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Abstract

The topic of this thesis are maximal almost disjoint (MAD) families and their relation to forcing. With forcing it is possible to add a set that is almost disjoint from every set in a MAD family, thereby destroying its maximality. We study when this does or does not happen and give a combinatorial characterization of MAD families that are indestructible by a given forcing. Using this we can easily see implications between indestructibility for different forcing notions. We then proof that these implications are the only ones, by constructing MAD families that are indestructible for one forcing while being destroyed by another.

Zusammenfassung

Das Thema der Arbeit sind maximale fast disjunkte Familien und ihre Relation zu Forcing. Es ist möglich MAD Familien durch Forcing zu zerstören, indem man eine neue Menge hinzufügt, die fast disjunkt von jedem Element der MAD Familie ist. Wir untersuchen wann dies passiert und wann nicht und geben eine kombinatorische Charakterisierung von MAD Familien, die von einem vorgegebenen Forcing nicht zerstört werden. Mithilfe dieser ist es einfach Implikationen zwischen Unzerstörbarkeit durch verschiedene Forcings zu zeigen. Wir zeigen dann, dass diese Implikationen die einzigen sind. Dafür konstruieren wir MAD Familien, die von einem Forcing zerstört werden, aber nicht von einem anderen.

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Contents

1 Introduction

Two subsets of the naturals $A, B \subseteq \omega$ are called almost disjoint, if their intersection $A \cap B$ is finite. The main interest of this notion lies in the study of infinite families of pairwise almost disjoint sets, called almost disjoint (AD) families. In contrast to disjoint families for every countable almost disjoint family there exists a set that is almost disjoint from every element of the family. On the other hand the axiom of choice implies that there are maximal almost disjoint families, short MAD families, that means an almost disjoint family such that no set is almost disjoint from all members of the family.

The goal of this thesis is to study the effect forcing can have on these maximal almost disjoint families. Of course it is not possible to change whether a family of sets is almost disjoint via forcing, but the maximality is different. It is quite easy to destroy every MAD family in the ground model, for example adding a dominating real has this effect. Preserving MAD families is harder, as there is a MAD family that is destroyed by every forcing that adds a real. Consider the almost disjoint family of reals in $2^{<\omega}$ and complete it to a MAD family. Then every new real is almost disjoint from all elements of this MAD family. Additionally it is not known whether ZFC proves that there exists a MAD family that is preserved by some forcing, see [10]. We will introduce a few standard forcing notions and show that there are well known statements, that are independent from ZFC, that imply the existence of MAD families that are indestructible by these forcings, for example Sacks indestructible MAD families exist if $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. We do this by giving a combinatorial characterization of $\mathbb{P}$-indestructibility for these forcings. For example for Sacks forcing we have the following.

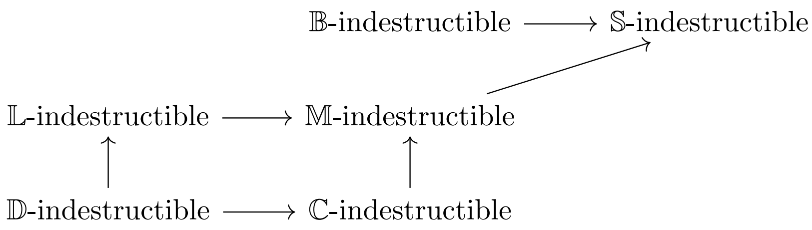
Theorem 1.0.1. *Let $\mathcal{A}$ be a MAD family. The following are equivalent:*

(1) $\mathcal{A}$ *is* $\mathbb{S}$*-indestructible.*

(2) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin \mathbf{cntble}$ $\forall f \colon B \to \omega$ *injective* $\exists I \in \mathcal{I}(\mathcal{A})$ *such that* $G_{f^{-1}[I]} \notin \mathbf{cntble}$.

(3) $\forall f \colon 2^{<\omega} \to \omega$ *injective* $\exists I \in \mathcal{I}(\mathcal{A})$ *such that* $G_{f^{-1}[I]} \notin \mathbf{cntble}$.

From this characterization we also directly get implications between indestructibility of MAD families. These are expressed in the following diagram, where arrows mean implication:



We then go on to construct MAD families, or close relatives, that are indestructible for some forcing while being destroyed by another. This will show that there are no other arrows in the diagram than the ones given.

The last chapter is devoted to partition forcing. This is a forcing that destroys an uncountable partition of 2^ω into closed sets. We show that the combinatorial characterization of indestructibility of MAD families also holds for partition forcing. Using this and assuming CH we can construct a maximal family of eventually different functions, a close relative of MAD families, that is partition forcing indestructible, both of these results are new. Additionally we give a property that is preserved under iterations and implies the indestructibility of certain MAD families, that are known to exist under the assumption $\mathfrak{b} = \mathfrak{c}$. That is in contrast to all the previous results as those only work for forcings that add a single real and are in general not preserved under iterations.

2 Preliminaries

We will generally work within ZFC set theory, with additional axioms specified if needed. Good references for set theory are [24] and [20]. Let X be a set and let κ be a cardinal. We denote by $[X]^\kappa$ the set of subsets of X of cardinality κ, by $[X]^{<\kappa}$ the set of subsets of X of cardinality less than κ. With X^κ we denote both the set of functions from κ to X and the cardinality of this set. We hope that it is always clear what is meant from the context. For a sequence $s \in X^{<\kappa}$ we define $[s] \subseteq X^\kappa$ to be the set of sequences extending s, by $|s|$ we mean the length of s and for $\alpha < |s|$ we let $s \restriction \alpha$ be the restriction of s to α. A tree T on X of height κ is a subset of $X^{<\kappa}$, such that if $t \in T$ also $t \restriction \alpha \in T$ for every $\alpha < |t|$. We let $[T] \subseteq X^\kappa$ be the set of branches of T, i.e. the set of all $x \in X^\kappa$ such that $x \restriction \alpha \in T$ for every $\alpha < \kappa$. If T is a tree and $s \in T$ is a node let $T(s) = \{t \in T |\ s \subseteq t \vee t \subseteq s\}$ be the subtree of T of all nodes that are compatible with s. The eventual dominance relation $\leq^*$ is defined on ω^ω by $g \leq^* f$ if and only if there is some $N \in \omega$ such that $g(n) \leq f(n)$ for all $n > N$.

An ideal on a set X is a set $\mathcal{A} \subseteq \mathcal{P}(X)$ such that if $A \subseteq B \in \mathcal{A}$, then $A \in \mathcal{A}$ and for every $A, B \in \mathcal{A}$ also $A \cup B \in \mathcal{A}$. We also assume that all ideals contain the finite subsets of X. An ideal is called σ-closed if additionally $\bigcup_{n\in\omega} A_n \in \mathcal{A}$, for every $\{A_n |\ n \in \omega\} \subseteq \mathcal{A}$. We denote by $\mathcal{A}^+$ the set of all subsets of X which are not in $\mathcal{A}$.

Forcing will be used throughout the thesis, but is to involved to introduce here. It is a technique to get from a model V of set theory another model $V[G]$ by adjoining a set to it, whose properties can be described in V via a partial order. A forcing notion $\mathbb{P}$ is a partial order, we call the elements conditions and say a condition p is stronger than q if $p \leq q$.

In a few proofs we will need the notion of a game on a set X. For an introduction to games see [23]. In a game two players will alternate playing to construct a sequence in X^ω, in the end we say player I wins if the end state is an element of some predetermined set $A \subseteq X^\omega$. The important property is that all games played with Borel payoff set are determined, that means one of the two players has a winning strategy.

Another thing we will use are cardinal characteristics. Those are statements that define a cardinal that lies somewhere between $\aleph_1$ and the continuum, also denoted $\mathfrak{c}$, but its value is not determined by ZFC. For example $\mathfrak{b}$ is the least size of a subset of ω^ω that is unbounded with respect to $\leq^*$. We will define all used cardinal characteristics, but will use the known relations between them freely. For more information look at [6].

3 Proper forcing

If we want to prove anything interesting about cardinal characteristics we need that the continuum has size bigger than $\aleph_1$, otherwise we will just have that all of them are equal to $\aleph_1 = \mathfrak{c}$. To get a forcing extension where the continuum hypothesis fails we have basically two methods. We can either start with a model with big continuum where all cardinal characteristics are big, a good example of this is a model of not CH and Martin's Axiom, and then force some of them to become small. The other one is to start with a model of CH and force the continuum and some characteristics to become big. Here we will mostly be concerned with the second approach. The question then becomes how we guarantee that the continuum really is big in the extension. If we collapse $\aleph_1$ to become countable we can add $\aleph_2$ many reals and still satisfy CH in the end. So we need some way to guarantee that $\aleph_1$ is not collapsed. The two most well known properties that assure this are the countable chain condition and σ-closedness. Of course σ-closed forcings do not add new reals, so they are not doing anything for us. Forcings with the c.c.c. are more interesting for our purposes, but there are still a lot of forcings that we want to look at that do not satisfy it, for example Sacks forcing. For this purpose Shelah [36] introduced proper forcing. All the proofs in this chapter are due to him. We will follow here the presentation given in [13], for other presentation see [1] and [14].

Definition 3.0.1. *Let $\mathbb{P}$ be a forcing notion, $\kappa > 2^{|\mathbb{P}|}$ a regular cardinal and let M be a countable elementary submodel of $H(\kappa)$ such that $\mathbb{P} \in M$. We say that $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic if for all dense sets $D \subseteq \mathbb{P}$ that belong to M we have $p \Vdash$ "$D \cap M \cap G \neq \emptyset$".*

We say that $\mathbb{P}$ is proper if for every $\kappa > 2^{|\mathbb{P}|}$ there is some $x \in H(\kappa)$ such that for every countable elementary submodel M of $H(\kappa)$ that contains x and $\mathbb{P}$ every $p \in \mathbb{P} \cap M$ has an $(M, \mathbb{P})$-generic extension.

This definition can be altered in a lot of ways to an equivalent definition. For example we only need this to hold for a single κ and omit the extra element x. As we often need a cardinal κ like in this definition we will just say that κ is "large enough" to mean $\kappa > 2^{|\mathbb{P}|}$ for all forcing notions $\mathbb{P}$ considered at that point. There are also many equivalent ways to state that a condition is generic.

Lemma 3.0.2. *Let $\mathbb{P}$ be a forcing notion, $\kappa > 2^{|\mathbb{P}|}$ a regular cardinal and let M be a countable elementary submodel of $H(\kappa)$ such that $\mathbb{P} \in M$. The following are equivalent:*

(1) p is $(M, \mathbb{P})$-generic.

(2) $M \cap D$ is predense below p for all dense $D \subseteq \mathbb{P}$ that belong to M.

(3) $p \Vdash$ "$M[\dot{G}] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$".

(4) $p \Vdash$ "$M[\dot{G}] \cap V = M \cap V$".

Proof. (4) $\Rightarrow$ (3) is trivial. The equivalence between (2) and (1) is simple.

(1) $\Rightarrow$ (4). Let $\dot{x} \in M$ be a $\mathbb{P}$-name. Define $D = \{q \leq p \mid \exists y\, (q \Vdash \text{"}y = \dot{x}\text{"})\}$ and let $f\colon D \to V$ be a function such that $q \Vdash$ "$\dot{x} = f(q)$", for all $q \in D$. Then $f, D \in M$ and D is dense below p. Thus $p \Vdash$ "$D \cap M \cap \dot{G} \neq \emptyset$". Let G be $\mathbb{P}$-generic with $p \in G$. Then there is $q \in D \cap M \cap G$. Now $f(q) \in M$ and $q \Vdash$ "$\dot{x} = f(q)$" so $\dot{x}^G \in M$.

(3) $\Rightarrow$ (1). Let $D \subseteq \mathbb{P}$ be a dense set contained in M. There are some α and $f\colon \alpha \to D$ surjective in M. Let $\dot{\tau}$ be the name for the minimal ordinal τ such that $f(\tau) \in G$. Let G be $\mathbb{P}$-generic with $p \in G$. Then by assumption $\dot{\tau}^G \in M$, so $f(\dot{\tau}^G) \in M \cap G \cap D$. □

As mentioned already the importance of proper forcing lies in the fact that it preserves $\aleph_1$. To show that this is the case we will prove the stronger statement that proper forcings preserve stationary subsets of $\aleph_1$.

Definition 3.0.3. *Let κ be an uncountable regular cardinal. A set $C \subseteq \kappa$ is called club, which stands for closed unbounded set, if it is closed and unbounded in κ. Closed means for every $C' \subset C$ either $\bigcup C' = \kappa$ or $\bigcup C' \in C$.*

A set $S \subseteq \kappa$ is called stationary if it has non empty intersection with every club subset of κ.

There are no stationary subsets of a countable set, as every cofinal sequence is a club. But $\aleph_1$ is stationary in itself, so forcings that collapse $\aleph_1$ can't preserve stationary subsets.

Theorem 3.0.4. *Let $\mathbb{P}$ be proper and let $S \subseteq \aleph_1$ be stationary. Then S is stationary in every forcing extension via $\mathbb{P}$.*

Proof. Fix a $\mathbb{P}$-generic filter G. Let $\dot{C}$ be a $\mathbb{P}$-name for a club set. Define

$$D = \left\{ M \cap \omega_1 \,\middle|\, M \preceq H(\kappa) \text{ is countable and } \dot{C}, \mathbb{P} \in M \right\}.$$

For $M \preceq H(\kappa)$ and a set $x \in M$ that M thinks is countable we have $x \subseteq M$, thus $M \cap \omega_1$ is a countable ordinal. Construct a club subset of D, by recursively constructing models M_α, $\alpha < \omega_1$. Let M_0 be some countable elementary submodel of $H(\kappa)$. If we have already constructed M_α, for $\alpha < \omega_1$, we can get an elementary submodel $M_{\alpha+1}$ of $H(\kappa)$ that contains $M_\alpha \cup \{M_\alpha \cap \omega_1\}$ with the Löwenheim-Skolem Theorem, then we also have $M_\alpha \prec M_{\alpha+1}$. At limit stages γ we take the direct union, which is still an elementary extension of all previous M_α and an elementary submodel of $H(\kappa)$. Then $\{M_\alpha \cap \omega_1 \mid \alpha < \omega_1\}$ is club. So there is a countable $M \preceq H(\kappa)$ with $\dot{C}, \mathbb{P} \in M$ such that $M \cap \omega_1 \in S$. Because $\mathbb{P}$ is proper we find $q \in G$ that is $(M, \mathbb{P})$-generic. Then $M[G] \cap \omega_1 = M \cap \omega_1 \in S$. Also $C \in M[G]$ is a club subset of $\omega_1^{M[G]} = \omega_1$ since $M[G] \preceq H(\kappa)[G]$. But then C is unbounded in $M \cap \omega_1$ and since it is closed this means $M \cap \omega_1 \in C$. So S intersects every club set in $V[G]$ and is therefore stationary. □

Corollary 3.0.5. *If $\mathbb{P}$ is proper then $\aleph_1$ is not collapsed by forcing with $\mathbb{P}$.*

Another nice property is that countable sets in the extension can be covered by countable ground model sets.

Lemma 3.0.6. *Let $\mathbb{P}$ be a proper forcing and let G be a $\mathbb{P}$-generic filter. Then for every countable set of ordinals $A \in V[G]$ there is a countable $B \in V$ such that $A \subseteq B$.*

Proof. Let κ be large enough. Let $\dot{A}$ be a $\mathbb{P}$-name for A. Then there is a countable elementary submodel $M \preceq H(\kappa)$ with $\dot{A} \in M$. So we have $A \in M[G]$ and because A is countable this implies $A \subseteq M[G]$. Since $\mathbb{P}$ is proper every generic filter contains a $(M, \mathbb{P})$-generic condition. Therefore $A \subseteq M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$, so $B = M$ works. □

3.1 Axiom A forcing

Showing properness directly is quite cumbersome, but many of the classical forcings one considers satisfy a stronger property which is rather easy to check. This property was introduced by Baumgartner [4].

Definition 3.1.1. *A forcing notion $\mathbb{P}$ satisfies Axiom A if there is a sequence of partial orders $\{ \leq_n \mid n \in \omega\}$ with the following properties:*

(1) $\leq_0 = \leq$,

(2) $\leq_{n+1} \subseteq \leq_n$,

(3) if $(p_n)_{n\in\omega}$ is a sequence of conditions such that $p_{n+1} \leq_n p_n$ for every n, then there is a condition p such that $p \leq_n p_n$ for every $n \in \omega$ and

(4) for every $p \in \mathbb{P}$, $n \in \omega$ and $\mathbb{P}$-name $\dot{\alpha}$ for an ordinal there exists $q \leq_n p$ and a countable B such that $q \Vdash$ "$\dot{\alpha} \in B$".

A sequence as in (3) is called a fusion sequence and the condition p is called fusion of the sequence.

Lemma 3.1.2. *If we replace (4) with*

(4') For every $p \in \mathbb{P}$, $n \in \omega$ and dense $D \subseteq \mathbb{P}$ there exists $q \leq_n p$ and a countable set $D' \subseteq D$ which is predense below q,

in the definition above we get an equivalent definition.

Proof. First we assume (4). Fix $p \in \mathbb{P}$, $n \in \omega$ and a dense set $D \subseteq \mathbb{P}$. Let $A = \{a_\alpha \mid \alpha < \lambda\}$ be a maximal antichain in D. Then there is a $\mathbb{P}$-name $\dot{\gamma}$ such that $a_\alpha \Vdash$ "$\dot{\gamma} = \alpha$", for all $\alpha < \lambda$. By (4) take $q \leq_n p$ and $B \subseteq \lambda$ such that $q \Vdash$ "$\dot{\gamma} \in B$". Then $A' = \{a_\alpha \mid \alpha \in B\}$ is predense below q. For $r \leq q$ there is $\alpha < \lambda$ such that a_α is compatible with s. Thus there is $s \leq r$ forcing $\dot{\gamma} = \alpha$ and hence $\alpha \in B$.

On the other hand assume (4') holds. Fix $p \in \mathbb{P}$, $n \in \omega$ and a $\mathbb{P}$-name $\dot{\alpha}$ for an ordinal. Let $D = \{r \in \mathbb{P} \mid \exists \alpha_r\, (r \Vdash \text{"}\dot{\alpha} = \alpha_r)\text{"}\}$. This set is dense, so there exists $q \leq_n p$ and a countable set $D' \subseteq D$ predense below q. Let $B = \{\alpha_r \mid r \in D'\}$. Every $s \leq q$ is compatible with some $r \in D'$, thus $q \Vdash$ "$\dot{\alpha} \in B$". □

Theorem 3.1.3. *A forcing $\mathbb{P}$ satisfying Axiom A is proper.*

Proof. Let X be the set of all dense subsets of $\mathbb{P}$ and let X' be the set of all countable subsets of $\mathbb{P}$. Since $\mathbb{P}$ satisfies Axiom A there exists a function

$$\sigma\colon \omega \times \mathbb{P} \times X \to \mathbb{P} \times X'$$

such that if $\sigma(n, p, D) = (q, D')$ then $q \leq_n p$ and $D' \subseteq D$ is predense below q. Now let κ be large enough and let $M \preceq H(\kappa)$ be countable such that $\mathbb{P}, \sigma \in M$. Fix $p_0 \in \mathbb{P} \cap M$. If we find a $(M, \mathbb{P})$-generic extension of p_0 we are done. Let $\{D_n |\ n \in \omega\}$ enumerate all dense subsets of $\mathbb{P}$ that are contained in M. Inductively define $(p_{n+1}, D'_n) = \sigma(n, p_n, D_n)$. Then $p_{n+1}, D'_n \in M$, because $\sigma \in M$. And from the definition of σ we have $p_{n+1} \leq_n p_n$ and $D'_n \subseteq D_n$ is predense below p_{n+1}. Let q be the fusion of the sequence $(p_n)_{n\in\omega}$. Then D'_n is predense below q. And since $D'_n \in M$ is countable we even have $D'_n \subseteq M$, thus $M \cap D_n \supseteq D'_n$ is predense below q. Since we enumerated all dense sets in M this means q is $(M, \mathbb{P})$-generic by condition (2) in 3.0.2. □

Theorem 3.1.4. *Forcings that have the countable chain condition or are σ-closed satisfy Axiom A.*

Proof. First consider a forcing $\mathbb{P}$ with the ccc. Define $\leq_n$, for $n > 0$ as $p \leq_n q$ if and only if $p = q$. Then fusion sequences are constant and the fusion is just this constant value. For the other condition every maximal antichain is countable which means (4') is satisfied by any maximal antichain in D.

Now let $\mathbb{P}$ be a σ-closed forcing. Then let $\leq_n = \leq$, for every $n \in \omega$. A fusion sequence is a decreasing sequence, which by σ-closeness has a lower bound which is a fusion for it. Every condition can be strengthened to a condition forcing a specific value which means (4) is true even if we require B to be a singleton. □

3.2 Iterating proper forcing

A property almost as important as that of preserving cardinals is the preservation of it under iterations. For example the countable chain condition is preserved under finite support iterations, but not under countable support iterations. For example if $p, q \in \mathbb{P}$ are two incompatible conditions, then in the countable support iteration of length ω_1 there is the uncountable antichain $\{r_\alpha |\ \alpha < \omega_1\}$, with $r_\alpha(\beta) = p$ if $\beta < \alpha$ and $r_\alpha(\alpha) = q$. In this section we will show that properness is preserved under countable support iterations.

The first step is to show that 2-step iterations of proper forcings are proper.

Lemma 3.2.1. *Let $\mathbb{P}$ be a forcing and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing. Let κ be large enough and let $M \preceq H(\kappa)$ be a countable elementary submodel with $\mathbb{P}, \dot{\mathbb{Q}} \in M$. Let $p \in \mathbb{P}$ be $(M, \mathbb{P})$-generic. Let $\dot{q}$ be a $\mathbb{P}$-name for an element of $\dot{\mathbb{Q}}$ such that $p \Vdash$ "$\dot{q}$ is $(M[\dot{G}], \dot{\mathbb{Q}})$-generic". Then $(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic.*

Proof. We will use (3) from 3.0.2. Let G be $\mathbb{P}$-generic with $p \in G$. Let H be $\dot{\mathbb{Q}}^G$-generic over $V[G]$ with $\dot{q}^G \in H$. Every $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter containing $(p, \dot{q})$ can be decomposed in this way, so it is enough to consider this case. Then $M \cap \mathrm{Ord} = M[G] \cap \mathrm{Ord} = M[G][H] \cap \mathrm{Ord}$, the first equality is because p is $(M, \mathbb{P})$-generic and the second one because q is $(M[G], \dot{\mathbb{Q}}^G)$-generic. If $\dot{\tau}$ is a $\mathbb{P} * \dot{\mathbb{Q}}$-name there is a definable $\mathbb{P}$-name $\dot{\tau}'$ for a $\dot{\mathbb{Q}}$-name that is interpreted in the same way. Then if $\dot{\tau} \in M$, then there is such $\dot{\tau}'$ also in M. So $M[G * H] \subseteq M[G][H]$ and therefore $M[G * H] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. □

Lemma 3.2.2. *Let $\mathbb{P}$ be a proper forcing and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a proper forcing. Let κ be large enough and let $M \preceq H(\kappa)$ be countable with $\mathbb{P} * \dot{\mathbb{Q}} \in M$. If q_0 is $(M, \mathbb{P})$-generic, $\dot{p}$ is a $\mathbb{P}$-name such that*

$$q_0 \Vdash \text{“}\dot{p} \in M \cap \mathbb{P} * \dot{\mathbb{Q}} \wedge \pi(\dot{p}) \in \dot{G}_0\text{”},$$

where π is the projection from $\mathbb{P} * \dot{\mathbb{Q}}$ onto the first coordinate and $\dot{G}_0$ is the name for the $\mathbb{P}$-generic filter, then there is a $\mathbb{P}$-name $\dot{q}_1$ such that $(q_0, \dot{q}_1)$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic and

$$(q_0, \dot{q}_1) \Vdash \text{“}\dot{p} \in \dot{G}\text{”},$$

where $\dot{G}$ is the name for the $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter.

Proof. Let $\dot{p}_0, \dot{p}_1$ be $\mathbb{P}$-names such that $\dot{p} = (\dot{p}_0, \dot{p}_1)$. Let G be a $\mathbb{P}$-generic filter containing q_0. Then $\dot{\mathbb{Q}}^G$ is proper, so there is $q_1 \leq \dot{p}_1^G$ which is $(M[G], \dot{\mathbb{Q}}^G)$-generic. Let $\dot{q}_1$ be a $\mathbb{P}$-name for this. By the previous lemma $(q_0, \dot{q}_1)$ is $(M, \mathbb{P} * \mathbb{Q})$-generic.

It remains to show that $(q_0, \dot{q}_1) \Vdash$ “$\dot{p} \in \dot{G}$”. Let $G * H$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter containing $(q_0, \dot{q}_1)$. By the assumption we have $\dot{p}_0^G = \pi(\dot{p}^G) \in G$. From the construction of $\dot{q}_1$ we have that $\dot{q}_1^G \leq \dot{p}_1^G$. Thus $\dot{q}_1^G \in H$. In total $\dot{p}^{G*H} \in G * H$. □

Lemma 3.2.3 (Properness Extension Lemma)**.** *Let $\left\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \middle| \alpha \leq \gamma\right\}$ be a countable support iteration of proper forcings, i.e. $\Vdash_\alpha$ “$\dot{Q}_\alpha$ is proper”, for all $\alpha < \gamma$. Let κ be large enough and $M \preceq H(\kappa)$ is countable with $\mathbb{P}_\gamma, \gamma \in M$. If $\gamma_0 \in \gamma \cap M$, q_0 is $(M, \mathbb{P}_{\gamma_0})$-generic and $\dot{p}_0$ is a $\mathbb{P}_{\gamma_0}$-name such that*

$$q_0 \Vdash_{\gamma_0} \text{“}\dot{p}_0 \in M \cap \mathbb{P}_\gamma \wedge \dot{p}_0 \restriction \gamma_0 \in \dot{G}_{\gamma_0}\text{”}$$

then there is a $(M, \mathbb{P}_\gamma)$-generic condition q such that $q \restriction \gamma_0 = q_0$ and

$$q \Vdash_\gamma \text{“}\dot{p}_0 \in \dot{G}_\gamma\text{”}.$$

Proof. The proof is by induction on γ. First consider the case where $\gamma = \gamma' + 1$ is a successor. As $\gamma \in M$ also $\gamma' \in M$. So we can use the induction hypothesis to get a $(M, \mathbb{P}_{\gamma'})$-generic q' with the required properties for γ'. Then the previous Lemma lets us extend q' to a $(M, \mathbb{P}_\gamma)$-generic q with the required properties.

Now assume γ is a limit ordinal and the Lemma holds for all $\gamma' < \gamma$. Let $\{\gamma_n | \, n \in \omega\} \subseteq \gamma \cap M$ be an unbounded increasing sequence. Let $\{D_n | \, n \in \omega\}$ enumerate all dense subsets of $\mathbb{P}_\gamma$ contained in M. We will recursively construct $\dot{p}_n, q_n$, $n \in \omega$, such that for every $n \in \omega$ the following holds:

(1) q_n is a $(M, \mathbb{P}_{\gamma_n})$-generic condition with $q_n \restriction \gamma_{n-1} = q_{n-1}$,

(2) $\dot{p}_n$ is a $\mathbb{P}_{\gamma_n}$-name such that

$$q_n \Vdash_{\gamma_n} \text{“}\dot{p}_n \in M \cap \mathbb{P}_\gamma \wedge \dot{p}_n \restriction \gamma_n \in G_{\gamma_n} \wedge \dot{p}_n \leq \dot{p}_{n-1} \wedge \dot{p}_n \in D_{n-1}\text{”}.$$

For $n = 0$ the conditions hold. Now assume we have defined $\dot{p}_n, q_n$ for some n. Let G_{γ_n} be a $\mathbb{P}_{\gamma_n}$-generic filter with $q_n \in G_{\gamma_n}$. Then there is $p_n \in \mathbb{P}_\gamma \cap M$ such that $p_n = \dot{p}_n^{G_{\gamma_n}}$. Define $D = \{r \restriction \gamma_n | \; r \in \mathbb{P}_\gamma \cap D_n \wedge r \leq p_n\}$. This set is contained in M, dense below p_n and $q_n \in G_{\gamma_n}$ is $(\mathbb{P}_{\gamma_n}, M)$-generic, thus $M \cap D \cap G_{\gamma_n} \neq \emptyset$. Therefore

$$H(\kappa)[G_{\gamma_n}] \models \text{“}\exists x\, (x \in D_n \wedge x \leq p_n \wedge x \restriction \gamma_n \in M \cap G_{\gamma_n})\text{”}.$$

So such x exists in $M[G_{\gamma_n}]$ and since $M[G_{\gamma_n}] \cap V = M \cap V$ we even have such x in M. Let $\dot{p}_{n+1}$ be a name for such an x. Now use the induction hypothesis with γ_{n+1}, q_n and $\dot{p}_{n+1} \restriction \gamma_{n+1}$ to get q_{n+1}. Then (1) is true and $q_{n+1} \Vdash_{\gamma_n} \text{“}\dot{p}_{n+1} \restriction \gamma_{n+1} \in \dot{G}_{\gamma_{n+1}}\text{”}$. All the other things in (2) are true by the choice of $\dot{p}_{n+1}$. This completes the construction of $\dot{p}_n$ and q_n.

Let $q = \bigcup_{n \in \omega} q_n$. This is well defined since the iteration has countable support and $q_n \restriction \gamma_m = q_m$, for all $m < n$. To finish the proof we will show that $q \Vdash_\gamma \text{“}\dot{p}_n \in \dot{G}_\gamma\text{”}$ for all $n \in \omega$. Then q is $(M, \mathbb{P}_\gamma)$-generic since the D_n enumerate all dense subsets of $\mathbb{P}_\gamma$ in M and $q \Vdash_\gamma \text{“}\dot{p}_n \in D_{n-1}\text{”}$. Let G_γ be a $\mathbb{P}_\gamma$-generic filter with $q \in G_\gamma$. Fix some $n \in \omega$. Take $p_n \in M$ such that $\dot{p}_n^{G_\gamma} = p_n$. Then there is some ordinal $\beta \in M$ containing the domain of p_n, since such an ordinal exists in $H(\kappa)$. Take $m \in \omega$ such that $\beta < \gamma_n$. Then $p_n \restriction \gamma_m = p_n$ and since $q_n \Vdash \text{“}\dot{p}_n \in \dot{G}_{\gamma_m}\text{”}$ and $q \leq q_n$ we have $p_n \in G_\gamma$. Thus $q \Vdash \text{“}\dot{p}_n \in G_\gamma\text{”}$. □

Theorem 3.2.4. *Let $\{\mathbb{P}_\alpha | \; \alpha \leq \gamma\}$ be a countable support iteration of proper forcings. Then $\mathbb{P}_\gamma$ is proper.*

Proof. Let κ be a large enough regular cardinal, M a countable elementary submodel of $H(\kappa)$ and $p \in \mathbb{P}_\gamma \cap M$. Without loss of generality $\mathbb{P}_0 = \{0\}$. Then 0 is a $(M, \mathbb{P}_0)$-generic condition. We can view p as a $\mathbb{P}_0$-name for an element of $\mathbb{P}_\gamma$. Then the conditions of the properness extension lemma are satisfied so we get a $(M, \mathbb{P}_\gamma)$-generic condition q such that $q \Vdash \text{“}p \in \dot{G}_\gamma\text{”}$. Since we can also assume that $\mathbb{P}_\gamma$ is seperative this implies that $q \leq p$. □

If we want to force the continuum to be bigger than $\aleph_1$ it is not enough to preserve $\aleph_1$, some cardinal bigger than $\aleph_1$ has to be preserved and we need to add at least that many new reals. The following theorem guarantees that if we iterate with small proper forcings no cardinals will be collapsed.

Theorem 3.2.5. *Assume CH. Let $\{\mathbb{P}_i | \; i \leq \gamma\}$ be a countable support iteration of length $\gamma \leq \omega_2$ of proper forcings of size $\aleph_1$. Then $\mathbb{P}_\gamma$ has the $\aleph_2$-cc.*

This theorem does present us with a new problem. The size of the intermediate forcings has to be $\aleph_1$, but this size has to be computed in the intermediate model. So it could happen that we want to iterate a forcing of size 2^ω, which is $\aleph_1$ in the ground model, but in some intermediate extension we blow up the continuum and the forcing has no longer size $\aleph_1$. That this does not happen is ensured by the next theorem.

Theorem 3.2.6. *Assume CH. Let $\{\mathbb{P}_i |\ i \leq \gamma\}$ be a countable support iteration of length $\gamma < \omega_2$ of proper forcings of size $\aleph_1$. Then CH holds in forcing extensions via $\mathbb{P}_\gamma$.*

4 Real forcing

This chapter will introduce a nice representation for forcings adding a single real and then introduces some of the common notions of forcing and show how they fit into this framework. This notion is called *real forcing* and was introduced by Zapletal, see [40] and [39]. They are forcings of the form $\mathcal{B}(\mathbb{R})/I_{\mathbb{P}}$, where $\mathcal{B}(\mathbb{R})$ is the set of Borel sets of $\mathbb{R}$ and $I_{\mathbb{P}}$ is a σ-ideal, ordered by inclusion modulo $I_{\mathbb{P}}$. By $\mathbb{R}$ we mean either 2^ω or ω^ω. Most general statements, i.e. not related to a specific forcing notion, will be stated only for 2^ω, but are also true for ω^ω with exactly the same proof. We will shorten $\mathcal{B}(\mathbb{R})$ to $\mathcal{B}$. To make work with them a little easier we notice the following, in light of which we will mostly be working with $\mathcal{B} \setminus I_{\mathbb{P}}$, to avoid dealing with equivalence classes.

Theorem 4.0.1. *Let $I_{\mathbb{P}} \subseteq \mathcal{B}$ be a σ-ideal. Then $\mathcal{B}/I_{\mathbb{P}}$ is forcing equivalent to $\mathcal{B} \setminus I_{\mathbb{P}}$, ordered by inclusion.*

Proof. The separative quotient of $\mathcal{B} \setminus I_{\mathbb{P}}$ is $\mathcal{B}/I_{\mathbb{P}}$. Therefore they are forcing equivalent. □

Real forcings add a new real that satisfies the following nice property. We will often call this real the generic real.

Theorem 4.0.2 (Zapletal, [39])**.** *There is a $\mathbb{P} = \mathcal{B} \setminus I_{\mathbb{P}}$-name $\dot{x}$ for a real such that for every generic filter $G \subseteq \mathbb{P}$ and Borel set B coded in V, $\dot{x}^G \in B$ if and only if $B \in G$. This real even has the property that $B \Vdash$ "$\dot{x} \in \dot{B}$" for every Borel set $B \in \mathbb{P}$ coded in V, where $\dot{B}$ is the name for the Borel set in the extension with the same definition as B.*

Proof. Without loss of generality we say $\mathbb{R} = 2^\omega$. Let $x\colon \omega \to 2$ be defined by $x(n) = m$ if and only if $\{f \in 2^\omega |\ f(n) = m\} \in G$. Since for any n the sets $\{f \in 2^\omega |\ f(n) = 0\}$ and $\{f \in 2^\omega |\ f(n) = 1\}$ form a maximal antichain, we have that this is indeed a total function. It might be the case that one of these sets is not in $\mathbb{P}$, but then the other one is a maximal condition. Now we will show the second statement by induction on the complexity of Borel sets, from this the first one follows. If B is closed or a basic open set then this is clear. Next assume $B = \bigcup_{n\in\omega} B_n$ and $B_n \Vdash$ "$\dot{x} \in \dot{B}_n$" for all $n \in \omega$ such that $B_n \in \mathbb{P}$. Because $I_{\mathbb{P}}$ is σ-closed we have $\bigcup \{B_n |\ B_n \in I_{\mathbb{P}}\} \in I_{\mathbb{P}}$. Thus $\{B_n |\ B_n \in \mathbb{P}\}$ is pre dense below B and therefore $B \Vdash$ "$\dot{x} \in \dot{B}$". Finally let $B = \bigcap_{n\in\omega} B_n$ and $B_n \Vdash$ "$\dot{x} \in \dot{B}_n$" for all $n \in \omega$, here we have that $B_n \in \mathbb{P}$ for all n, since $B \subseteq B_n$. Then $B \subseteq B_n$, for every $n \in \omega$, therefore $B \Vdash$ "$\dot{x} \in \dot{B}_n$". So $B \Vdash$ "$\dot{x} \in \bigcap_{n\in\omega} \dot{B}_n = \dot{B}$". □

We want our real forcings to have another property:

$$\begin{gathered}\text{If } B \subseteq 2^{<\omega} \text{ with } G_B \in \mathbb{P} \text{ and } E \leq G_B,\\ \text{then there is } B' \subseteq B \text{ such that } G_{B'} \in \mathbb{P} \text{ and } G_{B'} \leq E.\end{gathered} \tag{4.1}$$

This property follows from the continuous reading of names, see [40, Proposition 3.1.9]. There it is also shown that all real forcings generated by closed sets have the continuous reading of names. All the forcings we will consider are of this form.

4.1 Sacks forcing

We start our look at common forcing notions with Sacks forcing. As the name suggest it was introduced by Sacks [35]. Sacks forcing $\mathbb{S}$ is the set of perfect trees on $2^{<\omega}$, sometimes also called Sacks trees, ordered by inclusion. We say a node is a splitting node if both immediate successors are in the tree as well, the set of splitting nodes is denoted by $\operatorname{split}(T)$ and $\operatorname{split}_n(T)$ are all splitting nodes that have n splitting nodes below them. A perfect tree is a tree in which there is a splitting node above every node. To get a representation of Sacks forcing as real forcing we use the perfect set theorem, the proof given is due to Solovay, see [23, Exercise 29.2].

Theorem 4.1.1 (Souslin)**.** *For every analytic set $A \subseteq 2^\omega$, either A is countable, or it contains a perfect subset.*

Proof. Let $A \subseteq 2^\omega$ be an analytic set. Then there is a tree $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ such that $A = p[T]$, where p is the projection of branches of T onto the first coordinate. For trees $S \subseteq 2^{<\omega} \times \omega^{<\omega}$ define a tree

$$S' = \left\{(s,u) \in S \middle| \exists (t,v),(r,w) \in S\big((s,u) \subseteq (t,v),(r,w) \wedge t \perp r\big)\right\}.$$

Recursively define $T_0 = T$, $T_{\alpha+1} = (T_\alpha)'$ and for limit ordinals λ, $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$. Since T is countable there is some $\gamma < \omega_1$ such that $T_\gamma = T_{\gamma+1}$. Let $T_\infty = T_\gamma$. Consider two cases.

First assume $T_\infty \neq \emptyset$. Recursively construct a set $S = \{s_\sigma \mid \sigma \in 2^{<\omega}\}$ that generates a perfect tree and a set $\{u_\sigma \mid \sigma \in 2^{<\omega}\} \subseteq \omega^{<\omega}$ such that $(s_\sigma, u_\sigma) \in T_\infty$ for every $\sigma \in 2^{<\omega}$. Let $(s_0, u_0) \in T_\infty$ be arbitrary. If we have defined (s_σ, u_σ) there have to be $(s_{\sigma^\frown 0}, u_{\sigma^\frown 0}) \in T_\infty$ and $(s_{\sigma^\frown 1}, u_{\sigma^\frown 1}) \in T_\infty$ such that $(s_{\sigma^\frown 0}, u_{\sigma^\frown 0}), (s_{\sigma^\frown 1}, u_{\sigma^\frown 1}) \supseteq (s_{\sigma^\frown 1}, u_{\sigma^\frown 1})$ and $s_{\sigma^\frown 0} \perp s_{\sigma^\frown 1}$, since $(T_\infty)' = T_\infty$. Then $[S] \subseteq p[T] = A$ is a perfect set.

On the other hand assume that $T_\infty = \emptyset$. For $(s,u) \in T_\alpha \backslash T_{\alpha+1}$ all extensions $(t,v) \in T_\alpha$ have compatible first coordinates, thus there is at most one $x_{(s,u)} \in p[T_\alpha(s,u)]$. All $x \in p[T_\alpha] \setminus p[T_{\alpha+1}]$ are of this form. Thus $p[T_\alpha] \setminus p[T_{\alpha+1}]$ is countable. Since γ is countable this means that $A = p[T_0]$ is countable. □

Because of this Sacks forcing is a dense subset of $\mathcal{B}(2^\omega) \setminus \mathbf{cntble}$, where $\mathbf{cntble}$ is the ideal of countable sets. Thus they are forcing equivalent.

Theorem 4.1.2. *Sacks forcing satisfies Axiom A, in particular Sacks forcing is proper.*

Proof. For $n \in \omega$ and conditions $T, T' \in \mathbb{S}$ define $T \leq_n T'$ if and only if $T \leq T'$ and $\operatorname{split}_n(T) = \operatorname{split}_n(T')$. Then if $(T_n)_{n\in\omega}$ is a fusion sequence we have that $T = \bigcap_{n\in\omega} T_n$ is a condition, as the n-th splitting level of T is the same as the n-th splitting level of

T_n, and because of this it is also clear that it is a fusion of the sequence. Now let $T \in \mathbb{S}$, $n \in \omega$ and let $\dot{\alpha}$ be a name for an ordinal. For every $s \in \mathrm{split}_n(T)$ there is a condition $T_s \leq T(s)$ that decides $\dot{\alpha}$ to be some α_s. Define $S = \bigcup_{s \in \mathrm{split}_n(T)} T_s$, this is a perfect tree and the n-th splitting level is the same as in T, so $S \leq_n T$. Also $\{T_s |\ s \in \mathrm{split}_n(T)\}$ is dense below S, so $S \Vdash$ "$\dot{\alpha} \in \{\alpha_s |\ s \in \mathrm{split}_n(T)\}$". □

4.2 Miller forcing

Miller forcing $\mathbb{M}$ is the set of rational perfect trees ordered by inclusion. It was introduced by Miller [32].

Definition 4.2.1. *A tree $T \subseteq \omega^{<\omega}$ is called rational perfect if every node has an extension that has infinitely many immediate successors.*

A set $B \subseteq \omega^\omega$ is called σ-bounded if there is a countable set $X \subseteq \omega^\omega$ such that for all $y \in B$ there is some $x \in X$ such that $y \leq x$.

A node is called splitting if it has infinitely many direct successors in the tree, the set of splitting nodes is defined as for Sacks forcing. Let $\mathcal{K}_\sigma$ be the ideal of σ-bounded sets. The following result shows that Miller forcing is equivalent to $\mathcal{B}(\omega^\omega) \setminus \mathcal{K}_\sigma$.

Theorem 4.2.2 (Kechris, [22])**.** *Every analytic $A \subseteq \omega^\omega$ is either σ-bounded or it contains a rational perfect subset.*

Proof. This proof works very similar to the one for perfect sets. Let $A \subseteq \omega^\omega$ be an analytic set. Then there is a tree $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ such that $A = p[T]$. For trees $S \subseteq \omega^{<\omega} \times \omega^{<\omega}$ define a tree

$$S' = \{(s,u) \in S | \exists \{(t_n, v_n)|\ n \in \omega\} \subseteq S \exists l \in \omega$$
$$(t_n, v_n) \supseteq (s,u) \wedge |(t_n, s_n)| = l \wedge t_n(l-1) \geq n \wedge t_n \upharpoonright (l-1) = t_m \upharpoonright (l-1)\}.$$

Recursively define $T_0 = T$, $T_{\alpha+1} = (T_\alpha)'$ and for limit ordinals λ, $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$. Then there is $\gamma < \omega_1$ such that $T_\gamma = T_{\gamma+1}$. Let $T_\infty = T_\gamma$. In the case that $T_\infty \neq \emptyset$ construct a rational perfect set contained in A in the same way as in the proof for perfect sets. Thus assume now that $T_\infty = \emptyset$. For $(s,u) \in T_\alpha \setminus T_{\alpha+1}$ and $n \in \omega$ there is $g_{(s,u)}(n)$ such that all extensions $(t,v) \in T_\alpha$ of length n satisfy $t(n-1) < g_{(s,u)}(n)$. Thus all $x \in p[T_\alpha(s,u)]$ are dominated by $g_{(s,u)}$. Since γ is countable this means that we can build a countable set of such function that witnesses that A is σ-bounded. □

Theorem 4.2.3. *Miller forcing satisfies Axiom A, in particular Miller forcing is proper.*

Proof. For $n \in \omega$ and Miller trees T, S define $T \leq_n S$ if and only if $T \subseteq S$ and the first n splitting levels are the same. If $(T_n)_{n\in\omega}$ is a fusion sequence define the fusion $T = \bigcap_{n\in\omega} T_n$. For $t \in T$ on the $(n-1)$-th level we have $t \in T_n$ and there is a node $s \in T_n$ on the $(n-1)$-th splitting level extending t. Then there are infinitely many i such that $s^\frown i \in T_n$ and all of them have a node on the n-th splitting level extending them which are all in T, therefore s is a splitting node in T. Also $T \leq_n T_n$ for every

$n \in \omega$. Let $T \in \mathbb{M}$, $n \in \omega$ and let $\dot{\alpha}$ be an $\mathbb{M}$-name for an ordinal. Enumerate the n-th splitting level as $\{t_m |\ m \in \omega\}$. For every $m \in \omega$ choose $T_m \leq T(t_m)$ and an ordinal α_m such that $T_m \Vdash$ "$\dot{\alpha} = \alpha_m$". Then $S = \bigcup_{m\in\omega} T_m$ is a condition that is $\leq_n$ below T and forces that $\dot{\alpha}$ is one of the α_m. □

4.3 Laver Forcing

Laver forcing $\mathbb{L}$ is the set of Laver trees ordered by inclusion, it was introduced by Laver [27].

Definition 4.3.1. *A Laver tree is a tree $T \subseteq \omega^\omega$ such that there is a stem $t \in T$ with which all nodes are compatible and all nodes $s \supseteq t$ have infinitely many immediate successors in T.*

A set $B \subseteq \omega^\omega$ is called strongly dominating if for every function $\phi\colon \omega^{<\omega} \to \omega$ there is $g \in B$ such that for all but finitely many n, $g(n) \geq \phi(g \restriction n)$.

Let **not − dominating** be the ideal of not strongly dominating sets of reals. Then we have that Laver forcing is equivalent to $\mathcal{B}(\omega^\omega) \setminus$ **not − dominating**, because of the following theorem.

Theorem 4.3.2 ([8])**.** *Every Borel set $A \subseteq \omega^\omega$ is either not strongly dominating or contains a Laver tree.*

Proof. Let A be a Borel set. We will define a game. In turn m player II plays a natural number n_m. Player I plays as first move some sequence $s \in \omega^{<\omega}$ and in turn $m > 0$ some natural number $s_m > n_{m-1}$. Let $x = s^\frown s_0^\frown s_1 \ldots$. Then I wins if and only if $x \in A$. Because Borel sets are determined either I or II has a winning strategy.

If I has a winning strategy let T be the tree of all plays that I makes according to this strategy. Then if $s \in T$ player II can answer with any natural number n forcing I to play a continuation greater than n. Thus T is a Laver tree. Since all plays in T are made according to a winning strategy for I we have $[T] \subseteq A$.

On the other hand assume II has a winning strategy. We have to show that A is not strongly dominating. Define $\phi\colon \omega^{<\omega} \to \omega$ by sending s to the highest natural n that II plays if I plays s. Notice that there are different ways for I to play s as she can vary how much of it she plays in the first turn. Now for $x \in A$ and $n \in \omega$ there is a maximal play according to II's winning strategy where I starts by playing $x \restriction n$ and then continues to play x. The reason for this has to be that at his final turn II plays some natural number m bigger than the next number from x, say $x(k)$, that I would have to play. Then $\phi(x \restriction k) \geq m > x(k)$. Thus ϕ witnesses that A is not strongly dominating. □

Theorem 4.3.3. *Laver forcing satisfies Axiom A, in particular Laver forcing is proper.*

Proof. The proof is the same as for Miller forcing. In particular the ordering $T \leq_n S$ is given by $T \leq S$ and the n-th splitting level of T is the same as the n-th splitting level of S. □

4.4 Cohen forcing

Cohen forcing $\mathbb{C}$ is the set of finite partial functions $\omega \to 2$ ordered by reverse inclusion. This forcing was introduced by Cohen in his paper that introduced the method of forcing [11]. Let $\mathcal{M}$ be the ideal of meager sets, that is countable unions of nowhere dense sets. Then Cohen forcing is equivalent to $\mathcal{B}(2^\omega) \setminus \mathcal{M}$.

Theorem 4.4.1. *For every non meager Borel set $A \subseteq 2^\omega$ there is $s \in 2^{<\omega}$ such that $[s] \setminus A$ is meager.*

This theorem is due to Banach and Mazur, for a proof see [33] or [23].

Proof. We start by considering a game for arbitrary Borel sets A. Both players alternate playing finite sequences $s_n \in 2^{<\omega}$. Let $s = s_0^\frown s_1^\frown \dots$ and say that I wins if and only if $s \in A$.

Assume first that II has a winning strategy τ. We will show that then A is meager. For a partial play $p = (s_0, s_1, \dots, s_{2n-1})$ let $p_* = s_0^\frown s_1^\frown \dots ^\frown s_{2n-1}$ and define

$$D_p = \left\{ x \in 2^\omega \,\middle|\, p_* \subseteq x \to \exists t \in 2^{<\omega} \left(p_*^\frown t^\frown \tau(p^\frown(t)) \subseteq x \right) \right\}.$$

Claim. *D_p is open and dense.*

Proof. We have

$$D_p = (2^\omega \setminus [p_*]) \cup \bigcup \left\{ [p_*^\frown t^\frown \tau(p^\frown(t))] \,\middle|\, t \in 2^{<\omega} \right\},$$

so it is a union of open sets and therefore open. To see that it is dense take $s \in 2^{<\omega}$. If $s \not\supseteq p_*$ any extension of s that is not an extension of p_* is in D_p. And if $s \supseteq p_*$ take any $t \neq \emptyset$ such that $s \subseteq p_*^\frown t$. Then any extension of $p_*^\frown t^\frown \tau(p^\frown(t))$ is in D_p. □

If $x \in \bigcap_p D_p$, then we can recursively define a play $(s_0, s_i, \dots)$ according to τ such that $x = s_0^\frown s_1^\frown \dots$. As τ is a winning strategy for II this means $x \notin A$. Therefore $A \subseteq \bigcup_p (2^\omega \setminus D_p)$ and since the complement of a dense open set is nowhere dense this means that A is meager.

In the case that I has a winning strategy, let $s \in 2^{<\omega}$ be her first move. Then consider the game with payoff set $[s] \setminus A$. It is easy to see that II has a winning strategy by just doing what I would have done in the original game. Thus the other case shows that $[s] \setminus A$ is meager.

Finally consider a non meager Borel set A. Then the game with payoff set A is determined. I can't have a winning strategy as then A would be non meager. So II has a winning strategy, which means there is an $s \in 2^{<\omega}$ such that $[s] \setminus A$ is meager. □

Theorem 4.4.2. *Cohen forcing has the ccc. This also means it satisfies Axiom A and is proper.*

Proof. This is clear as Cohen forcing is countable. □

4.5 Hechler forcing

Hechler forcing is the poset

$$\mathbb{D} = \left\{ (s, f) \middle| \, s \in \omega^{\uparrow<\omega}, f \in \omega^{\uparrow\omega}, s \subseteq f \right\}$$

ordered by

$$(s, f) \leq (t, g) \iff s \supseteq t \wedge \forall n \, (f(n) \geq g(n)) \, ,$$

where $\omega^{\uparrow\omega}$ is the set of increasing functions from ω to ω, similarly define $\omega^{\uparrow<\omega}$. This forcing was introduced by Hechler [16]. Like Cohen forcing the ideal for Hechler forcing will be the set of Meager sets, but for a different topology.

Definition 4.5.1.

- *For $(s, f) \in \mathbb{D}$ let $U_{(s,f)} = \left\{ x \in \omega^{\uparrow\omega} \middle| \, s \subseteq x, x \geq f \right\}$.*
- *The dominating topology $\mathcal{D}$ on $\omega^{\uparrow\omega}$ is the topology generated by the sets $U_{(s,f)}$, where $(s, f) \in \mathbb{D}$.*

Let $\mathcal{M}_\mathcal{D}$ be the ideal of $\mathcal{D}$-meager subsets of $\omega^{\uparrow\omega}$, i.e. countable unions of nowhere $\mathcal{D}$-dense sets. Then Hechler forcing is a dense subset of $\mathcal{B}(\omega^{\uparrow\omega})/\mathcal{M}_\mathcal{D}$, so they are forcing equivalent. For more details see [26].

Theorem 4.5.2. *Hechler forcing has the ccc, in particular it satisfies Axiom A and is proper.*

Proof. For $s \in \omega^{\uparrow<\omega}$ let $D_s \subseteq \mathbb{D}$ be the set of all conditions that have s as fist coordinate. Then any two elements of D_s are compatible. Thus every element of an antichain is in a different D_s. As $\omega^{\uparrow<\omega}$ is countable this means every antichain is countable. □

4.6 Random forcing

Let $\mathcal{N}$ be the ideal of Lebesgue null sets. Then Random forcing $\mathbb{B}$ is $\mathcal{B}(2^\omega) \setminus \mathcal{N}$. This was introduced by Solovay [37]. Sadly he was good enough at naming things which is why this forcing notion is not named after its creator. We will state some general facts about Lebesgue measure that will be helpful, for proofs see [23]. Let μ be the Lebesgue measure.

Theorem 4.6.1. *For any measureable set $A \subseteq 2^\omega$*

$$\begin{aligned} \mu(A) &= \inf \{\mu(U) | \, U \supseteq A, U \text{ open}\} \\ &= \sup \{\mu(F) | \, F \subseteq A, F \text{ closed}\} \end{aligned}$$

This means we can approximate measurable sets from below with closed and from above with open sets.

Theorem 4.6.2. *Every analytic set is measurable.*

Theorem 4.6.3. *Random forcing has the ccc, in particular it satisfies Axiom A and is proper.*

Proof. If there were an uncountable antichain, then there is some n such that infinitely many member of this antichain have measure greater than $\frac{1}{n}$. From those we can take $n+1$ many and shrink them to have empty intersection, while preserving their measure. This contradicts the fact that the measure of the whole is 1. □

5 Forcing indestructibility of MAD families

5.1 MAD families and tall ideals

We now come to the main object of this thesis, maximal almost disjoint families, also called MAD families. An almost disjoint family is a set $\mathcal{A} \subseteq [\omega]^\omega$ such that any distinct $A, B \in \mathcal{A}$ have finite intersection. Then MAD families are infinite almost disjoint families that are maximal with respect to inclusion. It is an easy application of Zorn's Lemma that such families exist. A closely related notion is that of tall ideals. An ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called tall if the dual filter $\mathcal{I}^* = \{\omega \setminus I \mid I \in \mathcal{I}\}$ doesn't have a pseudo-intersection. A pseudo-intersection of a family $\mathcal{F} \subseteq [\omega]^\omega$ is a set $A \subseteq \omega$ that is almost contained in every element of $\mathcal{F}$, that means $A \setminus F$ is finite for every $F \in \mathcal{F}$.

Then we can associate to every family $\mathcal{A}$ of subsets of ω an ideal

$$\mathcal{I}(\mathcal{A}) = \left\{ A \subseteq \omega \,\middle|\, \exists n \exists \{A_i \mid i < n\} \subseteq \mathcal{A} \left(A \subseteq^* \bigcup_{i<n} A_i \right) \right\},$$

to get the following characterization of MAD families. Here $A \subseteq^* B$ stands for A is almost a subset of B, i.e. $A \setminus B$ is finite.

Theorem 5.1.1. *Let $\mathcal{A}$ be an almost disjoint family. Then $\mathcal{A}$ is MAD if and only if $\mathcal{I}(\mathcal{A})$ is tall.*

Proof. If $\mathcal{A}$ is not MAD then there is some $A \subseteq \omega$ that is almost disjoint from every element of $\mathcal{A}$. This set is a pseudo-intersection of $\mathcal{I}(\mathcal{A})^*$, so it is not a tall ideal. On the other hand if A is a pseudo intersection of $\mathcal{I}(\mathcal{A})^*$, then A is almost disjoint from every element of $\mathcal{A}$, so this is not a MAD family. □

Definition 5.1.2. *Let $A \subseteq 2^{<\omega}$. Define the G_δ closure of A as*

$$G_A = \left\{ x \in 2^\omega \,\middle|\, \exists^\infty n (x \restriction n \in A) \right\}.$$

We use this notion to give another characterization of tall ideals.

Theorem 5.1.3. *Let $\mathcal{I}$ be an ideal. The following are equivalent:*

(1) $\mathcal{I}$ is tall.

(2) For all $B \subseteq 2^{<\omega}$ and (injective) functions $f \colon B \to \omega$ the family $\left\{ G_{f^{-1}[I]} \,\middle|\, I \in \mathcal{I} \right\}$ covers G_B.

(3) For all (injective) functions $f \colon 2^{<\omega} \to \omega$ the family $\left\{ G_{f^{-1}[I]} \,\middle|\, I \in \mathcal{I} \right\}$ covers 2^ω.

Proof. Let us first show (1) implies (2). Assume that (2) does not hold and let $B \subseteq 2^{<\omega}$ and $f\colon B \to \omega$ witness this. Then there is some $x \in G_B \setminus \bigcup_{I\in\mathcal{I}} G_{f^{-1}[I]}$. Let $A = \{f(x \restriction n) |\ x \restriction n \in B\}$. If A were finite there would be $a \in A$ such that $f^{-1}[a]$ were infinite, but then $x \in G_{f^{-1}[\{a\}]}$ and as $\{a\} \in \mathcal{I}$ this is a contradiction. Thus A is infinite. If there is $I \in \mathcal{I}$ such that $A \cap I$ is infinite, then $x \in G_{f^{-1}[I]}$. So $A \cap I$ is finite, for every $I \in \mathcal{I}$. Thus A witnesses that $\mathcal{I}$ is not tall.

That (2) implies (3) is trivial. Assume $\mathcal{I}$ were not tall and let $A \in [\omega]^\omega$ be such that $A \cap I$ is finite for every $I \in \mathcal{I}$. Then there is an injective function $f\colon 2^{<\omega} \to \omega$ and a real $x \in 2^\omega$ such that $f(x \restriction n) \in A$, for every $n \in \omega$. But then x is not covered by $\{G_{f^{-1}[I]} |\ I \in \mathcal{I}\}$.

□

Lemma 5.1.4. *Let $\mathcal{A} \subseteq [\omega]^\omega$ be such that $\mathcal{F} = \{G_{f^{-1}[A]} |\ A \in \mathcal{A}\}$ is a disjoint covering of 2^ω, for every injective function $f\colon 2^{<\omega} \to \omega$. Then $\mathcal{A}$ is a MAD family.*

Proof. Assume towards a contradiction that $\mathcal{A}$ is not almost disjoint and take $A, B \in \mathcal{A}$ witnessing this. Let $f\colon 2^{<\omega} \to \omega$ be injective and $x \in 2^\omega$ such that f bijects $\{x \restriction n |\ n \in \omega\}$ onto $A \cap B$. Then $x \in G_{f^{-1}[A]} \cap G_{f^{-1}[B]}$, a contradiction. Thus $\mathcal{A}$ is an almost disjoint family and that it is maximal follows from 5.1.1 and 5.1.3. □

5.2 Weak fusion

The following notion was introduced by Jörg Brendle and Shunsuke Yatabe in [9]. Unless otherwise stated the results in this chapter are due to them.

Definition 5.2.1. *A real forcing $\mathbb{P} = \mathcal{B} \setminus I_\mathbb{P}$ with the continuous reading of names has weak fusion if the following holds:*

For every $E \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{C}$, such that $E \Vdash$ "$\dot{C} \in [\omega]^\omega$", there are

- *pairwise disjoint antichains $B_n \subseteq 2^{<\omega}$,*
- *antichains $\mathcal{A}_n \subseteq \mathbb{P}$,*
- *one-to-one functions $h_n\colon B_n \to \mathcal{A}_n$ and*
- *$g\colon \bigcup_{n<\omega} n \times \mathcal{A}_n \to \omega$ one-to-one such that $g(n, A) \geq n$*

such that

(1) $G_B \leq E$,

(2) $\forall B' \subseteq B$ *with* $G_{B'} \in \mathbb{P}\, \forall k \in \omega \exists n \geq k \exists s \in B_n \cap B'$
- $[s] \cap G_{B'} \in \mathbb{P}$,
- $[s] \cap G_{B'}$ *is compatible with* $h_n(s)$,

(3) $\forall n \forall A \in \mathcal{A}_n \big(A \Vdash$ "$g(n, A) \in \dot{C}$"$\big)$,

where $B = \bigcup_{n<\omega} B_n$.

Instead of (2) we can also require

(2') $\forall n \forall s \in B_n \big([s] \cap G_B \leq h_n(s)\big)$.

Assume this holds and let $B' \subseteq B$ with $G_{B'} \in \mathbb{P}$ and $k \in \omega$. Then $G_{B'} = \bigcup\{[s] \cap G_B | s \in B' \cap \bigcup_{k \geq n} B_k\}$. Now if neither element in this union is a condition, then $G_{B'}$ is also not a condition since $I_{\mathbb{P}}$ is σ-closed. So there is some $k > n$ and $s \in B_n \cap B'$ such that $[s] \cap G_B \in \mathbb{P}$. By (2') this condition is stronger than $h_n(s)$ and therefore compatible with it, so (2) holds.

Now we can give a purely combinatorial characterization of forcing indestructible tall ideals.

Theorem 5.2.2. *Let* $\mathbb{P} = \mathcal{B}/I_{\mathbb{P}}$ *be a real forcing that has weak fusion and the continuous reading of names. Let* $\mathcal{I}$ *be a tall ideal. Then the following are equivalent:*

(1) $\mathcal{I}$ *is* $\mathbb{P}$*-indestructible.*

(2) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin I_{\mathbb{P}}$ $\forall f \colon B \to \omega \exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$.

(3) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin I_{\mathbb{P}}$ $\forall f \colon B \to \omega$ *injective* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$.

Proof. We begin by showing (1) implies (2). Assume (2) is false and take $B \subseteq 2^{<\omega}$ and $f \colon B \to \omega$ witnessing this, i.e. $G_B \notin I_{\mathbb{P}}$ and $G_{f^{-1}[I]} \in I_{\mathbb{P}}$, for every $I \in \mathcal{I}$. Let $\dot{x}$ be a name for the generic real from theorem 4.0.2 and define $\dot{A} = \{f(\dot{x} \restriction n) | \; \dot{x} \restriction n \in B\}$. As $G_B \Vdash$ "$\dot{x} \in G_B$" it also forces that $\dot{A}$ is infinite. Take $I \in \mathcal{I}$. Since $G_{f^{-1}[I]} \in I_{\mathbb{P}}$ we have that $G_B \setminus G_{f^{-1}[I]} = G_B$, here we use the fact that $\mathcal{B} \setminus I_{\mathbb{P}}$ is forcing equivalent to $\mathcal{B}/I_{\mathbb{P}}$. Thus $G_B \Vdash$ "$\dot{x} \notin G_{f^{-1}[I]}$" and therefore it also forces that $\dot{A} \cap I$ is finite. So $\dot{A}$ witnesses the destruction of $\mathcal{I}$.

That (2) implies (3) is trivial.

To show (3) implies (1) we use weak fusion. Take $E \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{C}$ such that $E \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". We want to show that E does not force that $\dot{C}$ destroys $\mathcal{I}$. Let $B_n, \mathcal{A}_n, h_n$ and g be as in the definition of weak fusion. Define $f \colon B \to \omega$ by sending $s \in B_n$ to $g(n, h_n(s))$. This function is injective as both g and all the h_n are. So from (3) we get $I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$. To show that $G_{f^{-1}[I]}$ forces $\dot{C} \cap I$ to be infinite it is enough to show

$$\forall F \subseteq G_{f^{-1}[I]} \forall k \exists n > k \exists F' \leq F\big(F' \Vdash \text{“} n \in \dot{C} \cap I \text{”}\big).$$

Since forcings with the continuous reading of names have property 4.1 we can find $B' \subseteq f^{-1}[I]$ such that $G_{B'} \in \mathbb{P}$ and $G_{B'} \leq F$. Now from (2) in weak fusion we get $n > k$ and $s \in B_n \cap B'$ such that $[s] \cap G_{B'}$ is compatible with $h_n(s)$. Let F' be stronger than both of these. Then $F' \Vdash$ "$f(s) \in \dot{C}$", since $h_n(s)$ does this, but also $s \in f^{-1}[I]$ and therefore $f(s) \in I$. So there is a condition stronger than E that forces that $\dot{C}$ does not destroy $\mathcal{I}$. Thus E does not force that $\dot{C}$ destroys $\mathcal{I}$. $\square$

Now it just remains to show that all the forcing notions we introduced earlier do indeed satisfy weak fusion. We start with Sacks forcing, since most of the other arguments are just more complicated versions of this one.

Theorem 5.2.3. *Sacks forcing has weak fusion.*

Proof. Let $T \in \mathbb{S}$ be a condition and $\dot{C}$ an $\mathbb{S}$-name such that $T \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". Recursively construct a fusion sequence $(T_n)_{n\in\omega}$, such that for every $n > 0$ and $s \in \mathrm{split}_n(T_n)$ there is k_s such that $T_n(s) \Vdash$ "$k_s \in \dot{C}$". Let $T_0 = T$. Assume now we already constructed T_n and k_s, for $s \in \mathrm{split}_n(T_n)$. For $s \in \mathrm{split}_n(T_n)$ and $i < 2$ let $T_{s,i} \leq T_n(s^\frown i)$ be a Sacks tree and $k_{s,i} > n$ such that $T_{s,i} \Vdash$ "$k_{s,i} \in \dot{C}$". Since at every step we have only constructed finitely many $k_{s',i'}$ we can make sure that they are all different. Let $T_{n+1} = \bigcup_{s\in\mathrm{split}_n(T_n), i<2} T_{s,i}$. This tree has the same n-th splitting level as T_n and for every $s \in \mathrm{split}_{n+1}(T_{n+1})$ there is some $s' \in \mathrm{split}_n(T_n)$ and $i < 2$ such that $T_{n+1}(s) = T_{s',i}$ therefore $T_{n+1}(s) \Vdash$ "$k_s \in \dot{C}$", for $k_s = k_{s',i}$. This completes the construction. Let $T_\infty = \bigcap_{n\in\omega} T_n$. From this we easily get weak fusion. Let $B_n = \mathrm{split}_n(T_\infty)$, $\mathcal{A}_n = \{T_\infty(s) |\ s \in \mathrm{split}_n(T_\infty)\}$, $h(s) = T_\infty(s)$ and $g(n, T_\infty(s)) = k_s$. Then $G_B = T_\infty$, therefore (1) and (3). And for $s \in B_n$ we have $[s] \cap G_B = T_\infty(s)$, which gives (2'). □

Theorem 5.2.4. *Miller forcing has weak fusion.*

Proof. The difficulty in adapting the previous proof is that g has to be injective, which yields problem if we would add all infinitely many successors of splitting nodes directly. Thus at each stage we will only specify which nodes will be splitting nodes in the final tree and for each of them add one extra direct successor at each later step.

Let T be a Miller tree and $\dot{C}$ a name such that $T \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". Recursively construct $S_n \subseteq \omega^{<\omega}$ and Miller trees T_n. Let $T_0 = T$ and $S_0 = \{\mathrm{stem}(T)\}$. Assume we have constructed S_n and T_n. Fix $s \in S_n$. Let $i_s < \omega$ be the n-th number such that $s^\frown i_s \in T_n$. Let t_s be the stem of a tree $T_{t_s} \leq T_n(s^\frown i_s)$ such that $T_{t_s} \Vdash$ "$k_{t_s} \in \dot{C}$" for some $k_{t_s} \in \omega$ greater than the number of initial segments of s in S_n. Since we have only constructed finitely many integers so far, we can assume that k_{t_s} is distinct from all of them. Unfix s. Let $S_{n+1} = S_n \cup \{t_s |\ s \in S_n\}$ and $T_{n+1} = \bigcup_{s\in S_n} T_{t_s} \cup \left\{t \in T_n \middle|\ \forall s \in S_n \left(t \not\supseteq s^\frown i_s\right)\right\}$. This completes the construction. Let T_∞ be the downwards closure of $\bigcup_{n\in\omega} S_n$. The sets S_n are increasing and at every step we add a direct successor to every element of S_n, thus T_∞ is a Miller tree. Let $B_n = \mathrm{split}_n(T_\infty)$. This is a subset of $\bigcup_{n\in\omega} S_n$, so we can define $h_n(s) = T_s$ and $g(n, T_s) = k_s$. Then $\mathcal{A}_n = h_n[B_n]$ is an antichain, since B_n is an antichain and $\mathrm{stem}(h_n(s)) = s$, for every $s \in B_n$. This also implies that h_n is injective. That g is injective was made sure of in the choice of k_{t_s} in the construction. Also (1), (2') and (3) are satisfied. □

Theorem 5.2.5. *Laver forcing has weak fusion.*

Proof. Let $T \in \mathbb{L}$ be a Laver tree and let $\dot{C}$ be a $\mathbb{L}$-name such that $T \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". We will recursively construct pairwise disjoint antichains $B'_n \subseteq \omega^{\omega}$, antichains $\mathcal{A}'_n \subseteq \mathbb{L}$, bijective functions $h'_n \colon B'_n \to \mathcal{A}'_n$ and a function $g \colon \bigcup_{n\in\omega}\{n\} \times \mathcal{A}'_n \to \omega$ with $g(n, A) \geq n$ such that

- if $n < m$ and $s \in B'_m$, then $s \restriction k \in B'_n$ for some $k < |s|$,
- $h'_n(s)$ is a Laver subtree of T with stem s,
- for all $n \in \omega$ and $s \in B'_n$, $\bigcup \left\{ h'_{n+1}(t) \middle| \, s \subseteq t \in B'_{n+1} \right\}$ is a Laver subtree of $h'_n(s)$ with stem s,
- if $s^\frown i, s^\frown j \in B'_n$, then $g'_n(h'_n(s^\frown i)) \neq g'_n(h'_n(s^\frown j))$ and
- $h'_n(s) \Vdash$ "$g'_n(h'_n(s)) \in \dot{C}$".

For $n = 0$ let $B'_{-1} = \{\mathrm{stem}(T)\}$ and $h_{-1}(\mathrm{stem}(T)) = T$ and proceed as in the $n > 0$ case. Assume we already constructed everything for n. Fix $s \in B'_n$. Define a rank function on $t \supseteq s$ such that $t \in h_n(s)$ by recursion as follows.

- $\mathrm{rk}(t) = 0 \iff$ there is a Miller tree $h'_{n+1}(t) \leq h'_n(s)$ with stem t and $g'_{n+1}(h'_{n+1}(t)) > t(|s|)$ such that $h'_{n+1}(t) \Vdash$ "$g'_{n+1}(h'_{n+1}(t)) \in \dot{C}$".
- $\mathrm{rk}(t) \leq \alpha \iff \exists^\infty l \in \omega$ such that $t^\frown l \in h'_n(s)$ and $\mathrm{rk}(t^\frown l) < \alpha$.

In case this rank is undefined we say that it is ∞.

Claim. $\mathrm{rk}(t) < \infty$ *for all* $t \in h_n(s)$.

Proof. Assume this is not the case and $\mathrm{rk}(t) = \infty$. If $\mathrm{rk}(u) = \infty$, then for all but finitely many l, $\mathrm{rk}(u^\frown l) = \infty$. Thus we can build $S \leq h_n(s)$ with stem t such that $\mathrm{rk}(u) = \infty$, for every $u \in S$. But then there is $S' \leq S$ with stem u and $m > u(|s|)$ such that $S' \Vdash$ "$m \in \dot{C}$". So $\mathrm{rk}(s) = 0$, a contradiction. □

Thus we can find an antichain $X'_{n+1,s} \subseteq h_n(s)$ such that $\bigcup\{h'_{n+1}(t^\frown l) | t \in X'_{n+1,s}, \mathrm{rk}(u^\frown l) = 0\}$ is a Laver subtree of $h'_n(s)$ with stem s and $\mathrm{rk}(u) = 1$, for all $u \in X'_{n+1,s}$. For every $u \in X'_{n+1,s}$ there are infinitely many l such that $u^\frown l \in h'_n(s)$, $\mathrm{rk}(u^\frown l) = 1$ and $g'_{n+1}(h'_{n+1}(u^\frown l)) \neq g'_{n+1}(h'_{n+1}(u^\frown k))$, for $k \neq l$, otherwise $\mathrm{rk}(u) = 0$. Let $B'_{n+1,s}$ be the set where for every $u \in X'_{n+1,s}$ we add infinitely many of these $u^\frown l$. Unfix s. Let $B'_{n+1} = \bigcup_{s \in B'_n} B'_{n+1,s}$ and let $\mathcal{A}'_{n+1}$ be the image of B'_{n+1} under h'_{n+1}. This completes the construction. Now we can do an argument like in the proof for Miller forcing to get $B_n \subseteq B'_n$ and $\mathcal{A}_n \subseteq \mathcal{A}'_n$ such that $g = g' \restriction \bigcup_{n \in \omega} \{n\} \times \mathcal{A}_n$ is injective and all the properties above are still satisfied. Then this witnesses weak fusion. □

For Cohen forcing we will need the following lemma, which will also be useful to simplify the final characterization.

Lemma 5.2.6. *For* $B \subseteq 2^{<\omega}$ *the following are equivalent:*

- $G_B \in \mathcal{M}$.
- G_B *is nowhere dense.*
- B *is nowhere dense.*

Proof. The last two are clearly equivalent. If G_B is nowhere dense it is by definition meager. So let G_B be meager, i.e. there are nowhere dense D_n such that $G_B = \bigcup_{n\in\omega} D_n$. Assume G_B where dense above $s_0 \in B$. Recursively choose $s_{n+1} \supseteq s_n$ such that $s_{n+1} \in B$ and $[s_{n+1}] \cap D_n = \emptyset$. For $s_n \in B$ we can choose $s'_{n+1} \supseteq s_n$ such that $[s'_{n+1}] \cap D_n = \emptyset$ since D_n is nowhere dense, and then because G_B is dense above s there is $s_{n+1} \supseteq s'_{n+1}$ in B. Thus this construction is possible. But now $s = \bigcup_{n\in\omega} s_n$ can't be in any of the D_n, but is in G_B, a contradiction. So G_B is nowhere dense. □

Theorem 5.2.7. *Cohen forcing has weak fusion.*

Proof. Let $s \in \mathbb{C}$ and $\dot{C}$ be a $\mathbb{C}$-name such that $s \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". Enumerate all finite sequences extending s as $\{t_n |\ n \in \omega\}$. Recursively construct sets $\{s_n \in 2^{<\omega} |\ n \in \omega\}$ and $\{m_n \in \omega |\ n \in \omega\}$. Choose $s_n \supseteq t_n$ and m_n such that $s_n \Vdash$ "$m_n \in \dot{C}$", $m_n \geq n$ and s_n and m_n are distinct from all the previous s_m and m_m. Let $B_n = \{s_n\}$, $\mathcal{A}_n = \{s_n\}$ and $g(n, s_n) = m_n$. Then G_B is dense above s, so by the lemma we have that G_B is a condition strengthening s. Also $[t] \cap G_B \leq t = h_n(t)$, for every $t \in B_n$, so (2') holds. And (3) is true by the choice of s_n. □

Theorem 5.2.8. *Hechler forcing has weak fusion.*

Proof. In the same way as for Cohen forcing we have that if $A \subseteq \omega^{\uparrow<\omega}$ is somewhere $\mathcal{D}$-dense, then $G_A \notin \mathcal{M}_{\mathcal{D}}$, where A is $\mathcal{D}$-dense if for every $s \in \omega^{\uparrow<\omega}$ and $f \in \omega^{\uparrow\omega}$ there is $t \in A$ extending s and dominating f.

Let $E = (s, x) \in \mathbb{D}$ and let $\dot{C}$ be a name such that $E \Vdash$ "$\dot{C} \in [\omega]^{\omega}$". We say $t \in \omega^{\uparrow<\omega}$ is compatible with (u, y) if $u \subseteq t$ and $t(i) \geq x(i)$, for $|u| \leq i < |t|$, we will also write $t \leq y$ for the latter condition. Recursively construct sets $X_n, Y_n \subseteq \omega^{\uparrow<\omega}$ and for every $t' \in Y_n$ and $i \in \omega$ sequences $t_i^{t'} \in \omega^{\uparrow<\omega}$, natural numbers $m_i^{t'} \in \omega$ and conditions $A_i^{t'} \in \mathbb{D}$ such that the following conditions hold:

(1) X_n is a maximal antichain of $t \in \omega^{\uparrow<\omega}$ compatible with (s, x),

(2) Y_n is an antichain of $t \in \omega^{\uparrow<\omega}$ compatible with (s, x),

(3) $\forall t \in X_n \forall y \in \omega^{\uparrow\omega} \exists t' \in Y_{n+1}$ compatible with (t, y),

(4) $\forall t \in X_{n+1} \exists l \leq |t|$ such that $t \restriction l \in X_n$,

(5) $\forall t \in Y_{n+1} \exists l \leq |t|$ such that $t \restriction l \in X_n$,

(6) $\forall t' \in Y_n \exists l > |t'|$ such that for all i, $t' \subseteq t_i^{t'}, |t_i^{t'}| = l$ and $t_i^{t'}(|t'|) \geq i$,

(7) $\forall t' \in Y_n \forall i \neq j \left(m_i^{t'} \neq m_j^{t'}\right)$,

(8) $\forall t' \in Y_n \forall i \left(t_i^{t'} \in X_n\right)$,

(9) $\left\{A_i^{t'} \middle| \ t' \in Y_n, i \in \omega\right\}$ is an antichain,

(10) $A_i^{t'} \Vdash$ "$\dot{m}_i^{t'} \in \dot{C}$" and

(11) $A_i^{t'}$ is compatible with any condition of the form $(t_i^{t'}, y)$.

Let $X_{-1} = \{s\}$. Assume we already constructed X_n. Fix $t \in X_n$. Let $\dot{d}$ be a name for the Hechler real and take a name $\dot{m}_t$ for a natural number such that $\Vdash$ "$\dot{m}_t \geq \dot{d}(|t|) \wedge \dot{m}_t \in \dot{C}$". For $t' \supseteq t$ compatible with x and $m \in \omega$ recursively define a rank function

- $\mathrm{rk}_t^m(t') = 0 \Leftrightarrow \exists x' \supseteq t'$ such that $(t', x') \Vdash$ "$\dot{m}_t = m$",
- $\mathrm{rk}_t^m(t') \leq \alpha \Leftrightarrow \exists l > |t'| \exists \{t_n |\, n \in \omega\}$ such that $t' \subseteq t_n, |t_n| = l, t_n(|t'|) \geq n$ and $\mathrm{rk}_t^m(t_n) < \alpha$.

For $m \in \omega$ every $t' \supseteq t^\frown(m+1)$ satisfies $\mathrm{rk}_t^m(t') > 0$, otherwise there would be $x' \supseteq t'$ such that $(t', x') \Vdash$ "$m = \dot{m}_t \geq \dot{d}(|t|) = m + 1$". This implies that $\mathrm{rk}_t^m(t) = \infty$ for every $m \in \omega$. Now we define $\mathrm{rk}_t(t')$ recursively by

- $\mathrm{rk}_t(t') = 0 \Leftrightarrow \exists m$ such that $\mathrm{rk}_t^m(t') < \infty$,
- $\mathrm{rk}_t(t') \leq \alpha \Leftrightarrow \exists l > |t'| \exists \{t_n |\, n \in \omega\}$ such that $t' \subseteq t_n, |t_n| = l, t_n(|t'|) \geq n$ and $\mathrm{rk}_t(t_n) < \alpha$.

Claim. $\mathrm{rk}_t(t') < \infty$ *for every* $t' \supseteq t$ *compatible with* x.

Proof. Assume towards a contradiction that $\mathrm{rk}_t(u) = \infty$. Then for all but finitely many $l \in \omega$, $\mathrm{rk}_t(u^\frown l) = \infty$. Thus we can build a set $B \subseteq \omega^{\uparrow<\omega}$ that is $\mathcal{D}$-dense above t' and such that for every $u \in B$, $\mathrm{rk}_t(u) = \infty$. Now there is $(u, y) \leq G_B$ and $m \in \omega$ such that $(u, y) \Vdash$ "$m = \dot{m}_t$" and $u \in B$. Then $\mathrm{rk}_t^m(u) = 0$ so $\mathrm{rk}_t(u) = 0$, a contradiction. □

As we have already shown $\mathrm{rk}_t(t) > 0$. Unfix t. Define

$$Y_{n+1} = \left\{t' \middle| \exists t \in X_n \left(\mathrm{rk}_t(t') = 1 \wedge \forall l < |t'| \left(\mathrm{rk}_t(t' \restriction l) > 1\right)\right)\right\}.$$

Then (2) and (5) hold. For $y \in \omega^{\uparrow\omega}$ we can build B in the proof of the claim in a way that every $u \in B$ is compatible with y. Therefore (3) holds. For $t' \in Y_{n+1}$ choose $\left\{t_i^{t'} \middle| i \in \omega\right\} \subseteq \omega^{\uparrow<\omega}$ and $\left\{m_i^{t'} \middle| i \in \omega\right\} \subseteq \omega$ such that $t' \subseteq t_i^{t'}$, $|t_i^{t'}| = l$, $t_i^{t'}(|t'|) \geq i$ and $\mathrm{rk}_t^{m_i^{t'}}(t_i^{t'}) < \infty$. Since $\mathrm{rk}_t(t') > 0$ we can choose the $m_i^{t'}$ to be distinct. So (6) and (7) hold. Let X_{n+1} be a maximal antichain containing all $t_i^{t'}$ and satisfying (4). Then (1), (4) and (8) hold. Define $A_i^{t'} = \bigcup\{U_{(t'', x'')} | t'' \supseteq t_i^{t'}$ is compatible with (t, x), $\mathrm{rk}_t^{m_i^{t'}} = 0$ and $(t'', x'') \Vdash$ "$\dot{m}_t = m_i^{t'}$"$\}$. Then (9), (10) and (11) hold. This completes the construction.

With an argument like the one done for Miller forcing we can shrink the collection of $t_i^{t'}$ to get that all $m_i^{t'}$ are distinct for all $i \in \omega$, $n \in \omega$ and $t' \in Y_n$. Let $B_n = \left\{t_i^{t'} \middle| i \in \omega, t' \in Y_n\right\}$, $\mathcal{A}_n = \left\{A_i^{t'} \middle| i \in \omega, t' \in Y_n\right\}$, $h_n(t_i^{t'}) = A_i^{t'}$ and $g(n, A_i^{t'}) = m_i^{t'}$. Conditions (1), (3) and (6) imply that B is $\mathcal{D}$-dense below s, so $G_B \leq E$ is a condition. If $B' \subseteq B$ with $G_{B'} \in \mathbb{D}$ and $k \in \omega$ then there is $n \geq k$ and $t_i^{t'} \in B_n \cap B'$ such that $B' \cap \langle t_i^{t'} \rangle$ is $\mathcal{D}$-dense below $(t_i^{t'}, y)$ for some y. Then $A_i^{t'}$ is compatible with $[t_i^{t'}] \cap G_{B'}$, so condition (2) in the definition of weak fusion is satisfied. Condition (3) in the definition of weak fusion is just (10). □

Random forcing does not satisfy weak fusion as we defined it. But if we weaken the condition that g has to be injective we still get a true statement.

Theorem 5.2.9. *Random forcing satisfies everything in the definition of weak fusion if we replace g being injective with finite-to-one.*

Proof. Let $E \in \mathbb{B}$ and let $\dot{C}$ be a $\mathbb{B}$-name such that $E \Vdash$ "$\dot{C} \in [\omega]^\omega$". We will recursively construct disjoint finite antichains $B_n \subseteq 2^{<\omega}$, finite antichains $\mathcal{A}_n \subseteq \mathbb{B}$, conditions E_n, bijections $h_n\colon B_n \to \mathcal{A}_n$ and a function $g\colon \bigcup_{n\in\omega}\{n\} \times \mathcal{A}_n \to \omega$ such that

(1) $\forall m < n \forall \sigma \in B_n \exists k < |\sigma|$ such that $\sigma \restriction k \in B_m$,

(2) $\mu(E_n) \geq \mu(E)(\frac{1}{2} + \frac{1}{2^{n+2}})$,

(3) $E_{n+1} \leq E_n \leq E$,

(4) $E_n = \bigcup \mathcal{A}_n$,

(5) $\forall \sigma \in B_n \Big(h_n(\sigma) = [\sigma] \cap E_n\Big)$,

(6) $\forall A \in \mathcal{A}_n \Big(A \Vdash$ "$g(n, A) \in \dot{C}$"$\Big)$ and

(7) if $m < n$, $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then $g(m, A) < g(n, B)$.

Assume we already constructed these for $n-1$, where we set $E_{-1} = E$. Since $\mathcal{A}_{n-1}$ is finite there is $l_{n-1} = \max\{g(n-1, A) |\, A \in \mathcal{A}_{n-1}\}$, for $n = 0$ set $l_{-1} = 0$. For $l > l_{n-1}$ let $E^l \leq E_{n-1}$ be the biggest condition forcing $l \in \dot{C}$. Then there is l_n such that $\mu(\bigcup \{E^l |\, l_{n-1} < l \leq l_n\}) \geq \mu(E)\left(\frac{1}{2} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}}\right)$, since $\mu(E_{n-1}) \geq \mu(E)\left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)$. Now we can approximate measurable sets with basic open sets, so we can find $B^l \subseteq 2^{<\omega}$ such that $B_n = \bigcup \{B^l |\, l_{n-1} < l \leq l_n\}$ is an antichain satisfying (1) and if we define $h_n(\sigma) = [\sigma] \cap E^l$, for $\sigma \in B^l$, $\mathcal{A}_n = \{h_n(\sigma) |\, \sigma \in B_n\}$ and $E_n = \bigcup \mathcal{A}_n$, then $\mu(E_n) \geq \mu(E)\left(\frac{1}{2} + \frac{1}{2^{n+2}}\right)$. Then (1) through (5) hold. Define $g(n, h_n(\sigma)) = l$ for $\sigma \in B^l$. Then (6) holds by the choice of E^l and (7) holds by the choice of l_{n-1}. This completes the construction.

We have that g is finite-to-one because of the last condition and all $\mathcal{A}_n$ being finite. Conditions (3), (4) and (5) show that $G_B = \bigcap E_n$ and because of (2) and (3) this has measure greater or equal to $\frac{1}{2}$. Condition (5) and (6) are condition (2') and (3) in the definition of weak fusion respectively. □

This is enough to get 5.2.2 with "injective" replaced by "finite-to-one" in condition (3).

5.3 Characterizing indestructibility

Now that we have seen that all these forcings have weak fusion, we can use 5.2.2 to get characterization of forcing indestructibility of MAD families. The following property makes the characterization even simpler.

Definition 5.3.1. *An ideal $I_{\mathbb{P}} \subseteq \mathcal{B}$ is called strongly homogeneous if for all $B \subseteq 2^{<\omega}$ with $G_B \notin I_{\mathbb{P}}$ there is an injection $h\colon 2^{<\omega} \to B$ such that for all $C \subseteq B$, if $G_{h^{-1}[C]} \notin I_{\mathbb{P}}$, then $G_C \notin I_{\mathbb{P}}$.*

Lemma 5.3.2. *Let $\mathcal{I}$ be a tall ideal and assume $I_{\mathbb{P}}$ is strongly homogeneous, then the following are equivalent.*

(1) $\forall B \subseteq 2^{<\omega}$ such that $G_B \notin I_{\mathbb{P}}$ $\forall f\colon B \to \omega$ (injective) $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$.

(2) $\forall f\colon 2^{<\omega} \to \omega$ (injective) $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$

Proof. The downwards direction is trivial. For the upwards direction take $B \subseteq 2^{<\omega}$ and (injective) $f\colon B \to \omega$ such that $G_B \notin I_{\mathbb{P}}$. Let h be as in the definition of strongly homogeneous. If f is injective then so is $f \circ h$. Then there is $I \in \mathcal{I}$ such that $G_{(f\circ h)^{-1}[I]} \notin I_{\mathbb{P}}$. Now strong homogeneity gives that $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$. □

Theorem 5.3.3. **cntble**, $\mathcal{K}_\sigma$ *and $\mathcal{M}$ are strongly homogeneous.*

Proof. Let us begin with **cntble** and let $B \subseteq 2^{<\omega}$ be such that $G_B \notin$ **cntble**. For $B' = \{s \in B |\ G_B \cap [s] \in \textbf{cntble}\}$ we have $G_{B'} \notin$ **cntble**. Recursively define $h\colon 2^{<\omega} \to B'$ by sending $\emptyset$ to an arbitrary element of B' and if $h(s)$ is defined let $h(s^\frown 0)$ and $h(s^\frown 1)$ be two incompatible extensions of $h(s)$ in B'. Let $C \subseteq B$ with $G_{h^{-1}[C]} \in$ **cntble**. Then for all $s \in h^{-1}[C]$ there is some $s \subseteq t \in C$ such that no further extension of $h(t)$ is in C. Then there is also no further extension of $t \supseteq s$ in $h^{-1}[C]$, thus $G_{h^{-1}[C]}$ is countable.

For $\mathcal{K}_\sigma$ let $B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$. Then G_B contains a rational perfect tree so there is an injection $h\colon \omega^{<\omega} \to B$ that preserves the structure. Now if $C \subseteq B$ is such that $G_{h^{-1}[C]} \notin \mathcal{K}_\sigma$ there is a rational perfect tree in $G_{h^{-1}[C]}$ and the image of this tree under h is still a rational perfect tree, so $G_C \notin \mathcal{K}_\sigma$.

Finally let $B \subseteq 2^{<\omega}$ be such that $G_B \notin \mathcal{M}$. As we noticed earlier this means that B is somewhere dense, say above s. Then there is an injection $h\colon 2^{<\omega} \to B$ such that $h(t) \supseteq s^\frown t$. Now if $G_{h^{-1}[C]} \notin \mathcal{M}$ we have that $h^{-1}[C]$ is somewhere dense, say above t. Then C is dense above $h(t)$, so $G_C \notin \mathcal{M}$. □

Definition 5.3.4 (Katětov [21]). *Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. We say $\mathcal{J} \leq_K \mathcal{I}$ if there is a function $f\colon \omega \to \omega$ such that $f^{-1}[J] \in \mathcal{I}$ for every $J \in \mathcal{J}$. This is called the Katětov ordering and a function f like this is a Katětov morphism.*

Lemma 5.3.5. *Let $\mathbb{P}$ be a real forcing with weak fusion and the continuous reading of names such that $I_{\mathbb{P}}$ is strongly homogeneous. The following are equivalent for a tall ideal $\mathcal{I}$.*

(1) $\mathcal{I}$ is $\mathbb{P}$-destructible.

(2) $\mathcal{I} \leq_K \mathcal{I}_{\mathbb{P}} = \{I \subseteq 2^{<\omega} |\ G_I \in I_{\mathbb{P}}\}$.

Proof. From 5.3.2 and 5.2.2 we get that $\mathcal{I}$ being $\mathbb{P}$-destructible is equivalent to there being an injective $f\colon 2^{<\omega} \to 2^{<\omega}$ such that $G_{f^{-1}[I]} \in I_{\mathbb{P}}$ for every $I \in \mathcal{I}$. Such an f is exactly a Katětov morphism. □

Now we can state the final characterizations.

Theorem 5.3.6. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ is $\mathbb{S}$-indestructible.

*(2) $\forall B \subseteq 2^{<\omega}$ such that $G_B \notin$ **cntble** $\forall f\colon B \to \omega \exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin$ **cntble**.*

*(3) $\forall B \subseteq 2^{<\omega}$ such that $G_B \notin$ **cntble** $\forall f\colon B \to \omega$ injective $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin$ **cntble**.*

*(4) $\forall f\colon 2^{<\omega} \to \omega$ injective $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin$ **cntble**.*

(5) $\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{S}} = \{I \subseteq 2^{<\omega} | \, G_I \in \textbf{cntble}\}$.

(6) $\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ adding a new real.

Proof. The equivalence between (1), (2) and (3) is 5.2.2. That they are equivalent to (4) comes from the strong homogeneity of **cntble** and 5.3.2. The equivalence of (5) is 5.3.5. That (1) implies (6) is trivial. So all that remains is (6) implies any of the others. Assume towards a contradiction that $\mathcal{I}$ is not $\mathbb{S}$-indestructible, and let $f\colon 2^{<\omega} \to \omega$ witness not (4). Then if $x \in V^{\mathbb{P}}$ is a new real we have that it avoids all countable sets coded in the ground model, in particular $x \notin G_{f^{-1}[I]}$, for all $I \in \mathcal{I}$. So $\left\{G_{f^{-1}[I]} \middle| \, I \in \mathcal{I}\right\}$ doesn't cover 2^ω, which by 5.1.3 implies that $\mathcal{I}$ is not tall in the extension. □

Theorem 5.3.7. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ is $\mathbb{M}$-indestructible.

(2) $\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$ $\forall f\colon B \to \omega \exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin \mathcal{K}_\sigma$.

(3) $\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$ $\forall f\colon B \to \omega$ injective $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin \mathcal{K}_\sigma$.

(4) $\forall f\colon \omega^{<\omega} \to \omega$ injective $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin \mathcal{K}_\sigma$.

(5) $\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{M}} = \{I \subseteq \omega^{<\omega} | \, G_I \in \mathcal{K}_\sigma\}$.

(6) $\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ adding an unbounded real.

Proof. The proof is the same as for Sacks forcing. For (6) notice that no unbounded reals are added to sets in $\mathcal{K}_\sigma$ and all sets not in $\mathcal{K}_\sigma$ have an unbounded real added to them. This is clear for trees and due to the characterization of $\mathcal{K}_\sigma$ in 4.2.2 this is enough. □

Theorem 5.3.8. *Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ is $\mathbb{L}$-indestructible.

*(2) $\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin$ **not − dominating** $\forall f\colon B \to \omega \exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin$ **not − dominating**.*

(3) $\forall B \subseteq \omega^{<\omega}$ *such that* $G_B \notin \mathbf{not-dominating}$ $\forall f\colon B \to \omega$ *injective* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathbf{not-dominating}$.

(4) $\mathcal{I}$ *is* $\mathbb{P}$*-indestructible for some forcing* $\mathbb{P}$ *adding a dominating real.*

Proof. The equivalence of (1) - (3) is like the previous two proofs. That (1) implies (4) is trivial. Assume that (3) is false and let us show that $\mathcal{I}$ is destroyed by any $\mathbb{P}$ that adds dominating reals. Let $B \subseteq \omega^{<\omega}$ and $f\colon B \to \omega$ witness this. Call a real $x \in \omega^\omega$ strongly dominating if for every ground model function $\phi\colon \omega^{<\omega} \to \omega$ we have $x(n) \geq \phi(x \restriction n)$ for all but finitely many n. If $\mathbb{P}$ adds dominating reals it also adds strongly dominating reals. To see this take $\phi\colon \omega^{<\omega} \to \omega$ that eventually dominates every ground model function like this. Then we can recursively construct a real x such that $x(n) \geq \phi(x \restriction n)$, for all n. Such a real is strongly dominating. This recursive construction can be carried out in every Laver tree. Thus there is a strongly dominating real $x \in G_B$. Strongly dominating reals avoid every set $G_A \in \mathbf{not-dominating}$, as for every $s \in A$ there is a maximal antichain above s that consists only of $t \in A$ such that $t(|t|-1) < \phi(t \restriction (|t|-1))$, where ϕ is the function witnessing $G_A \in \mathbf{not-dominating}$. Thus $x \notin G_{f^{-1}[I]}$ for every $I \in \mathcal{I}$. Which means that $\mathcal{I}$ is no longer tall in the extension, by 5.1.3. □

Theorem 5.3.9. *Let* $\mathcal{I}$ *be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ *is* $\mathbb{C}$*-indestructible.*

(2) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin \mathcal{M}$ $\forall f\colon B \to \omega \exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{M}$.

(3) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin \mathcal{M}$ $\forall f\colon B \to \omega$ *injective* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{M}$.

(4) $\forall f\colon 2^{<\omega} \to \omega$ *injective* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{M}$.

(5) $\mathcal{I} \not\leq_K \mathcal{I}_{\mathbb{C}} = \{I \subseteq 2^{<\omega} | \, G_I \in \mathcal{M}\}$.

Proof. Exactly the same as for Sacks forcing. □

Due to 5.2.6 we can replace the G_δ-closure being in the meager ideal with being nowhere dense in this theorem.

Theorem 5.3.10. *Let* $\mathcal{I}$ *be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ *is* $\mathbb{D}$*-indestructible.*

(2) $\forall B \subseteq \omega^{\uparrow<\omega}$ *such that* $G_B \notin \mathcal{M}_{\mathcal{D}}$ $\forall f\colon B \to \omega \exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{M}_{\mathcal{D}}$.

(3) $\forall B \subseteq \omega^{\uparrow<\omega}$ *such that* $G_B \notin \mathcal{M}_{\mathcal{D}}$ $\forall f\colon B \to \omega$ *injective* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{M}_{\mathcal{D}}$.

Proof. Just like the previous times this follows directly from 5.2.2. □

Theorem 5.3.11. *Let* $\mathcal{I}$ *be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ *is* $\mathbb{B}$*-indestructible.*

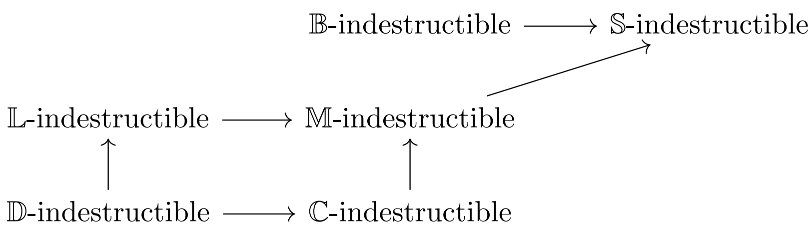
(2) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin \mathcal{N}$ $\forall f : B \to \omega \exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{N}$.

(3) $\forall B \subseteq 2^{<\omega}$ *such that* $G_B \notin \mathcal{N}$ $\forall f : B \to \omega$ *finite-to-one* $\exists I \in \mathcal{I}$ *such that* $G_{f^{-1}[I]} \notin \mathcal{N}$.

Proof. From the weakened version of weak fusion, that holds for Random forcing, we can proof 5.2.2 with "finite-to-one" instead of "injective". This works exactly the same as in the original proof. Then this theorem follows directly from that. □

5.4 Relations between indestructibility

Now that we have given nice descriptions of when a tall ideal, or MAD family, is forcing indestructible, we will go on and investigate how they relate to each other. The arrows in the following diagram are implications for the indestructibility of tall ideals.



That Sacks indestructibility is the weakest follows from 5.3.6 (6). The implication from $\mathbb{D}$ to $\mathbb{L}$ is 5.3.8(4). Since Hechler forcing adds a Cohen real we have the implication from $\mathbb{D}$ to $\mathbb{C}$. As $\mathbb{C}$ and $\mathbb{L}$ add an unbounded real 5.3.7(6) gives the implication from $\mathbb{C}$ and $\mathbb{L}$ to $\mathbb{M}$.

The goal of this section is to establish that these are the only implications between them. If we want to do this by constructing MAD families and not just tall ideals we usually need some assumption beyond ZFC. Often this will be the assumption that one of the following cardinal characteristics is equal to the continuum.

Definition 5.4.1. *For an ideal I on $\mathbb{R}$ define the following two cardinal invariants:*

- $\mathrm{cov}(I) = \min \left\{ |\mathcal{A}| \,\middle|\, \mathcal{A} \subseteq I, \mathbb{R} = \bigcup_{A \in \mathcal{A}} A \right\}$,
- $\mathrm{add}(I) = \min \left\{ |\mathcal{A}| \,\middle|\, \mathcal{A} \subseteq I, \bigcup_{A \in \mathcal{A}} A \notin I \right\}$.

The following theorem establishes a connection between these and the indestructibility of MAD families.

Theorem 5.4.2. *Let $\mathbb{P}$ be a real forcing with weak fusion and the continuous reading of names. Assume $\mathbb{P}$ is homogeneous, that means $\mathrm{cov}(I_{\mathbb{P}} \restriction G) = \mathrm{cov}(I_{\mathbb{P}})$ for all Borel sets $G \notin I_{\mathbb{P}}$. Then every MAD family of size less than $\mathrm{cov}(I_{\mathbb{P}})$ is $\mathbb{P}$-indestructible.*

Proof. Let $\mathcal{A}$ be a MAD family of size less than $\mathrm{cov}(I_{\mathbb{P}})$. Let $B \subseteq 2^{<\omega}$ be a set such that $G_B \notin I_{\mathbb{P}}$ and let $f\colon B \to \omega$ be an injective function. To show that $\mathcal{A}$ is $\mathbb{P}$ indestructible it is enough to find $I \in \mathcal{I}(\mathcal{A})$ such that $G_{f^{-1}[I]} \notin I_{\mathbb{P}}$, by 5.2.2. But by 5.1.3, $\left\{G_{f^{-1}[A]} \middle|\, A \in \mathcal{A}\right\}$ covers G_B. Since the size of $\mathcal{A}$ is smaller than $\mathrm{cov}(I_{\mathbb{P}} \restriction G) = \mathrm{cov}(I_{\mathbb{P}})$, this means that already one of the $G_{f^{-1}[I]}$ is in $I_{\mathbb{P}}$. □

Corollary 5.4.3. *In the situation above, if* $\mathfrak{a} < \mathrm{cov}(I_{\mathbb{P}})$*, then there is a* $\mathbb{P}$*-indestructible MAD family.*

Proof. As $\mathfrak{a} < \mathrm{cov}(I_{\mathbb{P}})$ there is a MAD family of size less than $\mathrm{cov}(I_{\mathbb{P}})$ and by the previous theorem such a MAD family is $\mathbb{P}$-indestructible. □

Now if we want to construct indestructible MAD families we can assume $\mathfrak{a} \geq \mathrm{cov}(I_{\mathbb{P}})$ without loss of generality.

5.4.1 Sacks indestructible MAD family

We start by constructing a Sacks indestructible MAD family. This does not help us finding out that the other arrows are impossible, but the proof gives an easier introduction to the ideas used in the other proofs. And of course it is still interesting to know when Sacks indestructible MAD families exist.

Theorem 5.4.4. *Assume* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*. There is a* $\mathbb{S}$*-indestructible MAD family.*

Proof. If $\mathfrak{a} < \mathfrak{c} = \mathrm{cov}(\mathbf{cntble})$ we can use 5.4.3 to get the existence of an $\mathbb{S}$-indestructible MAD family. So we can assume $\mathfrak{a} = \mathfrak{c}$. Let $\{f_\alpha\colon 2^{<\omega} \to \omega |\ \alpha < \mathfrak{c}\}$ enumerate all injective functions. We will recursively construct an almost disjoint family $\{A_\alpha |\ \alpha < \mathfrak{c}\}$ such that, for all $\alpha < \mathfrak{c}$,

- if $G_{f_\alpha^{-1}[A_\beta]}$ is countable, for all $\beta < \alpha$, then $G_{f_\alpha^{-1}[A_\alpha]}$ is uncountable.

Then 5.3.6 gives us that this is an $\mathbb{S}$-indestructible MAD family. Assume we have constructed all A_β for $\beta < \alpha$. If there is some $\beta < \alpha$ such that $G_{f_\alpha^{-1}[A_\beta]}$ is uncountable we can take A_α an arbitrary set almost disjoint from all A_β. This exist since $\alpha < \mathfrak{c} = \mathfrak{a}$. Now consider the case that $G_{f_\alpha^{-1}[A_\beta]}$ is countable, for all $\beta < \alpha$. Then $\bigcup_{\beta<\alpha} G_{f_\alpha^{-1}[A_\beta]}$ has cardinality less than $\mathfrak{c}$. Thus there is a perfect tree $T \subseteq 2^{<\omega}$ such that $[T] \cap \bigcup_{\beta<\alpha} G_{f_\alpha^{-1}[A_\beta]} = \emptyset$. Then the following Lemma with $f = f_\alpha$ gives us A_α. □

Lemma 5.4.5. *Assume* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$*. Let* $f\colon 2^{<\omega} \to \omega$ *be an injective function. Let* $\{A_\beta |\ \beta < \alpha\}$ *be an almost disjoint family such that* $G_{f^{-1}[A_\beta]} \in \mathbf{cntble}$*, for all* $\beta < \alpha$*. Let* $T \subseteq 2^{<\omega}$ *be a perfect tree such that* $[T] \cap G_{f^{-1}[A_\beta]} = \emptyset$*, for all* $\beta < \alpha$*. Then there is an* $A \subseteq \omega$ *almost disjoint from all* A_β*, such that* $G_{f^{-1}[A]} \notin \mathbf{cntble}$*.*

Proof. We already have that $f^{-1}[A_\beta] \cap T$ is an off branch set, for every $\beta < \alpha$. Meaning every branch of $2^{<\omega}$ contains only finitely many elements of $f^{-1}[A_\beta] \cap T$. We will construct a perfect subtree $S \subseteq T$ such that $S \cap f^{-1}[A_\beta]$ is finite, for every $\beta < \alpha$.

Then $A = f[S]$ is as desired. Let M be a model of ZFC with $T, f, \{A_\beta | \beta < \alpha\} \in M$ of cardinality $|\alpha|$. Notice that models of ZFC might not exist. In that case let M be a model of some big enough finite fragment of ZFC, then the proof works just the same. The following claim gives us that we can do Cohen forcing over M inside of V.

Claim. *There is a real $c \in V$ that is Cohen generic over M.*

Proof. There are at most $|\alpha|$ many meager sets in M. Since $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} > \alpha$ there is $c \in V$ not contained in any of them. Such a real is Cohen generic over M. □

Define a forcing $\mathbb{P}$ in M as follows. Conditions are finite subtrees of T such that all top nodes are on the same level, stronger conditions are end extensions. This forcing is countable, so it is equivalent to Cohen forcing which means there is $G \in V$ that is $\mathbb{P}$-generic over M. Let $S = \bigcup G$. An easy density argument shows that S is a perfect subtree of T. Let $\beta < \alpha$ and show that $S \cap f^{-1}[A_\beta]$ is finite. Let $p \in \mathbb{P}$. For a top node $s \in p$ there is $t_s \supseteq s$ such that $t_s \in T$ and no extension of t_s is in $f^{-1}[A_\beta]$, since $[T] \cap f^{-1}[A_\beta]$ is off branch. We can assume that all t_s are of the same length, since there are only finitely many of them and their defining property is preserved by extensions. Then $q = \bigcup\{t_s \restriction n | s$ is a top node of p and $n \leq |t_s|\}$ is a condition strengthening p. It forces that $S \cap f^{-1}[A_\beta] = q \cap f^{-1}[A_\beta]$ and therefore finite. So every condition can be extended to a condition forcing that $S \cap f^{-1}[A_\beta]$ is finite, so this is forced by every condition. □

Similarly it is possible to construct a MAD family that is $\mathbb{B}$-indestructible and $\mathbb{M}$-destructible, under the assumption $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$. This implies that there are no implications from $\mathbb{B}$ towards $\mathbb{L}, \mathbb{D}, \mathbb{C}$ and $\mathbb{M}$.

5.4.2 Eventually different functions

To show that there is no implication from $\mathbb{C}, \mathbb{M}$ and $\mathbb{S}$ to $\mathbb{B}$ we will consider a slight modification of MADness.

Definition 5.4.6. *Two infinite partial functions $f, g \in \omega^\omega$ are called eventually different if $f \cap g$ is finite, i.e. after some point they will always disagree with each other, if defined.*

An infinite family $\mathcal{A} \subseteq \omega^\omega$ is a family of eventually different partial functions if the elements of $\mathcal{A}$ are pairwise eventually different and $\mathcal{A}$ is maximal with this property.

The nice thing about these families is that they are all destroyed by random forcing, because it adds a real that is eventually different from all ground model reals. To show this we proof a characterization of when forcings add reals eventually different from all ground model reals. This proof is adapted from the proof of [3, Lemma 2.4.2] due to [2] and [31].

Theorem 5.4.7. *Let $\mathbb{P}$ be a forcing notion and let G be $\mathbb{P}$-generic. Then the following are equivalent:*

(1) $\omega^\omega \cap V$ is not meager in $V[G]$.

(2) $\forall f \in \omega^\omega \exists g \in \omega^\omega \cap V \exists^\infty n \in \omega$ *such that* $f(n) = g(n)$.

(3) $\forall F \in [\omega^\omega]^\omega \exists g \in \omega^\omega \cap V \forall f \in F \exists^\infty n \in \omega$ *such that* $f(n) = g(n)$.

(4) $\forall F \in [\omega^\omega]^\omega \forall G \in [[\omega]^\omega]^\omega \exists g \in \omega^\omega \cap V \forall f \in F \forall X \in G \exists^\infty n \in X$ *such that* $f(n) = g(n)$.

Proof. (1) ⇒ (2) Assume that there is an eventually different real in $V[G]$. Let $f \in \omega^\omega$ be eventually different from all ground model reals. Define $G = \{g \in \omega^\omega | \forall^\infty n\big(f(n) \neq g(n)\big)\}$. Because every ground model real is eventually different from f this set covers the ground model reals. It is also meager, since it is a countable union of nowhere dense sets $G = \bigcup_{m\in\omega} \{g \in \omega^\omega \big| \forall n > m\big(f(n) \neq g(n)\big)\}$. So the ground model reals are meager.

(2) ⇒ (3) Assume (2) holds. Let $F \in [\omega^\omega]^\omega$ and enumerate it as $\{f_\alpha | \alpha \in \omega\}$. Take a partition $\{A_\alpha | \alpha \in \omega\} \in V$ of ω into infinite sets. Let $\{a^n_\alpha | n \in \omega\}$ be the increasing enumeration of A_α. Define $f'_\alpha(n) = f_\alpha(a^n_\alpha)$. Then for every $\alpha \in \omega$ there is $g_\alpha \in \omega^\omega \cap V$ such that there are infinitely many n with $f'_\alpha(n) = g_\alpha(n)$. Define $g \in \omega^\omega$ by $g(a^n_\alpha) = g_\alpha(n)$. Then for every $f_\alpha \in F$ there are infinitely many n such that $f_\alpha(a^n_\alpha) = f'_\alpha(n) = g_\alpha(n) = g(a^n_\alpha)$.

(3) ⇒ (4) Assume (3) holds. Let $F \in [\omega^\omega]^\omega$ and let $G \in [[\omega]^\omega]^\omega$. Enumerate $F = \{f_\alpha | \alpha \in \omega\}$ and $G = \{X_\alpha | \alpha \in \omega\}$. Let $\{x^n_\alpha | n \in \omega\}$ be the increasing enumeration of X_α. For $\alpha, \beta < \omega$ define $h_{\alpha,\beta}(n) = f_\beta \restriction \{x^0_\alpha, x^1_\alpha, \ldots, x^{n+1}_\alpha\}$. We can code $[\omega]^{<\omega}$ by ω to get from (3) a function $h \in V$ such that for all $\alpha, \beta < \omega$ there are infinitely many $n \in \omega$ with $h_{\alpha,\beta}(n) = h(n)$. In V inductively choose $x_{n+1} \in \mathrm{dom}(h(n)) \setminus \{x_0, \ldots, x_n\}$. Define $g \in \omega^\omega \cap V$ by $g(x_n) = h(n)(x_n)$ and arbitrary everywhere else. Then for $f_\beta \in F$ and $X_\alpha \in G$ we have that if $h_{\alpha,\beta} = h(n)$, then $f_\beta(x_n) = g(x_n)$ and $x_n \in X_\alpha$. Since the condition happens infinitely often so does the conclusion and in that case $x_n \in X_\alpha$ and $f_\beta(x_n) = g(x_n)$.

(4) ⇒ (1) Assume (4) holds. Let $\{F_\alpha | \alpha \in \omega\} \subseteq \omega^\omega$ be a family of nowhere dense sets. Define functions $s_\alpha \colon \omega \to \omega^{<\omega}$ by $s_\alpha(n) = \min\{s \in \omega^{<\omega} \big| \forall t \in \omega^{<n}\big([t^\frown s] \cap F_\alpha = \emptyset\big)\}$, where the minimum is taken by some order on $\omega^{<\omega}$. We can again code $\omega^{<\omega}$ by ω to get from (4) a function $s \in V$ such that

$$\forall \alpha \in \omega \exists^\infty n\Big(s_\alpha(n) = s(n)\Big).$$

Let $X_\alpha = \{n | s_\alpha(n) = s(n)\}$.

Claim. *There is an increasing sequence* $\{k_n | n \in \omega\} \subseteq \omega$ *such that:*

- $\sum_{j \leq k_n} |s(j)| < k_{n+1}$ *and*
- $\forall \alpha \in \omega \exists^\infty n\Big(X_\alpha \cap [k_{2n}, k_{2n+1}] \neq \emptyset\Big)$.

Proof. For a finite set $A \subseteq \omega$ define $f_A \in \omega^\omega$ by $f_A(n) = \min\{m | \forall \alpha \in A\big([n, m) \cap X_\alpha \neq \emptyset\big)\}$. Let $f'_{A,k}(n) = f^{(n)}_A(k)$. These are countably many functions, so there is a single function $f \in \omega^\omega$ which eventually dominates all $f'_{A,k}$ and $\sum_{j \le f(n)} |s(j)| < f(n+1)$, for all $n \in \omega$. Then

$$\forall A \exists^\infty n \exists k \Big(f(n) \le k \le f_A(k) \le f(n+1) \Big).$$

Otherwise there would be an $A \in [\omega]^{<\omega}$ and $m \in \omega$ such that $f(n+1) < f_A(f(n))$ for all $n \ge m$, and therefore $f(n) \le f(n+m) < f_A(f(n+m-1)) < \cdots < f_A^{(n)}(f(m)) = f'_{A,f(m)}(n)$ contradicting that f eventually dominates $f'_{A,f(m)}$. There are infinitely many even or infinitely many odd n like this. Eventually replacing $f(n)$ by $f(n+1)$ we can assume that there are infinitely many even n like this. Thus $k_n = f(n)$ satisfies the claim. □

For $\alpha \in \omega$ define $f_\alpha(n) = \min(X_\alpha \cap [k_{2n}, k_{2n+1}))$, where the minimum over the empty set is said to be 0. Let $Y_\alpha = \{n |\, f_\alpha(n) \ne 0\}$. With (4) we get $g \in V \cap \omega^\omega$ such that for every $\alpha \in \omega$ there are infinitely many $n \in Y_\alpha$ with $g(n) = f_\alpha(n)$. Let $X = \{g(n) |\, n \in \omega\} \in V$. Then $X \cap X_\alpha$ is infinite for every $\alpha \in \omega$ and for all $n \in \omega$:

$$|X \cap [k_{2n}, k_{2n+1})| \le 1.$$

Let $\{x_n |\, n \in \omega\}$ be the increasing enumeration of X. Then $x = s(x_0)^\frown s(x_1)^\frown \cdots \in V$. To finish the proof we will show that $x \notin \bigcup_{\alpha\in\omega} F_\alpha$, showing that the ground model reals are not meager. Fix $\alpha \in \omega$. Then there is $x_n \in X \cap [k_{2n}, k_{2n+1})$ such that $s(x_n) = s_\alpha(x_n)$. We also have $\sum_{j<n} |s(x_j)| < k_{2n} < x_n$, and thus $[s(x_0)^\frown \ldots ^\frown s(x_n)] = [s(x_0)^\frown \ldots ^\frown s(x_{n-1})^\frown s_\alpha(x_n)]$ is disjoint from F_α. Therefore $x \notin F_\alpha$. □

Theorem 5.4.8. *Let G be a $\mathbb{B}$-generic filter. Then $2^\omega \cap V$ is meager in $V[G]$. In particular there is a real $r \in V[G]$ that is eventually different from all ground model reals.*

Proof. Work in V. We will first construct a null set $A \subseteq 2^\omega$ whose complement is meager. Let $\{q_n |\, n \in \omega\} \subseteq 2^\omega$ enumerate all functions that are eventually 0. Let $A_n = \bigcup_{j\in\omega} [q_j \restriction (n+j)]$. Then $A = \cap_{n\in\omega} A_n$ is such a set. It has measure 0 since the measure of A_n goes to 0 as n goes to infinity. Since A_n is open and dense the complement of A_n is nowhere dense. Thus $2^\omega \setminus A = \bigcup_{n\in\omega} (2^\omega \setminus A_n)$ is meager.

Let $B = 2^\omega \setminus A$. Let A' and B' in $V[G]$ be the sets with the same Borel code as A and B, respectively. Then A' is still of measure zero and B' is meager. Let $r = \bigcap G$ be the random real. For $s \in 2^\omega \cap V$ we have that $s + A$ has measure zero. Thus $r \notin s + A'$ and hence $s \in r - B'$. So $2^\omega \cap V \subseteq r - B'$ and therefore it is meager.

The in particular part follows directly from the previous theorem. □

Now the only thing we need to show is that there is a Cohen indestructible maximal family of eventually different functions. To do this we will proof an analog of 5.3.9 for families of eventually different functions and use that to construct a Cohen indestructible family of eventually different functions. This proof is similar to the construction of a Cohen indestructible MAD family given by Hrušák in [18]. First we need a modification of weak fusion.

Lemma 5.4.9. *Let $s \in \mathbb{C}$ and let $\dot{f}$ be a $\mathbb{C}$-name for an infinite partial function. Then there is $B \subseteq 2^{<\omega}$ and an injective function $g\colon B \to \omega \times \omega$ such that*

- *B is dense below s,*
- *$\forall t \in B\Big(t \Vdash \text{“}g(t) \in \dot{f}\text{”}\Big)$ and*
- *$g\,[B]$ is a partial function.*

Proof. Let $\{s_n |\, n \in \omega\}$ enumerate all extensions of s such that shorter sequences come before longer ones. Recursively construct $B = \{t_n |\, n \in \omega\}$ and a function $g\colon B \to \omega \times \omega$ such that $t_n \supseteq s_n$. At step n we can find (i_n, j_n) and $t_n \supseteq s_n$ such that $t_n \Vdash \text{“}(i_n, j_n) \in \dot{f}\text{”}$ and $i_n > i_m$ for every $m < n$. Let $g(t_n) = (i_n, j_n)$. This satisfies all conditions. $\square$

Theorem 5.4.10. *Let $\mathcal{A}$ be a maximal family of eventually different functions. The following are equivalent:*

(1) $\mathcal{A}$ is $\mathbb{C}$-indestructible.

(2) $\forall B \subseteq 2^{<\omega}$ such that $G_B \notin \mathcal{M}$ $\forall f\colon B \to \omega \times \omega$ injective such that $f\,[B]$ is a partial function $\exists x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin \mathcal{M}$.

(3) $\forall f\colon 2^{<\omega} \to \omega \times \omega$ injective such that $f\,[2^{<\omega}]$ is a partial function $\exists x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin \mathcal{M}$.

Proof. The implications (2) to (3) is trivial and that (3) implies (2) follows from the fact that Cohen forcing is strongly homogeneous. Now let us proof (1) implies (2). Assume that there is $B \subseteq 2^{<\omega}$ and $f\colon B \to \omega \times \omega$ such that $f\,[B]$ is a partial function, $G_B \notin \mathcal{M}$ and for every $x \in \mathcal{A}$, $G_{f^{-1}[x]} \in \mathcal{M}$. Let c be a Cohen real. Then $c \notin G_{f^{-1}[x]}$ for every $x \in \mathcal{A}$. Define a partial function $y = \{f(c \restriction n) |\, n \in \omega\}$. Then $y \cap x$ is finite for every $x \in \mathcal{A}$, as otherwise $c \in G_{f^{-1}[x]}$. Thus $\mathcal{A}$ is $\mathbb{C}$-destructible.

For the implication (2) to (1) we use the Lemma. Let $s \in \mathbb{C}$ and let $\dot{f}$ be a name for an infinite partial function. We will show that $\dot{f}$ does not destroy the maximality of $\mathcal{A}$. Take A and g as in the Lemma. Since A is dense below s it codes a condition G_A below s. So by (2) there is $x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin \mathcal{M}$. For every $n \in \omega$ and condition $t \leq G_{g^{-1}[x]}$ there is $u \in g^{-1}\,[x]$ extending t with $g(u)_0 > n$, the subscript means the first coordinate of $g(u)$. Then $u \leq t, G_{g^{-1}[x]}$ forces that $g(u) \in \dot{f} \cap x$ and also $g(u)_0 > n$. Thus $G_{g^{-1}[x]}$ forces that $\dot{f} \cap x$ is infinite, so $\dot{f}$ does not destroy the maximality of $\mathcal{A}$. $\square$

Theorem 5.4.11. *Assume $\mathfrak{b} = \mathfrak{c}$. There exists a $\mathbb{C}$-indestructible maximal family of eventually different functions.*

Proof. Let $\{f_\alpha\colon 2^{<\omega} \to \omega \times \omega |\, \alpha < \mathfrak{c}\}$ enumerate all injective functions whose range is a partial function. Recursively construct a family of eventually different functions $\{x_\alpha |\, \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$ either there is $\beta < \alpha$ with $G_{f_\alpha^{-1}[x_\beta]} \notin \mathcal{M}$ or $G_{f_\alpha^{-1}[x_\alpha]} \notin \mathcal{M}$. If we are in the first case we just need a partial function x_α that is eventually different from all x_β, $\beta < \alpha$. Since $\alpha < \mathfrak{b}$ we can find a function x_α that dominates all x_β, $\beta < \alpha$. Clearly such a function is eventually different. So assume $G_{f_\alpha^{-1}[x_\beta]} \in \mathcal{M}$ for all $\beta < \alpha$. By 5.2.6 we have $f_\alpha^{-1}[x_\beta]$ is nowhere dense. Let $\{s_i \mid i \in \omega\}$ enumerate $2^{<\omega}$. For $i \in \omega$ choose $\beta_i < \alpha$ such that $[s_i] \cap f_\alpha^{-1}[x_{\beta_i}]$ is infinite. If there is an i for which this is not possible let $x_\alpha = f_\alpha[[s_i]]$ and we are done. Since all $f_\alpha^{-1}[x_\beta]$ are nowhere dense we can get all β_i to be distinct. For $\beta < \alpha$ not one of the β_n define $g_\beta \colon \omega \to \omega$ by $g_\beta(n) = \max\operatorname{dom}(x_{\beta_n} \cap x_\beta)$. Since $\mathfrak{b} = \mathfrak{c}$ there is a function $g \colon \omega \to \omega$ dominating all the g_β, $\beta < \alpha$. For $n \in \omega$ choose $(a_n, b_n) \in x_{\beta_n} \setminus \bigcup_{m<n} x_{\beta_m}$ such that $f_\alpha^{-1}((a_n, b_n)) \supseteq s_m$ and $a_n > g(n)$, this is possible by the choice of β_n. Define $x_\alpha = \{(a_n, b_n) \mid n \in \omega\}$. For $n \in \omega$, $x_\alpha \cap x_{\beta_n} \supseteq \{(a_m, b_m) \mid m \leq n\}$, so it is eventually different. For $\beta < \alpha$ not one of the β_n there is $n \in \omega$ such that $g(m) > g_\beta(m)$ for $m > n$. Then for $m > n$, $(a_m, b_m) \notin x_\beta$ as otherwise $a_m \in \operatorname{dom}(x_{\beta_m} \cap x_\beta)$ and $a_m \geq g(m) > g_\beta(m) = \max\operatorname{dom}(x_{\beta_m} \cap x_\beta)$. Thus x_α is eventually different from all x_β, $\beta < \alpha$. By the choice of (a_n, b_n) we have that $f_\alpha^{-1}[x_\alpha]$ is dense, so $G_{f_\alpha^{-1}[x_\alpha]} \notin \mathcal{M}$. This concludes the construction and by the previous Theorem $\{x_\alpha \mid \alpha < \mathfrak{c}\}$ is a Cohen indestructible maximal family of eventually different functions. □

5.4.3 Maximal antichain families

To show that $\mathbb{B}$ and $\mathbb{C}$ destructibility does not imply $\mathbb{M}$-destructibility we do the same thing as in the last section, but for a different modification of MADness, which is due to Leathrum [28].

Definition 5.4.12. *An almost disjoint family $\mathcal{A} \subseteq [\omega^{<\omega}]^\omega$ is called a maximal antichain family if every element of $\mathcal{A}$ is an antichain and it is maximal with this property.*

First we show that Cohen and Random forcing destroy maximal antichain families. These results are also due to Leathrum.

Theorem 5.4.13. *Cohen forcing destroys all maximal antichain families.*

Proof. View Cohen forcing as finite sequences of natural numbers. This is forcing equivalent to finite $0-1$ sequences. Let c be a Cohen real and define $A = \{f(c \restriction n) \mid n \in \omega\}$, where f is the function taking a sequence and returning the sequence with 1 added to the last entry. We will show that A is almost disjoint from any antichain in the ground model, in particular it witnesses that no maximal antichain family from the ground model stays maximal. Let $B \in V$ be an antichain. Define $D = \left\{s \in \omega^{<\omega} \mid \forall t \in \omega^{<\omega} \left(f(s^\frown t) \notin B\right)\right\}$. This set is dense. For $s \in \omega^{<\omega}$ let $u \supseteq s$ be either in B or s if no such u exist in B. Then $u^\frown 0 \in D$, as for every $t \in \omega^{<\omega}$, $f(u^\frown 0^\frown t) \subseteq u$ and since B is an antichain no extension of u is in B. Since all $s \in D$ force that $f(c \restriction n) \notin B$ for all $n > |s|$, this means that $A \cap B$ is finite. □

Theorem 5.4.14. *Random forcing destroys all maximal antichain families.*

Proof. There are mappings from $2^{<\omega}$ to $\omega^{<\omega}$, and back, that send maximal antichain families on the one side to maximal antichain families on the other, see [28]. So we can consider maximal antichain families on $2^{<\omega}$. As for Cohen forcing we consider the set

$A = \{f(r \upharpoonright n) |\, n \in \omega\}$, where r is the random real and f is the function that flips the last entry of a sequence. Let B be an antichain and let us show that $A \cap B$ is finite. Enumerate $B = \{b_i |\, i \in \omega\}$. Consider the sets $\bigcup_{i \geq n}[\mathrm{pred}(b_i)]$, where $\mathrm{pred}(s) = s \upharpoonright (|s| - 1)$ is the predecessor of s. We have

$$\mu\left(\bigcup_{i \geq n}[\mathrm{pred}(b_i)]\right) \leq 2\mu\left(\bigcup_{i \geq n}[b_i]\right) = 2\sum_{i \geq n}\mu\left([b_i]\right).$$

Let $p \in \mathbb{B}$ be a condition. Since the right hand side goes to 0 as n goes to infinity there is an n such that $\mu(\bigcup_{i \geq n}[\mathrm{pred}(b_i)]) < \mu(p)$. Then $q = p \setminus \bigcup_{i \geq n}[\mathrm{pred}(b_i)]$ has positive measure, so it is a condition strengthening p. Now q forces for $i > n$, $\mathrm{pred}(b_i) \not\subseteq r$, in particular there is no m such that $f(r \upharpoonright m) = b_i$. So $A \cap B \subseteq \{b_i |\, i \leq n\}$. □

As in the last section we first proof a result similar to 5.3.7.

Theorem 5.4.15. *Let $\mathcal{A}$ be a maximal antichain family. The following are equivalent:*

(1) $\mathcal{A}$ is $\mathbb{M}$-indestructible.

(2) $\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$ $\forall f\colon B \to \omega^{<\omega}$ such that $f[B]$ is an antichain $\exists A \in \mathcal{A}$ such that $G_{f^{-1}[A]} \notin \mathcal{K}_\sigma$.

(3) $\forall B \subseteq \omega^{<\omega}$ such that $G_B \notin \mathcal{K}_\sigma$ $\forall f\colon B \to \omega^{<\omega}$ injective such that $f[B]$ is an antichain $\exists A \in \mathcal{A}$ such that $G_{f^{-1}[A]} \notin \mathcal{K}_\sigma$.

(4) $\forall f\colon \omega^{<\omega} \to \omega^{<\omega}$ injective such that $f[\omega^{<\omega}]$ is an antichain $\exists A \in \mathcal{A}$ such that $G_{f^{-1}[A]} \notin \mathcal{K}_\sigma$.

(5) $\mathcal{A}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ adding an unbounded real.

Proof. The implications (1) to (5), (2) to (3) and (3) to (4) are trivial.

Let us show (5) implies (2). Assume not (2) and let $B \subseteq \omega^{<\omega}$ and $f\colon B \to \omega^{<\omega}$ witness this. Let $x \in G_B$ be an unbounded real. Then $x \notin G_{f^{-1}[A]}$, for all $A \in \mathcal{A}$. Define $D = \{f(x \upharpoonright n) |\, n \in \omega\}$. As in the proof of 5.2.2 we get that D is almost disjoint from all $A \in \mathcal{A}$. And since the image of f is an antichain, D is an antichain, so $\mathcal{A}$ is not a maximal antichain family in the extension.

It remains to show that (4) implies (1). Assume (4) holds. Take a condition $T \in \mathbb{M}$ and an $\mathbb{M}$-name $\dot{C}$ such that $T \Vdash$ "$\dot{C} \subseteq \omega^{<\omega}$ is an infinite antichain". Let T', A and g be as in 5.4.17. Then we get $B \in \mathcal{A}$ such that $G_{g^{-1}[B]} \notin \mathcal{K}_\sigma$. This condition forces that B and $\dot{C}$ are almost disjoint. Thus T does not force that $\dot{C}$ destroys the maximality of $\mathcal{A}$, so it is $\mathbb{M}$-indestructible. □

Lemma 5.4.16. *Let $T \in \mathbb{M}$ be a Miller tree, $C \subseteq \mathrm{split}(T)$ a set and $h\colon C \to 2$ a function such that $G_C = [T]$. Then there is $T' \leq T$, $C' \subseteq C$ such that h is constant on C', $\mathrm{stem}(T) = \mathrm{stem}(T')$, $C' \subseteq \mathrm{split}(T')$ and $G_{C'} = [T']$.*

Proof. Let $\sigma = \operatorname{stem}(T)$. For $n \in \omega$ and $i < 2$ define $C_i^n = h^{-1}[i] \cap T(\sigma^\frown n)$. Then $G_{C_0^n} \cup G_{C_1^n} = [T(\sigma^\frown n)]$, for each n. Thus one of those sets contains a Miller tree. Let $C^n \subseteq C_{i_n}^n$ generate such a tree. Now there is at least one i such that infinitely many $i_n = i$. Let $C' = \bigcup \{C^n \mid n \in \omega, i_n = i\}$ and T' be the tree generated by C'. This satisfies the lemma. □

Lemma 5.4.17. *Let $T \in \mathbb{M}$ and let $\dot{C}$ be an $\mathbb{M}$-name such that $T \Vdash$ "$\dot{C} \subseteq \omega^{<\omega}$ is an infinite antichain". Then there is a tree $T' \leq T$, a set $A \subseteq \omega^{<\omega}$ and an injective function $g\colon A \to \omega^{<\omega}$ such that*

- $[T'] = G_A$,
- $\forall \sigma \in A \Big(T'(\sigma) \Vdash \text{“}g(\sigma) \in \dot{C}\text{”}\Big)$ *and*
- $g[A]$ *is an antichain.*

Proof. With a bijection between ω and $\omega^{<\omega}$ and weak fusion we can get $B_n, \mathcal{A}_n, h_n$ and g as in the definition of weak fusion for $\dot{C}$. Now if we look at the proof of weak fusion for Miller forcing we see that this actually gives us a Miller tree $S \leq T, B \subseteq \operatorname{split}(S)$ and $g\colon B \to \omega^{<\omega}$ such that $S(t) \Vdash$ "$g(t) \in \dot{C}$", for all $t \in B$. Without loss of generality we may assume that $S = T$. Notice that if $s \subseteq t$ then $g(s)$ and $g(t)$ are incompatible, as $T(t)$ forces that both of them are contained in the antichain $\dot{C}$. Let $\{\sigma_n \mid n \in \omega\}$ enumerate $\omega^{<\omega}$ in a way such that $\sigma_n \subseteq \sigma_m$ implies $n \leq m$. We will recursively construct a set $\{s_n \mid n \in \omega\} \subseteq \omega^{<\omega}$, Miller trees T_n and sets $B_n \subseteq \operatorname{split}(T_n)$ such that

(1) $T_0 = T, B_0 = B, \sigma_0 = \operatorname{stem}(T)$,

(2) $G_{B_n} = [T_n]$,

(3) $\{s_i \mid i \leq n\} \subseteq B_n$,

(4) $B_{n+1} \subseteq B_n$,

(5) $s_i \cap s_j = s_n$ if and only if $\sigma_i \cap \sigma_j = \sigma_n$ and

(6) $\forall t \in B_n \setminus \{s_n\} \Big(g(t) \perp g(s_n)\Big)$.

If we had this let $A = \{s_n \mid n \in \omega\}$. By condition (5) there is a Miller tree T' such that $G_A = [T]$. Because of condition (6), $g[A]$ is an antichain and we are done. Assume we have constructed this for n. We can find $s \in B_n$ such that $s \cap s_j = s_i$ if and only if $\sigma_{n+1} \cap \sigma_j = \sigma_i$, for all $i, j \leq n$. Recursively construct an increasing sequence $\left\{s^j \,\middle|\, j \leq n\right\} \subseteq B_n$.

For $j = 0$ let $s^{-1} = s$ and continue as in the case $j > 0$.

Assume we have already constructed s^{j-1}. If there is $t \in B_n, T^j \leq T_n(s_j)$ and $C^j \subseteq B_n$ such that $t \supseteq s^{j-1}, C^j \subseteq \operatorname{split}(T^j), G_{C^j} = [T^j], \operatorname{stem}(T^j) = s_j$ and $g(t) \subseteq g(u)$, for all $u \in T^j$, let $s^j = t$. Otherwise $s^j = s^{j-1}$.

Let $s_{n+1} \in B_n$ be a proper extension of s^n. For $j \leq n$ where the second alternative holds define $h^j : B_n \cap T_n(s_j) \to 2$ by $h^j(t) = 0$ if and only if $g(t) \perp g(s_{n+1})$. Then with 5.4.16 we can find $T^j \leq T_n(s_j)$ and $C^j \subseteq B_n \cap T_n(s_j)$ such that h^j is constant on C^j, $\text{stem}(T^j) = s_j, C^j \subseteq \text{split}(T^j)$ and $G_{C^j} = [T^j]$.

Assume towards a contradiction that $h^j \restriction C^j = 1$. Then for every $t \in C^j$, $g(t) \subseteq g(s_{n+1})$ or $g(t) \supseteq g(s_{n+1})$. If $g(t) \subseteq g(s_{n+1})$ were true for some $t \in C^j$ we would have for every $u \in B$, $g(t) \perp g(u)$ implies $g(u) \perp g(s_{n+1})$. But there are $u \in C^j$ extending t which by the earlier remark satisfy $g(u) \perp g(t)$ thus $g(u) \perp g(s_{n+1})$ and therefore $h^j(u) = 1$, a contradiction. Thus for every $t \in C^j$, $g(s_{n+1}) \subseteq g(t)$. But this means that s_{n+1}, T^j and C^j witness the first alternative in the construction of s^j, a contradiction.

Thus $h^j \restriction C^j = 0$ and therefore $g(t) \perp g(s_{n+1})$ for all $t \in C^j$. Now if the first alternative is true we also have that $g(t) \perp g(s_{n+1})$ for all $t \in C^j$. So if we define

$$B_{n+1} = \bigcup_{j \leq n} C^j \cup \{s_j | \ j \leq n+1\} \cup (B_n \cap T_n(s_{n+1})) \text{ and}$$

$$T_{n+1} = \bigcup_{j \leq n} T^j \cup T_n(s_{n+1}),$$

we have that (6) is satisfied. (5) comes from the choice of s in the beginning and the previous steps. The other ones are also clear. □

Theorem 5.4.18. *Assume $\mathfrak{b} = \mathfrak{c}$. There is an $\mathbb{M}$-indestructible maximal antichain family.*

Proof. Let $\{f_\alpha : \omega^{<\omega} \to \omega^{<\omega} | \ \alpha < \mathfrak{c}\}$ enumerate all injective functions whose range is an antichain. We will iteratively construct an antichain family $\{A_\alpha | \ \alpha < \mathfrak{c}\}$ such that if $G_{f_\alpha^{-1}[A_\beta]} \in \mathcal{K}_\sigma$, for all $\beta < \alpha$, then $G_{f_\alpha^{-1}[A_\alpha]}$ contains a rational perfect set. This condition guarantees that $\{A_\alpha | \ \alpha < \mathfrak{c}\}$ is a maximal antichain family, as for every antichain there is some α such that $f_\alpha [\omega^{<\omega}]$ is this antichain, then there is $\beta \leq \alpha$ such that $G_{f_\alpha^{-1}[A_\beta]} \notin \mathcal{K}_\sigma$, so in particular $f_\alpha^{-1} [A_\beta]$ is infinite and therefore A_β has infinite intersection with this antichain. Additionally 5.3.7 (4) says that $\{A_\alpha | \ \alpha < \mathfrak{c}\}$ is $\mathbb{M}$-indestructible.

Assume we already constructed A_β for all $\beta < \alpha$. The minimal size of a maximal antichain family is at least $\mathfrak{b}$, see [28]. So if there is some $\beta < \alpha$ such that $G_{f^{-1}[A_\beta]} \notin \mathcal{K}_\sigma$, then there is some antichain A_α that is almost disjoint from all A_β, for $\beta < \alpha$. Assume that $G_{f_\alpha^{-1}[A_\beta]} \in \mathcal{K}_\sigma$, for all $\beta < \alpha$. As $\text{add}(\mathcal{K}_\sigma) = \mathfrak{b} = \mathfrak{c}$ we have that $\bigcup_{\beta<\alpha} G_{f_\alpha^{-1}[A_\beta]} \in \mathcal{K}_\sigma$. So there is $h \in \omega^\omega$ that dominates every $x \in \bigcup_{\beta<\alpha} G_{f_\alpha^{-1}[A_\beta]}$. Recursively construct a set $\{\sigma_s | \ s \in \omega^{<\omega}\} \subseteq \omega^{<\omega}$ such that

- $\sigma_s \subseteq \sigma_t$ for all $s \subseteq t$,
- $\sigma_{s^\frown n}(|\sigma_s|) < \sigma_{s^\frown m}(|\sigma_s|)$ for all $n < m$,
- $\{n | \ f_\alpha(\sigma_{s^\frown n}) \in A_\beta\}$ is finite for every $\beta < \alpha$ and s and

- $\sigma_s(i) \geq h(i)$ for all $i < |\sigma_s|$.

Assume we have constructed σ_s. Let $\{x_n | \; n \in \omega\} \subseteq \omega^\omega$ be such that $\sigma_s \subseteq x_n$, $x_n \geq h$ and $x_n(|\sigma_s|) > x_m(|\sigma_s|)$ for all $n > m$. Then $x_n \notin \bigcup_{\beta<\alpha} G_{f_\alpha^{-1}[A_\beta]}$, so $\{m | \; f_\alpha(x_n \restriction m) \in A_\beta\}$ is finite. Let $k_\beta(n)$ be greater than all elements in this set. Since $\alpha < \mathfrak{b}$ there is $k \in \omega^\omega$ that dominates all k_β for $\beta < \alpha$. Let $\sigma_{s^\frown n} = x_n \restriction k(n)$. This completes the construction. For $\beta < \alpha$ define $l_\beta \colon \omega^{<\omega} \to \omega$ such that $f_\alpha(\sigma_{s^\frown m}) \notin A_\beta$ for all $m > l_\beta(s)$. This is possible by the third condition. Let l dominate all l_β, where dominating means that there is some length such that all longer sequences get bigger values. Let $A = \{\sigma_s | \; \forall i < |s| (s(i) > l(s \restriction i))\}$ and let T be the tree generated by A. The first two conditions guarantee that the σ_s generate a rational perfect tree, thus T is still a rational perfect tree. For every $\beta < \alpha$ there is some length n such that from then on l is bigger than l_β, then for all $\sigma_s \in A$ with $|s| > n$, $f_\alpha(\sigma_s) \notin A_\beta$. Thus $f_\alpha[A] \cap A_\beta$ is finite. So if we let $A_\alpha = f_\alpha[A]$ we have that A_α is almost disjoint from all A_β, for $\beta < \alpha$, and $G_{f_\alpha^{-1}[A_\alpha]}$ contains the rational perfect set $[T]$. This completes the construction. □

5.4.4 Indestructibility of ideals

Finally we have to look at the indestructibility of tall ideals instead of MAD families, because there can't be $\mathbb{D}$ or $\mathbb{L}$-indestructible MAD families. Which is also the reason why there are no arrows towards $\mathbb{D}$ or $\mathbb{L}$ from the other forcings.

Theorem 5.4.19. *The following are equivalent for any forcing $\mathbb{P}$:*

(1) $\mathbb{P}$ adds an unbounded real.

(2) $\mathbb{P}$ destroys $\mathcal{I}_\mathbb{M} = \{I \subseteq \omega^{<\omega} | \; G_I \in \mathcal{K}_\sigma\}$.

Proof. If $\mathcal{I}_\mathbb{M}$ is $\mathbb{P}$-indestructible and $\mathbb{P}$ adds an unbounded real, then 5.3.7 implies that $\mathcal{I}_\mathbb{M} \not\leq_K \mathcal{I}_\mathbb{M}$, a contradiction.

Now assume $\mathbb{P}$ does not add an unbounded real. Take $A \in [\omega^{<\omega}]^\omega$ in $V^\mathbb{P}$ and let us show that there is $I \in \mathcal{I}_\mathbb{M}^V$ such that $I \cap A$ is infinite.

If A contains a branch x then there is a ground model function f that dominates x. Then $I = \{s \in \omega^{<\omega} | \; s \leq f\} \in \mathcal{I}_\mathbb{M}^V$ and for all $n \in \omega$, $x \restriction n \in I$. So $A \cap I$ is infinite.

On the other hand assume A does not contain a branch. If for every $\sigma \in \omega^{<\omega}$ there are only finitely many $n \in \omega$ such that there exists a τ with $\sigma^\frown n^\frown \tau \in A$, then there is $f \in \omega^\omega$ such that $s \leq f$, for every $s \in A$. Then there is $g \in V$ dominating f and therefore $A \subseteq \{s \in \omega^{<\omega} | \; s \leq f\} \in \mathcal{I}_\mathbb{M}$. If this is not the case there is $\sigma \in \omega^{<\omega}$ and infinitely many $n \in \omega$ such that there exists τ_n with $\sigma^\frown n^\frown \tau_n \in A$. As $\mathbb{P}$ does not add an unbounded real there is a ground model function $\phi \colon \omega \to [\omega^{<\omega}]^{<\omega}$ such that for all n if there is τ with $\sigma^\frown n^\frown \tau \in A$ then there is such a τ with $\tau \in A$. To see this take a ground model function g dominating the function that sends n to $|\tau_n| + \max_{m<|\tau_n|} \tau(n)$ and let $\phi(n) = g(n)^{g(n)}$. Define $I = \{\sigma^\frown n^\frown \tau | \; \tau \in \phi(n)\}$. Then $G_I = \emptyset$ so $I \in \mathcal{I}_\mathbb{M}$ and $A \cap I$ is infinite. □

Something similar also holds for Cohen reals, for a proof see [9].

Theorem 5.4.20. *The following are equivalent for any forcing $\mathbb{P}$:*

(1) $\mathbb{P}$ adds a Cohen real,

(2) $\mathbb{P}$ destroys $\mathcal{I}_{\mathbb{C}} = \{I \subseteq 2^{<\omega} | \ G_I \in \mathcal{M}\}$,

(3) $\mathbb{P}$ destroys $\mathcal{I}_{\text{nwd}} = \{I \subseteq 2^{<\omega} | \ G_I \in \text{nwd}\}$.

Theorem 5.4.21. *There exists a tall ideal that is $\mathbb{C}$-destructible and $\mathbb{L}$-indestructible, namely $\mathcal{I}_{\mathbb{C}} = \{I \subseteq 2^{<\omega} | \ G_I \in \mathcal{M}\}$ is such an ideal.*

Proof. That $\mathcal{I}_{\mathbb{C}}$ is $\mathbb{C}$-destructible follows from 5.3.9 with $B = 2^{<\omega}$ and f the identity. And since Laver forcing does not add Cohen reals $\mathcal{I}_{\mathbb{C}}$ is $\mathbb{L}$-indestructible by the previous theorem. □

Theorem 5.4.22. *There exists a tall ideal that is $\mathbb{B}$-destructible and $\mathbb{D}$-indestructible, namely $\mathcal{I}_{\mathbb{B}} = \{I \subseteq 2^{<\omega} | \ G_I \in \mathcal{N}\}$ is such an ideal.*

Proof. That $\mathcal{I} = \mathcal{I}_{\mathbb{B}}$ is $\mathbb{B}$-destructible follows from 5.3.11 with $B = 2^{<\omega}$ and f the identity. To show that it is $\mathbb{D}$-indestructible we use 5.3.10 and show that for every injective partial function $f\colon \omega^{\uparrow<\omega} \to 2^{<\omega}$ with $\text{dom}(f) \notin \mathcal{M}_{\mathcal{D}}$ there is $I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin \mathcal{M}_{\mathcal{D}}$. We do so by constructing $I \subseteq 2^{<\omega}$ such that $|I \cap 2^n| \leq 1$ for all $n \in \omega$ and $G_{f^{-1}[I]} \notin \mathcal{M}_{\mathcal{D}}$. This implies that $G_I \in \mathcal{N}$. To see this enumerate $I = \{s_i | \ i \in \omega\}$ such that $|s_i| = i$, just enlarge I if there is no such s_i for some $i \in \omega$. Then $G_I \subseteq \bigcup_{i>n}[s_i]$ for all $n \in \omega$ and the measure of the set on the right is $\sum_{i>n} 2^{-i}$, which goes to zero as n goes to infinity.

Fix some f as required. Then there is $(s_0, h_0) \in \mathbb{D}$ such that $U_{(s_0,h_0)} \setminus G_{\text{dom}(f)} \in \mathcal{M}_{\mathcal{D}}$. Let $T_0 = \{s \in \omega^{\uparrow<\omega} \,|\, s \text{ is compatible with } (s_0, h_0)\}$, as a reminder s being compatible with (s_0, h_0) means that $s \supseteq s_0$ and $s(i) \geq h_0(i)$, for all $i \in \text{dom}(s)$. As usual for forcings adding a dominating real we define a rank function for $s \in T_0$.

$$\text{rk}(s) = 0 \Leftrightarrow \exists \{m_n | \ n \in \omega\} \subseteq \omega \text{ such that } s^\frown m_n \in T_0 \wedge m_n \geq n \wedge s^\frown m_n \in \text{dom}(f)$$

$$\text{rk}(s) \leq \beta \Leftrightarrow \exists \{m_n | \ n \in \omega\} \subseteq \omega \text{ such that } s^\frown m_n \in T_0 \wedge m_n \geq n \wedge \text{rk}(s^\frown m_n) < \beta$$

The rank is either undefined, in which case we say it is infinite, or it is less than ω_1.

Claim. $\text{rk}(s) < \infty$ *for all* $s \in T_0$.

Proof. Towards a contradiction assume $\text{rk}(s) = \infty$ for some $s \in T_0$. Recursively define a function $h \in \omega^\omega$ such that $s \subseteq h$, $h(i) \geq h_0(i)$, for all $i \in \omega$, and

$$\text{rk}(t) = \infty \text{ and } t \notin \text{dom}(f) \text{ for all } t \in T_0 \text{ compatible with } (s, h) \text{ and } s \subsetneq t. \qquad (\star)$$

If there were infinitely many $n \in \omega$ such that $\text{rk}(s^\frown n) < \infty$ or $s^\frown n \in \text{dom}(f)$ we would have $\text{rk}(s) < \infty$. Thus there are only finitely many n for which this is the case and we can define $h(|s|)$ to be bigger than all of them. Now assume we already defined $h \restriction m$ for $m > |s|$ such that $(\star)$ holds for t with $|t| \leq m$. Towards another contradiction assume we can't continue the construction of h. Then there is $\{t_n | \ n \in \omega\} \subseteq T_0$ such that $|t_n| = m + 1, t_n(m) \geq n, s \subseteq t_n, t_n(i) \geq h(i)$, for all $i < m + 1$ and

- either $\mathrm{rk}(t_n) < \infty$
- or $t_n \in \mathrm{dom}(f)$.

We can prune $\{t_n |\, n \in \omega\}$ to get $s' \in T_0$ such that $s' \supseteq s$ and $|s'| \leq m, t_n(|s'|) \geq n$, for all $n \in \omega$. Because of $(\star)$ we have $\mathrm{rk}(s') = \infty$ and $s' = s$ or $s' \notin \mathrm{dom}(f)$. By the definition of rank this implies that for almost all $n \in \omega$, $\mathrm{rk}(t_n) = \infty$ and $t_n \notin \mathrm{dom}(f)$. This contradicts the definition of $\{t_n |\, n \in \omega\}$. So we can define h. Now $U_{(s,h)} \subseteq U_{(s_0,h_0)}$ and $G_{\mathrm{dom}(f)} \cap U_{(s,h)} = \emptyset$, which contradicts $U_{(s_0,h_0)} \setminus G_{\mathrm{dom}(f)}$. □

For $s \in T_0$ with $\mathrm{rk}(s) = 0$ fix $\{m_n^s | n \in \omega\} \subseteq \omega$ as in the definition of rank. Since f is injective we can prune all the $\{m_n^s |\, n \in \omega\}$ to get $|I \cap 2^m| \leq 1$ for $I = \left\{ f(s^\frown m_n^s) \middle|\, s \in T_0, n \in \omega, \mathrm{rk}(s) = 0 \right\}$. We shall argue that $G_{f^{-1}[I]}$ is $\mathcal{D}$-comeager in $U_{(s_0,h_0)}$. Take an arbitrary $(s,h) \in \mathbb{D}$ such that $U_{(s,h)} \subseteq U_{(s_0,h_0)}$. Then there exists $s' \in T_0$ compatible with (s,h) such that $\mathrm{rk}(s') = 0$. To see this assume there are no such s' and take some $s' \in T_0$ compatible with (s,h) of smallest rank. It has rank less than infinity, so by the definition of rank there exists some $m > h(|s'|)$ such that $s'^\frown m \in T_0$ and $\mathrm{rk}(s'^\frown m) < \mathrm{rk}(s')$. But then $s'^\frown m$ is still compatible with (s,h), but of smaller rank, a contradiction. Since $\mathrm{rk}(s') = 0$ there exists some $m > h(|s'|)$ such that $s'^\frown m \in \mathrm{dom}(f)$. Then $s'^\frown m \in f^{-1}[I]$, so $G_{f^{-1}[I]} \cap U_{(s,h)} \neq \emptyset$. □

5.5 Tight MAD families

In this section we want to consider a strengthening of MAD families introduced by Malykhin [29].

Definition 5.5.1. *An almost disjoint family $\mathcal{A}$ is called tight if for every $\{X_n |\, n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite, for every $n \in \omega$.*

If an almost disjoint family is tight it is also maximal, which is why we call tight almost disjoint families tight MAD families. To see this take an infinite set $X \subseteq \omega$. If $X \in \mathcal{I}(\mathcal{A})$ then it has infinite intersection with some element of $\mathcal{A}$. And if $X \notin \mathcal{I}(\mathcal{A})$ the tightness condition for the sequence with every $X_n = X$ gives an element of $\mathcal{A}$ that has infinite intersection with X.

Kurilić [25] showed that tight MAD families can exist.

Theorem 5.5.2. *If $\mathfrak{b} = \mathfrak{c}$ then tight MAD families exist.*

Proof. Assume $\mathfrak{b} = \mathfrak{c}$. Enumerate all countable subsets of $[\omega]^\omega$ as $\{\{A_n^\alpha |\, n \in \omega\} |\, \alpha < \mathfrak{c}\}$. We will recursively construct a MAD family $\{A_\alpha |\, \alpha < \mathfrak{c}\}$ such that for all $\alpha < \mathfrak{c}$:

- either $\exists n \in \omega \Big(A_n^\alpha \in \mathcal{I}(\{A_\beta |\, \beta < \alpha\}) \Big)$,
- or $\forall n \in \omega \Big(A_n^\alpha \cap A_\alpha \Big)$ is infinite.

Assume we already constructed this up to α. If the "either" part is true we can let A_α be any set that is almost disjoint from all A_β, for $\beta < \alpha$, this exists since $\alpha < \mathfrak{c} = \mathfrak{b} = \mathfrak{a}$. So we can assume that $A_n^\alpha \notin \mathcal{I}(\{A_\beta |\ \beta < \alpha\})$, for all $n \in \omega$. As $\mathfrak{a} = \mathfrak{c}$ we even have that no restriction of $\{A_\beta |\ \beta < \alpha\}$ is MAD, in particular there are infinite sets $\bar{A}_n^\alpha \subseteq A_n^\alpha$ almost disjoint from all A_β. Without loss of generality $\bar{A}_n^\alpha = A_n^\alpha$. For $\beta < \alpha$ define $f_\beta \colon \omega \to \omega$ such that $A_\beta \cap A_n^\alpha \subseteq f_\beta(n)$. Since $\alpha < \mathfrak{b}$ there is a function $f \in \omega^\omega$ eventually dominating all f_β. Define $A_\alpha = \bigcup \{A_n^\alpha \setminus f(n) |\ n \in \omega\}$. Then $A_n^\alpha \cap A_\alpha \supseteq A_n^\alpha \setminus f(n)$ and therefore infinite. And for $\beta < \alpha, A_\beta \cap A_\alpha = \bigcup_{n\in\omega} (A_\beta \cap A_n^\alpha) \setminus f(n)$ which is finite for every n and will eventually become empty since f eventually dominates f_β. □

There is a close relationship between tight MAD families and Cohen indestructible MAD families.

Theorem 5.5.3 (Malykhin [29], Hruŝák, García Ferreira [19], Kurilić [25])**.**

(1) If $\mathcal{A}$ is tight, then $\mathcal{A}$ is Cohen indestructible.

(2) If $\mathcal{A}$ is Cohen indestructible, then there is $X \in \mathcal{I}(\mathcal{A})^+$ such that $\mathcal{A} \restriction X$ is tight.

Proof. Assume $\mathcal{A}$ is a tight MAD family. We will show 5.3.9(4). Fix an injective function $f \colon 2^{<\omega} \to \omega$. Enumerate $2^{<\omega} = \{s_i |\ i \in \omega\}$. If there is $i \in \omega$ with $f\,[[s_i]] \in \mathcal{I}(\mathcal{A})$ we are done. So assume there is no such i. Define $A_i = f\,[[s_i]]$. Then $\{A_i |\ i \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$. Since $\mathcal{A}$ is tight there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap A_i$ is infinite, for every $i \in \omega$. So $f^{-1}[B]$ is dense, so $G_{f^{-1}[B]} \notin \mathcal{M}$.

Assume now that no restriction of $\mathcal{A}$ is tight, we will show that then $\mathcal{A}$ is Cohen destructible. For $s \in \omega^{<\omega}$ we can recursively construct sets $X_s \in \mathcal{I}(\mathcal{A})^+$ such that $\{X_{s^\frown n} |\ n \in \omega\}$ witnesses that $\mathcal{A} \restriction X_s$ is not tight. Let $c \in \omega^\omega$ be a Cohen real. There is an infinite set X that is almost contained in $X_{c\restriction n}$, for all $n \in \omega$. Just take an increasing sequence of $x_n \in \bigcap_{m<n} X_{c\restriction m}$ and let $X = \{x_n |\ n \in \omega\}$. Fix $A \in \mathcal{A}$. Then $\{s \in \omega^{<\omega} |\ |A \cap X_s| < \omega\}$ is dense. So there exists $n \in \omega$ with $|A \cap X_{c\restriction n}| < \omega$, which means $A \cap X$ is finite. Thus X destroys the maximality of $\mathcal{A}$. □

5.5.1 Strong preservation of tight MAD families

So far we have only discussed preservation via a single forcing extension. And looking back at the definition of weak fusion there is a reason for that. The first one is that it only works for real forcing, which in a sense only add a single real. This can be circumvented by considering the Borel sets on $(2^\omega)^\alpha$ for some ordinal α. In that way it is possible to, for example, construct a MAD family that is indestructible by Sacks forcing, but destroyed by the 2-step iteration of Sacks forcing. But even then we run into the problem that g has to be an injective function with range ω, thus requiring that B is countable. So for iterations of uncountable length this notion will not work. That is why in this section we will discuss a preservation result for tight MAD families that is preserved under iterations. This notion and all the relating proofs are due to Guzmán, Hrusák and Téllez [15].

Definition 5.5.4. *Let $\mathcal{A}$ be a tight mad family. A proper forcing $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$ if for every $p \in \mathbb{P}$, M a countable elementary submodel of $H(\kappa)$ (where κ is a large enough regular cardinal) such that $\mathbb{P}, \mathcal{A}, p \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$, for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there is $q \leq p$ an $(M, \mathbb{P})$-generic condition such that $q \Vdash$ "$\forall \dot{Z} \in \left(\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]\right) |\dot{Z} \cap B| = \omega$" (where $\dot{G}$ denotes the name for the generic filter). We say that q is an $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition.*

We see a similarity to the definition of proper here, which will also be seen in most of the proofs regarding it.

Lemma 5.5.5. *Let $\mathcal{A}$ be an almost disjoint family, $\mathbb{P}$ a forcing, $p \in \mathbb{P}$ and $\dot{B}$ a name for a subset of ω such that $p \Vdash$ "$\dot{B} \in \mathcal{I}(\mathcal{A})^+$". Then $C = \left\{n \in \omega \middle| \exists q \leq p \left(q \Vdash \text{"} n \in \dot{B} \text{"}\right)\right\} \in \mathcal{I}(\mathcal{A})^+$.*

Proof. It is forced that $\dot{B}$ is a subset of C. □

Lemma 5.5.6. *Let $\mathcal{A}$ be a tight mad family, $\mathbb{P}$ a forcing, that strongly preserves the tightness of $\mathcal{A}$ and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a forcing which is forced to strongly preserve the tightness of $\mathcal{A}$. Then $\mathbb{P} * \dot{\mathbb{Q}}$ strongly preserves the tightness of $\mathcal{A}$. Furthermore, if $B \in \mathcal{I}(\mathcal{A})$, M a countable elementary submodel with $\mathcal{A}, \mathbb{P}, \dot{\mathbb{Q}} \in M$, $p \in \mathbb{P}$ a $(M, \mathbb{P}, \mathcal{A}, B)$-generic condition, and $\dot{q}$ is a $\mathbb{P}$-name for an element of $\dot{\mathbb{Q}}$ such that $p \Vdash$ "$\dot{q}$ is an $(M, \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition", then $(p, \dot{q})$ is an $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic condition.*

Proof. That $(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}})$-generic is just 3.2.1. Let G be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter such that $(p, \dot{q}) \in G$ and let H denote the projection of G to $\mathbb{P}$. Since p is $(M, \mathbb{P}, \mathcal{A}, B)$-generic we have that $|Z \cap B| = \omega$, for every $Z \in (\mathcal{I}(\mathcal{A})^+ \cap M[H])$. Also $\dot{q}$ is $(M, \mathbb{Q}, \mathcal{A}, B)$-generic in $M[H]$. Thus $|Z \cap B| = \omega$, for every $Z \in (\mathcal{I}(\mathcal{A})^+ \cap M[G])$. So $(p, \dot{q})$ is $(M, \mathbb{P} * \dot{\mathbb{Q}}, \mathcal{A}, B)$-generic.

□

Lemma 5.5.7. *Let $\left\{(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \middle| \alpha < \lambda\right\}$ be a countable support iteration of forcings that strongly preserve the tightness of $\mathcal{A}$, i.e $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ strongly preserves the thightness of $\mathcal{A}$", for all $\alpha < \lambda$. Let $B \in \mathcal{I}(\mathcal{A})$, M a countable elementary submodel of $H(\kappa)$, where κ is a large enough regular cardinal, with $\mathcal{A}, \left\{(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \middle| \alpha < \lambda\right\}, \lambda \in M$. Let $\alpha \in M \cap \lambda$, $p \in \mathbb{P}_\alpha$ a $(M, \mathbb{P}_\alpha, \mathcal{A}, B)$-generic condition and let $\dot{q}$ be a $\mathbb{P}_\alpha$-name such that $p \Vdash_\alpha$ "$\dot{q} \in \mathbb{P}_\lambda \cap M \wedge \dot{q} \restriction \alpha \in \dot{G}_\alpha$" . Then there is an $(M, \mathbb{P}_\lambda, \mathcal{A}, B)$-generic condition $\bar{p} \in \mathbb{P}_\lambda$ such that $\bar{p} \restriction \alpha = p$ and $\bar{p} \Vdash_\lambda$ "$\dot{q} \in \dot{G}$".*

Proof. We will prove this by induction on λ. For successor ordinal this follows easily from 5.5.6. So assume λ is a limit ordinal and the statement is true for every smaller ordinal. Let $\{\alpha_n | \, n < \omega\} \subseteq M \cap \lambda$ be cofinal and $\alpha_0 = \alpha$. Enumerate the dense sets of $\mathbb{P}_\lambda$ contained in M as $\{D_n | \, n < \omega\}$. Enumerate the $\mathbb{P}_\lambda$-names for elements of $\mathcal{I}(\mathcal{A})^+$ as $\left\{\dot{Z}_n \middle| n < \omega\right\}$, such that each name appears infinitely often. Recursively construct $(\dot{q}_n)_{n<\omega}$, $(p_n)_{n<\omega}$ and $(\dot{m}_n)_{n<\omega}$ such that the following holds:

(1) $p_0 = p$, $\dot{q}_0 = \dot{q}$,

(2) $p_n \in \mathbb{P}_{\alpha_n}$ is $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic, $p_n \restriction \alpha_{n-1} = p_{n-1}$,

(3) $\dot{q}_n$ is a $\mathbb{P}_{\alpha_n}$-name such that $p_n \Vdash$ "$\dot{q}_n \in \mathbb{P}_\lambda \cap M \cap D_{n-1} \wedge \dot{q}_n \restriction \alpha_n \in G_{\alpha_n}$",

(4) $p_{n+1} \Vdash$ "$\dot{q}_{n+1} \leq \dot{q}_n$" and

(5) $\dot{m}_n$ is a $\mathbb{P}_\lambda$-name such that $p_n \Vdash$ "$\dot{q}_n \Vdash$ "$\dot{m}_n \in \left(\dot{Z}_n \cap B\right) \setminus n$"".

Assume we have already defined this for n. Let $G_{\alpha_n} \subseteq \mathbb{P}_{\alpha_n}$ be generic with $p_n \in G_{\alpha_n}$. Work in $V[G_{\alpha_n}]$. Let $C = \left\{ m \in \omega \,\middle|\, \exists r \leq \dot{q}_n \left(r \restriction \alpha_n = \dot{q}_n \restriction \alpha_n \wedge r \Vdash \text{“}m \in \dot{Z}_n\text{”} \right) \right\} \in M[G_{\alpha_n}]$. By 5.5.5 this is in $\mathcal{I}(\mathcal{A})^+$. Since p_n is $(M, \mathbb{P}_{\alpha_n}, \mathcal{A}, B)$-generic, this means there is some $m \in (C \cap B) \setminus (n+1)$. Let r witness this. Again by genericity we can assume that $r \in M \cap D_{n-1}$. Also $r \restriction \alpha_n = \dot{q}_n \restriction \alpha_n \in G_{\alpha_n}$. Now work in V again. Let $\dot{q}_{n+1}$ and $\dot{m}_{n+1}$ be names for r and m that are forced by p_n to have these properties. Now use that the lemma holds for α_{n+1}, p_n and $\dot{q}_{n+1}$ to get p_{n+1} $(M, \mathbb{P}_{\alpha_{n+1}}, \mathcal{A}, B)$-generic such that $p_{n+1} \restriction \alpha_n = p_n$ and $p_{n+1} \Vdash$ "$\dot{q}_{n+1} \in \dot{G}$". Then all requirements are fulfilled and the construction is complete.

Let $\bar{p} = \bigcup_{n \in \omega} p_n$, this gives a condition since the iteration has countable support and the p_n form an increasing sequence where all initial segments agree. That $\bar{p} \Vdash$ "$\dot{q}_n \in \dot{G}$", for every n, is the same as in the proof for the iteration of proper forcings, see 3.2.3. This then gives us that $\bar{p}$ is $(M, \mathbb{P}_\lambda)$-generic, since for every dense set in M one of the $\dot{q}_n$ is in there. And $(M, \mathbb{P}_\lambda, \mathcal{A}, B)$-genericity is clear, as for every element of $\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]$ a name for it appears infinitely often among the $\dot{Z}_n$, and thus the $\dot{m}_n$ witness that it has infinite intersection with B.

□

Theorem 5.5.8. *Let* $\left\{ (\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha) \,\middle|\, \alpha < \lambda \right\}$ *be a countable support iteration of forcings that strongly preserve the tightness of* $\mathcal{A}$. *Then* $\mathbb{P}_\lambda$ *strongly preserves the tightness of* $\mathcal{A}$.

Proof. Let $q \in \mathbb{P}_\lambda$, M a countable elementary submodel of $H(\kappa)$ such that $\mathbb{P}_\lambda, \mathcal{A}, \dot{q} \in M$ and $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \omega$, for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. We have $\mathbb{P}_0 = \{1\}$ and 1 is $(M, \mathbb{P}_0, \mathcal{A}, B)$-generic, since $V[G_0] = V$. Thus we can use the previous lemma with $p = 1$ to get an $(M, \mathbb{P}_\lambda, \mathcal{A}, B)$-generic condition p' that forces "$q \in \dot{G}_\lambda$". As filters are directed there is $p \leq p', q$. Genericity is preserved by going to stronger conditions, so $p \leq q$ is $(M, \mathbb{P}_0, \mathcal{A}, B)$-generic.

□

6 Partition forcing

For the rest of this thesis we will discuss how the properties we have introduced so far relate to another forcing, which was introduced by Miller [30]. We want to also look at how this forcing behaves under iterations.

Definition 6.0.1. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Then $\mathbb{P}(\mathcal{C})$, called partition forcing, is the forcing consisting of perfect trees $T \subseteq 2^{<\omega}$ such that for every $C \in \mathcal{C}$, $C \cap [T]$ is nowhere dense in $[T]$, i.e. for every $t \in T$ there is $s \in T$ extending t such that $[T(s)] \cap C = \emptyset$, ordered by reverse inclusion.*

If we let x be the generic real added by this forcing, i.e. $x = \bigcap_{T \in G} T$, then $x \notin C$ for every $C \in \mathcal{C}$. So $\mathcal{C}$ is no longer a partition of 2^ω.

Definition 6.0.2. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. We say that $x, y \in 2^\omega$ are $\mathcal{C}$-different if they are in different elements of $\mathcal{C}$, conversely we say they are $\mathcal{C}$-equal if they are in the same element of $\mathcal{C}$. A tree $T \subseteq 2^{<\omega}$ is called $\mathcal{C}$-branching if for every $t \in T$ there are $\mathcal{C}$-different branches in $[T]$ extending t.*

Using this we can give some other characterization of elements of partition forcing that might seem more natural. This as well as the fusion later is due to Cruz-Chapital, Fischer, Guzmán and Šupina [12].

Lemma 6.0.3. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. The following are equivalent for a tree $T \subseteq 2^{<\omega}$:*

(1) $T \in \mathbb{P}(\mathcal{C})$.

(2) T is $\mathcal{C}$-branching.

(3) T is perfect and there is a countable dense set $D \subseteq [T]$ of $\mathcal{C}$-different branches.

(4) T is perfect and there is a dense set $D \subseteq [T]$ of $\mathcal{C}$-different branches.

Proof. ((1) ⇒ (3)) That T is perfect is included in the definition of $T \in \mathbb{P}(\mathcal{C})$. Enumerate $T = \{t_i |\, i < \omega\}$. Recursively construct $\mathcal{C}$-different branches $x_i \in T(t_i)$. For $i = 0$ let x_0 be an arbitrary branch in $T(t_0)$ and pick $C_0 \in \mathcal{C}$ such that $x_0 \in C_0$. For $i > 0$ we can construct an increasing sequence $\left(t_i^j\right)_{j<i} \subseteq T$ such that $[T(t_i^j)] \cap C_j = \emptyset$ for all $j < i$. Then $[T(t_i^{i-1})] \cap C_j = \emptyset$ for every $j < i$. So any branch $x_i \in [T(t_i^{i-1})]$ is $\mathcal{C}$-different from all previous branches, let $x_i \in C_i \in \mathcal{C}$. Then $\{x_i |\, i < \omega\}$ is a set of $\mathcal{C}$-different branches and by construction for every $t \in T$ there is some $i < \omega$ such that $t \subseteq x_i$, so it is dense.

((3) ⇒ (4)) Trivial.

((4) ⇒ (2)) Let $t \in T$. Since T is perfect there is a splitting node $s \in T$ above t. Since T is perfect and $D \subseteq [T]$ is dense there is some $x_i \in D$ that extends $s^\frown i$. Then $x_i \in C_i$ for some $C_i \in \mathcal{C}$. As $x, y \in D$ are different they are $\mathcal{C}$-different. Thus for every $t \in T$ there are $\mathcal{C}$-different branches extending t, thus T is $\mathcal{C}$-branching.

((2) ⇒ (1)) For $t \in T$ there are $\mathcal{C}$-different branches extending t, they have to differ at some smallest point, this gives a splitting node of T above t, so T is perfect. Let $t \in T$ and $C \in \mathcal{C}$. There are two $\mathcal{C}$-different branches extending t, thus one of them has to be outside of C, call this one x. Then $x \in [p] \setminus C$ and C is closed, thus there has to be some $t \subseteq s \subseteq x$ such that $[T(s)] \cap C = \emptyset$. Thus $T \in \mathbb{P}(\mathcal{C})$.

□

An important property we used throughout was the ability to "fuse together" many different condition. In its most pure form this is achieved by σ-closed forcings, but as we have seen with Axiom A it is also possible with a much wider variety of forcings. While Axiom A holds for partition forcing there have to be made choices of branches in the definition of the orderings, so we will make those explicit in the fusion ordering. We also get another way to fuse together conditions.

Lemma 6.0.4. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $\{p_i |\ i \in \omega\} \subseteq \mathbb{P}(\mathcal{C})$ be a decreasing sequence of conditions with increasing stems $s_i = \mathrm{stem}(p_i)$. Then $p = \bigcup_{i<\omega} p_i(s_i^\frown(1 - s_{i+1}(|s_i|))) \in \mathbb{P}(\mathcal{C})$.*

Proof. If $t \in p$, then there is some $i < \omega$ such that $t \in p_i(s_i^\frown(1 - s_{i+1}(|s_i|)))$. Since this is a condition there are $\mathcal{C}$-different branches extending t in there. These branches are also branches of p, thus p is $\mathcal{C}$-branching and therefore $p \in \mathbb{P}(\mathcal{C})$. □

Definition 6.0.5. *For $T \in \mathbb{P}(\mathcal{C})$ we say $X \subseteq [T]$ is a witness for the n-th level of T, if for every node s in the n-th splitting level of T there is some $x \in X$ extending s and X contains $\mathcal{C}$-different branches, i.e. they are all in different elements of $\mathcal{C}$.*

Let $n \in \omega$ $p, q \in \mathbb{P}(\mathcal{C})$ and X, Y witnesses for the $n+1$-th level of p, n-th level of q, respectively. Define $(p, X) \leq_n (q, Y)$ if $p \leq q$ and $X \supseteq Y$.

Lemma 6.0.6. *If $(p_{n+1}, X_{n+1}) \leq_n (p_n, X_n)$ for all n, then $p = \bigcap p_n \in \mathbb{P}(\mathcal{C})$. We say that $((p_n, X_n))_{n<\omega}$ is a fusion sequence with witnesses.*

Proof. Let $X = \bigcup_{n<\omega} X_n$. Then $X \subseteq [p]$ contains $\mathcal{C}$-different branches. It is also dense, for every $t \in p$ there is a splitting node $s \supseteq t$ and all splitting nodes are already splitting nodes in some p_n, thus there is $x \in X_n \subseteq X$ extending $s \supseteq t$. Thus $p \in \mathbb{P}(\mathcal{C})$. □

Let $I_{\mathcal{C}}$ be the σ-ideal generated by the elements of $\mathcal{C}$. Then partition forcing is equivalent to $\mathcal{B} \setminus I_{\mathcal{C}}$. This result is due to Zapletal [40], the proof given here is new.

Theorem 6.0.7. *Let $A \subseteq 2^\omega$ be analytic. Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Then A contains all branches of a tree in $\mathbb{P}(\mathcal{C})$ if and only if $A \notin I_{\mathcal{C}}$.*

Proof. Since A is analytic there is a tree $T \subseteq 2^{<\omega} \times \omega^{<\omega}$ such that $A = p\,[[T]]$. For a tree $S \subseteq 2^{<\omega} \times \omega^{<\omega}$ define

$$S' = \left\{(s,u) \in S \middle| \, \exists x, y \in [S]\Big((s,u) \subseteq x, y \wedge p\,[x]\,, p\,[y] \text{ are } \mathcal{C}\text{-different}\Big)\right\}.$$

For convenience we say $x, y \in [S]$ are $\mathcal{C}$-different if their projection is, similarly for other notions related to $\mathcal{C}$. Recursively define $T_0 = T, T_{\alpha+1} = T'_\alpha$ and for limit ordinals λ, $T_\lambda = \bigcap_{\alpha<\lambda} T_\alpha$. Since T is countable there is some $\gamma < \omega_1$ such that $T_\gamma = T_{\gamma+1}$, let $T_\infty = T_\gamma$.

First consider the case that $T_\infty = \emptyset$. If $(s,u) \in T_\alpha \setminus T_{\alpha+1}$, then all branches in T_α extending (s,u) are $\mathcal{C}$-equal. Since $T_\alpha \setminus T_{\alpha+1}$ is countable, $[T_\alpha] \setminus [T_{\alpha+1}]$ is covered by countably many elements of $\mathcal{C}$. Finally since α is countable and $[T_\alpha] = \emptyset$ this means $[T]$ is covered by countably many elements of $\mathcal{C}$ and therefore $A \in I_\mathcal{C}$.

Now consider the case that $T_\infty \neq \emptyset$. We will recursively construct sets $S = \{s_\sigma | \sigma \in 2^{<\omega}\}$ and $\{u_\sigma | \, \sigma \in 2^\omega\}$ such that $(s_\sigma, u_\sigma) \in T_\infty$ for every $\sigma \in 2^{<\omega}$ and the downwards closure of S is a tree in $\mathbb{P}(\mathcal{C})$. Let $(s_0, u_0) \in T_\infty$ be arbitrary. Pick some minimal σ such that not both $(s_{\sigma^\frown i}, u_{\sigma^\frown i})$, $i < 2$, are already constructed. Then there are two $\mathcal{C}$-different branches $x_0, x_1 \in [T_\infty]$ extending (s_σ, u_σ). Choose $(s_{\sigma^\frown i^\frown 0^n}, u_{\sigma^\frown i^\frown 0^n})$ to be the branch x_i, for $i < 2$. If one of the branches is already defined, keep it as it was and let the other one be either x_0 or x_1 depending on which of those two is $\mathcal{C}$-different to the already existing one. This completes the construction. As in every step we choose two $\mathcal{C}$-different branches to extend s_σ we have that S is $\mathcal{C}$-branching and therefore in $\mathbb{P}(\mathcal{C})$. Since $(s_\sigma, u_\sigma) \in T_\infty \subseteq T$ we have that $[S] \subseteq p\,[[T]] = A$.

It remains to show that there is no analytic $A \in I_\mathcal{C}$ that contains all branches of a tree in $\mathbb{P}(\mathcal{C})$. Take $T \in \mathbb{P}(\mathcal{C})$. Assume towards a contradiction that $\{C_i | \, i \in \omega\} \subseteq \mathcal{C}$ covers $[T]$. Recursively construct $s_n \in T$. Let $s_{-1} = \emptyset$. If we have defined s_n, by the definition of partition forcing, there is $s_{n+1} \in T$ extending s_n such that $[T(s_{n+1}] \cap C_{n+1} = \emptyset$. Then $s = s_0^\frown s_1^\frown \cdots \in [T]$, but can't be covered by any C_n. $\square$

6.1 Properties of partition forcing

Fusion arguments for partition forcing all go pretty similar. An easy example is the fact that partition forcing satisfies the Sacks property. This was proven by Spinas [38].

Definition 6.1.1. *A forcing notion $\mathbb{P}$ has the Sacks property, if for every name $\dot{f}$ and condition $p \in \mathbb{P}$, such that $p \Vdash$ "$\dot{f} \in \omega^\omega$", in the ground model there exists a tree $T \subseteq \omega^{<\omega}$ and a condition $q \leq p$, such that the n-th level of T has cardinality at most 2^n and $q \Vdash$ "$\dot{f} \in [T]$".*

Equivalently we can replace the restriction on the cardinality of the levels with any function that diverges to infinity. This property is interesting because it preserves unbounded sets from the ground model, as well as the smallness of many other cardinal characteristics. We will proof one such statement as an example.

Theorem 6.1.2. *Let $\mathbb{P}$ be a forcing notion with the Sacks property. Then every real x in a forcing extension via $\mathbb{P}$ is dominated by a ground model real.*

Proof. Let G be a $\mathbb{P}$-generic filter. Let $f \in V[G] \cap \omega^\omega$ and fix a name $\dot{f}$ for f. By the Sacks property there is a dense set of conditions q and ground model trees $T_q \subseteq \omega^{<\omega}$ such that the n-th level of T_q has cardinality at most 2^n and $q \Vdash$ "$\dot{f} \in [T_q]$". So there is one such condition $q \in G$ and therefore $f \in [T_q]$. Now T_q is in the ground model and every level is finite. Thus there exists a ground model function $g \in \omega^\omega$ such that $s < g$, for every $s \in T$. Then g dominates f. □

Theorem 6.1.3. *Let $\{\mathbb{P}_\alpha | \alpha < \lambda\}$ be a countable support iteration of proper forcings with the Sacks property. Then $\mathbb{P}_\lambda$ is proper and has the Sacks property.*

For a proof of this theorem see [36].

Now we come to the fusion argument. The first step is proving a lemma that will be used in the construction of the fusion sequence.

Lemma 6.1.4. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $\dot{f}$ be a name for a function in ω^ω, $p \in \mathbb{P}(\mathcal{C})$ a condition and $X \subseteq \omega^\omega$ a finite set of $\mathcal{C}$-different branches. Then there exists a condition $q \leq p$, a branch $x \in [q]$, which is $\mathcal{C}$-different from all elements of X, and a function $h \in \omega^\omega$ such that for every $m, n \in \omega$, if $x \restriction m \in \mathrm{split}_n(q)$ then $q(x \restriction m) \Vdash$ "$\dot{f} \restriction n = h \restriction n$".*

Notice that the conclusion stays true if we strengthen q.

Proof. We can ignore X without loss of generality since all C_α are nowhere dense in $[p]$. Recursively construct a decreasing sequence of conditions p_n and sequences $x_n \in \omega^{<\omega}$. Let $p_0 = p$ and $s_0 = stem(p_0)$. If we have defined p_n take $p_{n+1} \leq p_n(x_n^\frown 0)$ forcing $\dot{f}(n) = m_n$ and let $x_{n+1} = \mathrm{stem}(p_{n+1})$. Set $q = \bigcup_{n\in\omega} p_n(x_n^\frown 1)$, $h = \bigcup_{n\in\omega}(n, m_n)$ and $x = \bigcup_{n\in\omega} x_n$. This will satisfy the lemma. Let $m, n \in \omega$ such that $x \restriction m \in \mathrm{split}_n(q)$. Then $x \restriction m = x_n$ and therefore $q(x \restriction m) \leq \bigcup_{k>n} p_k(x_k^\frown 1) \leq p_l$, for every $l < n$. Since $p_l \Vdash$ "$\dot{f}(l) = h(l)$" we have $q(x \restriction m) \Vdash$ "$\dot{f} \restriction n = h \restriction n$".

□

Theorem 6.1.5. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Partition forcing $\mathbb{P}(\mathcal{C})$ has the Sacks property.*

Proof. Let $\dot{f}$ be a $\mathbb{P}(\mathcal{C})$-name and let $p \in \mathbb{P}(\mathcal{C})$ be such that $p \Vdash$ "$\dot{f} \in \omega^\omega$". Recursively define conditions $p_n \in \mathbb{P}(\mathcal{C})$, sets $X_n = \{x_\sigma | \sigma \in 2^n\} \subseteq [p_n]$, a tree $T = \{t_\sigma | \sigma \in 2^{<\omega}\}$, sets $S_n = \{s_\sigma | \sigma \in 2^n\}$ and a set $H = \{h_\sigma | \sigma \in 2^{<\omega}\} \subseteq \omega^\omega$ such that the following holds:

(1) $((p_n, X_n))_{n\in\omega}$ is a fusion sequence with witnesses,

(2) $\mathrm{split}_n(p_n) = S_n$,

(3) $s_\sigma \subseteq x_\sigma$,

(4) if $\sigma \in 2^n$, then $p_n(s_\sigma) \Vdash$ "$\dot{f} \restriction n = t_\sigma$" and

(5) h_σ, x_σ and $p_n(s_\sigma)$ satisfy the conclusion of the previous lemma.

For $n = 0$ use the lemma to get $x_\emptyset$ and p_0. Set $t_\emptyset = \emptyset$ and $s_\emptyset = \mathrm{stem}(p_0)$. Then all conditions are satisfied. Now assume we have already defined everything for n. Fix $\sigma \in 2^n$. For $i = x_\sigma(|s_\sigma|)$, set $x_{\sigma^\frown i} = x_\sigma$, $h_{\sigma^\frown i} = h_\sigma$. Let $s_{\sigma^\frown i} \in \mathrm{split}_{n+1}(p_n(s_\sigma))$ be the unique initial segment of x_σ Let $p_{\sigma^\frown i} = p_n(s_{\sigma^\frown i})$ and $t_{\sigma^\frown i} = h \restriction n+1$. For $1-i$ we use the lemma for $p_n(s_\sigma^\frown(1-i))$ to get $h_{\sigma^\frown(1-i)}$, $x_{\sigma^\frown(1-i)}$ and $p'_{\sigma^\frown(1-i)}$ and make sure that $h_{\sigma^\frown(1-i)}$ is $\mathcal{C}$-different from every h_τ constructed before. Let $s_{\sigma^\frown(1-i)} \in \mathrm{split}_{n+1}(p'_{\sigma^\frown(1-i)})$ be the unique initial segment of $x_{\sigma^\frown(1-i)}$. Let $p_{\sigma^\frown(1-i)} = p'_{\sigma^\frown(1-i)}(s_{\sigma^\frown(1-i)})$ and $t_{\sigma^\frown(1-i)} = h_{\sigma^\frown(1-i)} \restriction n+1$. Unfix σ. Let $p_{n+1} = \bigcup_{\sigma\in 2^{n+1}} p_\sigma$. Then all conditions are satisfied.

Since we have a fusion sequence with witnesses, $q = \bigcap_{n\in\omega} p_n$ is a condition. We show that $q \Vdash$ "$\dot{f} \in [T]$". As every level of T has cardinality at most 2^n, this will show that $\mathbb{P}(\mathcal{C})$ has the Sacks property. Let $q' \leq q$ and $n \in \omega$, we will show that there exists a condition $q'' \leq q'$ forcing that $\dot{f} \restriction n \in T$. By (2) we have $\mathrm{split}(q) = \bigcup_{n\in\omega} S_n$, so there is some $m > n$ and $s_\sigma \in S_m$ such that $s_\sigma \in q'$. Then $q'(s_\sigma) \leq q(s_\sigma) \leq p_n(s_\sigma)$ which forces $\dot{f} \restriction m = t_\sigma \in T$. Thus $q'' = q'(s_\sigma)$ is as desired. □

There is the trivial partition of 2^ω into singletons. It is easy to see that partition forcing for this partition is the same as Sacks forcing. Even more every partition that is analytic in the Vietoris topology gives rise to Sacks forcing, this was shown in [12]. Now the question becomes whether there exists a partition that gives a partition forcing that is different from Sacks forcing. The way we show this is by constructing for a given partition $\mathcal{C}$ another partition $\mathcal{C}'$ that is $\mathbb{P}(\mathcal{C})$-indestructible. Then $\mathbb{P}(\mathcal{C}')$ destroys $\mathcal{C}'$, so it has to be different from $\mathbb{P}(\mathcal{C})$. For Sacks forcing this result is due to Newelski [34], although the presentation here follows Hrušák [17].

Lemma 6.1.6. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $p \in \mathbb{P}(\mathcal{C})$ be a condition, τ a $\mathbb{P}(\mathcal{C})$-name for an element of 2^ω and $\{S_n | \, n \in \omega\}$ a set of trees on $2^{<\omega}$ such that $p \Vdash$ "$\tau \notin \bigcup_{n\in\omega}[S_n]$". Then there is a condition $p \leq q$, a branch $x \in [q]$ and sequences $s_m \in 2^{<\omega}$, for $m \in \omega$, such that $|s_m| \geq m$ and for all $m, n \in \omega$ if $x \restriction m \in \mathrm{split}_n(q)$, then $q(x \restriction m) \Vdash$ "$s_n \subseteq \tau \wedge s_m \notin S_m$"*

Proof. Let $p_0 = p$ and recursively construct a decreasing sequence of conditions $\{p_n | n \in \omega\}$ together with their stems $\{x_n | \, n \in \omega\}$. Choose $T_{n+1} \leq T_n(x_n^\frown 0)$ and $s_{n+1} \in 2^{<\omega}$ such that $T_{n+1} \Vdash$ "$s_{n+1} \subseteq \tau \wedge s_{n+1} \notin S_{n+1}$", this is possible since $T_n(x_n^\frown 0)$ forces that τ is not in $[S_{n+1}]$. Define $q = \bigcup_{n\in\omega} T_n(x_n^\frown 1)$ and $x = \bigcup_{n\in\omega} x_n$. If $x \restriction m \in \mathrm{split}_n(q)$ then $x \restriction m = x_n$. So $q(x \restriction m) = q(x_n) \leq T_n$ and the latter forces "$s_n \subseteq \tau \wedge s_n \notin S_n$", so $q(x \restriction m)$ does as well. □

Theorem 6.1.7. *Assume CH. Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. There is an uncountable partition of 2^ω into closed sets that is indestructible by $\mathbb{P}(\mathcal{C})$.*

Proof. Because of CH there is a set $\{(T_\alpha, \tau_\alpha) | \, \alpha < \omega_1\}$ such that $T_\alpha \in \mathbb{P}(\mathcal{C})$, τ_α is a $\mathbb{P}(\mathcal{C})$-name and if $T \Vdash$ "$\tau \in 2^\omega$" then there is $\alpha < \omega_1$ such that $T_\alpha \leq T$ and $T_\alpha \Vdash$ "$\tau = \tau_\alpha$". We will recursively construct nowhere dense trees $\{S_\alpha | \, \alpha < \omega_1\}$ such that for all $\alpha < \omega_1$

- $[S_\alpha] \cap \bigcup_{\beta<\alpha}[S_\beta] = \emptyset$ and
- $\exists T \leq T_\alpha \exists \beta \leq \alpha$ such that $T \Vdash$ "$\tau_\alpha \in [S_\beta]$".

This partition is $\mathbb{P}(\mathcal{C})$-indestructible as every new real is equal to one of the τ_α and they don't destroy the partition because of the second condition. For $\alpha = 0$ there is $T \leq T_0$ and a nowhere dense tree $S_0 \subseteq 2^{<\omega}$ such that $T \Vdash$ "$\tau_0 \in [S_0]$". From the Sacks property we can get a tree S_0, such that the n-th level has size at most n, such a tree has to be nowhere dense. Now assume we have completed the construction for all $\beta < \alpha$. If there is $T \leq T_\alpha$ and $\beta < \alpha$ such that $T \Vdash$ "$\tau_\alpha \in [S_\beta]$", we are done by choosing S_β to be an arbitrary nowhere dense tree that satisfies the first condition. Here we use that all our trees are nowhere dense, otherwise it could be the case that the constructed trees already cover all of 2^ω. If this is not the case we can find $p_{-1} \leq T_\alpha$ and a nowhere dense tree $S'_\alpha \subseteq 2^{<\omega}$ such that $p_{-1} \Vdash$ "$\tau_\alpha \notin \bigcup_{\beta<\alpha}[S_\beta] \wedge \tau_\alpha \in [S'_\alpha]$". Enumerate $\alpha = \{\alpha_n \mid n \in \omega\}$. Recursively construct conditions $p_n \in \mathbb{P}(\mathcal{C})$, branches $X_n = \{x_\sigma \mid \sigma \in 2^n\} \subseteq [p_n]$, sequences $S_n = \{s_\sigma \mid \sigma \in 2^n\} \subseteq 2^{<\omega}$ and sequences $\{t_\sigma \mid \sigma \in 2^n\} \subseteq p_n$ such that

(1) $\{(p_n, X_n) \mid n \in \omega\}$ is a fusion sequence with witnesses,

(2) $x_\sigma \supseteq t_\sigma$,

(3) $\mathrm{split}_n(p_n) = \{t_\sigma \mid \sigma \in 2^n\}$,

(4) if $\sigma \subseteq \sigma'$ then $t_\sigma \subseteq t_{\sigma'}$,

(5) $p_n(t_\sigma), x_\sigma$ and $\{s_{\sigma^\frown 0^i} \mid i \in \omega\}$ satisfy the conclusion of the lemma for the name τ_α and sequence $\left\{S_{\alpha_{|\sigma|+i}} \,\middle|\, i \in \omega\right\}$

The last condition in particular means $p_n(t_\sigma) \Vdash$ "$s_\sigma \subseteq \tau_\alpha \wedge s_\sigma \notin S_{\alpha_{|\sigma|}}$", which is the important part, the rest is just to continue the construction. For $n = 0$ use the lemma for p, τ_α and $\left\{S_{\alpha_{|\sigma|+i}} \,\middle|\, i \in \omega\right\}$, and assign everything correctly. Now assume we have constructed everything for $n \in \omega$. Fix $\sigma \in 2^n$. Without loss of generality $x_\sigma \supseteq t_\sigma^\frown 0$. Use the lemma for $p_n(t_\sigma^\frown 1), \tau_\alpha$ and $\left\{S_{\alpha_{|\sigma|+i+1}} \,\middle|\, i \in \omega\right\}$ to get $p_{\sigma^\frown 1}, x_{\sigma^\frown 1}$ and sequences s_m. Let $s_{\sigma^\frown 1^\frown 0^m} = s_m$ and $t_{\sigma^\frown 1} = \mathrm{stem}(p_{\sigma^\frown 1})$. Let $x_{\sigma^\frown 0} = x_\sigma$ and $t_{\sigma^\frown 0} = \mathrm{stem}(p_n(t_\sigma^\frown 0))$. This completes the construction.

Let $p = \bigcap_{n\in\omega} p_n$ be the fusion. Define $S_\alpha = \{t \in 2^{<\omega} \mid \exists \sigma \in 2^{<\omega}\, (t \subseteq s_\sigma)\}$. This is a nowhere dense tree as it is a subtree of S'_α. Also $p \Vdash$ "$\tau_\alpha \in [S_\alpha]$", as for every $n \in \omega$ and all $\sigma \in 2^n$, $p_n(t_\sigma) \Vdash$ "$\tau_\alpha \restriction n \subseteq s_\sigma$" and therefore $p_n \Vdash$ "$\tau \restriction n \in S_\alpha$". For $\beta < \alpha$ there is some $n \in \omega$ such that $\beta = \alpha_n$. Then $s_\sigma \notin S_\beta$, for all $\sigma \in 2^{<\omega} \setminus 2^{\leq n}$ and therefore $[S_\beta] \cap [S_\alpha] = \emptyset$. □

6.2 Partiton forcing and MAD families

The first thing we want to do is show that partition forcing has weak fusion.

Lemma 6.2.1. *Let $\mathcal{C} = \{C_\alpha |\ \alpha < \omega_1\}$ be an uncountable partition of 2^ω into closed sets. Let $T \in \mathbb{P}(\mathcal{C})$, $s \in T$, X a finite set of $\mathcal{C}$-different branches and $\dot{C}$ a $\mathbb{P}(\mathcal{C})$-name for an infinite subset of ω. Then there exists $T' \leq T$ and a branch $x \in [T']$ extending s and $\mathcal{C}$-different from all elements of X, such that*

$$\forall n \forall T'' \leq T' \text{ with } x \in [T''] \exists S \leq T'' \exists k \geq n \text{ such that } x \in [S] \wedge S \Vdash \text{“}k \in \dot{C}\text{”}.$$

Notice that the conclusion still holds for any strengthening of T'.

Proof. Since $C_\alpha \cap T(s)$ is nowhere dense in $T(s)$ for all α and X is finite we can assume that X and s is empty, by strengthening T if necessary.

Recursively define $m_n \in \omega, x_n \in \omega^{<\omega}$ and conditions $T_n \leq T_{n-1}$. Let $T_0 = T$ and $x_0 = \text{stem}(T_0)$. Assume we have already defined T_n and x_n. Then let $T_{n+1} \leq T_n(x_n^\frown 0)$ such that $T_{n+1} \Vdash$ “$m_{n+1} \in \dot{C}$”, where $m_{n+1} \geq n+1$. Set $x_{n+1} = \text{stem}(T_{n+1})$. Note that $x_{n+1}(|x_n|) = 0$.

Then $T' = \bigcup_{n<\omega} T_n(x_n^\frown 1) \in \mathbb{P}(\mathcal{C})$ and $x = \bigcup_{n<\omega} x_n$ is as desired. To see this let $n \in \omega$ and $T'' \leq T'$ with $x \in [T'']$. Then there is some $m > n$ such that x_m is a splitting node of T''. Let $S = T''(x_m)$. Then

$$S = T''(x_m) \leq T'(x_m) \leq \bigcup_{k \geq m} T_k(x_k^\frown 1) \leq T_m.$$

Therefore $S \Vdash$ “$m_m \in \dot{C}$”, since T_m already does and clearly $x \in [S]$.

□

Theorem 6.2.2. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Partition forcing $\mathbb{P}(\mathcal{C})$ has weak fusion.*

Proof. We will recursively construct the following:

- finite antichains $B_n = \{s_\sigma |\ \sigma \in 2^n\} \subseteq 2^{<\omega}$
- finite antichains $\mathcal{A}_n \subseteq \mathbb{P}(\mathcal{C})$
- $T_n \in \mathbb{P}(\mathcal{C})$
- $X_n = \{x_\sigma |\ \sigma \in 2^n\} \subseteq 2^\omega$ a witness for the n-th level of T_n
- bijections $h_n \colon B_n \to \mathcal{A}_n$ and
- a one-to-one function $g \colon \bigcup_{n<\omega} \{n\} \times \mathcal{A}_n \to \omega$

such that

(a) $\sigma \subseteq \tau \Rightarrow s_\sigma \subseteq s_\tau$,

(b) $(T_{n+1}, X_{n+1}) \leq_n (T_n, X_n) \leq_{n-1} \cdots \leq_0 (T_0, X_0) \leq T$,

(c) $T_n \leq \bigcup_{\sigma \in 2^n} [s_\sigma]$ and if $\sigma \in 2^n$, then $s_\sigma \in T_n$,

(d) $h_n(s_\sigma) \geq T_n \cap [s_\sigma]$,

(e) $\forall A \in \mathcal{A}_n$ we have that $A \Vdash$ "$g(n, A) \in \dot{C}$" and $g(n, A) \geq n$,

(f) $x_\sigma \supseteq s_\sigma$ and

(g) $h_n(s_\sigma)$ satisfies the conclusion of lemma 6.2.1 with $x = x_\sigma$.

For $n = 0$, let $T_0 \leq T$ and $x_\emptyset$ satisfy the lemma and forcing some $k \in \omega$ to be in $\dot{C}$. Let $s_\emptyset = \mathrm{stem}(T_0)$, $\mathcal{A}_0 = \{T_0\}$ and $g(0, T_0) = k$. All the conditions hold.

Assume we defined this for $n - 1$. From (c) we get that without loss of generality $\mathrm{split}_{n-1}(T_{n-1}) = \{s_\sigma \mid \sigma \in 2^{n-1}\}$.

Fix $\sigma \in 2^{n-1}$. Let $i = x_\sigma(|s_\sigma|)$. Then $T_{n-1}(s_\sigma \frown i) \leq h_{n-1}(s_\sigma)$ and $x_\sigma \in [T_{n-1}(s_\sigma \frown i)]$, so we can use that the conclusion of the lemma holds to get a condition $T_{\sigma\frown i} \leq T_{n-1}(s_{\sigma\frown i})$ and a natural number $k_{\sigma\frown i} \geq n$ such that $x_\sigma \in [T_{\sigma\frown i}]$ and $T_{\sigma\frown i} \Vdash$ "$k_{\sigma\frown i} \in \dot{C}$". Let $s_{\sigma\frown i} = \mathrm{stem}(T_{\sigma\frown i})$, $x_{\sigma\frown i} = x_\sigma$ and $h_n(\sigma\frown i) = T_{\sigma\frown i}$. The conclusion of the lemma still holds by the remark after the lemma. Use the lemma to get $h_n(\sigma\frown(1-i)) = T_{\sigma\frown(1-i)} \leq T_{n-1}(s_\sigma \frown (1-i))$ and $x_{\sigma\frown(1-i)}$, with X containing X_{n-1} and the previously constructed $x_{\tau\frown j}$ for $\tau \in 2^{n-1}$ and $j < 2$. We can strengthen $T_{\sigma\frown(1-i)}$ to get that it forces "$k_{\sigma\frown(1-i)} \in \dot{C}$" for some $k_{\sigma\frown(1-i)} \geq n$.

Unfix σ. Let $T_n = \bigcup_{\sigma \in 2^n} T_\sigma$, $\mathcal{A}_n = \{T_\sigma \mid \sigma \in 2^n\}$, $h_n(s_\sigma) = T_\sigma$ and $g(n, T_\sigma) = k_\sigma$. The properties hold.

Let $T_\infty = \cap_{n<\omega} T_n$. By (c) we have $[T_\infty] = G_B$, where $B = \bigcup_{n<\omega} B_n$. Thus (1) in the definition of weak fusion holds. Condition (e) is the same as (3). And finally (2') follows from (d) and $G_B \leq [T_n]$. □

This then immediately gives us the description of indestructibility from 5.2.2.

Theorem 6.2.3. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $\mathcal{I}$ be a tall ideal. The following are equivalent:*

(1) $\mathcal{I}$ is $\mathbb{P}(\mathcal{C})$-indestructible.

(2) $\forall B \subseteq 2^{<\omega}$ with $G_B \notin I_\mathcal{C} \forall f\colon B \to \omega \exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin I_\mathcal{C}$.

(3) $\forall B \subseteq 2^{<\omega}$ with $G_B \notin I_\mathcal{C} \forall f\colon B \to \omega$ injective $\exists I \in \mathcal{I}$ such that $G_{f^{-1}[I]} \notin I_\mathcal{C}$.

(4) $\mathcal{I}$ is $\mathbb{P}$-indestructible for some forcing $\mathbb{P}$ with the property that for every $G_B \notin I_\mathcal{C}$ there is a condition that forces the addition of an $I_\mathcal{C}$-quasi-generic real to G_B, a real is called $I_\mathcal{C}$-quasi-generic if it avoids every set of $I_\mathcal{C}$ coded in the ground model.

The notion of quasi-generic reals was introduced in [7]. Condition (4) is quite similar to the last condition in the characterizations of Sacks, Miller and Laver-indestructibility. For Sacks forcing every new real is quasi-generic, for Miller forcing the quasi-generic reals are the unbounded reals and for Laver forcing the quasi-generic reals are the strongly dominating reals. We even have this characterization for Random and Cohen forcing, it is just that quasi-generic reals for these are exactly Random or Cohen reals, so the characterization is trivial and therefore not explicitly stated.

Proof. The equivalence of (1)-(3) follows directly from 5.2.2. The implication (1) implies (4) is clear as partition forcing is a forcing as required in (4). Assume that $\mathcal{I}$ is $\mathbb{P}$-indestructible for $\mathbb{P}$ a forcing as in (4). We will show (3). Take $B \subseteq 2^{<\omega}$ with $G_B \notin I_{\mathcal{C}}$ and $f\colon B \to \omega$ injective. Then there is $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash$ "$\dot{x} \in G_B$ is $I_{\mathcal{C}}$-quasi-generic". Let $\dot{A}$ be the $\mathbb{P}$-name for the set $\{f(\dot{x} \restriction n)|\ n \in \omega, \dot{x} \restriction n \in B\}$. Let G be $\mathbb{P}$-generic with $p \in G$. As $\mathbb{P}$ does not destroy $\mathcal{I}$ there is $I \in \mathcal{I}$ such that $|\dot{A}^G \cap I| = \omega$. Thus $\dot{x}^G \in G_{f^{-1}[I]}$. This implies that $G_{f^{-1}[I]} \notin I_{\mathcal{C}}$, because $\dot{x}^G$ is $I_{\mathcal{C}}$-quasi-generic and therefore avoids all sets in $I_{\mathcal{C}}$. □

Similarly to the way we did it for Cohen forcing we can also construct a maximal family of eventually different reals that is partition forcing indestructible.

Lemma 6.2.4. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $T \in \mathbb{P}(\mathcal{C})$ and let $\dot{f}$ be a name for an infinite partial function from ω to ω. Then there is a set $B \subseteq T$ and an injective function $g\colon B \to \omega \times \omega$ such that*

- $G_B \notin I_{\mathcal{C}}$,
- $\forall t \in B \left(G_B \cap [t] \Vdash \text{“}g(t) \in \dot{f}\text{”}\right)$ *and*
- $g[B]$ *is a partial function.*

Proof. Redo the proof of weak fusion with $\dot{C}$ being the domain of $\dot{f}$ and whenever we choose a condition that forces some m to be in $\dot{C}$ make sure it also forces a single value for $\dot{f}(m)$. □

Theorem 6.2.5. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $\mathcal{A}$ be a maximal family of eventually different functions. The following are equivalent:*

(1) $\mathcal{A}$ is $\mathbb{P}(\mathcal{C})$-indestructible.

(2) $\forall B \subseteq 2^{<\omega}$ with $G_B \notin I_{\mathcal{C}} \forall f\colon B \to \omega \times \omega$ with image a partial function $\exists x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin I_{\mathcal{C}}$.

(3) $\forall B \subseteq 2^{<\omega}$ with $G_B \notin I_{\mathcal{C}} \forall f\colon B \to \omega \times \omega$ injective with image a partial function $\exists x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin I_{\mathcal{C}}$.

Proof. Let us start by proving (1) implies (2). Take $B \subseteq 2^{<\omega}$ with $G_B \notin I_{\mathcal{C}}$ and $f\colon B \to \omega \times \omega$ with image a partial function. Let G be $\mathbb{P}(\mathcal{C})$-generic with $G_B \in G$. Let x be the partition generic real. Then $x \in G_B$. Consider the partial function $y = f[x]$. Because $\mathcal{A}$ is $\mathbb{P}(\mathcal{C})$-indestructible there is some $z \in \mathcal{A}$ such that $y \cap z$ is infinite. Then $x \cap f^{-1}[z]$ is infinite, so $x \in G_{f^{-1}[z]}$. This implies $G_{f^{-1}[z]} \notin I_{\mathcal{C}}$.

The implication (2) to (3) is trivial.

Assume (3) holds. Take a condition $T \in \mathbb{P}(\mathcal{C})$ and a $\mathbb{P}(\mathcal{C})$-name $\dot{f}$ for a partial function. Let $B \subseteq T$ and $g\colon B \to \omega \times \omega$ be as in the lemma above. Then there is $x \in \mathcal{A}$ such that $G_{f^{-1}[x]} \notin I_{\mathcal{C}}$. Now for every $n \in \omega$ and $S \leq G_{f^{-1}[x]}$ we want to find $S' \leq S$ that forces some (m, k) into $\dot{f}$, where $m > n$. We can find $t \in S$ such that $g(t) = (m, k)$ where

$m > n$. Then $S' = S \cap [t]$ is as required. This means $G_{f^{-1}[x]}$ forces $x \cap \dot{f}$ to be infinite, which in turn means that $\dot{f}$ does not destroy $\mathcal{A}$. □

Theorem 6.2.6. *Assume CH. Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. There is a maximal family of eventually different functions that is $\mathbb{P}(\mathcal{C})$-indestructible.*

Proof. We proceed as usual for these arguments. Let $\{f_\alpha \colon B_\alpha \to \omega \times \omega | \ \alpha < \mathfrak{c}\}$ enumerate all injective function whose image is a partial function and $G_B \notin I_\mathcal{C}$. Recursively construct a maximal family of eventually different functions $\{x_\alpha | \ \alpha < \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$, if $G_{f_\alpha^{-1}[x_\beta]} \in I_\mathcal{C}$, then $G_{f_\alpha^{-1}[x_\alpha]} \notin I_\mathcal{C}$. Assume we already constructed this for all $\beta < \alpha$. If there is some $\beta < \alpha$ such that $G_{f_\alpha^{-1}[x_\beta]} \notin I_\mathcal{C}$ we can take x_α to be any partial function that is eventually different from all x_β, which exist because there are only countably many of them. Now consider the case that $G_{f_\alpha^{-1}[x_\beta]} \in I_\mathcal{C}$ for all $\beta < \alpha$. We can enumerate $\alpha = \{\beta_n | \ n \in \omega\}$. Let $T \in \mathbb{P}(\mathcal{C})$ be a tree such that $[T] \subseteq G_B$. Without loss of generality we can assume $B \subseteq T$. We will recursively construct $A = \{s_\sigma | \ \sigma \in 2^{<\omega}\} \subseteq B$, $\{C_\sigma | \ \sigma \in 2^{<\omega}\} \subseteq \mathcal{C}$ and $\{x_\sigma | \ \sigma \in 2^{<\omega}\}$. Assume we did the construction for all predecessors of σ and let τ be σ with the last entry removed. Then there is $t_0 \in B$ extending s_τ such that $[t_0] \cap C_\rho \cap [T] = \emptyset$ for every already constructed C_ρ. We have $G_{\bigcup_{m<n} f_\alpha^{-1}[x_{\beta_m}]} \in I_\mathcal{C}$. Thus there is $t_{n+1} \supseteq t_n$ with $s_{n+1} \in B$ and $t_{n+1} \notin \bigcup_{m<n+|\sigma|} f_\alpha^{-1}[x_{\beta_m}]$. Let $s_{\sigma^\frown 0^n} = t_n$ and choose $C_\sigma \in \mathcal{C}$ such that $x_\sigma = \bigcup_{n\in\omega} t_n \in C_\sigma$. So we have constructed a set of $\mathcal{C}$-different branches that is dense in G_A, so $G_A \notin I_\mathcal{C}$. We also made sure that $A \cap f_\alpha^{-1}[x_\beta]$ is finite for every $\beta < \alpha$. Thus $x_\alpha = f[A]$ continues the construction. □

6.3 Iterating Partition forcing

In this section we describe how partition forcing behaves under iterations. At first we show that partition forcing strongly preserves tight MAD families.

Theorem 6.3.1. *Let $\mathcal{C}$ be an uncountable partition of 2^ω into closed sets. Let $\mathcal{A}$ be a tight mad family. Then $\mathbb{P}(\mathcal{C})$ strongly preserves the tightness of $\mathcal{A}$.*

Proof. Let $T \in \mathbb{P}(\mathcal{C})$ and let M be a countable elementary submodel of $H(\kappa)$ such that $\mathbb{P}(\mathcal{C}), \mathcal{A}, T \in M$ and $B \in \mathcal{I}(\mathcal{A})$ for which $|B \cap Y| = \omega$, for every $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$. Let $\{D_n | \ n \in \omega\}$ be an enumeration of all open dense subsets of $\mathbb{P}(\mathcal{C})$ in M and $\left\{\dot{Z}_n \middle| \ n \in \omega\right\}$ be an enumeration of all $\mathbb{P}(\mathcal{C})$-names for elements of $\mathcal{I}(\mathcal{A})^+$ in M, such that each name appears infinitely often. We will recursively construct:

- conditions $T_n \in \mathbb{P}(\mathcal{C})$ and
- sets $X_n = \{x_\sigma | \ \sigma \in 2^n\}$

such that the following holds for every n:

(1) $((T_n, X_n)_{n\in\omega}$ is a fusion sequence with witnesses,

(2) $T_n \in M$,

(3) $T_n \Vdash$ "$\left(\dot{Z}_n \cap B\right) \setminus n \neq \emptyset$",

(4) $\forall s \in \text{split}_n(T_n)$ we have that $T_n(s) \in D_n$ and

(5) for every $k, m \in \omega$, if $x_\sigma \restriction k \in \text{split}_m(T_n)$, then $T_n(x_\sigma \restriction k) \Vdash$ "$\left(\dot{Z}_m \cap B\right) \setminus m \neq \emptyset$" and $T_n(x_\sigma \restriction k) \in D_m$.

For $n = 0$ set $T_{-1} = T$ and $X_{-1} = \emptyset$ and proceed as in the case $n > 0$. Assume we have already defined this for n. Fix $\sigma \in 2^n$ and pick $s_\sigma \in \text{split}_n(T_n)$ such that $s_\sigma \subseteq x_\sigma$. Let $i = x_\sigma(|s_\sigma|)$. Then there is some $k \in \omega$ such that $x_\sigma \restriction k \in \text{split}_{n+1}(T_n)$. Let $T_{\sigma^\frown i} = T_n(x_\sigma \restriction k)$ and $x_{\sigma^\frown i} = x_\sigma$. This is in $D_{n+1} \cap M$ by condition (5). For $1 - i$ we will recursively construct $\{s_m |\ m \in \omega\}$ and T_n^m such that:

- $m \leq m' \Rightarrow T_n^m \geq T_n^{m'}$,
- $m < m' \Rightarrow s_m \subsetneq s'_m$,
- $s_m = \text{stem}(T_n^m)$,
- $T_n^m \in D_m \cap M$ and
- $T_n^m \Vdash$ "$\left(B \cap \dot{Z}_m\right) \setminus m \neq \emptyset$".

For $m = 0$ let $s_{-1} \in T_n$ extend $s_\sigma^\frown(1-i)$ such that all branches in $[T_n(s_{-1})]$ are $\mathcal{C}$-different from the ones in X_n and the previously constructed ones at step $n+1$, this is possible since there are only finitely many branches it has to be $\mathcal{C}$-different from and the definition of being a condition. Then let $T_n^{-1} = T_n(s_{-1})$ and continue as in the $n > 0$ case. Assume we already constructed s_m and T_n^m. Then $C = \left\{k \in \omega \middle| \exists S \leq T_n^m \left(S \Vdash \text{“}k \in \dot{Z}_{m+1}\text{”}\right)\right\} \in \mathcal{I}(\mathcal{A})^+ \cap M$. So there is $j \in C \cap B$ such that $j > m+1$. Then we can find $T_n^{m+1} \leq T_n^m$ such that $T_n^{m+1} \in D_{m+1} \cap M$ and $T_n^{m+1} \Vdash$ "$j \in \dot{Z}_{m+1}$". Let $s_{m+1} = \text{stem}(T_n^{m+1})$, we can easily guarantee that this is a proper extension of s_m. Then all conditions are satisfied. Let $x_{\sigma^\frown(1-i)} = \bigcup_{m\in\omega} s_m$ and $T_{\sigma^\frown(1-i)} = \bigcup_{m>n} T_n^m\big(x_{\sigma^\frown(1-i)} \restriction (|s_m|+1)\big)$, lemma 6.0.4 gives that this is a condition. Unfix σ. Let $T_{n+1} = \bigcup_{\sigma\in 2^{n+1}} T_\sigma$. This extends the fusion sequence. That it is in M follows as all T_σ are in M and 2^{n+1} is finite and therefore also in M. We have, for every $s \in \text{split}_{n+1}(T_{n+1})$, some $\sigma \in 2^{n+1}$ such that $T_{n+1}(s) = T_\sigma \in D_n$ and this also forces "$\left(\dot{Z}_n \cap B\right) \setminus (n+1) \neq \emptyset$". So conditions (3) and (4) are satisfied. To see that condition (5) is satisfied we distinguish the two cases in the construction. In the first one we just remove k such that $x_\sigma \restriction k \in \text{split}_m(T_{n+1})$, thus the condition stays satisfied. In the other one we have that if $x_\sigma \restriction k \in \text{split}_m(T_{n+1})$, then $T_{n+1}(x_\sigma \restriction k) = \bigcup_{k>m} T_n^k(x_{\sigma^\frown(1-i)} \restriction (|s_m|+1))$ and by construction this is as required. This completes the construction.

Let $T_\infty = \bigcap_{n\in\omega} T_n$. This condition is $(M, \mathbb{P}(\mathcal{C}))$-generic as for every dense $D \in M$ there is some n such that $D = D_n$, then $T_\infty \leq T_n \Vdash$ "$\exists s \in \text{split}_n(T_n)\big(T_n(s) \in$

$M \cap D_n \cap \dot{G})$". It is also an $(M, \mathbb{P}(\mathcal{C}), \mathcal{A}, B)$-generic condition as for every element $\dot{Z} \in \mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]$ there are infinitely many n such that $\dot{Z}_n = \dot{Z}$ and then $T_n \Vdash$ "$\left(\dot{Z}_n \cap B\right) \setminus n \neq \emptyset$", so in total T_∞ forces that this intersection is infinite. □

The goal of iterating partition forcing is to destroy every partition of 2^ω in the ground model and every intermediate extension. If we are able to do this we would have that no partition of size $\aleph_1$ exists in the final model, as every such partition would already be in an intermediate extension and is therefore destroyed at some point. To do things like this there is the method of book keeping.

We start with a model of GCH and fix a function $\pi\colon \omega_2 \to \omega_2 \times \omega_2$ such that:

- if $\pi(\alpha) = (\beta, \gamma)$, then $\beta \leq \alpha$ and
- for all (β, γ) there exist arbitrarily large $\alpha < \omega_2$ such that $\pi(\alpha) = (\beta, \gamma)$.

We will recursively define a countable support iteration $\left\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \,\middle|\, \alpha \leq \omega_2\right\}$ of proper forcings of size $\aleph_1$. Assume we have already defined $\mathbb{P}_\alpha$. Fix a list of $\mathbb{P}_\alpha$-names $\left\{\dot{\mathcal{C}}^\gamma_\alpha \,\middle|\, \gamma < \omega_2\right\}$ that enumerate all partitions of 2^ω into closed sets in $V[G_\alpha]$. Let $\pi(\alpha) = (\beta, \gamma)$. Then $\beta \leq \alpha$, so $\dot{\mathcal{C}}^\gamma_\beta$ is defined, although it might no longer be a partition of 2^ω in $V[G_\alpha]$. If it still is we let $\dot{\mathbb{Q}}_\alpha = \mathbb{P}(\mathcal{C}^\gamma_\beta)$, otherwise we let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing. This completes the construction. Now let $\mathcal{C}$ be a partition of size $\aleph_1$ of 2^ω into closed sets. Then there is some $\beta < \omega_2$ such that $\mathcal{C} \in V[G_\beta]$. So there is some $\gamma < \omega_2$ such that $\mathcal{C}^\gamma_\beta = \mathcal{C}$. By the choice of π there is some $\alpha < \omega_2$ with $\pi(\alpha) = (\beta, \gamma)$. Then either $\mathcal{C}$ is not a partition of 2^ω in $V[G_\alpha]$, or $\mathbb{Q}_\alpha = \mathbb{P}(\mathcal{C})$. Both cases imply that $\mathcal{C}$ is not a partition of 2^ω in $V[G]$. With iteration of partition forcing we always mean this particular iteration, and the partition forcing model is the model generated by this forcing. So we have the following.

Theorem 6.3.2. *In the partition model there is a MAD family of size $\aleph_1$, the smallest partition of 2^ω into closed sets has size $\aleph_2$ and $\mathfrak{d} = \aleph_1$.*

Proof. Since in the ground model $\mathfrak{b} = \mathfrak{c} = \aleph_1$ there is a tight MAD family of size $\aleph_1$. All partition forcings strongly preserve tight MAD families and this property is preserved under countable support iterations. Thus this tight MAD family is still a tight MAD family in the partition model and it also still has size $\aleph_1$. By the same reasoning the iteration of partition forcing has the Sacks property, so every real in the partition model is dominated by a ground model real. Thus $\mathfrak{d} = \aleph_1$ in the partition model. The text before the theorem shows that there is no partition of 2^ω into closed sets of size less than $\aleph_2$ in the partition model. □

7 Open questions

The big open question is whether there exist indestructible MAD families in ZFC alone. Brendle and Yatabe [9] conjecture that they do, at least for Sacks forcing, which as we have seen is the weakest one.

Then of course we can ask where $\mathbb{P}$-indestructibility fits into the hierarchy for various other forcing notions. In particular partition forcing that we studied in the last chapter. Since it preserves tight MAD families and those are quite similar to Cohen indestructible MAD families, we conjecture that Cohen indestructibility implies partition forcing indestructibility. On the other hand we constructed partition forcings that are different from Sacks forcing, the question is are there MAD families that are destructible for these forcings, while being Sacks indestructible.

The next thing one could try is seeing whether the notion of weak fusion and these indestructibility results can be generalized to higher cardinals. This is done a bit in [5] by Baumhauer for higher random forcing.

And finally is it possible to adapt the methods of this thesis to other combinatorial objects, for example independent families or ultrafilter? For both of these examples an analog of 5.1.3 holds, but we do not know whether even just the non weak fusion direction of 5.2.2 holds.

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