MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit /
Title of the Master's Thesis

Margulis' Expanders and Ramanujan Graphs

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angestrebter akademischer Grad /
in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, im Monat März 2022 / Vienna, in March 2022

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
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Abstract

Expanders are sparse graphs with strong connectivity properties. They represent a central object of study in graph theory with various applications in both mathematics and computer science.

This thesis aims to give an introduction to expander graphs, focusing on the first explicit example of such a family of graphs due to Margulis as well as the optimal case of Ramanujan graphs.

After a recapitulation of the basic concepts of graph theory, we introduce expansion as a combinatorial property of a family of graphs and give an equivalent characterisation in more algebraic terms. The first explicit example of a family of expanders due to Margulis [Mar73] and some of its variants are studied, including a hands-on proof of their expansion property. A more conceptual proof of the same result arises as we investigate the connection of expanders and groups that have Kazhdan's property (T). Subsequently, we present a proof of the Alon-Boppana Theorem, which gives a theoretical limit to expansion. This motivates the definition of Ramanujan graphs as the best possible expanders. We give a reason why Margulis' expanders are not an example for a family of Ramanujan graphs and sketch the construction of the example given by Lubotzky, Phillip and Sarnak [LPS88] and, independently, Margulis [Mar88]. Finally, we give a brief insight to near-Ramanujan graphs, a topic that has received increased attention in recent years.

Zusammenfassung

Expander sind dünn besetzte Graphen mit starken Konnektivitätseigenschaften. Sie stellen einen zentralen Forschungsgegenstand im Gebiet der Graphentheorie mit vielfältigen Anwendungen in Mathematik und Informatik dar.

Die vorliegende Arbeit soll eine Einführung in die Theorie der Expander-Graphen geben, wobei insbesondere auf das erste explizite Beispiel einer solchen Familie von Graphen von Margulis sowie auf den Optimalfall der Ramanujan-Graphen genauer eingegangen werden soll.

Nach einer Wiederholung graphentheoretischer Grundbegriffe führen wir Expansion als kombinatorische Eigenschaft einer Familie von Graphen ein und geben eine äquivalente Charakterisierung in algebraischen Begriffen. Wir untersuchen das erste explizite Beispiel einer Familie von Expandern von Margulis [Mar88] sowie einige seiner Varianten und geben einen praktischen Beweis ihrer Expander-Eigenschaft. Ein konzeptuellerer Beweis desselben Resultats ergibt sich dann beim Studium der Verbindung zwischen Expandern und Gruppen, die Kazhdans Eigenschaft (T) erfüllen. In weiterer Folge stellen wir einen Beweis für den Satz von Alon-Boppana vor, der eine theoretische Grenze für Expansion angibt. Daraus motiviert sich die Definition von Ramanujan-Graphen als bestmögliche Expander. Wir begründen, warum Margulis' Expander kein Beispiel für eine Familie von Ramanujan-Graphen sind und skizzieren die Konstruktion des Beispiels von Lubotzky, Phillips und Sarnak [LPS88] bzw. Margulis [Mar88]. Abschließend geben wir einen kurzen Einblick in das Gebiet der Near-Ramanujan-Graphen, ein Thema, dem in den letzten Jahren verstärkt Aufmerksamkeit gewidmet wurde.

Contents

Chapter 1

Introduction

The purpose of this chapter is to recall basic concepts of graph theory and to settle the notation. In Section 1.1, most of the elementary graph theory background is introduced. In Section 1.2, we introduce Cayley and Schreier graphs, linking graphs and groups.

1.1 Graph theoretical preliminaries

In this section, we recall some basic definitions from graph theory.

Definition 1.1. A *graph* is a triple $X = (V, E, \varphi)$, where E and V are disjoint sets and $\varphi\colon E \to \{\{v, w\} : v, w \in V\}$ is a map. The elements of V are called *vertices* (or *nodes*) while those of E are called *edges*.
Given two vertices $v, w \in V$ and an edge $e \in E$ such that $\varphi(e) = \{v, w\}$, we say that v and w are *adjacent* vertices (or *neighbours*), denoted $v \sim w$ or $v \leftrightarrow w$, and that the edge e is *incident* to them.

Remark 1.2. (1) In general, we not only allow for X to have multiple edges (as φ need not be injective), but also self-loops (i.e. edges incident to a single node, making that node adjacent to itself) as we do not ask for v and w to be distinct in the image of φ.

(2) As usual, we depict a graph by plotting the vertices as points in the plane and the edges as lines connecting the vertices they are incident to.

(3) In most cases, we can (and will) omit the incidence function φ in the definition above by identifying the edge set E with its image under φ (as a multiset), and write $X = (V, E)$ instead of $X = (V, E, \varphi)$.
The main problem with this shorthand notation occurs when there are several edges incident to the same pair of vertices. Such edges cannot be distinguished (as they are just instances of the same element in the multiset E) if we omit φ.

Definition 1.3. A graph $X = (V, E)$ is said to be *finite* if both its vertex set V and its edge set E are finite. In that case, the *order* of G, denoted $|G|$, is the number number of vertices, that is $|X| \coloneqq |V|$.
Moreover, X is said to be *simple* if each $e \in E$ has multiplicity one (i.e. if e appears only once in the multiset E), and *loop-free* if E contains no edges of the form $\{v, v\}$.

Example 1.4. The following finite, simple, loop-free graphs are shown in Figure 1.1:

(a) The *cycle graph* on n vertices, denoted C_n, has vertex set $V = \{0, \dots, n-1\}$ and single edges joining each i with $i + 1 \pmod n$.

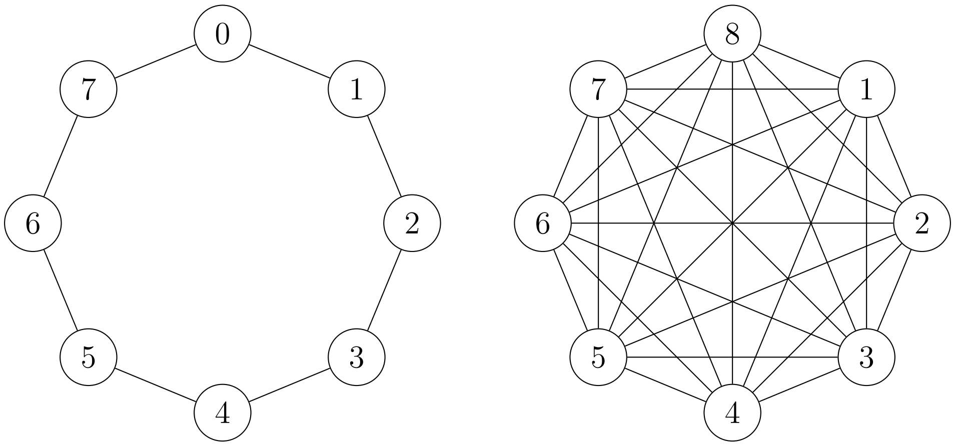

Figure 1.1: The graphs C_8 and K_8

(b) The *complete graph* on n vertices, denoted K_n, is given by $C_n = (\{1, \ldots n\}, \mathcal{P}(\{1, \ldots, n\}))$, that is, any two distinct vertices are joined by exactly one edge.

We introduce some convenient abbreviations:

Notation 1.5. Let $X = (V, E)$ be a graph, and $A, B \subseteq V$ be sets of vertices. Then we denote by $E(A, B)$ the multiset of all edges connecting vertices from A with vertices from B (that is, $E(A, B) = \{\{a, b\} \in E \mid a \in A,\ b \in B\}$), and by $e(A, B)$ its (multiset) cardinality.
If $A = \{a\}$ or $B = \{b\}$ are singletons, we omit brackets for notational convenience (e.g., we write $E(a, B)$ instead of $E(\{a\}, B)$).

As it is often the case with algebraic objects, we are interested in structure-preserving maps and sub-objects:

Definition 1.6. Let $X_1 = (V_1, E_1)$ and $X_2 = (V_2, E_2)$ be graphs.

(i) A *homomorphism of graphs* is a map $\varphi \colon V_1 \to V_2$ that preserves adjacency relations, that is, it satisfies $\{\varphi(v), \varphi(w)\} \in E_2$ whenever $\{v, w\} \in E_1$.
In this case, we can define an induced map $\varphi_E \colon E_1 \to E_2$ between the edge sets by setting $\varphi_E(\{v, w\}) := \{\varphi(v), \varphi(w)\}$.
An *isomorphism of graphs* is a homomorphism that is bijective and induces a bijection of the edge sets, that is, the map $\varphi_E \colon E_1 \to E_2$ is a bijection of multisets.

(ii) If $W \subseteq V_1$ is a set of vertices of X_1, then the *subgraph of* X_1 *induced by* W is the graph Y on W featuring all edges that appear between vertices from W in X_1, that is $Y = (W, E(W, W))$.

Definition 1.7. Let $X = (V, E)$ be a graph with vertex set V and edge set E.

(i) The number $\deg v := e(v, V)$ of edges incident to a vertex $v \in V$ is called the *degree* of v.

(ii) For an integer $d \geqslant 0$, the graph X is called *d-regular* if every vertex $v \in V$ has degree d.

Example 1.8. The cycle graph C_n is 2-regular for any $n \geqslant 3$, while the complete graph K_n is $(n-1)$-regular for any n.

Definition 1.9. Let $X = (V, E)$ be a finite graph with $V = \{v_1, \ldots, v_n\}$.

(i) The *degree matrix* D_X of X is defined by $D_X := (deg(v_i)\ \delta_{ij})_{i,j}$

(ii) The *adjacency matrix* A_X of X is defined by $A_X := (e(v_i, v_j))_{i,j}$.

Remark 1.10. (1) The definitions of D_X and A_X depend on the enumeration of the vertex set V. However, permuting the vertices will only result in conjugating D_X and A_X by a permutation matrix, which leaves algebraic properties such as determinant, trace and eigenvalues invariant. As we are interested in these properties rather than in the matrices themselves, we will not specify the chosen enumeration of vertices every time, but still refer to D_X and A_X as *the* degree and adjacency matrix of X (implicitly assuming some enumeration has been fixed).

(2) Despite its seemingly innocent definition, the adjacency matrix is actually a very potent tool: It allows us to represent a graph by a matrix, making the powerful toolkit of linear algebra applicable to graphs. We will encounter another instance of this concept of *linearisation* of algebraic objects in the form of group representations in Section 2.4.2.

Example 1.11. By definition, the degree matrix of a d-regular graph on n vertices is given by $d \cdot I_n$, where I_n denotes the $(n \times n)$-identity matrix. Thus, the degree matrices of the cycle graph and the regular graph are given by $D_{C_n} = 2I_n$ and $D_{K_n} = (n-1)I_n$.
The adjacency matrix of C_n has entries 1 in all positions of the form $(i, i+1)$ and $(i, i-1)$ $(\mathrm{mod}\ n)$ and 0 everywhere else, and the adjacency matrix of K_n has entries 0 on the main diagonal and 1 everywhere else.

Definition 1.12. Let $X = (V, E)$ be a graph.

(i) A *walk* in X is an alternating sequence $w = (v_1, e_1, v_2, \ldots, v_n, e_n, v_{n+1})$ of vertices $v_i \in V$ and edges $e_i \in E$ such that $e_i = \{v_i, v_{i+1}\}$ for every $1 \leqslant i \leqslant n$. The *length* of w, denoted $\mathrm{len}(w)$, is defined to be n.

(ii) A *trail* in X is a walk where all edges $e_1, \ldots, e_n$ are distinct.

(iii) A *path* in X is a trail where all vertices $v_1, \ldots, v_{n+1}$ are distinct, except possibly v_1 and v_{n+1}.
A path is said to be *closed* if $v_1 = v_{n+1}$, and *open* otherwise.

(iv) A *cycle* in X is a closed path of length at least one.

(v) Two vertices $u, w \in V$ are said to be *connected* if there exists a walk of the form $(u = v_1, e_1, v_2, \ldots, v_n, e_n, v_{n+1} = w)$.

Example 1.13. Consider the graph depicted in Figure 1.2.

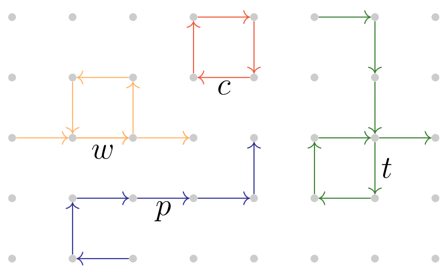

Figure 1.2

(a) w is a walk (but not a trail) of length 7,

(b) t is a trail (but not a path) of length 8,

(c) p is a path of length 6, and

(d) c is a cycle of length 4.

Remark 1.14. The connection relation $\leftrightsquigarrow$ on V defined by

$$v \leftrightsquigarrow w :\iff v \text{ is connected to } w$$

is an equivalence relation on V: reflexivity is guaranteed by allowing walks of length 0, symmetry follows from taking inverse walks and transitivity from concatenating walks.
Hence we can define:

Definition 1.15. The equivalence classes of V with respect to the relation $\leftrightsquigarrow$ of connection are called the *connected components* of X.
A subset S of V is said to be *connected* if it is contained in a single connected component. If the whole vertex set V is connected, we say that X is a *connected graph*.

Example 1.16. The cycle graph C_n and the complete graph K_n are connected for any n.

The length of the shortest path between two points gives a notion of distance on a graph:

Definition 1.17. Let $X = (V, E)$ be a graph.
The *distance* of two vertices $v, w \in V$ is defined by

$$d(v, w) := \inf \{\operatorname{len}(p) \mid p \text{ is a path connecting } v \text{ and } w\}.$$

For a fixed vertex $v \in V$ and an integer $r \geqslant 0$, the *ball* of radius r around v is given by

$$B_r(v) := \{w \in V \mid d(v, w) \leqslant r\}.$$

Moreover, the distance between two sets $A, B \subseteq V$ of vertices is defined by

$$d(A, B) := \inf \{d(a, b) \mid a \in A, b \in B\}.$$

Remark 1.18. If X is connected, d defines a metric on V.

Definition 1.19. A connected graph without cycles is called a *tree*. A vertex of degree 1 in a tree is called a *leaf*.

Example 1.20. The *d-regular infinite tree* $T_d = (V, E)$ can be formally described as follows: Fix a set $S = \{s_1, \ldots, s_d\}$ of d elements, and denote $S' = \{s_1, \ldots, s_{d-1}\}$. Then take V to be the set of all words over S' with an additional final letter from S, and add an edge between any two such words that can be obtained from each other by prescribing (or pre-deleting) a single letter.
One easily verifies that T_d is the only countable d-regular tree up to isomorphism.

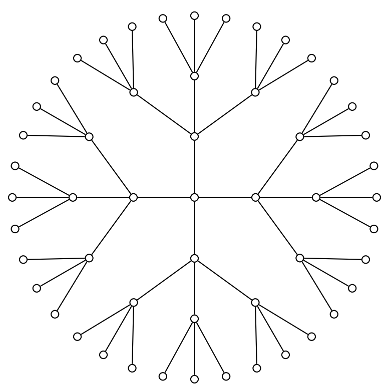

Figure 1.3: A ball of radius 3 in the 4-regular tree

We define two more combinatorial invariants of graphs:

Definition 1.21. The *girth* of a graph X is defined by

$$\operatorname{girth}(X) := \inf_c \operatorname{len}(c),$$

where the infimum is taken over all cycles in X.

Definition 1.22. The *diameter* of a graph $X = (V, E)$ is defined by

$$\operatorname{diam}(X) := \sup_{v,w \in V} d(v, w).$$

Example 1.23. The cycle graph C_n has girth n for $n \geqslant 2$, and the complete graph K_n has girth 3 for $n \geqslant 3$. They both have diameter $\lfloor n/2 \rfloor$.

We will also make use of the following:

Convention 1.24. If S is a countable set, we endow S with the counting measure μ defined by $\mu(A) := |A|$ for $A \subseteq S$ to obtain a measure space $(S, \mathcal{P}(S), \mu)$.

Denoting by $L^p(S, \mu)$ the Lebesgue space on S with respect to the measure μ and the parameter $p \geqslant 1$, we set

$$L^2(S) := L^2(S, \mu) = \left\{ f \colon S \to \mathbb{C} \;\middle|\; \int_S |f|^2 d\mu < \infty \right\}.$$

Since we have $\int_S f \; d\mu = \sum_{v \in S} f(v)$ by definition of the counting measure, $L^2(S)$ actually coincides with the space of square summable functions on S:

$$L^2(S) = \ell^2(S) = \left\{ f \colon S \to \mathbb{C} \;\middle|\; \sum_{s \in S} |f(s)|^2 < \infty \right\}.$$

Moreover, we set

$$L_0^2(S) := \left\{ f \in L^2(S) \;\middle|\; \int_S f d\mu = 0 \right\} = \left\{ f \in L^2(S) \;\middle|\; \sum_{s \in S} f(s) = 0 \right\}.$$

Remark 1.25. (1) If S is a finite set, then $L^2(S) = \mathbb{C}^S$ is the set of complex valued functions on S.

(2) Equipped with the inner product $\langle f, g \rangle := \sum_{v \in V} f(v)\overline{g(v)}$, $L^2(S)$ becomes a Hilbert space.

Notation 1.26. If X is a graph whose vertex set V is at most countable, we write $L^2(X)$ and $L_0^2(X)$ for $L^2(V)$ and $L_0^2(V)$, respectively.

1.2 Graphs and groups

An important application of graphs is that they can be used to visualise the structure of other algebraic objects, such as groups.

First, we fix some notational conventions for groups:

Notation 1.27. Whenever speaking of an abstract group G, we write G multiplicatively, that is, the group operation of two elements $g, h \in G$ will be denoted by $g \cdot h$ (multiplication) or gh (juxtaposition). The neutral element in G will be denoted by e.
Moreover, if $A, B \subseteq G$ are subsets and $n \geqslant 1$ is an integer, we use the following abbreviations:

(i) $A^{-1} := \{a^{-1} \mid a \in A\}$,

(ii) $AB := \{ab \mid a \in A, b \in B\}$, and

(iii) $A^n := \{a_1 \cdots a_n \mid a_1, \dots, a_n \in A\}$.

Definition 1.28. Let G be a group and $S \subseteq G$ a subset. We say that S is *symmetric* if $S^{-1} = S$, and that S is *symmetrically reduced* if $S \cap S^{-1} = \{s \in S \mid s = s^{-1}\}$.

Definition 1.29. If $S' \subseteq S \subseteq G$ are subsets, then we say that S' is a symmetric reduction of S if S' is symmetrically reduced and $g \in S$ implies $g \in S'$ or $g^{-1} \in S'$.

Remark 1.30. In other words, a symmetric reduction S' of S is obtained from S by removing either g or g^{-1} whenever both are in S.

Definition 1.31. Let G be a group and $S \subseteq G$ a subset.
The *Cayley graph* $\mathcal{G}(G,S)$ of G with respect to S is defined to be the undirected graph with vertex set $V = G$ and edge set

$$E = \{\{g, sg\} \mid g \in G, s \in S'\},$$

where S' is a symmetric reduction of S.

Remark 1.32. (1) There is no clear consent on the precise definition of the Cayley graph, and the one given here is far from being standard; the intended object however is usually the one defined above. The introduction of the "symmetric reduction" S' is a mere technicality to avoid double edges emerging from two generators that are inverses of each other.

(2) Clearly, the above definition only depends on G and S, but not on the chosen symmetric reduction S'.

(3) It sometimes (in particular when drawing a Cayley graph) makes sense to define $\mathcal{G}(G,S)$ as a directed graph (with edge set $E = \{(g, sg) : s \in S'\}$) and to equip it with the edge labelling $\lambda\colon E \to S$, $(g, sg) \mapsto s$.
In this case, there is a 1-to-1-correspondence between walks in $\mathcal{G}(G,S)$ and words in S' (consecutively read off the labels of the edges of the walk, taking inverses whenever walking against the direction of an edge), and multiplying the start point of a walk with this word will give the end point of the walk. In particular, a cycle in the Cayley graph corresponds to an equation of the form $g = s_1 \dots s_n g$, that is, labels of cycles correspond to relations in G. Thus, the directed, labelled Cayley graph allows to fully reconstruct the subgroup of G generated by S.

(4) The following properties of the Cayley graph $\Gamma = \mathcal{G}(G,S)$ are immediate:

(i) Γ is $2|S'|$-regular,

(ii) Γ is loop-free unless S contains the identity element,

(iii) Γ is simple unless S contains elements of order 2, and

(iv) Γ is connected if and only if S is a set of generators for G.

Example 1.33. (a) The Cayley graph of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with respect to $S = \{\overline{1}\}$ is isomorphic to the cycle graph C_n.

(b) The Cayley graph of $\mathbb{Z}/2n\mathbb{Z}$ with respect to $S = \{\overline{2}\}$ consists of two disjoint copies of C_n.

(c) The Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with respect to $S = \{\overline{1}, \ldots, \overline{\lfloor \frac{n}{2} \rfloor}\}$ is given by the complete graph K_n.

(d) The Cayley graph of the free group F_k on k generators $s_1, \ldots, s_k$ with respect to the generating set $S = \{s_1, \ldots, s_k\}$ is isomorphic to the $(2k)$-regular infinite tree T_{2k}. Indeed, the Cayley graph in question contains no cycles as the free group has no non-trivial relations, it is $2k$-regular, and it is connected as $s_1, \ldots, s_k$ generate F_k.

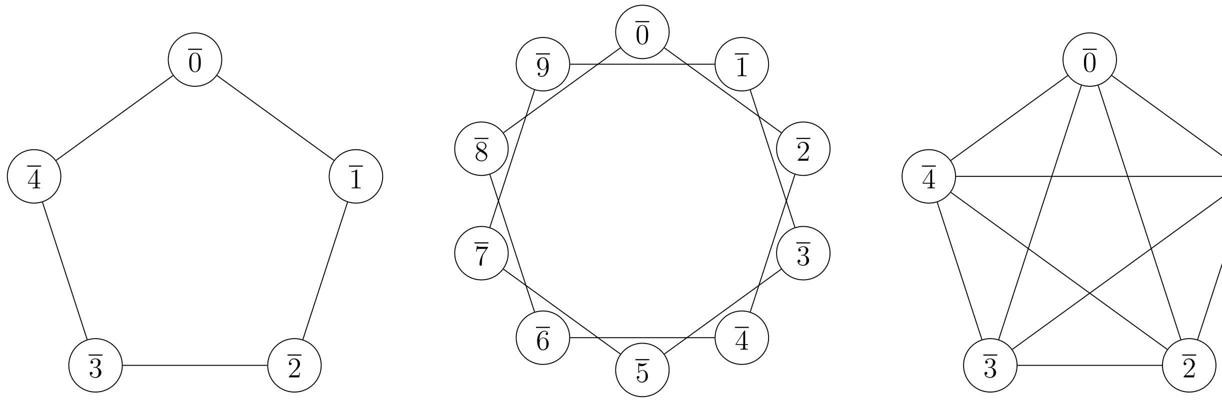

Figure 1.4: The Cayley graphs described in (a) - (c) above for $n = 5$.

Remark 1.34. As the above examples (a) - (c) illustrate, the Cayley graph of a group does depend on the set S, even if one requires S to be a set of generators.

We also define a more general notion:

Definition 1.35. Let G be a group that acts on a set X via an action $*$, and let $S \subseteq G$ be a subset. The *Schreier graph* of G with respect to X and S is the undirected graph $\mathcal{S}(G, X, S)$ with vertex set $V = X$ and edge set

$$E = \{\{x, s * x\} \mid x \in X, s \in S'\},$$

where S' is a symmetric reduction of S.

Remark 1.36. (1) Letting G act on itself by left multiplication, we obtain $\mathcal{S}(G, G, S) = \mathcal{G}(G, S)$, so Cayley graphs are special cases of Schreier graphs.

(2) Sometimes, Schreier graphs are defined in a different way, involving a subgroup $H \leqslant G$ rather than an action. In this case, the vertices are given by the left cosets $\{gH : g \in G\}$ of H (the resulting graph is therefore also called the *Schreier coset graph*), and edges are of the form $\{gH, sgH\}$ with $s \in S'$. These Schreier coset graphs correspond to the Schreier graphs in the above sense with G acting on the cosets via left multiplication. In fact, it has been shown by Annexstein, Baumslag and Rosenberg [ABR90, Theorem 3.2, p. 551] that every Schreier graph in the sense of Definition 1.35 can be realised as a Schreier coset graph (not necessarily with respect to the same group), so the two notions can be considered equivalent in a wide sense.

A concept that does not directly involve groups, but still links to them in a certain sense, is the notion of a graph covering.

Definition 1.37. Let $X = (V, E)$ be a graph, $v \in V$ a vertex. The *star* at v is the set

$$\mathrm{St}(X, v) := E(v, V)$$

of edges incident to v.

Definition 1.38. Let $X_1 = (V_1, E_1)$, $X_2 = (V_2, E_2)$ be two graphs. A map $\varphi\colon V_1 \to V_2$ is called a *covering* if it is a surjective homomorphism of graphs and a local bijection in the sense that the restriction of the induced map,

$$\varphi_E|_{\mathrm{St}(X_1,v))}\colon \mathrm{St}(X_1, v) \to \mathrm{St}(X_2, f(v)),$$

is bijective for any $v \in V$.
If such a covering exists, we say that X_1 *covers* X_2.

Example 1.39. (a) The canonical homomorphism $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, a \mapsto a + n\mathbb{Z}$ defines a covering $C_n = \mathcal{G}(\mathbb{Z}/n\mathbb{Z}, \{\bar{1}\}) \to \mathcal{G}(\mathbb{Z}, \{1\}) = T_2$.

(b) More generally, if G is a group generated by a finite, symmetrically reduced set S and $\varphi\colon G \to H$ is a surjective group homomorphism that is injective on S, then φ defines a covering of $\mathcal{G}(H, \varphi(S))$ by $\mathcal{G}(G, S)$.

(c) Any connected d-regular graph X on at most countably many vertices is covered by the d-regular tree. Indeed, fix vertices v of T_d and w of X and set $f(v) \coloneqq w$. Fix a bijection $\mathrm{St}(T_d, v) \to \mathrm{St}(X, w)$ and map vertices accordingly. Now suppose that f is already defined on $B_r(v)$, and that v' is a vertex of T_d at distance r from v. Then f is already defined on one edge incident to v', but not on the other $d - 1$. We can again fix a bijection of these edges to the remaining $d - 1$ edges incident to $f(v')$ and map vertices accordingly. In this way, we obtain a covering $f\colon T_d \to X$.

Chapter 2

Expander graphs

Expander graphs are sparse graphs with strong connectivity properties that are renowned for their various applications in both pure mathematics and computer science. The concept has been around since the late 1960s - for instance, Gromov and Guth [GG12] pointed out that Barzdin and Kolmogorov proved the existence of expanders via a probabilistic argument in 1967. The term *expanding graph* (which in later publications usually becomes *expander* graph) however was introduced by Pinsker [Pin73] in 1973. In the same year, Margulis [Mar73] gave the first explicit example of a family of expanders.

We will define expansion as a combinatorial property of graphs in Section 2.1 and give a more algebraic characterisation via the eigenvalues of the Laplacian matrix in Section 2.2. The subsequent Section 2.3 is dedicated to Margulis' example of an expander family, containing an elementary proof that verifies the expansion property. In Section 2.4, we discuss Kazhdan's property (T) and link it to expander graphs, yielding another proof of the fact that Margulis' graphs form a family of expanders.

2.1 The combinatorial property of expansion

In this section, we introduce the notion of expansion as a combinatorial property of graphs. Expander graphs are often described as "sparse graphs with strong connectivity properties" [Lub94, p. 1]. While "sparsity" means that the number of edges is reasonably low (which we will guarantee by imposing regularity), we still need to give some definitions to make sense of "strong connectivity properties".

Throughout this section, let $X = (V, E)$ be a graph with vertex set V and edge set E.

Definition 2.1. Let $S \subseteq V$ be a subset. The (*edge*, *Cheeger*) *boundary* ∂S of S is the set of edges connecting vertices of S with vertices outside S, that is

$$\partial S := E(S, S^C),$$

where $S^C := V \setminus C$ denotes the complement of S.

In analogy with a concept from differential geometry [cf. Lub94, p. 43, Def. 4.1.2], we define:

Definition 2.2. For any non-empty subset $S \subseteq V$, the *expansion constant* $h_X(S)$ of S with respect to X is given by

$$h_X(S) := \frac{|\partial S|}{|S|}.$$

Moreover, if V is finite, we define the expansion constant (also known as the *Cheeger constant*) of X by

$$h(X) := \min_{|S| \leqslant \frac{1}{2}|V|} h_X(S).$$

Here, the minimum is taken over all non-empty subsets S of V satisfying $|S| \leqslant \frac{1}{2}|V|$.

Remark 2.3. (1) Viewed geometrically, $h_X(S)$ describes the ratio of perimeter and volume of the set S. Finding the minimum of this ratio means to solve the isoperimetric problem for X, whence $h(X)$ is sometimes also called the *isoperimetric constant* of X.

(2) In a more combinatorial sense, the expansion constant of a graph measures how easy or hard it is to divide the graph into two disconnected subgraphs by removing edges. Graphs with large expansion constant are said to be well expanded while those with small expansion constant are said to be poorly expanded.

Example 2.4. (a) Any disconnected graph X has expansion constant 0: Take S to be a connected component satisfying $|S| \leqslant \frac{|V|}{2}$. Then the boundary ∂S is empty, and thus $h_X(S) = 0 = h(X)$.

(b) Consider the graph X depicted in Figure 2.1.

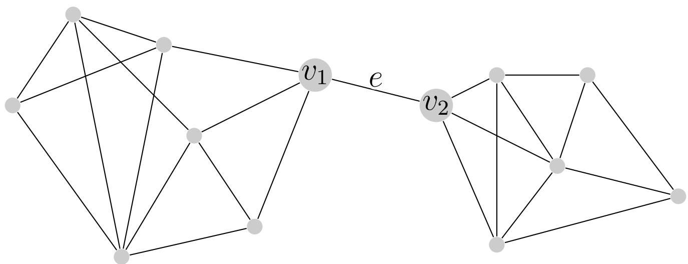

Figure 2.1: A poorly expanded graph

Here, if we take S to be the set of vertices to the right of (and including) v_2, then e is the only edge connecting S with its complement (e is a "bottleneck"). Thus, $h_X(S) = \frac{|\partial S|}{|S|} = \frac{1}{6}$.

We now introduce the protagonist of this thesis:

Definition 2.5. Let $d > 0$ be a fixed integer. Then an infinite family $\{X_n : n \in \mathbb{N}\}$ of finite $d-$regular graphs is called a *family of expanders* if the following two conditions are met:

(i) $|X_n| \to \infty$ as $n \to \infty$, and

(ii) there is a constant $C > 0$ such that $h(X_n) \geqslant C$ for all n.

Giving an explicit example of a family of expanders (and proving that it satisfies the definition) is a non-trivial task. We will be ready to do so in Section 2.3.

2.2 The combinatorial Laplacian

While in the last section, we have introduced the notion of expansion as a combinatorial property of graphs, we give an algebraic characterisation of this property in this section. More precisely, we will motivate and introduce the Laplacian matrix of a graph and present a proof of the Cheeger inequalities, which connect the expansion constant of a graph to the second eigenvalue of its Laplacian matrix.

Many of the objects and results that we will encounter in this section can be viewed as graph theoretical analoga of concepts from differential geometry. For a broad exposition of these analogies, the reader is referred to Chapter 4 in [Lub94]. Here, we will only adopt the geometrical viewpoint to motivate the definition of the combinatorial Laplacian [Lub94, p. 44].

Motivation. Let $X = (V, E)$ be a finite graph. Equip X with an arbitrary orientation: for any $e \in E$, declare one of the vertices that e is incident to its origin e^- and the other one its target e^+. One can think of e as a "tangent vector" of X at the point e^-, pointing towards e^+.
The space $L^2(E) = \{f : E \to \mathbb{C}\}$ of functions associating a number to each tangent vector can therefore be interpreted as the space of 1-forms of X.
The differential operator $d: L^2(V) \to L^2(E)$ is defined by

$$df(e) := f(e^+) - f(e^-).$$

Enumerating $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, the $(m \times n)$-matrix D representing the operator d with respect to the standard bases of $L^2(V)$ and $L^2(E)$ has entries

$$D_{i,j} = \begin{cases} 1 & \text{if } v_j = e_i^+ \\ -1 & \text{if } v_j = e_i^- \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.6. The $(n \times n)$-matrix $\Delta = D^T D$ is called the *Laplacian matrix* of X.

Remark 2.7. Just like adjacency and degree matrices, also the Laplacian depends on the chosen enumeration of V, which we will usually not mention explicitly but just assume to be fixed. However, the Laplacian is independent of the enumeration of E and the chosen orientation, as the next result shows.

Lemma 2.8 [Lub94, p. 44, Prop. 4.2.2]**.** *For any loop-free finite graph X with adjacency matrix A_X and degree matrix D_X, the Laplacian of X satisfies $\Delta = D_X - A_X$.*

Proof. By definition of D, we have

$$D_{k,i} \cdot D_{k,j} = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } v_i \text{ is incident to } e_k \\ -1 & \text{if } e_k = \{v_i, v_j\} \\ 0 & \text{otherwise} \end{cases}$$

for any $k \leqslant n$ and $i, j \leqslant m$.
Thus, we can compute the (i, j)-entry of Δ:

$$\Delta_{i,j} = \sum_{k=1}^{m} D_{k,i} D_{k,j} = \deg(v_i) \cdot \delta_{i,j} - e(v_i, v_j),$$

which is precisely the (i, j)-entry of $D_X - A_X$. □

In particular, the Laplacian matrix is symmetric and hence diagonalisable. Let us now investigate the eigenvalues of A_X and Δ:

Lemma 2.9 [Mur03, p. 2, Thm. 1]**.** *Let X be a finite graph with adjacency matrix A_X, and denote by δ the maximal degree of all vertices. Then any eigenvalue μ of A_X satisfies $|\mu| \leqslant \delta$.*

Proof. Let $x = (x_1, \dots, x_n)^T$ be an eigenvector corresponding to the eigenvalue μ, and let m be the index such that $|x_m| = \max_{1 \leqslant i \leqslant n} |x_i|$. Then we have

$$|\mu||x_m| = |(Ax)_m| = \sum_{i=1}^{n} e(v_m, v_i)|x_i| \leqslant \delta |x_m|,$$

so $|\mu| \leqslant \delta$. □

In particular, we have $|\mu| \leqslant d$ if X is d-regular. Moreover, if X is d-regular, then each row in D_X will contain entries summing up to d, so the vector $(1, \dots, 1)^T$ is an eigenvector of A_X, corresponding to the eigenvalue d.

Lemma 2.10 [Mur03, p. 3, Thm. 3]**.** *If X is a d-regular graph, then the multiplicity of d as an eigenvalue of A_X is equal to the number of connected components of X.*

Proof. Let X have connected components $C_1, \dots, C_k$. Sorting the vertex set accordingly, A_X becomes a block matrix of adjacency matrices of the induced subgraphs of X on the connected components. As each of these matrices is d-regular, the eigenvalue d of A_X appears with multiplicity at least k.

Conversely, it suffices to show that eigenvectors corresponding to the eigenvalue d are constant on connected components because then at most k of them can be linearly independent. So let $x = (x_1, \dots, x_n)^T$ be an eigenvector corresponding to the eigenvalue d. Let m be the index such that $|x_m| = \max_{1 \leqslant i \leqslant n} |x_i|$. Up to changing the sign of x, we may assume that x_m is positive. Now we have

$$dx_m = (Ax)_m = \sum_{i=1}^{n} e(v_i, v_m)x_i \geqslant \sum_{i=1}^{n} e(v_i, v_m)x_m = dx_m.$$

As equality holds, we have $x_i = x_m$ whenever $e(v_i, v_m) \neq 0$. Thus, x assumes value x_m on every node adjacent to v_m. Applying the same argument to a node adjacent to x_m and iterating, we conclude that x is constant on every connected component. □

Remark 2.11. If X is a d-regular graph, we have $\Delta = d \cdot I - A_X$, and an eigenvalue μ of A_X corresponds to the eigenvalue $d - \mu$ of Δ.

Notation 2.12. If X is a d-regular graph, we will denote the eigenvalues of A_X by $d = \mu_1 > \mu_2 > \cdots > \mu_k$, and the corresponding eigenvalues of Δ by $\lambda_i = d - \mu_i$, so that $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_k$.

While the multiciplity of the first eigenvalue λ_1 bears information about the connectivity of a d-regular graph, its value will always be zero. The first non-trivial eigenvalue is more interesting.

Definition 2.13. The least positive eigenvalue $\lambda_2 = d - \mu_2$ of the Laplacian matrix of a d-regular graph X is called the *spectral gap* (or simply the *second eigenvalue)* of X and denoted by $\lambda(X)$.

Remark 2.14. While we use $\lambda(X)$ to denote the second eigenvalue of the Laplacian of X, it is also not uncommon to use this notation for the second eigenvalue of the adjacency matrix of X. It is recommended to keep this ambiguity in mind and be extra cautious when dealing with this notation.

The second eigenvalue of a finite d-regular graph is closely related to its Cheeger constant:

Proposition 2.15 (Discrete Cheeger inequalities)**.** *Let X be a finite d-regular graph. Then we have*

$$\frac{\lambda(X)}{2} \leqslant h(X) \leqslant \sqrt{2d\lambda(X)}.$$

Before we start with the proof, we recall a useful tool from linear algebra:

Theorem 2.16 (Min-Max Theorem, Courant-Fischer, cf. [Mey08, Chapter 7.5, p. 550])**.** *Let A be an $n \times n$ Hermitian matrix, and for any vector $v \in \mathbb{C}^n$, denote by*

$$R_A := \frac{v^T A v}{v^T v}$$

the Rayleigh quotient of v with respect to A.
Then the eigenvalues $\lambda_1 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \leqslant \lambda_n$ of A are given by

$$\lambda_k = \min_{\dim(U)=k} \max_{v \in U,\ v \neq 0} R_A(v) = \max_{\dim(U)=n-k+1} \min_{v \in U,\ v \neq 0} R_A(v),$$

where U runs over all subspaces of the given dimension.
Moreover, if u_i, $1 \leqslant i \leqslant n$ form an orthonormal basis of eigenvectors corresponding to λ_i, we have

$$\lambda_k = \max_{v \in \operatorname{span}(u_1,\ldots,u_k), v \neq 0} R_A(v) = \min_{v \in \operatorname{span}(u_k,\ldots,u_n),\ v \neq 0} R_A(v).$$

Proof. Since A is Hermitian, we can fix a basis $\{u_1, \ldots, u_n\}$ of $\mathbb{C}^n$ consisting of eigenvalues of A, where each u_i corresponds to the eigenvalue λ_i. If $U \subseteq \mathbb{C}^n$ is a vector space of dimension k, then it intersects non-trivially with the $(n-k+1)$-dimensional span of the vectors $u_k, \ldots, u_n$. Let u be a non-zero vector in this intersection. Then we can compute the Rayleigh quotient $R_A(u)$ by writing $u = c_k u_k + \cdots + c_n u_n$ with coefficients $c_i = \langle u, u_i \rangle$:

$$R_A(u) = \frac{u^T A u}{u^T u} = \frac{\sum_{i,j \geqslant k} c_i c_j u_i^T A u_j}{\sum_{l,m \geqslant k} c_l c_m u_l^T u_m} = \frac{\sum_{i \geqslant k} c_i^2 \lambda_i}{\sum_{l \geqslant k} c_l^2}.$$

But this expression is just a weighted average of the eigenvalues $\lambda_k, \ldots, \lambda_n$, so we infer that $R_A(u) \geqslant \lambda_k$ for all $u \in U \setminus \{0\}$, and in particular, $\lambda_k \leqslant \max_{u \in U \setminus \{0\}} R_A(u)$. As this holds for any subspace U of dimension k, we have

$$\lambda_k \leqslant \min_{\dim(U)=k} \max_{v \in U,\ v \neq 0} R_A(v).$$

To establish the other inequality, consider the subspace U_0 spanned by $u_1, \ldots, u_k$. By the same argument as before, we have $R_A(u) \leqslant \lambda_k$ for any $u \in U_0 \setminus \{0\}$, and hence $\max_{u \in U_0 \setminus \{0\}} R_A(u) \leqslant \lambda_k$, yielding the other inequality. Thus, we have shown that

$$\lambda_k = \min_{\dim(U)=k} \max_{v \in U,\ v \neq 0} R_A(v) = \max_{v \in \operatorname{span}(u_1,\ldots,u_k),\ v \neq 0} R_A(v).$$

The other two equalities can be proved in a similar fashion. □

In particular, this allows us to compute the spectral gap:

Corollary 2.17 [cf. Lub94, Prop. 4.2.3, p. 45]**.** *Let X be a finite d-regular graph on n vertices with Laplacian Δ. Then*

$$\lambda(X) = \min_{v \perp 1} R_\Delta(v),$$

where $1 = (1, \ldots, 1)^T \in \mathbb{C}^n$.

Proof. Note that $u_1 = 1$ is an eigenvector to the eigenvalue $\lambda_1 = 0$. Choose suitable eigenvectors $u_2, \dots, u_n$ such that $\{u_1, \dots, u_n\}$ form an orthonormal basis. Now Theorem 2.16 gives that

$$\lambda(X) = \min_{v \in \operatorname{span}(u_2,\dots,u_n),\ v \neq 0} R_\Delta(v) = \min_{v \perp 1} R_\Delta(v).$$

□

Remark 2.18. We can rewrite the above expression further by viewing vectors in $\mathbb{C}^n$ as real-valued functions on the vertex set V. To this end, fix an enumeration $\{v_1, \dots, v_n\}$ of V, and identify a vector $u = (u_1, \dots, u_n)^T \in \mathbb{C}^n$ with the function f defined by $f(v_i) := u_i$. Then we have

$$\begin{aligned}
\lambda(X) = \min_{u \perp 1} R_X(U) = \min_{u \perp 1} \frac{u^T \Delta u}{u^T u} &= \min_{u \perp 1} \frac{u^T D_X u}{u^T u} - \frac{u^T A_X u}{u^T u} \\
&= \min_{f\colon V \to \mathbb{C},\ \sum f(v) = 0} \frac{\sum_{u \in V} d \cdot f(u)^2 - 2 \sum_{\{v,w\} \in E} f(v) f(w)}{\sum_{u \in V} f(u)^2} \\
&= \min_{f\colon V \to \mathbb{C},\ \sum f(v) = 0} \frac{2 \sum_{\{v,w\} \in E} f(v)^2 - 2 \sum_{\{v,w\} \in E} f(v) f(w)}{\sum_{u \in V} f(u)^2} \\
&= \min_{f\colon V \to \mathbb{C},\ \sum f(v) = 0} \frac{\sum_{\{v,w\} \in E} \left(f(v) - f(w)\right)^2}{\sum_{u \in V} f(u)^2} \\
&= \min_{f \in L_0^2(X)} \frac{\sum_{\{v,w\} \in E} \left(f(v) - f(w)\right)^2}{\|f\|^2}.
\end{aligned}$$

We will prove the two inequalities in Proposition 2.15 separately. The proof of the first one is rather straightforward.

Proposition 2.19 [cf. HLW06, Section 4.5.1, p. 475]**.** *Any finite d-regular graph X satisfies* $\lambda(X)/2 \leqslant h(X)$.

Proof. First, fix an enumeration $\{v_1, \dots, v_n\}$ of the vertex set V of X, and for any subset W of V denote by 1_W the vector $(w_1, \dots, w_n)^T$, where $w_i = 1$ if $v_i \in W$ and $w_i = 0$ otherwise. Now let S be an arbitrary subset of V, and denote $s := |S|$. Consider the vector $x := (n-s) \cdot 1_S - s \cdot 1_{S^C}$.
Then x satisfies

$$x^T x = s(n-s)^2 + (n-s)s^2 = sn(n-s)$$

and

$$x^T A_X\, x = 2 \cdot \left(\sum_{i,j : v_i \sim v_j} x_i x_j \right) = 2 \left(e(S,S) \cdot (n-s)^2 - e\left(S, S^C\right) \cdot (n-s) + e\left(S^C, S^C\right) \cdot s^2 \right).$$

By double-counting the edges of X starting from vertices in S, we get that $s \cdot d = 2e(S,S) + e(S,S^C)$ and, applying the same argument to S^C, that $(n-s) \cdot d = 2e(S^C, S^C) + e(S, S^C)$. Hence we obtain

$$\begin{aligned}
x^T A_X\, x &= (n-s)^2(ds - e(S,S^C)) - 2(n-s)se(S,S^C) + s^2((n-s)d - e(S,S^C)) \\
&= dns(n-s) - n^2 e(S,S^C).
\end{aligned}$$

By Corollary 2.17, we know that $\lambda(X) = \min_{u \in 1^\perp} R_\Delta(u)$.
Since $\langle x, 1\rangle = s(n-s) + (n-s)(-s) = 0$, we infer that

$$\lambda(X) \leqslant \frac{x^T \Delta\ x}{x^T x} = \frac{x^T D_X x - x^T A_X x}{x^T x} = d - \frac{dns(n-s) - n^2 e(S, S^C)}{sn(n-s)} = \frac{ne(S, S^C)}{s(n-s)}.$$

If S is of size $s \leqslant \frac{|V|}{2} = \frac{n}{2}$, we have

$$h_X(S) = \frac{|\partial S|}{|S|} = \frac{e(S, S^C)}{|S|} \geqslant \frac{n-s}{n}\lambda(X) \geqslant \frac{1}{2}\lambda(X),$$

which concludes the proof. $\square$

The proof of the other inequality is slightly more involved. To avoid notational ambiguities, we will use m for the regularity degree here and save the letter d for the differential operator.

Proposition 2.20 [Lub94, Prop. 4.2.4, p. 46]**.** *Any finite m-regular graph $X = (V, E)$ satisfies $\lambda(X) \geqslant \frac{h^2(X)}{2m}$.*

Proof. Fix an arbitrary orientation on X (so that we can use the differential operator d), and let $g \in L_0^2(V)$ be an eigenfunction of Δ corresponding to the eigenvalue $\lambda = \lambda(X)$ with $\|g\| = 1$. Then we have $\lambda = \langle \Delta g, g\rangle$.
Now set $V^+ := \{v \in V : g(v) > 0\}$ and define $f \in L^2(V)$ by

$$f(v) = \begin{cases} g(v) & \text{if } g(v) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Up to changing the sign of g, we may assume that $|V^+| \leqslant \frac{1}{2}|V|$. Observe that for every $v \in V^+$, we have

$$\Delta f(v) = mf(v) - \sum_{u \sim v} f(u) \leqslant mg(v) - \sum_{u \sim v} f(v) = \Delta g(v)$$

because $g \leqslant f$ and $g = f$ on V^+. Thus, we obtain

$$\langle \Delta f, f\rangle = \sum_{v \in V} \Delta f(v) \cdot f(v) = \sum_{v \in V^+} \Delta f(v) \cdot f(v) \leqslant \sum_{v \in V^+} \Delta g(v) \cdot g(v) = \lambda \sum_{v \in V^+} g(v)^2 = \lambda \|f\|^2.$$

Since $\langle \Delta f, f\rangle = \langle D^T D f, f\rangle = \langle df, df\rangle = \|df\|^2$, this means that $\|df\| \leqslant \sqrt{\lambda}\|f\|$.
Now set

$$A := \sum_{e \in E} |f^2(e^+) - f^2(e^-)|.$$

Then

$$\begin{aligned} A &= \sum_{e \in E} |f(e^+) + f(e^-)| \cdot |f(e^+) - f(e^-)| \\ &\leqslant \left(\sum_{e \in E} |f(e^+) + f(e^-)|^2\right)^{1/2} \left(\sum_{e \in E} |f(e^+) - f(e^-)|^2\right)^{1/2} \\ &\leqslant \left(2 \sum_{e \in E} f^2(e^+) + f^2(e^-)\right)^{1/2} \|df\| \\ &\leqslant \sqrt{2\lambda}\|f\| \left(\sum_{v \in V} m \cdot f^2(v)\right)^{1/2} \\ &= \sqrt{2m\lambda}\|f\|^2. \end{aligned}$$

Thus, we have proved that

$$\lambda \geqslant \frac{A^2}{2m\|f\|^4}.$$

It therefore suffices to show that $A \geqslant h(X)\|f\|^2$ to conclude.

To this end, write the image of f as $\mathrm{Im} f = \{\beta_0, \ldots, \beta_r\}$ with $0 = \beta_0 < \beta_1 < \cdots < \beta_r$. Moreover, set $L_i \coloneqq \{x \in V \mid f(x) = \beta_i\}$ for $0 \leqslant i \leqslant r$. Then $V = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_r$, and $|L_i| \leqslant \frac{1}{2}|V|$ for all $i \geqslant 1$ because $|V^+| \leqslant \frac{1}{2}|V|$.
We can rewrite A as

$$A = \sum_{j=1}^{r} \sum_{x : f(x)=\beta_j} \sum_{\substack{y:y\sim x,\\ f(y)<\beta_j}} f^2(x) - f^2(y).$$

Observe that if two vertices x and y are connected by an edge e and $f(x) = \beta_j > \beta_k = f(y)$, then we have

(i) $f^2(x) - f^2(y) = \beta_j^2 - \beta_k^2 = (\beta_j^2 - \beta_{j-1}^2) + (\beta_{j-1}^2 - \beta_{j-2}^2) + \cdots + (\beta_{k+1}^2 - \beta_k^2)$, and

(ii) $e \in \partial L_i \iff x \in L_i, y \notin L_i \iff f(y) < \beta_i \leqslant f(x) \iff \beta_k < \beta_i \leqslant \beta_j \iff k < i \leqslant j$.

Thus,

$$\begin{aligned} A &\overset{\text{(i)}}{=} \sum_{j=1}^{r} \sum_{x : f(x)=\beta_j} \sum_{\substack{y:y\sim x,\\ f(y)=\beta_k<\beta_j}} \sum_{i=k+1}^{j} \beta_i^2 - \beta_{i+1}^2 \\ &= \sum_{(i,e)\in I} \beta_i^2 - \beta_{i-1}^2, \end{aligned}$$

where I is given by the multiset

$$\begin{aligned} I &= \{(i, \{x,y\}) \mid 1 \leqslant i \leqslant r,\ \{x,y\} \in E,\ f(x) \geqslant \beta_i > f(y)\} \\ &\overset{\text{(ii)}}{=} \{(i,e) \mid 1 \leqslant i \leqslant r,\ e \in \partial L_i\}. \end{aligned}$$

Therefore, we can rewrite the above expression for A as

$$A = \sum_{i=1}^{r} \sum_{e\in\partial L_i} \beta_i - \beta_{i-1}^2 = \sum_{i=1}^{r} |\partial L_i| \left(\beta_i^2 - \beta_{i-1}^2\right).$$

Since $|L_i| \leqslant \frac{1}{2}|V|$ for $i > 0$, we have $h(X) \leqslant \frac{|\partial L_i|}{|L_i|}$ for $i > 0$.
Hence,

$$\begin{aligned} A &\geqslant \sum_{j=1}^{r} h(X)|L_i|(\beta_i^2 - \beta_{i-1}^2) \\ &= h(X) \cdot \left(|L_r|\beta_r^2 + \sum_{i=1}^{r-1} \beta_i^2 \left(|L_i| - |L_{i+1}|\right) \right). \end{aligned}$$

Finally, note that $v \in L_i \setminus L_{i+1}$ if and only if $f(v) = \beta_i$, so that $|\{v :\ f(v) = \beta_i\}| = |L_i| - |L_{i+1}|$ and hence

$$A \geqslant h(X) \cdot \sum_{v\in V} f(v)^2 = h(X)\|f\|^2.$$

So altogether, we have proved that

$$\lambda \geqslant \frac{A^2}{2m\|f\|^4} \geqslant \frac{h^2(X)}{2m}.$$

$\square$

As the Cheeger inequalities allow us to express expansion in terms of the second eigenvalue, we can reformulate the definition of a family of expanders by replacing the combinatorial condition (ii) in Definition 2.5 by an analogous spectral condition (ii)':

Definition 2.21. Let $d > 0$ be a fixed integer. Then an infinite family $\{X_n : n \in \mathbb{N}\}$ of finite d-regular graphs is called a *family of expanders*, if the following two conditions are met:

(i) $|X_n| \to \infty$ as $n \to \infty$, and

(ii)' there is a constant $C > 0$ such that $\lambda(X_n) \geqslant C$ for all n.

2.3 The Margulis expanders

In this section, we will give a slightly modified version of Margulis' [Mar73] example of an infinite family of expanders and discuss some of its variants (see Remark 2.25 (2 for a summary). Moreover, we will present an elementary proof of the fact that this family of graphs is indeed an expander family, following [Lee12].

We start by fixing the transformations that will be used to define said family of graphs.

Notation 2.22. Let ρ, σ, and τ be the maps $\mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$\rho(x,y) = (-y,x), \quad \sigma(x,y) = (x+y,y), \quad \text{and} \quad \tau(x,y) = (x,y+x).$$

We will identify them with the corresponding matrices

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tau = \sigma^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Construction 2.23 [BHV08; cf. Mar73]**.** For every integer $n > 0$, let $\Gamma_n \coloneqq (V_n, E_n)$, where $V_n \coloneqq (\mathbb{Z}/n\mathbb{Z})^2$ and E_n is defined by

$$(x,y) \leftrightarrow \begin{cases} (x \pm 1, y) \\ (x, y \pm 1) \\ (x \pm y, y) = \sigma^{\pm 1}(x,y) \\ (x, y \pm x) = \tau^{\pm 1}(x,y) \end{cases}$$

with all arithmetic carried out modulo n.

The hereby defined infinite family $\{\Gamma_n : n \in \mathbb{N}\}$ of 8-regular undirected graphs is called the *Margulis expander family* or the *Margulis expander*.

Remark 2.24. (1) In the above construction, the notation "$v \leftrightarrow w_1, \dots, w_n$" means that the vertices adjacent to v are precisely $w_1, \dots, w_n$ (and no more). This implies that the defining transformations above have to fulfil some symmetry property to ensure that "$v \leftrightarrow w$" and "$w \leftrightarrow v$" are equivalent.

(2) The original construction from [Mar73] differs from the one presented here in the point that Margulis' expanders are *bipartite*. Their vertex set is given by $V_n \sqcup V_n^*$, where V_n^* is a disjoint copy of V_n, and each vertex $v \in V_n$ is not adjacent to the vertices of V_n listed in Construction 2.23, but rather to their counterparts in V_n^*, so that all edges are between vertices from V_n and vertices from V_n^*. Moreover, every vertex $v \in V_n$ is also adjacent to its copy $v \in V_n^*$ in Margulis' original graphs.

The first four graphs in the Margulis expander as defined in Construction 2.23 are depicted in Figure 2.2.

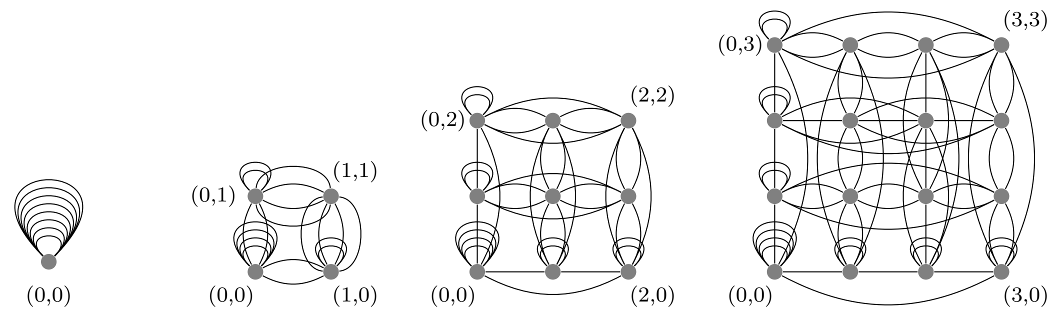

Figure 2.2: The first four Margulis expanders Γ_1, Γ_2, Γ_3, Γ_4

Remark 2.25. (1) We can actually view the graphs Γ_n as Schreier graphs:
Consider the semidirect product $G = \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. It acts on $V_n = (\mathbb{Z}/i\mathbb{Z})^2$ by letting

$$(v, A) \cdot p := Ap + v$$

for $v \in \mathbb{Z}^2$, $A \in \mathrm{SL}_2(\mathbb{Z})$ and $p \in V_n$.
This action is well-defined: We have

$$\mathrm{id}_G \cdot p = (0, \mathrm{id}_{\mathrm{SL}_2(\mathbb{Z})}) \cdot p = p$$

and

$$\begin{aligned}((v_1, A_1)(v_2, A_2)) \cdot p &= (v_1 + A_1 v_2, A_1 A_2) \cdot p \\ &= A_1 A_2 p + v_1 + A_1 v_2 \\ &= (v_1, A_1) \cdot (A_2 p + v_2) = (v_1, A_1) \cdot ((A_2, v_2) \cdot p).\end{aligned}$$

for all $A_1, A_2 \in \mathrm{SL}_2(\mathbb{Z})$, $v_1, v_2 \in \mathbb{Z}^2$ and $p \in V_n$.

Now consider the set of generators S of G consisting of the group elements $(0, \sigma)$, $(0, \tau)$, (e_1, Id), (e_2, Id) and their inverses, where

$$0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The action of the elements of S on a vertex $p = (x, y)^T$ in V_n produces exactly the vertices that are adjacent to p according to Construction 2.23. Hence, the corresponding Schreier graph $\mathcal{S}(G, V_n, S)$ is precisely the Margulis expander Γ_n.

(2) There are several families of graphs that are termed “Margulis expanders”, and in fact, also the original family of graphs presented in Margulis’ 1973 paper [Mar73] slightly differs from the one defined above. The general concept of connecting vertices in $V_n = (\mathbb{Z}/n\mathbb{Z})^2$ by applying certain transformations is always the same, but there is some variation in the exact choice of those transformations, and moreover, some of the graphs are bipartite (as sketched in Remark 2.24 (2)).

An overview of the different variants of Margulis expanders is given in Table 2.3.

notation	appears in	degree	$v \leftrightarrow$
Γ_n^M	[Mar73]*, [Lub94]	5/8	$\rho v, \sigma v, v+e_1, v+e_2$ (and inverse transformations)
Γ_n^J	[JM87]*, [HLW06]	5/8	$\sigma^2 v, \sigma^2 v+e_1, \tau^2 v, \tau^2 v+e_2$ (and inverse transformations)
Γ_n^G	[GG81]*	5	$\mathrm{Id}, \sigma v, \sigma v+e_1, \tau v, \tau v+e_2$
Γ_n	[BHV08], [Lee12]	8	$\sigma v, \tau v, v+e_1, v+e_2$ and inverse transformations

* bipartite expanders, with an extra edge obtained from the identity map in addition to the transformations listed

Table 2.3: The main variants of the Margulis expander family

A visualisation of these graphs is shown in Figure 2.4.

The main goal of this section is to prove the following:

Theorem 2.26. *The graphs $\{\Gamma_n : n \in \mathbb{N}\}$ form a family of expanders.*

In Section 2.4, we will see a result that verifies the expansion property for $(\Gamma_n)_{n\in\mathbb{N}}$ based on a rather deep representation-theoretic property of the group $G = \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

In this section however, we will follow a more hands-on approach and present an elementary proof based on Lee's talk [Lee12]. This approach goes back to a proof by Gabber and Galil [GG81] which has been further improved by Jimbo and Maruoka [JM87]. However, according to Hoory, Linial and Widgerson, even this elementary proof is "still subtle and mysterious" [HLW06, p. 503]. In contrast to this, Lee [Lee12] claims that his version of the proof (which is the one that we will see in this section) is "almost disillusioningly simple".

To begin with the proof, let us consider an infinite version of the Margulis expanders, without the "neighbour-edges" connecting (x, y) with $(x \pm 1, y)$ and $(x, y \pm 1)$.

Definition 2.27. Denote by Γ the graph with vertex set $\mathbb{Z}^2$ and edge set defined by

$$(x,y) \leftrightarrow \begin{cases} \sigma^{\pm 1}(x,y) = (x \pm y, y) \\ \tau^{\pm 1}(x,y) = (x, y \pm x). \end{cases}$$

This 4-regular infinite graph has strong expansion properties, as the following result indicates:

Lemma 2.28 [Lee12]**.** *In the above defined graph Γ, we have $|\partial A| \geqslant |A|$ for any $A \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$.*

Proof. Partition $\Gamma \setminus \{(0,0)\}$ into the four quadrants Q_1, Q_2, Q_3, Q_4, where $Q_1 = \{(x,y) \in \mathbb{Z}^2 : x > 0, y \geqslant 0\}$, and the other quadrants are obtained by rotating Q_1 by multiples of $\pi/2$.
Let $A_i := A \cap Q_i$. We first consider only Q_1 and claim that $e(A_1, A^C \cap Q_1) \geqslant |A_1|$.
To see this, first observe that $\sigma(A_1) \subseteq Q_1$ and $\tau(A_1) \subseteq Q_1$ (as σ and τ both map positive coordinates to positive coordinates again).
Moreover, note that $\sigma(A_1) \cap \tau(A_1) = \emptyset$ because σ maps elements of Q_1 below the diagonal $x = y$ while τ maps them above or onto the diagonal.
Thus, we have that

$$|\sigma(A_1) \cup \tau(A_1)| = |\sigma(A_1)| + |\tau(A_1)| = 2|A_1|,$$

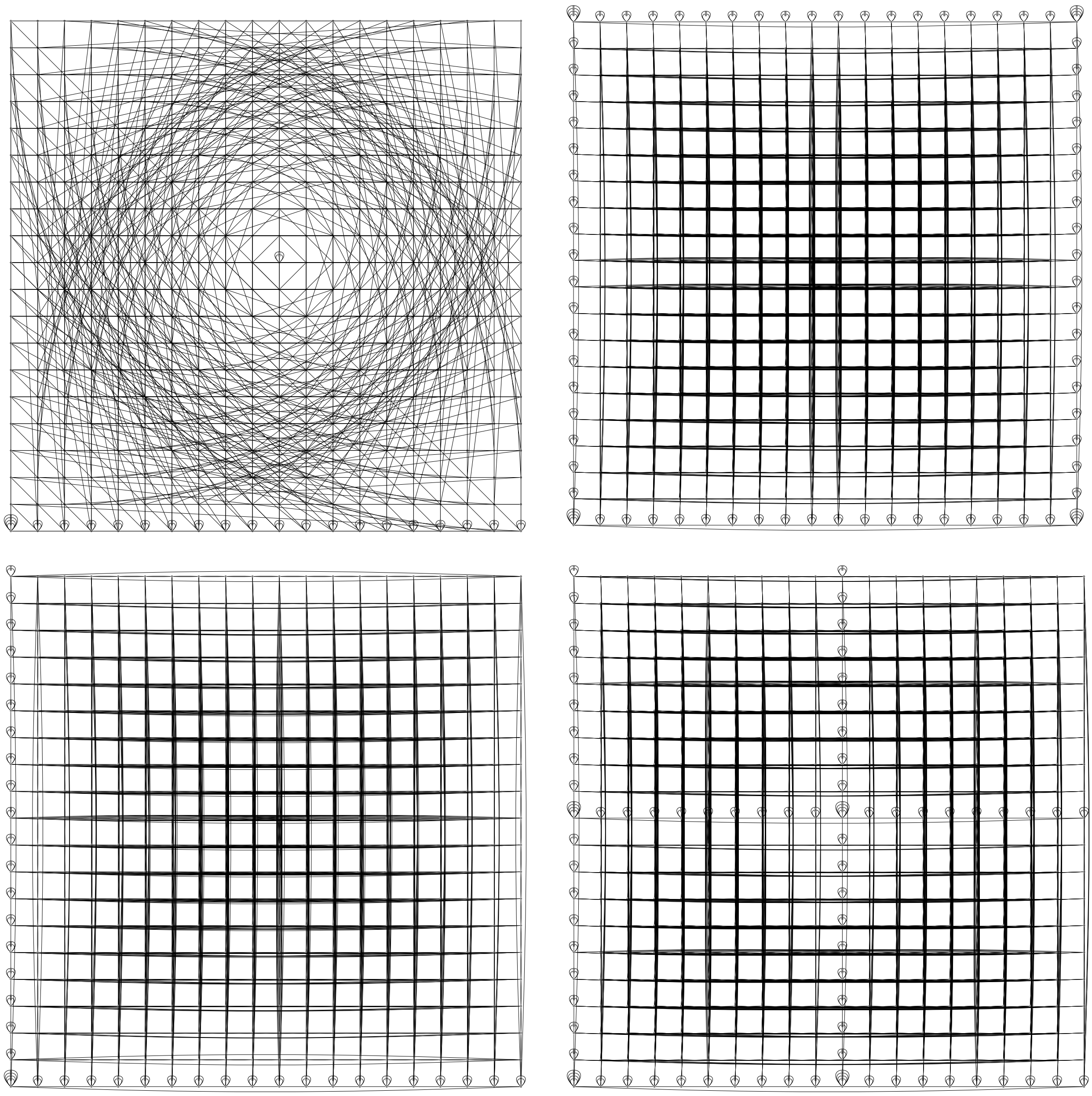

Figure 2.4: Margulis expanders (in their non-bipartite versions), visualised for $n = 20$. Clockwise, starting from top left: Γ_n^M, Γ_n^J, Γ_n^G, Γ_n.

so

$$|A_1| \leqslant |(\sigma(A_1) \cup \tau(A_1)) \setminus A_1| \leqslant |E(A_1, A^C \cap Q_1)|,$$

as claimed.
Note that Γ is invariant under rotation by $\pi/2$ (as rotating a vector in $\mathbb{Z}^2$ by $\pi/2$ corresponds to exchanging the entries of the coordinates and changing one sign), so the same property holds for the other quadrants. Hence, we obtain $|\partial A| \geqslant |A|$, which concludes the proof. $\square$

In order to verify that the Margulis expander family is indeed a family of expanders, it will turn out useful to study a "continuous" version: Consider the *torus* $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. In analogy to the formula for the second eigenvalue of a graph as given in Remark 2.18, we define

$$\lambda(\mathbb{T}^2) := \min_{f \in L_0^2(\mathbb{T}^2)} \left\{ \frac{\|f - f \circ \sigma\|^2 + \|f - f \circ \tau\|^2}{\|f\|^2} \right\},$$

where $L_0^2(\mathbb{T}^2) := \{f\colon \mathbb{T}^2 \to \mathbb{C} \mid \int_{\mathbb{T}^2} f\, d\lambda^2 = 0\}$ and $\|f\|$ denotes the L^2-norm $(\int_{\mathbb{T}^2} f^2\, d\lambda^2)^{1/2}$ of $f \in L^2(\mathbb{T}^2)$.

Lemma 2.29 [Lee12]**.** *There exists an $\varepsilon > 0$ such that $\lambda(\Gamma_n) \geqslant \varepsilon\lambda(\mathbb{T}^2)$ holds for all $n \in \mathbb{N}$.*

Proof. First, fix $n \in \mathbb{N}$ and a function $f\colon V_n \to \mathbb{R}$ satisfying $\sum_{v\in V_n} f(v) = 0$.
In order to be able to "embed" Γ_n, scale the torus up by a factor of n, i.e. consider $\widetilde{\mathbb{T}}^2 := n\mathbb{T}^2$ (note that this does not change the second eigenvalue of the torus).
Now extend f to a function $\tilde{f}$ defined on the whole torus by letting

$$\tilde{f}(z) := \frac{\sum_{i=1}^4 (1 - \|u_i - z\|_\infty) f(u_i)}{\sum_{i=1}^4 (1 - \|u_i - z\|_\infty)},$$

where $u_1, \ldots, u_4$ are the four vertices of Γ_n closest to z.
Note that $\tilde{f}(u_i) = f(u_i)$, so $\tilde{f}$ is indeed an extension of f. Also, if z is a point on the line between two vertices, say u_1 and u_2, then $\tilde{f}(z)$ depends only on u_1 and u_2, so $\tilde{f}$ is well-defined on all of $\widetilde{\mathbb{T}}^2$.

We claim that $\tilde{f}$ has the following properties:

(1) $\int_{\widetilde{\mathbb{T}}^2} \tilde{f}\, d\lambda^2 = 0$

(2) There exists a constant $c > 0$ such that

$$\int_{\widetilde{\mathbb{T}}^2} \tilde{f}^2\, d\lambda^2 \geqslant c \sum_{v\in V_n} f(v)^2.$$

(3) There exists a constant $\tilde{c}$, $0 < \tilde{c} < \infty$, such that

$$\|\tilde{f} - \tilde{f}\circ\sigma\|^2 + \|\tilde{f} - \tilde{f}\circ\tau\|^2 \leqslant \tilde{c} \sum_{\{u,v\}\in E} (f(u) - f(v))^2\ .$$

To see (1), we introduce some notation: for $0 \leqslant i, j < n$, denote by $\square_{i,j}$ the unit square with bottom left corner (i, j). Furthermore, denote the four corners of $\square_{i,j}$ by $u^1_{i,j}, \ldots, u^4_{i,j}$ (so that $u^1_{i,j} = (i, j)$, $u^2_{i,j} = (i + 1, j)$, $u^3_{i,j} = (i, j + 1)$, and $u^4_{i,j} = (i + 1, j + 1)$).
We then compute the integral by summing over all squares of the torus:

$$\begin{aligned}
\int_{\widetilde{\mathbb{T}}^2} \tilde{f}\, d\lambda^2 &= \sum_{1\leqslant i,j<n} \int_{\square_{i,j}} \tilde{f}\, d\lambda^2 \\
&= \sum_{1\leqslant i,j<n} \int_{\square_{i,j}} \frac{\sum_{k=1}^4 (1 - \|u^k_{i,j} - z\|_\infty) f(u^k_{i,j})}{\sum_{l=1}^4 (1 - \|u^l_{i,j} - z\|_\infty)}\, d\lambda^2 \\
&= \sum_{k=1}^4 \sum_{1\leqslant i,j<n} f(u^k_{i,j}) \int_{\square_{i,j}} \frac{1 - \|u^k_{i,j} - z\|_\infty}{\sum_{l=1}^4 (1 - \|u^l_{i,j} - z\|_\infty)}\, d\lambda^2
\end{aligned}$$

In the last line, note that the integrand no longer depends on i and j but only on k, so the integral can be viewed as a constant c_k. Since we assumed that f satisfies $\sum_{v\in V_n} f(v) = 0$, we obtain

$$\int_{\widetilde{\mathbb{T}}^2} \tilde{f}\, d\lambda^2 = \sum_{k=1}^4 \sum_{0\leqslant i,j<n} c_k f(u^k_{i,j}) = 0,$$

as claimed.

For (2), we proceed similarly: Choose $u_{i,j}^{\max}$ to be the corner of $\square_{i,j}$ that maximises the value of $|\tilde{f}|$, i.e. $u_{i,j}^{\max} = \operatorname{argmax}\{|\tilde{f}(v)| : v = u_{i,j}^1, \ldots, u_{i,j}^4\}$. Let $\widetilde{\square}_{i,j}$ denote a square inside $\square_{i,j}$ with corner $u_{i,j}^{\max}$, but smaller side length, e.g. side length $\frac{1}{8}$. Now for $z \in \widetilde{\square}_{i,j}$, we have $\frac{7}{8} \leqslant 1 - \|z - u_{i,j}^{\max}\|_\infty \leqslant 1$ and $0 \leqslant 1 - \|z - u_k^{i,j}\|_\infty \leqslant \frac{1}{8}$ for k such that $u_k^{i,j} \neq u_{i,j}^{\max}$.
Therefore, we have

$$\left|\tilde{f}(z)\right| \geqslant \left| \frac{\frac{7}{8} f(u_{i,j}^{\max}) - \frac{3}{8} f(u_{i,j}^{\max})}{1 + \frac{3}{8}} \right| = \frac{4}{11} \left|f(u_{i,j}^{\max})\right|$$

for all $z \in \widetilde{\square}_{i,j}$, and thus, we obtain

$$\int_{\widetilde{\mathbb{T}}^2} \tilde{f}^2 \, d\lambda^2 \geqslant \sum_{0 \leqslant i,j < n} \int_{\widetilde{\square}_{i,j}} \tilde{f}^2 \, d\lambda^2 \geqslant \sum_{0 \leqslant i,j < n} \operatorname{Vol}(\widetilde{\square}_{i,j}) \left(\frac{4}{11}\right)^2 \tilde{f}(u_{i,j}^{max})^2 \geqslant c \sum_{v \in V_n} \tilde{f}(v)^2.$$

To see (3), first note that it suffices to find a bound $\tilde{c}$ such that

$$\|\tilde{f} - \tilde{f} \circ \sigma\|^2 \leqslant \tilde{c} \sum_{\{u,v\} \in E} (f(u) - f(v))^2$$

because of the symmetry of the definitions of σ and τ.
Again we split the torus into the squares $\square_{i,j}$.
Fixing i, j and a point $z \in \square_{i,j}$, we try to find a bound for $f(z) - f(\sigma(z))$. Note that σ maps $\square_{i,j}$ to a parallelogram (one side of the original square is "shifted" by a distance of 1 with respect to the other). Thus, if we denote by $\square_{i,j}^\sigma$ the square that contains $\sigma(z)$, three of the corners of $\square_{i,j}^\sigma$ are of the form $\sigma(u_{i,j}^k)$ for some $k \in \{1, 2, 3, 4\}$.
Therefore, any corner of $\square_{i,j}$ is connected to any corner of $\square_{i,j}^\sigma$ by a path in Γ_n of length at most 4: Take a neighbour vertex and then take σ to land in some corner of $\square_{i,j}^\sigma$; from here, any other corner of $\square_{i,j}^\sigma$ can be reached in at most two steps.

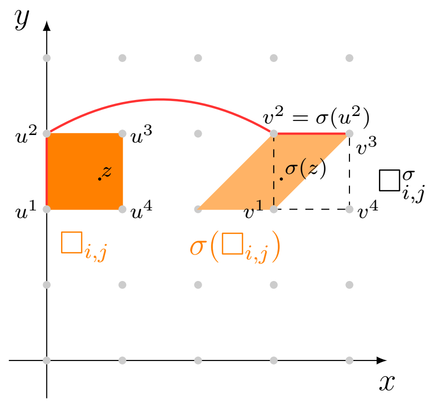

Figure 2.5: Any corner of $\square_{i,j}$ is connected to any corner of $\square_{i,j}^\sigma$ by a path of length at most 4.

By definition of $\tilde{f}$, the value of $\tilde{f}(z)$ is a convex combination of $f(u_{i,j}^1), \ldots, f(u_{i,j}^4)$, and, analogously, the value of $\tilde{f}(\sigma(z))$ is a convex combination of the values of f on the corners $v_{i,j}^1, \ldots, v_{i,j}^4$ of $\square_{i,j}^\sigma$. Thus, we have that

$$\tilde{f}(z) - \tilde{f}(\sigma(z))) \leqslant \max_{1 \leqslant k,l \leqslant 4} f(u_{i,j}^k) - f(v_{i,j}^l).$$

Let $u_{i,j}^{\max}$ and $v_{i,j}^{\max}$ denote the vertices where the maximum on the right hand side of the above equation is attained. As argued before, $u_{i,j}^{\max}$ and $v_{i,j}^{\max}$ are connected by a path in Γ_n of length at most four, say by the path

$$u_{i,j}^{\max} =: w_{i,j}^1 \leftrightarrow w_{i,j}^2 \leftrightarrow w_{i,j}^3 \leftrightarrow w_{i,j}^4 := v_{i,j}^{\max}.$$

Then, we have

$$\begin{aligned}\left(\tilde{f}(z)-\tilde{f}(\sigma(z))\right)^2 &\leqslant \left(f(u_{i,j}^{\max})-f(v_{i,j}^{\max})\right)^2\\ &=\left(\sum_{k=1}^{3} f(w_{i,j}^{k})-f(w_{i,j}^{k+1})\right)^2 \leqslant 3\sum_{k=1}^{3}\left(f(w_{i,j}^{k})-f(w_{i,j}^{k+1})\right)^2.\end{aligned}$$

Note that every edge $w_{i,j}^1 \leftrightarrow w_{i,j}^2$ can appear in this estimate for at most four different choices of i,j (as every corner is only adjacent to four different squares).
Also note that, given z in $\square_{i,j}$, there are two possibilities for $\square_{i,j}^{\sigma}$.
This yields

$$\begin{aligned}\|\tilde{f}-\tilde{f}\circ\sigma\| &= \int_{\widetilde{\mathbb{T}}^2}(\tilde{f}-\tilde{f}\circ\sigma)^2\,d\lambda^2=\sum_{i,j\leqslant n}\int_{\square_{i,j}}(\tilde{f}-\tilde{f}\circ\sigma)^2\,d\lambda^2\\ &\leqslant \sum_{i,j\leqslant n} 2\cdot 3\sum_{k=1}^{3}\left(f(w_{i,j}^{k})-f(w_{i,j}^{k+1})\right)^2 \leqslant 4\cdot 2\cdot 3\sum_{\{u,v\}\in E_n}\left(f(u)-f(v)\right)^2,\end{aligned}$$

finishing the proof of the claim.

Finally, we can put things together:

$$\lambda_2(\Gamma_n)=\min_{\substack{f\colon V_n\to\mathbb{R},\\ \sum f(u)=0}}\frac{\sum_{u\leftrightarrow v} f(u)f(v)}{\sum_{w\in V_n} f(w)^2}\geqslant\frac{c}{\tilde{c}}\min_{\substack{f\colon V_n\to\mathbb{R}\\ \sum f(u)=0}}\frac{\|\tilde{f}-\tilde{f}\circ\sigma\|^2+\|\tilde{f}-\tilde{f}\circ\tau\|^2}{\|\tilde{f}\|^2}\geqslant\frac{c}{\tilde{c}}\,\lambda(\mathbb{T}^2).$$

$\square$

So in order to show that Γ_n is an expander family, all we are left with is showing that $\lambda(\mathbb{T}^2)>0$. This will be the content of the next result.

Lemma 2.30 [Lee12]**.** *We have $\lambda(\mathbb{T}^2)>0$.*

Proof. Recall the Fourier transform $\mathcal{F}\colon L^2(\mathbb{T}^2)\to\ell^2(\mathbb{Z}^2)$, $f\mapsto\mathcal{F}f=\hat{f}$:
Consider the Fourier basis $\{\chi_{m,n}:\ m,n\in\mathbb{Z}\}$ of $L^2(\mathbb{T}^2)$ defined by $\chi_{m,n}(x,y):=\exp(2\pi i(mx+ny))$.
Then $\hat{f}\colon\mathbb{Z}^2\to\mathbb{C},\ (m,n)\mapsto\langle f,\chi_{m,n}\rangle_{L^2(\mathbb{T}^2)}$ is a function that outputs the coefficients of f in the Fourier basis, i.e. $\hat{f}$ has the property that

$$f=\sum_{(m,n)\in\mathbb{Z}^2}\hat{f}(m,n)\,\chi_{m,n}.$$

Observe that the Fourier transform has the following properties:

(1) $\mathcal{F}$ is a linear isometry (where $\ell^2(\mathbb{Z}^2)$ is equipped with the usual norm defined by $\|f\|_{\ell^2}:=\sum_{(i,j)\in\mathbb{Z}^2}f(i,j)^2)$;

(2) $\hat{f}(0,0)=\int_{\mathbb{T}^2}f\,d\lambda^2$;

(3) $\widehat{f\circ\sigma}=\hat{f}\circ\tau^{-1}$ and $\widehat{f\circ\tau}=\hat{f}\circ\sigma^{-1}$.

(1) is clear: $\mathcal{F}$ is linear by definition and an isometry by Plancherel's identity (e.g., see [Gra14, Prop. 3.2.7, p. 179]).
(2) is clear as well since by definition, $\hat{f}(0,0)=\langle f,1\rangle_{L^2(\mathbb{T}^2)}=\int_{\mathbb{T}^2}f\,d\lambda^2$.

To see (3), note that σ satisfies $\chi_{m,n} \circ \sigma(x,y) = \chi_{m,n}(x+y,y) = \exp(2\pi i(mx+my+ny)) = \chi_{m,m+n}(x,y)$, and therefore we have

$$f \circ \sigma = \sum_{m,n} \hat{f}(m,n)\chi_{m,n} \circ \sigma = \sum_{m,n} \hat{f}(m,n)\chi_{m,m+n} = \sum_{m,n} \hat{f}(m,n-m)\chi_{m,n} = \sum_{m,n} \hat{f} \circ \tau^{-1}(m,n)\chi_{m,n}.$$

Hence, $\widehat{f \circ \sigma} = \hat{f} \circ \tau^{-1}$, and analogously, one sees that $\widehat{f \circ \tau} = \hat{f} \circ \sigma^{-1}$.

Thus, we can rewrite the formula for $\lambda(\mathbb{T}^2)$:

$$\begin{aligned}
\lambda(\mathbb{T}^2) &= \min_{f \in L^2(\mathbb{T}^2)} \left\{ \frac{\|f - f\circ\sigma\|^2 + \|f - f\circ\tau\|^2}{\|f\|^2} : \quad \int_{\mathbb{T}^2} f \, d\lambda^2 = 0 \right\} \\
&\overset{(1)}{=} \min_{f \in L^2(\mathbb{T}^2)} \left\{ \frac{\|\hat{f} - \widehat{f\circ\sigma}\|^2 + \|\hat{f} - \widehat{f\circ\tau}\|^2}{\|\hat{f}\|^2} : \quad \int_{\mathbb{T}^2} f \, d\lambda^2 = 0 \right\} \\
&\overset{(2)+(3)}{\geqslant} \min_{\hat{f} \in \ell_2(\mathbb{Z}^2)} \left\{ \frac{\|\hat{f} - \hat{f}\circ\tau^{-1}\|^2 + \|\hat{f} - \hat{f}\circ\sigma^{-1}\|^2}{\|\hat{f}\|^2} : \quad \hat{f}(0,0) = 0 \right\} \\
&= \min_{\hat{f} \in \ell_2(\mathbb{Z}^2)} \left\{ \frac{\sum_{z\in\mathbb{Z}^2} \left(\hat{f}(z) - \hat{f}(\tau^{-1}(z))\right)^2 + \left(\hat{f}(z) - \hat{f}(\sigma^{-1}(z))\right)^2}{\sum_{z\in\mathbb{Z}^2} \hat{f}(z)^2} : \quad \hat{f}(0,0) = 0 \right\} \\
&= \min_{\hat{f} \in \ell_2(\mathbb{Z}^2)} \left\{ \frac{\sum_{\{x,y\}\in E(\Gamma)} \left(\hat{f}(x) - \hat{f}(y)\right)^2}{\sum_{z\in\mathbb{Z}^2} \hat{f}(z)^2} : \quad \hat{f}(0,0) = 0 \right\} \\
&= \min_{\hat{f} \in \ell_2(\mathbb{Z}^2)} \left\{ R_\Gamma(\hat{f}) : \quad \hat{f}(0,0) = 0 \right\} \\
&= \lambda(\Gamma \setminus \{(0,0)\}) \overset{\text{Cheeger inequality}}{\geqslant} \frac{h(\Gamma\setminus\{(0,0)\})^2}{2} \overset{\text{Lemma 2.28}}{\geqslant} \frac{1}{2}
\end{aligned}$$

□

This concludes the proof of Theorem 2.26. Even though there are some technicalities (mainly encountered in the proof of Lemma 2.29), the concept of the proof is indeed rather simple:

- Construct the underlying graph Γ and prove its expansion property;
- Estimate the second eigenvalue of Γ_n from below by the second eigenvalue of the torus (independently of n);
- Use the Fourier transform to show that the second eigenvalue of the torus is positive.

2.4 Kazhdan's property (T)

This section features a digression to the representation theory of locally compact groups, aiming to define and investigate Kazhdan's property (T), and finally to see how this deep group property can be used to construct expander families.

Property (T) was first introduced by Kazhdan in his three-page paper [Kaž67] in 1967 to show that a certain class of lattices are finitely generated, and it soon proved to be of interest in various areas. In 1973, Margulis [Mar73] used a "relative" version of property (T) to construct the first explicit example of a family of expander graphs, which was discussed in Section 2.3. We will present a variant of his result at the end of this section.

An extensive standard reference for property (T) is Bekka, de la Harpe and Valette's book [BHV08]. A very accessible treatment focussing on the connection to expansion can be found in Gardam's essay [Gar12].

As property (T) is defined in terms of unitary representations on locally compact groups, we first recall basic notions from representation theory and the theory of topological groups.

2.4.1 Locally compact topological groups

Definition 2.31. A *topological group* is a group $(G, \cdot)$ endowed with a topology such that the maps

$$\nu\colon \begin{cases} G & \to & G \\ g & \mapsto & g^{-1} \end{cases} \quad \text{(inversion)} \qquad \text{and} \qquad \mu\colon \begin{cases} G\times G & \to & G \\ (g,h) & \mapsto & g\cdot h \end{cases} \quad \text{(multiplication)}$$

are continuous.
Here, $G \times G$ is endowed with the product topology.

Remark 2.32. As ν is self-inverse, it is a homeomorphism. Moreover, for any fixed $g \in G$, the maps $l_g\colon x \mapsto gx$ and $r_g\colon x \mapsto xg$ of left and right multiplication by g are homeomorphisms from G to itself.

Example 2.33. (a) Any group is a topological group when endowed with the discrete topology.

(b) Any normed vector space (as an additive group), endowed with the topology induced by the norm, is a topological group. In particular, $(\mathbb{R}^n, +)$ and $(\mathbb{C}^n, +)$ are topological groups for any $n \in \mathbb{N}$.

(c) Viewing an $(n \times n)$-matrix as a vector with n^2 entries, we can identify $M_{n\times n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$ and equip it with the Euclidean topology. Then the matrix groups $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$, endowed with the subspace topology, are topological groups.

(d) The multiplicative groups $\mathbb{R}^*$ and $\mathbb{C}^*$ are topological groups.

Remark 2.34. If G, H are topological groups and $N \trianglelefteq G$ is a normal subgroup, then $G \times H$, $G \ltimes H$, $G \rtimes H$ and G/N are again topological groups when endowed with the product and quotient topology, respectively. From now on, groups of this form will always be assumed to be equipped with these natural topologies unless stated otherwise.

Definition 2.35. A topological group G is said to be *locally compact* if the associated topological space is Hausdorff (distinct points have disjoint neighbourhoods) and locally compact (every point has a compact neighbourhood).

Remark 2.36. The condition of the topology being Hausdorff is not a strong requirement: Indeed, it is easy to see that for topological groups, the separation axioms T_0 (two distinct points can be separated by an open set) and T_2 (Hausdorff) are equivalent, so that we actually just ask for G to be a T_0 space, which is a rather mild assumption.

A fundamental property of locally compact groups is that they are Baire spaces:

Theorem 2.37 (Baire, cf. [CRV17, Theorem 9.9, p. 130])**.** *In a locally compact Hausdorff space, the intersection of countably many dense open sets is dense. Equivalently, the union of countably many closed sets with empty interior has empty interior.*

Example 2.38. (a) All of the groups mentioned in Example 2.33 are actually locally compact topological groups.

(b) Any compact Hausdorff group is locally compact.

(c) The additive group $\mathbb{Q}$ of rational numbers, equipped with the topology inherited from $\mathbb{R}$, is an example for a topological group that is not locally compact:
For any $q \in \mathbb{Q}$, the set $\mathbb{Q} \setminus \{q\}$ is open and dense in $\mathbb{Q}$. But the countable intersection $\bigcap_{q\in\mathbb{Q}}(\mathbb{Q} \setminus \{q\}) = \emptyset$ is clearly not dense in $\mathbb{Q}$. Hence, $\mathbb{Q}$ is not locally compact.

2.4.2 Group representations

Definition 2.39. Let G be a group and V a vector space with automorphism group $\mathrm{Aut}(V)$. A *representation* of G on V is a group homomorphism

$$\rho\colon G \to \mathrm{Aut}(V).$$

Example 2.40. (a) The *trivial representation* is defined by $\rho_1(g) = \mathrm{Id}_V$ for any $g \in G$.

(b) If $G \leqslant \mathfrak{S}_n$ is a permutation group, the *sign representation* is defined by $\rho_{\mathrm{sgn}}(\sigma) = \mathrm{sgn}(\sigma) \cdot \mathrm{Id}_V$, where $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ is defined to be 1 if σ can be written as a composition of an even number of transpositions, and -1 otherwise.

(c) Let G be a topological group acting on some countable set X via a left action $*$. The *regular representation* $\rho_*\colon G \to \mathrm{Aut}(L^2(X))$ of G on $L^2(X)$ with respect to the action $*$ is defined by

$$\rho(g)f(x) := f(g^{-1} * x)$$

for $g \in G, x \in X, f \in L^2(X)$.

This indeed defines a representation because we have

$$\big(\rho_*(gh)f\big)(x) = f\left((gh)^{-1} * x\right) = f\left(h^{-1} * (g^{-1} * x)\right) = \big(\rho_*(h)f\big)(g^{-1} * x) = \big(\rho_*(g)\rho_r(h)f\big)(x)$$

for all $g, h \in G$, $x \in X$ and $f \in L^2(G)$.

In particular, if G is countable, we can let G act on itself via multiplication from the left (i.e. by setting $g * h := gh$) to obtain the *left-regular representation* ρ_l:

$$(\rho_l(g)f)(x) = f(g^{-1}x)$$

for $g, x \in G, f \in L^2(G)$.
Similarly, letting G act on itself via inverse multiplication from the right (i.e. by setting $g * h := hg^{-1}$), we obtain the *right-regular representation* ρ_r:

$$(\rho_r(g)f)(x) = f(xg)$$

for $g, x \in G, f \in L^2(G)$.

Definition 2.41. Let G be a topological group and V a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A representation ρ of G on V is said to be *unitary* if

(1) the automorphism $\rho(g) \in \mathrm{Aut}(V)$ is a *unitary operator* for any $g \in G$, that is, if

$$\langle \rho(g)f_1, \rho(g)f_2\rangle = \langle f_1, f_2\rangle$$

for all $g \in G$, $f_1, f_2 \in V$, and

(2) the representation ρ is *strongly continuous* in the sense that the mapping

$$\rho_v : G \to V, g \mapsto \rho(g)v$$

is continuous for any $v \in V$.

Here, V is equipped with the topology induced by the inner product.

Example 2.42. (a) The trivial and sign representations are clearly unitary whenever defined on a Hilbert space.

(b) If G acts on some countable set X via a left action $*$, then the corresponding regular representation ρ_* of G on the Hilbert space $L^2(X)$ defines a unitary operator: We have

$$\begin{aligned}\big\langle \rho_*(g)f_1, \rho_*(g)f_2 \big\rangle &= \sum_{x\in X} \big(\rho_*(g)f_1\big)(x) \cdot \overline{\big(\rho_*(g)f_2\big)(x)} \\ &= \sum_{x\in X} f_1(g^{-1} * x) \cdot \overline{f_2(g^{-1} * x)} \\ &= \sum_{x\in X} f_1(x) \cdot \overline{f_2(x)} \qquad\qquad = \langle f_1, f_2 \rangle\end{aligned}$$

for all $g \in G,\ f_1, f_2 \in L^2(G)$.

In particular, the left- and right-regular representations define unitary operators.

If moreover G is locally compact and the action $*$ is *strongly continuous*, i.e. the map

$$G \to X,\ g \mapsto g * x$$

is continuous for every $x \in X$, then ρ_*, is a strongly continuous representation.

This can be shown for continuous functions f by using strong continuity of the action and uniform convergence, and then for arbitrary L^2 functions via approximation by continuous functions.

2.4.3 Invariant vectors and property (T)

Before we can formulate property (T), we still need to give some preparatory definitions.

Definition 2.43. Let G be a locally compact topological group and $\rho\colon G \to \mathrm{Aut}(V)$ a unitary representation of G. Moreover, let $\varepsilon > 0$ be a constant, $Q \subseteq G$ a subset. A vector $v \in V$ is said to be

(i) *invariant*, if $\rho(g)v = v$ holds for all $g \in G$,

(ii) *Q-invariant*, if $\rho(g)v = v$ holds for all $g \in Q$, and

(iii) *(Q, ε)-invariant*, if $\|\rho(g)v - v\| < \varepsilon\|v\|$ holds for all $g \in Q$.

Definition 2.44. Let G be a locally compact topological group, $Q \subseteq G$ a subset and $\varepsilon > 0$ a constant. The pair (Q, ε) is said to be a *Kazhdan pair* for G if any unitary representation of G that has a (Q, ε)-invariant vector also has a non-zero invariant vector.

In this case, Q is said to be a *Kazhdan set* for G and ε is said to be a *Kazhdan constant* for G.

Remark 2.45. (1) One can replace Q by $\tilde{Q} \coloneqq Q \cup Q^{-1} \cup \{e\}$ in the above definition because whenever $\rho\colon G \to \mathrm{Aut}(V)$ is a unitary representation, we have

$$\|\rho(g)v - v\|^2 = \langle \rho(g)v - v, \rho(g)v - v\rangle = \left\langle v - \rho(g^{-1})v, v - \rho(g^{-1})v\right\rangle = \|\rho(g^{-1})v - v\|^2$$

for all $g \in G, v \in V$ as well as

$$\|\rho(e)v - v\|^2 = 0$$

for all $v \in V$.

(2) If (Q, ε) is a Kazhdan pair for G, then so is (Q', ε') for any $Q' \supseteq Q,\ \varepsilon' < \varepsilon$.

Finally, we can define property (T):

Definition 2.46. A locally compact group has *property* (T), also known as *Kazhdan's property*, if it admits a compact Kazhdan set.

Example 2.47. The infinite cyclic group $\mathbb{Z}$ does not have property (T). Indeed, fix $\varepsilon > 0$ and a compact set $Q \subseteq \mathbb{Z}$; as $\mathbb{Z}$ is discrete, Q is finite, so we can fix $N \geqslant 0$ with $Q \subseteq [-N, N]$. We will see that the left-regular representation ρ_l has (Q, ε)-invariant vectors, but no invariant vectors.

For $n > N$, consider the function $f_n \coloneqq c \cdot \mathbb{1}_{\{-n,\ldots,n\}}$, where $c = (2n+1)^{-1/2}$ is a normalizing factor, so that $\|f_n\| = 1$. Now for any $g \in Q, x \in \mathbb{Z}$ we have

$$\rho_l(g)f_n(x) - f_n(x) = f_n(g+x) - f_n(x) = \begin{cases} c, & \text{if } |g+x| \leqslant n < |x|, \\ -c, & \text{if } |g+x| > n \geqslant |x|. \end{cases}$$

Thus,

$$\|\rho_l(g)f_n(x) - f_n(x)\|^2 \leqslant 2|g|c^2 \leqslant \frac{2N}{2n+1},$$

which is less than ε for sufficiently large n.

However, ρ_g has no non-zero invariant vectors: Suppose $f \in L^2(\mathbb{Z})$ was $\rho(g)$-invariant; then $f(x) = f(g+x)$ for all $g, x \in \mathbb{Z}$, so f is constant. But as $\mathbb{Z}$ is infinite, the only constant function on $\mathbb{Z}$ with finite norm is $f \equiv 0$.
An analogous argument shows that $\mathbb{Z}^d$ does not have property (T) for any integer $d \geqslant 1$.

Lemma 2.48 [BHV08, Theorem 1.3.4, p. 43]**.** *Let G_1 and G_2 be locally compact groups, $\varphi\colon G_1 \to G_2$ a continuous epimorphism. If G_1 has property (T), then so does G_2.*

Proof. Let (Q_1, ε_1) be a Kazhdan pair for G_1, where $Q_1 \subseteq G_1$ is compact. Then $Q_2 \coloneqq \varphi(Q_1) \subseteq G_2$ is compact. We claim that (Q_2, ε) is a Kazhdan pair for G_2.
Indeed, suppose that $\rho\colon G_2 \to V$ is a unitary representation of G_2 and $v \in V$ is (Q_2, ε)-invariant. Then $\rho\circ\varphi\colon G_1 \to V$ is a unitary representation of G_1, and $\|(\rho\circ\varphi)(g)v - v\| = \|\rho(\varphi(g))v - v\| < \varepsilon$ holds for all $g \in Q_1$. Thus, there exists a non-zero vector $w \in V$ such that $(\rho\circ\varphi)(g)w = w$ for all $g \in G$, so w is invariant under $\mathrm{Im}\ \varphi = G_2$. Thus, G_2 has property (T). □

Corollary 2.49. *If G is a locally compact group with property (T) and $N \leqslant G$ is a closed normal subgroup, then G/N has property (T).*

Proof. Note that G/N is Hausdorff since N is closed [cf. Bou66, Section 3.2.5, Prop. 13, p. 231]. The quotient map $G \to G/N$, $g \mapsto gN$ is an epimorphism and continuous by definition of the quotient topology. □

Corollary 2.50. *The free group F_n on n generators does not have property (T) for any $n \geqslant 1$.*

Proof. The abelianisation $F_n^{\mathrm{ab}} = F_n/[F_n, F_n]$ is known to be isomorphic to $\mathbb{Z}^n$, which does not have property (T) (see Example 2.47). Thus, F_n does not have property (T) by Corollary 2.49. $\square$

Remark 2.51. Combining this corollary with the fact that property (T) is preserved under taking finite index subgroups of discrete groups [cf. BHV08, Thm. 1.7.1, p. 65], one can show that the group $\Gamma = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ does not have property (T).

Indeed, note that $\langle \sigma^2, \tau^2 \rangle \leqslant SL_2(\mathbb{Z})$ is a subgroup of index 12 that is free of rank 2 [cf. Löh17, Ex. 4.5.1, p. 72]. So Corollary 2.50 together with the above mentioned fact implies that $SL_2(\mathbb{Z})$ does not have property (T). But the projection map

$$\begin{aligned} \pi\colon \Gamma = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) &\to SL_2(\mathbb{Z}) \\ (v, A) &\mapsto A \end{aligned}$$

is clearly a continuous epimorphism, and therefore, Γ does not have property (T) by Lemma 2.48.

We introduce a slightly more general version of property (T) due to Margulis [Mar73]:

Definition 2.52. Let G be a locally compact group and $H \leqslant G$ a closed subgroup. Moreover, let $\varepsilon > 0$ be a constant and $Q \subseteq G$ a subset. The pair (G, H) is said to be a *relative Kazhdan pair* for (G, H) if every unitary representation that has a (Q, ε)-invariant vector also has a non-zero H-invariant vector.
In this case, Q is said to be a *relative Kazhdan set* for (G, H) and ε is said to be a *relative Kazhdan constant* for (G, H).

Definition 2.53. The pair (G, H) has *relative property (T)* if it admits a compact relative Kazhdan set.

Remark 2.54. In particular, G has property (T) if and only if the pair (G, G) has relative property (T).

Lemma 2.55 [cf. BHV08, Proposition F.1.7, p. 424]**.** *Let G be a locally compact group generated by a compact set Q, and let $K \subseteq G$ be a compact subset. Denote $\tilde{Q} := Q \cup Q^{-1} \cup \{e\}$.*
Then K is contained in $\tilde{Q}^N$ for some integer $N \geqslant 1$.

Proof. First note that $\tilde{Q}^n$ is closed for any $n \geqslant 1$: As Q is compact, so is its continuous image Q^{-1}, and thus, $\tilde{Q}$ is compact. By Tychonoff's Theorem, $\tilde{Q} \times \tilde{Q}$ is compact, and hence so is $\tilde{Q}^2 = \mu(\tilde{Q} \times \tilde{Q})$, and the same holds for $\tilde{Q}^n$ by induction. Hence, $\tilde{Q}^n$ is also closed as G is a Hausdorff space.
As Q is a set of generators for G, we have

$$G = \bigcup_{n \geqslant 1} \tilde{Q}^n.$$

This is a countable union of closed sets which has non-empty interior, so by Baire's Theorem, $\tilde{Q}^k$ has non-empty interior for some $k \in \mathbb{N}$.
Let $U \subseteq \tilde{Q}^k$ be open, $u \in U$ an interior point. For any $g \in G$, we have $g \in \tilde{Q}^m$ for some $m \in \mathbb{N}$, and hence, $l_{gu^{-1}}(U) \subseteq \tilde{Q}^{2k+m}$ is an open neighbourhood of g.
Thus, $\bigcup_{n \geqslant 1} \operatorname{int}(\tilde{Q}^n)$ is an open cover of $G \supseteq K$, and as K is compact, there is some N such that $K \subseteq \operatorname{int}(\tilde{Q}^N) \subseteq \tilde{Q}^N$. $\square$

Proposition 2.56 [cf. BHV08, Prop. 1.3.2, p. 42]**.** *Let G be a locally compact group and $H \leqslant G$ a closed subgroup such that the pair (G, H) has relative property (T). Moreover, assume that G is generated by a compact set Q.*
Then Q is a relative Kazhdan set for (G, H).

Proof. Let $K \subseteq G$ be a compact relative Kazhdan set for (G, H) and $\varepsilon > 0$ a corresponding relative Kazhdan constant. By Lemma 2.55, we have $K \subseteq \tilde{Q}^N$ for some integer $N \geqslant 1$.

We claim that $(\tilde{Q}, \varepsilon/N)$ is a relative Kazhdan pair:
Indeed, let ρ be a unitary representation of G, and suppose that v is a $(\tilde{Q}, \varepsilon/N)$-invariant vector. Then for any $g \in K \subseteq \tilde{Q}^N$, we can write $g = q_1 \cdots q_N$ with $q_i \in \tilde{Q}$.
Using the triangle inequality and that ρ is unitary, we obtain:

$$\begin{aligned}
\|\rho(g)v - v\| &= \|\rho(q_1 \cdots q_N)v - v\| \\
&\leqslant \|\rho(q_1 \cdots q_N)v - \rho(q_1 \cdots q_{N-1})v\| + \cdots + \|\rho(q_1 q_2)v - \rho(q_1)v\| + \|\rho(q_1)v - v\| \\
&= \|\rho(q_1 \cdots q_{N-1})(\rho(q_N)v - v)\| + \cdots + \|\rho(q_1)(\rho(q_2)v - v)\| + \|\rho(q_1)v - v\| \\
&= \|\rho(q_N)v - v\| + \cdots + \|\rho(q_2)v - v\| + \|\rho(q_1)v - v\| \\
&< N \cdot \frac{\varepsilon}{N}\|v\| = \varepsilon\|v\|
\end{aligned}$$

Hence there exists a non-zero H-invariant vector (because (K, ε) is a relative Kazhdan pair) and thus, $\tilde{Q}$ is a relative Kazhdan set. Thus, Q is a relative Kazhdan set by Remark 2.45 (1). □

2.4.4 Property (T) and expansion

Finally, we can connect Kazhdan's property and expansion:

Proposition 2.57 ([BHV08, Thm. 6.1.8, p. 272]; cf. [Mar73])**.** *Let G be a locally compact group, $H \leqslant G$ a closed subgroup such that (G, H) has relative property (T), and let S be a compact set of generators for G. Moreover, suppose that we have a sequence $(N_i)_{i \in \mathbb{N}}$ of finite index normal subgroups of H such that the index $[H : N_i]$ goes to infinity with i.*
If G acts on H in a way such that it induces a strongly continuous action on H/N_i, and for each i, the restriction of this action to H is transitive, then the Schreier graphs $X_i = \mathcal{S}(G, H/N_i, S/N_i)$ form a family of expanders.

Proof. As we have seen in Examples 2.40 (c) and 2.42 (b), the action of G on $V_i \coloneqq H/N_i$ induces a unitary representation of G on $L^2(V_i)$, the so-called regular representation ρ_*.
If a function $f \in L^2(V_i)$ is H-invariant, then we have

$$f(x) = (\rho_*(h)f)(x) = f(h^{-1} * x)$$

for all $h \in H$, $x \in V_i$.
Since the restriction of the action to H was assumed to be transitive, this means that any H-invariant function is constant.
Now replacing $L^2(V_i)$ by

$$L_0^2(V_i) = \left\{ f \in L^2(V_i) \;\middle|\; \sum_{x \in V_i} f(x) = 0 \right\}$$

in the definition of ρ_*, we obtain a unitary representation $\rho_0 \colon G \to \mathrm{Aut}(L_0^2(V_i))$.
This representation is well-defined as

$$\sum_{x \in V_i} (\rho_0(g)f)(x) = \sum_{x \in V_i} f(g^{-1} * x) = \sum_{x \in V_i} f(x) = 0$$

for all $f \in L_0^2(V)$.
As the only constant function in $L_0^2(V)$ is zero, there is no non-zero H-invariant function in

$L_0^2(V_i)$.
By Lemma 2.56, S is a relative Kazhdan set with respect to H, so there exists some $\varepsilon > 0$ and some $s \in S$ such that $\|\rho_0(s)f - f\| > \varepsilon\|f\|$ for all $f \in L_0^2(V_i)$.

We will use a special function to obtain expansion from this equality:
Arbitrarily partition $V_i = A \sqcup B$ into two sets and write $a = |A|$, $b = |B|$. Now define a function f_A on V_i by

$$f_A(x) = \begin{cases} b & \text{if } x \in A \\ -a & \text{if } x \in B. \end{cases}$$

Clearly, f_A is in $L_0^2(V_i)$, so we have $\|\rho_0(s)f_A - f_A\| > \varepsilon\|f_A\|$.
Observe that

$$(\rho_0(s)f_A)(x) = f_A(s^{-1} * x) = \begin{cases} b & \text{if } s^{-1} * x \in A \\ -a & \text{if } s^{-1} * x \in B \end{cases}$$

and hence

$$\rho_0(s)f_A - f_A = \begin{cases} a+b & \text{if } s^{-1} * x \in B \text{ and } x \in A \\ -(a+b) & \text{if } s^{-1} * x \in A \text{ and } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us estimate $e(A, B)$ in the corresponding Schreier graph X_i: since s is a generator, $e(A, B)$ can be bounded from below by the set of edges that are due to s (or equivalently, due to s^{-1}). The number of these edges is given by

$$\left|\left\{x \in B | s^{-1} * x \in A\right\} \cup \left\{x \in A | s^{-1} * x \in B\right\}\right| = \frac{1}{(a+b)^2} \|\rho_0(s)f_A - f_A\|^2$$

or half of that number if $s = s^{-1}$.
In either case, we have

$$e(A, B) \geqslant \frac{1}{2(a+b)^2} \|\rho_0(s)f_A - f_A\|^2 > \frac{\varepsilon^2}{2(a+b)^2} \|f_A\|^2.$$

Evaluating the right side of this inequality, we observe that

$$\|f_A\|^2 = ab^2 + ba^2 = ab(a+b).$$

Assume without loss of generality that $a \leqslant b$. Then $b/(a+b) \geqslant 1/2$, and we can put things together:

$$h_{X_i}(A) = \frac{e(A, B)}{|A|} > \frac{\varepsilon^2}{2a(a+b)^2} ab(a+b) \geqslant \frac{\varepsilon^2}{4}.$$

As the partition $V = A \sqcup B$ was chosen arbitrarily, we obtain that $h(X_i) > \varepsilon^2/4$ for any $i \in \mathbb{N}$. Since ε depends only on G, H and S, but not on N_i, we can conclude that $\{X_i\}_{i \in \mathbb{N}}$ is a family of expanders. □

Corollary 2.58. *Let G be a locally compact group with property (T), and let S be a compact set of generators. Moreover, let $(N_i)_{i \in \mathbb{N}}$ be a sequence of finite index normal subgroups of G such that the index $[H : N_i]$ goes to infinity with i. Then the Cayley graphs $X_i = \mathcal{G}(G/N_i, S/N_i)$ form a family of expanders.*

Proof. Setting $H = G$, we note that H is closed and (G, H) has relative property (T). Moreover, G acts transitively and in a strongly continuous way on each quotient G/N_i via left-multiplication, and the Schreier graph with respect to this action is just the Cayley graph of G/N_i. Hence, the previous proposition gives the statement. □

In order to apply these results, we will use the following without giving a proof:

Theorem 2.59 [cf. Mar73, Lemma 3.15]**.** *The pair $(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T).*

The proof of this theorem given in [Lub94] (Prop. 3.1.11, p. 24) relies on the usage of several other results. The proof in [BHV08] (Theorem 4.2.2, p. 207) is more elementary and bears close resemblances to the proof that Margulis' graphs $\{\Gamma_n : n \in \mathbb{N}\}$ form a family of expanders that was presented in Section 2.3. This does not come as a big surprise as the above result, combined with Proposition 2.57, just gives another proof for the expansion property of $\{\Gamma_n : n \in \mathbb{N}\}$:

Corollary 2.60. *The graphs $\{\Gamma_n : n \in \mathbb{N}\}$ and $\{\Gamma_n^M : n \in \mathbb{N}\}$ form families of expanders.*

Proof. Recall that in Remark 2.25, we saw that Γ_n can be described as a Schreier graph of $\Gamma = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ acting on $(\mathbb{Z}/n\mathbb{Z})^2$ with respect to the set $S = \{(0,\sigma), (0,\tau), (e_1, \mathrm{Id}), (e_2, \mathrm{Id})\}$. In the same way, we can view Γ_n^M as the Schreier graph of Γ acting on $(\mathbb{Z}/n\mathbb{Z})^2$ with respect to the set

$$S^M = \{(0,\rho), (0,\sigma), (e_1, \mathrm{Id}), (e_2, \mathrm{Id})\}.$$

Let us check the conditions of Proposition 2.57. We have just stated that $(\Gamma, (\mathbb{Z}/n\mathbb{Z})^2)$ has relative property (T) in Theorem 2.59. The action in question restricted to $\mathbb{Z}^2$ is the natural action of $\mathbb{Z}^2$ on $(\mathbb{Z}/n\mathbb{Z})^2$, which is clearly transitive, and trivially strongly continuous as Γ is discrete. Thus, it remains to check that the sets S and S^M generate $\Gamma = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$, respectively.

As S and S^M both contain the elements (e_1, Id) and (e_2, Id), they clearly generate the subgroup $\mathbb{Z}^2 \times \{\mathrm{Id}\}$, so the proof can be finished by showing that they also generate $SL_2(\mathbb{Z})$ as a subgroup of Γ. Since we have

$$\tau\sigma^{-1}\tau = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \rho,$$

it remains to check that $SL_2(\mathbb{Z})$ is generated by ρ and σ, which is a well-known statement (e.g., see [Ser73, Section VII, Thm. 2, p. 78]). □

Chapter 3

Ramanujan graphs

Ramanujan graphs are expanders whose spectral gap is as large as possible in an asymptotic sense. They can therefore be considered the best expander graphs that are theoretically possible. Besides the interest coming from the various applications of expander graphs, Murty [Mur03] also mentions a more aesthetic reason to study Ramanujan graphs, namely that they fuse different branches of mathematics, such as graph theory, representation theory and number theory.

The term *Ramanujan graph* was coined by Lubotzky, Phillips and Sarnak in their 1988 article [LPS88], in which they used the *Ramanujan conjecture* to construct an explicit family of Ramanujan graphs of degree $p+1$, where p is a prime number congruent to 1 modulo 4. The same construction was described independently by Margulis [Mar88]. Subsequently, the method was extended to yield families of Ramanujan graphs of degree $q+1$, where q is any prime power [Chi92; Mor94]. In more recent publications, so-called *near-Ramanujan graphs*, satisfying only a slightly weaker bound on their spectral gaps, have been studied.

In Section 3.1, we investigate random walks on graphs with the aim of giving an asymptotic upper bound on the spectral gap of a family of d-regular graphs. This leads to the definition of Ramanujan graphs in Section 3.2. There, we will also answer the question whether the examples for expander families encountered in Sections 2.3 and 2.4 are Ramanujan families, and present the historically first example of a Ramanujan family from [LPS88] and [Mar88]. Section 3.3 is devoted to the presentation of more recent results on near-Ramanujan graphs.

3.1 Random walks on regular graphs and the Alon-Boppana Theorem

In this section, we will present a proof of the Alon-Boppana Theorem, which gives an upper bound on the spectral gap of a family of d-regular graphs. This bound will motivate the definition of Ramanujan graphs as the best expanders that are theoretically possible.
This section follows Section 4.5 in Lubotzky's book [Lub94]. Unless stated otherwise, all results and proofs are taken from there.

In Section 2.2, we saw that the property of expansion is equivalent to the existence of a non-trivial upper bound for the spectral gap. In this section, we will investigate how small this bound can be made.
A rough answer to that question can be given immediately:

Lemma 3.1 [HLW06, Claim 2.8, p. 456]**.** *Let X be a connected, d-regular graph on n vertices whose adjacency matrix has second eigenvector $d-\lambda(X) =: \mu$. Then the following inequality holds:*

$$\mu \geqslant \sqrt{d} \cdot (1 - o_n(1)).$$

Proof. Let A be the adjacency matrix of X. Note that the entry of A^k corresponding to the pair $(v, w) \in V \times V$ equals the number of walks of length k from v to w. In particular, the diagonal entries of A^2 count the different closed walks of length 2 starting in a certain vertex v. As there are at least d such walks for any vertex (go back and forth along any edge incident to v), we have $\operatorname{tr}(A^2) \geqslant nd$. On the other hand,

$$\operatorname{tr}(A^2) = \sum \mu_i^2 \leqslant d^2 + (n-1)\mu^2.$$

Hence, we obtain that

$$\mu^2 \geqslant d\frac{n-d}{n-1},$$

concluding the proof. □

Remark 3.2. (1) As a consequence, if $\{X_n : n \in \mathbb{N}\}$ is a family of d-regular graphs whose number of vertices goes to infinity, then only finitely many second eigenvalues of the X_n can be bounded away from $\sqrt{d}$ from below (that is: for any $\varepsilon > 0$, only finitely many X_n satisfy $\mu \leqslant \sqrt{d} - \varepsilon$).

(2) The key observation of the proof is the relationship between walks in a graph and the adjacency matrix. To derive a stronger bound on $\lambda(X)$, we will refine this approach and consider random walks on graphs.

We now consider a random walk on V, where every vertex adjacent to the current vertex v has probability $1/\deg(v)$ of being the next step in the walk. The Markov chain obtained by this random walk defines an operator on $L^2(V)$:

Definition 3.3. Let X be a d-regular graph. The operator $M\colon L^2(X) \to L^2(X)$ defined by

$$Mf(v) := \frac{1}{d}\sum_{w\sim v} f(w)$$

is called the *Markov operator* of X.
Here, the sum really goes over all edges of the form $\{v, w\}$, that is, multiple edges are taken with their multiplicity.

Remark 3.4. If X is finite and d-regular, then its Markov operator is nothing else than $M = \frac{1}{d}A_X$, where A_X is the adjacency matrix.

Lemma 3.5. *The Markov operator M satisfies the following properties:*

(1) M is self-adjoint;

(2) $\|M\| \leqslant 1$, where $\|M\| := \sup_{x\in L^2(V),\ \|x\|=1} \|Mx\|$ denotes the operator norm of M.

Proof. (1) For all $f, g \in L^2(X)$, we have

$$\begin{aligned}\langle Mf, g\rangle &= \sum_{v\in V} Mf(v)\overline{g(v)} = \sum_{v\in V}\frac{1}{d}\sum_{w\sim v} f(w)\overline{g(v)}\\ &= \sum_{w\in V}\frac{1}{d}\sum_{v\sim w} f(w)\overline{g(v)} = \sum_{w\in V} f(w)\overline{Mg(w)} = \langle f, Mg\rangle.\end{aligned}$$

(2) We can rewrite the operator M as an integral operator:

$$Mf(v) = \frac{1}{d}\sum_{w\sim v} f(w) = \frac{1}{d}\sum_{v\in V} e(v,w)f(w) = \frac{1}{d}\int_V e(v,w)f(w)dw,$$

where we integrate with respect to the counting measure.
Applying the triangle and Cauchy-Schwarz equalities, we obtain

$$\begin{aligned}
|Mf(v)|^2 &= \left|\int_V \frac{1}{d}\, e(v,w)f(w)dw\right|^2 \\
&\leqslant \int_V \left(\frac{1}{d}\, e(v,w)\right)^2 |f(w)|^2\, dy \\
&\leqslant \int_V \frac{1}{d}\, e(v,w)dw \int_V \frac{1}{d}e(v,w)\ |f(w)|^2\, dw = \int_V \frac{1}{d}e(v,w)\,|\ f(w)|^2\, dw.
\end{aligned}$$

Integrating this inequality with respect to v and applying Fubini's theorem, we conclude:

$$\begin{aligned}
\|Mf\|^2 &= \int_V |Mf(v)|^2\, dv \\
&\leqslant \int_V \int_V \frac{1}{d}\, e(v,w)\,|f(w)|^2\ dw\ dv \\
&= \int_V \int_V \frac{1}{d}\, e(v,w)\ dv\ |f(w)|^2\, dw = \int_V |f(w)|^2\, dw = \|f\|^2.
\end{aligned}$$

Hence, $\|M\| \leqslant 1$.

□

Remark 3.6. The argument used in the proof of (2) is a special case of the so-called Schur test.

Definition 3.7. Let V, W be complex Banach spaces, $T\colon V \to W$ a bounded linear operator. Then we define:

(i) The *spectrum* of T is given by $\operatorname{spec}(T) := \{\lambda \mid T - \lambda\cdot \mathrm{Id} \text{ has no inverse}\}$.

(ii) The *spectral radius* of T is $\rho(T) := \sup\{|\lambda| \mid \lambda \in \operatorname{spec}(T)\}$.

Remark 3.8. In the finite dimensional case, we have

$$\lambda \in \operatorname{spec}(T) \Leftrightarrow \det(T - \lambda\ \mathrm{Id}) = 0 \Leftrightarrow \lambda \text{ is an eigenvalue of } T.$$

The Markov operator of a finite d-regular graph therefore always has spectral radius $\rho(M) = \rho(A_X)/d = d/d = 1$. Note that in the infinite case however, 1 need not be an eigenvalue of M as the corresponding eigenfunction $f \equiv 1$ is no longer in $L^2(X)$. Thus, we can hope for a connection between $\rho(M)$ and the second largest eigenvalue.

A result from functional analysis (which we state without proof) tells us that the spectral radius of M coincides with its operator norm:

Theorem 3.9 [MR19, Thm. 6.11.3, p. 218]**.** *Any self-adjoint bounded operator T on a Hilbert space satisfies $\rho(T) = \|T\|$.*

The next result gives yet another way of viewing $\|M\|$. The proof follows Buck's article [Buc86, Prop. 3.1, p. 290].

Lemma 3.10. *Let $X = (V, E)$ be an infinite graph, and let $v_0 \in V$ be a fixed vertex (call it the origin). Let r_n be the probability of the random walk being at the origin at time n having started there at time 0.*
Then,

$$\|M\| = \limsup_{n\to\infty} r_n^{1/n}.$$

Proof. For any $w \in V$, let $\delta_w \in L^2(X)$ be defined by

$$\delta_w(v) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

For notational convenience, abbreviate $\delta := \delta_{v_0}$. Then we have $r_n = \langle \delta, M^n \delta \rangle$:
Indeed, if $f : V \to [0,1]$ is a function such that for every $v \in V$, $f(v)$ indicates the probability of the random walk being at v at some time t, then the probability of the random walk being at a vertex w at time $t+1$ is given by $p = \sum_{v\sim w} \frac{1}{n} f(v) = Mf(v)$. Thus, $M\delta(v)$ gives the probability of the random walk being at some vertex v at time 0, and by induction, $M^n\delta(v_0) = \langle \delta, M^n \delta\rangle$ gives the probability of returning to the origin at time n.
Hence, we get $r_n \leqslant \|\delta\| \, \|M^n \delta\| \leqslant \|M\|^n \|\delta\| = \|M\|^n$ by Cauchy-Schwarz, so that $\|M\| \geqslant \sqrt[n]{r_n}$ and thus $\|M\| \geqslant \limsup r_n^{1/n}$.

To show the other inequality, it suffices to find a subsequence of $r_n^{1/n}$ converging to $\|M\|$. We will see that the subsequence determined by $2, 4, 8, \ldots, 2^k, \ldots$ works.
Let $f \in L^2(X)$ be of norm 1. As M is self-adjoint, we have

$$\|Mf\|^2 = \langle Mf, Mf\rangle = \left\langle M^2 f, f\right\rangle \leqslant \|M^2 f\|.$$

Iterating this argument and taking square roots gives

$$\|Mf\| \leqslant \left\|M^2 f\right\|^{1/2} \leqslant \cdots \leqslant \left\|M^{2^k} f\right\|^{1/2^k} \leqslant \ldots.$$

It follows that $\|M\| \leqslant \|M^2\|^{1/2} \leqslant \ldots$, so $\|M^2\| \geqslant \|M\|^2$. As we have $\|M^2 f\| \leqslant \|M\| \, \|Mf\| \leqslant \|M\|^2$ by definition of the operator norm, we infer that $\|M\|^2 = \|M^2\|$, and in general, $\|M\|^{2^k} = \|M^{2^k}\|$.
Letting $f = \delta$, we have that $r_{2^k}^{1/2^k} = \|M^{2^k}\delta\|^{1/2^k}$ is monotonously increasing in k and and bounded by $\|M\|$ from above as $\|M^{2^k}\delta\|^{1/2^k} \leqslant \|M^{2^k}\|^{1/2^k} = \|M\|$. Thus, it converges to a limit μ. We are left with showing that $\mu \geqslant \|M\|$.
As $\|M^n \delta\| = \langle M^n \delta, M^n \delta\rangle^{1/2} = \langle M^{2n}\delta, \delta\rangle^{1/2} = \sqrt{r_{2n}}$, we have $\|M^{2^k}\delta\| = \sqrt{r_{2^{k+1}}} \leqslant \mu^{2^k}$.
Now fix an arbitrary vertex $w \in V$. As X was assumed to be connected, we can fix N large enough that $\langle \delta_w, M^N \delta\rangle > 0$. Moreover, fix β satisfying $0 < \beta \leqslant \langle \delta_w, M^N \delta\rangle$.
We claim that $\|M^{2^k}\delta_w\| \geqslant \beta^{-1}\|M^{2^k}\delta\|$ for any $k \geqslant 1$.
Indeed, we know that $\delta_w \leqslant \beta^{-1} M^N \delta$ (as functions) because we have

$$\delta_w(w) = 1 = \frac{M^N \delta(w)}{\langle \delta_w, M^N \delta\rangle} \leqslant \beta^{-1} M^N \delta(w)$$

and

$$\delta_w(v) = 0 \leqslant \beta^{-1} M^N \delta(v)$$

for $v \neq w$.
As M has only positive entries, we can apply any power of M to both sides of this inequality of functions, so that in particular $M^{2^k}\delta_w \leqslant \beta^{-1} M^{2^k+N}\delta$ and thus

$$\|M^{2^k}\delta_w\| \leqslant \beta^{-1}\|M\|^N \|M^{2^k}\delta\|,$$

which yields the claim as we have $\|M\| \leqslant 1$ by the previous lemma.
Now to show that $\|M\| \leqslant \mu$, let f be a function of norm 1; we wish to prove $\|Mf\| \leqslant \mu$.
We will first prove this statement for finitely supported f and then generalise. If $X = \operatorname{supp} f$ is finite, we can choose $\beta > 0$ small enough that for every $x \in X$ there exists some $n \geqslant 0$ such that $\langle \delta_x, M^N \delta \rangle \geqslant \beta$, so we can use the inequality derived in the claim above for any $w \in X$.
As f is supported on the finite set X, we can write $f = \sum_{x \in X} f(x)\delta_x$; then we have

$$\|Mf\| \leqslant \|M^{2^k} f\|^{2^{-k}} \leqslant \left(\sum_{x \in X} \mid f(x) \mid \|M^{2^k} \delta_x\| \right)^{2^{-k}} \leqslant \left(\beta^{-1} \sum_{x \in X} \mid f(x) \mid \right)^{2^{-k}} \|M^{2^k}\|^{2^{-k}}$$

for any $k \geqslant 1$.
Taking $k \to \infty$, the expression on the right hand converges to μ, so that $\|Mf\| \geqslant \mu$.
Finally, it remains to show that the same holds for infinitely supported f of norm 1. As V was assumed to be at most countably infinite, so is $X = \operatorname{supp} f$. writing $X = \{x_n \mid n \in \mathbb{N}\}$, we can define a family $\{f_n\}_{n \in \mathbb{N}}$ of finitely supported functions by setting

$$f_n(x) = \begin{cases} f(x) & x \in \{x_1, \ldots, x_n\} \\ 0 & \text{otherwise.} \end{cases}$$

As f has finite norm $\|f - f_n\| = \sum_{i > n} f(x_i)^2$ goes to 0 as $n \to \infty$, so that the f_n converge to f. Since taking norms is continuous by the reversed triangle inequality and M is continuous as it is a bounded operator by Lemma 3.5, we can conclude:

$$\|Mf\| = \left\| M \left(\lim_{n \to \infty} f_n \right) \right\| = \lim_{n \to \infty} \|Mf_n\| \leqslant \mu.$$

Hence we have $\|M\| \leqslant \mu$, finishing the proof. □

Remark 3.11. By the Cauchy-Hadamard formula, $\|M\|$ can also be thought of as the reciprocal of the radius of convergence of the *return generating function* $R(z) = \sum_{n \geqslant 0} r_n z^n$. This point of view will be useful in the proof of Proposition 3.13.

A consequence of this result is that the norm of the Markov operator is well-behaved with respect to coverings:

Lemma 3.12 [GŻ99, Lemma 2, p. 5]**.** *Let $f\colon X_1 \to X_2$ be a covering of graphs with Markov operators M_1 and M_2, respectively. Then $\|M_1\| \leqslant \|M_2\|$.*

Proof. Fix any vertex v in X_1. Any cycle in X_1 starting in v is mapped to a cycle in X_2 starting in $f(v)$ under f because f preserves incidence relations, and different cycles in X_1 are mapped to different cycles in X_2 because f is locally bijective. Therefore, we have $r_n^{X_1} \leqslant r_n^{X_2}$, where $r_n^{X_i}$ denotes the probability of the random walk in X_i with starting point v or $f(v)$, respectively, returning after n steps. Thus $\|M_1\| \leqslant \|M_2\|$ by Lemma 3.10 above. □

Proposition 3.13 [Lub94, Prop. 4.5.2, p. 55]**.** *Let M be the Markov operator associated to the d-regular tree T_d.*
Then $\|M\| = \frac{2\sqrt{d-1}}{d}$.

Proof. Fix a vertex v_0 of T_d, and let w_0 be any adjacent vertex. Let q_n denote the probability that the random walk starting in v_0 returns to v_0 for the first time at time $n > 0$, and let t_n denote the probability that the random walk starting in w_0 reaches v_0 for the first time at time n. We will consider the complex power series

$$Q(z) := \sum_{n \geqslant 0} q_n z^n, \quad T(z) := \sum_{n \geqslant 0} t_n z^n$$

as well as the return generating function

$$R(z) = \sum_{n \geqslant 0} r_n z^n,$$

where r_n is the return probability of the random walk starting in v_0 after n steps.
Since the tree is symmetric, the value of t_n does not depend on the exact choice of w_0 (as long as it is adjacent to v_0). Therefore we have

$$Q(z) = \sum_{w \sim v_0} \frac{1}{d} zT(z) = zT(z).$$

Moreover, any returning walk is either of length 0 or can be thought of as a first returning walk combined with a returning walk (possibly of length 0). Therefore we have $R(z) = 1 + Q(z)R(z)$, so that

$$R(z) = \frac{1}{1 - Q(z)}.$$

Starting at a point w_1 at distance $m > 1$ instead of w_0, the generating function corresponding to the probability of arriving in v_0 is given by T^m, again because of the symmetry of the d-regular tree.
Starting from w_0, there is one edge leading to v_0 and $d-1$ edges leading to a point at distance 2 from v_0. Hence,

$$T(z) = \frac{1}{d} z + \frac{d-1}{d} zT^2(z),$$

that is

$$T(z) = \frac{d \pm \sqrt{d^2 - (d-1)z^2}}{2(d-1)z}.$$

Now note that for $0 \leqslant z \leqslant 1$, we can view $T(z)$ as the probability of ever returning to v_0 with an additional chance of "dying" in every step with probability $1 - z$. In particular, $T(z) \leqslant 1$ for $0 \geqslant z \geqslant 1$, and therefore the correct solution has to be

$$T(z) = \frac{d - \sqrt{d^2 - (d-1)z^2}}{2(d-1)z}.$$

We will now use this explicit formula for $T(z)$ to determine the radius of convergence of $R(z)$ and thereby the norm of the Markov operator. As the radius of convergence equals the least absolute value of a non-removable singularity, we need to investigate the singularities of R. Since $R(z) = \frac{1}{1 - zT(z)}$, the singularities of R are given by the singularities of T as well as all points $z \in \mathbb{C}$ that satisfy $zT(z) = 1$. Let us first examine the singularities of T.
First note that $z = 0$ is a zero of order 1 of the denominator of T, but a zero of order 2 of the numerator because the derivative of the numerator is given by

$$\frac{4(d-1)z}{\sqrt{k^2 - 4(k-1)z^2}},$$

which vanishes at $z = 0$. Therefore, the singularity of T in $z = 0$ is solvable. The only other singularity occurs at the "branch point" of the square root function, i.e. at the point where the term under the square root vanishes. This point is given by $z_0 := \frac{d}{2\sqrt{d-1}}$ (or $-z_0$).
Having identified the singularities of R originating from zeroes of T, we are still left with the points that satisfy $zT(z) = 1$, or equivalently $\sqrt{d^2 - 4(d-1)z^2} = k - 2$. Note that the right hand side of this equation is always real-valued while the left hand side is only real-valued for real z with $z \leqslant z_0$. Moreover, for $d \geqslant 2$, the right hand side is negative, while the left hand

side is positive, and for $d = 2$, the only solution occurs at z_0 itself. Therefore, the radius of convergence of R is z_0 in any case, and we can conclude:

$$\|M\| = \frac{1}{z_0} = \frac{2\sqrt{d-1}}{d}.$$

□

Remark 3.14. It is quite counter-intuitive that the Markov operator of the infinite tree has such a large spectral radius, given that finite trees are terrible expanders - they basically consist of “bottlenecks” only!

Corollary 3.15. *If X is a countable d-regular graph, then its Markov operator M satisfies*

$$|M| \geqslant \frac{2\sqrt{d-1}}{d}.$$

Proof. As X is covered by the d-regular tree T_d (cf. Example 1.39 (c)), Lemma 3.12 yields that

$$\|M\| \geqslant \|M(T_d)\| = \frac{2\sqrt{d-1}}{d}.$$

□

As Kesten showed, T_{2n} is actually the only graph with this property that appears as a Cayley graph of some finitely generated group:

Theorem 3.16 [Kes59, Theorem 3, p. 347]**.** *Let X be the Cayley graph of a group G with respect to the finite set of generators $\{s_1, \ldots, s_n\}$, and denote by M the associated Markov operator.*
Then

$$\|M\| = \frac{2\sqrt{2n-1}}{2n}$$

holds if and only if G is the free group on the free generators $s_1, \ldots, s_n$, that is, if and only if X is the $(2n)$-regular tree.

Proposition 3.17 [Lub94, Prop. 4.5.4, p. 56]**.** *Let $\{X_n = (V_n, E_n) \ : \ n \in \mathbb{N}\}$ be a family of d-regular graphs whose number of vertices goes to infinity as $n \to \infty$. Let M_n^0 be the Markov operator of X_n restricted to $L_0^2(X_n)$.*

Then

$$\liminf_{n\to\infty} \|M_n^0\| \geqslant \frac{2\sqrt{d-1}}{d}.$$

Proof. Fix n, and write $X = (V, E)$ instead of X_n. Moreover, fix an integer $r \geqslant 0$ such that $\operatorname{diam}(X) \geqslant 2r+1$ and choose vertices $v_1, v_2 \in V$ with $d(v_1, v_2) \geqslant 2r+1$. Then $B_r(v_1) \cap B_r(v_2) = \emptyset$.
Now define $\delta_i \in L^2(X)$ to be the indicator function with respect to $\{v_i\}$, and set $f := \delta_1 - \delta_2$. Note that $f \in L_0^2(X)$ and $\|f\|^2 = 2$.
Now consider the Markov operator M of X. Let g be some function in $L^2(X)$ and $v \in V$ a vertex. Note that the value of $Mg(v)$ is by definition the sum of the values that g attains at the vertices adjacent to v divided by d, and therefore in particular, we have $\operatorname{supp} Mg \subseteq B_1(\operatorname{supp} g)$. Iterating this argument, we see that $M^r\delta_i$ is supported on $B_r(v_i)$, so $M^r\delta_1$ and $M^r\delta_2$ are orthogonal. Thus,

$$\|M^r f\|^2 = \|M^r\delta_1 - M^r\delta_2\|^2 = \|M^r\delta_1\|^2 + \|M^r\delta_2\|^2.$$

Now denote by $\tilde{M}$ the Markov operator of the d-regular tree T_d, fix some vertex e of T_d and denote the indicator function with respect to $\{e\}$ by δ_e. Then we claim that $\|\tilde{M}^r(\delta_e)\| \leqslant \|M^r(\delta_i)\|$.
Indeed, fix a covering $\pi \colon T_d \to X$ that maps e to v_i. Then $\tilde{M}^r\delta_e(w)$ is the probability that the random walk starting in e ends up in w after r steps, and any walk connecting e and w is mapped to a walk in X connecting v_i and $\pi(w)$, with different walks being mapped to different walks. Hence $\tilde{M}\delta_e(w) \leqslant M^r\delta_i(\pi(w))$, and therefore $\|\tilde{M}^r\delta_e\| \leqslant \|M^r\delta_i\|$.
Now we can put everything together:

$$2\|M_0\|^{2r} = \|f\|^2\|M_0\|^{2r} \geqslant \|M_0^r f\|^2 = \|M^r f\|^2 = \|M^r\delta_1\|^2 + \|M^r\delta_2\|^2 \geqslant 2\|\tilde{M}^r\delta_e\|^2.$$

Thus $\|M_0\| \geqslant \|\tilde{M}^r(\delta_e)\|^{1/r}$.
As we have seen in the proof of Lemma 3.10, the right hand side of this inequality converges to $\|\tilde{M}\|$ as r goes to infinity. Now observe that $\operatorname{diam}(X_n) \to \infty$ as $n \to \infty$ because all X_n are d-regular, and therefore the above inequality holds for any r.
Thus, we can conclude:

$$\liminf_{n\to\infty} \|M_n^0\| \geqslant \lim_{r\to\infty} \|\tilde{M}^r(\delta_e)\|^{1/r} = \|\tilde{M}\| = \frac{2\sqrt{d-1}}{d}.$$

□

As a consequence, we obtain an asymptotic lower bound on the second eigenvalue of the adjacency matrix of a graph:

Theorem 3.18 Alon-Boppana**.** *Any family $\{X_n : \ n \in \mathbb{N}\}$ of d-regular graphs whose number of vertices goes to infinity as $n \to \infty$ satisfies*

$$\liminf_{n\to\infty}(d - \lambda(X_n)) \geqslant 2\sqrt{d-1}.$$

Proof. Recall that the Markov operator of a finite d-regular graph X is given by $M = \frac{1}{d}A$, where A denotes the adjacency matrix of X. As before, denote by M^0 the Markov operator restricted to L_0^2.
By a consequence of the Min-Max Theorem (Corollary 2.17), we have

$$\begin{aligned} d - \lambda(X) &= \min_{f\neq 0,\ f\perp 1} R_A(f) \\ &= \min_{f\neq 0,\ f\in L_0^2(X)} R_A(f) \\ &= d \cdot \text{largest eigenvalue of } M^0 = d\cdot\rho(M^0). \end{aligned}$$

Thus, we obtain

$$d - \liminf_{n\to\infty} \lambda(X_n) = d \cdot \liminf_{n\to\infty} \rho(M_n^0) \geqslant 2\sqrt{d-1}$$

by Proposition 3.17. □

3.2 Ramanujan graphs: (Non-) Examples

In Section 3.1, it was shown that an infinite family $\{X_n : n \in \mathbb{N}\}$ of d-regular graphs satisfies

$$\liminf_{n\to\infty}(d - \lambda(X_n)) \geqslant 2\sqrt{d-1}.$$

In this section, we will investigate graphs whose eigenvalues are optimal in this sense.

Definition 3.19. A d-regular graph X is said to be a *Ramanujan graph* if its second eigenvalue satisfies $d - \lambda(X) \leqslant 2\sqrt{d-1}$.

An infinite family $\{X_n : n \in \mathbb{N}\}$ of d-regular graphs is called a *Ramanujan family* if the following hold:

(i) $|X_n| \to \infty$ as $n \to \infty$, and

(ii) for any $n \in \mathbb{N}$, X_n is a Ramanujan graph.

Example 3.20. It is not hard to find examples for single Ramanujan graphs: for instance, the complete graph satisfies $(n-1) - \lambda(K_n) = 0$ for any $n \geqslant 2$ and is therefore Ramanujan. Coming up with an example of a Ramanujan family however is a very difficult task.

3.2.1 A second look at the Margulis expanders

As families of Ramanujan graphs are by definition also expander families, a question that comes up naturally at this point is whether the examples of expander families encountered in Chapter 2 even provide examples of Ramanujan families.

Regarding the expander families obtained as Cayley graphs of quotients of a Kazhdan group (as Corollary 2.58), the following result gives a negative answer:

Proposition 3.21 [Lub94, Prop. 4.5.7, p. 58]**.** *Let G be a group generated by a finite set S. Assume that there exist infinitely many finite index normal subgroups N_i of G such that $N_i \cap S = \emptyset$ and $\{\mathcal{G}(G/N_i, S) : i \in \mathbb{N}\}$ forms a family of Ramanujan graphs.*
Then G does not have property (T).

Although we will not provide a complete proof of this proposition, we will still show how it can be obtained given two other results that we use as “black boxes”. The first one is Kesten’s theorem on the Markov operator of the tree (Theorem 3.16). The second result is a generalisation of the Alon-Boppana Theorem due to Burger [Bur85]:

Theorem 3.22 [Lub94, Corollary 4.2.8, p. 48]**.** *Let X be a connected d-regular graph and $\{X_n : n \in \mathbb{N}\}$ a family of finite graphs covered by X with $|X_n| \to \infty$ as $n \to \infty$. Moreover, let M be the Markov operator of X, and denote by $\mu_1(X_n) \geqslant \cdots \geqslant \mu_k(X_n)$ the eigenvalues of the adjacency matrix of X_n. Then for every fixed integer $r \geqslant 1$, we have*

$$\liminf_{n\to\infty} \mu_r(X_n) \geqslant d\|M\|.$$

Proof of Proposition 3.21. Set $X := \mathcal{G}(G, S)$, $X_i := \mathcal{G}(G/N_i, S)$, and note that the canonical homomorphism $G \to G/N_i$ is a covering (cf. Example 1.39 (b)). Thus, the Markov operator M of X has norm

$$\|M(X)\| \leqslant \liminf_{n\to\infty} \frac{\mu_2(X_n)}{d} = \frac{2\sqrt{d-1}}{d}$$

by Theorem 3.22. As $\|M(X)\| \geqslant 2\sqrt{d-1}/d$ is always true (Corollary 3.15), we have $\|M(X)\| = 2\sqrt{d-1}/d$, and thus by Kesten’s result (Theorem 3.16), $X = \mathcal{G}(G, S)$ is a tree. Hence, G is a free group over a symmetric reduction S' of S. In particular, G does not have property (T) by Corollary 2.50. □

Remark 3.23. It is somewhat counter-intuitive that Kazhdan’s property (T) guarantees expansion (as of Proposition 2.57), and on the other hand also guarantees that the expansion property will not be too strong (as of Proposition 3.21 above).

As for the Margulis expanders, which can be viewed as Schreier graphs of quotients of a group with relative property (T), this result seems to hint that they might also not be Ramanujan. A generalisation of Proposition 3.21 to Schreier graphs however is not obvious - the main hole in the proof being that an analogue of Kesten's Theorem for Schreier graphs would be needed.

A numerical computation of eigenvalues yields that none of Γ_n, Γ_n^M and Γ_n^J is a Ramanujan graph for $5 \leqslant n \leqslant 200$ (see Figure 3.1), and the same holds for all larger values of n for which the computation was performed (for a selection of values, see Table 3.2).

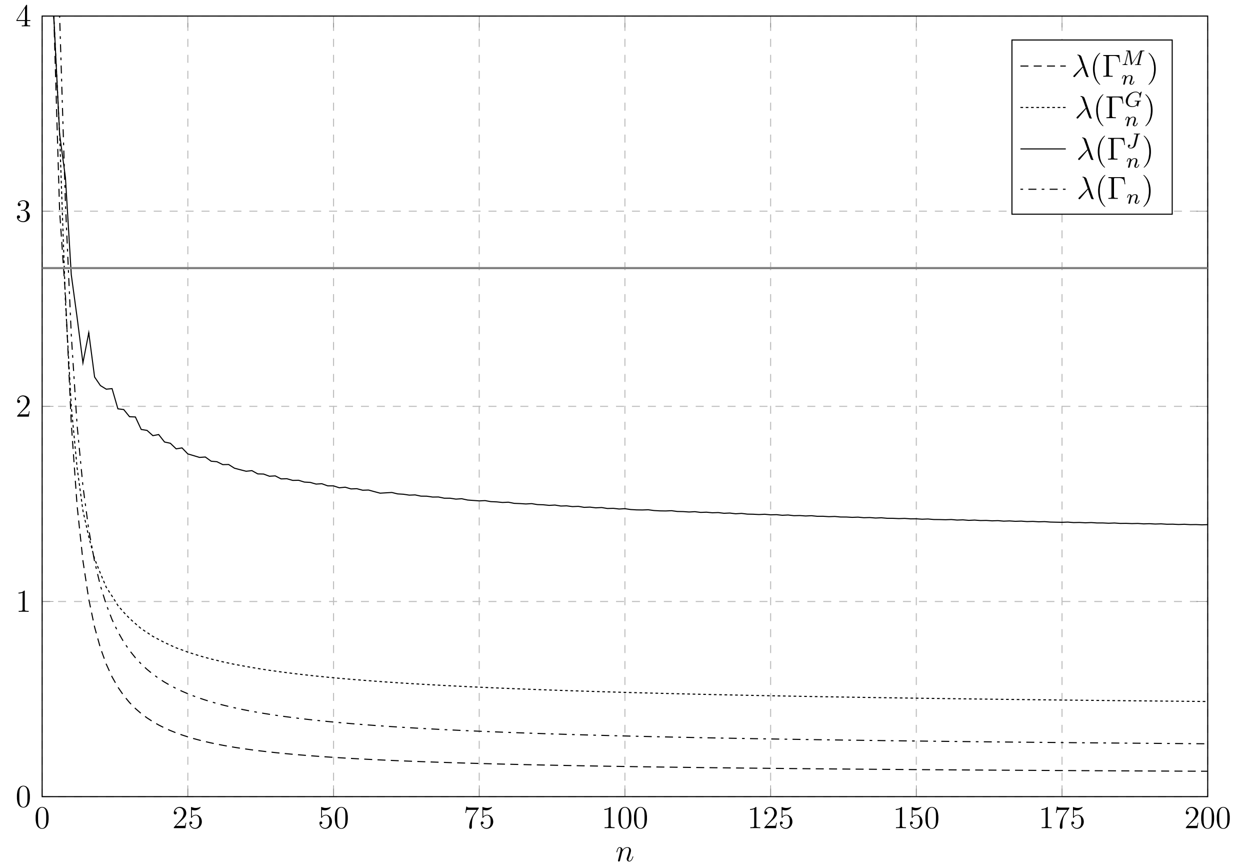

Figure 3.1: The values of the spectral gap in four (non-bipartite) Margulis expander variants for $n \leqslant 200$. The values fall below the critical bound $8 - 2\sqrt{7} \approx 2.708$, represented by the grey line, already at $n = 5$.

n	2	3	5	10	20	50	100	200	500	1000	1500
$\lambda(\Gamma_n^M)$	4.000	3.000	1.912	0.762	0.367	0.201	0.154	0.130	0.112	0.104	0.101
$\lambda(\Gamma_n^G)$	4.000	3.394	2.000	1.144	0.805	0.609	0.534	0.488	0.448	0.429	0.420
$\lambda(\Gamma_n^J)$	4.000	3.394	2.675	2.106	1.855	1.592	1.476	1.394	1.319	1.279	1.260
$\lambda(\Gamma_n)$	4.877	4.000	2.367	1.080	0.606	0.382	0.311	0.271	0.240	0.226	0.219

Table 3.2: A selection of values for the spectral gap in different (non-bipartite) variants of the Margulis expanders.

So the Margulis expanders are not Ramanujan families themselves, but a priori, they could still contain an infinite Ramanujan subfamily.

For Γ_n and Γ_n^M, this possibility is ruled out by the following result due to Abért, Glasner and Virág, which we present without proof:

Theorem 3.24 [AGV16, Theorem 2, p. 1602]. *Let $d \geqslant 2$ be an integer, and set $\beta := (30\log(d-1))^{-1}$. Then for any d-regular finite Ramanujan graph X, the proportion of vertices of X whose ball of radius $\beta \log\log|X|$ is a d-regular tree is at least $1 - c(\log|X|)^{-\beta}$, where c is a positive constant.*

Corollary 3.25. *The Margulis expander families $\{\Gamma_n : n \in \mathbb{N}\}$ and $\{\Gamma_n^M : n \in \mathbb{N}\}$ do not contain any Ramanujan subfamilies.*

Proof. Both Γ_n and Γ_n^M feature neighbour edges (in the sense that every vertex $v \in V_n$ is connected to its "neighbours" $v \pm e_1$ and $v \pm e_2$ modulo n). Thus, every vertex v is part of a cycle of length 4, e.g. the one given by $v \leftrightarrow v + e_1 \leftrightarrow v + e_1 + e_2 \leftrightarrow v + e_2 \leftrightarrow v$. Therefore, the proportion of vertices whose ball of radius 2 is a d-regular tree is zero, and thus Theorem 3.24 above tells us that Γ_n and Γ_n^M are non-Ramanujan for $n \geqslant \exp(7^{60} \cdot 1/2)$. In particular, only finitely many of the Γ_n and Γ_n^M can be Ramanujan graphs. $\square$

Remark 3.26. (1) Analogously to Corollary 3.25, the bipartite versions of Γ_n and Γ_n^M (as sketched in Remark 2.24 (2)) do not contain any Ramanujan subfamilies because the cycles of length 4 indicated in the proof above translate to cycles of length 6 in the bipartite versions of these graphs.

(2) The approach from Corollary 3.25 will not work for logarithmic girth graphs, i.e. for families $\{X_n : n \in \mathbb{N}\}$ of graphs satisfying $\operatorname{girth}(X_n) = O(\log|X_n|)$. In this case, the ball of radius $\beta \log\log|X_n|$ in X_n will always be a tree for large enough n, so that the conclusion of Theorem 3.24 holds for all except possibly finitely many n. Thus, one cannot exclude the possibility that $\{X_n : n \in \mathbb{N}\}$ contains an infinite Ramanujan subfamily.

3.2.2 LPS-expanders

Having seen that the families of expander graphs encountered in Chapter 2 are not good candidates for examples of Ramanujan graphs, we shall now sketch the first explicit construction of a family of Ramanujan graphs due to Lubotzky, Phillips and Sarnak [LPS88] and, independently, Margulis [Mar88]. This subsection mostly follows the presentation in [DSV03].

We first recall some definitions from algebra.

Definition 3.27. Let R be a commutative ring with identity. The *quaternion algebra* $\mathbb{H}(R)$ over R is the free R-module over the symbols $1, i, j, k$, that is,

$$\mathbb{H}(R) = \{a_0 + a_1 i + a_2 j + a_3 k \mid a_0, a_1, a_2, a_3 \in R\},$$

endowed with a multiplication defined by the following relations:

(i) 1 is the multiplicative identity,

(ii) $i^2 = j^2 = k^2 = -1$, and

(iii) $ij = -ji = k$, $ki = -ik = j$, $jk = -kj = i$.

Remark 3.28. (1) The ring R naturally embeds into $\mathbb{H}(R)$ via $r \mapsto r + 0i + 0j + 0k$, and can therefore be viewed as a subring of $\mathbb{H}(R)$.

(2) For the sake of notational simplicity, we will omit symbols with coefficient 0 from now on.

Definition 3.29. Let R be a commutative unital ring, and let $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ be an element of the quaternion algebra. The element $\bar{\alpha} := a_0 - a_1 i - a_2 j - a_3 k$ of $\mathbb{H}(R)$ is called the *conjugate* of α. Moreover, $N(\alpha) := \alpha \cdot \bar{\alpha} = a_0^2 + a_1^2 + a_2^2 + a_3^2 \in R$ is called the *norm* of α.

Remark 3.30. It is easily checked that the norm $N\colon \mathbb{H}(R) \to R$ is multiplicative. In particular, the norm of an invertible element of $\mathbb{H}(R)$ is invertible in R and the norm of a zero divisor is a zero divisor in R. As a consequence of the latter, if R is an integral domain, then so is $\mathbb{H}(R)$.

Let now p, q be unequal prime numbers congruent to 1 modulo 4, and consider the field $R = \mathbb{F}_q = \mathbb{Z}/q\mathbb{Z}$. A convenient aspect of the associated quaternion algebra $\mathbb{H}(\mathbb{F}_q)$ is that its elements can be viewed as matrices:

Lemma 3.31 [DSV03, Proposition 2.5.2, p. 589]**.** *If q is an odd prime number and $u \in \mathbb{F}_q$ satisfies $u^2 = -1$, then the map*

$$\begin{aligned} \psi_q\colon \qquad \mathbb{H}(\mathbb{F}_q) &\longrightarrow M_2(\mathbb{F}_q) \\ a_0 + a_1 i + a_2 j + a_3 k &\longmapsto \begin{pmatrix} a_0 + ua_1 & a_2 + ua_3 \\ -a_2 + ua_3 & a_0 - ua_1 \end{pmatrix} \end{aligned}$$

is an algebra isomorphism.
Moreover, ψ_q satisfies

(i) $N(\alpha) = \det \psi_q(\alpha)$ and

(ii) $\psi_q(\alpha)$ is a scalar matrix if and only if $\alpha = \bar{\alpha}$

for any $\alpha \in \mathbb{H}(\mathbb{F}_q)$.

Proof. It is easily verified that ψ_q is an algebra homomorphism that fulfils the stated properties. To see that ψ_q is an isomorphism, it suffices to prove injectivity since both $\mathbb{H}(\mathbb{F}_q)$ and $M_2(\mathbb{F}_q)$ have dimension 4 as vector spaces over $\mathbb{F}_q$. But $\psi_q(\alpha) = 0$ means that the coefficients of $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ solve a system of equations with determinant

$$\det \begin{pmatrix} 1 & u & 0 & 0 \\ 0 & 0 & 1 & u \\ 0 & 0 & -1 & u \\ 1 & -u & 0 & 0 \end{pmatrix} = (-u - u) \cdot (u + u) = -4u^2 \equiv 4 \not\equiv 0.$$

Thus, $\alpha = 0$, so ψ_q is injective. □

Remark 3.32. As -1 is a quadratic residue modulo any prime q congruent to 1 modulo 4, we can always find u as above and apply the Lemma.

We will make use of this way of viewing the quaternion algebra of $\mathbb{F}_q$ and define the LPS-graphs as Cayley graphs of a group of matrices over $\mathbb{F}_q$ with respect to a generating set obtained from a certain set of quaternions.

More precisely, let us consider the algebra of *integral quaternions* $H(\mathbb{Z})$. Given an odd prime number p, we investigate the set of integral quaternions of norm p.

Lemma 3.33. *Up to associates, there are $p + 1$ integral quaternions of norm p.*

Proof. An integral quaternion α of norm p corresponds to a quadruple (a_0, a_1, a_2, a_3) of integers satisfying $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$. By a famous theorem of Jacobi [cf. DSV03, Thm. 2.4.1, p. 52], there are $8(p + 1)$ solutions to this equation, so there are $8(p + 1)$ integral quaternions of norm p.

By multiplicativity of the norm, any associate of an element of norm p has norm p. The units of $\mathbb{H}(\mathbb{Z})$ are $\{\pm 1, \pm i, \pm j, \pm k\}$, so if $\alpha \in \mathbb{H}(\mathbb{Z})$ is of norm p, then so are $-\alpha, \pm i\alpha, \pm j\alpha, \pm k\alpha$. Moreover, any two associates are distinct because otherwise, we would have $\varepsilon_1 \alpha = \varepsilon_2 \alpha$ for two units $\varepsilon_1, \varepsilon_2$, so that $(\varepsilon_2^{-1} \varepsilon_1 - 1)\alpha = 0$ and therefore $\varepsilon_1 = \varepsilon_2$ as $\mathbb{H}(\mathbb{Z})$ is an integral domain (see Remark 3.30). Hence, there are $p + 1$ elements of norm p up to associates. □

Construction 3.34. We will now fix one representative of each associate class. To do so, consider the equation $a_0^2 + a_1^2 + a_2^2 + a_3^2 = p$ modulo 4, and note that the square of a number is always congruent to 0 or 1 modulo 4, according to its parity. Now if $p \equiv 1 \mod 4$, then exactly one of the coefficients a_0, a_1, a_2, a_3 is odd while the others are even; denote this coefficient by a_i^0. Exactly one associate of $\alpha = a_0 + a_1 i + a_2 j + a_3 k$ has $|a_i^0|$ as coefficient of 1, and we call this associate *distinguished*. Note that if α is distinguished, then so is $\bar{\alpha}$.

We denote by S_p the set of distinguished integral quaternions of norm p.

As S_p contains one representative of every class of associates, we have $|S_p| = p+1$ by Lemma 3.33, and fixing another prime $q > p$, we can consider the image of S_p under the reduction map $\tau_q \colon \mathbb{H}(\mathbb{Z}) \to \mathbb{H}(\mathbb{F}_q)$, which is still a set of elements of norm p.
Considering now the image of this set under the map ψ_q, we obtain a subgroup of $GL_2(\mathbb{F}_q)$ as for any $\alpha \in S_p$, $\det \psi_q(\tau_q(\alpha)) = N(\tau_q(\alpha)) = p$ is invertible. Finally, we mod out by constants, that is, we concatenate further with the homomorphism $\varphi \colon GL_2(\mathbb{F}_q) \to GL_2(\mathbb{F}_q)/(\mathbb{F}_q^\times \cdot \mathrm{Id}) = PGL_2(\mathbb{F}_q)$ mapping $A \in GL_2(\mathbb{F}_q)$ to $A \cdot (\mathbb{F}_q^\times \cdot \mathrm{Id})$.
Summing up, we consider the image of S_p under the following maps:

$$\mathbb{H}(\mathbb{Z}) \xrightarrow{\tau_q} \mathbb{H}(\mathbb{F}_q) \xrightarrow{\psi_q} M_2(\mathbb{F}_q) \supseteq GL_2(\mathbb{F}_q) \xrightarrow{\varphi} PGL_2(\mathbb{F}_q).$$

Definition 3.35. We set

$$S_{p,q} := (\varphi \circ \psi_q \circ \tau_q)(S_p).$$

Remark 3.36. Note that $S_{p,q}$ is symmetric as S_p is closed under conjugates and for $\alpha \in S_p$, $(\psi_q \circ \tau_q)(\bar{\alpha}\alpha) = \psi_q(p)$ is a scalar matrix by Lemma 3.31 and therefore is mapped to the identity under φ.

Choosing q large enough guarantees that the above mapping is injective:

Lemma 3.37 [DSV03, Lemma 4.2.1, p. 1139]**.** *If q satisfies $q > 2\sqrt{p}$, then we have $|S_{p,q}| = p + 1$.*

Proof. Let $\alpha = a_0 + a_1 i + a_2 j + a_3 j$ and $\beta = b_0 + b_1 i + b_2 j + b_3 k$ be distinct elements of S_p. Then $a_n \neq b_n$ for some $n \in \{0, 1, 2, 3\}$. Since $N(\alpha) = N(\beta) = p$, we have $a_n, b_n \in (-\sqrt{p}, \sqrt{p})$. As we assumed that $p > 2\sqrt{q}$, this means that $a_n \not\equiv b_n \mod q$ and therefore $\tau_q(\alpha) \neq \tau_q(\beta)$. Denoting $A = (\psi_q \circ \tau_q)(\alpha), B = \psi_q \circ \tau_q)(\beta)$, we have $A \neq B$ in $GL_2(\mathbb{F}_q)$. Now suppose that $\varphi(A) = \varphi(B)$, that is that $A = \lambda B$ for some $\lambda \in \mathbb{F}_q^\times$, $\lambda \neq 1$. Taking determinants gives $\det A = \lambda^2 \det B$, that is $p = \lambda^2 p$ by Lemma 3.31. Thus, $\lambda^2 = 1$, so that $\lambda = -1$ and $A = -B$. This implies that $\alpha \equiv \beta \mod q$, and therefore that $\alpha = -\beta$ (again using that $q > 2\sqrt{p}$). But since α and β are distinct elements of S_p, they are not associated, so in particular, $\alpha \neq -\beta$. □

Remark 3.38. Note that $S_{p,q} \subseteq PSL_2(\mathbb{F}_q)$ holds if and only if p is a quadratic residue modulo q. Indeed, if $\alpha \in \mathbb{H}(\mathbb{F}_q)$ is of norm p, then $A := (\psi_q \circ \tau_q)(\alpha)$ has determinant p, and

$$\begin{aligned}
\varphi(A) \in PSL_2(\mathbb{F}_q) &\iff \exists \lambda \in \mathbb{F}_q^\times : \det(\lambda A) = 1 \\
&\iff \exists \lambda \in \mathbb{F}_q^\times : \lambda^2 p = 1 \\
&\iff \exists \lambda \in \mathbb{F}_q^\times : p = (\lambda^{-1})^2 \\
&\iff p \text{ is a quadratic residue modulo } q.
\end{aligned}$$

We now proceed to define the LPS-graphs $X^{p,q}$:

Construction 3.39. Let p, q be two distinct primes congruent to 1 modulo 4 such that $q > 2\sqrt{p}$. Then we set

$$X^{p,q} := \begin{cases} \mathcal{G}(PSL_2(\mathbb{F}_q), S^{p,q}) & \text{if } (\frac{p}{q}) = 1 \\ \mathcal{G}(PGL_2(\mathbb{F}_q), S^{p,q}) & \text{if } (\frac{p}{q}) = -1. \end{cases}$$

Here we use the Legendre symbol

$$\left(\frac{p}{q}\right) := \begin{cases} 1 & \text{if } p \text{ is a quadratic residue modulo } q \\ -1 & \text{if } p \text{ is not a quadratic residue modulo } q. \end{cases}$$

Remark 3.40. By Lemma 3.37, $X^{p,q}$ is a $(p+1)$-regular graph. Its order is

$$|X^{p,q}| = \begin{cases} |PSL(\mathbb{F}_q)| = \frac{q(q^2-1)}{2} & \text{if } \left(\frac{p}{q}\right) = 1 \\ |PGL(\mathbb{F}_q)| = q(q^2-1) & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

The main result of this section is the following Theorem:

Theorem 3.41 [LPS88, Theorem 4.1, p. 269]**.** *If p and q are primes congruent to* 1 *modulo* 4 *with $p < q$, then $X^{p,q}$ is a Ramanujan graph.*

The proof of this theorem relies on deep results from number theory (more precisely, it uses a consequence of proofs of the *Ramanujan conjecture*, which inspired the name “Ramanujan graphs”), and presenting it would be beyond the scope of this thesis. In fact, even proving that the graphs $X^{p,q}$ are connected requires some work. For a self-contained presentation of the LPS-construction that at least contains a proof of the expansion property, the interested reader is referred to [DSV03].

Remark 3.42. (1) In particular, for fixed p, the construction yields an infinite family $\{X^{p,q} : q > 2\sqrt{p},\ q \text{ prime},\ q \equiv 1 \mod 4\}$ of $(p+1)$-regular Ramanujan graphs.

(2) Moreover, it can be shown that the following girth estimates hold [LPS88, Theorem 3.2, p. 267]:

$$\mathrm{girth}(X^{p,q}) \geqslant \begin{cases} 2\log_p q & \text{if } \left(\frac{p}{q}\right) = 1 \\ 4\log_p q - \log_p 4 & \text{if } \left(\frac{p}{q}\right) = -1. \end{cases}$$

Similar estimates can be obtained for any family of Ramanujan graphs $\{X_n : n \in \mathbb{N}\}$ consisting of Cayley graphs via Theorem 3.24: Recall that a cycle in a Cayley graph corresponds to a word in the generators that evaluates to the identity element. Therefore, any vertex of a Cayley graph will be contained in a cycle of minimal length (multiply the word representing the cycle by the group element representing the vertex), so that the proportion of vertices of X_n whose ball of radius $\mathrm{girth}(X_n)$ is a tree is equal to zero. Thus, Theorem 3.24 yields a bound of the form

$$\mathrm{girth}(X_n) > \frac{\log\log|X_n|}{30\log(d-1)}.$$

In general, however, large girth is not a necessary condition for Ramanujan graphs. For instance, Glasner [Gla03] gave an example of a Ramanujan family of girth 1, consisting of Schreier coset graphs.

(3) Davidoff, Sarnak and Valette [DSV03] present a slight adaptation that allows to generalise the construction to any odd prime p: Instead of fixing a single integer u satisfying $u^2 \equiv -1 \mod p$, we fix two integers x and y such that $x^2 + y^2 \equiv -1 \mod p$, which is possible also for primes p congruent to 3 modulo 4. Then we can use the map

$$\begin{aligned} \tilde{\psi}_q\colon \qquad \mathbb{H}(\mathbb{F}_q) &\longrightarrow M_2(\mathbb{F}_q) \\ a_0 + a_1 i + a_2 j + a_3 k &\longmapsto \begin{pmatrix} a_0 + a_1 x + a_3 y & -a_1 y + a_2 + a_3 x \\ -a_1 y - a_2 - a_3 x & a_0 - a_1 x - a_3 y \end{pmatrix} \end{aligned}$$

instead of ψ_q and otherwise proceed as before (with an extra bit of care to be taken in the definition of S_p) to obtain a Ramanujan graph $X^{p,q}$. A visualisation of the construction for $p = 3, q = 5$ and $p = 3, q = 5$ is shown in Figure 3.3.

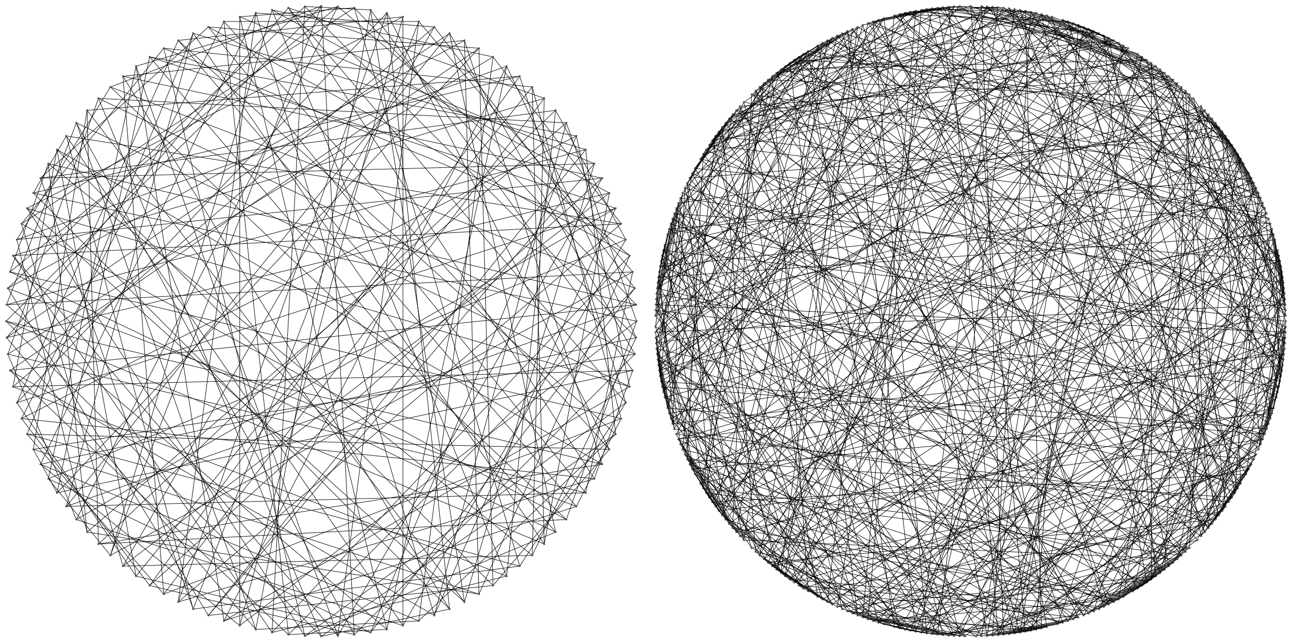

Figure 3.3: The construction from [LPS88] and [Mar88], visualised for $p = 3$, $q = 5$ and $p = 3$, $q = 7$.

(4) Chiu [Chi92] gave an example of a family of 3-regular Ramanujan graphs. Morgenstern [Mor94] extended the LPS-construction further and gave examples of $(k + 1)$-regular Ramanujan graphs for any odd prime power k. Further constructions of Ramanujan graphs have been given by Pizer [Piz90], Glasner [Gla03], and Jo-Yamasaki [JY18].

While the existence of Ramanujan graphs of any degree $d \geqslant 3$ has been shown by Marcus, Spielman and Srivastava [MSS15], no explicit constructions have been given for any value of d other than the above mentioned.

3.3 Near-Ramanujan graphs

While the last section sketched the classical construction of a family of Ramanujan graphs due to Lubotzky-Philips-Sarnak [LPS88] and Margulis [Mar88], this section is dedicated to the more recent topic of Near-Ramanujan graphs. Although the LPS-graphs are optimal expanders in the sense that they satisfy the strongest eigenvalue bound that one can hope for, they also come with some disadvantages (e.g., they are not yet available for any degree), and in some scenarios, one might prefer to slightly relax the eigenvalue bound in exchange for other desirable properties, such as free choice of the degree or computational advantages. Constructing such graphs is a topic that has received scientific attention in the last 15 years; Table 3.4 gives an overview of publications dealing with this subject. While expressions such as "quasi-Ramanujan" or "almost-Ramanujan" have been used as well, "near-Ramanujan" is the prevailing terminology in most recent publications (see Table 3.4).

In this section, we take a closer look at the construction from [AGS21], yielding a family of "near-Ramanujan" graphs with large girth and "localised eigenvectors". The main result of this paper reads as follows:

Theorem 3.43 [AGS21, Theorem 1.2, p. 2]**.** *For any prime p and any $\alpha \in (0, 1/6)$, there are infinitely many integers m such that there exists a $(p + 1)$-regular graph $X_m = (V_m, E_m)$ on m vertices satisfying*

Publication(s)	terminology	eigenvalue bound*	special properties
[BL06]	quasi-Ramanujan	$O(\sqrt{d \log^3 d})$	any d, polynomial-time computable
[Fri08]	nearly Ramanujan	$2\sqrt{d-1}+\varepsilon$	any odd d (non-constructive)
[CM08]	almost-Ramanujan	$(2+\varepsilon)\sqrt{d-1}$	almost all $d^{\dagger}$
[BATS11; SZ09]	almost-Ramanujan	$d^{1/2+O(1/\sqrt{\log d})}$	any d, purely combinatorial construction
[Dud15]	almost-Ramanujan	$2(d-1)^{1/2+O(\log\log d/\log d)\ddagger}$	any $d \geqslant 3$
[MOP20]	near-Ramanujan	$2\sqrt{d-1}+\varepsilon$	any $d \geqslant 3$, polynomial-time computable
[MM21]	near-Ramanujan	$2\sqrt{d-1}+O(1/\log_{d-1} n)$	large girth, bounded vertex expansion on sublinear sets
[AGS21]	near-Ramanujan	$\frac{3}{\sqrt{2}}\sqrt{d-1}$	large girth, fully localized eigenvectors

* Here, we denote by d the degree and by n the number of vertices. The bounds are upper bounds for the eigenvalues of the adjacency matrices, so using the notation introduced in Chapter 2, this means "$d-\lambda(X) \leqslant$ eigenvaluebound". Whenever an ε appears in the bound, this is to be read as "for any fixed $\varepsilon > 0$, the construction yields a family of graphs whose eigenvalues satisfy the bound".

† in the sense of natural density

‡ assuming the Riemann hypothesis

Table 3.4: A selection of papers dealing with graphs that are close to being Ramanujan in different ways.

(1) $p+1-\lambda(X_m) \leqslant \frac{3}{\sqrt{2}}\sqrt{p}$

(2) $\mathrm{girth}(X_m) \geqslant 2\alpha \log_p(m)(1-O(1))$

(3) There is a set $S_m \subseteq V_m$ of size $O(m^{\alpha})$ such that at least $\lfloor \alpha \log_p(m) \rfloor$ eigenvalues λ of X_m have corresponding eigenvectors that are supported entirely on S_m.

Remark 3.44. (1) Rephrasing properties (1)-(3), Theorem 3.43 gives a family of near-Ramanujan graphs with high girth and localised eigenvectors. However, both the eigenvalue and the girth estimate are worse than the corresponding properties of the LPS-graphs, which are the starting point for the construction. The real value of this result thus lies in the "localised eigenvectors" property (3), combined with the fact that the losses in expansion and girth are still reasonably small.

(2) Theorem 3.43 is not merely an existence result, but witnessed by an explicit construction (see Construction 3.47).

(3) The restriction of the degree to successors of prime numbers is solely caused by the fact that the construction relies on the usage of Ramanujan graphs satisfying certain girth estimates as a starting point. As the same girth estimates are also satisfied by the Ramanujan graphs in [Mor94], the result above even holds for degrees of the form $k+1$, where k is a prime power.

The strategy of proving Theorem 3.43 is the following: We begin with a Ramanujan graph as constructed in [LPS88]. As these graphs have high girth (see Remark 3.42 (2)), they will locally look like trees. Now the idea is that by inserting another tree, we can achieve that certain eigenvalues of these trees will appear as eigenvalues of the whole graph corresponding to eigenvectors whose support is contained in the trees. One needs to make sure that this procedure can be done in a way that maintains high girth. Finally, it remains to be checked that the resulting graphs' eigenvalues still satisfy the inequality stated in the theorem.

At first, we will concentrate on the problem of ensuring high girth. Some terminology is useful:

Definition 3.45. Let d and D be positive integers, and let u be a vertex in the infinite $(d+1)$-regular tree T_{d+1}. The subgraph of T_{d+1} induced by the ball of radius D around u is called the *d-ary tree of depth D*, and the vertex u is said to be its *root*.

The main tool to ensure high girth will be the following result:

Lemma 3.46 [AGS21, Lemma 2.1, p. 7]**.** *Let T^1, T^2 be two d-ary trees of depth D with leaf sets L_1 and L_2, respectively. Then there exists a bijection $\pi\colon L_1 \to L_2$ such that the graph X obtained from the disjoint union of T^1 and T^2 by identifying each $v \in L_1$ with $\pi(v) \in L_2$ satisfies*

$$\operatorname{girth}(X) \geqslant 2\log_{2d-1}(n),$$

where $n = (d+1)d^{D-1}$ is the number of leaves of T^1 and T^2, respectively.

The proof of this Lemma is elementary, but non-constructive and rather lengthy. We will not present it here.

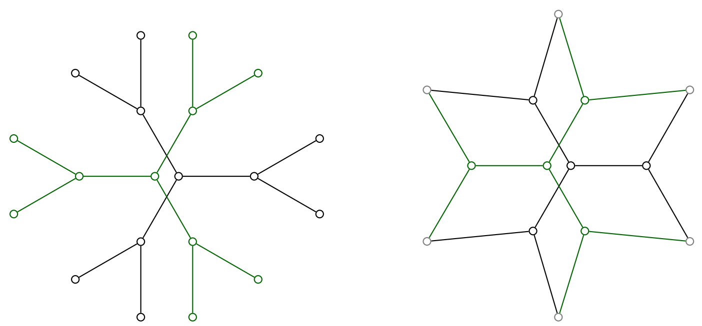

Figure 3.5: A possible way of combining two trees to obtain a (connected) graph of relatively large girth, as in Lemma 3.46; illustrated for $D = 2$ and $d = 6$.

Construction 3.47 [AGS21]**.** Fix a prime p and some $\alpha \in (0, 1/6)$. Let $H = (V, E)$ be a $(p+1)$-regular Ramanujan graph with m vertices and girth no less than $2/3\log_p(m)$. Now set $r = \lfloor \alpha \log_p m \rfloor$, so that

$$2/3\log_p m \geqslant 4r.$$

Fix a vertex u of H, and consider the induced subgraph T^1 of H consisting of all vertices of distance at most r from u. As $2r < \operatorname{girth}(H)$, it is a tree. Now to each vertex v_i of distance exactly r from u, we choose an adjacent vertex w_i of distance $r+1$ from u. Let H' denote the graph obtained from H by deleting all edges of the form $\{v_i, w_i\}$. Now take another tree T^2 isomorphic to T^1 on new vertices, and identify its leaves with those of T^1 using Lemma 3.46. Finally, take yet another copy T^3 of T^1 and identify its leaves with the w_i. Denote the resulting graph by X_m.

Remark 3.48. The existence of the Ramanujan graphs used in Construction 3.47 for infinitely many m follows from the results in [LPS88] and [Mar88]: Indeed, if $(\frac{p}{q}) = 1$, then the Ramanujan graph $X^{p,q}$ has $m = q(q^2-1)/2$ vertices (Remark 3.40), so that $q \geqslant \sqrt[3]{2m}$. Thus, the girth estimates for the LPS-graphs (Remark 3.42 (2)) yield

$$\operatorname{girth}(X^{p,q}) \geqslant 2\log_p(q) \geqslant 2\log_p(\sqrt[3]{2m}) = 2/3\log_p(2m).$$

Also note that assuming $(\frac{p}{q}) = 1$ is not problematic as for any fixed number p, there are infinitely many primes q so that p is a quadratic residue modulo q.

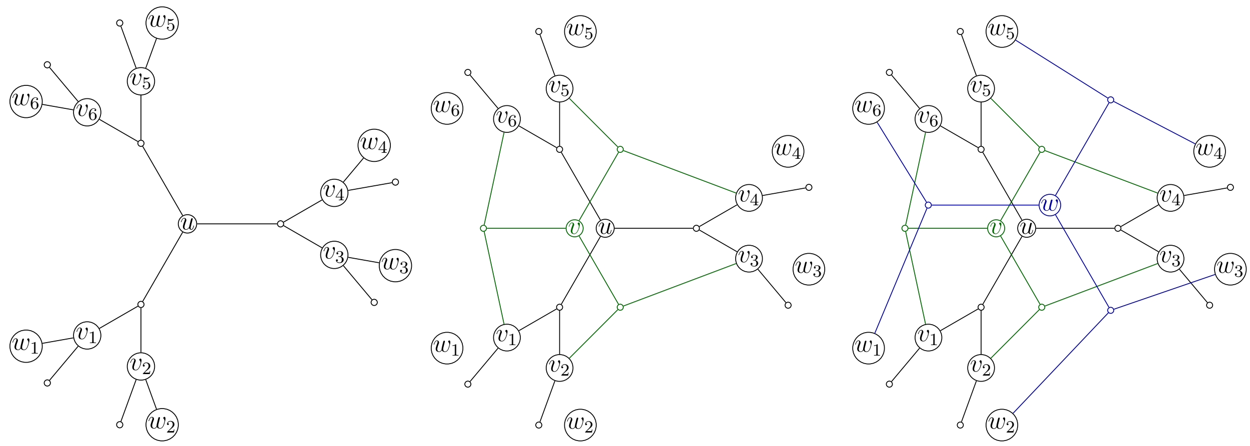

Figure 3.6: Construction 3.47, illustrated for p=2. Fix a root u and a "matching" $\{v_i, w_i\}$, remove the matching and apply Lemma 3.46 to the v_i and, finally, insert another tree whose leaves are the w_i.

From Lemma 3.46, Part (2) of Theorem 3.43 follows directly:

Proof of Part (2). As the girth of H is at least $4r$, we only need to consider cycles that involve vertices from $T := T^1 \cup T^2 \cup T^3$. Note that two distinct v_i have distance at least $2r$ in $H \setminus T$ because $\operatorname{girth}(H) > 4r$, and the same holds for the w_i. Thus, it remains to show that the girth estimate holds inside T, where we can use Lemma 3.46:

$$\operatorname{girth}(T) \geqslant 2\log_{2p-1}(n) = 2\log_{2p-1}\left((p+1)p^{r-1}\right) \geqslant 2\log_{2p-1}(p^r) = 2r\log_{2p-1}(p).$$

As $\log_{2p-1}(p)$ monotonously decreases to 0 as $p \to \infty$, we obtain

$$\operatorname{girth}(X) \geqslant 2r\,(1 - O(1)) = 2\alpha\log_p(m)\,(1 - O(1))\,.$$

□

Let us now sketch the proof of Part (3) of the theorem. As mentioned above, the idea is to find eigenvectors of trees that can be "extended" to eigenvectors of our graph X_m.

Definition 3.49. Let $T = (V, E)$ be a d-ary tree of depth D with root u. An eigenvector f of T is said to be a *radial eigenvector* if it is constant on all sets of the form

$$L_r := \{v \in V :\ d(v,u) = r\}$$

for $0 \leqslant r \leqslant D$ (the value of the constant may depend on r, however).

An eigenvalue of T is said to be a *radial eigenvalue* if it corresponds to a radial eigenvector of T.

The next lemma, which we state without proof, tells us that radial eigenvalues are actually rather common:

Lemma 3.50 [GS21, Lemma 3.1, p. 5777]**.** *The d-ary tree of depth D has exactly $D+1$ radial eigenvalues counting multiplicities.*

The following result shows that radial eigenvalues of a tree correspond to localised eigenvectors of the graphs X_m.

Lemma 3.51. *Let p, X_m and r be as in Construction 3.47. Moreover, denote by V_1 and V_2 the sets of non-leaves of the trees T^1 and T^2 from Construction 3.47, respectively. Then for any radial eigenvalue λ of a $(p+1)$-ary tree of depth $r-1$ there exists an eigenvector of X_m corresponding to λ that is supported only on $V_1 \cup V_2$.*

Proof. Let λ be such an eigenvalue of the $(p+1)$-ary tree of depth $r-1$ and f a corresponding eigenvector. As V_1 and V_2 induce such trees, f can be viewed as a function on $V_1 \cup V_2$. Now define $h \in L^2(V_m)$ by

$$h(v) := \begin{cases} f(v) & \text{if } v \in V_1 \\ -f(v) & \text{if } v \in V_2 \\ 0 & \text{otherwise.} \end{cases}$$

We claim that h is an eigenvector of X_m corresponding to the eigenvalue λ. Indeed, every neighbour of a vertex $v \in V_1$ is either in V_1 or outside $V_1 \cup V_2$, where h assumes the value 0. Therefore, we have

$$A_{X_m} h(v) = \sum_{w \in V_m} e(w,v) h(w) = \sum_{w \in V_1} e(w,v) f(v) = \lambda f(v) = \lambda h(v)$$

for v in V_1, and for the same reason

$$A_{X_m} h(v) = -\lambda f(v) = \lambda h(v)$$

for v in V_2.
As the eigenvalue equation is trivially fulfilled for vertices whose neighbours all lie outside $V_1 \cup V_2$, it only remains to check the vertices $v_1, \dots, v_n$. But each of these vertices has precisely one neighbour x_1 in V_1 and one neighbour x_2 in V_2 while the remaining neighbours lie outside $V_1 \cup V_2$. As f is a radial eigenvector, it assumes the same value on x_1 and x_2. Thus,

$$A_{X_m} h(v_j) = h(x_1) + h(x_2) = f(x_1) - f(x_2) = 0 = \lambda h(v_j)$$

for any $j \in \{1, \dots, n\}$.
So h is an eigenvector of X_m which is only supported on $V_1 \cup V_2$. □

Part (3) of Theorem 3.43 follows immediately:

Proof of Part (3). Take $S_m := V_1 \cup V_2$; then we have

$$|S_m| = 2(p+1)p^{r-2} \geqslant 2p^{r-1} = 2p^{\lfloor \alpha \log_p(m) \rfloor - 1} = O(m^\alpha).$$

Moreover, combining Lemmata 3.50 and 3.51 we infer that counting multiplicities, there are at least $r = \lfloor \alpha \log_p(m) \rfloor$ radial eigenvalues of X_m that have eigenvectors supported in S_m. □

As the proof of Part (1), the near-Ramanujan property, is not as straight-forward as for the other parts, we do not present it here, but rather content ourself with a brief sketch. The main idea of the proof is to view the graphs H, T^2, T^3 from Construction 3.47 as graphs on the vertex set of X_m (each with many isolated vertices), allowing to decompose the adjacency matrix of X_m as a sum of the adjacency matrices of those graphs (where the adjacency matrix of the "matching" $\{v_i, w_i\}$ needs to be subtracted). The fact that H is Ramanujan combined with estimates for the spectrum of the trees T^2 and T^3 then yields an upper bound for the second eigenvalue of the adjacency matrix of X_m, which in turn can be shown to be bounded from above by $\frac{3}{\sqrt{2}}\sqrt{p}$.

Remark 3.52. It is not quite obvious whether the methods from [AGS21] described here can be extended to expanders instead of Ramanujan graphs in order to obtain localised eigenvectors. In particular, it would be interesting to see whether the graphs obtained by applying Construction 3.47 to a family of expanders instead the LPS-graphs still form a family of expanders, and if so, how the expansion constant changes.

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