_\beta}\frac{\pi}{\sin\left(\pi\left(\frac{\delta\tau_\beta}{\tau_\alpha}+i\sigma\beta\right)\right)}\\
&\cdot\frac{1}{\Gamma\left(a\beta+\frac{\delta\tau_\beta}{\tau_\alpha}+i\sigma\beta+\gamma\tau_\beta+1\right)}\\
&\cdot\frac{1}{\Gamma\left(b\beta+\frac{\delta\tau_\beta}{\tau_\alpha}+i\sigma\beta+\gamma\tau_\beta+1\right)}
\end{aligned}\tag{A.91}$$

We introduce:

$$\begin{aligned}\mathcal{Z}^{\zeta\gg 0}_{2,reg}(\sigma,z,\delta,\gamma) = &\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+n\alpha+j\beta)} z^{i\sigma+a+\frac{\gamma}{\kappa_1}+\frac{\delta}{\alpha}+q}\\ &\cdot\frac{\Gamma\left(ad_1+\frac{\delta d_1}{\alpha}+i\sigma d_1+\frac{\gamma d_1}{\kappa_1}+1\right)}{\Gamma\left(a+\frac{\delta}{\alpha}+i\sigma+\frac{\gamma}{\kappa_1}+1\right)^{5+k-n-j}}\\ &\cdot\frac{\Gamma\left(ad_2+\frac{\delta d_2}{\alpha}+i\sigma d_2+\frac{\gamma d_2}{\kappa_1}+1\right)^k}{\Gamma\left(a\alpha+i\sigma\alpha+\delta+\gamma\tau_\alpha+1\right)^n}\\ &\cdot\frac{1}{\Gamma\left(a\beta+\frac{\delta\tau_\beta}{\tau_\alpha}+i\sigma\beta+\gamma\tau_\beta+1\right)^j},\end{aligned} \tag{A.92}$$

and

$$\begin{aligned}\mathcal{Z}^{\zeta\gg 0}_{2,sing}(\sigma,\delta,\gamma) = \pi^4 &\frac{\sin\left(\pi\left(\frac{\delta d_1}{\alpha}+i\sigma d_1+\frac{\gamma d_1}{\kappa_1}\right)\right)}{\sin\left(\pi\left(\frac{\delta}{\alpha}+i\sigma+\frac{\gamma}{\kappa_1}\right)\right)^{5+k-n-j}\sin\left(\pi\left(i\sigma\alpha\right)\right)^n}\\ &\cdot\frac{\sin\left(\pi\left(\frac{\delta d_2}{\alpha}+i\sigma d_2+\frac{\gamma d_2}{\kappa_1}\right)\right)^k}{\sin\left(\pi\left(\frac{\delta\tau_\beta}{\tau_\alpha}+i\sigma\beta\right)\right)^j}.\end{aligned} \tag{A.93}$$

We collect all the minus signs appearing in (A.91) and find

$$(-1)^{1+k+5+k-n-j+n+j} = (-1)^{6+2k} = 1. \tag{A.94}$$

The integral can now be written in the following compact form:

$$\begin{aligned}Z^{\zeta\gg 0}_{2,S^2} = &-\frac{1}{2\pi}\sum_{\delta=1}^{\tau_\alpha-1}\sum_{\gamma=0}^{\kappa_1-1}(-1)^{n\delta}(-1)^{n\gamma\tau_\alpha}(-1)^{j\gamma\tau_\beta}\\ &\cdot\oint \mathrm{d}\sigma\,\mathcal{Z}^{\zeta\gg 0}_{2,sing}(\sigma,\delta,\gamma)|\mathcal{Z}^{\zeta\gg 0}_{2,reg}(\sigma,z,\delta,\gamma)|^2.\end{aligned} \tag{A.95}$$

The final task is to analyse the pole structure of (A.93). We find for our models of interest, that (A.95) is always zero:

$$Z^{\zeta\gg 0}_{2,S^2} = 0. \tag{A.96}$$

Appendix B

Evaluation of the Sphere Partition Function - Two Parameter Abelian

In Appendix A we discussed the one-parameter abelian models. Here we consider the two parameter models. In contrast to the previous discussion we encounter multi-dimensional residues and we need a more sophisticated algorithm to evaluate the sphere partition function. We use the procedure developed in [118], which extends the results of [134]. We will refer to the two $U(1)^2$ models by their geometric phase, which is either $\mathbb{P}_{11222}[8]$ or $\mathbb{P}_{11169}[18]$. In the following, in order to get used to the algorithm of [118], we will describe key features for an abstract two parameter model with field content:

	p	x_6	x_1	x_2	x_3	x_4	x_5	FI
$U(1)_1$	$-d$	1	0	0	α_3	α_4	α_5	ζ_1
$U(1)_2$	0	$-\beta_6$	1	1	β_3	0	0	ζ_2
$U(1)_V$	$2-2dq_1$	$2q_1-2\beta_6 q_2$	$2q_1$	$2q_1$	$2\alpha_3 q_1+2\beta_3 q_2$	$2\alpha_4 q_1$	$2\alpha_5 q_1$	
$\beta_6 U(1)_1+U(2)_2$	$-\beta_6 d$	0	1	1	$\beta_6\alpha_3+\beta_3$	$\beta_6\alpha4$	$\beta_6\alpha_5$	$\beta_6\zeta_1+\zeta_2$

, (B.1)

with

$$d,\alpha_3,\alpha_4,\alpha_5,\beta_3,\beta_6\in\mathbb{Z}_{\geq 0}, \tag{B.2}$$

(B.3)

and

$$\alpha_3=d-1-\alpha_4-\alpha_5, \qquad \beta_3=\beta_6-2. \tag{B.4}$$

We further assume:

$$\alpha_5\geq\alpha_4. \tag{B.5}$$

For later convenience we introduce:

$$\beta_1 = 1. \tag{B.6}$$

Let us mention that in the models studied in this thesis:

$$(\alpha_3, \beta_3) \in \{(1,0), (0,1)\} \tag{B.7}$$

The complete field content for the model $\mathbb{P}_{11222}[8]$ and $\mathbb{P}_{11169}[18]$ is given in (6.143) and (6.201), respectively. The superpotential in these models is of the form

$$W = pG_{(d,0)}(x_i), \tag{B.8}$$

where $G_{(d,0)}(x_i)$ is weighted homogenous polynomial of degree $(d,0)$ under the $U(1)$ actions. We suppose that $G_{(d,0)}$ is chosen such that

$$\frac{\partial G_{(d,0)}(x_j)}{\partial x_i} = 0 \quad \forall i \quad \Leftrightarrow \quad x_j = 0 \quad \forall j. \tag{B.9}$$

The columns in (B.1) which encode the $U(1)$ charges are referred to as charge vectors.

To study the phase structure of the model (B.1) we look at the D-term and F-term equations, as done in Section 6.4. For our class of models it is enough to focus on the D-terms to determine the phase boundaries and therefore we can apply a method given in [161]. In this method one embeds the charge lattice into the space spanned by the FI-theta parameters $\mathbb{R}^k$ in the case of k different parameters. The phase boundaries encoded in the D-terms are given by hypersurfaces in the positive linear span of $(k-1)$ charge vectors of some of the fields. For our models of interest the phase diagram is given in Figure B.1, where the phase boundaries are given by the dotted grey half-lines. Observe, that we drew charge vectors with label $x_{3\alpha}$ and $x_{3\beta}$, as x_3 lies either on the $\zeta_1 > 0$ line for $(\alpha_3, \beta_3) = (1,0)$ or on the $\zeta_2 > 0$ line for $(\alpha_3, \beta_3) = (0,1)$.

We identify the following phases

$$\begin{aligned} &\mathbf{I} : \zeta_1 > 0, \zeta_2 > 0, && \text{Geometric phase,} \\ &\mathbf{II} : \zeta_2 < 0, \beta_6\zeta_1 + \zeta_2 > 0, && \text{Orbifold phase,} \\ &\mathbf{III} : \zeta_2 < 0, \beta_6\zeta_1 + \zeta_2 < 0, && \text{Landau-Ginzburg orbifold phase,} \\ &\mathbf{IV} : \zeta_1 < 0, \zeta_2 > 0, && \text{Hybrid phase.} \end{aligned} \tag{B.10}$$

The characteristics of the above phases are discussed in Section 6.4.

From Figure B.1 the deleted set for each phase can be determined following the approach of [161]. For this purpose one draws a hyperplane through the origin such that the phase of interest lies completely on one side. All fields, whose charge vectors lie on the same side as the phase form a subset of the deleted set. This procedure is repeated for every possible hyperplane such that

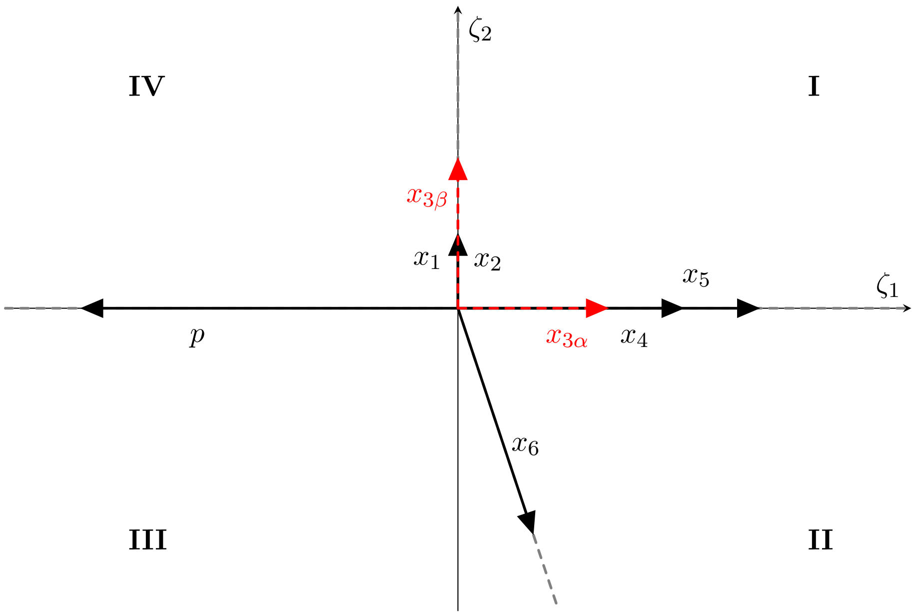

Figure B.1: Phase diagramm of the two paramter models

the phase of interest lies on one side. The deleted set of the phase is then the union of all obtained subsets. For our model the deleted sets are given by:

$$\begin{aligned} F_{\mathbf{I}} &= \{x_1 = x_2 = x_{3\beta} = 0\} \cup \{x_{3\alpha} = x_4 = x_5 = x_6 = 0\}, \\ F_{\mathbf{II}} &= \{x_1 = x_2 = \cdots = x_4 = x_5 = 0\} \cup \{x_6 = 0\}, \\ F_{\mathbf{III}} &= \{p = 0\} \cup \{x_6 = 0\}, \\ F_{\mathbf{IV}} &= \{p = 0\} \cup \{x_1 = x_2 = x_{3\beta} = 0\}. \end{aligned} \tag{B.11}$$

Not all fields appearing in the deleted set are forced to have a non-zero vacuum expectation value (VEV) in the respective phase. To decide which fields are classically zero one has to take a look at the D- and F-term equations (see 4.2). For example in the phase **II** it is possible to set $x_1 = x_2 = x_{3\beta} = 0$.

The sphere partition function of the model with field content (B.1) reads:

$$\begin{aligned} Z_{S^2} = \frac{1}{(2\pi)^2} \sum_{m_1, m_2} \iint_{\mathbb{R}^2} \mathrm{d}\sigma^2 Z_p Z_6 Z_1^2 Z_3 \\ \cdot Z_4 Z_5 e^{-4\pi i(\zeta_1\sigma_1 + \zeta_2\sigma_2) - i(\theta_1 m_1 + \theta_2 m_2)}, \end{aligned} \tag{B.12}$$

with

$$\begin{aligned}
Z_p &= \frac{\Gamma\left(d(i\sigma_1 - q_1 + \frac{1}{d}) + \frac{dm_1}{2}\right)}{\Gamma\left(1 - d(i\sigma_1 - q_1 + \frac{1}{d}) + \frac{dm_1}{2}\right)}, \\
Z_6 &= \frac{\Gamma\left(-\left((i\sigma_1 - q_1) - \beta_6(i\sigma_2 - q_2)\right) - \frac{m_1 - \beta_6 m_2}{2}\right)}{\Gamma\left(1 + \left((i\sigma_1 - q_1) - \beta_6(i\sigma_2 - q_2)\right) - \frac{m_1 - \beta_6 m_2}{2}\right)}, \\
Z_1 &= \frac{\Gamma\left(-(i\sigma_2 - q_2) - \frac{m_2}{2}\right)}{\Gamma\left(1 + (i\sigma_2 - q_2) - \frac{m_2}{2}\right)}, \\
Z_3 &= \frac{\Gamma\left(-\alpha_3(i\sigma_1 - q_1) - \beta_3(i\sigma_2 - q_2) - \frac{\alpha_3 m_1 + \beta_3 m_2}{2}\right)}{\Gamma\left(1 + \alpha_3(i\sigma_1 - q_1) + \beta_3(i\sigma_2 - q_2) - \frac{\alpha_3 m_1 + \beta_3 m_2}{2}\right)}, \\
Z_4 &= \frac{\Gamma\left(-\alpha_4(i\sigma_1 - q_1) - \frac{\alpha_4 m_1}{2}\right)}{\Gamma\left(1 + \alpha_4(i\sigma_1 - q_1) - \frac{\alpha_4 m_1}{2}\right)}, \\
Z_5 &= \frac{\Gamma\left(-\alpha_5(i\sigma_1 - q_1) - \frac{\alpha_5 m_1}{2}\right)}{\Gamma\left(1 + \alpha_5(i\sigma_1 - q_1) - \frac{\alpha_5 m_1}{2}\right)}.
\end{aligned} \tag{B.13}$$

Our goal is to evaluate (B.12) in the phases (B.10) by a residue integration. In a first step we make the following transformation of variables:

$$\sigma_i = -ix_i + q_i, \tag{B.14}$$

which results in

$$\begin{aligned}
Z_{S^2} = \frac{1}{(2\pi)^2} \sum_{m_1, m_2} \iint_{i\mathbb{R}^2 + \gamma} &\mathrm{d}x^2 Z_p Z_6 Z_1^2 Z_3 \\
&\cdot Z_4 Z_5 e^{-\mathbf{p}\cdot\mathbf{x} + \mathbf{p}\cdot\gamma - i(\theta_1 m_1 + \theta_2 m_2)},
\end{aligned} \tag{B.15}$$

with:

$$\gamma = -\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \mathbf{p} = 4\pi \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{B.16}$$

The different contributions of (B.15) read:

$$\begin{aligned}
Z_p &= \frac{\Gamma\left(1+dx_1+\frac{dm_1}{2}\right)}{\Gamma\left(-dx_1+\frac{dm_1}{2}\right)},\\
Z_6 &= \frac{\Gamma\left(-\left(x_1-\beta_6 x_2\right)-\frac{m_1-\beta_6 m_2}{2}\right)}{\Gamma\left(1+\left(x_1-\beta_6 x_2\right)-\frac{m_1-\beta_6 m_2}{2}\right)},\\
Z_5 &= \frac{\Gamma\left(-\alpha_5 x_1-\frac{\alpha_5 m_1}{2}\right)}{\Gamma\left(1+\alpha_5 x_1-\frac{\alpha_5 m_1}{2}\right)},\\
Z_4 &= \frac{\Gamma\left(-\alpha_4 x_1-\frac{\alpha_4 m_1}{2}\right)}{\Gamma\left(1+\alpha_4 x_1-\frac{\alpha_4 m_1}{2}\right)},\\
Z_3 &= \frac{\Gamma\left(-\alpha_3 x_1-\beta_3 x_2-\frac{\alpha_3 m_1+\beta_3 m_2}{2}\right)}{\Gamma\left(1+\alpha_3 x_1+\beta_3 x_2-\frac{\alpha_3 m_1+\beta_3 m_2}{2}\right)},\\
Z_1 &= \frac{\Gamma\left(-x_2-\frac{m_2}{2}\right)}{\Gamma\left(1+x_2-\frac{m_2}{2}\right)}.
\end{aligned} \tag{B.17}$$

For later convenience we introduce

$$h(\mathbf{x}) = \frac{1}{(2\pi)^2} Z_p Z_6 Z_1^2 Z_3 Z_4 Z_5 Z_{class}, \tag{B.18}$$

with

$$Z_{class} = e^{-\mathbf{p}\cdot\mathbf{x}+\mathbf{p}\cdot\gamma} e^{-i\theta_1 m_1 - i\theta_2 m_2}. \tag{B.19}$$

Notice that in (B.16) we already started to introduce quantities used in the procedure of [118]. To evaluate (B.15) we need to determine the location of the poles and deform the integration contour. The location of the poles is governed by hyperplanes, which are defined by the arguments of the Γ-functions. Similar to [118] we refer to the hyperplanes as divisors. The contributing poles to the integral are then given by the intersection of two such divisors. For our models of interest we can determine the divisors, by taking into account the fact that the poles come from the Γ functions in (B.17) whenever their argument is a negative integer:

$$
\begin{aligned}
&D_p = 1 + dx_1 + \frac{dm_1}{2} + n_p, && n_p \geq \max[0, -dm_1],\\
&D_6 = -(x_1 - \beta_6 x_2) - \frac{m_1 - \beta_6 m_2}{2} + n_6, && n_6 \geq \max[0, m_1 - \beta_6 m_2],\\
&D_5 = -\alpha_5 x_1 - \frac{\alpha_5 m_1}{2} + n_5, && n_5 \geq \max[0, \alpha_5 m_1],\\
&D_4 = -\alpha_4 x_1 - \frac{\alpha_4 m_1}{2} + n_4, && n_4 \geq \max[0, \alpha_4 m_1],\\
&D_3 = -(\alpha_3 x_1 + \beta_3 x_2) - \frac{\alpha_3 m_1 + \beta_3 m_2}{2} + n_3, && n_3 \geq \max[0, \alpha_3 m_1 + \beta_3 m_2],\\
&D_1 = -x_2 - \frac{m_2}{2} + n_1, && n_1 \geq \max[0, m_2],
\end{aligned}
\tag{B.20}
$$

where $n_i \in \mathbb{Z}_{\geq 0}$ and the restrictions on them come from cancellations by poles of the denominators in (B.17). If we draw the charge vectors of the fields in $\mathbb{R}^2$ spanned by the real part of the x_is, we see that the charge vectors of the fields are orthogonal to the respective hyperplanes given by (B.20).

Next we need to determine a contour of integration. We found that the poles are located at hyperplanes of $\mathbb{R}^2$. Further, the asymptotic behaviour of (B.15) is governed by the exponential and for convergence we need to choose a path such that

$$\mathbf{p} \cdot \mathrm{Re}(\mathbf{x}) > \mathbf{p} \cdot \gamma. \tag{B.21}$$

In a similar way as [118] we introduce the halfspace H:

$$H = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{p} \cdot \mathbf{x} - \mathbf{p} \cdot \gamma > 0\}, \tag{B.22}$$

and its boundary:

$$\partial H = \{\mathbf{x} \in \mathbb{R}^2 | \mathbf{p} \cdot \mathbf{x} - \mathbf{p} \cdot \gamma = 0\}. \tag{B.23}$$

The vector $\mathbf{p}$ is orthogonal to ∂H and points in the direction of the halfspace H. We see that γ gives a point on ∂H which splits ∂H into two rays: $\partial H_{\pm}$. The ray ∂H_+ is chosen such that $(\partial H_+, \mathbf{p})$ has positive orientation with respect to the standard basis of $\mathbb{R}^2$. The next task is to determine which ray $\partial H_{\pm}$ the divisors (B.20) intersect. We define an ordering on the divisors:

$$D_p \succ D_6 \succ D_5 \succ D_4 \succ D_1. \tag{B.24}$$

Afterwards we group the contributing intersection points into two sets:

$$\Pi_{\pm} = \{q \in \mathbb{R}^2 | D_i \succ D_j, \quad D_i \cap \partial H \in \partial H_{\pm}, \quad D_j \cap \partial H_{\mp}, \quad D_i \cap D_j \in H\}. \tag{B.25}$$

The residue integral is then given by

$$Z_{S^2} = 2\pi i \sum_{p\in\Pi_+} \mathrm{Res}_p\left(h(\mathbf{x})\right) - 2\pi i \sum_{p\in\Pi_-} \mathrm{Res}_p\left(h(\mathbf{x})\right), \tag{B.26}$$

with $h(\mathbf{x})$ introduced in (B.18). We want to remark that we only gave the procedure for the generic case. In principle certain complications can occur. For example a divisor could be parallel to ∂H or a pole could lie on more than 2 divisors. How to proceed in such cases is explained in the appendix of [118].

Let us next determine the intersection points of the divisors (B.20) with the hypersurface ∂H. The hypersurface ∂H can be parameterized by

$$\mathbf{x}(t) = t\mathbf{h} + \gamma, \tag{B.27}$$

with:

$$\mathbf{h} = \begin{pmatrix} \zeta_2 \\ -\zeta_1 \end{pmatrix}. \tag{B.28}$$

Next, we determine the determinant of the change of basis matrix T from the standard basis in $\mathbb{R}^2$ to $(\pm\mathbf{h}, \mathbf{p})$. We find

$$\det T = \pm 4\pi\left(\zeta_1^2 + \zeta_2^2\right). \tag{B.29}$$

We see that $(+\mathbf{h}, \mathbf{p})$ forms a right-handed coordinate basis and so $\mathbf{h}$ gives the direction of the positive ray ∂H_+. Further, this choice of $\mathbf{h}$ guarantees that we do not see an orientation flip if we consider different phases. In order to determine which ray $\partial H_\pm$ a divisor intersects, we first calculate the intersection point with ∂H and afterwards insert the result into the right hand side of (B.27). We then solve for t. If $t > 0$ then the divisor intersects ∂H_+ otherwise the intersection point lies on the negative ray. We find for t:

$$\begin{aligned}
D_p \cap \partial H : t &= -\frac{dm_1 - 2dq_1 + 2n_p + 2}{2d\zeta_2}, \\
D_6 \cap \partial H : t &= \frac{\beta_6\left(m_2 - 2q_2\right) - m_1 + 2\left(n_6 + q_1\right)}{2\left(\beta_6\zeta_1 + \zeta_2\right)}, \\
D_1 \cap \partial H : t &= \frac{m_2 - 2\left(n_1 + q_2\right)}{2\zeta_1}, \\
D_3 \cap \partial H : t &= \frac{\alpha_3\left(m_1 - 2q_1\right) + \beta_3\left(m_2 - 2q_2\right) - 2n_3}{2\beta_3\zeta_1 - 2\alpha_3\zeta_2}, \\
D_4 \cap \partial H : t &= \frac{2n_4 - \alpha_4\left(m_1 - 2q_1\right)}{2\alpha_4\zeta_2}, \\
D_5 \cap \partial H : t &= \frac{2n_5 - \alpha_5\left(m_1 - 2q_1\right)}{2\alpha_5\zeta_2}.
\end{aligned} \tag{B.30}$$

Next, we use the restrictions on the n_is given in (B.20) to determine if the intersection point has $t \lessgtr 0$. We find

$$
\begin{aligned}
D_p \cap \partial H &\in \begin{cases} \partial H_- & \zeta_2 > 0, \\ \partial H_+ & \zeta_2 < 0, \end{cases} 1 - dq_1 \geq 0, \\
D_6 \cap \partial H &\in \begin{cases} \partial H_+ & \beta_6\zeta_1 + \zeta_2 > 0, \\ \partial H_- & \beta_6\zeta_1 + \zeta_2 < 0, \end{cases} q_1 - \beta_6 q_2 \geq 0, \\
D_1 \cap \partial H &\in \begin{cases} \partial H_- & \zeta_1 > 0, \\ \partial H_+ & \zeta_1 < 0, \end{cases} q_2 \geq 0, \\
D_3 \cap \partial H &\in \begin{cases} \partial H_+ & -\beta_3\zeta_1 + \alpha_3\zeta_2 > 0, \\ \partial H_- & -\beta_3\zeta_1 + \alpha_3\zeta_2 < 0, \end{cases} \alpha_3 q_1 + \beta_3 q_2 \geq 0. \\
D_4, D_5 \cap \partial H &\in \begin{cases} \partial H_+ & \zeta_2 > 0, \\ \partial H_- & \zeta_2 < 0, \end{cases} q_1 \geq 0.
\end{aligned}
\tag{B.31}
$$

The conditions on the intersection of D_3 might look strange, but observe that in the case $(\alpha_3, \beta_3) = (0,1)/(1,0)$ it boils down to the conditions on the intersection of $D_1/D_4, D_5$ as expected. For the subsequent discussion we introduce $D_{3\alpha}/D_{3\beta}$ which signals that we set $(\alpha_3, \beta_3) = (1,0)/(0,1)$. Let us next give the position of the intersection with ∂H in the different phases under the assumption that the conditions on the q_i found in (B.31) are fulfilled. We find

Phase	D_p	D_6	$D_{5,4,3\alpha}$	$D_{3\beta,1}$
I	$-$	$+$	$+$	$-$
II	$+$	$+$	$-$	$-$
III	$+$	$-$	$-$	$+$
IV	$-$	$+/-$	$+$	$+$

(B.32)

where $\pm$ stands for intersection point at $\partial H_\pm$. Note that for D_6 in the phase **IV** we need to separately take into account the conditions found in (B.31). Further, at first the divisor D_6 looks like it could in principle be parallel to ∂H, but we are interested in the evaluation deep in a phase, where this is not the case. The non-empty intersection points of the divisors are given by:

$D_i \cap D_j$	x_1	x_2	
(D_p, D_6)	$-\frac{dm_1+2n_p+2}{2d}$	$-\frac{\beta_6 dm_2+2dn_6+2n_p+2}{2\beta_6 d}$	
$(D_p, D_{3\beta,1})$	$\frac{-dm_1-2n_p-2}{2d}$	$\frac{2n_l-\beta_l m_2}{2\beta_l}$	$l \in \{3,1\}$
$(D_6, D_{5,4,3\alpha})$	$-\frac{\alpha_l m_1-2n_l}{2\alpha_l}$	$-\frac{\alpha_l\beta_6 m_2+2\alpha_l n_6-2n_l}{2\alpha_l\beta_6}$	$l \in \{5,4,3\}$
$(D_6, D_{3\beta,1})$	$-\frac{\beta_l m_1-2\beta_l n_6-2\beta_6 n_l}{2\beta_l}$	$-\frac{\beta_l m_2-2n_l}{2\beta_l}$	$l \in \{3,1\}$
$(D_{5,4,3\alpha}, D_{3\beta,1})$	$\frac{2n_k-\alpha_k m_1}{2\alpha_k}$	$\frac{2n_l-\beta_l m_2}{2\beta_l}$	$\substack{k\in\{5,4,3\} \\ l\in\{3,1\}}$ $k \neq l$

(B.33)

By using (B.32) and (B.33) we can determine which divisor intersections contribute in a chosen phase. For this purpose we insert the position of the intersection point (B.33) into the halfspace equation (B.22) and check if it is fulfilled. We first list the possible intersections in the different phases:

Phase	(D_p, D_6)	$(D_p, D_{3\beta,1})$	$(D_6, D_{5,4,3\alpha})$	$(D_6, D_{3\beta,1})$	$(D_{5,4,3\alpha}, D_{3\beta,1})$
I	$-$			$+$	$+$
II		$+$	$+$	$+$	
III	$+$			$-$	$-$
IV	$-$	$-$	$-$	$-$	
	$\beta_6\zeta_1+\zeta_2>0$		$\beta_6\zeta_1+\zeta_2<0$	$\beta_6\zeta_1+\zeta_2<0$	

(B.34)

by $\pm$ we denote the overall sign, introduced in (B.26), of the residue contribution. We proceed by inserting the possible intersections into the left-hand side of the halfspace equation (B.22). We denote this by $H((D_i, D_j))$ and derive an inequality by using the restrictions on the n_is given in (B.20). We find:

Phase	$H((D_p, D_6))$	$H((D_p, D_{3\beta,1}))$
I	$\leq -4\pi\left(\left(\frac{1}{d}-q_1\right)\zeta_1+\left(\frac{1}{d\beta_6}-q_2\right)\zeta_2\right)$	
II		$\leq -4\pi\left(\left(\frac{1}{d}-q_1\right)\lvert\zeta_1\rvert+q_2\lvert\zeta_2\rvert\right)$
III	$\geq 4\pi\left(\left(\frac{1}{d}-q_1\right)\lvert\zeta_1\rvert+\left(\frac{1}{d\beta_6}-q_2\right)\lvert\zeta_2\rvert\right)$	
IV	$\leq -4\pi\left(\left(-\frac{1}{d}+q_1\right)\lvert\zeta_1\rvert+\left(\frac{1}{d\beta_6}-q_2\right)\lvert\zeta_2\rvert\right)$	$\geq 4\pi\left(\left(\left(\frac{1}{d}-q_1\right)\lvert\zeta_1\rvert+q_2\lvert\zeta_2\rvert\right)$

, (B.35)

and for the remaining intersection points the conditions read:

Phase	$H((D_6, D_{5,4,3\alpha}))$	$H((D_6, D_{3\beta,1}))$	$H((D_{5,4,3\alpha}, D_{3\beta,1}))$
I		$\geq 4\pi(q_1\zeta_1+q_2\zeta_2)$	$\geq 4\pi(q_1\zeta_1+q_2\zeta_2)$
II	$\geq 4\pi(q_1\lvert\zeta_1\rvert-q_2\lvert\zeta_2\rvert)$	$\geq 4\pi(q_1\lvert\zeta_1\rvert-q_2\lvert\zeta_2\rvert)$	
III		$\leq -4\pi(q_1\lvert\zeta_1\rvert+q_2\lvert\zeta_2\rvert)$	$\leq -4\pi(q_1\lvert\zeta_1\rvert+q_2\lvert\zeta_2\rvert)$
IV	$\leq -4\pi(q_1\lvert\zeta_1\rvert-q_2\lvert\zeta_2\rvert)$	$\leq -4\pi(q_1\lvert\zeta_1\rvert-q_2\lvert\zeta_2\rvert)$	

. (B.36)

We see that (B.35) and (B.36) single out sensible choices of q_1 and q_2 in the respective phase. By sensible we mean choices for which all intersection points coming from a single (D_i, D_j) lie completely in a halfspace and do not switch from the positive halfspace (B.22) to the negative halfspace (replace $>$ by $<$ in (B.22)). The sensible q_1 and q_2 choices are compatible with (B.31). In addition a sensible choice in the above sense is also obtained if we take q_1 and q_2 such that fields with a non-zero VEV in a phase have zero $U(1)_V$ charge. This is also the preferred choice from physics. In our model this choices are given by

Phase	q_1	q_2
I	0	0
II	0	0
III	$\frac{1}{d}$	$\frac{1}{d\beta_6}$
IV	$\frac{1}{d}$	0

. (B.37)

If we insert the choices (B.37) into (B.35), (B.35) we find the following contributing intersections:

$$\begin{array}{rll} \mathbf{I}: & +(D_6, D_{3\beta,1}) & +(D_{5,4,3\alpha}, D_{3\beta,1}) \\ \mathbf{II}: & +(D_6, D_{5,4,3\alpha}) & +(D_6, D_{3\beta,1}) \\ \mathbf{III}: & +(D_p, D_6) & \\ \mathbf{IV}: & -(D_p, D_{3\beta,1}). & \end{array} \tag{B.38}$$

In the phases were we see multiple contributing intersections we need to check for possible overlaps to avoid an overcounting. For this purpose we equate the intersection point coordinates (B.20) for two different intersections and express the n_is of one contribution in terms of the n_js of the other point. Afterwards we need to show that the conditions on the dependent n_is are fulfilled if we use the conditions on the n_js (B.33). If we take into account (B.7) we find that:

- **I**: We first sum over the poles at $(D_5, D_{3\beta})$ and afterwards sum over the poles at $(D_4, D_{3\beta})$ with $n_4 = \tau_{\alpha_4} k + \delta \quad \delta = 1, \ldots, \tau_{\alpha_4} - 1 \quad k \in \mathbb{Z}_{\geq 0}$.
- **II**: Here we sum over the poles at (D_6, D_5) and after that we sum over the poles at (D_6, D_4) with $n_4 = \tau_{\alpha_4} k + \delta \quad \delta = 1, \ldots, \tau_{\alpha_4} - 1 \quad k \in \mathbb{Z}_{\geq 0}$.
- **III**: We have only one contributing intersection, so this phase is straightforward.
- **IV**: In this phase it is enough to sum over the poles at $(D_p, D_{3\beta})$.

We introduced:

$$\tau_{\alpha_4} = \frac{\alpha_4}{\mathrm{GCD}(\alpha_5, \alpha_4)}. \tag{B.39}$$

Let us mention that in the case with $\beta_3 = 0$ we make the following replacement in the discussion above:

$$D_{3\beta} \to D_1. \tag{B.40}$$

It is interesting to note that in a phase the contributing intersections come always from contributions Z_i (B.13), whose corresponding fields have a nonzero VEV in the phase under consideration. We have now gathered everything to write (B.15) as sum over residue integrals:

$$Z_{S^2} \approx \frac{1}{2\pi} \sum_{n_i, n_j} \sum_{m_1, m_2} \oint \mathrm{d}^2\varepsilon h(\varepsilon_1 + \tilde{x}_1, \varepsilon_2 + \tilde{x}_2), \tag{B.41}$$

where $\tilde{x}_i$ is one of the intersection points (B.33) contributing in the respective phase. Further, the correct sign of the intersection (B.34) has to be taken into account in (B.41). There is the possibility that in a phase there are multiple contributions to Z_{S^2} of the same type as (B.41). For example this could happen in the phase **I**, because we have two contributing intersections.

In the following we will in addition to Section 6.4 give some steps on how to transform Z_{S^2} into a form similar to (B.41) in the respective phases. Our focus lies on the phases studied in Section 6.4 and different to the discussion above we will focus on a specific model at a time.

B.1 The $\mathbb{P}_{11222}[8]$ Model

In the model $\mathbb{P}_{11222}[8]$ we have fewer different contributions than displayed in (B.17), because:

$$Z_5 = Z_3 = Z_4. \tag{B.42}$$

It follows that (B.12) simplifies to

$$Z_{S^2} = \frac{1}{(2\pi)^2} \sum_{m_1,m_2} \iint_{\mathbb{R}^2} \mathrm{d}\sigma^2 Z_p Z_6 Z_1^2 Z_5^3 e^{-4\pi i(\zeta_1\sigma_1+\zeta_2\sigma_2)-i(\theta_1 m_1+\theta_2 m_2)}. \tag{B.43}$$

Phase I

In this phase the contributing intersections come from (D_5, D_1) We perform the following shift:

$$x_1 = \epsilon_1 + \frac{-m_1+2n_5}{2}, \qquad x_2 = \epsilon_2 + \frac{-m_2+2n_1}{2}, \tag{B.44}$$

and get for the unequal contributions (B.17):

$$\begin{aligned}
Z_p &= \frac{\Gamma\left(4n_5+4\epsilon_1+1\right)}{\Gamma\left(4m_1-4n_5-4\epsilon_1\right)},\\
Z_6 &= \frac{\Gamma\left(-n_5-\epsilon_1+2\left(n_1+\epsilon_2\right)\right)}{\Gamma\left(-m_1+2m_2-2n_1+n_5+\epsilon_1-2\epsilon_2+1\right)},\\
Z_5 &= \frac{\Gamma\left(-n_5-\epsilon_1\right)}{\Gamma\left(-m_1+n_5+\epsilon_1+1\right)},\\
Z_1 &= \frac{\Gamma\left(-n_1-\epsilon_2\right)}{\Gamma\left(-m_2+n_1+\epsilon_2+1\right)}.
\end{aligned} \tag{B.45}$$

The next step is to simplify the sums over the n_is and the m_is appearing in (B.41). We know from (B.20) that

$$n_5 \geq m_1, \qquad n_1 \geq m_2, \tag{B.46}$$

and therefore we introduce

$$n_5' = n_5 - m_1 \geq 0, \qquad n_1' = n_1 - m_2 \geq 0. \tag{B.47}$$

This transformation simplifies the sums to

$$\sum_{n_5}\sum_{n_1}\sum_{m_1}\sum_{m_2} \to \sum_{n_5,n_5',n_1,n_1'\geq 0} . \tag{B.48}$$

All the remaining steps are described in the main text Section 6.4. We further comment on the transformation of the residue integral.
We start from (6.149), which is of the schematic form:

$$Z_{S^2}^{geom} \sim \oint \mathrm{d}H^2 \frac{4H_1 f(H_1,H_2)}{H_2^2 H_1^3 (H_1 - 2H_2)}. \tag{B.49}$$

Due to the polynomial in the denominator of (B.49) the evaluation of the residue integral is not straightforward. Our goal is to transform the residue integral in such a way, that we can evaluate it as a product of univariate residue integrals. For this purpose we apply the transformation formula for multivariate residues (see [119]). Let $\{f_1(z_i),\dots,f_n(z_i)\}$ and $\{g_1(z_i),\dots,g_n(z_i)\}$ be holomorphic functions in the n variables $z_1,\dots,z_n$ such that

$$g_k(z_i) = T_{kj} f_j(z_i), \tag{B.50}$$

where T is a holomorphic matrix. Then

$$\mathrm{Res}\left(\frac{h(z_i)\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n}{f_1(z_i)\cdot \dots \cdot f_n(z_i)}\right) = \mathrm{Res}\left(\mathrm{det}T \frac{h(z_i)\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n}{g_1(z_i)\cdot \dots \cdot g_n(z_i)}\right). \tag{B.51}$$

The question is how to choose the $g_k(z_i)$ and how to identify the $f_j(z_i)$? We follow the approach given in [136], who modelled their procedure after [162]. To apply their approach we need to have as many factors in the denominator as we have integration variables, so we need to group the denominator of (B.49) into two factors. In general this procedure is ambiguous. A discussion of the origin of this ambiguity is given in [136]. In our case we choose the following grouping:

$$f_1(H_2) = H_2^2, \qquad f_2(H_1,H_2) = H_1^3(H_1 - 2H_2), \tag{B.52}$$

which can be motivated by remembering how the contributing divisors intersect:

$$(D_6, D_1) \subset (D_5, D_1), \tag{B.53}$$

and how their resulting pole contributions looks:

$$D_6 \to (H_1 - 2H_2), \qquad D_1 \to H_2^2, \qquad D_5 \to H_1^3. \tag{B.54}$$

We will find a further justification for the choice (B.52) a posteriori. To obtain univariate polynomials $g_k(z_i)$ we apply the following procedure. We calculate a Gröbner basis for the polynomials $\{f_1, f_2\}$ in lexicographic monomial order.

If we specify the variable order by $H_1 > H_2$ the first element of the Gröbner basis will depend only on H_2. If we cycle the order of the variables and extract always the first term of the obtained Gröbner basis, we can construct the univariate $g_k(z_i)$s. For our case of interest we find:

$$g_1(H_2) = H_2^2 \qquad\qquad g_2(H_1) = H_1^5. \tag{B.55}$$

The transformation matrix T is then given by:

$$\begin{pmatrix} 1 & 0 \\ 4H_1^3 & H_1 + 2H_2 \end{pmatrix} \tag{B.56}$$

and

$$\det T = H_1 + 2H_2. \tag{B.57}$$

We can apply (B.51) to (B.49) and find

$$Z_{S^2}^{geom} \approx \oint \mathrm{d}H^2 \left(\frac{4}{H_1^3 H_2^2} + \frac{8}{H_1^4 H_2} \right) f(H_1, H_2). \tag{B.58}$$

This is exactly the result which we would expect from the triple-intersection numbers (6.152) and gives a further justification of (B.52).

Phase III

In this phase we transform by:

$$x_1 = \epsilon_1 + \frac{1}{8}\left(-4m_1 - 2n_p - 2\right), \tag{B.59}$$

$$x_2 = \epsilon + \frac{1}{16}\left(-8m_2 - 2n_p - 8n_6 - 2\right). \tag{B.60}$$

In the next step we make the additional transformation:

$$\varepsilon_1 \to -\frac{\varepsilon_1}{4}, \qquad\qquad \varepsilon_2 \to -\frac{\varepsilon_1 + 4\varepsilon_2}{8}, \tag{B.61}$$

which gives a Jacobi-determinant of $\frac{1}{8}$. After the transformations the different contributions (B.17) are given by:

$$\begin{aligned} Z_p &= \frac{\Gamma\left(-n_p - \epsilon_1\right)}{\Gamma\left(4m_1 + n_p + \epsilon_1 + 1\right)}, \\ Z_6 &= \frac{\Gamma\left(-n_6 - \epsilon_2\right)}{\Gamma\left(-m_1 + 2m_2 + n_6 + \epsilon_2 + 1\right)}, \\ Z_5 &= \frac{\Gamma\left(\frac{1}{4}\left(n_p + \epsilon_1 + 1\right)\right)}{\Gamma\left(\frac{1}{4}\left(-4m_1 - n_p - \epsilon_1 + 3\right)\right)}, \\ Z_1 &= \frac{\Gamma\left(\frac{1}{8}\left(4n_6 + n_p + \epsilon_1 + 4\epsilon_2 + 1\right)\right)}{\Gamma\left(\frac{1}{8}\left(-8m_2 - 4n_6 - n_p - \epsilon_1 - 4\epsilon_2 + 7\right)\right)}. \end{aligned} \tag{B.62}$$

In this phase we need to guarantee:

$$n_p \geq -4m_1, \qquad n_6 \geq m_1 - 2m_2, \tag{B.63}$$

if we sum over the n_i and m_i. How to take this into account is discussed in Section 6.4.

Phase IV

We transform by:

$$x_1 = \varepsilon_1 + \frac{1}{8}\left(-4m_1 - 2n_p - 2\right), \qquad x_2 = \varepsilon_2 + \frac{1}{2}\left(2n_1 - m_2\right). \tag{B.64}$$

The second transformation we apply in this phase is given by

$$\varepsilon_1 \to -\frac{\varepsilon_1}{4}, \qquad \varepsilon_2 \to \varepsilon_2. \tag{B.65}$$

This transformation results in a Jacobi-determinant of $\frac{1}{4}$. After the above transformations the integral contributions (B.17) have the form:

$$\begin{aligned} Z_p &= \frac{\Gamma\left(-n_p - \epsilon_1\right)}{\Gamma\left(4m_1 + n_p + \epsilon_1 + 1\right)}, \\ Z_6 &= \frac{\Gamma\left(\frac{1}{4}\left(n_p + \epsilon_1 + 8\left(n_1 + \epsilon_2\right) + 1\right)\right)}{\Gamma\left(\frac{1}{4}\left(-4m_1 + 8m_2 - 8n_1 - n_p - \epsilon_1 - 8\epsilon_2 + 3\right)\right)}, \\ Z_5 &= \frac{\Gamma\left(\frac{1}{4}\left(n_p + \epsilon_1 + 1\right)\right)}{\Gamma\left(\frac{1}{4}\left(-4m_1 - n_p - \epsilon_1 + 3\right)\right)}, \\ Z_1 &= \frac{\Gamma\left(-n_1 - \epsilon_2\right)}{\Gamma\left(-m_2 + n_1 + \epsilon_2 + 1\right)}. \end{aligned} \tag{B.66}$$

The summations appearing in (B.41) have to fulfil

$$n_p \geq -4m_1, \qquad n_1 \geq m_2. \tag{B.67}$$

We introduce:

$$n_p' = n_p + 4m_1 \geq 0, \qquad n_1' = n_1 - m_2, \tag{B.68}$$

and the summations become:

$$\sum_{n_p}\sum_{n_1}\sum_{m_1}\sum_{m_2} \to \sum_{n_p, n_p', n_1, n_1' \geq 0}. \tag{B.69}$$

After this change of summation variables we have to guarantee, that

$$m_1 = \frac{n_p' - n_p}{4} \in \mathbb{Z}. \tag{B.70}$$

This is the reason for the second transformation given in (6.193) and the sum over δ appearing in (6.192). The remaining steps are given in Section 6.4.

B.2 The $\mathbb{P}_{11169}[18]$ Model

In this model (B.12) is given by

$$\begin{aligned} Z_{S^2} = \frac{1}{(2\pi)^2} \sum_{m_1,m_2} \iint_{\mathbb{R}^2} \mathrm{d}\sigma^2 Z_p Z_6 Z_1^3 \\ \cdot Z_5 Z_4 e^{-4\pi i(\zeta_1\sigma_1+\zeta_2\sigma_2)-i(\theta_1 m_1+\theta_2 m_2)}. \end{aligned} \tag{B.71}$$

Phase IV

In this phase the contributing intersection is given by (D_p, D_1). We perform the following transformation:

$$x_1 = \varepsilon_1 + \frac{1}{12}\left(-6m_1 - 2n_p - 2\right), \qquad x_2 = \varepsilon_2 + \frac{1}{2}\left(2n_1 - m_2\right). \tag{B.72}$$

Another transformation we make is given by

$$\varepsilon_1 \to -\frac{\varepsilon_1}{6}, \qquad \varepsilon_2 \to \varepsilon_2, \tag{B.73}$$

which gives a Jacobi-determinant of $\frac{1}{6}$. The above transformations result in the following form of (B.17):

$$\begin{aligned} Z_p &= \frac{\Gamma\left(-n_p-\epsilon_1\right)}{\Gamma\left(6m_1+n_p+\epsilon_1+1\right)}, \\ Z_6 &= \frac{\Gamma\left(\frac{1}{6}\left(n_p+\epsilon_1+18\left(n_1+\epsilon_2\right)+1\right)\right)}{\Gamma\left(-m_1+3m_2-3n_1-3\epsilon_2-\frac{n_p}{6}-\frac{\epsilon_1}{6}+\frac{5}{6}\right)}, \\ Z_5 &= \frac{\Gamma\left(\frac{1}{2}\left(n_p+\epsilon_1+1\right)\right)}{\Gamma\left(\frac{1}{2}\left(-6m_1-n_p-\epsilon_1+1\right)\right)}, \\ Z_4 &= \frac{\Gamma\left(\frac{1}{3}\left(n_p+\epsilon_1+1\right)\right)}{\Gamma\left(\frac{1}{3}\left(-6m_1-n_p-\epsilon_1+2\right)\right)}, \\ Z_1 &= \frac{\Gamma\left(-n_1-\epsilon_2\right)}{\Gamma\left(-m_2+n_1+\epsilon_2+1\right)}. \end{aligned} \tag{B.74}$$

In the summations appearing in (B.41) we need to make sure that

$$n_p \geq -6m_1, \qquad n_1 = m_2. \tag{B.75}$$

This motivates the following transformations of the summation variables:

$$n_p' = n_p + 6m_1 \geq 0, \qquad n_1' = n_1 - m_2, \tag{B.76}$$

which simplifies the summations to

$$\sum_{n_p}\sum_{n_1}\sum_{m_1}\sum_{m_2} \to \sum_{n_p,n_p',n_1,n_1'\geq 0}. \tag{B.77}$$

Similar to Section B.1 the requirement that

$$m_1 = \frac{n'_p - n_p}{6} \in \mathbb{Z}, \tag{B.78}$$

leads to another transformation with:

$$n'_p = 6k + \delta, \qquad n_p = 6l + \delta, \qquad \delta = 1, \dots 5, \quad k, l \in \mathbb{Z}_{\geq 0}. \tag{B.79}$$

This further modifies the summations to

$$\sum_{n_p, n'_p, n_1, n'_1 \geq 0} \to \sum_{\delta=1}^{5} \sum_{k,l \geq 0} \sum_{n_1, n'_1 \geq 0} . \tag{B.80}$$

The remaining transformations are straightforward and the outcome is given in Section 6.4.

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